Chapter-2-

Topological Spaces

2.1 Definition and some examples of a topological space

Def'n A topology $\mathcal{J}$ on a nonempty set X is a collection of subsets of X with the following properties;

a) ϕ and X are in $\mathcal{J}$

b) The union of any family of members of $\mathcal{J}$ is a member of $\mathcal{J}$. (i.e., $U_i \in \mathcal{J}$ $(i \in I) \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{J}$

c) The intersection of any finite family of members of $\mathcal{J}$ is a member of $\mathcal{J}$. (i.e., $U_i \in \mathcal{J}$ $(i = 1, 2, \cdots, n)$ $\Rightarrow \bigcap_{i=1}^{n} U_i \in \mathcal{J}$)

Def'n Let $\mathcal{J}$ be a topology on X, then the members of $\mathcal{J}$ are called open sets and the ordered pair $(X, \mathcal{J})$ is called a topological space or simply a space.

Remark Using the term "open set" instead of "members of $\mathcal{J}$", the def'n of topology may be restated as:

a) Both X and ϕ are open sets

b) The union of any family of open sets is an open set.

c) The intersection of any finite family of open sets is an open set.

Def'n Let $(X, \mathcal{J})$ be a topological space. Then $U \subseteq X$ is called open set if $U \in \mathcal{J}$

Examples

1) Let X be any nonempty set. Then $\mathcal{J} = \{\phi, X\}$ is a topology on X, called the trivial or indiscrete topology.

2) Let $X = \{a, b\}$. Then

$J_1 = \{\phi, X\}$, $J_2 = \{\phi, \{a\}, X\}$, $J_3 = \{\phi, \{b\}, X\}$ and

$J_4 = \{\phi, \{a\}, \{b\}, X\}$ are topologies on X.

↳ which is called discrete topology

3) Let $X = \{a, b, c\}$. Then

$J_1 = \{\phi, X, \{a\}, \{b\}\}$ and $J_2 = \{\phi, X, \{a,b\}, \{b,c\}\}$

are not topologies on X.

4) Let X be a none empty set

Let $J = \{U \subseteq X : U = \phi \text{ or } X \setminus U \text{ is finite}\}$

→ $X \setminus U = X$

↗ if it is ϕ or $\exists N \in \mathbb{N}$ for which there is 1-1 correspondance b/n A and $\{1, 2, \dots, N\}$

Then J is a topology on X, called the finite complemen topology on X.

Pf a) ϕ, X are in J, since $X / X = \phi$ is finite and $\Rightarrow X \in J$

$X / \phi = X$ is all of X

b) Let $\{U_i\}_{i \in I}$ be ~~non empty~~ non empty collection of elements of J.

Then $X \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} (X / U_i)$

since each $X | U_i$ is finite, then $\bigcap_{i \in I} (X | U_i)$ is finite.

$\Rightarrow \bigcup_{i \in I} U_i \in J$

c) Suppose $U_1, U_2, \dots, U_n$ be nonempty elements of J.

Then $X \setminus \bigcap_{i=1}^{n} U_i = \underbrace{\bigcup_{i=1}^{n} (\underbrace{X | U_i}_{\text{finite}})}_{\text{finite union of finite set is fine}}$

$\Rightarrow \bigcap_{i=1}^{n} U_i \in J$

$X \cap (\cap U_i)^c$

$\cup (X \cap U_i^c)$

Remark A topology on X is not unique.

Def'n A subset G of a topological space X is closed provided that its complement $X \setminus G$ is an open set.

Theorem Let X be topological space. Then

a) X and ϕ are closed sets

b) The intersection of any family of closed sets is closed set.

c) The union of any finite family of closed sets is a closed set.

Pf EX

Def'n: Let $(X, \mathcal{J})$ be topological space and $A \subseteq X$. A point $x \in X$ is a limit point, cluster point or accumulation point of A if every open set containing x contains a point of A distinct from x. The set of limit points of A is called the derived set of A, denoted by A'.

eg Consider the topological space $(\mathbb{R}, \mathcal{J})$, where $\mathcal{J}$ is the finite complement topology.

a) For any infinite subset A of $\mathbb{R}$, $A' = \mathbb{R}$

To see this, $\mathcal{J} = \{U \subseteq \mathbb{R} : U = \phi \text{ or } \mathbb{R} \setminus U \text{ finite}\}$

