Exercise 4 PROVE or DISPROVE : Through any point P off a line $\mathcal{L}$, there passes a unique line parallel to the given line $\mathcal{L}$.

Exercise 5 Show that three points $P_1 = (x_1, y_1), P_2 = (x_2, y_2)$ and $P_3 = (x_3, y_3)$ are *collinear* if and only if

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 0.$$

1.2 Transformations

One of the most important concepts in geometry is that of a *transformation*.

NOTE : Transformations are a special class of functions. Consider two sets $\mathbf{S}$ and $\mathbf{T}$. A *function* (or *mapping*) α from $\mathbf{S}$ to $\mathbf{T}$ is a rule that associates with each element s of $\mathbf{S}$ a unique element $t = \alpha(s)$ of $\mathbf{T}$; the element $\alpha(s)$ is called the *image* of s under α, and s is a *preimage* of $\alpha(s)$. The set $\mathbf{S}$ is called the *domain* (or *source*) of α, and the set $\mathbf{T}$ is the *codomain* (or *target*) of α. The set of all $\alpha(s)$ with $s \in S$ is called the *image* (or *range*) of α and is denoted by $\alpha(\mathbf{S})$. If any two different elements of the domain have different images under α (that is, if $\alpha(s_1) = \alpha(s_2)$ implies that $s_1 = s_2$), then α is *one-to-one* (or *injective*). If all elements of the codomain are images under α (that is, if $\alpha(\mathbf{S}) = \mathbf{T}$), then α is *onto* (or *surjective*). If a function is injective and surjective, it is said to be *bijective*.

Exercise 6 If there exists a one-to-one mapping $f : \mathbf{A} \to \mathbf{A}$ which is *not* onto, what can be said about the set $\mathbf{A}$?

When both the domain and codomain of a mapping are "geometrical" the mapping may be referred to as a transformation. We shall find it convenient to use the word *transformation* ONLY IN THE SPECIAL SENSE of a bijective mapping of a set (space) onto itself. We make the following definition.

1.2.1 DEFINITION. A **transformation** on the plane is a bijective mapping of $\mathbb{E}^2$ onto itself.

Transformations will be denoted by lowercase Greek letters.

For a given transformation α, this means that for every point P there is a unique point Q such that $\alpha(P) = Q$ and, conversely, for every point S there is a unique point R such that $\alpha(R) = S$.

Note : Not every mapping on $\mathbb{E}^2$ is a transformation. Suppose a mapping α is given by $(x, y) \mapsto (\alpha_1(x, y), \alpha_2(x, y))$. Then α is a bijection (i.e. a transformation) if and only if, given the *equations* (of α)

$$\begin{aligned} x' &= \alpha_1(x, y) \\ y' &= \alpha_2(x, y), \end{aligned}$$

one can solve *uniquely* for (the "old" coordinates) x and y in terms of (the "new" coordinates) x' and y' : $x = \beta_1(x', y')$ and $y = \beta_2(x', y')$.

1.2.2 Examples. The following mappings on $\mathbb{E}^2$ are transformations:

1. $(x, y) \mapsto (x, y)$ (identity);
2. $(x, y) \mapsto (-x, y)$ (reflection);
3. $(x, y) \mapsto (x - 1, y + 2)$ (translation);
4. $(x, y) \mapsto (-y, x)$ (rotation);
5. $(x, y) \mapsto (2x, 2y)$ (dilation);
6. $(x, y) \mapsto (x + y, y)$ (shear);
7. $(x, y) \mapsto (-x + \frac{y}{2}, x + 2)$ (affinity);
8. $(x, y) \mapsto (x, x^2 + y)$ (generalized shear);
9. $(x, y) \mapsto (x, y^3)$;
10. $(x, y) \mapsto (x + |y|, y)$.

1.2.3 Examples. The following mappings on $\mathbb{E}^2$ are *not* transformations:

1. $(x, y) \mapsto (x, 0)$;

2. $(x, y) \mapsto (xy, xy)$;

3. $(x, y) \mapsto (x^2, y)$;

4. $(x, y) \mapsto \left(-x + \frac{y}{2}, 2x - y\right)$;

5. $(x, y) \mapsto (e^x \cos y, e^x \sin y)$.

