CHAPTER II

Orthogonal Transformations

The concept of transformation in geometry first arose from a consideration of displacement—the movement of rigid bodies from one place to another. A characteristic of such motion, and the most important one from the point of view of geometry, is the preservation of the size and the shape of a body. Throughout its displacement, a moving body preserves its shape and dimensions and is the same at the end of the displacement as at the beginning. Thus, if we consider only the initial and final moments of the motion, we can establish a correspondence between the points of the body in its initial and in its final positions. To the point M in space occupied by a certain point P in the body at the start of the displacement we make correspond the point M' occupied by P at the end of the displacement. If M goes into M' and N into N', then the lengths of the segments MN and $M'N'$ are equal, each segment being equal to the distance between two fixed points of the rigid body. In geometry, as opposed to kinematics, a displacement is not regarded as an actual process of motion from one point to another but merely as a correspondence between the points occupied by the figure in its initial and final positions: such an approach allows us to regard displacements in geometry as mappings that take intervals into equal intervals (that is, mappings that "preserve distance"). From the geometric point of view, such mappings are the simplest, since they preserve both the dimensions and the shapes of figures and change only their position. We shall start our study of geometric transformations in the plane and in space with transformations of this type. We shall not call these transformations *displacements*, since there are distance-preserving transformations which are not displacements (for example, reflections) but rather *orthogonal mappings* (or *orthogonal transformations*). The reason for the use of this terminology will appear later. Throughout this book we shall regard mappings and transformations as defined on the whole plane or the whole of space. Transformations and mappings of figures will be regarded as induced by such mappings.

4. Orthogonal Mappings

Definition. *An orthogonal mapping of a plane π into a plane* π' is a mapping under which line segments of π are carried into equal line segments of π'. More precisely, the mapping α of π into π' is said to be orthogonal if, for any two points M, N of π, the distance between M and N is equal to the distance (in π') between $\alpha(M)$ and $\alpha(N)$. We take the notion of distance in the plane to be fundamental.

Orthogonal mappings of π into π' are one-one and onto. For suppose M_1 and M_2 are distinct points of π. Then their images M_1' and M_2' must also be distinct, since the line segments M_1M_2 and $M_1'M_2'$ are equal. Suppose, next, M' is any point of π'. We show that it has an inverse image M in π. Let A, B, C be the vertices of a triangle in π, and let A', B', C' be their respective images in π'. Then A', B', C' are the vertices of a triangle. For otherwise B', say, would be between A' and C', and $A'C' = A'B' + B'C'$. But then $AC = AB + BC$, which is a contradiction, since the total length of two sides of a triangle is always greater than the length of the third side. Since $A'B'C'$ is a triangle, the point M' does not lie on at least one of its sides, say the side $A'B'$. Let M'' be the reflection in $A'B'$ (perhaps extended) of M'; then the triangles $A'B'M'$ and $A'B'M''$ are congruent. Let us construct points M_1 and M_2 in the plane π such that the triangles ABM_1 and ABM_2 are both congruent to $A'B'M'$ and $A'B'M''$. The distances from the point M_1 to A and B are M_1A and M_1B, respectively. So the image of M_1 must be the same distances from A' and B' and must therefore be either M' or M''. Similarly the image of M_2 must be either M' or M''. And since M_1 and M_2 cannot both have the image M'' (for they have distinct images), one of them has the image M'. Thus M' has an inverse image, and, in fact, a unique inverse image. Since an orthogonal mapping is one-one and onto, it has an inverse mapping, and as the inverse mapping also clearly preserves distances, the inverse of an orthogonal mapping is itself an orthogonal mapping.

Definition. An orthogonal mapping of a plane onto itself is called an orthogonal transformation of the plane.

It is clear that the product of any two orthogonal transformations is itself an orthogonal transformation, and we have already seen that the same holds for the inverse of an orthogonal transformation; it follows that the set of all orthogonal transformations of the plane forms a group, which we call the *orthogonal group* (of the plane).

In a similar way we may define orthogonal transformations of space and show that they form a group.

5. Properties of Orthogonal Mappings

Theorem 1. *Under an orthogonal mapping, any three collinear points are taken into three collinear points, and any three noncollinear points are taken into three noncollinear points.*

Proof. Let P, Q, R be three collinear points, and suppose, for example, that Q lies between P and R.

Then

$$PQ + QR = PR.$$

Suppose the respective images of P, Q, R are P', Q', R'. Then by the definition of orthogonality, $P'Q' = PQ$, etc., and so

$$P'Q' + Q'R' = P'R'$$

But this is possible only if P', Q', R' lie on a line, with Q' in the middle; otherwise we should have

$$P'Q' + Q'R' > P'R'$$

Let P, Q, R be noncollinear points, and suppose their images are collinear. Then the inverse mapping which takes P' into P, etc., would take the collinear points P', Q', R' into collinear points, by what we have already proved (since the inverse of an orthogonal mapping is orthogonal). But P, Q, R are not collinear; this contradiction shows that the images are not collinear. ▼

Theorem 2. *Let α be an orthogonal map of the plane π onto the plane π'. Then the image under α of a line l in π is a line l' in π'. More precisely: given a line l in π, there is a line l' in π' such that every point of l is mapped onto some point of l', and moreover every point of l' has precisely one point of l mapped onto it. We may say more concisely that α induces a one-one mapping of l onto l'.*

Proof. Let A and B be any two distinct points of l, and let A' and B' be their (distinct) images. Let l' be the line of π' through A' and B'. Then, by Theorem 1, any point C of the line l is mapped into a point of l'. For C is collinear with A and B, so that its image must be collinear with A' and B'.

Conversely let C' be any point of l'. Then, by the same argument, its image under the inverse mapping α^{-1} of π' onto π must lie on l, so that every point of l' has an inverse image on l.

We have shown that the line l is mapped onto the line l'. That the mapping of l is one-one follows from the fact that α is one-one. ▼

As in the definitions of Section 2 (Chapter I), we call l' the image of l and l the inverse image of l' under α.

Theorem 3. *Under an orthogonal mapping α of space into itself, the image of a plane π is a plane π'. Moreover, the mapping of π onto π' is itself an orthogonal mapping.*

Proof. Let A, B, C be three noncollinear points of π, and A', B', C' their images under α. By Theorem 1, A', B', and C' are not collinear.

Let π' be the plane passing through A', B', and C'. Suppose M is an arbitrary point of π. If it lies on one of the lines BC, CA, or AB, then by Theorem 1 its image lies on $B'C'$ or $C'A'$ or $A'B'$, as the case may be. If not, suppose MA meets BC in P (Fig. 11). Then the image P' of P lies on $B'C'$ and thus lies in the plane π'. Since A, M, P are collinear, so too are their images A', M', P'. But A' and P' lie in π', so that the whole line $A'P'$ and, in particular, M' lie in π'. We have shown that the image of the plane π lies *in* the plane π'. But the inverse transformation of space must clearly map π' into π (by what we have already shown), which means in particular that α maps π *onto* π'. For every point in π' has an inverse image (an image under the inverse mapping) in π. That the mapping of π onto π' is one-one and orthogonal follows at once from the corresponding properties of α. ▼

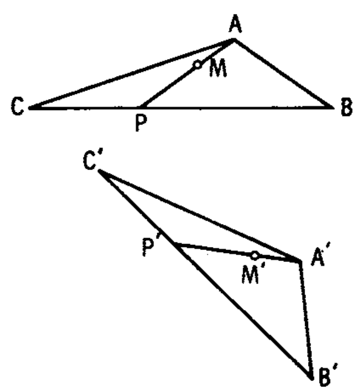


Fig. 11

Theorem 4. *Under an orthogonal mapping of a plane π onto a plane π', the image of two parallel lines of π is two parallel lines of π'.*

Proof. By Theorem 2, two parallel lines of π go into two lines of π'. If these two lines had a point in common, the inverse image of this point would be a point common to the two parallel lines of π, which is impossible. Thus the lines in π' have no common point; that is, they are parallel. ▼

Theorem 5. *Under an orthogonal mapping of space:*
1. *the image of two parallel lines is two parallel lines;*
2. *the image of two parallel planes is two parallel planes;*
3. *the image of a plane and a line parallel to it is a plane and a line parallel to it.*

The proofs of these propositions are left to the reader.

Theorem 6. *Under an orthogonal mapping, the order of points on a line is preserved. That is to say, if P', R' are the images of two points P, R, then the interior points of the segment PR go into the interior points of the segment $P'R'$, while the exterior points of PR go into the exterior points of $P'R'$.*

We have already given a proof in our proof of Theorem 1.

Corollary. *If the points P, Q lie on opposite sides of a line l, then their images P', Q' lie on opposite sides of the image l' of l.*

Let PQ meet l in R. Then R is an interior point of PQ, so that its image R' is an interior point of $P'Q'$. But R' lies on l', so that P' and Q' must lie on opposite sides of l'.

If the points P and Q lie on the same side of l, then their images lie on the same side of l'.

Theorem 7. *Orthogonal mappings preserve angles.*

Proof. Let a and b be two rays through a point O. Choose points A, B on a, b respectively, neither being the point O. Let O', A', B' be the images of the three points under the orthogonal mapping. Then $O'A'$, $O'B'$ will be the images of a and b respectively (by Theorem 6).

By the orthogonality of the mapping, the triangles OAB and $O'A'B'$ are congruent (three pairs of equal sides). So the respective angles are equal, and, in particular, $\angle AOB = \angle A'O'B'$. ▼

Theorem 8. *Let A, B, C be three noncollinear points of the plane π, and A', B', C' three points of the plane π' such that $B'C' = BC$, $C'A' = CA$, $A'B' = AB$. Then there exists one and only one orthogonal mapping of the plane π onto the plane π' such that the images under it of A, B, C are A', B', C', respectively.*

Proof. We construct a mapping as follows: we make A, B, C correspond to A', B', C', respectively. If P is a point of AC, we make it correspond to the point P' of $A'C'$ such that $A'P' = AP$; if P lies on the extension of AC, we let its image P' be the point on the extension of $A'C'$ such that (1) $AP = A'P'$, and (2) the points P', A', C' lie in the same order along the line $A'C'$ as do P, A, and C along the line AC.

It is easy to see that if P and P_1 are any points of AC, and P', P_1' their images, then $PP_1 = P'P_1'$ and that the order of the points P', P_1', A, C along the line $A'C'$ is the same as the order of P, P_1, A, C along the line AC. We place the points Q of AB in correspondence with the points Q' of $A'B'$ in just the same way (Fig. 12).

Suppose now that M is a point of the plane not lying on either of the lines AB or AC. We draw parallels through M to

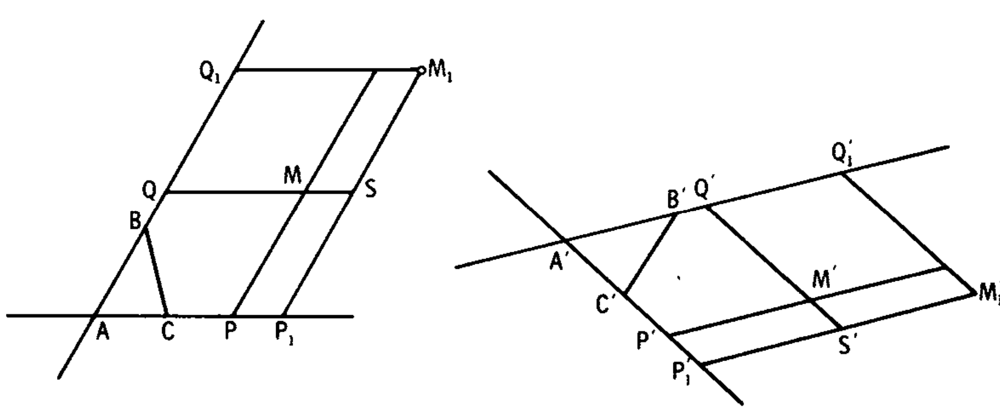


Fig. 12

meet AB and AC in Q and P, respectively. Let Q' and P' be the images of Q and P on $A'B'$ and $A'C'$. Through Q' and P' draw parallels to $A'C'$ and $A'B'$ respectively, and suppose these parallels meet in M'. Then we put M in correspondence with M'. We have now said what we put in correspondence with every point of π. Let us show that the mapping we have defined is orthogonal. Let M and M_1 be two points of π and M', M_1' their respective images. If M and M_1 both lie on AB, or both on AC, then we already know $MM_1 = M'M_1'$. If M and M_1 both lie on a line parallel to AC (say), then $MM_1 = PP_1 = P'P_1' = M'M_1'$ (where the notation is obvious). In the general case, let MQ meet M_1P_1 in S, so that $M'Q'$ meets $M_1'P_1'$ in the image S' of S (in case M, for example, lies on AB, we interpret MQ to be the line through M parallel to AC, and $Q = M$). Then $MS = PP_1 = P'P_1' = M'S'$, and $SM_1 = QQ_1 = Q'Q_1' = S'M_1'$. Next, the sides of the angles BAC and M_1SM are parallel, so that the angles must be equal or supplementary. If they are equal, then so are the angles $B'A'C'$ and $M_1'S'M'$, but if BAC and MSM_1 are supplementary, $B'A'C'$ and $M'S'M_1'$ will be too. But $\angle BAC = \angle B'A'C'$, so that $\angle MSM_1 = \angle M'S'M_1'$. Thus the triangles MSM_1 and $M'S'M_1'$ are congruent (two sides and included angle), and, in particular, $MM_1 = M'M_1'$. We have shown that the mapping we have constructed is orthogonal. ▼

Note. The reader should check that our proof still holds when one of M, M_1 lies on AB or AC, and even when one of them lies on one of AB, AC and the other on the other.

We have proved that there is an orthogonal mapping of the plane π onto the plane π' in which A, B, C have A', B', C' for their images. It remains to prove that the mapping is unique. Let α be the mapping we have constructed and β any orthogonal mapping with the required properties. Then β carries any point P of AC onto a point $\beta(P)$ of $A'C'$ such that $AP = A'\beta(P)$ and $CP = C'\beta(P)$. But this means that $\beta(P) = P' = \alpha(P)$, so that α and β coincide for points of AC and similarly on AB. Suppose now M is a point of π not on AB or AC, and P and Q are defined as before. Then the image of PM under β is a line parallel to $A'B'$ (Theorem 4) and through P' (since P' is the image of P under β). Similarly, the image under β of MQ is the line through Q' parallel to $A'C'$. But these lines intersect in M', so that we must have $\beta(M) = M'$. Thus the effect of β is the same as that of α for every point of π, and so $\beta = \alpha$. We have proved the uniqueness as well as the existence of an orthogonal mapping taking A, B, C into A', B', C'.

