# Chapter Five

# Second order partial differential equations

Jimma Univerity , April 2020

# TOPICS

5.1 Definition and classification of second order PDEs

5.2 Method of separation of variables

5.3 One dimensional heat and their solutions by using methods of Fourier transform

5.4 One dimensional wave equations and their solutions by using methods of Fourier transform

5.5 The potential (Laplace) equation

5.6 Fourier and Laplace transforms, applied to other PDEs

## 5.1 Definition and classification of second order PDEs

At the end of this section the reader will

1) Define second order linear PDE.
2) Be able to distinguish between the 3 classes of 2nd order, linear PDE's. Know the physical problems each class represents and the physical/mathematical characteristics of each

# Definition

## Classification of second order PDEs

The linear second-order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0,$$

where $A$, $B$, $C$, $D$, $E$, and $F$ are real constants, is said to be

$$\begin{aligned} &\text{hyperbolic if} \quad B^2 - 4AC > 0, \\ &\text{parabolic if} \quad B^2 - 4AC = 0, \\ &\text{elliptic if} \quad B^2 - 4AC < 0. \end{aligned}$$

# Examples

Classify the following equations:

(a) $3\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$ (b) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$ (c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

**SOLUTION** (a) By rewriting the given equation as

$$3\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0,$$

we can make the identifications $A = 3$, $B = 0$, and $C = 0$. Since $B^2 - 4AC = 0$, the equation is parabolic.

(b) By rewriting the equation as

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,$$

we see that $A = 1$, $B = 0$, $C = -1$, and $B^2 - 4AC = -4(1)(-1) > 0$. The equation is hyperbolic.

(c) With $A = 1$, $B = 0$, $C = 1$, and $B^2 - 4AC = -4(1)(1) < 0$ the equation is elliptic. ■

# 5.2 Method of separation of variables

**Objective:**

At the end of this unit reader will know:

- The method of separation of variables
- How to obtain the solution of P.D.E by the method of separation of variables.

# Continued

This method consists of the following steps

1. If x and y are indepent variables and u is the dependent variable, we find a solution of the given equation in the form $u=XY$, where $X=X(x)$ is a function of x alone and $Y=Y(y)$ is a function of y alone.

# Continued .

Then, we substitute for u and its partial derivative (computed from u=XY) in the equation and rewrite the equation in such a way that the L.H.S involves X and its derivatives and the R.H.S involves Y and its derivatives.

2. We equate each side of the equation obtained in step 1 to a constant and solve the resulting O.D.E for X and Y.

# Continued

3. Finally we substitute the expression for X and Y obtained in step 2 in u=XY. The resulting expression is the general solution for u.

Examples :Using the method of separation of variables solve

a. $$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y},$$

# Solution :

Substituting $u(x,y) = X(x)Y(y)$ the partial differential equation yields

$$X''Y = 4XY'.$$

After dividing both sides by 4*XY, we have separated the variables:*

$$\frac{X''}{4X} = \frac{Y'}{Y}.$$

Both sides of the equation are independent of *x and y. i.e.* each side of the equation must be a constant

# continued

In practice it is *convenient to write this real* ***separation constant as*** $-\lambda$(using $\lambda$ would lead to the same solutions)

From the two equalities $\frac{X''}{4X} = \frac{Y'}{Y} = -\lambda$

we obtain the two linear ordinary differential equations:

$X'' + 4\lambda X = 0$ and $Y' + \lambda Y = 0$ ----------(1)

we consider three cases for $\lambda$ : zero, negative, or positive,

# Continued

**CASE I : If $\lambda = 0$, then the two ODEs in (1) are:**

$$X'' = 0 \quad \text{and} \quad Y' = 0$$

Solving each equation ,we find $X = c_1 + c_2 x$ and $Y = c_3$

Thus, a particular product solution of the given PDE is $u(x,y) = X(x)Y(y) = (c_1 + c_2 x)c_3$

**CASE II : If $\lambda = -\alpha^2, \alpha > 0$ , then the ODEs in (1 ) are:**

$$X'' - 4\alpha^2 X = 0 \quad \text{and} \quad Y' - \alpha^2 Y = 0$$

**From their general solutions :**

$$X = c_4 \cosh 2\alpha x + c_5 \sinh 2\alpha x \quad \text{and} \quad Y = c_6 e^{\alpha^2 y}$$

