8.1 Collineations

We now turn to transformations that were first introduced by Leonhard Euler (1707-1783).

Affine transformations (as collineations)

8.1.1 Definition. An **affine transformation** (or **affinity**) is a collineation that preserves parallelness among lines.

So, if $\mathcal{L}$ and $\mathcal{M}$ are parallel lines and α is an affine transformation, then lines $\alpha(\mathcal{L})$ and $\alpha(\mathcal{M})$ are parallel. It is easy to prove the following result.

8.1.2 Proposition. *A collineation is an affine transformation and, conversely, an affine transformation is a collineation.*

Proof : An affine transformation is by definition a collineation. If β is any collineation and $\mathcal{L}$ and $\mathcal{M}$ are distinct parallel lines, then $\beta(\mathcal{L})$ and $\beta(\mathcal{M})$ cannot contain a common point $\beta(P)$, as point P would then have to be on both $\mathcal{L}$ and $\mathcal{M}$. Therefore, every collineation is an affine transformation. □

Note : Affine transformations and collineations are *exactly* the same thing for the Euclidean plane. The choice between the terms *affine transformation* and *collineation* is sometimes arbitrary and sometimes indicates a choice of emphasis on parallelness of lines or on collinearity of points. Loosely speaking, *affine geometry* is what remains after surrendering the ability to measure length (isometries) and surrendering the ability to measure angles (similarities), but maintaining the incidence structure of lines and points (collineations).

8.1.3 Example. Similarities preserve parallelness and hence are affine transformations. In particular, isometries are also affine transformations.

8.1.4 Example. The mapping

$$\alpha : \mathbb{E}^2 \to \mathbb{E}^2 , \quad (x, y) \mapsto (2x, y)$$

is an affine transformation that is not a similarity.

Note : The word *symmetry* brings to mind such general ideas as balance, agreement, order, and harmony. We have been exceedingly conservative in our use of the word *symmetry*; for us, symmetries are restricted to isometries. With a broader mathematical usage of the term, we would certainly be saying that the similarities are the symmetries of similarity geometry and that the collineations are the symmetries of affine geometry. In the most broad usage, *the group of all transformations on a structure that preserves the essence of that structure constitutes the symmetries (also called the automorphisms) of the structure.*

A collineation preserves collinearity of points. We wish to show that, conversely, a transformation such that the image of every three collinear points are themselves collinear must be a collineation.

8.1.5 Proposition. *A transformation such that the images of every three collinear points are themselves collinear is an affine transformation.*

Proof : We suppose α is a transformation that preserves collinearity and aim to show $\alpha(\mathcal{L})$ is a line whenever $\mathcal{L}$ is a line. Let A and B be distinct points on $\mathcal{L}$, and let $\mathcal{M}$ be the line through $\alpha(A)$ and $\alpha(B)$. By the definition of α, all the points of $\alpha(\mathcal{L})$ are on $\mathcal{M}$. However, are all the points of $\mathcal{M}$ on $\alpha(\mathcal{L})$? Suppose C' is a point on $\mathcal{M}$ distinct from $\alpha(A)$ and $\alpha(B)$, and let C be the point such that $\alpha(C) = C'$. To show C must be on $\mathcal{L}$, we assume C is off $\mathcal{L}$ and then obtain a contradiction. Now the image of all the points of $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{AC}$ are on $\mathcal{M}$ since collinearity is preserved under α. However, any point P in the plane is on a line containing two distinct points of $\triangle ABC$. Since the images of these two points lie on $\mathcal{M}$, then the image of P lies on $\mathcal{M}$. Therefore, the image of every point lies on $\mathcal{M}$, contradicting the fact that α is an onto mapping. Hence, C must lie on $\mathcal{L}$, $\mathcal{M} = \alpha(\mathcal{L})$, and α is a collineation, as desired. □

Are the affine transformations the same as those transformations for which the images of any three noncollinear points are themselves noncollinear ? The answer is "Yes".