6. Orientation

For a more detailed investigation of orthogonal transformations and the establishment of the connection between them and displacements, we shall need to introduce the important geometric concept of *orientation*. A graphic illustration of this concept is provided by a comparing two figures whose boundaries are traversed in a definite sense. Thus (Fig. 13), we say that the triangles ABC and $A'B'C'$ have the same orientation, since in both cases the vertices are traversed the same way round (clockwise). On the other hand, the triangles ABC and $A''B''C''$ have opposite orientations.

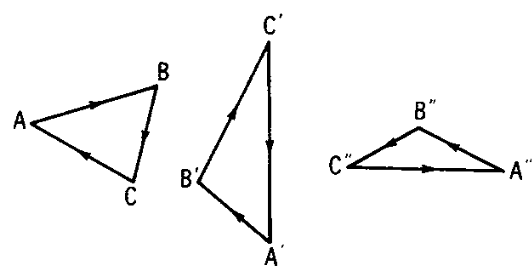


Fig. 13

The concept of orientation arises when we measure angles or discuss the areas of figures bounded by complicated curves (in particular, self-intersecting curves) and also in a number of questions of higher mathematics (topology). We now give a mathematical definition of orientation.

Definition I. An *oriented triangle* is an ordered triple of noncollinear points. Here the points are the vertices of the triangle, and the orientation is given by the order in which the vertices appear.

Definition 2. A *chain of triangles* joining the oriented triangle ABC with the oriented triangle $A'B'C'$ is a finite sequence of oriented triangles, the first triangle being ABC, the last $A'B'C'$, such that each pairing of adjacent triangles (in the sequence) differs either by the order of the vertices alone or by one vertex which occupies the same place (first, second, or third) in each of the triangles.

Theorem I. *Any two oriented triangles ABC and $A'B'C'$ can be joined by a chain.*

Proof. One such chain is

$$ABC, \quad ABQ, \quad APQ, \quad A'PQ, \quad A'B'Q, \quad A'B'C',$$

where Q is any point not on AB or $A'B'$ and P is any point not on AQ or $A'Q$ (Fig. 14). ▼

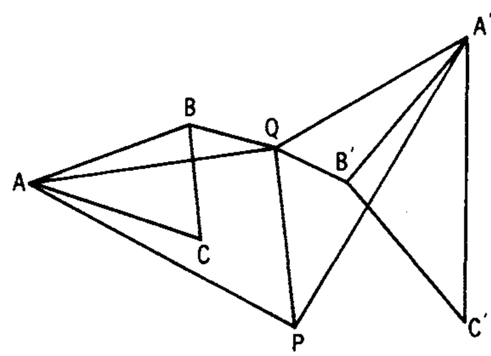


Fig. 14

Definition 3. We say two oriented triangles with the same vertices are *co-oriented* if the vertices of one of them can be obtained by a cyclic permutation of the vertices of the other. If not, we call them *anti-oriented.* (This cumbersome terminology will only be required for a couple of pages.)

Thus the triangles ABC, BCA, CAB are co-oriented in pairs, as are also the triangles ACB, CBA, BAC, while each of the latter is anti-oriented with each of the former.

Definition 4. We say two oriented triangles differing in one vertex that occupies the same position in each of them are *co-oriented* if these vertices lie on the same side of the line joining the other two vertices and otherwise *anti-oriented*. Thus if C and D lie on the same side of the line AB, the triangles ABC and ABD are co-oriented If C and D are on opposite sides of AB, then these triangles are anti-oriented. (Fig. 15).

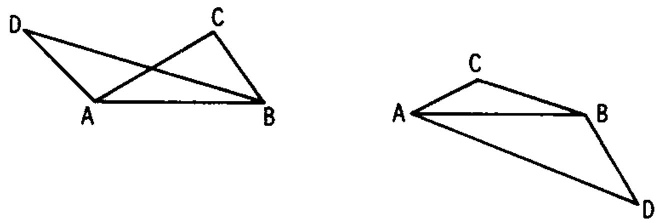
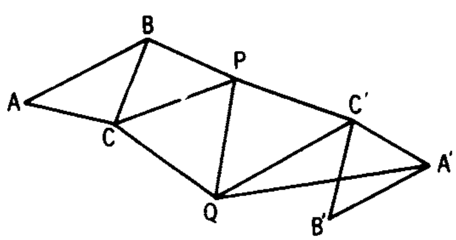


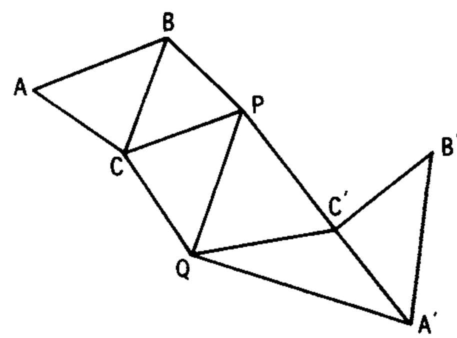
Fig. 15

Definition 5. Given two oriented triangles ABC and $A'B'C'$ and a chain joining them, we say ABC and $A'B'C'$ have the *same orientation* if the number of pairs of adjacent triangles (in the chain) that are anti-oriented is even, and otherwise we say ABC and $A'B'C'$ have *opposite orientations* (Figs. 16 and 17). In order to show that this is a meaningful definition, we need to establish:



ABC, PBC, PQC, PQC', A'QC', A'B'C'

Fig. 16



ABC, PBC, PQC, PQC', A'QC', A'B'C'

Fig. 17

Theorem 2. *Given two oriented triangles ABC and $A'B'C'$, the number of pairs of adjacent triangles in a chain joining ABC to $A'B'C'$ that are anti-oriented is either always even or always odd.*

If we prove this theorem, we shall have shown that the property of two triangles of having the same or opposite orientation is independent of the choice of a chain between them.

Theorem 2 is a consequence of:

Theorem 3. Let (x_i, y_i) *and* (x_i', y_i') *be the coordinates of the vertices of the oriented triangles* ABC *and* $A'B'C'$, *respectively* $(i = 1, 2, 3)$. *In order that the triangles have the same orientation (with respect to a given chain), it is necessary and sufficient that the determinants*

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix}$$

have the same sign.

Let us first see why this theorem entails Theorem 2. Suppose ABC and $A'B'C'$ have the same orientation with respect to a chain D. Then, by the "necessary" part of Theorem 3, the determinants (1) have the same sign. Let D' be any chain joining the triangles. Then by the "sufficient" part of Theorem 3 the triangles have the same orientation with respect to D'. We see, therefore, that if the triangles have the same orientation with respect to one chain, they have it with respect to every chain, and it follows that if they have the opposite orientation with respect to one chain, they have it with respect to every chain. So the property of pairs of oriented triangles of having the same or opposite orientations is independent of the connecting chains.

Proof. Consider a pair of adjacent triangles in the given chain joining ABC to $A'B'C'$, and suppose first that they differ in one vertex. If the triangles are MNS and MNT, we show that the determinants

$$\delta_1 = \begin{vmatrix} x_M & y_M & 1 \\ x_N & y_N & 1 \\ x_S & y_S & 1 \end{vmatrix} \quad \text{and} \quad \delta_2 = \begin{vmatrix} x_M & y_M & 1 \\ x_N & y_N & 1 \\ x_T & y_T & 1 \end{vmatrix}$$

(the notation being obvious) have the same or opposite sign, according as S and T lie on the same or opposite sides of MN.

Now the equation of the line MN is

$$\begin{vmatrix} x_M & y_M & 1 \\ x_N & y_N & 1 \\ x & y & 1 \end{vmatrix} = 0 .$$

It is known from analytic geometry that S and T lie on the same side of MN if and only if the substitution of their coordinates in the left side of the equation for MN gives us two numbers with the same sign, that is, if and only if δ_1 and δ_2 have the same sign. Thus, by Definition 4, δ_1 and δ_2 have the same sign if and only if MNS and MNT are co-oriented.

If two adjacent triangles of the chain differ only in the order of their vertices, then the corresponding determinants (constructed as was δ_1) have the same sign if and only if the vertices of one triangle are obtained from those of the other by a cyclic permutation; that is, if and only if these triangles are co-oriented (Definition 3).

Thus the number of sign changes in the sequence of determinants corresponding to the successive triangles of our chain is equal to the number of pairs of adjacent triangles which are anti-oriented. So if the determinants (1) have the same sign, the number of sign changes in the sequence of determinants must be even, and the number of pairs of adjacent triangles which are anti-oriented must be even. Similarly, if the determinants (1) have opposite signs, then the number of sign changes, and therefore also the number of pairs of adjacent anti-oriented triangles, must be odd. This proves the theorem. ▼

Note. It follows from this theorem that co-oriented triangles have the same orientation, and anti-oriented ones opposite orientations. Therefore, we no longer need to talk of co- or anti-oriented triangles.

The concept of orientation may be extended to three-dimensional space:

Definition 6. An *oriented tetrahedron* is an ordered quadruple of points in space (the vertices) not all lying in one plane.

Definition 7. A *chain of tetrahedra* connecting an oriented tetrahedron $A = A_1A_2A_3A_4$ with an oriented tetrahedron $A'A_1'A_2'A_3'A_4'$ is a finite sequence of oriented tetrahedra, the first of which is A and the last A', such that two adjacent tetrahedra of the sequence differ either only by the order of their vertices or by a single vertex occupying the same place in both of them.

It can easily be shown that any two tetrahedra can be joined by a chain.

Definition 8. We say that two oriented tetrahedra with the same vertices are *co-oriented* if the vertices of one of them can be obtained from those of the other by an even permutation.

We say that a permutation taking a sequence of elements into a different sequence of the same elements is *even* if the second sequence can be obtained from the first by an even number of transpositions of pairs of elements. It can be shown that whatever sequence of transpositions we choose in order to take one given sequence into another, we shall either always need an odd number or always an even number. A permutation in which an odd number of transpositions is required is called an *odd* permutation. For example, the permutation $\begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 4 & 3 \end{pmatrix}$ is even, since we may go from the sequence 2341 to the sequence 1243 by means of the following two transpositions:

$$2341 \to 1342; \qquad 1342 \to 1243.$$

On the other hand, the permutation $\begin{pmatrix} 2 & 3 & 1 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$ is odd, since we may reach 1243 from 2314 by means of three transpositions as follows: $2314 \to 1324 \to 1234 \to 1243$. So, for example, the oriented tetrahedra $A_2A_3A_4A_1$ and $A_1A_2A_4A_3$ are co-oriented, but $A_2A_3A_1A_4$ and $A_1A_2A_4A_3$ are anti-oriented.

Definition 9. Two oriented tetrahedra differing in a single vertex that occupies the same position in each of them are said to be *co-oriented* if these vertices lie on the same side of the plane through the other three vertices, and otherwise *anti-oriented*.

Definition 10. If a chain joining the oriented tetrahedra A and A' is such that the number of pairs of adjacent tetrahedra in it that are anti-oriented is even, then A and A' are said to have the *same orientation*, and otherwise they are said to have *opposite orientations.*

This definition is justified by the following theorem, which, like Theorem 2, may be proved by a consideration of determinants:

Theorem 4. *In any two chains joining the given oriented tetrahedra A and A', the respective numbers of pairs of anti-oriented adjacent tetrahedra will be both even or both odd.*

The definition we have given of orientation in Euclidean three-dimensional space can be generalized to n-dimensional space.

7. Orthogonal Transformations of the First and Second Kinds

Theorem I. *If a triangle ABC and its image $A'B'C'$ under an orthogonal transformation of the plane have the same orientation, then so also do any triangle and its image. Conversely, if ABC and $A'B'C'$ have opposite orientations, so also do any triangle and its image.*

Proof. Suppose the triangles ABC and $A'B'C'$ have the same orientation, and let PQR be any triangle and $P'Q'R'$ its image. We show that PQR and $P'Q'R'$ have the same orientation. We start by making the following remark: If D_1, D_2, ..., D_n is any sequence of triangles, then D_1 and D_n will have the same orientation if the number of pairs of adjacent triangles in the sequence having opposite orientation is even, and otherwise they will have opposite orientation. The proof is given by supplying a chain of triangles between each pair D_r, D_{r+1}, and we leave it to the reader. Suppose now S is a chain of triangles joining PQR to ABC. It will be mapped under the orthogonal transformation α into a chain S' joining $P'Q'R'$ and $A'B'C'$. The number of anti-oriented pairs in S' is the same as the number in S, so that $A'B'C'$ and $P'Q'R'$ will have the same orientation if and only if ABC and PQR do. On considering the sequence of triangles PQR, ABC, $A'B'C'$, $P'Q'R'$, the reader will see that in any case PQR and $P'Q'R'$ have the same orientation.

The second part may be proved analogously, but it follows from what we know already. Thus, suppose ABC and $A'B'C'$ have opposite orientations and PQR is any triangle, If PQR and $P'Q'R'$ have the same orientation, then, by what we have already proved, every triangle has the same orientation as its image; in particular, the triangle ABC. This contradiction shows that PQR and $P'Q'R'$ must have opposite orientations. ▼

Definition 1. An orthogonal transformation will be said to be *of the first kind* if it preserves the orientation of every triangle. If the transformation changes the orientation of every triangle, it will be said to be *of the second kind*. Theorem 1 shows that every orthogonal transformation is either of the first or of the second kind. The classification into transformations of the first and second kind can be extended into space, and even into n-dimensional Euclidean space for any positive integer n.

Definition 2. An orthogonal transformation of the first kind is called a *displacement*.

In mechanics and physics a displacement is commonly regarded as a process in which a body moves from one position to another. During the motion, lengths of segments and sizes of angles, and also orientation, are preserved. In a number of questions in geometry we are interested only in the initial and final positions of the body. So, instead of thinking of a body that moves through space from one position to another, we think of an orthogonal transformation such that the image of a plane figure (the " body " in its initial position) is another plane figure (the body in its final position). The transformation has to be orthogonal, since we want the image to be congruent with the original figure, and it has to be of the first kind, since we wish the orientation of the image to be the same as that of the original. The set of all orthogonal transformations of the plane of the first kind is a subgroup of the full orthogonal group. For it is clear that the product of two orientation-preserving maps also preserves orientation, and so does the inverse of an orientation-preserving map.

