A short note on Numerical Analysis II meant for self-taught students until face to face class will resume

Complied by

Hailu Muleta (Assistant Professor of Mathematics)

April, 2020

Jimma, Ethiopia

Chapter One

Revision on Numerical Integration

This chapter is highly devoted to find the numerical integration for a given set of data points. The topics covered here are

- Numerical Integration
- Newton-Cote's quadrature formula
- Trapezoidal Rule
- Simpson's one-third Rule
- Simpson's three-eighths Rule
- Weddle's rule

Consider the definite integral $\int_a^b f(x)dx$.

This integral represents the area between $y = f(x)$, the x –axis and the lines $x = a$ & $x = b$.

This integration is possible as far as $f(x)$ is explicitly given and the function is integrable.

Now suppose set of (n+ 1) paired values are given. First as we did in the case of numerical differentiation, we find $f(x)$ by an interpolating polynomial $P_n(x)$ and obtain $\int_a^b p_n(x)dx$ which can approximate the value for $\int_a^b p_n(x)dx$.

A **General Quadrature Formula for Equidistant Spacing (*Newton-Cote's Formula*)**

For equally spaced intervals, we have Newton's forward difference formula as

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots (1)$$

Here, $u = \frac{x - x_0}{h}, h$ is the interval of differencing.

Now, instead of $f(x)$, we will replace it by this interpolating polynomial $y(x)$ of Newton.

Since $x_n = x_0 + nh \ and \ u = \frac{x - x_0}{h} \Rightarrow u = n = \frac{x - x_0}{h}$

Thus, $\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_0+nh} f(x)dx$

$= \int_{x_0}^{x_0+nh} P_n(x)dx$. Where $P_n(x)$ is interpolating polynomial of degree n.

$$= \int_0^n \left(y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots \right) hdu$$

$$= h\int_0^n \left(y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u^3 - 3u^2 + 2u}{3!}\Delta^3 y_0 + \ldots \right) du$$

$$= h\left[uy_0 + \frac{u^2}{2}\Delta y_0 + \frac{\frac{u^3}{3} - \frac{u^2}{2}}{2!}\Delta^2 y_0 + \frac{\frac{u^4}{4} - u^3 + u^2}{3!}\Delta^3 y_0 + \ldots \right]_0^n$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \ldots (2)\right.$$

This equation is called ***Newton-cote's quadrature formula*** and is a general quadrature formula.

By taking different values for n we get a number of special formulas. Here we try to look for some values of n for which their practical application is very important in different disciplines of science. Detail information and results are explained below.

Trapezoidal Rule

Put n = 1, in the quadrature formula

$$\int_{x_0}^{x_0+nh} f(x)dx = \int_{x_0}^{x_0+h} f(x)dx \approx h\left[1.y_0 + \frac{1}{2}\Delta y_0\right]$$

Since other differences do not exist if n = 1.

$$\approx h\left(y_0 + \frac{1}{2}(y_1 - y_0)\right)$$

$$\approx \frac{h}{2}(y_0 + y_1)$$

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_0+nh} f(x)dx$$

$$= \int_{x_0}^{x_0+nh} f(x)dx + \int_{x_0+h}^{x_0+2h} f(x)dx + \ldots + \int_{x_0+(n-1)h}^{x_0+nh} f(x)dx$$

$$= \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \ldots + \frac{h}{2}(y_{n=1} + y_n)$$

$$= \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})]$$

This is known as a ***trapezoidal rule***.

Even though this method is very simple for calculation, the error in this case is significant.

Truncation Error in Trapezoidal Rule

In the neighborhood of $x - x_0$, we can expand $y = f(x)$ by Taylor series in powers of $x - x_0$.

That is,

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \ldots \qquad (1)$$

$$\int_{x_0}^{x_1} y\,dx = \int_{x_0}^{x_1}\left[y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \ldots\right]dx$$

$$= \left[y_0 x + \frac{(x-x_0)^2}{2!}y_0' + \frac{(x-x_0)^3}{3!}y_0'' + \ldots\right]_{x_0}^{x_1}$$

$$= y_0(x_1-x_0) + \frac{(x_1-x_0)^2}{2!}y_0' + \frac{(x_1-x_0)^3}{3!}y_0'' + \ldots$$

$$= hy_0 + \frac{h^2}{2!}y_0' + \frac{h^3}{3!}y_0'' + \ldots \qquad (2)$$

If h is the equidistant length, then also

$\int_{x_0}^{x_1} y\,dx = \frac{h}{2}(y_0 + y_1) =$ Area of the first trapezium = A_0 say

Putting $x = x_1$ in (1), we get

$$y(x_1) = y_1 = y_0 + (x_1-x_0)y_0' + \frac{(x_1-x_0)^2}{2!}y_0'' + \ldots$$

$$= y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \ldots$$

$$A_0 = \frac{h}{2}(y_0 + y_1) = \frac{h}{2}\left(y_0 + y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \ldots\right)$$

$$= hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{2.2!}y_0'' + \ldots$$

Subtracting A_0 from (2), we obtain

$$\int_{x_0}^{x_1} y\,dx - A_0 = h^3 y_0''\left(\frac{1}{3!} - \frac{1}{2.2!}\right) + \ldots$$

$$= -\frac{1}{2}h^3 y_0'' + \dots$$

$\therefore$ The error made in the first interval (x_0, x_1) is $-\frac{1}{2}h^3 y_0'' + \dots$

Similarly the error in the i[th] interval $= -\frac{1}{2}h^3 y_{i-1}''$

Hence, the total cumulative error E is

$$E \approx -\frac{1}{2}h^3\left(y_0'' + y_1'' + y_2'' + \dots + y_{n-1}''\right)$$

$$\Rightarrow |E| < \frac{nh^3}{12}M \quad where \quad M = \max\left\{\left|y_0''\right|, \left|y_1''\right|, \left|y_2''\right|, \dots\right\}$$

$$< \frac{(b-a)h^2}{12}M \text{ if the interval is } (a,b) \text{ and } h = \frac{b-a}{n}$$

$\therefore$ The error in the trapezoidal rule is of order h^2.

The accuracy of the result can be improved by increasing the number of intervals and decreasing the value of h.

Simpson's One-Third Rule

Setting n = 2 in Newton-cotes quadrature formula, we have

$$\int_{x}^{x_2} f(x)dx = h\left[2y_0 + 4\Delta y_0 + \frac{1}{2}\left(\frac{8}{3} - \frac{4}{2}\right)\Delta^2 y_0\right]$$ since the other terms vanish (become zero).

$$= h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(E-1)^2 y_0\right]$$

$$= h\left[2y_0 + 2y_1 - 2y_0 + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= h\left[\frac{1}{3}y_2 + \frac{4}{3}y_1 + \frac{1}{3}y_0\right]$$

$$= \frac{h}{3}\left[y_2 + 4y_1 + y_0\right]$$

Similarly, $\int_{x_2}^{x_4} f(x)dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$ and $\int_{x_i}^{x_{i+2}} f(x)dx = \frac{h}{3}\left[y_i + 4y_{i+1} + y_{i+2}\right]$

If n is an even integer, then the last integral will be

$$\int_{x_{n-2}}^{x_n} f(x)dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

Adding all these integrals, if n is even positive integer, then $y_0, y_1, y_2, y_3, ..., y_n$ are odd in number; we have

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + ... + \int_{x_{n-2}}^{x_n} f(x)dx$$

$$= \frac{h}{3}\left[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + ... + (y_{n-2} + 4y_{n-1} + y_n)\right]$$

$$= \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + ... + y_{n-2}) + 4(y_1 + y_3 + ... + y_{n-1})\right]$$

Simpson's Three-Eighths Rule

Putting n = 3 in Newton-cotes formula, we get

$$\int_{x_0}^{x_3} f(x)dx = h\left[3y_0 + \frac{9}{2}\Delta y_0 + \frac{1}{2}\left(\frac{9}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{81}{4} - 27 + 9\right)\Delta^3 y_0\right]$$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(E-1)^2 y_0 + \frac{3}{8}(E-1)^3 y_0\right]$$

$$= h\left[3y_0 + \frac{9}{2}y_1 - \frac{9}{2}y_0 + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}(y_3 + 3y_2 + 3y_1 + y_0)$$

If n is a multiple of 3,

$$\int_{x_0}^{x_0+nh} f(x)dx = \int_{x_0}^{x_0+3h} f(x)dx + \int_{x_0+3h}^{x_0+6h} f(x)dx + ... + \int_{x_0+(n-3)h}^{x_0+nh} f(x)dx$$

$$= \frac{3h}{8}\left[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + ... + (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)\right]$$

$$= \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + ... + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + y_n)\right]$$

This is ***Simpson's three-eighths rule*** and is applicable only when n is a multiple of 3.

Weddle's Rule

Putting n = 6 in Newton-cotes formula

$$\int_{x_0}^{x_0+6h} f(x)dx = h\left[6y_0 + 18\Delta y_0 + \frac{1}{2}(72-18)\Delta^2 y_0 + \frac{1}{6}(324-216+36)\Delta^3 y_0 + ...\right]$$

$$= h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0\right]$$

Now replace the term $\frac{41}{140}\Delta^6 y_0$ *by* $\frac{42}{140}\Delta^6 y_0$, by doing this, the error introduced is only $\frac{h}{140}\Delta^6 y_0$

which is negligible when h and $\Delta^6 y_0$ are small.

Using $\Delta = E - 1$ and replacing all differences in terms of y 's,

we get

$$\int_{x_0}^{x_0+6h} f(x)dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

Similarly, $\int_{x_0+6h}^{x_0+12h} f(x)dx = \frac{3h}{10}\left[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}\right]$

and $\int_{x_0+(n-6)h}^{x_0+nh} f(x)dx = \frac{3h}{10}\left[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n\right]$

Adding all these integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{10}\left[\left(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5\right) + \left(2y_6 + 5y_7 + y_8 + 6y_9 + y_0 + 5y_{11}\right) + \ldots + \left(2y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n\right)\right]$$

This equation is called ***Weddle's rule***.

Truncation Error in Simpson's Formula

By Taylor expansion of $y = f(x)$ in the neighborhood of $x = x_0$, we obtain

$$y = y_0 + (x - x_0)y_0^{'} + \frac{(x-x_0)^2}{2!}y_0^{''} + \frac{(x-x_0)^3}{3!}y_0^{'''} + \ldots \quad \ldots(1)$$

$$\int_{x_0}^{x_2} ydx = \int_{x_0}^{x_2}\left(y_0 + \frac{(x-x_0)}{1!}y_0^{'} + \frac{(x-x_0)^2}{2!}y_0^{''} + \frac{(x-x_0)^3}{3!}y_0^{'''} + \ldots\right)dx$$

$$= \left[y_0 x + \frac{(x-x_0)^2}{21!}y_0^{'} + \frac{(x-x_0)^3}{3!}y_0^{''} + \frac{(x-x_0)^4}{4!}y_0^{'''} + ..\right]_{x_0}^{x_2}$$

$$= y_0(x_2 - x_0) + \frac{(x_2-x_0)^2}{2!}y_0^{'} + \frac{(x_2-x_0)^3}{3!}y_0^{''} + \frac{(x_2-x_0)^4}{4!}y_0^{'''} + \ldots$$

$$= 2hy_0 + \frac{4h^2}{2!}y_0^{'} + \frac{8h^3}{3!}y_0^{''} + \frac{16}{4!}h^2 y_0^{'''} + \ldots$$

$$= 2hy_0 + 2h^2 y_0^{'} + \frac{4}{3}h^3 y_0^{''} + \frac{2h^4}{3}y_0^{'''} + \frac{4h^5}{15}y_0^{[4]} + \ldots$$

Now let $A_1 = \text{area} = \int_{x_0}^{x_2} ydx = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$, by Simpson's rule

Putting $x = x_1$ in (1), to get

$$y_1 = y_0 + (x_1 - x_0)y_0^{'} + \frac{(x_1 - x_2)}{2!}y_0^{''} + \ldots$$

$$= y_0 + hy_0^{'} + \frac{h^2}{2!}y_0^{''} + \frac{h^3}{3!}y_0^{'''} + \ldots$$

Putting $x = x_2$ in (1), we have

$$y_2 = y_0 + 2hy_0' + \frac{4h^2}{2!}y_0'' + \frac{8h^3}{3!}y_0''' + ...$$

$$\Rightarrow A_1 = \frac{h}{3}\left[y_0 + 4\left(y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + ...\right) + \left(y_0 + 2hy_0' + 2h^2 y_0'' + ...\right)\right]$$

$$= 2hy_0 + 2h^2 y_0' + \frac{4}{3}h^3 y_0'' + \frac{2}{3}h^4 y_0''' + \frac{5}{18}h^5 y_0^{[4]} + ...$$

$$\Rightarrow \int_{x_0}^{x_2} y\,dx - A_1 = \left(\frac{4}{15} - \frac{5}{18}\right)h^5 y_0^{[4]} + ...$$

$= \frac{-h^5}{90}y_0^{[4]} + ...$ Omitting the remaining terms involving h^6 and higher powers of h.

This means that the error made in (x_0, x_2) is $= \frac{-h^5}{90}y_0^{[4]} + ...$

Similarly, the error made in (x_2, x_4) *is* $\frac{-h^5}{90}y_2^{[4]}$ and so on.

Hence the total error E is (x_0, x_n) $is = -\frac{h^5}{9}\left(y_0^{[4]} + y_2^{[4]} + ...\right)$

$\therefore |E| < \frac{nh^5}{90}M$, where M is the maximum value of $y_0^{[4]}, y_2^{[4]} + ..., y_{2n-2}^{[4]}$

Since (x_{2n}, y_{2n}) is the last paired value because we require odd number of ordinates to apply Simpson's one-third rule.

If the interval is (a,b), then $b - a = h(2n)$, using this

$$|E| < \frac{(b-a)h^4}{180}M$$

Hence, the error in Simpson's one-third rule is of the order h^4.

Examples

1. Evaluate $\int_1^7 x^2 dx$ by using

a) Trapezoidal rule

b) Simpson's rule and verify your results by actual integration

Solution

Here $y(x) = x^2$ and the interval length is 7-1 = 6 so we divide this interval into 6 equal parts with $h = \frac{6}{6} = 1$

x:	1	2	3	4	5	6	7
y:	1	4	9	16	25	36	49

a) By Trapezoidal rule

$$\int_1^7 x^2 dx = \frac{h}{2}[(y_1 + y_7) + 2(y_2 + y_3 + y_4 + y_5 + y_6)]$$

$$\frac{1}{2}((1+49)+2(4+9+16+25+36)) = 115$$

b) By Simpson's one-third rule

$$\int_1^7 x^2 dx = \frac{1}{3}((y_1 + y_7) + 2(y_3 + y_5) + 4(y_2 + y_4 + y_6))$$

$$= \frac{1}{3}[(1+49)+2(9+25)+4(4+16+36)]$$

$$= 114$$

Since n = 6, we can also use Simpson's three-eighth rule.

So $\int_1^7 x^2 dx = \frac{3}{8}((y_1 + y_7) + 3(y_2 + y_3 + y_5 + y_6) + 2y_4)$

$$= \frac{3}{8}((1+49)+3(4+9+25+36)+2(16))$$

$= 114$

$$\int_1^7 x^2 dx = \left.\frac{x^3}{3}\right|_1^7 = \frac{1}{3}7^3 - \frac{1}{3} = \frac{342}{3} = 114$$

2. Evaluate $\int_0^1 \frac{dx}{1+x^2}$, using Trapezoidal rule with $h = 0.2$, and hence obtain an approximate value of π .

Solution

$$Let \quad y(x) = \frac{1}{1+x^2}$$

x :	0	0.2	0.4	0.6	0.8	1
y :	1	0.961538461	0.862068965	0.735294117	0.609756097	0.5

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)\right]$$

$$= \frac{0.2}{2}\left[(1 + 0.5) + 2(0.961538461 + 0.862068965 + 0.735294117 + 0.6097566097)\right]$$

$$= 0.783731528$$

But by actual integration

$$\int_0^1 \frac{dx}{1+x^2} = \left.\tan^{-1} x\right|_0^1 = \frac{\pi}{4}$$

$$\Rightarrow \frac{\pi}{4} \approx 0.783731528$$

$$\therefore \pi \approx 3.134926112$$

To compare this approximated value of π with its actual value using calculator 3.141592654,

the error is 0.00666654159 which is 0.6%.

3. From the table below, find the area bounded by the curve and

the x-axis from $x = 7.47$ to $x = 7.53$

x :	7.47	7.48	7.49	7.50	7.51	7.52	7.53
y :	1.93	1.95	1.98	2.01	2.03	2.06	2.08

Solution

Let us compare the results obtained by different methods

i) By Trapezoidal rule

$$\int_{7.47}^{7.53} f(x)dx = \frac{0.01}{2}\left[(1.93 + 2.08) + 2(1.95 + 1.98 + 2.01 + 2.03 + 2.06)\right]$$

$$= 0.12035$$

ii) By Simpson's one-third rule

$$\int_{7.47}^{7.53} f(x)dx \approx \frac{0.01}{3}\left[(1.93 + 2.08) + 2(1.98 + 2.03 + 4) + 4(1.95 + 2.01 + 2.06)\right]$$

$$\approx 0.120366667$$

iii) By Simpson's three-eighths rule

$$\int_{7.47}^{7.53} f(x)dx \approx \frac{3(0.01)}{8}\left[(1.93 + 2.08) + 3(1.95 + 1.98 + 2.03 + 2.06) + 2(2.01)\right]$$

$$\approx 0.1203375$$

iv) By Weddle's rule

$$\int_{7.47}^{7.53} f(x)dx \approx \frac{3(0.01)}{10}\left[1.93 + 5(1.95)1.98 + 6(2.01) + 2.03 + 5(2.06) + 2.08\right]$$

$$\approx 0.12039$$

As we can see from these rules the area is 0.1203 (correct to four decimal places)

4. Evaluate the integral $I = \int_{4}^{5.2} \ell n x dx$ using the rules so far developed.

Solution

Since $b - a = 5.2 - 4 = 1.2$ let us divide the interval into 6 equal parts, i.e.; $h = \frac{1.2}{6} = 0.2$

x	$\ln x$	x	$\ln x$
4	1.386294361	4.6	1.526056303
4.2	1.435084525	4.8	1.568615918
4.4	1.481604541	5.0	1.609437912
		5.2	1.648658626

i) **By Trapezoidal rule**

$$\int_{4}^{5.2} \ell n x dx = \frac{0.2}{2}\left[(1.386294361 + 1.648658626) + 2(1.435084525 + 1.481604541 + 1.526056303 + 1.568615918 + 1.609437912)\right]$$

$$= 1.827655139$$

ii) By Simpson's one-third rule

$$\int_{4}^{5.2} \ell n x dx \approx \frac{0.2}{3}\left[(1.386294361 + 1.648658626) + 2(1.481604541 + 1.568615918) + 4 (1.435084525 + 1.526056303 + 1.609437912)\right]$$

$$\approx 1.827847258$$

iii) By Simpson's three-eighths rule

$$\int_{4}^{5.2} \ell n x dx \approx \frac{0.2}{3}\left[(1.386294361 + 1.648658626) + 3(1.435084525 + 1.481604541) + 3 (1.568615918 + 1.609437912) + 2(1.526056303)\right]$$

$$\approx 1.827847258$$

iv) By Weddle's Rule

$$\int_4^{5.2} \ell nx dx \approx \frac{3(0.2)}{10}[1.386294361 + 5(1.435084525) + 1.481604541 + 6(1.526056303) +$$

$$6(1.526056303) + 1.1.568615918 + 5(1.609437912) + 1.648658626$$

$$\approx 1.827847407$$

By actual integration,

$$\int_4^{5.2} \ell nx dx = x(\ell nx - 1) \Big|_4^{5.2} = 1.827847409$$

Here Weddle's rule best approximates the exact value.

