CHAPTER III

Similarity Transformations

As we saw in the previous chapter, orthogonal transformations leave invariant both the shape and the dimensions of geometric figures. If we discard the demand that our transformations preserve dimension but still insist that shapes be preserved, the set of transformations we get is the group of similarity transformations (of the plane or of space). We shall see that such transformations increase or decrease all lengths in the same ratio but leave shapes unchanged.

Elementary geometry studies those properties of figures that are preserved under orthogonal transformations and also those properties that are preserved under similarity transformations. For example, such properties of a triangle as its area and the lengths of its sides are invariant under orthogonal transformations, but, in general, are not invariant under similarity transformations. On the other hand such properties as its angles or the position of its center of gravity are invariant under similarity transformations as well as under orthogonal transformations.

14. Similarity Mappings

A mapping α of a plane π into a plane π' is called a *similarity mapping* with coefficient $k > 0$, or simply a *similarity*, provided it has the following property: if A and B are any two points of

π, and A', B' are their images under α, then $A'B' = kAB$. If $k = 1$, α is an orthogonal mapping; and, conversely, any orthogonal mapping of π onto π' is a similarity with coefficient $k = 1$. We shall show first that any mapping of π into π' that preserves all shapes must be a similarity.

Let α be a mapping of the plane π into the plane π' such that the shape of the image triangle $A'B'C'$ is the same as the shape of any given triangle ABC in π. This is a weaker requirement than that α preserve *all* shapes, but we shall see that it is equivalent and also equivalent to the requirement that α be a similarity. Consider first the case where α maps every point of π onto a single point O of π'. Since the image of any figure in π is a single point, its shape is undefined, and we may, if we like, say that α preserves shapes by definition. But then we can also say that α is a (degenerate) similarity with coefficient $k = 0$. Conversely, any mapping α of π into π', which is a similarity except that the coefficient $k = 0$, maps every point of π onto a single point of π'. The proof is left to the reader. In the future, we exclude such degenerate mappings.

Suppose, then, that A, B are two points of π whose images A', B' under α are distinct. Let C, D be any two points of π, not both on AB. Then at least one of A and B does not lie on the line CD, say, for instance, A. Since not both of C and D lie on AB, suppose, for example, that C does not. Then ABC is a triangle similar to its image $A'B'C'$ in π'. We define the (positive) number k by the equation $A'B' = kAB$. Since the sides of similar triangles are in proportion, $A'C' = kAC$. Next, since A does not lie on CD, the triangle ACD is similar to its image $A'C'D'$, and since $A'C' = kAC$, we also have $C'D' = kCD$. We have thus shown that all segments not on AB have their lengths changed by a factor k under the mapping α. Now choose two points C, D not lying on AB and repeat the whole argument with C, D in place of A, B. We find that all lengths of segments not lying on CD, and, in particular, of all segments lying on AB, are changed by a factor k. We have thus shown that α is a similarity, and, since the image plane π' is a scale model of π (the scale factor being k), that α preserves *all* shapes.

It is clear that a similarity is one-one, and we may prove it is onto almost exactly as we proved the same for orthogonal maps (Section 4). So a similarity α has an inverse α^{-1}, and it is clear that α^{-1} is itself a similarity of π' onto π, with coefficient $1/k$.

A similarity α of π onto itself is called a *similarity transformation*. The product of two similarity transformations with coefficients k_1 and k_2 is a similarity transformation with coefficient $k_1 k_2$. We regard the identity transformation as a similarity transformation with coefficient 1. It is clear then that the set of all similarity transformations (of a plane or of space) is a group, of which the orthogonal group is a subgroup.

15. Properties of Similarity Transformations

Under a similarity transformation, the images of three collinear points A, B, C are three collinear points A', B', C'. For, if this is not so, then $A'B'C'$ is a triangle, and its image under the inverse transformation must be a similar triangle—a contradiction, since A, B, C are collinear. It is clear that if B lies between A and C, then B' lies between A' and C'. As in the previous chapter, we may show that the image of a line is a line and in a similarity transformation of space the image of a plane is a plane.

