Chapter Three: Orthogonal Transformations

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Motivation

Meaning is important in Mathematics and Geometry is important source of that meaning!!!!

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Presentation Outline

- Orthogonal Transformation
- Properties of Orthogonal Transformation
- Fundamental Types of Orthogonal Transformation
- Orientation Preserving and Reversing Orthogonal Transformation

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3. Orthogonal Transformations

Preliminaries

* Distance is a real valued non-negative function denoted by which assigns to any pair of points in the plane or space a non-negative real number satisfying the following conditions:

$i)\ d(P,Q) = d(Q,P)$

$ii)\ d(P,Q) \geq 0 d(P,Q) = 0 \Leftrightarrow P = Q$

$iii)\ d(P,R) \leq d(P,Q) + d(Q,R)$

Here, the third property is known as *triangle inequality* and equality occurs if and only if the points P, Q, R are collinear points.

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Note: The notation $d(P,Q)$ stands to mean the distance from P to Q and equivalently denoted by $d(P,Q) = |PQ| = \overline{PQ}$. It is the length of the line segment between P and Q which shows that line segment is the shortest path between two points.

In Euclidean geometry, distance between two points P and Q in a plane is given by $d(P,O) = \overline{PQ} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, where $P = (x_1, y_1)$ and $Q = (x_2, y_2)$

It can be easily verified that this distance formula satisfies the above three conditions. Throughout this text, the writer uses $\overline{f(P)f(Q)}$ to mean the distance between $f(P)$ and $f(Q)$, $\overline{PQ}$ to mean the distance between P and Q.

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Definition. *An orthogonal mapping of a plane π into a plane π' is a mapping under which line segments of π are carried into equal line segments of π'.* More precisely, the mapping α of π into π' is said to be orthogonal if, for any two points M, N of π, the distance between M and N is equal to the distance (in π') between $\alpha(M)$ and $\alpha(N)$. We take the notion of distance in the plane to be fundamental.

Definition: An isometric transformation or orthogonal transformation of a plane is a transformation from a plane on to itself which preserves distances.

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Example: Verify whether the following transformation is isometries or mot. $f: R^2 \to R^2$ given by $f(x, y) = (2y - 9, 2x + 9)$

Solution

For any two points

$P = (x, y)$ and $Q = (z, w)$

$f(P) = f(x, y) = (2y - 9, 2x + 9) = P'$

$f(Q) = f(z, w) = (2w - 9, 2z + 9) = Q'$

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✓ $d(P,Q) = \sqrt{(x-z)^2 + (y-w)^2}$

✓ $d(P,Q) = \sqrt{(z-x)^2 + (w-y)^2}$ *

↳ $d(P',Q') = \sqrt{(2w-g-2y+g)^2 + (2z+g-2x-g)^2}$

↳ $d(P',Q') = \sqrt{4(w-y)^2 + 4(z-x)^2}$

↳ $d(P',Q') = 2\sqrt{(z-x)^2 + (w-y)^2}$ **

From * and **

$d(P,Q) \neq d(P',Q')$

Therefore, f is NOT Isometry.

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Properties of Orthogonal Transformations

Propositions 3.1: The inverse of an isometry is an isometry.

Proof: Let f be an isometry. Now let P and Q be any two points. We need to show $\|f^{-1}(P)-f^{-1}(Q)\|=\|P-Q\|$. Since f is an isometry,

$$\|f^{-1}(P)-f^{-1}(Q)\|=\|f(f^{-1}(P))-f(f^{-1}(Q))\|$$

$$=\|(f\circ f^{-1})(P))-(f\circ f^{-1})(Q))\|=\|i(P)-i(Q)\|=\|P-Q\|$$

$$\Rightarrow\|f^{-1}(P)-f^{-1}(Q)\|=\|P-Q\|$$

Hence, for any isometry f, f^{-1} is also an isometry.

Proposition 3.2: The composition of any two isometries is again an isometry.

Proof: Let f and g be any two isometries. We need to show their composition

$f \circ g$ is also an isometry. Let P and Q be any two points.

Since f is an isometry, $\|f(P)-f(Q)\| = \|P-Q\|$.

But g is also an isometry, so $\|g(f(P))-g(f(Q))\| = \|f(P)-f(Q)\|$.

Combining these two results together, we get that

$\|g \circ f(P) - g \circ f(Q)\| = \|g(f(P))-g(f(Q))\| = \|f(P)-f(Q)\| = \|P-Q\|$.

Hence, the composition $f \circ g$ is an isometry.

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Theorem I. *Under an orthogonal mapping, any three collinear points are taken into three collinear points, and any three noncollinear points are taken into three noncollinear points.*

Proof. Let P, Q, R be three collinear points, and suppose, for example, that Q lies between P and R.

Then

$$PQ + QR = PR.$$

Suppose the respective images of P, Q, R are P', Q', R'. Then by the definition of orthogonality, $P'Q' = PQ$, etc., and so

$$P'Q' + Q'R' = P'R'$$

But this is possible only if P', Q', R' lie on a line, with Q' in the middle; otherwise we should have

$$P'Q' + Q'R' > P'R'$$

Let P, Q, R be noncollinear points, and suppose their images are collinear. Then the inverse mapping which takes P' into P, etc., would take the collinear points P', Q', R' into collinear points, by what we have already proved (since the inverse of an orthogonal mapping is orthogonal). But P, Q, R are not collinear; this contradiction shows that the images are not collinear. ▼

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Theorem 2. *Let* α *be an orthogonal map of the plane* π *onto the plane* π'. *Then the image under* α *of a line* l *in* π *is a line* l' *in* π'. *More precisely: given a line* l *in* π, *there is a line* l' *in* π' *such that every point of* l *is mapped onto some point of* l', *and moreover every point of* l' *has precisely one point of* l *mapped onto it. We may say more concisely that* α *induces a one-one mapping of* l *onto* l'.

