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FINITE ORDERED SETS

Concepts, Results and Uses

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and Bernard Monjardet

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FINITE ORDERED SETS
Concepts, Results and Uses

Ordered sets are ubiquitous in mathematics and have significant applications in computer science, statistics, biology, and the social sciences. As the first book to deal exclusively with finite ordered sets, this book will be welcomed by graduate students and researchers in all of these areas.

Beginning with definitions of key concepts and fundamental results (Dilworth's and Sperner's theorem, interval and semiorders, Galois connection, duality with distributive lattices, coding and dimension theory), the authors then present applications of these structures in fields such as preference modeling and aggregation, operational research and management, cluster and concept analysis, and data mining. Exercises are included at the end of each chapter with helpful hints provided for some of the most difficult examples. The authors also point to further topics of ongoing research.

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Finite Ordered Sets

Concepts, Results and Uses

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Contents

Preface

The notions of order, classification, and ranking exist in numerous human activities and situations: administrative or social hierarchies, organization charts, scheduling of sports tournaments, precedence, succession or preference orders, agendas, school, audiovisual or webpage rankings, alphabetical and lexicographic orders, etc. It would be endless to enumerate all the situations where orders appear.

It is thus not surprising, considering the development of the use of mathematics in the modeling of multiple phenomena, to find a great number of fields where order mathematics occur. Nevertheless, the latter are relatively recent. Of course, in mathematics, the notion of the order of magnitude has been known for a long time and in the sixteenth century the symbols “<” and “>” appeared for the first time to express “less than” and “greater than.”[1] Yet, the abstract notion of an order defined as a particular type of transitive relation was developed only between 1880 and 1914 by mathematicians and/or logicians such as Peirce, Peano, Schröder, Cantor, Dedekind, Russell, Huntington, Scheffer, and Hausdorff, in the context of the formalization of the “algebra of logic” (that is, Boolean algebra) and also of the creation of set theory (with the study of “order types”). Lattices, which are particular orders since they can be defined algebraically, were also considered as early as the later part of the nineteenth century by Schröder and Dedekind, and then fell into oblivion before arising again during the 1930s thanks to Birkhoff, Öre, and several other eminent mathematicians. For a long time, lattices were the main studied orders. Lattice theory – as well as universal algebra, which is its natural extension – is still extremely active. Besides, the most fundamental result of the theory of finite orders, namely Dilworth’s Theorem, was proved only in 1950 in relation to a problem on lattices. However, since the 1970s, the situation has evolved significantly. Researches on order structures have increased widely to answer internal motivations of “pure mathematics” as well as problems raised by the use of these structures in “applied mathematics” (in fields such as operations research, microeconomics, data analysis, data mining, biology, robotics,

[1] In *Artis Analyticae Praxis ad Aequationes Algebraicas Resolvendas*, by the mathematician Thomas Harriot.

theoretical computer science, algorithmics, etc).[2] Today, no less than a treatise of at least 1000 pages would be necessary to present a mere synthesis of the existing results.

This is of course not the purpose of this book, which is limited to some aspects and to the following three main goals:

- to define the concepts and to expound the fundamental results on finite ordered sets;
- to present their uses in various fields;
- to point out a number of results and current works.

The choice to remain within the scope of *finite* ordered sets is in part justified by our concern to publish a reasonably sized book. It also places the book in the field of discrete mathematics, the importance of which nowadays is clear. In this (still very wide) scope, we have given greater importance to the notions and results which seemed essential to us because of their uses in a great number of modelings: linear extensions of an order, isotony, closure operators and closure systems, residual maps and Galois connections, chains and antichains with the Dilworth and Sperner theorems, the duality between ordered sets and distributive lattices, order codings and dimensions, interval orders and semiorders, Arrowian results on orders, etc. And actually, in every chapter of this book, the reader will find examples of uses of order structures in various fields. The last and longest chapter develops some of these uses in (often interdisciplinary) contexts such as preference modeling, data analysis, and scheduling.

At last, in order to cover up the fact that we present only the most fundamental results, each chapter is enriched with a "Further topics and references" section. There, we point out numerous themes that could not be developed in the body of the chapter, and we sometimes give some historical elements, often useful for a better understanding of the subjects.

There is a more general motivation for writing this book. We have mentioned the important development of lattice theory, which has been so much written about – several tens of volumes. On the contrary, books on general ordered sets are very rare and, most often, deal with particular aspects (see Appendix D). A consequence of this situation is that some results are too often unused or rediscovered, which goes against the good use of mathematics.

We continue this preface with a description of the contents of the chapters and appendices.

The principal aim of Chapter 1 is to define and illustrate the fundamental notions used to describe, study, and work on ordered sets, these notions being used and/or developed in the following chapters. Consequently, this chapter presents few results

[2] Another expression of this development was the creation in 1985 of the journal *Order* by Ivan Rival.

and gives no proof. It nevertheless contains examples of uses of ordered sets in fields going from mathematics to operations research, biology, computer science or social sciences.

If orders are binary relations satisfying strong properties, it is however a fact that there exist few results concerning the class of all orders. Actually, like in some other mathematical theories, one is often concerned with classes of orders verifying some particular properties. Chapter 2 describes the most important classes of ordered sets: ranked ordered sets, semimodular and bipartite ordered sets, ordered sets defined by forbidden configurations, semilattices and lattices, linearly ordered sets.

Chapter 3 is devoted to the important question of morphisms between ordered sets; that is, maps between two ordered sets, which preserve or reverse their order. Among these morphisms, one finds codings, closure and dual closure operators, residuated, residual or Galois maps. The latter are the components of Galois connections, the fundamental tool which allows us to set the duality between two ordered sets and, in particular, to define a Galois lattice – the use of which goes from the search for "Guttman scales" (in questionnaires analysis) to the generation of "concepts" (in data analysis or artificial intelligence). This is also the chapter in which we develop the important notions of irreducible elements of an ordered set and of arrow relations between these elements.

Dilworth's Theorem (1950) sets the equality in any ordered set between the maximum number of its pairwise incomparable elements and the minimum number of chains in a chain partition of the ordered set. This is a central result since, on the one hand, it holds for any ordered set and allows us to solve a problem met in various situations (for instance, in operations research, computer science or plane geometry) and, on the other hand, it is related – and in fact, often equivalent – to many other famous results in combinatorics, namely, for example, the König–Hall, Menger, and Ford and Fulkerson theorems. Chapter 4 is devoted to Dilworth's Theorem, and also to the generalizations of another famous result due to Sperner, which gives the maximum number of incomparable subsets (for the inclusion relation) of a given set. Sperner orders, studied in this chapter, are the ranked orders for which the maximum number of incomparable elements of the order can be obtained from the consideration of its rank-sets.

The representation theorem of distributive lattices owing to Birkhoff provides a set representation of a distributive lattice by means of its irreducible elements. This leads to a fundamental duality between distributive lattices and ordered sets, which implies that every result on a distributive lattice can be translated into a result on an ordered set, and conversely. This duality is studied in Chapter 5, where we prove that it is the consequence of a Galois connection between binary relations and families of subsets. We also present another duality between orders and some particular sets of linear orders.

Szpilrajn's 1930 result states that every order can be extended into a linear order (called a linear extension of the order) and allows us to prove that every order is the intersection of all its linear extensions. The dimension of an order is then the minimum number of linear extensions of which it is the intersection. The dimension parameter has been studied intensively for theoretical reasons, but also because it was used in a number of modelings. For instance, it was used as an explanatory model of a preference relation in microeconomics: the partial preference order of an economic agent on a set of commodity bundles is interpreted as resulting from the consideration of several criteria modelized by linear orders. Moreover, trying to determine the dimension of an order is also equivalent to searching for the minimum number of linear orders in the direct product of which it can be coded (that is, in which one can find an isomorphic image of the order). More generally, one can be interested in coding an order in a direct product of chains the length of which is given. If these chains have length 1, the latter operation is equivalent to coding the order by some subsets of a set (in other words, by sequences of 0 and 1), and we then talk about a Boolean coding and the Boolean dimension, notions that were introduced and most studied in computer science. Chapter 6 sets out the fundamental results on these codings and dimensions.

All these chapters are illustrated with examples of uses of ordered sets in various contexts. Our last chapter develops some of these uses. The first two sections of Chapter 7 focus on the notion of a preference, which concerns, among others, cognitive science, microeconomics, operations research, artificial intelligence, and also databases (in which it helps to define effective request languages). We first deal with the modeling of a preference relation, for instance that of an economic agent, when we release the strong hypothesis that the indifference relation should be transitive; the suitable models are then interval orders and semiorders. We next consider the problem of the aggregation of several preference relations into a global preference relation, which should be an order, and we establish a number of "Arrowian" theorems that give prominence to the difficulty of getting a satisfactory result. We carry on with the presentation of ordered models used in mathematical taxonomy: hierarchies, valued hierarchies, median semilattices, partition lattices. The next section focuses on the use of Galois lattices in relational data analysis. We show how, from such a lattice or from its associated closure operator, we can deduce an implicational system allowing us to answer questions like: do subjects having such-and-such characteristics have – or not have – such-and-such other characteristics? As for the fifth section, it presents scheduling problems and some ordinal tools used to deal with them.

Each one of the seven chapters, after its "Further topics and references" section, contains as a last section a list of exercises, which illustrates the notions presented in the chapter and the solutions that have to be sought by anyone who really wants to become familiar with ordered set mathematics. For the majority of these exercises, the solutions will easily be found from the results inside the chapter. For the others, hints and references are provided.

The practical use of the notions and results presented in this book requires being able to answer questions asked on an ordered set modeling some situations, which will in general be done by resorting to a program implementing a resolving algorithm. Appendix A gives some basic notions on the theory of algorithmic complexity, and numerous complexity results for order algorithmics. Appendices B and C are concerned with small orders and the counting of orders. Appendix D provides various documentary indications, while the list of references will allow the reader to get to the results mentioned in the "Further topics and references" section of the chapters. As expected, we also provide an index and a list of symbols.

In each chapter, the definitions, theorems, propositions, corollaries, lemmas, examples, and remarks are numbered $n.p$ where n is the chapter number and p the appearance number in the chapter. For instance, Definition 4.14 is the fourteenth numbered item in Chapter 4.

In each chapter, all figures (respectively, tables) are labeled Figure $n.p$ (respectively, Table $n.p$), where n is the chapter number and p the appearance number in the chapter. For instance, Figure 3.6 is the sixth figure in Chapter 3 and Table 7.3 is the third table in Chapter 7.

1

Concepts and examples

This first chapter covers all basic notions of the theory of finite ordered sets and gives an idea of the various domains in which they are encountered. Beware! It would be fastidious and unproductive to approach this book with a linear reading of this chapter. The reader is invited to use it as a reference text in which he will find the definitions and illustrations of the notions used in the next chapters. In particular, we do not provide in this chapter the proofs of the few stated results (the reader will find these proofs in other chapters and/or in exercises). In Section 1.1 we give the concepts and the vocabulary allowing us to define, represent, and describe an ordered set. We also introduce several graphs (comparability, incomparability, covering, neighborhood graphs) associated with an ordered set. Section 1.2 presents some examples of ordered sets that appear in various disciplinary fields from mathematics themselves to social sciences and ranging from biology to computer science. We define the notions of an ordered subset, a chain, an antichain, and of an extension of an ordered set in Section 1.3 and the notions of a join and a meet, of irreducible elements, and of downsets or upsets in Section 1.4. Finally, Section 1.5 describes the basic construction rules (linear sum, disjoint union, substitution, direct product, etc.) that form new ordered sets from given ones.

1.1 Ordered sets

In the very beginning there was the order... or the strict order! This section therefore begins with the definition of these two order notions, with their associated terminology. Then we present different graphs (comparability, incomparability, covering, neighborhood graphs) associated with an ordered set. Section 1.1.1 is devoted to a very useful representation of an ordered set called its diagram. Finally, we end the section by the standard mathematical notion of an isomorphism between ordered structures together with the also very significant notion of a dual isomorphism (or duality).

1.1.1 Orders and strict orders

What is an order in mathematics? This question, raised in 1903 by the logician and philosopher Bertrand Russell, has essentially received two answers (generally) named *order* and *strict order* (see Section 1.6 for the history of these notions). On the other hand, the notations and terms used to refer to the same ordinal notions have been – and remain – very diverse. In this book we do not refrain from using several different symbols or terms to denote or to name the same notions since, from experience, we know that using a unique notation system may cause more disadvantages than advantages. However in this section and in order to compensate for these possible disadvantages, we specify (in a very thorough and thus somewhat tedious way) the two fundamental notions of order with the various notation systems that we will use.

A *binary relation* on a set X is a subset R of the set X^2 of the ordered pairs of X. The notation $(x,y) \in R$ (or xRy) means that the ordered pair (x,y) belongs to the relation R. We write $(x,y) \notin R$ – or $xR^c y$ – if not.

Definition 1.1 A binary relation O on a set X is an *order* if it satisfies the following three properties:

1. Reflexivity: for each $x \in X, xOx$.
2. Antisymmetry: for all $x,y \in X$, xOy and yOx imply $x = y$.
3. Transitivity: for all $x,y,z \in X$, xOy and yOz imply xOz.

The order O is *linear* (or *total*[1]) if, for all $x,y \in X$, $xO^c y$ implies yOx.

An *ordered set* (or a *partially ordered set* or a *poset*) is an ordered pair $P = (X,O)$ where X is a set and O an order on X (sometimes to avoid ambiguity, we will find it useful to denote O_P the order of the ordered set P). If O is a linear order, $P = (X,O)$ is then called a *linearly ordered set* (or a *totally ordered set* or a *chain*). The symbols $\underline{n}$ or C_n denote a chain of size n.

Example 1.2 Let $X = \{a,b,c,d,e\}$ and $P = (X,O)$ be the ordered set where O is the following order on X:

$$O = \{(a,b),(a,e),(c,b),(c,d),(c,e),(d,e),(a,a),(b,b),(c,c),(d,d),(e,e)\}$$

An ordered set P can be represented by a network, the points of which correspond to the elements of X and the arcs (or directed edges) to the ordered pairs of O, the loops representing the ordered pairs of the form (x,x) (cf. Figure 1.1). We can also represent it by tables (see Table 1.1). The cells of these tables correspond to all ordered pairs of X and a 1 or a $\times$ in a cell (respectively, a 0 or an empty cell) means that the corresponding ordered pair belongs (respectively, does not belong) to O. Yet we will

[1] A binary relation R on X is said to be *total* (respectively, *weakly total*) if, for all $x,y \in X$, $xR^c y$ implies yRx (respectively, $x \neq y$ and $xR^c y$ imply yRx).

Table 1.1 Two kinds of table representing the ordered set P in Example 1.2

	a	b	c	d	e
a	×	×			×
b		×			
c		×	×	×	×
d				×	×
e					×

	a	b	c	d	e
a	1	1	0	0	1
b	0	1	0	0	0
c	0	1	1	1	1
d	0	0	0	1	1
e	0	0	0	0	1

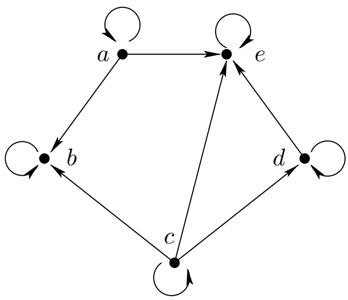


Figure 1.1 An ordered set P represented by a network.

see in Section 1.1.3 a much more economical way to represent an ordered set: the *(Hasse) diagram*.

We now give the definition of a strict order.

Definition 1.3 Let O be a binary relation on a set X.

- O is a *strict order* if it is irreflexive (for each $x \in X$, xO^cx) and transitive. A *strictly ordered* set is an ordered pair $P = (X,O)$, where X is a set and O a strict order on X.
- A strict order O is *strictly linear* if, for all $x,y \in X$, $x \neq y$ and xO^cy imply yOx. We then say that $P = (X,O)$ is a *strictly linearly ordered set*. If $|X| = n$, P or the corresponding strictly linear order may be denoted by $\underline{n}_s$.

Note A strict order O on X is an *asymmetric* relation, i.e., such that yO^cx for all $x,y \in X$ satisfying xOy (prove it).

Since there exists an obvious one-to-one correspondence between the set of orders and the set of strict orders defined on a set X (what is it?), there are two equivalent ways to formalize the notion of an order (see the further topics in Section 1.6). So, to each particular class of orders corresponds a particular class of strict orders (for example, strictly linear orders correspond to linear orders). In order to simplify the terminology, it will sometimes be preferable to use the same terms to name the orders of these two corresponding classes. Thus, in Section 7.1 of Chapter 7, where we will consider several models of strict preferences represented by strict orders, the qualifier "strict" will be systematically omitted.

So far we have used the notation O for an order but, in most cases, this notation is favorably replaced by the symbol "$\leq$" which is read "less than or equal to." We thus use the traditional symbol of the order between numbers for an arbitrary order. An ordered set is then denoted by $P = (X, \leq_P)$ or, more simply, by $P = (X, \leq)$.

Likewise, a strict order will often be denoted by the symbol "$<$," which is read "less than" (or "smaller than"), and we will write $P = (X, <_P)$ or, more simply, $P = (X, <)$ for a strictly ordered set.

Let $P = (X, \leq)$ (or (X, O)) be an ordered set.

- The size of P is the size of X and we may denote it by $|P|$, $|X|$ or simply n according to the context.
- The expression "x belongs to P" means $x \in X$ and we also write $x \in P$.
- The expression "(x,y) belongs to P" means $x \leq y$ (or $(x,y) \in O$) and we also write $x \leq_P y$, xO_Py or simply xOy depending on the notations used for P.
- The number of the ordered pairs belonging to O is denoted by $|O|$, $m(P)$ or simply m.

Let x,y be two elements of an ordered set $P = (X, \leq)$.

- If $x \leq y$, we say that x is *less than or equal to* y, or that y is *greater than or equal to* x. We also say that x is a *lower bound* of y and that y is an *upper bound* of x. The set $\{t \in P : t \leq x\}$ of lower bounds of x is denoted by $(x]$ or Px. The set $\{t \in P : x \leq t\}$ of upper bounds of x is denoted by $[x)$ or xP.
- If $x \leq y$ does not hold, we say that x *is not less than or equal to* y and we write $x \not\leq y$. This relation is also denoted by $\leq^c$ (since it is the complementary relation of the relation $\leq$).
- If $x \leq y$ and $x \neq y$, we say that x is *less than* y, or that y is *greater than* x, and we write $x < y$. We also say that x is a *strict lower bound* of y and that y is a *strict upper bound* of x. The relation $<$ is the strict order relation associated with the relation $\leq$. The set of strict lower bounds (respectively, strict upper bounds) of x is denoted by $(x[$ (respectively, $]x)$).
- If $x \leq y$ or $y \leq x$, we say that x and y are *comparable*. If not, i.e., if $x \not\leq y$ and $y \not\leq x$, we say that x and y are *incomparable* and we write $x||y$ (or $xInc_Oy$ if the order is denoted by O).

In Example 1.2, a and b are comparable whereas a and c are incomparable.

One will observe that $y \not\leq x$ is equivalent to ($x < y$ or $x||y$). It results from the above definitions that a chain is an ordered set in which any two elements are always comparable. Conversely, we define the notion of an antichain.

Definition 1.4 An *antichain* is an ordered set such that any two distinct elements are always incomparable. We write A_n for an antichain of size n.

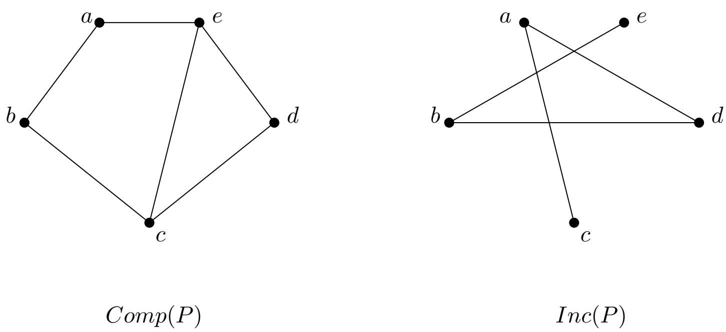


Figure 1.2 The comparability and incomparability graphs of the ordered set in Example 1.2.

1.1.2 Graphs associated with an ordered set

Several graphs are naturally associated with an ordered set,[2] such as, in particular, the comparability, incomparability, covering, or neighborhood graph. Each of these graphs corresponds to some particular aspects of the ordered set and may be important for its study. We define and illustrate these graphs below.

Definition 1.5 Let $P = (X, O)$ be an ordered set. The *comparability graph* of P is the undirected graph $Comp(P) = (X, Comp_P)$, the vertices of which are the elements of P and where the edges are the pairs $\{x, y\}$ of comparable elements in P. The relation $Comp_P$, also written $Comp_O$, is called the *comparability relation of P*.

The *incomparability graph* $Inc(P) = (X, Inc_P)$ of P is defined similarly, with its edges equal to the pairs $\{x, y\}$ of incomparable elements in P. The relation Inc_P, also written Inc_O, is called the *incomparability relation of P*.

Figure 1.2 shows the comparability and incomparability graphs of the ordered set in Example 1.2.

The graphs $Comp(P)$ and $Inc(P)$ are obviously complementary to each other in the sense that the pair $\{x, y\}$ is an edge of one of them if and only if it is not an edge of the other one. Then, to study one of them is equivalent to studying the other.

We now define the covering relation associated with an ordered set. This relation is generally not an order but is, on the other hand, the most economical way (with respect to the number of ordered pairs) to describe an order. It will be used constantly throughout the book.

[2] An *undirected graph* is an ordered pair $G = (X, E)$ where X is a set and E a set of pairs of distinct elements of X, called the *edges* of G; a *directed graph* is an ordered pair $G = (X, A)$ where X is a set and A a set of ordered pairs of X, called the *arcs* of G.

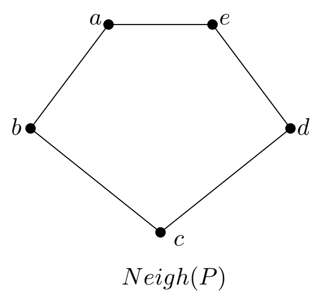


Figure 1.3 The neighborhood graph of the ordered set P in Example 1.2.

Definition 1.6 The *covering relation* of an ordered set $P = (X, \leq)$, denoted by $\prec_P$ or simply $\prec$, is defined by $x \prec y$ if $x < y$ and $x \leq z < y$ imply $x = z$. We then say that x *is covered by* y or that y *covers* x. We write $xP^+ = \{t \in P : x \prec t\}$ and $P^- x = \{t \in P : t \prec x\}$.

The directed graph $Cov(P) = (X, \prec)$ associated with the covering relation is called the *covering graph* of P.

In other words, x is covered by y in P if $x < y$ and if there does not exist in P any element z greater than x and less than y.

The ordered set P in Example 1.2 has five covering ordered pairs: $a \prec b, a \prec e$, $c \prec b, c \prec d$, and $d \prec e$.

The covering relation of a chain defined on a set of size n is written $x_1 \prec x_2 \prec ... \prec x_n$, which we more simply denote by $x_1x_2 \ldots x_n$. This particularly economical notation of linear orders will often be used. Conversely, any sequence of n distinct elements (or, equivalently, any permutation on these elements) can be seen as defining a linear order on these elements, namely the apparition order in the series. This implies that the number of linear orders on a set of size n is equal to $n!$.

With the covering relation of P is also associated an undirected graph called the *neighborhood graph of* P, denoted by $Neigh(P) = (X, N_P)$, where the pair $\{x,y\} \in N_P$ if $(x \prec y$ or $y \prec x)$. For the ordered set in Example 1.2, this graph is given by Figure 1.3.

A number of notions on ordered sets may be defined by means of the neighborhood graph. It is the case for the notion of the connectivity:

Definition 1.7 An ordered set P is *connected* if its neighborhood graph is connected, i.e., if, for any pair of distinct vertices $\{x,y\}$ of P, there exists a sequence $x = x_0, x_1, ..., x_i, x_{i+1}, ..., x_p = y$ of vertices such that $x_i N x_{i+1}$, for any $i = 0, ..., p-1$.

A non-connected ordered set is partitioned into maximal connected ordered subsets (see Definition 1.26), called its *connected components*. Since a problem on a non-connected ordered set most often comes back to a problem on its connected components, it is generally enough to consider connected ordered sets.

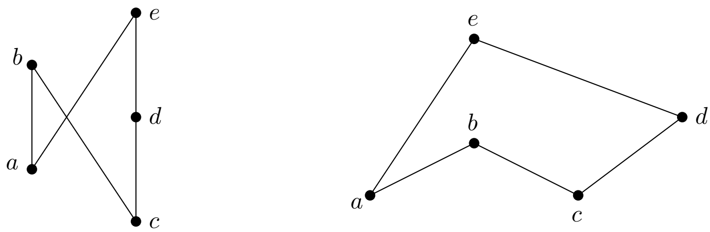


Figure 1.4 Two diagrams of the ordered set in Example 1.2.

1.1.3 Diagram of an ordered set

At the beginning of the chapter we saw that one can consider an ordered set as a network, which is a "geometrical" representation of the latter (see Figure 1.1). However, as soon as one considers a "big" ordered set, the network risks becoming quite inextricable. It is possible to do better thanks to the notion of a diagram of an ordered set, which provides a much more economical representation of an ordered set. First, we observe that to know the covering ordered pairs of $P = (X, \leq)$ allows us to find all ordered pairs of the order $\leq$. Indeed, we have $x < y$ if and only if there exists a sequence $x_0, x_1, \ldots, x_p$ of elements of X such that $x = x_0 \prec x_1 \prec \ldots \prec x_p = y$. Thus we can represent an ordered set thanks to its covering ordered pairs, which is done by using the (Hasse) diagram.

Definition 1.8 The *diagram* (or *Hasse diagram*) of an ordered set $P = (X, \leq)$ is a representation of its covering graph in which the elements x of P are represented by points $p(x)$ of the plane, with the following two rules:

- If $x < y$ (the horizontal line going through) $p(x)$ is below (the horizontal line going through) $p(y)$.
- $p(x)$ and $p(y)$ are linked by a line segment if and only if $x \prec y$.

Clearly, there exist an infinity of possible diagrams for a given ordered set. Yet, just like we did in the above definition, we will generally talk about "the" diagram of an ordered set P, instead of specifying "one of the diagrams" of P. The choice of the position of the points allows us to obtain some diagrams that are easier to read than others. Figure 1.4 shows two possible diagrams for the ordered set in Example 1.2, and Figure 1.5 shows diagrams of the chain C_4, of the antichain A_4, and of the "cube" B_3 (the letter "B" stands for "Boolean," see Example 1.12 further on).

Later on, all figures representing an ordered set will show a diagram of the latter. Let us observe that, since the diagram of an ordered set does not represent the reflexive ordered pairs, it may as well represent the associated strict ordered set.

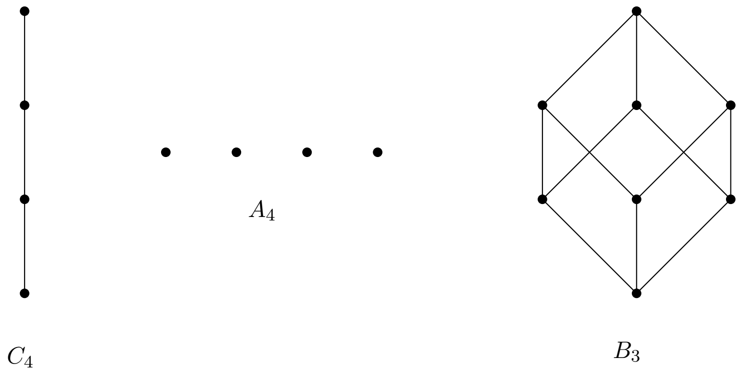


Figure 1.5 The diagrams of the chain C_4, of the antichain A_4, and of the cube B_3.

1.1.4 Isomorphism and duality

In mathematics, the notion of an *isomorphism* between two structures is fundamental. It allows us to prove that two sets of objects of different nature may satisfy the same properties. When considering order structures, we have to consider also the other very significant notion of a *dual isomorphism* (or of a *duality*). Recall that a map f from X to Y is called a *bijection* – or a *one-to-one correspondence* – if it is injective – or one-to-one, i.e., $x \neq y$ implies $f(x) \neq f(y)$, and surjective – or onto, i.e., $f(X) = Y$.

Definition 1.9 Two ordered sets $P = (X, \leq_P)$ and $Q = (Y, \leq_Q)$ are said to be *isomorphic* (or *of the same type*) if there exists a bijection f from X to Y such that:

$$x \leq_P y \iff f(x) \leq_Q f(y)$$

The bijection f is called an *order isomorphism* between P and Q and we write $P \equiv Q$. When $P = Q$, we say that f is an *automorphism* of P.

In other terms, two ordered sets are isomorphic if they are identical up to the denomination of their elements. Thus we obtain an ordered set isomorphic to that in Figure 1.4 (Example 1.2) by replacing a, b, c, d, e with 1, 2, 3, 4, 5.

The isomorphism relation between ordered sets is an equivalence relation the classes of which, in accordance with Definition 1.9, are called the types of ordered sets. Then two isomorphic ordered sets are said to be "of the same type." In order to illustrate the difference between order and order type, we can note that there exist 130 023 distinct orders defined on a set of size 6 whereas there are only 318 different order types on such a set[3] (for easy countings, see Exercise 1.1). Appendix B provides the diagrams of the 58 connected order types of size at most equal to 5.

[3] In general, counting all orders (respectively, all order types) on a set of size n is very difficult and, at the present time, the answer is known only for $n \leq 18$ (respectively, $n \leq 16$). These numbers increase very quickly (see Appendix C).

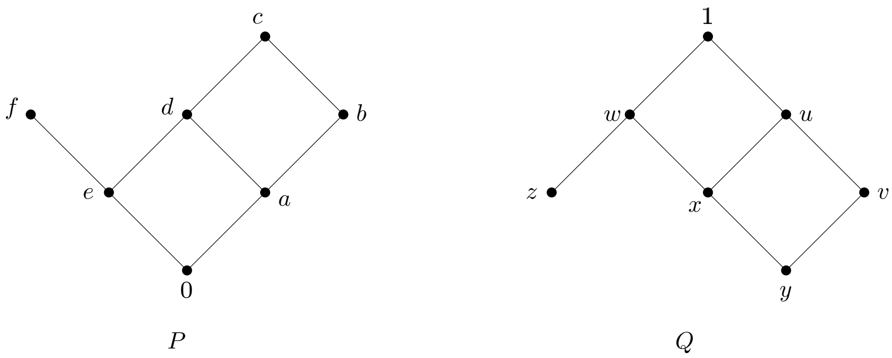


Figure 1.6 Two dual ordered sets P and Q.

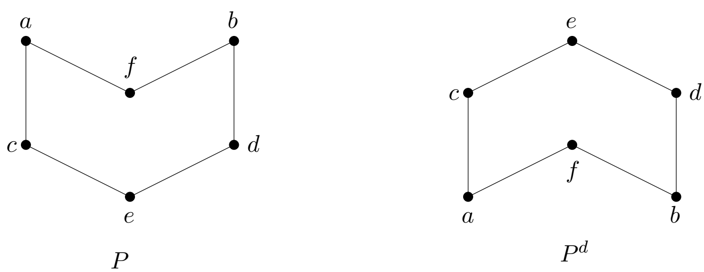


Figure 1.7 An ordered set P and its dual ordered set P^d.

Definition 1.10 Two ordered sets $P = (X, \leq_P)$ and $Q = (Y, \leq_Q)$ are said to be *dually isomorphic* (or simply *dual*) if there exists a bijection f from X to Y such that, for all $x, y \in X$:

$$x \leq_P y \iff f(x) \geq_Q f(y)$$

The bijection f is called an *(order) dual isomorphism* (or a *dual isomorphism*) between P and Q and we write $P \equiv_d Q$.

See Figure 1.6 for an example of dual ordered sets.

A particularly interesting case of a dual isomorphism is obtained by considering the ordered set $P^d = (X, \leq^d)$, *dual* of an ordered set $P = (X, \leq)$ and defined by:

$$x \leq^d y \iff y \leq x$$

The reader will check that $\leq^d$ is an order and that P and P^d are dually isomorphic. The order $\leq^d$ is called the *dual* (sometimes the *reverse*) of the order $\leq$ and we also denote it by $\geq$. A diagram of P^d is obtained by turning a diagram of P "upside down" (Figure 1.7).

We note that, in the case of a linearly ordered set L which is written $L = x_1x_2...x_n$, the linearly ordered set L^d is written $L^d = x_n...x_2x_1$.

From the existence of the dual order for any order follows the so-called *duality principle* for ordered sets, which states as follows:

If a property using symbols $\leq$ and $\geq$ holds in any ordered set, so does the dual property obtained by permuting these symbols.

For instance, in any ordered set, any element is less than or equal to at least one maximal element (see page 23). Dually, in any ordered set, any element is greater than or equal to at least one minimal element (see the same page).

More generally, the *dual class* $\mathcal{E}^d$ of a class $\mathcal{E}$ of ordered sets is formed from all ordered sets P^d with $P \in \mathcal{E}$. If a property holds in any ordered set of $\mathcal{E}$, the dual property holds in any ordered set of $\mathcal{E}^d$.

A class $\mathcal{E}$ of ordered sets is said to be *ipsodual* (or *autodual*) if any ordered set of $\mathcal{E}$ has its dual in $\mathcal{E}$, i.e., if $\mathcal{E} = \mathcal{E}^d$ (the class of linearly ordered sets on the one hand, and that of all ordered sets on the other hand, are two examples of ipsodual classes). For such a class, the duality principle then states as follows:

If a property holds in any ordered set of an ipsodual class of ordered sets, so does the dual property.

1.2 Examples of uses

Classifying, comparing, and hierarchizing activities are consubstantial to cognitive activity, so it is not surprising that mathematical models of order are present in a great number of fields, ranging from mathematics to biology, computer science or social sciences. In this section we present a sample of examples where orders – or order notions – appear in the latter fields. Chapter 7 will develop several of these uses. Of course, one could find many other examples: e.g. the uses of orders in quantum theory (see, for instance, Marlow, 1978) and in environmental sciences and chemistry (Brüggemann and Carlson, 2006).

1.2.1 Mathematics

Example 1.11 The notation $\leq$ used for an arbitrary order is the classic notation of the order defined on a set of numbers, for instance on the set $\mathbb{N}$ of non-negative integers.

Another order defined on $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ playing a significant role in number theory is the *divisibility order* on positive integers, denoted by $|$ and defined by: $a|b$ if a divides b (the reader can check that this relation is indeed an order). Any set of positive integers is an ordered set for this divisibility order (see Exercise 1.10 in Section 1.7).

Example 1.12 We denote by $P(E)$ the set of all subsets of a set E. In this book, the notation $\underline{2}^E$ will often stand for the set $(P(E), \subseteq)$ of all subsets of E ordered by set inclusion.[4] If $|E| = n$, the latter ordered set may also be denoted by B_n ("B" for *Boolean*, see Figure 1.5 and Definition 2.19 of Chapter 2) or $\underline{2}^n$ (see footnote 4). It is linearly ordered only for $n = 1$.

When E is equal to the set X^2 of all ordered pairs of a set X, we obtain the ordered set $(P(X^2), \subseteq)$, also denoted by $\underline{2}^{X^2}$, of all binary relations defined on X. For $R, S \in P(X^2)$ with $R \subseteq S$, it is also said that the relation R *implies* the relation S – or that S is *compatible* with R.

When E is equal to the set $P(X)$ of all subsets of X, we obtain the ordered set $(P(P(X)), \subseteq)$, also denoted by $(P^2(X), \subseteq)$ or $\underline{2}^{P(X)}$. Its elements, which are the subsets of $P(X)$, will be called *families of subsets of X* (so in this book a family is a set). A fundamental correspondence between the two ordered sets $\underline{2}^{X^2}$ and $\underline{2}^{P(X)}$ will be studied in Chapter 5.

Example 1.13 We denote $\mathcal{O}_E$ (or $\mathcal{O}_n$) the set of all orders defined on a set E of size n. For two such orders $\leq$ and $\leq'$, we say that $\leq$ *implies* $\leq'$ if, for all $x, y \in E$, $x \leq y$ implies $x \leq' y$. This relation – which is nothing but the inclusion relation on orders – is an order. The notation $\mathcal{O}_E$ will also be used for the set of all orders on E ordered by inclusion (it will be considered in Chapter 5).

1.2.2 Biology

Example 1.14 A *partition* $\mathbf{P} = \{C_1, C_2, ..., C_p\}$ of a set E is a set of non-empty subsets of E, called its *classes*, pairwise disjoint and the union of which is equal to E. The relation defined on the set $\mathcal{P}_E$ of partitions of E by $\mathbf{P} \leq \mathbf{P}'$ if any class of $\mathbf{P}$ is included in a class of $\mathbf{P}'$ is an order (which is moreover linear for $|E| < 3$) and we say that "$\mathbf{P}$ is finer than $\mathbf{P}'$." This order is called the *refinement order* on partitions. The search for a partitioning of a set into classes appears in numerous fields, like classification or discrimination problems. For instance, the recognition of several microbial strains related to an infectious disease (the contamination processes of which we are interested in) leads to the study of the joining of the infectious agent found on each patient to one of the latter strains, hence a partitioning of the set of these observations. A new distinction between the strains identified until then will correspond to a finer partition of the data.

Example 1.15 A family $\mathcal{H}$ of subsets of a set E is a *tree of subsets* if $E \in \mathcal{H}$, $\emptyset \notin \mathcal{H}$, and if $A, B \in \mathcal{H}$ implies $A \cap B \in \{\emptyset, A, B\}$. A tree $\mathcal{H}$ of subsets is a *hierarchy* on E if, moreover, $\{e\} \in \mathcal{H}$ for any $e \in E$. If $\mathcal{H}$ is a family of subsets of E ordered by set inclusion, then $\mathcal{H}$ is a tree of subsets, called a *tree-ordered set* (see Definition 2.12) if and only if $E \in \mathcal{H}$ and, for any $A \in \mathcal{H}$, the set of elements of $\mathcal{H}$ which

[4] This notation is very classic: it results from the well-known fact – consequence of a more general result proposed in Exercise 3.5 – that the ordered set $(P(E), \subseteq)$ is isomorphic to the ordered set of isotone maps (see Definition 3.1) from E to $\underline{2}$, which is denoted by $\underline{2}^E$ (see Definition 3.4 and its comments).

include A is a chain. $\mathcal{H}$ is a hierarchy if and only if $E \in \mathcal{H}$ and if, for any $A \in \mathcal{H}$ of size at least 2, the set of the subsets covered by A (in the inclusion order) forms a partition of A. Hierarchies are the basic model for classification trees considered in phylogeny (and, more generally, in data analysis). For instance, if E is a set of present species, an element A of $\mathcal{H}$ which is a subset of E with at least two elements corresponds to a set of species of E that have a common ancestor and then represents this hypothetical ancestor. The hierarchy $\mathcal{H}$ thus represents a set of – present or hypothetical – species, endowed with a filiation order (dual of inclusion). The set of all ancestors of a given species is linearly ordered by filiation. Hierarchies as well as other ordinal classification models will be studied in Section 7.3 of Chapter 7.

Example 1.16 It is possible to recognize the presence of homologous genes on the DNA corresponding to two close enough species. However, these common genes often appear in a different order. A commonly accepted hypothesis is that the elementary transformation leading from a DNA to another one is the reversal of an interval of this order. To evaluate the "distance" between the two DNAs, one then searches for the minimum number of elementary transformations, and hence of interval reversals, that is necessary to go from a linear order to another one over the same ground set X of genes. This distance is called the *reversal distance*; let us notice that, in the following Example 1.17, these interval reversals are the commutations if we consider only the two-element intervals. Obviously, the biological situation is much more complex. A sign ($+$ or $-$) is assigned to any element of X according to the DNA strand on which it is located (let us recall that DNA is structured as a double helix) and the reversal of an interval goes with the sign change of its elements. In return, the linear algorithmic complexity of the latter "signed" problem makes it quite treatable, whereas the "non-signed" problem is $\mathcal{NP}$-hard (see Appendix A).

1.2.3 Computer science

Example 1.17 We denote by Σ_n the set of the $n!$ permutations of $\{1,...,i,...,n\}$. We say that we make an *inversion* (or an *increasing commutation*) on the permutation $s = s_1...s_i...s_n$ of Σ_n if we exchange two consecutive elements s_i and s_{i+1} of s satisfying $s_i < s_{i+1}$ (with $i < n$). We write $s < s'$ if the permutation s' can be obtained from s by a sequence of inversions. We then obtain an order (linear for $n < 3$) on Σ_n, called *permutoedre order* or *(weak) Bruhat order*, defined by means of the notion of an inversion. This notion is naturally used for measuring the degree of order or disorder of a sequence of elements forming an information to sort. Numerous sorting algorithms are thus intrinsically related to the number of inversions present in this information, often representable as a permutation of elements (see Knuth, 1973 for the study of such methods). The notation $(\Sigma_n, <)$, or Σ_n if there is no ambiguity, will stand for the set of all permutations on $\{1,...,i,...,n\}$ endowed with the permutoedre order defined above. We have already noticed in Section 1.1.1 that there exists a bijection between permutations and linear orders defined on a set. This bijection involves an order on the set of linear orders isomorphic to the permutoedre order.

Some properties of this order are given in Chapter 5 (Sections 5.5 and 5.6) and at the very end of Section 6.5.

Example 1.18 In object-oriented programming languages with multiple inheritance, one has a type hierarchy which is an ordered set. We denote by T the set of types and by $\leq$ the inclusion between types. In order to efficiently implement this type hierarchy, one uses a "coding" of the ordered set $(T, \leq)$ in the ordered set of all subsets of a set S. A type is represented by a subset of S and is included in another one if and only if the subset of S that codes the first type is included in the set of S that codes the second one; in other words, the images of the types form an ordered subset of $\underline{2}^S$ (set of all subsets of S ordered by set inclusion) isomorphic to $(T, \leq)$. Such a coding is called *Boolean* since $(T, \leq)$ is coded in the Boolean lattice $\underline{2}^S$ (see Example 1.12). On the other hand, one searches to obtain an "optimal" Boolean coding, in the sense that the size of the set S must be minimum. This minimum size is called the 2-*dimension* of $(T, \leq)$. More generally, the 2-*dimension of an ordered set* $(P, \leq)$ is defined as the minimum size of a set S such that it is possible to code P in $\underline{2}^S$. This notion will be studied in Chapter 6.

Example 1.19 Some multiprogramming operating systems maintain up to date a dynamical graph called the *resources allocation graph*, which describes the use of all resources of the system (CPU(s), memories, peripheral devices, semaphores, etc.) by threads and processes and detects the possible arrival of blocking situations. In these systems, threads and processes are competing with one another for all the non-shareable resources (that is, which cannot be used at the same time by more than one thread/process) and the system is generally not supposed to pre-empt a resource used by a thread/process. A set of threads/processes is *deadlocked* if each one of them currently uses some resources and is blocked, waiting for another resource, which itself is used by another thread/process of this set, and if this blocking situation cannot improve even if some threads/processes release some resources. The simplest example is illustrated by the case of two processes P_0 and P_1 such that P_i has and uses resource R_i and is blocked, waiting for process R_{1-i} in order to go on running, for $i = 0, 1$.

The allocation graph G of the system can be used to detect such a situation. Its vertices are of two types: a vertex of G is either a process P or a *class of resources* R of the system (that is, a set of resources "equivalent" for the processes, for instance the class of printers, that of hard disks, etc.). The arcs of G are of two types: a *request* arc $P \longrightarrow R$ appears in G as soon as an instance of resource class R is needed for the execution of P. This arc is transformed into an allocation arc $R \longrightarrow P$ as soon as this request is satisfied by the system. The allocation arc disappears when R is released by P after being used.

Regularly, a system program makes sure that G is *cycle-free*, which amounts to checking that its reflexo-transitive closure (see Example 1.20) is an order (see Theorem 2.23). If it is not the case, all processes involved in the cycle are deadlocked and the system then starts to run the associated correction program.

1.2.4 Social sciences

Example 1.20 A *preordered set* is an ordered pair (X,R) where X is a set and R a *preorder*, i.e., a transitive and reflexive relation defined on X. When the preorder is also total, (X,R) is called a *totally preordered set*. The reader will prove that, if R is a preorder on X, the relation O defined on X by $(x,y) \in O$ if $[(x,y) \in R$ and $(y,x) \notin R]$ is a strict order, and that the relation I on X defined by $(x,y) \in I$ if $[(x,y) \in R$ and $(y,x) \in R]$ is an equivalence relation. By definition, the *classes of the preorder* R are the classes of the associated equivalence relation I. For two classes C and C' of R, we write $C \leq_R C'$ if there exists $x \in C$ and $y \in C'$ such that $(x,y) \in O$. The reader will prove that $\leq_R$ is an order relation on the set of classes (by first showing that one has $C \leq_R C'$ if and only if, for any $x \in C$ and any $y \in C'$, $(x,y) \in O$). The order $\leq_R$ is called the *quotient order* of the preorder R. When the preorder is total, the quotient order is a linear order on its classes.

The construction of a quotient order from a preorder is quite frequent. In particular, if U is an arbitrary binary relation, one associates a preorder R by writing xRy if $x = y$ or if there exists a *path* from x to y in U, that is a sequence $x_0, x_1, ..., x_p$ of *distinct elements* of X such that $x = x_0$, $y = x_p$, and $x_0 U x_1$, $x_1 U x_2, ..., x_{p-1} U x_p$. Then (X,R) is the *reflexo-transitive closure* of the directed graph $G = (X,U)$. An ordered pair (x,y) is an arc of this closure if $x = y$ or if there exists a path from x to y in G. The classes of the preorder R are called the *strongly connected classes* of the graph G. These classes are then ordered by the associated quotient order. This construction is used, for instance, in the study of a social network modeled by a graph. It allows us to divide the set of the individuals of the network into a set of equivalence classes provided with an ordinal structure. If, for instance, the graph (X,U) represents a "domination" relation between individuals, an equivalence class is then greater than another one if any individual of this class dominates – directly (by the relation U) or indirectly (by the relation R) – any individual of the other class (and the converse never holds).

Example 1.21 In the previous example, one could see that a strict order is associated with any preorder. The strict order O associated with a total preorder R (i.e., its asymmetric part) is called a *(strict) weak order*. This class of orders appears in numerous situations. Indeed, any numerical function f defined on X induces a weak order O on X defined by xOy if $f(x) < f(y)$. Thus, in microeconomics, the *preference* of an economical agent over a set of bundles of goods is generally described by means of a numerical utility function u: the bundle y is preferred to the bundle x if $u(x) < u(y)$. The preference of the agent is then a weak order, and its *indifference relation* (which is nothing but the incomparability relation of the weak order) is an equivalence relation. This modeling implies that the indifference relation of the agent is transitive, which in fact does not necessarily hold. We will see in Section 7.1 of Chapter 7 some ordinal modelings of preference which do not imply that the indifference relation be transitive (see also Example 1.22 below) and, in this context, we will go back to weak orders.