Let us note that the product of an orthogonal transformation of the first and second kinds is of the second kind and that the product of two transformations of the second kind is of the first kind. Compare Example 7 at the end of Chapter I, where Γ_1 is the set of orthogonal transformations of the line of the first kind, B those of the second, and D the full orthogonal group of the line.

We showed above that there exists a unique orthogonal transformation carrying three given points ABC into three given points $A'B'C'$ such that the triangles ABC and $A'B'C'$ are congruent. We can now sharpen this result.

Theorem 2. *Given two distinct points A, B and two points A', B' such that $A'B' = AB$, there exists a unique orthogonal transformation of the first kind and a unique orthogonal transformation of the second kind (each defined on the plane) such that the images of A and B are A' and B', respectively.*

Proof. Choose an arbitrary point C not lying on the line AB, and let C' and C'' be the two points of the plane for which the triangles $A'B'C'$ and $A'B'C''$ are congruent with the triangle ABC. It is clear that any orthogonal transformation of the plane which takes A and B into A' and B' must take C into either C' or C''. Moreover, there is precisely one transformation α taking A, B, C into A', B', C', respectively, and precisely one orthogonal transformation β taking A, B, C into A', B', C'', by Theorem 8 of Section 5. Now C' and C'' lie on opposite sides of $A'B'$, so that the triangles $A'B'C'$ and $A'B'C''$, have opposite orientations. So just one of them, say $A'B'C'$, has the same orientation as ABC. But then α is of the first kind, by Theorem 1 and β is of the second. Thus, exactly one of the two possible orthogonal transformations taking A, B into A', B' is of the first kind, and one is of the second. ▼

8. The Fundamental Types of Orthogonal Transformation (Translation, Reflection, Rotation)

In this section we consider the fundamental types of orthogonal transformation, in terms of which every such transformation can be expressed.

8.1. Translation

Suppose we are given a vector $\mathbf{a}$ of the plane π. We make correspond to each point M of π the point M' for which $MM' = \mathbf{a}$ (Fig. 18). This correspondence is a transformation of the plane called a *translation*. Thus, under a translation, every point is carried a given distance in a given direction.

Translations are orthogonal transformations of the plane. For suppose M_1 and M_2 are given points and M_1', M_2' their images under the given translation (Fig. 19). Then, by the definition of a translation, $\overrightarrow{M_1M_1'} = \overrightarrow{M_2M_2'} = \mathbf{a}$. Adding to both sides the vector $\overrightarrow{M_1'M_2}$, we find that $\overrightarrow{M_1M_1'} + \overrightarrow{M_1'M_2} = \overrightarrow{M_1'M_2} + \overrightarrow{M_2M_2'}$, or $\overrightarrow{M_1M_2} = \overrightarrow{M_1'M_2'}$, so that the segments

M_1M_2 and $M_1'M_2'$ are equal (actually, we have even shown that they are parallel).

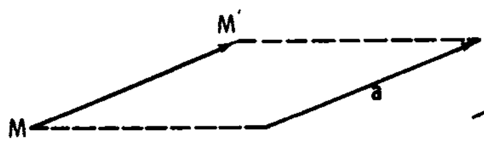


Fig. 18

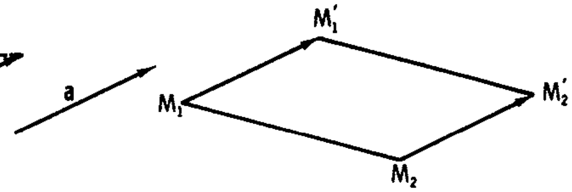


Fig. 19

A translation is an orthogonal transformation of the first kind. For suppose the translation takes some point A into A', and A' into A''. Let C be any point not lying on AA', and suppose its image is C'. We show that the oriented triangles $AA'C$ and $A'A''C'$ have the same orientation. It will follow at once from Theorem 1 that the translation is a transformation of the first kind. Consider the chain of triangles:

$$AA'C, \quad C'A'C, \quad C'A'A'', \quad A'A''C'.$$

Since $ACC'A'$ is a parallelogram, A and C' lie on opposite sides of CA', so that the first pair of triangles in the chain is anti-oriented. Similarly, $A'CC'A''$ is a parallelogram, so that C and A'' lie on opposite sides of $C'A'$, and the second pair in the chain is anti-oriented. Finally, the last pair is related by a cyclic permutation and so is co-oriented. Thus in the chain there are two changes of orientation, and therefore $AA'C$ and its image $A'A''C$ have the same orientation.

Let us note that any transformation of the plane in which vectors are transformed into equal vectors is a translation. For (with the obvious notation) if $\overrightarrow{M_1M_2} = \overrightarrow{M_1'M_2'}$ it follows that $\overrightarrow{M_1M_2} + \overrightarrow{M_2M_1'} = \overrightarrow{M_2M_1'} + \overrightarrow{M_1'M_2'}$ so that $\overrightarrow{M_1M_1'} = \overrightarrow{M_2M_2'} = \mathbf{a}$, where $\mathbf{a}$ is a constant vector not depending on our choice of M.

We have already seen (Example 3 at the end of Chapter I) that the set of all translations (including the identity transformation) is a group.

8.2. Reflection in a Line

Suppose in the plane we are given a line l. Let us make correspond to each point M of the plane its reflection M' in l (we say M' is the reflection of M in l if l is the perpendicular bisector of the segment MM'). We make the points of l correspond to themselves. This correspondence is called a reflection (Fig. 20).

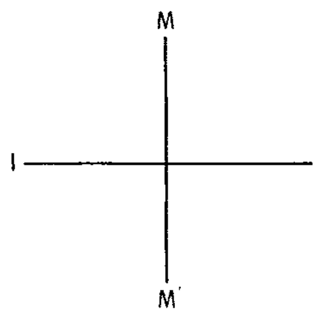


Fig. 20

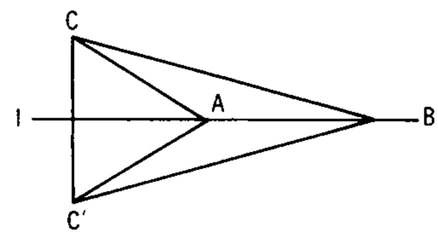


Fig. 21

A reflection is an orthogonal transformation of the second kind. It is clear that lengths are not altered by a reflection, so that it is an orthogonal transformation. To show it is of the second kind, let A and B be two points of l and C a point not on l. Let C' be the reflection of C in l (Fig. 21). Then the triangle ABC is mapped by the reflection into the triangle ABC'. These two triangles are anti-oriented by definition, so that they have opposite orientations (consider the chain whose only two members are ABC, ABC'). So, by Theorem 1, Section 7, the reflection changes the orientation of *every* triangle and thus is of the second kind.

A reflection may be defined as the unique transformation, other than the identity, that leaves fixed at least two given points.

Suppose we are given that the transformation α leaves fixed the points A and B. We already know of two transformations that leave A and B fixed: the identity and the reflection in the line AB. By Theorem 2, Section 7, these are the only two, and since α is not the identity by hypothesis, it must be the reflection.

8.3. Reflection in a Point

Suppose we are given a point O of the plane. Let us make correspond to each point M of the plane the point M' symmetrically opposite it with respect to O. That is, M' is that point of the plane for which O is the midpoint of MM'. We make O correspond to itself. The transformation we have defined is called the *reflection in O*. We show that reflection in O is an orthogonal transformation of the first kind. Let A', B' be the images under the reflection of two given points A, B, respectively. Then, if A, B, O are not collinear, the triangles AOB and $A'OB'$ are congruent (two sides and included angle), so that $A'B' = AB$. We leave the case where A, B, O are collinear to the reader. This shows that reflection is an orthogonal transformation. To see that it is of the first kind, choose A and B not collinear with O and consider the sequence

$$AOB, \quad A'OB, \quad A'OB'.$$

The triangles AOB and $A'OB$ have opposite orientation, since A and A' are on opposite sides of OB; and $A'OB$ and $A'OB'$ have opposite orientation, since B and B' are on opposite sides of OA'. So AOB and $A'OB'$ have the same orientation, and the transformation is of the first kind.

Under a reflection in a point, each segment is transformed into an equal segment having the opposite direction: strictly speaking, $\overrightarrow{A'B'} = -\overrightarrow{AB}$, for every pair of points A, B. If A and B are collinear with O, then $A'B'$ is collinear with AB, and otherwise it is parallel (but pointing the other way).

Conversely, it is true that any transformation of the plane in which vectors are taken into their negatives is a reflection in a point.

For it is first clear that the transformation is orthogonal. It is not the identity, so choose a point A whose image A' is distinct from it. Let O be the midpoint of AA' and M any point not on the line AA'. We show that the image M' of M is its reflection in O. We know that $\overrightarrow{AM} = -\overrightarrow{A'M'} = \overrightarrow{M'A'}$, so that $AMA'M'$ form the vertices of a parallelogram. Its diagonals bisect each other in O, so that O is the midpoint of MM'. The proof that the image of a point on AA' is its reflection in O is left to the reader.

8.4. Rotation

Suppose that we fix in the plane a point O. We wish to define the transformation consisting of a rotation about O through a given angle. The reader will probably have a fairly clear idea without any explanation what such a transformation should be. We need to give a rigorous definition, which is best done by first considering the concept of an *oriented angle*.

Let S be the circle with center O and radius 1. If M and N are any points on S, the ordinary angle $\angle MON$ may be defined as the pair of rays (OM, ON) and its (radian) measure as the shorter distance between M and N measured around S.

For our purposes, this notion is inadequate. We wish to distinguish between the rotation about O that carries M into N and the rotation that carries N into M. We shall thus need to distinguish between the angle $\angle MON$ (the angle from M to N) and $\angle NOM$ (the angle from N to M). The distinction is exactly parallel to the one we must make for translations; we need to distinguish between the translation that carries M into N (translation through the vector $\overrightarrow{MN}$), and that which carries N into M (translation through the vector $\overrightarrow{NM}$). The concept corresponding to "line segment MN" here is "ordinary angle $\angle MON$," and the concept corresponding to "vector $\overrightarrow{MN}$" will be "oriented angle $\angle MON$."

The formal definition of an oriented angle is quite easy: it is an *ordered* pair of rays OM and ON from a given origin O. This definition is exactly similar to the definition of a vector or a directed line segment as an ordered pair of points. Thus the notation we should perhaps use is (OM, ON), where we take note of the order; this angle is to be considered distinct from $(ON, OM$.) However, it is usual in practice to use the same notation as for an unoriented angle. Of course, we still have to take note of the order; $\angle MON$ is different from $\angle NOM$.

We define the oriented angles $\angle MON$ and $\angle M'O'N'$ to be *equal* provided there is an orthogonal transformation *of the first kind* carrying OM onto $O'M'$ and ON onto $O'N'$. An equivalent definition is as follows. Since M and N are only used to indicate the two rays, we may as well take $OM = 1 = ON$ and $O'M' = 1 = O'N'$. Then the oriented angles $\angle MON$ and $\angle M'O'N'$ are equal if and only if the triangles MON and $M'O'N'$ are congruent *and have the same orientation.*

Our next task is to define the measure of an oriented angle $\angle MON$, just as we would have to define lengths if we were talking about segments. We start by defining a counterclockwise arc from M to N (we are assuming as before that M and N both lie on the circle S with center O). We fix an oriented triangle ABC in the plane. Then a certain one of the arcs from M to N will be said to be the counterclockwise arc if the triangle MPN has the same orientation as the triangle ABC, where P is some point of the arc we are considering. Of course, we must verify that this gives us an unambiguous answer. Verification follows from the facts that if P and Q lie on the same one of the arcs from M to N, the oriented triangles MPN and MQN will have the same orientation, whereas if they lie on opposite arcs, these triangles will have opposite orientations (Fig. 22). To make this definition accord with our usual idea of what "counterclockwise arc" should mean, we merely have to choose the orientation of triangle ABC suitably. Our choice does not make any difference to the mathematics involved—all it affects is the appropriateness of our terminology. Of course, once we have chosen our triangle, we must remain with our choice.

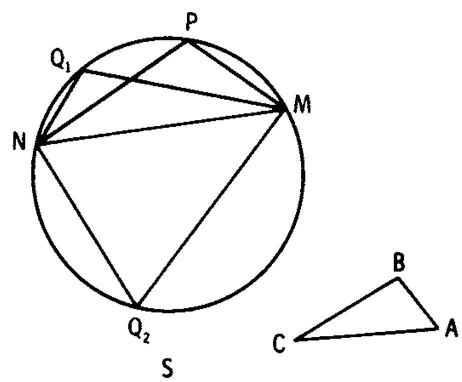


Fig. 22

We now define the measure of the oriented angle $\angle MON$ to be the length α of the counterclockwise arc from M to N. To ensure that this definition is respectable, we must check that equal angles have the same measure. But this is obvious. We shall write $\angle MON = \alpha$ to mean that the angle $\angle MON$ has measure α.

We next define the sum of two oriented angles. We define $\angle MON + \angle N'O'P'$ to be the angle $\angle MOP$, where $\angle NOP = \angle N'O'P'$. That is, we construct an angle $\angle NOP$ with initial ray ON, and equal to the angle $\angle N'O'P'$ (Fig. 23). We may

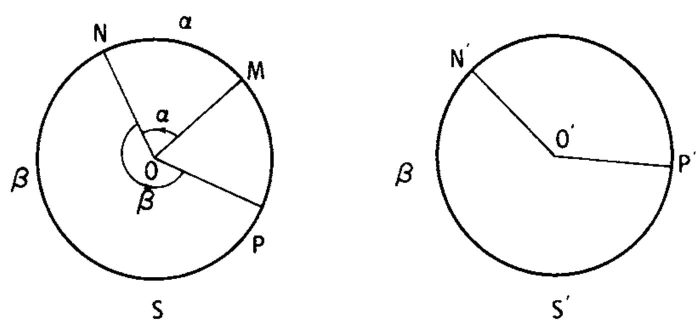


Fig. 23

easily verify that the sums of angles equal in pairs are themselves equal (according to our definition of equality of oriented angles). We must now establish the connection between the sum of two angles (an angle) and the sum of their measures (a number). A little thought will convince the reader that what we would like to say is: if $\angle AOC = \alpha$ and $\angle B'O'D' = \beta$, then their sum $\angle AOC + \angle B'O'D'$ has measure $\alpha + \beta$. After all, we can make exactly corresponding assertion about line segments, and it is just this correspondence between numbers and segments (congruent segments have equal lengths, and the sum of two segments has length equal to the sum—different meaning of sum!—of the lengths of its components) that makes numbers relevant to calculations with lengths.