Let $A \subseteq \mathbb{R}$ & A is infinite set

let $x \in \mathbb{R}$ and O is open set such that $x \in O$

$\Rightarrow O \in \mathcal{J}$

$\Rightarrow \mathbb{R} \setminus O$ is finite

$\Rightarrow O$ must contains all but a finite number of member of A

In particular, O must contains at least one point of A distinct from x. Hence $x \in A'$

so $A' = \mathbb{R}$

b) A finite subset B of $\mathbb{R}$ has no limit points

To see this, let $x \in \mathbb{R}$ and $x \notin B$

then $\mathbb{R} | B$ is an open set containing x which contains no point of B (b/c $\mathbb{R} | \mathbb{R} | B = B$ is finite)

Let $x \in \mathbb{R}$ and $x \in B$, then

$\{x\} \cup (\mathbb{R} | B)$ is an open set containing x which contains no point of B different from x. (b/c $\mathbb{R} / \{x\} \cup (\mathbb{R} | B) = B | \{x\}$ is finite).

eg let $X = \{a, b\}$. consider $\mathcal{J} = \{\phi, X\}$. Then a is limit pt of $\{b\}$ since X is the only open set containing a that contains elmts of $\{b\}$ dl from a.

Theorem
A subset A of a topological space X is closed iff A contains all its limit points.

Def^n
Let X be a topological space and $\{x_n\}_{n=1}^{\infty}$ a sequence of points of X. Then $\{x_n\}_{n=1}^{\infty}$ converges to the point $x \in X$, or x is a limit of the sequence, if for each open set O containing x there is a positive integer N such that $x_n \in O$, $\forall n \geq N$.

eg Consider the set $\mathbb{R}$ with the finite complement topology. Then, every sequence $\{x_n\}_{n=1}^{\infty}$ of distinct points converges to every point.

To see this, let $x \in \mathbb{R}$ and O be open set containing x

$\Rightarrow$ $\mathbb{R} | O$ is finite

$\Rightarrow$ O must contains all but a finite number of terms of the sequence.

In particular, $\exists N \in \mathbb{N} \ni$ if $n \geq N$, then $x_n \in O$

$\therefore$ $\{x_n\}$ converges to every member of $\mathbb{R}$, and therefore, a sequence may have more than one limit.

2.2 Interior, Closure and Boundary

Def'n Let A be a subset of a topological space X.

(1) A point x in A is an interior point of A if there is an open set O containing x and contained in A. Equivalently A is called a nbhd of x.

(2) The interior of A, denoted $\text{int } A$, is the set of all interior points of A.

(3) The closure $\bar{A}$ of A is the union of A with its set of limit points: $\bar{A} = A \cup A'$, where A' is the derived set of A.

(4) A point $x \in X$ is a boundary point of A if x belongs to both $\bar{A}$ and $\overline{X \setminus A}$. The set of boundary points of A is called the boundary of A and is denoted $\text{bdy } A$.

Theorem For any subsets A, B of a topological space X

EX
(1) The interior of A is the union of all open sets contained in A and is therefore the largest open set contained in A.

(2) A is open iff $A = \text{int } A$

(3) If $A \subseteq B$, then $\text{int } A \subseteq \text{int } B$.

(4) $\text{int}(A \cap B) = \text{int } A \cap \text{int } B$.

Proof Statements (1) and (2) carry over from chater-1. and (3) is an immediate consequence of the def'n of interior.

To prove (4), since $A\cap B \subseteq A$ and $A\cap B\subseteq B$, then $\operatorname{int}(A\cap B)\subseteq \operatorname{int} A\cap \operatorname{int} B$ by (3).

open by (1)

For the reverse inclusion, note that $\underbrace{\operatorname{int} A}_{\subseteq A,\ \text{open}} \cap \underbrace{\operatorname{int} B}_{\subseteq B,\ \text{open}}$ is an open set and is a subset of $A\cap B$.

Since $\operatorname{int}(A\cap B)$ is the largest open set contained in $A\cap B$, then $\operatorname{int} A\cap \operatorname{int} B \subseteq \operatorname{int}(A\cap B)$.

Remark in general, $\operatorname{int}(A\cup B) \neq \operatorname{int} A\cup \operatorname{int} B$.

eg Consider the real line with $A=[0,1]$ and $B=[1,2]$.