The ratio of the lengths of any two line segments is equal to the ratios of the lengths of their images under a similarity transformation. For let AB and CD be any two segments, and let $A'B'$ and $C'D'$ be their images (the image of a line segment is a line segment by what we said above). Then, for some positive k, $A'B' = kAB$, and $C'D' = kCD$, whence (since $k \neq 0$)

$$\frac{A'B'}{C'D'} = \frac{AB}{CD}.$$

Under a similarity, parallel lines are taken into parallel lines, for the image of a line is a line, and the two image lines can have no point in common, since a similarity is one-one. Similarities also preserve angles. For let A be the vertex of an angle and B and C points on the two arms. Let A', B', C' be the respective images. Then the triangles $A'B'C'$ and ABC are similar, and so they have corresponding angles equal. In particular, a similarity takes perpendicular lines into perpendicular lines.

Suppose that we are given three noncollinear points A, B, C of the plane π and three points A', B', C' of the plane π', which are such that $A'B' = kAB$, $B'C' = kBC$, $C'A' = kCA$. Then there exists one, and only one, similarity of π onto π' that takes A, B, C into A', B', C', respectively.

To prove this, let us choose points B^* and C^* on the rays $A'B'$ and $A'C'$ such that $A'B^* = AB$ and $A'C^* = AC$. Then the triangles ABC and $A'B^*C^*$ are congruent (two sides and included angle), so that $B^*C^* = BC$. By Theorem 8 of Section 5, there exists an orthogonal transformation of π onto π', say β, that takes A, B, C into A', B^*, C^*, respectively. Let us define a transformation γ of the plane π' by making each point M of π' correspond to the point M' for which $A'M' = kA'M$ and make A' correspond to itself. We shall prove in the next section that γ is a similarity transformation of π' with coefficient k, and it is clear that β is a similarity of π onto π' with coefficient 1. So the composite mapping $\gamma\beta$ of π onto π' that takes each point M of π to M' *via* M^* is a similarity with coefficient k of π onto π'. Moreover, it is clear that $\gamma\beta$ takes A, B, C into A', B', C', respectively.

To prove uniqueness, suppose that α and β are two similarities of π onto π', each having the same effect on A, B, and C. Consider the composite mapping $\beta^{-1}\alpha$ of π onto itself. Since α and β have the same coefficient k, β^{-1} has coefficient k^{-1}, and $\beta^{-1}\alpha$ has coefficient 1. So $\beta^{-1}\alpha$ is an orthogonal mapping of π onto itself, and it is clear that it leaves invariant the points A, B, C. By Theorem 8, Section 5, it must be the identity transformation, so $\alpha = \beta$.

Under a similarity, the image of a circle is a circle (we leave the proof to the reader). It is not hard to show that a similarity either preserves the orientation of every triangle (that is, every triangle of the plane has the same orientation as its image) or reverses the orientation of every triangle. We say a similarity transformation is of the first or second kind, depending on which of these cases holds.

It may be shown (just as for orthogonal transformations) that, given two points A, B and two points A', B', there exists precisely one similarity of each kind that takes A, B into A', B', respectively, where A, B are distinct points of a plane π and A', B' are distinct points of a plane π'.

16. Homothetic Transformations

Let O be any point of a given plane and k a given positive number. Then the *homothetic transformation* of the plane with center O and coefficient k is that transformation γ of the plane which leaves O fixed and takes every other point M into the point M' for which $\overrightarrow{OM'} = k\overrightarrow{OM}$ (O, M, M' are collinear, with M and M' on the same side of O). The transformation γ which we introduced on the previous page was a homothetic transformation of the plane π' with center A' and coefficient k.

The set of all homothetic transformations of the plane with a given center O forms a group of transformations. For if α and β are the homothetic transformations with center O and coefficients k_1 and k_2, then $\alpha\beta = \beta\alpha$ is the homothetic transformation with center O and coefficient k_1k_2, and α^{-1} is the homothetic transformation with center O and coefficient $1/k_1$. The identity transformation is the one with coefficient $k = 1$. Note that the identity transformation may be regarded as the homothetic transformation with coefficient 1 and *any* center.

Theorem 1. *A homothetic transformation γ with center O and coefficient k is a similarity transformation with coefficient k and is of the first kind.*

Proof. Let A and B be any two distinct points of the plane and A', B' their images. Suppose first that A and B lie on a line through O. If A and B lie on the same side of O, then so do A' and B', and $A'B' = |OB' - OA'| = |kOB - kOA| =$

$k|OB - OA| = kAB$; while if A and B lie on opposite sides of O, $A'B' = OA' + OB' = kOA + kOB = k(OA + OB) = kAB$.