5. Evaluate $\int_0^1 e^x dx$ by Simpson's one-third rule correct to five decimal places.

Solution

The interval $b - a = 1$

Since error $|E| < \frac{(b-a)}{180} h^4 M$, where M = Max (e^x) in the range

$$< \frac{1}{180} h^4 e$$

Now we require $|E| < 10^{-6}$

$$\Rightarrow \frac{h^4 e}{180} < 10^{-6}$$

$$\Rightarrow h < \left(\frac{180 \times 10^{-6}}{e}\right)^{1/4} = 0.090207886 \approx 0.1$$

Hence we take $h = 0.1$

$$\therefore \int_0^1 e^x dx \approx \frac{0.1}{3}\left[(1+e)+2\left(e^{0.2}+e^{0.4}+e^{0.6}+e^{0.8}\right)+4\left(e^{0.1}+e^{0.3}+e^{0.5}+e^{0.7}+e^{0.9}\right)\right]$$
$$\approx 1.718282782$$

By the actual integration,

$$\int_0^1 e^x dx = e^x\Big|_0^1 = e-1 = 1.71828182\text{8}$$

Correct to five decimal places $\int_0^1 e^x dx = 1.71828$ which is the same as the exact value.

6. A curve passes through the points $(1,2),(1.5,2.4),(2.0,2.7),(2.5,2.8),\ (3,3),(3.5,2.6)$ and $(4,2.1)$. Obtain the area bounded by the curve, the x-axis, $x=1$ and $x=4$.

Area = $\int_a^b y dx = \int_1^4 y dx$

$$= \frac{0.5}{3}\left[(2+2.1)+2(2.7+3)+4(2.4+2.8+2.6)\right]$$

$$= \frac{1}{6}(4.1+11.4+31.2) = 7.7833 \text{ sq. units}$$

Volume = $\pi\int_a^b y^2 dx = \pi\int_1^4 y^2 dx$

$$= \frac{0.5\pi}{3}\left(\left(2^2+(2.1)^2\right)+2\left((2.7)^2+3^2\right)+4\left((2.4)^2+(2.8)^2+(2.6)^2\right)\right]$$

$$= \frac{\pi}{6}(8.41+32.58+81.44) = 20.405\pi = 64.1041981 \text{ cubic units.}$$

7. A river is 80 meters wide. The depth 'd' in meters at a distance x meters from one bank is given by the following table.

x:	0	10	20	30	40	50	60	70	80
d:	0	4	7	9	12	25	14	8	3

Calculate the area of cross-section of the river using Simpson's rule.

Solution

Area of cross-section = $\int_0^{80} y dx$

$$= \frac{10}{3}\left[(0+3)+2(7+12+14)+4(4+9+15+8)\right] = 710 \text{ sq. meters}$$

8. The table below gives the velocity v of a moving particle at time t second. Find the distance (S) covered by the particle in 12 seconds and also the acceleration at $t=2$ seconds.

t:	0	2	4	6	8	10	12
v:	4	6	16	34	60	94	136

Solution

We know that $v = \frac{ds}{dt}$ and $a = \frac{dv}{dt}$

To get S

$$\therefore S = \int_0^{12} v dt = \frac{2}{3}\left((4+136)+2(16+60)+4(6+34+94)\right)$$

$= 552$ meters

To find a, $a = \left(\frac{dv}{dt}\right)_{t=2}$ first form the difference table

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$
0	4			
		2		
			8	
2	6			0
		10		
			8	
4	16			0
		18		
			8	
6	34			0
		26		
			8	
8	60			
		34		
				0

10 94

42 8

12 136

$$\left(\frac{dv}{dt}\right)_{t=2} = \frac{1}{h}\left[\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0\right]$$

Taking $v_0 = 6$

$$= \frac{1}{2}\left(10 - \frac{1}{2}(8)\right] = 3m/\sec^2$$

Exercise

1. Evaluate $\int_1^2 \frac{dx}{1+x^2}$ taking $h = 0.2$, using Trapezoidal rule. Can you use Simpson's rule? Justify your reasons.

2. Compute the value of $\int_0^1 \sqrt{\sin x + \cos x}\,dx$ correct to four decimal places taking $h = 0.8$.

3. Find the value of $\log 2^{\frac{1}{3}}$ from $\int_0^1 \frac{x^2}{1+x^3}\,dx$ using Simpson's one-third rule with $h = 0.25$.

4. When a train is moving at $30m/\sec.$ steam is shut off and brakes are applied. The speed of the train per second after t seconds is given by

Time t:	0	5	10	15	20	25	30	35	40
Speed v:	30	24	19.5	16	13.6	11.7	10	8.5	7.0

Using Simpson's rule, determine the distance moved by the train in 40 seconds.

5. Evaluate $\int_0^1 e^{-x^2}\,dx$ a) dividing the range into four equal parts

b) dividing the range into ten equal parts by

i) Trapezoidal rule and

ii) Simpson's one-third rule

6. Evaluate $\int_0^{\frac{\pi}{2}} e^{\sin x} dx$ taking $h = \frac{\pi}{6}$.

7. Calculate $\int_0^{\pi} \sin^3 x dx$ taking $h = \frac{\pi}{6}$.

8. Evaluate $\int_3^7 x^2 \log x dx$ taking four strips.

9. Calculate $\int_{0.5}^{0.7} e^{-x} \sqrt{x} dx$ taking 5 ordinates by Simpson's rule.

10. Evaluate $\int_0^{0.5} \frac{dx}{\sqrt{1-x^2}} dx$ by Weddle's rule, dividing the range into six parts.

11. Evaluate $\int_0^{\pi} \frac{\sin x}{x} dx$ dividing into six equal parts using Simpson's rule, Weddle's rule and Trapezoidal rule.

Chapter Two

Curve Fitting

Fitting of curves to a set of numerical data is of considerable importance- theoretical as well as practical. Theoretically it is useful in the study of correlation and regression. In practice it enables us to represent the relationship between two variables by simple algebraic expressions (polynomials, exponential or logarithmic functions or any). Besides, it may be used to estimate the values of one variable which would correspond to the specified values of the other variable(s).

This chapter covers how to fit a curve for a given set of data points using different methods and it focuses on the following points:

Regression

- Linear regression
- quadratic regression
- polynomial regression
- multiple regression
- fitting an exponential curve
- curve fitting with Sinusoidal Functions

In most of the fields of engineering and science, we come across experiments which involve many variables, and most of the time data is collected or given for discrete values along a continuum; the relation between these variables can be discussed so easily and for many of these variables it is very difficult to identify the relation unless we can model the system mathematically. When the system is explained in terms of mathematical models we have the following relationships about the variables:

1. The relationship between these variables is given in terms of

mathematical rules, formulae if any, to determine the quantities of these variables. Actually it is simple to use these rules for application.

2. The quantities/ variables are given so that we will be interested in finding the relationships between these variables. This process is a little bit difficult because to write one variable in terms of the other variables (called empirical equation). Most of the time we may not be able to get an exact relation between these variables and we may get only an approximate relation or curve.

This approximating curve is an empirical equation and the method of finding such an approximating curve is called ***curve fitting***.

Suppose (x_i, y_i), $i = 1,2,3,...,n$ be n sets of observations and the law relating x and y can be determined by different mathematical systems that clearly explains the relationship between these sets of n observations (x_i, y_i). Actually, here we may have different approaches to fit the given data, and one system may approximate better than the other system on the same given set of data points.

Now we will see some of these different approaches:

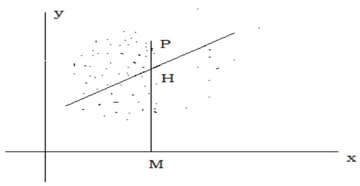
<u>REGRESSION</u>

1. LINEAR REGRESSION

Suppose that the relationship is given

$$y_i = a + bx_i, \quad i = 1,2,3,...,n. \tag{1}$$

Equation (1) represents a family of straight lines for different values of the arbitrary constants 'a' and 'b'. The problem now is to determine 'a' and 'b' so that the line (1) is the line of "best fit". The term best fit is interpreted in accordance with the Legendre's principle of least squares which consists of the deviations of the actual values as given by the line of best fit. As a matter of chance all the points may lie on a straight line and in this case the line is a 'perfect fit' and the sum of the squares of the deviations is zero.



Let $P_i(x_i, y_i)$ be any point in the scatter diagram. Draw P_iM perpendicular to the x-axis meeting the line $y = a + bx$ in H_i. The coordinates of H_i are $(x_i, a + bx_i)$.

$$P_iH_i = P_iM - H_iM$$
$$= y_i - (a + bx_i)$$

P_iH_i is called the error of estimates or the residual for y_i.

According to the principle of least squares, we have to determine a and b so that

$$E = \sum_{i=1}^{n} P_iH_i^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2 \text{ is the minimum.}$$

Using the principle of maxima and minima what we have studied in calculus, the partial derivatives of E with respect to a and b should vanish separately.

That is, $\dfrac{\partial E}{\partial a} = -2\sum_{i=1}^{n} (y_i - a - bx_i) = 0$

$$\Rightarrow \quad \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} a + b\sum_{i=1}^{n} x_i$$
$$= na + b\sum_{i=1}^{n} x_i \qquad (i)$$

.

$$\frac{\partial E}{\partial b} = -2\sum_{i=1}^{n} x_i(y_i - a - bx_i) = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^{\,2} \qquad (ii)$$

Equations (i) and (ii) are called ***normal equations***.

Solving for a and b from (i) and (ii), we get the values of a and b, and with these values of a and b so obtained, equation (1) is the line of best fit to the given set of points (x_i, y_i), $i = 1,2,3,\ldots,n$.

Now let us see some examples to illustrate the above discussion.

Example 1. By the method of least squares find the best fitting straight line to the data given below:

x:	5	10	15	20	25
y:	15	19	23	26	30

Solution

Let the line of best be $y = a + bx$

The normal equations are

$$\sum_{i=1}^{n} y_i = 5a + b\sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^{\,2}$$

We calculate $\sum x, \sum y, \sum x^2, \sum xy$ and form the table below

x	y	x^2	xy
5	15	25	75
10	19	100	190
15	23	225	345
20	26	400	520
25	30	625	750
75	114	1375	1885

Using these values in the normal equations, we get

$$5a + 75b = 114$$
$$75a + 1375b = 1885$$

Solving for a and b we get a=12.3 and b=0.7 and thus the line of best fit is $y = 12.3 + 0.7x$

Example 2. Find the best fitting straight line to the data given below

by the method of least squares and also estimate y when

x is 70.

x:	71	68	73	69	67	65	66	67
y:	69	72	70	70	68	67	68	64

Solution

First transform the values of x and y to $X = x - 68$ and $Y = y - 70$

and the normal equations are

$$b\sum X + 8a = \sum Y$$
$$b\sum X^2 + a\sum X = \sum XY$$

Calculations:

x	y	X	Y	X^2	XY
71	69	3	-1	9	-3
68	72	0	2	0	0

73	70	5	0	25	0
69	70	1	0	1	0
67	68	-1	-2	1	2
65	67	-3	-3	9	9
66	68	-2	-2	4	4
67	64	-1	-6	1	6
		2	-12	50	18

Substituting these values in the normal equations, we get

$$2b + 8a = -12$$
$$50b + 2a = 18$$

Solving for a and b, we get $b = \frac{35}{99}$ and $a = \frac{16}{99}$

Thus the line of best fit is of the form $Y = \frac{35}{99}X + \frac{16}{99}$

This implies $y - 70 = \frac{35}{99}(x - 68) + \frac{16}{99}$

$$y = \frac{35}{99}x + \frac{4566}{99}$$

When $x = 70 \Rightarrow y = 70.87$

1. Quadratic Regression (Fitting Of Second Degree Parabola)

Let $y = a + bx + cx^2$ be the second degree parabola of best fit to set of n points (x_i, y_i),

$i = 1,2,3,..., n$.

Using the principle of least squares, we have to determine $a, b,$ and c so that

$$E = \sum_{i=1}^{n}\left(y_i - a - bx_i - cx_i^2\right)^2 \text{ is minimum.}$$

Equating to zero the partial derivatives of E with respect to $a, b,$ and c separately, we get the normal equations for estimating $a, b,$ and c as

$$\frac{\partial E}{\partial a} = -2\sum_{i=1}^{n}\left(y_i - a - bx_i - cx_i^2\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} x_i^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{\partial E}{\partial b} = -2\sum_{i=1}^{n}\left(y_i - a - bx_i - cx_i^2\right)x_i = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i^3 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\frac{\partial E}{\partial c} = -2\sum_{i=1}^{n}\left(y_i - a - bx_i - cx_i^2\right)x_i^2 = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i^2 y_i = a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^4 \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Solving for $a, b,$ and c from (1), (2) and (3), we get with these values of $a, b,$ and c the parabola of best fit.

Example 1 Fit a parabola of second degree to the following data

X:	0	1	2	3	4
Y:	1	1.8	1.3	2.5	6.3

Solution

X	Y	X^2	X^3	X^4	XY	X^2 Y
0	1	0	0	0	0	0
1	1.8	1	1	1	1.8	1.8
2	1.3	4	8	16	2.6	5.2

3	2.5	9	27	81	7.5	22.5
4	6.3	16	64	256	25.2	100.8
10	12.9	30	100	354	37.1	130.3

Substituting these values in the normal equations, we get

$$12.9 = 5a + 10b + 30c$$
$$37.1 = 10a + 30b + 100c$$
$$130.3 = 30a + 100b + 354c$$

Solving for $a, b,$ and c, we get

$a = 1.42, b = -1.07$, and $c = 0.55$

$\therefore\ y = 1.42 - 1 - 07x + 0.55x^2$ is the best fit.

Exercise

1. Find the best fitting parabola to the data given below by the

   method of least squares and also estimate y when x is 70.

x:	71	68	73	69	67	65	66	67
y:	69	72	70	70	68	67	68	64

2. Polynomial Regression Fitting Of A Polynomial Of kth Degree

If $y = a_0 + a_1 x + a_2 x^2 + ... + a_k x^k$ is the kth degree polynomial of best fit to the set of points (x_i, y_i); $i = 1,2,3,..., n$ the constants $a_0, a_1, a_2, ..., a_n$ are to be obtained so that

$$E = \sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_i - a_2 x_i^2 ... a_k x_i^k \right)^2 \text{ is minimum.}$$

Thus the normal equations for estimating $a_0, a_1, a_2, ..., a_n$ are obtained on equating to zero the partial derivatives of E with respect to $a_0, a_1, a_2, ..., a_n$ separately.

$$\frac{\partial E}{\partial a_0} = 0 \Rightarrow \sum y_i = na_0 + a_1 \sum x_i + a_2 \sum x_i^2 + ... + a_k \sum x_i^k$$

$$\frac{\partial E}{\partial a_1} = 0 \Rightarrow \sum_{i=1}^{n} x_i y_i = a_0 \sum x_i + a_1 \sum x_i^2 + ... + a_k \sum x^{k+1}$$

$$\cdots$$

$$\frac{\partial E}{\partial a_k} = 0 \Rightarrow \sum_{i=1}^{n} x_i^k y_i = a_0 \sum x_i^k + a_1 \sum x_i^{k+1} + ... + a_k \sum x_i^{2k}$$

Exercise

1. Find the polynomial of degree three that best fits the data given below by the method of least squares and also estimate y when x is 70.

x:	71	68	73	69	67	65	66	67
y:	69	72	70	70	68	67	68	64

4. Multiple Regressions

There are different multiple regression forms. For the sake of discussion let us see the following regression type.

Suppose we want to fit the set of data points by the relation

$Z = ax + bxy + cy$, here we need the points to be of the form (x_i, y_i, z_i).

We determine the values of $a, b,$ and c, from

$$E = \sum_{i=1}^{n} (z_i - ax_i - bx_i y_i - cy_i)^2$$

$$\frac{\partial E}{\partial a} = -2 \sum_{i=1}^{n} (z_i - ax_i - bx_i y_i - cy_i) x_i = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i z_i = a \sum_{i=1}^{n} x_i^2 + b \sum x_i^2 y_i + c \sum x_i y_i \text{..............................} (1)$$

$$\frac{\partial E}{\partial b} = -2\sum_{i=1}^{n}(z_i - ax_i - bx_iy_i - cy_i)x_iy_i$$

$$\Rightarrow \sum_{i=1}^{n} z_i x_i y_i = a\sum_{i=1}^{n} x_i^2 y_i + b\sum_{i=1}^{n} x_i^2 y_i^2 + c\sum_{i=1}^{n} x_i y_i^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\frac{\partial E}{\partial b} = -2\sum_{i=1}^{n}(z_i - ax_i - bx_iy_i - cy_i)y_i$$

$$\Rightarrow \sum_{i=1}^{n} z_i y_i = a\sum_{i=1}^{n} x_i y_i + b\sum_{i=1}^{n} x_i y_i^2 + c\sum_{i=1}^{n} y_i^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

By solving (1), (2) and (3) for $a, b,$ and c, we get the best approximation.

5. **Fitting an Exponential Curve**

Let (x_i, y_i), $i = 1,2,3,\ldots,n$ be the n sets of observations of related data and let $y = ab^x$ be the best fit for the data.

Then taking logarithm on both sides,

$$\log_{10}^{y} = \log_{10}^{a} + x\log_{10}^{b} \qquad (*)$$

Let $Y = \log_{10}^{y}$, $A = \log_{10}^{a}$, and $B = \log_{10}^{b}$, then (*) reduces to

$Y = A + Bx$ which is linear in x and Y, we can find A, B since x and Y are known, and from A, B we can get a, b and hence $y = ab^x$ is found out.

Fitting a curve of the form $y = ax^b$

$$y = ax^b$$

$$\Rightarrow \log_{10}^{y} = \log_{10}^{a} + b\log_{10}^{x}$$

$$\Rightarrow Y = A + bx, \quad letting \quad y = \log_{10}^{y}, \quad A = \log_{10}^{a}, \quad x = \log_{10}^{x}$$

Using this linear fit, we find A, b .

$\Rightarrow\ a, b$ are known and thus $y = ax^b$ is found out.

Example 1. From the table given below, find the best values of

a and b in the law $y = ae^{bx}$ by the method of least squares.

x :	0	5	8	12	20
y :	3.0	1.5	1.0	0.55	0.18

Solution

Let $y = ae^{bx}$ be the approximating curve.

$\Rightarrow Y = A + bx\log_{10}^{e} \Rightarrow Y = A + Bx$ where B = b $\log_{10}^{e}$

So the normal equations are

$$B\sum x + 5A = \sum Y$$
$$B\sum x^2 + A\sum x = \sum xY$$

x	y	Y	x^2	xY
0	3.0	0.4771	0	0
5	1.5	0.1761	25	0.8805
8	1.0	0	64	0
12	0.55	-0.2596	144	-3.1152
20	0.18	-0.7447	400	-14.894
45		-0.3511	633	-17.1287

Substituting these values, we get

$$5A + 45B = -0.3511$$
$$45A + 633B = -17.1287$$

Solving for A and B, we get

$A = 0.4815$

$B = -0.0613$

So $a = 10^A = 10^{0.4815} = 3.0304$

$B = b\log_{10}^{e} = -0.0613$

$\Rightarrow b = (-0.0613)\log_{e}^{10} = -0.1411$

Hence the curve is $y = 3.0304e^{-0.1411x}$

Exercise

1. Fit a straight line to the following data and hence find $y(x = 25)$

x:	0	5	10	15	20
y:	7	11	16	20	26

2. Fit a straight line to the data

x:	0.5	1.0	1.5	2.0	2.5	3.0
y:	0.31	0.82	1.29	1.85	2.51	3.02

3. Fit a parabola to the data

x:	1	2	3	4	5
y:	2	3	5	8	10

4. Fit a curve of the form $y = ae^{bx}$ to the data given below:

x:	1	2	3	4	5	6	7	8
y:	15.3	20.5	27.4	36.6	49.1	65.6	87.8	117.6

5. Fit a curve of the form $z = ax + by + cxy$ to the data given below

(0,0,1), (0,1,2), (1,0,4), (1,1,1), (2,0,4), (1,2,5)

6. It is given that x, and y are related by $y = \frac{a}{x} + bx$ to the data below and obtain the best values of a and b.

x :	1	2	4	6	8
y :	5.43	6.28	10.32	14.86	19.5

Chapter Three

Numerical Solution of Ordinary Differential Equations

In this chapter we are highly interested in finding the solution of ordinary differential equation numerically using different methods.