8.1.6 Proposition. *A transformation is an affine transformation if and only if the images of any three noncollinear points are themselves noncollinear.*

Proof : Suppose α is an affine transformation. Then α^{-1} is an affine transformation and can't take three noncollinear points to three collinear points. Therefore, affine transformation α must take any three noncollinear points to three noncollinear points.

Conversely, suppose β is a transformation such that the images of any three noncollinear points are themselves noncollinear. Assume β is *not* an affine transformation. Then β^{-1} is not an affine transformation. By the contrapositive of the preceding result, then there are three collinear points whose images under β^{-1} are not collinear. Hence, since β is the inverse of β^{-1}, then there are three noncollinear points whose images under β are collinear, contradiction. Therefore, β is an affine transformation. □

An affine transformation preserves betweenness

The result above does *not* state that the image of a triangle under an affine transformation is necessarily a triangle, but states only that the images of the vertices of a triangle are themselves vertices of a triangle. We do not know the image of a segment is necessarily a segment. More fundamental, we do not know that an affine transformation necessarily preserves betweenness. It will take some effort to prove this. We begin by showing that *midpoint is actually an affine concept*; that is, an affine transformation carries the midpoint of two given points to the midpoint of their images.

8.1.7 Proposition. *If α is an affine transformation and M is the midpoint of points A and B, then $\alpha(M)$ is the midpoint of $\alpha(A)$ and $\alpha(B)$.*

Proof : Suppose A and B are distinct points and α is an affine transformation. Let P be any point off $\overleftrightarrow{AB}$. Let Q be the intersection of the line through A that is parallel to $\overleftrightarrow{PB}$ and the line through B that is parallel to $\overleftrightarrow{PA}$. So $\square APBQ$ is a parallelogram. Let $A' = \alpha(A)$, $B' = \alpha(B)$, $P' = \alpha(P)$, and $Q' = \alpha(Q)$. Since two parallel lines go to two parallel lines under α, then $\Box A'P'B'Q'$ is a parallelogram. (We are *not* claiming that $\alpha(\Box APBQ) = \Box A'P'B'Q'$ but only that A', P', B', Q' are vertices in order of a parallelogram.) Further, M, the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{PQ}$, must go to M', the intersection of $\overleftrightarrow{A'B'}$ and $\overleftrightarrow{P'Q'}$. However, since the diagonals of a parallelogram bisect each other, then M is the midpoint of A and B while M' is the midpoint of A' and B'. Hence, α preserves midpoints. $\Box$

8.1.8 PROPOSITION. *If α is an affine transformation, the $n+1$ points $P_0, P_1, P_2,$ $\dots, P_n$ divide the segment $\overline{P_0P_n}$ into n congruent segments $\overline{P_{i-1}P_i}$, and $P_i' = \alpha(P_i)$, then the $n+1$ points $P_0', P_1', P_2', \dots, P_n'$ divide the segment $\overline{P_0'P_n'}$ into n congruent segments $\overline{P_{i-1}'P_i'}$.*

PROOF : Suppose α is an affine transformation and the $n+1$ points $P_0, P_1, P_2, \dots,$ P_n divide the segment $\overline{P_0P_n}$ into n congruent segments $\overline{P_{i-1}P_i}$. Let $P_i' = \alpha(P_i)$. Since $P_0P_1 = P_1P_2$, $P_1P_2 = P_2P_3$, $\dots$, then P_1 is the midpoint of P_0 and P_2, point P_2 is the midpoint of P_1 and P_3, etc. Hence, P_1' is the midpoint of P_0' and P_2', point P_2' is the midpoint of P_1' and P_3', etc. So the images $P_0', P_1', P_2', \dots, P_n'$ divide the segment $\overline{P_0'P_n'}$ into n congruent segments $\overline{P_{i-1}'P_i'}$. $\Box$

It follows from this last result that *P between A and B implies $\alpha(P)$ between $\alpha(A)$ and $\alpha(B)$ provided that $\underline{AP}/\underline{PB}$ is rational.*

NOTE : It would have to be a very strange collineation that allowed betweenness not to be preserved in general although preserving midpoints. Early geometers avoided such a "monster transformation" simply by incorporating the preservation of betweenness within the definition of an affine transformation. In 1880 GASTON DARBOUX (1842-1917) showed that the "monster transformation" does not exist. Thus the following result holds (but the proof wil be omitted).

