Further, since $a+c=c+a$ and $b+d=d+b$, it follows that

$$\tau_{O,T}\tau_{O,S} = \tau_{O,S}\tau_{O,T}.$$

So the translations form a commutative group (of transformations). $\square$

2.1.9 PROPOSITION. *The dilatations form a group $\mathfrak{D}$, called the **dilatation group**.*

PROOF : Dilatations are collineations.

By the symmetry of parallelness for lines (i.e., $\mathcal{L} \parallel \mathcal{L}' \Rightarrow \mathcal{L}' \parallel \mathcal{L}$), the inverse of a dilatation is a dilatation.

By the transitivity of parallelness for lines (i.e., $\mathcal{L} \parallel \mathcal{L}'$ and $\mathcal{L}' \parallel \mathcal{L}'' \Rightarrow \mathcal{L} \parallel \mathcal{L}''$), the product of two dilatations is a dilatation.

So the dilatations form a group (of transformations). $\square$

2.2 Halfturns

A *halfturn* turns out to be an involutory rotation; that is, a rotation of $180°$. So, a halfturn is just a special case of a rotation. Although we have not formally introduced rotations yet, we look at this special case now because halfturns are nicely related to translations and have such easy equations. Informally, we observe that if point A is rotated $180°$ about point P to point A', then P is the midpoint of A and A'. Hence, we need only the midpoint formulas to obtain the desired equations. From equations

$$\begin{cases} \dfrac{x+x'}{2} = a \\[2ex] \dfrac{y+y'}{2} = b \end{cases}$$

we can make our definition as follows.

2.2.1 DEFINITION. If $P=(a,b)$, then the **halfturn σ_P about point P** is the mapping

$$\sigma_P : \mathbb{E}^2 \to \mathbb{E}^2, \quad (x,y) \mapsto (-x+2a, -y+2b).$$

Such a halfturn σ_P has equations

$$\begin{cases} x' = -x + 2a \\ \\ y' = -y + 2b. \end{cases}$$

Note : For the halfturn about the origin we have

$$\sigma_O((x, y)) = (-x, -y).$$

Under transformation σ_O does (x, y) go to $(-x, -y)$ by going directly through O, by rotating counterclockwise about O, by rotating clockwise about O, or by taking some "fanciful path" ? Either the answer is "None of the above" or, perhaps, it would be better to ask whether the question makes sense. Recall that transformations are just one-to-one correspondences among points. There is actually no physical motion being described. (That is done in the study called *differential geometry*.) We might say *we are describing the end position of physical motion*. Since our thinking is often aided by language indicating physical motion, we continue such usage as the customary "P goes to Q" in place of the more formal "P coresponds to Q".

What properties of a halfturn follow immediately from the definition of σ_P ? First, for any point A, the midpoint of A and $\sigma_P(A)$ is P. From this simple fact alone, it follows that *σ_P is an involutory transformation*. Also from this simple fact, it follows that σ_P fixes exactly the one point P. It even follows that *σ_P fixes line $\mathcal{L}$ if and only if P is on $\mathcal{L}$*.

2.2.2 Proposition. *A halfturn is an involutory dilatation. The midpoint of points A and $\sigma_P(A)$ is P. Halfturn σ_P fixes point A if and only if $A = P$. Halfturn σ_P fixes line $\mathcal{L}$ if and only if P is on $\mathcal{L}$.*

Proof : We shall show that σ_P is a collineation.

Suppose that line $\mathcal{L}$ has equation $ax + by + c = 0$. Let $P = (h, k)$. Then σ_P has equations

$$\begin{cases} x' = -x + 2h \\ \\ y' = -y + 2k. \end{cases}$$

Then

$$ax + by + c = 0 \iff ax' + by' + c - 2(ah + bk + c) = 0.$$

So $\sigma_P(\mathcal{L})$ is the line $\mathcal{M}$ with equation

$$ax + by + c - 2(ah + bk + c) = 0.$$

Therefore, not only σ_P is a collineation, but a dilatation as $\mathcal{L} \parallel \mathcal{M}$.

Finally, $\mathcal{L}$ and $\mathcal{M}$ are the same if and only if $ah + bk + c = 0$, which holds if and only if (h, k) is on $\mathcal{L}$. $\square$

Since a halfturn is an involution, then $\sigma_P\sigma_P = \iota$. What can be said about the product of two halfturns in general ?

Let $P = (a, b)$ and $Q = (c, d)$. Then

$$\begin{aligned} \sigma_Q\sigma_P((x, y)) &= \sigma_Q((-x + 2a, -y + 2b)) \\ &= (-(-x + 2a) + 2c, -(-y + 2b) + 2d) \\ &= (x + 2(c - a), y + 2(d - b)). \end{aligned}$$

Since $\sigma_Q\sigma_P$ has equations

$$\begin{cases} x' = x + 2(c - a) \\ \\ y' = y + 2(d - b) \end{cases}$$

then $\sigma_Q\sigma_P$ is a translation. This proves the important result that *the product of two halfturns is a translation.*

2.2.3 PROPOSITION. *If Q is the midpoint of points P and R, then*

$$\sigma_Q\sigma_P = \tau_{P,R} = \sigma_R\sigma_Q.$$

PROOF : We have

$$\sigma_Q\sigma_P(P) = \sigma_Q(P) = R \quad \text{and} \quad \sigma_R\sigma_Q(P) = \sigma_R(R) = R.$$

Since there is a unique translation taking P to R, then each of $\sigma_Q\sigma_P$ and $\sigma_P\sigma_R$ must be $\tau_{P,R}$. $\square$

NOTE : A product of two halfturns is a translation and, conversely, a translation is a product of two halfturns. Also, notice that $\sigma_Q\sigma_P$ moves each point *twice* the directed distance from P to Q.