Example 1.22 We consider a set of finite intervals on $\mathbb{R}$. We write $[i_1, i_2] < [j_1, j_2]$ if $i_2 < j_1$. The reader can check that one obtains a strict order relation on the set of the considered intervals. All the orders that can be obtained that way are called *interval orders* and have appeared in extremely varied contexts (statistical estimation, psychophysics, utility theory, multi-criteria decision analysis, seriation, scheduling, temporal logic, combinatorics, etc.). Their origin dates back to Wiener's 1914 paper, where he introduces (with a different name) interval orders in order to answer a question from Russell on the possibility of defining the notion of an instant in time from the notion of an event in time. They were rediscovered in the 1950s concerning problems of modeling preferences, mentioned in the previous example. We will find them again in another form in Chapter 2 (see Definition 2.12) and they will be studied in the first section of Chapter 7 in the context of preference modeling.

1.2.5 Operations research

Example 1.23 Let us consider a group having at its disposal a budget for some investment taken in a set $X = \{1, ..., i, ..., n\}$ of possible investments. Each investment i has a cost c_i and a potential utility p_i. The group searches for a set A of investments whose cost $(\Sigma c_i, i \in A)$ does not exceed the total capacity of investment and which maximizes the utility $(\Sigma p_i, i \in A)$. This optimization problem is nothing else but the famous "knapsack problem," a very difficult problem (it is $\mathcal{NP}$-hard, see Appendix A). If, moreover, some investments may be made only if some others have been made (for instance, foundings for company settling following the creation of an industrial area), there exists a precedence order $<$ over the n investments. Then the set A must satisfy the following: $i \in A, j \in X$, and $j < i$ imply $j \in A$. Such a set A is what we will call a downset of the ordered set $(X, <)$ (see Definition 1.42). The optimization problem then becomes the *ordered knapsack problem*, in general as difficult as that without order constraint. However, for some ordered set classes such as 2-dimension orders (see Section 6.3) or bipartite orders (Definition 2.6), one may find good procedures of approximation – or even of determination – of the solution.[5] This problem is also linked to scheduling problems such as those mentioned in Example 1.25 below and in Section 7.5.

Example 1.24 The budget project of a territorial community, e.g., a county, is generally prepared by a commitee composed of representatives of the different districts. For a list of the n possible projects to budget for, each representative has his/her priority order for their carrying out. The priority order proposed at last then has to realize a consensus between the latter different orders. One may (more or less rightly) assume that each representative is able to give a linear priority order over the projects. In order to obtain the consensus order, a possible procedure could consist in searching

[5] See Kolliopoulos and Steiner (2002).

for a linear order L (called *median order*) that maximizes the sum of the "agreements" between L and the different orders of the representatives; the ordered pair (y,x) is an agreement between L and the order of a representative if project x is prefered to project y in these two orders. This optimization problem, easy to set and which, in a different context, was thought easy to solve, is actually very difficult (in fact $\mathcal{NP}$-hard, see Appendix A) in the general case where the orders of the representatives may be arbitrary. It may, however, be brought back to the application of the *majority rule* of aggregation and then become easy, if the orders to be aggregated respect a given structure, which may be the case in the considered example. We will go back to these problems of consensus orders and to the *majority* and the *median procedures* in Sections 7.2 and 7.3.[6]

Example 1.25 Consider a set of jobs that can be performed on several identical machines working in parallel, that is, such that each of them takes the same time to do the same job. On the other hand, there is a precedence order between jobs, coming from the fact that some jobs may be done only after the end of some others. A *scheduling* of these jobs consists of allocating them to the different machines taking into account the precedence constraints. One searches for an *optimal scheduling*, i.e., a scheduling which minimizes the total duration of the execution of these jobs. We will see in Section 7.5 how the ordinal modeling of this problem allows us to find an efficient algorithm for solving some of its instances.

1.3 Ordered subsets and extensions

Three types of problem frequently arise when an ordered set modelizes a given situation.

First, one can search for a subset of this set satisfying some given properties. Consider, for instance, an ordered set that represents a set of projects of setting-up companies, ordered according to a number of criteria. This defines an order, generally not linear, on these projects. One is led to search for a set of pairwise incomparable projects, and hence an antichain, which separates the "positive" projects (that is, considered as acceptable by a credit institution) and the "negative" ones (considered as not acceptable).

On the contrary, in other situations, one will search for a set of maximum size of pairwise comparable elements (and hence a chain) of an ordered set.

[6] The problem of searching for median orders is a particular case known in the literature as the "linear ordering problem." The latter is equivalent to the classic combinatorics optimization problem consisting of searching for a cycle-free subgraph with maximum weight in a weighted directed graph. These problems have appeared in different forms and in different fields, for instance in economics with the problem of triangulation of the input–output tables. Reinelt's 1985 book is a basic reference on these problems.

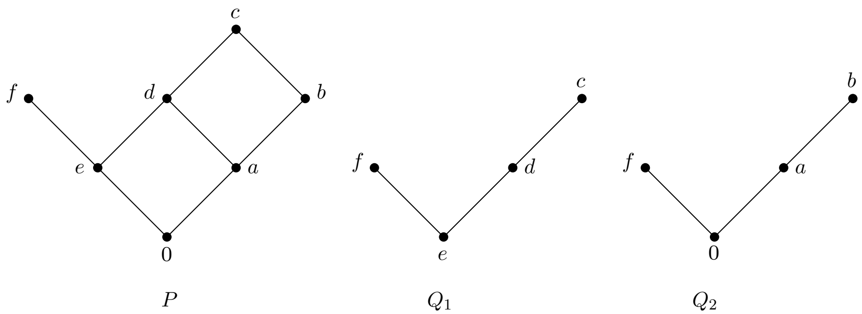


Figure 1.8 Q_1 is a covering ordered subset of P and Q_2 is a non-covering one.

In many jobs scheduling problems one can also search for a linear order on the jobs, compatible with the temporal constraints initially given on the latter: such-and-such job must be ended before such-and-such other job may be started (for instance, a file which must be compiled before another one). We will tackle some aspects of these problems in Section 7.5.

In this section, we make explicit the notions allowing us to modelize such problems.

1.3.1 Ordered subsets

Definition 1.26 Let $P = (X, \leq)$ be an ordered set and Y a subset of X. The restriction of the order $\leq$ to the subset Y is an order, denoted by $\leq_Y$ and called a *suborder* of $\leq$. We then say that $Q = (Y, \leq_Y)$ is an *ordered subset* of P, denoted by $Q \sqsubseteq P$.

If, moreover, $x \prec y$ in Q implies $x \prec y$ in P, we say that Q is a *covering ordered subset* of P.

See Figure 1.8 for an example of covering and non-covering ordered sets.

The ordered subset restricted to the subset $X \setminus x$ (respectively, $X \setminus A$) of X is denoted by $P \setminus x$ (respectively, $P \setminus A$).

In Section 1.3.2, we define two important types of ordered subset of an ordered set, namely its *chains* and its *antichains*. Other examples equally important are its *intervals* and its *convex subsets*.

Let x and y be two elements of the ordered set $P = (X, \leq)$ with $x \leq y$. We write:

$$[x,y] = \{z \in X : x \leq z \leq y\}$$

This set – or the corresponding ordered subset of P – is called the *interval between x and y*. A subset Y of P is said to be *convex* if it contains the interval $[x,y]$ as soon as it contains x and y with $x \leq y$.

1.3.2 Chains, antichains, and associated parameters

Definition 1.27 Let Q be an ordered subset of an ordered set P. If Q is linearly ordered, we say that Q is a *chain of* P. We write $x_0 < x_1 < ... < x_p$ for a chain of P or, simply, $x_0x_1...x_p$. The element x_0 is the *origin* of the chain, x_p its *extremity*, and its *length* is the number of its elements minus one (hence p for the chain $x_0 < x_1 < ... < x_p$, and 0 for the chain x_0). The chain $x_0x_1...x_p$ will be called a *chain of size* $p+1$ or a $(p+1)$-*element chain*.

A chain $Q = x_0x_1...x_p$ of P is said to be:

- *covering* if it satisfies $x_0 \prec x_1 \prec ... \prec x_p$ (where $\prec$ is the covering relation of P) with $p \geq 1$;
- *maximal* if it is not included in any other chain of P;
- *extended* if it contains a minimal element and a maximal element of P (see Definition 1.38).

Remark 1.28 Observe that in this terminology, the term "chain" may stand either for a linearly ordered set (the chain $\underline{n}$ in Definition 1.1) or as above, a linearly ordered subset of an ordered set. The context allows us to distinguish between these two uses, which besides are non-contradictory (since a set is a subset of itself). The same remark applies to the term "antichain" (Definitions 1.4 and 1.29).

A maximal chain is covering but the converse is in general not true. However, if Q is an extended chain of an ordered set P, Q is maximal if and only if it is covering.

The ordered set in Figure 1.4 (Example 1.2) has four maximal chains, ab, ae, cb, cde (the first three have length 1 and the last one has length 2) and eight non-maximal chains, two of which are covering (the reader can check that). The chains $0acf1$, $0bf1$, and 01 are three extended chains of the ordered set L given in Figure 1.10, and only the first one is maximal.

These different notions of chains apply to any ordered subset of P. Thus, in the case of an interval $[x,y]$ of P, a chain from x to y is maximal (for the ordered subset $[x,y]$) if and only if it is covering.

Definition 1.29 Let Q be a subset of an ordered set P. If the elements of Q are pairwise incomparable, we say that Q is an *antichain* of P.

An antichain of P is *maximal* if it is not contained in any other antichain of P. The ordered set in Figure 1.4 (Example 1.2) contains four maximal antichains and five other antichains (of size 1).

Notice that the only ordered subsets of P that are both a chain and an antichain of P are the singletons of P. On the other hand, all antichains of P are clearly convex (what about its chains?).

With the notions of a chain and an antichain of an ordered set P are associated four fundamental parameters:

Definition 1.30 Let P be an ordered set. The maximum size of an antichain of P is called the *width* of P and is denoted by $\alpha(P)$. The *range* $\kappa(P)$ of P is the maximum size of a chain of P. Also, we denote $\gamma(P)$ the minimum number of antichains in a partition of P into antichains and $\theta(P)$ the minimum number of chains in a partition of P into chains.

Note In the definitions of $\theta(P)$ and $\gamma(P)$, the expression "partition of P into chains" (respectively, into antichains) stands for a partition of the set of elements of P into chains (respectively, into antichains).

Instead of the range of P, one also considers its *height* $\lambda(P) = \kappa(P) - 1$. This parameter is then the maximum length of a chain of P.

In Figure 1.4 (Example 1.2), $\{ae, cb, d\}$ forms a partition into three chains and $\{ac, d, be\}$ a partition into three antichains of P (let us recall that here, for the sake of simplicity, the notation *ae* (for instance) stands for the set $\{a, e\}$). The reader can check that, in this example, the equalities $\kappa(P) = \gamma(P) = 3$ and $\theta(P) = \alpha(P) = 2$ hold.

These equalities are not fortuitous. Actually we will prove that any ordered set P satisfies the following equalities:

- $\gamma(P) = \kappa(P)$,
- $\theta(P) = \alpha(P)$.

The first equality is easy to prove. Obtaining the second one is the purpose of the most famous theorem of the theory of ordered sets, Dilworth's Theorem (1950). The latter has numerous applications and will be proved in Chapter 4 (Theorem 4.2). We can however notice that the inequalities $\kappa(P) \leq \gamma(P)$ and $\alpha(P) \leq \theta(P)$ are obvious (why?).

Note We immediately observe that the chains (respectively, the antichains) of P correspond to the *cliques* (respectively, to the *independent sets*) of the comparability graph of P. The parameters $\kappa(P)$, $\alpha(P)$, $\theta(P)$, and $\gamma(P)$ defined above then coincide with four fundamental parameters of the comparability graph of P. For instance, $\gamma(P)$ is the *chromatic number* of this graph. Since $\kappa(P)$, $\alpha(P)$, $\theta(P)$, and $\gamma(P)$ depend only on $Comp(P)$, one says that they are *comparability invariants*. Later we will see other – much less obvious – examples of such invariants (see Section 1.6, "Further topics and references").

1.3.3 Extensions

In this section, we consider the possibilities of extension of an order O into an order O' (we use the literal notations O and O' for simplicity).

Definition 1.31 Let O and O' be two orders defined on a set X. We say that O' is an *extension* of O if $O \subseteq O'$(in other words, for all $x,y \in X$, xOy implies $xO'y$).

An extension O' of O is said to be *linear* if O' is a linear order.

When O' is an extension (respectively, a linear extension) of O, we will also say that the ordered set $Q = (X,O')$ is an extension (respectively, a linear extension) of the ordered set $P = (X,O)$. The ordered pairs of P are ordered pairs of the extension Q, which we denote by $P \subseteq Q$ (and which must be distinguished from the notation $P \sqsubseteq Q$ used to express the fact that P is an ordered subset of Q).

Observe that if $O \subseteq O'$ then $Inc_{O'} \subseteq Inc_O$.

In the next chapter, we will prove (Theorem 2.29) that any order has at least one linear extension (using Exercise 1.7 the reader can make sure that this is the case). The search for arbitrary or for particular linear extensions occurs in multiple situations and especially in scheduling problems (see Section 7.5 of Chapter 7). We denote $\mathcal{L}(O)$ (respectively, $\mathcal{L}(P)$) the set of all linear extensions of the order O (respectively, of the ordered set $P = (X,O)$). The determination of this set is generally not easy (see Appendix A).

A linear extension of the ordered set in Figure 1.4 (Example 1.2) is *acbde* (that is, $a < c < b < d < e$). The reader can check that there are seven others.

Definition 1.32 Let $L = x_1x_2...x_n$ be a linear extension of an ordered set P of size n. The ordered pair (x_i,x_{i+1}) is called a *jump* of L if x_i and x_{i+1} are incomparable in P. We denote $s(P,L)$ the number of jumps of L and we call the *jump number of* P the integer $s(P) = min\{s(P,L), L$ a linear extension of $P\}$.

We can also consider the number $o(P,L)$ of *covers* of L, that is, the number of ordered pairs (x_i,x_{i+1}) of L such that x_iPx_{i+1}, and call the *number of covers of* P the integer $o(P) = \max\{o(P,L), L$ a linear extension of $P\}$. It is clear that $s(P,L) + o(P,L) = n-1$ and so $s(P) + o(P) = n-1$. Using the equality $\alpha(P) = \theta(P)$ (see Section 1.3.2), the reader can prove that $s(P) \geq \alpha(P) - 1$ always holds (and can check that this inequality is true for the ordered set in Figure 1.4).

In a problem of scheduling defined on a set of jobs provided with a (partial) anteriority order, it may be interesting to find a linear extension of this order which minimizes the jump number, when these jumps have a cost. We will go back to these scheduling problems in Section 7.5 of Chapter 7.

In the next chapter we will prove (Theorem 2.29) that an order O is equal to the intersection of all its linear extensions. In other terms, for all $x,y \in X, xOy$ if and only if xLy for any $L \in \mathcal{L}(O)$. This is written:

$$O = \bigcap \{L : L \in \mathcal{L}(O)\}$$

and also induces the following definitions.

Definition 1.33 Let $P = (X, O)$ be an ordered set. Any set of linear orders on X the intersection of which is O is called a *realization* of O (or of P). We call the *dimension* of O (or of P) the minimum number of linear orders the intersection of which is O (or P). This number is denoted by $\dim O$ (or $\dim P$). A realization of P formed with $\dim P$ linear orders is called a *basis* of P.

The dimension of an ordered set is a parameter that has been widely studied and Chapter 6 is mainly devoted to its study.

The reader can show that, in Figure 1.4 (Example 1.2), the two linear extensions *acbde* and *cdaeb* form the unique basis of P and so $\dim P = 2$. He can also prove that the dimension of the ordered set B_3 given in Figure 1.5 is 3 (more generally $\dim B_n = n$ holds, see Chapter 6).

Any linear extension of an order O is a maximal extension (with respect to the number of added ordered pairs). In order to present the symmetric notion of a *minimal extension* of an order (into an order), we define the notion of a critical ordered pair.

Definition 1.34 Let $P = (X, O)$ be an ordered set and $x, y \in X$. The ordered pair (x,y) is said to be *P-critical* (or *O-critical* or simply *critical*) if $(x,y) \notin P$ and $P + (x,y)$ is still an ordered set.

In other words, an ordered pair (x,y) is P-critical if $P + (x,y)$ is a minimal extension of P (since we add a unique ordered pair to P). Simple results prove that any ordered set different from a chain has at least one critical ordered pair; these results use the additional notions of a *forcing relation* and of the *arrow relations*.

Definition 1.35 Let $P = (X, O)$ be an ordered set. The *forcing relation* associated with P is the binary relation F_P defined on the ordered pairs of incomparable elements of P by $(x,y)F_P(z,t)$ if zOx and yOt.

In other words, $(x,y)F_P(z,t)$ holds if (z,t) belongs to the transitive closure of $P + (x,y)$.

We immediately observe that the forcing relation associated with an ordered set is still an order, which implies in particular that it always has maximal elements (see Definition 1.38 for the notions of maximal and of minimal elements).

Let us recall that P^-x (respectively, yP^+) denotes the set of elements covered by x (respectively, covering y) in P, and Px (respectively, xP) that of lower bounds (respectively, upper bounds) of x.

Definition 1.36 Let x, y be two elements of an ordered set $P = (X, \leq)$. We say that:

- x and y are in the *downarrow relation*, which is written $x \downarrow y$, if x is minimal among all elements z of P such that $z \nleq y$. Equivalently, $x \downarrow y$ holds if and only if ($x \nleq y$ and $P^-x \subseteq Py$).

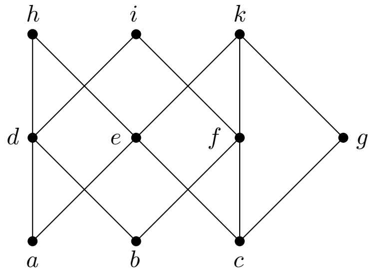


Figure 1.9 Example 1.37.

- x and y are in the *uparrow relation*, which is written $x \uparrow y$, if y is maximal among all elements z of P such that $x \not\leq z$. Equivalently, $x \uparrow y$ holds if and only if ($x \not\leq y$ and $xP \supseteq yP^{+}$).
- x and y are in the *double-arrow relation*, which is written $x \updownarrow y$, if $x \downarrow y$ and $x \uparrow y$.

Example 1.37 In the ordered set given in Figure 1.9, $\{a,d,e,g,h,i,k\}$ is the set of the elements which are not lower bounds of f. Since a and g are the minimal elements of this set, we find $a \downarrow f$ and $g \downarrow f$. The set of elements that are not upper bounds a is $\{b,c,f,g\}$ and f and g are maximal in this set, hence $a \uparrow f$ and $a \uparrow g$. As a result we obtain $a \updownarrow f$. The arrow relations of this ordered set are given in Table 3.3 presented in Chapter 3 (page 96).

The above-mentioned fact that there always exists a P-critical ordered pair for P different from a chain results from the following fact, that the reader will verify: *an ordered pair* (x,y) *is P-critical if and only if* $(x,y) \in MaxF_P$, *and if and only if* $x \updownarrow y$. These two equivalences belong to a larger set of equivalent conditions, that are the subject of Exercise 1.2. As for the arrow relations, they prove to be a particularly useful tool that we will meet very often throughout Chapter 3.

1.4 Particular elements and subsets

Let P be an ordered set that modelizes, for instance, the hierarchical structure observed in a given animal society, a tribe, a company... Does there exist one or several leaders? More generally, if two individuals are at the same hierarchical level, does there exist some common superiors and, if so, is there one unique superior "closest" to them, that can settle their possible disagreements. The answer to these questions involves the consideration of some particular elements of the ordered set P. This section defines a number of such elements, as well as some subsets of an ordered set, such as its downsets and its upsets, that will be constantly encountered subsequently.

1.4.1 Meets, joins, and irreducible elements

Definition 1.38 Let $P = (X, \leq)$ be an ordered set.

- The *minimum* of P if $x \leq y$ for any $y \in P$.
- The *maximum* of P if $y \leq x$ for any $y \in P$.
- A *minimal element* of P if there does not exist $y \in P$ such that $y < x$.
- A *maximal element* of P if there does not exist $y \in P$ such that $x < y$.

Rather than "minimum" (respectively, "maximum"), we also say *the smallest* – or *the least* – *element* (respectively, *the greatest element*) of P. These elements are necessarily unique when they exist and are often respectively denoted by 0_P and 1_P (or, simply, 0 and 1) and we then say that P is *bounded*.

The set of minimal (respectively, maximal) elements of P is denoted by $MinP$ (respectively, $MaxP$).

These definitions extend to any ordered subset Q of P. For instance, $x \in Q$ is a minimal element of Q if there does not exist $y \in Q$ with $y < x$. If Q has a minimum (respectively, a maximum), it can be denoted by 0_Q (respectively, 1_Q).

The ordered set in Figure 1.4 (Example 1.2) has two minimal elements, a and c, and two maximal elements, b and e. The ordered subset defined by $\{a,c,e\}$ has a greatest element, e, but has no minimum. The ordered set B_3 in Figure 1.5 has a minimum and a maximum.

If Y is a subset of $P = (X, \leq)$, a *lower bound* (respectively, an *upper bound*) of Y is an element m of P satisfying $m \leq x$ (respectively, $m \geq x$) for any $x \in Y$. The subset Y is said to be *lower bounded* (respectively, *upper bounded*) if it has at least one lower bound (respectively, one upper bound).

Given a subset Y of P, we denote $LowerY$ the set of lower bounds of Y and $UpperY$ that of its upper bounds.

The following notions allow us to define important classes of ordered sets.

Definition 1.39 Let Y be a subset of an ordered set P. We say that $r \in P$ is the *meet* – or the *greatest lower bound* – of Y if Y is lower bounded and if the set of its lower bounds has r as its maximum. We denote it by $\bigwedge Y$. Similarly, Y has a *join* – or a *least upper bound* – t if Y is upper bounded and if the set of its upper bounds has t as its minimum. We denote it by $\bigvee Y$.

Thus, for instance, $r = \bigwedge Y$ if the following two conditions hold:

- $r \leq z$ for any $z \in Y$,
- if x satisfies $x \leq z$ for any $z \in Y$, then $x \leq r$.

If $Y = \{x,y\}$, its meet (respectively, its join) is denoted by $x \wedge y$ (respectively, $x \vee y$).

The meet and the join of an arbitrary subset Y of an ordered set P do not necessarily exist (for instance, when the set of lower bounds of Y has several maximal elements,

Y has no meet). When any subset Y of P has a meet (respectively, a join), P is called a *meet-semilattice* (respectively, a *join-semilattice*). A *lattice* is an ordered set that is a meet- and a join-semilattice.

We can prove that $\bigvee \emptyset$ exists in an ordered set P if and only if P has a minimum 0. Then we have $\bigvee \emptyset = 0$. Similarly, $\bigwedge \emptyset$ exists in P if and only if P has a maximum 1 and then we have $\bigwedge \emptyset = 1$ (see the proof of these results in Example 3.2 in Chapter 3).

Example 1.40 Any linearly ordered set is a lattice. Moreover, for any set E, the ordered set $\underline{2}^E$ of the subsets of E is a lattice the meet of which is equal to the set intersection and the join of which is equal to the set union. Such a lattice belongs to the class of *Boolean* lattices (see Chapter 2, Definition 2.19).

The class of (semi)lattices is presented in Section 2.3 of Chapter 2. The very important particular case of *distributive* lattices is presented in the same section and is then developed in Chapter 5 in the context of Birkhoff's fundamental theorem that establishes a one-to-one correspondence between ordered sets and distributive lattices.

All previous definitions obviously extend to the case where Y is a multiset, i.e., when an element may have several occurences in Y.

From these definitions it follows that, if Y has a minimum 0_Y (respectively, a maximum 1_Y), one has $\bigwedge Y = 0_Y$ (respectively, $\bigvee Y = 1_Y$). In particular, for $Y = \{x,y\}$:

$$x \leq y \iff x \wedge y = x \iff x \vee y = y$$

and so

$$x = y \iff x \wedge y = x \vee y = x$$

A subset Y of P is said to be *meet-closed* – or *meet-stable* – (respectively, *join-closed* – or *join-stable*) – if, for any subset B of Y with a meet (respectively, a join), the latter belongs to Y.

In the ordered set P in Figure 1.10, $e \wedge f = c$ and $d \wedge f = 0 = \bigwedge\{d,c,f\}$ hold. Yet $e \wedge d$ does not exist (why?), neither $d \vee e$ nor $a \vee b$. Besides, $b \vee c = e = \bigvee\{a,b,c\}$. The subsets $\{d,e\}$ and $\{0,c,d,e,f\}$ are meet-closed but the subset $\{d,f\}$ is not.

Definition 1.41 Let $P = (X,\leq)$ be an ordered set.

1. An element $x \in P$ is *join-irreducible* if it is not the join of any subset not containing it. Equivalently, x is join-irreducible if the subset $(x[$ has at least two minimal upper bounds.
2. An element $x \in P$ is *meet-irreducible* if it is not the meet of any subset not containing it. Equivalently, x is meet-irreducible if the subset $]x)$ has at least two maximal lower bounds.
3. An element $x \in P$ is said to be *irreducible* if it is join- or meet-irreducible; it is called *doubly irreducible* if it is join- and meet-irreducible.

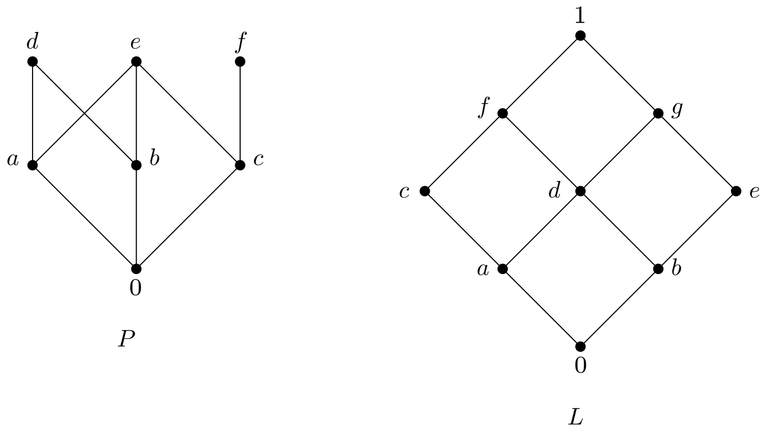


Figure 1.10 P is not lattice ordered; L is a lattice.

Any non-join-irreducible element of P is then the join of a subset of P not containing it. Let us remark that if P has a minimum, the latter is not join-irreducible since we can prove that it is the join of the empty set (see the proof given in Example 3.2, on page 69). Similarly, if P has a maximum, the latter is not meet-irreducible as the meet of the empty set $\emptyset$.

For the ordered set P in Figure 1.10, all elements – except for 0 and e – are join-irreducible, whereas 0 and c are the only non-meet-irreducible elements. For the lattice L in the same figure, the join-irreducible elements are a,b,c,e; the meet-irreducibles are c,e,f,g; and the doubly irreducibles are c and e. The example of the element d of P (still in Figure 1.10) shows that a join-irreducible may cover several elements. Yet, an element x which covers a unique element – then denoted by x^- – is necessarily join-irreducible (why?). Similarly, if x is covered by a unique element – then denoted by x^+ – it is necessarily meet-irreducible.

Irreducible elements may also be characterized by the arrow relations (see Definition 1.36 and Propositions 3.8 and 3.17 in Chapter 3).

We will respectively denote J_P or $J(P)$, M_P or $M(P)$, IR_P or $IR(P)$, and DIR_P or $DIR(P)$ the set of join-irreducible, meet-irreducible, irreducible, and doubly irreducible elements of an ordered set P.

Writing J_x for the set of join-irreducibles less than or equal to x and M^x for the set of meet-irreducibles greater than or equal to x, we will show that any element x of P satisfies $x = \bigvee J_x = \bigwedge M^x$ (see Proposition 3.11 in Chapter 3).

We say that a subset Y of P is a *join-generating set (of P)* if any element of P is the join of a subset of Y. Since it is clear that such a subset must contain all the join-irreducibles of P, we find that a subset of P is join-generating if and only if it contains all the join-irreducibles of P. Dually a subset Y of P is a *meet-generating set (of P)* if any element of P is the meet of elements of Y or, equivalently, if it contains all the meet-irreducibles of P. These notions will be widely studied in Section 3.2, Chapter 3.

1.4.2 Downsets and upsets (ideals and filters)

The subsets of an ordered set that we are now going to consider play a fundamental role, especially specified in Chapter 5.

Definition 1.42 Let $P = (X, \leq)$ be an ordered set. A subset D of P is a *downset* – or an *ideal* – if, for all $t \in P$ and $y \in D$, $t \leq y$ implies $t \in D$. A subset U of P is an *upset* – or a *filter* – if, for all $t \in P$ and $y \in U$, $y \leq t$ implies $t \in U$.

We write $\mathcal{D}(P)$ (respectively, $\mathcal{U}(P)$) for the set of downsets (respectively, upsets) of P (the reader can search for the subsets of an ordered set P which are downsets as well as upsets). A downset (respectively, an upset) of P different from $\emptyset$ and P is called a *proper downset* (respectively, *upset*).

The downset $(x]$ formed from all the lower bounds of x will be called a *principal downset*, or a *principal ideal*. Dually the upset $[x)$ formed from all the upper bounds of x will be called a *principal upset*, or a *principal filter*.

All downsets and upsets are obviously convex. On the other hand, the convex subsets of an ordered set P are the intersections of a downset and of an upset of P (why?).

The following properties are easy to prove (the reader can check them), but nevertheless particularly important:

- the union and the intersection of downsets (respectively, upsets) are downsets (respectively, upsets);
- any downset (respectively, upset) is the union of principal downsets (respectively, principal upsets);
- the complementary subset of a downset (respectively, an upset) of an ordered set P is an upset (respectively, a downset).

The reader may use these properties to show that the ordered set in Figure 1.4 (Example 1.2) has 10 downsets (for instance, $\{a, c, d\}$ is such a subset) and as many upsets (but he should not be tempted to believe that the determination of these subsets will always be that easy for an arbitrary ordered set).

Note As already said, the terminology of the theory of ordered sets is far from fixed. Thus, instead of *downset* or *ideal* (respectively, *upset* or *filter*), one also finds *order ideal*, *initial segment*, *hereditary subset* (respectively, *order filter*, *final segment*, *cohereditary subset*), etc.

In the case where the ordered set is $(P(E), \subseteq)$, its downsets are also called *hereditary families*, *abstract simplicial complexes*, or *independence systems*.

1.5 Constructing ordered sets from given ones

If $P = (X, O)$ and $Q = (X, O')$ are two ordered sets on the same ground set X, we will denote by $P \cap Q$ the set X endowed with the relation $O \cap O'$. The reader can check that $O \cap O'$ is an order relation and so that $P \cap Q$ is an ordered set. It is generally not the case for the union of P and Q as soon as $|X| \geq 2$.

In this section, we are going to consider more general cases of ordered sets, constructed from some others that do not necessarily have the same ground set. These operations may be used to solve the converse decomposition problem: how to bring a complex ordered set back to a number of components, each of which is a simpler ordered set (this allows us to solve some problems on the given ordered set, see Section 1.6).

All operations defined below are illustrated in Figure 1.11 at the end of the section.

1.5.1 Substitution, disjoint union, linear sum, lexicographic product

We first consider the fundamental operation called the *substitution*, which has the *lexicographic sum*, the *disjoint union*, the *linear sum*, and the *lexicographic product* as particular cases. Let $Q = (Y, \leq_Q)$ be an ordered set and let us consider $h \geq 1$ ordered sets $P_i = (X_i, \leq_i)$ (with $i = 1, ..., h \leq |Y|$) such that the sets $Y, X_1, ..., X_h$ are pairwise disjoint (and non-empty). Given h distinct elements $y_1, ..., y_h$ of Y, we write:

$$P = Q^{P_1 \dots P_h}_{y_1 \dots y_h}$$

to denote the ordered set obtained by "substituting" the ordered set P_i for each element y_i of Y. More precisely, if $P = (X, \leq_P)$ then:

$$X = (Y \setminus \{y_1, ..., y_h\}) \cup (\bigcup_{1 \leq i \leq h} X_i)$$

and

$$a \leq_P b \iff \begin{cases} a, b \in Y \setminus \{y_1, ..., y_h\} \text{ and } a \leq_Q b, \text{ or} \\ \exists i : a \in X_i,\ b \in Y \setminus \{y_1, ..., y_h\} \text{ and } y_i <_Q b, \text{ or} \\ \exists\, i : a \in Y \setminus \{y_1, ..., y_h\},\ b \in X_i \text{ and } a <_Q y_i, \text{ or} \\ \exists\, i : a, b \in X_i \text{ and } a \leq_i b, \text{ or} \\ \exists\, i \neq j : a \in X_i,\ b \in X_j \text{ and } y_i <_Q y_j \end{cases}$$

We let the reader make sure that $\leq_P$ is actually an order on X.

This substitution operation is also sometimes called *X-join*, *series–parallel composition*, etc. The diagram of P is obtained as follows: for $i = 1, ..., h$, we substitute the diagram of P_i for the point that represents y_i. If y_j is covered by y_i in Q, the (points representing the) maximal elements of X_j are linked to the (points representing the) minimal elements of X_i in P. Likewise, if $y \in Y \setminus \{y_1,...,y_h\}$ is covered by y_i in Q, y is linked to the minimal elements of X_i in P. We proceed dually if y_i is covered by y.

When $h = |Y|$, this operation is often called *lexicographic sum* and $P = (X, \leq_P)$ is then defined by:

$$X = \bigcup_{1 \leq i \leq h} X_i$$

and

$$a \leq_P b \iff \begin{cases} \exists\, i : a, b \in X_i \text{ and } a \leq_i b, \text{ or} \\ \exists\, i \neq j : a \in X_i,\ b \in X_j \text{ and } y_i <_Q y_j \end{cases}$$

A first particular case of the lexicographic sum is obtained by assuming that Q is the antichain A_h of size h. We then say that the ordered set P is the *disjoint union* (or the *parallel sum*, or the *parallel composition*, or the *horizontal sum*) of the h ordered sets P_i, which we denote:

$$P = \sum_{1 \leq i \leq h} P_i$$

(or simply $P = \sum_h P_i$).

If we set $P = (X, \leq_P)$, we have:

$$X = \bigcup_{1 \leq i \leq h} X_i$$

and

$$a \leq_P b \iff \exists i \text{ such that } a, b \in X_i \text{ and } a \leq_i b$$

The diagram of P is simply obtained by "juxtaposing" those of the P_i's.

We notice that, if the P_i's are connected, they form the connected components of P and that, conversely, any ordered set is the disjoint union of its connected components. The disjoint union of two ordered sets P_1 and P_2 is denoted by $P_1 + P_2$. The notation $\sum_h R$ stands for the disjoint union of h ordered sets all isomorphic to the ordered set R.

Now, assume that Q is equal to the chain $C_h = y_1 < ... < y_h$ of size h. We then say that P is the *linear sum* (or the *ordinal sum*, or the *series composition*, or the *vertical sum*) of the P_i's, which we write:

$$P = \bigoplus_{1 \leq i \leq h} P_i$$

That is, with $P = (X, \leq_P)$:

$$X = \bigcup_{1 \leq i \leq h} X_i$$

and

$$a \leq_P b \iff \begin{cases} a \in X_i, b \in X_j \text{ and } a \leq_i b & \text{if } i = j \\ a \in X_i, b \in X_j \text{ and } y_i <_Q y_j & \text{otherwise} \end{cases}$$

As for the diagram of P, the linear sum comes back to drawing the diagrams of the P_i's above one another in the order of Q, and linking, for any $i < h$, each maximal element of P_i to each minimal element of P_{i+1}.

The linear sum of P_1 and P_2 is denoted by $P_1 \oplus P_2$, whereas the notation $\oplus_h R$ stands for the linear sum of h ordered sets all isomorphic to the ordered set R.

Remark 1.43 When P_1 has a maximum u_1 and P_2 has a minimum 0_2, we also use a variant of the linear sum, obtained from $P_1 \oplus P_2$ by identifying the elements u_1 and 0_2. It is called the *glued linear sum* – or sometimes the *vertical sum* – and is denoted by $P_1 \oplus' P_2$.

We now define the particular substitution operation that constructs the lexicographic product of several ordered sets. We start with the case of two ordered sets $Q = (Y, \leq_Q)$ and $R = (Z, \leq_R)$. The *lexicographic product of Q by R* is the ordered set $P = (X, \leq_P)$, which we denote:

$$P = Q \otimes R$$

and that we define by:

$$X = Y \times Z$$

and

$$(y,z) \leq_P (y',z') \iff [y <_Q y' \text{ or } (y = y' \text{ and } z \leq_R z')]$$

This is actually a substitution operation: indeed, if $Y = \{y_1, ..., y_n\}$, to construct $Q \otimes R$ comes back, up to isomorphism, to making the substitution $Q^{R,...,R}_{y_1,...,y_n}$ where we substitute a copy of R for each element y_i of Y.

Let us now define this product in the more general case of an arbitrary number of ordered sets. The lexicographic product of $h \geq 1$ ordered sets $P_1 = (X_1, \leq_1), ..., P_h = (X_h, \leq_h)$ is the ordered set $P = (X, \leq_P)$, which we denote:

$$P = \otimes_{1 \leq i \leq h} P_i$$

and that we define by:

$$X = \Pi_{1 \leq i \leq h} X_i$$

and

$$(x_1, ..., x_i, ..., x_h) \leq_P (x'_1, ..., x'_i, ..., x'_h)$$
$$\text{if and only if}$$
$$(x_1, ..., x_i, ..., x_h) = (x'_1, ..., x'_i, ..., x'_h) \text{ or there exists } k \in \{1, ..., h\} \text{ such that, for any } i < k, (x_i = x'_i \text{ and } x_k < x'_k)$$

Obviously, since $P = (\otimes_{1 \le i \le h-1} P_i) \otimes P_h$, this product may be recursively defined from the substitution operation.

1.5.2 Direct product

We now define another product which, contrary to the previous one, is not derived from the substitution operation. For $h \ge 1$ ordered sets $P_1 = (X_1, \le_{P_1}), ..., P_h = (X_h, \le_{P_h})$, we call the *direct product* of the P_i's – and we write

$$P = \Pi_{1 \le i \le h} P_i$$

the ordered set $P = (X, \le_P)$ defined by:

$$X = \Pi_{1 \le i \le h} X_i$$

and

$$(x_1, ..., x_i, ..., x_h) \le_P (x'_1, ..., x'_i, ..., x'_h) \iff \forall i = 1, ..., h,\ x_i \le_{P_i} x'_i.$$

Here again, the reader can make sure that $\le_P$ is actually an order on X. The direct product of P_1 and P_2 is denoted $P_1 \times P_2$. One has $(x_1, x_2) \prec_P (x'_1, x'_2)$ if and only if $x_1 = x'_1$ and $x_2 \prec x'_2$ or $x_1 \prec x'_1$ and $x_2 = x'_2$. The diagram of $P_1 \times P_2$ is then obtained by replacing each element x_1 of P_1 with $\{x_1\} \times P_2$ and linking the ordered pairs (x_1, x_2) to the ordered pairs (x'_1, x_2) with $x_1 \prec x'_1$. The reader can generalize this construction to the case of an arbitrary number of ordered sets. The direct product of n times the ordered set P is denoted by P^n.

The notion of a direct product order is useful in multiple situations. Let us consider, for instance, the case of some "objects" of arbitrary nature described by a set of descriptors. As soon as each of these descriptors induces an order on the objects, for instance, as soon as it attributes them some numerical values (for example, the marks obtained by some students at several examinations), the latter are naturally ordered by the direct product of these different orders. Thus, in a study for classifying 65 insecticides according to their degree of dangerousness for the human race or nature (Carlsen, 1984), the latter have first been partially ordered by the direct product order of several descriptors (such as their lifetime or their toxicity). In order to obtain a ranking (i.e., a total preorder) of the insecticides, the author has then considered the linear extensions of this direct product order and used an (approximate) formula giving the mean of the ranks (Definition 2.1) obtained in these linear extensions for each insecticide.

Remark 1.44 1. Another important operation, called "exponentiation," associates with two ordered sets P and Q the ordered set Q^P and will be considered in Section 3.1 of Chapter 3 (Definition 3.4).

2. The operations dealt with in this section have been defined as operations on ordered sets endowed with a reflexive order. It is clear that one could also define them in terms of ordered sets endowed with a strict order, and we will sometimes use them in this case. On the other hand, if no ambiguity is possible, we will sometimes talk about the direct product of two orders instead of the direct product of two ordered sets (for instance).

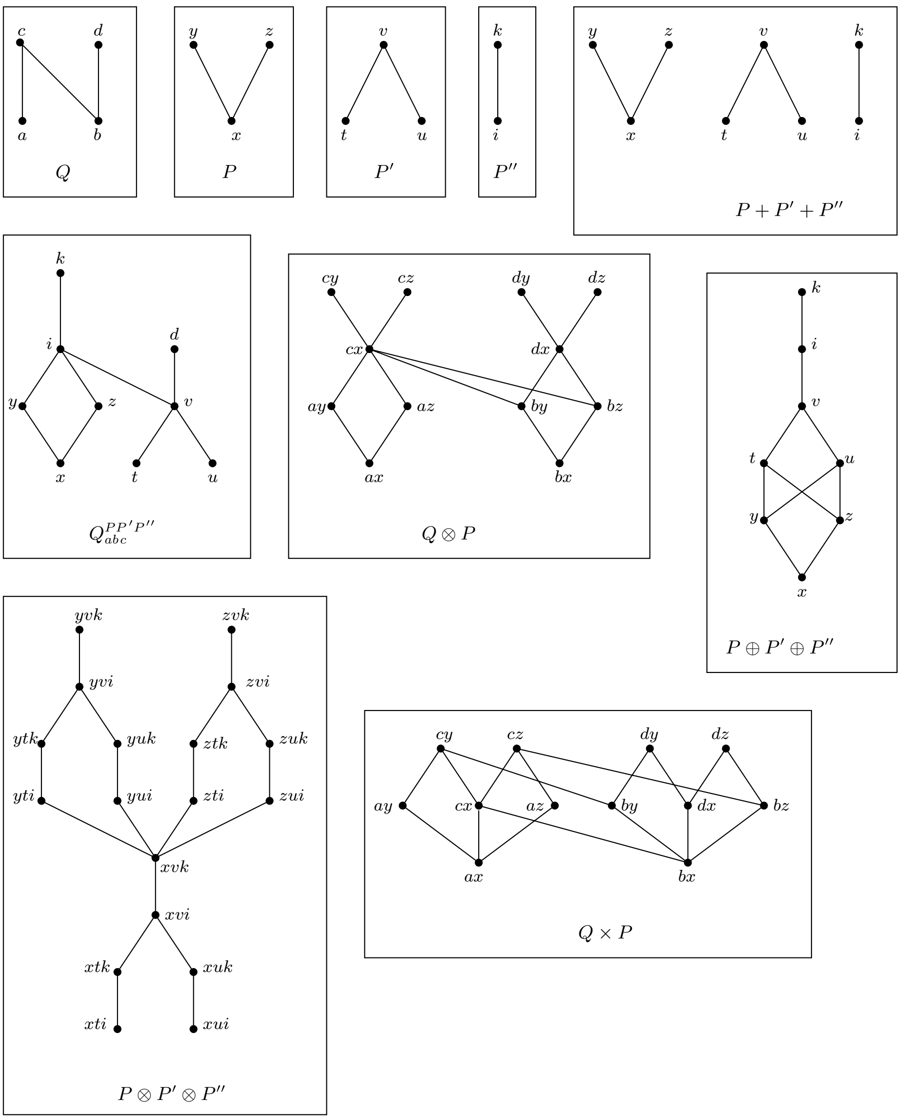


Figure 1.11 Examples of operations on four ordered sets Q, P, P', and P''.

1.6 Further topics and references

To the question "what is an order?" raised by Russell in 1903, we have given two answers. The first one is that of the Bourbakist vulgate (anticipated by Peirce in 1880) and says that an order is a transitive, antisymmetric, and reflexive relation. The second one says that an order is a transitive and irreflexive relation (and hence asymmetric) then generally called a strict order. The latter conception first appeared in the form of a strictly linear order with Russell and Huntington, who introduced it, as early as the beginning of the twentieth century and with the name of "series" or "simple order," from the works of several authors (such as Cantor, de Morgan, Peano, etc.). It was then extended to the general case by Hausdorff (1914), often considered as the first author to define an abstract notion of an ordered set, that he called "teilweise geordnete Menge."[7] In fact, although the reflexivity property of an order may be convenient (for instance, to consider orders and equivalences as two particular cases of preorders), it is often useless. On the other hand, and contrary to what one may think, the two notions of an order and of a strict order are not quite equivalent. Certainly, it is sufficient to add (respectively, to remove) reflexivity to a strict order (respectively, from an order) so as to make it an order (respectively, a strict order). Yet, the case of strict interval orders, studied in Section 7.1 of Chapter 7, shows that some simple characterizations of these orders are no longer valid in the case of (non-strict) interval orders.

Historically, the study of ordered sets began with that of lattices, first in the late nineteenth century then from the 1930s (see Section 2.5 of Chapter 2). It was only at the beginning of the 1960s that the theory of – especially finite – ordered sets began to develop significantly, under various impulses: links with combinatorics and discrete mathematics, relations with algorithmics and theoretical computer science, various applications for example in operations research and in social sciences (see for instance Barbut and Monjardet, 1970). One may find evidence of this developement in the reports of the numerous conferences that have been devoted to ordered sets since the 1980s: *Ordered Sets* (Rival, 1982), *Orders: Descriptions and Roles* (Pouzet and Richard, 1984), *Graphs and Orders* (Rival, 1985a), *Combinatorics and Ordered Sets* (Rival, 1986), *Algorithms and Orders* (Rival, 1989a), *Combinatorics of Ordered Sets* (Aigner and Wille, 1991), as well as, since 1984, in the dedicated journal *Order*.

As for the links with discrete mathematics, we will first observe that some properties or some tools of study of ordered sets may remain relevant on some more general structures. Thus, some properties of comparability graphs result from the fact that these graphs are "perfect" (see Section 4.5), and the substitution operation of Section 1.5 may be defined for many other discrete structures. We will also note that some properties of ordered sets may facilitate the study of some graphs. This is, for instance, the case for the "interval graphs," which are the incomparability graphs of interval

[7] A story of the origin of the notions of order and, more generally, of the notions of set theory, may be found in a paper of Cegielski (1987).

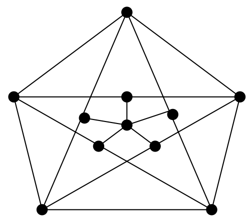


Figure 1.12 The smallest triangle-free graph which is not a neighborhood graph.

orders (see Chapter 7, Sections 7.1 and 7.6). Finally, we will remark that many sets of combinatorial structures of a given type (for instance, the partitions of a set or of a positive integer) are naturally provided with an order that may play an important role in their study – this viewpoint is excellently presented in Aigner's 1979 book *Combinatorial Theory*.

In Sections 1.1.2 and 1.1.3 we have defined several graphs associated with an ordered set. A natural question is to provide characterizations of such graphs.

Exercise 1.6 supplies one with a characterization of the covering directed graph of an order. Yet there is no characterization of neighborhood graphs of ordered sets, that is, of the undirected graphs which may be directed in order to become covering graphs of some ordered set. The obvious necessary condition of being triangle-free is far from enough, as testified by the example in Figure 1.12, which is the smallest triangle-free graph that is not a neighborhood graph. This characterization problem remains open (see Rival, 1985b).