Then $\operatorname{int}(A\cup B) = \operatorname{int}[0,2] = (0,2)$

while $\operatorname{int} A \cup \operatorname{int} B = (0,1)\cup(1,2)$.

So, $1\in \operatorname{int} A\cup B$ but $1\notin \operatorname{int} A\cup \operatorname{int} B$.

Ex prove that $\operatorname{int} A\cup \operatorname{int} B \subseteq \operatorname{int}(A\cup B)$

Theorem For any subsets A, B of a space X.

Ex
1. The closure of A is the intersection of all closed sets containing A and therefore the smallest closed set containing A.
2. A is closed iff $A=\bar{A}$
3. If $A\subseteq B$, then $\bar{A}\subseteq \bar{B}$
4. $\overline{A\cup B} = \bar{A}\cup\bar{B}$

Proof Statements (1) and (2) carry over from Chapter-1.
For (3), note that if $A\subseteq B$, then $A'\subseteq B'$ (by def'n of limit point)
Then, $\bar{A} = A\cup A' \subseteq B\cup B' = \bar{B}$

For (4), note that $\bar{A}\cup\bar{B}$ is a closed set which contains $A\cup B$. Since $\overline{A\cup B}$ is the smallest closed set containing $A\cup B$, then $\overline{A\cup B}\subseteq \bar{A}\cup\bar{B}$. For the reverse inclusion use (3) and the fact that A and B are subsets of $A\cup B$
i.e., $A\subseteq A\cup B \wedge B\subseteq A\cup B \Rightarrow \bar{A}\subseteq\overline{A\cup B} \wedge \bar{B}\subseteq \overline{A\cup B} \Rightarrow \bar{A}\cup\bar{B}\subseteq \overline{A\cup B}$

Remark In general, $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$

of let $A = (0,1)$ and $B = (1,2)$ on the real line. The $\overline{A \cap B} = \bar{\phi} = \phi$ but

$\bar{A} \cap \bar{B} = [0,1] \cap [1,2] = \{1\}$

EX Show that $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

Theorem Let A be a subset of a topological space X.

1. $\text{bdy } A = \bar{A} \cap (\overline{X \setminus A}) = \text{bdy}(X \setminus A)$
2. $\text{bdy } A$, $\text{int } A$ and $\text{int}(X \setminus A)$ are pairwise disjoint sets whose union is X.
3. $\text{bdy } A$ is a closed set
4. $\bar{A} = \text{int } A \cup \text{bdy } A$.
5. A is open iff $\text{bdy } A \subseteq (X \setminus A)$
6. A is closed iff $\text{bdy } A \subseteq A$
7. A is open and closed iff $\text{bdy } A = \phi$

Proof properties (1) through (4) follow immediately from the definitions.

To prove (5), note that if A is open, then $A = \text{int } A$. Since $\text{int } A$ and $\text{bdy } A$ are disjoint (by (2)), then A and $\text{bdy } A$ are disjoint, so $\text{bdy } A$ must be a subset of $X \setminus A$.

For the reverse implication, suppose $\text{bdy } A \subseteq X \setminus A$. Then no point of A is a boundary point of A, so every point of A is an interior point.

Thus $A = \text{int } A$, so A is open.

To prove (6), follows from the duality b/n open and closed sets!

A is closed iff $X \setminus A$ is open

By (5), this is equivalent to

bdy $(X \setminus A) \subseteq X \setminus (X \setminus A)$ or

bdy $A \subseteq A$.

To prove (7), combining (5) and (6):
A is both open and closed iff bdy A is contained in both A and $X \setminus A$.
Since A and $X \setminus A$ are disjoint, this occurs iff bdy $A = \phi$.

Def^n A subset A of a space X is dense in X provided that $\bar{A} = X$. If X has a countable dense subset, (a finite or countably infinite (e.g. $\mathbb{N}$ to $\mathbb{Z}^+$)) then X is called separable space.

Remark $A \subseteq X$ is dense in X iff every nonempty open set in X contains at least one point of A.

eg ① The real line $\mathbb{R}$ is separable
② the set of rational numbers is countable and dense in $\mathbb{R}$.