1.2.4 Example. Consider the mapping

$$\beta : \mathbb{E}^2 \to \mathbb{E}^2 , \quad (x, y) \mapsto (x', y') = (x^2 - y^2,\, 2xy).$$

Let us first use *polar coordinates* r, t so that

$$x = r \cos t \,, \quad y = r \sin t \,, \quad 0 \leq t \leq 2\pi.$$

By using some trigonometric identities, we can express $\beta((x, y))$ as

$$\beta((r \cos t, r \sin t)) = (r^2 \cos 2t, r^2 \sin 2t) \,, \quad 0 \leq t \leq 2\pi.$$

From this it follows that under β the image curve of the circle of radius r and center at the origin counterclockwise once is the circle of radius r^2 and center at the origin counterclockwise twice. Thus *the effect of β is to wrap the plane $\mathbb{E}^2$ smoothly around itself, leaving the origin fixed,* since $\beta((0, 0)) = (0, 0)$, and therefore β is surjective but not injective.

Exercise 7 Verify that the mapping

$$(x, y) \mapsto \left(x - \frac{2a}{a^2 + b^2}(ax + by + c),\; y - \frac{2b}{a^2 + b^2}(ax + by + c) \right)$$

is a transformation.

Collineations

1.2.5 Definition. A transformation α with the property that if $\mathcal{L}$ is a line, then $\alpha(\mathcal{L})$ is also a line is called a **collineation**.

NOTE : We take the view that *a line is a set of points* and so $\alpha(\mathcal{L})$ is the set of all points $\alpha(P)$ with point P on line $\mathcal{L}$; that is,

$$\alpha(\mathcal{L}) = \{\alpha(P) \mid P \in \mathcal{L}\} \subset \mathbb{E}^2.$$

Clearly, $\alpha(P) \in \alpha(\mathcal{L}) \iff P \in \mathcal{L}$.

1.2.6 EXAMPLE. The mapping

$$\alpha : \mathbb{E}^2 \to \mathbb{E}^2, \quad (x, y) \mapsto (x, y^3)$$

is a transformation as $(u, \sqrt[3]{v})$ is the unique point sent to (u, v) for given numbers u and v (given the equations $u = x$ and $v = y^3$, one can solve uniquely for x and y in terms of u and v). However, α is *not* a collineation, since the line with equation $y = x$ is not sent to a line, but rather to the *cubic curve* with equation $y = x^3$.

1.2.7 EXAMPLE. The mapping

$$\beta : \mathbb{E}^2 \to \mathbb{E}^2, \quad (x, y) \mapsto \left(-x + \frac{y}{2}, x + 2\right)$$

is a collineation. Indeed, from (the equations of β)

$$\begin{aligned} x' &= -x + \frac{y}{2} \\ y' &= x + 2 \end{aligned}$$

we get (uniquely)

$$\begin{aligned} x &= y' - 2 \\ y &= 2x' + 2y' - 4. \end{aligned}$$

Hence β is a transformation.

Now consider the line $\mathcal{L}$ with equation $ax+by+c = 0$, and let $P' = (x', y')$ denote the image of the (arbitrary) point $P = (x, y)$ under (the transformation) β. Recall that

$$P' = (x', y') \in \beta(\mathcal{L}) \iff P = (x, y) \in \mathcal{L}.$$

Then

$$a(y' - 2) + b(2x' + 2y' - 4) + c = 0$$

or, equivalently,

$$(2b)x' + (a + 2b)y' + c - 4b - 2a = 0.$$

(Observe that $(2b)^2 + (a + 2b)^2 \neq 0$ since $a^2 + b^2 \neq 0$.) So the line $\mathcal{L}$ with equation $ax + by + c = 0$ goes to the line with equation $(2b)x + (a + 2b)y + c - 4b - 2a = 0$. Hence β is a collineation.

Exercise 8 PROVE or DISPROVE : Collineations preserve *parallelness* among lines (i.e. the images of two parallel lines under a given collineation are also parallel lines).

1.3 Properties of Transformations

Various sets of transformations correspond to important geometric properties. We will look at properties of sets of transformations that make them algebraically interesting. Let $\mathfrak{G}$ be a set of transformations.

Sets of transformations will be denoted by uppercase Gothic letters.

1.3.1 DEFINITION. The transformation defined by

$$\iota : \mathbb{E}^2 \to \mathbb{E}^2, \quad P \mapsto P$$

is called the **identity transformation**.

NOTE : No other transformation is allowed to use the Greek letter *iota*. The identity transformation may seem of little importance by itself, but its presence simplifies investigations about transformations, just as the number 0 simplifies addition of numbers.