8.1.9 THEOREM. *If α is an affine transformation and point P is between points A and B, then point $\alpha(P)$ is between $\alpha(A)$ and $\alpha(B)$.*

As an immediate consequence of THEOREM 8.1.9, we know that *an affine transformation preserves all those geometric entities whose definition goes back only to the definition of betweenness.* Thus, an affine transformation preserves segments, rays, triangles, quadrilaterals, halfplanes, interiors of triangles, etc. In particular, the following result holds :

8.1.10 PROPOSITION. *If* A', B', C' *are the respective images of three non-collinear points* A, B, C *under affine transformation* α*, then*

$$\alpha(\overline{AB}) = \overline{A'B'} \quad \textit{and} \quad \alpha(\triangle ABC) = \triangle A'B'C' .$$

8.1.11 PROPOSITION. *An affine transformation fixing two points on a line fixes that line pointwise.*

PROOF : Suppose affine transformation α fixes two points A and B. Assume there is a point C on $\overleftrightarrow{AB}$ such that $C' \neq C$ with $C' = \alpha(C)$. Without loss of generality, we may assume C is on $AB^{\rightarrow}$. As an intermediate step, we shall show C is between two fixed points A and D. Let $B_0 = B$ and define B_{i+1} so that B_i is the midpoint of A and B_{i+1} for $i = 0, 1, 2 \ldots$. Since A and B_0 are given as fixed by α, then each of $B_1, B_2, B_3, \ldots$ in turn must be fixed by α since α preserves midpoints. Let $D = B_k$ where k is an integer such that

$$AB_k = 2^k AB > AC .$$

Then C lies between fixed points A and D. So $\overline{AD}$ is then fixed and both C and C' lie in $\overline{AD}$. Now, let n be an integer large enough so that $nCC' > AD$. Let $P_0 = A$, $P_n = D$, and the $n+1$ points $P_0, P_1, \ldots, P_n$ divide the segment $\overline{AD}$ into n congruent segments $\overline{P_{i-1}P_i}$. Each of the points P_i is fixed by α by PROPOSITION 8.1.7. So each $\overline{AP_i}$ and $\overline{P_iD}$ is fixed by α. However, integer n was chosen large enough so that for some integer j point P_j is between C and C'. So C and C' are in different fixed segments $\overline{AP_j}$ and $\overline{P_jD}$, contradiction. Therefore, $\alpha(C) = C$ for all points on $\overleftrightarrow{AB}$, as desired. □

8.1.12 COROLLARY. *An affine transformation fixing three noncollinear points must be the identity. Given $\triangle ABC$ and $\triangle DEF$, there is at most one affine transformation α such that $\alpha(A) = D$, $\alpha(B) = E$, and $\alpha(C) = F$.*

NOTE : In the next section we shall see that there is also at least one affine transformation α as described in the corollary above. Thus *an affine transformation is completely determined once the images of any three noncollinear points are known.*

8.2 Affine Linear Transformations

We start by making an "ad hoc" definition.

8.2.1 DEFINITION. An **affine linear transformation** is any mapping

$$\alpha : \mathbb{E}^2 \to \mathbb{E}^2, \quad (x, y) \mapsto (ax + by + h, cx + dy + k) \quad \text{where } ad - bc \neq 0\,.$$

The number $ad - bc$ is called the **determinant** of α.

An affine linear transformation is actually a transformation since a given (x, y) obviously determines a unique (x', y') and, conversely, a given (x', y') determines a unique (x, y) precisely because the determinant is nonzero. As we might expect, affine linear transformations are related to affine transformations.