2.3 Basis and sub-basis

Def^n Let $(X, \mathcal{T})$ be a topological space. A base or basis $\mathcal{B}$ for $\mathcal{T}$ is a subcollection of $\mathcal{T}$ such that each member of $\mathcal{T}$ is a union of members of $\mathcal{B}$.
The members of $\mathcal{B}$ are called basic open sets, and $\mathcal{T}$ is the topology generated by $\mathcal{B}$.

eg 1) The collection $\mathcal{B}$ of all open intervals is a basis for the usual topology of $\mathbb{R}$.
↳ topology generated by usual metric

2) For any space $(X, \mathcal{T})$, the topology $\mathcal{T}$ is a basis of itself.

3) Let $\mathcal{B}$ = the set of all circular regions (interior of circles) in the plane. Then $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{R}^2$ where $U \in \mathcal{T}$ if for every $x \in U$, $\exists$ some circular region O: $x \in O \subseteq U$

Def'n Let $(X,\mathcal{T})$ be a space and let a be a member of X. A local base or local basis at a is a subcollection $\mathcal{B}_a$ of $\mathcal{T}$ such that

1) a belongs to each member of $\mathcal{B}_a$, and
2) each open set containing a contains a member of $\mathcal{B}_a$.

eg 1) For $a \in \mathbb{R}$, the collection $\mathcal{B}_a$ of all open intervals of the form $(a-\varepsilon, a+\varepsilon)$, $\varepsilon > 0$, is a local basis at a.

2) For any metric space (X,d) and $a \in X$, the collection $\mathcal{B}_a$ of all open balls centered at a is a local base at a.

3) For any space $(X,\mathcal{T})$ and $a \in X$, the collection of all open sets containing a is a local basis at a.

Def'n ① A space X is first countable or satisfies the first axiom of countability provided that there is a countable local basis at each point of X.

② A space X is second countable or satisfies the second axiom of countability provided that the topology of X has a countable basis.

Remark Every second countable space is first countable b/c If X has a countable basis, that is, a base $\mathcal{B}$ consisting of a countable family of open sets, then the members of $\mathcal{B}$ which contain a particular point a form a countable local basis at a.

Theorem

(1) Every second countable space is separable.

(2) Every metric space is first countable

(3) Every separable metric space is second countable

Corollary Euclidean n-space $\mathbb{R}^n$ is second countable for each positive integer n.

Theorem A family $\mathcal{B}$ of subsets of a set X is a basis for some topology X iff both the following condition hold:

a) The union of the members of $\mathcal{B}$ is X

b) for each $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there is a member B_x of $\mathcal{B}$ such that $x \in B_x \subseteq B_1 \cap B_2$

Def'n Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies τ and τ' for a set X. Then $\mathcal{B}$ and $\mathcal{B}'$ are equivalent bases provided that the topologies τ and τ' are identical.

Theorem Bases $\mathcal{B}$ and $\mathcal{B}'$ for topologies on a set X are equivalent iff both of the following conditions hold:

a) for each $B \in \mathcal{B}$ and $x \in B$, there is a member $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

b) for each $B' \in \mathcal{B}'$ and $x \in B'$, there is a member $B \in \mathcal{B}$ such that $x \in B \subseteq B'$.

Def'n Let (X, τ) be a space. A subcollection S of τ is a subbasis or subbase for τ if the family $\mathcal{B}$ of all finite intersections of members of S is a basis for τ

eg The collection S of all open intervals of the form (a, ∞) and $(-\infty, b)$ $a, b \in \mathbb{R}$ is a subbase for the usual topology for $\mathbb{R}$.

2.4. Continuity and topological equivalence

Def'n Let (X, τ) and (Y, τ') be topological space, $f: X \to Y$ be a function and $a \in X$. Then f is continuous at a if for each open set V in Y containing $f(a)$ there is an open set U in X containing a such that $f(U) \subseteq V$.
The function f is continuous if it is continuous at each point of its domain.

Alternate definition: A function $f: (X, \tau) \to (Y, \tau')$ is continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

recall $f^{-1}(V) = \{x \in X : f(x) \in V\}$

Theorem Let $f: X \to Y$ be a function and $a \in X$. The following statements are equivalent:

1) f is continuous at a
2) for each open set V in Y containing $f(a)$, there is an open set U in X such that $a \in U$ and $U \in f^{-1}(V)$.
3) for each nbhd V of $f(a)$, $f^{-1}(V)$ is a nbhd of a
4) for each subset V of Y with $f(a) \in \text{int } V$, a belongs to $\text{int } f^{-1}(V)$.