If ι is in the set $\mathfrak{G}$, then $\mathfrak{G}$ is said to have the **identity property**.

Recall that α is a transformation if (and only if) for every point P there is a unique point Q such that $\alpha(P) = Q$ and, conversely, for every point S

there is a point R such that $\alpha(R) = S$. From this definition we see that the mapping $\alpha^{-1} : \mathbb{E}^2 \to \mathbb{E}^2$, defined by

$$\alpha^{-1}(A) = B \iff \alpha(B) = A$$

is a transformation, called the **inverse** of α.

NOTE : We read "α^{-1}" as "alpha inverse". If (the transformation) α is given by

$$(x, y) \mapsto (x', y') = (\alpha_1(x, y), \alpha_2(x, y))$$

with $x = \beta_1(x', y')$ and $y = \beta_2(x', y')$, then (the transformation)

$$\beta : (x, y) \mapsto (\beta_1(x, y), \beta_2(x, y))$$

is the inverse of a; that is, $\beta = \alpha^{-1}$.

If α^{-1} is also in $\mathfrak{G}$ for every transformation α in our set $\mathfrak{G}$ of transformations, then $\mathfrak{G}$ is said to have the **inverse property**.

Whenever two transformations are brought together they might form new transformations. In fact, one transformation might form new transformations by itself, as we can see by considering $\alpha = \beta$ below.

1.3.2 DEFINITION. Given two transformations α and β, the mapping

$$\beta\alpha : \mathbb{E}^2 \to \mathbb{E}^2, \quad P \mapsto \beta(\alpha(P))$$

is called the **product** of the transformation β by the transformation α.

NOTE : Transformation α is applied first and then transformation β is applied. We read "$\beta\alpha$" as "the product beta-alpha".

1.3.3 PROPOSITION. *The product of two transformations is itself a transformation.*

PROOF : Let α and β be two transformations. Since for every point C there is a point B such that $\alpha(B) = C$ and for every point B there is a point A such that $\alpha(A) = B$, then for every point C there is a point A

such that $\beta\alpha(A) = \beta(\alpha(A)) = \beta(B) = C$. So $\beta\alpha$ is an onto mapping. Also, $\beta\alpha$ is one-to-one, as the following argument shows. Suppose $\beta\alpha(P) = \beta\alpha(Q)$. Then $\beta(\alpha(P)) = \beta(\alpha(Q))$ by the definition of $\beta\alpha$. So $\alpha(P) = \alpha(Q)$ since β is one-to-one. Then $P = Q$ as α is one-to-one. Therefore, $\beta\alpha$ is both one-to-one and onto. $\square$

If our set $\mathfrak{G}$ has the property that the product $\beta\alpha$ is in $\mathfrak{G}$ whenever α and β are in $\mathfrak{G}$, then $\mathfrak{G}$ is said to have the **closure property**. Since both $\alpha^{-1}\alpha(P) = P$ and $\alpha\alpha^{-1}(P) = P$ for every point P, we see that

$$\alpha^{-1}\alpha = \alpha\alpha^{-1} = \iota.$$

Hence *if* $\mathfrak{G}$ *is a nonempty set of transformations having both the inverse property and the closure property, then* $\mathfrak{G}$ *must necessarily have the identity property.*

Our set $\mathfrak{G}$ of transformations is said to have the **associativity property**, as any elements α, β, γ in $\mathfrak{G}$ satisfy the *associativity law* :

$$\gamma(\beta\alpha) = (\gamma\beta)\alpha.$$

Indeed, for every point P,

$$(\gamma(\beta\alpha))(P) = \gamma(\beta\alpha(P)) = \gamma(\beta(\alpha(P))) = (\gamma\beta)(\alpha(P)) = ((\gamma\beta)\alpha)(P).$$

Groups of transformations

The important sets of transformations are those that simultaneously satisfy the closure property, the associativity property, the identity property, and the inverse property. Such a set is called a **group** (**of transformations**).

NOTE : We mention all four properties because *it is these four properties that are used for the definition of an abstract group in algebra.* However, when we want to check that a nonempty set $\mathfrak{G}$ of transformations forms a group, we need check *only* the closure property and the inverse property.

1.3.4 PROPOSITION. *The set of all transformations forms a group.*

PROOF : The closure property and the inverse property hold for the set of all transformations. □

Exercise 9 Let α be a collineation. Show that, given a line $\mathcal{L}$, there exists a line $\mathcal{M}$ such that $\alpha(\mathcal{M}) = \mathcal{L}$.

