Chapter - 4

Compactness

4.1 Compact spaces and subspaces

Def'n (1) An open cover of a set E in a topological space X is a collection $\{G_\alpha\}$ of open subsets of X such that $E \subseteq \bigcup_\alpha G_\alpha$.

(2) Let O be an open cover of a set E in a topological space X. A subcollection O^* of O whose union contains E is called subcover.

(3) An open cover of a space X is a collection of open subsets of X whose union is X.

eg Consider the family G of open intervals of the form $(a-\frac{1}{n}, b+\frac{1}{n})$, $n=1,2,3,\ldots$,

Then, each open interval is an open set in $\mathbb{R}$ and we've $[a,b] \subseteq \bigcup_{n=1}^{\infty} (a-\frac{1}{n}, b+\frac{1}{n})$

Hence G is an open covering of $[a,b]$.

Since any one of the open intervals of the covering contains $[a,b]$, we've a finite subcovering for $[a,b]$.

Def'n A space X is compact if every open cover of X contains a finite subcover.

i.e., X is compact if for every collection O of open sets whose union equals X, there is a finite subcollection $\{O_i\}_{i=1}^{N}$ of O whose union equals X.

Def'n A subspace A of a space X is compact if A is a compact topological space in its subspace topology.

i.e., A subspace A of a space X is compact iff every open cover of A by open sets in X has a finite subcover.

Examples

① The real line $\mathbb{R}$ is not compact.

Let $\mathcal{C} = \{(n, n+2) : n \in \mathbb{Z}\}$

Then $\mathcal{C}$ is an open covering of $\mathbb{R}$ but has no finite sub cover.

② $(0,1)$ is not compact

Let $\mathcal{C} = \{(\frac{1}{n}, 1) : n \in \mathbb{Z}^+\}$

Then $\mathcal{C}$ is an open covering of $(0,1)$ but doesn't admit a finite sub cover.

In general, an open interval (a,b) is not compact.

③ Any space consisting of a finite no of points is compact

To show this, If X is a finite having n-elements and if $\{G_\alpha\}$ is an open cover of X, then we can find atmost n sets $G_{\alpha_1}, \ldots, G_{\alpha_n}$ $\ni$ $X \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n}$.

Def'n A collection $\mathcal{C}$ of subsets of X is said to satisfy the finite intersection property if for every finite sub collection $\{C_1, C_2, \ldots, C_n\}$ of $\mathcal{C}$, the intersection $C_1 \cap C_2 \cap \cdots \cap C_n$ is nonempty.

eg The collection $\{(\frac{1}{n}, 1)\}_{n=2}^{\infty}$ is a family of open subsets of $\mathbb{R}$ with the finite intersection property.

N.B the intersection of any finite collection these open intervals is the interval of the largest index involved in the intersection.

Theorem A space X is compact iff every family of closed sets in X with the finite intersection property has non-empty intersection.

Proof ($\Rightarrow$) suppose X is compact

Let $\mathcal{C}$ be any collection of closed sets in X satisfying the finite intersection property.

wts $\bigcap_{C \in \mathcal{C}} C \neq \phi$

Assume the contrary, i.e., $\bigcap_{C \in \mathcal{C}} C = \phi$

Then $\bigcup_{C \in \mathcal{C}} C' = X$

$\therefore$ The collection $\{C' : C \in \mathcal{C}\}$ is the an open cover of X

Since X is compact, there is a finite subcollection $\{C_1', \ldots, C_n'\}$ that covers X.

i.e., $X = \bigcup_{i=1}^{n} C_i'$

$\therefore \bigcap_{i=1}^{n} C_i = \phi \rightarrow\leftarrow$ to the finite intersection property.

Hence $\bigcap_{C \in \mathcal{C}} C \neq \phi$

($\Leftarrow$) suppose that any collection $\mathscr{C}'$ of closed sets which satisfy the finite intersection property also satisfies

$$\bigcap_{C \in \mathscr{C}'} C \neq \phi$$

<u>WTS</u> X is compact.

let $\mathscr{C}$ be any open covering of X

Suppose $\mathscr{C}$ doeson't admite finite subcover of X.