In our case, however, this equivalence will not work completely. For the measure of an oriented angle is the length of an arc of the unit circle and thus is a number lying between 0 and 2π. Yet $\alpha + \beta$ in the assertion above might be greater than 2π (Fig. 24).

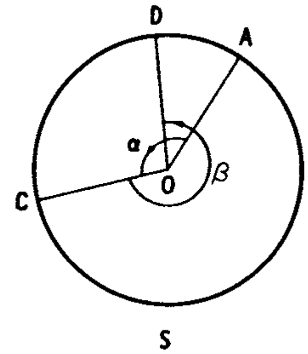


Fig. 24

Intuitively we would say there is no difficulty; a rotation counterclockwise through $2\pi + \alpha$ brings us to the same point as a rotation through α. So perhaps we might say that an oriented angle $\angle MON$ should be allowed measure α whenever a point moving counterclockwise around S through a distance α starting from M would end up at N. This would allow the point to make a number of revolutions before stopping and would also allow a number of different measures to the same angle. However, as far as a formal definition is concerned, it is somewhat difficult to capture this concept. The dynamic concept of a point moving around a circle is much more elusive than the static concept of a counterclockwise arc. So we will content ourselves with two possible formulations that can be made rigorous, and the reader (and author!) need not stop thinking about moving points in deference to the formulations if he does not wish.

The first course is to allow an infinity of values for the measure of the angle $\angle MON$. If α is the value we have already defined (the "principal value"), we also allow all the values $\alpha + 2\pi$, $\alpha + 4\pi$, $\alpha + 6\pi$, ... It is convenient, in addition, to allow the values $\alpha - 2\pi$, $\alpha - 4\pi$, ... Intuitively, these would correspond to the idea of a point reaching N from M by going so many revolutions clockwise after first going α counterclockwise.

The second course is to perform all our arithmetic with the measure of oriented angles not in the real numbers, but in the real numbers "modulo 2π." Since we are concerned only with addition of angles, we are working in the additive group of real numbers modulo 2π. It is a group because every element has an additive inverse; the inverse of α is $2\pi - \alpha$, since the sum of these is 2π, which we agree to identify with 0. There is no difficulty in a corresponding definition of subtraction for angles; we may define $-\angle MON = \angle NOM$, and then define $\angle AOC$

$-\angle B'O'D'$ to be the same as $\angle AOC + (-\angle B'O'D')$ (Fig. 25).

The question of which definition to adopt is one of convenience. If we are working, for example, with a problem concerning lengths of cable unrolling from a drum, the first definition is the more appropriate. If we are working with rotations of the plane, as we will be, the second is the more appropriate, because the rotation through α and the rotation through $2\pi + \alpha$ are identical in their effect on the plane, even if they are not identical on a spool of thread.

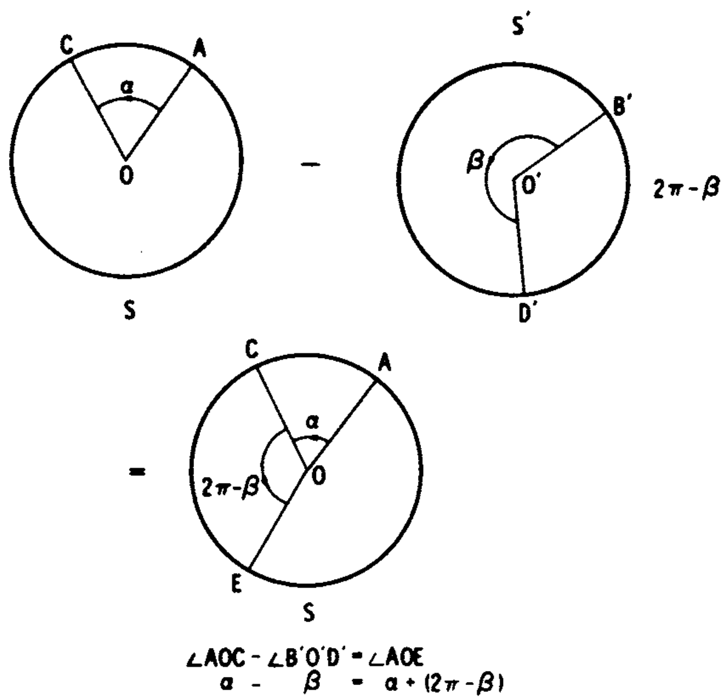


Fig. 25

Whichever course we adopt, we can now safely say that if $\angle AOC = \alpha$ and $\angle B'O'D' = \beta$, then $\angle AOC + \angle B'O'D' = \alpha + \beta$. In the first case this says that if α is one of the measures of the angle $\angle AOC$ and β is one of the measures of $\angle B'O'D'$, then $\alpha + \beta$ (the ordinary sum of the ordinary signed numbers) is one of the measures of the sum angle $\angle AOC + \angle B'O'D'$ (constructed according to the definition we gave earlier).

In the second case this says that if α is *the* measure of the angle $\angle AOC$ and β is *the* measure of the angle $\angle B'O'D'$, then $\alpha + \beta$ is *the* measure of the sum angle. However, in this case, α and β are not strictly real numbers, but elements of the group of real numbers modulo 2π, and $\alpha + \beta$ is the result of performing the operation of addition *as defined in this group*. If we represent α and β as real numbers lying between 0 and 2π, then their sum will be represented either as $\alpha + \beta$ (the ordinary sum) or $\alpha + \beta - 2\pi$. For example, $\pi/4 + \pi/2 = 3\pi/4$, but $\pi/4 + 3\pi/4 = 0$, $\pi/2 + 3\pi/4 = \pi/4$.

We are now in a position to define a rotation. Given the *center of rotation* O, and an oriented angle $\angle AOC = \alpha$, the *angle of rotation*, we make correspond to each point M of the plane the point M' for which $OM = OM'$ and the oriented angle $\angle MOM' = \alpha$. Of course, $O' = O$.

A rotation about O is completely defined by its effect on a single point $M \neq O$. For if the image of M is M', then the rotation is the one through the angle $\alpha = \angle MOM'$.

A rotation is an orthogonal transformation of the first kind. For given M and N, let the oriented angle $\angle MON$ have measure θ (principal value). Then

$$\begin{aligned}\angle M'ON' &= \angle M'OM + \angle MON + \angle NON' \\ &= -\alpha + \theta + \alpha \\ &= \theta.\end{aligned}$$

We examine what this sequence of equalities says. The first line asserts the equality (according to our definition of equality of oriented angles) of the left-hand angle and the sum angle of the three angles on the right. Actually the two are not merely equal; they are identical. We have already discussed the step from the first equality to the second. The last step is purely algebraic. In the first interpretation, it is an equality in the additive group of reals; in the second, it is an equality in the additive group of reals mod 2π.

We have thus shown that the triangles MON and $M'ON'$ have the same principal value for their oriented angle at O. But then they also have the same ordinary angle at O. It will be

θ if $O \leq \theta \leq \pi$, and $2\pi - \theta$ otherwise. Now $OM = OM'$ and $ON = ON'$, by definition of the rotation. So the triangles are congruent (side, angle, side). We do not exclude the case where M, O, and N are collinear ($\theta = 0$ or π); the congruence still follows. A reference to our original definition of equality of oriented angles shows more; the two angles have the same orientation. We have thus proved our assertion (Fig. 26).

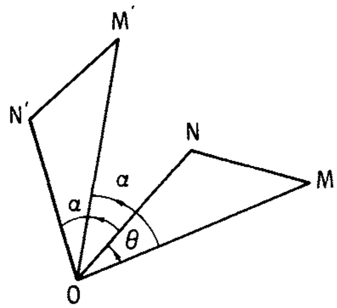


Fig. 26

9. Representations of Orthogonal Transformations as Products of the Fundamental Orthogonal Transformations: Translations, Reflections, and Rotations

We have examined three special types of orthogonal transformation: translation, reflection, and rotation. In this section we shall show that any plane orthogonal transformation may be represented as a product of such special transformations.

Theorem 1. *Any (plane) orthogonal transformation of the first kind is either a translation or a rotation (including the possibility of a rotation through π, that is, reflection in a point).*

Proof. Let A be any point of the plane, B its image under the transformation α, and C the image of B under α.

There are three possible cases to consider:

Case 1. The line segments AB and BC lie on the same line and point the same way (Fig. 27). In this case, the translation by the vector $\overrightarrow{AB}$ $(=\overrightarrow{BC})$ has the same effect on A and B as does α. So by Theorem 2 of Section 7, α, in fact, *is* this translation.

Case 2. The line segments AB and BC lie on the same line but point in opposite directions. In this case, the points C and A coincide (Fig. 28).

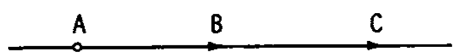


Fig. 27

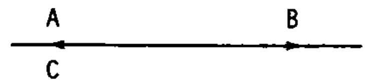


Fig. 28

α takes A into B and B into A, but so too does the reflection in the midpoint O of AB. Since the reflection too is a transformation of the first kind, again by Theorem 2 (Section 7), α must be this reflection.

Case 3. The line segments AB and BC do not lie on the same line (Fig. 29). Let O be the point of intersection of the perpendicular bisectors to AB and BC. Then $AO = BO = CO$, so that the triangles ABO, BCO are congruent. But then the rotation β about O that carries A into B also carries B into C, so that by the usual argument $\beta = \alpha$, and α is a rotation. ▼

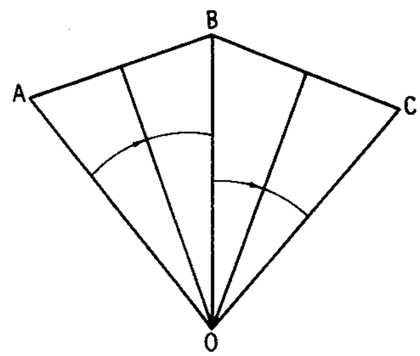


Fig. 29

Note. If in the plane we are given two equal line segments AB and $A'B'$, then we may give a direct description of the orthogonal transformation of the first kind that takes A into A' and B into B'. For, if AB and $A'B'$ are parallel and point in the same direction, the transformation is the translation through $\overrightarrow{AA'} = \overrightarrow{BB'}$. If AB and $A'B'$ are parallel but point in opposite directions, the transformation is the reflection in the midpoint O of AA' (or BB'). (These assertions remain true even when AB and $A'B'$ are collinear.) Suppose now AB and $A'B'$ are not parallel. We know (Theorem 2 of Section 7) that there exists a unique transformation of the first kind taking A to A' and B to B'. Now this transformation cannot be either a translation or a reflection in a point (since this would take the line segment AB into a parallel line segment); it must therefore be a rotation.

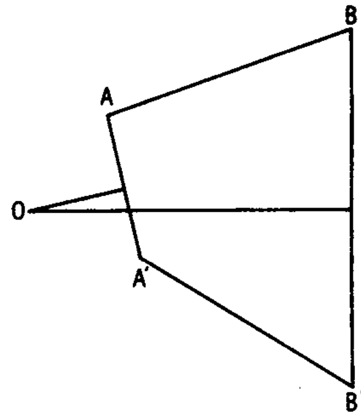


Fig. 30

If the center of this rotation is O, then $OA = OA'$, so that O lies on the perpendicular bisector of AA', and similarly O lies on the perpendicular bisector of BB'. Thus O is the point of intersection of these (nonparallel!) lines. The reader should give a direct proof for himself that the rotation about O that takes A into A' will also take B into B' (Fig. 30).

Theorem 2. *Any plane orthogonal transformation α of the second kind can be represented uniquely as the product of a reflection σ in some line l and a translation τ parallel to l. The line l is uniquely defined by α, and $\sigma\tau = \tau\sigma$.*

Proof. We distinguish the same three cases as we did for Theorem 1. Let B be the image of the point A under α, and let C be the image of B.

Case 1. AB and BC lie on the same line and point the same way (Fig. 31). Let τ be the translation determined by the vector $\overrightarrow{AB}$ $(=\overrightarrow{BC})$ and σ the reflection in the line AB.

Then $\beta = \sigma\tau = \tau\sigma$, like α, carries A into B and B into C. Since α and β are both of the second kind, Theorem 2 (Section 7) shows that they are the same.

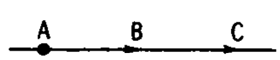


Fig. 31

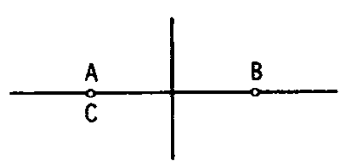


Fig. 32

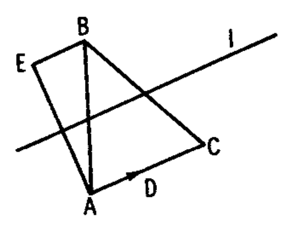


Fig. 33

Case 2. C coincides with A (Fig. 32). Then α is, by the familiar argument, the reflection σ in the perpendicular bisector of AB.

In this case we take τ to be the identity translation, and have $\sigma\tau = \tau\sigma = \sigma$.

Case 3. AB and BC do not lie on the same line (Fig. 33).

Let l be the line through the midpoints of AB and BC, and let D be the midpoint of AC. Let σ be the reflection in l and τ the translation along $\overrightarrow{AD} = \overrightarrow{DC}$. Then σ takes A to E, and τ takes E to B, so that $\tau\sigma$ takes A to B. Similarly, σ takes B to D, and τ takes D to C, so that $\tau\sigma$ takes B to C. By the usual argument, $\alpha = \tau\sigma$. It is easily checked that $\sigma\tau = \tau\sigma$.

We now show that such a representation of α is unique. Suppose $\alpha = \sigma\tau$, where τ is not the identity. Then l is the only line mapped into itself by α. For if m meets l in P, its image m' meets l in $P' \neq P$ (Fig. 34), so that $m' \neq m$. If n is parallel to l then $n' \neq n$ (Fig. 35).

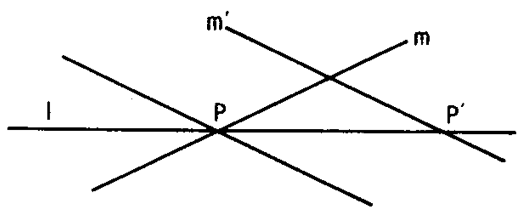


Fig. 34

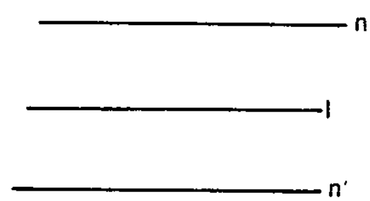


Fig. 35

Suppose now that $\alpha = \sigma'\tau'$, where σ' and τ' have axis l'. By what we have proved, $l' = l$ (since l' is invariant under α). Thus $\sigma' = \sigma'$, and, by cancellation (end of Chapter I), $\tau' = \tau$.

