Chapter One: Transformation and Collineation

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Presentation Outline

- Mapping and Types of Mapping
- Transformation
- Properties of Transformations
- Collineation and Dilatation
- Examples of Groups of Transformation

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Motivation

Imagination is more important than knowledge.

Albert Einstein

Do not just pay attention to the words;

Instead pay attention to meaning behind the words.

But, do not just pay attention to meanings behind the words;

Instead pay attention to your deep experience of those meanings.

Tenzin Gyatso, The 14th Dalai Lama

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1. Transformations and Collineations

1.1 Preliminaries

Definition: Let X and Y be nonempty sets. Then, a *mapping* f from X to Y is a rule which assigns to every element x in X exactly one (unique) value $f(x)$ in Y, here, $f(x)$ is called the *image* of x under f. The set X is said to be the *domain of* f and Y is the co-domain of f. The set of all images of f is called *range* of f. In this definition of mappings, the word unique (exactly one) refers to the idea of *well definedness*. A rule which assigns to every element in the domain (in X) some value in the co domain (in Y) is said to be a mapping if it is well defined.

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To show well-defined ness, it suffices to show that $f(x) = y,\ f(x) = z \Rightarrow y = z$.

Notation: The mapping f from X to Y is denoted symbolically by $f : X \to Y$.

Examples

1. Let $g : R^2 \to R^2$ be given by $g(x, y) = (2x, 3y)$. Show that g is a mapping.

Solution: Clearly g is a rule which assigns to each value in R^2 a value in R^2.

Now, let's show that g is *well-defined*. Suppose $g(x, y) = (a, b) \wedge g(x, y) = (c, d)$

$$\begin{aligned} g(x, y) = (a, b) \wedge g(x, y) = (c, d) &\Rightarrow (2x, 3y) = (a, b) \wedge (2x, 3y) = (c, d) \\ &\Rightarrow 2x = a, 3y = b \wedge 2x = c, 3y = d \\ &\Rightarrow a = c \wedge b = d \Rightarrow (a, b) = (c, d) \end{aligned}$$

This implies that the image of any point (x, y) in R^2 is unique and hence g is well defined and it is a mapping.

2*. Let Z be set of integers. Consider the set $S = \{x \in Z : |x-1| \le 2\}$. Define $h : Z \to S$ by $h(x) = x^2$. Is h a mapping or not?

Solution: Here, $S = \{x \in Z : |x-1| \le 2\} = \{x \in Z : -2 \le x-1 \le 2\}$

$$= \{x \in Z : -1 \le x \le 3\} = \{-1, 0, 1, 2, 3\}$$

As we see, $x = -1$ is in S. But there is no integer in the domain such that $h(x) = -1$. This means $x = -1$ has no pre-image. Hence, h is not a mapping.

Definitions:

a) One-to-one (Injective) mapping: A mapping $f : X \to Y$ is said to be a *one-to-one (injective)* mapping if and only if f sends distinct elements of X in to distinct elements of Y. This means $x \ne y \Rightarrow f(x) \ne f(y)$. In other words, f is one to one if and only if $f(x) = f(y) \Rightarrow x = y$.

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b) Onto (Surjective) mapping: A mapping $f: X \to Y$ is said to be *onto* mapping if and only if for every point y in Y, there exists an element x in X such that

$y = f(x)$. Or if the image of f is the whole of Y. That is every element of Y has at least one pre-image in X.

c) Bijective mapping: A mapping is said to be bijective if and only if it is both one to one and onto mapping.

EXAMPLE

- Verify the whether the following mappings are injective, surjective or bijective

a) $f: R^2 \to R^2$ given by $f(x,y) = (2x, y-1)$

b) $f: R^2 \to R^2$, $f(x,y) = (\sqrt[3]{x-1}, y^3+3)$

c) $f: R^2 \to R^2$ given by $f(x,y) = (x+y, x-y)$

d) $f: R \to R^+$ given by $f(x) = e^x$

Solution:

a) Assume $f(x,y)=f(z,w)$ for any two points (x,y) and (z,w) in R^2. Then,

$(2x,y-1)=(2z,w-1)$. But from equality of order pairs, this equality is true if and only if $\begin{cases}2x=2z\\ y-1=w-1\end{cases} \Rightarrow x=z,\ y=w \Rightarrow (x,y)=(z,w)$. So, f is one-to-one.

ii) Let $(a,b)\in R^2$ be in the co-domain of f. Then, if $\exists(x,y)\in R^2$ in the domain of f, such that $f(x,y)=(a,b)$, then f is on to.