1.3.5 PROPOSITION. *The set of all collineations forms a group.*

PROOF : We suppose α and β are collineations. Suppose $\mathcal{L}$ is a line. Then $\alpha(\mathcal{L})$ is a line since α is a collineation, and $\beta(\alpha(\mathcal{L}))$ is then a line since β is a collineation. Hence, $\beta\alpha(\mathcal{L})$ is a line, and $\beta\alpha$ is a collineation. So the set of collineations satisfies the closure property. There is a line $\mathcal{M}$ such that $\alpha(\mathcal{M}) = \mathcal{L}$. So

$$\alpha^{-1}(\mathcal{L}) = \alpha^{-1}(\alpha(\mathcal{M})) = \alpha^{-1}\alpha(\mathcal{M}) = \iota(\mathcal{M}) = \mathcal{M}.$$

Hence, α^{-1} is a collineation, and the set of all collineations satisfies the inverse property. The set is not empty as the identity is a collineation. Therefore, the set of all collineations forms a group. □

If every element of transformation group $\mathfrak{G}'$ is an element of transformation group $\mathfrak{G}$, then $\mathfrak{G}'$ is a **subgroup** of $\mathfrak{G}$. *All of our groups will be subgroups of the group of all collineations.* These transformation groups will be a very important part of our study of geometry.

NOTE : The word *group* now has a technical meaning and should never be used as a general collective noun in place of the word *set*.

Transformations α and β may or may not satisfy the *commutativity law* : $\alpha\beta = \beta\alpha$. If the commutativity law is always satisfied by the elements from a group, then that group is said to be **commutative** (or **Abelian**). The term *Abelian* is after the Norwegian mathematician N.H. ABEL (1801-1829).

Orders and generators

Given a transformation α, the product $\alpha\alpha\ldots\alpha$ (n times) is denoted by α^n. As expected, we define α^0 to be ι. Also, we write

$$\left(\alpha^{-1}\right)^n = \alpha^{-n}, \quad n \in \mathbb{Z}.$$

If group $\mathfrak{G}$ has exactly n elements, then $\mathfrak{G}$ is said to be **finite** and have **order** n; otherwise, $\mathfrak{G}$ is said to be **infinite**. Analogously, if there is a smallest positive integer n such that $\alpha^n = \iota$, then transformation α is said to have **order** n; otherwise α is said to have **infinite order**.

1.3.6 EXAMPLE. Let ρ be a *rotation* of $\frac{360}{n}$ degrees about the origin with n a positive integer and let

$$\tau : \mathbb{E}^2 \to \mathbb{E}^2, \quad (x, y) \mapsto (x + 1, y).$$

Then

- ρ has order n,
- the set $\{\rho, \rho^2, \ldots, \rho^n\}$ forms a group,
- τ has infinite order,
- the set $\{\tau^k : k \in \mathbb{Z}\}$ forms an infinite group.

If every element of a group containing α is a power of α, then we say that the group is **cyclic** with **generator** α and denote the group as $\langle\alpha\rangle$.

1.3.7 EXAMPLE. If ρ is a rotation of $36°$, then $\langle\rho\rangle$ is a cyclic group of order 10. Note that this same group is generated by β where $\beta = \rho^3$. In fact, we have

$$\langle\rho\rangle = \langle\rho^3\rangle = \langle\rho^7\rangle = \langle\rho^9\rangle.$$

So *a cyclic group may have more than one generator.*

NOTE : Since the powers of a transformation always commute (i.e. $\alpha^m\alpha^n = \alpha^{m+n} = \alpha^n\alpha^m$ for integers m and n), we see that *a cyclic group is always Abelian.*

If $\mathfrak{G} = \langle \alpha, \beta, \gamma, \ldots, \rangle$, then every element of group $\mathfrak{G}$ can be written as a product of powers of $\alpha, \beta, \gamma, \ldots$ and $\mathfrak{G}$ is said to be **generated** by $\{\alpha, \beta, \gamma, \ldots\}$.

Involutions and multiplication tables

Among the particular transformations that will command our attention are the *involutions.*

1.3.8 DEFINITION. A transformation α is an **involution** if $\alpha^2 = \iota$ but $\alpha \neq \iota$.

NOTE : The identity transformation is *not* an involution by definition.