Suppose next that A and B do not lie on a line through O. Then the triangles AOB and $A'OB'$ are similar (they have two pairs of corresponding sides with a common ratio, since $OA' = kOA$ and $OB' = kOB$, and the included angle $AOB = A'OB'$ equal). So the third pair of corresponding sides is also in the same ratio; that is, $A'B' = kAB$. We have thus shown that for any two points A, B, $A'B' = kAB$, and γ is a similarity transformation with coefficient k.

We show next that it is a similarity transformation of the first kind. Let A and B be points not on a line through O and A', B' their images under γ. In the chain of triangles

$$OAB, \quad OA'B, \quad OA'B',$$

both pairs of adjacent triangles have the same orientation, so that the same is true of OAB and $OA'B'$. ▼

It is easy to see that under a homothetic transformation the image of a line is the same or a parallel line, and, moreover, that the sense along a line is preserved; that is, if A, B are distinct points, then the vectors $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ are parallel and point in the same direction. In fact, we have $\overrightarrow{A'B'} = k\overrightarrow{AB}$.

Conversely, if α is a similarity transformation with coefficient $k \neq 1$ under which each line l is taken into the same or a parallel line l', and, moreover, such that sense is preserved, then α is a homothetic transformation. The hypothesis may be restated in the form: if $\overrightarrow{A'B'} = k\overrightarrow{AB}$ for all A and B and some fixed positive constant $k \neq 1$, then α is a homothetic transformation.

Proof. If the image of every line is the same line, then α is the identity. For given any point O, choose distinct lines l, m through it. Then O' is the point of intersection of l' and m' and since $l' = l$ and $m' = m$, we have $O' = O$. Suppose then that A and B are two points whose images A', B' do not lie on AB. Since $A'B'$ is parallel but not equal to AB, $ABB'A'$ is not a parallelogram, and therefore AA' and BB' meet in a point O (note that AA' and BB' are well-defined lines, since we may easily see that $A' = A$ or $B' = B$ is impossible). Consider the image of the line OAA' under α. It is the same or a parallel line through the image A' of A, and so it must be the same line. Similarly, OBB' is invariant under α. So the point O of intersection of these lines is invariant under α. Next, the triangles OAB and $OA'B'$ are similar, so that $OA' : OA = OB' : OB = A'B' : AB = k$. Thus $OA' = kOA$ and $OB' = kOB$, and since A and A', and B and B', are on the same side of O, the homothetic transformation with center O and coefficient k, like α, leaves O fixed and takes A to A' and B to B'. By a result in Section 15, this means that α *is* the homothetic transformation with center O. ▼

Note 1. There is nothing to stop us from allowing a *negative* coefficient k in the definition of a homothetic transformation. In this case M is taken to a point M' lying on OM but on the other side of O from where M is located. Thus a homothetic transformation with negative coefficient $-k$ and center O is the product in either order of the (ordinary) homothetic transformation with center O and coefficient k, and the reflection in O. The set of "homothetic" transformations with center O is a commutative group, of which our previous group is a subgroup. We now have Note 2:

Note 2. In our proof of the converse to Theorem 1 we used the fact that $\overrightarrow{A'B'}$ pointed the same way as $\overrightarrow{AB}$ only in order to deduce that A and A' (and B and B') lay on the same side of O. If we allow homothetic transformations to have negative coefficients, we may restate the converse to Theorem 1 as follows:

If α is a transformation such that for some $k \neq 0, 1$ we have $\overrightarrow{A'B'} = k\overrightarrow{AB}$ for all A and B, then α is a homothetic transformation with coefficient k.

This result can be improved still further:

If α is a similarity transformation carrying every vector into a *parallel vector, then α is either a homothetic transformation or a translation.*

If α has coefficient k, this amounts to saying that if for every A, B we have $\overrightarrow{A'B'} = \pm k\overrightarrow{AB}$, then we must either always take the + sign, or always take the − sign.

This result suggests that a translation might be regarded as a homothetic transformation with center at infinity and coefficient 1.

