4.1 Isometries as Product of Reflections

A product of reflections is clearly an isometry. The converse is also true; that is, *every isometry is a product of reflections*. We prove now this fact in seven (small) steps. (Actually, we shall do better than that by showing the product has *at most* three factors, not necessarily distinct.)

Looking at the fixed points of isometries turns out to be very rewarding in general.

4.1.1 PROPOSITION. *If an isometry fixes two distinct points on a line, then the isometry fixes that line pointwise.*

PROOF : Knowing point P is on the line through distinct points A and B and knowing the nonzero distance AP, we do not know which of the two possible points is P. However, if we also know the distance BP, then P is uniquely determined. It follows that an isometry fixing both A and B must also fix the point P, since an isometry is a collineation that preserves distance. In other words, an isometry fixing distinct points A and B must fix every point on the line through A and B.

$\square$

4.1.2 PROPOSITION. *If an isometry fixes three noncollinear points, then the isometry must be the identity.*

PROOF : Suppose that an isometry fixes each of three noncollinear points A, B, C. Then the isometry must fix every point on $\triangle ABC$ as the isometry fixes every point on any one of the lines $\overleftrightarrow{AB}, \overleftrightarrow{BC}, \overleftrightarrow{CA}$. Every point Q in the plane lies on a line that intersects $\triangle ABC$ in two distinct points. Hence the point Q is on a line containing two fixed points and, therefore, must also be fixed. So an isometry that fixes three noncollinear points must fix every point Q in the plane. $\square$

4.1.3 Proposition. *If α and β are isometries such that*

$$\alpha(P) = \beta(P)\,, \quad \alpha(Q) = \beta(Q)\,, \quad \textit{and} \quad \alpha(R) = \beta(R)$$

for three noncollinear points P, Q, R, then $\alpha = \beta$.

Proof : Multiplying each of the given equations by β^{-1} on the left, we see that $\beta^{-1}\alpha$ fixes each of the noncollinear points P, Q, R. Then $\beta^{-1}\alpha = \iota$ by Proposition 4.1.2. Multiplying this last equation by β on the left, we have $\alpha = \beta$. □

4.1.4 Proposition. *An isometry that fixes two points is a reflection or the identity.*

Proof : Suppose isometry α fixes distinct points P and Q on line $\mathcal{L}$. We know two possibilities for α, namely ι and $\sigma_{\mathcal{L}}$. We shall show these are the only two possibilities by supposing $\alpha \neq \iota$ and proving $\alpha = \sigma_{\mathcal{L}}$. If $\alpha \neq \iota$, then there is a point R not fixed by α. So R is off $\mathcal{L}$, and P, Q, R are three noncollinear points. Let $R' = \alpha(R)$. So $PR = PR'$ and $QR = QR'$, as α is an isometry. Therefore, $\mathcal{L}$ is the perpendicular bisector of $\overline{RR'}$ as each of P and Q is in the locus of all points equidistant from R and R'. Hence,

$$\alpha(R) = R' = \sigma_{\mathcal{L}}(R) \quad \text{as well as} \quad \alpha(P) = P = \sigma_{\mathcal{L}}(P) \quad \text{and} \quad \alpha(Q) = Q = \sigma_{\mathcal{L}}(Q)\,.$$

By Proposition 4.1.3 we have $\alpha = \sigma_{\mathcal{L}}$. □

4.1.5 Proposition. *An isometry that fixes exactly one point is a product of two reflections.*

Proof : Suppose isometry α fixes exactly one point C. Let P be a point different from C, let $\alpha(P) = P'$, and let $\mathcal{L}$ be the perpendicular bisector of $\overline{PP'}$. Since $CP = CP'$ as α is an isometry, then C is on $\mathcal{L}$. So $\sigma_{\mathcal{L}}(C) = C$ and $\sigma_{\mathcal{L}}(P') = P$. Then

$$\sigma_{\mathcal{L}}\alpha(C) = \sigma_{\mathcal{L}}(C) = C \quad \text{and} \quad \sigma_{\mathcal{L}}\alpha(P) = \sigma_{\mathcal{L}}(P') = P\,.$$

By PROPOSITION 4.1.4

$$\sigma_{\mathcal{L}}\alpha = \iota \quad \text{or} \quad \sigma_{\mathcal{L}}\alpha = \sigma_{\mathcal{M}}\,, \quad \text{where } \mathcal{M} = \overleftrightarrow{CP}\ .$$