1.3.9 EXAMPLE. The following transformations are involutions :

1. $(x, y) \mapsto (y, x)$;
2. $(x, y) \mapsto (-x + 2a, -y + 2b)$;
3. $(x, y) \mapsto \left(\frac{1}{2}(x + \sqrt{3}\, y),\ \frac{1}{2}(\sqrt{3}\, x - y)\right)$.

1.3.10 PROPOSITION. *A nonidentity transformation α is an involution if and only if $\alpha = \alpha^{-1}$.*

PROOF *:* ($\Rightarrow$) Assume the nonidentity transformation α is an involution. Then $\alpha^2 = \iota$. By multiplying both sides by α^{-1}, we get

$$\alpha^{-1}(\alpha\alpha) = \alpha^{-1}\iota \iff (\alpha^{-1}\alpha)\alpha = \alpha^{-1} \iff \iota\alpha = \alpha^{-1} \iff \alpha = \alpha^{-1}.$$

($\Leftarrow$) Conversely, assume the nonidentity transformation α is such that $\alpha = \alpha^{-1}$. Then by multiplying both sides by α, we get

$$\alpha^2 = \alpha\alpha = \alpha\alpha^{-1} = \iota.$$

□

Exercise 10 Determine whether the transformation

$$(x, y) \mapsto \left(x - \frac{2a}{a^2 + b^2}(ax + by + c),\ y - \frac{2b}{a^2 + b^2}(ax + by + c) \right)$$

is an involution.

A multiplication table for a finite group is often called a **Cayley table** for the group. This is in honour of the English mathematician A. CAYLEY (1821-1895). In a Cayley table, the product $\beta\alpha$ is found in the row headed "β" and the column headed "α".

1.3.11 EXAMPLE. Consider the group $\mathfrak{C}_4$ that is generated by a rotation ρ of $90°$ about the origin. The Cayley table for $\mathfrak{C}_4$ is given below :

$\mathfrak{C}_4$	ι	ρ	ρ^2	ρ^3
ι	ι	ρ	ρ^2	ρ^3
ρ	ρ	ρ^2	ρ^3	ι
ρ^2	ρ^2	ρ^3	ι	ρ
ρ^3	ρ^3	ι	ρ	ρ^2

Clearly, $\mathfrak{C}_4$ is a group of order 4 (it is easy to check the closure property and the inverse property). Group $\mathfrak{C}_4$ is cyclic and is generated by ρ. Since

$$(\rho^3)^2 = \rho^6 = \rho^2\,, \quad (\rho^3)^3 = \rho^9 = \rho\,, \quad \text{and} \quad (\rho^3)^4 = \rho^{12} = \iota\,,$$

then $\mathfrak{C}_4$ is also generated by ρ^3. So

$$\mathfrak{C}_4 = \langle \rho \rangle = \langle \rho^3 \rangle\,.$$

Note, also, that group $\mathfrak{C}_4$ contains the one involution ρ^2.

1.3.12 EXAMPLE. Consider the group $\mathfrak{V}_4 = \{\iota, \sigma_O, \sigma_h, \sigma_v\}$, where

$$\iota((x, y)) = (x, y)\,, \quad \sigma_O((x, y)) = (-x, -y)\,,$$

$$\sigma_h((x, y)) = (x, -y)\,, \quad \sigma_v((x, y)) = (-x, y).$$

The Cayley table for $\mathfrak{V}_4$ can be computed algebraically without any geometric interpretation.

$\mathfrak{V}_4$	ι	σ_h	σ_v	σ_O
ι	ι	σ_h	σ_v	σ_O
σ_h	σ_h	ι	σ_O	σ_v
σ_v	σ_v	σ_O	ι	σ_h
σ_O	σ_O	σ_v	σ_h	ι

Group $\mathfrak{V}_4$ *is Abelian but not cyclic.* Every element of $\mathfrak{V}_4$ except the identity is an involution.

1.4 Exercises

Exercise 11 Let P, Q, and R be three distinct points. Prove that

$$PQ + QR = PR \iff Q = (1-t)P + tR \quad \text{for some } 0 < t < 1.$$

(The *line segment* $\overline{PR}$ consists of P, R and all points between P and R. Hence

$$\overline{PR} = \{(1-t)P + tR \,|\, 0 \leq t \leq 1\}.)$$

Exercise 12 Which of the following mappings defined on the Euclidean plane $\mathbb{E}^2$ are transformations ?

(a) $(x, y) \mapsto (x^3, y^3)$.