The topics covered here is

- Taylor series method
- Taylor series method for simultaneous first order DE
- Taylor series method for second order differential equation
- Picard's method of successive approximations
- Euler's Method
- Runge-Kutta method
- Predictor-corrector method

In the fields of Engineering and Science, we come across through the natural phenomena that can be represented by mathematical models which happen to be in the form of differential equations; for instance, the equation of motion, the equation of deflection of a beam, etc. The solution of these differential equations is very essential in the studies of such phenomena.

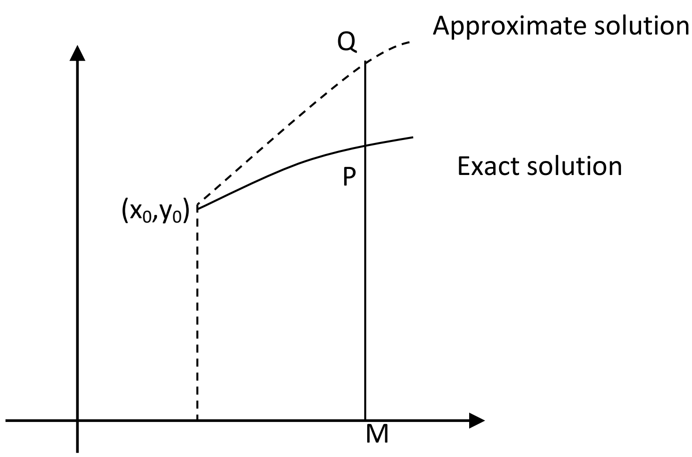
While finding the solution of these differential equations there are number of differential equations that we cannot solve analytically; however, in such situations, depending on the nature of the model, we go for numerical solutions of these differential equations. In many researches, especially after the advent of modern computers, the numerical solutions of the differential equations have become very easy for manipulation.

Thus, in this part we try to look some of the methods of numerical solutions that are approximate solutions and in many cases these solutions are in the required (desired) degree of accuracy and are quite sufficient.

Suppose we want to solve $\frac{dy}{dx} = f(x, y)$ with the initial condition $y(x_0) = y_0$.

Let $y(x_0) = y_0$, $y(x_1) = y_1$, $y(x_2) = y_2$, … be the solution of y at $x = x_0, x_1, x_2, ...$

Let $y = y(x)$ be the exact solution. If we plot and draw the graph of $y = y(x)$, (the exact curve) and also draw the approximate curve by plotting $(x_0, y_0), (x_1, y_1), (x_2, y_2), ...$ (the approximated solution graph) we get two curves.



QM = approximate value

PM = exact value at $x = x_i$

Then QP = QM-PM

$= y_i - y(x_i) = \epsilon_i$ is called

Suppose $y' = \frac{dy}{dx} = f(x, y)$........(1)

The equation $y' = f(x, y)$ subject to the initial condition $y(x_0) = y_0$ is called an **initial-value problem**.

Using Taylor series we can expand $y(x)$ in the neighborhood of x_0 as a power series of $x - x_0$. That is, if x is close to x_0, then by Taylor's series, we have

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)(x - x_0)^2}{2!} + \frac{y'''(x_0)(x - x_0)^3}{3!} + ... \qquad (*)$$

where $y'(x_0) = \left(\frac{dy}{dx}\right)_{x=x_0}$, $y''(x_0) = \left(\frac{d^2y}{dx^2}\right)_{x=x_0}$, etc

If $x = x_1$ is close to x_0, substitute $x = x_1$ in ($*$) and get $y_1 = y(x_1)$

Again starting from x_1, express $y(x)$ in a power series of $x - x_1$ and then substitute $x = x_2$ to get $y_2 = y(x_2)$. In this way we can get the sequence of y values $y_0, y_1, y_2,...$

If $x = x_0 = 0$, we get the Maclaurin's series expansion,

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!} y''(0) + ...$$

Example 1. Evaluate the solution of the differential equation $\frac{dy}{dx} = y^2 + 1$

by taking four terms of its Maclaurin's series for $x = 0$, $x = 0.2$, $x = 0.6$ given $y(0) = 0$ and compare this result with its exact solution.

Solution

$y' = y^2 + 1$	$y'(0) = 1$
$y'' = 2y\,y'$	$y''(0) = 2y(0)y'(0) = 0$
$y''' = 2y'^2 + 2yy''$	$y'''(0) = 2$
$y^{[4]} = 2yy''' + 6y'\,y''$	$y^{[4]}(0) = 0$

By Maclaurin's series, we have

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + ...$$

$$= 0 + x + 2\frac{x^3}{3!} + \frac{16x^5}{5!} + ...$$

$y(0) = 0$

$$y(0.2) = 0.2 + \frac{(0.2)^3}{3} + \frac{2}{15}(0.2)^5 \approx 0.2 + \frac{0.008}{3} + \frac{2}{15}(0.00032) \approx 0.4213$$

$$y(0.6) = 0.6 + \frac{(0.6)^3}{3} + \frac{2}{15}(0.6)^5 \approx 0.6720$$

Exact solution $\int \frac{dy}{y^2+1} = \int dx \Rightarrow \tan^{-1} y = x + c \quad \Rightarrow y = \tan x$

$$\tan 0 = 0, \tan o.2 = 0.2927, \tan 0.6 = 0.6942$$

Let us compare the actual value with the approximate value

Values of x	0	0.2	0.4	0.6
Actual value of y	0	0.2027	0.4228	0.6841
Approximate value of y	0	0.2027	0.4213	0.6720
Error	0	0	0.0015	0.0121
Percentage of error	0	0	0.35	1.77

This table shows that when the distance of x from x_0 increases the error also increases.

In this example, we have expanded $y(x)$ in the neighborhood of $x = 0$ and used the same result to find $y(x)$ when $x = 0.2$, and $x = 0.6$.

Now instead of doing this, after getting $y(x_1) = y(0.2)$, expand $y(x)$ again in the neighborhood of $x = 0.2$ and use this result to get $y(0.6)$. In doing so, we can minimize the error.

Thus in the neighborhood of $x = 0.2$

$$y(x) = y(0.2) + y'(0.2)(x - 0.2) + \frac{y''(0.2)(x-0.2)^2}{2!} + \frac{y'''(0.2)(x-0.2)^3}{3!} + \ldots$$

$$y(0.2) = 0.2027$$

$$y'(0.2) = (y(0.2))^2 + 1 = (0.2027)^2 + 1 = 1.0411$$

$$y''(0.2) = 2y(0.2)y'(0.2) = 2(0.2027)(1.0411) = 0.4221$$

$$y'''(0.2) = 2[y'(0.2)]^2 + 2y(0.2)y''(0.2)$$

$$= 2[1.0411]^2 + 2(0.2027)(0.4221) = 2.3389, \quad \ldots, \text{etc.}$$

Putting these values and using $x = 0.4$ in (1), we get

$$y(0.4) = 0.2027 + 1.0411(0.4 - 0.2) + \frac{0.4221}{2}(0.4-0.2)^2 + \frac{2.3389}{6}(0.4-0.2)^3 + \ldots$$

$$= 0.2027 + (0.2)(1.0411) + (0.04)(0.21105) + (0.008)(0.389871) + \ldots$$

$$= 0.422480536 \approx 0.4225$$

When we compare this value with the actual one i.e. 0.4228, we see that the error is only 0.0003, nearly 0.07%

The error has decreased from 0.35% to 0.07%

Therefore, to reduce the error, each time obtain the power series of $y(x)$ at $x = x_{i+1}$ and use this to get $y(x_{i+2})$ and so on.

This method is called the **method of starting the solution.**

Point Wise Methods

Consider the previous example $y' = y^2 + 1, \quad y(0) = 0$

First we got $y(x) = x + \frac{x^3}{3} + \ldots$ in terms of x and then we substituted

$x = x_1 = 0.2$. Instead, without getting $y(x)$ as a function of x we can directly get $y(x_1) = y(0.2)$ as

$$y(0.2) = y(0) + y'(0)(0.2) + \frac{y''(0)}{2!}(0.2)^2 + \frac{y'''(0)(0.2)^3}{3!} + \ldots$$

That is we get $(x_1, y_1), (x_2, y_2)$ directly. So, a point wise solution is a series of points $(x_1, y_1), (x_2, y_2), \ldots$ which satisfy approximately a pre-assigned but not known particular solution.

Solution Using Taylor Series

AIM- To find the numerical solution of the equation

$$\frac{dy}{dx} = f(x, y) \text{ given the initial condition } y(x_0) = y_0 \ldots\ldots\ldots\ldots (1)$$

Now, we expand $y(x)$ about the point $x = x_0$ using Taylor's series in powers of $x - x_0$.

That is,

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \ldots$$

Where $y^{[r]}(x_0) = \left(\frac{d^r y}{dx^r}\right)_{x=x_0}$

$$y_1 = y(x_1) = y_0 + hy_0^1 + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots \qquad (2)$$

, where $h = x_1 - x_0$ or $x_1 = x_0 + h$

To find $y_0', y_0'', \ldots$ we use (1) and its derivatives at $x = x_0$.

Even though the series in (2) is an infinite series, we can truncate it at any convenient term, if h is small, and the accuracy can be obtained.

Now, once if we get y, we can calculate $y_1', y_1'', \ldots$, etc by using

$$y' = f(x, y)$$

Again, expanding $y(x)$, in a Taylor's series about the point

$x = x_1$, we get

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \ldots$$

Proceeding in the same way, we get

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \ldots$$

Since this is an infinite series, to get an approximate value we have to truncate it at some term to have a calculated numerical value.

Now, let us consider the terms up to and including h^n and neglect terms involving h^{n+1} and higher powers of h. The Taylor algorithm used this way is said to be of ***n^th^ order***.

Thus, the truncation error is $O(h^{n+1})$. If h is small enough we can neglect terms after the n^{th} term and get the error as

$$\frac{h^n}{n!} f^{[n]}(\theta) \text{ where } x_0 < \theta < x_1 \quad if \quad x_1 - x_0 = h$$

Example 1. Solve $\frac{dy}{dx} = x + y$, given $y(1) = 0$, and find $y(1.1), y(1.2)$ by using Taylor's method.

Solution

Here $x_0 = 1$ and $y_0 = 0, \quad h = 0.1$

$$\frac{dy}{dx} = y' = x + y \qquad\qquad y_0 = y(x = 1) = 0$$

$y'' = y' + 1 \qquad y_0' = x_0 + y_0 = 1 + 0 = 1$

$y''' = y'' \qquad y''(0) = 1 + 1 = 2$

$y'''(0) = 2, \quad etc$

Thus by Taylor's method, we have

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots$$

$$y(1.1) = 0 + (0.1)(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(2) + \frac{(0.1)^5}{5!}(2) + \ldots$$

$$= 0.1 + 0.01 + 0.00033 + 0.00000833 + 0.000000166 + \ldots$$

$$\approx 0.11033847$$

Now again take $x_0 = 1.1$ and $h = 0.1$

$$y_1' = x_1 + y_1 = 1.1 + 0.11033847 = 1.21033847$$

$$y_1'' = 1 + y_1' = 2.21033847$$

$$y_1''' = y_1'' = 2.2103384$$

$$y_1''' = y_1^{[4]} = y_1^{[5]} = \ldots$$

$$\therefore y_2 = 0.11033847 + (0.1)(1.21033847) + \frac{(0.1)^2}{2}(2.21033847) + \frac{(0.1)^3}{3!}(2.21033847) + \frac{(0.1)^4}{4!}(2.21033847) + \ldots$$

$$= 0.11033847 + 0.121033847 + 2.21033847 + (0.005) + (0.0016667)(2.21033847) + \ldots$$

$$\approx 0.2461077$$

Let us check for its exact solution

$$\frac{dy}{dx} = x + y$$

Let $x + y = v \Rightarrow \frac{dv}{dx} = 1 + \frac{dy}{dx}$

$$\Rightarrow \frac{dv}{dx} = v + 1 \quad \text{and} \quad \Rightarrow dx = \frac{dv}{v+1}$$

Integrating both sides,

$$x + c = \ell n|v + 1|$$

$$\Rightarrow x + c = \ell n|x + y + 1|$$

$$\Rightarrow x + y + 1 = e^{x+c}$$

Since $y(1) = 0$

$$\Rightarrow 2 + 0 = e^{c+1}$$

$$\Rightarrow c + 1 = \ell n 2 \quad \Rightarrow c = \ell n 2 - 1$$

$$\therefore y = -x - 1 + 2e^{x-1}$$

Hence $y(1.1) = -1.1 - 1 + 2e^{1.1-1} = 0.110341836$

$$y(1.2) = -1.2 - 1 + 2e^{1.2-1} = 0.242805516$$

Example 2. Using Taylor series method, compute $y(0.1)$ correct to four decimal places, given $\frac{dy}{dx} = x^2 + y^2$ and $y(0) = 1$.

Solution

$\frac{dy}{dx} = y' = x^2 + y^2$ here $x_0 = 0$ and $y_0 = 1$, $h = 0.1$

$$y'(0) = 0^2 + 1^2 = 1$$

$$y'' = 2x_0 + 2y_0 y_0' = 2$$

$$y_0''' = 2 + 2y_0' + 2y_0 y_0'' = 8$$

$$y_0^{[4]} = 4y_0' y_0'' + 2y_0' y_0'' + 2y_0 y_0''' = 6y_0' y_0'' + 2y_0' y_0'' = 28$$

$$y(0.1) = 1 + \frac{0.1}{1}(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{3!}(8) + \frac{(0.1)^4}{4!}(28) + \ldots$$

$$= 1 + 0.1 + 0.01 + 0.0013333333 + 0.000116667 \approx 1.11144999$$

$$\approx \underline{\underline{1.11145}}$$

Example 3. Using Taylor method, compute $y(0.2)$ and $y(0.4)$

correct to four decimal places given

$$\frac{dy}{dx} = 1 - xy \text{ and } y(0) = 0$$

Solution

$y' = 1 - 2xy$	$y_0' = 1 - 2x_0 y_0 = 1$
$y'' = -2(y + xy')$	$y_0'' = -2(0 + 0(1)) = 0$
$y''' = -2(y' + y' + xy'')$	$y_0''' = -2(2 + 0) = -4$
$y^{[4]} = -2(3y'' + xy''')$	$y_0^{[4]} = 0$
$y^{[5]} = -2(4y''' + xy^{[4]})$	$y_0^{[5]} = 32,$ etc

By Taylor's series, we have

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{[4]} + \ldots$$

$$y(0.2) = 0 + (0.2)(1) + \frac{(0.2)^2}{2!}(0) + \frac{(0.2)^3}{3!}(-4) + \frac{(0.2)^4}{4!}(0) + \frac{(0.2)^5}{5!}(33) + \ldots$$

$$= 0.2 - 0.00533333 + 0.000085333$$

≈ 0.194752003

Now again starting with $x = 0.2$, we have

$$x = 0.2,\; y_0 = 0.194752003,\; h = 0.2$$

$$y_0' = 1 - 2x_0y_0 = 1 - 2(0.2)(0.194752003) = 0.9220992$$

$$y_0'' = -2(x_0y_0' + y_0) = -2[(0.2)(0.9220992) + 0.194752063] = -0.758343806$$

$$y_0''' = -2(2y_0' + x_0y_0'') = -2[(0.2)(-0.7583436806) + 2(0.9220992)]$$

$$= -3.38505933$$

$$y_0^{[4]} = -2[(0.2)(-3.38505933) + (-0.758343686)] = 5.90408585$$

$$\therefore\; y(0.4) = 0.194752003 + (0.2)(0.9220992) + \frac{(0.2)^2}{2}(-0.758343686) + \frac{(0.2)^3}{3!}(-3.38505933) + \frac{(0.2)^4}{4!}(5.90408585) + \dots$$

$$\approx 0.359883723$$

Example 4. Using Taylor series method, find y at $x = 0.1, x = 0.2$ and $x = 0.4$ given $\frac{dy}{dx} = x^2 - y,\; y(0) = 1$ (correct to five decimal places)

Solution

Here $x_0 = 0, y_0 = 1, h = 0.1, x_1 = 0.1, x_2 = 0.2, \dots$

$y' = x^2 - y$ $\qquad$ $y_0' = -1$

$y'' = 2x - y'$ $\qquad$ $y_0'' = 1$

$y''' = 2 - y''$ $\qquad$ $y_0''' = 2 - 1 = 1$

$y^{[4]} = -y'''$ $\qquad y_0^{[4]} = -1$

$y^{[5]} = -y^{[4]}$ $\qquad 0y_0^{[5]} = 1, \quad etc$

$$y(0.1) = 1 + (0.1)(-1) + \frac{(0.1)^2}{2}$$

$$= 1 + (0.1)(-1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{3!}(1) + \frac{(0.1)^4}{4!}(-1) + \frac{(0.1)^5}{5!}(1) + \ldots$$

= 1+ (-0.1) + 0.005 + 0.0001666 + (-0.0000416) +…

= 0.905125

Now again using $x_1 = 0.1$ and $y_1 = 0.905125$, we have

$y_1' = x_1^2 - y_1 = 0.01 - 0905125 = -0.895125$

$y_1'' = 2x_1 - y_1' = 0.2 - (-0.895125) = 1.095125$

$y_1''' = 2 - y_1'' = 2 - 0905125 = 0.904875$

$y_1^{[4]} = -y_1''' = -0.904875$, etc.

$$\therefore y(0.2) = 0.905125 + (0.1)(-0.895125) + \frac{(0.1)^2}{2}(1.095125) +$$

$$\frac{(0.1)^3}{3!}(0.904875) + \frac{(0.1)^4}{4!}(-0.904875) + \ldots = 0.821235167$$

Similarly, $y(0.3) = 0.7492$ and $y(0.4) = 0.6897$

Taylor series method for simultaneous first order differential equations

The equation of the type

$$\frac{dy}{dx} = f(x, y, z) \text{ and } \frac{dz}{dx} = g(x, y, z)$$

with initial conditions

$$y(x_0) = y_0, \quad z(x_0) = z_0$$

can be solved by Taylor series method as given below:

Example 5. Solve $\frac{dy}{dx} = z - x, \frac{dz}{dx} = y + x$ with $y(0) = 1, z(0) = 1$ by taking

$h = 0.1$, to get $y(0.1)$ and $z(0.1)$.