But, $f(x,y)=(2x,y-1)=(a,b)\Rightarrow 2x=a,\ y-1=b\Rightarrow x=\frac{a}{2},\ y=b+1$. Thus, we can find

$(x,y)=(\frac{a}{2},b+1)\in R^2$ such that $f(x,y)=f(\frac{a}{2},b+1)=(a,b),\ \forall(a,b)\in R^2$.

So f is on to. Therefore, the given map is bijective.

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Now, we are at the position to define transformation as follows.

Definition: *Transformation* is a one-to-one mapping from a set X onto itself. In other words, the map $f: X \to X$ is said to be a transformation if and only if it is one to one and onto. This means that for every point P in the domain there is a unique point Q such that $f(P) = Q$ and conversely, for every point R in the range there is a unique point S in the domain such that $f(S) = R$.

Transformation are denoted by Greek letters like

α	alpha
β	beta
γ	gamma
δ	delta

But, for the sake of simplicity we use letters for function like *f*, *g*, *h* in some cases.

Example: Verify the whether the following mappings are transformation or not.

$a)$ $g: R^2 \to R^2$ given by $g(x,y)=(x+y+1, x-y-1)$

$b)$ $f: R^2 \to R^2$ given by $f(x,y)=(x, x-y)$

Solution:

a) i) Assume $g(x,y)=g(z,w)$ for any two points (x,y) and (z,w) in R^2. Then,

$(x+y+1, x-y-1)=(z+w+1, z-w-1)$.

But from equality of order pairs, this equality is true if and only if

$\begin{cases} x+y+1=z+w+1 \\ x-y-1=z-w-1 \end{cases} \Rightarrow 2x=2z \Rightarrow x=z,\ y=w$. This gives $(x,y)=(z,w)$ So, g is one-to-one.

ii) Let $(a,b)\in R^2$ be in the co-domain of g .Then, if $\exists(x,y)\in R^2$ in the domain of g , such that $g(x,y)=(a,b)$, then g is on to.

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Equality of Transformations: Two transformations f and g on the same set from X to X are said to be equal if and only if they have the same value for each x in X. That is, $f = g \Leftrightarrow f(x) = g(x), \forall x \in X$.

Examples:

1. Let f and g be transformations on R^2 given by $f(x, y) = (2ax^5 - 3, 4y)$ and $g(x, y) = (6x^5 + 2b, 4y)$. If $f = g$, find the constants a and b.

Solution: By definition of equality,

$$\begin{aligned} f = g &\Leftrightarrow f(x, y) = g(x, y), \forall (x, y) \in R^2 \\ &\Leftrightarrow (2ax^5 - 3, 4y) = (6x^5 + 2b, 4y) \\ &\Leftrightarrow 2ax^5 - 3 = 6x^5 + 2b \\ &\Leftrightarrow 2a = 6, -3 = 2b \Leftrightarrow a = 3, b = -3/2 \end{aligned}$$

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Properties of Transformation

Let $f: X \to Y$ and $g: Y \to Z$ be mappings. Then for each $x \in X$, $f(x) \in Y$. Thus, there exists $y \in Y$ such that $f(x) = y$. Besides, as g is a mapping from Y to Z for each $y \in Y$, there exists $z \in Z$ such that $g(y) = z$. Thus, $g(y) = g(f(x)) = z$ which makes sense to write $g(f(x))$ as a mapping from X to Z. This mapping is evaluated by applying f first on the elements of X followed by g. This is defined as $g \circ f(x) = g(f(x))$ for each $x \in X$. So, the mapping $gof\,(\text{"}f \text{ follwed by g"})$ is called the composition mapping. In such cases, one has to remember that the range of the first mapping is a subset of the domain of the second mapping.