This is not the case for the *comparability graphs*, that is, the (undirected and loop-free) graphs G for which there exists an ordered set P with $G = Comp(P)$. In other words, the edges of a comparability graph may be directed so that the set of obtained arcs is the set of ordered pairs of a (strict) order relation. Such an orientation of the edges is called a *transitive orientation* of the graph. For instance, complete graphs and bipartite graphs are comparability graphs (why?), as well as any graph with less than 5 vertices. Comparability graphs have been characterized by Ghouila-Houri (1962) and Gilmore and Hoffman (1964); a sequence $x_0, \ldots, x_i, \ldots, x_p = x_0$ of vertices of an undirected graph G, where any two consecutive vertices x_i and x_{i+1} form an edge of G, is a *pseudo-cycle* of length p of G, which is said to be odd if p is odd; a *triangular chord* of this pseudo-cycle is an edge of the form $x_i x_{i+2}$ (the addition of indices is made modulo p). The previously mentioned authors have shown that an undirected graph G is a comparability graph if and only if any odd pseudo-cycle of G has a triangular chord. Figure 1.13 shows two examples of graphs that do not satisfy this condition (in the second one, the triangular chord-free pseudo-cycle is *abcdcefeba*, of length 9).

Another characterization of comparability graphs, by means of forbidden subgraphs, has been provided by Gallaï (1967). The links between these characterizations or others, as well as the recognition problems on these graphs, are dealt with in

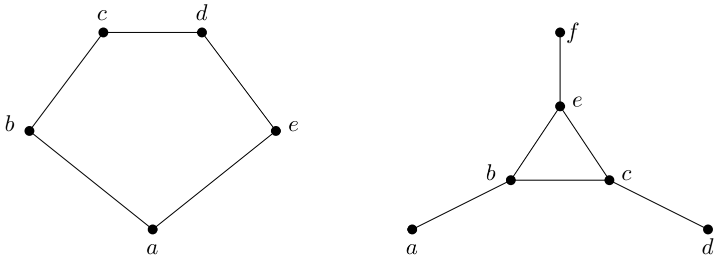


Figure 1.13 Two graphs that are not comparability graphs.

Golumbic's work (1980) – see also Golumbic *et al.*, (1983) and the reports of Kelly (1985) and of Möhring (1984).

Given a comparability graph, one may wonder how to characterize the ordered sets corresponding to its transitive orientations. The answer is supplied by means of the substitution operation defined in Section 1.5. Actually, the following result holds (Dreesen *et al.*, 1985; Kelly, 1986), as a consequence of Gallaï's results (1967) on the decomposition by substitution of comparability graphs: two ordered sets P and Q have the same comparability graph if and only if Q may be obtained by a finite sequence of transformations of P, each of which consists in replacing a homogeneous ordered subset H of P with its dual H^d. Such an ordered subset is that defined by a *P-homogeneous* subset, that is, by a subset H of elements of P that have the same "behavior" relative to any exterior element [formally, for all $x, y \in H$ and any $z \notin H$, z is less than (respectively, greater than, incomparable to) x if and only if z is less than (respectively, greater than, incomparable to) y]. Substituting H^d for H consists in reversing in P all the order relations between the elements of H and preserving all the other order relations of P. A consequence of this result is that the comparability graph of P has only two transitive orientations (that correspond with P and with its dual P^d) if and only if P has only itself and some antichains as homogeneous ordered subsets.

The previous result of characterization of the ordered sets that have the same comparability graph could be stated differently. Let us call *ordered set parameter* any map π defined on the class of all types of ordered sets, and *property* any parameter the values of which belong to the set {true, false}. Generalizing a little bit the definition given on page 19, we say that such a parameter is a *comparability invariant* if its value depends only on the (type of) comparability graph $Comp(P)$ of the ordered set P. In other words, if $Comp(P)$ is isomorphic to $Comp(Q)$, $\pi(P) = \pi(Q)$ holds. The previous result then becomes: an ordered set parameter π is a comparability invariant if and only if, for any ordered pair (P, R) of ordered sets and any element x of P, $\pi(P_x^R) = \pi(P_x^{R^d})$ holds (where P_x^R is the ordered set obtained by substituting the ordered set R for the element x of P).

Using this result, one easily proves that the dimension and the number of linear extensions of an ordered set are comparability invariants (these results are respectively due to Arditti (1978) and Trotter *et al.* (1976), Golumbic (1980), Habib (1984) and Stanley (1986b)), just as the properties of being a series–parallel, a covering N-free or an interval ordered set (see Definition 2.12, Chapter 2), as well as the fixed point property defined in Exercise 3.3 (Dreesen *et al.*, 1985; Kelly, 1986).

In Sections 1.3.2 and 1.3.3 we have defined the fundamental parameters associated with an ordered set. Of course many other parameters have been considered. We mention one of them, which is widely studied (see, for instance, Tanenbaum *et al.* (2001), or Keller *et al.* (2010)), namely the *linear discrepancy*. It expresses how far an ordered set P is from being a linear order: P has linear discrepancy k if $k = \min\{d_L(P), L \in \mathcal{L}(P)\}$ with, for a linear extension $L \in \mathcal{L}(P)$, $d_L(P) = \max|r_L(a) - r_L(b)|$, for all incomparable elements a, b of P.

To each of the operations on ordered sets defined in Section 1.5 corresponds a notion of decomposability. We describe the most important one, related to the substitution and often called *modular decomposition*. An ordered set P is said to be *decomposable* (with respect to the substitution, or modularly) if there are $h+1$ ordered sets $Q, P_1, ..., P_h$, with $|Q| > 1$ and $|P_i| > 1$ for at least one i, and some elements $y_1, ..., y_h$ of Q, such that: $P = Q^{P_1 \ldots P_h}_{y_1 \ldots y_h}$ (in that case, Q is called a *quotient* of P). Otherwise, P is said to be *indecomposable* or *prime* (for instance, A_2 is prime but C_2 is not).

One proves that, if P is decomposable, one (and only one) of the following cases holds:

- Q is an antichain, that is, P is the disjoint union of the P_i's;
- Q is a chain, which amounts to saying that P is the linear sum of the P_i's;
- Q is a prime ordered set (uniquely determined): P is said to be *of prime type* and Q is called the prime ordered set associated with P.

Repeating the decomposition operation on all the obtained P_i's that are decomposable, while taking the quotients Q of maximum size if they are chains or antichains, one finally obtains a *canonical decomposition tree*. The root of this tree corresponds to P and its leaves correspond to the elements of P. The maximum number of vertices covered by a vertex of this tree is called the *decomposition diameter* of P. For some classes of ordered sets, namely that of the ordered sets the decomposition diameter of which is bounded, the use of this canonical decomposition tree allows us to efficiently solve many algorithmic problems (see on these points Möhring's brilliant surveys (1984, 1989), as well as Appendix A, Section A.2.3).

The substitution operation, defined here for ordered sets, may be defined for many discrete structures and then leads to a decomposition theory of these structures, that has in particular important algorithmic applications. A more general decomposition theory of discrete structures was introduced by Cunningham and Edmonds (the "split decomposition") in 1980 and applied to ordered sets by Wagner (1990).

The possibilities of simplification for the operations on ordered sets have been the purpose of many works. Thus, the following results have been proved:

- $P \times Q \equiv P \times R$ implies $Q \equiv R$;
- P is not an antichain and $P^Q \equiv P^R$ imply $Q \equiv R$;
- P and Q are connected such that $P^P \equiv Q^Q$ or $P^Q \equiv Q^P$ imply $P \equiv Q$;
- $P^R \equiv Q^R$ implies $P \equiv Q$.

(See the survey papers of Jónsson (1982), Jónsson and McKenzie (1982), Davey and Duffus (1982), Duffus (1984), as well as McKenzie's 2000 paper).

Let us notice that, if the operations studied in Section 1.5 always lead to an order, the set union of orders may lead to an arbitrary relation. Fishburn and Spencer (1971) have studied the minimum number of (strict) orders whose union is a given (irreflexive) relation, as well as the maximum value of this number for the relations defined on a set of size n.

As already said in the Preface, this book concentrates on the fundamental concepts and results on finite ordered sets and on some important uses. So, unfortunately, we have been obliged to leave out significant topics. To end this section, we give some information on three of them: sorting, random orders, and enumeration problems.

(1) Sorting: a classic sorting problem consists of determining an unknown linear order L over a set X of n elements by asking a series of questions of the form "is the element x less than the element y in the order L?" This problem has generated a great deal of research and many sorting algorithms have been provided, the typically best ones allowing us to determine L in $O(n\log n)$ time complexity (see Appendix A and, for instance, Volume 3: *Sorting and Searching* in Knuth (1973)).

Sometimes, some information about L is already known in the form of an ordered set P on X. Then, the set of possible linear orders is the set $\mathcal{L}(P)$ of the linear extensions of P and – without any other information – it is reasonable to consider that they are all equally likely. Therefore, $\mathcal{L}(P)$ becomes a (finite) probability space endowed with the uniform probability $\mathrm{pr}(L) = 1/|\mathcal{L}(P)|$. In such a space, an event is a set of linear extensions of P and we can consider the event $\{L \in \mathcal{L}(P) : (x,y) \in L\}$ denoted by $[x < y]$ and whose probability is $\mathrm{pr}\,[x < y] = |\{L \in \mathcal{L}(P) : (x,y) \in L\}|/|\mathcal{L}(P)|$. This quantity is fundamental in deciding the expected complexity of sorting algorithms. Assume now that x,y,z are three incomparable elements in P. Are the two events $[x < y]$ and $[x < z]$ independent or correlated?[8] The answer is given by the Fishburn–Shepp inequality (Shepp,1982; Fishburn, 1984), also known as the *XYZ inequality*:

$$\mathrm{pr}([x < y].\mathrm{pr}([x < z]) < \mathrm{pr}([x < y] \text{ and } [x < z])$$

So these two events are positively correlated, which can also be written, for instance, as $\mathrm{pr}([x < y]|[x < z]) > \mathrm{pr}([x < y] \text{ and } [x < z])$ and means that, if $[x < z]$ occurs then

[8] In a probability space, two (non-empty) events A and B are *independent* if $\mathrm{pr}(A).\mathrm{pr}(B) = \mathrm{pr}(A$ and $B)$ or, equivalently, if the conditional probabilities $\mathrm{pr}(A|B)(= \mathrm{pr}(A$ and $B)/\mathrm{pr}(B))$ and $\mathrm{pr}(B|A) = \mathrm{pr}(A$ and $B)/\mathrm{pr}(A))$ are (respectively) equal to $\mathrm{pr}(A)$ and $\mathrm{pr}(B)$. The two events are *positively correlated* if $\mathrm{pr}(A).\mathrm{pr}(B) < \mathrm{pr}(A$ and $B)$ or, equivalently, if $\mathrm{pr}(A|B)$ and $\mathrm{pr}(B|A)$ are (respectively) strictly greater than $\mathrm{pr}(A)$ and $\mathrm{pr}(B)$.

$[x < y]$) is more likely to occur. Shepp and Fishburn's proofs used a result known as the Ahlswede–Daykin four functions inequality about functions defined on a distributive lattice. Later, Brightwell and Trotter (2002) provided a combinatorial proof.

Always in connection with this problem of sorting in the presence of partial information and with the same uniform probability space on $\mathcal{L}(P)$, there is a long-standing Kislitsyn's 1/3–2/3 conjecture (1968): if P is a non-linearly ordered set then P has two incomparable elements x and y (forming a so-called *balancing pair*) such that $1/3 \leq \mathrm{pr}([x < y]) \leq 2/3$.

Equivalently, if this conjecture is true, it means that at each step of a sorting algorithm, it is possible to add an ordered pair that reduces the number of possible linear extensions by a factor of at worst 2/3. This has been proved true only for some classes of ordered sets (for instance semiorders). For arbitrary ordered sets, Kahn and Saks (1984) have proved the existence of two incomparable elements x and y such that $3/11 \leq \mathrm{pr}([x < y]) \leq 8/11$, and then one had to wait more than 10 years before Brightwell *et al.* (1996) (slightly) improved this result by proving the existence of two incomparable elements x and y such that $1/2 - \sqrt{5}/10 \leq \mathrm{pr}([x < y]) \leq 1/2 + \sqrt{5}/10$.

On these topics of the correlation inequalities and the 1/3–2/3 conjecture, one will find some general reports in Graham (1982), Rival (1984), Saks (1985), Winkler (1986), Brightwell and Graham (1999), and Fishburn and Shepp (2001).

Notice finally that the sorting problem has been generalized to the case where one must discover an unknown partial order (see, for instance, Daskalakis *et al.*, 2009).

(2) Random Orders: there are several ways to define such orders on a set X of size n and, following the terminology in Brightwell's survey (1993a), we consider three of them.

- Model 1: *random partial orders*. The probability space is the set of all n-element orders endowed with the uniform probability distribution.
- Model 2: *random k-dimensional orders*. Introduced by Winkler (1985), they are obtained by taking the intersection of k linear orders chosen independently and uniformly at random.
- Model 3: *random graph orders* (also known as *transitive percolation processes*) and introduced by Albert and Frieze (1989). Here one takes a random graph on $(\{1, 2, ..., n\}, <)$ and one puts i below j if there is a path $i = i_1 ... i_k = j$ in the graph with $i_1 < ... < i_k$.

According to the considered model, one studies the structure or relevant parameters of these random orders (connectedness, height, width, dimension, number of extremal elements, number of incomparable pairs, number of linear extensions, etc.). For instance, in the case of Model 1, Kleitman and Rothschild (1975) proved that almost every n-element order has height 3.[9] Also, in the case of Model 2, Winkler (1985)

[9] This means that the probability that an n-element order has height 3 tends to 1 when $n \to \infty$.

proved that the width of almost every random k-dimensional order lies between $e^{-1}n^{(k-1)/k}$ and $n^{(k-1)/k}\log(n)$. And, in the case of Model 3, Bollobás and Brightwell (1997) showed that, if the random order has many elements comparable with all others, it decomposes as a linear sum of smaller orders.

Since we do not know any report more recent than Brightwell's 1993 survey quoted above, we mention some more recent papers. On Model 1, Brightwell *et al.* (1996); on Model 2, Bollobás and Brightwell (1995); on Model 3, Bollobás and Brightwell (1997) and Brightwell and Georgiou (2010).

(3) Enumeration: an important branch of combinatorial theory – now called enumerative combinatorics – is devoted to counting the number of finite structures of a given type. Exercise 1.1 proposes counting the order types defined on a set of size n at most equal to 4. Yet, for an increasing value of n, these numberings become harder and harder. One will find in Appendix C some tables giving the numbers of orders and order types that have been known so far, as well as asymptotic bounds for the number of orders of size n given by the Kleitman–Rothschild Theorem. The latter problem is a particular case of the problem of computing the number $|\mathcal{L}(P)|$ of linear extensions of an ordered set P (here the antichain A_n), which is in general difficult (see Rival (1984) and Brightwell and Winkler (1991)). It may however be solved for some particular ordered sets (see Atkinson, 1989). In some cases the problem is equivalent to other classic problems of combinatorial counting. Thus, when P is a downset of $(\mathbb{N}^2, \leq)$, the number of its linear extensions is the number of associated standard Young tableaux; on this subject one may consult Chapter 3 in Aigner's book (1979) or Stanley's book (1986a), two basic references for anything that concerns the use of ordered sets (particularly, the theory of Möbius functions) in counting problems. Problems and results about the enumeration of particular classes of orders are presented in Quackenbush (1982) and El Zahar (1989).

1.7 Exercises

Exercise 1.1 [Counting small orders] Count and represent by diagrams all order types of size 1, 2, 3, and 4. *Hint*: see Appendix B.

Prove that any two linearly ordered sets of size n are isomorphic.

Show that there exist 5 (respectively, 16) non-isomorphic types of ordered sets of size 3 (respectively, 4).

Exercise 1.2 [P-critical ordered pairs] Let x,y be two incomparable elements of an ordered set P and F the forcing order associated with P (see Definition 1.35). Prove that the following conditions are equivalent:

- (x,y) is P-critical;
- $(x,y) \in MaxF$;

- $(y,x) \in MinF$;
- $(x[\subset (y[$ and $]y) \subset]x)$ (that is, $x \updownarrow y$);
- $P^- x \subset (y[$ and $yP^+ \subset]x)$.

Denoting by $Crit(P)$ the set of all P-critical ordered pairs of P, deduce the following inclusions: $(MaxP \times MinP) \cap Inc_P \subseteq Crit(P) \subseteq (M(P) \times J(P)) \cap Inc_P$.

Show that, in the set $\mathcal{O}_n$ of all orders on n elements, ordered by set inclusion (see Example 1.13), P is covered by Q if and only if $|Q| = |P| + 1$.

Finally, show that, unless P is linearly ordered, there exist at least two ordered sets covering it.

Note The latter point amounts to showing that, in that case, P has at least two P-critical ordered pairs.

Exercise 1.3 [Orders on words] Let $(V, \leq)$ be a linearly ordered set. A *word* (*of length* $p \geq 1$) on V is a sequence $a_1 a_2 ... a_p$ of p elements of V. We denote V^* the set of all words on V and we define a relation $\leq_1$ on V^* by writing $a_1 a_2 ... a_p \leq_1 b_1 b_2 ... b_q$ if there exists i with $0 \leq i \leq p$ such that $a_1 = b_1, ..., a_i = b_i$ and such that ($i = p \leq q$ or ($i < p$ and $a_{i+1} < b_{i+1}$)). Prove that $\leq_1$ is a linear order – called *lexicographic order* – on the set of words (this is the usual order of dictionaries).

We define a second relation $\leq_2$ on words by writing $a_1 a_2 ... a_p \leq_2 b_1 b_2 ... b_q$ if $p < q$ or if ($p = q$ and $a_1 a_2 ... a_p \leq_1 b_1 b_2 ... b_q$). Show that $\leq_2$ is a linear order (this is the order of crossword dictionaries). We define a third relation $\leq_3$ on words by writing $a_1 a_2 ... a_p \leq_3 b_1 b_2 ... b_q$ if $p \leq q$ and if, for $j = q - p + 1$, the equality $a_1 a_2 ... a_p = b_j b_{j+1} ... b_{j+p-1}$ holds. Prove that $\leq_3$ is an order, called *right factor order* on V^*. How would one define a left factor order on V^*? Same question for a factor order.

We write $V = \{a < b\}$ and $V_3^* = \{$words of V^* of length at most 3$\}$. Draw some diagrams of V_3^*, provided with each of the previous orders.

Exercise 1.4 [Lectic order] Let $S = \{1, ..., i, ..., n\}$. For all $A, B \subseteq S$, we write $A < B$ if there exists $i \in B \setminus A$ such that $A \cap \{1, ..., i-1\} = B \cap \{1, ..., i-1\}$.

Prove that $<$ is an order relation on $\underline{2}^S$.

Prove that this order is linear.

Note This order is used for the enumeration of the closed sets of a Galois lattice in Ganter's algorithm (see Appendix A on page 270 and Ganter (1984)).

Exercise 1.5 [Orders on integer partitions, Aigner (1979)] An (*integer*) *partition* of the positive integer n is a decreasing sequence of n positive integers $n_1 \geq n_2 \geq ... \geq n_n \geq 0$, called the *partition parts* and the sum of which is equal to n. Show that the following three relations define three orders on the set of partitions of n. Let $\mu = (m_1 \geq m_2 \geq ... \geq m_n)$ and $\nu = (n_1 \geq n_2 \geq ... \geq n_n)$ be two partitions of n.

- μ is *lexicographically less than or equal to* ν if $\mu = \nu$ or if, for $i = 1, 2, ..., k$ ($k < n$), $m_i = n_i$, and $m_{k+1} < n_{k+1}$.

- μ is *finer than* ν if there exists a (set) partition on the partition parts of μ such that the sums of the numbers in the classes of this partition are the partition parts of ν.
- μ is *dominated* by ν if, for $i = 1,2,...,n$, $m_1 + m_2 + ...m_i \leq n_1 + n_2 + ... + n_i$ holds (this relation, called *dominance order*, is important in combinatorics and in many other fields, such as inequalities theory).

Draw some diagrams of these orders for the set of partitions of the numbers 5, 6, and 7.

Exercise 1.6 [Covering graph] In an irreflexive directed graph G, the path $x_0x_1...x_p$ (with $p \geq 2$) is a *cycle* if it satisfies $x_0 = x_p$; it is a *quasi-cycle* if it is a path such that x_0x_p is an arc of G (so it is sufficient to reverse this arc to obtain a cycle). Show that a directed graph G is the covering graph of an order if and only if G contains neither a cycle nor a quasi-cycle.

Exercise 1.7 [Reflexo-transitive closure] Show that the reflexo-transitive closure of a directed graph $G = (X,U)$ (see the definition given in Example 1.20) is the smallest preordered set (X,R) "including" G (that is, such that xUy implies xRy).

Prove that, if G is cycle-free, its reflexo-transitive closure is an ordered set. Show that the covering graph of an ordered set P is the smallest directed graph the reflexo-transitive closure of which is equal to P.

Let x,y be two incomparable elements of an ordered set P. Prove that the reflexo-transitive closure of the graph of $P + (x,y)$ is an ordered set that is an extension of P. Then deduce that there exists a linear extension of P which contains the ordered pair (x,y) and that the intersection of all linear extensions of P is P (see Theorem 2.29).

Exercise 1.8 [Downsets and upsets] Draw diagrams of the set, ordered by set inclusion, of all downsets of the ordered set in Figure 1.4, of the three ordered sets in Figure 1.5, and of P in Figure 1.7 (the numbers of these downsets are respectively 10, 5, 16, 20, and 15).

Same question replacing "downset" with "upset." What do you notice?

Show that there exists a bijection between downsets and upsets of an ordered set. Define a bijection between downsets and antichains of an ordered set. Deduce an order relation between antichains of an ordered set.

Exercise 1.9 [Constructing a linear extension of an order] Show that any ordered set has (at least) one minimal element. Define a linear order $x_1x_2...x_n$ on the ordered set $(X,\leq)$ of size n by taking as x_1 a minimal element of $X_1 = X$ and, for any x_i (with $1 < i \leq n$), one minimal element of $X_i = X_{i-1} \setminus \{x_{i-1}\}$. Show that this linear order is a linear extension of the order $\leq$ and that any linear extension of $(X,\leq)$ may be obtained likewise.

Exercise 1.10 [Divisibility order on positive integers] Show that the divisibility relation defined on the set of positive integers is an order relation.

Draw some diagrams of the following sets of positive integers, ordered by the divisibility order ($i \leq j$ if i divides j):

- {positive integers from 1 to 10},
- {positive integers dividing 16},
- {positive integers dividing 30},
- {positive integers dividing 36}.

Among those ordered sets, are there direct products of chains? Generalize this observation.

Exercise 1.11 [Preorders of sections] Let R be a binary relation on a set X and x an element of X. We write $xR = \{z \in X : xRz\}$ and $Rx = \{z \in X : zRx\}$, these two sets being respectively called *right section* and *left section of basis* x. We define three preorders on X, respectively called *right trace preorder*, *left trace preorder*, and *trace preorder*, by writing $xT_r y$ if $xR \supseteq yR$, $xT_l y$ if $Rx \subseteq Ry$, and $T = T_r \cap T_l$.

A binary relation R on X is a *tournament* if it is total and antisymmetric (Definition 2.25). Prove that, for a tournament R, the two preorders T_r and T_l are two identical orders contained in R. Show that x is a maximal element of the ordered set (X,T) if and only if, for any $y \in X$, there exists $z \in X$ such that $(x,z) \in R$ and $(z,y) \in R$ (z is then called a *center* of the tournament). What do these results become if R is a transitive tournament?

2

Particular classes of ordered sets

This book contains results such as Dilworth's or Hiraguchi's Theorems (Chapters 4 and 6 respectively) that hold for any ordered set. However, this type of general result is rather rare. The notion of an order, although very restrictive compared to the notion of a relation, actually remains very general, a fact revealed by the huge number of different orders that can be obtained on a set with a small size (more than two million types of order on a set with 10 elements! See Appendix C). Yet in practice, the orders that naturally appear in many contexts most often belong to some particular classes of orders. These classes may be defined in many ways. They are obtained, for instance, by setting the value of a parameter (for instance, orders of dimension 2, studied in Section 6.3), by forbidding the presence of some given configurations (for instance, interval orders mentioned in Example 1.22 and in Section 2.2), by constructing the class by iteration of some given operations on a family of initial orders (for instance, series–parallel orders, defined in Section 2.2). In this chapter, we present some of the most frequent classes of orders, that will be regularly encountered all throughout the book. Although we define them in a unique way here, we will see in the exercises and later in the text that these classes often have several alternative definitions. This explains the fact that they have sometimes appeared independently in various contexts and reinforces their interest.

The first section develops the case of ranked ordered sets and, in particular, of those that are semimodular or bipartite. In Section 2.2, we present a number of ordered sets defined by forbidden configurations, that is, such that their order – or an associated graph – does not contain a given determined substructure. Section 2.3 consists of an introduction to lattices and semilattices. We give the basic definitions but also a number of results and their dual expressions; the latter result from the ipsodual nature of the class of lattices. Within this class, the subclass of distributive lattices proves to be particularly important. The next section presents the relations existing between linearly ordered sets and "tournament" relations (the name of which comes from the fact that they model the results of some sports tournaments, such as the Six Nations Rugby Tournament).

2.1 Ranked, semimodular, and bipartite ordered sets

Definition 2.1 A *ranked ordered set* is an ordered set $P = (X, \leq)$ for which there exists a *rank function* (or, more simply, a *rank*), i.e., a function r from X to the ordered set $(\mathbb{N}, \leq)$ of non-negative integers, which preserves the covering relation:

$$x \prec y \Longrightarrow r(y) = r(x) + 1$$

In other words, if y covers x, its rank is equal to the rank of x plus 1. A rank is sometimes also called a *graduation* and a ranked ordered set a *graded* ordered set.

Any ranked ordered set has an infinity of ranks, since adding any fixed positive integer to a rank defines another rank. The rank of P is said to be *normalized* if $r(x) = 0$ for at least one minimal element of P,[1] i.e., if the smallest possible value of the rank is reached. It may be proved that, if P is connected, the normalized rank is unique. In particular, if P is ranked and has a minimum 0_P, P has a unique normalized rank, which thus satisfies $r(0_P) = 0$. In the case of a ranked ordered set P with a unique normalized rank, denoted by r, we write:

$$r(P) = \max_{x \in P} r(x)$$

$r(P)$ is then called the *rank* of P.

Note The expression "rank of P" can sometimes refer to the parameter $r(P)$ and sometimes to the rank function defined on P, the context allowing us to determine its meaning.

Figure 2.1 shows the normalized ranks of three ranked ordered sets. The reader can check that the ordered set in Figure 1.4 (see page 7), on the contrary, is not ranked.

Henceforth and unless explicitly mentioned in the text, the word "rank" will stand for "normalized rank."

Note that, if P is ranked, its rank is not always equal to its height – that is, the maximum length of its chains (the reader can try to find an example).

A linearly ordered set (X, L) is ranked and its normalized rank is given by $r_L(x) = |\{y \in X : y \neq x \text{ and } yLx\}|$. Then it is easy to check that xLy if and only if $r_L(x) \leq r_L(y)$.

The ordered sets $\underline{2}^E$, $\mathcal{O}_n$, Σ_n, and $\mathcal{P}_E$ (Examples 1.12, 1.13, 1.17, and 1.14 in Chapter 1) are ranked and their normalized ranks are, respectively, the size of a subset of E, the number of ordered pairs (x,y) of an order with $x \neq y$, the number of increasing commutations needed to go from the bottom permutation (the smallest one with respect to the weak Bruhat order on permutations) to a permutation s, the size of the set E minus the number of classes in a partition of E. Distributive lattices presented in Section 2.3 are also ranked (and an expression of their rank is given in Proposition 5.14 in Chapter 5).

[1] Observe that such a minimal element cannot be arbitrarily chosen, see for instance Figure 2.1(b).

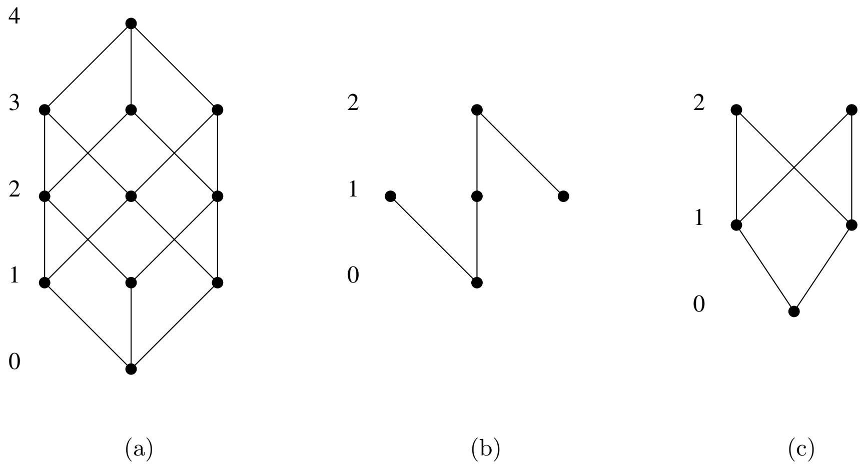


Figure 2.1 Three ranked ordered sets with the indication of their normalized rank.

If P is ranked and k is a non-negative integer, we write $R_k = \{x \in P : r(x) = k\}$. The non-empty R_k's are called the *rank-sets (of rank k) of P* and their sizes $n_k = |R_k|$ are called the *Whitney numbers of P*. Observe that they form an antichain partition of P (why?). For any ranked ordered set P, we write:

$$\nu(P) = \max_{0 \leq k \leq r(P)} |R_k|$$

Thus $\nu(P)$ is the maximum number of elements of a rank-set of P and, since these rank-sets are antichains (and recalling that $\alpha(P)$ is the width of P, see Definition 1.30), we obtain:

$$\nu(P) \leq \alpha(P)$$

Remark 2.2 A ranked ordered set P is called a *Sperner ordered set* if it satisfies the equality $\nu(P) = \alpha(P)$. This term will be justified in Chapter 4, where Section 4.3 is devoted to the study of these ordered sets.

For any element x of an ordered set P, we define the *height* of x, denoted by $h(x)$, as the maximum length of a chain of P the greatest element of which is x. The height function h thus defined from P to $\mathbb{N}$ satisfies:

$$x \prec y \Longrightarrow h(y) \geq h(x) + 1$$

(why?), but h is in general not a rank, even though P is ranked (see Figure 2.1(b)). Note that the integer $h(P) = \max_{x \in P} h(x)$ is equal to what we have called the *height of P* in Chapter 1 (see page 19).

The following definition and result will allow us to obtain a class of ordered sets in which the height function is a rank.

Definition 2.3 An ordered set P satisfies *(the) Jordan–Dedekind (property)* if, for all $x, y \in P$ such that $x < y$, all maximal chains from x to y have the same length (depending only on x and y).

Theorem 2.4 *Let P be an ordered set. Each of the conditions below implies the following one:*

1. *The height h defined on P is a rank.*
2. *P is ranked.*
3. *P satisfies Jordan–Dedekind.*

If, moreover, P has a minimum, these three conditions are equivalent.

Proof The implication of (2) by (1) is immediate. Let us prove that (2) implies (3). Let x and y be two comparable elements of a ranked ordered set P (with $x < y$) and let $x = x_0 \prec x_1 \prec \ldots \prec x_p = y$ be a maximal chain from x to y. The length p of this chain is equal to $r(y) - r(x)$ and thus does not depend on the considered chain (since it only depends on the ranks of x and y). It follows that all maximal chains from x to y have this same length. This fact holds for any pair of comparable elements and, consequently, we get the Jordan–Dedekind property.

Now assume that P has a minimum 0_P. We prove that, under this condition, (3) implies (1). Let x and y be two elements of P with $x \prec y$. According to Jordan–Dedekind, all maximal chains from 0_P to x have the same length, which is equal to the height $h(x)$ of x. A particular maximal chain from 0_P to y is formed with a maximal chain from 0_P to x followed by the covering relation $x \prec y$. Now, this maximal chain from 0_P to y has length $h(x) + 1$ which, still with respect to Jordan–Dedekind, implies $h(y) = h(x) + 1$ as required. □

Remark 2.5 1. The reverse implications of this theorem are in general not satisfied, as illustrated by the ordered sets P and P' in Figure 2.2. As a matter of fact, the ordered set P (which is the same as in Figure 1.4) "trivially" satisfies Jordan–Dedekind (between any two comparable elements, there exists only one maximal chain) but is not ranked. As for P', it is ranked but its height function is not a rank.

2. Theorem 2.4 has a dual version which uses the notion of the *depth* $p(x)$ of x defined as the maximum length of a chain with minimum x. Then, the dual of Theorem 2.4 is obtained by replacing the height with the function f from P to $\mathbb{N}$ defined by $f(x) = |P| - p(x)$ and where we replace "minimum" with "maximum."

3. Finally, the reader can prove that the height h of a ranked ordered set P is a rank of P if and only if all minimal elements of P have the same rank (which is then equal to 0).

Within the class of ranked ordered sets, one finds particular subclasses among which the most important ones are the classes of *bipartite* ordered sets and of *semimodular* ordered sets, which we now define. Whereas bipartite ordered sets are trivially ranked,

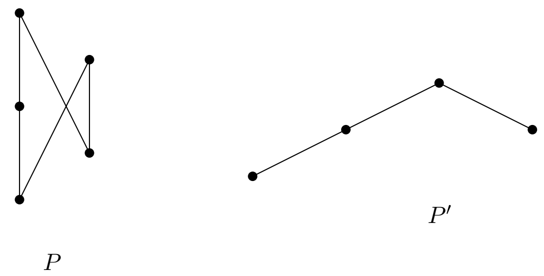
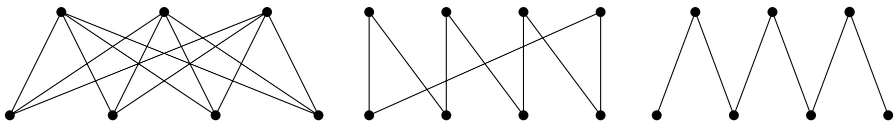


Figure 2.2 Counterexamples to the converse conditions of Theorem 2.4.



(a) $K_{4,3}$ (b) CR_4 (c)

Figure 2.3 (a) $K_{4,3}$, (b) CR_4, and (c) a fence.

the fact that semimodular ordered sets are also ranked (Theorem 2.10) follows from two non-trivial lemmas.

Definition 2.6 An ordered set P is *bipartite* if the range $\kappa(P)$ of P is equal to 2, that is, if the height of P is 1.

A connected bipartite ordered set is clearly ranked, the set A of its minimal elements and the set B of its maximal elements forming its two rank-sets, respectively, of rank 0 and of rank 1. Such an ordered set will generally be denoted by $(A+B,\leq)$. Figure 2.3 shows three examples of bipartite ordered sets. The first one is the complete bipartite ordered set $K_{4,3}$ – more generally, we denote $K_{p,q}$ the complete bipartite ordered set $A_p \oplus A_q$ in which each one of the p minimal elements is less than each one of the q maximal elements. The second example is the crown CR_4 – more generally, we call *crown* CR_p the ordered set formed with p minimal elements $a_1,...,a_p$ and p maximal elements $b_1,...,b_p$, satisfying $a_i \prec b_{i-1}, b_i$ for any $1 < i \leq p$ and $a_1 \prec b_p, b_1$. The last example is a particular case of a fence, the class of which is an important class of bipartite ordered sets (see, for instance, the beginning of Section 3.6 in Chapter 3).

Definition 2.7 An ordered set $P = (X,\leq)$ is *upper semimodular* (*USM*) if, for all $x,y,z \in P$ with $z \prec x$ and $z \prec y$, there exists $t \in P$ such that $x \prec t$ and $y \prec t$. Similarly P is *lower semimodular* (*LSM*) if the dual of P is upper semimodular, that is, if, for all $x,y,z \in P$ with $x \prec z$ and $y \prec z$, there exists $t \in P$ such that $t \prec x$ and $t \prec y$. At last P is *modular* if it is upper and lower semimodular.

The ordered set in Figure 2.1(c) is lower but not upper semimodular.

In order to establish Theorem 2.10 on upper semimodular ordered sets, we begin with the proof of the following two lemmas.

Lemma 2.8 *Let P be an upper semimodular ordered set. If P is connected, then it has a maximum 1_P.*

Proof Let P be upper semimodular and connected. Assume that P has two distinct maximal elements x and y. Since P is connected, there exists a sequence $x = x_0, ..., x_i, x_{i+1}, ..., x_k = y$ of elements of P such that, for any $i < k$, we have either $x_i \prec x_{i+1}$ or $x_{i+1} \prec x_i$. Besides, since x and y are maximal in P (hence incomparable), there exists at least one integer $0 < i < k$ for which $x_i \prec x_{i+1}$ (if not, one would have $y < x$). If i_0 is the smallest integer satisfying this condition, we have $x_{i_0} \prec x_{i_0-1}$ and $x_{i_0} \prec x_{i_0+1}$ which imply (since P is upper semimodular) the existence of some z_1 of P covering x_{i_0-1} and x_{i_0+1}. The equality $x_{i_0-1} = x$ is impossible since x is maximal in P, so there exists an element z_2 in P covering x_{i_0-2} and z_1 (still according to the upper semimodularity of P). Again, we must have $x_{i_0-2} \neq x$. Iterating this argument, we conclude that there exists z in P covering x, which contradicts the maximality of x in P. Finally, P has a maximum 1_P, as required. □

Lemma 2.9 *Let P be an upper semimodular ordered set. If P is connected, then it is ranked.*

Proof To prove that P is ranked, we are going to prove that it satisfies Jordan–Dedekind. Indeed, Lemma 2.8 then implies that P has a maximum and the dual version of Theorem 2.4 (see item (2) in Remark 2.5) will imply that P is ranked. We first prove by induction that Jordan–Dedekind is true on any interval $[x, 1_P]$ of P. To do so consider, for any $m > 0$, the following property $P(m)$:

For any $x \neq 1_P$ of P, if there exists a maximal chain of length m from x to 1_P, then all maximal chains from x to 1_P have length m.

$P(1)$ clearly holds (since if there exists a maximal chain C of length 1 from an element x to 1_P, then x is covered by 1_P and C is the unique chain joining x to 1_P). Now assume $P(m)$ holds for $m > 1$ and let $x \in P$ be such that there exists a maximal chain $C = x \prec x_1 \prec \cdots \prec x_m \prec 1_p$ of length $m+1$ from x to 1_P. Three cases may occur: (a) if C is the unique maximal chain from x to 1_P, $P(m+1)$ holds for x. (b) If all maximal chains from x to 1_P contain x_1, they are formed with the covering pair $x \prec x_1$ followed by a maximal chain from x_1 to 1_P, of length m by applying $P(m)$. All maximal chains from x to 1_P are thus of length $m+1$. (c) If there exists at least one other maximal chain from x to 1_P that does not contain x_1, let us denote by $C' = x \prec x_1^{'} \prec \cdots \prec x_q^{'} \prec 1_P$ this chain, the length of which is $q+1$ and let us prove that $q = m$. Since P is upper semimodular and since $x \prec x_1, x_1^{'}$, there exists $y \in P$ such that $x_1, x_1^{'} \prec y$. Denote by C_y a maximal chain from y to 1_P. The part of C from x_1

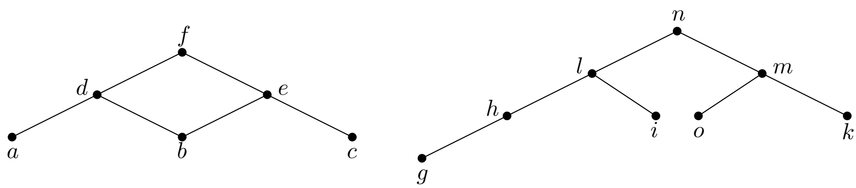


Figure 2.4 An upper semimodular ordered set with two connected components.

to 1_P is a maximal chain from x_1 to 1_P of length m so, according to the induction hypothesis, all maximal chains from x_1 and 1_P have this length. Thus the maximal chain formed by the covering pair $x_1 \prec y$ followed by C_y has length m, which implies that the length of C_y is $m-1$. Therefore, the maximal chain from $x_1^{'}$ to 1_P formed by the covering pair $x_1^{'} \prec y$ followed by C_y has length m and, here again by the induction hypothesis, the part of C' from $x_1^{'}$ to 1_P has length m. Finally the length of C' is $m+1$ and $P(m+1)$ holds.

We can now conclude that the Jordan–Dedekind property holds on P. Indeed, let x and y be two elements of P with $x < y$. All maximal chains from x to 1_P have the same length – equal to the depth of x (see page 45). It is the same for maximal chains from y to 1_P – the length of which is equal to the depth of y. The length of all maximal chains from x to y is thus equal to the difference of these two depths, which depends only on x and y. □

The following theorem is obtained as a direct corollary of Lemma 2.9 and by observing that an ordered set is ranked (respectively, upper semimodular) if and only if all its connected components are. See Figure 2.4 for an example.

Theorem 2.10 *Any upper semimodular ordered set P is ranked.*

Remark 2.11 1. Lemmas 2.8 and 2.9 and Theorem 2.10 have dual expressions obtained by replacing "upper semimodular" by "lower semimodular" and "maximum" by "minimum."

2. We will see in Chapter 5 that distributive lattices are upper (and lower) semimodular and, thus, that they are also ranked. Moreover, since they have a minimum, their height is a rank.

2.2 Ordered sets with forbidden configurations

There exist other important classes of ordered sets defined by the fact that their order (or one of their associated graphs) does not contain a suborder (or a subgraph) of a given type.

Figure 2.5 gives the seven types of ordered sets of size 2 or 3 (C_2, A_2, C_3, A_3, $A_2 \oplus A_1$, $A_1 \oplus A_2$, and $C_1 + C_2$) together with three particular types of ordered sets of

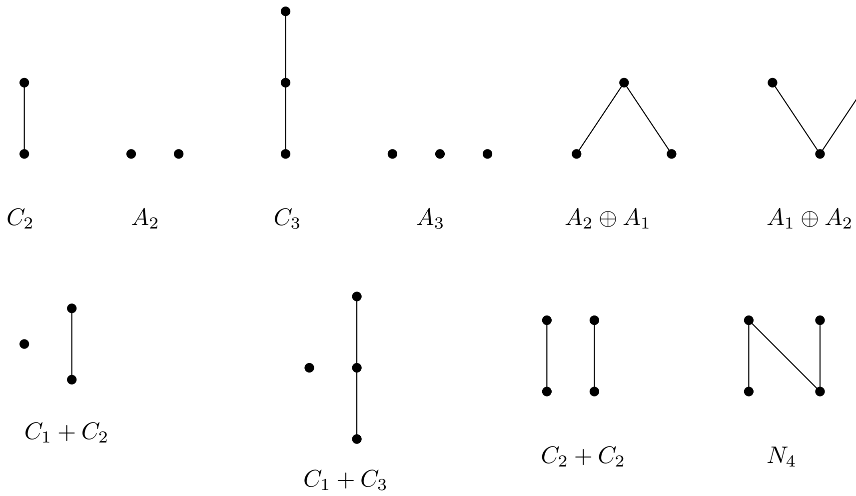


Figure 2.5 10 particular types of ordered sets with two, three or four elements.

size 4 ($C_1 + C_3$, $C_2 + C_2$, and N_4). We will say that an ordered set *does not contain one of these types* if it contains no ordered subset of this type.

One easily observes that an ordered set P contains no C_2 (respectively, A_2) if and only if it is an antichain (respectively, a chain). Similarly, P contains no C_3 if and only if it is an antichain or is bipartite.

The classes of ordered sets that we now consider are defined by forbidden configurations of one of the last seven types.

Definition 2.12 Let P be an ordered set.

1. P is a *2-chain* if it contains no A_3 and is not a chain.
2. P is a *tree-ordered set* if it is connected and either does not contain any $A_1 \oplus A_2$ or does not contain any $A_2 \oplus A_1$.
3. P is a *weakly ordered set* if it contains no $C_1 + C_2$.
4. P is an *interval ordered set* if it contains no $C_2 + C_2$.
5. P is a *semiordered set* if it contains neither $C_1 + C_3$ nor $C_2 + C_2$.
6. P is *N-free* if it contains no N_4.
7. P is *covering N-free* if it contains no N_4 as a covering ordered subset.

Figure 1.4 of Chapter 1, the ordered set P' in Figure 2.2, and Figures 2.3(a), 1.4, 2.1(b), 2.3(a), and 2.1(b) provide examples of a 2-chain, a tree-ordered set, a weakly ordered set, an interval ordered set, a semiordered set, an N-free, and a covering N-free ordered set, respectively.

The names of these ordered sets are justified by some of the properties that will be studied in the exercises and in some following chapters. These ordered sets are endowed with reflexive orders to which correspond strict orders (that is, irreflexive ones). For instance, a (strict) interval order is a strict order which does not contain any $\underline{2}_s + \underline{2}_s$ (see Definition 1.3). The strict order called "strict weak order" (respectively, strict interval order) associated with a weakly ordered set (respectively, an interval ordered set) was first introduced in Example 1.21 (respectively, Example 1.22) in another form. Section 7.1 in Chapter 7 will develop some uses of (strict) interval orders, of (strict) weak orders, and of (strict) semiorders for the modeling of preference relations.

Let us also specify the following points, some of which are clear (the other ones will be proved later or proposed as exercises):

1. A 2-chain is an ordered set of width 2.
2. A tree-ordered set has a minimum or a maximum. Let us also observe that a tree-ordered set with a maximum (respectively, a minimum) is upper (respectively, lower) semimodular and series–parallel (why?).
3. An ordered set is a weakly ordered set if and only if it is a linear sum of antichains.
4. The class $\mathcal{SP}$ of series–parallel ordered sets is defined as follows:
   - the 1-element ordered set belongs to $\mathcal{SP}$;
   - if P_1 and P_2 belong to $\mathcal{SP}$, so do $P_1 + P_2$ and $P_1 \oplus P_2$.

   The class $\mathcal{SP}$ is therefore obtained by use of the two particular substitution operations of cardinal and linear sums. Then one can show that an ordered set is N-free if and only if it is series–parallel.
5. The following classes of ordered sets satisfy the next inclusion relations (and the reader will find examples to prove that these inclusions are strict): {chains} $\subset$ {weakly ordered sets} $\subset$ {semiordered sets} $\subset$ {interval ordered sets} and, besides, {chains} $\subset$ {weakly ordered sets} $\subset$ {series–parallel ordered sets} $\subset$ {covering N-free ordered sets}.

2.3 Semilattices and lattices

These ordered sets have already been mentioned in the previous chapter, after the definitions of the notions of a meet and of a join (Definition 1.39).

Definition 2.13 An ordered set P is a *meet-semilattice* if any pair $\{x,y\}$ of its elements has a meet $x \wedge y$. It is a *join-semilattice* if any pair $\{x,y\}$ of its elements has a join $x \vee y$. It is a *lattice* if any pair of its elements has a meet and a join, so if it is a meet- and a join-semilattice.

Proposition 2.14 *Let L be a meet-semilattice $(X, \leq, \wedge)$ (respectively, a join-semilattice $(X, \leq, \vee)$). Any subset of L has a meet (respectively, a join).*

Proof Let A be a 3-element subset of a meet-semilattice $(X, \leq, \wedge)$. It is easy to check that $\bigwedge A$ exists and is equal, for any element L of A, to $(\bigwedge(A \setminus t)) \wedge t$. By

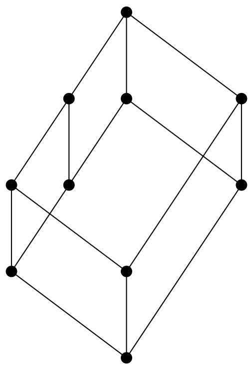


Figure 2.6 A lattice.