Exercise 126 If $P = (p_1, p_2)$, $Q = (q_1, q_2)$, and $R = (r_1, r_2)$ are vertices of a triangle, show that the area of $\triangle PQR$ is

$$\frac{1}{2}|(q_1 - p_1)(r_2 - p_2) - (q_2 - p_2)(r_1 - p_1)|.$$

(Hence the area of a triangle with vertices $(0, 0)$, (a, b), (c, d) is half the absolute value of $ad - bc$.)

8.2.2 PROPOSITION. *An affine linear transformation is an affine transformation and, conversely, an affine transformation is an affine linear transformation.*

Proof : Let α be an affine linear transformation and suppose line $\mathcal{L}$ has equation $px+qy+r=0$. Since p and q are not both zero, then $ap+cq$ and $bp+dq$ are not both zero. So there is a line $\mathcal{M}$ with equation

$$(ap+cq)x+(bp+dq)y+r+hp+kq=0.$$

Line $\mathcal{M}$ is introduced because each of the following implies the next, where $\alpha((x,y))=(x',y')$:

(1) (x',y') is on line $\mathcal{L}$.

(2) $px'+qy'+r=0$.

(3) $p(ax+by+h)+q(cx+dy+k)+r=0$.

(4) $(ap+cq)x+(bp+dq)y+r+hp+kq=0$.

(5) (x,y) is on line $\mathcal{M}$.

We have shown that α^{-1} is a transformation that takes any line $\mathcal{L}$ to some line $\mathcal{M}$. So α^{-1} is a collineation. Hence, α is itself a collineation.

Conversely, suppose α is an affine transformation. Let

$$\alpha((0,0))=(p_1,p_2)=P, \quad \alpha((1,0))=(q_1,q_2)=Q, \quad \text{and} \quad \alpha((0,1))=(r_1,r_2)=R\,.$$

Since $(0,0),(1,0),(0,1)$ are noncollinear, then P,Q,R are noncollinear. Hence the mapping β with equations

$$\begin{cases} x'=(q_1-p_1)x+(r_1-p_1)y+p_1 \\ \\ y'=(q_2-p_2)x+(r_2-p_2)y+p_2 \end{cases}$$

is an affine linear transformation, since the absolute value of its determinant is twice the area of $\triangle PQR$ and therefore nonzero (see **Exercise 126**). Further,

$$\beta((0,0))=\alpha((0,0)), \quad \beta((1,0))=\alpha((1,0)), \quad \text{and} \quad \beta((0,1))=\alpha((0,1)).$$

Therefore (Corollary 8.1.11), we have $\alpha=\beta$. So α is an affine linear transformation. □

NOTE : Chosing the term *affine linear transformation* over its equivalents *collineation* and *affine transformation* can emphasize a coordinate viewpoint.

Given $\triangle ABC$ and $\triangle DEF$, we know that there is at most one affine transformation α such that $\alpha(A) = D$, $\alpha(B) = E$, and $\alpha(C) = F$. We can now show that there is at least one such transformation α.

8.2.3 PROPOSITION. *Given $\triangle ABC$ and $\triangle DEF$, there is a unique affine transformation α such that*

$$\alpha(A) = D, \quad \alpha(B) = E, \quad \textit{and} \quad \alpha(C) = F.$$

PROOF : Given $\triangle ABC$ and $\triangle DEF$, we know (COROLLARY 8.1.12) there is at most one affine transformation α such that $\alpha(A) = D$, $\alpha(B) = E$ and $\alpha(C) = F$. We now show there is at least one such affine transformation α. From the preceding paragraph, we see how to find the equations for an affine linear transformation β_1 such that

$$\beta_1((0,0)) = A, \quad \beta_1((1,0)) = B, \quad \text{and} \quad \beta_1((0,1)) = C.$$

Repeating the process, we see there is an affine linear transformation β_2 such that

$$\beta_2((0,0)) = D, \quad \beta_2((1,0)) = E, \quad \text{and} \quad \beta_2((0,1)) = F.$$

The transformation $\beta_2\beta_1^{-1}$ is the desired affine transformation α that takes points A, B, C to points D, E, F, respectively. $\square$

Matrix representation

Let $\alpha : \mathbb{E}^2 \to \mathbb{E}^2$ be a transformation given by

$$(x, y) \mapsto (ax + by + h, cx + dy + k).$$

(α is an affine linear transformation.)