However, $\sigma_{\mathcal{L}}\alpha \neq \iota$ as otherwise α is $\sigma_{\mathcal{L}}$ and fixes more points than C. Thus $\sigma_{\mathcal{L}}\alpha = \sigma_{\mathcal{M}}$ for some line $\mathcal{M}$. Multiplying this equation by $\sigma_{\mathcal{L}}$ on the left, we have $\alpha = \sigma_{\mathcal{L}}\sigma_{\mathcal{M}}$. □

4.1.6 PROPOSITION. *An isometry that fixes a point is a product of at most two reflections.*

PROOF : Since $\iota = \sigma_{\mathcal{L}}\sigma_{\mathcal{L}}$ for any line $\mathcal{L}$, the result follows as a corollary of PROPOSITION 4.1.5. □

We are now prepared to prove the main result.

4.1.7 THEOREM. *Every isometry is a product of at most three reflections.* (We count the number of factors even though the factors themselves may not be distinct.)

PROOF : The identity is a product of two reflections. Suppose nonidentity isometry α sends point P to different point Q. Let $\mathcal{L}$ be the perpendicular bisector of $\overline{PQ}$. Then $\sigma_{\mathcal{L}}\alpha$ fixes point P. We have just seen that $\sigma_{\mathcal{L}}\alpha$ must be a product β of at most two reflections. Hence $\alpha = \sigma_{\mathcal{L}}\beta$ and α is a product of at most three reflections. □

Congruence

Suppose $\triangle PQR \cong \triangle ABC$. We know there is *at most* one isometry α such that

$$\alpha(P) = A,\ \alpha(Q) = B, \quad \text{and} \quad \alpha(R) = C.$$

The question is whether there exists *at least* one such isometry α. It is possible to construct effectively such an isometry (as a product of at most three reflections).

4.1.8 Proposition. *If $\triangle PQR \cong \triangle ABC$, then there is a unique isometry α such that*

$$\alpha(P) = A\,, \quad \alpha(Q) = B\,, \quad \textit{and} \quad \alpha(R) = C.$$

Proof : Suppose $\triangle PQR \cong \triangle ABC$. So $AB = PQ$, $AC = PR$, and $BC = QR$. If $P \neq A$, then let $\alpha_1 = \sigma_{\mathcal{L}}$, where $\mathcal{L}$ is the perpendicular bisector of $\overline{PA}$. If $P = A$, then let $\alpha_1 = \iota$. In either case, then $\alpha_1(P) = A$. Let $\alpha_1(Q) = Q_1$ and $\alpha_1(R) = R_1$. If $Q_1 \neq B$, then let $\alpha_2 = \sigma_{\mathcal{M}}$, where $\mathcal{M}$ is the perpendicular bisector of $\overline{Q_1B}$. In this case, point A is on $\mathcal{M}$ as $AB = PQ = AQ_1$. If $Q_1 = B$, then let $\alpha_2 = \iota$. In either case, we have $\alpha_2(A) = A$ and $\alpha_2(Q_1) = B$. Let $\alpha_2(R_1) = R_2$. If $R_2 \neq C$, then let $\alpha_3 = \sigma_{\mathcal{N}}$, where $\mathcal{N}$ is the perpendicular bisector of $\overline{R_2C}$. In this case, $\mathcal{N} = \overleftrightarrow{AB}$ as $AC = PR = AR_1 = AR_2$ and $BC = QR = Q_1R_1 = BR_2$. If $R_2 = C$, then let $\alpha_3 = \iota$. In any case, we have $\alpha_3(A) = A$, $\alpha_3(B) = B$, and $\alpha_3(R_2) = C$. Let $\alpha = \alpha_3\alpha_2\alpha_1$. Then

$$\begin{aligned}
\alpha(P) &= \alpha_3\alpha_2\alpha_1(P) = \alpha_3\alpha_2(A) = \alpha_3(A) = A \\
\alpha(Q) &= \alpha_3\alpha_2\alpha_1(Q) = \alpha_3\alpha_2(Q_1) = \alpha_3(B) = B \\
\alpha(R) &= \alpha_3\alpha_2\alpha_1(R) = \alpha_3\alpha_2(R_1) = \alpha_3(R_2) = C
\end{aligned}$$

as desired. □

4.1.9 Corollary. *Two segments, two angles, or two triangles are, respectively, congruent if and only if there is an isometry taking one to other.*

Note : In elementary plane geometry there are three different relations indicated by the same words "is congruent to", one for segments, one for angles, and a third for triangles. All three can be combined under a generalized definition that applies to arbitrary sets of points as follows. If $\mathbf{S}_1$ and $\mathbf{S}_2$ are sets of points, then $\mathbf{S}_1$ and $\mathbf{S}_2$ are said to be **congruent** if there is an isometry α such that

$$\alpha(\mathbf{S}_1) = \mathbf{S}_2.$$

Exercise 51 Give a reasonable definition for $\square\, ABCD \cong \square\, PQRS$.