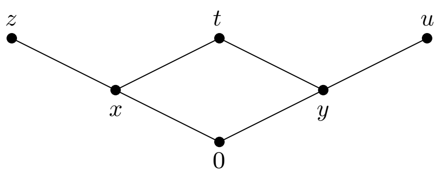


Figure 2.7 A meet-semilattice.

iterating this process, we deduce that any (non-empty) subset A of X has a meet – in particular X has a least element (equal to $\bigwedge X$).

The proof is similar in the case of a join-semilattice. □

A lattice could be denoted by $L = (X, \leq, \wedge, \vee)$ but we will often use the simpler notation L. The expression "semilattice" will stand for either a meet-semilattice or a join-semilattice depending on the context.

The ordered sets of Figures 2.1 and 2.3 are not semilattices (only the case of 2.1(a) is not immediate). Each connected component of the ordered set in Figure 2.4 is a join-semilattice and the ordered set in Figure 2.6 is a lattice (observe that the latter is not ranked).

Definition 2.15 Let $L = (X, \leq, \wedge)$ be a meet-semilattice. A subset A of X is a *sub-meet-semilattice of* L if it is meet closed, that is, if $x, y \in A$ implies $x \wedge y \in A$.

Consider the meet-semilattice L in Figure 2.7. Its restriction to $\{x, z, t\}$ is a sub-meet-semilattice of L. We moreover note that its restriction to $\{0, z, t\}$, which is not a sub-meet-semilattice of L, is nevertheless a meet-semilattice (with $z \wedge t = 0$ whereas $z \wedge t = x$ in L). Therefore the two notions of "sub-meet-semilattice of L" and "ordered subset of L which is a meet-semilattice for the order of L" do not coincide.

Likewise, we define the notion of a *sub-join-semilattice* of a join-semilattice L as any subset of L closed for the join operation of L, thus as any join closed subset of

L. A *sublattice* of a lattice L is a subset of L which is closed for the join and the meet operations of L.

It was already seen in Example 1.40 that the ordered set $\underline{2}^E = (P(E), \subseteq)$ of all subsets of a set E ordered by set inclusion is a lattice. In fact it is clear that, if A and B are two subsets of E, we have:

$$A \wedge B = A \cap B \qquad\qquad A \vee B = A \cup B$$

It is not always so simple to prove that an ordered set is a lattice. In particular, the reader can compare the cases of the order on permutations (Example 1.17), of the refinement order between partitions (Example 1.14 and Section 7.3), and of the dominance and refinement orders between integer partitions (Exercise 1.5) that all define lattices. However, a general result on semilattices allows us to characterize those among them that are lattices. Thus, we obtain the important result of Theorem 2.17, the proof of which arises directly from the following proposition:

Proposition 2.16 *Let A be a subset of a meet-semilattice. A has a join if and only if it is upper bounded.*

Proof Since the join (when it exists) of a subset A of a meet-semilattice is a particular upper bound of A, the necessary condition is immediate (this fact more generally holds in any ordered set P).

Conversely, let A be an upper bounded subset of a meet-semilattice L, B the set of the upper bounds of A, and $x = \bigwedge B$ the meet of B (which exists by Proposition 2.14). Observe that every element a of A is a lower bound of B, which implies $a \leq x$. So $x \in B$ and, since moreover $x = \bigwedge B$, then x is the minimum of B and so $x = \bigvee A$, which completes the proof. □

Theorem 2.17 *A meet-semilattice with a maximum is a lattice.*

Proof Let L be a meet-semilattice with maximum 1_L. Any subset of L is upper bounded by 1_L and, by Proposition 2.16, any subset of L has a join. L is therefore a join-semilattice and thus a lattice. □

Remark 2.18 Proposition 2.16 and Theorem 2.17 have dual versions:

- *A subset of a join-semilattice has a meet if and only if it is lower bounded. This meet is the join of its lower bounds.*
- *Any join-semilattice with a minimum is a lattice.*

As illustrated in this remark, the class of join-semilattices is the dual of the class of meet-semilattices. As a matter of fact, if $P = (X, \leq, \wedge)$ is a meet-semilattice, then $P^d = (X, \geq, \vee^d)$ is a join-semilattice with $\vee^d = \wedge$. As a result we can apply the duality principle between these two classes (Section 1.1.4): if a condition using the symbols $\leq$, $\geq$, and $\wedge$ holds in any meet-semilattice, the condition obtained by replacing $\leq$

with $\geq$, $\geq$ with $\leq$, and $\wedge$ with $\vee$ holds in any join-semilattice. In other words, it is sufficient to study the properties of one of these classes of ordered sets and to deduce the properties of the other class by duality. This duality principle applies to lattices – that form an ipsodual class – where it is expressed as follows:

If a condition holds in any lattice, the condition obtained by exchanging the symbols $\leq$ and $\geq$ on the one hand, and $\wedge$ and $\vee$ on the other hand, also holds in any lattice.

For instance, in any lattice L, $x \leq y$ implies $x \wedge z \leq y \wedge z$ for any $z \in L$ (why?). By duality it follows that $x \geq y$ implies $x \vee z \geq y \vee z$ for any $z \in L$.

If $A = \{x,y,z\}$, we can write:

$$\bigwedge\{x,y,z\} = (x \wedge y) \wedge z = x \wedge (y \wedge z)$$

Thus, the operation which, with any two elements of a meet-semilattice, associates their meet is associative. It is also commutative ($x \wedge y = y \wedge x$) and idempotent ($x \wedge x = x$). It is easy to see that, conversely, every set X equipped with an associative, commutative, and idempotent operation $\perp$, can be equipped with a meet-semilattice order $\leq$, with $x \wedge y = x \perp y$ (Exercise 2.6). Thus meet-semilattices can be defined as particular algebraic structures, which allows their study by means of algebraic methods. By duality, all these results apply to join-semilattices and, then, to lattices. Thus a lattice has a minimum and a maximum and can be algebraically defined using two operations (Exercise 2.7).

Let x_1, x_2,..., x_p and y_1, y_2,..., y_q be $p+q$ elements of a lattice L and such that, for all $1 \leq i \leq p$ and $1 \leq j \leq q$, $x_i \leq y_j$ holds. Then we have $\bigvee_{1 \leq i \leq p} x_i \leq \bigwedge_{1 \leq i \leq q} y_j$ (why?). It follows that any three elements x,y,z of a lattice always satisfy the following inequalities:

1. $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$,
2. $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$,
3. $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) \leq (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$.

The following result, the proof of which will be given in Section 5.1 of Chapter 5, is particularly significant:

In a lattice the inequality (1) is always an equality if and only if the inequality (2) is always an equality and if and only if the inequality (3) is always an equality.

Definition 2.19 Let L be a lattice and 0_L and 1_L its minimum and maximum respectively.

- L is *distributive* if any triple (x,y,z) of L satisfies $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.
- L is *complemented* if any $x \in L$ has at least a *complement*, that is, an element $x' \in L$ satisfying $x \wedge x' = 0_L$ and $x \vee x' = 1_L$.
- L is *Boolean* if it is distributive and complemented.

According to what has been said, a lattice is distributive if one of the inequalities (1), (2) or (3) (hence, also the three of them) is always an equality. The name of these lattices obviously comes from the fact that each of their two operations is distributive with respect to the other one:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad\qquad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad (D)$$

In the case of a distributive lattice, the identity $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ shows that the element obtained by taking the join of the pairwise meets of three elements is equal to the one obtained by taking the meet of their pairwise joins. This particular element is called the *median of the three elements* x, y, z, a name that will be justified in Section 7.3 of Chapter 7, where we will more generally study the medians of any number of elements of a "median semilattice."

The study of ordered sets cannot be separated from that of distributive lattices on account of the existence of a one-to-one correspondence between these two classes of structures. This correspondence will be presented in Chapter 5 with a lot of properties of distributive lattices. We give here some rather obvious ones.

It is easy to check that a chain $\underline{k}$ is a distributive lattice. On the other hand, the reader will easily show the following property:

Proposition 2.20 *The direct product of distributive lattices is a distributive lattice.*

Any direct product of chains is therefore a distributive lattice. In particular, this is the case for the direct product $\underline{k}^n$ of n chains, each of which is isomorphic to the chain $\underline{k}$. In Chapter 4, Section 4.4, we will study some properties of the direct product $\underline{k}^n$ and, in Chapter 6, we will encode any given ordered set within such a product (the lattice L in Figure 1.10 represents the diagram of $\underline{3}^2$). If $k = 2$, we get $\underline{2}^n$ and thus we reobtain the notation of the ordered set $\underline{2}^E = (P(E), \subseteq)$ of all subsets of a set E of size n (since this ordered set is isomorphic to the direct product of n chains of size 2). Besides, $\underline{2}^E$ is obviously complemented (by the map which associates the subset $E \setminus A$ of E to any $A \subseteq E$), which makes it a Boolean lattice (see also Exercise 5.11 in Chapter 5).

The fact that $\underline{2}^n$ is distributive is expressed by the well-known distributivity of both set union and set intersection operations:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \qquad\qquad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

The distributivity identities (D) still hold in any sublattice of a distributive lattice, so that the following result is immediate:

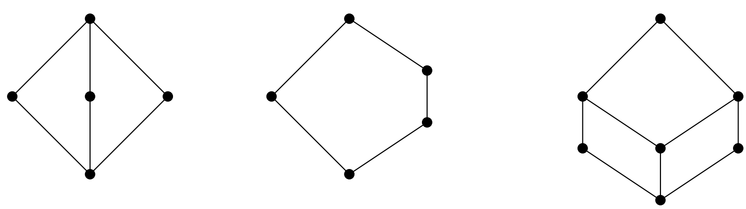
Proposition 2.21 *Any sublattice of a distributive lattice is a distributive lattice.*

In particular, any sublattice of $\underline{2}^E$ – that is, any family $\mathcal{F}$ of subsets of E such that, if A and B belong to $\mathcal{F}$, so do $A \cap B$ and $A \cup B$ – forms a distributive lattice. Such a family $\mathcal{F}$ of subsets of E is called a *distributive family of subsets (of E)*. If, moreover, $\emptyset$ and E belong to $\mathcal{F}$, $\mathcal{F}$ is called a *topology*. It was observed in Section 1.4.2 that the intersection or the union of downsets of an ordered set P is still a downset. From this it results that the family $\mathcal{D}(P)$ of all downsets of P is a topology (and it is the same for the family $\mathcal{U}(P)$ of all upsets of P). In Chapter 5, Birkhoff's Representation Theorem (Theorem 5.9) will state that any distributive lattice is isomorphic to a topology.

We define other classes of semilattices (or of lattices) by imposing additional conditions that were previously already considered for ordered sets. In some cases the obtained definitions can be simplified: for example, since a meet-semilattice has a minimum, it is ranked if and only if $x \prec y$ implies $h(y) = h(x) + 1$, or if and only if, for all x and y with $x < y$, all maximal chains from x to y have the same length.

The expression of semimodularity is also simplified in the lattice case: a meet-semilattice is lower semimodular if and only if, for all x,y such that $x \vee y$ exists, $x \prec x \vee y$ and $y \prec x \vee y$ imply $x \wedge y \prec x$ and $x \wedge y \prec y$. Likewise a lattice is modular (that is, lower and upper semimodular) if and only if ($x \prec x \vee y$ and $y \prec x \vee y$) is equivalent to ($x \wedge y \prec x$ and $x \wedge y \prec y$).

Several characterizations of semimodular or of modular semilattices and lattices are given in Exercises 2.9 and 2.10. In particular, a modular lattice L can be defined by the fact that, for any triple (x,y,z) of elements of L, $x \leq z$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$. From this it follows directly that any distributive lattice is modular. Another characterization of modular lattices (given in Exercise 2.9) is obtained by excluding the lattice N_5 represented in Figure 2.8. If we add the condition of "exclusion" of the lattice M_3 represented in the same figure, we obtain a classic characterization of distributive lattices (but more difficult to prove, see for instance Barbut and Monjardet (1970)). We can give a typical form to these characterizations by forbidden substructures, noticing (and the reader can check that fact) that M_3 is the smallest modular non-distributive lattice (with respect to the number of elements) and that, likewise, N_5 is the smallest non-modular lattice. As for the lattice in Figure 2.8(c), it is the smallest lower semimodular non-modular lattice. Observe that it contains a sublattice isomorphic to N_5, so a non-lower semimodular one, which proves that this property is not always preserved in a sublattice.



(a) M_3 (b) N_5 (c)

Figure 2.8 Three particular lattices.

2.4 Linearly ordered sets and tournaments

In this section, we focus on linearly ordered sets and we study their relations with the so-called *tournament* binary relations.

We recall that, for a relation R defined on a set X, an element $x \in X$ is maximal with respect to R if there does not exist any $y \in X$ distinct from x and such that xRy. We also recall that a *path* from x_1 to x_p in R is a sequence $(x_1,...,x_p)$ of distinct elements (except possibly for x_1 and x_p) of X such that $x_1Rx_2,...,x_{p-1}Rx_p$, and that a *cycle* of R is a path $(x_1,...,x_p)$ of R with $x_1 = x_p$. A path (respectively, a cycle) of R is called *hamiltonian* if any element of X appears exactly once in this path (respectively, except for its origin and its extremity which are equal). The lemma below is easily obtained by using antisymmetry and transitivity.

Lemma 2.22 *If O is an order on a set X, O is cycle-free.*

The following theorem is important and is moreover used in the proof of Theorem 2.27, which characterizes linear orders among tournaments.

Theorem 2.23 *Let R be a binary relation on a set X. The following properties are equivalent:*

1. *R is cycle-free.*
2. *The reflexo-transitive closure of R is an order.*
3. *R is included in a linear order.*
4. *For any subset Y of X, the set of maximal elements of the restriction $R_{|Y}$ of R to Y is non-empty.*

Proof (1) $\Longrightarrow$ (2): we prove that, if the reflexo-transitive closure $\pi(R)$ of R is not an order, then R has a cycle. If $\pi(R)$ is not an order, it is not antisymmetric. So there exists an ordered pair (x,y) of distinct elements of X satisfying both $x(\pi(R))y$ and $y(\pi(R))x$. Then we have:

(a) either xRy and yRx,
(b) or xRy and $y(\pi(R) \setminus R)x$,
(c) or yRx and $x(\pi(R) \setminus R)y$,
(d) or, at last, $x(\pi(R) \setminus R)y$ and $y(\pi(R) \setminus R)x$.

In Case (a), the sequence (x,y,x) is a cycle of R. In Case (b), $y(\pi(R) \setminus R)x$ implies the existence of a path $(y,t_1,...,t_p,x)$ in R which, with the ordered pair (x,y), forms a cycle in R. Case (c) is symmetric to Case (b). Finally, Case (d) implies the existence of two paths $(x,t_1,...,t_p,y)$ and $(y,t'_1,...,t'_q,x)$ of R, the concatenation of which contains a cycle in R.

(2) $\Longrightarrow$ (3): denote by $O = \pi(R)$ the reflexo-transitive closure of R, which is assumed to be an order. If O is a linear order, the implication is immediate. If not, there exist two elements x and y that are incomparable in O. Let $O' = \pi(O + (x,y))$ be the reflexo-transitive closure of $O + (x,y)$. We are going to show that $O + (x,y)$ is

cycle-free which, according to what has been said, allows us to prove that O' is an order including O. The result will then be obtained by iteration. If there exists a cycle C in $O+(x,y)$, C necessarily contains the ordered pair (x,y) and so, it also contains a path from y to x in O. By transitivity of O, yOx follows, a contradiction with the incomparability of x and y in O.

(3) $\Longrightarrow$ (1): since the property of being cycle-free is preserved in any subrelation of R, this implication is immediate according to Lemma 2.22.

(1) $\Longrightarrow$ (4): according to the argument used in the previous implication, it suffices to prove that, if R is cycle-free, the set of its maximal elements is not empty. Let $x_1 \in X$. If x_1 is not maximal in R, there exists $x_2 \in X$, distinct from x_1 and such that $x_1 R x_2$. Iterating this argument, we find either an element of X, maximal in R, or a previously encountered one, which involves the existence of a cycle (since X is finite), a contradiction with the hypothesis.

(4) $\Longrightarrow$ (1): if R has a cycle $(x_1,...,x_k,x_1)$, the restriction of R to the subset $\{x_1,...,x_k\}$ has no maximal element. □

Remark 2.24 This theorem has many applications. For instance, in computer science, a *topological sorting* on a set X endowed with a cycle-free binary relation R is a (bijective) numbering f of the elements of X such that xRy implies $f(x) < f(y)$. In other words, a topological sorting defines a linear order which includes R. In microeconomics, the theory of "rationalizable" choice functions uses the equivalence between conditions (1) and (4) of Theorem 2.23 (see on this subject Section 2.5 on page 64).

Definition 2.25 A binary relation T on a set X is a *tournament* if it is total (that is, if, for all $x,y \in X$, xT^cy implies yTx) and antisymmetric. We define the *rank (function)* r_T on any tournament T on X by $r_T(x) = |\{y \in X : y \neq x \text{ and } yTx\}|$.

The word "tournament" used for such a relation comes from the fact that it can model the results of a sports tournament, in which every team meets every other team exactly once and where there does not exist any draw (as is often the case in the Six Nations Rugby Tournament). In such a situation, if yTx means that Team y was beaten by Team x, one notices that the rank $r_T(x)$ of x turns out to be the number of teams that were beaten by x (and, in the directed graph $G = (X,T)$ associated with the tournament, $r_T(x)$ is thus equal to the indegree $d_G^-(x)$ of x).

If a tournament T is transitive, it is a linear order and, in Theorem 2.27, we will provide necessary and sufficient conditions for this to happen. The following lemma will be useful for the proof of this theorem.

Lemma 2.26 *The number of 3-cycles (that is, of 3-element cycles) of a tournament T defined on a set X of size n is equal to:*

$$\frac{n(n-1)(2n-1)}{12} - \frac{1}{2}\sum (r_T(x))^2$$

Proof Let us denote $c_3(T)$ the number of 3-cycles of T, that is, the number of triples (x,y,z) satisfying xTy, yTz, and zTx. With $\overline{c_3(T)}$ denoting the number of transitive triples of T (that is, the number of triples (x,y,z) of distinct elements such that yTz, yTx, and zTx), we have:

$$c_3(T) + \overline{c_3(T)} = \binom{n}{3} = \frac{n(n-1)(n-2)}{6}$$

Now, for a given element x, the number of transitive triples with yTx and zTx is equal to:

$$\frac{(r_T(x))(r_T(x)-1)}{2} \text{ (why?)}$$

so we have:

$$\overline{c_3(T)} = \sum_{x\in X} \frac{[(r_T(x))(r_T(x)-1)]}{2} = \frac{1}{2}\sum_{x\in X}(r_T(x))^2 - \frac{1}{2}\sum_{x\in X} r_T(x)$$

We have besides: $\sum_{x\in X} r_T(x) = \frac{n(n-1)}{2}$ (why?).
Therefore:

$$\overline{c_3(T)} = \frac{1}{2}\sum_{x\in X}(r_T(x))^2 - \frac{n(n-1)}{4}$$

Finally:

$$\begin{aligned} c_3(T) &= \frac{n(n-1)(n-2)}{6} + \frac{n(n-1)}{4} - \frac{1}{2}\sum_{x\in X}(r_T(x))^2 \\ &= \frac{n(n-1)(2n-1)}{12} - \frac{1}{2}\sum_{x\in X}(r_T(x))^2 \end{aligned}$$

□

Figure 2.9 shows a non-transitive tournament on four elements, on which the reader can check the equality given in Lemma 2.26. We will find again this tournament in Figure 7.3 (Section 7.2, Chapter 7), where it is obtained by application of the Condorcet majority rule.

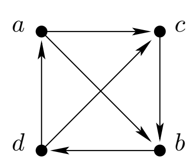


Figure 2.9 A non-transitive tournament on four elements.

Theorem 2.27 *Let T be a tournament on a set X of size n. The following properties are equivalent:*

1. *T is a linear order on X.*
2. *T is 3-cycle-free.*
3. *T is cycle-free.*
4. *T has a unique hamiltonian path.*
5. *The ranks of the n elements of X are the integers from 0 to $n-1$.*

Proof (1) $\Longrightarrow$ (2): immediate according to Lemma 2.22.

(2) $\Longrightarrow$ (1): obvious since xTy and yTz involve zT^cx and so xTz.

(2) $\Longrightarrow$ (3): suppose that T has a cycle but no 3-cycle. By definition, T is antisymmetric, so the length of this cycle is at least 4. Let $C = (x_1,x_2,...,x_p,x_1)$ be a cycle of T with minimum length (so $p \geq 4$). Since T is a linear order then – for instance for x_1 and x_3 – either x_3Tx_1, or x_1Tx_3. If x_3Tx_1, (x_1,x_2,x_3,x_1) is a 3-cycle of T; a contradiction. Otherwise, $C' = (x_1,x_3,x_4,...,x_p,x_1)$ is a cycle of T with length $p-1$; a contradiction with the minimality of the length p.

(3) $\Longrightarrow$ (1): if the tournament T was not a linear order, it would necessarily have a non-transitive triple (x,y,z), so such that xTy, yTz, and xT^cz. Since T is total then zTx and T would have a cycle.

(3) $\Longrightarrow$ (4): by induction on n. Condition (3) holds for $n = 1,2$. Assume now that the condition holds for any tournament of size $n-1$. Now let $n \geq 3$ and consider a tournament T with n elements and which is moreover cycle-free. According to Theorem 2.23, T has at least one maximal element. Moreover, it can have only one such element since T is a linear order (by (3) $\Longrightarrow$ (1)). If we denote this element x_n, we have yTx_n for every $y \in T$. Now, $T \setminus x_n$ is a cycle-free tournament of size $n-1$ so, by the induction hypothesis, it has a unique hamiltonian path, which we denote $(x_1,...,x_{n-1})$. The sequence $(x_1,x_2,...,x_n)$ is thus a hamiltonian path of T. Its uniqueness comes from the fact that any hamiltonian path of T necessarily ends with x_n and that, if there were more than one hamiltonian path in T, there would be as many in $T \setminus x_n$.

(4) $\Longrightarrow$ (5): by induction on n. The implication holds for $n = 1,2$. Assume that the condition is satisfied for any tournament of size $n-1$ and now consider a tournament T with $n \geq 3$ elements and with a unique hamiltonian path $(x_1,x_2,...,x_n)$. It is enough to prove that x_jTx_n holds for any $j \leq n$ and then to apply the induction hypothesis on $T \setminus x_n$. Indeed, in this case, the ranks of the elements $x_1,...,x_{n-1}$ are the integers from 0 to $n-2$ and, since x_jTx_n holds for any $j \leq n$, the rank of x_n in T is $n-1$. Now x_1Tx_n (if not, $(x_1,x_2,...,x_n,x_1)$ would be a hamiltonian cycle of T, which would then possess more than one hamiltonian path). Likewise, x_2Tx_n (if not, $(x_1,x_n,x_2,...,x_{n-1})$ would be a second hamiltonian path of T). This argument applied iteratively allows us to deduce that T satisfies x_jTx_n for any $j \leq n$.

(5) $\Longrightarrow$ (2): if the ranks for T of the n elements of X are the integers from 0 to $n-1$, then Lemma 2.26 implies

$$c_3(T) = \frac{n(n-1)(2n-1)}{12} - \frac{1}{2}\sum_{i=0}^{n-1} i^2 = 0$$

□

Remark 2.28 In Section 1.1.2 of Chapter 1 we introduced the notation $x_1x_2...x_n$ to represent a linear order on a set of size n in which, for any $i < n$, x_i is covered by x_{i+1}. This is equivalent to writing the hamiltonian path of the linear order. On the other hand, this is also equivalent to writing the elements of X in the increasing order of their ranks, by taking as the index of an element its normalized rank plus one unit (the normalized rank of x_1 is 0), which is in general more natural and practical.

We end this section with a significant theorem concerning linear extensions (see Definition 1.31), since its first part (already stated in Chapter 1) leads to the concept of the dimension of an ordered set (see Definition 1.33 and Chapter 6). We prove this theorem, using some parts of the proof of Theorem 2.23.

Theorem 2.29 1. *Every order has a linear extension and is the intersection of all its linear extensions.*
2. *If a total preorder includes an order, it also includes a linear extension of this order.*

Proof (1) The first assertion of this item is obtained thanks to the implication of (3) by (2) in Theorem 2.23. The proof of the latter implies moreover that whenever x and y are incomparable in an order O, there exists a linear extension of O containing the ordered pair (x,y) and a linear extension containing the ordered pair (y,x). We immediately infer that O is the intersection of all its linear extensions.

(2) Let R be a total preorder including an order O and let x,y be two elements incomparable in O. Since R is total, we have either xRy or yRx. If, for instance, xRy, we then consider the reflexo-transitive closure $\pi(O+(x,y))$ of $O+(x,y)$, which is an order (still according to the proof of the implication of (3) by (2) in Theorem 2.23). It is thus enough to show that this order is still in R, then to iterate the argument. Now, (z,t) belongs to $\pi(O+(x,y)) \setminus (O+(x,y))$ if and only if we have zOx and yOt. But then we also have zRx, xRy, and yRt and, by transitivity of R, zRt as required. □

2.5 Further topics and references

The classes of ordered sets presented in this chapter have often appeared in different contexts and with different names, a consequence of their many equivalent definitions. This is, for instance, the case for series–parallel orders (see page 50) and for the subclass of these orders formed by threshold orders. A threshold order is defined by the following property: there exists $n = |P|$ real numbers $p_1, p_2, ..., p_n$ and a real number s such that A is a maximal antichain of P if and only if $\Sigma_{i \in A} p_i \leq s$. Threshold orders are particular interval orders and their comparability graphs are the so-called "threshold graphs," which were studied intensively in graph theory (see Mahadev and Peled, 1995). Series–parallel orders and in particular threshold orders were moreover used in the study of electric circuits, in scheduling problems, or for modeling in parallelism (see, for instance, Lawler (1978), Faigle *et al.* (1986) or Möhring (1989)). Likewise, weak orders, semiorders or interval orders have appeared in various contexts and in different forms. Their use in the modeling of preference relations will be described in Section 7.1 of Chapter 7, but the reader can find in Section 7.6.1 other situations in which these orders are encountered. Another interesting case is that of covering N-free orders. Several equivalent definitions are given in Exercise 2.3, among which the one "every maximal chain meets every maximal antichain" had led to their initial name "chain–antichain-complete" given by Grillet (1969). Yet, they are also obtained by a construction generalizing that of series–parallel orders, hence the name "quasi-series–parallel orders" (Habib and Möhring, 1987). These orders are useful in task analysis techniques by means of PERT networks (see, for instance, Sysło (1984) or Radermacher (1986)), as well as in problems of planar representations of graphs (de Fraysseix and de Mendez, 1997).

Beyond the classes of ordered sets presented in this chapter, many others exist that are interesting and we are going to evoke some of them before going back to lattices.

For instance, concern over getting an expressive geometrical representation of the diagram of an order has in particular led to defining the classes of planar and of dismantable ordered sets (as well as other classes; refer to Rival (1989b)). A diagram of an ordered set is planar if its lines cross only at the points of the plane that represent the elements of the set. An ordered set is called *planar* if it has a planar diagram. This is the case for Example 1.2 in Chapter 1, two diagrams of which are represented in Figure 1.4, one planar and the other not. The ordered set P in Figure 2.10 is planar (why?), but is no longer planar when adding a greatest element.

The complete bipartite ordered set $K_{3,3}$ is not planar but becomes so if one removes any ordered pair (find a planar diagram for this ordered set). The first observation made (by Birkhoff (1940)) on these ordered sets is that a planar ordered set with a minimum and a maximum is necessarily a lattice. Of course, not every lattice is planar and those that are have been characterized as free of some particular sublattices (Kelly and Rival, 1975). There is a similar characterization for planar modular lattices (Kelly, 1980) and many characterizations for planar distributive lattices (Monjardet, 1976). Planar lattices form a subclass of *dismantable* lattices. The latter can be defined by the fact that one can go from such a lattice to the 1-element lattice by deleting step by step a doubly irreducible element from the lattice previously obtained. One can also characterize them as CR_n-free (where CR_n is the crown of size n) for $n \geq 2$ (Kelly and

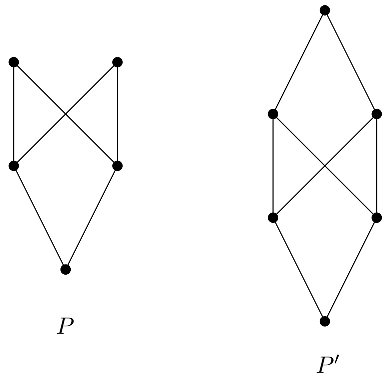


Figure 2.10 P is planar, contrary to P'.

Rival, 1974). A dismantable ordered set is defined just like a dismantable lattice by replacing the doubly irreducible elements with the elements covered by, or covering, a unique element. The CR_n-free ordered sets are dismantable but the converse is not true (Duffus and Rival, 1976).

We have already mentioned in Example 1.12 that any family of subsets of a set, ordered by set inclusion, is an ordered set that can be arbitrary. Indeed, the one-to-one correspondence associating with any element of an ordered set its principal downset shows that any ordered set is representable by (that is, isomorphic to) the family of these downsets. Different motivations, and in particular the expansion of algorithmic geometry, have motivated the study of the ordered sets representable by different families of geometric sets: real intervals or, more generally, n-dimensional boxes, (regular) convex polygons of the plane, discs or angles of the plane, spheres of $\mathbb{R}^n$, etc. All orders are representable by convex polygons and those which are representable by n-dimensional intervals are orders of dimension $2n$ (see Definition 1.33). As for the other families, they lead to new classes of orders such as "circular," angular" or "spherical" orders. The reader will find a panorama of these "geometric" orders in Urrutia (1989) and in Fishburn and Trotter (1999a). Still in the field of algorithmic geometry, one has also become interested in "directional" orders generated by an "obstruction" relation between plane objects (see the presentation by Rival (1989) and, for instance, Bouchitté *et al.* (1993)). Finally, with the recent development of ordered set algorithmics, attention has turned to those for which one knows "efficient" algorithms (see Appendix A) of recognition and/or of construction of some parameters, which is the case for all the ordered sets defined by forbidden configurations in Section 2.2. This concern has led us to consider, among others, the following order classes: *orders of dimension 2* (see Section 6.3), *orders with bounded width* (that is, such that $\alpha(P) \leq k$, for some k), *Dilworth orders* (that is, for which the number $s(P)$ of jumps is equal to $\alpha(P) - 1$), *covering W-free orders* (that is, the covering relation of which does not include the order on 5 elements, the diagram of which can be represented by the letter W), *alternating-cycle-free orders* (an example of such a cycle is given by CR_4 in Figure 2.3), *greedy orders* (see Section 6.5). All these classes of ordered sets are presented in detail in Möhring (1989) and Bouchitté and Habib (1989). Let us also notice that there exist enumeration results for some of them (see in particular El Zahar (1989)).

The third section of this chapter was devoted to semilattices and lattices. As a matter of fact, the study of ordered sets has for a long time been almost reduced to that of lattices, that is, of those ordered sets that have the advantage of being also “strong" algebraic structures. The notion of a lattice defined by Schröder and Dedekind in the late nineteenth century, then almost forgotten, re-emerged in the 1930s thanks to many mathematicians, most particularly Birkhoff, Öre, and Klein. In fact, it was noticed that lattices were present in many – new or old – mathematical fields (topology, general algebra, measure theory, geometry, combinatorics, etc.) as well as in mathematical logics or in theoretical physics (quantum and wave mechanics). Since then there have been considerable developments in lattice theory, which still continue – all the more so since it got extended within the “universal algebra” theory. The basic references on lattice theory are the books by Birkhoff (1940) and Grätzer (1998), whereas those by Szasz (1971) and by Davey and Priestley (2001) are excellent introductions. Even though the present book is not devoted to lattices, let us notice that they appear many times. Indeed, there exist many relations between ordered sets and lattices. First of all there is the duality between ordered sets and distributive lattices that will be studied in Chapter 5. Then any ordered set may be embedded into an associated lattice, called its “MacNeille” completion (see Section 3.5.3). Conversely, any lattice may be represented by the bipartite ordered set of its irreducible elements (see the table of a lattice in Section 3.5.2). This duality and these relations make possible the translation of ordered set properties into lattice properties, and conversely. On the other hand, a natural idea consists in defining some classes of ordered sets by generalizing some lattice theoretic notions. A standard means to do so is to say that an ordered set has property (P) if its MacNeille completion has property (P). In this way one defines, for example, the significant class of *distributive* (also called *dissective*) ordered sets (see Exercises 3.18 and 5.15 in Chapters 3 and 5 respectively and, for instance, Niederle (1995) or Reading (2002)).

This chapter ends with the characterization of linear orders among tournaments and provides the formula that gives the number of 3-cycles of a tournament (Lemma 2.26). This formula was obtained by Kendall and Babington Smith (1940), about the *pairwise comparison* method used in psycho-sociology. A subject is asked to express his preferences between different options (for instance, for a child, between the jobs he would like to practice) by choosing, for each possible pair of different options, the one he prefers. If the choice is “forced” (neither *ex-aequo* nor empty choice), the result of all pairwise comparisons is a tournament on the set of all options. One observes that, in a not insignificant number of cases, the obtained tournament T is not a linear order; that is, it includes some 3-cycles. In order to evaluate the intransitivity degree of T, Kendall and Babington Smith then proposed the following intransitivity index: the ratio $c_3(T)/Maxc_3$, where $c_3(T)$ is the number of 3-cycles of the tournament T and $Maxc_3$ the maximum number of 3-cycles on all the tournaments with the same size as T. Since simple formulas giving the value of this maximum are known, this index is easily computable. Nevertheless, it can achieve a relatively high value for a tournament in which it suffices to reverse one arc to obtain a linear order. Hence the proposition made by Slater (1961) of another intransitivity index: the ratio (minimum number of arcs to reverse in the tournament T to make it transitive)/(maximum number of this minimum on all the tournaments having the same size as T. Yet, these numbers are sometimes difficult to compute, just as the linear orders called "Slater orders" obtained by reversing a minimum number of arcs in the tournament (see on this subject Bermond (1972), Charon *et al.* (1992), and Charon and Hudry (2010)).

In microeconomics, the choice made by a consumer on a set X of commodity bundles is described by his choice function. The latter associates with any subset Y of X the choice $c(Y)$ with $\emptyset \subset c(Y) \subseteq Y$. In the classic theory of choice functions, the choice of the consumer is called *rationalizable* if the consumer has a preference relation R on X that "explains" his choice in the sense that, for any subset Y of X, the choice $c(Y)$ is the set of maximal elements of the relation R restricted to Y. Theorem 2.23 thus proves that the preference relation R of the consumer must be cycle-free. The choice functions rationalizable by a (cycle-free) relation or by various types of order relations may be axiomatically characterized (see for instance Aleskerov *et al.* (2007), where the case of a possible empty choice is also considered).

2.6 Exercises

Exercise 2.1 Prove that, if an ordered set P has at most 4 elements, it is ranked. Show that there exist two types of non-ranked ordered sets of size 5. Observe moreover that they are semiordered sets.

Exercise 2.2 [Rank, sums, and products] Prove that, if the ordered sets P and Q are ranked, so are their disjoint union $P+Q$ and their direct product $P \times Q$. What about their lexicographic product $P \otimes Q$ and their linear sum $P \oplus Q$ (see the definitions given in Section 1.5 of Chapter 1)?

Exercise 2.3 [Covering N-free ordered sets] Show that a covering N-free ordered set can be defined by each one of the following equivalent conditions:

1. P does not include any ordered subset $\{a,b,c,d\}$ with $a < b$, $c \prec b$, $c < d$ and $a \parallel d$.
2. Every maximal chain of P meets every maximal antichain of P.
3. $P^-x \cap P^-y \neq \emptyset$ implies $P^-x = P^-y$ (where $P^-x = \{z \in X : z \prec x\}$).
4. $xP^+ \cap yP^+ \neq \emptyset$ implies $xP^+ = yP^+$ (where $xP^+ = \{z \in X : x \prec z\}$).

Exercise 2.4 [Ranked covering N-free ordered sets; Leclerc and Monjardet (1995)] Show that an ordered set P is covering N-free and ranked if and only if any ordered subset of P defined by two consecutive rank-sets is a disjoint union of complete bipartite ordered sets.

Prove that the class of ranked, connected, and covering N-free ordered sets includes both classes of weakly ordered sets and of tree-ordered sets. Characterize the ranked covering N-free lattices. When are they distributive? Provide examples of non-modular covering N-free lattices.

Exercise 2.5 [Tree-ordered sets] Let P be an ordered set with a minimum, denoted by 0. Prove that the following conditions are equivalent:

1. P is a tree-ordered set (see Definition 2.12).
2. Every element distinct from 0 is join-irreducible.
3. For every x of P, the interval $[0,x]$ is a chain.
4. For any upper bounded subset $\{x,y\}$ of P, x and y are comparable.
5. For all comparable elements x,y of P, there exists a unique chain between these two elements.

Prove that such an ordered set is a meet-semilattice.

Show that a meet-semilattice P is a tree-ordered set if and only if, for all elements x,y,z of P, $|\{x \wedge y, y \wedge z, z \wedge x\}| < 3$.

Prove that the left factor and the right factor orders on words, defined in Exercise 1.3, are tree meet-semilattices.

Exercise 2.6 [Semilattice algebra] Let X be a set on which is defined an associative, commutative, and idempotent operation $\perp$. Prove that, if we set $x \leq y \iff x \perp y = x$ (respectively, $x \leq y \iff x \perp y = y$) then $(X, \leq)$ is a meet-semilattice with $x \wedge y = x \perp y$ (respectively, a join-semilattice with $x \vee y = x \perp y$).

Exercise 2.7 [Lattice algebra] Prove that in a lattice L, $x \vee (x \wedge y) = x = x \wedge (x \vee y)$ always holds (*absorption laws*).

Show that, if a set X is endowed with two associative, commutative, and idempotent operations $\perp$ and $\top$ which are moreover such that, for all x,y, $x \perp (x \top y) = x = x \top (x \perp y)$, then it can be endowed with a lattice order by writing $x \perp y = x \wedge y$ and $x \top y = x \vee y$.

Exercise 2.8 [Lower directed ordered sets] An ordered set P is *lower directed* if, for all $x,y \in P$, the set of all common lower bounds of x and y is not empty. Show that this condition is equivalent to the existence of a minimum in P.

Prove that P is a meet-semilattice if and only if it is lower directed and has no ordered subset of the type $A_2 \oplus A_2$ (that is, isomorphic to the crown CR_2).

Exercise 2.9 [Semimodular semilattices; Birkhoff (1940)] Let L be a ranked meet-semilattice such that, for all x,y,z with $z \geq x,y$, $r(x)+r(y) \leq r(x \wedge y)+r(z)$. Prove that L is lower semimodular.

Conversely, show that, if L is lower semimodular (and thus ranked, see Theorem 2.10), its rank satisfies the above inequality, which is equivalent to saying that the height of the interval $[x \wedge y, y]$ is less than or equal to that of the interval $[y,z]$. *Hint*: consider the image of a maximal chain $y \prec t_1 \prec ... \prec z$ by the map $t_i \rightarrow t_i \wedge x$.

Write the dual result characterizing upper semimodular join-semilattices.

Prove that a lattice is lower (respectively, upper) semimodular if and only if it is ranked, with the rank satisfying, for all x,y, $r(x)+r(y) \leq r(x \wedge y)+r(x \vee y)$ (respectively, $r(x)+r(y) \geq r(x \wedge y)+r(x \vee y)$).

Deduce that a lattice is lower (respectively, upper) semimodular if and only if, for all x,y such that $x \prec x \vee y$, $x \wedge y \prec x$ (respectively, for all x,y such that $x \wedge y \prec x$, $x \prec x \vee y$) holds.

Exercise 2.10 [Modular lattices; Birkhoff (1940)] Show that, if x,y,z are three elements of a lattice with $x \leq z$, then $x \vee (y \wedge z) \leq (x \vee y) \wedge z$. Prove that, for any lattice L, the following are equivalent:

1. L is modular.
2. L is ranked and, for all $x,y \in L$, $r(x)+r(y) = r(x \wedge y)+r(x \vee y)$.
3. For all x,y,z of L such that $x \leq y$, $x \vee (y \wedge z) = (x \vee y) \wedge z$ holds.
4. For all $x,y \in L$, the maps $L \longmapsto x \vee t$ and $s \longmapsto s \wedge y$ define two inverse isomorphisms between $[x \wedge y, y]$ and $[x, x \vee y]$.
5. L has no sublattice of the type N_5.

Indication: prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1) and use the results in Exercise 2.9.

Show that a distributive lattice is modular.

3

Morphisms of ordered sets

Let P be an ordered set modeling, for instance, a scheduling problem (see Section 7.5 in Chapter 7). The determination of some characteristics of this ordered set, for example its linear extensions, requires the implementation of an algorithm where P is represented by means of an appropriate data structure. In particular, the elements of P may be suitably represented by sequences of symbols 0 and 1 of fixed length r. A condition for this to hold is that, if $c(x)$ and $c(y)$ are the r-sequences representing two elements x and y of P, then $c(x) < c(y)$ if and only if $x < y$, where the former is the order of the direct product $\underline{2}^r$. In particular, the map c from P to this direct product must be order-preserving. This is an example among many others where order-preserving or reversing maps between two ordered sets are needed. This chapter is devoted to the study of such maps, called *morphisms*.[1] We define several fundamental types of morphisms, such as *codings* (or *embeddings*, or *mergings*), *closure* and *dual closure operators*, *residuated*, *residual*, and *Galois maps*. We are concerned with relations between these various types of maps, canonical examples, and natural developments.

Several types of morphisms between ordered sets are defined in Section 3.1, namely *isotone* (or *strictly isotone*) maps, *antitone* maps, and *codings*, which make a copy of their domain appear in their range set. Such maps will frequently appear throughout this book. For instance, Chapter 6 is devoted to codings from ordered sets to direct products of chains, leading to the important notions of order *dimension*.

A canonical coding from any ordered set P to a Boolean lattice is provided by the join-irreducible representation of the elements of P presented in Section 3.2. It is particularly efficient in many situations where P is a set of objects to study and where the join-irreducible elements are "elementary" such objects. Then, we get an efficient tool for reducing any element of P to a set of elementary objects. In the same section, various properties and characterizations concerning irreducible elements are obtained, the use of the so-called *arrow relations* allowing us to obtain particularly simple proofs.

Section 3.3 is devoted to *closure* (and *dual closure*) *operators*. Many mathematical theories and models involve specific closures, namely logic, inductive, topological,

[1] For some authors, the term morphism must be restricted to isotone maps.

algebraic, convex, Kleene, transitive closures, etc. Here, the study of closures in an arbitrary ordered set situates them in their most general framework, emphasizing the properties common to all the above-mentioned special cases. Nevertheless, the particular case of closures in the lattice of subsets of a set is the most frequently encountered one. Such closures are associated with *Moore families* (also called *closure systems*), which are their lattices of *closed sets*. We will go back to these topics in Section 7.4, in relation to implication relations and the association rules that have an important role in fields such as artificial intelligence or databases.

Residuated maps may be defined in several ways, for instance by strengthening a characteristic property of isotone maps. The existence of a residuated (or residual) map between two ordered sets P and Q implies the existence of an ordered subset "common" to P and Q (in fact, of two isomorphic ordered subsets). This common structure is obtained as the image of a closure on P as well as of a dual closure on Q. In many contexts, especially in knowledge representation or data analysis, an order duality gives these maps the equivalent (and fundamental) form of a *Galois connection*. Various aspects of residuated, residual, and Galois maps are presented in Section 3.4. A direct and somewhat unexpected use of residual maps in classification will be considered in Section 7.3.

The Galois connections associated with binary relations and the related *Galois* (or *concept*) *lattices* are particularly popular. This comes from the fact that these lattices appear in the extremely diverse fields where objects are described by binary attributes. They turned out to be particularly rich of applications. Section 3.5, which is divided into three subsections, is devoted to this essential type of correspondence. There, we make use of notions coming from each of the previous sections. Section 3.5.1 presents the Galois lattice of a binary relation. It is shown in Section 3.5.2 that every lattice L is obtained, up to isomorphism, as the Galois lattice of a relation between its join- and its meet-irreducibles, called the *lattice table of* L; this table may be completed with the *arrow relations* into an *arrowed table*. Finally, Section 3.5.3 describes the procedure of the *completion* of an ordered set into a lattice, which is precisely a canonical coding from any ordered set into a lattice.

3.1 Isotone and antitone maps: exponentiation

We first consider several types of map between ordered sets, preserving (more or less) their order structure.

Definition 3.1 Let $P = (X, \leq_P)$ and $Q = (Y, \leq_Q)$ be two ordered sets and f a map from P to Q. The map f is called:

1. *isotone* (or *increasing*) if it satisfies: for all $x, x' \in P$, $x \leq_P x' \Longrightarrow f(x) \leq_Q f(x')$;
2. *strictly isotone* if it satisfies: for all $x, x' \in P$, $x <_P x' \Longrightarrow f(x) <_Q f(x')$;
3. a *coding* (or an *embedding*) *from P to Q* if it satisfies: for all $x, x' \in P$, $x \leq_P x' \Longleftrightarrow f(x) \leq_Q f(x')$.

Property (3) above implies Property (2) which, in turn, implies Property (1). On the other hand, if f is isotone and injective then it is strictly isotone (see also Exercise 3.1). If f is a bijective coding from P to Q, we find again the notion of an *order isomorphism* defined in Chapter 1 (Definition 1.9). In fact, a coding f is an isomorphism between P and the ordered subset of Q induced by the image $f(P)$.

The case of a "Boolean" coding – that is, a coding from P to the lattice of subsets of a set – has been illustrated with a type hierarchy in Example 1.18. More generally, the codings from an ordered set to a direct product of chains will be studied in Chapter 6.

Considering the previous conditions for maps from P to the dual Q^d of Q, we obtain dual definitions. For instance, a map f from P to Q is *antitone* (or *decreasing*) if it is isotone from P to Q^d: that is, such that $x \leq_P x'$ implies $f(x) \geq_Q f(x')$. A map f from P to Q is called *monotone* if it is isotone or antitone. The dual isomorphisms introduced in Chapter 1 (Definition 1.10) are particular antitone maps.

In the sequel, we generally omit the subscripts refering to a particular ordered set and simply write, for instance, $\leq$ instead of $\leq_P$.

Example 3.2 Let $P = (X, \leq)$ be an ordered set. For any subset Y of X, we denote *Upper*Y the set of upper bounds of Y in P (as in Section 1.4.1). Clearly, the set *Upper*X of upper bounds of X reduces to the greatest element of P when it exists and is empty otherwise. On the other hand, observe that *Upper*$\emptyset = X$ since, otherwise, there would exist $x \in P$ and $y \in \emptyset$ with $y \not\leq x$ (of course, such a y cannot exist!). Then, with this remark, $Y \subseteq Y'$ implies *Upper*$Y' \subseteq$ *Upper*Y, for all $Y, Y' \subseteq X$, and so *Upper* is an antitone map on $\underline{2}^X$. A second antitone map *Lower* on $\underline{2}^X$ maps any subset Y of X to the set *Lower*Y of lower bounds of Y in P, with similar properties. It follows from these considerations that the join $\bigvee \emptyset$ exists in P if and only if P has a minimum 0_P, which is then equal to $\bigvee \emptyset$. Similarly, $\bigwedge \emptyset$ exists in P if and only if P has a maximum 1_P, which then satisfies $\bigwedge \emptyset = 1_P$. In Section 3.5.3, the maps *Upper* and *Lower* will be used to define the lattice completion of an ordered set P.