NOTE : Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

Hence the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ defines a mapping $(x, y) \mapsto (ax + by, cx + dy)$. Indeed, we write the pair (x, y) as a column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ (in fact, we identify points with geometric vectors) and so we get

$$(x, y) = \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = (ax + by, cx + dy).$$

This mapping is *linear* (i.e. preserves the vector structure of $\mathbb{E}^2$) and is invertible if (and only if) the matrix is invertible.

When the coefficients h and k vanish, α is linear and hence admits a matrix representation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

We say that the (invertible) matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *represents* the (linear) transformation α. In order to extend this representation to the general case, of affine linear transformations, we need to accomodate translations.

Exercise 127

(a) Verify that

$$\begin{bmatrix} 1 & 0 & 0 \\ h & a & b \\ k & c & d \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ ax + by + h \\ cx + dy + k \end{bmatrix}.$$

(b) Show that the matrix $\begin{bmatrix} 1 & 0 & 0 \\ h & a & b \\ k & c & d \end{bmatrix}$ is *invertible* if and only if $ad - bc \neq 0$, and then find its inverse.

If we "redefine" the concept of point – and write the pair (x, y) as a column matrix $\begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$ (this identification is more than just a "clever" notation) – then we have

$$(x,y) = \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ h & a & b \\ k & c & d \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ ax+by+h \\ cx+dy+k \end{bmatrix} = (ax+by+h, cx+dy+k).$$

We see that the 3×3 matrix

$$[\alpha] = \begin{bmatrix} 1 & 0 & 0 \\ h & a & b \\ k & c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mathbf{v} & A \end{bmatrix}$$

(where $\mathbf{v} = \begin{bmatrix} h \\ k \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$) *represents* the transformation

$$\alpha : \mathbb{E}^2 \to \mathbb{E}^2, \quad (x,y) \mapsto (ax+by+h, cx+dy+k).$$

Exercise 128 Use matrix representation to show that the set of all linear affine transformations forms a group. (This group consists of all collineations, and is usually denoted by $\mathfrak{Aff}$.)

8.2.4 Example. The *identity transformation* ι is represented by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Thus

$$[\iota] = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}.$$

8.2.5 Example. Consider the point $P = (h,k)$ and let $\mathbf{v} = \begin{bmatrix} h \\ k \end{bmatrix}$. The *translation* $\tau = \tau_{O,P}$ is represented by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$. Thus

$$[\tau] = \begin{bmatrix} 1 & 0 \\ \mathbf{v} & I \end{bmatrix}.$$

8.2.6 EXAMPLE. Again, consider the point $P = (h, k)$. The *halfturn* $\sigma = \sigma_P$ is represented by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 2h & -1 & 0 \\ 2k & 0 & -1 \end{bmatrix}$. Thus

$$[\sigma] = \begin{bmatrix} 1 & 0 \\ 2\mathbf{v} & -I \end{bmatrix}.$$

Exercise 129 Let $P = (h, k)$ be a point. Determine the matrix which represents the dilation $\delta_{P,r}$ (of ratio $r \neq 0$) and hence verify the relations :

(a) $\delta_{P,-r} = \sigma_P \delta_{P,r}$.

(b) $\delta_{P,1} = \iota$.

(c) $\delta_{P,-1} = \sigma_P$.

(d) $\delta_{P,s}\delta_{P,r} = \delta_{P,rs} \quad (r, s \neq 0)$.