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Proof. Let A and B be any two distinct points of l, and let A' and B' be their (distinct) images. Let l' be the line of π' through A' and B'. Then, by Theorem 1, any point C of the line l is mapped into a point of l'. For C is collinear with A and B, so that its image must be collinear with A' and B'.

Conversely let C' be any point of l'. Then, by the same argument, its image under the inverse mapping α^{-1} of π' onto π must lie on l, so that every point of l' has an inverse image on l.

We have shown that the line l is mapped onto the line l'. That the mapping of l is one-one follows from the fact that α is one-one. ▼

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Theorem 3. *Under an orthogonal mapping α of space into itself, the image of a plane π is a plane π'. Moreover, the mapping of π onto π' is itself an orthogonal mapping.*

Proof: Exercise!

Theorem 4. *Under an orthogonal mapping of a plane π onto a plane π', the image of two parallel lines of π is two parallel lines of π'.*

Proof. By Theorem 2, two parallel lines of π go into two lines of π'. If these two lines had a point in common, the inverse image of this point would be a point common to the two

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parallel lines of π, which is impossible. Thus the lines in π' have no common point; that is, they are parallel. ▼

Theorem 5. *Under an orthogonal mapping of space:*

1. *the image of two parallel lines is two parallel lines;*
2. *the image of two parallel planes is two parallel planes;*
3. *the image of a plane and a line parallel to it is a plane and a line parallel to it.*

Theorem 6. *Under an orthogonal mapping, the order of points on a line is preserved. That is to say, if P', R' are the images of two points P, R, then the interior points of the segment PR go into the interior points of the segment $P'R'$, while the exterior points of PR go into the exterior points of $P'R'$.*

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Theorem 7. *Orthogonal mappings preserve angles.*

Proof. Let a and b be two rays through a point O. Choose points A, B on a, b respectively, neither being the point O. Let O', A', B' be the images of the three points under the orthogonal mapping. Then $O'A'$, $O'B'$ will be the images of a and b respectively (by Theorem 6).

By the orthogonality of the mapping, the triangles OAB and $O'A'B'$ are congruent (three pairs of equal sides). So the respective angles are equal, and, in particular, $\angle AOB = \angle A'O'B'$. ▼

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Theorem 8. *Let A, B, C be three noncollinear points of the plane π, and A', B', C' three points of the plane π' such that $B'C' = BC$, $C'A' = CA$, $A'B' = AB$. Then there exists one and only one orthogonal mapping of the plane π onto the plane π' such that the images under it of A, B, C are A', B', C', respectively.*

Proof. We construct a mapping as follows: we make A, B, C correspond to A', B', C', respectively. If P is a point of AC, we make it correspond to the point P' of $A'C'$ such that $A'P' = AP$; if P lies on the extension of AC, we let its image P'

be the point on the extension of $A'C'$ such that (1) $AP = A'P'$, and (2) the points P', A', C' lie in the same order along the line $A'C'$ as do P, A, and C along the line AC.

It is easy to see that if P and P_1 are any points of AC, and P', P_1' their images, then $PP_1 = P'P_1'$ and that the order of the points P', P_1', A, C along the line $A'C'$ is the same as the order of P, P_1, A, C along the line AC. We place the points Q of AB in correspondence with the points Q' of $A'B'$ in just the same way (Fig. 12).

Suppose now that M is a point of the plane not lying on either of the lines AB or AC. We draw parallels through M to

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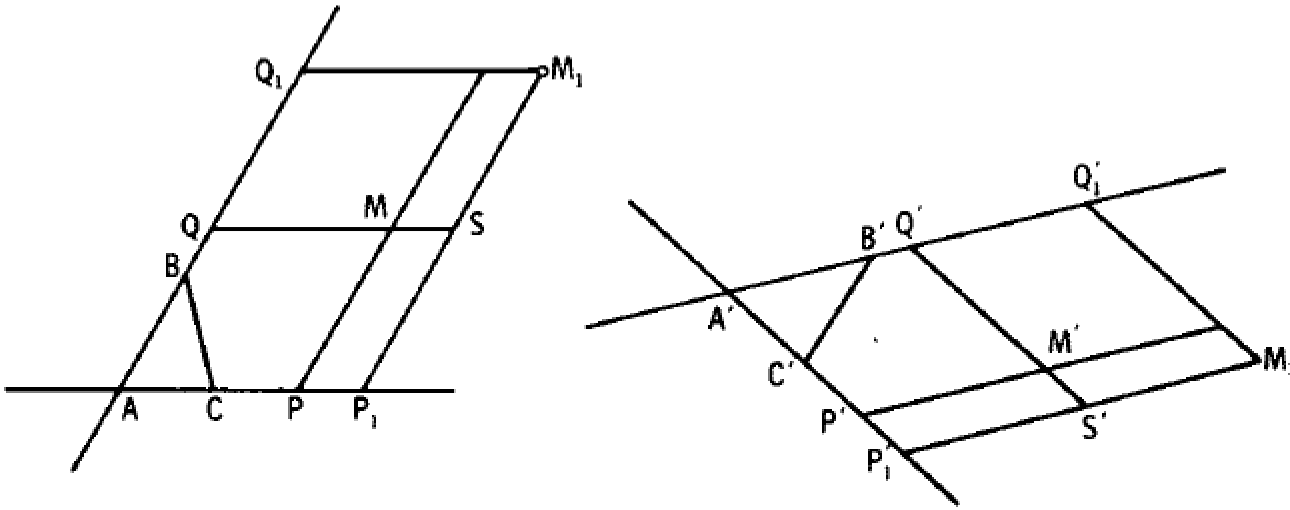

Fig. 12

meet AB and AC in Q and P, respectively. Let Q' and P' be the images of Q and P on $A'B'$ and $A'C'$. Through Q' and P' draw parallels to $A'C'$ and $A'B'$ respectively, and suppose these parallels meet in M'. Then we put M in correspondence with M'.