(b) $(x, y) \mapsto (\cos x, \sin y)$.

(c) $(x, y) \mapsto (x^3 - x, y)$.

(d) $(x, y) \mapsto (2x, 3y)$.

(e) $(x, y) \mapsto (-x, x+3)$.

(f) $(x, y) \mapsto (3y, x+2)$.

(g) $(x, y) \mapsto (\sqrt[3]{x}, e^y)$.

(h) $(x, y) \mapsto (-x, -y)$.

(i) $(x, y) \mapsto (x+2, y-3)$.

Exercise 13 Which of the transformations in the exercise above are collineations ? For each collineation, find the image of the line with equation $ax + by + c = 0$.

Exercise 14 Find the image of the line with equation $y = 5x+7$ under collineation α if $\alpha((x,y))$ is :

(a) $(-x, y)$.

(b) $(x, -y)$.

(c) $(-x, -y)$.

(d) $(2y - x, x - 2)$.

Exercise 15 TRUE or FALSE ? Suppose α is a transformation on the plane.

(a) If $\alpha(P) = \alpha(Q)$, then $P = Q$.

(b) For any point P there is a unique point Q such that $\alpha(P) = Q$.

(c) For any point P there is a point Q such that $\alpha(P) = Q$.

(d) For any point P there is a unique point Q such that $\alpha(Q) = P$.

(e) For any point P there is a point Q such that $\alpha(Q) = P$.

(f) A collineation is necessarily a transformation.

(g) A transformation is necessarily a collineation.

(h) A collineation is a mapping that is one-to-one.

(i) A collineation is a mapping that is onto.

(j) A transformation is onto but not necessarily one-to-one.

Exercise 16 Give three examples of transformations on the plane that are not collineations.

Exercise 17 Find the preimage of the line with equation $y = 3x + 2$ under the collineation

$$\alpha : \mathbb{E}^2 \to \mathbb{E}^2 , \quad (x, y) \mapsto (3y, x - y).$$

Exercise 18 If

$$\begin{cases} x' = ax + by + h \\ \\ y' = cx + dy + k \end{cases}$$

are the equations for mapping $\alpha : \mathbb{E}^2 \to \mathbb{E}^2$, then what are the necessary and sufficient conditions on the coefficients for α to be a *transformation* ? Is such a transformation always a collineation ?

Exercise 19 Let $\mathbf{P} = \{P_1, \ldots, P_n\}$ be a finite set of points (in the plane), and let C be its **centre of gravity**, namely

$$C := \frac{1}{n}\left(P_1 + \cdots + P_n\right).$$

Consider a transformation $\alpha : \mathbb{E}^2 \to \mathbb{E}^2$ of the form

$$(x, y) \mapsto (ax + by + h,\ cx + dy + k) \quad \text{with } ad - bc \neq 0$$

and let $P_i' = \alpha(P_i)$, $i = 1, 2, \ldots n$ and $C' = \alpha(C)$. Show that

$$C' = \frac{1}{n}\left(P_1' + \cdots + P_n'\right).$$

Exercise 20 Sketch the image of the unit square under the following transformations :

(a) $(x, y) \mapsto (x, x + y)$.

(b) $(x, y) \mapsto (y, x)$.

(c) $(x, y) \mapsto (x, x^2 + y)$.

(d) $(x, y) \mapsto (-x + \frac{y}{2}, x + 2)$.

Exercise 21 Prove that if α, β, and γ are elements in a group, then

(a) $\beta\alpha = \gamma\alpha$ implies $\beta = \gamma$;

(b) $\beta\alpha = \beta\gamma$ implies $\alpha = \gamma$;

(c) $\beta\alpha = \alpha$ implies $\beta = \iota$;

(d) $\beta\alpha = \beta$ implies $\alpha = \iota$;

(e) $\beta\alpha = \iota$ implies $\beta = \alpha^{-1}$ and $\alpha = \beta^{-1}$.

Exercise 22 TRUE or FALSE ?

(a) If α and β are transformations, then $\alpha = \beta$ if and only if $\alpha(P) = \beta(P)$ for every point P.

(b) Transformation ι is in every group of transformations.

(c) If $\alpha\beta = \iota$, then $\alpha = \beta^{-1}$ and $\beta = \alpha^{-1}$ for transformations α and β.

(d) "$\alpha\beta$" is read "the product beta-alpha".

(e) If α and β are both in group $\mathfrak{G}$, then $\alpha\beta = \beta\alpha$.