We have shown that if α has one representation $\alpha = \sigma\tau$, where $\tau \neq \varepsilon$, then the representation is unique. The only remaining possibility is that for *every* representation, $\tau = \varepsilon$, But then $\alpha = \sigma$ has the unique representation $\sigma\varepsilon$. ▼

Theorem 3. *Any orthogonal transformation of the first kind may be represented as the product of two reflections in lines; any orthogonal transformation of the second kind either is itself a reflection in a line or can be represented as the product of three such reflections.*

Proof. Let α be an orthogonal transformation of the first kind. Then, by Theorem 1, it is either a rotation or a translation or a reflection in a point.

(1) Suppose first that α is a translation, and that $\alpha(A) = A'$ (Fig. 36). Let l_1 be the perpendicular bisector of AA' and l_2 the perpendicular to AA' through A'. Let σ_1 and σ_2 be the reflections in these two lines, respectively.

Under the transformation $\sigma_2\sigma_1$, the point A will go into A', and any point B such that AB is parallel to l_1 will go into the same point B' as it would under the transformation α. So, by the usual argument, $\alpha = \sigma_2\sigma_1$.

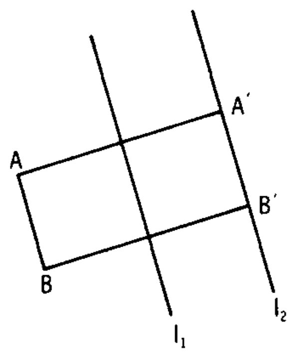


Fig. 36

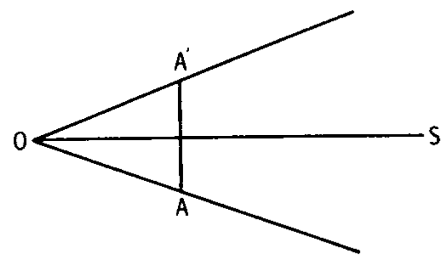


Fig. 37

(2) Suppose now that α is a rotation. Let O be its center and A' the image of some point A (Fig. 37). Let OS be the perpendicular bisector of AA' and σ_1 and σ_2 the reflections in OS and OA', respectively. Then, under $\sigma_2\sigma_1$, O remains in place, and A is taken to A'. So, by the usual argument, $\alpha = \sigma_2\sigma_1$.

The reader should check that this construction works also for a reflection in O; in this case, OS will be perpendicular to AOA'.

Suppose now that α is an orthogonal transformation of the second kind. Then either it is itself a reflection or it can be represented as the product of a reflection and a translation (this is part of the content of Theorem 2). But a translation can itself be represented as the product of two reflections, as we have just seen. So α can be represented as the product of three reflections. ▼

Note that the representation of an orthogonal transformation as a product of reflections is not unique. For transformations of the first kind, we make the situation clear in Theorem 4; for transformations of the second kind, the situation is more complicated.

Theorem 4. *Consider the translation $\tau = \sigma_2\sigma_1$, where σ_1 and σ_2 are reflections. Let l_1 and l_2 be the axes of σ_1 and σ_2, and suppose that τ is the translation associated with the vector* **a**. *Then l_1 and l_2 are both perpendicular to* **a**. *Subject to this condition, we may choose either σ_1 or σ_2 arbitrarily, but our choice then fixes the other.*

Consider next the rotation $\rho = \sigma_2\sigma_1$, where σ_1 and σ_2 are reflections. Let l_1 and l_2 be the axes of σ_1 and σ_2, and let the center of the rotation be the point O. Then l_1 and l_2 both pass through O. Subject to this condition, we may choose either σ_1 or σ_2 arbitrarily, but our choice then fixes the other.

We leave the proofs of these statements, which are quite easy, to the reader.

10. Orthogonal Transformations of the Plane in Coordinates

Let us introduce in the plane a rectangular Cartesian system of coordinates xOy with unit points E_1 and E_2. Let $M(x, y)$ be any point of the plane and $M'(x', y')$ its image under the orthogonal transformation α. In this section we shall derive formulas expressing the coordinates x', y' of M' in terms of the coordinates x, y of M (all in the given coordinate system).

10.1. Translation

Let τ be the plane translation determined by the vector **t**. Suppose that in the given coordinate system **t** has the coordinates a, b (Fig. 38). Let x, y, x', y' be as above, where $\alpha = \tau$.

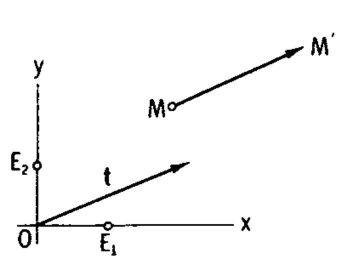


Fig. 38

Then, by definition, $\overrightarrow{MM'} = \mathbf{t}$. This means that the coordinates of $\overrightarrow{MM'}$ are a and b (since two vectors are equal if and only if they have the same coordinates). But the coordinates of $\overrightarrow{MM'}$ are the differences between the coordinates of its endpoint and its initial point: that is, $x' - x$, $y' - y$.

So we have $x' - x = a$; $y' - y = b$; and

$$x' = x + a; \qquad y' = y + b.$$

This is the expression for a translation, written coordinatewise. We may also write it $\tau(x, y) = (x + a, y + b)$.

10.2. Reflection in a Line

We shall consider only reflection in a line through O, for a reason that will appear in Section 10.5.

Let l be a line through O making an oriented angle γ with the x axis. Let M be an arbitrary point other than O, and let (r, θ) be its polar coordinates. That is, $OM = r$, and the oriented angle $xOM = \theta$. Then (Fig. 39), the polar coordinates of M' are $(r, 2\gamma - \theta)$. So

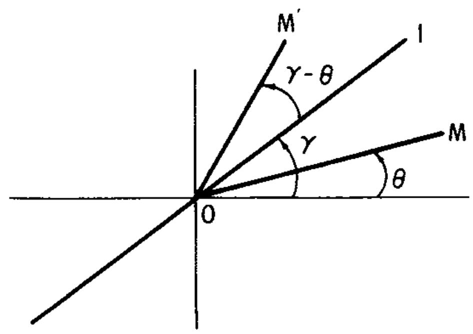


Fig. 39

$$x' = r\cos(2\gamma - \theta) = r\cos 2\gamma\cos\theta + r\sin 2\gamma\sin\theta;$$

$$y' = r\sin(2\gamma - \theta) = r\sin 2\gamma\cos\theta - r\cos 2\gamma\sin\theta.$$

But

$$x = r\cos\theta,$$

$$y = r\sin\theta.$$

So

$$\begin{aligned} x' &= x\cos 2\gamma + y\sin 2\gamma, \\ y' &= x\sin 2\gamma - y\cos 2\gamma. \end{aligned} \tag{1}$$

This is the equation of the reflection in the line l. Note that the addition formulas for sin and cos that we used, as well as the formulas $x = r\cos\theta$, $y = r\sin\theta$ for the point $M(x, y)$ with polar coordinates (r, θ), are true *only* when we take α and θ to be oriented angles.

We did not consider the formula for the point into which O is taken, but we see at once that (1) is valid for it too.

Note the special cases where l is the x axis or the y axis. In the first case, $\gamma = 0$, and (1) reduces to

$$x' = x, \qquad y' = -y.$$

In the second case, $\gamma = \pi/2$, and the formula reduces to

$$x' = -x, \qquad y' = y.$$

10.3. Reflection in a Point

We take the point to be the origin O (Fig. 40). Then, for any point $M(x, y)$, the point M' symmetrically opposite it with respect to the origin is $M(x', y')$, where

$$x' = -x, \qquad y' = -y. \tag{2}$$

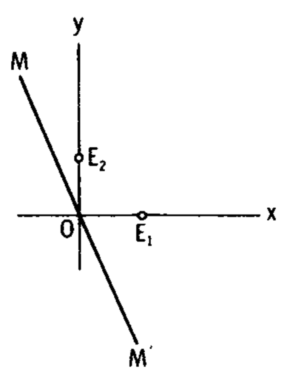


Fig. 40

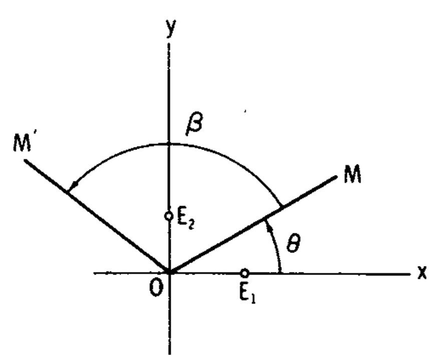


Fig. 41

10.4. Rotation

We take the center of the rotation to be the origin (Fig. 41). Let ρ be the rotation about O through the oriented angle β. Let M be any point of the plane other that O, and (r, θ) its polar coordinates. If $\rho(M) = M'$, then clearly the polar coordinates of M' are $(r, \theta + \beta)$. So if $M' = M'(x', y')$, we have

$$x' = r\cos(\theta + \beta) = r\cos\theta\cos\beta - r\sin\theta\sin\beta,$$
$$y' = r\sin(\theta + \beta) = r\cos\theta\sin\beta + r\sin\theta\cos\beta.$$

But $x = r\cos\theta$, $y = r\sin\theta$, so that

$$\begin{aligned} x' &= x\cos\beta - y\sin\beta; \\ y' &= x\sin\beta + y\cos\beta. \end{aligned} \tag{3}$$

This formula for the result of a rotation clearly holds also for the point O. Note that the case of Section 10.3 (that of reflection in O) is obtained from (3) by taking $\beta = \pi$; that is, reflection in a point is the same as rotation through two right angles.

10.5. The General Case

I. Suppose now that α is any orthogonal transformation of the first kind, and $\alpha(O) = O'(a, b)$. We introduce a new system of coordinates with origin at O' and axes parallel to the old. Let us use the notation $M(x, y) = M'(x^*, y^*)$ to mean that the point whose coordinates in the old system are x, y has coordinates x^*, y^* in the new system. Then $M(x, y) = M'(x - a, y - b)$. Let τ be the translation which takes O into O'. Then $\alpha = \alpha'\tau$, where α' is another orthogonal transformation of the first kind, which leaves O' fixed. By a result in Section 8, α' is the rotation about O' through an oriented angle of, say, β.

Then, for any point $M(x, y)$, we have

$$\begin{aligned} \alpha(M(x, y)) &= \alpha'\tau(M(x, y)) = \alpha'(M(x + a, y + b)) \\ &= \alpha'(M'(x, y)) \\ &= M'(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta) \\ &= M(x\cos\beta - y\sin\beta + a, x\sin\beta + y\cos\beta + b). \end{aligned}$$

Thus we have

$$\begin{aligned} x' &= x\cos\beta - y\sin\beta + a; \\ y' &= x\sin\beta + y\cos\beta + b. \end{aligned} \tag{4}$$

Note that we have incidentally proved the following theorem:

Theorem 1. *A given orthogonal transformation α of the first kind may be expressed in the form $\alpha = \rho\tau$ $(\alpha = \tau'\rho')$, for some translation τ (τ') and rotation ρ (ρ'), if and only if ρ (ρ') is a rotation through a certain fixed angle β, determined by α.*

If the center of ρ is O', then τ is the translation along the vector $\overrightarrow{OO'}$, where O is the inverse image under α of O'; if the center of ρ' is O, then τ' is the translation through $\overrightarrow{OO'}$, where $O' = \alpha(O)$.

II. Suppose that α is an orthogonal transformation of the second kind. Let $\alpha(O) = O'(a, b)$, and let τ be the translation through the vector $\overrightarrow{OO'}$. Then $\alpha = \alpha'\tau$, where α' is a transformation of the second kind leaving O' fixed. By a result in Section 8, α' is a reflection in a line through O'. We introduce new coordinates as before, and since in these coordinates α' is given by Eq. (1), we find, as before, that $\alpha(M(x, y)) = (x', y')$, where

$$\begin{aligned} x' &= x \cos 2\gamma + y \sin 2\gamma + a, \\ y' &= x \sin 2\gamma - y \cos 2\gamma + b. \end{aligned} \tag{5}$$

We also have:

Theorem 2. *A given orthogonal transformation α of the second kind may be expressed in the form $\alpha = \sigma\tau(\alpha = \tau'\sigma')$, for some translation $\tau(\tau')$ and reflection $\sigma(\sigma')$, if and only if $\sigma(\sigma')$ is the reflection in some line $l(l')$ parallel to a given line m determined by α.*

Here m is any line making an oriented angle γ with the x axis, in our notation. If O' is any point on the axis l of σ, then τ is the translation through $\overrightarrow{OO'}$, where O is the inverse image of O'. Similarly, if O is any point on l', τ' is the translation through the vector $\overrightarrow{OO'}$, where O' is the image of O.

Note that the general equation [Eq. (4) or (5)] is linear in the coordinates and that, if α is of the first kind, its determinant is equal to $+1$:

$$\begin{vmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{vmatrix} = +1;$$

whereas if it is of the second kind, its determinant is -1:

$$\begin{vmatrix} \cos 2\gamma & \sin 2\gamma \\ \sin 2\gamma & -\cos 2\gamma \end{vmatrix} = -1.$$

It may be suggested that the sign indicates whether the transformation preserves orientation, and its absolute value, 1, indicates that areas are multiplied by a factor of 1.

11. Orthogonal Transformations in Space

Orthogonal transformations in space are defined in exactly the same way as for the plane and, like those in the plane, fall into two classes—transformations of the first and second kind—according to whether they do or do not preserve orientation.

If A, B, C, D are four non-coplanar points and if A', B', C', D' are four points such that $AB = A'B'$, $AC = A'C'$, $AD = A'D'$, $BC = B'C'$, $BD = B'D'$, and $CD = C'D'$, then there is a unique orthogonal transformation taking A, B, C, D to A', B', C', D', respectively.

If A, B, C are any three noncollinear points and A', B', C' three points such that the triangles ABC and $A'B'C'$ are congruent, then there is a unique orthogonal transformation of each kind (first and second) taking A into A', etc. The proofs of these theorems may be carried out similarly to the planar case (Theorem 8, Section 5, and Theorem 2, Section 7).

The set of all orthogonal transformations of space is a group, and the subset of all transformations of the first kind is a subgroup of this group.