In particular, composition of transformation is defined as $(g \circ f)(x) = g(f(x))$ where f and g are transformations on the same set X.

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Proposition 1.1: Composition of mappings is associative

If $h: X \to Y, g: Y \to Z$ and $f: Z \to W$ are mappings, then the compositions $(f \circ g) \circ h$ and $f \circ (g \circ h)$ represent the same mapping from X in to W. That is $(f \circ g) \circ h = f \circ (g \circ h)$. Particularly, $(f \circ g) \circ h = f \circ (g \circ h)$ holds if f, g and h are transformations on the same set X.

Proof: Since the domain of h is X, by definition of composition of mappings we can see that the domain of $(f \circ g) \circ h$ is also X. But, the domain of $f \circ (g \circ h)$ is the same as the domain of $g \circ h$ and the domain of $g \circ h$ is X. Hence, the domain of $(f \circ g) \circ h$ is the same as that of $f \circ (g \circ h)$, that is, X.

So, $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are mappings with the same domain.

Furthermore, for any $x \in X$,

$$((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x))), \forall x \in X$$

$$\text{Also,} (f \circ (g \circ h))(x) = f((g \circ h)(x)) = f(g(h(x))), \forall x \in X$$

Hence, by definition of equality of mappings $(f \circ g) \circ h = f \circ (g \circ h)$.

Note: Composition of two transformations need not be commutative. That is even though both $f \circ g$ and $g \circ f$ exists and have the same domain and co-domain, $f \circ g$ and $g \circ f$ may not be equal.

Example: Let $f(x, y) = (ax + 8, 3y - 5)$ and $g(x, y) = (7x, 4y + b)$ be the transformations.

If $(g \circ f)(x, y) = (14x + 8, 12y + 23 - b)$, then, find the constants a and b.

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If $(g \circ f)(x, y) = (14x + 8, 12y + 23 - b)$, then, find the constants a and b.

Solution:

Here, $(f \circ g)(x, y) = (14x + 8, 12y + 23 - b)$

$$\Rightarrow f(g(x, y)) = (14x + 8, 12y + 23 - b)$$

$$\Rightarrow f(7x, 4y + b) = (14x + 8, 12y + 23 - b)$$

$$\Rightarrow (7ax + 8, 12y + 3b - 5) = (14x + 8, 12y + 23 - b)$$

$$\Rightarrow 7ax + 8 = 14x + 8,\ 12y + 3b - 5 = 12y + 23 - b$$

$$\Rightarrow 7a = 14,\ 3b - 5 = 23 - b$$

$$\Rightarrow a = 2,\ b = 7$$

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Proposition 1.2: The composition of transformations on the same set are again transformations.

Proof: Let $f : X \to X, g : X \to X$ be any two transformations on set X. We need to show that $f \circ g$ is also a transformation. That means we need to verify that $f \circ g$ is one to one and onto. By definition of composition $f \circ g$ is a mapping from X into X. To show $f \circ g$ is one to one, let x and y be arbitrary elements of X such that $f \circ g(x) = f \circ g(y)$.

Then, $f \circ g(x) = f \circ g(y) \Rightarrow f(g(x)) = f(g(y)) \Rightarrow g(x) = g(y) \Rightarrow x = y$ (because both f and g are one to one). Thus, $f \circ g$ is one to one.

To show $f \circ g$ is onto, let t be any element in X (considering X as co-domain of f), since f is onto there exists an element y in X (in the domain) such that $f(y) = t$.

Again, g is onto corresponding to the element y in X there is an element x in X suchthat $g(x) = y$. As a result, $(f \circ g)(x) = f(g(x)) = f(y) = t$.

Thus, $f \circ g$ is onto. Hence, we have got that $f \circ g$ is one to one and onto on the set X. Therefore, $f \circ g$ is a transformation whenever f and g are transformations on X.

Definition: A transformation from a set X into X denoted by i is said to be *identity transformation* if and only if $i(x) = x, \forall x \in X$.

Any two transformations f and g from X to X are said to be *inverse of each other* if both $g \circ f$ and $f \circ g$ are identity transformations.