4.2 The Product of Two Reflections

Every isometry is a product of at most three reflections (see THEOREM 4.1.7). So each isometry is of the form

$$\sigma_{\mathcal{L}}\,, \quad \sigma_{\mathcal{M}}\sigma_{\mathcal{L}}\,, \quad \text{or} \quad \sigma_{\mathcal{N}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}.$$

We shall examine now the case $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$. Since a reflection is an involution, we know that $\sigma_{\mathcal{L}}\sigma_{\mathcal{L}} = \iota$ for any line $\mathcal{L}$.

Thus we are concerned with the product of two reflections in distinct lines $\mathcal{L}$ and $\mathcal{M}$. There are two cases : either $\mathcal{L}$ and $\mathcal{M}$ are parallel lines or else $\mathcal{L}$ and $\mathcal{M}$ intersect at a unique point.

Case 1 : $\mathcal{L}$ and $\mathcal{M}$ are (distinct) parallel lines

4.2.1 PROPOSITION. *If lines $\mathcal{L}$ and $\mathcal{M}$ are parallel, then $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$ is the translation through twice the directed distance from $\mathcal{L}$ to $\mathcal{M}$.*

PROOF : Let $\mathcal{L}$ and $\mathcal{M}$ be distinct parallel lines. Suppose $\overleftrightarrow{LM}$ is a common perpendicular to $\mathcal{L}$ and $\mathcal{M}$ with L on $\mathcal{L}$ and M on $\mathcal{M}$. The directed distance from $\mathcal{L}$ to $\mathcal{M}$ is the directed distance from L to M. (We are going to use PROPOSITION 4.1.3.) With K a point on $\mathcal{L}$ distinct from L, let $L' = \sigma_{\mathcal{M}}(L)$ and $K' = \tau_{L,L'}(K)$. Then (by PROPOSITION 2.1.2 and PROPOSITION 2.1.4) we have $\tau_{K,K'} = \tau_{L,L'}$ and $\square LKK'L'$ is a rectangle with $\mathcal{M}$ the common perpendicular bisector of $\overline{LL'}$ and of $\overline{KK'}$. So

$$\sigma_{\mathcal{M}}(K) = K'.$$

Now, let $J = \sigma_{\mathcal{L}}(M)$. Then, since L is the midpoint of $\overline{JM}$ and M is the midpoint of $\overline{LL'}$, we have

$$\tau_{J,M} = \tau_{L,L'}$$

where $\tau_{L,L'}$ is the translation through *twice* the directed distance from $\mathcal{L}$ to

$\mathcal{M}$. Hence

$$\begin{aligned}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}(J) &= \sigma_{\mathcal{M}}(M) = M = \tau_{L,L'}(J)\\ \sigma_{\mathcal{M}}\sigma_{\mathcal{L}}(K) &= \sigma_{\mathcal{M}}(K) = K' = \tau_{L,L'}(K)\\ \sigma_{\mathcal{M}}\sigma_{\mathcal{L}}(L) &= \sigma_{\mathcal{M}}(L) = L' = \tau_{L,L'}(L).\end{aligned}$$

Since an isometry is determined by any three noncollinear points (see Proposition 4.1.2), the equations above give the desired result

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \tau_{L,L'} = \tau^2_{L,M}.$$

□

4.2.2 Proposition. *If line $\mathcal{A}$ is perpendicular to line $\mathcal{L}$ at L and to line $\mathcal{M}$ at M, then*

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \tau^2_{L,M} = \sigma_M\sigma_L.$$

Proof : In the proof above we have $\tau_{L,L'} = \sigma_M\sigma_L$ (Proposition 2.2.3). □

Is every translation a product of two reflections (in parallel lines) ? The answer is "Yes". The following result holds.