Then, for any finite subcollection $\{U_1, U_2, \dots, U_n\}$ of $\mathscr{C}$,

$$X - \bigcup_{i=1}^{n} U_i \neq \phi$$

Consider $\mathcal{B} = \{X - U : U \in \mathscr{C}\}$

Then $\mathcal{B}$ is a collection of closed sets in X.

let $\{C_1, C_2, \dots, C_n\}$ be any finite subcollection of $\mathcal{B}$, then

$$\bigcap_{i=1}^{n} C_i = \bigcap_{i=1}^{n} (X - U_i) \text{ , where } C_i = X - U_i \text{ for some } U_i \in \mathscr{C}$$

But $\displaystyle\bigcap_{i=1}^{n} C_i = X - \bigcup_{i=1}^{n} U_i \neq \phi$

$\therefore$ $\mathcal{B}$ satisfies the finite intersection property

$$\phi \neq \bigcap_{C \in \mathcal{B}} C = \bigcap_{U \in \mathscr{C}} (X - U) = X - \bigcup_{U \in \mathscr{C}} U \quad \rightarrow\!\leftarrow \text{ to the assumption that } \mathscr{C} \text{ cover } X$$

Hence X is compact

$Y \subseteq X$ $U_{\alpha} \subseteq Y$

Theorem Every closed subset of a compact space is compact.

Proof Let Y be a closed subset of the compact space X.

Let $\mathcal{C}$ be a covering of Y by open sets in X.

Then $\mathcal{B} = \mathcal{C} \cup \{X - Y\}$ is an open covering of X

(since Y is closed, then $X - Y$ is open)

Since X is compact, $\exists$ a finite subcover of X

If that finite sub-cover contains $X - Y$, then we discard $X - Y$ and the remaining collection covers Y.

If the finite subcover doesn't contain $X - Y$, it also covers Y.

Hence, Y is compact.

Theorem Each compact subset of Hausdorff space is closed.

for each dlt points of X $\exists$ disjoint open set U & V $\ni$ $a \in U$ & $b \in V$.

Corollary Let X be a compact Hausdorff space. $A \subseteq X$ is compact iff it is closed.

not compact

e.g. $(a, b]$ and $[a, b)$ are not compact in $\mathbb{R}$ b/c they are not closed.

which one of these are? / some of

- $[a, b]$ in $\mathbb{R}$ is compact.

Remark Any compact set need not be closed.

4.2 Compactness and Continuity

Theorem Let X be a compact space, Y a space and $f: X \to Y$ a continuous function from X onto Y. then Y is compact.

Proof Let $\mathcal{O}$ be an open cover of Y.

$\Rightarrow$ for each $U \in \mathcal{O}$, $f^{-1}(U)$ is open in X

so, the collection $\mathcal{O}^* = \{f^{-1}(U) : U \in \mathcal{O}\}$ of inverse images of members of $\mathcal{O}$ is an open cover of X.

Since X is compact, $\mathcal{O}^*$ has a finite subcover $\{f^{-1}(O_i)\}_{i=1}^{n}$ for X corresponding to a finite subcollection $\{O_i\}_{i=1}^{n}$ of $\mathcal{O}$.

Since $X = \bigcup_{i=1}^{n} f^{-1}(O_i)$ and f maps X onto Y, then

$$Y = f(X) = f\left(\bigcup_{i=1}^{n} f^{-1}(O_i)\right) = \bigcup_{i=1}^{n} f(f^{-1}(O_i))$$

$$\subseteq \bigcup_{i=1}^{n} O_i$$

Thus, $\{O_i\}_{i=1}^{n}$ is a finite subcover for Y derived from $\mathcal{O}$.

$\therefore$ Y is compact.

Corollary The image of a compact space under a continuous map is compact.

Corollary Compactness is a topological invariant.

Theorem Let $f: X \to Y$ be a bijection continuous fun. Let X be compact and Y be Hausdorff. Then, f is a homeomorphism.

Proof Here is suffices to show that f^{-1} is continuous.

Let A be closed in X

$\Rightarrow$ A is compact since every closed subset of

(2) compact space is compact.

(3) Since A is compact, $f(A)$ is compact in Y

Since Y is Hausdorff, $f(A)$ is closed in Y b/c every compact subset of Hausdorff space is closed.

$\therefore f^{-1}$ is continuous.

(5) Hence, f is a homeomorphism.

Def^n Let (X, d) and (Y, d') be metric spaces and $f: X \to Y$ a function. Then f is uniformly cont. if for each positive number ε there is a positive number δ such that if x_1, x_2 are points of X for which $d(x_1, x_2) < \delta$, then $d'(f(x_1), f(x_2)) < \varepsilon$.

m. space

Theorem (Uniform Continuity Theorem)

Let $f: X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y). Then f is uniformly continuous.

4.3 Properties related to Compactness

Theorem A subset A of $\mathbb{R}^n$ is compact iff it is closed and bounded in Euclidean metric.

Defn A topological space X is called countably compact if every countable open cover of X has a finite subcover.

Remark ① Every compact space is countably compact.

② Countable compactness is a topological property.