That is $(g \circ f)(x) = (f \circ g)(x) = i(x) = x, \ \forall x \in X$, then f is called the inverse of g and g is called the inverse of f. We denote the inverse of a transformation f by f^{-1} (Read as "the inverse of f" or f – inverse).

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Finding Inverse of a transformation: Now let's see how can we find the inverse of transformations. Since every transformation f is bijective, its inverse denoted by f^{-1} always exists. But there is no hard and fast rule on how to find f^{-1} from the formula of f. Any way, one can use the following hints on how to find f^{-1} whenever the formula of f is given. Let $f: S \to S$ be a transformation such that $Y = f(X)$. Then, to find f^{-1}:

Step-1: Interchange X and Y in the formula of f

Step-2: Solve for Y (for coordinates of Y) in terms of X (coordinates of X).

Step-3: Equate $f^{-1}(X) = Y$ from $Y = f(X)$. That will be the formula of f^{-1}.

Example: Find the inverse of the transformation $g: R^2 \to R^2$, $g(x,y) = (2x-1, y+5)$

Step-1: Interchange coordinates of X and Y:

$$g(Y) = X \Rightarrow g(z,w) = (x,y) \Rightarrow (2z-1, w+5) = (x,y)$$

Step-2: Solve for coordinates of Y in terms of coordinates of X .That is

$$(2z-1, w+5) = (x,y) \Rightarrow 2z-1 = x,\ w+5 = y \Rightarrow z = \frac{x}{2} + \frac{1}{2},\quad w = y-5$$

Step-3: Equate the coordinates of Y obtained in step 2 with $f^{-1}(x,y)$.

Hence, $g^{-1}(X) = Y \Rightarrow g^{-1}(x,y) = (z,w) \Rightarrow g^{-1}(x,y) = (\frac{x}{2} + \frac{1}{2}, y-5)$

Thus, $g(x,y) = (2x-1, y+5) \Leftrightarrow g^{-1}(x,y) = (\frac{x}{2} + \frac{1}{2}, y-5)$

Preposition 1.3: The inverse of a transformation is unique. Besides, $(f^{-1})^{-1} = f$.

Proof: Let f be a transformation whose inverses are g and h. That is $f^{-1} = g$ and $f^{-1} = h$. We need to show $g = h$. Here, $f^{-1} = g \Rightarrow f \circ g = g \circ f = i$ and $f^{-1} = h \Rightarrow f \circ h = h \circ f = i$. But, $g = i \circ g = (h \circ f) \circ g = h \circ (f \circ g) = h \circ i = h$ (Because composition is associative as well as g and h are inverses of f).

From this we can conclude that the inverse of a transformation is unique.

Preposition 1.4: The inverse of a transformation is again a transformation.

***Proof*:** Let $f: X \rightarrow X$ be any transformation on set X. Then, f^{-1} also exists as f is bijective. Now, we need to show f^{-1} is also a transformation.

(i) **One-to-ones:** Let a and b be arbitrary elements in X. Since f is bijective, there exists unique $x, y \in X, \ni f(x) = a, f(y) = b$.

But, $f(x) = a, f(y) = b \Rightarrow x = f^{-1}(a), y = f^{-1}(b)$.

Now, assume that $f^{-1}(a) = f^{-1}(b)$.

But, $f^{-1}(a) = f^{-1}(b) \Rightarrow x = y \Rightarrow f(x) = f(y) \Rightarrow a = b$

Thus, $f^{-1}(a) = f^{-1}(b) \Rightarrow a = b$. Hence, f^{-1} is one to one.

(ii) **Onto ness:** Let x be arbitrary element in X. Since f is onto, $f(x) \in X$, so, for every $x \in X, \exists f(x) \in X, \ni f^{-1}(f(x)) = x$. Hence, f^{-1} is onto. Therefore, from (i) and (ii), whenever f is a transformation on set X and so is f^{-1}.

Proposition 1.5: ***(Reverse Law of Inverse)***

For any two transformations f and g, $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Definition: Let $f : S \to T$ be a mapping. Then, a point $x_0 \in S$ (in the domain of f) is said to be a fixed point of f if and only if $f(x_0) = x_0$. Generally, the set of fixed points of a mapping f is the set given by $S = \{x : f(x) = x\}$.