4.2.3 Theorem. *Every translation is a product of two reflections in parallel lines, and, conversely, a product of two reflections in parallel lines is a translation.*

Proof : Given nonidentity translation $\tau_{L,N}$, then $\tau_{L,N} = \sigma_M\sigma_L$, where M is the midpoint of $\overline{LN}$. With $\mathcal{L}$ the perpendicular to $\overleftrightarrow{LM}$ at L and $\mathcal{M}$ the perpendicular to $\overleftrightarrow{LM}$ at M, we have

$$\sigma_M\sigma_L = \sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$$

by Proposition 4.2.2. So

$$\tau_{L,N} = \sigma_{\mathcal{M}}\sigma_{\mathcal{L}} \quad \text{with} \quad \mathcal{L} \parallel \mathcal{M}.$$

□

4.2.4 PROPOSITION. *If lines $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are perpendicular to line $\mathcal{A}$, then there are unique lines $\mathcal{P}$ and $\mathcal{Q}$ such that*

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{P}} = \sigma_{\mathcal{Q}}\sigma_{\mathcal{N}} \,.$$

Further, the lines $\mathcal{P}$ and $\mathcal{Q}$ are perpendicular to $\mathcal{A}$.

PROOF : The equations

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{P}} = \sigma_{\mathcal{Q}}\sigma_{\mathcal{N}}$$

have unique solutions for lines $\mathcal{P}$ and $\mathcal{Q}$, given lines $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are parallel. To show this, let line $\mathcal{A}$ be perpendicular to lines $\mathcal{L}, \mathcal{M}, \mathcal{N}$ at points L, M, N, respectively. Let P and Q be the unique points on $\mathcal{A}$ such that

$$\sigma_M\sigma_L = \sigma_N\sigma_P = \sigma_Q\sigma_N.$$

Let line $\mathcal{P}$ be perpendicular to $\mathcal{A}$ at P, and let line $\mathcal{Q}$ be perpendicular to $\mathcal{A}$ at Q. Then

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_M\sigma_L = \sigma_N\sigma_P = \sigma_{\mathcal{N}}\sigma_{\mathcal{P}} \quad \text{and} \quad \sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_M\sigma_L = \sigma_Q\sigma_N = \sigma_{\mathcal{Q}}\sigma_{\mathcal{N}}.$$

The uniqueness of these lines $\mathcal{P}$ and $\mathcal{Q}$ that satisfy the equations follows from the *cancellation laws*. (For example, $\sigma_{\mathcal{N}}\sigma_{\mathcal{P}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{P}'}$ implies $\sigma_{\mathcal{P}} = \sigma_{\mathcal{P}'}$, which implies $\mathcal{P} = \mathcal{P}'$.) □

NOTE : In the proposition above, $\mathcal{P}$ is just the unique line such that directed distance from $\mathcal{P}$ to $\mathcal{N}$ equals the directed distance from $\mathcal{L}$ to $\mathcal{M}$ and that $\mathcal{Q}$ is just the unique line such that the directed distance from $\mathcal{N}$ to $\mathcal{Q}$ equals the directed distance from $\mathcal{L}$ to $\mathcal{M}$.

4.2.5 COROLLARY. *If $P \neq Q$, then $\tau_{P,Q}$ may be expressed as $\sigma_{\mathcal{B}}\sigma_{\mathcal{A}}$, where either one of $\mathcal{A}$ or $\mathcal{B}$ is an arbitrarily chosen line perpendicular to $\overleftrightarrow{PQ}$ and the other is then a uniquely determined line perpendicular to $\overleftrightarrow{PQ}$.*

Observe that

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{P}} \quad \text{and} \quad \sigma_{\mathcal{N}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{P}}$$

are equivalent equations.

4.2.6 COROLLARY. *If lines $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are perpendicular to line $\mathcal{A}$, then $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$ is a reflection in a line perpendicular to $\mathcal{A}$.*

Case 2 : $\mathcal{L}$ and $\mathcal{M}$ are (distinct) intersecting lines

$\mathcal{L}$ and $\mathcal{M}$ are lines intersecting at a point C. We shall follow much the same path as we did for parallel lines.

