Chapter One

Fourier series and orthogonal functions

1.1 Orthogonal functions

1.2 Fourier series

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1.2.3 Fourier series of odd and even functions

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NTRODUCTION:

- We know that Taylor's series representation of functions are valid only for those functions which are continuous and differentiable. But there are many discontinuous periodic functions of practical interest which requires to express in terms of infinite series containing "sine" and "cosine" terms

- Fourier series, which is an infinite series representation in term of “sine” and “cosine” terms , is a useful tool here. Thus, Fourier series is, in certain sense, more universal than Taylor’s series as it applies to all continuous, periodic functions and discontinuous functions
- Fourier series is a very powerful method to solve ordinary and partial differential equations, particularly with periodic functions.
- Fourier series has many applications in various fields like Approximation Theory, Digital Signal Processing, Heat conduction problems, Wave forms of electrical field, Vibration analysis, etc.

- Fourier series was developed by Jean Baptiste Joseph Fourier in 1822.
- **Dirichlet Condition For Existence Of Fourier Series of f(x):**

i. f(x) is bounded.

ii. f(x) is single valued.

iii. f(x) has finite number of maxima and minima in the interval.

iv. f(x) has finite number of discontinuity in the interval.

Definition 4 (Periodic) Let $T > 0$.

1. A function f is called T-periodic or simply periodic if $f(x + T) = f(x)$------------------- (2) for all x.
2. The number T is called a period of f.
3. If f is non-constant, then the smallest positive number T with the above property is called the fundamental period or simply the period of f.

- For a *T*-periodic function

$$f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = \dots = f(x + nT)$$

If *T* is a period then *nT* is also a period for any integer $n > 0$. *T* is called a fundamental period.

Definition 2 (Orthogonal Functions) Two functions f and g are said to be orthogonal over the interval [a, b] if

$$\int_a^b f(x)g(x)\,dx = 0$$

Theorem 2 The functions in the trigonometric system 1, cos x, cos 2x, ..., cosmx, ..., sin x, sin 2x, ..., sin nx, ... are orthogonal over the interval [c, c+2π] in other words, if m and n are two nonnegative integers, then

a. $\int_{c}^{c+2\pi} \cos mx \cos nx \, dx = \begin{cases} 0, if\ m \neq n \\ \pi, if\ m = n \end{cases}$

b. $\int_{c}^{c+2\pi} \sin mx \sin nx \, dx = \begin{cases} 0, if\ m \neq n \\ \pi, if\ m = n \end{cases}$

c. $\int_{c}^{c+2\pi} \cos mx \sin nx \, dx = 0 \ \forall \ m, n$

Proof. To prove this theorem use the identities

$$\sin\alpha \cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

$$\cos\alpha \sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

$$\sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]$$

$$\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

1.2 Fourier Series of 2π-Periodic Functions

Proposition The Fourier series representation of f(x) over the interval $c < x < c + 2\pi$ *is given by*

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) - - - - - - *$$

Then the coefficients a_0 , a_n , b_n for n = 1, 2, ... are called the Fourier coefficients of f and are given by the Euler's formulas

$$a_0 = \frac{1}{2\pi} \int_c^{c+2\pi} f(x)\,dx \quad , \quad a_n = \frac{1}{2\pi} \int_c^{c+2\pi} f(x) \cos nx\,dx$$

and $$b_n = \frac{1}{2\pi} \int_c^{c+2\pi} f(x) \sin nx\,dx$$

To determine the coefficient a_0

Integrate both sides of Eq. over the interval [c, c+2π] with respect to x

$$\int_c^{c+2\pi} f(x)\,dx = \int_c^{c+2\pi} a_0\,dx + \sum_{n=1}^{\infty}\int_c^{c+2\pi} (a_n\cos nx + b_n \sin nx)\,dx$$

Since $\int_c^{c+2\pi} \sin nx\,dx = \int_c^{c+2\pi} \cos nx\,dx = 0, n = 1,2,3,\cdots$

$$\int_c^{c+2\pi} f(x)\,dx = \int_c^{c+2\pi} a_0\,dx = 2\pi a_0 \Rightarrow a_0 = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)\,dx$$

To determine the coefficient a_n

Multiplying both sides of Eq. With Cosnx and integrating the resulting Eq. over the interval [c, c+2π] with respect to x ,we get $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$

Similarly, the coefficient b_n is determined by multiplying both sides of Eq. ( ) with sin *nx* and integrating the resulting equation over the interval [c, c+2π] with respect to x, we get

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

Euler's Formulae for Different Intervals

Case (i): If C = 0, then the interval for the above series ( ) become $0<x< 2\pi$ and the Euler's formulas reduce to

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\;dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\;dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

- Case (ii): If C = -π , then the interval for the above series ([illegible]) become -π <x< π and the Euler's formulas reduce to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Example 1 Given the step function

$$\mathrm{f(x)} = \begin{cases} -1, if -\pi < x < 0 \\ 1, if\ 0 < x < \pi \end{cases}$$

a. Show that f has a Fourier series $\mathrm{f(x)} = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$

b. Find a series for $\frac{\pi}{4}$ or show that

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

Solution :Let the series be of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$,where

$$a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right] = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -dx + \int_{0}^{\pi} dx\right] = 0$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right] = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\cos nx\,dx + \int_{0}^{\pi} \cos nx\,dx\right] = 0$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right] = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin nx\,dx + \int_{0}^{\pi} \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin nx\,dx + \int_{0}^{\pi} \sin nx\,dx\right] = \frac{2}{\pi n}[1-(-1)^n] = \begin{cases} 0, if\ n\ is even \\ \frac{4}{n\pi}, if\ n\ is\ odd \end{cases}$$

a. Thus,the Fourier series of the given function is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} \qquad \text{----------1}$$

b. Put $x = \frac{\pi}{2}$ into Eq.1 , we get $\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$

π

Example 2.Show that the Fourier series representation for the function

$$f(x) = \begin{cases} 1, if \ 0 < x < \pi \\ 0 \ \ , if \ \pi < x < 2\pi \end{cases}$$

is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$$

Solution : Let the Fourier series representation f is of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right] = \frac{1}{2\pi}\left[\int_0^{\pi} dx\right] = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\cos nx\,dx + \int_{\pi}^{2\pi} f(x)\cos nx\,dx\right] = \frac{1}{\pi}\int_0^{\pi} \cos nx\,dx = 0$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\sin nx\,dx + \int_{\pi}^{2\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin nx\,dx$$

$$= \frac{-1}{\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi} = \frac{-((-1)^n - 1)}{\pi n} = \begin{cases} 0, if\ n\ is\ even \\ \frac{2}{\pi n}, if\ n\ is\ odd \end{cases}$$

- It follows that the Fourier series representation of the given function is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$$

1.2.2 Fourier Series of Functions With Arbitrary Period

In many of the engineering problems (i.e. electrical engineering problems) the period of the function is not always 2pi but it is different say 2L or T.

Fourier series representation of f(x) over the interval $c \le x \le c+2L$

The Fourier series expansion of f(x) in the interval

$c \le x \le c+2L$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \text{------(c)}$$

Where

$$a_0 = \frac{1}{2L}\int_c^{c+2L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Euler's Formulae for Different Intervals

- Case (i): If C = 0, then the interval for the above series (c) become 0<x< 2L and the Euler's formulas reduce to

$$a_0 = \frac{1}{2L}\int_0^{2L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

For n=1,2,3,...

- Case (ii): If C = -L, then the interval for the above series (c) become -L<x< L and the Euler's formulas reduce to

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

,for n=1,2,3,...

Example 1 Find the Fourier series expansion of f(x) if

$$\mathrm{f(x)} = \begin{cases} 1, if\ \ 0 \le x \le 1 \\ 0\ \ , if\ \ 1 \le x \le 2 \end{cases}$$

Solution

- Here 2l=2 and hence L=1

Let the series be

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

Since L=1,we have

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos(n\pi x) + b_n \sin(n\pi x)]$$

and

$$a_0 = \frac{1}{2L}\int_0^{2L} f(x)\,dx$$

$$a_0 = \frac{1}{2}\left[\int_0^1 1\,dx + \int_1^2 2\,dx\right] = \frac{3}{2}$$

To find a_n we use the formula

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$a_n = \int_0^1 \cos(n\pi x)\,dx + \int_1^2 2\cos(n\pi x)\,dx$$

$$a_n = \left[\frac{\sin(n\pi x)}{n\pi}\right]_0^1 + 2\left[\frac{\sin(n\pi x)}{n\pi}\right]_1^2$$

$$a_n = [0-0] + 2[0-0] = 0$$

To find b_n we use the formula

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

since L=1, $a_n = \int_0^1 \sin(n\pi x)\,dx + 2\int_1^2 \sin(n\pi x)\,dx$

$$a_n = \frac{-1}{n\pi}[1-(-1)^n] = \begin{cases} \frac{-2}{n\pi}, if\ n\ is\ odd \\ 0, if\ n\ is\ even \end{cases}$$

Thus, the Fourier series of the given function is

$$f(x) = \frac{3}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}$$

- Note: The Fourier series converges to f (x) if f is continuous at x and $\frac{f(x+)+f(x-)}{2}$ other wise.

1.2.3 Fourier series of odd and even functions

Definition 1 (Even and Odd) Let f be a function defined on an interval I (finite or infinite) centred at x = 0.

1 . f is said to be even if f (−x) = f (x) for every x in I.

2. f is said to be odd if f (−x) = −f (x) for every x in I.

Examples of even functions are

$$x^2, x^n, if\ n\ is\ even, \cos x\ , |x|, \ldots$$

Examples of Odd functions are

$$x, x^n, if\ n\ is\ odd, \sin x\ , \ldots$$

Theorem 2 Let f be a function which domain includes [−a, a] where a > 0.

1. If f is an even function , then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
2. If f is an odd function ,then $\int_{-a}^{a} f(x)\,dx = 0$

Theorem 3 When adding or multiplying even and odd functions, the following is true:

a. even + even = even

b. even × even = even

c. odd + odd = odd

d.odd × odd = even

e. even × odd = odd

Theorem Suppose that f is 2pi-periodic and has the Fourier series representation

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) - - - - - *$$

Then :

1. f is an even function if and only if $b_n = 0$ for all n and in this case $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ where

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx \quad \text{and} \quad a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx$$

2. f is an odd function if and only if $a_n = 0$ for all n and in this case $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\, dx$$

Theorem Suppose that f is 2L-periodic and has the Fourier series representation

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

Then :

1. f is an even function if and only if $b_n = 0$ for all n and in this case $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$ where $a_0 = \frac{1}{L}\int_0^L f(x)\,dx$ and $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$
2. f is an odd function if and only if $a_n = 0$ for all n and in this case $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Example 1. Obtain the Fourier series for f(x) = IxI in the interval $-\pi < x < \pi$ and deduce that $$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$$

Solution. we have f(x) = I x I

since f (-x) = I-xI = I x I = f(x) , f(x) is an even function

Therefore ,f(x) contain only cosine terms and we have $b_n = 0$

Let $$f(x)=|x|=a_0+\sum_{n=1}^{\infty} a_n \cos nx$$

we have $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx=\frac{1}{\pi}\int_{0}^{\pi} x\,dx=\frac{\pi}{2}$ and

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx=\frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

*

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Contu'd-----

$$a_n = \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right] = \begin{cases} \frac{-4}{\pi n^2}, if\ n\ is\ odd \\ 0, if\ n\ is\ even \end{cases}$$

There fore, the required Fourier series expansion is

$$f(x) = |x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} ------*$$

Putting x=0 in Eq. star, we get

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Class Activities

1. Let f(x) = x for $-\pi \le x \le \pi$. Write the Fourier series of f on $[-\pi \;\; \pi]$.
2. Obtain the Fourier Series expansion for the function $f(x) = x^2$ in $-\pi < x < \pi$ .Hence, deduce that

   a. $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

   b. $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

   c. $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Example Find the Fourier series representation of $f(x) = x$ on the interval $-2 \leq x \leq 2$.