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Involution: A non-identity transformation α is said to be an *involution* if and only if $\alpha^2 = \alpha \circ \alpha = i$. That means $\alpha^2(x) = (\alpha \circ \alpha)(x) = \alpha(\alpha(x)) = i(x) = x$ for all x in the domain of α.

Examples: Verify whether the following transformations are involution or not.

$a)$ $\beta : R \to R$ given by $\beta(x) = 1 - x$

$b)$ $h : R^2 \to R^2$, $h(x, y) = (-x + 7, -y - 2)$

$c)$ $\alpha : R \to R$ given by $\alpha(x) = x + 3$

$d)$ $g : R^2 \to R^2$, $g(x, y) = (x - 3, y + 5)$

Solution:

$a)$ $\beta^2(x) = \beta \circ \beta(x) = \beta(\beta(x)) = \beta(1 - x) = x = i(x) \Rightarrow \beta^2 = i$.

So, β is an involution.

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$b)\ h^2(x,y) = h(h(x,y)) = h(-x+7,-y-2) = (x,y) = i(x,y) \Rightarrow h^2 = i$.

So, h is an involution.

$c)\ \alpha^2(x) = \alpha(\alpha(x)) = \alpha(x+3) = x+6 \neq x = i(x) \Rightarrow \alpha^2 \neq i \Rightarrow \alpha$ is not an involution .

$d)\ g^2(x,y) = g(g(x,y)) = g(x-3, y+5) = (x-6, y+10) \neq (x,y) = i(x,y) \Rightarrow g^2 \neq i$.

Hence, g is not an involution.

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Definition: A transformation f is said to be a *collineation* if and only if the image of any line l under f is again a line. In other words, for any point $P \in l$ the image $f(P) \in f(l)$. Further more; f is said to be a *dilatation* if and only if the image of any line l under f is a line parallel to l. That is $f(l) // l$ whenever f is a collineation then f is said to be a dilatation.

Examples:

1. Let $\alpha : R^2 \to R^2$ be given by $\alpha(x, y) = (x+1, y-2)$. Show that α is a dilatation.

Solution: First let's show that α is a transformation. But, to show that α is a transformation, we need to show that it is one to one and onto.

One- to- one: Assume $\alpha(x,y)=\alpha(z,w)$ for any two points (x,y) and (z,w) in R^2. Then, $(x+1,y-2)=(z+1,w-2)$. But from equality of order pairs, this equality is true if and only if $x+1=z+1$ and $y-2=w-2$. This gives $x=z$ and $y=w$ which implies $(x,y)=(z,w)$. So, α is one-to-one.

On to ness: Let $(a,b)\in R^2$ be in the co-domain of α. Then, if $\exists(x,y)\in R^2$ in the domain of α, such that $\alpha(x,y)=(a,b)$, then α is on to. But,

$\alpha(x,y)=(x+1,y-2)=(a,b)\Rightarrow x+1=a,\ y-2=b\Rightarrow x=a-1,\ y=b+2$. Thus, we can find $(x,y)=(a-1,b+2)\in R^2$ such that $\alpha(x,y)=(a,b)$. So α is onto. Therefore, the given map α is a transformation. To show that α is a collineation we need to show the image of an arbitrary line $l: ax+by+c=0$ is again a line.

Let (x, y) be any point on l. Then, the image $(x', y') = \alpha(x, y) = (x+1, y-2)$.

Solving this for x and y we get $x = x'-1$, $y = y'+2$. So,the image line will be

$l': a(x'-1) + b(y'+2) + c = 0 \Rightarrow ax'+by'+2b+c-a = 0$ and this is equation of a line.

Hence we can say that α is a collineation. Besides, l' has the same slope to that of l which means l' is parallel to l. In other words, $l \,//\, \alpha(l)$. Therefore, α is a dilatation.

Exercise: Let $\alpha : R^2 \to R^2$ be given by $\alpha(x, y) = (-y, x+1)$. Then, find

a) The image of the line $l : x + 3y = 6$ under α.

b) The pre- image of the line $l' : 2x + y + 3 = 0$ under α.