Note 3. The use of geometric transformations allows us, in many cases, to give a simple solution of geometric problems that would otherwise be much more difficult. We give an example by proving the following theorems, due to Euler:

Theorem 2. *Let ABC be any triangle, H its orthocenter (the point of intersection of its altitudes), G its centroid or center of gravity (the point of intersection of its medians), and O its circumcenter (the center of the circumscribed circle S or the point of intersection of the perpendicular bisectors of the sides). Then O, G, H are collinear, with G between O and H, and*

$$OG : GH = 1 : 2.$$

In particular, if two of O, G, H coincide, then they all do.

Theorem 3. *The following nine points lie in a circle known as Euler's circle s: the midpoints of the sides of ABC, the bases of its altitudes, and the midpoints of the line segments joining H with the vertices. The center E of s is the midpoint of OH, and its radius is half that of the circumscribed circle.*

Proof. Consider the "homothetic" transformation γ with center G and coefficient $-\frac{1}{2}$. Since G lies a third of the way between each side and the opposite vertex, γ will take A, B, C into the midpoints A', B', C' of the opposite side. Consider the altitude of ABC through A. Its image is a parallel line through the image A' of A, that is, the perpendicular bisector of BC, and similarly for the other altitudes. So the image of the point H of intersection of the altitudes is the point O of intersection of the perpendicular bisectors of the sides. It follows that H and O lie on opposite sides of G and that $OG : GH = 1 : 2$. ▼.

The image of the circumcircle S of ABC is a circle s, whose center is the image of O under γ and which passes through the images A', B', C' of the vertices. Since the image of O is the midpoint E of OH, we see that the circle s through the midpoints of the sides has center E. Since the coefficient of γ (as a similarity transformation) is $\frac{1}{2}$, the radius of s will be half that of S. Now E is equidistant from O and H, and so it is also equidistant from the projections of O and H onto any line, in particular, the sides of ABC. But these projections are just the midpoints of the sides and the bases of the altitudes. Since s with center E passes through the former, it must also pass through the latter.

Consider now the homothetic transformation β with center H and coefficient $\frac{1}{2}$. Just as for γ, the point O is taken into E, and S is taken into a circle with center E and radius half that of S; that is, the circle s. But under β the vertices of ABC go into the midpoints of AH, BH, CH, so that these points lie on s.

The circle s is known as the nine-point, or Euler, circle associated with the triangle ABC. ▼

Note 4. Let A_0, B_0, C_0 be the second points in which the altitudes of ABC meet S. Under β, these points must go into points of s also on the altitudes (since the altitudes pass through the center H of β) and on the same side of H as are A_0, B_0, C_0. So their images are the feet of the altitudes. It follows that BC is the perpendicular bisector of HA_0, and so on; in other words, the reflections of H in the three sides of ABC lie on the circumcircle.

Note 5. In Sections 17 and 18 we shall use the expression "homothetic transformation" to apply only to those transformations which have positive coefficients.

17. Representation of a Similarity Transformation as the Product of a Homothetic Transformation and an Orthogonal Transformation

Theorem 1. *Any similarity transformation α with coefficient k can be represented as the product of the homothetic transformation β with coefficient k and prescribed center O, and an orthogonal transformation γ.*

Proof. If $k = 1$, we take the homothetic transformation to be the identity (regarded as the homothetic transformation with center at the given point O and coefficient 1). Let β be the homothetic transformation with coefficient k and center the given point O, and let $\gamma = \beta^{-1}\alpha$. Then $\alpha = \beta\gamma$. By results in Section 14, γ is a similarity transformation with coefficient $k^{-1}k = 1$ and is thus an orthogonal transformation. ▼

We leave it to the reader to show that β and γ are uniquely determined by α and O and the requirement that γ be an orthogonal transformation.

