Chapter - 3

Connectedness

3.1 Definition and ~~theorems~~ theorems on connectedness

Def: ① A topological space X is disconnected or separated if it is the union of two disjoint, nonempty open sets. Such a pair A, B of subsets of X is called a separation of X.

② A space X is connected if it is not disconnected i.e., X is connected if there do not exists open subsets A and B of X such that $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$, $A \cup B = X$.

Remark A subspace Y of X is connected provided that it is a connected space when assigned the subspace topology.

Examples

① A discrete space with more than one point is disconnected.

② Any trivial space is connected since there fail to exist two non empty open sets.

3) Let Y be the set of non-zero real numbers with the subspace topology of $\mathbb{R}$. Then Y is disconnected since $(-\infty, 0)$ and $(0, \infty)$ form a separation.

4) Let $Z = \mathbb{R}^2 / \mathbb{R}$ denote the plane minus the real axis, with the subspace topology. Then Z is disconnected since The upper half plane $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ and lower half plane $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 < 0\}$ form a separation

(5) Let $X = [0,1] \cup [2,3]$ with the subspace topology of $\mathbb{R}$.
Then $A = [0,1]$ and $B = [2,3]$ are disjoint, non empty open subsets of X for which $X = A \cup B$.
So, X is disconnected.

6) Let $X = \mathbb{R} \cup [0,1)$ and
let the topology $\mathcal{J}$ for X consists of ϕ and all subsets of X which contain $(0,1)$.
Then X is connected since there do not exist two disjoint, non-empty open sets.

Example The real line $\mathbb{R}$ with the usual topology is connected

Proof let $\mathbb{R}$ be disconnected,
Then, $\mathbb{R} = A \cup B$, for some disjoint, nonempty open sets A and B of $\mathbb{R}$.
Since $A = \mathbb{R} \setminus B$ and $B = \mathbb{R} \setminus A$, then A and B are close & as well as open.
Consider two points a and b with $a \in A$ and $b \in B$.
without loss of generality, assume $a < b$.
Let $A' = A \cap [a,b]$.
Now, A' is closed and bdd subset of $\mathbb{R}$.
$\Rightarrow A'$ contains its least upper bound, say C.
Since A & B are disjoint, $C \neq b$
$\Rightarrow C < b$
Since A contains not point of $(C,b]$, then $(C,b] \subseteq B$
and hence $C \in \bar{B}$, But B is closed, so $C \in B$.
$\Rightarrow C$ belongs to both A and B $\rightarrow \leftarrow$

Theorems on Connectedness

Def'n Non-empty subsets A and B of a space X are separated sets if $\bar{A} \cap B = A \cap \bar{B} = \phi$

Theorem Let X be a topological space. Then the following statements are equivalent.

1. X is disconnected
2. X is the union of two disjoint, non-empty closed sets.
3. X is the union of two separated sets.
4. There is a continuous function from X onto a discreate two-point space $\{a, b\}$.
5. X has a proper subset A which is both open and closed.
6. X has a proper subset A such that $\bar{A} \cap \overline{(X \setminus A)} = \phi$.

Corollary The following statements are equivalent for a topological space X.

1. X is Connected,
2. X is not the union of two disjoint, non empty closed sets.
3. X is not the union of two separated sets.
4. There is no continuous function from X onto a discrete two-point space $\{a, b\}$.
5. The only subsets of X which are both open and closed are X and ϕ.
6. X has no proper subset A for which $\bar{A} \cap \overline{(X \setminus A)} = \phi$

3.2 Connectedness and Continuity

* **Theorem** Connectedness is preserved by continuous function.

Proof Let X be a connected space and $f: X \to Y$ a continuous function from X onto a space Y.

nts Y is connected.

using the contrapositive form, assume that Y is disconnected.

There are disjoint, non-empty open sets A and B in Y such that $Y = A \cup B$. Then the sets $f^{-1}(A)$ and $f^{-1}(B)$

(1) are open sets since f is cont.

(2) are disjoint since f is a fun.

(3) are non-empty since f is surjective.

(4) have union on X since $X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}($

* **Corollary** If $f: X \to Y$ is a continuous function on the indicated spaces and X is connected, then the image $f(X)$ is a connected subspace of Y.

Theorem : A subspace Y of a space X is disconnected iff there exist open sets U and V in X such that $U \cap Y \neq \phi$, $V \cap Y \neq \phi$, $U \cap V \cap Y = \phi$, $Y \subset U \cup V$.