Solution

$y' = z - x$ and $z' = x + y$

$x_0 = 0$ and $y_0 = 1$ $x_0 = 0$, $z_0 = 1$ and $h = 0.1$

$y_1 = y(0.1) = ?$ $z_1 = z(0.1) = ?$

$y' = z - x$ $z'' = 1 + y'$

$y'' = z' - 1$ $z''' = y''$ *etc*

$y''' = z''$

Using Taylor series, for y_1 and z_1, we have

$$y_1 = y(0.1) = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + ... \quad \ldots\ldots \quad (1)$$

and $$z_1 = z(0.1) = y_0 + hy_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + ... \quad ..... \quad (2)$$

$y_0 = 1$ $z_0 = 1$

$y_0^1 = z_0 - x_0 = 1$ $z_0' = x_0 + y_0 = 0 + 1 = 1$

$y_0'' = z_0' = 1 - 1 = 0$ $z_0'' = 1 + y_0' = 1 + 1 = 2$

$y_0''' = z_0'' = 2$ $z_0''' = y_0'' = 0$

$z_0^{[4]} = y_0''' = 2$

Substituting these in (1) and (2), we get

$$y_1 = 1 + 0.1 + \frac{0.01}{2}(0) + \frac{0.001}{6}(2) + \frac{0.0001}{24}(0) + ...$$

= 1+0.1+0.000333+…

= 1.1003

$$z_1 = 1 + 0.1 + \frac{0.01}{2}(2) + \frac{0.001}{6}(0) + \frac{0.0001}{24}(2) + ...$$

=1+ 0.1 + 0.01+0.0000083+… = 1.1100

Example 6. Find $y(0.1), y(0.2), z(0.2)$ given

$$\frac{dy}{dx} = x + z, \quad \frac{dz}{dx} = x - y^2 \text{ and } y(0) = 2,\ z(0) = 1$$

Solution

Here $x_0 = 0,\ y_0 = 2$ and $z_0 = 1$

$y' = x + z$ $\qquad z' = x - y^2$

$y'' = 1 + z'$ $\qquad z'' = 1 - 2yy'$

$y''' = z''$ $\qquad z''' = -2(y')^2 - 2yy''$

$y^{[4]} = z'''$ *etc* $\qquad z^{[4]} = -4y'y'' = 2y'y'' - 2yy''' = -6y'y'' - 2yy''$ etc

$y_0' = y'(0) = x_0 + z_0 = 1$ $\qquad z_0' = 0 - 2^2 = -4$

$y_0' = 1 + (-4) = -3$ $\qquad z_0'' = 1 - 2(2)(1) = -3$

$y_0'' = -3$ $\qquad z_0''' = -2(1)^2 - 2(2)(-3) = 10$

$y_0^{[4]} = 10$ $\qquad z_0^{[4]} = -6(1)(-3) - 2(2)(-3) = 30$

$y_0^{[5]} = 30$ $\qquad z_0^{[5]} = 74$

$$y(0.1) = 2 + (0.1)(1) + \frac{(0.1)^2}{2}(-3) + \frac{(0.1)^3}{6}(-3) + \frac{(0.1)^4}{24}(10) + \frac{(0.1)^5}{120}(30) + ...$$

$= 2.084544167$

$$z(0.1) = 1 + (0.1)(-4) + \frac{(0.1)^2}{2}(-3) + \frac{(0.1)^3}{6}(10) + \frac{(0.1)^4}{24}(30) + \frac{(0.1)^5}{120}(-74) + \ldots$$

$= 2.084544167+0.267132966-0.016226621-0.0003105449027$

$- (0.000003091501458) +\ldots$

$$y(0.2) = 0.5867855 + (0.1)(-4.245324384) + \frac{(0.1)^2}{2}(-1.863269416) + \frac{(0.1)^3}{6}(-0.74196035) + \frac{(0.1)^4}{24} + \ldots$$

$=0.152817221$

Taylor series method for second order differential equation

Any differential equation of the second order or higher can be solved by reducing it to a lower order differential equation. A second order differential equation can be reduced to a first order differential equation by transforming $y' = z$ and then the given equation can be solved so easily.

Suppose $\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$ …… (1)

$y'' = f(x, y, y')$ is the given differential equation together with the given initial conditions

$y(x_0) = y_0$ and $y'(x_0) = y_0'$ where y_0, y_0' are known values at x_0.

Setting $y' = p$, we get $y'' = p'$ and (1) becomes

$p' = f(x, y, p)$

with initial condition $y(x_0) = y_0$ and $p(x_0) = p_0 = y_0'$

Using Taylor series method, we get

$p_1 = p_0 + hp_0' + \frac{h^2}{2} p_0'' + \frac{h^3}{3!} y_0''' + \ldots$ where $p_1 = p(x = x_1)$ & $x_1 - x_0 = h$ …… (*)

$$y_1 = y_0 + hy_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{3!} y_0''' + \ldots \quad \text{becomes}$$

$$y_1 = y_0 + hp_0 + \frac{h^2}{2!} p_0{}' + \frac{h^3}{3!} p_0{}'' + \ldots \quad \ldots(**)$$

Since $p' = f(x, y, p)$ and taking the derivative of p' again and again with respect to x, we get

$p'', \ p'''$, etc.

Hence $p_0', p_0'', p_0''', \ldots$ can be solved using (**) and (*), so that we can get

y_1 & p_1 .

Once knowing y_1 and p_1 we can get $p_1', p_1'', p_1''', \ldots \quad at \quad (x_1, y_1)$

Again using

$p_2 = p_1 + hp_1' + \frac{h^2}{2!} p_1'' + \frac{h^3}{3!} p_1''' + \ldots$ we get some value for p_2 and using

$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots$ we get still some value for y_2 .

Example 7. Evaluate the values of $y(0.1)$ and $y(0.2)$ given

$y''' - x(y')^2 + y^2 = 0, \ y(0) = 1, y'(0) = 0$ by using Taylor series method.

Solution

First put $y' = z$ and hence the equation reduces to $z' - xz^2 + y^2 = 0$

$$\Rightarrow z' = xz^2 - y^2$$

Using the initial condition $y(0) = 1, z_0 = y_0' = 0$

Now $z' = xz^2 - y^2$ can be solved given that $z_0 = z(0) = 0, \quad and \ x_0 = 0$

Here, $z_1 = z_0 + hz_0' + \frac{h^2}{2} z_0'' + \ldots \quad \ldots(1)$

$z' = xz^2 - y^2$ $\qquad\qquad\qquad\qquad y' = z$

$z'' = z^2 + 2xzz' - 2yy'$ $\qquad\qquad\qquad y'' = z', y''' = z'', ...$

$z''' = 2zz' + 2\left(xzz'' + zz' + x(z')^2\right) - 2\left(yy'' + (y')^2\right)$

$\therefore z_0' = x_0 z_0^2 - y_0^2 = -1$

$z_0'' = 0$ and $z_0''' = 2$

Substituting, these values in (1), we get

$$z_1 = 0 + (0.1)(-1) + \frac{(0.01)}{2}(0) + \frac{0.001}{6}(2) + ...$$

= -0.0997

By Taylor series for y_1, we get

$$y_1 = y(0.1) = y_0 + hy_0' + \frac{h^2}{2}y_0'' + ...$$

$$= 1 + (0.1)(z_0) + \frac{(0.1)^2}{2}\left(z_0'\right) + \frac{(0.1)^3}{6}\left(z_0''\right) + ...$$

$$= 1 + 0.1(0) + \frac{0.01}{2}(-1) + \frac{0.001}{6}(0) + ... = 0.995$$

Similarly,

$$y_2 = y(x_2) = y_1 + hy_1' + \frac{h^2}{2}y_1'' + ...$$

$$= 0.995 + 0.1(z_1) + \frac{0.01}{2}z_1' + \frac{0.001}{6}z_1'' + ... \quad ...(2)$$

$$z_1' = x_1 z_1^2 - y_1^2 = (0.1)(-0.0997)^2 - (0.995)^2 = -0.9890$$

$$z_1'' = -0.1687$$

Using (2),

$$y_2 = 0.995 + (0.1)(-0.0997) + \frac{0.01}{2}(-0.9890) + \frac{0.001}{6}(-0.1687) + ...$$

$= 0.9801$

Example 8. Solve $y'' = y + xy' \quad given \quad y(0) = 0$ and calculate $y(0.1)$.

Solution

Here $x_0 = 0, y_0 = 1, y_0^{'} = 0$ and $y'' = y + xy'$

Differentiating with respect to x,

$$y''' = y' + y' + xy'' = 2y' + xy'' \qquad y_0^{''} = y_0 + x_0 y_0^{'} = 1$$

$$y^{[4]} = 2y'' + y'' + xy''' = 3y'' + xy''' \qquad y_0^{'''} = 2y_0^{'} + x_0 y_0^{''} = 0$$

$$y^{[5]} = 4y''' + xy^{[4]} \qquad y_0^{[4]} = 3$$

$$y^{[6]} = 5y^{[4]} + xy^{[5]} \qquad y_0^{[5]} = 0 \qquad y_0^{[6]} = 15, ...$$

Here $y(x) = y_0 + xy_0^{'} + \frac{x^2}{2} y_0^{''} + \frac{x^3}{3!} y_0^{'''} + ...$

$$= 1 + 0 + \frac{x^2}{2!}(1) + 0 + \frac{x^4}{4!}(3) + ...$$

$$= 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + ...$$

$$= 1 + \frac{(0.1)^2}{2} + \frac{(0.1)^4}{8} + \frac{(0.1)^6}{48} + ... = 1.00501252$$

Exercise

Using Taylor series method, find the values required in each problem.

1. Find $y(0.1)$ given $\frac{dy}{dx} = x + y, \quad y(0) = 1$

2. Find $y(0.1)$ given $y' = x^2 y - 1,\ y(0) = 1$

3. Obtain $y(4.2)$ and $y(4.4)$ given $\frac{dy}{dx} = \frac{1}{x^2 + y},\ y(4) = 4$ taking $h = 0.2$.

4. Find $y(0.1), y(0.2), y(0.3)$ given $y' = \frac{x^3 + xy^2}{e^x},\quad y(0) = 1$.

5. Find $y(0.1), y(0.2), z(0.1), z(0.2)$ given

$$\frac{dy}{dx} = x + z,\ \frac{dz}{dx} = x - y^2 \text{ and } y(0) = 2, z(0) = 1$$

6. Evaluate $x(0.1), y(0.1), x(0.2), y(0.2)$ given

$$\frac{dx}{dt} = ty + 1,\ \frac{dy}{dt} = -tx \text{ given } x = 0, y = 1 \text{ at } t = 0$$

7. Solve for x and y $\frac{x}{dt} = x + y + t,\ \frac{dy}{dt} = 2x - t$ given $x = 0, y = 1$ at $t = 1$.

8. Find y at $x = 1.1,\quad x = 1.2,\quad \text{x} = 1.3$ given $y'' + y^2 y' = x^3,\quad y(1) = 1,\quad y'(1) = 1$

9. Express y as a power series given, $y' = (0.1)(x^2 + y^2),\quad y(0) = 1$

Picard's Method of Successive Approximations

AIM: To solve $\frac{dy}{dx} = f(x, y)$ subject to $y(x_0) = y_0$

Now $\frac{dy}{dx} = f(x, y) \Rightarrow dy = f(x, y)dx$

Integrating, $y = \int^x f(x, y)dx + c$

Setting $x = x_0$, we have

$$y_0 = \int^{x_0} f(x, y)dx + c$$

$$\Rightarrow y - y_0 = \int_{x_0}^{x} f(x, y)dx$$

$\therefore\ y = y_0 + \int_{x_0}^{x} f(x, y)dx$ this type of equation is called an **integral equation**.

As the integration is not possible as it is, we will solve it numerically by successive approximation. Now substitute the initial values of y namely y_0 in the integrand $f(x, y)$ in place of y and then integrate it to get an approximate value of y.

i.e. $y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)dx$

After getting the first approximation $y^{(1)}$ for y, use this value of $y^{(1)}$ in place of y in $f(x, y)$ and then again integrate to get the second approximation of y namely $y^{(2)}$.

Thus $y^{(2)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(1)}\right)dx$

Repeating this procedure again and again, we eventually get

$$y^{(n)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(n-1)}\right)dx$$

This equation is called ***Picard's iteration*** formula. This formula gives the general iterative formula for y.

The sequence $y^{(1)}, y^{(2)}, \ldots, y^{(n)}$ should converge to $y(x)$; otherwise the process is not valid.

The condition for the convergence of the sequence is both $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous.

i.e. $\left|f(x, y)\right| \le k_1 \text{ and } \left|\frac{\partial f}{\partial y}\right| \le k_2$ in a region containing the point (x_0, y_0) where k_1, k_2 are constants.

Example 1. Solve $y' = y - x_2$, $y(0) = 1$, by Picard's method up to the third approximation. Hence find the value of $y(0.1), y(0.2)$.

Solution

$$y' = y - x^2$$

$$\therefore y = y_0 + \int_{x_0}^{x} \left(y - x^2\right) dx....$$

Hence, $x_0 = 0$, $y_0 = 1$ and $f(x, y) = y - x^2$

$$\Rightarrow y = 1 + \int_0^x \left(y - x^2\right) dx \;.....(1)$$

$$y^{(1)} = 1 + \int_0^x \left(1 - x^2\right) dx = 1 + x - \frac{x^3}{3}$$

Using $y^{(1)}$ again in (1), we get

$$y^{(2)} = 1 + \int_0^x \left(y^{(1)} - x^2\right) dx$$

$$= 1 + \int_0^x \left(1 + x - \frac{x^3}{3} - x^2\right) dx$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{12}$$

Using again this result, we get

$$y^{(3)} = 1 + \int_0^x \left(y^{(2)} - x^2\right) dx$$

$$= 1 + \int_0^x \left(1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{12} - x^2\right) dx$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{60}$$

Now put $x = 0.1$

$$y(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} - \frac{(0.1)^3}{6} - \frac{(0.1)^4}{12} - \frac{(0.1)^5}{60}$$

$= 1.104824833$

$$y(0.2) = 1 + 0.2 + \frac{(0.2)^2}{2} - \frac{(0.2)^3}{6} - \frac{(0.2)^4}{12} - \frac{(0.2)^5}{60}$$

$= 1.218528$

Note- In getting the value $y(0.2)$ we could have started with $x_0 = 0.1$ and

$y_0 = 1.104824833$ to get a closer value of $y(0.2)$

i.e. $y = 1.104824833 + \int_{0.1}^{x} (y_0 - x^2)dx$

$$y^{(1)} = 1.104824833 + \left(y_0 x - \frac{x^3}{3} \right)_{0.1}^{x}$$

$$= 1.104824833 + 1.104824833x - \frac{x^3}{3} - (1.104824833(0.1)) + \frac{0.1^3}{3}$$

$$= 0.99467574 + 1.104824833x - \frac{x^3}{3}$$

$$y^{(2)} = 1.104824833 + \int_{0.1}^{x} (y^{(1)} - x^2)dx$$

$$= 1.104824833 + \int_{0}^{x} \left(0.99467574 + 1.104824833x - \frac{x^3}{3} - x^2 \right) dx$$

$$= 1.104824833 + 0.99467574(x - 0.1) + 1.104824833(x^2 - (0.1)^2)$$

$$- \frac{1}{12}(x^4 - (0.1)^4) - \frac{1}{3}(x^3 - (0.1)^3)$$

$y^{(2)}(0.2) = 1.2184066$

Example 2. Solve $\frac{dy}{dx} = x + y$, given $y(0) = 1$. Obtain the values of

$y(0.1)$ and $y(0.2)$ using Picard's method and check your result with the exact solution.

Solution

Here $f(x, y) = x + y, \quad x_0 = 0$ and $y_0 = 1$

$$\therefore y = y_0 + \int_{x_0}^{x} f(x, y) dx$$

$$= 1 + \int_0^x f(x, y) dx$$

$$y^{(1)} = 1 + \int_0^x f(x, y) dx$$

$$= 1 + \int_0^x (x, 1) dx$$

$$= 1 + \int_0^x (x + 1) dx = 1 + x + \frac{x^2}{2}$$

$$y^{(2)} = 1 + \int_0^x \left(1 + x + \frac{x^2}{2} + x\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{6}$$

$$y^{(3)} = 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{6}\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

$$\therefore y(x) = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24} + \ldots$$

Now put $x = 0.1$

$$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{24} + \ldots$$

$$= 1 + 0.1 + 0.01 + \frac{0.001}{3} + \frac{0.0001}{24} + \dots$$

$$= 1.1103374$$

To find $y(0.2)$ we use $x_0 = 0.1$ and $y_0 = 1.1103374$

$$y = y_0 + \int_{0.1}^{x} f(x, y)dx$$

$$= 1.1103374 + \int_{0.1}^{x} (x + y)dx$$

$$y^{(1)} = 1.1103374 + \int_{0.1}^{x} (x + 1.1103374)dx$$

$$= 1.1103374 + 1.1103374(x - 0.1) + (x^2 - (0.1)^2)$$

$$= 0.98930366 + 1.1103374x + x^2$$

$$y^{(2)} = 1.1103374 + \int_{0.1}^{x} \left(x + 0.98930366 + 1.1103374x + x^2\right)dx$$

$$= 1.1103374 + 0.98930366(x - 0.1) + 2.1103374(x^2 - (0.1)^2) + \frac{1}{3}\left(x^3 - (0.1)^3\right)$$

$$= 0.989970326 + 0.98930366x + 2.1103374x^2 + \frac{1}{3}x^3$$

$$y^{(3)} = 1.1103374 + \int_{0.1}^{x} \left(0.989970326 + 1.98930366x + 2.1103374x^2 + \frac{1}{3}x^3\right)dx$$

$$= 1.1103374 + 0.989970326(x - 0.1) + 1.98930366(x^2 - (0.1)^2)$$

$$+ 2.1103374\left(x^3 - (0.1)^3\right) + \frac{1}{12}\left(x^4 - (0.1)^4\right)$$

$$\therefore y^{(3)}(0.2) = 1.1103374 + 0.989970326(0.1) + 1.98930366(0.03)$$

$$+ 2.1103374(0.007) + \frac{1}{12}(0.0015)$$

$$= 1.283910904$$

Solving $\frac{dy}{dx} = x + y$ analytically, we get $y = 2e^x - x - 1$

$y(0.1) = 1.110341836$ and

$y(0.2) = 1.242805516$

Example 3. Solve $\frac{dy}{dx} = x^2 + y^2, y(0) = 1$ by Picard's method

Solution

Here $x_0 = 0,\ y_0 = 1$

$$\therefore y = y_0 + \int_{x_0}^{x} f(x, y)dx = 1 + \int_0^x \left(x^2 + y^2\right)dx$$

Now

$$y^{(1)} = 1 + \int_0^x \left(x^2 + 1\right)dx = 1 + x + \frac{x^3}{3}$$

$$y^{(2)} = 1 + \int_0^x \left(x^2 + \left(1 + x + \frac{x^3}{3}\right)^2 \right)dx$$

$$= 1 + x + x^2 + \frac{2x^3}{3} + \frac{1}{6}x^4 + \frac{1}{63}x^7 + \frac{2}{15}x^5$$

$$= 1 + x + x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4 + \frac{2}{15}x^5 + \frac{1}{63}x^7 + \ldots$$

Example 4. Solve $y' + y = e^x, y(0) = 0$, by Picard's method

Solution

By Picard's method $y = y_0 + \int_{x_0}^{x} f(x, y)dx = \int_0^x \left(e^x - y\right)dx$

$$y^{(1)} = \int_0^x \left(e^x - 0\right) dx = e^x - 1$$

$$y^{(2)} = \int_0^x \left(e^x - \left(e^x - 1\right)\right) dx = x$$

$$y^{(3)} = \int_0^x \left(e^x - x\right) dx = e^x - \frac{x^2}{2} - 1$$

$$y^{(4)} = \int_0^x \left(e^x - \left(e^x - \frac{x^2}{2} - 1\right)\right) dx$$

$$= \frac{x^3}{6} + x$$

$$y^{(5)} = \int_0^x \left(e^x - \left(\frac{x^3}{6} + x\right)\right) dx$$

$$= e^x - \frac{x^2}{2} - \frac{x^4}{24} - 1$$

$$\therefore y(x) = e^x - \frac{x^2}{2} - \frac{x^4}{24} - 1$$

Exercise

1. Using Picard's iterative formula,

   a) Solve $\frac{dy}{dx} = x + y^2 + 1$, given $y(0) = 0$.

   b) Obtain $y(0.1)$ given $y' = \frac{y - x}{y + x}$ and $y(0) = 1$

   c) Solve $y' = 1 + 2xy$ given $y(0) = 0$

2. Find the values of y for $x = 0, x = 0.1$ and $x = 0.5$ given $y' = x^2 + y^2$ and which passes through $(0,1)$.

3. Given $y' = \frac{x^2}{1 + y^2}$ and $y(0) = 0$, find $y(0.25), y(0.5)$

Euler's Method

In solving a first order differential equation by numerical methods, we have two types of solutions:

i) Values of y at specified values of x

ii) A series solution of y in terms of x, which will yield the value of y at a particular value of x by direct substitution in the series solution.