Remarks:

1. The main difference between collineation and dilatation is that any collineation maps a pair of parallel lines to a pair of parallel line but a dilatation maps every line to a line parallel to the given line. This means a transformation α is a collineation if and only if for any two lines m and n, $m // n \Rightarrow \alpha(m) // \alpha(n)$. A transformation α is a dilatation if and only if for any line m, $m // \alpha(m)$.

2. If $\alpha(x, y) = (x', y')$ where $x' = ax + by + h,\ y' = cx + dy + k$, then the necessary and sufficient conditions on the coefficients of x and y such that α to be a transformation is that $ad - bc \neq 0$.(This is known as transformation test).

Examples:

1. Define $\alpha : R^2 \to R^2$ by $\alpha(x,y)) = (-y,x)$. Show that α is a collineation but not a dilatation.

Solution: Clearly, α is a transformation. Besides, for any two arbitrary parallel lines $m : ax+by+c=0$ and $n : ax+by+k=0$ (Parallel lines differ by a constant),

$m' = \alpha(m) : ay-bx+c=0$, $n' = \alpha(n) = ay-bx+k=0$. Still, m' and n' have the same slope which means they are parallel. $i.e\ m // n \Rightarrow \alpha(m) // \alpha(n)$. Thus, α is a collineation. But, if we consider only the single line $m : ax+by+c=0$ separately, $m' = \alpha(m) : ay-bx+c=0$. In this case, slope of line m is $-\frac{a}{b}$ while that of m' is $\frac{b}{a}$ which gives the product of their slope is -1. This means $i.e\ m \perp \alpha(m) = m'$. In other words, m and $\alpha(m) = m'$ are not parallel lines. Particularly, take the line $m : 6x-2y+5=0$. Then, its image under α is $\alpha(m) = m' : 2x+6y+5=0$. Consequently, α is a collineation but not a dilatation.

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Examples of Transformation Groups

Definition: Let G be the set of all transformations on a non empty set S. Then the system $(G,\circ)$ is said to be a transformation group if and only if the following conditions are satisfied:

i) For all f,g in G, $g\circ f$ is in G.

ii) For all f in G, $\exists g\in G \ni f\circ g = g\circ f = i$ and denoted by $f^{-1}=g$.

iii) For all f in G, $\exists i\in G \ni f\circ i = i\circ f = f$.

Examples

1. Let $f_{ab}:R\to R$ be defined by $f_{ab}(x)=ax+b, a\neq 0$.

Let $G=\{f_{ab}/a,b\in R, a\neq 0\}$. Show that $(G,\circ)$ forms a transformation group.

Theorem 1.1 (Test for a transformation groups):

Let G be a nonempty set of transformations on a set S. Then, G with composition is a transformation group if and only if the following conditions are satisfied:

$$a)\ f \in G \Rightarrow f^{-1} \in G,\ \forall f \in G$$

$$b)\ f, g \in G \Rightarrow f \circ g \in G,\ \forall f, g \in G$$

Proof: Suppose $(G,\circ)$ is a transformation group. We need to show conditions (a) and (b) hold true. Since $(G,\circ)$ is a transformation group, from the definition $\forall f, g \in G$, $f^{-1} \in G$ and $f \circ g \in G$ whish implies that conditions (a) and (b) are true. Conversely, suppose conditions (a) and (b) hold true. We need to show $(G,\circ)$ is a transformation group.

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i) Existence of Inverse: Since $G \neq \Phi$, $\exists f \in G$ but $\forall f \in G, f^{-1} \in G$ from condition (a). So, G contains inverse transformation.

ii) Closure property: From condition (b), $f, g \in G \Rightarrow f \circ g \in G, \forall f, g \in G$.

iii) Existence of Identity: $\forall f \in G$, $f^{-1} \in G$ from condition (a) and from condition (b), $f^{-1} \circ f \in G$ and $f \circ f^{-1} \in G$.

But, $f^{-1} \circ f = i \in G$ and $f \circ f^{-1} = i \in G$.

Thus, G contains identity transformation. Therefore, by definition, $(G, \circ)$ forms a transformation group.

Example: Let $f_a : R^2 \to R^2;\ f_a(x,y) = (2a - x, y),\ \forall a \in R$ such that $G = \{f_a : \forall a \in R\}$.