Solution. we have $f(x) = x$ since $f(-x) = -x = -f(x)$. There fore ,f(x) is an odd function. Hence $a_0 = a_n = 0$ for all n

Let $f(x) = x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ where $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$$b_n = \frac{1}{2}\int_0^2 x\sin\left(\frac{n\pi x}{2}\right)dx$$ since 2L=4,we have

$$b_n = \left[\frac{-2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$b_n = \frac{-4}{n\pi}(-1)^n = \frac{4(-1)^{n+1}}{n\pi}$$

Thus, $$f(x) = x = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{2}\right)$$ for $-2 \leq x \leq 2$.

Class Activities

1. Find the Fourier series representation of f (x) = |x| in the interval –L ≤ x ≤ L.
2. Obtain the Fourier series expansion of $f(x) = x^2$ in the interval $(-L, L)$

   Find the sum of $\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right]$

3. Obtain the Fourier series for f(x) defined in $(-1, 1)$ by

$$f(x) = \begin{cases} k_1, if -1 < x < 0 \\ k_2, if \quad 0 < x < 1 \end{cases}$$

Half-range series ,Period 0 or L (o to π)

Let $f(x)$ be defined on the interval $0 \le x \le L$. Then the **sine series** representation or half-range sine Fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

and the cosine series or half –range Cosine Fourier series representation of *f (x) is given by*

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

Where

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$ and $$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Let $f(x)$ be defined on the interval $0 \leq x \leq \pi$. Then the **cosine series** representation or half-range cosine Fourier series is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

Where

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx \quad \text{and} \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\, dx$$

and the sine series or half –range sine Fourier series representation of *f (x) is given by*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

Where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\, dx$$

Example 1. Find the sine and cosine representations of *f (x) = x for 0 ≤ x ≤ π.*

Solution The sine series representation is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

Where

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin(nx)\, dx$$ integrating ,we find that

$$b_n = (-1)^{n+1}\frac{2}{n}$$

so the required half–range sine Fourier series or Fourier sine representation is

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

The cosine series representation is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

Where $a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2}$ and

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx\,dx = \frac{2}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right] = \begin{cases} \frac{-4}{\pi n^2}, if\ n\ is\ odd \\ 0, if\ n\ is\ even \end{cases}$$

so the cosine series representation is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

Example 2. Find the sine and cosine representations of $f(x) = x$ for $0 \le x \le L$.

Solution The sine series representation is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Where

$$b_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2L(-1)^{n+1}}{n\pi}$$

So the required sine Fourier series of the given function is

$$f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right)$$

The cosine series representation is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

Where

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}$$ and

$$a_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx$$

$$a_n = \frac{-2L}{n^2\pi^2}[1-(-1)^n] = \begin{cases} \frac{-4L}{n^2\pi^2}, if\ n\ is\ odd \\ 0, if\ n \neq 0, is\ even \end{cases}$$

So the required cosine series is

$$f(x) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi x}{L}\right)$$

1.3 Complex form of Fourier series

. Let the real function *f (x) be defined on the interval* $c < x < c + 2\pi$ *Then the* complex Fourier series representation of *f (x) is*

$$f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{inx}$$

Where

$$c_n = \frac{1}{2\pi} \int_c^{c+2\pi} f(x) e^{-inx}\, dx$$ *for all* $n = 0, \pm 1, \pm 2, \cdots$

How do you get this formulae ? : Here is the answer for a function f with period 2π *It Fourier series representation is given by*

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) - - - - ■$$

- From Euler's formula , we have

$$e^{inx} = \cos nx + i \sin nx \quad \text{and} \quad e^{-inx} = \cos nx - i \sin nx$$

Hence

$$\cos nx = \left[\frac{e^{inx} + e^{-inx}}{2}\right] \quad \text{and} \quad \sin nx = \left[\frac{e^{inx} - e^{-inx}}{2}\right]$$

Substituting these into Eq- ■,we have

$$c_{-n} = \frac{1}{2}[a_n + ib_n] = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)e^{inx}\,dx$$, where $a_0 = c_0 = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)\,dx$
,

$$f(x) = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{inx} + c_{-n}e^{-inx}\right) \quad \text{and} \quad c_n = \frac{1}{2}[a_n - ib_n] = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)e^{-inx}\,dx$$

Hence, $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ with

$$c_n = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)e^{-inx}\,dx \quad \text{for all} \quad n = 0, \pm 1, \pm 2, \cdots$$

Note that If f(x) is a period function of period 2L,then the complex form of the Fourier series is given by $f(x)=\sum_{n=-\infty}^{\infty} c_n e^{\frac{i\pi n x}{L}}$ where

$$c_n=\frac{1}{2\pi}\int_{C}^{C+2L} f(x)e^{\frac{-i\pi n x}{L}}dx \quad , n=0,\pm1,\pm2,\cdots$$

Example1: Find the complex Fourier series representation of $f(x)=\begin{cases}0, if\ 0<x<1\\1, if\ 1<x<4\end{cases}$

***Solution** : The function f (x) is defined on the interval*
***0 ≤ x ≤ 2L, with 2L= 4, so** L= 2.Thus, the complex Fourier coefficients c_n are given by*

$$c_n=\frac{1}{2\pi}\int_{C}^{C+2L} f(x)e^{\frac{i\pi n x}{L}}dx \quad n=0,\pm1,\pm2,\cdots$$

$$c_n = \frac{1}{4}\int_0^4 f(x)e^{\frac{i\pi n x}{2}}\,dx = \frac{1}{4}\left[\int_0^1 f(x)e^{\frac{i\pi n x}{2}}\,dx + \int_1^4 f(x)e^{\frac{i\pi n x}{2}}\,dx\right]$$

$$c_n = \frac{i}{2\pi n}\left(1 - e^{-\frac{in\pi}{2}}\right) \quad n \neq 0$$ and $n = \pm 1, \pm 2, \cdots$

But, n=0 $c_0 = \frac{1}{4}\int_1^4 1\,dx = \frac{3}{4}$ and hence the complex Fourier series of f is

$$f(x) = \frac{3}{4} + \frac{1}{2\pi}\sum_{n=-\infty, n\neq 0}^{\infty}\left(\frac{1 - e^{-\frac{in\pi}{2}}}{n}\right)e^{\frac{in\pi}{2}}$$

Example 2: Find the complex Fourier series representation of $f(x) = \sin ax$ where a is not an integer in $-\pi < x < \pi$.<u>Ans</u> $\sin ax = \frac{i \sin a\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1} n e^{inx}}{a^2 - n^2}$

Solution :

The Complex form of the Fourier series of the given function f is of the form $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \quad n = 0, \pm 1, \pm 2, \cdots$$

But,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin ax \;\; e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{e^{iax} - e^{-iax}}{2i}\right) e^{-inx}\, dx$$

$$c_n = \frac{1}{4\pi i}\left[\frac{1}{(a-n)i} e^{(a-n)ix} + \frac{1}{(a+n)i} e^{(a+n)ix}\right]_{x=-\pi}^{x=\pi}$$

$$c_n = \frac{-1}{4\pi}\left[\frac{e^{ia\pi}(-1)^n - e^{-ia\pi}(-1)^n}{(a-n)} - \frac{e^{ia\pi}(-1)^n - e^{-ia\pi}(-1)^n}{(a+n)}\right]$$

$$c_n = \frac{(-1)^{n+1}}{4\pi}\left[\frac{\left(e^{ia\pi} - e^{-ia\pi}\right)(a+n) - \left(e^{ia\pi} - e^{-ia\pi}\right)(a-n)}{a^2 - n^2}\right]$$

$$c_n = \frac{(-1)^{n+1}}{4\pi}\left[\frac{\left(ae^{ia\pi} + ne^{ia\pi} - ae^{-ia\pi} - ne^{-ia\pi}\right) + -ae^{ia\pi} + ae^{-ia\pi} + ne^{ia\pi} - ne^{-ia\pi}}{a^2 - n^2}\right]$$

$$c_n = \frac{(-1)^{n+1}}{4\pi}\left[\frac{\left(e^{ia\pi} - e^{-ia\pi}\right)(a+n) - \left(e^{ia\pi} - e^{-ia\pi}\right)(a-n)}{a^2 - n^2}\right]$$

$$c_n = \frac{(-1)^{n+1}}{4\pi}\left[\frac{\left(ae^{ia\pi} + ne^{ia\pi} - ae^{-ia\pi} - ne^{-ia\pi}\right) + -ae^{ia\pi} + ae^{-ia\pi} + ne^{ia\pi} - ne^{-ia\pi}}{a^2 - n^2}\right]$$

$$f(x) = \frac{i \sin a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} n}{(a^2 - n^2)} e^{inx}$$

$$c_n = \frac{(-1)^{n+1}}{4\pi}\left[\frac{2ne^{ia\pi} - 2ne^{-ia\pi}}{a^2 - n^2}\right] = \frac{(-1)^{n+1} ni}{\pi(a^2 - n^2)}\left[\frac{e^{ia\pi} - e^{-ia\pi}}{2i}\right] = \frac{(-1)^{n+1} ni}{\pi(a^2 - n^2)} \sin a\pi$$

Thus, the Fourier series of this function is

Parseval's Identity :

Let f(x) be a periodic function with period 2pi defined in the interval $-\pi < x < \pi$. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x)]^2\, dx = {a_0}^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left[{a_n}^2 + {b_n}^2\right]$$

Where a_0, a_n and b_n are Fourier coefficients.