Taylor and Picard's method studied so far belong to the first category in finding the numerical solution of differential equations; the methods due to Euler, Runge-kutta, Adam-Bash Forth and Milne come under the second category.

The methods of second category are called ***step-by-step methods*** because the values of y are calculated by short steps ahead of equal interval h of the independent variable x.

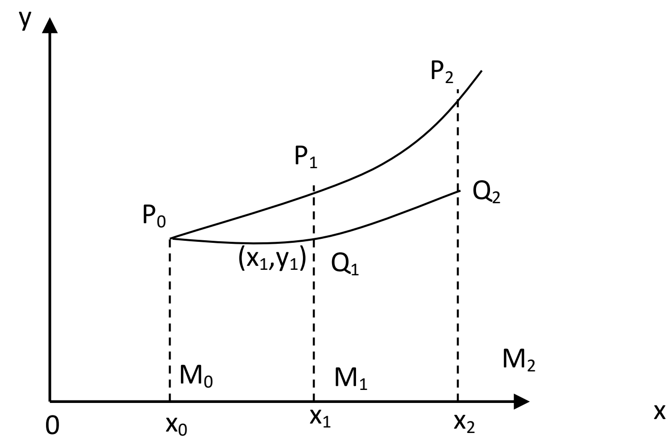
Euler's method

Suppose we want to solve $\dfrac{dy}{dx} = f(x,y)$ with initial conditions $y(x_0) = y_0$

Let us take the points

$x = x_0, x_1, x_2, ...$ where $x_i - x_{i-1} = h$

i.e. $x_i = x_0 + ih$, $i = 0,1,2,...$



Let the actual solution of the differential equation be denoted by the graph $P_i = P_i(x_i, y_i)$ lies on the curve. We require the value of y the curve at $x = x_i$.

The equation of the tangent line at (x_0, y_0) to the curve is

$$y - y_0 = y'_{(x_0, y_0)}(x - x_0)$$

$$= f(x_0, y_0)(x - x_0)$$

$$\Rightarrow y = y_0 + f(x_0, y_0)(x - x_0)$$

In the interval (x_0, x_1), the curve is approximated by the tangent.

$\therefore$ The value of y on the curve is approximately equal to the value of y on the tangent at (x_0, y_0) corresponding to $x = x_i$.

$$\therefore y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

i.e. $y_1 = y_0 + hy'_0$

Again, we approximate the curve by the line through (x_1, y_1) and whose slope is f (x_1, y_1), we get

$$y_2 = y_1 + f(x_1, y_1)h$$

$$= y_1 + hy'_1$$

Thus, $y_{n+1} = y_n + hf(x_n, y_n), \quad n = 0,1,2,...$

This formula is called **Euler's algorithm**.

In other words, $y(x+h) = y(x) + hf(x, y)$

In this method, the actual curve is approximated by a sequence of short straight lines. As the intervals increase the straight line deviates much from the actual curve. Hence the accuracy cannot be obtained as the number of intervals increase.

Referring to the above graph

$Q_1P_1 =$ error at $x = x_1$

$= \dfrac{(x_1 - x_0)^2}{2!} y''(x_1, y_1) = \dfrac{h^2}{2} y''(x_1, y_1)$ it is of order h^2.

Example 1. Given $y' = -y$ and $y(0) = 1$, determine the values of y at

$x = 0.01, x = 0.02$ and $x = 0.04$ by Euler method.

Solution

$y' = -y$ and $y(0) = 1$; $f(x, y) = -y$

Here $x_0 = 0,\quad y_0 = 1,\quad x_1 = 0.01,\quad x_2 = 0.02,\quad x_3 = 0.03,\quad x_4 = 0.04$

We have to find y_1, y_2, y_3, y_4. Take $h = 0.01$

By Euler algorithm,

$$y_{n+1} = y_n + hy'_n = y_n + hf(x_n, y_n)$$

$$y_1 = y_0 + hf(x_0, y_0) = 1 + (0.01)(-1) = 1 - 0.01 = 0.99$$

$$y_2 = y_1 + hy'_1 = 0.99 + (0.01)(-y_1)$$

$$= 0.99 + (0.01)(-0.99) = 0.9801$$

$$y_3 = y_2 + hf(x_2, y_2) = 0.9801 + (0.01)(-0.9801) = 0.9606$$

$$y_4 = y_3 + hf(x_3, y_3) = 0.9703 + (0.01)(-0.9703) = 0.9606$$

Let us compare the results

x	0	0.01	0.02	0.03	0.04
y	1	0.9900	0.9801	0.9703	0.9606
$y = e^{-x}$	1	0.9900	0.9802	0.9704	0.9608

Example 2. Using Euler's method, solve numerically the equation,

$y' = x + y, y(0) = 1$ for $x = 0, x = 0.2$ and $x = 1.0$.

Check your answer with the exact solution

Solution

Here $h = 0.2$, $f(x, y) = x + y$, $x_0 = 0, y_0 = 1$

$x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1.0$

By Euler algorithm,

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h(x_0 + y_0)$$

$$= 1 + 0.2(0+1) = 1.2$$

$$y_2 = y_1 + h(x_1 + y_1) = 1.2 + (0.2)(0.2 + 1.2)$$

$$= 1.48$$

$$y_3 = y_2 + h(x_2 + y_2) = 1.48 + (0.2)(0.4 + 1.48)$$

$$= 1.856$$

$$y_4 = y_3 + h(x_3 + y_3) = 1.856 + (0.2)(0.6 + 1.856)$$

$$= 2.3472$$

$$y_5 = 2.3472 + 0.2(0.8 + 2.3472) = 2.94664$$

Exact solution is $y = 2e^x - x - 1$

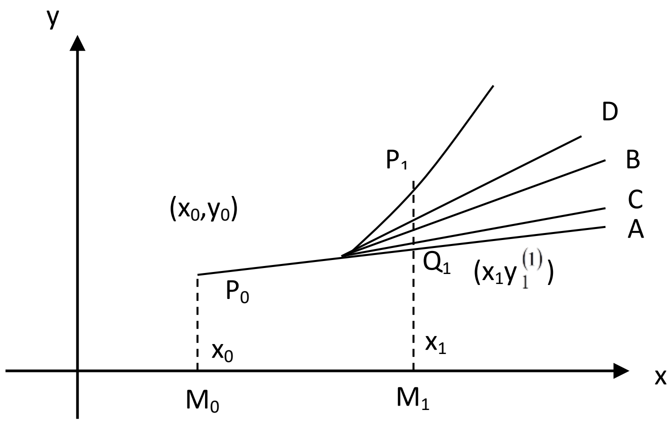
	x	0	0.2	0.4	0.6	0.8	1.0
Euler	y	1	1.2	1.48	1.856	2.3472	2.94664
Exact	y	1	1.2428	1.5836	2.0442	2.6511	3.4366

As you can see from the table, the values of y deviate from the exact values as x increases. To avoid this discrepancy we need to improve Euler's method.

Improved Euler Method

Let the tangent at (x_0, y_0) to the curve be P_0A. In the interval (x_0, x_1) by the previous Euler's method, we approximate the curve by the tangent P_0A.

$$\therefore y_1^{(1)} = y_0 + hf(x_0, y_0) \quad \text{where} \quad y_1^{(1)} = M_1Q_1$$



Let Q_1C be the line at Q_1 whose slope is $f\left(x_1, y_1^{(1)}\right)$

Now take the average of the slopes at P_0 and Q_1

i.e. $\frac{1}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$

Draw a line P_0D through $P_0(x_0, y_0)$ with this as the slope.

That is,

$$y - y_0 = \frac{1}{2}\left[f(x_0, y_0) f\left(x_1, y_1^{(1)}\right)\right](x - x_0)$$

and this line intersects $x = x_1$ at

$$y_1 = y_0 + \frac{1}{2}h\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$= y_0 + \frac{1}{2}h\left[f(x_0, y_0) + f(x_1, y_0 + hf(x_0, y_0))\right]$$

In general,

$$y_{n+1} = y_n + \frac{1}{2}h\left[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))\right]$$

This is what is known as **improved Euler's method.**

Notice that the difference between Euler's method and the improved Euler's method is that in the improved one we take the average of the slopes at (x_0, y_0) and $\left(x_1, y_1^{(1)}\right)$ instead of the slope at (x_0, y_0) in the former method.

Modified Euler's Method

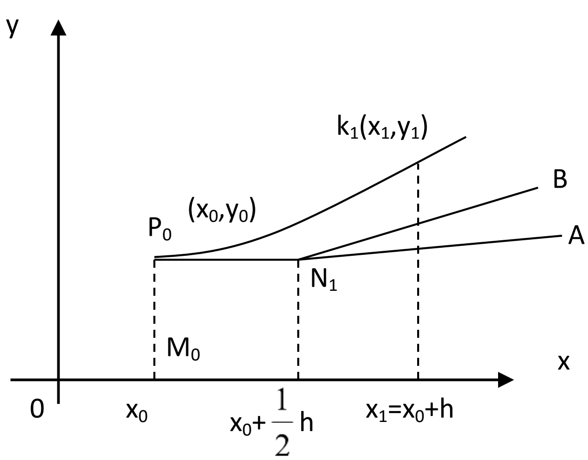
In the improved Euler method, we arranged the slopes, whereas in modified Euler method, we will average the points.

Let $P_0\,(x_0, y_0)$ be the point on the solution curve

Let P_0A be the tangent at (x_0, y_0) to the curve. Now let this tangent meet the ordinate at

$x = x_0 + \frac{1}{2}h$ at N_1 and y coordinate of $N_1 = y_0 + \frac{1}{2}hf(x_0, y_0)$. Calculate the slope at N_1 i.e.

$f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hf(x_0, y_0)\right)$.



Let this line meet $x = x_1$ at $k_1\left(x_1, y_1^{(1)}\right)$.

This $y_1^{(1)}$ is taken as the approximate value of y at $x = x_1$

$$\therefore y_1^{(1)} = y_0 + h\left[f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hf(x_0, y_0)\right)\right]$$

In general,

$$y_{n+1} = y_n + h\left[f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)\right)\right]$$

or $y(x+h) = y(x) + h\left[f\left(x + \frac{1}{2}h, y + \frac{1}{2}hf(x, y)\right)\right]$

This is called ***modified Euler's formula***.

Note: The Euler predictor is $y_{n+1} = y_n + hy_n^{'}$ and the corrector is $y_{n+1} = y_n + \frac{h}{2}\left(y_n^{'} + y_{n+1}^{'}\right)$ in the improved Euler method.

When you read some literature there is some confusion among the authors. Some take the improved Euler method as the modified Euler method and the modified Euler method is not mentioned at all.

Example 3 Solve numerically $y' = y + e^x$, $y(0) = 0$ for $x = 0.2$, $x = 0.4$ by improved Euler's method.

Solution

$y' = y + e^x$, $y(0) = 0$ $x_0 = 0$, $y_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$ and h = 0.2.

By improved Euler method,

$$y_1 = y_0 + \frac{1}{2}h\left[f(x_0, y_0) + f(x_1, y_0 + hf(x_0, y_0))\right]$$

$$\therefore y_1 = 0 + \frac{0.2}{2}\left[y_0 + e^{x_0} + y_0 + h\left(y_0 + e^{x_0}\right) + e^{x_0 + h}\right]$$

$$= (0.1)\left(0 + 1 + 0 + 0.2(0+1) + e^{0.2}\right)$$

$$y(0.2) = y_1 = 0.24214$$

$$y^2 = y_1 + \frac{1}{2}h\left[f(x_1, y_1) + f\left(x_1 + h, y_1 + hf(x_1, y_1)\right)\right]$$

Here $f(x_1, y_1) = y_1 + e^{x_1} = 0.24214 + e^{0.2} = 1.46354$

$$y_1 + hf(x_1, y_1) = 0.24214 + 0.2(1.46354) = 0.53485$$

$$f\left(x_1 + h, y_1 + hf(x_1, y_1)\right) = 0.53485 + e^{0.4} = 2.02667$$

$$y_2 = 0.24214 + (0.1)(1.46354 + 2.02667)$$

$$y(0.4) = y_2 = 0.59116$$

Example 4. Compute y at $x = 0.25$ by Modified Euler's method

$$y' = 2xy, \; y(0) = 1.$$

Solution

$f(x, y) = 2xy; \quad x_0 = 0, \; y_0 = 1$

Take $h = 0.25, \quad x_1 = 0.25$

By modified Euler method,

$$y_{n+1} = y_n + h\left[f\left(x_n + \frac{h}{2}, y_n + \frac{1}{2}hf(x_n, y_n)\right)\right]$$

$$\therefore y_1 = y_0 + h\left[f\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}hf(x_0, y_0)\right)\right]$$

$$f(x_0, y_0) = f(0,1) = 0$$

$$y_1 = 1 + (0.25)\, f(0.125, 1)$$

$$= 1 + (0.25)(2)(0.125)(1)$$

$$y(0.25) = 1.0625$$

$$y' = 2xy \quad \Rightarrow \quad \frac{dy}{y} = 2xdx$$

Using $y(0) = 1$

$$y(0) = c = 1$$

$$\Rightarrow y = e^{x^2}$$

$$\therefore y(0.25) = e^{(0.25)^2} = 1.064494459$$

The error is only $0.0019994458 91$.

Example 5. Solve the equation $\frac{dy}{dx} = 1 - y$, given $y(0) = 0$ using modified and Euler's method and tabulate the solutions at $x = 0.1, x = 0.2$, and $x = 0.3$. Compare these results with the exact solutions.

Solution

Here $x_0 = 0, y_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, h = 0.1$

$$y' = 1 - y$$

$$\therefore f(x, y) = 1 - y, \quad \Rightarrow f(x_0, y_0) = 1 - y_0 = 1$$

i) Modified Euler method

$$y_1 = y_0 h f\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2} h f(x_0, y_0) \right)$$

$$x_0 + \frac{1}{2}h = \frac{1}{2}(0.1) = 0.05 \quad \text{and} \quad y_0 + \frac{1}{2} hf(x_0, y_0) = 0.05$$

$$\therefore y_1 = 0 + (0.1)[f(0.05, 0.05)] = 0.1(1 - 0..05) = 0.095$$

Now again $f(x_1, y_1) = 1 - y_1$ = 1- 0.95 = 0.905

$$y_2 = y_1 + h\, f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}h\, f(x_1, y_1)\right)$$

$$= 0.095 + (0.1)(f(0.15)0.14025)$$

$$= 0.095 + (0.1)(1 - 0.14025)$$

$$= 0.180975$$

$$y_3 = y_2 + h\, f\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}h\, f(x_2, y_2)\right)$$

$$= 0.180975 + (0.1) f\left(0.2 + \frac{0.1}{2}, 0.180975 + \frac{0.1}{2} f(0.2, 0.180975)\right)$$

$$= 0.180975 + (0.1) f(0.25, 0.22192625)$$

$$= 0.25878237$$

ii) Improved Euler method

$$y_{n+1} = y_n + \frac{1}{2}h \; [f(x_n, y_n) + f(x_n + h, y_n + h\, f(x_n, y_n))]$$

$$\therefore y_1 = y_0 + \frac{1}{2}h\, [f(x_0, y_0) + f(x_0 h, y_0 + h\, f(x_0, y_0))]$$

$$f(x_0, y_0) = 1 - y = 1$$

$$f(x_0 + h, y_0 + h\, f(x_0, y_0)) = f(0.1, 0 + 0.1(1)) = f(0.1, 0.1)$$

$$= 1 - 0.1 = 0.9$$

$$\therefore y_1 = y(0.1) = 0 + \frac{0.1}{2}(1 + 0.9) = 0.095$$

$$y_2 = y_1 + \frac{1}{2}h\, [f(x_1, y_1) + f\ (x_1 + h, y_1 + h\, f(x_1, y_1))]$$

$$f(x_1, y_1) = 1 - y_1 = 1 - 0.095 = 0.905$$

$$f(x_1 + h, y_1 + hf(x_1, y_1)) = f(0.2, 0{,}905 + (0.1)(1 - 0.095))$$

$$= f(0.2, 0.1855) = 0.8145$$

$$\therefore y_2 = 0.095 + \frac{0.1}{2}(0.8145 + 0.905) = 0.180975$$

$$y_3 = y_2 + \frac{1}{2}h\ [f(x_2, y_2) + f\ (x_2 + h, y_2 + h\ f(x_2, y_2))]$$

$$= 0.180975 + \frac{0.1}{2}((1 - 0.180975) + f(0.3, 0.180975 + (0.1)(1 - 0.180975)))$$

$$= 0.180975 + 0.05(0.819025 + 1 - 0.2628775)$$

$$= 0.258782375$$

Exact solution

$$\frac{dy}{dx} = 1 - y \qquad \Rightarrow \frac{dy}{1 - y} = dx$$

$$-\ell n(1 - y) = x + c_1 \quad \Rightarrow \quad 1 - y = ce^{-x}$$

At $x = 0, \quad 1 - 0 = ce^0 \Rightarrow c = 1$

$$\therefore y = 1 - e^{-x}$$

$$\Rightarrow y(0.1) = 0.095162581$$

$$y(0.2) = 0.181269246$$

$$y(0.3) = 0.259181779$$

x	Modified	improved	exact
0.1	0.095	0.095	0.095162581
0.2	0.180975	0.180975	0.181269246

0.3	0.258782375	0.258782375	0.259181779

Example 6. Find, correct to four decimal places, the value of $y(0.1)$

Given $y' = x^2 - y$, $y(0) = 1$ by using improved Euler method.