③ every countable compact spaces may not be compact.

Defn A space X has the Lindelof property or is a Lindlof space if every opencover of X has a countable subcover.

Theorem Let X is a Lindlof space. X is compact iff it is countably compact.

Proof (⇒) Compactness always implies countable compactness.

(⇐) Let O be an open cover of a Lindelof space X. Then O has a countable subcover O^* for X. Since X is countably compact, O^* has a finite subcover O^{**} for X.

Then O^{**} is a finite subcover for X derived from O.

So, X is compact.

Theorem Every second countable space is Lindelof. (iff it has countable basis)

Proof Let $\mathcal{B}$ be a countable basis (each member of $\mathcal{T}$ is union of member of $\mathcal{B}$) for a second countable space X.

Let $\mathcal{O}$ be an open cover of X.

$\forall x \in X$, Let O_x be a member of $\mathcal{O}$ containing x and B_x a member of $\mathcal{B}$ such that $x \in B_x \subseteq O_x$

Since $\mathcal{B}$ is a countable basis, the set $\{B_x : x \in X\}$ is a countable open cover of X.

For each B_x, let O_x' be a member of $\mathcal{O}$ such that $B_x \subseteq O_x'$

$\Rightarrow$ The collection $\mathcal{O}'$ of open sets O_x' is a countable subcover for X derived from $\mathcal{O}$.

eg $\mathbb{R}^n$ is Lindlof, since it is 2^{nd} countable.

Def'n A space X has the Bolzano-weierestrass property if every infinite subset of X has a limit point.

Remark The Bolzano weirestrass property is a topological invariant.

Theorem Every compact space has Bolzano-weierstrass property.

eg (1) $[a,b]$ has Bolzano-weierstrass property b/c it is compact.

(2) $\mathbb{R}$ has no Bolzano-weierstrass property b/c $\mathbb{Z}$, for example, has no limit point.

Def^n Let X be a metric space and $\mathcal{O}$ an open cover of X. A Lebesgue number for $\mathcal{O}$ is a +ve number ε with the property that every subset of X of diameter less than ε is contained in some member of $\mathcal{O}$.

let $A \subseteq (X, d)$ then $\text{lub}\{d(x,y) : x, y \in A\} = D(A)$

Theorem If X is compact space, then every open cover of X has a Lebesgue number.

Theorem Let $A \subseteq \mathbb{R}^n$. The following statements are equivalent.

1) A is compact
2) A has the Bolzano Weierstrass property
   every infinite subsets has limit pt
3) A is countably compact
   every countable open cover has finite subcover
4) A is closed and bounded.

Def^n A compact, connected, locally connected metric space is called a Peano space or a Peano continuum.

eg Closed and bdd intervals in $\mathbb{R}$ are Peano spaces.

4.4 One-point compactification

Def^n ① A space X is locally compact at a point x in X if there is an open set U containing x for which $\bar{U}$ is compact.

② A space X is locally compact if it is locally compact at each point of its points.

eg ① $\mathbb{R}$ is locally compact. The point x lies in some open interval (a,b) for which $\overline{(a,b)} = [a,b]$ is is compact subspace.

② The subspace $\mathbb{Q}$ of rational numbers is not locally compact.

Remarks a) Locally compactness is a topological invariant.

b) Every compact space is locally compact since if X is compact, then X itself is an open set with compact closure $\Rightarrow \bar{X} = X$ since X is closed ($X^c = \emptyset$ is open)

c) locally compactness do not emply compactness

Def^n Let X be a topological space and ∞, called the point at infinity, an object not in X. Let $X_\infty = X \cup \{\infty\}$ and define a topology $\mathfrak{T}_\infty$ on X_∞ by the collection of open sets of X_∞ to consist of

a) the open sets of X

b) all sets of the form X_∞ / C, where C is a closed compact subspace of X.

c) The set X_∞.

Then, the space (X_∞, J_∞) is called one point compactification of X.

Remark J_∞ is a topology for the set X_∞. (EX!)

Theorem let (X, J) be a space and (X_∞, J_∞) its one point compactification. Then

a) (X_∞, J_∞) is compact.

b) (X, J) is a subspace of (X_∞, J_∞).

c) X_∞ is Hausdorff iff X is Hausdorff and locally compact.

d) X is a dense subset of X_∞ iff X is not compact

4.5 The Cantor set

Def'n The Cantor set is the subset of $I = [0,1]$ defined as follows: let F_{n+1} be the subset of F_n obtained by removing the open middle third of the components of F_n. Then the set $C = \bigcap_{n=1}^{\infty} F_n$ is the Cantor set.

(2)