Theorem 2. *Given any similarity transformation α of the plane, exactly one of the following holds:*

(1) *α is an orthogonal transformation.*

(2) *α is not orthogonal and of the first kind. In this case, α has a unique representation $\alpha = \gamma\rho$ such that γ is a homothetic transformation and ρ is a rotation about the center O of γ. Moreover, $\alpha = \rho\gamma$. We allow the special cases where ρ is the rotation through* 0 *(that is, the identity) or π (that is, the reflection in O).*

(3) *α is not orthogonal and of the second kind. In this case, α has a unique representation $\alpha = \gamma\sigma$ such that γ is a homothetic transformation and σ is the reflection in some line through the center O of γ. Moreover, $\alpha = \sigma\gamma$.*

Proof. Case 1. Let α be a similarity transformation with coefficient k. If $k = 1$, we have the first case. So we assume from now on that $k \neq 1$.

Case 2. Let A be any point, B its image under α, and C the image of B. Then $BC/AB = k$. We distinguish three cases.

i. A, B, C are collinear with B in the middle. There is a unique point O lying on AB, outside the segment AB, such that $OB/OA = k$.

If $k > 1$ then O lies on BA produced in the direction of A and so outside the segment AC. So

$$\frac{OC}{OB} = \frac{OB + BC}{OA + AB} = \frac{kOA + kAB}{OA + AB} = k\,.$$

If $k < 1$, then O lies on AB produced beyond B, and since

$$\frac{OB}{OA} = \frac{OB}{OB + BA} = k$$

and

$$\frac{BC}{AB} = k,$$

we must have $OB > BC$, and therefore O lies on BC produced beyond C. So

$$\frac{OC}{OB} = \frac{OB - BC}{OA - AB} = \frac{kOA - kAB}{OA - AB} = k\,.$$

Consider the homothetic transformation γ with center O and coefficient k. Like α, γ takes A into B and B into C, and since α and γ are both of the first kind, $\alpha = \gamma$. To prove part 2 of the theorem, we take ρ to be the identity (rotation through 0 about O).

ii. A, B, C are collinear, and B is not in the middle. Suppose $k > 1$. Choose the point O inside the segment AB for which

$$\frac{OB}{OA} = k.$$

Since $BC/AB = k$, C lies on BA produced beyond A, and therefore

$$\frac{OC}{OB} = \frac{BC - OB}{AB - OA} = \frac{kAB - kOA}{AB - OA} = k.$$

Also A and B, and B and C, lie on opposite sides of O. Let γ be the homothetic transformation with center O and coefficient k, and let σ be the reflection in O. Then, by the usual argument, we see that $\alpha = \gamma\sigma = \sigma\gamma$.

If $k < 1$, consider the inverse transformation α^{-1}. It takes C into B and B into A; B is not in the middle, and the coefficient is >1. So, by what we have just proved, $\alpha^{-1} = \gamma\sigma = \sigma\gamma$, where γ is a homothetic transformation with coefficient k^{-1}, and σ is the reflection in its center. But then $\alpha = \sigma\gamma^{-1} = \gamma^{-1}\sigma$, where γ^{-1} is a homothetic transformation with coefficient $(k^{-1})^{-1} = k$, and σ is the reflection in its center (which is the same as that of γ).

iii. A, B, C are not collinear. Let S be the circle through A and B tangent to BC at B, and let T be the circle through B and C tangent to AB at B. S and T intersect in two points, one of which is B, the other O, say. Produce CB to a point P and AB to Q. Then $\angle AOB = \angle PBA$ (both are equal to half the arc AB), and similarly $\angle BOC = \angle QBC$. But $\angle PBA = \angle QBC$ (vertically opposite), so that

$$\angle AOB = \angle BOC.$$

Furthermore, $\angle OAB = \angle OBC$. It follows that the triangles OAB and OBC are similar, the scale factor being $BC : AB = k$. Let ρ be the rotation about O through the angle β which takes the ray OA into OB and OB into OC. Then it is clear that $\rho\gamma$, like α, takes A into B and B into C, where γ is the homothetic transformation with coefficient k and center O. Since α and $\rho\gamma$ are both of the first kind, they are equal, and it is clear that also $\rho\gamma = \gamma\rho$.

We now show that if α is a similarity transformation of the first kind, then its representation as the product of a rotation and a homothetic transformation with center at the center of the rotation is unique.

Suppose then that $\alpha = \rho\gamma$, where γ is a rotation about some point O and γ is a homothetic transformation with center O. Then O is fixed under α and is the only such point. For if A also were fixed, then $OA = O'A' = kOA$, so that $k = 1$, contrary to the hypothesis that α is not orthogonal. So if α also can be represented in the form $\rho^*\gamma^*$, where ρ^* and γ^* have the same center, this center must be O.