Theorem let $f: X \to Y$ be a function. The following statements are equivalent:

1) f is continuous.
2) for each closed subset C of Y, $f^{-1}(C)$ is closed in X.
3) for each subset A of X, $f(\bar{A}) \subseteq \overline{f(A)}$
4) There is a basis $\mathcal{B}$ for the topology of Y such that $f^{-1}(B)$ is open in X for each basic open set B in $\mathcal{B}$.
5) There is a subbasis S for the topology of Y such that $f^{-1}(s')$ is open in X for each subbasic open set s' in S.

Theorem If $f: X \to Y$ and $g: Y \to Z$ are continuous functions, ~~an~~ then the composite function $g \circ f: X \to Z$ is continuous.

eg ① Consider the function $f: \mathbb{R} \to Y$ from the real line to a discrete two-point space $Y = \{a, b\}$

the largest possible collection of open subsets of X (power set of X)

defined by $f(x) = \begin{cases} a & \text{if } x \le 0 \\ b & \text{if } x > 0 \end{cases}$

$x \le 0$ or $x > 0$

If $x \in \mathbb{R}$, $(x-\varepsilon, x+\varepsilon)$ is open in $\mathbb{R}$

Then, f maps open sets in $\mathbb{R}$ to open sets in Y b/c every subset of Y is open. But f is not continuous, $\{a\}$ is open in Y but $f^{-1}(a) = (-\infty, 0]$ is not open in $\mathbb{R}$.

(2) Let $f:(\mathbb{R}, \mathcal{T}) \to (\mathbb{R}_l, \mathcal{T}')$ where $\mathcal{T}$ is a topology generated by (a,b) (called usual or standard) topology and $\mathcal{T}'$ is a topology generated by $[a,b)$ which is called lower limit topology

such that $f(x) = x, \forall x \in \mathbb{R}$. Then f is not a cont. function: b/c $\underbrace{f^{-1}([a,b))}_{\text{open in } \mathbb{R}_l} = \underbrace{[a,b)}_{\text{not open in } \mathbb{R}}$.

However, $g: \mathbb{R}_l \to \mathbb{R}$ given by $g(x) = x$ is cont. b/c $g^{-1}(a,b) = (a,b)$ which is open in $\mathbb{R}_l$.

<u>Def'n</u> Let $f: X \to Y$ be a function on the indicated spaces. Then f is an open function or open mapping if for each open set O in X, $f(O)$ is open in Y. The function f is a closed function or closed mapping if for each closed set C in X, $f(C)$ is closed in Y.

<u>Remarks</u>
- A closed mapping may not be cont.
- An open " " " " be closed, and conversely.
- A cont. fun ~~map~~ may be neither open nor closed.
- A mapping that is both open and closed may not be cont.

e.g. Consider eg (1) above, f is open mapping but not cont.

Def^n Topological spaces X and Y are topologically equivalent or homeomorphic if there is a 1-1 and onto fn $f: X \to Y$ such that f and f^{-1} are cont. The function f is called a homeomorphism.

eg For every space $(X, \mathcal{J})$, the identity map $i: X \to X$ is a homeomorphism.

Remark Topological equivalence is an equivalence relation.

Remark A homeomorphism $f: X \to Y$ is a bijection which is also an open, continuous function b/c $(f^{-1})^{-1} = f$ for a bijection and continuity of f^{-1} can be expressed for each open set O in X, $f(O)$ is open in Y

Remark f is homeomorphism iff it is closed, continuous bijection, since continuity of f^{-1} can be expressed as for each closed set C in X, $f(C)$ is closed in Y.

Def^n A property P of topological spaces is a topological property or topological invariant if X has property P, then so does every space Y which is topologically equivalent to X.

Theorem Separability is a topological property.

Theorem First countability and second countability are topological properties.

2.5 Subspaces

Def'n Given a topological space $(X, \mathcal{T})$ with $Y \subseteq X$, then the family $\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$ is a topology on Y, called the subspace topology or relative topology. With this topology Y is called a subspace of X. i.e., $(Y, \mathcal{T}_Y)$ is a subspace of $(X, \mathcal{T})$. The members of $\mathcal{T}_Y$ are called relatively open sets or open sets in Y.