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We have now said what we put in correspondence with every point of π. Let us show that the mapping we have defined is orthogonal. Let M and M_1 be two points of π and M', M_1' their respective images. If M and M_1 both lie on AB, or both on AC, then we already know $MM_1 = M'M_1'$. If M and M_1 both lie on a line parallel to AC (say), then $MM_1 = PP_1 = P'P_1' = M'M_1'$ (where the notation is obvious). In the general case, let MQ meet M_1P_1 in S, so that $M'Q'$ meets $M_1'P_1'$ in the image S' of S (in case M, for example, lies on AB, we interpret MQ to be the line through M parallel to AC, and $Q = M$). Then $MS = PP_1 = P'P_1' = M'S'$, and $SM_1 = QQ_1 = Q'Q_1' = S'M_1'$. Next, the sides of the angles BAC and M_1SM are parallel, so that the angles must be equal or supplementary. If they are equal, then so are the angles $B'A'C'$ and $M_1'S'M'$, but if BAC and MSM_1 are supplementary, $B'A'C'$ and $M'S'M_1'$ will be too.

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But $\angle BAC = \angle B'A'C'$, so that $\angle MSM_1 = \angle M'S'M_1'$. Thus the triangles MSM_1 and $M'S'M_1'$ are congruent (two sides and included angle), and, in particular, $MM_1 = M'M_1'$. We have shown that the mapping we have constructed is orthogonal. ▼

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The fundamental types of orthogonal transformations

1. Translation

Definition: A mapping $T: R^2 \to R^2$ is called translation if there exists a vector $\vec{v}$ such that $T(P) = P + \vec{v}$ for every point P in R^2.

In other word, for $P = (x, y), v = (a, b), T(P) = T(x, y) = (x', y')$ where $\begin{cases} x' = x + a \\ y' = y + b \end{cases}$

The vector v is called **translator** vector. The translation with translator vector v is sometimes denoted by T_v. In translation problem, whenever any two of $(x, y), (x', y')$, or (a, b) (the pre image, the image or the translator vector) are given, the third can be *uniquely* determined from the translation equation.

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Properties of Translations

a) The translator vector of a translation is unique.

b) The composition of translation T_v and T_w is again a translation by $v+w$.

c) The inverse of a translation is a gain a translation with opposite vector.

d) The image of a line under a translation is a line parallel to the given line.

e) The image of a vector under a translation is an equal vector.

Examples: Let T be a translation by the vector $(1,2)$. Find the image of

a) ΔABC whose vertices are $A(0,0)$, $B(3,0)$ and $C(0,4)$

b) the line $l:\ x-2y=6$.

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Solution: Let $P=(x,y)$ by any object in the plane containing ΔABC. Then,

$T(P)=T(x,y)=(x+1,y+2)$ by definition of translation. Thus,

a) $A'=T(A)=(1,2), B'=T(B)=(4,2), C'=T(C)=(1,6)$. Hence, the image of ΔABC under T will be $\Delta A'B'C'$ with vertices $A'=(1,2,), B'=(4,2)$ and $C'=(1,6)$.

Theorem: Any translation is a dilatation. **Proof!!**

For any four points P, Q, R and S, if $T_{P,Q}(R)=S$, then $T_{P,Q}=T_{R,S}$.

Proof: Let v be a translator vector of the translation T. Then,

$$T_v(P)=v+P=Q \Rightarrow v=Q-P=\overrightarrow{PQ}$$
$$\Rightarrow T_v(R)=Q-P+R=S$$
$$\Rightarrow Q-P=S-R \Rightarrow \overrightarrow{PQ}=\overrightarrow{RS} \Rightarrow T_{\overrightarrow{PQ}}=T_{\overrightarrow{RS}}$$

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2. Reflection

Definition: Given a line l and a point P. Then P' is said to be the reflection image of P on the line l if and only if $\overline{PP'}$ is perpendicular to l and $\overline{PM} = \overline{P'M}$, where M is the point of intersection of $\overline{PP'}$ and the line l. In other words, P and P' are located on different sides of l but at equal distances from the line l. In this case, P' is said to be the mirror image of P and the line l is said to be line of reflection or axis of symmetry.

Notation: Reflection on l is usually denoted by S_l.

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Theorem 3.2: (The Generalized Reflection Theorem): Let $l : ax + by + c = 0$ be any line and S_l be a reflection on line l.

Then, for any point (x, y), $S_l(x, y) = (x', y')$ where

$$\begin{cases} x' = x - \dfrac{2a(ax+by+c)}{a^2+b^2} \\ y' = y - \dfrac{2b(ax+by+c)}{a^2+b^2} \end{cases}$$

Proof: From the definition of reflection, the line through $P(x, y)$ and $P'(x', y')$ is perpendicular to the given line and the midpoint of (x, y) and (x', y') is on the line l. Refer the figure 3.3.