The reader can check that the types of map whose definitions follow are isotone. Yet, an isotone map between two lattices is not always a lattice – or even a semilattice – morphism (find an example).

Definition 3.3 Let P and Q be two ordered sets and f a map from P to Q.

1. If P and Q are meet-semilattices, then f is a *meet-morphism* if it is meet-preserving, i.e., if, for all $x, x' \in P$, $f(x \wedge x') = f(x) \wedge f(x')$.
   If, moreover, f is injective, it is a *meet-coding* from P to Q.
2. If P and Q are join-semilattices, then f is a *join-morphism* if it is join-preserving, i.e., if, for all $x, x' \in P$, $f(x \vee x') = f(x) \vee f(x')$.
   If, moreover, f is injective, it is a *join-coding* from P to Q.
3. If P and Q are lattices, then f is a *lattice morphism* if it is both a meet- and a join-morphism.
   If, moreover, f is injective, it is a *lattice coding* from P to Q and, if f is bijective, it is a *lattice isomorphism*.

Recall that the pointwise order on the set of maps from an ordered set P to an ordered set Q is given by:

$$f \leq g \iff \forall x \in P, f(x) \leq g(x)$$

Definition 3.4 Let P and Q be two ordered sets. The operation which associates with the pair (P,Q) the set of all isotone maps from P to Q, endowed with the pointwise order $\leq$ on maps, is called *exponentiation*. The latter ordered set is denoted Q^P and its order is called the *exponentiation order*.

Therefore, in this book, we restrict the notation Q^P to the set of all *isotone* maps from an ordered set P to an ordered set Q. Here, we do not follow the common use where this notation stands for the set of all maps from P to Q. Observe that, when P is an antichain (i.e., $x \leq x'$ if and only if $x = x'$), all maps from P to Q are isotone and Q^P is equal to the set of all maps from P to Q.

Exercise 3.4 consists of showing that, if Q is a meet-semilattice (respectively, a join-semilattice, a lattice), so is Q^P.

3.2 Join- and meet-generating sets

The search for "good" codings from ordered sets to some particular ordered sets is a classic problem and a number of related results will be found in this book. Now, in order to illustrate this type of problem with an important example, let us consider some canonical codings from an ordered set P to a Boolean lattice $\underline{2}^E$ (see Example 1.18 in Chapter 1), where E is a finite set to be determined. To do so, we first define the join-generating sets of P and, dually, its meet-generating sets. In the case of a lattice, the more general notion of a *generating set* will appear in Chapter 5 (Definition 5.17).

Definition 3.5 A subset G of an ordered set P is said to be *join-generating* if every element of P is the join of a subset of G, and it is *meet-generating* if every element of P is the meet of a subset of G.

In the sequel, we mainly consider the case of join-generating sets. We first show that a coding of the ordered set P preserving all the existing meets is associated with such a subset (Proposition 3.6). The main result of this section is that any ordered set P has a unique minimal join-generating set which is the set of its join-irreducible elements (Proposition 3.11 and Corollary 3.12). It is proved by using preliminary results on join-irreducibles and the downarrow relation $\downarrow$ (Proposition 3.8 and Lemma 3.10), completed with particular results in the case of a meet-semilattice (Proposition 3.15). The section ends with the statements of dual results about meet-generating sets and meet-irreducible elements.

If G is a join-generating set of P, so is every subset G' of P including G. Every ordered set P has at least one join-generating set which is P itself, since the equality $x = \bigvee(x]$ always holds for every $x \in P$. Given a subset G of P, let us denote G_x the set of lower bounds of x in G.

Proposition 3.6 *Let G be a join-generating set of an ordered set P. For all $x, y \in P$, the following properties hold:*

1. $x = \bigvee G_x$.
2. $x \leq y$ *if and only if* $G_x \subseteq G_y$.
3. *If x and y have a meet $x \wedge y$, then $G_{x \wedge y} = G_x \cap G_y$.*

Proof (1) Since G is a join-generating set, there exists $A \subseteq G_x$ such that $x = \bigvee A$. Every upper bound y of G_x is also an upper bound of A. Then, the fact that x is the join of A – that is, its least upper bound – implies $x \leq y$, and x is also the least upper bound of G_x, that is to say $x = \bigvee G_x$.

(2) $G_x \subseteq G_y$ implies $Upper G_y \subseteq Upper G_x$, which in turn implies $y \in Upper G_x$ and, finally, $x \leq y$. The converse implication is immediate.

(3) is also immediate with, for any $g \in G$, the following equivalences: $g \in G_{x \wedge y}$ if and only if $g \leq x \wedge y$, if and only if ($g \leq x$ and $g \leq y$), if and only if $g \in G_x \cap G_y$. $\square$

Remark 3.7 When the join $x \vee y$ of x and y exists, the inclusion $G_x \cup G_y \subseteq G_{x \vee y}$ holds but, in general, not the equality. Consider the lattice M_3 whose diagram is given in Figure 3.1, and its join-generating set $G = M_3$. Then $G_{z \vee t} = G$, with $G_z = \{0, z\}$ and $G_t = \{0, t\}$.

Given a join-generating set G of an ordered set P, Property (2) in Proposition 3.6 shows that the map $x \longmapsto G_x$ is a coding from P to $\underline{2}^G$. Moreover, according to Property (3) in the same proposition, this coding preserves all the meets existing in P. Especially, in a meet-semilattice, it is a meet-coding from P to $\underline{2}^G$. In order to obtain "economical" such codings, we search for minimal generating sets. First

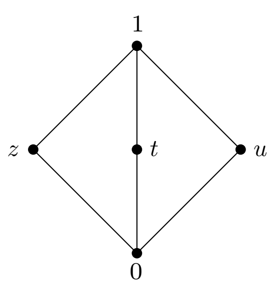


Figure 3.1 The lattice M_3.

observe that any join-generating set of P must include the set J_P of join-irreducibles of P (Chapter 1, Definition 1.41) since a join-irreducible cannot be obtained as the join of other elements. We are going to show that, in fact, J_P is itself a join-generating set (Proposition 3.11), so the unique minimal one. We first give two characterizations of join-irreducibles in an ordered set. One of them is based on the very important arrow relations defined in Chapter 1, Definition 1.36 (recall that $P^{-}x$ denotes the set of elements covered by x in P):

Proposition 3.8 *Let P be an ordered set and $j \in P$. The following three conditions are equivalent:*

1. *j is a join-irreducible element of P.*
2. *j is not the join of $P^{-}j$.*
3. *There exists an element y of P such that $j \downarrow y$.*

Proof (1) $\Longrightarrow$ (2): immediate.

(2) $\Longrightarrow$ (3): if j is a minimal element of P, then $P^{-}j = \emptyset$. If j was the unique minimal element, that is, the minimum of P, the equality $j = \bigvee \emptyset = \bigvee (P^{-}j)$ would hold (see Example 3.2), a contradiction with (2). So there is another minimal element y in P with, clearly, $j \downarrow y$ (i.e., with j minimal among all elements z of P such that $z \not\leq y$). If j covers a unique element y, then it satisfies (2) with $P^{-}j = \{y\}$ and $j \downarrow y$. Finally, if j covers at least two elements and satisfies (2), it is not the unique minimal upper bound of $P^{-}j$. So, there exists such an upper bound y of $P^{-}j$ which is not comparable with j and the property $j \downarrow y$ still holds.

(3) $\Longrightarrow$ (1): if j is minimal then it is not minimum since it is not less than or equal to y. Then, the empty set has no join and $\{j\}$ is the only subset A of P satisfying $j = \bigvee A$ that is, j is join-irreducible. Otherwise, if $j = \bigvee A$ for a non-empty subset A with $j \notin A$, all elements of A are less than j and – since $j \downarrow y$ – smaller than y. Then $j \leq y$, a contradiction. □

Remark 3.9 The above proof provides a classification of the join-irreducible elements of an ordered set P into three types. For $j \in J_P$, one has:

1. either $|P^{-}j| = 0$, where j is minimal in P but is not its minimum;
2. or $|P^{-}j| = 1$, where j covers a unique element, denoted by j^{-};
3. or $|P^{-}j| \geq 2$, where $P^{-}j$ has no join.

The join-irreducibles of the first two types are easy to find in the diagram of P, whereas recognizing those of the third type is less immediate. It requires, for instance, the search for the minimal upper bounds of $P^{-}j$. In the ordered set given in Figure 3.2, a, b, and c are join-irreducibles as minimal elements, g because it covers a unique element, d, e, and f because, for $j \in \{d,e,f\}$, the set of upper bounds of $P^{-}j$ has two minimal elements (for instance, d and k are the minimal upper bounds of $P^{-}d = \{a,b\}$). The elements h, i, and k are not join-irreducible.

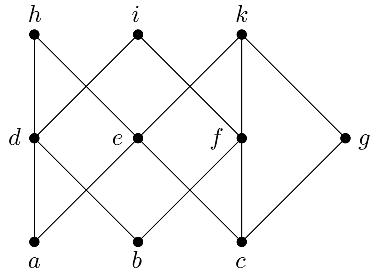


Figure 3.2 An ordered set with 7 join-irreducible elements.

Proposition 3.11 is an essential result, implying that the set J_P of join-irreducibles of any ordered set P forms a join-generating set of P. Lemma 3.10 leads to a very simple proof of this fact.

Lemma 3.10 *Let P be an ordered set. For all $x, y \in P$ such that $x \not\leq y$, there exists $j \in J_P$ such that $j \leq x$ and $j \downarrow y$.*

Proof Assume $x \not\leq y$ and consider the set $A = \{z \in P : z \leq x \text{ and } z \not\leq y\}$. First, $A \neq \emptyset$ since $x \in A$. Now, we prove that a minimal element j of A is also minimal in $\{z \in P : z \not\leq y\}$. Indeed, assume that j is minimal in A and that there exists $t < j$ with $t \not\leq y$. So, since $t < x$, then $t \in A$ with $t < j$; a contradiction. Thus $j \downarrow y$ holds and, by Proposition 3.8, $j \in J_P$. □

Proposition 3.11 *For any element x of an ordered set P, the equality $x = \bigvee J_x$ holds, where $J_x = \{j \in J_P : j \leq x\}$.*

Proof If x is join-irreducible, it is the maximum of J_x and the property is satisfied. Otherwise, it is clear that x is an upper bound of J_x. Assume that there exists another upper bound y of J_x, with $x \not\leq y$. According to the previous lemma, there exists $j \in J_P$ such that $j \leq x$ and $j \downarrow y$. So, $j \in J_x$ and $j \not\leq y$, and y cannot be an upper bound of J_x. Finally, x is the join of J_x, as required. □

The two items in the following corollary are respectively derived from Propositions 3.11 and 3.6.

Corollary 3.12 *Let P be an ordered set.*

1. *A subset G of P is join-generating if and only if it includes the set J_P of join-irreducibles of P.*
2. *If P is a lattice, then the map $x \longmapsto J_x$ from P to the lattice of subsets of J_P is a meet-coding.*

Definition 3.13 When an ordered set P has a minimum 0_P, any element covering this minimum is called an *atom*. An ordered set P with a minimum is said to be *atomistic* if all its join-irreducibles are atoms.

Indeed, an atom a is always a join-irreducible element (with $P^{-}a = \{0_P\}$, see Remark 3.9) and, according to the previous proposition and corollary, an ordered set is atomistic if and only if every element is a join of atoms.

Example 3.14 Let $\mathcal{F} \subseteq 2^E$ be a family of subsets of a set E such that $\{e\} \in \mathcal{F}$, for any $e \in E$. From the equality $F = \bigcup\{\{e\} : e \in F\}$ satisfied by any $F \in \mathcal{F}$, we observe that the 1-element subsets of E are the join-irreducibles of the ordered set $(\mathcal{F}, \subseteq)$, which is atomistic if it contains the empty set. Thus, in the particular case where E is the set of ordered pairs of a set I, and $\mathcal{F}$ the set $\mathcal{O}_I^s$ of strict orders on I, it follows that the ordered set $(\mathcal{O}_I^s, \subseteq)$ is atomistic and that its atoms are the strict orders on I with a unique ordered pair of distinct elements.

Similarly, for the family $\mathcal{F}' = \{F \cup K : F \in \mathcal{F}\}$ where K is a fixed subset of E, the join-irreducibles are the elements of the form $\{e\} \cup K$, with $e \in E \setminus K$. In the case where $\mathcal{F}'$ is the set $\mathcal{O}_I$ of orders on I (the subset K being the diagonal $D = \{(i,i), i \in I\}$), the join-irreducibles are the orders on I containing a unique ordered pair of distinct elements. All of them are atoms in the ordered set $(\mathcal{O}_I, \subseteq)$, which is atomistic.

In a meet-semilattice L, where then every upper bounded subset has a join (Chapter 2, Proposition 2.16), there is a very simple characterization of the join-irreducibles in terms of the covering relation since, in this case, all join-irreducibles belong to the second type described in Remark 3.9. A simple characterization of the downarrow relation $j \downarrow x$ also follows.

Proposition 3.15 *An element j of a meet-semilattice L is join-irreducible if and only if it covers a unique element of L, denoted by j^{-}. Moreover, for any $x \in L$, $j \downarrow x$ holds if and only if $j \wedge x = j^{-}$ holds.*

Proof Let j be a join-irreducible of a meet-semilattice L. The latter has a minimum which, as already observed, is not join-irreducible and so $|L^{-}j| = |\{x \in L : x \prec j\}| \geq 1$. Assume $|L^{-}j| \geq 2$. Since $L^{-}j$ has j as an upper bound, it has a join y (Proposition 2.16), with $y \leq j$. Then, for any $x \in L^{-}j$, the inequalities $x \leq y \leq j$ hold. Since $x \prec j$ (i.e., x is covered by j), then either $x = y$ or $y = j$, but the first equality cannot hold since there exists in $L^{-}j$ an element distinct from x and necessarily less than y. The second equality $j = y = \bigvee L^{-}j$ is in contradiction with the assumption that j is join-irreducible. Finally, $|L^{-}j| = 1$ holds. The converse implication is Item (2) in Remark 3.9.

Let $x \in L$ satisfying $j \downarrow x$. By definition, $j^{-} \leq x$ and, clearly, any common lower bound of j and x is less than or equal to j^{-}, and so $j \wedge x = j^{-}$. Conversely, if $j \wedge x = j^{-}$ then $j \not\leq x$ holds, whereas every lower bound of j^{-} is a lower bound of x. Finally, $j \downarrow x$ as required. $\square$

Applying the duality principle (see page 10) to meet-generating sets (Definition 3.5) and to meet-irreducible elements (Definition 1.41), we obtain the following properties and notions, dual of those presented from Proposition 3.6 to Proposition 3.15 (and where H^x denotes the set of the elements of H greater than or equal to x).

Proposition 3.16 *If H is a meet-generating set of an ordered set P, then the following properties hold for all $x,y \in P$:*

1. $x = \bigwedge H^x$.
2. $x \leq y$ *if and only if* $H^y \subseteq H^x$.
3. *If x and y have a join $x \vee y$, then* $H^{x \vee y} = H^x \cap H^y$.

Let M_P be the set of meet-irreducible elements of P and M^x the set of the elements of M_P greater than or equal to x (recall that xP^+ is the set of the elements covering x in P).

Proposition 3.17 *Let P be an ordered set and $m \in P$. The following three conditions are equivalent:*

1. *m is a meet-irreducible element of P.*
2. *m is not the meet of mP^+.*
3. *There exists an element x of P satisfying $x \uparrow m$.*

Lemma 3.18 *Let P be an ordered set. For all $x,y \in P$ such that $x \not\leq y$, there exists $m \in M_P$ such that $y \leq m$ and $x \uparrow m$.*

Proposition 3.19 *For any element x of an ordered set P, the equality $x = \bigwedge M^x$ holds.*

Then, a subset H of P is meet-generating if and only if it includes the set M_P of meet-irreducibles of P (we leave to the reader the less straightforward dualization of Item (2) in Corollary 3.12). Finally, Remark 3.9 dualizes as follows: an element m of P is meet-irreducible if it is either maximal but not maximum, or covered by a unique element, then denoted by m^+, or if it satisfies $|mP^+| \geq 2$ and is not the only minimal upper bound of mP^+.

Definition 3.20 When an ordered set P has a maximum, any element covered by this maximum is called a *coatom*. An ordered set P with a maximum is said to be *coatomistic* if all its meet-irreducibles are coatoms.

Thus, if P is coatomistic, each of its elements is a meet of coatoms.

Proposition 3.21 *An element m of a join-semilattice L is meet-irreducible if and only if it is covered by a unique element of L, denoted by m^+. Moreover, for any $x \in L$, one has $x \uparrow m$ if and only if $x \vee m = m^+$.*

3.3 Closure and dual closure operators

As mentioned in the introduction to this chapter, closure and dual closure operators defined on an ordered set are found everywhere in (pure or applied) mathematics. These two notions are dual and, here, we mainly study closures. We leave to the reader the dualizations of some of the obtained results. These results concern the equivalence between closure operators and closure nets, and the characterization of the latter (Propositions 3.26 and 3.27). This characterization is particularized to the case of lattices (Proposition 3.28), then to the case of lattices of subsets of a set, where it leads to the important notion of Moore families (Definition 3.29).

Let P, Q, and V be three ordered sets. The composition of two maps $f : P \longmapsto Q$ and $g : Q \longmapsto V$ is denoted by gf. Thus, for any $x \in P$, $gf(x) = g(f(x))$; if $P = Q = V$, f^2 denotes ff, f^3 denotes $f^2 f$, etc.

Definition 3.22 Let P be an ordered set and φ a map *on* P (that is, a map from P to P). The map φ is said to be:

- *Idempotent* if $\varphi^2 = \varphi$, that is, if $\varphi(\varphi(x)) = \varphi(x)$, for any $x \in P$.
- *Extensive* if $id_P \leq \varphi$ (where id_P is the identity map on P), that is, if $x \leq \varphi(x)$, for any $x \in P$.
- *Reductive* if $\varphi \leq id_P$, that is, if $x \geq \varphi(x)$ for any $x \in P$.
- A *retract* if it is isotone and idempotent.
- A *closure* (or a *closure operator*) if it is an extensive retract.
- A *dual closure* (or a *dual closure operator*) if it is a reductive retract.

In the literature, a closure is also sometimes called a *hull* operator and a dual closure is sometimes called an *anticlosure*, an *interior* or a *kernel* operator.

A fixed point of a map φ on P is an element x such that $\varphi(x) = x$. When φ is idempotent, the set of fixed points of φ is equal to the image $\varphi(P)$ of P by φ.

Definition 3.23 Let φ be a map on P.

- If φ is a closure, its fixed points are called the *closed elements* of φ (such an element is also said to be *closed by* φ).
- If φ is a dual closure, its fixed points are called the *open elements* of φ (such an element is also said to be *open by* φ).

Example 3.24 It may be checked that the map φ defined on the ordered set P whose diagram is given in Figure 3.3 is a closure: $\varphi(a) = \varphi(c) = c$, $\varphi(b) = b$, $\varphi(d) = \varphi(f) = f$, and $\varphi(e) = \varphi(1) = 1$.

In the sequel, we obtain some properties of closures (and the dual properties for dual closures). Let us denote Φ_φ (or simply Φ) the set of closed elements of a closure φ defined on $P = (X, \leq)$.

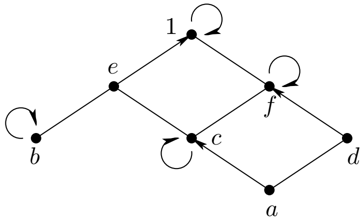


Figure 3.3 A closure on an ordered set P.

Definition 3.25 Let $P = (X, \leq)$ be an ordered set. A *closure net* – or a *Moore subset* – of P is a subset Φ of P such that, for any $x \in P$, the set $\{h \in \Phi : x \leq h\}$ of upper bounds of x in Φ has a minimum, denoted by $\varphi_\Phi(x)$.

So, when we have a closure net a map Φ, we also have a map φ_Φ.

Let us denote $\mathbb{B}$ the set of all closure nets of $P = (X, \leq)$ and $\mathbb{C}$ the set of all closure operators on P. The set $\mathbb{B}$ is endowed with the inclusion order ($\mathbb{B}$ is then an ordered subset of $\underline{2}^{P(X)}$) and – since closures are isotone maps – the set $\mathbb{C}$ is endowed with the restriction of the exponentiation order on P^P (Definition 3.4). According to the next proposition, there exists a dual isomorphism between these two ordered sets. In other terms, the notions of a closure operator and a closure net are equivalent.

Proposition 3.26 *The maps $\varphi \longmapsto \Phi_\varphi$ and $\Phi \longmapsto \varphi_\Phi$ are dual isomorphisms between the ordered set $\mathbb{C}$ of closure operators on P and the ordered set $\mathbb{B}$ of closure nets of P.*

Proof Let φ be a closure on P. For all $x \in P$, $h \in \Phi_\varphi$, $x \leq h$ implies $x \leq \varphi(x) \leq \varphi(h) = h$, so $\varphi(x)$ is the minimum of the subset $\{h \in \Phi_\varphi : x \leq h\}$ of P, and Φ_φ is a closure net of P. Conversely, if Φ is a closure net of P, the map φ_Φ defined above is a closure: it is extensive and idempotent by definition, and also isotone since, for all $x \leq y$, the minimum $\varphi_\Phi(y)$ of $\{h \in \Phi : y \leq h\}$ belongs to the set $\{h \in \Phi : x \leq h\}$. Finally, given a closure net Φ of P, it is easy to check that the set of closed elements of the closure φ_Φ is again Φ, and that the closure associated with the closure net Φ_φ is again φ.

Let φ and φ' be two closures on P. We show that $\varphi \leq \varphi'$ implies $\Phi_{\varphi'} \subseteq \Phi_\varphi$. Let $h' \in \Phi_{\varphi'}$ be a closed element of φ'. By assumption, $\varphi(h') \leq \varphi'(h') = h'$ and also $h' \leq \varphi(h')$ since the closure φ is extensive. So, $h' = \varphi(h')$ and every closed element h' of φ' is a closed element of φ. The inclusion $\Phi_{\varphi'} \subseteq \Phi_\varphi$ follows.

Let Φ and Φ' be two closure nets of P. We show that $\Phi \subseteq \Phi'$ implies $\varphi_{\Phi'} \leq \varphi_\Phi$. For any $x \in P$, $\Phi \subseteq \Phi'$ implies $\{h \in \Phi : x \leq h\} \subseteq \{h' \in \Phi' : x \leq h'\}$ and so $\varphi_{\Phi'}(x) = \min\{h' \in \Phi' : x \leq h'\} \leq \varphi_\Phi(x) = \min\{h \in \Phi : x \leq h\}$. The inequality $\varphi_{\Phi'} \leq \varphi_\Phi$ for the exponentiation order follows. □

It is also easy to check that, for any $x \in P$, $\varphi(x)$ is the maximum of the set $\{x' \in P : \varphi(x') = \varphi(x)\}$. Recall (see Section 1.4.1) that a subset Q of P is said to be *meet-closed* if, as soon as two elements h and h' of Q have a meet $h \wedge h'$, the latter belongs to Q. The following properties hold for any closure net Φ.

Proposition 3.27 *If Φ is a closure net of an ordered set P then Φ is a meet-closed subset of P containing every maximal element of P.*

Proof Let h and h' be two elements of a closure net Φ of P. If $h \wedge h'$ exists in P then, by isotony of the closure φ associated with Φ, we have $\varphi(h \wedge h') \leq \varphi(h) = h$; similarly $\varphi(h \wedge h') \leq h'$, and so $\varphi(h \wedge h') \leq h \wedge h'$ which, by extensivity of φ, leads to $\varphi(h \wedge h') = h \wedge h'$, i.e., $h \wedge h' \in \Phi$. The second part is obvious: $h \leq \varphi(h)$ with h maximal in P, implies $h = \varphi(h)$. □

The reader can investigate the validity of the converse result of this proposition.

The notion of a dual closure net of $P = (X, \leq)$, equivalent to that of a dual closure on P, is dually defined. A *dual closure net of P* is a subset Ψ of P such that, for any $x \in P$, the set of lower bounds of x in Ψ has a maximum; then it is a join-closed subset of P containing every minimal element of P.

Proposition 3.27 implies that a closure net Φ of a lattice L is a sub-meet-semilattice of L and contains the greatest element 1_L of L. Conversely, if Φ is a subset of L satisfying these conditions then, for any $x \in L$, the set $\{h \in \Phi : x \leq h\}$ is non-empty, lower bounded by x and has a minimum. According to Theorem 2.17 in Chapter 2, we then obtain:

Proposition 3.28 *A subset Φ of a lattice L is a closure net if and only if it is meet-closed and contains the maximum of L. Then Φ is a lattice, sub-meet-semilattice of L.*

The meet operation of the lattice Φ is simply the restriction to Φ of the meet operation of L. Yet, the join (in L) $h \vee h'$ of two closed elements h and h' does not in general belong to Φ. With the notation $\triangledown$ for the join operation of the lattice Φ, the equalities $h \triangledown h' = \bigwedge\{x \in \Phi : h \vee h' \leq x\} = \varphi_\Phi(h \vee h')$ hold.

The result of Proposition 3.28 obviously dualizes to dual closures. The dual closure nets on a lattice L are characterized as join-closed subsets of L containing the least element of L and are lattices sub-join-lattices of L.

When L is the lattice $\underline{2}^E$ of subsets of a set E, the previous characterization of closure nets corresponds with the definition of an important class of families of subsets, called Moore families.

Definition 3.29 A *Moore family* (or a *closure system*) on a set E is a subset $\mathcal{F}$ of $\underline{2}^E$ satisfying the following two conditions:

- $E \in \mathcal{F}$.
- $F, F' \in \mathcal{F}$ implies $F \cap F' \in \mathcal{F}$.

When φ is a closure on $\underline{2}^E$, its closed elements are called the *closed sets of φ*. Now the following results hold.

Proposition 3.30 *Let E be a set.*

1. *Let φ be a closure on $\underline{2}^E$. Then the set of its closed sets is a Moore family $\mathcal{F}_\varphi$.*
2. *Let $\mathcal{F}$ be a Moore family on E. The map $\varphi_\mathcal{F}$ defined by $A \longmapsto \varphi_\mathcal{F}(A) = \bigcap\{F \in \mathcal{F} : A \subseteq F\}$ is a closure on $\underline{2}^E$. Ordered by inclusion, $\mathcal{F}$ is a lattice with, for all $A, B \subseteq E$, $A \wedge B = A \cap B$ and $A \vee B = \varphi_\mathcal{F}(A \cup B)$.*
3. *The set $\mathbb{F}$, ordered by inclusion, of Moore families on E and the set $\mathbb{C}$, ordered by exponentiation, of closures on $\underline{2}^E$ are two dual lattices by the maps $\mathcal{F} \longmapsto \varphi_\mathcal{F}$ and $\varphi_\mathcal{F} \longmapsto \mathcal{F}$.*

This proposition results from the previous definitions and propositions, except for the fact that $\mathbb{F}$ and $\mathbb{C}$ are two dual lattices, which is easy to see. The former lattice will be considered in Section 7.4 (starting from Theorem 7.68).

The results given in Proposition 3.30 can obviously be dualized to the case of dual closures on $\underline{2}^E$. The notion of a Moore family is then replaced with that of a union-stable family of subsets containing the empty set, that can be called a *dual closure system*.

In the sequel, when a map ϕ is a closure on $\underline{2}^E$, we will frequently say that ϕ is a closure on E.

Example 3.31 Let L be a lattice and G a join-generating set of L. Consider the family of subsets $\mathcal{G} = \{G_x, x \in L\}$ (with $G_x = G \cap (x]$). According to Property (3) in Proposition 3.6, the family $\mathcal{G}$ is intersection-stable; moreover $G_{1_L} = G$, where 1_L is the maximum of L. So, the family $\mathcal{G}$ is a Moore family on G and the lattices $(\mathcal{G}, \subseteq)$ and L are obviously isomorphic. We have then shown that every lattice L is obtained, up to isomorphism, as a lattice of closed sets, which forms a representation of L by a family of subsets. The closure φ associated with $\mathcal{G}$ is given, for any $H \subseteq G$, by $\varphi(H) = \{g \in G : g \leq \bigvee H\}$.

Especially, for such a representation of a lattice L by a family of subsets, one had better use the smallest join-generating set of L, i.e., the set J_L of its join-irreducibles. The family of closed sets obtained in this way and the corresponding closure will appear in Theorem 3.52.

Example 3.32 Let E be a finite set and $\mathcal{D}$ an arbitrary family of subsets of E. A family $\mathcal{F} = \{\bigcap \mathcal{B} : \mathcal{B} \subseteq \mathcal{D}\}$, denoted $\mathbf{m}(\mathcal{D})$, may be derived from $\mathcal{D}$. The family $\mathcal{F}$ is a Moore family since it contains E (obtained with $\mathcal{B} = \emptyset$) and is obviously intersection-stable. Since $\mathcal{D}$ is a meet-generating set of the lattice $\mathcal{F}$, it contains all its meet-irreducibles. Then, observe that restricting $\mathcal{D}$ to the set of these meet-irreducibles does not change the obtained Moore family $\mathcal{F}$.

We will often find this construction of the Moore family $\mathbf{m}(\mathcal{D})$ again. In Section 3.5 we define two families of closed sets associated with a binary relation and, in Proposition 3.46, show that they are obtained in this way. In Section 7.4 we consider the closure associated with the family $\mathbf{m}(\mathcal{D})$. Another interesting example is obtained when $\mathcal{D}$ is a *hyperplane arrangement*, i.e., a finite family of hyperplanes in a linear, affine or projective space. The subspaces belonging to $\mathbf{m}(\mathcal{D})$, i.e., obtained by intersection of hyperplanes in $\mathcal{D}$, are called the *flats* of the arrangement. Ordered by dual inclusion they form – when the intersection of all the hyperplanes is empty – a geometric lattice (see Exercise 7.16).

Example 3.33 Given an ordered set $P = (X, \leq)$ and a subset Y of P, we write $(Y] = \{x \in P$: there exists $y \in Y$ such that $x \leq y\}$ and $[Y) = \{x \in P$: there exists $y \in Y$ such that $y \leq x\}$ (these notations are extensions of the notations $(x]$ and $[x)$ used for principal downsets and upsets). It may be verified that the maps $Y \longmapsto (Y]$ and $Y \longmapsto [Y)$ are closures on $\underline{2}^X$, the closed sets of which are, respectively, the downsets and the upsets of P. These maps will respectively be called the *down* and the *up closure* of P (these closures are obtained in a different way in Exercise 3.13).

3.4 Residuated, residual, and Galois maps

Three types of map between two ordered sets are studied in this section. Each of them may serve to define the other two (see Definition 3.35 and Theorem 3.36 with its comments). These maps are as important as closures or dual closures, which moreover may be derived from the former (Theorem 3.37). Especially, they are consubstantially related to the – fundamental – notion of a Galois connection (Definition 3.40). After the study of their properties in arbitrary ordered sets, we consider the case of lattices (Proposition 3.42 and Corollary 3.43). In the next section a particularization of the latter case will lead to the notion of the Galois lattice of a relation.

Let P and Q be two ordered sets and f a map from P to Q. For any subset Y of Q, the inverse image of Y by f is the subset $f^{-1}(Y) = \{x \in P : f(x) \in Y\}$. Isotone maps may be characterized in terms of inverse images; the very simple proof of the next result is left to the reader.

Proposition 3.34 *Let f be a map from an ordered set P to an ordered set Q. Then f is isotone if and only if one of the following equivalent conditions holds:*

1. *The inverse image $f^{-1}(A)$ of any downset A of Q is a downset of P.*
2. *The inverse image $f^{-1}((y])$ of any principal downset $(y]$ of Q is a downset of P.*
3. *The inverse image $f^{-1}(A)$ of any upset A of Q is an upset of P.*
4. *The inverse image $f^{-1}([y))$ of any principal upset $[y)$ of Q is an upset of P.*

It is then natural to consider Condition (1) below, more restrictive than Condition (2) in the above Proposition 3.34. With two different dualizations, we define three particularly interesting classes of maps.

Definition 3.35 A map f from an ordered set P to an ordered set Q is said to be:

1. *Residuated* if, for any $y \in Q$, the inverse image $f^{-1}((y])$ of the principal downset $(y]$ of Q is a principal downset of P.
2. *Residual* if, for any $y \in Q$, the inverse image $f^{-1}([y))$ of the principal upset $[y)$ of Q is a principal upset of P.
3. A *Galois* map if it is a *residuated* map from P to the dual Q^d of Q.

A more intuitive way to state Condition (1) in the latter definition is the following: the map f is residuated if, for any y in Q, the inequation $f(x) \leq y$ has a greatest solution.

The following results provide two characterizations of residuated and of residual maps, showing in particular that these two types of map always make a pair.

Theorem 3.36 *Let f be a map from an ordered set P to an ordered set Q. The following three conditions are equivalent:*

1. *The map f is residuated.*
2. *There exists a map g from Q to P such that, for all $x \in P$ and $y \in Q$, $x \leq g(y)$ if and only if $f(x) \leq y$.*
3. *The map f is isotone and there exists an isotone map g from Q to P such that the map $\varphi = gf$ is extensive and the map $\psi = fg$ is reductive.*

Proof (1) $\Longrightarrow$ (2): define g by $g(y) = \max f^{-1}((y])$ for any $y \in Q$, to obtain (2).

(2) $\Longrightarrow$ (1): if (2) is satisfied then, for any $y \in Q$, the equality $f^{-1}((y]) = (g(y)]$ holds, that is, (1).

(1) $\Longrightarrow$ (3): if f satisfies (1) then, according to Condition (2) in Proposition 3.34, it is isotone. Moreover, since f also satisfies (2) then, for g, the equality $g^{-1}([x)) = [f(x))$ holds for any $x \in P$ and, according to the last condition in Proposition 3.34, g is isotone. The last part of (3) is obtained by taking successively $y = f(x)$ and $x = g(y)$ in the double implication of (2).

(3) $\Longrightarrow$ (1): assume that f satisfies (3) and let $y \in Q$ and $x \in f^{-1}((y])$, i.e., such that $f(x) \leq y$. Since gf is extensive and g isotone, the inequalities $x \leq gf(x) \leq g(y)$ hold and so $f^{-1}((y]) \subseteq (g(y)]$. For any $x' \in P$ such that $x' \leq g(y)$, the inequalities $f(x') \leq fg(y) \leq y$ hold and so $(g(y)] \subseteq f^{-1}((y])$, which completes the proof. □

The map g from Q to P associated with the residuated map f is in fact a residual map and is characterized by the equivalent Conditions (2′) and (3′), dual of Conditions (2) and (3) in Theorem 3.36:

1′. *The map g is residual.*

2′. *There exists a map f from P to Q such that, for all $x \in P$ and $y \in Q$, $x \leq g(y)$ if and only if $f(x) \leq y$.*

3′. *The map g is isotone and there exists an isotone map f from P to Q such that the map $\varphi = gf$ is extensive and the map $\psi = fg$ is reductive.*

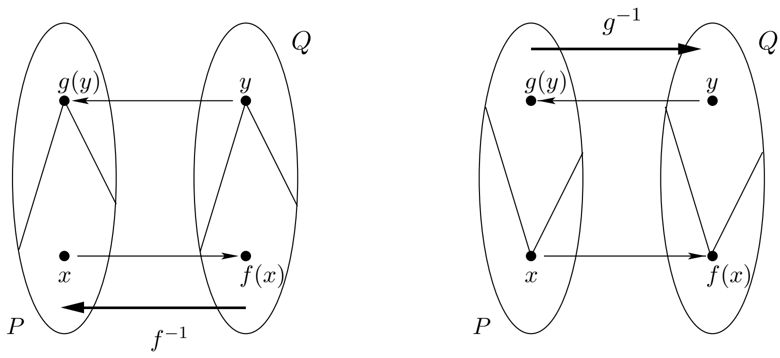


Figure 3.4 f is a residuated map from P to Q and g is a residual map from Q to P.

Dually, the map f associated with the residual map g is a residuated map (see Figure 3.4). Now, Condition (3) in Theorem 3.36 allows us to show that the composition of a residuated map by its associated residual is a closure, while the composition of this residual by the residuated is a dual closure.

Theorem 3.37 *Let P and Q be two ordered sets, f a residuated map from P to Q and g the associated residual map. The following properties hold:*

1. *$fgf = f$ and $gfg = g$.*
2. *The composition map $\varphi = gf$ is a closure on P and the composition map $\psi = fg$ is a dual closure on Q.*
3. *The ordered subset $\Phi = \varphi(P)$ of closed elements of φ in P is equal to $g(Q)$ and the ordered subset $\Psi = \psi(Q)$ of open elements of ψ in Q is equal to $f(P)$. The ordered subsets Φ and Ψ are isomorphic, by the restrictions to the latter of f and g.*

Proof (1) If f is a residuated map and g is its associated residual map, then the following inequalities hold: $gf(g(y)) \geq g(y)$ since gf is extensive, $fg(y) \leq y$ since fg is reductive, and $g(fg(y)) \leq g(y)$ since g is isotone. So $gfg = g$ and, similarly, $fgf = f$.

(2) The map $\varphi = gf$ is extensive, isotone as the composition of two isotone maps, and idempotent since, according to Item (1) above, $\varphi^2 = gfgf = gf = \varphi$; hence it is a closure on P. Similarly, $\psi = fg$ is isotone, reductive, and idempotent, hence a dual closure on Q.

(3) According to Item (1), for any $x \in P$, $\psi(f(x)) = f(x)$ and so $f(P) \subseteq \Psi$. Since any open element h' of Ψ satisfies $h' = \psi(h') = fg(h')$, the inclusion $\Psi \subseteq f(P)$ holds, and so $\Psi = f(P)$. Similarly, $g(Q) = \Phi$. The equalities $h = gf(h)$ and $h' = fg(h')$ for all $h \in \Phi, h' \in \Psi$, together with the equalities $\Psi = f(P)$ and $g(Q) = \Phi$, imply that the restrictions of f to Φ and of g to Ψ are inverse bijections. $\square$

Table 3.1 Table of the maps f and g

$x \in P$	1	2	3	4	5	6	7	8	9
$f(x)$	a	e	a	d	a	j	k	a	k

$y \in Q$	a	b	c	d	e	h	i	j	k
$g(y)$	1	2	6	4	2	7	7	6	7

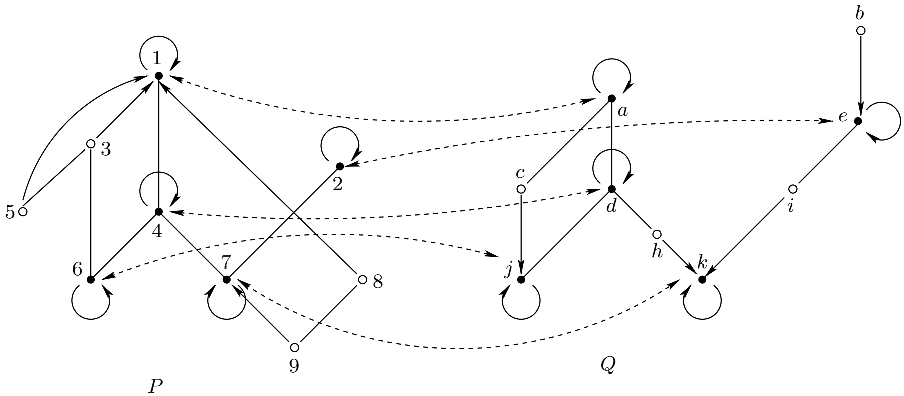


Figure 3.5 The closure $\varphi = gf$ on P and the dual closure $\psi = fg$ on Q.

Example 3.38 Figure 3.5 shows two ordered sets P and Q together with a pair (f, g) of maps, where f is a residuated map from P to Q and g is the corresponding residual map from Q to P (these maps are given in Table 3.1). The figure also shows the closure $\varphi = gf$ on P, the dual closure $\psi = fg$ on Q, and the isomorphism between the images $\varphi(P)$ and $\psi(Q)$. The closure $\varphi(x)$ of an element x of P is represented by arrowing the covering arc if $\varphi(x)$ covers x, or with an arrow from x to $\varphi(x)$ otherwise. The representation of the dual closure ψ on Q is similar. The dotted arrows give the isomorphism between $\varphi(P)$ and $\psi(Q)$. The closed elements of P, like the open elements of Q, are represented by loops and black points.

From Definition 3.35, a map f from P to Q is a *Galois* map if the inverse image by f of any principal upset of Q is a principal downset of P. Theorem 3.36 may then be rewritten as follows.

Theorem 3.39 *Let P and Q be two ordered sets and f a map from P to Q. Then f is a Galois map if and only if it satisfies one of the following equivalent three conditions:*

1. *For any $y \in Q$, the inverse image $f^{-1}([y))$ of the principal upset $[y)$ of Q is a principal downset of P.*
2. *There exists a map g from Q to P such that, for all $x \in P$ and $y \in Q$, $x \le g(y)$ if and only if $y \le f(x)$.*

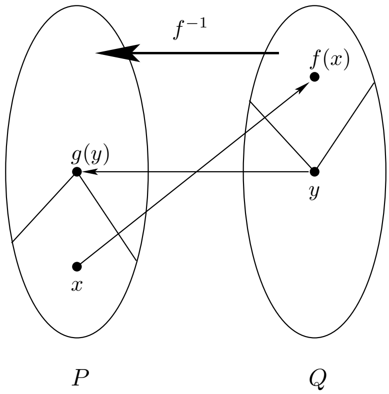


Figure 3.6 f is a Galois map from P to Q.

3. *The map f is antitone and there exists an antitone map g from Q to P such that both maps $\varphi = gf$ and $\psi = fg$ are extensive.*

The equivalence in Condition (2) underlines the fact that f and g play symmetric roles. Indeed, the map g in Theorem 3.39 is also a Galois map and is called the Galois map *associated with f* (similarly f is the Galois map associated with g). See Figure 3.6.

Definition 3.40 A *Galois connection* between two ordered sets P and Q is a pair (f, g), where f is an antitone map from P to Q, g is an antitone map from Q to P, and where both composition maps gf and fg are extensive.

From Item (3) in Theorem 3.39, a Galois connection is characterized by the fact that f is a Galois map from P to Q; g is then the Galois map from Q to P associated with f (it is also the residual map from Q^d to P associated with the residuated map f from P to Q^d). Theorem 3.37 (and Proposition 3.28 in the lattice case) allows us to state:

Theorem 3.41 *Let (f, g) be a Galois connection between two ordered sets P and Q. The following properties hold:*

1. *$fgf = f$ and $g = gfg$.*
2. *The composition maps $\varphi = gf$ and $\psi = fg$ are two closures, respectively on P and on Q, satisfying $\varphi(P) = g(Q)$ and $\psi(Q) = f(P)$.*
3. *The closure nets $\varphi(P)$ and $\psi(Q)$ are dual isomorphic, by the restrictions to these ordered sets of f and g. If, moreover, P and Q are two lattices, then $\varphi(P)$ and $\psi(Q)$ are two dual lattices.*

It follows from what precedes that the notions of a residuated, a residual, a Galois map, and a Galois connection between two ordered sets are equivalent. They are very frequently encountered and it may be more convenient to use one or another of these notions according to the properties that will be particularly useful in a given situation. For instance, it is easy to see that the composition of two residuated (respectively, residual) maps is still residuated (respectively, residual) – see Exercise 3.7 – a property which is no longer valid for Galois maps. On the other hand, the two maps in a Galois connection play a symmetric role, which may be an interesting feature. It will appear in the following section that these connections are fundamental for the construction of classes of objects defined by their common properties.

One may wonder whether there always exists some map of one of the above types between any two given ordered sets P and Q. The answer is negative: if, for instance, Q has a greatest element 1_Q then $(1_Q] = Q$. It follows that $f^{-1}(Q) = P$, for any map f from P to Q. Then, if P has no maximum, Condition (1) in Definition 3.35 cannot be satisfied and there is no residuated map from P to Q. Yet, the situation is not the same when both P and Q are lattices.

Proposition 3.42 *Let L and L' be two lattices and f a map from L to L'. Then, f is residuated if and only if it is a join-morphism satisfying $f(0_L) = 0_{L'}$.*

Proof Assume that f is a residuated map. Since the minimum 0_L of L belongs to the principal ideal $f^{-1}((0_{L'}]) = f^{-1}(0_{L'})$, we have $f(0_L) = 0_{L'}$. Let $x,x' \in L$ and $y = f(x) \vee f(x')$; from the isotony of f, $f(x) \le f(x \vee x')$ and $f(x') \le f(x \vee x')$. Thus $y \le f(x \vee x')$. Now, let us show $f(x \vee x') \le y$. From the isotony of the residual map g associated with f and the extensivity of gf, $f(x) \le y$ implies $x \le gf(x) \le g(y)$. The same holds for x' and $x \vee x' \le g(y)$. Thus, by isotony of f and since fg is reductive, $f(x \vee x') \le fg(y) \le y$. The equality $f(x \vee x') = f(x) \vee f(x')$ follows.

For the converse, assume that the map f is a join-morphism satisfying $f(0_L) = 0_{L'}$. It is then isotone since $x \le x'$ implies $f(x \vee x') = f(x') = f(x) \vee f(x')$, and so $f(x) \le f(x')$. For $y \in L'$, the set $\{x \in L : f(x) \le y\}$ is never empty since $f(0_L) = 0_{L'}$. Write $g(y) = \bigvee\{x \in L : f(x) \le y\} = \bigvee f^{-1}((y])$. By definition, $f^{-1}((y]) \subseteq (g(y)]$. Since f is a join-morphism, we also have $fg(y) = \bigvee\{f(x) : x \in L, f(x) \le y\} \le y$. Thus, $g(y) \in f^{-1}((y])$ and, by isotony of f, $(g(y)] \subseteq f^{-1}((y)]$. Finally, $(g(y)] = f^{-1}((y)]$ for any $y \in L'$, and f satisfies Item (1) in Definition 3.35, a characterization of residuated maps. □

Corollary 3.43 *Let L and L' be two lattices and f a map from L to L'. Then f is:*

- *residual if and only if it is a meet-morphism satisfying $f(1_L) = 1_{L'}$;*
- *a Galois map if and only if $[f(0_L) = 1_{L'}$ and, for all $x,x' \in L$, $f(x \vee x') = f(x) \wedge f(x')]$.*

It follows from Theorem 3.41 that a Galois connection (f,g) between two lattices L and L' defines two dual lattices of closed elements $f(L) = fg(L') \subseteq L'$ and $g(L') = gf(L) \subseteq L$. Denoting $\triangledown$ and $\triangledown'$ the join operations of these lattices of closed sets, and $\wedge$ and $\wedge'$ the meet operations of L and L', the following holds for all closed elements

h, k of L:

$$f(h \wedge k) = f(h) \nabla' f(k) \text{ and } f(h \nabla k) = f(h) \wedge' f(k)$$

Example 3.44 Let E and E' be two sets and $R \subseteq E \times E'$ a relation (and R^c the complementary relation of R). Define a map $f_0 : \underline{2}^E \longmapsto \underline{2}^{E'}$ by $f_0(A) = \{b \in E' : aRb$ for at least one element a of $A\}$, for any subset A of E. Then, $f_0(\emptyset) = \emptyset$ and it is clear that $f_0(A \cup A') = f_0(A) \cup f_0(A')$, for all $A, A' \subseteq E$. Thus, from Proposition 3.42, f_0 is a residuated map from $\underline{2}^E$ to $\underline{2}^{E'}$. The residual map g_0 associated with f_0 is defined by $g_0(B) = \{a \in E : aR \subseteq B\}$, for any subset B of E' (with $aR = \{b \in E' : aRb\}$). For a use of such maps to define "upper" or "lower" concepts, see Section 3.6 (Further topics and references).