It follows from

$$\alpha = \rho\gamma = \rho^*\gamma^*$$

that

$$\gamma\gamma^{*-1} = \rho^{-1}\rho^*.$$

But the right side is the product of two rotations about O, so it is itself a rotation about O, and the left side is the product of two homothetic transformations with center O, so it is itself a homothetic transformation with center O. However, a homothetic transformation cannot be a rotation unless it is the identity. Thus $\gamma\gamma^{*-1} = \rho^{-1}\rho^* = \varepsilon$, and $\gamma = \gamma^*$, $\rho = \rho^*$.

Case 3. Let A be any point, B its image, and C the image of B. Then $BC/AB = k$. We distinguish cases, as before.

i. A, B, C all lie on a line l, with B in the middle.

We have already shown (Case 2,i) that there exists a homothetic transformation γ with center O on l taking A to B and B to C. Let σ be the reflection in l. Then σ leaves the points of l invariant, and so it also takes A to B and B to C. By the usual argument, $\alpha = \gamma\sigma = \sigma\gamma$, and this is a representation in the required form.

ii. A, B, C are collinear, with B not in the middle. In this case, we know that there is a homothetic transformation γ with coefficient k and center O on l, such that $\sigma\gamma$ takes A into B and B into C, where σ is the reflection in O. Let m be the perpendicular to l through O, and let σ' be the reflection in m. Then σ' has the same effect on the points of l as does σ. Thus $\sigma'\gamma$ takes A into B and B into C, and, being of the second kind, it must be α. Also $\alpha = \gamma\sigma'$.

iii. A, B, C are not collinear. Choose points P and Q on the line segments AB and BC, respectively, such that $BP : PA = CQ : QB = k$. Let A^* and B^* be the reflections of A and B in the line $l = PQ$. A and C lie on the same side of l, which is the opposite side from B. Thus on one side of l lie A^* and B and on the other A, B^*, and C. The ratio of the distances to l from B and A^* is $k \neq 1$, so that A^*B intersects l in some point O lying outside the segments A^*B, since A^* and B are on the same side of l.

Since

$$\frac{CQ}{QB} = k,$$

we have

$$QB = \frac{BC}{k+1} = \frac{k}{k+1} AB,$$

and since

$$\frac{BP}{PA} = k,$$

we have

$$PB = \frac{k}{k+1} \cdot AB,$$

so that $BQ = PB$, and $\angle BPQ = \angle BQP = \beta$, say. Since B and B^* are symmetrically opposite l, $\angle B^*PQ = \beta$, so that PB^* is parallel to $BQ = BC$ (alternate angles). Since A^*, P, B^* are the images of A, P, B under the reflection in l, they are collinear. Thus A^*B^* is parallel to BC.

We now show that B^*C passes through O. Let C' be the point of intersection of OB^* and BC (they are not parallel, since then OB^* would also be parallel to A^*B^*). Since A^*B^* is parallel to BC, $C'Q : QB = B^*P : PA^*$. But $B^*P : PA^* = PB : PA = k$. Thus $C'Q : QB = k = CQ : QB$, so that C' coincides with C. Now the triangles OA^*B^* and OBC are similar (since they have the same angle at O and parallel bases), so that

$$\frac{OB}{OA^*} = \frac{BC}{A^*B^*} = \frac{BC}{AB} = k.$$

Let σ be the reflection in l. It takes A and B into A^* and B^* respectively. The homothetic transformation γ with center O and coefficient k takes A^* to B and, therefore, also takes B^*

into C (for the image D of B^* must lie on OB^* and also be such that BD is parallel to A^*B^*). Thus $\gamma\sigma = \sigma\gamma$, like α, takes A into B and B and C, and, since both are of the second kind, we have

$$\alpha = \gamma\sigma = \sigma\gamma.$$

To prove the uniqueness of this representation, note that the center O of γ is the only point invariant under α. So if also $\alpha = \sigma^*\gamma^*$, where γ^* is a homothetic transformation with center O^* and σ^* the reflection in a line through O^*, then we must have $O^* = O$.