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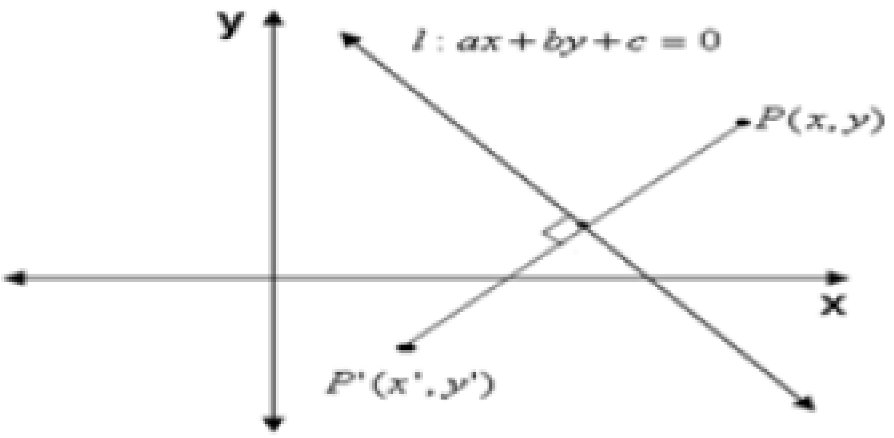

Figure 3.3: Reflection on arbitrary line l

As the slope of the given line is $m=-\frac{a}{b}$, the slope of the line through $P(x,y)$

and $P'(x',y')$ is $m'=\frac{b}{a}$.

Thus, the equation of the line through $P(x, y)$ and $P'(x', y')$ is given by

$$\frac{y'-y}{x'-x} = \frac{b}{a} \Rightarrow a(y'-y) = b(x'-x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Now, the midpoint of $P(x, y)$ and $P'(x', y')$ is on l means $\left(\frac{x+x'}{2}, \frac{y+y'}{2}\right)$ is on l.

$$\text{So, } a\left(\frac{x+x'}{2}\right) + b\left(\frac{y+y'}{2}\right) + c = 0 \Rightarrow ax' + by' = -2c - ax - by \ldots\ldots\ldots\ldots (ii)$$

Combining these two equations gives us

$$\begin{cases} bx' - ay' = bx - ay \\ ax' + by' = -2c - ax - by \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$$

Now, solve these equations for x' and y'. In this equation, by multiplying the first equation by $'b'$, the second by $'a'$ and adding them we obtain,

$$a^2x'+b^2x'=b(bx-ay+a(-2c-ax-by)$$

$$\Rightarrow x'=\frac{b(bx-ay)+a(-2c-ax-by)}{a^2+b^2}$$

$$=\frac{b^2x+a^2x-2a^2x-2aby-2ac}{a^2+b^2}$$

$$=\frac{x(a^2+b^2)-2a(ax+by+c)}{a^2+b^2}$$

$$=x-\frac{2a(ax+by+c)}{a^2+b^2}$$

Similarly, multiplying the first equation by $'a'$, the second by $'b'$ and adding the result gives,

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$$y' = \frac{b(-2c - ax - by) - a(bx - ay)}{a^2 + b^2}$$

$$= \frac{a^2y + b^2y - 2b^2y - 2abx - 2bc}{a^2 + b^2}$$

$$= \frac{y(a^2 + b^2) - 2b(ax + by + c)}{a^2 + b^2}$$

$$= y - \frac{2b(ax + by + c)}{a^2 + b^2}$$

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Examples

1. Find the image of the point $(2,3)$ by a reflection on the line $l: 3x - 2y + 5 = 0$

Solution: Given $(x, y) = (2,3)$ and from $l: 3x - 2y + 5 = 0$, $a = 3, b = -2$, $c = 5$.

Then,

$$\begin{cases} x' = x - \dfrac{2a(ax+by+c)}{a^2+b^2} = 2 - \dfrac{6(6-6+5)}{9+4} = -\dfrac{4}{13} \\ y' = y - \dfrac{2b(ax+by+c)}{a^2+b^2} = 3 + \dfrac{4(6-6+5)}{9+4} = \dfrac{59}{13} \end{cases}$$

Therefore, $S_l(2,3) = (-\dfrac{4}{13}, \dfrac{59}{13})$.

2. Given $S_l(a,b) = (2,5)$ where $l: x - y + 1 = 0$. Find the value of the point (a,b).

Solution: Using the generalized reflection equation derived in the above theorem,

$$S_l(a,b)=(2,5)\Rightarrow\begin{cases}2=a-\dfrac{2(a-b+1)}{1+1}=a-(a-b+1)\Rightarrow b=3\\[2ex]5=b+\dfrac{2(a-b+1)}{1+1}=b+a-b+1\Rightarrow a=4\end{cases}$$

3. Given the lines $m: y=2x+1$ and $n: y=2x-3$. Find the image of the point $(1,1)$ by a product of reflection on line m followed by line n.

Solution: We need to find $S_n\circ S_m(1,1)$

First calculate $S_m(1,1)$ using reflection equation as

$$S_m(1,1)=(x',y')\Rightarrow\begin{cases}x'=1-\dfrac{4(2-1+1)}{4+1}=-\dfrac{3}{5}\\[2ex]y'=1+\dfrac{2(2-1+1)}{4+1}=\dfrac{9}{5}\end{cases}$$

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Now, $S_n \circ S_m(1,1) = S_n(S_m(1,1)) = S_n(-\frac{3}{5}, \frac{9}{5}) \Rightarrow \begin{cases} x'' = -\frac{3}{5} - \dfrac{4(-\frac{6}{5} - \frac{9}{5} - 3)}{4+1} = \frac{21}{5} \\ y'' = \frac{9}{5} + \dfrac{2(\frac{-6}{5} - \frac{9}{5} - 3)}{4+1} = -\frac{3}{5} \end{cases}$

Therefore, $S_n \circ S_m(1,1) = (\frac{21}{5}, -\frac{3}{5})$.