Proof: The Fourier series representation of f(x) in the interval $-\pi < x < \pi$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$ ------------(1) where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad \text{or} \quad \int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0$$ ---------(2)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad \text{or} \quad \int_{-\pi}^{\pi} f(x)\cos nx\,dx = a_n\pi$$ -----(3)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad \text{or} \quad \int_{-\pi}^{\pi} f(x)\sin nx\,dx = b_n\pi$$ ------(4)

Multiplying both sides of Eq-1 by f(x) and integrating term by term from $-\pi$ to π, we have

$$\int_{-\pi}^{\pi} [f(x)]^2\,dx = a_0\int_{-\pi}^{\pi} f(x)dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} f(x)\cos nx\,dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} f(x)\sin nx dx$$ ------(5)

Using (2),(3),and (4) in to Eq-5,we get

$$\int_{-\pi}^{\pi}[f(x)]^2\,dx = 2\pi {a_0}^2 + \sum_{n=1}^{\infty}\pi {a_n}^2 + \sum_{n=1}^{\infty}\pi {b_n}^2$$ or equivalently

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = {a_0}^2 + \frac{1}{2}\sum_{n=1}^{\infty}[{a_n}^2 + {b_n}^2].$$

Activies

1. Show that the **Parseval's relation for a function *f (x) defined*** on the interval −L ≤ x ≤ L takes the form

$$\frac{1}{2L}\int_{-L}^{L}[f(x)]^2\,dx = {a_0}^2 + \frac{1}{2}\sum_{n=1}^{\infty}[{a_n}^2 + {b_n}^2]$$

2. Use the sine series together with the Orthogonality of the functions $\sin\left(\frac{n\pi x}{L}\right)$, for n=1,2,3,...,on the interval **0 ≤ x ≤ L** to show that the **Parseval relation** for sine series takes the form $\frac{2}{L}\int_{0}^{L}[f(x)]^2\,dx = \sum_{n=1}^{\infty}{b_n}^2$

Example1: Find the Fourier representation of $f(x) = x^2$ in $(-\pi, \pi)$ and using Parseval's identity show that $\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$

Example 2 Find the cosine series for $f(x) = x$ in $(0, \pi)$. Use parseval's identity to show that

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$$

Review exercise for chapter one

1. Suppose that f is T-periodic. Then show that for any real number a,

$$\int_0^T f(x)\,dx = \int_a^{a+T} f(x)\,dx$$

2. Prove that $\int_{-a}^{a} f(x)\,dx = 0$ if f is odd on $[-a,a]$.

3. Prove that $\int_{-a}^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$ if f is even on $[-a,a]$.

4. Let f(x) be a function of period 2π such that

$$f(x) = \begin{cases} 0, -\pi < x < 0 \\ x, 0 < x < \pi \end{cases}$$

   a) Sketch a graph of f(x) in the interval $-3\pi < x < 3\pi$.

   b) Show that the Fourier series for f(x) in the interval $-\pi < x < \pi$ is

$$\frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right] + \left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\right]$$

   c) By giving appropriate values to x, show that

      i. $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

      ii. $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} \ldots$

Continued

5. Let f(x) be a function of period 2π such that

$$f(x) \doteq x^2$$

a) Sketch a graph of f(x) in the interval $-3\pi < x < 3\pi$.

b) Show that the Fourier series for f(x) in the interval $-\pi < x < \pi$ is

$$\frac{\pi^2}{3} - 4\left[\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \ldots\right]$$

c) By giving appropriate values to x, show that

$$\frac{\pi^2}{6} \doteq \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

6. Find the Fourier series expansion for the function

a) $f(x) \doteq |x|, -1 \le x \le 1 \; and \; f(x+2) \doteq f(x)$ to obtain the result

$$\frac{\pi^2}{8} \doteq 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} \ldots$$

b) $f(x) \doteq 2 - x^2, -2 < x < 2 \; and \; f(x+4) \doteq f(x)$

c) $$f(x) \doteq \begin{cases} 0, -3 < x < -1 \\ 1, \; -1 < x < 1 \\ 0, \; 1 < x < 3 \end{cases} \; and \; f(x+6) \doteq f(x)$$

Continued

7. Show that the complex form of the Fourier series for

   a) $f(x) = e^{-x}$ in $-1 < x < 1$ is given by $e^{-x} = \sinh 1 \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1 - in\pi)}{1 + n^2\pi^2} e^{inx}$

   b) $f(x) = \cos ax$ in $-\pi < x < \pi$ where a is not an integer is given by

   $$f(x) = \frac{a \sin(a\pi)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 - n^2} e^{inx}$$

8. Show that the Parseval's relation for a function *f (x) defined* on the interval $-L \le x \le L$ takes the form $\frac{1}{2L} \int_{-L}^{L} [f(x)]^2 dx = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_0^2 + b_0^2)$

9. Assume that f has a cosine series $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx), 0 \le x \le \pi$.

   a) Show formally that $\frac{2}{\pi} \int_0^{\pi} [f(x)]^2 dx = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2$

   b) Apply the result of part (a) to the cosine series for $f(x) = x$ in $(0, \pi)$, and thereby show that $\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$

Chapter Two

2. Introduction to Partial Differential Equations

2.1 Definitions and basic concepts

2.2 Classification of PDEs

2.3 Definition of initial/boundary value problems

2.4 Well-posedness of a problem

2.5 Modelling some physical problems using PDEs

2.1 Definitions and basic concepts

Note that . Partial differential equation is an equation involving an unknown function (possibly a vector-valued) of two or more variables and a finite number of its partial derivatives.

❖ In the sequel we reserve the following terminology and notations:

- Independent variables: denoted by

$$x=\left(x_1,x_2,x_3,\cdots,x_n\right)\in\Omega\subseteq\mathbb{R}^n\left(n\geq 2\right)$$

- Dependent variables: denoted by $u=\left(u_1,u_2,u_3,\cdots,u_n\right)\in\mathbb{R}^n$ also called **unknown function**.

- Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n) \in (\mathrm{N} \cup \{o\})^n$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n$

Then $D^\alpha u$ denotes

$$D^\alpha u = \frac{\partial^\alpha u}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \partial^{\alpha_3} x_3 \cdots \partial^{\alpha_n} x_n}$$

We define a PDE more formally as

Definition 2.1 (PDE).Let $\Omega \subset \mathbb{R}^n$ and $m \in \mathbb{N}$

$F : \Omega \times \mathbb{R}^p \times \mathbb{R}^{np} \times \mathbb{R}^{n^2 p} \times \cdots \times \mathbb{R}^{n^m p} \to \mathbb{R}^q$ be a function. A system of Partial differential equations of order m is defined by the equation

$F(x, u, Du, D^2 u, \cdots, D^m u) = 0$ where some m^{th} order partial derivative of the vector function u appears in the system of equations

Examples of PDEs

1. Laplace Equation $\Delta u \equiv \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0$
2. Heat Equation $\frac{\partial u}{\partial t} - \Delta u = 0$
3. Wave Equation $\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$
4. Burgers' Equation $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2} \quad t > 0, x \in \square \, , \mu \geq 0$

2.2 Classification of PDEs

Partial differential equations can be classified in at least three ways. They are

1. Order of PDE.
2. Linear, Semi-linear, Quasi-linear, and fully non-linear.
3. Homogenous and non homogeneous

1. Order of PDE

Definition: The **order of a PDE** is the order of the highest partial derivative in the equation

Examples: Find the order of each of the following partial differential equations:

i. $u_{xx} + 2xu_{xy} + u_{yy} = e^y$;Order is two

ii. $u_{xxx} + xu_{xy} + yu^2 = x$;Order is three

iii. $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2} \quad t > 0, x \in \mathbb{R}, \mu \geq 0$; Order is two

2 Linear, Semi-linear, Quasi-linear, and fully non-linear.

Definitions :

2.1 PDE of order m is called Quasi-linear if it is linear in the derivatives of order m with coefficients that depend on the independent variables and derivatives of the unknown function or order strictly less than m.

2.2 Quasi-linear PDE where the coefficients of derivatives of order m are functions of the independent variables alone is called a Semi-linear PDE.

2.3 A PDE which is linear in the unknown function and all its derivatives with coefficients depending on the independent variables alone is called a Linear PDE.

2.4 A PDE which is not Quasi-linear is called a Fully nonlinear PDE.

Remark : The classification first order PDE as Linear, Semi-linear, Quasi-linear, and fully non-linear

Definition :A first order partial differential equation is called Quasi-linear if it can be written in the form

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \ldots (*)$$

Note that :

1. If $a(x,y,u)=\alpha(x,y)$ and $b(x,y,u)=\beta(x,y)$,then $(*)$ is called semi- linear .

2. If $c(x,y,u)=\gamma(x,y)u+\delta(x,y)$,then $(*)$ is called linear.

Or A partial differential equation is said to be a linear if

i) it is linear in the unknown function and

ii) all the derivatives of the unknown functions with constant coefficients or the coefficients depends on the independent variables.

or A PDE is linear if the dependent variable and all its derivatives appear in a linear fashion (i.e. they are not multiplied together or squared)

Definition: A partial differential equation that is not linear is called non-linear.

Examples :**Determine whether the given PDE is linear, Quasi-linear, semi-linear, or non-linear**

a. $xu_x + yu_y = x^2 + y^2$

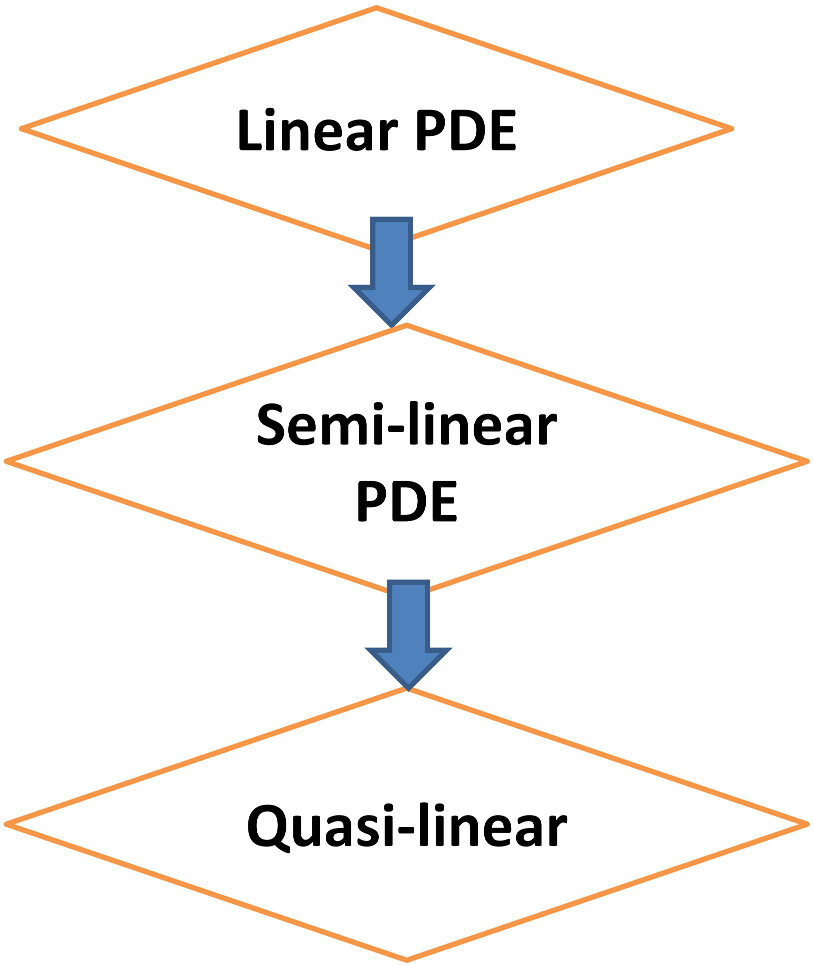
b. $uu_x + u_y = 2$

c. $u_x^2 + u_y^2 = 2$

Answer :

a. Linear, quasi-linear, Semi-linear.
b. Quasi-linear, non-linear.
c. non-linear

Remark: 1.



2. But, the converses may not hold

Examples:

i. A semi linear PDE need not be Linear

$xu_x + yu_y = (x+y)u^2 + x^2 + y^2$. Is Semi linear PDE but not Linear as the power u is not one .

ii. A Quasi-linear PDE need not be semi linear

$x^2u^3u_x + xyuu_y = x^2yu^4 \overset{u_x}{+} x^3yu^3$ .Is PDE Quasi-linear PDE but not Semi linear as the coefficient of u_x and u_y involves terms of u.

3. Homogeneous and Non- Homogenous

Definition : A Partial differential equation is said to be a homogeneous partial differential equation if its all terms contain the unknown functions or its derivatives otherwise non-homogeneous .

Examples :

a. $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = u$; Homogeneous

b. $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$; Homogeneous

c. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$; Non -Homogeneous if $f(x, y) \neq 0$

2.3 Definition of Initial/Boundary Value Problems

Definition .