Remark 3.45 The constant map that sends each element of the lattice L to the least element $0_{L'}$ of the lattice L' is obviously residuated. On the other hand, from the representation of any element of L as a join of join-irreducibles, a residuated map is determined by its restriction to the set J_L of join-irreducibles of L. Actually, if L' is distributive the reader can check that any isotone map from J_L to L' is the restriction to J_L of a residuated map from L to L'. Thus, the ordered set (subset of L'^L) of the residuated maps from L to L' is isomorphic to the ordered set L'^{J_L} which, according to Exercise 3.4, is a lattice (see also Exercises 3.11 and 3.12).

3.5 The Galois connection associated with a binary relation

We have seen in the previous section that one of the main features of a Galois connection is to reveal two dual substructures in two different ordered sets. When, moreover, the latter are lattices, many further properties are obtained. In this section, the particular case where these lattices are the lattices of subsets of two sets is considered. This case leads, on the one hand, to many applications and, on the other hand, to general results on lattice representation. It also allows us to establish the existence of a canonical coding of any ordered set in a lattice by the *completion* procedure.

3.5.1 Galois lattice

A *binary relation between two sets E and E'* is a subset R of the set $E \times E'$ of ordered pairs $\{(x,y) : x \in E, y \in E'\}$. In Example 3.44, a residuated map together with its associated residual – so a Galois connection – is associated with such a relation. Yet, the standard Galois connection (f_R, g_R) associated with R is defined in a different way. First, for all $e \in E$ and $e' \in E'$, write $eR = \{e' \in E' : eRe'\}$ and $Re' = \{e \in E : eRe'\}$ and, for all $A \subseteq E, B \subseteq E'$:

$$f_R(A) = \{b \in E' : aRb \text{ for any } a \in A\} = \bigcap\{eR : e \in A\}$$
$$g_R(B) = \{a \in E : aRb \text{ for any } b \in B\} = \bigcap\{Re' : e' \in B\}$$

In other words, the image of A by f_R (respectively, of B by g_R) is the set of all the elements of E' in relation to all the elements of A (respectively, all the elements of E in relation to all the elements of B).

It is easy to check that both maps f_R and g_R are antitone and that the inclusions $A \subseteq g_R f_R(A)$ and $B \subseteq f_R g_R(B)$ always hold. Hence the pair (f_R, g_R) satisfies the conditions in Definition 3.40 and so is a Galois connection between the lattices $\underline{2}^E$ and $\underline{2}^{E'}$.

Let $\varphi_R = g_R f_R$ and $\psi_R = f_R g_R$ be the closures, respectively on $\underline{2}^E$ and $\underline{2}^{E'}$, associated with this connection. Then we have:

$$\varphi_R(A) = \{e \in E : \text{for any } e' \in E', [A \subseteq Re' \text{ implies } eRe']\} = \bigcap\{Re' : A \subseteq Re'\}$$
$$\psi_R(B) = \{e' \in E' : \text{for any } e \in E, [eR \supseteq B \text{ implies } eRe']\} = \bigcap\{eR : eR \supseteq B\}$$

Below, and in general, we simply write f, g, φ, and ψ instead of f_R, g_R, φ_R, and ψ_R.

Proposition 3.46 *Let $R \subseteq E \times E'$ be a relation between two sets E and E', F a subset of E and H a subset of E'. With the above notations, the following four conditions are equivalent:*

1. *F is closed by φ and $H = f(F)$.*
2. *H is closed by ψ and $F = g(H)$.*
3. *$H = f(F)$ and $F = g(H)$.*
4. *$F \times H \subseteq R$, where both F and H are maximal with this property.*

Moreover, if we write $\mathcal{D} = \{Re' : e' \in E'\}$, $\mathcal{D}' = \{eR : e \in E\}$, $\mathcal{F} = \varphi(\underline{2}^E)$, and $\mathcal{G} = \psi(\underline{2}^{E'})$, then $\mathcal{F} = \mathbf{m}(\mathcal{D}) = \bigcap_{B \in \underline{2}^{E'}} \{Re' : e' \in B\}$ and $\mathcal{G} = \mathbf{m}(\mathcal{D}') = \bigcap_{A \in \underline{2}^E} \{eR : e \in A\}$.

Proof The equivalence of Items (1), (2), and (3) follows from the properties of Galois connections. If F and H satisfy (3) then, by definition, aRb holds for any ordered pair $(a,b) \in F \times H$. Moreover, taking for instance $b' \in E' \setminus H$, there exists at least one element a of F such that $aR^c b'$ (otherwise $b' \in f(F)$) and Item (4) follows.

Conversely, if (4) is satisfied then $H = \{b \in E' : aRb \text{ for any } a \in F\} = f(F)$ and, similarly, $F = g(H)$.

The fact that, for example, the set $\mathcal{F}$ of closed sets of the closure φ is the set of intersections of the family of the sets (Re') results from the equality $\varphi(\underline{2}^E) = g(\underline{2}^{E'})$ shown in Theorem 3.41 (Item (2)) for an arbitrary Galois connection (f, g). □

A *concept* of the relation R is an ordered pair $(F, H) \in \underline{2}^E \times \underline{2}^{E'}$, where F and H satisfy the equivalent conditions in Proposition 3.46. The associated relation $F \times H \subseteq R$ is called a *prime submatrix* or a *maximal rectangle*, among other denominations. We denote by $Gal(E, E', R)$ the set of concepts (F, H) of R. From the properties of Galois connections, if (F, H) and (F', H') are two concepts, then $F \subseteq F'$ if and only $H' \subseteq H$. The set $Gal(E, E', R)$ is then ordered by the relation:

$$\begin{aligned}(F,H) \leq (F',H') &\iff F \subseteq F' \\ &\iff H' \subseteq H\end{aligned}$$

Definition 3.47 The ordered set $(Gal(E,E',R),\leq)$ is called the *Galois lattice* (or the *concept lattice*) of R.

The ordered set $Gal(E,E',R)$ is indeed a lattice since, by definition, it is isomorphic to $\mathcal{F}=\varphi(\underline{2}^E)$ and dual of $\mathcal{G}=\psi(\underline{2}^{E'})$. It follows that its join and meet operations are given by:

$$(F,H)\wedge(F',H')=(F\cap F',\psi(H\cup H'))$$

and

$$(F,H)\vee(F',H')=(\varphi(F\cup F'),H\cap H')$$

Example 3.48 (Guénoche, 1993) Table 3.2 gives a binary relation R from $E=\{1,2,3,4,5,6,7,8\}$ to $E'=\{A,B,C,D,E,F,G\}$ and Figure 3.7 shows its Galois lattice, which has 15 elements. The meanings of the elements of E and E' are:

1	ostrich	A	lays eggs
2	canary	B	has feathers
3	duck	C	has scales
4	shark	D	has a naked skin
5	salmon	E	has teeth
6	frog	F	flies
7	crocodile	G	swims
8	barracuda	H	breathes in air

Table 3.2 Table of the relation R

	A	B	C	D	E	F	G	H
1	×	×						×
2	×	×				×		×
3	×	×				×	×	×
4	×			×	×		×	
5	×		×				×	
6	×			×			×	×
7	×			×	×		×	×
8	×		×		×		×	

Proposition 3.46 and the previous example state and illustrate what a Galois connection of this type brings out when a set E of objects is described by a set E' of binary "presence/absence" attributes. For such an attribute, presence and absence do not play equivalent roles since the possession of attribute e' by object e is always significant but not necessarily its absence. A closed set F then forms the "class" of all the objects sharing all attributes that belong to the closed set $G=f(F)$; that is, the description in *extent* of the concept $C=(F,G)$. The latter classes form the Moore family $\mathcal{F}$. On the other hand, the closed set G is the set of all the attributes shared by

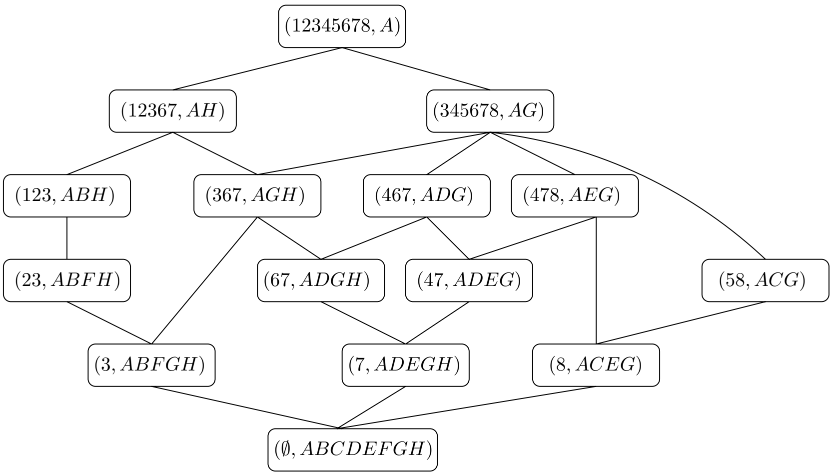


Figure 3.7 The Galois lattice of the relation R.

all elements of F; that is, the description in *intent* of the concept C. The latter descriptions form the dual Moore family $\mathcal{G}$. Finally, the Galois lattice of R is the ordered set of all concepts generated by the relation R. Further developments on this subject will be considered in Section 3.6 of this chapter and in Section 7.4 of Chapter 7. We may already observe that the terminology used for the elements of the Galois lattice points out the relation between some laws of thought and "Galois classification" (see Section 7.4 in Chapter 7).

3.5.2 Table of an ordered set

In this section, we define the table of an ordered set and show that, in the case of a lattice, the table allows us to reconstruct the lattice (Theorem 3.52). We then give some properties of the table of a lattice (Proposition 3.54), which will be extended at the end of Section 3.5.3 to the table of an arbitrary ordered set.

Let P be an ordered set, J_P and M_P the sets of its join- and of its meet-irreducibles, and $\downarrow$ and $\uparrow$ its downarrow and uparrow relations (Chapter 1, Definition 1.36). Let $\downarrow_{|J_P \times M_P}$ and $\uparrow_{|J_P \times M_P}$ be the respective restrictions of these arrow relations to $J_P \times M_P$. For the sake of simplicity, we use in general the same notation for the arrow relations of P and for their restrictions to $J_P \times M_P$ – Exercise 3.15 justifies this use.

Definition 3.49 The *table* of an ordered set P is the relation $R_P \subseteq J_P \times M_P$ defined by jR_Pm if $j \leq m$. The *arrowed table* of P is the triple $\mathcal{R}_P = (R_P, \downarrow, \uparrow)$ of relations on $J_P \times M_P$.

Remark 3.50 Several representations of a relation by "tables" were used in Chapter 1, for instance Table 1.1 in Section 1.1.1. Clearly, in the present expression "table of an ordered set," the term "table" has a different meaning since it refers to a particular relation defined between two subsets of the ordered set. Of course, the relation itself and the arrow relations may be represented by a table in the usual sense. Indeed, this is done further (on page 96) with Table 3.3, which represents the above-defined arrowed table of the ordered set in Figure 3.2 (and thus three relations).

From now on, we study the particular case of a lattice L. A decomposition of the Galois connection (f,g) associated with the table R_L as in the previous section leads to a proof of Theorem 3.52, which is the finite case of what is sometimes called "the fundamental theorem of concept theory." Then, we partly characterize those binary relations which are tables of lattices (this characterization is completed in Exercise 3.16) and, finally, we obtain some properties of the arrowed table of L.

Let L be a lattice and J_L, M_L, and R_L be respectively the set of its join-irreducibles, the set of its meet-irreducibles, and its table. Since R_L is a binary relation between two sets, we may consider the Galois connection (f,g) associated with R_L (as at the beginning of Section 3.5.1). Then:

$$f(A) = \{b \in M_L : a \leq b \text{ for any } a \in A\}$$
$$g(B) = \{a \in J_L : a \leq b \text{ for any } b \in B\}$$

Recall that J_x and M^x respectively denote the set of join-irreducibles of L less than or equal to x and the set of meet-irreducibles of L greater than or equal to x (Section 3.2).

We define the maps $f_1 : P(J_L) \longmapsto L$, $g_1 : L \longmapsto P(J_L)$, $f_2 : L \longmapsto P(M_L)$, and $g_2 : P(M_L) \longmapsto L$ as follows: for all $x \in L, A \subseteq J_L, B \subseteq M_L$,

$$f_1(A) = \bigvee A, \;\; g_1(x) = J_x, \;\; f_2(x) = M^x, \;\; g_2(B) = \bigwedge B$$

Endowing – as usual – $P(J_L)$ with the inclusion order and – just for the following proposition and proof – $P(M_L)$ with the dual inclusion order $\supseteq$, we obtain:

Proposition 3.51 *Let L be a lattice. With the above notations and definitions, the following properties hold:*

1. $f_1 g_1 = g_2 f_2 = id_L$.
2. $f = f_2 f_1$ *and* $g = g_1 g_2$.
3. f_1 *and* f_2 *are residuated and* g_1 *and* g_2 *are their respective associated residual maps.*

Proof The equalities in (1) are obtained as direct consequences of Propositions 3.11 and 3.19. Property (2) comes from the following equalities: for all $A \subseteq J_L$, $B \subseteq M_L$,

$$f(A) = \{b \in M_L : a \leq b, \text{ for any } a \in A\} = \{b \in M_L : \bigvee A \leq b\} = f_2 f_1(A)$$

$$\begin{tikzcd}
& L \arrow[dl, shift left, "g_1"] \arrow[dr, shift left, "f_2"] & \\
P(J_L) \arrow[ur, shift left, "f_1"] \arrow[rr, shift left, "f"] & & P(M_L) \arrow[ul, shift left, "g_2"] \arrow[ll, shift left, "g"]
\end{tikzcd}$$

Figure 3.8 The decomposition of a Galois connection (f,g).

and

$$g(B) = \{a \in J_L : a \leq b, \text{ for any } b \in B\} = \{a \in J_L : a \leq \bigwedge B\} = g_1 g_2(B)$$

In order to prove (3), we show, for any $x \in L$, the following two equivalences: $A \subseteq g_1(x)$ if and only if $x \geq f_1(A)$, and $x \leq g_2(B)$ if and only if $f_2(x) \supseteq B$; that is, $A \subseteq J_x$ if and only if $\bigvee A \leq x$, and $x \leq \bigwedge B$ if and only if $B \subseteq M^x$. The first equivalence is true since $A \subseteq J_x$ implies $\bigvee A \leq \bigvee J_x = x$ and $\bigvee A \leq x$ implies $j \leq x$ for any j in A. The proof of the second equivalence is similar. Thus, each of the maps f_1 and f_2 satisfies Condition (2) in Theorem 3.36 characterizing residuated maps, while g_1 and g_2 happen to be their associated residual maps. Figure 3.8 illustrates this proposition. □

We are now able to prove the next theorem (Barbut, 1965).

Theorem 3.52 *Every lattice L is isomorphic to the Galois lattice of its table R_L (by the map $x \longmapsto (J_x, M^x)$).*

Proof According to Item (3) in the previous proposition and Theorem 3.37 applied to the residuated map f_1 and its associated residual map g_1, $g_1 f_1$ is a closure φ_1 and the lattice of closed sets $\varphi_1((P(J_L), \subseteq))$ is isomorphic to $f_1 g_1(L) = id_L(L) = L$. We denote by φ the closure operator obtained as the composition gf of the two maps of the Galois connection (f,g). According to Items (1) and (2) in the previous proposition, $\varphi = gf = g_1 g_2 f_2 f_1 = g_1 f_1 = \varphi_1$, so the Galois lattice $Gal(J_L, M_L, R_L)$, which is isomorphic to the lattice of closed sets $\varphi((P(J_L), \subseteq))$, is also isomorphic to the lattice L. □

A direct consequence of this result is expressed in the following corollary.

Corollary 3.53 *Any lattice is isomorphic to the lattice of closed sets of a closure operator.*

In other words, any lattice has a set representation by a Moore family.

According to Theorem 3.52, the table of a lattice provides a condensed representation. A natural question is then to characterize those binary relations which are tables of lattices. To do so, we now consider two sets E and E', an arbitrary relation $R \subseteq E \times E'$ and its Galois lattice $Gal(E, E', R)$. We define a reduction procedure which, up to isomorphism, does not change the Galois lattice of R and leads to its table. The consequence of this procedure is that a reduced relation is the table of a lattice. Conversely, the purpose of Exercise 3.16 is to show that the table of a lattice is reduced. So, a relation is the table of a lattice if and only if it is reduced.

We use the notations given in Section 3.5.1 for the Galois connection (f,g) associated with R. The Moore family $\mathcal{F} = \varphi(\underline{2}^E)$ is obtained as in Example 3.32 by intersections of elements of the family $\mathcal{D} = \{Re' : e' \in E'\}$ of subsets of E, i.e., is equal to $\mathbf{m}(\mathcal{D})$. Thus, every meet-irreducible of $\mathcal{F}$ is necessarily equal to Re' for some $e' \in E'$ (by duality, the closure $\psi(e')$ of the latter is a join-irreducible of $\mathcal{G} = \psi(\underline{2}^{E'})$). In Example 3.32, it was observed that the family $\mathcal{F}$ remains unaltered when restricting $\mathcal{D}$ to the set of meet-irreducibles of $\mathcal{F}$. This comes down to selecting a subset E'_1 of E' and considering the restriction R_1 of the relation R to $E \times E'_1$. In other terms, an element e'_0 of E' is removed from E' in each of the following cases:

- $Re'_0 = E$ (since E is not a meet-irreducible of $\mathcal{F}$).
- There exists a subset B of $E' \setminus e'_0$ such that $Re'_0 = \bigcap\{Re' : e' \in B\}$ (since then Re'_0 is not a meet-irreducible of $\mathcal{F}$).
- $Re'_0 = Re'_1$, for some e'_1 already kept in E'_1 (since, then, Re'_0 is redundant in $\mathcal{F}$).

This first reduction does not affect the Moore family $\mathcal{F}$ whereas, in the Galois connection (f',g') associated with the relation R_1, the Moore family $\mathcal{G} = \psi(\underline{2}^{E'})$ on E' is replaced with a Moore family $\mathcal{G}_1$ on E'_1 which, ordered by set inclusion, is a lattice, dual of the lattice $\mathcal{F}$, and so isomorphic to the lattice $\mathcal{G}$.

A second reduction similarly selects a subset E_2 of E, leading to a binary relation $R_2 \subseteq E_2 \times E'_1$ which is then said to be *reduced* (the term *reduced table* is also used). In the Galois connection (f'',g'') associated with R_2, the Moore family $\mathcal{F}$ on E is replaced with a Moore family $\mathcal{F}_2$ on E_2, ordered by inclusion as a lattice dual of $\mathcal{G}_1$ and isomorphic to $\mathcal{F}$, the Moore family $\mathcal{G}_1$ remaining the same. An important fact is that, for any $e \in E_2$, the subset $eR_2 = f''(e)$ of E'_1 is a meet-irreducible of $\mathcal{G}_1$ (and $g''f''(e)$ is a join-irreducible of $\mathcal{F}_2$). Similarly, for any $e' \in E'_1$, the subset $R_2e' = g''(e')$ of E_2 is a meet-irreducible of $\mathcal{F}_2$ (and $f''g''(e')$ a join-irreducible of $\mathcal{G}_1$). In other terms, the relation R_2 is the table of its Galois lattice. Exercise 3.16 proposes to show that, conversely, the table of any lattice cannot be reduced by the previous procedure.

As a consequence, all the relations which have the same reduced relation also have, up to isomorphism, the same Galois lattice. In fact, up to isomorphisms and labelings, reduced relations and lattices are in one-to-one correspondence.

The arrowed table of a lattice brings even more information, since its properties allow us to recognize whether a lattice belongs to some important class of lattices. This is, for instance, the case for distributive lattices (as will be pointed out by Characterization (5) of these lattices in Theorem 5.1, Chapter 5). The following result gives two properties which are always satisfied by the arrowed table of a lattice. We will see in Section 3.5.3 that they remain valid in any ordered set.

Proposition 3.54 *Let L be a lattice. The following properties hold:*

1. *For any $j \in J_L$, there exists $m \in M_L$ such that $j \updownarrow m$.*
2. *For any $m \in M_L$, there exists $j \in J_L$ such that $j \updownarrow m$.*

Proof (1) We first show that, for any join-irreducible j, there exists a meet-irreducible m satisfying $j \downarrow m$. According to Proposition 3.8, there exists an element x with $j \downarrow x$. If x is not meet-irreducible then, according to Proposition 3.19, $x = \bigwedge M^x$. Then, j is not a lower bound of M^x since this would imply $j \leq x$. So there exists a meet-irreducible $m \in M^x$ such that $j \not\leq m$. Moreover, for the unique element j^- covered by j (Proposition 3.15), the inequality $j^- \leq x \leq m$ holds, and the property $j \downarrow m$ follows.

Let m be a meet-irreducible, maximal with the property that $j \downarrow m$. The unique element m^+ covering m satisfies $j \leq m^+$ (otherwise, there would exist as above a meet-irreducible m' greater than m and satisfying $j \downarrow m'$). Thus, $j \uparrow m$ and finally $j \updownarrow m$.

Property (2) is obtained dually. □

3.5.3 Completion of an ordered set

In Sections 3.5.1 and 3.5.2 we studied the Galois lattice of an arbitrary relation R between two sets E and E'. In particular, we may consider the case where $E = E'$, i.e., where R is a relation on E. The aim of this section is to study the Galois lattice $Gal(X, X, \leq)$ of an ordered set $P = (X, \leq)$. Let us go back to Example 3.2 where two antitone maps *Upper* and *Lower* were defined on $\underline{2}^X$. It is easy to see that the composition maps *LowerUpper* and *UpperLower* are extensive: for instance, for $A \subseteq A' \subseteq X$, $x \in A$ implies $x \leq y$ for any $y \in UpperA$, hence $x \in Lower(UpperA)$ and $A \subseteq Lower(UpperA)$. In fact, the pair $(Upper, Lower)$ is nothing else but the Galois connection associated with the relation $\leq$ on X and which corresponds with the Galois lattice of P.

Definition 3.55 For any ordered set $P = (X, \leq)$, the Galois lattice $Gal(X, X, \leq)$, denoted by $Gal(P)$ in the sequel, is called the *completion* (or *MacNeille completion*, or *Dedekind–MacNeille completion*) *of* P.

The reasons for the term "completion" will appear in Proposition 3.56. We now establish some properties of the completion of $P = (X, \leq)$ and, to do so, it may be convenient to work on the isomorphic and/or the dual isomorphic lattice of closed sets (see Section 3.5.1). The corresponding two closures are the composition maps $\mu = LowerUpper$ and $\mu' = UpperLower$ with, for any $A \subseteq X$:

$$\mu(A) = \bigcap\{(x], x \in UpperA\} \text{ and } \mu'(A) = \bigcap\{[x), x \in LowerA\}$$

and, in particular, for any $x \in P$:

$$\mu(\{x\}) = Lower\{x\} = (x] \text{ and } \mu'(\{x\}) = Upper\{x\} = [x)$$

Since a subset of X is closed by μ if and only if it is equal to X or to an intersection of some principal downsets of P, it is a downset of P. It follows that the lattice $\mathcal{F} = \mu(\underline{2}^X)$ of closed sets of μ is included in the distributive lattice $\mathcal{D}(P)$ of downsets of P (this lattice is studied in Section 5.2); up to isomorphism, the same holds for the completion of P. Similarly, the lattice $\mathcal{G} = \mu'(\underline{2}^X)$ of the subsets closed by μ' (dual of the lattice $\mathcal{F}$ and of the completion of P) is included in the distributive lattice of upsets of P.

The closure μ is defined on $\underline{2}^X$. The next proposition states that its restriction to X satisfies strong properties. For the sake of simplicity, we write $\mu(\{x\}) = \mu(x)$ for any $x \in P$.

Proposition 3.56 *Let $P = (X, \leq)$ be an ordered set and $\mu = LowerUpper$ the closure defined on $\underline{2}^X$ by $\mu(A) = \bigcap\{(x], x \in UpperA\}$. The restriction of μ to P is a coding from P to the lattice $\mathcal{F} = \mu(\underline{2}^X)$ satisfying the following properties:*

1. *If x and y have a meet $x \wedge y$ in P, then $\mu(x \wedge y) = \mu(x) \cap \mu(y)$.*
2. *If x and y have a join $x \vee y$ in P, then $\mu(x \vee y)$ is the join of $\mu(x)$ and $\mu(y)$ in the lattice $\mathcal{F}$.*

Proof If $x \leq y$ then $z \leq x$ implies $z \leq y$, and so $(x] = \mu(x) \subseteq (y] = \mu(y)$. Conversely, $(x] = \mu(x) \subseteq (y] = \mu(y)$ implies $x \in (y]$ that is, $x \leq y$. Then, the map μ is a coding from P to $\mathcal{F}$.

(1) If x and y have a meet $x \wedge y$, then:

$$\begin{aligned} z \in \mu(x \wedge y) = (x \wedge y] &\iff z \leq x \wedge y \\ &\iff z \leq x \text{ and } z \leq y \\ &\iff z \in (x] \cap (y] \\ &\iff z \in \mu(x) \cap \mu(y) \end{aligned}$$

and Property (1) holds.

(2) Similarly, if x and y have a join $x \vee y$, and since $\mu' = UpperLower$, then:

$$\begin{aligned} z \in \mu'(x \vee y) = [x \vee y) &\iff z \geq x \vee y \\ &\iff z \geq x \text{ and } z \geq y \\ &\iff z \in [x) \cap [y) \\ &\iff z \in \mu'(x) \cap \mu'(y) \end{aligned}$$

The lattice $\mathcal{F}$ of closed sets of μ and the lattice $\mathcal{G}$ of closed sets of μ' are dual isomorphic by the restrictions of *Upper* and *Lower*, while $[x \vee y)$, $[x)$ and $[y)$ are closed by μ', and $[x \vee y)$ is the intersection of $[x)$ and $[y)$. Thus, in the lattice $\mathcal{F}$ (dual of $\mathcal{G}$ by the restriction of *Lower*), $\mu(x \vee y) = Lower[x \vee y) = (x \vee y]$ is the join of $Lower[x) = (x]$ and $Lower[y) = (y]$, which is Property (2). □

Then, the coding μ from P to the lattice $\mathcal{F}$ (and so, up to isomorphism, to the completion of P) "preserves" all the joins and meets of P. In fact, it may be shown that the completion of P is the smallest lattice with these properties (see Section 3.6). Moreover, the coding μ also "preserves" the arrow relations as well as the irreducible elements of P, as will be established in the final results of this section.

Proposition 3.57 *Let x and y be two elements of an ordered set $P = (X, \leq)$ and $\mathcal{F} = \mu(\underline{2}^X)$, where $\mu = LowerUpper$ is the closure on $\underline{2}^X$ defined in Proposition 3.56. Then:*

1. *$x \downarrow_P y$ if and only if $\mu(x) \downarrow_{\mathcal{F}} \mu(y)$.*
2. *$x \uparrow_P y$ if and only if $\mu(x) \uparrow_{\mathcal{F}} \mu(y)$.*

Proof (1) Assume $x \downarrow_P y$, then $x \not\leq y$ and so $\mu(x) \not\subseteq \mu(y)$. Moreover, by assumption, x is the only element of $\mu(x) = (x]$ not less than or equal to y in P. Then a closed set of μ – that is, an ideal of P – strictly included in $\mu(x)$ cannot contain x and is included in $(y] = \mu(y)$, which implies $\mu(x) \downarrow_{\mathcal{F}} \mu(y)$.

Conversely, assume $\mu(x) \downarrow_{\mathcal{F}} \mu(y)$. Then, in the lattice $\mathcal{F}$, $\mu(x)$ is minimal among the closed sets not included in $\mu(y)$. Since the map μ is a coding, x is not less than or equal to y in P, and $z < x$ implies $z < y$. The relation $x \downarrow_P y$ follows.

(2) The equivalence between $x \downarrow_P y$ in P and $\mu'(x) \downarrow_{\mathcal{G}} \mu'(y)$ in the lattice $\mathcal{G}$ of closed sets of $\mu' = UpperLower$ may be obtained similarly and leads – by duality of the lattices $\mathcal{F}$ and $\mathcal{G}$ – to the equivalence for the uparrow relation. □

Proposition 3.58 *Let $P = (X, \leq)$ be an ordered set and $\mathcal{F} = \mu(\underline{2}^X)$. An element F of $\mathcal{F}$ is a join-irreducible (respectively, a meet-irreducible) of $\mathcal{F}$ if and only if $F = (j]$ (respectively, $F = (m]$) is the image by μ of a join-irreducible j (respectively, a meet-irreducible m) of P.*

Proof From the formula giving $\mu'(A)$ at the beginning of this section, the set of principal upsets of P is a meet-generating set of the lattice $\mathcal{G}$. According to the dual version of Corollary 3.12, all meet-irreducibles of $\mathcal{G}$ belong to this set. Since the restriction of the map $Lower$ to $\mathcal{G}$ is a dual isomorphism between $\mathcal{G}$ and $\mathcal{F}$, it follows that any join-irreducible F of $\mathcal{F}$ is equal to $F = (x] = Lower[x)$ for some x in P. Moreover, Property (2) in Proposition 3.56 implies that, if $(x]$ is a join-irreducible of $\mathcal{F}$, then x is a join-irreducible of P.

Conversely, if j is a join-irreducible of P then it follows from Proposition 3.8 that there exists an element x of P satisfying $j \downarrow_P x$. Thus, by Proposition 3.57, we have $\mu(j) \downarrow_{\mathcal{F}} \mu(x)$ which, again by Proposition 3.8, implies that $\mu(j)$ is a join-irreducible of $\mathcal{F}$.

The result on meet-irreducibles is obtained in a similar way. □

In Section 3.5.2, the considered tables were essentially those of a lattice. Yet, Definition 3.49 dealt with arbitrary ordered sets. Any element x of an ordered set P may be identified with its image $\mu(x)$ in the lattice $\mathcal{F}$ (isomorphic to the completion of P). Then, Propositions 3.56 to 3.58 allow us to state the first part of the next corollary, which uses this identification. The isomorphism result of the second part then follows from Theorem 3.52.

Corollary 3.59 *Let $P = (X, \leq)$ be an ordered set and $Gal(P)$ its completion. The following holds:*

1. *P and $Gal(P)$ have the same ordered sets of join-irreducibles and the same arrowed table.*
2. *$Gal(P)$ is isomorphic to the Galois lattice of $R_P = (J_P, M_P, \leq)$.*

It is then immediate that Proposition 3.54 extends to the (arrowed) table of any ordered set; that is, for any $j \in J_P$ (respectively, any $m \in M_P$), there exists $m \in M_P$ (respectively, $j \in J_P$) such that $j \updownarrow m$.

Let J and M be two given sets and $R \subseteq J \times M$ a reduced relation (see Section 3.5.2). The lattice $L = Gal(J, M, R)$ is, up to isomorphism, the unique lattice having R as its table. On the other hand, we have just observed that all the ordered sets the completion of which is isomorphic to L have this table and, moreover, have the same arrowed table as L. Exercise 3.15 proposes to determine all these ordered sets and, in particular, to show that the smallest one is the ordered subset $IR(L)$ of irreducible elements of L.

Example 3.60 The join-irreducible (respectively, meet-irreducible) elements of the ordered set in Figure 3.2 are a, b, c, d, e, f, and g (respectively, d, e, f, g, h, i, and k). Table 3.3 and Figure 3.9 show, respectively, the arrowed table and the completion of this ordered set.

Table 3.3 The arrowed table of the ordered set in Figure 3.2

	d	e	f	g	h	i	k
a	$\times$	$\times$	$\updownarrow$	$\updownarrow$	$\times$	$\times$	$\times$
b	$\times$	$\updownarrow$	$\times$	$\updownarrow$	$\times$	$\times$	$\times$
c	$\updownarrow$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
d	$\times$				$\times$	$\times$	$\updownarrow$
e		$\times$		$\uparrow$	$\times$	$\updownarrow$	$\times$
f			$\times$	$\uparrow$	$\updownarrow$	$\times$	$\times$
g		$\downarrow$	$\downarrow$	$\times$	$\updownarrow$	$\updownarrow$	$\times$

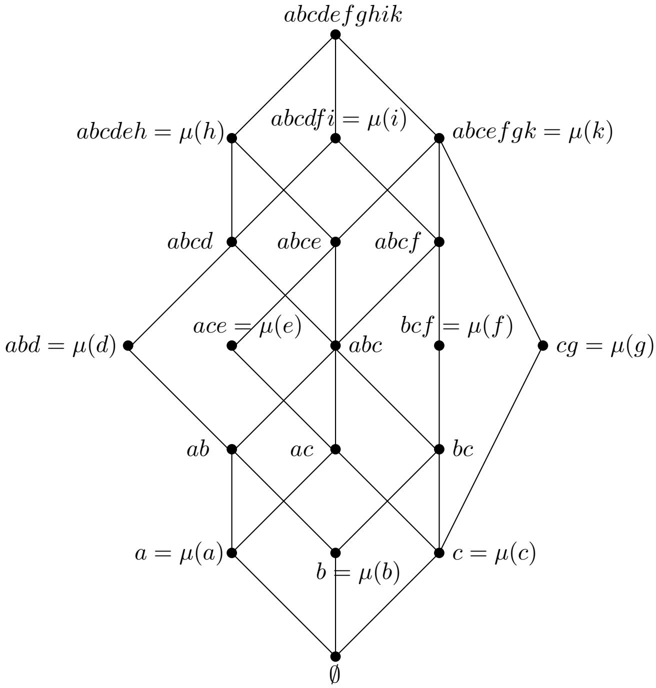


Figure 3.9 The completion of the ordered set in Figure 3.2.

3.6 Further topics and references

The notion of an isotone map between two ordered sets is essential, but the following result shows that it is too general to induce a convenient classification of ordered sets. Duffus and Rival (1981) have shown that any connected ordered set P of size $n \geq 3$ is the image by an isotone map of a fence (see page 46) of size at most $2n-3$. In this paper, they develop a theory of the classification of ordered sets based on the notion of a retract.

The representations of any element of an ordered set as a join of join-irreducibles and as a meet of meet-irreducibles given in Section 3.2 are fundamental examples of codings from arbitrary ordered sets to Boolean lattices. Nevertheless, they do not guarantee that the latter are of minimal size, and we will go back to this question in Chapter 6.

The notions of closure operators and closure systems appeared at the beginning of the twentieth century with the birth of general topology as an axiomatic theory. In this context, Kuratowski was apparently the first to take the notion of a closure operator (on a set) as the primitive notion. In his famous 1922 paper, he observed that the operation of the "Analysis Situs" associating with a subset A of a Euclidean space the closed set $\overline{A}$ formed of A and the accumulation points of A is extensive, idempotent, and satisfies $\overline{\emptyset} = \emptyset$ and $\overline{A \cup B} = \overline{A} \cup \overline{B}$ (so it is also isotone and defines what is now called a *topological closure*). In this set context, Sierpinski (1927) obtained the equivalence between the notions of an (arbitrary) closure operator and an (arbitrary) Moore family by showing that both of them are equivalent to a *derivation operator* satisfying some properties. But, in fact, the name "Moore family" appeared later and is due to Birkhoff (1937a), who mentioned the equivalence between closure operators on a set and families containing this set and intersection-stable as already pointed out in two of Moore's papers (1909, 1910). In fact, closure operators appear almost everywhere in mathematics, logics, and computer science. For instance, as early as the 1930s, Tarski (1930) used them to define *logical deductive systems*.

The significance of Moore families also comes from the fact – see Corollary 3.53 – that they are the set representations of lattices (which remains true for complete lattices). Then, particular classes of Moore families correspond to particular classes of lattices. For instance, the so-called convex geometries correspond to lower locally distributive lattices (see the end of Section 5.6).

The generalization of closure systems and closure operators to arbitrary lattices or ordered sets promptly followed Birkhoff's 1940 book (see, for instance, Monteiro and Ribeiro (1942)).

We will go back to Moore families in Section 7.4 (Chapter 7), where we study the lattice of all such families and its duality with the lattice of implicational systems. The latter is isomorphic to the lattice of closure operators, a lattice thoroughly studied as early as 1943 by Öre.

Let us simply add that the number of Moore families on a set of size n is only known up to $n = 7$ (A102896 in the On-Line Encyclopedia of Integer Sequences, and Colomb *et al*. (2010)).

The Galois connection associated with a binary relation (studied in Section 3.5) was exhibited as early as 1940, the year of the first edition of Birkhoff's treatise, with the name of *polarity*. The term and the abstract notion of a Galois connection are due to Öre (1944), who made a deep study of the latter. He especially pointed out that such a connection between subgroups and subfields appears in Galois theory of equations. The close notion of a residuated (or a residual) map seems to explicitly appear for the first time in the works of Dubreil and Croisot (1954), followed by Croisot's 1956 paper. The maps that we prefer to call Galois maps are frequently known in the literature as "polarized maps" (Shmuely, 1974). Some authors (see Denecke *et al*., 2004) call a *covariant* (respectively, a *contravariant*) *Galois connection* any pair (f, g) where f is a residuated (respectively, Galois) map and g the associated residual (respectively, Galois) map. Others – or the same – also use the term *adjunction*, f being called the *left* (or *inferior*) *adjunction* and g the *right* (or *superior*) *adjunction*. This terminology comes from category theory, where one finds a notion of an adjunction which generalizes the previous situation. The book Residuation Theory by Blyth and Janowitz (1972) and the treatise edited by Denecke *et al*. (2004) emphasize the ubiquity of these notions in mathematics, with an extensively developed history in the latter book (see the chapter "Adjunctions and Galois connections: origins, history and developments" by Erné). Observing the number of times they have been rediscovered, one may only regret that they are not included in the basic training of mathematicians.

The fundamental objects called Galois lattices or concept lattices were introduced in Section 3.5. We have kept the last term, popularized by Wille since 1982, to name a concept (F,G), whereas the subrelation $F \times G$ was previously called in French a "rectangle maximal" (see for instance, Kaufmann and Pichat (1977) and Barbut and Monjardet (1970)). The term "concept" refers to some philosophical considerations developed, for instance, in Arnauld and Nicole's *Logique de Port-Royal* (1662) – see, for example, Duquenne (1987). In the latter, F and G are respectively called the "range" ("étendue") and the "comprehension" ("compréhension") of the "idea" ("idée") $F \times G$; that is, in modern language, the *extent* and *intent* of the concept.

The use of the Galois lattice of a relation in data analysis goes back to a text of Barbut (1965) published in a book on questionnaire analysis. Indeed, the answers of a set of subjects to dichotomic questions (answers are "yes" or "no") define a binary relation between subjects and questions. The (rare) case where the Galois lattice of this relation is a chain corresponds to the case where the answers to the questionnaire form a *Guttman scale* revealing two dual linear hierarchical structures on subjects and questions: there exists a total preorder on questions such that, if a subject answers "yes" to some question, he also answers "yes" to all the dominated questions in this preorder. In this case, encoding the "yes" answers by 1 and the "no" by 0, the relation may be represented by a "step-type" table as in Table 7.1 (yet here, lines and columns are labeled according to two different sets). In the general case, each of the maximal chains of the Galois lattice associated with the relation forms a *Guttman subscale* allowing us to obtain the table of a "partial scale" and a total preorder on a subset of subjects in correspondence with a total preorder on a subset of questions. This is illustrated in Figure 3.10 and Table 3.4, with the Galois lattice given in Figure 3.7.

Table 3.4 The table of the Guttman subscale obtained from the maximal chain in Figure 3.10(b)

	A	G	D	E	H	B	C	F
1	×							
2	×							
3	×	×						
5	×	×						
8	×	×						
6	×	×	×					
4	×	×	×	×				
7	×	×	×	×	×			

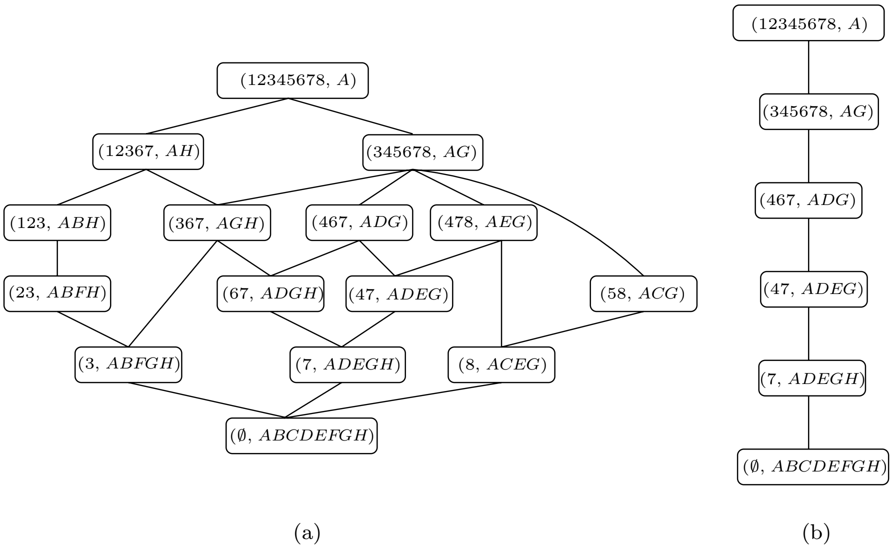


Figure 3.10 (a) The Galois lattice given in Figure 3.7; (b) a maximal chain of the lattice giving the Guttman subscale in Table 3.4.

In the same 1965 text, Barbut, developing a Schützenberger observation (on page 14 in his 1956 paper), shows that every lattice is a Galois lattice (Theorem 3.52) and gives an algorithm for constructing such a lattice (see also Barbut and Monjardet, 1970). This result, which has been rediscovered several times in various forms, leads to a nice synthesis of the set representation results of a lattice due to Campbell (1943) and Birkhoff and Frink (1948). Other developments on the use of Galois lattices in questionnaire analysis may be found in the works of Flament, Degenne, and Vergès (see, for instance, Flament *et al.* (1979)). This use in data analysis has known a considerable growth since the 1980s, thanks to the works of the Darmstadt team (see, for instance, Ganter and Wille's book (1999)) and those of Duquenne (1987, 1993). The determination of all concepts (i.e., the elements of the Galois lattice) is an important problem of exponential complexity in the size of the considered relation (see Appendix A.2.2, where several algorithms proposed in the literature are mentioned).

In Example 3.44, with a binary relation between two sets E and E' was associated a pair (f, g), where f is a residual map and g the associated residual map, in other terms a "covariant" Galois connection. In Exercises 4.15 and 5.8 of their 1972 book, Blyth and Janowitz propose studying the properties of such connections when $E = E'$. In the general case, such a connection – sometimes called an *axiality* – induces a lattice, the elements of which – i.e., the pairs (A, B) with $A = g(B) \subseteq E$ and $B = f(A) \subseteq E'$ – were called *upper concepts*. Applying the same connection to the relation R^d defined between E' and E leads to the lattice of *lower concepts*. These two connections induce a closure and a dual closure on E (as well as a dual closure and a closure on

E'). Considering the equivalence relation on E defined by $xR = yR$, this closure and dual closure are nothing else but the *upper* and the *lower approximation operators* associated with this equivalence and used in the "rough set analysis" (see, for instance, Düntch and Gediga (2003) or Wolski (2004)).

Another interesting example of a covariant Galois connection may be found in mathematical morphology, i.e., in an approach to image processing based on geometric transformations of images structured as a lattice (see, for instance, Serra, 1988 or Ronse, 2011). In the case of binary images, the latter are subsets of a set E of points ordered by inclusion. Two operators on $P(E)$ are associated with a given subset B of E; they associate with B the Minkowski difference and the Minkowski sum of any subset X of E, respectively. The former operator is a residual map called *dilatation*, while the latter is the corresponding residuated map called *erosion*. These maps, together with their composition closure and dual closure, are the basic morphological operations in this image processing.

It is also worthwhile pointing out, although they essentially concern infinite but *complete* lattices (i.e., lattices any non-empty subset of which has a join and a meet), the use of covariant Galois connections to describe the correctness of implementations. Here, there are an ordered set P of "abstract" values, an ordered set Q of "concrete" values, an implementation (or concretization) map from P to Q, and a verification (or abstraction) map from Q to P. The existence of interrelating properties between these two maps, guaranteeing correctness of the implementation, is insured if they form a Galois connection (see, for instance, Melton *et al.* (1987)).

Section 3.5 ends with an exposition of the MacNeille completion, also called *normal* completion or *Dedekind–MacNeille* completion, of an ordered set P. This completion easily generalizes to the case where P is no longer finite. The origin of the latter denomination is triple: first, this procedure associates a complete lattice with any ordered set (finite or not); then, the construction of the complete chain of real numbers from the chain of rationals by means of "Dedekind cuts" is an example of this completion; finally, this procedure has been defined for any ordered set P by MacNeille (1937). The characterization of the completion of P as the smallest lattice L for which there exists a coding from P to L preserving all the joins and meets of P is due to Banaschewski (1956) and Schmidt (1956).

3.7 Exercises

Exercise 3.1 Consider the lattice M_3 in Figure 3.1 and the ordered set P in Figure 3.3. Find two maps f and f' from M_3 to P satisfying the following conditions:

- f is strictly isotone and is not injective.
- f' is isotone and injective but is not a coding.

Exercise 3.2 Show that a map f from a lattice L to a lattice L' is an order isomorphism if and only if it is a lattice isomorphism.

State the corresponding result for a dual isomorphism.

Exercise 3.3 [FPP] Let P be an ordered set. One says that P has the *fixed point property* (FPP) if, for any isotone map f on P, there exists a *fixed point*; that is, some $x \in P$ such that $f(x) = x$. Show that any lattice L satisfies FPP. *Hint*: consider the join of the elements x such that $x \leq f(x)$.

Note The unsolved problem to characterize the ordered sets which satisfy FPP have induced many researches (see, for instance, Schröder (2002)).

Exercise 3.4 [Semilattice of maps] Consider an ordered set P, a meet-semilattice Q, and the set $\mathcal{A}$ of all maps from P to Q, endowed with the usual pointwise order: for $f, g \in \mathcal{A}$, $f \leq g$ if $f(x) \leq g(x)$ for any $x \in P$. Given $f, g \in \mathcal{A}$, consider the map $f \wedge g \in \mathcal{A}$ defined by $(f \wedge g)(x) = f(x) \wedge g(x)$ for any $x \in P$. Show that $f \wedge g$ is the meet of f and g in $\mathcal{A}$.

Now consider the ordered subset Q^P of $\mathcal{A}$ formed by all isotone maps from P to Q. Show that Q^P is meet-closed ($f, g \in Q^P$ implies $f \wedge g \in Q^P$) and that Q^P is a sub-meet-semilattice of $\mathcal{A}$.