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Examples: Find the images of the circle $C: x^2 + y^2 + 2x - 6y + 6 = 0$ and the ellipse $E: 4x^2 + 9y^2 = 36$ under a reflection on the line $l: y = x + 1$.

Solution: For clarity, let's follow the above procedure directly.

First: Identify the center and radius of the circle $C: x^2 + y^2 + 2x - 6y + 6 = 0$.

By completing square, we get $x^2 + y^2 + 2x - 6y + 6 = 0 \Rightarrow (x+1)^2 + (y-3)^2 = 4$.

Hence, the center is $O = (-1,3)$ and its radius is $r = 2$.

Second : Find the image of the center $O = (-1,3)$ by a reflection on $l: y = x + 1$

Using, reflection formula we get the image of the center to be

$O' = S_l(-1,3) = (2,0)$

Third: Write the equation of the image circle using the image center and the radius of the given circle.

That is the image circle is $C'':(x-2)^2+(y-0)^2=4 \Rightarrow x^2+y^2=4x$

In standard form, the ellipse is written as $E:4x^2+9y^2=36 \Rightarrow \frac{x^2}{9}+\frac{y^2}{4}=1.$

Thus, the major axis is $a=9$, the minor axis is $b=2$ and the center is $C(0,0)$.

Besides, the image of the center is $C''(-1,1)$. Therefore, the image of the ellipse becomes $E':\frac{(x+1)^2}{9}+\frac{(y-1)^2}{4}=1 \Rightarrow 4(x+1)^2+9(y-1)^2=36.$

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3. Rotation

Definition: A rotation is a transformation in which a figure is turned about a fixed point through an angle of θ in a specific direction. In other words, rotation about a point C through directed angle θ is a transformation that fixes the point C and sends every other point P to P' such that P and P' have the same distance from the fixed point C. Here, the fixed point C is called the center of rotation and the angle θ measured from $\overrightarrow{CP}$ to $\overrightarrow{CP'}$ is called direction of the rotation. The rotation may happen either clockwise or counter clockwise direction, usually clockwise rotation will have negative measure of angle, whereas counter clockwise rotation will have positive measure of angle. Rotation with center C through an angle of θ is usually denoted by $\rho_{C,\theta}$.

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So, the image of any point P under $\rho_{C,\theta}$ is given as:

$$\rho_{C,\theta}(P)=\begin{cases} C, \text{ if } P=C \\ P', \text{ if } P\neq C, \text{ s.t. } \overline{CP}=\overline{CP'} \end{cases}$$

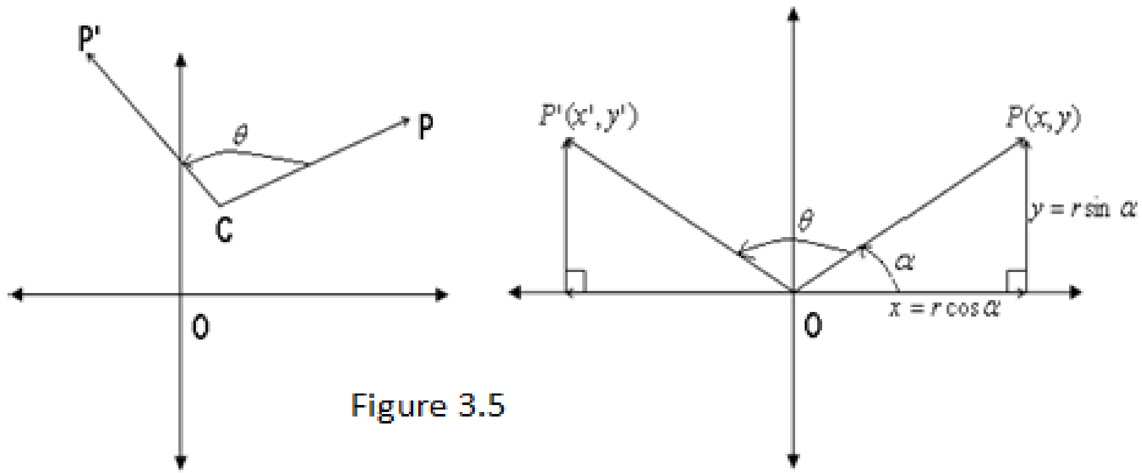

Figure 3.5

Theorem 3.3: A rotation through an angle of θ, about the origin which takes each point $P(x,y)$ in to $P'(x',y')$ is given by $\rho_{O,\theta}(x,y)=(x',y')$, where

$$\begin{cases} x'=x\cos\theta-y\sin\theta \\ y'=x\sin\theta+y\cos\theta \end{cases}$$

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Theorem 3.4 (Generalized Rotation Theorem):

The image of any point $P(x, y)$ under a rotation about arbitrary center $C(h,k)$ through an angle of θ is given by $\rho_{O,\theta}(x, y) = (x', y')$ where

$$\begin{cases} x' = (x-h)\cos\theta - (y-k)\sin\theta + h \\ y' = (x-h)\sin\theta + (y-k)\cos\theta + k \end{cases}$$

Example: The image of the point $(1,2)$ by a counter clockwise rotation about the center $C = (2,3)$ is $(2, 3-\sqrt{2})$. Find the angle of rotation.