Remark ① ϕ and Y in $\mathcal{T}_Y$

b/c $\phi = Y \cap \phi$ and $Y = Y \cap X$ where ϕ and X are elements of $\mathcal{T}$. (1)

② Let $V_1, V_2, \ldots, V_n \in \mathcal{T}_Y$

Then $V_i = Y \cap U_i$, for some $U_i \in \mathcal{T}$, $i = 1, \ldots, n$

$$\Rightarrow \bigcap_{i=1}^{n} V_i = \bigcap_{i=1}^{n} (Y \cap U_i) = Y \cap \underbrace{\left(\bigcap_{i=1}^{n} U_i\right)}_{\in \mathcal{T} \because \mathcal{T} \text{ is a top.}} \in \mathcal{T}_Y$$

i.e $\mathcal{T}_Y$ is closed under finite intersection (3)

③ Let $\{V_i\}_{i \in I}$ be a family of elements of $\mathcal{T}_Y$.

Then, $$\bigcup_{i \in I} V_i = \bigcup_{i \in I} (Y \cap U_i) = Y \cap \underbrace{\left(\bigcup_{i \in I} U_i\right)}_{\in \mathcal{T} \because \mathcal{T} \text{ is top.}} \in \mathcal{T}_Y$$

i.e., $\mathcal{T}_Y$ is closed under arbitrary unions.

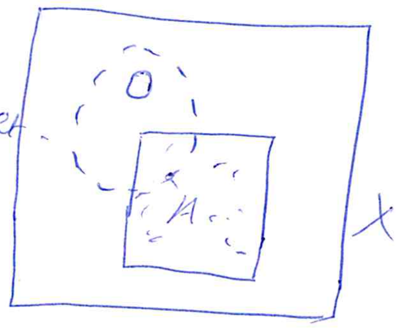
Hence, $\mathcal{T}_Y$ is a topology on Y. (5)

Def's Let $(A, \mathcal{J}')$ be a subspace of a topological space $(X, \mathcal{J})$. $D \subseteq A$ is relatively closed if it is a closed set in the subspace topology for A; i.e., D is relatively closed iff $A \setminus D = O \cap A$ for some open set O in X.

Theorem Let (A, τ') be a subspace of a topological space (X, τ). $D \subseteq A$ is closed in the subspace topology for A iff $D = C \cap A$ for some closed subset C of X.

Proof

$(\Rightarrow)$ Suppose D is relatively closed set. Then, $A \setminus D = O \cap A$ for some open set O in X, and



$D = A \setminus (A \setminus D) = A \setminus (O \cap A) = (X \setminus O) \cap A$

$\Rightarrow$ D is the intersection of A with the closed set $C = X \setminus O$ in X.

$(\Leftarrow)$ Suppose that $D = C \cap A$ for some closed set C in X. Then

$O = X \setminus C$ is open in X and

$A \setminus D = A \setminus (C \cap A) = (X \setminus C) \cap A = O \cap A$

So, $A \setminus D$ is open in the subspace topology of A and D is relatively closed set.

Def'n A property P of topological spaces is hereditary if X has property P, then every subspace of X has property P.

Def'n (2 pts) A topological space X is a Hausdorff space if for each pair a and b of distinct points of X there exist disjoint open sets U and V such that $a \in U$ and $b \in V$.

eg Every metric space (X, d) is Hausdorff.

To see this, let $a \neq b$ be dist points of X.

$$\Rightarrow \quad r = d(a, b) > 0 \quad (1)$$

Thus, $U = B(a, r/2)$ and $V = B(b, r/2)$ are disjoint open sets containing a and b, respectively. (3)

eg Consider the subset X of $\mathbb{R}^2$ consisting of the real line axis $\mathbb{R}$ and point $a = (0,1)$. Define a topology $\mathcal{J}$ on X to consist of the empty set ϕ and all subsets of X which contain a. Then $(X, \mathcal{J})$ is not a Hausdorff since every ~~open~~ non empty open set in X contains the point $(0,1)$.

Theorem

1. The property of being a Hausdorff space is a topological and hereditary property.
2. A sequence $\{x_n\}_{n=1}^{\infty}$ in a Hausdorff space cannot converge to more than one limit.

Def^{n}: If X is a space which is homeomorphic to a subspace A of a space Y, then X is called embedded in Y. The homeomorphism $f: X \to A$ is called an embedding of X in Y.