- Initial value problem (IVP): When all of the constraints are specified at the same value of x, the problem is called an initial value problem.
- Boundary value problem (BVP): When constraints are specified at two, or more, different values of x, for example at each end of an interval I, then the problem is called a boundary value problem.

Contu'd------

- Example1: As a simple example, we suppose that our unknown function u is dependent on one variable x. Then the following problem is known as initial value problem $u_{xx} + u_x - 2u = 0, u(0) = 3, u_x(0) = 7$
- Example2: Now we suppose that our unknown function u is dependent on two variable t; x. Then the following problem is known as initial value problem$u_{xx} + u_t - 2u = 0, u(0,x) = 3x, u_t(0,x) = \sin x$

- Example1: As a simple example, we suppose that our unknown function u is dependent on one variable x. Then the following problem is known as Boundary value problem

$$u_{xx} + u_x - 2u = 0, u(0) = 3, u_x(1) = 7$$

- Example2: Now we suppose that our unknown function u is dependent on two variable t; x. Then the following problem is known as Boundary value problem $u_{xx} + u_t - 2u = 0,$

$$u_{xx} + u_t - 2u = 0, u(0,x) = 3x, u_t(0,x) = \sin x$$

- Contu’d----

2.4 Well-posedness of A Problem

A problem (PDE + side condition) is said to be well-posed if it satisfies the following criteria:

1. The solution must exist.
2. The solution should be unique.
3. The solution should depend continuously on the initial and/or boundary data.

❖ If one or more of the conditions above does not hold, we say that the problem is ill-posed.

2.5 Modelling some physical problems using PDEs

- Many PDE models come from a basic balance or conservation law, which states that a particular measurable property of an isolated physical system does not change as the system evolves. Any particular conservation law is a mathematical identity to certain symmetry of a physical system.
- Here are some examples

❖ conservation of mass (states that the mass of a closed system of substances will remain constant)

❖ conservation of energy (states that the total amount of energy in an isolated system remains constant, first law of thermodynamics)

Continued

- conservation of linear momentum (states that the total momentum of a closed system of objects – which has no interactions with external agents – is constant)
- conservation of electric charge (the total electric charge of an isolated system remains constant)

Chapter 3

3.1 Solution of first order PDEs with constant coefficients

3.2 Solution of a first order PDEs with variable coefficients

3.3 Charpit’s method

3.4 Application of a first order PDEs to fluid flow problems

In this chapter z will be taken as dependent variable and x,y are independent variables so that z=f(x,y). We will use the following standard notations to denote the partial derivatives

$$\frac{\partial z}{\partial x}=p,\frac{\partial z}{\partial y}=q,\frac{\partial^2 z}{\partial x^2}=r,\frac{\partial^2 z}{\partial x\partial y}=s,\frac{\partial^2 z}{\partial y^2}=t$$

Formation of partial differential equation:

There are two methods to form a partial differential equation.

i. By elimination of arbitrary constants.

ii. By elimination of arbitrary functions.

i. By elimination of arbitrary constants.

Consider two parameters family of surface described by the equation *f(x, y, z, a, b) = 0*----------(1)

Where a and b are arbitrary constants.

Differentiating (1) with respect to *x and y, we obtain*

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0$$ -----------(2)

$$\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0$$ ------------(3)

Eliminate the constants *a, b from equations(1), (2),and (3) ,we obtain a first-order PDE* of the form

f(x, y, z, p, q) = 0. ------- (4)

This is Equ.(4)a partial differential of first order.

Example Form the partial differential equation by eliminating the arbitrary constants a and b from the following equation

a) $z = (x+a)(y+b)$.

Solution :Let $z = (x+a)(y+b)$ ----------(1)

Differentiating equation (1) partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = p = (y+b) \text{ and } \frac{\partial z}{\partial y} = q = (x+a)$$

Substituting in (1) we have z = pq which is the required differential equation.

b) $z = (x^2+a)(y^2+b)$

Solution : let $z = (x^2+a)(y^2+b)$---------------(2)

Differentiating equation (2) partially with respect to x and y, we get $\frac{\partial z}{\partial x} = p = 2x(y^2 + b)$ and $\frac{\partial z}{\partial y} = q = 2y(x^2 + a)$

Therefore, $(x^2 + a) = \frac{q}{2y}$ and

$$(y^2 + b) = \frac{p}{2x}$$

Substituting these in (2),we get $z = \frac{q}{2y}\frac{p}{2x}$

i.e. $4xyz = pq$

Note

1. If the number of arbitrary constants equal to the number of independent variables in (1) ,then the P.D.E obtained is of first order.
2. If the number of arbitrary constants is more than the number of independent variables, then the P.D.E obtained is of 2nd or higher orders.

Class activity

1. Find the differential equations of all spheres whose centres lie on z-axis.(Hint : the equation of these sphere s is $x^2 + y^2 + (x - c)^2 = r^2$)
2. Find the differential equations of all spheres of radius 3 units having centres on the x y-plane .(Hint : the equation of these sphere s whose centre is (a,b,0) $(x-a)^2 + (y-b)^2 + z^2 = 9$)

Answer

1. $qx - py = 0$
2. $z^2(p^2 + q^2 + 1) = 9$

II. By elimination of arbitrary functions.

Consider $z = f(u)$ ----------------(5), where $f(u)$ is an arbitrary function of u and $u = u(x, y, z)$

Differentiating (5) partially w.r.t x, y by chain rule

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} \quad \text{-----------------(6)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial y} \quad \text{-----------------(7)}$$

By eliminating the arbitrary function f from (5), (6), (7) we get a P.D.E of first order.

Note: If the partial differential equation is obtained by elimination of arbitrary functions, then the order of the partial differential equation, in general, equals to the number of arbitrary functions eliminated.

Example. Form the partial differential equation by eliminating the arbitrary function from

a) $z = f\left(\frac{y}{x}\right)$ ---------------(1)

Solution : Differentiating partially (1) with respect to x and y, we get

$\frac{\partial z}{\partial x} = p = f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$ -----------(2) and

$\frac{\partial z}{\partial y} = q = f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$------------(3)

Dividing (2) by (3),we get $\frac{p}{q} = \frac{-y}{x}$ or $px + yq = 0$ is the required partial differential equation.

b) $z = f(x + ay) + g(x - ay)$------------------(a)

Solution :Differentiating (a) partially with respect x and y ,we get

$$\frac{\partial z}{\partial x} = p = f'(x + ay) + g'(x - ay)\text{-------------(b)}$$

$$\frac{\partial z}{\partial y} = q = af'(x + ay) - ag'(x - ay)\text{ -------------(c)}$$

Again differentiating partially (b) with respect to x and (c) with respect y , we get

$$\frac{\partial^2 z}{\partial x^2} = p^2 = f''(x + ay) + g''(x - ay)\text{ ---- ----------(d)}$$

$$\frac{\partial^2 z}{\partial y^2} = q^2 = a^2 f''(x + ay) + a^2 g''(x - ay)\text{---------------(e)}$$ .Substituting (d) In to (e) $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$ which is the required partial differential equation

Exercise :Form the partial differential equation by eliminating the arbitrary function from

$f(xy + z^2, x + y + z) = 0$ .ANS. $P(x - 2z) + q(2x - y) = y - x$

Solutions of Partial Differential Equations of First Order

Solutions of Partial Differential Equations of First order with constant coefficients

Definition: a) The **general solution** of a linear partial differential equation is a linear combination of all linearly independent solutions of the equation with as many arbitrary functions as the order of the equation

b) A **particular** solution of a differential equation is one that does not contain arbitrary functions or constants

c) Any equation of the type $F(x,y,u,c_1,c_2)=0$, where c_1 and c_2 are arbitrary constants, which is a solution of a partial differential equation of first-order is called a **complete solution** or a **complete integral** of that equation.

Definition: The most general form of linear partial differential equations of first order with constant coefficients is

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y), u = u(x,y) \text{-----------(1)}$$

Where a, b , and c are constants .

To find the general solution of Equ-1,we apply two cases.

Case I:Assume b=0 $(a \neq 0)$,then Equ-(1)takes the form

Characteristics line of the partial differential (1).

To find the appropriate change of variables, we choose (w , z) such that w=b x-a y and z=y. Then we define a new function v by

$$v(w,z) \equiv u(x,y) = u\left(\frac{w+az}{b}, z\right) \quad \text{-----(4)}$$

Thus, Eq-(1) can be rewritten in terms of the variables (w , z) as

$$b\frac{\partial v}{\partial z} + cv = f\left(\frac{w+az}{b}, z\right) \text{-------------(5)}$$

and hence ,the general solution of Eq-(1) is given by

U(x, y)=v (bx-ay, y).

First order PDEs with variable coefficients

They have the form

$$a u_x + b u_y + c u = f(x,y), u = u(x,y) \qquad (1)$$

Where a, b , and ,c are constants

Assume that $a^2 + b^2 > 0$ (at least one of constants a and b is not zero; if they are both zero ,we do not have a Pde any more)

Here we consider the vector $\vec{g} = (a,b)$ that indicates the direction in which information "propagates "

To solve equ.(1) we consider two cases

Case I: Either $a=0$ or $b=0$ (but not both)

say $a=0, b\neq 0$ .In this case the vector $\vec{g}=(0,b)$ and the Pde becomes:

$$bu_y + cu = f(x,y), u = u(x,y)$$ or

$$u_y + \frac{c}{b}u = \frac{1}{b}f(x,y) \qquad (2)$$

Treating x as a constant ,we may see it as first order linear differential in variable y.

here the integrating factor of equ.(2) is $e^{\frac{c}{b}y}$ and

The general solution of equ.(2) is

$$u(x,y)=e^{\frac{-c}{b}y}\left(\frac{1}{b}\int e^{\frac{c}{b}y}f(x,y)\,dy+c(x)\right)$$

(The case $a\neq 0$ and $b=0$ is completely similar).

Case II: Both $a,b\neq 0$

Note that the lines that are parallel to $\vec{g}=(a,b)$,is called " the characteristic lines of the pde " ,have equation: $bx-ay=k$,where k is arbitrary constant .

Now if we perform the change of variables :

Now if we perform the change of variables :

$$\begin{cases} w = bx - ay \\ z = y \end{cases}$$ whose inverse is $$\begin{cases} x = \frac{1}{b}(w + az) \\ y = z \end{cases}$$

And define the function $v(w,z) = u\left(\frac{1}{b}(w+az), z\right)$

equ.(1) takes the form

$$bv_z + cv = f\left(\frac{1}{b}(w+az), z\right) \qquad (3)$$

Is first order linear differential equation in variable z and where a, w are constants .

Example 1: Find the general solution of the PDE

a) $3u_x - 2u_y + u = x, u = u(x,y)$

Solution : This equation is equation of the form

$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y), u = u(x,y)$ with a=3,b=-2, c=1 ,and $f(x,y) = x$.

Thus, $b\frac{\partial v}{\partial z} + cv = f\left(\frac{w+az}{b}, z\right)$ takes the form

$-2\frac{\partial v}{\partial z} + v = f\left(\frac{w+3z}{-2}, z\right)$ -----------------(1) or equivalently

$\frac{\partial v}{\partial z} - \frac{1}{2}v = \frac{w+3z}{4}$ -----------(2).The general solution of Eq-(2) $v(w,z) = \frac{-w}{2} - \frac{3}{2}z - 3 + c(w)e^{\frac{z}{2}}$ .Hence, the general solution of the given equation is

$$u(x,y) = v(-2x-3y, y) = x - 3 + c(-2x-3y)e^{\frac{y}{2}}$$

B) $u_x - u_y + 2u = 1$

Solution : This equation is equation of the form

$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y), u = u(x,y)$ with a=1,b=-1,andc= 2

So, with a,b, and c values in Eq $b\frac{\partial v}{\partial z} + cv = f\left(\frac{w+az}{b}, z\right)$

give $\frac{\partial v}{\partial z} - 2v = 1$ .The general solution this Eq- is

$v(w,z) = \frac{1}{2} + c(w)e^{2y}$ .Thus, the general solution of the given p d e is $u(x,y) = v(x+y,y) = \frac{1}{2} + c(x+y)e^{2y}$.