Solution

Here, $h = 0.1$, $x_0 = 0$, $x_1 = 0.1$ and $y_0 = 1$

By the improved Euler method,

$$y_{n+1} = y_n + \frac{1}{2}h\left[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))\right]$$

$$y_1 = y_0 + \frac{1}{2}h\left[f(x_0, y_0) + f(x_0 + h, y_0 + hf(x_0, y_0))\right]$$

$$= 1 + \frac{0.1}{2}\left(x_0^2 - y_0 + f\left(0.1, y_0 + 0.1\left(x_0^2 - y_0\right)\right)\right)$$

$$= 1 + 0.05\left(-1 + (0.1)^2 - (1 + (0.1)(-1))\right) = 0.9055$$

Example 7. Using improved Euler method find y at $x = 0.1$ and at

$x = 0.2$ given $\frac{dy}{dx} = y - \frac{2x}{y}$, $y(0) = 1$

Solution

By improved Euler theorem,

$$y_{n+1} = y_n + \frac{1}{2}h\left[f(x_n, y_n) + f(x_n + h, y_n + h\, f(x_n, y_n))\right]$$

Here $x_0 = 0, y_0 = 1, h = 0.1$

$$\therefore y_1 = y_0 + \frac{1}{2}h\left[f(x_0, y_0) + f(x_0 + h, y_0 + hf(x_0, y_0))\right]$$

$$= 1 + \frac{0.1}{2}\left[1 - \frac{2(0)}{1} + 1 + \frac{(0.1)(1)(2)}{1.1} - 0\right]$$

$$f\left(x_0+h, y_0+hf\left(x_n, y_n\right)\right)= f\left(0.1, 1+0.1(1)\right)= f(0.1, 1.1)$$

$$=1.1-\frac{2(0.1)}{1.1}=0.918181818..$$

$$=0.918182$$

$$\therefore y_1=1+\frac{0.1}{2}(1+0.918182)=1.095909091$$

$$y_2 = y_1+\frac{h}{2}\left(f\left(x_1, y_1\right)+hf\left(x_1+h, y_1+hf\left(x_1, y_1\right)\right)\right)$$

$$f\left(x_1, y_1\right)=1.095909091-\frac{2(0.1)}{1.09590909}$$

$$=0.913412201$$

$$f\left(x_1+h, y_1+hf\left(x_1, y_1\right)\right)= f\left(0.2, 1.095909091+(0.1)(0.913412201)\right)$$

$$= f(0.2, 1.187250311)$$

$$=1.187250311-\frac{2(0.2)}{1.187250311}$$

$$=0.850337365$$

Thus $y_2=1.095909091+\frac{0.1}{2}(0.913412201+0.850337365)$

$$=\underline{\underline{1.184096569}}$$

Example 8. Using modified Euler method, find $y(0.1),\ y(0.2)$

given $\frac{dy}{dx}=x^2+y^2,\quad y(0)=1$

Solution

Here, $x_0=0, y_0=1, h=0.1, x_1=0.1$, $f(x, y)=x^2+y^2$

By modified Euler method,

$$y_1 = y_0 + hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{h}{2}f(x_0, y_0)\right)$$

$$= 1 + (0.1)\left(0.05^2 + (1 + 0.05(1))^2\right)$$

$$y(0.1) = 1.1105$$

$$y_2 = y_1 + hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{h}{2}f(x_1, y_1)\right)$$

$$= 1.1105 + (0.1)\left(0.15^2 + 1.172660513^2\right)$$

$$= \underline{\underline{1.250263268}}$$

Exercise

1. Use Euler's method to find

   a) $y(0.4)$ given $y' = xy, y(0) = 1$

   b) $y(1.5)$ taking $h = 0.5$ given $y' = y - 1, y(0) = 1.1$

2. Compute $y(0.3)$ taking $h = 0.1$ given $\frac{dy}{dx} = y - \frac{2x}{y}, y(0) = 1$ using improved Euler method.

3. Find $y(0.6), y(0.8)$ and $y(1)$ given $y' = x + y, y(0) = 0$ taking $h = 0.2$ by improved Euler method.

4. Use improved Euler method to find $y(0.1)$ given $y' = \frac{y - x}{y + x}, y(0) = 1$.

5. Using improved Euler method find $y(0.2), y(0.4)$ given $\frac{dy}{dx} = x + \left|\sqrt{y}\right|, y(0) = 1$.

6. Use improved Euler method to calculate $y(0.5)$,taking $h = 0.1$ and

   $y' = y + \sin x, y(0) = 2$

7. Use modified Euler method and obtain $y(0.2)$ given $y' = \log(x+y), y(0) = 1, h = 0.2$.

8. Use improved and modified Euler method, to get $y(1.6)$ if $\frac{dy}{dx} = y^2 - \frac{y}{x}$, if $y(1) = 1$.

9. Solve $y' = 3x^2 + y$ given $y(0) = 4$ if $h = 0.25$ to obtain $y(0.25), y(0.5)$.

10. Given $y' = \frac{y}{x} - \frac{5}{2}x^2 y^3, y(1) = \frac{1}{\sqrt{2}}$ find $y(2)$ if $h = 0.125$.

11. Find $y(0.2)$ by improved Euler method, given $y' = -xy^2, y(0) = 2$

    if $h = 0.1$.

Runge-Kutta Method

i) Second order Runge-kutta method

Suppose $\frac{dy}{dx} = f(x,y)$ given $y(x_0) = y_0$ ....(1)

By Taylor series, we have

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + 0(h^3) \quad \text{..........(2)}$$

Differentiating (1) with respect to x,

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = f_x + y' f_y = f_x + ff_y \quad \text{........(3)}$$

Using the values of y' and y'' derived from (1) and (3), in (2) we get

$$y(x+h) - y(x) = hf + \frac{1}{2}h^2\left(f_x + ff_y\right) + O(h^3)$$

$$\Rightarrow \Delta y = hf + \frac{1}{2}h^2\left(f_x + ff_y\right) + O(h^3) \quad \text{........(4)}$$

Let $\Delta_1 y = hf(x,y) = f(x,y)\Delta x = k_1$

$\Delta_2 y = hf(x + mh,\ y + mk_1) = k_2$

and let $\Delta y = ak_1 + bk_2$ where a, b and m are constants to be determined to get the better accuracy of Δy.

Now expand k_2 and Δy in powers of h, by Taylor series for two variables, we have

$$k_2 = h\, f\left(x + mh, y + mk_1\right)$$

$$= h\left[f(x, y) + \left(mh\,\frac{\partial}{\partial x} + mk_1\,\frac{\partial}{\partial y}\right) f + \frac{\left(mh\frac{\partial}{\partial x} + mk_1\frac{\partial}{\partial y}\right)^2 f}{2!} + \ldots \right]$$

$$= h\left[f + mhf_x + mhff_y + \frac{\left(mh\frac{\partial}{\partial x} + mk_1\frac{\partial}{\partial y}\right)^2 f}{2!} + \ldots \right]$$

$= h\, f + mh^2 f_x + mh^2 ff_y + \ldots$ higher powers of h.

Substituting k_1, k_2 in Δy, we get

$$\Delta y = ahf + b\left(hf + mh^2\left(f_x + ff_y\right) + O\left(h^3\right)\right)$$

$$= (a+b)hf + bmh^2\left(f_x + ff_y\right) + O\left(h^3\right) = ak_1 + bk_2 \ldots\ldots(5)$$

Equating (4) and (5), we have

$$a + b = 1, \quad mb = \frac{1}{2}$$

$$\Rightarrow a = 1 - b \quad and \quad m = \frac{1}{2b}$$

$$\therefore \Delta y = (1-b)k_1 + b\,k_2$$

Where $k_1 = hf(x, y)$

$$k_2 = h\, f\left(x + \frac{h}{2b}, y + \frac{h\, f}{2b} \right)$$

Now $\Delta y = y(x+h) - y(x)$

$$\Rightarrow y(x+h) = y(x) + (1-b)hf + bhf\left(x + \frac{h}{2b}, y + \frac{hf}{2b} \right)$$

i.e. $y_{n+1} = y_n + (1-b)h\, f(x_n, y_n) + b\, h\, f\left(x_n + \frac{h}{2b}, y_n + \frac{h}{2b}\, f(x_n, y_n) \right) + O(h^3)$

From this general second order Runge-kutta formula, setting $a = 0$, $b = 1$, $m = \frac{1}{2}$, we get the second order Runge-kutta algorithm.

$$k_1 = hf(x, y)$$
$$k_2 = hf\left(x + \frac{1}{2}h, y + \frac{1}{2}k_1 \right)$$

and $\Delta y = k_2$ where $h = \Delta x$

ii) Third order Runge-kutta Method

For n = 3, a similar derivation to the one as the second-order method can be performed. Since the derivation is tedious we state simply the formula.

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

, where $k_1 = h\, f(x, y)$

$$k_2 = hf\left(x + \frac{1}{2}h, y + \frac{1}{2}k_1 h \right)$$

$$k_3 = hf(x + h, y - k + 2k_2)$$

iii) Fourth Order Runge-kutta method

The most popular and commonly used form is the classical fourth-order Runge-kutta method.

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

, where $k_1 = hf(x, y)$

$$k_2 = hf\left(x + \frac{1}{2}h, y + \frac{1}{2}k_1 h\right)$$

$$k_3 = hf\left(x + \frac{h}{2}, y + \frac{1}{2}k_2 h\right)$$

$$k_4 = hf(x + h, y + k_3 h)$$

Note 1. The second order Runge-kutta method,

$$\Delta y_0 = k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1\right)$$

$$= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}hf(x_0, y_0)\right)$$

$$\therefore y_1 = y_0 + hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}hf(x_0, y_0)\right)$$

this is exactly the Modified Euler method.

Thus, the second order Runge-kutta method is simply the modified Euler method.

2. If $f(x, y) = f(x)$, a function of x alone, then the fourth order Runge-kutta method reduces to

$$k_1 = hf(x_0)$$

$$\Delta y = \frac{1}{6}h\left[f(x_0) + 4f\left(x_0 + \frac{h}{2}\right) + f(x_0 + h)\right]$$

$$= \frac{h/2}{3}\left(f(x_0) + 4f\left(x_0 + \frac{h}{2} \right) + f(x_0 + h) \right)$$

= the area of $y = f(x)$ between $x = x_0$ and $x = x_0 + h$ with two equal intervals of length $\frac{h}{2}$ by Simpson's one-third rule. i.e. Δy reduces to the area by Simpson's one-third rule.

Example 1. Apply the fourth order Runge-kutta method to find

$y(0.2)$ given that $y' = x + y, y(0) = 1$

Solution

Since h is not mentioned, we can take $h = 0.1$

$$\therefore f(x, y) = x + y, x_0 = 0, \quad y_0 = 1, \quad x_1 = 0.1, \quad x_2 = 0.2$$

By fourth-order Runge-kutta method, for the first interval

$$k_1 = hf(x_0, y_0) = (0.1)(0 + 1) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1 \right) = (0.1)(0.05 + 1 + 0.05) = 0.11$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hk_2 \right) = (0.1)(0.05 + 1 + 0.055) = 0.1105$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)(0.1 + 1 + 0.1105) = 0.12105$$

$$\therefore y(0.1) = y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0.1 + 2(0.11) + 2(0.1105) + 0.12105)$$

$$= \underline{\underline{1.110341667}}$$

Now starting from (x_1, y_1) we get (x_2, y_2). Again apply Runge-kutta algorithm replacing (x_0, y_0) by (x_1, y_1).

$$k_1 = hf(x_1, y_1) = (0.1)(0.1 + 1.110341667) = 0.121034166$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.1)(0.15 + 1.176384605)$$

$$= 0.13263846$$

$$\therefore\ y(0.2) = 1.110341667 + \frac{1}{6}\left[0.121034166 + 2(0.132085875 + 0.13263846) + 0.144298012\right]$$

$$= 1.242805142$$

Remember that the exact solution is $y = 2e^x - x - 1$

$$y(0.2) = 1.242805516$$

The difference between the exact solution and the fourth order Runge-kutta method is 0.00000037432.

As compared to other methods, this method is the best one.

Example 2. Find the values of y at $x = 0.1,\ 0.2$ using Runge-kutta method of i) second order ii) third order and iii) fourth order for the given that $y' = -y$ and $y(0) = 1$.

Solution

$$f(x, y) = -y, x_0 = 0, y_0 = 1, x_1 = 0.1, x_2 = 0.2, h = 0.1$$

i) Second order

$$k_1 = hf(x_0, y_0) = (0.1)(-1) = -0.1$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1)(-(1-0.05)) = -0.095$$

$$\therefore y_1 = y_0 + k_2$$

$$= 1 - 0.095 = 0.905$$

Again let $(x_1, y_1) = (0.1, 0.905)$

$$k_1 = (0.1)(-0.905) = -0.0905$$

$$k_2 = (0.1)\left[-\left(0.905 + \frac{1}{2}(-0.0905)\right)\right] = -0.085975$$

$$y(0.2) = 0.905 + (-0.085975) = 0.819025$$

ii) Third order

$$k_1 = hf(x_0, y_0) = -0.1$$

$$k_2 = -0.095$$

$$k_3 = hf(x_0 + h,\ y_0 + 2k_2 - k_1)$$

$$= (0.1)(-(1 + 2(-0.095)\ (-0.1)) = -0.091$$

$$y(0.1) = 1 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

$$= 1 + \frac{1}{6}(-0.1 + 4(-0.095) + (-0.091))$$

$$= 0.904833333$$

Now take $(x_1, y_1) = (0.1, 0.904833333)$ and repeat the process

$$k_1 = hf(x_1, y_1) = (0.1, 0.904833333) = -0.0904833333$$

$$k_2 = (0.1)\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}k_1\right)$$

$$= (0.1)\left(-\left(0.904833333 + \frac{1}{2}(-0.0904833333)\right)\right)$$

$$= -0.085959166$$

$$k_3 = (0.1)(-[(0.904833333) + 2(-0.085959166) - (-0.0904833333)])$$

$$= -0.082339833$$

$$\therefore y_2 = y_1 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

$$= 0.904833333 + \frac{1}{6}(-0.0904833333 + 4(-0.082339833) + (-0.082339833))$$

$$= 0.821136249$$

iii) Fourth order

$$k_1 = -0.1$$

$$k_2 = -0.095$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1)(-(1 + (-0.0475))) = 0.09525$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)(-(1 + (-0.09525))) = -0.090475$$

$$\therefore y_1 = 1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.9048375$$

Again taking $(x_1, y_1) = (0.1, 0.9048375)$ and repeating the process, we get

$$k_1 = (0.1)(-0.9048375) = -0.09048375$$

$$k_2 = (0.1) f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$= (0.1)(-(0.9048375 - 0.045241875)) = -0.085959562$$

$$k_3 = (0.1) f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$= (0.1)\left(-\left(0.9048375 + \frac{1}{2}(-0.085959562)\right)\right) = -0.086185771$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= (0.1)(-0.9048375 + 0.086185771 = -0.081865172$$

$$\therefore y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.9048375 + \frac{1}{6}(-0.09048375 + 2(-0.085959562 + -0.086185771))$$
$$+ (-0.081865172) = \underline{\underline{0.818730902}}$$

x	2nd order	3rd order	4th order	exact $y = e^{-x}$
0.1	0.905	0.904833333	0.9048375	0.904837418
0.2	0.819025	0.821136249	0.818730902	0.818730753

As we can see from the table fourth order values are closer to the exact values.

Example 3. Compute $y(0.3)$ given $\frac{dy}{dx} + y = -xy^2, y(0) = 1$ by taking $h = 0.1$ using Runge-kutta method of fourth order.

Solution

$$y' = -y - xy^2 = f(x, y) = -(y + xy^2)$$

$$x_0 = 0, y_0 = 1, h = 0.1, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3$$

Now k_1 = (0.1) (-1) = -0.1

$$k_2 = (0.1)\left(-\left[\left(1+\frac{(-0.1)}{2}\right)+\left(\left[1+\frac{(-0.1)}{2}\right]^2\left(\frac{0.1}{2}\right)\right)\right]_1\right)$$

$$= -(0.1)\left[0.95 + (0.05)(0.95)^2\right]$$

$$= -0.0995125$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2\right)$$

$$= (0.1) f(0.05, 0.95024375)$$

$$= -(0.1)\left[0.95024375 + (0.05)(0.95024375)^2\right]$$

$$= -0.09953919$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= -(0.1) f(0.1, 0.900460809)$$

$$= -0.098154377$$

$$\therefore\ y_1 = 1 + \frac{1}{6}(-0.1 + 2(-0.0995125) + 2)(-0.09953919) + (-0.098154377)$$

$$y(0.1) = 0.900623707$$

Now again taking (x_1, y_1) in place of (x_0, y_0), we get

$$k_1 = hf(x_1, y_1) = (0.1) f(0.1, 0.900623707)$$

$$= 0.098173601$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= (0.1) f (0.15, 0.851536906)$$

$$= -0.096030417$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= -(0.1)\left[0.852608498 + (0.15)(0.852608498)^2\right]$$

$$= -0.096164968$$

$$k_4 = hf(x_1 + h,\ y_1 + k_3)$$

$$= (0.1) f (0.2, 0.804458738)$$

$$= -0.093388951$$

$$\therefore\ y_2 = 0.900623707 + \frac{1}{6}\big[-0.098173601 + 2(-0.096030417) + 2(-0.096164968) + (-0.093388951)\big]$$

$$= 0.804631486$$

Again, using $(x_2, y_2) = (0.2, 0.804631486)$ we find $y_3 = y(0.3)$

$$k_1 = hf(x_2, y_2)$$

$$= -(0.1)\left[0.804631486 + (0.2)(0.804631486)^2\right]$$

$$= -0.093411785$$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = (0.1) f (0.25, 0.757925593)$$

$$= -0.090153839$$

$$k_3 = hf\left(x_2 + h, y_2 + \frac{k_2}{2}\right) = (0.1)f(0.25, 0.759554566)$$

$$= -0.090378535$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = (0.1)f(0.25, 0.714252951)$$

$$= -0.084179227$$

$$\therefore y_3 = 0.804631486 + \frac{1}{6}[-0.093411785 + 2(-0.090153839) + 2$$
$$(-0.090378535) + (-0.084179227)]$$

$$= \underline{\underline{0.714855526}}$$

Example 4. Using Runge-kutta method of fourth order, find $y(0.8)$

if $y' = y - x^2, y(0.6) = 1.7379$

Solution

Here $x_0 = 0.6, y_0 = 1.7379, h = 0.1, x_1 = 0.7, x_2 = 0.8$ and $f(x, y) = y - x^2$

$$k_1 = hf(x_0, y_0) = (0.1)(1.7379 - 0.6^2) = 0.13779$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= (0.1)(0.65, 1.806795) = 0.1384295$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= (0.1)(1.80711475 = 0.65^2) = 0.38461475$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$\therefore\ y(0.7) = 1.7379 + \frac{1}{6}\left[0.13779 + 2(0.1384295) + 2(0.138461475) + 0.138636147\right]$$

$$= 1.876268016$$

Now using $(x_1, y_1) = (0.7, 1.876268016)$ we continue to find $y(0.8)$

$$k_1 = hf(x_1, y_1) = (0.1)\left(1.876268016 - 0.7^2\right) = 0.138626801$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= (0.1)f(0.75, 1.945422087) = 0.138292208$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= (0.1)f(0.8, 2.014560224) = 0.137456022$$

$$y(0.8) = 1.876268016 + \frac{1}{6}\left[0.138626801 + 2(0.138308141) + 2(0.138292208) + 0.13746022\right]$$

$$= 2.014481936$$

Example 5. Using Runge-kutta method of fourth order, solve

$$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2} \text{ given } y(0) = 1 \text{ at } x = 0.2, 0.4$$

Solution

$x_0 = 0, y_0 = 1, h = 0.2, x_1 = 0.2, x_2 = 0.4$ and $f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2}$

$$k_1 = hf(x_0, y_2) = (0.2)\left(\frac{1-0}{1+0}\right) = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= (0.2)\left(\frac{1.1^2 - 0.1^2}{1.1^2 + 0.1^2}\right) = 0.196721311$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= (0.2)\left(\frac{1.098360656^2 - 0.1^2}{1.098360656^2 + 0.1^2}\right) = 0.196711597$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= (0.2)\left(\frac{1.196711598^2 - 0.2^2}{1.196711598^2 + 0.2^2}\right) = 0.18913131$$

$$\mathrm{y}(0.2) = 1 + \frac{1}{6}(0.2 + 2(0.196721311) + 2(0.196711597) + 0.18913131)$$

$$= 1.195999521$$

To find $y(0.4)$ we use $(x_1, y_1) = (0.2, 1.195999521)$

$$k_1 = (0.2) f(x_1, y_1) = (0.2)\left(\frac{1.195999521^2 - 0.2^2}{1.195999521^2 + 0.2^2}\right)$$

$$= 0.189118717$$

$$k_2 = (0.2) f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.2) f(0.3, 1.29055888)$$

$$= 0.179493515$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.2)f(0.3, 1.285746279)$$

$$= 0.179347655$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.2)f(0.4, 1.375347176)$$

$$= 0.168804528$$

$$\therefore y(0.4) = 1.195999521 + \frac{1}{6}[0.189118717 + 2(0.179493515) + 2(0.179347655) + 0.168804528]$$

$$= 1.375267119$$

Runge-kutta method for simultaneous first order differential equations

Aim- To solve numerically the simultaneous equations

$\frac{dy}{dx} = f_1(x, y, z)$ *and* $\frac{dz}{dx} = f_2(x, y, z)$ given the initial conditions

$y(x_0) = y_0, z(x_0) = z_0$

Now starting from (x_0, y_0, z_0) the increments Δy *and* Δz in y and z respectively are given by formulae

$k_1 = hf_1(x_0, y_0, z_0)$ $\qquad$ $m_1 = hf_2(x_0, y_0, z_0)$

$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{m_1}{2}\right)$ $\qquad$ $m_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{m_1}{2}\right)$

$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{m_2}{2}\right)$ $\qquad$ $m_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{m_2}{2}\right)$

$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + m_3)$ $\qquad$ $m_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + m_3)$

and $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ $\qquad$ $\Delta z = \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4)$

$\therefore\ y_1 = y_0 + \Delta y \quad and \quad z_1 = z_0 + \Delta z$

By repeating this algorithm once again we can find (x_2, y_2, z_2) starting from (x_1, y_1, z_1).