$Y \subseteq U \cup V$

$Y \cap (U \cup V) = Y$

$(Y \cap U) \cup (Y \cap V) = Y$

Proof ($\Rightarrow$) suppose that Y is disconnected.

$\Rightarrow$ There are disjoint, non-empty open sets A and B in the subspace topology for Y such that $Y = A \cup B$.

By def'n of relatively open sets, there must be open sets U and V in X such that

$A = U \cap Y$, $B = V \cap Y$ $Y = A \cup B = (U \cap Y) \cup (V \cap Y) = Y \cap (U \cup V) \Rightarrow Y \subseteq U \cup V$

Then, the U and V have the required properties

($\Leftarrow$) suppose that U and V are open subsets of X such that $U \cap Y \neq \phi$, $V \cap Y \neq \phi$, $U \cap V \cap Y = \phi$

$Y \subseteq U \cup V$.

Then, $A = U \cap Y$, $B = V \cap Y$ are non-empty, disjoint relatively open sets whose union is Y

$\Rightarrow Y$ is disconnected.

Remark. The above thm holds for closed sets.

Pf EX.

eg ① Each open interval on the real line, being homeomorphic to $\mathbb{R}$, is connected.

② Every interval on the real line is connected

Theorem If Y is a connected subspace of a space X, then $\bar{Y}$ is connected.

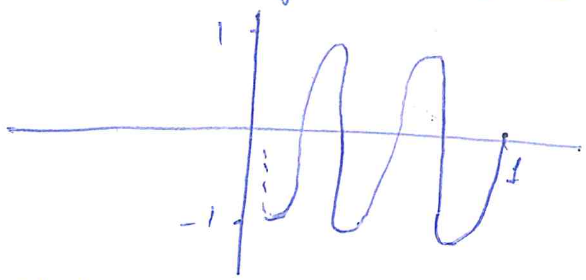
Corollary Let Y be a connected subspace of a space X and Z a subspace of X such that $Y \subset Z \subset \bar{Y}$. Then Z is connected.

eg The Topologist's sine curve.

let $A = \{(0,y) \in \mathbb{R}^2 : -1 \le y \le 1\}$

$B = \{(x,y) \in \mathbb{R}^2 : 0 < x \le 1, \ y = \sin \pi/x \}$

The subspace $T = A \cup B$ of $\mathbb{R}^2$, is called the Topologist's sine curve.



The topologist's sine curve.

Note that: B is connected since it is the image of $(0,1]$ under a continuous function.

since $T = \bar{B}$, T is connected.

Theorem : let X be a space and $\{A_\alpha : \alpha \in I\}$ a family of connected subsets of X for which $\bigcap_{\alpha \in I} A_\alpha$ is non empty. Then $\bigcup_{\alpha \in I} A_\alpha$ is connected.

proof let $Y = \bigcup_{\alpha \in I} A_\alpha$

Suppose that U and V are open sets in X for which $U \cap Y \neq \phi$, $U \cap V \cap Y = \phi$, and $Y \subseteq U \cup V$

NTS $V \cap Y = \phi$

now, $U \cap Y \neq \phi$, so U contains some point in $A_{\alpha'}$, for some $\alpha' \in I$.

Since $A_{\alpha'}$ is connected, then $A_{\alpha'} \subseteq U$.

If $b \in \bigcap_{\alpha \in I} A_\alpha$, then $b \in A_{\alpha'}$, so $b \in U$.

Thus U contains a point b in each A_α, $\alpha \in I$

Since A_α is connected, then $A_\alpha \subseteq U$ for each $\alpha \in I$

Thus $Y = \bigcup_{\alpha \in I} A_\alpha \subseteq U$

So, $V \cap Y = \phi$ b/c $U \cap Y \cap V = \phi$ $\Rightarrow Y \cap V = \phi$ $\Rightarrow Y \subseteq U$

<u>Corollary</u> Let X be a space, $\{A_\alpha : \alpha \in I\}$ a family of connected subsets of X, and B a connected subset of X such that, for each $\alpha \in I$, $A_\alpha \cap B \neq \phi$. Then $B \cup \{\bigcup_{\alpha \in I} A_\alpha\}$ is connected.

<u>Theorem</u> Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of connected subsets of a space X such that for each integer $n \geq 1$, A_n has at least one point in common with one of the preceding sets $A_1, A_2, \cdots, A_{n-1}$. Then $\bigcup_{n=1}^{\infty} A_n$ is connected.