The fundamental types of orthogonal transformation in space are translation, reflection in a plane, rotation about a line (including rotation through two right angles, which is reflection in the line), and reflection in a point.

11.1. Translation

A translation is defined exactly as in the plane: each point M is taken into the point M' for which $\overrightarrow{MM'} = \mathbf{a}$, where $\mathbf{a}$ is a given fixed vector. Translation is an orthogonal transformation of the first kind. That it is an orthogonal transformation may be proved as in the case of the plane; we now show that it is of the first kind. Let A be any point, A' its image, and π any plane through AA'. Then α induces an orthogonal transformation on π. Let ABC be any triangle in π, and let $A'B'C'$ be its image under α. Then $A'B'C'$ lies in π and has the same orientation as ABC. Let S be any point not in π and S' its image under α. Consider any chain D of triangles joining ABC to $A'B'C'$; the number of pairs of adjacent triangles in D that have opposite orientations is even. Let D' be the chain of tetrahedra joining $ABCS$ to $A'B'C'S$, obtained from D by letting S be the fourth vertex of each tetrahedron, the other vertices being those of the triangles of D in order. The reader should verify that D' is a chain and that adjacent tetrahedra in the sequence have the same orientation if and only if the corresponding triangles in D have the same orientation. It follows that the number of pairs of adjacent tetrahedra that have opposite orientation is even, so that $ABCS$ and $A'B'C'S$ have the same orientation. But $A'B'C'S$ and $A'B'C'S'$ have the same orientation, since S and S' lie on the same side of π, and therefore so do $ABCS$ and $A'B'C'S'$. This shows that α is of the first kind.

In order that an orthogonal transformation be a translation, it is necessary and sufficient that every vector $\overrightarrow{AB}$ be transformed into an equal vector $\overrightarrow{A'B'}$. The proof is just as for the case of the plane. The set of all translations in space (including the identity transformation, which is the translation by the zero vector) forms a subgroup of the group of orthogonal transformations of the first kind.

11.2. Reflection in a Plane

Suppose we are given a plane π in space. We place in correspondence with each point M of space its reflection M' in π. That is to say, π is the perpendicular bisector of MM'. The points of π itself are placed in correspondence with themselves.

Such a transformation is called the *reflection in* π. As in the planar case, we may show that reflection in a plane is an orthogonal transformation of the second kind. It is also true that α is the reflection in a plane if and only if there are three noncollinear points which remain invariant under α, and α is not the identity. The proof is as in the planar case, the plane π of reflection being the plane through the given three points.

11.3. Reflection in a Line

Suppose we are given a line l in space. We place in correspondence with each point M of space its reflection M' in l. That is to say, the lines MM' and l intersect in the midpoint of the former and at right angles. The points of l are put in correspondence with themselves. Such a transformation is called a *reflection* (the reflection in the line l), and l is called its *axis*.

Reflection in a line is an orthogonal transformation of space of the first kind. For any segment parallel to the axis is transformed into an equal segment also parallel to it. Any segment lying in a plane π perpendicular to l is transformed into an equal and parallel segment lying in π (since α induces in π the transformation that reflects each point of π in the point O of intersection of π with l). Suppose now MN is a segment neither parallel nor perpendicular to l. Let π be the plane through M perpendicular to l, and let P be the base of the perpendicular from N to π. Let M', N', P' be the reflections in l of M, N, P, respectively. Then, by what we have already said, $M'P' = MP$, and $N'P' = NP$. Since MPN and $M'P'N'$ are both right-angled triangles, it follows that $M'N' = MN$.

We show now that reflection in l is a transformation of the first kind. Let O and S be any points of l and A and B be points of the plane π through O and perpendicular to l but not collinear with O. Let A', B' be the reflections in l of A and B, respectively. Then the triangles AOB and $A'OB'$ both lie in π and have the same orientation, since they are reflections of each other in O. It follows, by the same argument as in the section on translations, that the tetrahedra $SAOB$ and $SA'OB'$ also have the same orientation.

11.4. Rotation

Let l be any line of space and β a fixed oriented angle. For an arbitrary point M not on l, let π be the plane through M and perpendicular to l, and suppose that π intersects l in O. Then we put in correspondence with M its image M' under the rotation of π with the center O and through the oriented angle β. We put each point of l in correspondence with itself. This transformation of space is called the *rotation about l through β*, and it is an orthogonal transformation of the first kind. To prove this, we note first that, if MP is a segment parallel to l, then its image under α is an equal segment also parallel to l (see Fig. 42). For the triangles OPP' and O^*MM' are congruent (two equal sides and the included angle β), and since OO^* and PM are perpendicular to π, so too must $P'M'$ be. It is clear also that the image of a line segment PN perpendicular to l is an equal segment. The proof that the image of any line segment MN is an equal segment is now completed as in Section 11.3. The proof that the rotation is of the first kind is identical to the proof for the case of a reflection in l (11.3).

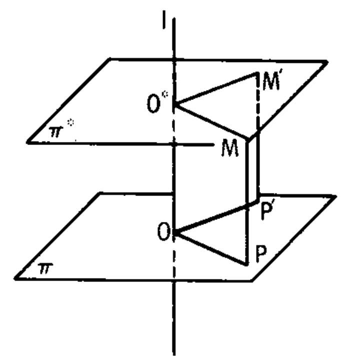


Fig. 42

A rotation about a line is uniquely determined by its axis l and a pair of corresponding points A and A' not on l. For if π is the plane through A and A' perpendicular to l, and P is the point of intersection of l and π, then the rotation is through $\beta = \angle APA'$. Conversely, given a line l and points A and A' not on l, there exists a rotation with axis l taking A to A' if and only if A and A' are equidistant from l, and $AA' \perp l$.

The set of all rotations about a given line l forms a group (of course, we include the rotation through π, that is, the reflection in l; and the rotation through 0, that is, the identity transformation). In fact, each rotation is associated with an oriented angle in exactly the same way as the case of a plane rotation about a given point, and the group of space rotations about a line and plane rotations about a point are effectively the same. These groups are infinite and commutative.

11.5. Reflection in a Point

Let O be a fixed point of space. Let us make correspond to each point M of space its reflection M' in O. That is, O is the midpoint of the line segment MM'. We made O correspond to itself. The transformation we have thus defined is called the *reflection in* O, and O is called its *center*.

A reflection of space in a point is an orthogonal transformation of the second kind. For let M and N be any two points not collinear with O and π the plane through O, M, N. Then the reflection induces a reflection about O in the plane π, and since we already know this reflection is an orthogonal map, we conclude that $M'N' = MN$ (of course, M' and N' lie in π). If O lies on MN, we leave the proof to the reader.

To show that the transformation is of the second kind, let $OABC$ be any tetrahedron, and consider the chain of tetrahedra

$$OABC, \quad OABC', \quad OAB'C', \quad OA'B'C'.$$

The members of any successive pairing of these tetrahedra have opposite orientations, a total of three orientation changes. Thus $OA'B'C'$ has the opposite orientation from $OABC$.

12. Representation of an Orthogonal Transformation of Space as a Product of Fundamental Orthogonal Transformations

Theorem 1. (Chasles). *Any orthogonal transformation of the first kind having at least one fixed point is a rotation about an axis l passing through this point (in particular, it may be the reflection in l or the identity transformation).*

Proof. Let α be an orthogonal transformation of the first kind, having the fixed point O. It may be that α is the identity transformation, and we exclude this case. Let A be a point whose image B under α does not coincide with it, and let C be the image of B. Then $C \neq B$, since $BC = AB$. If $C = A$, and O, A, B are not collinear, then α has the same effect on O, A, B as the reflection σ in the line OD, where D is the mid-point of the segment AB. (Note that $OA = OB$.) Since α and σ are both of the first kind and have the same effect on three noncollinear points, they coincide, and α is a reflection. If, on the other hand, $OA = OB$ lie on the same line for every choice of A, then α would clearly be the reflection in O—a contradiction, since this reflection is of the second kind.

Suppose finally that A, B, C are all distinct. A and B are not reflections of each other in O, since otherwise the line AOB would be mapped into BOC, so that the distinct lines AB and BC would have the two common points O and B. A, B, C cannot be collinear, since they are equidistant from O and are distinct in pairs. Let π and π' be the planes through O and perpendicular to AB and BC, respectively, and let l be their line of intersection. Then l is perpendicular to AB and to BC and, therefore, to the plane ABC. Let l meet this plane in O^*. Consider the tetrahedron OO^*AB. Its faces OO^*A and OO^*B are both right triangles, and since $OA = OB$ and OO^* is common, they are congruent. So $O^*A = O^*B$. Similarly $O^*B = O^*C$, and the triangles AO^*B and BO^*C are congruent (since they have three equal sides), and in particular $\gamma = \angle AO^*B = \angle BO^*C$. Let ρ be the rotation about l through the angle γ. Then clearly ρ takes A into B and B into C, and since (Section 11.4) it is of the first kind, we have $\alpha = \rho$. ▼

Theorem 2. *Any orthogonal transformation of the first kind either is the identity, a translation, or a rotation; or can be represented uniquely as the product $\gamma\beta$ of a rotation γ about some axis l and a translation β parallel to l. Moreover, $\gamma\beta = \beta\gamma$.*

Proof. Let α be an orthogonal transformation of the first kind.

If α is the identity, $\beta = \varepsilon = \gamma$. So suppose that it is not, and let A be a point such that $\alpha(A) = A' \neq A$. Let β be the translation which takes A into A'. Set $\gamma = \alpha\beta^{-1}$. Then γ is an orthogonal transformation of the first kind that leaves A' fixed, and so, by Theorem 1, it is the rotation about some axis l through A' (or the identity). If A lies on l, then $\alpha = \gamma\beta$ is the required representation of α. It is clear that $\gamma\beta = \beta\gamma$.

Suppose next that AA' does not lie on l. Set $\overrightarrow{AA'} = \overrightarrow{AP} + \overrightarrow{AQ}$, where $\overrightarrow{AP}$ is parallel to l and $\overrightarrow{AQ}$ is perpendicular to it (Fig. 43).

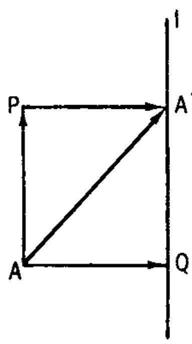


Fig. 43

Let β_1 and β_2 be the translations defined by the vectors $\overrightarrow{AP}$ and $\overrightarrow{AQ}$, respectively. Then $\beta = \beta_2\beta_1$, so that $\alpha = \gamma\beta_2\beta_1$. Set $\gamma\beta_2 = \gamma_1$. Then $\alpha = \gamma_1\beta_1$. We assert that this is the required representation of α. The transformation $\gamma_1 = \gamma\beta_2$ leaves every plane π perpendicular to l invariant as a whole. Thus it induces in each such plane an orthogonal transformation $\bar{\gamma}_1$, which is clearly of the first kind, since it is the product of a translation $\bar{\beta}_2$ through $\overrightarrow{PA'}$ and a rotation $\bar{\gamma}$ about the point A' of intersection of π with l (see Fig. 43). By Theorem 1 of Section 9, this is either a translation or a rotation. But if $\bar{\gamma}_1$ is a translation, then $\bar{\gamma} = \bar{\gamma}_1\bar{\beta}_2^{-1}$ is the product of two translations, and so is itself a translation, whereas we know that $\bar{\gamma}$ is a rotation. (We write a bar over a transformation to denote its restriction to the plane π.) Thus γ induces a rotation, indeed the same rotation, in every plane π perpendicular to l (where points of these planes are identified if one lies vertically above or below the other); the locus of the centers of all these rotations is a vertical line m, and γ_1 is a rotation about m.

Consider now the equation $\alpha = \gamma_1\beta_1$. If β_1 reduces to the identity (which happens when AA' is perpendicular to l), then $\alpha = \gamma_1$ is the rotation about m. If not, γ_1 is the rotation about a line parallel to l, and β_1 is a translation parallel to l, and so also parallel to m. In any case $\alpha = \gamma_1\beta_1$ is the required representation of α. It is clear that $\gamma_1\beta_1 = \beta_1\gamma_1$.

We must now prove the uniqueness of this representation. Under the transformation $\alpha = \gamma_1\beta_1$, the line m is carried into itself. We show that m is the only line with this property. We shall assume that neither γ_1 nor β_1 is the identity.

Let p be an arbitrary line, and suppose first it is skew to m. Let SP be the common perpendicular to m and p (S on m and P on p). Under α, m and p are transformed into skew lines m' and p', and SP goes into $S'P'$, which is distinct from SP, since S is taken by β_1 into a point S' distinct from it, and S' remains invariant under γ_1. Inasmuch as angles are preserved under orthogonal maps, $S'P'$ is the common perpendicular to m and p', so that, as it is not SP, p and p' must be distinct.

Suppose now that p intersects m in S. Then, under β_1, S goes into S', while under γ_1, S' remains in place. So, under α, S goes into the distinct point S'. But S' is the point of intersection of m and p', so that p' cannot be the same line as p.

Suppose finally that p is parallel to m. Then under β_1 it is transformed into itself, and under γ_1 it goes into a line p' parallel to p and m, but distinct from p.

We have thus shown that m is the only line invariant under α. Suppose that $\alpha = \gamma_0\beta_0$ for some reflection γ_0 in a line m_0 and translation β_0. Then, by what we have proved, the line m_0 (which is evidently invariant under $\alpha = \gamma_0\beta_0$) must be the line m. So γ_0 is a rotation about m, and β_0 is a translation along m. Now $\alpha = \gamma_1\beta_1 = \gamma_0\beta_0$, and $\gamma_0^{-1}\gamma_1 = \beta_0\beta_1^{-1}$. Since γ_0 and γ_1 are both rotations about m, so too are γ_0^{-1} and $\gamma_0^{-1}\gamma_1$, and since β_0 and β_1 are both translations parallel to m, β_1^{-1} and

$\beta_0\beta_1^{-1}$ are so also. Thus we find that a rotation about m is equal to a translation parallel to m. Under the rotation, every point of m remains fixed, so that the translation is the identity. But then the rotation is also the identity; that is, $\gamma_0^{-1}\gamma_1 = \beta_0\beta_1^{-1} = \varepsilon$. Hence $\gamma_0 = \gamma_1$ and $\beta_0 = \beta_1$.