Using criteria of transformation group, determine whether $(G,\circ)$ is transformation group or not.

Solution: For any $f_a \in G$, $f_a(x,y) = (2a - x, y)$. First of all, we need to show that every element of G is transformation.

Now suppose, $f_a(x,y) = f_a(z,w)$.

$$
\begin{aligned}
f_a(x,y) = f_a(z,w) &\Rightarrow (2a - x, y) = (2a - z, w) \\
&\Rightarrow 2a - x = 2a - z,\ y = w \\
&\Rightarrow x = z,\ y = w \Rightarrow (x,y) = (z,w)
\end{aligned}
$$

This shows that each f_a is one to one.

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Besides, to each $(z,w) \in R^2$ (in the co-domain), $\exists (x,y) = (2a - z, w) \in R^2$ such that $f_a(x,y) = f_a(2a - z, w) = (z,w)$.

So, each f_a is also onto.

Therefore, from these explanations every element of G is a transformation.

For any $f_a \in G$,

$$f_a(x,y) = (2a - x, y) \Rightarrow f_a^{-1}(x,y) = (2a - x, y) \Rightarrow f_a^{-1} = f_a \Rightarrow f_a^{-1} \in G.$$

But for any two elements $f_a,\ f_b \in G$,

$$f_a \circ f_b(x,y) = f_a(2b - x, y) = (x + 2a - 2b, y) \neq f_r(x,y),\ \forall r \in R \Rightarrow f_a \circ f_b \notin G.$$

This means the second condition of the above theorem (test for a transformation group) fails.

As a result, $(G,\circ)$ does not form transformation group.

Theorem 1.2 (Cancellation Laws on Transformation Groups):

Let G be a transformation group. Then, for α, β, σ in G

a) $\alpha \circ \beta = \alpha \circ \sigma \Rightarrow \beta = \sigma$ (This is called Left Cancellation Law)

b) $\alpha \circ \beta = \sigma \circ \beta \Rightarrow \alpha = \sigma$ (This is called Right Cancellation Law)

Proof: Let G be a transformation group. Then, for α, β, σ in G

a) $\alpha \circ \beta = \alpha \circ \sigma \Rightarrow \alpha^{-1} \circ (\alpha \circ \beta) = \alpha^{-1} \circ (\alpha \circ \sigma) \Rightarrow \beta = \sigma$

b) $\alpha \circ \beta = \sigma \circ \beta \Rightarrow (\alpha \circ \beta) \circ \beta^{-1} = (\sigma \circ \beta) \circ \beta^{-1} \Rightarrow \alpha = \sigma$

Theorem 1.3: In any transformation group G, for any α, β in G, the equation

$\alpha \circ \sigma = \beta$ has a unique solution for σ in G which is given by $\sigma = \alpha^{-1} \circ \beta$.

Proof: For α, β in G, $\alpha \circ \sigma = \beta \Rightarrow \alpha^{-1} \circ (\alpha \circ \sigma) = \alpha^{-1} \circ \beta \Rightarrow \sigma = \alpha^{-1} \circ \beta \in G$

Hence, the equation has a solution in G. For the uniqueness of this solution, assume there are two different solutions say σ, θ, then

$\alpha \circ \sigma = \beta, \ \alpha \circ \theta = \beta \Rightarrow \alpha \circ \sigma = \alpha \circ \theta \Rightarrow \sigma = \theta$ (by Left cancellation)

Example: Let $(G, \circ)$ be a transformation group such that α, β, σ in G. If $\alpha(x, y) = (3 - x, 5 - y), \beta(x, y) = (3 - 2x, 5 - 3y)$. Find σ such that $\alpha \circ \sigma = \beta$

Solution: By the above theorem, $\alpha \circ \sigma = \beta$ has a unique solution for σ in G which is given by $\sigma = \alpha^{-1} \circ \beta$.

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But after some ups and downs we get $\alpha^{-1}(x,y)=(3-x,5-y)$.