Give similar results in the case where Q is a join-semilattice, then a lattice.

Exercise 3.5 [Isotone maps, total preorders, and chains; Stanley (1986a)] Show that, if a map f from an ordered set P to an ordered set Q is isotone, then the classes $C(x)$ of the canonical partition of P associated with f are ordered by writing $C(x) \leq C(y)$ if and only if $f(x) \leq_Q f(y)$.

Show that the ordered set of these classes, denoted P/f, is isomorphic to the ordered subset $f(P)$ of Q.

Find a one-to-one correspondence between:

1. the set of the isotone maps from P to $\underline{k}$ (with $1 \leq k \leq n = n(P)$);
2. the set of the total preorders with at most k classes including the order $\leq_P$;
3. the set of the extended chains of length at most k of upsets of P.

Conclude that the ordered set $\underline{2}^P$ of isotone maps from P to $\underline{2}$ is isomorphic (respectively, dual isomorphic) to the ordered set of upsets (respectively, downsets) of P.

Find a one-to-one correspondence between:

1. the set of the surjective isotone maps from P to $\underline{k}$ (with $1 \leq k \leq n = n(P)$);
2. the set of the total preorders with k classes including the order $\leq_P$;
3. the set of the extended chains of length k of upsets of P.

Conclude that the number of surjective isotone maps from P to $\underline{n}$ is equal to the number of linear extensions of P.

Exercise 3.6 [Morgado relation] Show that a map f on an ordered set P is a closure if and only if it satisfies, for any $x \in P, f^{-1}((x]) = f^{-1}((f(x)])$.

This is equivalent to the Morgado relation: for all $x, y \in P$, $y \leq f(x)$ if and only if $f(y) \leq f(x)$.

Write and prove the dual result characterizing dual closures.

Exercise 3.7 [Composition of residuated maps] Let P, Q, and R be three ordered sets and $f: P \longmapsto Q$ and $g: Q \longmapsto R$ two residuated maps together with their corresponding residual maps f^r and g^r. Show that gf is residuated, with $f^r g^r$ as its associated residual map.

Exercise 3.8 [Surjective or injective residuated map; Croisot (1956)] Let f be a residuated map from an ordered set P to an ordered set Q and g the associated residual. Show that the following properties are equivalent and imply that g is a coding from Q to P:

1. $fg = id_Q$.
2. f is surjective.
3. g is injective.
4. For any ordered set R and all maps h and k from Q to R, $hf = kf$ implies $h = k$.

Show that the following properties are equivalent and imply that f is a coding from P to Q:

1′. $gf = id_P$.
2′. f is injective.
3′. g is surjective.
4′. For any ordered set R and all maps h and k from R to P, $fh = fk$ implies $h = k$.

Conclude that the bijective residuated maps from P to Q are isomorphisms between P and Q.

Exercise 3.9 [Residuation and closure; Croisot (1956)] Let f be a residuated map on an ordered set P and g its residual map. Show that the following properties are equivalent:

1. f is a dual closure.
2. g is a closure.
3. $f = fg$.
4. $g = gf$.

Exercise 3.10 Show that an ordered set P is a meet-semilattice if and only if, for any x in P, the canonical injective map m from $(x]$ to P defined by $m(y) = y$ is a residuated map.

Exercise 3.11 [Lattice of residual maps] Let L and L' be two lattices and f_1 and f_2 two isotone maps from L to L'. According to the result in Exercise 3.4, the meet of f_1 and f_2 for the exponentiation order L'^L is the isotone map $f_1 \wedge f_2$ defined by $(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x)$ for any $x \in L$, while the join of f_1 and f_2 is dually defined. Assume that f_1 and f_2 are residual, g_1 and g_2 denoting the associated residuated maps. Using Characterization (2′) of residual maps on page 81, show that $f_1 \wedge f_2$ is still residual and that $g_1 \vee g_2$ is the associated residuated map.

Show that the constant map $f_{\max}$ which sends any $x \in L$ to the maximum of L' is residual. *Hint*: use Definition 3.35 of residual maps.

Deduce that the set of all residual maps from L to L' is a closure net of L'^L and is endowed with a lattice order. What about the set of residuated maps?

Exercise 3.12 [Biclosed relations; Domenach and Leclerc (2001)] Let E, E' be two finite sets, $\mathcal{F}$ and φ (respectively $\mathcal{F}'$ and φ') a Moore family on E (respectively, on E') and its associated closure. A relation $R \subseteq E \times E'$ is said to be *biclosed* if, for all $e \in E$ and $e' \in E'$, $eR = \{e' \in E' : eRe'\} \in \mathcal{F}'$ and $Re' = \{e \in E : eRe'\} \in \mathcal{F}$.

(1) Show that the set $\mathcal{R}_{\mathcal{F}\mathcal{F}'}$ of all biclosed relations between E and E' is a Moore family on $\underline{2}^{E \times E'}$.

(2) Consider the map $t : \underline{2}^{E \times E'} \longmapsto \mathcal{F}'^{\mathcal{F}}$ which sends the relation R to the map tR defined by: for any $F \in \mathcal{F}$, $tR(F) = \varphi'(f_R(F)) = \varphi'(\bigcap_{e \in F} eR)$. Show that, if R is biclosed, then tR is a Galois map from the lattice $\mathcal{F}$ to the lattice $\mathcal{F}'$.

(3) Consider the map $u : \mathcal{F}'^{\mathcal{F}} \longmapsto \underline{2}^{E \times E'}$ which sends a map f to the relation uf defined by: $uf = \{(e, e') \in E \times E' : e' \in f(\varphi(e))\}$. Show that, if f is a Galois map, then uf is biclosed. Does the pair (t, u) form a residuated/residual one?

(4) Denote by $\mathcal{F} \otimes \mathcal{F}'$ the set of Galois maps from $\mathcal{F}$ to $\mathcal{F}'$ endowed with the pointwise order. Show that the ordered set $\mathcal{R}_{\mathcal{F}\mathcal{F}'}$ ordered by inclusion and $\mathcal{F} \otimes \mathcal{F}'$ are isomorphic by the restrictions of t and u to these ordered sets.

Note As a consequence of these results, biclosed relations form a set-coding of the Galois maps between two finite lattices. On the other hand, they also happen to be a particular case of relational Galois connections, as considered by Ganter (2007). The latter form a generalization of Galois connections between ordered sets, where orders are replaced with relations of a more general type.

Exercise 3.13 Let $P = (X, \leq)$ be an ordered set. Show that a subset A of X is an element of the lattice $Gal(X, X, \not>)$ if and only if A is a maximal antichain of P.

Show that a subset $A \times B$ of X^2 is a concept of the lattice $Gal(X, X, \not\geq)$ if and only if A is a downset and B an upset of P.

Deduce that the sets $\mathcal{D}(P)$ and $\mathcal{U}(P)$, ordered by inclusion, are two dual lattices (a sophisticated proof for an obvious result!). What about the lattice $Gal(X, X, \not\leq)$?

Exercise 3.14 [Maximal antichains of a bipartite ordered set; Morvan and Nourine (1996)] Let $R \subseteq X \times Y$ be a relation between two sets X and Y and let R^c be the complementary relation of R in $X \times Y$. Show that a subset $A \times B$ of $X \times Y$ is a concept of the lattice $Gal(X,Y,R)$ if and only if $A+B$ is a maximal antichain of the bipartite ordered set $(X+Y,R^c)$.

Show that the order on maximal antichains induced by this correspondence is the same as the inclusion order on the associated upsets.

Using Exercise 3.13, prove that the lattice of downsets of an ordered set P, the lattice of maximal antichains of P, and the completion of P are isomorphic to a lattice of maximal antichains of an appropriate bipartite ordered set.

Let L be a lattice. From the above results and Theorem 3.52, show that L is isomorphic to the lattice of maximal antichains of a bipartite ordered set.

Exercise 3.15 [Arrow relations of an ordered subset] Show that the arrow relations are preserved in an ordered subset; that is, for instance, given an ordered set P and an ordered subset Q of P, $x \downarrow y$ in P and $x,y \in Q$ imply $x \downarrow y$ in Q.

Recall that, for any ordered set P, $IR(P) = J_P \cup M_P$ denotes the set of irreducible elements of P. Consider an ordered subset Q of P including $IR(P)$. Derive from the previous results that any join-irreducible of P is still a join-irreducible of Q.

Show that an element $x \in Q \setminus J_P$ cannot be a join-irreducible of Q and, then, that $J_Q = J_P$. Show that, similarly, $M_Q = M_P$.

Conclude that Q has the same arrowed table as P.

Exercise 3.16 [Properties of a lattice table] Let J_L and M_L be the sets of join-irreducibles and of meet-irreducibles of a lattice L, and $J_m = \{j \in J_L : j \leq m\}$. Show that none of the following three situations can occur:

1. There exists $m \in M_L$ such that $J_m = J_L$.
2. There exist $m,m' \in M_L$ such that $m \neq m'$ and $J_m = J_{m'}$.
3. There exist $m \in M_L$ and $M' \subset M_L$ with $m \notin M'$ and $J_m = \bigcap\{J_{m'} : m' \in M'\}$.

Show the dual results by exchanging join-irreducibles and meet-irreducibles.

Exercise 3.17 [Strict completions of an ordered set] Let $P = (X, \leq)$ be an ordered set and $\mathcal{F} = \mu(\underline{2}^X)$, isomorphic to the completion of P (see Section 3.5.3). A lattice $\mathcal{K}$ of some downsets of P is called a *strict completion* of P if the join-irreducibles of $\mathcal{K}$ are all the principal downsets of P. For which ordered sets P is $\mathcal{F}$ a strict completion of P? If $\mathcal{F}$ is not a strict completion of P, which downsets have to be added to $\mathcal{F}$ in order to get a strict completion of P? The set of strict completions of P has a maximum: what is it? Find an ordered set which has a unique strict completion (which is or is not equal to the completion of P).

Note The set of strict completions of P, ordered by inclusion, is itself a lattice, studied by Bordalo and Monjardet (2002, 2003).

Exercise 3.18 [Dissectors] If an ordered set P may be obtained as the disjoint union of a principal upset $[x)$ and of a principal downset $(y]$, we say that x (respectively, y)

is an *upper dissector* (respectively, a *lower dissector*) *of* P. The set of upper dissectors of P is denoted by $Dis(P)$.

Use the examples of ordered sets given in this chapter to show that an ordered set may have no dissectors. Find ordered sets the dissectors of which are all its join-irreducibles.

Characterize (upper) dissectors by an arrow property (see Definition 1.36) and show that they are join-irreducible. Find all ordered sets of size less than or equal to 4 such that $Dis(P) = J(P)$ (there are 12 such ordered sets).

Let $Gal(P)$ be the MacNeille completion of P. Show that the two ordered sets $Dis(P)$ and $Dis(Gal(P))$ are isomorphic.

4

Chains and antichains

The problems of sorting, searching, and scheduling encountered, for instance, in computer science and operations research frequently involve the determination of the width of some ordered set; that is, the maximum size of its antichains. Two illustrations of this general observation are given (Example 4.28 and Exercise 4.2). Thus, this chapter is devoted to the study of the width and to some related topics. First, Dilworth's Theorem states that, in any ordered set P, the minimum number of chains in a chain partition of P is equal to its width. This is one of the most famous results in the field of combinatorics, and the subject of the first two following sections. The theorem is stated and proved in Section 4.1, together with its close relatives. Section 4.2 is devoted to its consequences in the special case of bipartite ordered sets and points out its equivalence to König–Egerváry's Theorem on matchings and transversals in such a structure. In Section 4.5 the importance of this equivalence is emphasized. Especially it leads to an algorithmic determination of the width, by means of results on flows in graph theory. It is moreover recalled that Dilworth's Theorem is also equivalent to three fundamental results of combinatorics, namely the König–Hall, Menger, and Ford and Fulkerson theorems. These results are quite essential and have many practical applications, for instance on binary matrices or allocation problems for the first one (Exercise 4.6) and on transportation networks for the other two.

The above-mentioned flow algorithms provide a computation of the width of any ordered set. Nevertheless, in the case of a ranked ordered set P, the width may be equal to the maximum size of a rank-set of P. An ordered set satisfying this property is called a *Sperner ordered set*. In such an ordered set, there exists an alternative – and generally more direct – way to determine the width. The previous designation comes from the fact that this property was established by Sperner in 1928 for the Boolean lattice $\underline{2}^E$ of subsets of a set E. The generalizations of Sperner's Theorem, which are considered in Section 4.3, form another major topic in the combinatorics of ordered sets. In particular, products of chains are an important example of Sperner ordered sets, that frequently appear in the modeling of problems pertaining to domains such as multicriterion decision or data analysis. Section 4.4 is devoted to the determination of the sizes of the rank-sets of products of chains and, so, to the determination of their width.

4.1 Dilworth's decomposition Theorem

In Section 1.3.2 of Chapter 1, we defined four fundamental parameters associated with any ordered set P:

- $\kappa(P)$, the maximum size of a chain of P, is its *height*.
- $\alpha(P)$, the maximum size of an antichain of P, is its *width*.
- $\theta(P)$ is the minimum number of chains in a chain partition of P.
- $\gamma(P)$ is the minimum number of antichains in an antichain partition of P.

If Q is an ordered subset of P, then the inequality $\alpha(Q) \leq \alpha(P)$ holds, since an antichain A of Q of size $\alpha(Q)$ is still an antichain of P but not necessarily of maximum size. Likewise, the inequality $\kappa(Q) \leq \kappa(P)$ is obtained when considering chains.

The inequalities $\kappa(P) \leq \gamma(P)$ and $\alpha(P) \leq \theta(P)$ in any ordered set P are straightforward, since two elements of P cannot belong simultaneously to a chain and an antichain. The purpose of this section is to show that these inequalities are in fact equalities (Theorems 4.1 and 4.2) and to deduce some consequences. We easily obtain the first equality.

Theorem 4.1 *For any ordered set P, the equality $\kappa(P) = \gamma(P)$ holds.*

Proof We construct a partition of P into h antichains $A_1, A_2, ..., A_h$ as follows: A_1 is the set of all minimal elements of P and, for $k > 1$, A_k is the set of all minimal elements of the ordered subset $P_k = P \setminus (\bigcup_{1 \leq j < k} A_j)$. Each A_k is an antichain of P_k, so an antichain of P and, for any $k > 1$, every element x_k of A_k covers at least one element x_{k-1} of A_{k-1} (otherwise there would exist some $x_k \in A_{k-1}$). So we are able to construct a covering chain $C = x_1 \prec x_2 \prec ... \prec x_h$ of P of size h. Then, we obtain on the one hand $\gamma(P) \leq h$, since P is partitioned into h antichains and on the other hand $h \leq \kappa(P)$, since the size of the chain C is h. The inequality $\gamma(P) \leq \kappa(P)$ follows and, since the converse inequality has already been observed, the equality $\gamma(P) = \kappa(P)$ holds. □

This proof is constructive in the sense that a partition of P into $\gamma(P)$ antichains is actually obtained, together with a chain of P of maximum size $\kappa(P)$. This chain corresponds to a sequence $x_1, x_2, ..., x_{\kappa(P)}$ with, for $k = 1, ..., \kappa(P)$, $x_k \in A_k$ and, for $k = 2, ..., \kappa(P)$, $x_{k-1} \prec x_k$.

The situation is different for the equality $\alpha(P) = \theta(P)$, which is the subject of Dilworth's Theorem, and which is obtained below in a non-constructive way. In the sequel, a chain C of P is said to *meet* an antichain A of P if their intersection is not empty.

Theorem 4.2 (Dilworth, 1950) *For any ordered set P, the equality $\alpha(P) = \theta(P)$ holds.*

Proof We proceed by induction on the size $n = n(P)$ of P. For $n = 1$, the equality $\theta(P) = \alpha(P) = 1$ is immediate. Assume that the result is true for any ordered set P' of size $n' < n$ and consider a maximal chain C of P. Two cases may then occur.

Case 1: if C meets every antichain of P of size $\alpha(P)$, then $\alpha(P \setminus C) = \alpha(P) - 1$ and, by the induction hypothesis, $P \setminus C$ may be partitioned into $\theta(P) - 1 = \alpha(P) - 1$ chains. So, adding the chain C, we obtain that P is partitioned into $\alpha(P)$ chains.

Case 2: there exists a maximum size antichain $A = \{a_1, a_2, ..., a_{\alpha(P)}\}$ of P which does not meet C. The antichain A determines two ordered subsets of P, namely its down closure $(A] = \{x \in P$: there exists $a \in A$ such that $x \leq a\}$ and its up closure $[A) = \{x \in P$: there exists $a \in A$ such that $a \leq x\}$ (Chapter 3, Example 3.33). Both of them are still of width $\alpha(P)$ since they include the antichain A.

Let x be an element of $P \setminus A$. If x is incomparable to every element of A, then $A + \{x\}$ is still an antichain, a contradiction with the maximum size of A in P. If x is simultaneously greater than some element a and less than another element a' of A, then $a \leq x \leq a'$, which is impossible since A is an antichain. Thus $(A] \cup [A) = P$ and $(A] \cap [A) = A$.

Observe that the greatest element of C does not belong to $(A]$ since it would then be less than some element of A and the chain C would not be maximal. So $(A] \subset P$ and the induction hypothesis applies to the ordered subset $(A]$, allowing us to partition it into $\alpha(P)$ chains $C_{11}, C_{12}, ..., C_{1\alpha(P)}$, such that, for $i = 1, 2, ..., \alpha(P)$, the element a_i of A belongs to the chain C_{1i}. Moreover, a_i is the greatest element of C_{1i}, since otherwise there would exist a maximal element a of $(A]$ – so an element of A – with $a_i < a$, and A would not be an antichain. Similarly, there exists a partition of the ordered subset $[A)$ into $\alpha(P)$ chains $C_{21}, C_{22}, ..., C_{2\alpha(P)}$ such that, for $i = 1, 2, ..., \alpha(P)$, a_i is the least element of C_{2i}.

Finally, for $i = 1, ..., \alpha(P)$, $C_i = C_{1i} \oplus' C_{2i}$ (Chapter 1, Remark 1.43) is a chain of P, and the C_i's form a partition of P into $\alpha(P)$ chains, as required. □

Example 4.3 It is easy to check that the width $\alpha(P)$ of the ordered set P – a diagram of which is given in Figure 4.1(a) – is equal to 2. So, according to Theorem 4.2, P may be partitioned into two chains; Figure 4.1(b) shows such a partition. To obtain a partition into a minimum number of chains is frequently not straightforward: for instance, beginning with the chain *abcd* suggested by the alphabetic order on the labels of the elements of P makes it necessary to keep the other two chains e and f (Figure 4.1(c)).

Remark 4.4 Both Theorems 4.1 and 4.2 above are of "min–max" type: they state that the minimum of a considered set of values (for instance, the number of chains in chain partitions of P) is equal to the maximum of another set of values (for instance,

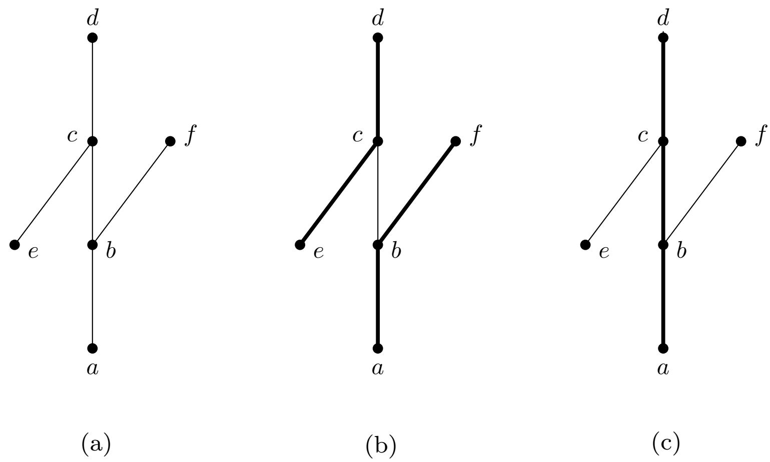


Figure 4.1 (b) and (c) Two partitions of the ordered set in (a) into, respectively, 2 and 3 chains.

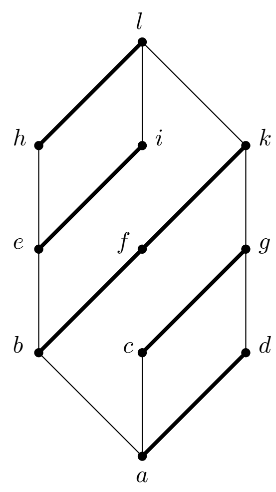


Figure 4.2 An ordered set P with $\alpha(P) = \theta(P) = 5$.

the sizes of the antichains of P). Then finding instances of the first and the second sets which achieve equality proves that this minimum and this maximum are actually reached. For example, to obtain in an ordered set P – a partition of P into p chains together with an antichain of size p implies $\alpha(P) = \theta(P) = p$. So, observing that the ordered set P – the diagram of which is given in Figure 4.2 – has a partition into five chains (bolded edges) and one antichain $\{c,d,f,h,i\}$ of size 5 implies the equalities $\theta(P) = \alpha(P) = 5$.

As will be shown in the next proposition, several statements on ordered sets are – almost directly – equivalent to Dilworth's Theorem. A set $\mathcal{R} = \{C_1, ..., C_h\}$ of chains of an ordered set P is called a *covering* of P if every element x of P belongs to at least one chain of $\mathcal{R}$.

Proposition 4.5 *Each of the following four statements is equivalent to Dilworth's Theorem:*

1. *For any ordered set P, every chain of a partition of P into $\theta(P)$ chains meets every antichain of P of size $\alpha(P)$.*
2. *For any ordered set P, there exists a chain of P which meets every antichain of P of size $\alpha(P)$.*
3. *For any ordered set P, there exists an antichain A of P and a chain partition* **P** *of P such that A meets every chain of* **P**.
4. *For any ordered set P, there exists a covering of P with $\alpha(P)$ maximal chains.*

Proof We first give a circular proof of the equivalence of Items (1) and (2) with Dilworth's Theorem, then with Item (3), and finally the equivalence of Item (4) with Dilworth's Theorem.

Dilworth's Theorem $\Longrightarrow$ (1): if (1) is not satisfied by an ordered set P, there exists an antichain A of P of size $\alpha(P)$, a partition $\mathbf{P}$ of P into $\theta(P)$ chains, and a chain C of $\mathbf{P}$ such that C does not meet A. Then, the partition $\mathbf{P}$ is formed of $\alpha(P)$ chains each of which contains an element of A and of the chain C; this implies $\alpha(P) < \theta(P)$, a contradiction.

(1) $\Longrightarrow$ (2): choose a chain of a partition of P into $\theta(P)$ chains.

(2) $\Longrightarrow$ Dilworth's Theorem: if (2) is true, then Case 1 in the proof of Dilworth's Theorem always occurs and may be used to prove the latter by induction.

(1) $\Longrightarrow$ (3): choose a partition of P into $\theta(P)$ chains and an antichain of P of size $\alpha(P)$.

(3) $\Longrightarrow$ Dilworth's Theorem: let A and $\mathbf{P}$ be respectively an antichain and a chain partition of P such that A meets every chain of $\mathbf{P}$. Then $|\mathbf{P}| \leq |A|$ which, together with the inequalities $|A| \leq \alpha(P) \leq \theta(P) \leq |\mathbf{P}|$, implies $\alpha(P) = \theta(P)$; that is, Dilworth's Theorem.

Dilworth's Theorem $\Longrightarrow$ (4): if $\mathbf{P}$ is a partition of P into k chains, then extend each of these chains into a maximal chain in order to obtain a covering of P with k maximal chains. Take $k = \alpha(P)$ to see that Dilworth's Theorem implies (4).

(4) $\Longrightarrow$ Dilworth's Theorem: if (4) holds, then there exists a covering $\mathcal{R}$ of P with $\alpha(P)$ maximal chains. Let $C_1, ..., C_{\alpha(P)}$ be an arbitrary numbering of the elements of $\mathcal{R}$. Write $C'_k = C_k \setminus (\bigcup_{1 \leq j < k} C_j)$ for $k = 2, ..., \alpha(P)$. The set $\{C_1, C'_2, ..., C'_{\alpha(P)}\}$ forms a partition of P into $\alpha(P)$ chains (none of the C'_k's is empty since it would imply the

existence of a partition of P into less than $\alpha(P)$ chains). So (4) implies Dilworth's Theorem and the equivalence follows. □

4.2 Matchings and transversals in a bipartite ordered set

A bipartite ordered set was defined in the first section of Chapter 2 as an ordered set P of height $\kappa(P) = 2$. Such an ordered set is denoted $P = (Y + Z, \leq)$, with Y (respectively, Z) the set of its minimal (respectively, maximal) elements. In this section, only bipartite ordered sets without isolated elements are considered; that is, every element is either minimal or maximal but not both. Then, the notations $y \leq z$ with $y \in Y$ and $z \in Z$, $y \prec z$ or $y < z$ will be equivalent. For any subset X' of $X = Y + Z$, we write $X'_Y = X' \cap Y$ and $X'_Z = X' \cap Z$. Write $p = |Y|$ and $q = |Z|$ (with $n = p + q$). The inequality $p \leq q$ is assumed in this section, without loss of generality since P may always be replaced with its dual.

There are several equivalent ways to consider bipartite ordered sets:

- the binary relation R between Y and Z defined by $R = \{(y,z) \in Y \times Z : y < z\}$;
- the bipartite graph (Y,Z,R), with R as above;
- the 0/1 matrix corresponding to the map μ from $Y \times Z$ to $\{0,1\}$ defined by $\mu(y,z) = 1$ if $y < z$ and $\mu(y,z) = 0$ otherwise;
- the family $\mathcal{F} = \{yP : y \in Y\}$ of subsets of Z, or the *"dual" family* $\mathcal{F}_d = \{Pz : z \in Z\}$ *of subsets of* Y.

The first two are just different terminologies, whereas the last two are different but equivalent structures.

Thus the combinatorial results obtained below with an ordinal approach also apply to each of these types of structure (and are frequently presented in these terms in the literature). Let us first introduce some definitions:

Definition 4.6 Let $P = (Y + Z, \leq)$ be a bipartite ordered set with $X = Y + Z$, $p = |Y|$ and $q = |Z|$.

- A *matching* of P is a set of pairwise disjoint chains of P of length 1. Let $\sigma(P)$ be the maximum number of chains in a matching of P. A matching of P with p chains is called a *matching from* Y *into* Z.
- A *transversal of the chains of length* 1 *of* P is a subset T of $X = Y + Z$ such that, for all $y,z \in X$ with $y < z$, either $y \in T$ or $z \in T$. The minimum size of such a transversal (which will simply be called a *transversal of* P in the sequel) is denoted $\tau(P)$.
- Let Y' be a subset of Y and $]Y')$ the set of all the elements $z \in Z$ such that $y < z$ for at least one element y of Y'. The *deficiency* $\delta(Y')$ of Y' is equal to $|Y'| - |]Y')|$ if $|]Y')| \leq |Y'|$ and to 0 otherwise. The *deficiency of* P is defined by $\delta(P) = \max\{\delta(Y') : Y' \subseteq Y\}$.

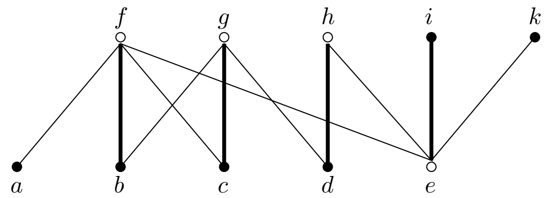


Figure 4.3 A maximum size matching and a minimum size transversal in a bipartite ordered set P.

Example 4.7 The bipartite ordered set P in Figure 4.3 satisfies $aP = \{f\}$, $bP = cP = \{f,g\}$, $dP = \{g,h\}$, and $eP = \{f,h,i,k\}$, with $p = q = 5$, $\sigma(P) = \tau(P) = 4$, and $\delta(P) = 1$ (check these values). The bolded lines correspond to a matching of maximum size of P and the white circles to a transversal of P of minimum size.

Proposition 4.8 below shows that the three parameters $\sigma(P)$, $\tau(P)$, and $\delta(P)$ associated with a bipartite ordered set P are related to the parameters $\alpha(P)$ and $\theta(P)$ of P (see page 108). Then, in Theorem 4.9, an expression of two classic combinatorial results will be derived from Dilworth's Theorem. The same Proposition 4.8 will also allow us to obtain Theorem 4.10, the famous König–Hall result on the existence of a matching from Y into Z in a bipartite graph (or ordered set), also stated in its most classic form in Corollary 4.11.

Notice that, in a bipartite ordered set $P = (Y+Z, \leq)$, the antichains Y and Z are transversals of P, and that a transversal of P must contain at least one element of each chain of a matching of P. The sequence of inequalities $\sigma(P) \leq \tau(P) \leq p \leq q \leq \alpha(P)$ is then straightforward. Here are some other relations between these parameters.

Proposition 4.8 *For any bipartite ordered set P of size $n = p+q$, the following equalities hold:*

1. $\tau(P) + \alpha(P) = n$,
2. $\alpha(P) = q + \delta(P)$,
3. $\tau(P) = p - \delta(P)$,
4. $\theta(P) + \sigma(P) = n$.

Proof The equality in (1) comes from the fact that the transversals of $P = (Y+Z, \leq)$ are exactly the complements in $Y+Z$ of the antichains of P.

For the proof of (2), first consider an antichain A of P of maximum size $\alpha(P)$. Observe that the maximality of A implies $|]A_Y)| = |Z \setminus A_Z|$. So $\delta(P) \geq \delta(A_Y) = |A_Y| - |]A_Y)| = |A_Y| - (|Z| - |A_Z|) = \alpha(P) - q$. Also $\alpha(P) \geq q + \delta(P)$ since, for any $Y' \subseteq Y$ with $\delta(Y') = \delta(P)$, the subset $Y' \cup (Z \setminus]Y'))$ is an antichain of P of size $|Y'| + |Z| - (|Y'| - \delta(Y')) = q + \delta(Y') = q + \delta(P)$. We thus have (2), which we subtract term by term from (1) to obtain (3).

A partition of P into $c' + (n-2c') = n-c'$ chains is associated with any matching of P with c' chains, completed with 1-element chains. Thus, starting from a matching of P with $\sigma(P)$ chains, the inequality $\theta(P) \leq n-\sigma(P)$ holds. Conversely, a partition of P into k chains is formed of c' chains of a matching together with $n-2c'$ 1-element chains; so $c' = n-k$. Starting from a partition into $\theta(P)$ chains, we find $\sigma(P) \geq n-\theta(P)$ and finally the equality in (4). $\square$

The proof of the above proposition does not use the equality $\alpha(P) = \theta(P)$ of Dilworth's Theorem. From the equalities in (1) and (4) above, taking the latter theorem into account leads to the first expression of the parameter $\sigma(P)$ given in Theorem 4.9 below. This expression, called the *König–Egerváry Theorem*, is again a "min–max" type result. The second expression of $\sigma(P)$ in this theorem comes from the previous one, together with Item (1) in Proposition 4.8 and the equalities $\alpha(P) = \max_{A\in\mathcal{A}(P)}\{|A_Y|+|A_Z|\} = \max_{Y'\subseteq Y}\{|Y'|+|Z\backslash]Y')|\}$, where $\mathcal{A}(P)$ is the set of all antichains of P. This expression of $\sigma(P)$ is known as the *König–Öre Theorem*.

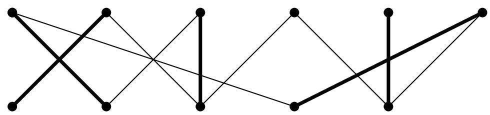
Theorem 4.9 (König–Öre and König–Egerváry) *In any bipartite ordered set $P = (Y+Z,\leq)$, the equalities $\sigma(P) = \tau(P) = \min_{Y'\subseteq Y}\{|Y \setminus Y'|+|]Y')|\}$ hold.*

Conversely, the equality $\sigma(P) = \tau(P)$ may be obtained in other ways, then used to derive Dilworth's Theorem. Indeed, searching for a partition of some ordered set P (not necessarily bipartite) into $\theta(P)$ chains is equivalent to the search for a matching into $\sigma(P')$ chains in a bipartite ordered set P' associated with P. Similarly, to search for an antichain of P of size $\alpha(P)$ is equivalent to searching for a transversal of P' of size $\tau(P')$. In Section 4.5 (Further topics and references), we mention how these considerations lead to the practical determination of these parameters (see also Exercise 4.5).

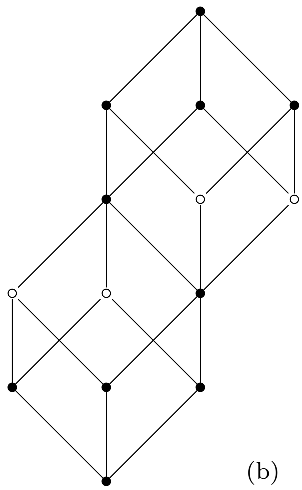
The next theorem characterizes the particular situation where the parameter $\alpha(P)$ reaches its smallest possible value q, which means that the set Z is an antichain of P of maximum size. The equivalence of Conditions (1) and (3) is nothing but the *König–Hall Theorem* – one of the most famous theorems in combinatorics – on the existence of a matching from Y into Z. The classic forms of this result are given in Corollary 4.11 and Exercise 4.6, while Figure 4.4(a) shows an example of such a matching. Each element y of Y belongs to one of the chains of such a matching, the other element of the chain (belonging to Z) being denoted $\iota(y)$. This defines an injective map ι from Y to Z which is extensive since $y < \iota(y)$, for any $y \in Y$. Clearly, the existence of such a matching is equivalent to the equality $\sigma(P) = p$.

Theorem 4.10 *For any bipartite ordered set P, the following three conditions are equivalent:*

1. $\sigma(P) = \tau(P) = p$,
2. $\alpha(P) = \theta(P) = q$,
3. $\delta(P) = 0$.



(a)



(b)

Figure 4.4 (a) A Sperner bipartite ordered set and (b) a non-Sperner distributive lattice.

Proof If Condition (1) is satisfied then, according to Items (1) and (4) in Proposition 4.8, the equalities $\alpha(P) = n - p = q$ and $\theta(P) = n - p = q$ hold, hence Condition (2). Likewise, the implications of (3) by (2) and (1) by (3) come from Items (2) and (3) in Proposition 4.8. □

Since $\sigma(P) = p$ means that there exists a matching from Y into Z, we obtain:

Corollary 4.11 (König, 1931) *Let $P = (Y + Z, \leq)$ be a bipartite ordered set. There exists a matching from Y into Z if and only if the deficiency $\delta(P)$ of P is null; that is, if $|Y'| \leq |]Y')|$ for any $Y' \subseteq Y$.*

Example 4.12 In the bipartite ordered set P in Figure 4.3, there exists no matching from Y into Z since $\delta(P) = 1$ (observe that the "strict" up closure $]\{a,b,c,d\})$ is equal to $\{f,g,h\}$).

Obviously, the complete bipartite ordered set $K_{Y,Z}$ has a matching from Y into Z. This observation extends as follows: a bipartite ordered set $P = (Y + Z, \leq)$ is said to be *regular* when each element of Y is covered by the same number k of elements of Z, while each element of Z covers the same number k' of elements of Y. The equality $kp = k'q$ is obtained by counting in two different ways the ordered pairs (y,z) such that $y < z$. Thus, $k \geq k'$ (since $p \leq q$). For $Y' \subseteq Y$, we also have $k|Y'| \leq k'|]Y')|$ since the ordered pairs with the lower element in Y' form a subset of those with the greater element in $]Y')$; so $k|Y'| \leq k|]Y')|$ and, according to Corollary 4.11, we conclude:

Corollary 4.13 *If $P = (Y + Z, \leq)$ is a regular bipartite ordered set, then it has a matching from Y into Z and the equality $\alpha(P) = |Z| = q$ holds.*

4.3 The Sperner property

In this section all considered ordered sets are ranked, with the normalized rank function r. In such an ordered set P of rank $r(P)$, for $k = 0,...,r(P)$, the *rank-set* $R_k = \{x \in P : r(x) = k\}$ *of rank* k and the Whitney numbers $n_k = |R_k|$ of P have been defined in Section 2.1. Denote by $\nu(P) = \max_{0\leq k\leq r(P)} n_k$ the maximum Whitney number. Rank-sets are particular antichains, so $\nu(P) \leq \alpha(P)$ holds.

Definition 4.14 A ranked ordered set P is called a *Sperner ordered set* – or is simply said to be *Sperner* – if it satisfies the *Sperner property*; that is, every rank-set of maximum size is a maximum size antichain (i.e., the equality $\nu(P) = \alpha(P)$ holds).

We will recall in Section 4.5 that the search for a maximum size antichain of P is a problem treatable in any ordered set since it is equivalent to the search for a maximum flow in a graph associated with P. Nevertheless, the solution of this problem is much more immediate in a Sperner ordered set as soon as the determination of the rank of any element is easy.

The equality $\nu(P) = \alpha(P)$ in Boolean lattices, now called Sperner's Theorem, was established in 1928 by Sperner. In this section several ways of recovering this result are presented, each of them leading to a class of Sperner ordered sets. First, two conditions equivalent to the Sperner property are given in Theorem 4.17, then several conditions on a ranked ordered set which, alone or combined, imply the Sperner property. We finally give three proofs of Sperner's Theorem (Theorem 4.20).

Remark 4.15 Sperner bipartite ordered sets are those having a matching from $Y = R_0$ into $Z = R_1$ or from R_1 into R_0 (the inequality $|Y| \leq |Z|$ is no longer assumed), i.e., those satisfying the conditions in Theorem 4.10.

Example 4.16 The ordered set P in Figure 4.1(a) is Sperner since it satisfies $\nu(P) = \alpha(P) = 2$; the one in Figure 4.2, with $\nu(P) = 3$ and $\alpha(P) = 5$, is not Sperner. According to the above remark, Figure 4.4(a) shows a Sperner bipartite ordered set since it has a matching from Y into Z. Figure 4.4(b), where the white elements form a maximum size antichain, shows the diagram of a non-Sperner distributive lattice L (see Chapter 2, Definition 2.19) (check that $\alpha(L) \geq \nu(L)$).

We start with two characterizations of Sperner ordered sets, given in Theorem 4.17 below. Both reveal that the Sperner property is linked to the existence of some relations between the chains and the rank-sets of an ordered set. Condition (N) is a direct consequence of Dilworth's Theorem together with Remark 4.4.

Consider a covering $\mathcal{R}$ of an ordered set P with h maximal chains $C_1,...,C_h$, not necessarily all distinct. For any element x of P, set $\rho_{\mathcal{R}}(x) = |\{j \in \{i,...,h\} : x \in C_j\}|$ and $\rho = \min_{x\in P} \rho_{\mathcal{R}}(x)$. A rank-set R is said to be $\mathcal{R}$*-regular* if any element x of R belongs to the same number $\rho_{\mathcal{R}}(x) = \frac{h}{|R|}$ of these chains. It is *minimum-*$\mathcal{R}$*-regular* if, moreover, for any x of R, the equality $\rho_{\mathcal{R}}(x) = \rho$ holds.

Theorem 4.17 *Let P be a ranked ordered set. The following three conditions are equivalent:*

(S) P is Sperner.
(N) There exists a partition of P into $\nu(P)$ chains.
(MR) There exists a covering $\mathcal{R}$ of P with maximal chains and a rank-set R of P such that R is minimum-$\mathcal{R}$-regular.

Proof Condition (S) means that $\nu(P)$ is equal to the width $\alpha(P)$ of P which, according to Dilworth's Theorem, is the minimum number of chains in a chain partition of P. So (S) implies (N).

To show that (N) implies (MR), consider a partition of P into $\nu(P)$ chains and complete each of these chains into a maximal one if it is not already so. Obviously the obtained set of chains constitutes a covering $\mathcal{R}$ of P with maximal chains, while every rank-set R of size $\nu(P)$ is minimum-$\mathcal{R}$-regular, with $\rho_{\mathcal{R}}(x) = 1$ for any $x \in R$.

Finally, if (MR) is true and if R is a minimum-$\mathcal{R}$-regular rank-set for a covering $\mathcal{R}$ of P with h maximal chains, then, for any antichain A of P, the inequalities $h \geq \Sigma_{x\in A}\rho_{\mathcal{R}}(x) \geq \rho|A|$ hold. Thus $|A| \leq \frac{h}{\rho} = |R|$ and P is Sperner. □

As a particular case, a condition implying (MR) and thus implying that P is Sperner, is obtained by taking as $\mathcal{R}$ the set $\mathcal{C}$ of all maximal chains of P:

(MCR) There exists a minimum-$\mathcal{C}$-regular rank-set of P.

Let P be a ranked ordered set and $P_k = (R_k + R_{k+1}, \leq)$ the bipartite ordered sets, restrictions of P to two consecutive rank-sets (with $0 \leq k \leq r(P) - 1$). Consider the following conditions, that we first state then explain and comment on:

(REG) For any $k = 0, ..., r(P) - 1$, the bipartite ordered set P_k is regular.
(SBR) For any $k = 0, ..., r(P) - 1$, the bipartite ordered set P_k is Sperner.
(UNI) The ordered set P is unimodal.
(RSU) The Whitney numbers of P satisfy $n_0 = n_{r(P)} \leq n_1 = n_{r(P)-1} \leq ... \leq n_k = n_{r(P)-k} \leq ...$, for $k \leq \frac{r(P)+1}{2}$.
(SYM) The ordered set P is symmetric chain.

These conditions, or some combinations of them, will allow us to determine several classes of Sperner ordered sets. Conditions (REG) of *regularity* and (SBR) – for *Sperner by rank-sets* – concern the bipartite ordered sets P_k.

A finite integer sequence $m_0, m_1, ..., m_q$ is *unimodal* if it is the concatenation of an increasing sequence $m_0 \leq ... \leq m_p$ and a decreasing sequence $m_{p+1} \geq ... \geq m_q$. A ranked ordered set is *unimodal* if the sequence of its Whitney numbers n_k, $k = 0, 1, ..., r(P)$, is unimodal (it is obvious or well-known that chains and Boolean lattices are unimodal); this condition is denoted by (UNI). A particular case of Condition (UNI) is Condition (RSU) of *rank symmetry–unimodality*. The latter implies $\nu(P) = n_{\frac{r(P)}{2}}$ if $r(P)$ is even and $\nu(P) = n_{\frac{(r(P)-1)}{2}} = n_{\frac{(r(P)+1)}{2}}$ if $r(P)$ is odd.

A chain C of P is said to be *symmetric* if it is covering and if $r(maxC)+r(minC) = r(P)$, where $maxC$ and $minC$ are respectively the greatest and the least elements of C. A ranked ordered set is *symmetric chain* if it has a symmetric chain partition, i.e., a chain partition of P in which every chain is symmetric.

Proposition 4.18 *Let P be a ranked ordered set.*

1. *If P is regular, then it satisfies* (SBR).
2. *If P satisfies* (RSU)*, then it satisfies* (UNI).
3. *If P satisfies* (SYM)*, then it satisfies* (RSU).
4. *If P satisfies* (SYM)*, then it satisfies* (UNI) *and* (SBR).
5. *If P satisfies* (UNI) *and* (SBR)*, then it is Sperner.*

Proof (1) This implication is a direct consequence of Corollary 4.13.

(2) and (3) come from the definitions of the properties (RSU), (UNI), and (SYM).

(4) Assume that P has a symmetric chain partition $\mathbf{P}$. It follows from (2) and (3) that P satisfies (UNI). Now we prove that P also satisfies (SBR). For any rank-set R_k of P, n_k is the number of chains of $\mathbf{P}$ which have an element in R_k. Assume $k \leq \frac{r(P)}{2}$. By the symmetry of these chains, if one of them has an element in the rank-set R_{k-1}, it also meets the rank-set $R_{r(P)-(k-1)}$ and, since it is covering, it has an element in the rank-set R_k. Thus there exists a matching from R_{k-1} into R_k and so P_{k-1} is Sperner and satisfies $n_{k-1} \leq n_k$ (see Remark 4.15). A similar situation occurs for $k \geq \frac{r(P)}{2}$, with $n_k \geq n_{k+1}$, which implies (SBR).

(5) We show that, if P satisfies (UNI) and (SBR), then it may be partitioned into $\nu(P)$ chains. Let R_m be a rank-set of size $\nu(P)$. If $m \neq 0$, then, by (UNI), $n_{m-1} \leq n_m$ and, by (SBR) and Remark 4.15, there exists a matching from R_{m-1} into R_m, corresponding to an injective and extensive map ι_{m-1} from R_{m-1} into R_m. Similarly, injective and extensive maps $\iota_k : R_k \to R_{k+1}$ are obtained for any k such that $0 \leq k < m$. The sequences $x, \iota_{m-1}^{-1}(x), \iota_{m-2}^{-1}(\iota_{m-1}^{-1}(x)), \ldots$, for each x in the rank-set R_m, form a partition of the downset $(R_m]$ into $\nu(P)$ chains, each of them with a distinct element of R_m as its maximum. The upset $[R_m)$ being partitioned likewise, the pairwise concatenations of the obtained chains provide a partition of P into $\nu(P)$ chains. □

The lattice in Figure 4.4(b) satisfies (UNI) but not (SBR) and is not Sperner. Likewise, the reader will search for a non-Sperner ordered set satisfying (SBR) and not (UNI). Thus, Conditions (UNI) and (SBR) are independent and, separately, none of them implies (S).

Definition 4.19 A family $\mathcal{F}$ of subsets of a set E is called a *Sperner family* (or a *clutter*) *on* E if it is an antichain of the lattice $\underline{2}^E$; that is, none of the subsets in $\mathcal{F}$ is strictly included in another one.

The Sperner Theorem proved below states that the maximum size of such a family is equal to the maximum size of a rank-set of the Boolean lattice $\underline{2}^E$ (Chapter 1, Example 1.40). This result accounts for the denomination "Sperner ordered set" in Definition 4.14. Exercise 4.9 is based on a nice application of this result to a question pertaining to numerical analysis.

Theorem 4.20 (Sperner, 1928) *Let E be a set of size n. The Boolean lattice $\underline{2}^E$ is a Sperner ordered set, with $\alpha(\underline{2}^E) = \nu(\underline{2}^E) = \binom{n}{\lceil \frac{n}{2} \rceil}$.*

Proof It is well-known that $\nu(\underline{2}^E) = \binom{n}{\lceil \frac{n}{2} \rceil}$. The previous results provide us with three ways of showing that $\underline{2}^E$ is Sperner.

1. It may be proved that $\underline{2}^E$ satisfies Condition (SYM) by the direct construction of a symmetric chain partition (Exercise 4.10). Then Proposition 4.18 applies.

2. It is well-known that $\underline{2}^E$ satisfies Condition (RSU) and, so, (UNI). Since each of its bipartite ordered sets $P_k = (R_k + R_{k+1}, \leq)$ is regular, the ordered set $\underline{2}^E$ satisfies (REG) and, so, (SBR). Then Item (5) in Proposition 4.18 applies.

3. The ordered set $\underline{2}^E$ satisfies Condition (MCR) since all its rank-sets are $\mathcal{C}$-regular (compute the number of maximal chains including a subset of E of size k). So it satisfies Condition (MR) in Theorem 4.17. □

4.4 Direct products of chains

An important class of ordered sets, which is an immediate generalization of Boolean lattices, is that of direct products of chains; that is, products $\underline{c_1} \times ... \times \underline{c_i} \times ... \times \underline{c_m}$ of linearly ordered sets where, for any $i = 1,...,m$, c_i is an integer greater than or equal to 2 and $\underline{c_i}$ is the chain $\{0 < 1 < ... < c_{i-1}\}$ of size c_i.