<u>Def'n</u> A <u>Component</u> of a topological space X is a connected subset C of X which is not a proper subset of any connected subset of X.

<u>Remarks</u> (1) Let $a, b \in X$, then the components C_a and C_b are either identical or disjoint.

(2) Every connected subset of X is contained in a component.

③ Each component of X is a closed set.

④ X is connected iff it has only one component.

⑤ If C is a component of X and A, B form a separation of X, then C is a subset of A or a subset of B.

eg ① Consider the subspace $X = (0,1) \cup (2,3)$ of $\mathbb{R}$

Then, $(0,1)$ and $(2,3)$ are components of X.

② In a discrete space, each component contains only one point.

Def'n A space X is totally disconnected provided that each component of X contains a single point.

eg discrete spaces of more than one point is totally disconnected spaces.

3.3 Connected Subsets of the real line

~~Theorem~~

Remark From the previous examples, the real line and all intervals on the real line are connected.

- This section shows that there are no other connected subsets of $\mathbb{R}$.

Theorem The connected subsets of $\mathbb{R}$ are precisely the intervals.

-9- **Lemma** $\phi \neq A \subseteq \mathbb{R}$ is an interval iff $c, d \in A$, every real number b/n c and d is in A.

Proof We know that every interval is connected, NTS a subset B of $\mathbb{R}$ which is not an interval is disconnected.

Let $B \subseteq \mathbb{R}$ and B is not an interval.

By the above lemma, there are members c and d in B and $y \in \mathbb{R}$ with $c < y < d$ for which $y \notin B$.

Then the open sets $U = (-\infty, y)$, $V = (y, \infty)$ have the following properties:

- $c \in U \cap B \Rightarrow U \cap B \neq \phi$

  $d \in V \cap B \Rightarrow V \cap B \neq \phi$
- $U \cap V = \phi \Rightarrow U \cap V \cap B = \phi$
- $B \subseteq U \cup V$

$\therefore$ B is disconnected.

Hence every connected subset of $\mathbb{R}$ must be a interval.

3.4 Applications of Connectedness

Theorem (The intermediate value theorem)

Let $f : [a, b] \to \mathbb{R}$ be a continuous function from $[a, b]$ into $\mathbb{R}$ and $y_0 \in \mathbb{R}$ such that y_0 is b/n $f(a)$ and $f(b)$. Then there is a number $c \in [a, b]$ for which $f(c) = y_0$.

Proof $[a,b]$ is connected, since f is cont,

Then $f([a,b])$ is connected subset of $\mathbb{R}$.

$\Rightarrow f([a,b])$ must be an interval

Thus, any number y_0 b/n $f(a)$ and $f(b)$ must be in the image $f([a,b])$.

This means that $y_0 = f(c)$ for some real number c b/n a and b.

Corollary let $f:[a,b] \to \mathbb{R}$ be a continuous functi for which one of $f(a)$ and $f(b)$ is positive and the other negative. then the eqn $f(x)=0$ has a root b/n a and b.

Theorem Let $f:[a,b] \to [a,b]$ be cont. fun. The there is a member $c \in [a,b]$ such that $f(c)=c$.

Def'n ① A fixed point of a function $f: X \to X$ is a point x for which $f(x)=x$.

② A topological space X has the fixed-point property if every continuous function from X into itself has at least one fixed point.

eg from the above Thm, every closed and bdd interval has fixed-point property.

Theorem The fixed point property is a topological invariant (i.e., If X is homeomorphic to a space with the fixed point property, then X has the fixed point property).

Proof Let X be a space which has fixed point property.
Let Y is a space homeomorphic to X.
Let $h: X \to Y$ a homeomorphism.
(1) Let $f: Y \to Y$ be cont. fn.
since the composite fn $h^{-1} f h: X \to X$ is cont. and,
(3) it has at least one fixed point x_0 : $h^{-1} f h(x_0) = x_0$.
then, $f(h(x_0)) = h \underbrace{h^{-1} f h(x_0)}_{x_0} = h(x_0)$
$\Rightarrow h(x_0)$ is a fixed point for f.
(5) Thus, Y has fixed-point property.

eg - The real line does not have fixed-point property since the fn $f(x) = x+1$, $x \in \mathbb{R}$ has no fixed point.
- Since open interval is homeomorphic to $\mathbb{R}$, the above Theorem shows that no open interval has the fixed-point property.