# Continued

we obtain another particular product solution of the PDE,

$$u(x,y)=X(x)Y(y)=\left(c_4\cosh 2\alpha x+c_5\sinh 2\alpha x\right)c_6e^{\alpha^2 y}$$ or

$$u(x,y)=X(x)Y(y)=A_1\cosh 2\alpha x\, e^{\alpha^2 y}+A_2\sinh 2\alpha x\, e^{\alpha^2 y}$$

Where $A_1=c_4c_6$ and $A_2=c_5c_6$

**CASE III : If $\lambda=\alpha^2$ , then the ODEs takes the form**

$X''+4\alpha^2X=0$ **and** $Y'+\alpha^2Y=0$

# continued

and their general solutions are:

$X = c_7 \cos 2\alpha x + c_8 \sin 2\alpha x$ and $Y = c_9 e^{-\alpha^2 y}$

give yet another particular solution

$$u(x,y) = X(x)Y(y) = (c_7 \cos 2\alpha x + c_8 \sin 2\alpha x) c_9 e^{-\alpha^2 y}$$ or

$$u(x,y) = X(x)Y(y) = A_3 \cos 2\alpha x \, e^{-\alpha^2 y} + A_4 \sin 2\alpha x \, e^{-\alpha^2 y}$$

Where $A_3 = c_7 c_9$ and $A_4 = c_8 c_9$

Exercise check each values of u obtained in all cases satisfies equation a.

# continued

$$b. \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Exercise

# 5.3 One dimensional heat and their solutions by using methods of Fourier transform

# Derivation Heat Equation in 1D

Exercise

# Example :Using Fourier transform

Solve the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty, t > 0$$ subject to $u(x,0) = f(x)$

Where $f(x) = \begin{cases} u_0, if\ |x| < 1 \\ 0, if\ |x| > 1 \end{cases}$

Solution :The problem can be interpreted as finding the temperature u(x, t) in an infinite rod. Because the domain of x is the infinite interval $(-\infty, \infty)$ .use Fourier transform and define $F(u(x,t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{i\alpha x} dx = U(\alpha, t)$

# Continued

If we transform the partial differential equation and use properties Fourier transform,

$$F\left\{\frac{\partial u}{\partial t}\right\} = F\left\{k\,\frac{\partial^2 u}{\partial x^2}\right\}$$

Yields

$$\frac{dU}{dt} = -k\alpha^2 U(\alpha,t)$$

or

$$\frac{dU}{dt} + k\alpha^2 U(\alpha,t) = 0$$

Solving the last equation gives

$U(\alpha,t) = e^{-k\alpha^2 t}$ .Now the transform of the initial condition

# Continued

Now the transform of the initial condition

$$F\left(u(x,0)\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} u_0 e^{i\alpha x}dx$$

$$=u_0\frac{1}{\sqrt{2\pi}}\frac{e^{i\alpha}-e^{-i\alpha}}{i\alpha}=U(\alpha,0)$$

Thus,

$$U(\alpha,0)=u_0\frac{1}{\sqrt{2\pi}}\frac{e^{i\alpha}-e^{-i\alpha}}{i\alpha}=u_0\sqrt{\frac{2}{\pi}}\;\frac{\sin\alpha}{\alpha}$$

Applying this condition to the solution $U(\alpha,t)$

# Continued

gives $U(\alpha,0)=c=u_0\sqrt{\frac{2}{\pi}}\frac{\sin\alpha}{\alpha}$ ,so

$$U(\alpha,t)=u_0\sqrt{\frac{2}{\pi}}\frac{\sin\alpha}{\alpha}e^{-k\alpha^2 t}$$

Then it follows from the inverse Fourier transform that

$$F^{-1}\left(F\{u(x,t)\}\right)=u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}U(\alpha,t)e^{-i\alpha x}d\alpha$$

$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}U(\alpha,t)e^{-i\alpha x}d\alpha$ and hence

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u_0\sqrt{\frac{2}{\pi}}\frac{\sin\alpha}{\alpha}e^{-k\alpha^2 t}e^{-i\alpha x}d\alpha$$