Example 2 Solve the following PDE with the given condition: $u_x - u_y + 2u = 1$ $u(x,0) = x^2$

Definition: The most general form of linear partial differential equations of first order with variable coefficients is

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y), u = u(x,y) \quad \text{-------(1)}$$

where a,b,c,and f continuous function .

Definition :The characteristic curve of pde (1) is a curve on the xy-plane that ,at each point, is tangent to the vector field $g(x,y)$.

To solve (1) at each point (x,y) the slope of the vector

$$g(x,y) = \left(a(x,y), b(x,y)\right)$$

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} \quad \text{------(2)}$$ is an ode

- Then put the solution to (2) in " implicit form"

$$h(x,y)=d, constant$$

Finally ,define the change of variables

$$\begin{cases} w=h(x,y) \\ z=y \end{cases} \overset{and \quad invert\ it}{\Longrightarrow} \begin{cases} x=k(w,z) \\ y=z \end{cases}$$

And the function $v(w,z)$,which is nothing but $u(x,y)$

expressed in a new coordinates

$v(w,z)=u\left(k(w,z),z\right)$;equivalently we have :

$$u(x,y)=v\left(h(x,y),y\right)$$

Now

$$a u_x + b u_y = \left(a h_x + b h_y\right) v_w + b v_z \qquad \text{-------------(3)}$$

- But, $ah_x + bh_y = 0 ------(4)$ .Thus, $au_x + bu_y = bv_z$ using the equation (1) takes the form

$$bv_z + v = f(k(w,z),z)$$

Example 1: consider the simple $xu_x - yu_y = 0$ subject to the boundary condition $u = x^4$ on the line $y = x$.

Answer: $u(x,y) = x^2y^2$

Example 2: show that the pde

$$yu_x - 3x^2yu_y = 3x^2u.$$

Has the general solution $yu(x,y) = f(x^3 + y)$ where f is arbitrary function.

i. if you are given that $u(0,y) = y^{-1}\tanh y$ on the line $x = 0$,show that $yu(x,y) = \tanh(x^3 + y)$

Ii. If we are given that $u(x,1) \mathrm{B}\ x^6$ show that

$$yu(x,y) \mathrm{B} \left(x^3 + y - 1\right)^2$$

Theorem 3.1 . The general solution of first order quasi-linear partial differential equation $Pp + Qq = R$

can be written in the form F(u; v) = 0; where F is an arbitrary function, and $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ form a solution of the equation

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)} \quad \text{-----(3.1)}$$

The curves defined by $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ are called the families of characteristics curves of equation $Pp + Qq = R$.

Method of obtaining the general solution:

1. Rewrite the given equation in the standard form Pp+Qq=R.

2. Form the Lagrange's auxiliary equation (A.E)

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}$$

3. $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ are said to be the complete solution of the system of the simultaneous equations (3.1) provided u and v are linearly independent .

4. To find these u and v we can apply the following 4 cases

Case1: One of the variables is either absent or cancels out from the set of auxiliary equations

Case2: If $u = c_1$ is known but $v = c_2$ is not possible by case 1, then use $u = c_1$ to get $v = c_2$.

Case3: Introducing Lagrange's multipliers P_1 , Q_1 , R_1 which are functions of x, y, z or constants , each fraction in (3.1) is equal to

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R} \quad \text{----------(3.2)}$$

P_1 ,Q_1 ,R_1 are chosen that $P_1 P + Q_1 Q + R_1 R = 0$,then $P_1 dx + Q_1 dy + R_1 dz = 0$ which can be integrated.

Case 4: Multipliers may be chosen (more than once) such that the numerator $P_1 dx + Q_1 dy + R_1 dz$ is an exact differential equation of the denominator $P_1 P + Q_1 Q + R_1 R$,Combining (3.2) with a fraction of (3.1) to get an integral.

4. General solution of (1) is F(u, v)=0 or v=Φ(u).

Example: Solve

a) $y^2p - xyq = x(z-2y)$ c) $p + 3q = 5z + \tan(y-3x)$

b) $xyp+y^2q=xyz-2x^2$ d) $\frac{y-z}{yz}p + \frac{z-x}{zx}q = \frac{x-y}{xy}$

e) $(y+z)p+(z+x)q=x+y$

Solution : a) $y^2p - xyq = x(z-2y)$

$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$ is an auxiliary equation

Consider $\frac{dx}{y^2} = \frac{dy}{-xy}$ on integration give $u(x,y,z)=x^2+y^2=c_1$

Similarly by considering $\frac{dy}{-y} = \frac{dz}{z-2y}$ we get $v(x,y,z)=yz-y^2=c_2$

The required general solution is $f(x^2+y^2, yz-y^2)=0$, f being an arbitrary differentiable function.

b) $p + 3q = 5z + \tan(y - 3x)$

Solution :The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan\ (y - 3x)}$$

Taking the first two relations we get, $v(x,y,z) = y - 3x = c_1$

Taking first and last member, $\frac{dx}{1} = \frac{dz}{5z + tanc_1}$

$u(x,y,z) = 5x - \ln(5z + \tan(y - 3x)) = c_2$. The general solution

$f(y - 3x, 5x - \ln(5z + \tan(y - 3x))) = 0$ is where f is an arbitrary differentiable function.

c) $y^2 p - xyq = x(z - 2y)$

solution : $\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z - 2y)}$ .The general solution of the left pair is $x^2+y^2=c_1$ and the last two $yz-y^2=c_2$.Thus,the general solution of the given pde is $f(x^2+y^2,yz-y^2)=0$.

d) $\frac{y-z}{yz}p+\frac{z-x}{zx}q=\frac{x-y}{xy}$

Solution : The given equation is of the form $Pp+Qq=R$

The auxiliary equation is given by $\frac{dx}{\frac{y-z}{yz}}=\frac{dy}{\frac{z-x}{zx}}=\frac{dz}{\frac{x-y}{xy}}$.

Here we choose the multipliers x, y,z $(\because xdx+ydy+zdz=0)$

$\Rightarrow x^2+y^2+z^2=c_1, where\ c_1=2c_0$ and again we choose the multipliers *yz, zx, xy* $(\because yzdx+xzdy+xydz=0)$

$\Rightarrow d(xyz)=0 \Rightarrow xyz=c_2$. The general solution is given by $f(x^2+y^2+z^2, xyz)=0$

e)(y+z)p+(z+x)q=x+y. Answer

General solution is f((x-y)^2(x+y+z) , $\frac{x-y}{y-z}$) =0.

DEFINITION 1. *(Compatible systems of first-order PDEs) A system of two first-order PDEs*

$f(x, y, z, p, q) = 0$---------------(1)

and $g(x, y, z, p, q) = 0$----------------(2)

are said to be compatible if they have a common solution.

THEOREM 2. *The equations* $f(x, y, z, p, q) = 0$ *and* $g(x, y, z, p, q) = 0$ *are compatible on a domain D if*

i. $J \square \frac{\partial(f,g)}{\partial(p,q)} \square \begin{vmatrix} f_p & f_q \\ g_p & g_p \end{vmatrix} \neq 0$ on D

ii. *p and q can be explicitly solved from (1) and (2) as* $p = \phi(x, y, z)$ *and* $q = \psi(x, y, z)$. *Further, the equation*

$dz = \phi(x, y, z)dx + \psi(x, y, z)dy$ *is integrable.*

THEOREM 3. *A necessary and sufficient condition for the integrability of the equation* $dz = \phi(x, y, z)dx + \psi(x, y, z)dy$ *is*

$$[f,g] \equiv \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,q)} \square 0 \quad \text{--(3)}$$

In other words, the equations (1) and (2) are compatible iff (3) holds.

Example :*Show that the equations* $xp - yq = 0$, $z(xp + yq) = 2xy$ *are compatible and solve them*

Solution: Take $f \equiv xp - yq = 0$, $g \equiv z(xp + yq) - 2xy = 0$.

Note that $f_x = p$, $f_y = -q$, $f_z = 0$, $f_p = x$, $f_q = -y$.

and

$g_x = zp - 2y$, $g_y = zq - 2x$, $g_z = xp + yq$, $g_p = zx$, $g_q = zy$.

Compute

$$J=\frac{\partial(f,g)}{\partial(p,q)}=\begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix}=\begin{vmatrix} x & -y \\ zx & xy \end{vmatrix}=2zxy\neq 0 \text{ for } x,y,z\neq 0$$

Further ,

$$\frac{\partial(f,g)}{\partial(x,p)}=\begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix}=\begin{vmatrix} p & x \\ zp-2y & zx \end{vmatrix}=2xy \quad \text{----(1)}$$

$$\frac{\partial(f,g)}{\partial(y,q)}=\begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix}=\begin{vmatrix} -q & -y \\ zq-2x & zy \end{vmatrix}=-2xy \quad \text{--------(2)}$$

$$\frac{\partial(f,g)}{\partial(z,p)}=\begin{vmatrix} f_z & f_p \\ g_z & g_p \end{vmatrix}=\begin{vmatrix} 0 & x \\ xp+yq & zx \end{vmatrix}=-x^2p-xyq \quad \text{----(3)}$$

$$\frac{\partial(f,g)}{\partial(z,q)}=\begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix}=\begin{vmatrix} 0 & -y \\ xp+yq & zy \end{vmatrix}=y^2q+xyp \quad \text{----(4)}$$

Thus substituting (1),(2),(3),and (4)into the equation

$$[f,g] \equiv \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,p)}$$

$$= 2xy - x^2p^2 - xypq - 2xy + y^2q^2 + xypq$$

$$= 0$$

so the equations are compatiable

From the two equations $\mathrm{xp} - \mathrm{yq} = 0$, $\mathrm{z(xp + yq)} = 2\mathrm{xy}$ solving for p and q, we get

$$\mathrm{p} = \frac{y}{\mathrm{z}} = \phi(x, y, \mathrm{z}) \text{ and } \mathrm{q} = \frac{x}{\mathrm{z}} = \psi(x, y, \mathrm{z})$$

Substituting p and q in $dz \,\square\, p\,dx + q\,dy$, *we get*

$$zdz = ydx + xdy = d(xy) \textit{ and on integrating, it gives you } z^2 = 2xy + c$$

Example2: Show that the following partial differential equations are compatible

$$xp - yq = x, x^2 p + q = xz$$

and, hence, find their solution.

Charpit's Method:

It is a general method for finding the complete integral of a nonlinear PDE of first-order of the form

$$f(x, y, z, p, q) = 0.\text{-------------}(1)$$

The basic idea of this method is to introduce another partial differential equation of the first order

$$g(x, y, z, p, q, a) = 0\text{------------}(2)$$

which contains an arbitrary constant *a and is such that*

i. Equations (1) and (2) can be solved for *p and q to obtain* $p = p(x, y, z, a)$, $q = q(x, y, z, a)$.

ii. The equation $dz = p(x, y, z, a)dx + q(x, y, z, a)dy$---(3) is integrable.