We now consider a product of three halfturns. By thinking about the equations, it should almost be obvious that $\sigma_R\sigma_Q\sigma_P$ is itself a halfturn. We shall prove that and a little more.

2.2.4 PROPOSITION. *A product of three halfturns is a halfturn. In particular, if points P, Q, R are not collinear, then $\sigma_R\sigma_Q\sigma_P = \sigma_S$ where $\square PQRS$ is a parallelogram.*

PROOF : Suppose $P = (a, b)$, $Q = (c, d)$, and $R = (e, f)$. Let $S = (a - c + e, b - d + f)$. In case P, Q, R are not collinear, then $\square PQRS$ is a parallelogram. (This is easy to check as opposite sides of the quadrilateral are congruent and parallel.) We calculated $\sigma_Q\sigma_P((x, y))$ above. Whether P, Q, R are collinear or not, we obtain

$$\begin{aligned} \sigma_R\sigma_Q\sigma_P((x, y)) &= (-x + 2(a - c + e), -y + 2(b - d + f)) \\ &= \sigma_S((x, y)). \end{aligned}$$

$\square$

2.2.5 EXAMPLE. Given any three of the not necessarily distinct points A, B, C, D, then the fourth is uniquely determined by the equation $\tau_{A,B} = \sigma_D\sigma_C$.

PROOF : We can solve the equation $\tau_{A,B} = \sigma_D\sigma_C$ for any one of A, B, C, D in terms of the other three. Knowing C, D and one of A or B, we let the other be defined by the equation $\sigma_D\sigma_C(A) = B$ or the equivalent equation $\sigma_C\sigma_D(B) = A$. In either case, product $\sigma_D\sigma_C$ is the unique translation taking A to B, and so $\sigma_D\sigma_C = \tau_{A,B}$. When we know both A and B, we let M be the midpoint of A and B. So $\tau_{A,B} = \sigma_M\sigma_A$. Knowing A, B, D, we have C is the unique solution for Y in the equation $\sigma_D\sigma_M\sigma_A = \sigma_Y$ as then

$\tau_{A,B} = \sigma_M\sigma_A = \sigma_D\sigma_Y$. Knowing A, B, C, we have D is the unique solution for Z in the equation $\sigma_M\sigma_A\sigma_C = \sigma_Z$ as then $\tau_{A,B} = \sigma_M\sigma_A\sigma_Z\sigma_C$. □

NOTE : In general, halfturns *do not* commute. Indeed, if $\sigma_Q\sigma_P = \tau_{P,R}$, then $\tau_{P,R}^{-1} = \sigma_P\sigma_Q$. So

$$\sigma_Q\sigma_P = \sigma_P\sigma_Q \iff P = Q.$$

2.2.6 PROPOSITION. *$\sigma_R\sigma_Q\sigma_P = \sigma_P\sigma_Q\sigma_R$ for any points P, Q, R.*

PROOF : For any points P, Q, R, there is a point S such that

$$\sigma_R\sigma_Q\sigma_P = \sigma_S = \sigma_S^{-1} = (\sigma_R\sigma_Q\sigma_P)^{-1} = \sigma_P^{-1}\sigma_Q^{-1}\sigma_R^{-1} = \sigma_P\sigma_Q\sigma_R.$$

□

NOTE : The halfturns *do not* form a group by themselves.

2.2.7 PROPOSITION. *The union of the translations and the halfturns forms a group $\mathfrak{H}$.*

PROOF : The product of two halfturns is a translation. Since a translation is a product of two halfturns, then the product in either order of a translation and a halfturn is a halfturn.

Recall that the inverse of a translation is a translation, and that a halfturn is an involutory transformation.

So the union of the translations and the halfturns forms a group. □

NOTE : A product of an even number of halfturns is a product of translations and, hence, is a translation.

A product of an odd number of halfturns is a halfturn followed by a translation and, hence, is a halfturn.

2.3 Exercises

Exercise 28 If τ is the product of halfturns about O and O', what is the product of halfturns about O' and O ?

Exercise 29 Prove that

$$\tau_{A,B}\sigma_P\tau_{A,B}^{-1} = \sigma_Q\,, \quad \text{where} \quad Q = \tau_{A,B}(P).$$

Exercise 30 TRUE or FALSE ?

(a) A product of two involutions is an involution or ι.

(b) $\mathfrak{D} \subset \mathfrak{H} \subset \mathfrak{T}$.