Solution: By the generalized rotation theorem,

$$\begin{cases} x' = (x-h)\cos\theta - (y-k)\sin\theta + h \\ y' = (x-h)\sin\theta + (y-k)\cos\theta + k \end{cases}$$

So, for $(x, y) = (1,2)$, $C = (h,k) = (2,3)$ and $(x', y') = (2,3-\sqrt{2})$, we get

$$\begin{cases} x' = -\cos\theta + \sin\theta + 2 = 2 \\ y' = -\sin\theta - \cos\theta + 3 = 3-\sqrt{2} \end{cases} \Rightarrow \begin{cases} x' = -\cos\theta + \sin\theta = 0 \\ y' = -\sin\theta - \cos\theta = -\sqrt{2} \end{cases}$$

$$\Rightarrow \begin{cases} \cos\theta = \sin\theta \\ -\sin\theta - \cos\theta = -\sqrt{2} \end{cases} \Rightarrow \sin\theta = \cos\theta = \frac{\sqrt{2}}{2}$$

Here, both $\sin\theta$ and $\cos\theta$ are positive.

But this is true if and only if θ is in the first quadrant.

Thus, an angle in the first quadrant with $\sin\theta = \cos\theta = \frac{\sqrt{2}}{2}$ is $\frac{\pi}{4}$.

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Suppose the general equation of a rotation about any center $C=(h,k)$ with angle of rotation θ is given by $\rho_{C,\theta}(x,y)=(x',y')$ where

$$\begin{cases} x' = x\cos\theta - y\sin\theta + r \\ y' = x\sin\theta + y\cos\theta + t \end{cases}$$ with r, t, θ being real numbers.

Then, the center $C=(h,k)$ of this rotation is given

$$\begin{cases} h = \dfrac{r}{2} - \dfrac{t}{2\tan\frac{\theta}{2}} \\ k = \dfrac{t}{2} + \dfrac{r}{2\tan\frac{\theta}{2}} \end{cases}$$

Proof! Exercise

Example: Suppose $R_{C,\theta}$ is a counterclockwise rotation with center $C=(h,k)$ whose equations are given by $\begin{cases} x' = \frac{1}{2}x - \frac{\sqrt{3}}{2}y + 2 + 4\sqrt{3} \\ y' = \frac{\sqrt{3}}{2}x + \frac{1}{2}y + 4 - 2\sqrt{3} \end{cases}$. Find the angle and center of this rotation.

Answer!

Here, $r = 2 + 4\sqrt{3}, t = 4 - 2\sqrt{3}$. Besides, $\cos\theta = \frac{1}{2}, \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 60^\circ$.

Therefore, the center of the rotation is $C = (h,k) = (4,8)$.

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Theorem 3.6: Let R be a counter clock wise rotation by a given angle θ about the origin. Then,

a) $R_\theta \circ R_\beta = R_{\theta+\beta}$, for any two angles

b) $R_\theta^{-1} = R_{-\theta}$, the inverse of a rotation by θ is a rotation by $-\theta$

c) $R_\theta = i \Leftrightarrow \theta = 2n\pi,\ n \in Z$ (Where i is identity rotation)

Proof: a) Let $P(x, y)$ be any point. Then,

$$R_\theta R_\beta(x, y) = R_\theta(R_\beta(x, y))$$

$$= R_\theta(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta) = (x', y'), \text{ where}$$

$$\begin{cases} x' = (x\cos\beta - y\sin\beta)\cos\theta - (x\sin\beta + y\cos\beta)\sin\theta \\ y' = (x\cos\beta - y\sin\beta)\sin\theta + (x\sin\beta + y\cos\beta)\sin\theta \end{cases}$$

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Rearranging these equations and using angle sum theorem, we get

$$\begin{cases} x' = x\cos(\theta+\beta) - y\sin(\theta+\beta) \\ y' = x\sin(\theta+\beta) + y\cos(\theta+\beta) \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

On the other hand, using $\varphi = \theta + \beta$ as angle of rotation, we get

$$R_{\theta+\beta}(x,y) = (x\cos(\theta+\beta) - y\sin(\theta+\beta), x\sin(\theta+\beta) + y\cos(\theta+\beta)) = (x', y') \ldots\ldots\ldots\ldots (ii)$$

Comparing equations (i) and (ii), one can conclude that $R_\theta R_\beta = R_{\theta+\beta}$.

Examples:

1. Find the image of the point $(2,5)$ by a product of rotations through an angle of $\theta = 15^\circ$ and $\beta = 75^\circ$ in counterclockwise direction about the same center $C = (7,2)$.

Solution: From the above theorem, $\rho_{C,\theta} \circ \rho_{C,\beta}(x, y) = \rho_{C,\theta+\beta}(x, y)$ where

$(x, y) = (2,5)$, $\theta = 15^\circ$, $\beta = 75^\circ$ and $C = (7,2)$.

Thus, $\rho_{C,15^\circ} \circ \rho_{C,75^\circ}(2,5) = \rho_{C,90^\circ}(2,5) \Rightarrow \begin{cases} x' = (2-7)\cos 90^\circ - (5-2)\sin 90^\circ + 7 = 4 \\ y' = (2-7)\sin 90^\circ + (5-2)\cos 90^\circ + 2 = -3 \end{cases}$

2. Suppose R_θ is a counterclockwise rotation about the origin whose equations are given by $\begin{cases} x' = \frac{\sqrt{3}}{2}x - \frac{1}{2}y \\ y' = \frac{1}{2}x + \frac{\sqrt{3}}{2}y \end{cases}$. Find the equations for the inverse, $R^{-1}{}_\theta$, of this rotation.