When such a function *g is found, the solution* $F(x, y, z, a, b) = 0$ *of (4)* containing two arbitrary constants *a, b will be the solution of (1).*

Note: Notice that another PDE *g is introduced so that the equations f and g are compatible* and then common solutions of *f and g are determined in the Charpit's method.* The equations (6) and (7) are compatible if $[f,g] \equiv \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,q)}$.

Expanding it, we are led to the linear PDE

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

. These equations are known as Charpit's equations.

To solve non-linear p de of first order(By Charpit's method)

Step I :Write the given equation in the form of

$f(x, y, z, p, q) = 0$.

Step II: Find f_x, f_y, f_z, f_p, and f_q .

Step III: Consider the Charpit's Auxiliary equations as

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

Step Iv: Find p(or q) from step III and use the given equation to find q(or p).

Step V. Use $dz = pdx + qdy$ and integrate to find general solution

Example: Solve the following partial differential equations by Charpit's method.

a) $px + qy = pq$ b) $z^2 = pqxy$ c) $(p^2 + q^2)y = qz$

Solution: let $f(x, y, z, p, q) \equiv px + qy - pq = 0$----------(1)

Then $f_x = p, f_y = q, f_p = x - q, f_q = y - q, f_z = 0$ .Thus,

$pf_p + qf_q = p(x-q) + q(y-q), -(f_x + pf_z) = -p, -(f_y + qf_z) = -q$

and the Charpit's auxiliary equations are

$\frac{dx}{x-q} = \frac{dy}{y-q} = \frac{dz}{p(x-q)+q(y-q)} = \frac{dp}{-p} = \frac{dq}{-q}$ from the last two fractions ,we get $\frac{p}{q} = a \Rightarrow p = aq$ substituting this in Eq-(1),we get $q = \frac{ax+y}{a}$ .Therefore, $p = ax + y$ .and

$dz = (ax+y)dx + \left(\frac{ax+y}{a}\right)dy$ or $adz = a(ax+y)dx + (ax+y)dy = (ax+y)(adx+dy)$

Integrating ,we get $az = \frac{(ax+y)^2}{2} + c$

b) $z^2 = pqxy$ .Solution :Let $f(x, y, z, p, q) \equiv z^2 - pqxy = 0$ -----(a)

.Then $f_x = -pqy, f_y = -pqx, f_p = -xyq, f_q = -pxy, f_z = 2z$

$pf_p + qf_q = -2pqxy, -(f_x + pf_z) = p(qy-2z), -(f_y + qf_z) = q(px-2z)$ and, thus, the Charpit's auxiliary equations are

$$\frac{dx}{-xyq} = \frac{dy}{-pxy} = \frac{dz}{-2pqxy} = \frac{dp}{p(qy-2z)} = \frac{dq}{q(px-2z)} \quad \text{------------------(b)}$$

From Eq-(b),we get $\frac{dz}{z} = \frac{d(px+qy)}{px+qy}$ on integrating both sides, we obtain

z=a(px+qy),where a is arbitrary constant and substituting this in Eq-(a)

$p = \frac{z}{x}c, where\ c = \frac{1}{2a} \pm \frac{\sqrt{1-4a^2}}{2a}$ and $q = \frac{z}{cy}$ substituting these values of p and q in to eq- $dz = pdx + qdy$,we have

$dz = \frac{z}{x}c\,dx + \frac{z}{cy}dy \Rightarrow \frac{dz}{z} = c\frac{dx}{x} + \frac{1}{c}\frac{dy}{y}$ on integrating this we get

$z = bx^c y^{\frac{1}{c}}$,where c is arbitrary constant of integration is the required solution

C) $(p^2+q^2)y = qz$

solution :Let $f(x,y,z,p,q) = (p^2+q^2)y - qz = 0$ -------------(*)

Then $f_x = 0, f_y = p^2+q^2 \quad , f_p = 2py, f_q = 2qy - z, f_z = -q$ and

$pf_p + qf_q = \; 2(p^2+q^2)y - qz, , -(f_x + pf_z) = pq, -(f_y + qf_z) = -p^2$

Thus, the Charpit's auxiliary equations are,

$\frac{dx}{2py} = \frac{dy}{2qy-z} = \frac{dz}{2(p^2+q^2)y - qz} = \frac{dp}{pq} = \frac{dq}{-p^2}$ .Form the last two fractions i.e. $\frac{dp}{pq} = \frac{dq}{-p^2}$,we obtain $p^2+q^2=a^2$ substituting this in Eq-(*) and solving for p and q,we get $q=\frac{a^2y}{z}$ and $p^2=a^2-\frac{a^4y^2}{z^2}$ and substituting these values of p and q in to $dz = pdx + qdy$ this equation takes the form

$dz = \frac{a\sqrt{z^2-a^2y^2}}{z}dx + \frac{a^2y}{z}dy \Rightarrow \frac{z\,dz - a^2y\,dy}{\sqrt{z^2-a^2y^2}} = adx$ on integrating ,we obtain

$z^2 - a^2y^2 = (ax+b)^2$ which is the required solution.

Special Types of First Order partial differential equation .

Type I f(p , q)=0[Equations involving only p and q]

The auxiliary equations are $\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{0}=\frac{dq}{0}$.

Solving $\frac{dp}{0}=\frac{dq}{0}$, we get either p=a or q=a. Then we solve f(a,q)=0 [or f(a,p)=0] for q=Q(a)[or p=P(a)]. Then dz=a dx+Q(a) dy implying z=ax+Q(a)y+b.

Example 1: Find a general solution of p + q=pq.

Solution : Put q=a. Then p=$\frac{a}{a-1}$. Then dz=$\frac{a}{a-1}$ dx+ a dy. Hence z=$\frac{ax}{a-1}$+ay +b, is the required general solution.

Example 2:Find a general solution of $pq=1$.

Answer : $z=ax+\frac{y}{a}+b$

Type II (Equations not involving the independent variables i.e .(not involving X and y):

For the equation of the type

$$f(z, p, q) = 0 \text{ ------------------(a)}$$

Charpit's equation becomes $\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-pf_z} = \frac{dq}{-qf_z}$ and taking the last two fractions i.e. $\frac{dp}{-pf_z} = \frac{dq}{-qf_z}$ gives p=aq-------(b)

Solving (a) and (b) for p and q , we obtain

$$q = Q(a,z) \Rightarrow p = aQ(a,z)$$

Now $dz = aQ(a,z)dx + Q(a,z)dy$ on integrating both sides we get $\int \frac{dz}{Q(a,z)} = ax + y + b$ constant of integration.

Example 1 : Find a general solution of $p^2z^2 + q^2 = 1$.

Solution. Putting *p = a q in the given PDE, we obtain*

$p^2z^2 + q^2 = 1 \Rightarrow (aq)^2z^2 + q^2 = 1$ and $q = \dfrac{1}{\sqrt{1 + a^2z^2}}$

Type III:(Separable equations)

A first-order PDE is separable if it can be written in the form

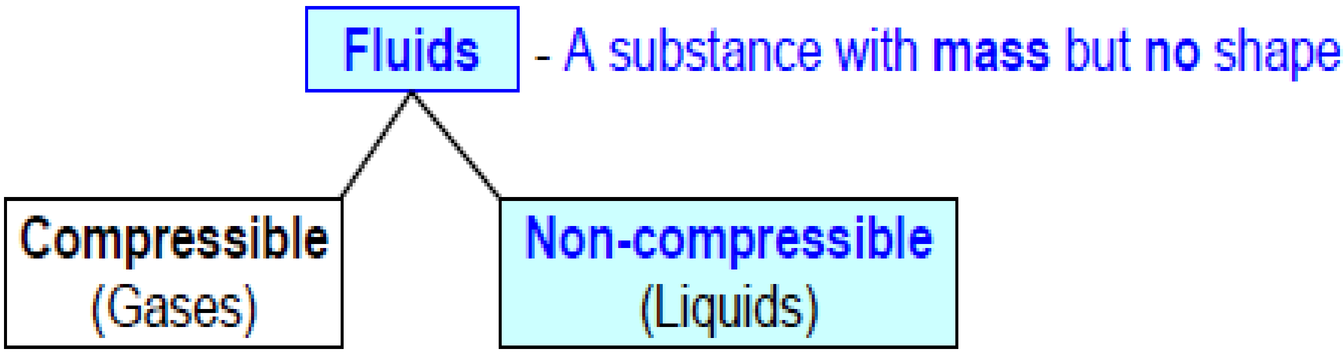
$$f(x,p) = g(y,q) \text{ -----------------(a)}$$

That is, a PDE in which z is absent and the terms containing x and p can be separated from those containing y and q

For this type of equation, Charpit's equations become $\frac{dx}{f_p} = \frac{dy}{-g_q} = \frac{dz}{pf_p - qg_q} = \frac{dp}{-f_x} = \frac{dq}{-g_y}$ from the relation $\frac{dx}{f_p} = \frac{dp}{-f_x}$,we obtain $f_x dx + f_p dy = 0$ which may be solved to yield p as a function of x and an arbitrary constant a

3.4 Application of a first order PDEs to fluid flow problems

Fundamental Principles of Fluid Mechanics Analysis

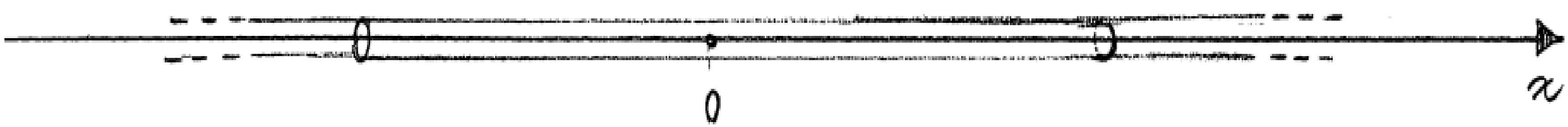


Moving of a fluid requires:

- A conduit, e.g., tubes, pipes, channels
- Driving **pressure**, or by gravitation, i.e., difference in "head"
- Fluid flows with a **velocity** v from higher pressure (or elevation) to lower pressure (or elevation)

Application to first order linear equation to <u>GAS FLOW</u>.

We will only consider the one-dimensional case. So we assume we have a thin pipe, with a coordinate system (the x-axis):



The pipe is filled with gas, with a density that is a function of space and time:

$\rho(x,t)$: density of gas at position x and time t (measured in Kg/m).

Typically, $\rho(x,t)$ is the UNKNOWN function. What is known is the initial condition, i.e. the initial density of gas along the pipe: $\rho(x,0) = \rho_0(x)$ (which is some assigned function of x).

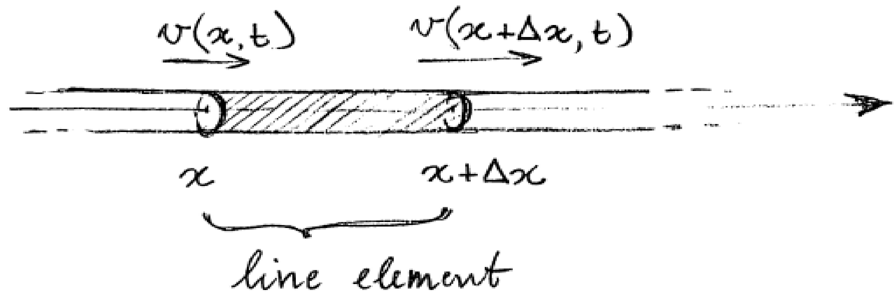
What is also known is the velocity of the gas, as a function of space and time:

$v(x,t)$: velocity of gas at position x and time t (measured in m/s, i.e. meters per second).