Since

$$\alpha = \sigma\gamma = \sigma^*\gamma^*,$$

we have

$$\sigma^{*-1}\sigma = \gamma^*\gamma^{-1}.$$

But the left side, the product of reflections in two lines through O, is a rotation about O, while the right side is a homothetic transformation with center O. This means that both sides are the identity, so that $\sigma^* = \sigma$, $\gamma^* = \gamma$. ▼

18. Similarity Transformations of the Plane in Coordinates

18.1. Homothetic Transformations

Let γ be the homothetic transformation with center O and coefficient k. We introduce a system of rectangular coordinates with origin at O. Let (x, y) be the coordinates of a point M of the plane and (x', y') those of its image M' under γ. Drop perpendiculars MP and MQ from M onto the x and y axes and perpendiculars $M'P'$ and $M'Q'$ from M' (Fig. 44). Then

$$\frac{OM'}{OM} = \frac{OP'}{OP} = \frac{OQ'}{OQ} = k. \qquad (3)$$

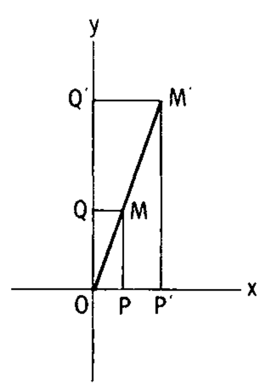

Fig. 44

So

$$OP' = kOP,$$

$$OQ' = kOQ.$$

Now P and P' lie on the same side of the same ray Ox through O, and similarly for Q and Q'. It follows that

$$x' = kx,$$

$$y' = ky.$$

18.2. The General Case

Let α be a similarity transformation with coefficient k. We introduce rectangular coordinates with origin at any point O. By Theorem 1 of the last section, we may write $\alpha = \omega\gamma$, where γ is the homothetic transformation with center O and coefficient k, and ω is an orthogonal transformation.

By Section 18.1, γ takes the point $M(x, y)$ into the point $M^*(x^*, y^*)$, where

$$x^* = kx, \qquad y^* = ky. \tag{1}$$

If α is of the first kind, so is ω, and, by Eq. (4) in Section 10.5, if $\omega(M^*) = M'(x', y')$,

$$x' = x^* \cos\beta - y^* \sin\beta + a, \qquad y' = x^* \sin\beta + y^* \cos\beta + b. \tag{2}$$

Here (a, b) are the coordinates of the image O' of O, and β is the angle through which every vector is rotated by α (or ω).

It follows from (1) and (2) that

$$x' = k(x\cos\beta - y\sin\beta) + a,$$

$$y' = k(x\sin\beta + y\cos\beta) + b.$$

If α is of the second kind, then so is ω and, as before, we have

$$x' = k(x \cos 2\gamma + y \sin 2\gamma) + a,$$
$$y' = k(x \sin 2\gamma - y \cos 2\gamma) + b,$$

where (a, b) are the coordinates of the image O' of O under α, and γ is the angle that the axis of any reflection σ, such that $\omega = \sigma\tau$ for some translation τ, makes with the x axis. Or we may say that 2γ is the angle between Ox and its image under α or ω.

19. Similarity Transformations in Space

Similarity transformations of space are defined just as for the plane. Under them, lines go into lines, the order of points along lines is preserved, planes go into planes, the images of two parallel lines or planes are two parallel lines or planes, angles between lines or planes are preserved, and the ratio between the lengths of segments is preserved.

If A, B, C, D are any four noncoplanar points, and A', B', C', D' are four points such that the tetrahedra $ABCD$ and $A'B'C'D'$ are similar, then there exists a unique similarity transformation taking A, B, C, D into A', B', C', D', respectively, and the coefficient of this transformation is the ratio between any pair of corresponding sides of the two tetrahedra (for example, $A'B' : AB$). Just as for plane transformations, similarity transformations of space can be divided into those of the first and second kinds.

If A, B, C are three noncollinear points and A', B', C' are three points such that the triangles ABC and $A'B'C'$ are similar, then there exists a unique similarity transformation of the first kind and a unique one of the second kind, taking A, B, C into A', B', C', respectively. These two transformations have the same coefficient $k = B'C' : BC$.

The set of all similarity transformations of space forms a group, of which the orthogonal group of space is a subgroup.