(c) If δ is a dilatation and lines $\mathcal{L}$ and $\mathcal{M}$ are parallel, then $\delta(\mathcal{L})$ and $\delta(\mathcal{M})$ are parallel to $\mathcal{L}$.

(d) Given points A, B, C, there is a D such that $\tau_{A,B} = \tau_{D,C}$.

(e) Given points A, B, C, there is a D such that $\tau_{A,B} = \sigma_D\sigma_C$.

(f) If $\tau_{A,B}(C) = D$, then $\tau_{A,B} = \tau_{C,D}$.

(g) If $\sigma_Q\sigma_P = \tau_{P,R}$, then $\sigma_P\sigma_Q = \tau_{R,P}$.

(h) $\sigma_A\sigma_B\sigma_C = \sigma_B\sigma_C\sigma_A$ for points A, B, C.

(i) A translation has equations $x' = x - a$ and $y' = y - b$.

(j) $\sigma_Q\sigma_P = \tau^2_{P,Q}$ for any points P and Q.

Exercise 31

$$\begin{cases} x' = -x + 3 \\ \\ y' = -y - 8 \end{cases}$$

are the equations for which transformation ?
What are the equations for $\tau_{S,T}^{-1}$ if $S = (a, c)$ and $T = (g, h)$?

Exercise 32 PROVE or DISPROVE : $\sigma_P\tau_{A,B}\sigma_P = \tau_{C,D}$, where $C = \sigma_P(A)$ and $D = \sigma_P(B)$.

Exercise 33 If $P_i = (a_i, b_i)$, $\quad i = 1, 2, 3, 4, 5$, then what are the equations for the product

$$\tau_{P_4,P_5}\tau_{P_3,P_4}\tau_{P_2,P_3}\tau_{P_1,P_2}\tau_{O,P_1} \text{ ?}$$

Exercise 34 What is the image of the line with equation $y = 5x + 7$ under σ_P, when $P = (-3, 2)$?

Exercise 35 If α is a translation, show that $\alpha\sigma_P$ is the halfturn about the midpoint of points P and $\alpha(P)$. What is $\sigma_P\alpha$?

Exercise 36 Draw line $\mathcal{L}$ with equation $y = 5x+7$ and point P with coordinates $(2, 3)$. Then draw $\sigma_P(\mathcal{L})$.

Exercise 37 Show that $\tau_{P,Q}$ has infinite order if $P \neq Q$.

Exercise 38 Suppose that $\langle\tau_{P,Q}\rangle$ is a subgroup of $\langle\tau_{R,S}\rangle$. Show there is a positive integer n such that $PQ = n\,RS$.

Exercise 39 PROVE or DISPROVE : $\langle\tau_{P,Q}\rangle = \langle\tau_{R,S}\rangle$ implies $\tau_{P,Q} = \tau_{R,S}$ or $\tau_{P,Q} = \tau_{S,R}$.

Exercise 40 Consider the points $A = (-1, -1)$, $B = (0, 0)$, $C = (1, 0)$, $D = (1, 1)$, and $E = (0, 1)$. Find points X, Y, Z such that :

(a) $\sigma_A\sigma_E\sigma_D = \sigma_X$.

(b) $\sigma_D\tau_{A,C} = \sigma_Y$.

(c) $\tau_{B,C}\tau_{A,B}\tau_{E,A}(A) = Z$.

DISCUSSION : In the Euclidean plane $\mathbb{E}^2$, for each line $\mathcal{L}$ and point $P \notin \mathcal{L}$ there is a unique line $\mathcal{L}'$ through P which does not meet $\mathcal{L}$. The line $\mathcal{L}'$ is called the *parallel* to $\mathcal{L}$ through P. Parallels provide us with a global notion of *direction* in the Euclidean plane. Each member of a family of parallel lines has the same direction, measured by the angle a member of the family makes with the x-axis, and parallels are a constant distance appart. A translation slides each member of a family of parallels along itself a constant distance. Consequently, translations always commute.

The situation changes in other spaces (with "non-euclidean" geometries). For example, in the *sphere* $\mathbb{S}^2$ (viewed as a surface of positive constant curvature in Euclidean 3-dimensional space) the "lines" are *great circles* (i.e. intersections of the sphere with planes through the origin), and hence any two of them intersect. Thus, there are *no* parallels, no global notion of direction (which way is north at the north pole ?), and no translations. Each rotation slides just *one* line (great circle) along itself, together with the curves at constant distance from this line. These "equidistant curves", however, are not lines.

Another example is the *hyperbolic plane* $\mathbb{H}^2$ (viewed as a surface of negative constant curvature in Euclidean 3-dimensional space, the *pseudosphere*). In this case, there are many lines $\mathcal{L}'$ through a point $P \notin \mathcal{L}$ which do not meet $\mathcal{L}$. (This is typical of the way hyperbolic geometry departs from Euclidean – in the opposite way from spherical geometry.) Translations exist, but each translation slides just one line along itself, together with the curves at constant distance from this line. These "equidistant curves" are also no lines, and translations with different invariant lines do not commute.

The most suggestive and notable achievement of the last [19th] century is the discovery of non-Euclidean geometry.

David Hilbert