The question is: given the initial density $\rho_0(x)$ and the velocity $v(x,t)$, how does the density evolve in space and time? I.e. what is $\rho(x,t)$?

Let's first derive a PDE for $\rho(x,t)$, based on physical principle

Here is a sketch of the reasoning:



Suppose Δx small and Δt small.

- Amount of gas <u>entering</u> the line element at point x during the time interval $[t, t+\Delta t]$:

(*) $\rho(x,t)\, v(x,t)\, \Delta t$ $\left(\text{measured in } \frac{kg}{m} \cdot \frac{m}{s} \cdot s = kg\right)$

- Amount of gas exiting the line element at point $x+\Delta x$ during the time interval $[t, t+\Delta t]$:

(**) $\rho(x+\Delta x, t)\, v(x+\Delta x, t)\,\Delta t$, also measured in Kg

- Therefore, the change of mass (amount) of gas in the line element $[x, x+\Delta x]$ during the time interval $[t, t+\Delta t]$ is:

(***)
$$\begin{aligned}\Delta m &= (*) - (**)\\ &= \rho(x,t)\, v(x,t)\,\Delta t - \rho(x+\Delta x, t)\, v(x+\Delta x, t)\,\Delta t\\ &= -\left[\rho(x+\Delta x, t)\, v(x+\Delta x, t) - \rho(x,t)\, v(x,t)\right]\Delta t\end{aligned}$$

- Howewer, if we define $m(t)$ as the mass (amount) of gas in the line element $[x, x+\Delta x]$ at time t, we also have:

$$m(t) \simeq \rho(x,t)\,\Delta x \qquad \text{(since } \Delta x \text{ is small)}$$

so $m(t+\Delta t) = \rho(x, t+\Delta t)\,\Delta x$, and therefore (***) may also be expressed as:

(****) $\Delta m = m(t+\Delta t) - m(t) = \left[\rho(x, t+\Delta t) - \rho(x,t)\right]\Delta x$.

Now, equating the right-hand sides of (***) & (****) yields:

$$[\rho(x, t+\Delta t) - \rho(x,t)]\,\Delta x$$
$$= -\,[\rho(x+\Delta x, t)\,v(x+\Delta x, t) - \rho(x,t)\,v(x,t)]\,\Delta t$$

At this point divide by $\Delta x \Delta t$, which yields:

(5*) $$\frac{\rho(x, t+\Delta t) - \rho(x,t)}{\Delta t} = -\,\frac{\rho(x+\Delta x, t)\,v(x+\Delta x, t) - \rho(x,t)\,v(x,t)}{\Delta x}$$

Now take the limits for $\Delta t \to 0$ and $\Delta x \to 0$. We have:

$$\lim_{\Delta t \to 0} \frac{\rho(x, t+\Delta t) - \rho(x,t)}{\Delta t} = \frac{\partial \rho}{\partial t}(x,t)$$ (by definition of partial derivative)

whereas if we define $f(x,t) =$

$$\lim_{\Delta x \to 0} \frac{\rho(x+\Delta x, t)\, v(x+\Delta x, t) - \rho(x,t)\, v(x,t)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x, t) - f(x,t)}{\Delta x} = \frac{\partial f}{\partial x}(x,t)$$

$$= \frac{\partial}{\partial x}\left[\rho(x,t)\, v(x,t)\right].$$

Therefore (5*) becomes:

$$\frac{\partial \rho}{\partial t}(x,t) = -\frac{\partial}{\partial x}\left[\rho(x,t)\, v(x,t)\right],$$

or, in a more compact form, $\boxed{\rho_t + (v\rho)_x = 0}$,

which is known as the "CONTINUITY EQUATION" or the "CONSERVATION OF MASS" equation.

Remarks:

- typically $v(x,t)$ and $\rho(x,0) = \rho_0(x)$ are assigned, and you have to solve for $\rho(x,t)$.

- We may rewrite the continuity equation as $\rho_t + v_x \rho + v \rho_x = 0$, or more explicitly:

$$v(x,t)\,\rho_x(x,t) + \rho_t(x,t) + v_x(x,t)\,\rho(x,t) = 0$$

which is a linear first order PDE, i.e. of the type:

$$a(x,t)\,\rho_x(x,t) + b(x,t)\,\rho_t(x,t) + c(x,t)\,\rho(x,t) = f(x,t)$$

where:

$$\left.\begin{aligned} a(x,t) &= v(x,t) \\ b(x,t) &= 1 \\ c(x,t) &= v_x(x,t) \\ f(x,t) &= 0 \end{aligned}\right\}$$

note that two out of the four coefficients are determined by $v(x,t)$

- Why is $\rho_t = -(v\rho)_x$ called "conservation of mass" equation?

If we define: $M(t) = \int_{-\infty}^{\infty} \rho(x,t)\, dx$, which is the total mass of gas along the infinitely long pipe, we have:

$$\frac{d}{dt} M(t) = \frac{d}{dt} \int_{-\infty}^{\infty} \rho(x,t)\, dx = \int_{-\infty}^{\infty} \frac{\partial \rho}{\partial t}(x,t)\, dx =$$

$$= \int_{-\infty}^{\infty} \rho_t(x,t)\, dx = \int_{-\infty}^{\infty} (v\rho)_x\, dx = \Big[v(x,t)\, \rho(x,t) \Big]_{x=-\infty}^{x=+\infty}$$

and assuming(•) $\lim_{x\to\infty} \rho(x,t) = 0$ and $\lim_{x\to-\infty} \rho(x,t) = 0$,

we have: $\frac{d}{dt} M(t) = 0$, i.e. $M(t) =$ constant

In other words, the total mass of gas is conserved.

(•) These assumptions mean that there is no density at $x = \pm\infty$, i.e. the mass does not escape the pipe.

- <u>One last remark</u>: in 3D, the density is a function of 4 variables (3 spatial variable, plus time: $\rho(x,y,z,t)$. The <u>velocity</u> is a 3-dimensional vector field that depends on 4 variables:

$$\bar{v}(x,y,z,t) = \big(v_1(x,y,z,t),\ v_2(x,y,z,t),\ v_3(x,y,z,t)\big)$$

(<u>note</u>: ρ is still a scalar, so $\rho\bar{v}$ is a vector). In this case the continuity equation is: $\boxed{\rho_t + \operatorname{div}(\rho\bar{v}) = 0}$,

or, more explicitly: $\rho_t + (\rho v_1)_x + (\rho v_2)_y + (\rho v_3)_z = 0.$

For a derivation, see any textbook on fluid mechanics (we will only treat the 1D case).

<u>Example</u>: solve $\rho_t + (v\rho)_x = 0$, $\rho = \rho(x,t)$
with initial condition $\rho(x,0) = \rho_0(x)$ (given function)
and CONSTANT VELOCITY: $\boxed{v(x,t) = v_0 > 0}$.

$(x,t) = v_0$

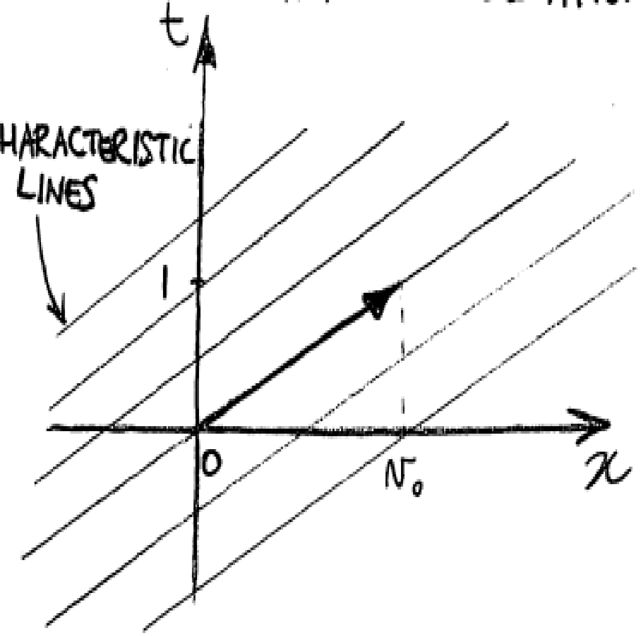
In this case: $\rho_t + (v\rho)_x = 0$ becomes: $\rho_t + v_0\, \rho_x = 0$,

or (if we consider x as the first variable & t as the 2^{nd} variable):

$$\boxed{v_0\, \rho_x + \rho_t = 0}$$

"TRANSPORT EQUATION"

. The vector field $\bar{g} = (v_0, 1)$ in the (x,t)-plane is a constant one:



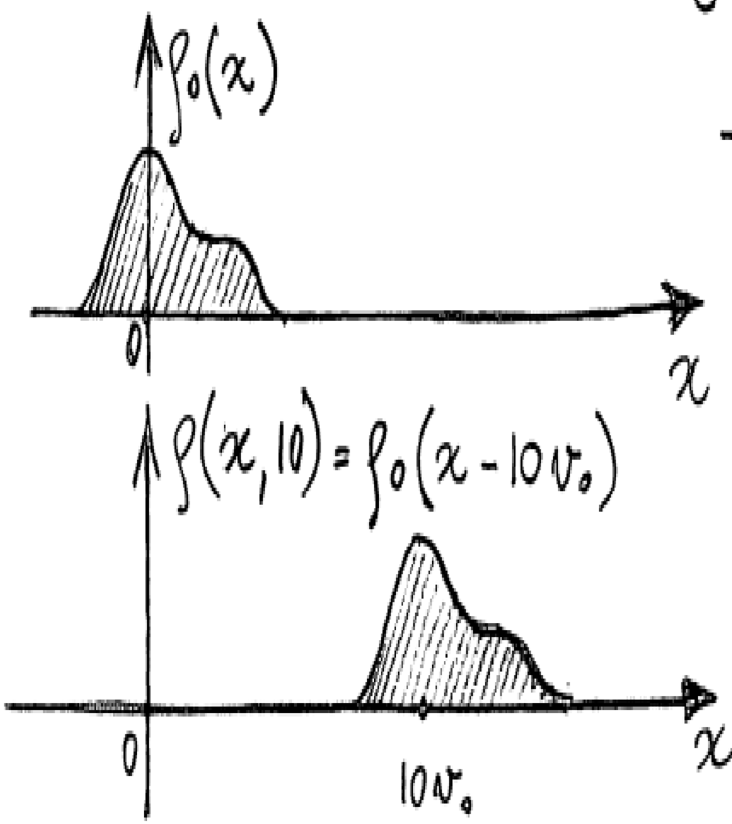
It is simple to verify that the general solution to the PDE is:

$$(\bullet\bullet) \quad \rho(x,t) = C(x - v_0 t),$$

where $C(r)$ is an arbitrary function.

From $(\bullet\bullet)$ we get: $\rho(x,0) = C(x)$, so $C(x) = \rho_0(t)$ and the particular solution is

$$\rho(x,t) = \rho_0(x - v_0 t)$$



The mass is "TRANSPORTED" to the right as t increases. ∎

Chapter 4: Fourier transforms

Theorem [Fourier integral representation]

Suppose that f is piece wise smooth on every finite interval and that $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$ .Then f has the following Fourier integral representation

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\ e^{is(x-t)}\, dt\, ds$$ or equivalently

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\, dt\, d\lambda$$

Alternatively,

Definition. (Fourier integrals) Let f be a function and let $A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(t\lambda)\,dt$ and $B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(t\lambda)\,dt$

Then the integral $\int_0^{\infty}[A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)]d\lambda = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda$

i s called the Fourier integral formula for f(x).