# Continued

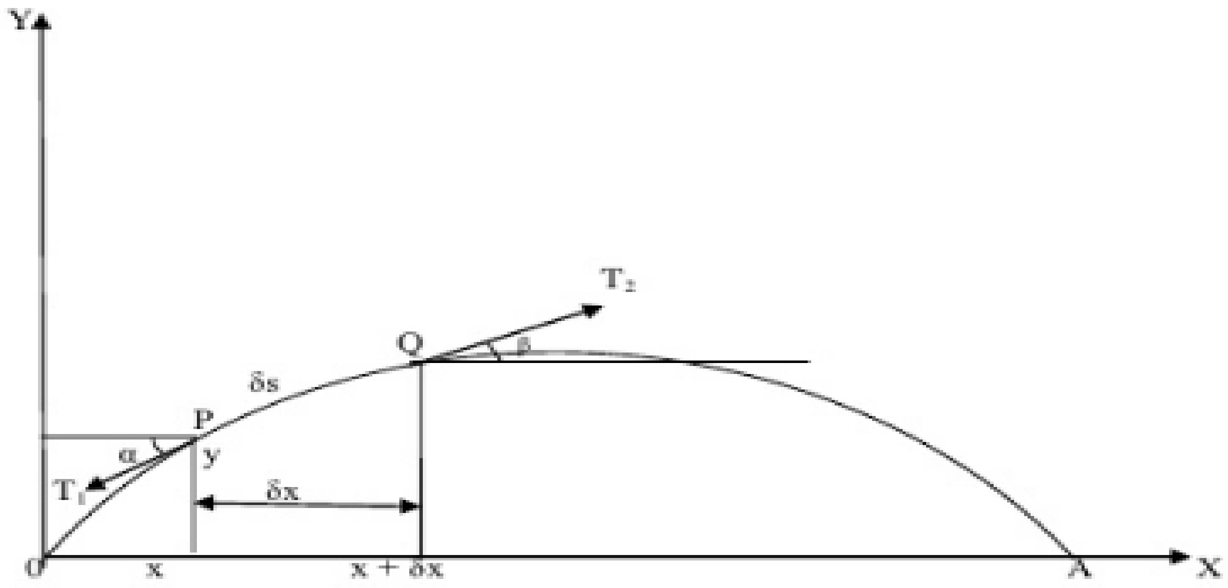
Thus,

$$u(x,t) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin\alpha \cos\alpha x}{\alpha} e^{-k\alpha^2 t}\, d\alpha$$

# Derivation Wave Equation in 1D

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

(Vibrations of a stretched string)




Consider a uniform elastic string of length *l* stretched tightly between points O and A and displaced slightly from its equilibrium position OA. Taking the end O as the origin, OA as the axis and a perpendicular line through O as the y-axis, we shall find the displacement y as a function of the distance x and time t.

displacement y as a function of the distance x and time t.

**Assumptions**

(i) Motion takes places in the XY plane and each particle of the string moves perpendicular to the equilibrium position OA of the string.

(ii) String is perfectly flexible and does not offer resistance to bending.

(iii) Tension in the string is so large that the forces due to weight of the string can be

Let m be the mass per unit length of the string. Consider the motion of an element PQ of length $\delta s$. Since the string does not offer resistance to bending(by assumption), the tensions $T_1$ and $T_2$ at P and Q respectively are tangential to the curve.

Since there is no motion in the horizontal direction, we have

$$T_1 \cos\alpha = T_2 \cos\beta = T(\text{constant}) \qquad .....(1)$$

Mass of element PQ is $m\delta s$.

By Newton's second law of motion, the equation of motion in the vertical direction is

$$m\delta s \frac{\partial^2 y}{\partial x^2} = T_2 \sin\beta - T_1 \sin\alpha$$

Mass of element PQ is mδs.

By Newton's second law of motion, the equation of motion in the vertical direction is

$$m\delta s\frac{\partial^2 y}{\partial x^2} = T_2 \sin\beta - T_1 \sin\alpha$$

$$m\delta s\frac{\partial^2 y}{\partial x^2} = \frac{T}{\cos\beta}\sin\beta - \frac{T}{\cos\alpha}\sin\alpha \qquad (from(1))$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m\delta s}(\tan\beta - \tan\alpha)$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m\delta x}\left[\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_{x}\right]$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left[\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_{x}}{\delta x}\right]$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2}, \qquad as\ \delta x \to 0$$

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \quad where\ c^2 = \frac{T}{m}$$

This is the partial differential equation giving the transverse vibrations of the string. It is also called the one-dimensional wave equation.