We have now shown that if α has one representation $\alpha = \beta\gamma$ of the required type, and if neither of β and γ is the identity, then this representation is unique. There remains only the following case: in *every* representation of α of the required type, either β or γ is the identity. But then the other must be α, so that the representation is $\alpha = \alpha\varepsilon = \varepsilon\alpha$. In this case either α is a rotation, and ε is thought of as the identity translation, or α is a translation and ε is thought of as the identity rotation. ▼

Theorem 3. *Any orthogonal transformation α of the second kind either is a reflection in a plane or can be represented as the product of a reflection in a plane and a rotation about a line perpendicular to this plane or can be represented as the product of a reflection in a plane and a translation in some direction parallel to this plane, according as α has more than one fixed point, just one fixed point, or no fixed point.*

Such a representation is unique except in the second case when the rotation in the line is through two right angles. In this (and only in this) case, α is the reflection in a point O (the point of intersection of the line and the plane), and then it may be represented as the product of the reflection in any plane π through O and the rotation through two right angles about the line through O perpendicular to π.

Proof. Case 1. Let us suppose first that the orthogonal transformation α of the second kind has a fixed point O. Since α is not the identity (which is of the first kind), we may choose a point A whose image B under α does not coincide with it. Let C be the image of B. Then clearly C does not coincide with B.

If, for any point A, its image B lies on the line OA, then $OA = OB$, and therefore α is the reflection in the point O. The reflection can certainly be represented in the manner described in the second paragraph of the theorem. Moreover, it cannot be represented in any of the other ways there listed. For reflection in a plane followed by rotation in a perpendicular line is the only type of representation which leaves invariant a single point: the point of intersection of the line and the plane. Furthermore, reflection in the plane π, followed by rotation through an angle β about the perpendicular line l, is a reflection in the point O of intersection of π and l only if $\beta =$ two right angles.

Suppose next that B does not lie on OA, and suppose that C coincides with A. Then α is the reflection in the plane π through O and perpendicular to AB, since, like this reflection, it takes O, A, B into O, B, C, respectively, and both transformations are of the second kind.

Suppose next that B does not lie on OA, that C does not coincide with A, and that the four points O, A, B, C are coplanar. Then the transformation is the product of the reflection in this plane π and the rotation about the perpendicular l to π through O that takes A into B. For the reflection leaves O, A, B, C invariant, and the rotation leaves O invariant and takes A into B (such a rotation exists, since $OA = OB$), and also takes B into C, since the triangles AOB and BOC are congruent (three equal sides) and have the same orientation.

Suppose finally that B does not lie on OA, that C does not coincide with A, and that the four points $OABC$ are not coplanar. Let D be the midpoint of AB, E the midpoint of BC, and π the plane through ODE. Let l be the perpendicular through O to π. We show that α is the product of the reflection σ in π and a rotation ρ about l. For suppose that σ takes B into B^*. Then A, B^*, and C will all be on the same side of π and at equal distances from it. Let A_0, B_0^*, and C_0 be the projections of A, B^* and C onto π. Then, since $OA = OB^* = OC$, we have $OA_0 = OB_0^* = OC_0$. Next, AB^*B and CB^*B are both right triangles, since, for example, A and B^* are the same distance from π, so that AB^* is parallel to π, while BB^* is perpendicular to it. Also $AB = BC$. Thus the triangles AB^*B and CB^*B are congruent (since they are right triangles with a common side and equal hypotenuses). $AB^* = B^*C$, and therefore also $A_0B_0^* =$

B^*C_0 (since, for example, $AB^*B_0{}^*A_0$ is a rectangle). Finally, the triangles $OA_0B_0{}^*$ and $OB_0{}^*C_0$ are congruent (since they have three equal sides), and it follows that $\angle A_0OB_0{}^* = \angle B_0{}^*OC_0$.

Let ρ be the rotation about l that takes A_0 into $B_0{}^*$ (this rotation exists, since π is perpendicular to l and $OA_0 = OB_0{}^*$). Then, by what we have just shown, ρ will take $B_0{}^*$ into C_0. Since A, B^*, C are vertically "above" A_0, $B_0{}^*$, and C_0 and in a parallel plane, ρ will take A into B^* into C. Thus $\sigma\rho$ will, like α, take O, A, B into O, B, C, respectively, so that, since both these transformations are of the second kind, $\alpha = \sigma\rho$. It is clear also that $\sigma\rho = \rho\sigma$.

Case 2. Suppose now that α is an orthogonal transformation of the second kind, and that it has no fixed point. We shall need the following two lemmas:

Lemma 1. *Let σ be the reflection in a plane π and β_2 a translation perpendicular to π. Then $\sigma\beta_2$ is the reflection in a plane π' parallel to π.*

Proof. Let A, B, C be any three noncollinear points of π, and A', B', C' their images under $\sigma\beta_2$. Then AA', BB', CC' are equal segments, all perpendicular to π. Let π' be the plane that passes through the midpoints of these segments. Then it is clear that π' is parallel to π and that the reflection σ_1 in the plane π' also takes A, B, C into A', B', C', respectively. Since $\sigma\beta_2$ and σ_1 are both of the second kind, they are equal.

Lemma 2. *Let ρ be a rotation (other than the identity) about a line l and β_1 a translation parallel to a plane π perpendicular to l. Then $\rho\beta_1$ is a rotation about a line n parallel to l.*

Proof. Let l meet π in O, and let P be the point (of π) such that β_1 is the translation associated with $\overrightarrow{PO}$. Let m be that perpendicular bisector of OP which lies in the plane π. Suppose that ρ is the rotation about l through an oriented angle δ. Choose the point R on m for which the oriented angle $\angle PRO = \delta$. Then $\rho\beta_1$ is the rotation through δ about the line n through R and perpendicular to π (and so parallel to l). To prove this, we can consider the restrictions of all the maps to the plane π. The restriction of $\rho\beta_1$ is an orthogonal map of the first kind. Now β_1 takes R into the point Q for which $POQR$ is a parallelogram, and ρ takes Q through the oriented angle $\delta = \angle PRO = \angle QOR$ to the point R (since $QO = RP = RO$). Thus the restriction map is of the first kind and leaves R invariant, so it is the rotation about R. The same argument applies in every plane parallel to π, and the result follows immediately.

Let us return now to α and suppose that $\alpha(A) = A'$ $(\neq A)$. Let β be the translation through the vector $\overrightarrow{AA'}$. Then β takes A into A', and $\gamma = \alpha\beta^{-1}$ leaves A' invariant. But it is clear that γ is an orthogonal map of the second kind. So, by the first part of this proof, $\gamma = \sigma\rho$ is the product of the reflection σ in a plane π through A' and a rotation ρ about the perpendicular l to π through A'. (A representation of this form need not be unique, and ρ may be the identity.) Let us represent β in the form $\beta = \beta_1\beta_2$, where β_1 is a translation parallel to π, and β_2 is a translation perpendicular to π (and so parallel to l). Such a representation for β is always possible (in fact, is uniquely possible). Then $\alpha = \sigma\rho\beta_1\beta_2$.

Suppose now that ρ is not the identity. Then, by Lemma 2, $\rho\beta_1$ is a rotation ρ_1 about a line n perpendicular to π. So

$$\alpha = \sigma\rho\beta_1\beta_2 = \sigma\rho_1\beta_2 = \rho_1\sigma\beta_2.$$

Next, by Lemma 1, $\sigma\beta_2$ is a reflection σ_1 in a plane π' parallel to π, and $\alpha = \rho_1\sigma_1$. But then the point S of intersection of π' and n is invariant under α, which is contrary to hypothesis.

So ρ is the identity, and $\alpha = \sigma\beta_1\beta_2 = \sigma\beta_2\beta_1 = \sigma_1\beta_1$. This is the required representation of α.

Note that in each case in this theorem where α is represented as the product of two simple transformations, these transformations commute. Thus, in Case 1 we have $\alpha = \sigma\rho = \rho\sigma$, and, in Case 2, $\alpha = \sigma_1\beta_1 = \beta_1\sigma_1$.

We now have to prove the "uniqueness" part of the theorem. That is to say, we must prove that if α is an orthogonal transformation of the second kind, then its representation as the product of a reflection in a plane and a rotation about it (if it has fixed points), or of a reflection in a plane and a translation parallel to it (if not), is unique except for the case in which α is the reflection in a point.

Suppose first that $\alpha = \sigma\rho$, where σ is the reflection in a plane π and ρ is a rotation—not through two right angles and not the identity—about a line l perpendicular to π. Then π is the only plane invariant (as a whole) under α. For if λ is a parallel plane, its image is a plane λ' parallel to λ but on the other side of it from π, whereas if λ intersects π in the line m, then the image of m is a line m' distinct from m (this is where we use the fact that ρ is not a rotation through two right angles, for otherwise $m' = m$ if m passes through the point O of intersection of l with π). So λ', which intersects $\pi' = \pi$ in m', cannot be the same plane as λ.

Suppose now that α can also be represented in the form $\alpha = \sigma^*\rho^*$, where σ^* is the reflection in a plane π^*, and ρ^* is a rotation about a line perpendicular to π^*. Then π^* is invariant under $\sigma^*\rho^* = \alpha$, and so is equal to π. But then $\sigma^* = \sigma$, and so also $\rho^* = \rho$.

Suppose now that $\alpha = \beta\sigma$, where σ is the reflection in a plane π and β is a translation parallel to π. We allow the possibility that β is the identity. Then π is the only plane invariant under α whose orientation is preserved. For suppose first that the plane λ is parallel to π. Then its image under α is a plane λ' parallel to λ but on the other side of π from it. Suppose next that λ intersects π in the line m. Then if λ is invariant under α (which happens provided that λ is perpendicular to π and, if β is not the identity, m is parallel to the vector associated with β), α induces in λ the reflection in m, which is an orthogonal transformation of the second kind. In π, α induces the translation β, so that our assertion is proved.

Suppose that α has another representation $\alpha = \beta^*\sigma^*$, where σ^* is the reflection in a plane π^* and β^* is a translation parallel to it. Then, by what we have just proved, π^* is the only plane invariant under α whose orientation is preserved. But this means that $\pi^* = \pi$ and hence also that $\sigma^* = \sigma$ and $\beta^* = \beta$. ▼

Theorem 4. *Any orthogonal transformation of the first kind can be represented as the product of two or four reflections in planes, and any orthogonal transformation of the first kind either is itself a reflection in a plane or can be represented as the product of three such.*

Proof. Let α be an orthogonal transformation of the first kind. Then, by Theorem 1, it either is a rotation or a translation or can be represented as the product of a rotation and a translation.

(1) Suppose first that α is a rotation about a line l. Let π_1 be any plane through l, and let π_2 be its image under α. Let π be the bisector of these planes lying inside the oriented angle $\pi_1 l \pi_2$. Let σ be the reflection in π and σ_2 the reflection in π_2. Then $\sigma_2\sigma$ takes π_1 into π_2, and since it, like α, is of the first kind, $\alpha = \sigma_2\sigma$.

(2) Suppose next that α is the translation determined by the vector $\overrightarrow{AA'}$. Let π be the plane that bisects AA' at right angles and π' the plane through A' parallel to π. Let σ and σ' be the reflections in these two planes. It is clear that $\alpha = \sigma'\sigma$.

(3) Suppose finally that α can be represented as the product $\rho\tau$ of a rotation and a translation. On substituting the product of two reflections for each of ρ and τ (by parts 1 and 2 of this proof), we find that α can be represented as the product of four reflections.

Suppose now that α is of the second kind. Then, by Theorem 3, either it is the reflection in some plane (in which case we are through) or it can be represented as the product of a reflection and a rotation or a translation. But either a translation or a rotation can be represented as the product of two reflections, so α can be represented as the product of three. ▼

Note 1. Since the product of two transformations of the same kind is of the first kind, while the product of two transformations of different kinds is of the second kind (we may think of transformations of the first kind as positive and those of the second kind as negative), and since reflection in a plane is of the second kind, any representation of a transformation of the first kind by reflections must have an even number of reflections, while any representation of a transformation of the second kind by reflections must have an odd number. Theorem 4 is "best possible" in the sense that, although any transformation that can be represented as a product of n reflections can also be represented as a product of $(n + 2m)$ reflections for any positive integer m, we cannot reduce the number of reflections required below that which is stated in Theorem 4. This is clear for transformations of the second kind and for rotations and translations (a product of 0 reflections may be taken to mean the identity transformation), but if α is of the first kind and not a rotation or a translation, it cannot be represented as a product of two reflections (the only possible number less than 4). For if these reflections are in planes π and π', and π and π' are parallel, then α is a translation in the direction of their common perpendicular, whereas if π and π' intersect in the line l, then α is a rotation about l.

Note 2. The representation of a transformation as a product of reflections is not unique. If α is a rotation or a translation, the situation is analogous to that of Theorem 4, Section 9, but in general it is more complicated.

13. Orthogonal Transformations of Space in Coordinates

Let us introduce in space a system of rectangular Cartesian coordinates with origin O and unit points E_1, E_2, E_3.

Let us place in correspondence with each point $M(x, y, z)$ of space the point $M'(x', y', z')$ whose coordinates are given by the linear equations

$$x' = a_{11}x + a_{12}y + a_{13}z + a,$$
$$y' = a_{21}x + a_{22}y + a_{23}z + b,$$
$$z' = a_{31}x + a_{32}y + a_{33}z + c,$$

where

$$\left.\begin{aligned} a_{11}^2 + a_{21}^2 + a_{31}^2 &= 1,\\ a_{12}^2 + a_{22}^2 + a_{32}^2 &= 1,\\ a_{13}^2 + a_{23}^2 + a_{33}^2 &= 1,\\ a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} &= 0,\\ a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} &= 0,\\ a_{13}a_{11} + a_{23}a_{21} + a_{33}a_{31} &= 0. \end{aligned}\right\} \qquad (2)$$

We shall show that this mapping of space into itself is an orthogonal transformation. To do so, we need to show that, for any two points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$, the distance $M_1'M_2'$ between their images $M_1'(x_1', y_1', z_1')$ and $M_2'(x_2', y_2', z_2')$ is the same as the distance M_1M_2 between them.