Example 6. Find $y(0.1)$, $z(0.1)$ from the system of equations,

$\frac{dy}{dx} = x + z, \frac{dz}{dx} = x - y^2$ given $y(0) = 2, z(0) = 1$ using Runge-kutta method of fourth order.

Solution

Here $x_0 = 0, y_0 = 2, z_0 = 1, h = 0.1, f_1(x,y,z) = x + z, f_2(x,y,z) = x - y^2$

So $k_1 = hf_1(x_0, y_0, z_0)$ $\qquad m_1 = hf_2(x_0, y_0, z_0)$

$= (0.1)(0+1) = 0.1$ $\qquad = (0.1)(0 - 2^2) = -0.4$

$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{m_1}{2}\right)$ $\qquad m_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{m_1}{2}\right)$

$= (0.1)(0.05 + 1 + (-0.2))$ $\qquad = (0.1)(0.05 - 4.2025)$

$= 0.085$ $\qquad = -0.41525$

$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{m_2}{2}\right)$ $\qquad m_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{m_2}{2}\right)$

$= (0.1)(0.05 + 0.792375)$ $\qquad = (0.1)(0.05 - (2.0425)^2)$

$= 0.0842375$ $\qquad = -0.412180625$

$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + m_3)$ $\qquad m_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + m_3)$

$= (0.1)(0.1 + 0.587819375)$ $\qquad = (0.1)(0.1 - (2.0842375)^2)$

$= 0.068781937$ $\qquad = -0.424404595$

$\therefore\ y_1 = 2 + \frac{1}{6}(0.1 + 2(0.085) + 2(0.0842375) + 0.068781937)$

$= 2.084542823$

$$z_1 = 1 + \frac{1}{6}\left[-0.4 + 2(-0.41525) + 2(-0.412180625) + (-0.424404595)\right]$$

$= 0.586789025$

Runge-kutta method for second order differential equation

Aim – To solve $y'' = f(x, y, y')$ given $y(x_0) = y_0, y''(x_0) = y_0'$

Now, let $y' = z$ and so $y'' = z'$

$\Rightarrow y'' = z' = f(x, y, y')$ and $y' = z$ are the two simultaneous equations with $f_1(x, y, z) = z$ and $f_2(x, y, z) = f(x, y, z)$ itself.

Once again by applying Runge-kutta method for solving simultaneous first order differential equations we can get the solution of the given problem.

Example 7. Given $y'' + xy' + y = 0, y(0) = 1, y'(0) = 0$, find $y(0.1)$ by using Runge-kutta method of fourth order.

Solution

$y'' + xy' + y = 0 \Rightarrow y'' = -y - xy'$

Let $y' = z \quad \Rightarrow z' = -y - xz$

$$\therefore \frac{dy}{dx} = z = f_1(x, y, z)$$

$$\frac{dz}{dx} = -y - xz = f_2(x, y, z)$$

given $y_0 = 1, z_0 = 0, y_0', h = 0.1, x_0 = 0$

$k_1 = hf_1(x_0, y_0, z_0) = (0.1)(0) = 0 \qquad m_1 = hf(x_0, y_0, z_0) = -0.1$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{m_1}{2}\right) \qquad m_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{m_1}{2}\right)$$

$$= (0.1)(0.05) = -0.005 \qquad = (0.1)(0.05, 1, -0.05)$$

$$k_3 = hf_1\left(0.05, 1 - + \frac{0.005}{2}, -0.049875\right) \qquad = (0.1)[-1 - (0.05)(-005)]$$

$$= (0.1)(-0.049875) \qquad = -0.09975$$

$$= -0.0049875 \qquad m_3 = hf_2(0.05, 0.9975, -0.049875)$$

$$k_4 = hf_1(x_1, y_0 + k_3, z_0 + m_3) \qquad = (0.1)((0.05)(0.049875)0.9975)$$

$$= (0.1)(-0.099500625) \qquad = -0.099500625$$

$$= -0.0099500625 \qquad m_4 = hf_2(x_i, y_0 + k_3, z_0 + m_3)$$

$$= (0.1)(0.1, 0.09950125, -0.099500625)$$

$$= (0.1)((0.1)(0.09950125) - 0.9950125)$$

$$= -0.098506248$$

$$\therefore\ y_1 = 1 + \frac{1}{6}[0 + 2(-0.005) + 2(-0.0049875) + (-0.0099500625)]$$

$$= 0.995012489$$

Exercise

In the exercise below, unless specified use fourth order Runge-Kutta method.

1. Find $y(0.2)$ given $y' = y - x, y(0) = 2$ taking $h = 0.1$.
2. Obtain the value of y at $x = 0.2$ if y satisfies $\frac{dy}{dx} - x^2 y = x, y(0) = 1$ taking $h = 0.1$.
3. Solve $\frac{dy}{dx} = xy$ for $x = 1.4$, taking $y(1) = 2, h = 0.2$.

4. Solve $\frac{dy}{dx} = \frac{y-x}{y+x}$ given $y(0)=1$, to obtain $y(0.2)$.

5. Solve the initial-value problem $\frac{du}{dt} = -2tu^2, u(0)=1$ with $h=0.2$ on the interval $(0,0.6)$.

6. Evaluate $y(0.1), y(0.2), y(0.3)$ given $y' = \frac{1}{2}(x+1)y^2, y(0)=1$.

7. Solve $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, y(1)=1$ for $y(1.1)$ taking $h=0.05$.

8. Find $y(0.1), y(0.2)$ given $y' = x-2y, y(0)=1$ taking $h=0.1$ by

   i) second order ii) third order

   iii) fourth order Runge-Kutta methods.

9. Solve $y' = xy+1$ as $x=0.2, x=0.4, x=0.6$ given $y(0)=2$ taking $h=0.2$.

10. Solve the system: $\frac{dy}{dx} = xz+1, \frac{dz}{dx} = -xy$ for $x=0.3, x=0.6, x=0.9$ taking $x=0, y=0, z=1$.

11. Solve $\frac{dy}{dx} = x+z, \frac{dz}{dx} = x-y$, given $y(0)=0, z(0)=1$ for $x=0.0$ to 0.2 taking $h=0.1$.

12. Evaluate $y(1.1), z(1.1)$ given $\frac{dy}{dx} = xyz, \frac{dz}{dx} = \frac{xy}{z}, y(1)=0.5, z(1)=1$.

13. Using Runge-Kutta method determine $x(0.1), y(0.1)$ given $\frac{dx}{dt} = xy+t, x(0)=1, \frac{dy}{dt} = ty+x, y(0)=-1$.

14. Solve $y'' - x(y')^2 + y^2 = 0$ using Runge-Kutta method for $x=0.2$ given $y(0)=1, y'(0)=0$ taking $h=0.2$.

15. Find $y(0.1)$ given $y'' = y^2, y(0)=10, y'(0)=5$ by Runge-Kutta method.

16. Find $y(0.1, y(0.2)$ given $y'' - x^2y' - 2xy = 1, y(0)=1, y'(0)=0$.

17. Obtain the value of $x(0.1)$ given $\frac{d^2x}{dt^2} = \frac{tdx}{dt} - 4x, x(0) = 3, x'(0) = 0$.

18. Compute the value of $y(0.2)$ given $y'' = -y, y(0) = 1, y'(0) = 0$.

Predictor-Corrector Method

The methods so far discussed are called single step methods because they use only the information from the last step computed. But now we try to discuss multi-step methods.

Consider $\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$

We have used Euler's formula to solve differential equation of the form

$$y_{i+1} = y_i + hf'(x_i, y_i), \ i = 0,1,2, \qquad ...(1)$$

, and we have improved the Euler method by

$$y_{i+1} = y_i + \frac{1}{2}h[f(x_i, y_i) + f(x_{i+1}, y_{i+1})] \qquad ...(2)$$

In (2), to get the value of y_{i+1} we require y_{i+1} on the right hand side.

To overcome this difficulty, we calculate y_{i+1} using Euler's formula (1) and then we use it on the right hand side of (2), to get the left hand side of (2). This y_{i+1} can be used further to get refined y_{i+1} on the left hand side. Here, we predict the value of y_{i+1} from the rough formula (1) and use in (2) to correct the value. Every time, we improve using (2). Hence equation (1) Euler's formula is a predictor and (2) is a corrector.

A predictor formula is used to predict the value of y at y_{i+1} and a corrector formula is used to correct the error and to improve that value of y_{i+1}.

Milne's Predictor- Corrector Formulae

Suppose our aim is to solve $\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$ numerically.

$y_1 = y_0(x_0+h) = y(x_1) = y(x_1),\ y_2 = y(x_0+2h),\ ..., y_n = y(x_0+nh)$, where h is a suitable accepted spacing, which is very small.

By Newton's forward interpolation formula, we have

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + ...$$

, where $u = \frac{x-x_0}{h} \Rightarrow x = x_0 + uh$

Changing y to y'

$$y' = y_0' + u\Delta y_0' + \frac{u(u-1)}{2!}\Delta^2 y_0' + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0' + ... \quad (2)$$

Integrating both sides from x_0 to x_4,

$$\int_{x_0}^{x_4} y'\,dx = \int_{x_0}^{x_0+4h}\left(y_0' + u\Delta y_0' + \frac{u(u-1)}{2!}\Delta^2 y_0' + ... \right)dx = y\,|_{x_0}^{x_0+4h}$$

$$= h\int_0^4\left(y_0' + u\Delta y_0' + \frac{u(u-1)}{2}\Delta^2 y_0' + ... \right)du$$

Since $x = x_0 + uh \Rightarrow dx = hdu$

Now $y_4 - y_0 = h\left[uy_0' + \Delta y_0'\frac{u^2}{2} + \frac{1}{2}\Delta^2 y_0'\left(\frac{u^3}{3} - \frac{u^2}{2}\right) + \frac{1}{6}\Delta^3 y_0'\left(\frac{u^4}{4} - u^3 + u^2\right) + ... \right]$

$$= h\left[4\Delta y_0' + 8\Delta y_0' + \frac{1}{2}\left(\frac{64}{3} - 8\right)\Delta^2 y_0' + \frac{1}{6}\Delta^3 y_0'(64 - 64 + 16) + ... \right]$$

$$= h\left[4\Delta y_0' + 8\Delta y_0' + \frac{20}{3}\Delta^2 y_0' + \frac{8}{3}\Delta^3 y_0' + ... \right]$$

$$= h\left[4\Delta y_0' + 8(E-1)\Delta y_0' + \frac{20}{3}(E-1)^2 y_0' + \frac{8}{3}(E-1)^3 y_0' + \frac{14}{45}\Delta^4 y_0' + ... \right]$$

$$= h\left[4\Delta y_0' + 8(y_1' - y_0') + \frac{20}{3}(y_0' - 2y_1' + y_0') + \frac{8}{3}(y_3' - 3y_2' + 3y_1' - y_0') + \frac{14}{45}\Delta^4 y_0' + ... \right]$$

$$= h\left[\left(4 - 8 + \frac{20}{3} - \frac{8}{3}\right)y_0' + \left(8 - \frac{40}{3} + 8\right)y_1' + \left(\frac{20}{3} - 8\right)y_2 + \frac{8}{3}y_3'\right] + \frac{14}{45}h\Delta^4 y_0' + ..$$

$$= h\left[\frac{8}{3}y_1' - \frac{4}{3}y_2' + \frac{8}{3}y_3'\right] + \frac{14}{45}h\Delta^4 y_0' + ...$$

$$= \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right] + \frac{14h}{45}\Delta^4 y_0' + ... \quad ....(3)$$

Taking into account only up to the third order equation, (3) gives

$$\begin{aligned} y_4 &= y_0 + \frac{4h}{3}\left(2y_1' - y_2' + 2y_3'\right) \\ &= y_0 + \frac{4h}{3}\left(2f_1 - f_2 + 2f_3\right) \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots (4)$$

The error happened in (4) is $\frac{14h}{45}\Delta^4 y_0' + ...$ and this can be proved to be $\frac{14h}{45} y^{(v)}(\xi)$, where $\xi(x_0, x_4)$.

Since $\Delta = E - 1 = e^{hD} - 1 = hD$ for small values of h.

$\therefore$ The error is $\frac{14h^5}{45} y^{(v)}(\xi)$ and hence (3) becomes

$$y_4 = y_0 + \frac{4h}{3}\left(2y_1' - y_2' + 2y_3'\right) + \frac{14h}{45} y^{(v)}(\xi) \quad \ldots\ldots..(5)$$

In general,

$$y_{n+1} = y_{n-3} + \frac{4h}{3}\left(2y_{n-2}' - y_{n-1}' + 2y_n'\right) + \frac{14h^5}{45} y^{(v)}(\xi_1), \text{ where } \xi_1 \in \left(x_{n-3}, x_{n+1}\right) \quad \ldots\ldots(6)$$

This equation is called ***Milne's predictor formula***.

To get Milne's corrector formula, integrate equation (2) between the limits x_0 to $x_0 + 2h$.

$$\therefore \int_{x_0}^{x_0+2h} y'\,dx = \int_{x_0}^{x_0+2h}\left(y_0' + u\Delta y_0' + \frac{u(u-1)}{2}\Delta^2 y_0' + ...\right)dx$$

$$y_2 - y_0 = h\int_0^2\left(y_0' + u\Delta y_0' + \frac{u(u-1)}{2}\Delta^2 y_0' + \ldots \right)du$$

$$= h\int_0^2\left(y_0' u + \frac{u^2}{2}\Delta y_0' + \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Delta^2 y_0' + \ldots \right) \Big|_0^2$$

$$= h\left(2y_0' + 2(E-1)y_0' + \frac{1}{2}\left(\frac{8}{3} - 2\right)(E-1)^2 y_0' - \frac{4}{15}\cdot\frac{1}{24}\Delta^4 y_0' + \ldots \right)$$

$$= h\left(2y_0' + 2\left(y_1' - y_0'\right) + \frac{1}{3}\left(y_2' - 2y_1' + y_0'\right) - \frac{1}{90}\Delta^4 y_0' + \ldots \right)$$

$$= \frac{h}{3}\left(y_0' + 4y_1' + y_2'\right) - \frac{h}{90}\Delta^4 y_0' + \ldots \qquad \ldots\ldots\ldots(7)$$

Again here taking into account only up to third order, we get

$$y_2 = y_0 + \frac{h}{3}\left(y_0' + 4y_1' + y_2'\right) \qquad \ldots(8)$$

Here the error is $-\frac{h}{90}\Delta^4 y_0' + \ldots$ and this can be proved to be $-\frac{h^5}{90} y^{(v)}(\xi)$

where $x_0 < \xi < x_2$; and thus (7) becomes

$$y_2 = y_0 + \frac{h}{3}\left(y_0' + 4y_1' + y_2'\right) - \frac{h^5}{90} y^{(v)}(\xi)$$

In general,

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left(y_{n-1}' + 4y_n' + y_{n+1}'\right) - \frac{h^5}{90} y^{(v)}(\xi_2) \text{ where } x_{n-1} < \xi_2 < x_{n+1}.$$

This equation is called ***Milne's corrector formula***.

Example 1. Find $y(2)$ if $y(x)$ is the solution of $\frac{dy}{dx} = \frac{1}{2}(x+y)$ given

$$y(0) = 0, y(0.5) = 2.636, y(1) = 3.595 \text{ and } y(1.5) = 4.968$$

Solution

Here $x_0 = 0, x_1 = 0.5, x_2 = 1, x_3 = 1.5, x_4 = 2, h = 0.5$

$$y_0 = 2, y_1 = 2.636, y_2 = 3.595, y_3 = 4.968$$

$$f(x,y) = \frac{1}{2}(x+y) = y'$$

By Milne's predictor formula,

$$y_{n+1} = y_{n-3} + \frac{4h}{3}\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

$$\therefore y_4 = y_0 + \frac{4h}{3}\left(2y'_1 - y'_2 + 2y_3\right)$$

Now, $y'_1 = \frac{1}{2}(x_1 + y_1) = \frac{1}{2}(0.5 + 2.636) = 1.568$

$$y'_2 = \frac{1}{2}(x_2 + y_2) = \frac{1}{2}(1 + 3.595) = 2.2975$$

$$y'_3 = \frac{1}{2}(x_3 + y_3) = \frac{1}{2}(1.5 + 4.968) = 3.234$$

$$\therefore y_4 = 2 + \frac{4(0.5)}{3}\left(2(1.568) - 2.2975 + 2(3.234)\right)$$

$$= 6.871$$

Using Milne's corrector formula, we get,

$$y_4 = y_2 + \frac{h}{3}\left(y'_2 + 4y'_3 + y'_4\right)$$

$$y'_4 = \frac{1}{2}(x_4 + y_4) = \frac{1}{2}(2 + 6.871) = 4.4355$$

$$\therefore y_4 = 3.595 + \frac{0.5}{3}\left(2.2975 + 4(3.234) + 4.4355\right)$$

$$= 6.873166667$$

Example 2. Using Milne's method find $y(4.4)$ given $5xy' + y^2 - 2 = 0$

given $y(4) = 1, y(4.1)1.0049, y(4.2) = 1.0097$ and $y(4.3) = 1.0143$.