**Boundary conditions**

For every value of t, $y = 0$ when $x = 0$

$y = 0$ when $x = l$

**Initial conditions**

If the string is made to vibrate by pulling it into a curve $y=f(x)$ and then releasing it, the initial conditions are

(i) $y = f(x)$ when $t = 0$

(ii) $\frac{\partial y}{\partial t} = 0$ when $t = 0$

# Poisson’s and Laplace Equations

A useful approach to the calculation of electric potentials
Relates potential to the charge density.
The electric field is related to the charge density by the divergence relationship

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

$E$ = electric field
$\rho$ = charge density
$\varepsilon_0$ = permittivity

The electric field is related to the electric potential by a gradient relationship

$$E = -\nabla V$$

Therefore the potential is related to the charge density by Poisson's equation

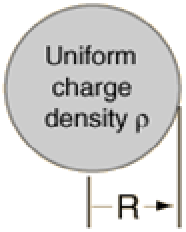
$$\nabla \cdot \nabla V = \nabla^2 V = \frac{-\rho}{\varepsilon_0}$$

In a charge-free region of space, this becomes Laplace's equation

$$\nabla^2 V = 0$$

# Continued

## Potential of a Uniform Sphere of Charge




Total charge

$$Q = \frac{4}{3}\pi R^3 \rho$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta} = \frac{-\rho}{\varepsilon_0}$$

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} = \frac{-\rho}{\varepsilon_0}$$

outside $\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} = 0$ , solution of form $\frac{a}{r} + b$ $\quad a = \frac{Q}{4\pi\varepsilon_0} = kQ \quad V = \frac{Q}{4\pi\varepsilon_0 r}$

inside $V = cr^2 + d \qquad 2c + 4c = \frac{-\rho}{\varepsilon_0}$ giving $c = \frac{-\rho}{6\varepsilon_0}$

$$\frac{-\rho R^2}{6\varepsilon_0} + d = \frac{Q}{4\pi\varepsilon_0 R}$$ giving $$d = \frac{Q}{4\pi\varepsilon_0 R} + \frac{\rho R^2}{6\varepsilon_0}$$

$$V = \frac{\rho}{6\varepsilon_0}\left[R^2 - r^2\right] + \frac{Q}{4\pi\varepsilon_0 R} = \frac{\rho}{6\varepsilon_0}\left[R^2 - r^2\right] + \frac{\rho R^2}{3\varepsilon_0}$$

# Continued

## Poisson's and Laplace Equations

From the point form of Gaus's Law

Del_dot_D = $\rho_V$

Definition D

D = εE

and the gradient relationship

E = −DelV

Del_D = Del_(εE) = −Del_dot_(εDelV) = $\rho_V$

Del_DelV = $\frac{-\rho_V}{\varepsilon}$ Poisson's Equation

Laplace's Equation

if $\rho_V = 0$

Del_dot_D = $\rho_V$

Del_Del = Laplacian

The divergence of the gradient of a scalar function is called the Laplacian.

# Continued

$$V = \frac{\rho}{6\varepsilon_0}\left[R^2 - r^2\right] + \frac{Q}{4\pi\varepsilon_0 R} = \frac{\rho}{6\varepsilon_0}\left[R^2 - r^2\right] + \frac{\rho R^2}{3\varepsilon_0}$$

From the point form of Gaus's Law

Del_dot_ D = $\rho_V$

Definition D

D = εE

and the gradient relationship

E = −DelV

Del_D = Del_(εE) = −Del_dot_(εDelV) = $\rho_V$

Del_DelV = $\frac{-\rho_V}{\varepsilon}$

Laplace's Equation

if $\rho_V = 0$

Del_dot_D = $\rho_V$

Del_Del = Laplacian

The divergence of the gradient of a scalar function is called the Laplacian.

# Continued

## Poisson's and Laplace Equations

$$LapR := \left[\frac{d}{dx}\left(\frac{d}{dx}V(x,y,z)\right) + \frac{d}{dy}\left(\frac{d}{dy}V(x,y,z)\right) + \frac{d}{dz}\left(\frac{d}{dz}V(x,y,z)\right)\right]$$

$$LapC := \frac{1}{\rho}\cdot\frac{d}{d\rho}\left(\rho\cdot\frac{d}{d\rho}V(\rho,\phi,z)\right) + \frac{1}{\rho^2}\cdot\left[\frac{d}{d\phi}\left(\frac{d}{d\phi}V(\rho,\phi,z)\right)\right] + \frac{d}{dz}\left(\frac{d}{dz}V(\rho,\phi,z)\right)$$

$$LapS := \left[\frac{1}{r^2}\cdot\frac{d}{dr}\left(r^2\cdot\frac{d}{dr}V(r,\theta,\phi)\right)\right] + \frac{1}{r^2\cdot\sin(\theta)}\cdot\frac{d}{d\theta}\left(\sin(\theta)\cdot\frac{d}{d\theta}V(r,\theta,\phi)\right) + \frac{1}{r^2\cdot\sin(\theta)^2}\cdot\frac{d}{d\phi}\frac{d}{d\phi}V(r,\theta,\phi)$$