Strains and shears

Some specific, basic affine transformations are introduced next.

8.2.7 DEFINITION. For number $k \neq 0$, the affine transformation

$$\varepsilon_{\mathcal{X},k} : (x, y) \mapsto (x, ky)$$

is called a **strain of ratio k about the x-axis**.

8.2.8 DEFINITION. For number $k \neq 0$, the affine transformation

$$\varepsilon_{\mathcal{Y},k} : (x, y) \mapsto (kx, y)$$

is called a **strain of ratio k about the y-axis**.

For fixed k, the product of the two affine transformations above is the familiar dilation about the origin $(x, y) \mapsto (kx, ky)$. Thus

$$\varepsilon_{\mathcal{X},k}\varepsilon_{\mathcal{Y},k} = \delta_{O,k}.$$

Note : The concept of a strain of ratio k about a given line $\mathcal{L}$ can be defined analogously. However, one can prove that *any dilation is the product of two strains about perpendicular lines*.

8.2.9 Example. The strain with equations

$$\begin{cases} x' = 2x \\ \\ y' = y \end{cases}$$

fixes the y-axis pointwise and stretches out the plane away from and perpendicular to the y-axis.

Note : As with similarity theory, the terminology here is *not* standardized. Each of the following words has been used for a strain or for a strain with positive ratio : enlargement, expansion, lengthening, stretch, compression.

8.2.10 Definition. For number $k \neq 0$, the affine transformation

$$\zeta_{\mathcal{X},k} : (x, y) \mapsto (x + ky, y)$$

is called a **shear along the x-axis**.

Here the x-axis is fixed pointwise and every point is moved "horizontally" a directed distance proportional to its directed distance from the x-axis. We shall see below that *a shear has the property of preserving area.*

8.2.11 Definition. An affine transformation that preserves area is said to be **equiaffine**.

8.2.12 Proposition. *An affine transformation is the product of a shear, a strain, and a similarity.*

Proof : We can see that the general affine linear transformation with equations

$$\begin{cases} x' = ax + by + h \\ \\ y' = cx + dy + k \end{cases} \quad \text{with} \quad ad - bc \neq 0$$

can be factored into the *similarity* with equations

$$\begin{cases} x' = ax - cy + h \\ \\ y' = cx + ay + k \end{cases}$$

following the *strain* with equations

$$\begin{cases} x' = x \\ \\ y' = \dfrac{ad - bc}{a^2 + c^2} y \end{cases}$$

following the *shear* with equations

$$\begin{cases} x' = x + \dfrac{ab + cd}{a^2 + c^2} y \\ \\ y' = y. \end{cases}$$

Indeed, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ h & a & b \\ k & c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ h & a & -c \\ k & c & a \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{ad-bc}{a^2+c^2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{ab+cd}{a^2+c^2} \\ 0 & 0 & 1 \end{bmatrix}.$$

□

8.2.13 PROPOSITION. *An affine transformation is a product of strains.*

PROOF : First, we see that the shear ($\zeta_{\mathcal{X},1}$) with equations

$$\begin{cases} x' = x + y \\ \\ y' = y \end{cases}$$

can be factored into the *similarity* with equations

$$\begin{cases} x' = \frac{5-\sqrt{5}}{20} x + \frac{5-3\sqrt{5}}{20} y \\ \\ y' = \frac{-5+3\sqrt{5}}{20} x + \frac{5-\sqrt{5}}{20} y \end{cases}$$

following the *strain* with equations

$$\begin{cases} x' = \frac{3+\sqrt{5}}{2}x \\ \\ y' = y \end{cases}$$

following the *similarity* with equations

$$\begin{cases} x' = 2x + (1+\sqrt{5})y \\ \\ y' = -(1+\sqrt{5})x + 2y. \end{cases}$$