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$$R^{-1}{}_{\theta}(x,y)=(x',y') \text{ where } \begin{cases} x'=x\cos(-30^\circ)-y\sin(-30^\circ)=\dfrac{\sqrt{3}}{2}x+\dfrac{1}{2}y \\ y'=x\sin(-30^\circ)+y\cos(-30^\circ)=-\dfrac{1}{2}x+\dfrac{\sqrt{3}}{2}y \end{cases}$$

3. If $R_\theta = R_{-\theta}$, then what should be the possible values of θ? Particularly for $0<\theta<2\pi$.

Solution: Using part (c) of the above theorem, we have

$$R_\theta = R_{-\theta} \Leftrightarrow R_\theta(P)=R_{-\theta}(P) \Leftrightarrow R_\theta \circ R_\theta(P) = R_\theta \circ R_{-\theta}(P)$$
$$\Leftrightarrow R_{2\theta}(P)=R_0(P)=i(P)=P \Leftrightarrow R_{2\theta}=i$$
$$\Leftrightarrow 2\theta = 2\pi n,\ n\in Z \Leftrightarrow \theta=\pi n,\ n\in Z$$

Particularly for $0<\theta<2\pi$, $\theta=\pi$, when $n=1$.

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Half-turns

Definition: *A half turn* is a rotation by 180°. A half turn about a point P is denoted by H_P. If A is rotated by 180° about point $P(a,b)$, then $A'P = AP$. In other words P is the mid point of A and A'.

Thus using the midpoint formula, $\frac{A'+A}{2} = P \Rightarrow \begin{cases} \frac{x'+x}{2} = a \\ \frac{y'+y}{2} = b \end{cases} \Rightarrow x' = -x + 2a,\ y' = -y + 2b$.

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Examples:

1. Find the image of a point $(2,-7)$ by a half-turn about the point $P=(5,-3)$.

Solution: By definition,

$$H_P(x,y)=(x',y') \text{ where } \begin{cases} x'=-x+10 \\ y'=-y-6 \end{cases}. \text{ Therefore, } H_P(2,-7)=(8,1).$$

2.If the image of $(-2,3)$ by a half-turn is $(10,11)$, find the center of the half-turn.

Solution: Let the center be $P=(a,b)$. Then, using the definition, we have

$$\begin{aligned} H_P(-2,3)=(10,11) &\Rightarrow (2a+2,2b-3)=(10,11) \\ &\Rightarrow 2a+2=10, 2b-3=11 \\ &\Rightarrow a=4, b=7 \Rightarrow P=(4,7) \end{aligned}$$

3. Let H_P be a half turn about $P=(-3,2)$. Find,

a) The image of the line $l: y=5x+7$

b) The pre-image of the line $m: y=2x+17$

Example: Let $PQRT$ be a parallelogram with vertices $P(1,2)$, $Q(6,2)$, $T(1,4)$.

Find the vertex R.

Solution: From the above corollary, $PQRT$ is a parallelogram if and only if

$H_R \circ H_Q \circ H_P = H_T$. Let $R=(a,b)$ and $X=(x,y)$ be any point.

Then,

$$
\begin{aligned}
& H_R \circ H_Q \circ H_P = H_T \\
& \Leftrightarrow H_R \circ H_Q \circ H_P(X) = H_T(X) \\
& \Leftrightarrow H_R \circ H_Q(-x+2,-y+4) = (-x+2,-y+8) \\
& \Leftrightarrow H_R(x+10,y) = (-x+2,-y+8) \\
& \Leftrightarrow (-x-10+2a,-y+2b) = (-x+2,-y+8) \Leftrightarrow (2a-10,2b) = (2,8) \\
& \Leftrightarrow (2a,2b) = (12,8) \Leftrightarrow (a,b) = (6,4)
\end{aligned}
$$

Thus, the unknown vertex is $R=(a,b)=(6,4)$.

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5. Glide Reflection

Definition: A glide reflection g is the composition of a reflection S_l over a line l followed by a translation T_v with *non-zero* vector v where the line l is parallel to the direction of the translation or parallel to the translator vector v. The vector v in this case is called *glide vector* and the line l is called *axis of the glide reflection*. Here, the vector v is required to be non-zero otherwise translation by a zero vector will be identity map and the composition also will be the usual reflection but not glide reflection. We can easily justify that the same result is obtained by first reflecting and then translating or vice versa. As a result, the order of the two transformations (Translation and reflection) is immaterial. So, $g = T_v \circ S_l = S_l \circ T_v$.

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General Equations of Glide-Reflections:

Let g be a glide reflection with axis $l: ax+by+c=0$ and glide vector $\vec{v}=(\mathbf{d},\mathbf{e})$ with the condition $a.\mathbf{d}+b.\mathbf{e}=0$ .Then, the general equation of g is given by

$$g(x,y)=T_{\vec{v}}\circ S_l(x,y)=S_l(x,y)+\vec{v}=(x',y')\text{ where }\begin{cases} x'=x-\dfrac{2a(ax+by+c)}{a^2+b^2}+\mathbf{d} \\ y'=y-\dfrac{2b(ax+by+c)}{a^2+b^2}+\mathbf{e}\end{cases}$$

Conversely, if g is a glide reflection given by $g(x,y)=(x',y')$ where

$$\begin{cases} x'=ax+by+c \\ y'=bx-ay+d \end{cases}$$

Then, the axis of g is given by $l: 2bx-2(a+1)y+ad-bc+d=0$ or $2x=c$.