# Examples of the Solution of Laplace's Equation

Given

$$V(x,y,z) := \frac{4\cdot y\cdot z}{x^2+1}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\varepsilon o := 8.85410^{-12}$$

$$V(x,y,z) = 12$$

Find: V @ and ρv at P

$$LapR := \left[\frac{d}{dx}\left(\frac{d}{dx}V(x,y,z)\right) + \frac{d}{dy}\left(\frac{d}{dy}V(x,y,z)\right) + \frac{d}{dz}\left(\frac{d}{dz}V(x,y,z)\right)\right]$$

$$LapR = 12$$

$$\rho v := LapR\cdot\varepsilon o$$

$$\rho v = 1.062\times 10^{-10}$$

# continued

## Uniqueness Theorem

Given is a volume V with a closed surface S. The function V(x,y,z) is completely determined on the surface S. There is only one function V(x,y,z) with given values on S (the boundary values) that satisfies the Laplace equation.

Application: The theorem of uniqueness allows to make statements about the potential in a region that is free of charges if the potential on the surface of this region is known. The Laplace equation applies to a region of space that is free of charges. Thus, if a region of space is enclosed by a surface of known potential values, then there is only one possible potential function that satisfies both the Laplace equation and the boundary conditions.

Example: A piece of metal has a fixed potential, for example, V = 0 V. Consider an empty hole in this piece of metal. On the boundary S of this hole, the value of V(x,y,z) is the potential value of the metal, i.e., V(S) = 0 V. V(x,y,z) = 0 satisfies the Laplace equation (check it!). Because of the theorem of uniqueness, V(x,y,z) = 0 describes also the potential inside the hole

# Example : Using Laplace transforms

## Solve the Wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

subject to $w(0,t) = f(t)$ and $\lim_{x\to\infty} w(x,t) = 0$ (for $t \geq 0$)

initial conditions $w(x,0) = 0$,

$$\left.\frac{\partial w}{\partial t}\right|_{t=0} = 0$$

FIRST, we take the LT with respect to $t$ :

$$s^2 W(s) - sw(x,0) - \left.\frac{\partial w}{\partial t}\right|_{t=0} = c^2 L\left(\frac{\partial^2 w}{\partial x^2}\right)$$

The initial conditions mean that the second and third terms drop out

# Continued

$$L\left(\frac{\partial^2 w}{\partial x^2}\right) = \int_0^\infty e^{-st} \frac{\partial^2 w}{\partial x^2} dt$$

Exchanging the order of integration and differentiation :

$$L\left(\frac{\partial^2 w}{\partial x^2}\right) = \frac{\partial^2}{\partial x^2} \int_0^\infty e^{-st} w(x,t) dt = \frac{\partial^2}{\partial x^2} L(w) = \frac{\partial^2 W}{\partial x^2}$$

It follows that :

$$s^2 W = c^2 \frac{\partial^2 W}{\partial x^2}$$

$$\frac{\partial^2 W}{\partial x^2} - \frac{s^2}{c^2} W = 0$$

so

$$W(x,s) = A(s) e^{\frac{sx}{c}} + B(s) e^{-\frac{sx}{c}}$$

# Continued

$$F(s) = L(f(t)) = W(0,s)$$

Exchanging integration & differentiation again :

$$\lim_{x\to\infty} W(x,s) = \lim_{x\to\infty} \int_0^\infty e^{-st} w(x,t)dt = \int_0^\infty e^{-st} \lim_{x\to\infty} w(x,t)dt = 0$$

and so

$$A(s) = 0$$
$$W(0,s) = B(s) = F(s)$$
$$W(x,s) = F(s)e^{-sx/c}$$

From HLT or from Kreysig page 296 (line 11), we have :

$$L\left(f\left(t-\frac{x}{c}\right)u\left(t-\frac{x}{c}\right)\right) = F(s)e^{-as} \text{ (second shifting theorem)}$$

and so $$w(x,t) = f\left(t-\frac{x}{c}\right)u\left(t-\frac{x}{c}\right)$$

that is :

$$w(x,t) = \begin{cases} \sin\left(t-\frac{x}{c}\right) & \text{if } \frac{x}{c} < t < \frac{x}{c} + 2\pi \\ 0 & \text{otherwise} \end{cases}$$