Indeed, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{5-\sqrt{5}}{20} & \frac{5-3\sqrt{5}}{20} \\ 0 & \frac{-5+3\sqrt{5}}{20} & \frac{5-\sqrt{5}}{20} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3+\sqrt{5}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1+\sqrt{5} \\ 0 & -(1+\sqrt{5}) & 2 \end{bmatrix}.$$

Secondly, we see that the nonidentity shear ($\zeta_{\mathcal{X},k}$, $k \neq 0$) with equations

$$\begin{cases} x' = x + ky \\ \\ y' = y \end{cases}$$

can be factored into the *strain* of ratio k about the y-axis ($\varepsilon_{\mathcal{Y},k} : (x,y) \mapsto (kx, y)$) following the *shear* that just factored above following the *strain* of ratio $\frac{1}{k}$ about the y-axis ($\varepsilon_{\mathcal{Y},\frac{1}{k}} : (x,y) \mapsto (\frac{1}{k}x, y)$). The relation

$$\zeta_{\mathcal{X},k} = \varepsilon_{\mathcal{Y},k}\zeta_{\mathcal{X},1}\varepsilon_{\mathcal{Y},\frac{1}{k}}$$

holds since

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{k} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Putting these results together with Proposition 8.2.13, we see that *an affine transformation is a product of strains and similarities.* Since a similarity is an isometry following a dilation about the origin (PROPOSITION 7.1.12), and since a dilation about the origin is a product of two strains, then an affine transformation is a product of strains and isometries. However, isometries are products of reflections, which are special cases of strains. Thus affine transformations are products of strains.

□

NOTE : One can also prove that *an affine transformation is the product of a strain and a similarity.*

8.2.14 THEOREM. *Suppose affine transformation* α *has equations*

$$\begin{cases} x' = ax + by + h \\ y' = cx + dy + k \end{cases} \quad \textit{with} \quad ad - bc \neq 0.$$

Transformation α *is equiaffine if and only if*

$$|ad - bc| = 1\,.$$

Transformation α *is a similarity (of ratio* r *) if and only if*

$$a^2 + c^2 = b^2 + d^2 = r^2 \quad \textit{and} \quad ab + cd = 0.$$

Transformation α *is an isometry if and only f*

$$a^2 + c^2 = b^2 + d^2 = 1 \quad \textit{and} \quad ab + cd = 0.$$

PROOF : Suppose

$$\begin{cases} x' = ax + by + h \\ y' = cx + dy + k \end{cases}$$

are equations for affine transformation α. So the determinant $ad - bc$ of α is nonzero. What are the necessary and sufficient conditions for α to be equiaffine? In other words, when is area preserved by α ? Suppose P, Q, R are noncollinear points with

$$P = (p_1, p_2), \quad Q = (q_1, q_2), \quad R = (r_1, r_2),$$

$$P' = \alpha(P) = (p_1', p_2'), \quad Q' = \alpha(Q) = (q_1', q_2'), \quad R' = \alpha(R) = (r_1', r_2').$$

Recall that the area PQR of $\triangle PQR$ is given by

$$PQR = \pm\frac{1}{2}|(q_1 - p_1)(r_2 - p_2) - (q_2 - p_2)(r_1 - p_1)|$$

and similarly the area $P'Q'R'$ of $\triangle P'Q'R'$ is given by

$$P'Q'R' = \pm\frac{1}{2}|(q_1' - p_1')(r_2' - p_2') - (q_2' - p_2')(r_1' - p_1')|.$$

Substitution shows that

$$P'Q'R' = \pm(ad - bc)PQR\,.$$

Thus, *under an affine transformation with determinant t, area is multiplied by $\pm t$.* This result answers our question about preserving area : area is preserved by α when the determinant of α is ± 1.