4.2.7 PROPOSITION. *If lines $\mathcal{L}$ and $\mathcal{M}$ intersect at point C and the directed angle measure of a directed angle from $\mathcal{L}$ to $\mathcal{M}$ is $\frac{r}{2}$, then $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \rho_{C,r}$.*

PROOF : We first show that $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$ is a rotation about C by using the fact that three noncollinear points determine an isometry. Suppose $\frac{r}{2}$ is the directed angle measure of one of the two directed angles from $\mathcal{L}$ to $\mathcal{M}$. We may as well suppose $-90 < \frac{r}{2} \leq 90$. (Note that the notation suggests correctly that we are going to encounter *twice* the directed angle from $\mathcal{L}$ to $\mathcal{M}$ in our conclusion.) Let L be a point on $\mathcal{L}$ different from C. Let point M be the intersection of line $\mathcal{M}$ and circle C_L such that the directed angle measure from $\overrightarrow{CL}$ to $\overrightarrow{CM}$ is $\frac{r}{2}$. We have $\mathcal{L} = \overleftrightarrow{CL}$ and $\mathcal{M} = \overleftrightarrow{CM}$. Let $L' = \rho_{C,r}(L)$. Then L' is on circle C_L, and $\mathcal{M}$ is the perpendicular bisector of $\overline{LL'}$. So $L' = \sigma_{\mathcal{M}}(L)$. Let $J = \sigma_{\mathcal{L}}(M)$. Then $\mathcal{L}$ is the perpendicular bisector of $\overline{JM}$. So J is on circle C_L, and the directed angle measure from $\overleftrightarrow{CJ}$ to $\overleftrightarrow{CM}$ is r. Hence, $M = \rho_{C,r}(J)$. Therefore,

$$\begin{aligned}
\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}(C) &= \sigma_{\mathcal{M}}(C) = C = \rho_{C,r}(C)\\
\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}(J) &= \sigma_{\mathcal{M}}(M) = M = \rho_{C,r}(J)\\
\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}(L) &= \sigma_{\mathcal{M}}(L) = L' = \rho_{C,r}(L).
\end{aligned}$$

Since points C, J, L are not collinear, we conclude

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \rho_{C,r}.$$

So $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$ is the rotation about C through *twice* a directed angle from $\mathcal{L}$ to $\mathcal{M}$. □

The following result (analogue of THEOREM 4.2.3) holds.

4.2.8 THEOREM. *Every rotation is a product of two reflections in intersecting lines, and, conversely, a product of two reflections in intersecting lines is a rotation.*

PROOF : Suppose $\rho_{C,r}$ is given. Let $\mathcal{L}$ be any line through C, and let $\mathcal{M}$ be the line through C such that a directed angle from $\mathcal{L}$ to $\mathcal{M}$ has directed angle measure $\frac{r}{2}$. Then $\rho_{C,r} = \sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$, and this completes the proof. □

4.2.9 PROPOSITION. *If lines $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are concurent at point C, then there are unique lines $\mathcal{P}$ and $\mathcal{Q}$ such that*

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{P}} = \sigma_{\mathcal{Q}}\sigma_{\mathcal{N}}.$$

Further, the lines $\mathcal{P}$ and $\mathcal{Q}$ are concurent at C.

PROOF : Given rays $CL^{\rightarrow}$, $CM^{\rightarrow}$, and $CN^{\rightarrow}$, there are unique rays $CP^{\rightarrow}$ and $CQ^{\rightarrow}$ such that the directed angle from $CL^{\rightarrow}$ to $CM^{\rightarrow}$, the directed angle from $CP^{\rightarrow}$ to $CN^{\rightarrow}$, and the directed angle from $CN^{\rightarrow}$ to $CQ^{\rightarrow}$, all have the same directed angle measure. With $\mathcal{N} =\overleftrightarrow{CN}$, $\mathcal{P} =\overleftrightarrow{CP}$, and $\mathcal{Q} =\overleftrightarrow{CQ}$, we have solutions $\mathcal{P}$ and $\mathcal{Q}$ to the equations

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{P}} = \sigma_{\mathcal{Q}}\sigma_{\mathcal{N}}.$$

when $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are given lines concurent at C. The uniqueness of such lines $\mathcal{P}$ and $\mathcal{Q}$ follows from the cancellation laws. □

4.2.10 COROLLARY. *Rotation $\rho_{C,r}$ may be expressed as $\sigma_{\mathcal{B}}\sigma_{\mathcal{A}}$, where either one of $\mathcal{A}$ or $\mathcal{B}$ is an arbitrarily chosen line through C and the other is then a uniquely determined line through C.*