A homothetic transformation of space is defined in the same way as a plane homothetic transformation. A given similarity can be represented in the form $\gamma\beta$, where γ is a homothetic transformation with prescribed center O, and β is an orthogonal transformation of space.

Any similarity transformation α of space either is an orthogonal transformation or can be represented uniquely as the product of a homothetic transformation γ and a rotation ρ about an axis l passing through the center O of γ, if α is of the first kind, or as a product $\gamma\rho\sigma$ where γ and ρ satisfy the conditions above and σ is the reflection in the plane through O perpendicular to l, if α is of the second kind. These transformations can be taken in any order. In particular, a similarity transformation of space with coefficient $k \neq 1$ has a unique fixed point.

If we introduce rectangular coordinates in space, then a given similarity transformation α with coefficient k is specified in coordinates by a system of equations of the following form:

$$\begin{aligned} x' &= k(a_{11}x + a_{12}y + a_{13}z) + a, \\ y' &= k(a_{21}x + a_{22}y + a_{23}z) + b, \\ z' &= k(a_{31}x + a_{32}y + a_{33}z) + c, \end{aligned} \tag{1}$$

where (x', y', z') are the coordinates of the image M' of the point M with coordinates (x, y, z). The image O' of O has coefficients (a, b, c), and the a_{ij} are the direction cosines of the angles that the images of the three coordinates axes make with the coordinate axes. In particular, the a_{ij} satisfy Eqs. (1) and (2) of Section 13. We may prove (1) above exactly as we proved the corresponding result for the plane in Section 18, by using the results of Section 13. As a particular case of (1), the analytic expression for the homothetic transformation with center O and coefficient k is

$$x' = kx,$$

$$y' = ky,$$

$$z' = kz.$$

We may use (1) to prove that a similarity transformation of space with coefficient $k \neq 1$ has a unique fixed point. The coordinates of a fixed point $M(x, y, z)$ must satisfy (1) when we substitute x for x', etc. Thus we have

$$(a_{11}k - 1)x + a_{12}ky + a_{13}kz + a = 0,$$

$$a_{21}kx + (a_{22}k - 1)y + a_{23}kz + b = 0,$$

$$a_{31}kx + a_{32}ky + (a_{33}k - 1)z + c = 0.$$

To show that this system of equations has a unique solution it is sufficient (and necessary) to show that the determinant associated with it is nonzero, so that

$$\begin{vmatrix} a_{11}k - 1 & a_{12}k & a_{13}k \\ a_{21}k & a_{22}k - 1 & a_{23}k \\ a_{31}k & a_{32}k & a_{33}k - 1 \end{vmatrix} \neq 0.$$

But this determinant is zero if and only if there exist numbers p, q, r such that

$$a_{11}p + a_{12}q + a_{13}r = \frac{1}{k}p,$$

$$a_{21}p + a_{22}q + a_{23}r = \frac{1}{k}q, \qquad (2)$$

$$a_{31}p + a_{32}q + a_{33}r = \frac{1}{k}r.$$

Now the left-hand sides of these equations are the coordinates of the image of the point $P(p, q, r)$ under the orthogonal transformation α whose analytic expression is given by (1) of Section 13 (with $a = b = c = 0$). If this is the point P', then we must have $OP' = OP$. But the coordinates of P' are given by the right-hand side of (2), and we see from them that $OP' = k^{-1}OP$. So the determinant cannot be zero if $k \neq 1$, and the system (1) above has a unique solution.

We could establish in exactly the same way that a similarity transformation of the plane has exactly one fixed point; having done so, Theorem 2 of Section 17 becomes very easy, and the result on the representation of a similarity transformation of space as a product of elementary transformations (given above) becomes equally easy. For a direct proof (not using analytic methods to establish the existence of a fixed point), see Jacques Hadamard, "Leçons de géométrie élémentaire," Vol. I, p. 142, A. Colin, Paris, 1898.

We may define similarity transformations in the obvious manner for spaces of higher dimensions than three, and all the obvious analogs of previous results continue to hold. In particular, the analytic expression for such a transformation is the obvious generalization of (1), subject to conditions generalizing (2) in Section 13. A similarity transformation with coefficient $k \neq 1$ of a space of any number of dimensions has exactly one fixed point.