Therefore, $\sigma(x,y)=\alpha^{-1}\circ\beta(x,y)=\alpha^{-1}(\beta(x,y))$

$$=\alpha^{-1}(3-2x,5-3y)=(2x,3y)$$

Review Problems on Chapter One

1.1. Which of the mappings defined on the Cartesian plane by the equations below are transformations?

$\alpha((x, y)) = (x^3, y^3)$, $\beta((x, y)) = (\cos x, \sin y)$, $\gamma((x, y)) = (x^3 - x, y)$,

$\delta((x, y)) = (2x, 3y)$, $\varepsilon((x, y)) = (-x, x + 3)$, $\eta((x, y)) = (3y, x + 2)$,

$\rho((x, y)) = (\sqrt[3]{x}, e^y)$, $\sigma((x, y)) = (-x, -y)$, $\tau((x, y)) = (x + 2, y - 3)$.

1.2. Which of the transformations in the exercise above are collineations? For each collineation, find the image of the line with equation $aX + bY + c = 0$.

1.3. Without looking back in the text, write in your own words definitions for *transformation* and *collineation*. Then compare to see whether your definitions are equivalent to those in the text.

1.4. Find the image of the line with equation $Y = 5X + 7$ under collineation α if $\alpha((x, y))$ is:

(a) $(-x, y)$, (b) $(x, -y)$, (c) $(-x, -y)$, (d) $(2y - x, x - 2)$.

Review Problems on Chapter One

1.5. Fill the following table with yes or No

$\alpha((x, y))$	$(\lvert x\rvert, \lvert y\rvert)$	(e^x, e^y)	$(x^3 - x^2, y)$	(x^3, y^3)	$(y/2, 2x)$	$(x/2, y/2)$	$(x - 2, 3y)$	$(-y, -x)$
α is 1–1								
α is onto								
α is a trans-formation								
α is a collineation								

1.6. Find the preimage of the line with equation $Y = 3X + 2$ under the collineation α where $\alpha((x, y)) = (3y, x - y)$.

1.7. Show the lines with equations $aX + bY + c = 0$ and $dX + eY + f = 0$ are parallel iff $ae - bd = 0$ and are perpendicular iff $ad + be = 0$.

Review Problems on Chapter One

1.8. Find the order of the following tranformations.

1. $(x, y) \mapsto (y, x)$;

2. $(x, y) \mapsto (-x + 2a, -y + 2b)$;

3. $(x, y) \mapsto \left(\frac{1}{2}(x + \sqrt{3}\, y),\ \frac{1}{2}(\sqrt{3}\, x - y)\right)$.

1.9. Consider the group $\mathfrak{V}_4 = \{\iota, \sigma_O, \sigma_h, \sigma_v\}$, where

$$\iota((x, y)) = (x, y), \quad \sigma_O((x, y)) = (-x, -y),$$

$$\sigma_h((x, y)) = (x, -y), \quad \sigma_v((x, y)) = (-x, y).$$

Review Problems on Chapter One

Show that: i. It forms abelian group

ii. Every element except identity is involution

1.10. Prove that if α, β, and γ are elements in a group, then

(a) $\beta\alpha = \gamma\alpha$ implies $\beta = \gamma$;

(b) $\beta\alpha = \beta\gamma$ implies $\alpha = \gamma$;

(c) $\beta\alpha = \alpha$ implies $\beta = \iota$;

(d) $\beta\alpha = \beta$ implies $\alpha = \iota$;

(e) $\beta\alpha = \iota$ implies $\beta = \alpha^{-1}$ and $\alpha = \beta^{-1}$.

Review Problems on Chapter One

1.11. Find all a and b such that the transformation

$$(x, y) \mapsto \left(ay, \frac{x}{b}\right)$$

is an involution.

1.12. Suppose α, β, σ are transformation such that $\alpha \circ \sigma(X) = \beta(X)$ where

$\alpha(x, y) = (-y, x)$ and $\beta(x, y) = (1 - x, -y - 10)$. Find equation of σ.

1.13. Let α, β, σ be elements of a transformation group G such that

$\alpha \circ \sigma(x, y) = \beta(x, y), \forall (x, y) \in R^2$ where $\alpha(x, y) = (-3y, 2x + 1)$ and

$\beta(x, y) = (9y, 4x + 1)$. Find the equation of σ.

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