Examples:

1. Let g be a glide reflection with axis $l: 3x-4y-2=0$ and glide vector $\vec{v}=(-4,-3)$. Find the equation of g and calculate the image of the point $(0,0)$

Solution: Here, $g(x,y)=T_{\vec{v}}\circ S_l(x,y)=S_l(x,y)+\vec{v}=(x',y')$ where

$$\begin{cases} x'=x-\dfrac{2a(ax+by+c)}{a^2+b^2}+\mathbf{d}=x-\dfrac{6(3x-4y-2)}{9+16}-4=\dfrac{7x}{25}+\dfrac{24y}{25}-\dfrac{88}{25} \\ y'=y-\dfrac{2b(ax+by+c)}{a^2+b^2}+\mathbf{e}=y+\dfrac{8(3x-4y-2)}{9+16}-3=\dfrac{24x}{25}-\dfrac{32y}{25}-\dfrac{91}{25} \end{cases}$$

Therefore the image of the point $(0,0)$ is given by

$g(0,0)=(x',y')$ where $x'=\dfrac{7(0)}{25}+\dfrac{24(0)}{25}-\dfrac{88}{25}=-\dfrac{88}{25}$, $y'=\dfrac{24(0)}{25}-\dfrac{32(0)}{25}-\dfrac{91}{25}=-\dfrac{91}{25}$

Hence, $g(0,0)=(-\dfrac{88}{25},-\dfrac{91}{25})$.

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Exercise! 2. Suppose g is a glide reflection with axis $l: 5x - ky + 7 = 0$ and glide vector $\vec{v} = (6,10)$. Then, find the value of the constant k.

3. Let g be a glide reflection with axis $l: 2x + 7y - 9 = 0$ and glide vector $\vec{v} = (d,4)$. Then, find the value of the constant d.

4. Let g be a glide reflection with axis $l: 2x + y - 5 = 0$ and glide vector $\vec{v} = (d,e)$. If $g(0,0) = (6,-2)$, then find the glide vector $\vec{v}$.

5. Let g be a glide reflection with axis $l: x - y + 1 = 0$ and glide vector $\vec{v} = (3,3)$. If $g(p,q) = (5,8)$, then find the point (p,q).

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Orientation Preserving and Orientation Reversing Orthogonal transformation

Definitions: Let g be any orthogonal transformation. Then, we say that g preserves orientation if and only if for any positively oriented vectors X and Y, their images $X' = g(X)$, $Y' = g(Y)$ are again positively oriented vectors. In this case, g is said to be orientation preserving orthogonal transformation. In general, if the pair (X, Y) and the pair $(g(X), g(Y))$ have the same orientation, then g preserves orientation. But, if they have opposite orientation, then g reverses (changes) orientation. In this case, g is said to be orientation reversing (changing) orthogonal transformation.

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Examples:

1. Determine whether the following isometries preserve or reverse orientation.

$a)\ g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$ $\qquad b)\ g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$ $\qquad c)\ g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+3 \\ y-2 \end{pmatrix}$

Solution: Let $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $Y = \begin{pmatrix} z \\ w \end{pmatrix}$ be positively oriented vectors. Then,

$$\det(X,Y) = \begin{vmatrix} x & z \\ y & w \end{vmatrix} > 0$$

a) From the given formula,

$$X' = g(X) = g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix},\ Y' = g(Y) = g\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} -w \\ z \end{pmatrix}$$

$$\Rightarrow \det(X',Y') = \begin{vmatrix} -y & -w \\ x & z \end{vmatrix} = xw - yz = \begin{vmatrix} x & z \\ y & w \end{vmatrix} > 0$$

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Thus, the pair (X', Y') is positively oriented, has the same orientation to the pair (X, Y), which shows that g preserves orientation.

b) Similarly as in part (a), $X' = g(X) = \begin{pmatrix} x \\ -y \end{pmatrix}$, $Y' = g(Y) = \begin{pmatrix} z \\ -w \end{pmatrix}$

$$\Rightarrow \det(X', Y') = \begin{vmatrix} x & z \\ -y & -w \end{vmatrix} = -xw + yz = yz - xw = -\begin{vmatrix} x & z \\ y & w \end{vmatrix} < 0$$

Thus, the pair (X', Y') is negatively oriented, has opposite orientation to the pair (X, Y), which implies that g reverses or changes orientation.

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Examples: Determine whether the following transformations are orientation preserving or orientation reversing.

a) $g : R^2 \to R^2$ given by $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x - 2y + 7 \\ x + 5y - 11 \end{pmatrix}$

b) $\alpha : R^2 \to R^2$ given by $\alpha\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 1 \\ y - 11 \end{pmatrix}$

Left for the reader!

Translations and rotations (including identity) are the only types of isometries preserving orientation. Reflections and glide-reflections are the only types of isometries reversing (changing) orientation.

Prove is an exercise!!!

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Fixed Points of Isometries

i) Exactly one fixed point: Isometries that have *exactly one* fixed point are only

Rotations: Any rotation has exactly one fixed point and the fixed point is exactly the center of the rotation.

ii) Two or more fixed points:

Any isometry that has two fixed points but not identity is **a reflection** over a line and the whole points on the line of reflection are also fixed points.

As a result the line of reflection is a fixed line.

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iii) Three non-collinear fixed points:

An isometry that has three non-collinear fixed points is an **identity**.

iv) No fixed poin:

Isometries that have *no* fixed point at all- This category includes

translation

and **glide reflection.**

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