Solution

$$y' = \frac{2 - y^2}{5x}, x_0 = 4,\ x_1 = 4.1,\ x_2 = 4.2,\ x_3 = 4.3,\ x_4 = 4.4,\ h = 0.1$$

$$y_0 = 1,\ y_1 = 1.0049,\ y_2 = 1.0097,\ y_3 = 1.0143$$

$$y_1' = \frac{2 - y_1^2}{5x_1} = \frac{2 - 1.0049^2}{5(4.1)} = 0.048301267$$

$$y_2' = \frac{2 - y_1^2}{5x_2} = \frac{2 - 1.0097^2}{5(4.2)} = 0.046690757$$

$$y_3' = \frac{2 - y_3^2}{5x_2} = \frac{2 - 1.0143^2}{5(4.3)} = 0.045171884$$

By Milne's predictor formula,

$$y_4 = y_0 + \frac{4h}{3}\left(2y_1' - y_2' + 2y_3'\right)$$

$$= 1 + \frac{4(0.1)}{3}\left(2(0.048301267 - 0.046690757 + 2(0045171884))\right)$$

$$= 1.018700739$$

$$y_4' = \frac{2 - y_4^2}{5x_4} = \frac{2 - 1.018700739^2}{5(4.4)} = 0.043738581$$

Using the corrector formula, we get

$$y_4 = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

$$= 1.0097 + \frac{0.1}{3}\left(0.046690757 + 4(0.045171884) + 0.043738581\right)$$

$$= 1.018737229$$

Example 3. Determine the value of $y(0.4)$ using Milne's method given $y' = xy + y^2$, $y(0) = 1$; use Taylor series to get the values of $y(0.1), y(0.2)$, and $y(0.3)$

Solution

Here $x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, x_4 = 0.4, y_0 = 1, h = 0.5$

$$y' = xy + y^2 \quad \Rightarrow y_0^{'} = x_0 y_0 + y_0^2 = 1$$

$$y'' = xy' + y + 2yy' \Rightarrow y_0^{''} = x_0 y_0^{'} + y_0 + 2y_0 y_0^{'} = 3$$

$$y''' = xy'' + 2y' + 2yy'' + 2(y')^2 \Rightarrow y_0^{'''} = 10$$

$$y_1 = y_0 + hy_0^{'} + \frac{h^2}{2} y_0^{''} \frac{h^3}{3!} y_0^{'''} + \ldots$$

$$= 1 + (0.1)(1) + \frac{0.01}{2}(3) + \frac{0.001}{6}(10) + \ldots = 1.116666667$$

Now $y_1^{'} = x_1 y_1 + y_1^2 = 1.358611111$

$$y_1^{''} = x_1 y_1^{'} + y_1 + 2y_1 y_1^{1} = 4.28675926$$

$$y_1^{'''} = x_1 y_1^{''} + 2y_1 + 2y_1 y_1^{1'} + 2\left(y_1^{'}\right)^2 = 16.4113088$$

$$\therefore y_2 = 1.11666667 + (0.1)(1.358611111) + \frac{(0.01)}{2}(4.28675926) + \frac{0.001}{6}(16.4113088) + \ldots$$

$$= 1.276696798$$

Once again $y_2^{'} = x_2 y_2 + y_2^2 = 1.885294059$

$$y_2^{''} = x_2 y_2^{'} + y_2 + 2y_2 y_2^{'} = 6.467653362$$

$$y_2^{'''} = x_2 y_2^{''} + 2y_2^{'} + 2y_2 y_2^{''} + 2\left(y_2^{'}\right)^2 = 28.68725078$$

$$\therefore\ y_3 = 1.276696793 + (0.1)(1.885294059) + \frac{0.01}{2}(6.467653362) + \frac{0.001}{6}(28.68725078)$$

$$= 1.502345674$$

By Milne's predictor formula

$$y_4 = y_0 + \frac{4h}{3}\left(2y_1^{'} - y_2^{'} + 2y_3^{'}\right)$$

Since $y_1^{'} = 1.358611111$

$$y_2^{'} = 1.885294059$$

$$y_3^{'} = x_3 y_3 + y_3^2 = 2.707746226$$

$$\therefore\ y_4 = 1 + \frac{4(0.1)}{3}\left(2(1.358611111) - 1.885294059 + 2(2.70774626)\right)$$

$$= 1.832989424$$

$$y_4^1 = x_4 y_4 + y_4^2 = 4.093046$$

Now using Milne's corrector formula,

$$y_4 = y_2 + \frac{h}{3}\left(y_2^{'} + 4y_3^{'} + y_4^{'}\right)$$

$$= 1.276696793 + \frac{0.1}{3}\left(1.885294059 + 4(2.70774626) + 4.093046\right)$$

$$= \underline{\underline{1.83700763}}$$

Example 4. Given $\frac{dy}{dx} = \frac{1}{2}\left(1 + x^2\right)y^2$ and $y(0) = 1, y(0.1) = 1.06, y(0.2) = 1.12, y(0.3) = 1.21$,

evaluate $y(0.4)$ by Milne's predictor-corrector method.

Solution

$$x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, x_4 = 0.4$$

$$y_0 = 1, y_1 = 1.06, y_2 = 1.12, y_3 = 1.21, h = 0.1$$

$$y' = \frac{1}{2}\left(1 + x^2\right)y^2$$

$$y_0' = \frac{1}{2}\left(1 + x_0\right)^2 y_0^2 = \frac{1}{2}\left(1 + 0\right)\left(1\right) = \frac{1}{2}$$

$$y_1' = \frac{1}{2}\left(1 + x_1^2\right)^2 y_1^2 = \frac{1}{2}\left(1 + 0.1^2\right)\left(1.06\right)^2 = 0.567418$$

$$y_3{}' = \frac{1}{2}(1 + x_3^2)y_3^2 = \frac{1}{2}(1 + 0.3^2)(1.21)^2 = 0.7979345$$

By Milne's predictor formula

$$y_4 = y_0 + \frac{4h}{3}\left(2y_1' - y_2' + 2y_3'\right)$$

$$= 1 + \frac{4(0.1)}{3}\left(2(0.367418) - 0.652288 + 2(0.7979345)\right)$$

$$= 1.277122267$$

By Milne's corrector formula

$$y_4 = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right) \qquad \left(y_4' = 0.946003944\right)$$

$$= 1.12 + \frac{0.1}{3}\left(0.652288 + 4(0.7979345) + 0.946003944\right)$$

$$= 1.279667665$$

Once again if we use this value of y_4, we get

$$y_4' = \frac{1}{2}\left(1 + 0.4^2\right)\left(1.279667665\right)^2 = 0.949778612$$

So $y_4 = 1.12 + \frac{0.1}{3}\left(0.652288 + 4(0.7979345) + 0.949778612\right)$

$= 1.279793487$

If we repeat this procedure once again using this y_4,

$$y_4' = \frac{1}{2}\left(1 + 0.4^2\right)(1.279793487)^2 = 0.949965394$$

By Milne's corrector formula

$$y_4 = 1.12 + \frac{0.1}{3}\left(0.652288 + 4(0.7979345) + 0.949965394\right)$$

$= 1.279799713$

Continuing this process again and again, after some steps we get

$$y_4 = 1.279800037$$

Example 5. Given $y' = 1 - y$, and $y(0) = 0$, and

i) $y(0.1)$ by Euler method; using that value obtained

ii) $y(0.2)$ by Modified Euler method

iii) Obtain $y(0.3)$ by Improved Euler method and find

iv) $y(0.4)$ by Milne's method.

Solution

By Euler method $y_1 = y_0 + hf(x_0, y_0) = 0 + (0.1)(1 - 0) = 0.1$

By Modified Euler

$$y_2 = y_1 + hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}hf(x_1, y_1)\right)$$

$$= 0.1 + (0.1)\left(1 - \left(0.1 + \frac{1}{2}(0.1)(1 - 0.1)\right)\right) = 0.1855$$

By improved Euler method

$$y_3 = y_2 + \frac{1}{2}h\left[f\left(x_2, y_2\right) + f\left(x_3, y_2 + hf\left(x_2, y_2\right)\right)\right]$$

$$= 0.1855 + \frac{0.1}{2}\left[(1 - 0.1855) + 1 - \left(0.1855 + (0.1)(1 - 0.1855)\right)\right]$$

$$= 0.2628775$$

Now using Milne's predictor formula

$$y_4 = y_0 + \frac{4h}{3}\left(2y_1' - y_2' + 2y_3'\right)$$

$$= y_0 + \frac{4h}{3}\left(2(1 - y_1) - (1 - y_2) + 2(1 - y_3)\right)$$

$$= 0 + \frac{4(0.1)}{3}\left(2(1 - 0.1) - (1 - 0.1855) + 2(1 - 0.2628775)\right)$$

$$= 0.327966$$

$$y_4' = 1 - y_4 = 1 - 0.327966 = 0.672034$$

By Milne's corrector formula

$$y_4 = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

$$= y_2 + \frac{h}{3}\left((1 - y_2) + 4(1 - y_3) + y_4'\right)$$

$$= 0.1855 + \frac{0.1}{3}((1 - 0.1855) + 4(0.2628775) + 0.672034$$

$$= 0.33333413$$

Exercise

1. Using Milne's method, find $y(0.2)$ given

$$\frac{dy}{dx} = 0.2x + 0.1y,\ y(0) = 2,\ y(0.05) = 2.0103,\ y(0.1) = 2.0211,\ y(0.15) = 2.0323$$

2. Find $y(0.8)$ given $y' = y - x^2, y(0) = 1, y(0.2) = 1.12186, y(0.4)1.46820, y(0.6) = 1.7379$.

3. Using Runge-Kutta method of fourth order, find y at $x = 0.1, x = 0.2, x = 0.3$ given $y' = xy + y^2, y(0) = 1$. Continue your work to get $y(0.4)$ by Milne's method.

4. Solve $y' = \frac{1}{2}(1+x)y^2, y(0) = 1$ by Taylor series method at $x = 0.2, x = 0.4, x = 0.6$ and hence find $y(0.8)$ and $y(1)$ by Milne's method.

5. If $\frac{dy}{dx} = 2e^x - y, y(0) = 2, y(0.1) = 2.010, y(0.2) = 2.040, y(0.3) = 2.090$, find $y(0.4)$ and $y(0.5)$ by Milne's method.

6. Estimate $y(0.8)$ and $y(1)$ using Milne's method correct to three decimal places, given $y' = 1 + y^2, y(0) = 0, y(0.2) = 0.2027, y(0.4) = 0.4228, y(0.6) = 0.6841$.

7. Solve $y' = x - y^2, y(0) = 1$ to obtain $y(0.4)$ by Milne's method. Obtain the data you require by any method you like.

8. Using both predictor-corrector methods, estimate $y(1.4)$ if y satisfies $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$, and $y(1) = 1, y(1.1) = 0.996, y(1.2) = 0.986, y(1.3) = 0.972$.

9. Given $y' = \frac{1}{x+y}, y(0) = 2, y(0.2) = 2.0933, y(0.4) = 2.1755, y(0.6) = 2.2493$, find $y(0.8)$ by Milne's predictor-corrector method.

10. Compute $y(0.6)$ by Milne's method given $y' = x + y, y(0) = 1$ with $h = 0.2$. Obtain the required data by Taylor series method.

11. Given $y' = 3e^x + 2y, y(0) = 0$, find $y(0.1)$ by Euler method; $y(0.2)$ by Taylor series method; $y(0.3)$ by Runge-Kutta method and $y(0.4)$ by Milne's method.

12. Find $y(0.2)$ by Taylor series method; $y(0.4)$ by modified Euler method; $y(0.6)$ by Runge-Kutta method and $y(0.8)$ by Milne's method, given $y' = 1 + y^2, y(0) = 0$.

13. Given $y' = 1 + xy, y(0) = 1$ obtain $y(0.1)$ by Picard's method; $y(0.2)$ by modified Euler method; $y(0.3)$ by Runge-Kutta method; $y(0.4)$ by Milne's predictor-corrector method.

14. Given $y' == 2 - xy^2, y(0) = 10$, obtain power series by Picard's method; using Milne's method, estimate and show that $y(1) = 1.6505, h = 0.2$.

Estimate $y(0.5), y(0.4)$ given $y' = x + y^2, y(0) = 1$ using $h = 0.1$.

Boundary-Value Problem (BVP)

BVPs can be solved numerically by using either the Shooting Method or the Finite-Difference Method (FDM). Here we consider the numerical solution of PVP using FDM. The former method is left for the students as a reading assignment.

Some simple examples of two-point linear BVPs are:

$$y''(x) + f(x)y'(x) + g(x)y(x) = r(x) \tag{1}$$

with boundary conditions

$$y(x_0) = a \text{ and } y(x_n) = b \tag{2}$$

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + O(h^2) \qquad y^{iv}(x) = p(x)y(x) + q(x)$$

(3)

with boundary conditions

$$y(x_0) = y'(x_0) = A \text{ and } y(x_n) = y'(x_n) = B. \tag{4}$$

Problems of the type in Eq. (3) and Eq. (4), which involve the fourth-order differential equation, are much involved and will not be discussed here. There exist many methods for solving second-order BVPs of the type in Eq. (1) and Eq. (2). Of these, the Finite-Difference Method is popular one and will be discussed.

Finite-Difference Method (FDM)

The FDM for the solution for a two-point BVP consists in replacing the derivatives occurring in the differential equation (and the boundary conditions as well) by means of their finite-difference approximations and then solving the resulting linear system of equations by a standard procedure.

To obtain the approximate finite-difference approximations to the derivatives, we proceed as follows:

Expanding $y(x+h)$ in Taylor's series, we have

$$y(x+h) = y(x) + hy'(x) + \tfrac{h^2}{2} y''(x) + \tfrac{h^3}{6} y'''(x) + \dots \tag{5}$$

from which we obtain

$$y'(x) = \frac{y(x+h) - y(x)}{h} - \frac{h}{2} y''(x) - \dots$$

Thus we have

$$y'(x) = \frac{y(x+h) - y(x)}{h} + O(h) \tag{6}$$

Which is forward difference approximation for $y'(x)$.

Similarly, expansion of $y(x-h)$ in Taylor's series gives

$$y(x-h) = y(x) - hy'(x) + \tfrac{h^2}{2} y''(x) - \tfrac{h^3}{6} y'''(x) + \dots \tag{7}$$

from which we obtain

$$y'(x) = \frac{y(x) - y(x-h)}{h} + O(h) \tag{8}$$

Which is the backward difference approximation for $y'(x)$.

A central difference approximation for $y'(x)$ can be obtained by subtracting Eq. (7) from Eq. (5). We thus have

$$y'(x) = \frac{y(x+h) - y(x-h)}{2h} + O(h^2) \tag{9}$$

It is clear that Eq. (9) is better approximation to $y'(x)$ than either Eq. (6) or Eq. (8). Again, adding Eq. (5) and Eq. (7) we get an approximation for $y''(x)$ as

$$y''(x) = \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} + O(h^2) \tag{10}$$

In a similar manner, it is possible to derive finite-difference approximations to higher derivatives.

To solve the BVP defined by Eq. (1) and Eq. (2), we divide the range $[x_0, x_n]$ into n equal subintervals of width h so that

$$x_i = x_0 + ih, \quad i = 0, 1, 2, ..., n.$$

The corresponding values of y at these points are denoted by

$$y(x_i) = y_i = y(x_0 + ih), \ i = 0, 1, 2, ..., n.$$

From Eq. (9) and Eq. (10), values of $y'(x)$ & $y''(x)$ at the point $x = x_i$ can now be written as

$$y_i' = \frac{y_{i+1} - y_i}{2h} + O(h^2)$$

and $$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + O(h^2)$$

In many applied problems; however, derivative boundary conditions may be prescribed, and this requires a modification of the procedures described above. The following examples illustrate the application of the FDM.

Example 1: A boundary-value problem is defined by

$$y^{ii} + y + 1 = 0, \qquad 0 \le x \le 1,$$

where $$y(0) = y(1) = 0, \; and \;\; h = 0.5$$

Use the FDM to determine the value of y(0.5). Its exact solution is given by

$$y(x) = \cos x + \frac{1-\cos 1}{\sin 1}\sin x - 1,$$

from which, we obtain

$$y(0.5) = 0.139493927.$$

Here $nh = 1$. The difference equation is approximated as

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + y_i + 1 = 0$$

, and this gives after simplification

$$y_{i-1} - \left(2 - h^2\right)y_i + y_{i+1} = -h^2, \;\; i = 1, 2, ..., n-1,$$

which together with the boundary conditions $y_0 = y_n = 0$, comprises a system of $(n+1)$ equations for $(n+1)$ unknowns $y_0, y_1, ..., y_n$.

Choosing $h = \frac{1}{2}$ (i.e. $n = 2$), the above system becomes

$$y_0 - \left(2 - \frac{1}{4}\right) y_1 + y_2 = -\frac{1}{4}.$$

With $y_0 = y_2 = 0$, this gives

$$y_1 = y(0.5) = 0.142857142.$$

Comparing with exact solution given above shows that the error in the computed solution is 0.00336.

On the other hand, if we choose $h = \frac{1}{4}$ (i.e. n=4), we obtain the three equations:

$$y_0 - \frac{31}{16} y_1 + y_2 = -\frac{1}{16}$$
$$y_1 - \frac{31}{16} y_2 + y_3 = -\frac{1}{16}$$
$$y_2 - \frac{31}{16} y_3 + y_4 = -\frac{1}{16}$$

Where $y_0 = y_4 = 0$. Solving the system we obtain $y_2 = y(0.5) = 0.140311804$, the error in which 0.00082. Since the ration of the error is about 4, it follows that the order of convergence is h^2.

These results show that the accuracy obtained by the finite difference method depends upon the width of the subinterval chosen and also on the order of approximations. As h is reduced, the accuracy increases but the number equations to be solved also increases.

Example 2: Solve the boundary-value problem

$$\frac{d^2y}{dx^2} - y = 0, \quad 0 \le x \le 2,$$

with $y(0) = 0$ & $y(2) = 3,62686$.

The exact solution of this problem is $y = \sinh x$. The finite-difference approximation is given by

$$\frac{1}{h^2}\left(y_{i-1} - 2y_i + y_{i+1}\right) = y_i \qquad (i)$$

We subdivide the subinterval $[0,2]$ the four equal parts so that $h = 0.5$. Let the values of y at the five points be $y_0, y_1, y_2, y_3, \& \, y_4$. We are given that $y_0 = 0$, and $y_4 = 3.62686$.

Writing the difference equations at the three interval points (which are the unknowns), we obtain

$$\left.\begin{aligned} 4\left(y_0 - 2y_1 + y_2\right) &= y_1 \\ 4\left(y_1 - 2y_2 + y_3\right) &= y_2 \\ 4\left(y_2 - 2y_3 + y_3\right) &= y_3 \end{aligned}\right\} \qquad (ii)$$

, respectively. Substituting for $y_0 \,\&\, y_4$ and rearranging, we get the system

$$\left.\begin{aligned} -9y_1 + 4y_2 &= 0 \\ 4y_1 - 9y_2 + 4y_3 &= 0 \\ 4y_2 - 9y_3 &= -14.50744 \end{aligned}\right\} \qquad (iii)$$

The solution of (iii) is given in the table below.

x	Computed solution of y	Exact value $y = \sinh x$	Error
0.5	0.52635	0.52110	0.00525
1.0	1.18428	1.17520	0.00908
1.5	2.13829	2.12928	0.00901

Exercise

1. Solve the boundary value defined by

$$y'' - y = 0, y(0) = 0, y(1) = 1,$$

by finite-difference method. Compare the computed solution at $y(0.5)$ with the exact value. Take $h = 0.25$.

2. Project work. Shooting Method. This is a popular method for the solution of two-point boundary-value problems. If the problem is defined by

$$y''(x) = f(x), y(x_0) = 0 \, and \, y(x_1) = A,$$

Then it is first transformed into the initial value problem

$y'(x) = z$, $z'(x) = f(x)$ with $y(x_0) = 0$ and $z(x_0) = m_0$, where m_0 is a guess for the value $y'(x_0)$. Let the solution corresponding to $x = x_1$ be Y_0. If Y_1 is the value obtained by another guess m_1 for $y'(x_0)$, then $Y_0 \, and \, Y_1$ and are related linearly. Thus, linear interpolation can be carried out between the values (x_0, y_0) and (m_1, y_1).

Obviously, the process can be repeated till we obtain the value for $y(x_1)$ is close to A.

Apply the Shooting method to solve the BVP

$$y''(x) = y(x), \; y(0) = 0, y(1) = 1$$