Theorem 1. Assume that f a piecewise smooth function on every finite interval [a, b] ⊆ R and assume that $\int_{-\infty}^{\infty}|f(x)|\,dx$ converges. Then the Fourier integral of f converges to $\frac{f(x+)+f(x-)}{2}$ for all x ∈ R; .i.e.

$\int_0^{\infty}[A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)]d\lambda = \frac{f(x+)+f(x-)}{2}$ for all x ∈ R.

Example 1: Find the Fourier integral re-presentation of the function

$$f(x) = \begin{cases} 1, if\ |x| < 1 \\ 0, if\ |x| > 1 \end{cases}$$ and find the value of $\int_0^{\infty} \frac{\sin\lambda}{\lambda} d\lambda$.

Solution : since $\int_{-\infty}^{\infty} |f(x)|dx = \int_{-1}^{1} dx = 2 < \infty$ .Thus, the Fourier integral representation of f is given by

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda$$ i.e

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\int_{-1}^{1} 1\cos\lambda(t-x)\right]dtd\lambda = \frac{1}{\pi}\int_0^{\infty}\left[\frac{\sin\lambda(t-x)}{\lambda}\right]_{t=-1}^{t=1} d\lambda = \frac{2}{\pi}\int_0^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda$$

.

Thus,

$$\int_0^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda = \begin{cases} \frac{\pi}{2}, & \text{if } 0 \le x < 1 \\ \frac{\pi}{4}, & \text{if } x = 1 \\ 0, & \text{if } x > 1 \end{cases}$$ ----------(*).When x=0

Eq.(*) gives $\int_0^{\infty}\frac{\sin\lambda}{\lambda}d\lambda = \frac{\pi}{2}$

Example 2:Find the Fourier integral representation of $f(x) = e^{-|x|}$

Solution :since $\int_{-\infty}^{\infty}|f(x)|dx = \int_{-\infty}^{\infty}|e^{-|x|}|dx = 2\int_0^{\infty} e^{-x}\,dx = 2$,and

Hence the given function is absolutely convergent .Therefore, the function has a Fourier integral representation $f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty e^{-|t|} \cos\lambda(t-x)\, dt\, d\lambda$ and .Thus, $f(x) = \frac{1}{\pi}\int_0^\infty \left[\cos\lambda x \int_{-\infty}^\infty e^{-|t|}\cos\lambda t\, dt + \sin\lambda x \int_{-\infty}^\infty e^{-|t|} \sin\lambda t\, dt\right] d\lambda$ .But,

$\int_{-\infty}^\infty e^{-|t|}\cos\lambda t\, dt = 2\int_0^\infty e^{-t}\cos\lambda t\, dt$ since $e^{-|t|}\cos\lambda t$ is even.

$\int_{-\infty}^\infty e^{-|t|} \sin\lambda t\, dt = 0$ since $e^{-|t|}\sin\lambda t$

Hence, $f(x) = e^{-|x|} = \frac{2}{\pi}\int_0^\infty \frac{\cos\lambda x}{1+\lambda^2}\, d\lambda$.

1. $\int_0^\infty e^{-ax}\cos bx\ dx = \frac{a}{a^2+b^2}$.
2. $\int_0^\infty e^{-ax}\sin bx\ dx = \frac{b}{a^2+b^2}$

Fourier sine and cosine integral

1. Fourier sine integral of a function f is given by

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(t)\sin(\lambda x)\sin(t\lambda)\,dt\,d\lambda$$

and

2. Fourier cosine integral of a function f is given by

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(t)\cos\lambda x\,\cos\lambda t\,dt\,d\lambda$$

Note that Fourier integral of an even function is known as Fourier cosine integral where as Fourier integral of an odd function is known as Fourier sine integral.

Example 1:Express $f(x) = \begin{cases} 1, if\ 0 < x < \pi \\ 0, if\ x > \pi \end{cases}$ as a Fourier sine series and hence evaluate $\int_0^{\infty} \frac{1-\cos\lambda\pi}{\lambda}\sin\lambda\pi\,d\lambda$.

Solution : The Fourier sine integral of f(x) is given by

$$f(x)=\frac{2}{\pi}\int_0^\infty[\sin(\lambda x)]\int_0^\infty f(t)\sin(t\lambda)\,dt\,d\lambda=\frac{2}{\pi}\int_0^\infty[\sin(\lambda x)]\left(\int_0^\pi \sin(t\lambda)\,dt\right)d\lambda$$ since $f(t)=\begin{cases}1, if\ 0<t<\pi\\ 0, if\ t>\pi\end{cases}$

$$f(x)=\frac{2}{\pi}\int_0^\infty[\sin(\lambda x)]\left[\frac{-\cos\lambda t}{\lambda}\right]_{t=0}^{t=\pi}d\lambda=\frac{2}{\pi}\int_0^\infty\frac{1-\cos\lambda\pi}{\lambda}\sin\lambda\pi\, d\lambda$$

$$\therefore\quad \int_0^\infty\frac{1-\cos\lambda\pi}{\lambda}\sin\lambda\pi\, d\lambda=\frac{\pi}{2}f(x)=\begin{cases}\frac{\pi}{2}, 0\le x<\pi\\ \frac{\pi}{4}\quad, x=\pi\\ 0\ , x>\pi\end{cases}$$

Example 2: Find the Fourier cosine integral of the function e^{-ax} .Hence deduce the value of the integral $\int_0^\infty\frac{\cos\lambda x}{1+\lambda^2}\,d\lambda$.

Answer :The Fourier cosine integral of the function

$$f(x)=e^{-ax}=\frac{2a}{\pi}\int_0^\infty\frac{\cos\lambda x}{a^2+\lambda^2}\,d\lambda$$ and $$\int_0^\infty\frac{\cos\lambda x}{a^2+\lambda^2}\,d\lambda=\frac{\pi}{2a}e^{-ax}$$

Fourier transform [Complex Fourier transform]

The complex form of Fourier integral is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\; e^{is(x-t)}\,dt\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\left(\int_{-\infty}^{\infty} f(t)e^{-ist}dt\right)ds$$

The function $F(s) = F(f(t)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ist}\,dt$ is the complex Fourier transform of f(x).

The function $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]e^{isx}\,ds$ is called the inversion formula for the complex Fourier transform of F[f(x)] and it is denoted by $f(x) = F^{-1}[F(f(x))]$.

Example : Find the Fourier transform of

a. $f(x) = \begin{cases} 2-|x|, if\ |x| < 2 \\ 0 \quad , if\ |x| > 2 \end{cases}$ and hence prove that

$$\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

b. $f(x) = \begin{cases} 1 - x^2, if\ |x| \le 1 \\ 0 \quad , if\ |x| > 1 \end{cases}$ Hence deduce that

$$\int_0^\infty \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$$

Fourier sine transform and Fourier Cosine transform

1. The Fourier sine transform of the function f is given by $F_S[(f(t))] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\lambda t)\, dt$ and the inversion formula for Fourier sine transform is given by $f(x) = F_S^{-1}[F_S(f(x))] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_S[f(x)] \sin(sx)\, ds$.
2. The Fourier cosine transform of the function f is given by $F_C(f(t)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos\lambda t\, dt$ and the inversion formula for Fourier sine transform is given by

$$f(x) = F_C^{-1}[F_C(f(x))] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_C(f(x)) \cos sx\, ds$$

Example 1:Find f(x),if its Fourier sine transform is $\frac{e^{-as}}{s}$. Hence find $F_S^{-1}\left(\frac{1}{s}\right)$.

Solution: Inversion formula, we have

$$f(x)=F_S^{-1}\left[\frac{e^{-as}}{s}\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-as}}{s}\sin(sx)\,ds \Rightarrow \frac{df(x)}{dx}=\frac{d\left(\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-as}}{s}\sin(sx)\,ds\right)}{dx}$$

$$\frac{df(x)}{dx}=\sqrt{\frac{2}{\pi}}\int_0^{\infty}e^{-as}\cos sx\,ds=\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+x^2}\right)$$

Integrating both sides with respect to x ,we get

$$f(x)=\sqrt{\frac{2}{\pi}}\int\left(\frac{a}{a^2+x^2}\right)dx=\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{a}\right)+c$$

.But,

$$F_S^{-1}\left[\frac{e^{-as}}{s}\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-as}}{s}\sin(sx)\,ds$$

Example 2:Solve for f(x) from the integral equation

$$\int_0^{\infty}f(x)\cos\lambda x=\begin{cases}1-\lambda, if\ 0\le\lambda\le 1\\ 0\ \ , if\ \ \lambda>1\end{cases}$$

and hence evaluate $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$

The main operational properties of Fourier transforms

Theorem : Linearity of the Fourier transform Let the functions *f (x) and g(x) have the respective* Fourier transforms *F(ω) and G(ω), and let a and b be arbitrary constants. Then*

$$F\{a f(x) + bg(x)\} = aF\{f(x)\} + bF\{g(x)\}.$$

Theorem : Fourier transform of a derivative of *f (x)*

Let *f (x) be a continuous function* of *x with the property that* $\lim_{|x|\to\infty} f(x) = 0$, *and such that f (x) is absolutely* integrable over $(-\infty,\infty)$. *Then:*

(a) *F{ f ‘(x)} = iωF(ω).*

(b) For all *n such that the derivatives f (r)(x) with r = 1, 2, . . . , n satisfy Dirichlet* conditions, are absolutely integrable over (−∞,∞)*, and lim|x|→∞ f (n−1)(x) = 0, F{ f (n)(x)} = (iω)nF(ω),* where *f (n)(x) = dn f/dxn.*

Theorem : Fourier transform of *xn f (x)* .*Let f (x) be a continuous and differentiable function* with an *n times differentiable Fourier transform F(ω). Then*

i. $F\left(xf\left(x\right)\right) \square\, i\dfrac{d}{dw}\left(F\left(w\right)\right)$ *and*

ii. $F\left(x^n f\left(x\right)\right) \square\, i^n\dfrac{d^n}{dw^n}\left(F\left(w\right)\right)$, for all n such that $\lim\limits_{|x|\to\infty} F^{(n)}\left(w\right) \square\, 0$

Theorem : **The convolution theorem for Fourier transforms** Let the functions *f* (*x*) and *g*(*x*) be piecewise continuous, bounded, and absolutely integrable over (−∞, ∞) with the respective Fourier transforms *F*(*ω*) and *G*(*ω*). Then

a. $F((f*g)(x)) = F(f(x))F(g(x)), or\ F(f*g) = F(\omega)G(\omega)$ and ,conversely ,

b. $(f*g)(x) = \int_{-\infty}^{\infty} F(\omega)G(\omega)e^{i\omega x}d\omega$

Proof :exercise

Theorem : **The Parseval relation for the Fourier transform .** If *f (x) has the Fourier transform F(ω), then* $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$

Theorem : **Fourier transforms involving scaling *x by a*, *shifting x by a, and shifting ω by λ*** *. If f (x) has a Fourier transform F(ω), then*

a) $F\{(f(ax))\} \mathrm{B} \frac{1}{a} F\left(w/a\right)$

b) $F\left(f(x - a)\right) \mathrm{B}\, e^{-iwx} F(w)$

c) $F\left(e^{ilx} f(x)\right) \mathrm{B}\, F(w - l)$

Read the remaining properties of Fourier transform from advanced engineering mathematics