4.2.11 Corollary. *Halfturn σ_P is the product (in either order) of the two reflections in any two lines perpendicular at P.*

4.2.12 Corollary. *If lines $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are concurent at point C, then $\sigma_\mathcal{N}\sigma_\mathcal{M}\sigma_\mathcal{L}$ is a reflection in a line through C.*

4.2.13 Theorem. *A product of two reflections is a translation or a rotation; only the identity is both a translation and a rotation.*

Proof : Clearly,

$$\sigma_\mathcal{L}\sigma_\mathcal{L} = \tau_{P,P} = \rho_{P,0} = \iota$$

for any line $\mathcal{L}$ and any point P. Also, a rotation has a fixed point while a nonidentity translation does not. From these observations and the fact that lines $\mathcal{L}$ and $\mathcal{M}$ must be parallel or intersect, we have the result. □

4.3 Fixed Points and Involutions

We have not considered products of three reflections, except in the very special cases where the reflections are in lines that are parallel or in lines that are concurent. Therefore, it would be fairly surprising if we could at this stage *classify all the isometries that have fixed points* and *classify all the isometries that are involutions.* Such is the case, however.

4.3.1 Proposition. *An isometry that fixes exactly one point is a nonidentity rotation. An isometry that fixes a point is a rotation or a reflection.*

Proof : An isometry with a fixed point is a product of at most two reflections (Proposition 4.1.6). Of course, the identity and a reflection have fixed points. Otherwise, an isometry with a fixed point must be a translation or a rotation (Theorem 4.2.13). Since a nonidentity translation has no fixed points and a nonidentity rotation has exactly one fixed point, the desired result follows. □

The involutions come next.

4.3.2 PROPOSITION. *The involutory isometries are the reflections and the halfturns.*

PROOF : Suppose α is an involutory isometry. Since α is not the identity, there are points P and Q such that $\alpha(P) = Q \neq P$. Since $P = \alpha^2(P) = \alpha(Q)$, then α interchanges distinct points P and Q. Hence (PROPOSITION 3.2.3), α must fix the midpoint of $\overline{PQ}$. Therefore, α must be a rotation or a reflection by PROPOSITION 4.3.1. Since the involutory rotations are halfturns (see **Exercise 44**), we obtain the desired result. □

Exercise 52 Do involutory isometries form a group ?

Although we know that halfturn σ_P fixes line $\mathcal{L}$ if and only if point P is on line $\mathcal{L}$ (see PROPOSITION 2.2.2), we have not considered the fixed lines of an arbitrary rotation. We do so now.

4.3.3 PROPOSITION. *A nonidentity rotation that fixes a line is a halfturn.*

PROOF : Suppose nonidentity rotation $\rho_{C,r}$ fixes line $\mathcal{L}$. Let $\mathcal{M}$ be the line through C that is perpendicular to $\mathcal{L}$. Then (COROLLARY 4.2.10), there is a line $\mathcal{N}$ through C and different from $\mathcal{M}$ such that $\rho_{C,r} = \sigma_{\mathcal{N}}\sigma_{\mathcal{M}}$. Since $\mathcal{L}$ and $\mathcal{M}$ are perpendicular, then (PROPOSITION 3.1.2) we have

$$\mathcal{L} = \rho_{C,r}(\mathcal{L}) = \sigma_{\mathcal{N}}\sigma_{\mathcal{M}}(\mathcal{L}) = \sigma_{\mathcal{N}}(\mathcal{L}).$$

So $\sigma_{\mathcal{N}}$ fixes line $\mathcal{L}$. Then $\mathcal{N} = \mathcal{L}$ or $\mathcal{N} \perp \mathcal{L}$. Lines $\mathcal{M}$ and $\mathcal{N}$ cannot be two intersecting lines and both perpendicular to $\mathcal{L}$. Hence, $\mathcal{N} = \mathcal{L}$. So $\mathcal{M}$ and $\mathcal{N}$ are perpendicular at C and $\rho_{C,r}$ is the halfturn σ_C. □

4.4 Exercises

Exercise 53 Given $\triangle ABC \cong \triangle DEF$, where $A = (0,0)$, $B = (5,0)$, $C = (0,10)$, $D = (4,2)$, $E = (1,-2)$, and $F = (12,-4)$, find equations of lines such that the product of reflections in these lines takes $\triangle ABC$ to $\triangle DEF$.