But

$$\begin{aligned} M_1'M_2'^2 &= (x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2\\ &= [(a_{11}x_2 + a_{12}y_2 + a_{13}z_2 + a)\\ &\qquad - (a_{11}x_1 + a_{12}y_1 + a_{13}z_1 + a)]^2\\ &\quad + [(a_{21}x_2 + a_{22}y_2 + a_{23}z_2 + b)\\ &\qquad - (a_{21}x_1 + a_{22}y_1 + a_{23}z_1 + b)]^2\\ &\quad + [(a_{31}x_2 + a_{32}y_2 + a_{33}z_2 + c)\\ &\qquad - (a_{31}x_1 + a_{32}y_1 + a_{33}z_1 + c)]^2\\ &= [a_{11}(x_2 - x_1) + a_{12}(y_2 - y_1) + a_{13}(z_2 - z_1)]^2\\ &\quad + [a_{21}(x_2 - x_1) + a_{22}(y_2 - y_1) + a_{23}(z_2 - z_1)]^2\\ &\quad + [a_{31}(x_2 - x_1) + a_{32}(y_2 - y_1) + a_{33}(z_2 - z_1)]^2 \end{aligned}$$

$$= (a_{11}^2 + a_{21}^2 + a_{31}^2)(x_2 - x_1)^2 + (a_{12}^2 + a_{22}^2 + a_{32}^2)(y_2 - y_1)^2$$
$$+ (a_{13}^2 + a_{23}^2 + a_{33}^2)(z_2 - z_1)^2$$
$$+ 2(a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})(x_2 - x_1)(y_2 - y_1)$$
$$+ 2(a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33})(y_2 - y_1)(z_2 - z_1)$$
$$+ 2(a_{13}a_{11} + a_{23}a_{21} + a_{33}a_{31})(z_2 - z_1)(x_2 - x_1)$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = M_1M_2{}^2,$$

by the relations (2).

We shall now show that if α is any orthogonal transformation taking the typical point $M(x, y, z)$ into $M'(x', y', z')$, then α is given by formulas of the form (1) and that, moreover, the relations (2) are satisfied. Suppose that $O'(a, b, c)$, $E_1'(p_1, q_1, r_1)$, $E_2'(p_2, q_2, r_2)$, $E_3'(p_3, q_3, r_3)$ are the images of $O(0, 0, 0)$, $E_1(1, 0, 0)$, $E_2(0, 1, 0)$, $E_3(0, 0, 1)$, respectively.

Let us define the numbers a_{ij} by

$$\begin{array}{lll} a_{11} = p_1 - a, & a_{12} = p_2 - a, & a_{13} = p_3 - a, \\ a_{21} = q_1 - b, & a_{22} = q_2 - b, & a_{23} = q_3 - b, \\ a_{31} = r_1 - c, & a_{32} = r_2 - c, & a_{33} = r_3 - c. \end{array} \tag{3}$$

Then

$$a_{11}^2 + a_{21}^2 + a_{31}^2 = (p_1 - a)^2 + (q_1 - b)^2 + (r_1 - c)^2$$
$$= O'E_1'^2 = OE_1{}^2 = 1,$$

and, similarly,

$$a_{12}^2 + a_{22}^2 + a_{23}^2 = 1,$$
$$a_{13}^2 + a_{23}^2 + a_{33}^2 = 1.$$

Next, since α is orthogonal, $E_1'O'E_2'$ is a right triangle and also $O'E_1' = O'E_2' = 1$. So,

$$E_1'E_2'^2 = O'E_1'^2 + O'E_2'^2 = 2$$

or

$$(p_2 - p_1)^2 + (q_2 - q_1)^2 + (r_2 - r_1)^2 = 2$$

or

$$(a_{11} - a_{12})^2 + (a_{21} - a_{22})^2 + (a_{31} - a_{32})^2 = 2,$$

$$a_{11}^2 + a_{21}^2 + a_{31}^2 - 2(a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32}) \\ + a_{12}^2 + a_{22}^2 + a_{32}^2 = 2.$$

But

$$a_{11}^2 + a_{21}^2 + a_{31}^2 = a_{12}^2 + a_{22}^2 + a_{32}^2 = 1,$$

so that

$$a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0.$$

The last two equations in (2) are proved similarly.

Let the orthogonal transformation β be defined by the formulas (1), where the a_{ij} are given by (3). Then it is clear that β has the same effect as α on the four non-coplanar points O, E_1, E_2, E_3. By a result stated at the beginning of Section 11, $\beta = \alpha$. We have thus proved that a transformation α of space is orthogonal if and only if its expression in terms of coordinates is given by (1) and the relations (2) are satisfied.

The reader may feel that the treatment we have given in this section is somewhat artificial; we seem to have pulled Eqs. (1) and the relations (2) out of a hat, and then proved that any orthogonal transformation can be expressed in this form, so to speak, backwards. The reader with a little knowledge of vector algebra may find the treatment that follows more natural.

Suppose we are given a system of rectangular Cartesian coordinates in space, with origin O. We identify each point $M(x_1, x_2, x_3)$ of space with the vector $\overrightarrow{OM} = \mathbf{x}$, so that $\mathbf{x}$ is the vector whose coordinates are (x_1, x_2, x_3). We shall say that a mapping α of space is *linear* provided the following conditions are satisfied:

$$\alpha(a\mathbf{x}) = a(\alpha(\mathbf{x})), \tag{4}$$

$$\alpha(\mathbf{x} + \mathbf{y}) = \alpha(\mathbf{x}) + \alpha(\mathbf{y}), \tag{5}$$

for all vectors $\mathbf{x}$ and $\mathbf{y}$ and all numbers a. In particular, a linear mapping takes the origin into itself.

Suppose now that α is an orthogonal transformation leaving O invariant. We show that it is a linear mapping. We shall typically write $\mathbf{x}'$ for the image under α of the vector $\mathbf{x}$.

Let $\mathbf{x}$ be any vector and a any real number. Then O, $M(\mathbf{x})$, $N(a\mathbf{x})$ are collinear, so that their images O, $M'(\mathbf{x}')$, $N'((a\mathbf{x})')$ are also collinear. Suppose first that a is positive. Then $ON = aOM$, and M and N lie on the same side of O. But then $\overrightarrow{ON'} = aOM'$, and N' and M' lie on the same side of O. Since N' lies on OM', we have $(a\mathbf{x})' = a\mathbf{x}'$. If a is negative, $ON' = |a|OM'$, and since in this case M' and N' lie on the same line through O but on opposite sides of it, $\overrightarrow{ON'} = -|a|\overrightarrow{OM'} = a\overrightarrow{OM'}$; that is, $(a\mathbf{x})' = a\mathbf{x}'$. We have thus proved (4), the case where $a = 0$ being trivial.

Suppose next that $M(\mathbf{x})$ and $N(\mathbf{y})$ are any vectors (points). Then the midpoint P of MN is $P(\frac{1}{2}(\mathbf{x} + \mathbf{y}))$. Under α, P goes into the midpoint P' of $M'N'$; that is, $P'(\frac{1}{2}(\mathbf{x}' + \mathbf{y}'))$. So

$$\tfrac{1}{2}(\mathbf{x}' + \mathbf{y}') = (\tfrac{1}{2}(\mathbf{x} + \mathbf{y}))'.$$

Using (4) with $a = \frac{1}{2}$, we find that (5) follows from this. We have thus shown that an orthogonal transformation leaving the origin fixed is a linear mapping.

Suppose now that the orthogonal transformation α with fixed point O takes the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ into the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, where $\mathbf{a}_1$, for example, is the vector (a_{11}, a_{21}, a_{31}). Let $M(x, y, z)$ be a general point of space. Then $M = M(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3)$. Since α is a linear transformation, $\alpha(M) = \alpha(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3) = x\mathbf{a}_1 + y\mathbf{a}_2 + z\mathbf{a}_3$ [where we have used (4) and (5)]. On rewriting this in coordinates, we have (1), with $a = b = c = 0$.

Next, the length of $\mathbf{a}_i$ is $\mathbf{a}_i^2 = 1$, since $\mathbf{a}_i$ is the image of the unit vector $\mathbf{e}_i$ ($i = 1, 2, 3$). This gives us the first three relations of (2). Since $\mathbf{e}_i$ and $\mathbf{e}_j$ are perpendicular ($i = j$), so are $\mathbf{a}_i$ and $\mathbf{a}_j$. That is to say, $\mathbf{a}_i \cdot \mathbf{a}_j = 0$. This gives us the last three relations of (2). We have thus shown that an orthogonal transformation leaving the origin fixed is given by (1) (with $a = b = c = 0$) subject to the conditions (2). If α is an arbitrary orthogonal transformation, and $\alpha(O) = O'$, we set $\alpha = \beta\gamma$, where β is the translation through the vector $\overrightarrow{OO'} = (a, b, c)$. Then γ is an orthogonal transformation leaving O fixed, and so it is of the form we have just described. The translation β then takes $M(x, y, z)$ into $\gamma M(x, y, z) + \overrightarrow{OO'}$, which, written coordinatewise, is just (1). We have thus proved that any orthogonal transformation can be expressed in the form (1), subject to (2).

Suppose next that α is the mapping of space given by (1) and that (2) is satisfied. We show that α is an orthogonal transformation. Let β be the translation through the vector (a, b, c); then $\alpha = \gamma\beta^{-1}$, where γ is the transformation given by (1) subject to (2), except that a, b, c have been deleted. It may be checked immediately that this map is linear, that is, satisfies (4) and (5) above. Also the images under γ of the $\mathbf{e}_i$ are the $\mathbf{a}_i$ defined as before ($i = 1, 2, 3$). The relations (2) then state that the $\mathbf{a}_i$ are mutually perpendicular unit vectors. We show now that this implies that γ is an orthogonal map. Let $P(\mathbf{p})$ and $Q(\mathbf{q})$ be any two points (vectors) and $P'(\mathbf{p}')$, $Q'(\mathbf{q}')$ their images under γ. If $\mathbf{p} = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3$, then $\mathbf{p}' = p_1\mathbf{a}_1 + p_2\mathbf{a}_2 + p_3\mathbf{a}_3$, since γ is a linear map. Because of the relations $\mathbf{a}_i{}^2 = 1$ and $\mathbf{a}_i \cdot \mathbf{a}_j = 0$ $(i = j)$, we have $\mathbf{p}' \cdot \mathbf{q}' = p_1q_1 + p_2q_2 + p_3q_3 = \mathbf{p} \cdot \mathbf{q}$ (since the $\mathbf{e}_i$ satisfy the same relations). In particular, taking $\mathbf{q} = \mathbf{p}$, we find $\mathbf{p}'^2 = \mathbf{p}^2$, and similarly $\mathbf{q}'^2 = \mathbf{q}^2$. Now

$$PQ^2 = (\mathbf{p} - \mathbf{q})^2 = \mathbf{p}^2 - 2\mathbf{p} \cdot \mathbf{q} + \mathbf{q}^2 = \mathbf{p}'^2 - 2\mathbf{p}' \cdot \mathbf{q}' + \mathbf{q}'^2$$

$$= (\mathbf{p}' - \mathbf{q}')^2 = P'Q'^2.$$

Thus γ preserves all lengths; that is, it is an orthogonal transformation. And since the translation β is also an orthogonal transformation, so is $\alpha = \beta\gamma$. We have finally shown that the transformation α is orthogonal if and only if it can be represented in the form (1), subject to (2).

Note 1. Since the $\mathbf{a}_i$ are unit vectors, a_{11}, a_{21}, a_{31} are the direction cosines of $O'E_1'$ in the given system of coordinates, that is, the cosines of the angles that $O'E_1'$ makes with the respective axes—and similarly for the coordinates of $\mathbf{a}_2$ and $\mathbf{a}_3$. Since the $\mathbf{a}_i$ are mutually orthogonal, we could take them as a new system of rectangular coordinates, and the direction cosines in this system of Ox, Oy, Oz are clearly (a_{11}, a_{12}, a_{13}), and so on. Let us forget for the moment about the translation part of α, taking $a = b = c = 0$. Then $\alpha = \gamma$ takes E_i into E_i' and leaves O fixed. It follows that the inverse transformation γ^{-1} takes E_i' into E_i and leaves O fixed. But, in the new system of coordinates, the equation for γ^{-1} is just (1) with the rows and columns of the array (a_{ij}) interchanged (by the remark we just made about direction cosines). Since γ^{-1} is certainly an orthogonal transformation, we conclude that

$$a_{i1}^2 + a_{i2}^2 + a_{i3}^2 = 1 \qquad (i = 1, 2, 3),$$

and

$$a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3} = 0 \qquad (ij = 12, 23, 31).$$

It is a remarkable algebraic fact that the relations (2) imply in this way these "reciprocal" relations, and the reader is invited to deduce them directly. The theory of orthogonal transformations of space treated coordinatewise can be further developed by the use of matrix theory: the interested reader is referred to Leonid Mirsky, "An Introduction to Linear Algebra," Oxford Univ. (Clarendon) Press, 1955, or any book on linear algebra.

Note 2. If α is an orthogonal transformation of the first kind, the tetrahedra $OE_1E_2E_3$ and $OE_1'E_2'E_3'$ have the same orientation, and, in this case,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 1, \tag{6}$$

while if α is an orthogonal transformation of the second kind, $\Delta = -1$. In fact, Δ is related to the expression for the (oriented!) volume of the oriented tetrahedron $OE_1'E_2'E_3'$. In general, the volume of the oriented tetrahedron $ABCD$ is the absolute value of the determinant

$$E = \begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix} \tag{7}$$

(with the obvious notation), and in the case where one of the vertices is O, this expression reduces to (6) above. The sign of E is positive or negative according to whether $ABCD$ has the same orientation as $OE_1E_2E_3$ or the opposite orientation.

Note 3. The formulas and theorems we have proved for orthogonal transformations in three-dimensional space have very obvious extensions to spaces of higher dimension; we can define points or vectors with n coordinates instead of three, define distance by the obvious extension of Pythagoras' rule, and define orthogonal transformations. Then, for example, the generalizations of (1) and (2) will hold, and we can define orientation and "volume" of "simplexes" (generalized tetrahedra) by the analogs of (7) above. It will also turn out that the orthogonal transformation given by the array (a_{ij}) (with i and j running from 1 to n) will be of the first or second kind according to whether the determinant analogous to Δ above is equal to $+1$ or -1. The reader should check that Eqs. (4) and (5) in Section 10.5 satisfy the two-dimensional analog of Eqs. (1) and (2) in this section and that the corresponding statement about the analog of Δ is true.

Below we give the array (a_{ij}) associated [as in Eq. (1)] with the following orthogonal transformations, respectively: rotation through β about Oz, reflection in the plane $x \sin \alpha -$

$y \cos \alpha = 0$, reflection in the plane xOy, and reflection in O.

$$(1)\quad \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (2)\quad \begin{pmatrix} \cos 2\alpha & \sin 2\alpha & 0 \\ \sin 2\alpha & -\cos 2\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$(3)\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad (4)\quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Note added in proof. We conclude this chapter by calling attention to a very elegant result on length-preserving mappings, due to Peter Zvergnowski, of the University of Chicago. For details, see Appendix (p. 152).