Continuing with the same notation for affine transformation α, we recall that α is a similarity if and only if there is a positive number r such that

$$P'Q' = rPQ \quad \text{for all points } P \text{ and } Q\,.$$

With substitution, this equation becomes

$$\sqrt{(a^2 + c^2)(q_1 - p_1)^2 + (b^2 + d^2)(q_2 - p_2)^2 + 2(ab + cd)(q_1 - p_1)(q_2 - p_2)} =$$

$$r\sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}\,.$$

This equation can hold for all p_1, p_2, q_1, q_2 if and only if

$$a^2 + c^2 = b^2 + d^2 = r^2 \quad \text{and} \quad ab + cd = 0\,.$$

Since a similarity of ratio r is an isometry if and only if $r = 1$, we obtain the last result. □

NOTE : The matrix representing the given affine transformation α is

$$[\alpha] = \begin{bmatrix} 1 & 0 \\ \mathbf{v} & A \end{bmatrix}$$

where $\mathbf{v}$ is arbitrary and A is invertible (i.e. $ad - bc \neq 0$).

Transformation α is *equiaffine* if and only if $\det A = \pm 1$.

Transformation α is a *similarity* (of ratio r) if and only if $AA^T = r^2 I$.

Transformation α is an *isometry* if and only if $AA^T = I$ (such a matrix is called *orthogonal).*

8.3 Exercises

Exercise 130

(a) For a given nonzero number k, find all fixed points and fixed lines for the affine transformations α_k and β_k with respective equations

$$\begin{cases} x' = kx \\ y' = y \end{cases} \quad \text{and} \quad \begin{cases} x' = x + ky \\ y' = y. \end{cases}$$

(b) If $P = (-2, -1), Q = (1, 2)$, and $R = (3, -6)$, what is the area of $\triangle PQR$?

(c) What are the areas of the images of $\triangle PQR$ under the collineations α_k and β_k, respectively ?

Exercise 131 TRUE or FALSE ?

(a) An affine transformation is a collineation; a collineation is an affine linear transformation; and an affine linear transformation is an affine transformation.

(b) An affine transformation is determined once the images of three given points are known.

(c) Strains and shears are equiaffine.

(d) A shear is a product of strains and similarities.

(e) A collineation is a product of strains and similarities.

(f) A collineation is a product of strains and isometries.

(g) A dilatation is a product of strains; a strain is a product of dilations.

Exercise 132 Given nonzero number k and line $\mathcal{L}$, give a definition for the *strain* of ratio k about line $\mathcal{L}$. Using your definition, show that a dilation is a product of two strains.

Exercise 133 If $x' = ax + by + h$ and $y' = cx + dy + k$ are the equations of (affine linear) transformation α, find the equations of its inverse α^{-1}. Hence determine the matrices $[\alpha]$ and $[\alpha^{-1}]$ representing α and α^{-1}, respectively.

Exercise 134 PROVE or DISPROVE : If affine linear transformation α has determinant t, then α^{-1} has determinant t^{-1}.

Exercise 135 Suppose any affine transformation is the product of a strain and a similarity. Then show that an affine transformation is a product of two strains about perpendicular lines and an isometry. (To see that the perpendicular lines cannot be chosen arbitrarily, see the next exercise.)

Exercise 136 Show the shear with equations $x' = x + y$ and $y' = y$ is not the product of strains about the coordinate axes followed by an isometry.

Exercise 137 Show that the shears do not form a group.

Exercise 138 PROVE or DISPROVE : An equiaffine similarity is an isometry.

Exercise 139 PROVE or DISPROVE : An involutory affine transformation is a reflection or a halfturn.

Exercise 140 Give an example of an equiaffine transformation that is neither an isometry nor a shear.

DISCUSSION : Perhaps the most fundamental concept of the earlier books of EUCLID's *Elements* is that of *congruence*. Intuitively, two plane geometrical figures (i.e. arbitrary subsets of the plane) are congruent if they differ only in the position they occupy in the plane; that is, if they can be made to coincide by the application of some "rigid motion" in the plane. Somewhat more precisely, two figures F_1 and F_2 are said to be congruent if there is a mapping α of the plane onto itself that leaves invariant the distance between each pair of points (i.e. $\alpha(\mathsf{F}_1) = \mathsf{F}_2$ and