Exercise 54 Suppose lines $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ have, respectively, equations $x = 2$, $y = 3$, and $y = 5$. Find the equations for $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$ and $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}$.

Exercise 55 PROVE or DISPROVE : Every isometry is either a product of five reflections or a product of six reflections.

Exercise 56 PROVE or DISPROVE : The images of a triangle under two distinct isometries cannot be identical.

Exercise 57 TRUE or FALSE ?

(a) $(\sigma_{\mathcal{Z}}\sigma_{\mathcal{Y}}\sigma_{\mathcal{X}} \cdots \sigma_{\mathcal{C}}\sigma_{\mathcal{B}}\sigma_{\mathcal{A}})^{-1} = \sigma_{\mathcal{A}}\sigma_{\mathcal{B}}\sigma_{\mathcal{C}} \cdots \sigma_{\mathcal{X}}\sigma_{\mathcal{Y}}\sigma_{\mathcal{Z}}$ for all lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}, \ldots,$ $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$.

(b) If $A_B = C_D$, then $AD = BC$.

(c) A product of four reflections is an isometry.

(d) The set of all rotations *generates* a commutative group.

(e) The set of all reflections *generates* $\mathfrak{Isom}$.

(f) If A and B are two distinct points, $PA = PB$ and $QA = QB$, then $P = Q$.

(g) An isometry that fixes a point is an involution.

(h) If isometry α fixes points A, B, and C, then $\alpha = \iota$.

(i) If α and β are isometries and $\alpha^2 = \beta^2$, then $\alpha = \beta$ or $\alpha = \beta^{-1}$.

Exercise 58 Prove the if $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}$ fixes point P and $\mathcal{M} \neq \mathcal{N}$, then P is on both $\mathcal{M}$ and $\mathcal{N}$.

Exercise 59 PROVE or DISPROVE : If α is an involution, then $\beta\alpha\beta^{-1}$ is an involution for any transformation β.

Exercise 60 If $\mathcal{M} \parallel \mathcal{N}$, find points M and N such that

$$\sigma_{\mathcal{N}}\sigma_{\mathcal{M}} = \sigma_N\sigma_M.$$

Exercise 61 What are the equations for $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}$ if line $\mathcal{M}$ has equation $y = -2x + 3$ and line $\mathcal{N}$ has equation $y = -2x + 8$?

Exercise 62 Show that $\sigma_{\mathcal{L}}\rho_{C,r}\sigma_{\mathcal{L}} = \rho_{C,-r}$ if point C is on line $\mathcal{L}$.

Exercise 63 TRUE or FALSE ?

(a) If a directed angle from line $\mathcal{L}$ to line $\mathcal{M}$ is $240°$, then $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$ is a rotation of $120°$.

(b) $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \tau^2_{L,M} = \sigma_M\sigma_L$ if point L is on line $\mathcal{L}$ and point M is on line $\mathcal{M}$.

(c) An isometry has a unique fixed point if and only if the isometry is a nonidentity rotation.

(d) An isometry that is its own inverse must be a halfturn, a reflection, or the identity.

(e) If $L' = \sigma_{\mathcal{M}}(L)$ and $K' = \tau_{L,L'}(K)$, then $\mathcal{M}$ is the perpendicular bisector of $\overline{KK'}$.

(f) Given points L, M, N, there is a point P such that $\sigma_M\sigma_L = \sigma_N\sigma_P$.

(g) Given lines $\mathcal{L}, \mathcal{M}, \mathcal{N}$, there is a line $\mathcal{P}$ such that $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{P}}$.

(h) If lines $\mathcal{L}$ and $\mathcal{M}$ intersect at point C and a directed angle from $\mathcal{L}$ to $\mathcal{M}$ is $r°$, then $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \rho_{C,2r}$.

(i) An isometry that fixes a point must be a rotation, a reflection, or the identity.

(j) Isometry $\alpha\beta\alpha^{-1}$ is an involution for any isometry α if and only if isometry β is an involution.

Exercise 64 Given nonparallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, show there is a rotation ρ such that
$\rho(\overrightarrow{AB}) = \overrightarrow{CD}$.

Exercise 65 PROVE or DISPROVE : Every translation is a product of two noninvolutory rotations.

Exercise 66 PROVE or DISPROVE : If $P \neq Q$, then there is a unique translation taking point P to point Q but there are an infinite number of rotations that take P to Q.

Exercise 67 What lines are fixed by rotation $\rho_{C,r}$?