In this section, the term *product of chains* (or *chain product*) always stands for *direct product of chains* and a number of properties of such products are stated. To do this we first consider the more general case of an ordered set $P' = P \times \underline{c}$, which is the direct product of a *ranked* ordered set P with the chain $\underline{c} = \{0 < ... < c-1\}$ of size c. We observe that P' inherits several properties of P (Propositions 4.21 to 4.24) and the results of this type are applied to chain products (Corollary 4.25).

An element of P' is denoted by $x' = (x,j)$, with $x \in P$ and $j \in \underline{c}$. Let r be the rank function of P. The ordered set P' is ranked too, with the rank function r' given by $r'(x') = r'((x,j)) = r(x) + j$ for any $x' \in P'$. If $r(P)$ is the rank of P, then the rank (parameter) of P' is $r(P') = r(P) + c - 1$ (as in Chapter 2, Section 2.1, we distinguish the rank *parameter*, denoted by $r(P')$, of the ordered set P' from the *rank function* r' defined on P'). As above, the size of the rank-set R_k is denoted by n_k, while the size of the rank-set R'_k of P' is denoted by n'_k. This sequence of numbers is extended by writing $n_k = 0$ (respectively, $n'_k = 0$) for any integer $k \notin [0, r(P)]$ (respectively, $k \notin [0, r(P')]$). It is then easy to obtain the equalities (1) and (2) in Proposition 4.21 (draw a figure).

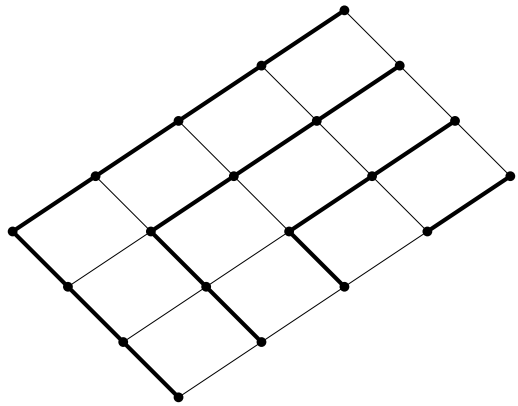


Figure 4.5 A symmetric chain partition of the direct product $\underline{4} \times \underline{5}$.

Proposition 4.21 *Let P be a ranked ordered set, $\underline{c}$ a chain of size c, and $P' = P \times \underline{c}$. For any integer $k \geq 0$, the numbers n_k and n'_k satisfy the following equalities:*

1. $n'_k = \Sigma_{k-c+1 \leq i \leq k} n_i$.
2. $n'_{k+1} = n'_k + n_{k+1} - n_{k-c+1}$.

Conditions (UNI) of unimodality of P, (RSU) of rank symmetry–unimodality of P, and (SYM) meaning that P is symmetric chain have been defined in the previous section. In the sequel, we also consider the following condition: a ranked ordered set P is called *strongly unimodal* if it is unimodal and, moreover, if $n_{k+1} = n_k$ implies either $n_k = 0$ or $n_k = \nu(P)$. In other words two consecutive rank-sets of such an ordered set do not have the same size unless this size is maximum. An example of such an ordered set is given in Figure 4.5.

The implication of Conditions (UNI) and (RSU) by (SYM) was shown in the previous section. A generalization of the construction appearing in Figure 4.5 for $\underline{4} \times \underline{5}$ allows us to observe that any product of two chains satisfies Condition (SYM). The proof of the next result is proposed in Exercise 4.12.

Proposition 4.22 *Let P be a symmetric chain ordered set, $\underline{c}$ a chain, and $P' = P \times \underline{c}$. Then the direct product $P' = P \times \underline{c}$ is still symmetric chain.*

As mentioned above, we now infer other properties of the Whitney numbers of $P' = P \times \underline{c}$ from those of the Whitney numbers of P.

Proposition 4.23 *Let P be a unimodal ranked ordered set, $\underline{c}$ a chain, and $P' = P \times \underline{c}$. Then the following three properties hold:*

1. *P' is unimodal.*
2. *If P satisfies Condition (RSU), so does P'.*
3. *If P satisfies (RSU) and is strongly unimodal, so does P'.*

Proof (1) We first show that, if P is unimodal, so is P'. From Item (2) in Proposition 4.21, the first part of the sequence of the n'_i is increasing: $n'_0 = n_0$; $n'_1 - n'_0 = n_1 > 0$. Now, as soon as $n'_{k+1} < n'_k$ for an index value k, the sequence must decrease. Let k be an index value such that $n'_{k+1} - n'_k < 0$ and so, by Item (2) again, $n_{k+1} < n_{k-c+1}$. In that case, since P is unimodal, n_{k+1} is necessarily in the non-increasing part of the sequence of the n_i and thus $n_{k+2} \leq n_{k+1}$.

If n_{k-c+1} is still in the non-decreasing part of the sequence of the n_i, then $n_{k-c+1} \leq n_{k-c+2}$ and, so, $n'_{k+2} - n'_{k+1} = n_{k+2} - n_{k-c+2} \leq n_{k+1} - n_{k-c+1} < 0$. Otherwise, the inequalities $n_{k-c+1} \geq n_{k-c+2} \geq n_{k+2}$ hold and imply $n'_{k+2} - n'_{k+1} \leq 0$. Since the latter inequalities remain true for the subsequent differences of Whitney numbers, the sequence of the n'_i cannot increase again. So, the ordered set P' is unimodal.

(2) If P satisfies (RSU), then the equalities $n'_k = \Sigma_{0\leq i\leq c-1} n_{k-i} = \Sigma_{0\leq i\leq c-1} n_{r(P)-k+i} = \Sigma_{0\leq i\leq c-1} n_{r(P)+(c-1)-k+(i-c+1)} = \Sigma_{0\leq i\leq c-1} n_{r(P)-k+i} = n'_{r(P')-k}$ hold. Together with the unimodality of P' obtained above, this implies Condition (RSU) for P'.

(3) Now assume that P satisfies (RSU) and is strongly unimodal. We show that P' is still strongly unimodal; that is, $n'_{k+1} = n'_k$ implies either $n'_k = 0$ or $n'_k = \nu(P')$. From the recurrence formula (2) in Proposition 4.21, the equality $n'_{k+1} = n'_k$ implies $n_{k+1} = n_{k-c+1}$. We examine the three cases where the latter equality may occur:

- If $n_{k+1} = 0$, then $n_i = 0$ for any i less than k and, so, by Item (1) in Proposition 4.21, $n'_{k+1} = n'_k = 0$.
- If $n_{k+1} = n_{k-c+1} = \nu(P)$, then, since P is unimodal, $n_i = \nu(P)$ for any i between $k-c+1$ and $k+1$. So, by Item (1) in Proposition 4.21, $n'_{k+1} = n'_k = c\nu(P) = \nu(P')$.
- If $n_{k+1} = n_{k-c+1}$ is different from 0 or $\nu(P)$, then by strong unimodality, the equality $k-c+1 = r(P)-k-1$ holds. Then $r(P') = r(P)+c-1 = 2k+1$ is odd and we find $k = \frac{r(P')-1}{2}$ and $k+1 = \frac{r(P')+1}{2}$ which, with the symmetry property, implies $n'_{k+1} = n'_k = \nu(P')$.

□

Let q (respectively, q') be the number of the rank-sets of P (respectively, of P') of maximum size $\nu(P)$ (respectively, $\nu(P')$). The next proposition allows us to derive q' from q and c in some cases.

Proposition 4.24 *Let P be a ranked ordered set, $\underline{c}$ a chain of size c, and $P' = P \times \underline{c}$. The following properties hold:*

1. *If P is unimodal with $c \leq q$, then P' has exactly $q-c+1$ rank-sets of maximum size $\nu(P')$. In this case, the equality $\nu(P') = c\nu(P)$ holds.*
2. *If P satisfies Condition (RSU) and is strongly unimodal, and if $c > q$, then P' has exactly one (if $r(P')$ is even) or two (if $r(P')$ is odd) rank-sets of maximum size $\nu(P')$.*

Proof (1) Let j be the smallest integer such that $n_j = \nu(P)$. Since P is unimodal, the greatest integer i such that $n_i = \nu(P)$ is $j+q-1$. Now, with $c \leq q$, we observe that Item (1) in Proposition 4.21 gives $n'_k = \Sigma_{k-c+1 \leq i \leq k} n_i = c\nu(P)$ if and only if $j+c-1 \leq k \leq j+q-1$. From Item (1) in Proposition 4.21 again, it is clear that this value $c\nu(P)$ is the greatest possible one for the size of a rank-set of P'.

(2) First note that, by Proposition 4.23, if P satisfies (RSU), so does P'. So, if there are several rank-sets of size $\nu(P')$ in P', their ranks are "centered" on the value $\frac{r(P')}{2} = \frac{r(P)+c-1}{2}$. Assume $r(P')$ is even and write $j = \frac{r(P')}{2} = \frac{r(P)+c-1}{2}$. We have $n'_j = \nu(P')$ and $n'_{j+1} - n'_j = n_{j+1} - n_{j-c+1}$ (Item (2) in Proposition 4.21). Since there are $q < c$ rank-sets of size $\nu(P)$ in P, the value $\nu(P)$ cannot be simultaneously that of n_{j+1} and n_{j-c+1}. So, $n'_{j+1} - n'_j = n_{j+1} - n_{j-c+1} \neq 0$ and, since $n'_j = \nu(P')$, then $n'_{j+1} < n'_j$. By Condition (RSU), $n'_{j+1} = n'_{j-1} < n'_j$, and so $j = \frac{r(P')}{2}$ is the unique rank value for which $n'_j = \nu(P')$.

The case where $r(P')$ is odd is similar, starting from $j = \frac{r(P')+1}{2}$. □

Clearly, any chain satisfies Condition (SYM) and is strongly unimodal. The following properties of a product of m chains are immediately derived from the above results (more particularly, from Proposition 4.18 and Propositions 4.21 to 4.23) by induction on m.

Corollary 4.25 *Let $P = \underline{c_1} \times ... \times \underline{c_m}$ be a product of m chains. The ordered set P is ranked with rank $r(P) = (\Sigma_{1\leq i\leq m} c_i) - m$. It satisfies Condition (SYM) and is strongly unimodal. P is Sperner and its width $\alpha(P) = \nu(P)$ is the size of the rank-set $R_{\frac{r(P)}{2}}$ if $r(P)$ is even, or the size of the rank-sets $R_{\frac{r(P)-1}{2}}$ and $R_{\frac{r(P)+1}{2}}$ if $r(P)$ is odd.*

Proposition 4.21 provides recurrence formulas for the computation of Whitney numbers and, especially, the width of a chain product P. Likewise, the properties listed in the above corollary allow us to use Proposition 4.24 to determine the number of rank-sets of maximum size in a chain product $\underline{c_1} \times ... \times \underline{c_m}$ from this number in $\underline{c_1} \times ... \times \underline{c_{m-1}}$.

Example 4.26 A chain product $P = \underline{c_1} \times ... \times \underline{c_m}$ is a *Post lattice* if $c_1 = c_2 = ... = c_m = c$. For $c = 2$, P is a Boolean lattice and the recurrence formula (1) in Proposition 4.21 becomes $n'_k = n_k + n_{k-1}$, the formula of binomial numbers. For any c, the Whitney numbers of the ordered set $\underline{c}^m$ generalize these numbers. For instance, when $c = 6$, the size n_k of the rank-set R_k corresponds to the number of different ways to obtain the score $k+m$ in a throw of m (non-loaded) dice. Then $\frac{n_k}{6^m}$ is the probability of obtaining this score in a "fair" throwing. Table 4.1 gives the corresponding numbers n_k for $1 \leq m \leq 4$.

In Table 4.1, the transition from the lines $m = 1$ to $m = 2$ – i.e., from the chain $\underline{6}$ to $\underline{6}^2$ – illustrates Item (1) in Proposition 4.24 (with $P = \underline{6}$ and so $c = q = 6$). For $m \geq 2$, the other transitions illustrate Item (2).

Table 4.1 Whitney numbers of the Post lattices $P = \underline{6}^m$ for $m \leq 4$

$m \backslash k$	0	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1					
2	1	2	3	4	5	6	5	4	3	2	1
3	1	3	6	10	15	21	25	27	27	25	21
4	1	4	10	20	35	56	80	104	125	140	146

$m \backslash k$	11	12	13	14	15	16	17	18	19	20
1										
2										
3	15	10	6	3	1					
4	140	125	104	80	56	35	20	10	4	1

In general, the rank of a Post lattice $P = \underline{c}^m$ is equal to $m(c-1)$. Applying Proposition 4.24 allows us to see that, for $m > 1$, the ordered set P has one rank-set of size $\nu(P)$ if this product is even, and two otherwise.

Example 4.27 Consider an integer $p = p_1^{a_1} p_2^{a_2} ... p_m^{a_m}$, where $p_1, p_2, ..., p_m$ are the prime factors of p. The set P of all divisors of p, endowed with the divisibility order, is isomorphic to the chain product $(\underline{a_1+1}) \times (\underline{a_2+1}) \times ... \times (\underline{a_m+1})$. Thus the previous results apply to this ordered set.

Example 4.28 In the field of knowledge extraction (and, more precisely, of the extraction of the decision rules of an expert), Pichon *et al.* (1994) assume that objects are described by m attributes, each corresponding to an ordinal scale (a chain) $\underline{c_i}$ of size c_i (with $i = 1, ..., m$). So, the universe of all possible descriptions corresponds to the chain product $P = \underline{c_1} \times ... \times \underline{c_m}$ of rank $r(P) = (\Sigma_{1 \leq i \leq m} c_i) - m$. An expert selects or rejects objects in such a way that, if object e is selected, so are all objects e' greater than e in P whereas, if e is rejected, so are all objects e' less than e in P. In other terms, the set of all selected objects is an upset F of P, while the rejected ones correspond to the complementary downset. The determination of the antichain A of all minimal elements of F will reveal the decision rules of the expert (who, in some cases, would hardly be able to make these rules explicit by himself). It is then possible, for instance, to set specifications for the realization of an expert system.

It may be important to organize the questions addressed to the expert in order to reduce the number of these questions. To do so, one may first consider a partition of P into $\alpha(P)$ chains, each of size at most $r(P)+1$. Then one finds the limit element of the chain by a dichotomic research, with at most $\lceil \log_2 c \rceil$ questions. The upper bound $\alpha(P)\lceil \log_2 r(P)+1 \rceil$ for the number of objects submitted to the expert is then obtained (in fact, this bound is valid in every ranked ordered set). The authors of the paper quoted above also seek to improve the bound by procedures more specific to chain products and a choice of questions which evolves depending on the steps of the research.

4.5 Further topics and references

While studying the dimension of distributive lattices, a question dealt with in Chapter 6, Dilworth was led to state his theorem in 1950. The simple proof of this result given in Section 4.1 is due to Tverberg (1967). The Dilworth and Sperner Theorems were the starting points of a particularly broad literature, including a number of brilliant results and also several outstanding syntheses providing all the needed references: the survey, emphasizing proof methods, written by Greene and Kleitman (1978), the very complete historical presentation by West (1982), and Engel's book (1997), dense and technical.

Here we have essentially presented some basic results pertaining to ordered set theory. However, the bipartite ordered sets of Section 4.2 are clearly equivalent to bipartite graphs and, as mentioned, to other combinatorial structures. As a matter of fact, Dilworth's Theorem is at the very center of combinatorial theory, as is testified by its relation with integer linear programming. This relation is implicit in its equivalence with the Ford and Fulkerson "max flow–min cut" Theorem (1962) and is explicit as a corollary of a duality theorem by Dantzig and Hoffman (1956). Moreover, the equivalence between Dilworth's Theorem and several other fundamental results of combinatorics has been proved in the literature. For instance, it was shown in Section 4.2 how Dilworth's Theorem implies the König–Hall Theorem. The latter implies Menger's Theorem in graph theory which, in turn, implies that of Ford and Fulkerson. The latter result leads to a proof of the König–Egerváry Theorem on bipartite graphs, which finally implies Dilworth's Theorem. Below we go back to the last two of these implications between theorems. For the other (and for variants of Menger's Theorem), the reader may refer to Chapter 8 in Aigner (1979) and to Chapter 13 in Welsh (1976) for the implication of Menger's Theorem by that of König–Hall.

Exercise 4.5 proposes a proof of the implication of Dilworth's Theorem by the König–Egerváry Theorem (the converse was established in Section 4.2). To do so, a bipartite ordered set $P' = (X + X', \leq')$ is associated with any ordered set $P = (X, \leq)$ as follows: $X' = \{x'_1, ..., x'_n\}$ is a copy of $X = \{x_1, ..., x_n\}$, with $x_i <' x'_j$ if and only if $x_i < x_j$. On the other hand, the proof of the implication of the König–Egerváry Theorem by that of Ford and Fulkerson involves an efficient algorithm for determining the width of an ordered set P (and, as a consequence, this determination is a polynomial problem – see Appendix A). In fact, the search for a partition of any ordered set P into $\theta(P)$ chains is equivalent to the search for a maximum flow in the directed graph G_P defined below, while the search for an antichain of P of size $\alpha(P)$ is equivalent to the search for a minimum cut of G_P. We now make explicit the equivalence between Dilworth's Theorem and the Ford and Fulkerson max flow–min cut Theorem. Let us first recall the latter.

For each vertex v of a directed graph $G = (V, U)$, denote by Uv (respectively, vU) the set of arcs of extremity (respectively, of origin) v. For two distinct vertices

y and z, a *flow* from y to z is a map $f : U \longmapsto \mathbb{N}$ such that, for any $v \in X \setminus \{y,z\}$, $\Sigma_{u \in Uv} f(u) = \Sigma_{u \in vU} f(u)$, while, if $u \in Uy$ (respectively, $u \in zU$), then $f(u) = 0$. The *value of the flow* f is equal to $f(y,z) = \Sigma_{u \in yU} f(u) = \Sigma_{u \in Uz} f(u)$. To be *admissible*, a flow f must also satisfy, for each arc u, the inequality $f(u) \leq \chi(u)$, where the integer $\chi(u)$ is the "capacity" of the arc u. To maximize $f(y,z)$ subject to these constraints may then be seen as a problem of maximum conveyance through a transportation network (represented by the graph G). A *cut* (*separating* y *and* z) is a partition of V into two classes Y and Z such that $y \in Y$ and $z \in Z$. Consider the set $D(Y,Z)$ of the arcs (y',z') of G such that $y' \in Y$ and $z' \in Z$, and the *capacity of the cut* $\chi(Y,Z) = \Sigma_{u \in D(Y,Z)} \chi(u)$. The Ford and Fulkerson Theorem states the equality:

$$\max\{f(y,z), f \text{ an admissible flow}\} = \min\{\chi(Y,Z), (Y,Z) \text{ a cut separating } y \text{ and } z\}$$

This theorem is associated with efficient algorithms for the determination of such flow and cut. It applies to the graph $G_{P'} = (V,U)$ derived from the above ordered set P' by adding a source y and a sink z. We write $V = X \cup X' \cup \{y,z\}$ and $U = \{(y,x) : x \in X\} \cup \{(x,x') : x \in X, x' \in X', x < x'\} \cup \{(x',z) : x' \in X'\}$. This graph is endowed with the capacities $\chi(y,x) = \chi(x',z) = 1$, for all $x \in X, x' \in X'$, and $\chi(x,x') = n+1$ for all other arcs (x,x') (with $n = |X|$).

A matching C from X into X' with $\sigma(P')$ chains then corresponds to a maximum flow from y to z, with $f(y,z) = \sigma(P')$, while a cut (Y,Z) of G_P, with $y \in Y$ and $z \in Z$, has a minimum capacity $\chi(Y,Z)$ if and only if it satisfies the following properties: $Y \cap X = Z \cap X' = \emptyset$ and $(Y \cap X') \cup (Z \cap X)$ is a transversal of P' of minimum size. So, the determination of a maximum flow of G_P implies that of a matching of P' with $\sigma(P')$ chains and of a partition of P into $\theta(P)$ chains, while the determination of a minimum cut of G_P implies that of a transversal of P' of size $\tau(P')$ and of an antichain of P of size $\alpha(P)$.

Problems of cuts have been generalized. And, for example, Dilworth's Theorem is crucial in the proof of the following result: the multicut problem parameterized by a solution size k is fixed-parameter-tractable (FPT) (see Bousquet *et al.*, 2010).

Another property of ordered sets is that their comparability graphs (and their complementary incomparability graphs, see Definition 1.5 in Chapter 1) form one of the main classes of perfect graphs, the definition of which is recalled below. In an undirected graph $G = (V,U)$, the *independence number* $\alpha(G)$ is the maximum size of an independent subset (that is, without adjacent elements) of V, the *chromatic number* $\gamma(G)$ is the minimum number of independent subsets in a partition of V into independent subsets, the number $\kappa(G)$ is the maximum size of a clique (i.e., a subset of pairwise adjacent elements) of V, and the number $\theta(G)$ is the minimum number of cliques in a clique partition of V. The graph G is said to be *perfect* if the equality $\alpha = \theta$ is satisfied by G and by any induced subgraph of G. Equivalently, G is perfect if $\kappa = \gamma$ holds in G and in any induced subgraph of G (see the surveys by Toft (1995) and Chudnovski *et al.* (2003) on the proofs of Berge's 1961 conjectures on these graphs). When G is the comparability graph $Comp(P)$ of an ordered set P, it is immediate to see that the cliques of $Comp(P)$ correspond exactly to the chains of P and that, in fact, these graph parameters coincide with the order parameters studied throughout this chapter. Then, Dilworth's Theorem and/or Theorem 4.1 imply that any comparability – or incomparability – graph is perfect. The particular ordered sets whose comparability graphs are also incomparability graphs are the 2-dimensional ordered sets and they will be characterized in Chapter 6, Section 6.4.

Many generalizations of Dilworth's Theorem have also been obtained. A particularly remarkable one concerns the *k-antichains* of maximum size, in relation to the existence of particular chain partitions (Greene and Kleitman, 1976). Exercise 4.4 presents two equivalent definitions of these k-antichains.

There are other relations between chains and antichains. For instance, Howard and Trotter (2010) proved that there exist relations between the number of pairwise disjoint maximal chains (respectively, antichains) of an ordered set and the size of all its maximal antichains (respectively, the lengths of all its maximal chains).

In another direction, Shum and Trotter Jr. (1996) have studied the problem of partitioning an ordered set into a minimal number of chains of length bounded by a fixed integer and shown that the corresponding decision problem is $\mathcal{NP}$-complete. A generalization of this problem where the elements of the ordered set are weighted is considered in Moonen and Spieksma (2008).

Sperner's Theorem may be seen as the starting point of the "extremal set theory," the purpose of which is to find, given a set E, the maximum (or minimum) number of elements of a family $\mathcal{F}$ of subsets of E that satisfies some properties. For instance, if the property is the incomparability for the inclusion order, the problem is to determine the width of the ordered set $(\mathcal{F}, \subseteq)$. The interested reader may refer to Anderson's book (1987).

The problem of recognizing whether an order is or is not Sperner is central in the study of ranked ordered sets. It induces a specific classification of these ordered sets: for instance, neither distributive lattices (Figure 4.4(b) provides a counter-example) nor modular lattices nor geometric lattices (Dilworth and Greene, 1971) are Sperner in general. The case of the (geometric) partition lattice has been a famous problem for a long time until Canfield (1978) proved that this lattice is not Sperner as soon as its size is large enough.

On the other hand, according to Proposition 4.18, the existence of a symmetric chain partition implies the Sperner property. It also has many other consequences which make the search for such partitions particularly interesting (see Griggs (1988) and Exercise 4.10).

Since chain products generalize Boolean lattices, the Whitney numbers of the former generalize binomial numbers. Indeed, we have observed in Example 4.26 that the recurrence formulas of Proposition 4.21 generalize Pascal's triangle. Yet, the "factorial" formula for binomial numbers hardly extends (see Exercise 4.13 for the case $m = 3$). However, besides recurrence formulas, probabilities provide efficient tools for an approximative evaluation of the number of elements whose rank lies between two given values in a "large" chain product. With the notations in Example 4.27, write $V = \frac{1}{12}\Sigma_{1\leq i\leq m} a_i(a_i+2)$ and $s = \sqrt{V}$. We may consider the rank $r(x)$ of an element x, chosen at random in P with uniform probability, as a random variable of mean $\frac{r(P)}{2}$ and variance V. If the sequence $a_1, a_2, ..., a_m, ...$ is such that $\lim_{m\to\infty} \frac{a_m}{s} = 0$, the distribution of $r(x)$ tends toward the normal one. In particular, this leads to the asymptotic formula

$$\alpha(P) \approx \frac{1}{\sqrt{2\pi}} \frac{n(P)}{s}$$

(see, for instance, Leclerc (1990)). This is the case for the chain product $\underline{2} \times \underline{3} \times ... \times \underline{m}$, which was shown by Le Conte de Poly-Barbut (1990a) to have the same Whitney numbers as the weak Bruhat order on Σ_m defined in Chapter 1, Example 1.17.

4.6 Exercises

Exercise 4.1 Show that a ranked lattice the rank-sets of which have size at most 2 may be partitioned into two chains (the example in Figure 4.4(b) shows that the situation is different when the size of the rank-sets is at most 3).

Exercise 4.2 [Fleet optimization] A company has scheduled n flights departing from an airport. To each flight i corresponds a departure time t_i and the duration d_i of the absence of the plane until its return. Two flights i and j are incompatible if $[t_i, t_i + d_i] \cap [t_j, t_j + d_j] \neq \emptyset$. Show that the minimum number of required planes to realize the schedule is equal to the maximum number of pairwise incompatible flights.

Exercise 4.3 [The lattice of maximum size antichains; Dilworth (1960)] Let A and A' be two antichains of size $\alpha(P)$ of an ordered set P. Write $Y = A \cup A'$ and let B and B' be respectively the sets of the minimal and the maximal elements in the ordered subset $(Y, \leq)$.

(1) Show that $Y = B \cup B'$ and deduce that B and B' are also antichains of P, of size $\alpha(P)$.

(2) Show that, for the order defined on antichains of P by $A_1 \leq A_2$ if $(A_1] \subseteq (A_2]$, B (respectively, B') is the meet (respectively, the join) of A and A' (this order is considered in Chapter 5, on page 137).

Exercise 4.4 [k-Antichains] Let $P = (X, \leq)$ be an ordered set. A subset Y of X is called a *k-antichain of P* if Y is the union of at most k antichains of P. Show that Y is a k-antichain if and only if $\kappa(P') \leq k$, where P' is the ordered subset $(Y, \leq)$ of P.

Exercise 4.5 [From König–Egerváry to Dilworth; Fulkerson (1956)] Let $P = (X, \leq)$ be an ordered set of size n. Consider the associated bipartite ordered set $P' = (X + X', \leq^{'})$, where $X' = \{x'_1, ..., x'_n\}$ is a copy of $X = \{x_1, ..., x_n\}$ and $x_i <^{'} x'_j$ if and only if $x_i < x_j$.

(1) Show that a partition $\mathbf{P}$ of P into $n - c'$ chains corresponds to a matching of P' into c' chains and conversely. *Indication:* start from an empty matching C of P' and from the partition $\mathbf{P}$ of P into n 1-element chains. Then add ordered pairs (x_i, x'_j) with $x_i <^{'} x'_j$ to C and reduce at the same time the number of chains in $\mathbf{P}$ by putting x_i and x_j in the same chain.

(2) Show that an antichain A of size $n - t'$ of P corresponds to a transversal T of size t' of P' and conversely. *Indication:* x_i is an element of A if and only if neither x_i nor x'_i are elements of T.

(3) Conclude that the equality $\sigma(P') = \tau(P')$ implies the equality $\theta(P) = \alpha(P)$.

Exercise 4.6 [Systems of distinct representatives] Let $\mathcal{F} = (F_i)_{i \in I}$ be a family of subsets of a set E. A *system of distinct representatives* (*SDR*) of $\mathcal{F}$ is a set $\mathcal{R} = \{r_i, i \in I\}$ of (distinct) elements of E such that, for any $i \in I$, $r_i \in F_i$. Show that Corollary 4.11 is equivalent to the following statement: *a necessary and sufficient condition for the existence of a SDR of a family $\mathcal{F}$ of subsets of E is that, for any $J \subseteq I$, the inequality $|\bigcup_{i \in J} F_i| \geq |J|$ holds*. *Hint*: use the equivalence between structures pointed out at the beginning of Section 4.2.

Note This well-known result was obtained in this form by Hall (1935). It applies, among others, to assignment problems. If, for instance, E is the set of pilots and I the set of planes of a flight company, F_i being the set of pilots qualified for plane i, then the obtained condition concerns the possibility of simultaneous flights for all planes.

Exercise 4.7 Consider the Boolean lattice $\underline{2}^E$ of subsets of a set E and two rank-sets R_k and $R_{k'}$ of this lattice, with $k < k' < \frac{n}{2} + 1$. Let $P = (R_k + R_{k'}, \leq)$ be the bipartite ordered subset of $\underline{2}^E$ induced by these rank-sets. Does there exist a matching from R_k into $R_{k'}$? Determine the parameters $\sigma(P)$, $\tau(P)$, $\gamma(P)$, $\alpha(P)$, and $\delta(P)$.

Exercise 4.8 Characterize the ranked ordered sets in which any antichain is included in a rank-set. *Clue:* they were considered in Chapter 2.

Exercise 4.9 [The Littlewood–Offord problem] Let $r_1, ..., r_n$ be a collection of real numbers, all greater than or equal to 1 and $I = [t, t+1[$ a real interval. Consider the set E of the numbers of form $r = \Sigma_{1 \leq i \leq n} \varepsilon_i r_i$, where the ε_i's are equal to 0 or 1.

Show that the number of elements of E belonging to the interval I is at most equal to $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. *Indication:* consider the n-tuple of the ε_i's as an element of the Boolean lattice $\underline{2}^n$ and show that the set of the r's belonging to the interval I corresponds to an antichain of $\underline{2}^n$.

Note Chapter 11 in Anderson (1987) is devoted to the many extensions of this result.

Exercise 4.10 [Symmetric chain partition] Give a symmetric chain partition of the Boolean lattice $\underline{2}^E$ of subsets of a set E. *Indication:* choose an element e of E and derive two chains of lengths $\ell+1$ and $\ell-1$ of $\underline{2}^E$ from each chain of length ℓ of a symmetric chain partition of $\underline{2}^{E-e}$.

Extend the result in Exercise 4.7 to all k, k', and generalize it to any pair of rank-sets of an ordered set satisfying Condition (SYM).

Exercise 4.11 [Symmetric chain partition of the permutoedre] Find a symmetric chain partition of the permutoedre order on the set Σ_4 of the commutations of a set of size 4 (Chapter 1, Example 1.17; the diagram of this ordered set is given in Chapter 5, Figure 5.7).

Note The existence of a symmetric chain partition of the permutoedre order on Σ_n for any integer n remains an open problem (see Leclerc (1994c) on this topic, where this existence is shown for $n = 5$).

Exercise 4.12 Let $\{C_1, ..., C_k\}$ be a partition of a ranked ordered set P into k symmetric chains and let C be a chain. Show that any element x of the direct product $P \times C$ belongs to a unique ordered subset $C_i \times C$, for $i \in \{1, ..., k\}$.

Deduce that, if P is a symmetric chain ordered set and C is a chain, then $P \times C$ is still symmetric chain.

Exercise 4.13 [The width of the product of two or three chains; Leclerc (1990a)] Consider three chains $\underline{c_1}$, $\underline{c_2}$, and $\underline{c_3}$, with $c_1 \geq c_2 \geq c_3 \geq 2$.

(1) Show that $\alpha(\underline{c_1} \times \underline{c_2}) = c_2$.

(2) Show that $\alpha(\underline{c_1} \times \underline{c_2} \times \underline{c_3}) = c_2 c_3 - \frac{K}{4}$, with:

- $K = 0$ if $c_1 > c_2 + c_3 - 2$;
- $K = (c_2 + c_3 - c_1 - 1)(c_2 + c_3 - c_1 + 1)$ if $c_1 \leq c_2 + c_3 - 2$ and if $c_2 + c_3 - c_1$ is odd;
- $K = (c_2 + c_3 - c_1)^2$ if $c_1 \leq c_2 + c_3 - 2$ and if $c_2 + c_3 - c_1$ is even.

5

Ordered sets and distributive lattices

The particular ordered sets called *lattices* have been defined in Chapter 1. In Chapter 2, we introduced some particular classes of lattices such as distributive, modular, or semimodular lattices. The class of distributive lattices is the most significant for several reasons. First, the distributivity properties between the two operations make the algebraic handling of such lattices easier. Then, many natural orders in pure or applied mathematics are distributive lattices, to begin with chains and lattices of subsets of a set. The latter are isomorphic to direct products of 2-element chains; more generally, any product of chains is a distributive lattice. Then, when in a multicriteria decision problem the possible options are assessed according to several linearly ordered criteria, these options are elements of the distributive lattice, which is the product of these orders. Finally and above all, there exists a fundamental correspondence between ordered sets and distributive lattices allowing any property or question on ordered sets to be translated into a property or question on distributive lattices (and conversely). For instance, in scheduling problems where one must search for a linear extension of an ordered set, considering the corresponding problem on an associated distributive lattice turns out to be profitable (see Section 7.5).

In Section 5.1 we give several characterizations of distributive lattices (Theorem 5.1) and examples of such lattices. In the following section, Theorem 5.6 describes the properties of a distributive lattice associated with an ordered set, namely the lattice of its downsets (ordered by inclusion). Section 5.3 shows that, conversely, any distributive lattice can be represented by sets, a fundamental Birkhoff's result, known as the Fundamental Theorem on Finite Distributive Lattices (FTFDL). More precisely, any distributive lattice is isomorphic to the lattice of downsets of an ordered set, namely the ordered set of its join-irreducible elements (Theorem 5.9). A canonical one-to-one correspondence is then obtained between the class of ordered sets and the class of distributive lattices. This one-to-one correspondence can be interpreted as resulting from a one-to-one correspondence between preorders and topologies, the latter being itself a consequence of a Galois connection between binary relations and families of subsets (defined on the same set). In Section 5.4 (Theorem 5.24), we present this Galois connection, which implies several dualities. First, the duality between preorders and topologies (Corollary 5.25), then that between total preorders and linear topologies (Proposition 5.28), and finally that between orders and quasi-separated topologies or, equivalently, between orders and distributive lattices (Corollary 5.30).

Beforehand, in Section 5.3, we use Birkhoff's representation Theorem to prove several significant properties of distributive lattices such as the isomorphism between their ordered sets of join-irreducible and of meet-irreducible elements. We also study the problem of determining the minimum size of a generating set (for both join and meet operations) of a distributive lattice. Indeed, the result will be useful in Chapter 6 for the determination of the Boolean dimension of an ordered set. The main result states that to find this minimum number amounts to finding a minimum size transversal of a family of intervals of the lattice (Theorem 5.21 and Corollary 5.22).

5.1 Distributive lattices

In Section 2.3 (Definition 2.19) a distributive lattice was defined as a lattice satisfying one of the two equivalent distributivity properties of one of its operations with regard to the other (Properties (1) and (2) below). In this section, we prove the equivalence of these two properties and their equivalence with four others.

We recall the following notations for a lattice $L : J_L$ (respectively, M_L) is the set of its join-irreducible (respectively, meet-irreducible) elements; J_x (respectively, M^x) is the set of the join-irreducible elements less than (respectively, meet-irreducible elements greater than) or equal to an element x of L; for $j \in J_L$ and $m \in M_L$, $j \uparrow m$ means that $j \vee m = m^+$ (Chapter 3, Proposition 3.21).

Theorem 5.1 *A lattice L is distributive if and only if it satisfies any of the following properties:*

1. *For all $x,y,z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.*
2. *For all $x,y,z \in L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.*
3. *For all $x,y,z \in L$, $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$.*
4. *For all $j \in J_L$ and $X \subseteq L$, $j \leq \bigvee X$ implies $j \leq x$ for at least one element x of X.*
5. *For every $j \in J_L$, there exists a unique $m \in M_L$ such that $j \uparrow m$.*
6. *For all $x,y \in L$, $J_{x \vee y} = J_x \cup J_y$.*

Proof We show that Properties (1) to (6) are equivalent.

(1) $\Longrightarrow$ (2): assume (1) is true and let $x,y,z \in L$. Then $(x \vee y) \wedge (x \vee z) = [(x \vee y) \wedge x] \vee [(x \vee y) \wedge z]$ (by (1)) $= x \vee [z \wedge (x \vee y)]$ (by absorption and commutativity) $= x \vee [(z \wedge x) \vee (z \wedge y)]$ (by (1)) $= [x \vee (z \wedge x)] \vee (z \wedge y)$ (by associativity) $= x \vee (y \wedge z)$ (by absorption and commutativity). The implication (2) $\Longrightarrow$ (1) is shown dually.

(2) $\Longrightarrow$ (3): $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = ([(x \wedge y) \vee (y \wedge z)] \vee z) \wedge ([(x \wedge y) \vee (y \wedge z)] \vee x)$ (by (2)) $= [(x \wedge y) \vee z] \wedge [(x \vee (y \wedge z)]$ (by absorption and commutativity)

$= [(x \vee z) \wedge (y \vee z)] \wedge [(x \vee y) \wedge (x \vee z)]$ (by (2)) $= (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ (by associativity and idempotence).

(3) $\Longrightarrow$ (1): if (3) holds, it is easy to see that L satisfies the following property (M) of modularity: $\forall x, y, z \in L$, $x \leq z \Longrightarrow (x \vee y) \wedge z = x \vee (y \wedge z)$ (see Exercise 2.10).

Now by (3) $x \wedge [(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)] = x \wedge [(x \vee y) \wedge (y \vee z) \wedge (z \vee x)]$. Reorganizing the first member of this equality and simplifying the second, we obtain: $([(x \wedge y) \vee (z \wedge x)] \vee (y \wedge z)) \wedge x = x \wedge (y \vee z)$. The inequality $(x \wedge y) \vee (z \wedge x) \leq x$ allows us to use (M) to write the first member of the equality as: $([(x \wedge y) \vee (z \wedge x)] \vee ((y \wedge z) \wedge x) = [(x \wedge y) \vee (z \wedge x)] \vee (y \wedge z \wedge x) = (x \wedge y) \vee (x \wedge z)$. Hence finally, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, as required.

(1) $\Longrightarrow$ (4): if $j \in J_L$ and $j \leq \bigvee X$, then $j = j \wedge (\bigvee X) = \bigvee_{x \in X} (j \wedge x)$ by distributivity and, j being join-irreducible, $j = j \wedge x$ holds for at least one element x of X.

(4) $\Longrightarrow$ (5): assume that there exist $j \in J_L$ and two distinct elements m_1 and $m_2 \in M_L$ with $j \uparrow m_1$ and $j \uparrow m_2$. Thus m_1 and m_2 are incomparable and so (for example) $j \leq m_1^+ \leq m_1 \vee m_2$ (see Proposition 3.21). Then (4) implies $j \leq m_1$ or $j \leq m_2$; a contradiction.

(5) $\Longrightarrow$ (6): in any lattice L, $J_x \cup J_y \subseteq J_{x \vee y}$ (since $j \leq x$ or $j \leq y$ implies $j \leq x \vee y$). Assume that there exists $j \in J_L$ with $j \leq x \vee y$, $j \not\leq x$ and $j \not\leq y$. Thus x and y are incomparable and there exist m_1 and $m_2 \in M_L$ with $x \leq m_1$, $y \leq m_2$, $j \uparrow m_1$ and $j \uparrow m_2$ (see Definition 1.36 of the uparrow relation). But then (5) implies $m_1 = m_2$, hence $j \leq x \vee y \leq m_1$; a contradiction.

(6) $\Longrightarrow$ (1): in Chapter 3 (Corollary 3.12) we have shown that, for any lattice L, the map $x \longmapsto J_x$ is a meet-coding (Definition 3.3) from L to the lattice $\underline{2}^{J_L}$. (6) implies that this map is an injective lattice morphism and thus that L is isomorphic to a sublattice of the Boolean lattice $\underline{2}^{J_L}$. Since we have observed in Section 2.3 that $\underline{2}^{J_L}$ is distributive and that a sublattice of a distributive lattice is distributive, the result follows. □

Properties (1) and (2) of the above theorem are dual whereas Property (3) is ipsodual. Since they characterize distributive lattices, it follows that the dual of a distributive lattice is distributive, so that we may state:

Corollary 5.2 *The class of distributive lattices is ipsodual.*

Properties (1) and (2) have been used in Chapter 1 as definitions of a distributive lattice. Property (3) states that the join of the pairwise meets of three elements x, y, z of a distributive lattice is equal to the meet of their pairwise joins; this element is called the *median* of x, y, and z. More generally, the median m of a tuple $(x_1, x_2, \ldots, x_k)$ of elements of a lattice L, with k odd, is defined by $m = \bigvee_{I \subseteq \{1, \ldots, k\}, 2|I| > k} (\bigwedge_{i \in I} x_i)$. In a distributive lattice, one may show that the median of a tuple $(x_1, x_2, \ldots, x_k)$ is also equal to $\bigwedge_{I \subseteq \{1, \ldots, k\}, 2|I| > k} (\bigvee_{i \in I} x_i)$. In Section 7.3 of Chapter 7 we will see that the operation associating with a tuple its median is a lattice formalization of the majority rule and that this operation may also be defined in a “metric” way, the median also being an element whose distance from the tuple is minimum.

An element x of a lattice L is called *join-prime* (respectively, *meet-prime*) if, for any $X \subseteq L$, $x \leq \bigvee X$ implies $x \leq y$ for at least one element $y \in X$ (respectively, $x \geq \bigwedge X$ implies $x \geq y$ for at least one element $y \in X$). The reader will easily check that a join-prime element is join-irreducible (and that a meet-prime element is meet-irreducible). Then Characterization (4) may be written: *a lattice is distributive if and only if all its join-irreducible elements are join-prime*.

It is suggestive to give Characterization (5) another form. Let us call *cleavage* of a lattice a bipartition of its elements into a principal filter and a principal ideal. Figure 5.4(a) shows an example of such a cleavage: the distributive lattice L is divided into the principal filter of the elements greater than or equal to the join-irreducible a and the principal ideal of the elements less than or equal to the meet-irreducible i. Observe that a lattice may have no cleavage (search for examples). On the contrary, Property (5) in Theorem 5.1 means that, in a distributive lattice L, given a join-irreducible j and the unique meet-irreducible m such that $j \uparrow m$, $[j) + (m]$ forms a cleavage. Since this property characterizes distributive lattices, we may write: *a lattice L is distributive if and only if, for every join-irreducible j of L, the set of the elements greater than or equal to j and the complementary set of the elements not greater than or equal to j form a cleavage of L.*

Let us also observe that, if $[j) + (m]$ is a cleavage of L then $j \updownarrow m$ holds (indeed, $m^+ \not\leq m$ implies $m^+ \geq j$ and thus $j \vee m = m^+$; and likewise, $j \not\leq j^-$ implies $j \wedge m = j^-$).

Remark 5.3 Since the class of distributive lattices is ipsodual (Corollary 5.2) we may use the duality principle (see page 10): therefore, with any result on the join-irreducible elements of a distributive lattice corresponds a dual result on its meet-irreducible elements. Particularly, we may state other characteristic properties of distributive lattices obtained by taking the duals of the characteristic Properties (4) to (6):

7. *Every meet-irreducible of L is meet-prime.*
8. *For any $m \in M_L$, there exists a unique $j \in J_L$ such that $j \downarrow m$.*
9. *For all $x, y \in L$, $M^x \cup M^y = M^{x \wedge y}$.*

There exist many other characterizations of distributive lattices different from those given in Theorem 5.1 and in the above remark. Some will be found in Section 5.6 (Further topics and references) of this chapter. Here, we simply mention two classic characterizations, respectively by sublattices exclusion and by "simplification" rule (the reader may check that the second is easily obtained from the first):

10. *A lattice is distributive if and only if it does not include sublattices of type M_3 or N_5 (see Figure 5.1).*
11. *A lattice L is distributive if and only if, for all x, y, z in L, $x \wedge y = x \wedge z$ and $x \vee y = x \vee z$ imply $y = z$.*

Example 5.4 Let us recall several examples of distributive lattices already given in Chapter 2. First, we can mention linearly ordered sets, with the operations *max*

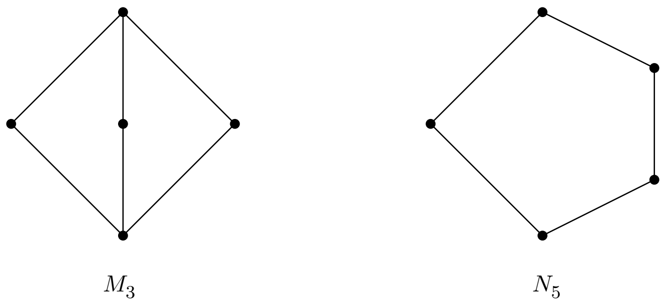


Figure 5.1 The lattices M_3 and N_5.

as the join and *min* as the meet. Then, direct products of linearly ordered sets are distributive lattices since, more generally, the direct product of distributive lattices is a distributive lattice (Proposition 2.20). The Boolean lattice B_n is a particular case of such a product, since it is isomorphic to the direct product $\underline{2}^n$ of n linearly ordered sets $\underline{2} = \{0 < 1\}$. In Proposition 2.21, we have shown that a sublattice of a distributive lattice is a distributive lattice, which allows us to provide many other examples. In particular, we have called *distributive family of sets* a set $\mathcal{H}$ of subsets of a set X which is a sublattice of $\underline{2}^X$ (i.e., such that $A, B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$ and $A \cap B \in \mathcal{H}$) and we have pointed out the particular case of topologies. The latter playing a significant role in the fourth section of this chapter, we give again their definition.

Definition 5.5 A *topology* on a set X is a family $\mathcal{T}$ of subsets of X satisfying the following conditions:

1. $A, B \in \mathcal{T}$ implies $A \cup B \in \mathcal{T}$ and $A \cap B \in \mathcal{T}$.
2. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

In other words, a topology on X is a distributive family of subsets of X containing the empty set and the set X. We observe that topologies are exactly those families of sets that are both a closure system and a dual closure system. An example of a topology on X is the set of downsets (respectively, upsets) of an ordered set $P = (X, \leq)$. Indeed, in Section 1.4.2 we mentioned that the set of downsets (respectively, upsets) of an ordered set $P = (X, \leq)$ is stable for the union and the intersection operations and that it contains the empty set and X. Some particular classes of topologies will be considered in Definition 5.27.

5.2 The distributive lattice associated with an ordered set

In the previous section we gave as an example of a distributive lattice the lattice of downsets of an ordered set. The following theorem develops this example; in particular we specify the covering relation and the irreducible elements of this lattice.

Let us recall (see Definition 1.26) that a sublattice L' of a lattice L is said to be *covering* if its covering relation $\prec'$ is the restriction to L' of the covering relation $\prec$ of L, i.e., if, for all $x,y \in L'$, $x \prec' y$ if and only if $x \prec y$.

Theorem 5.6 *Let $P = (X, \leq)$ be an ordered set of size n.*

1. *The set $\mathcal{D}(P)$ of downsets of P is a distributive lattice of height n, covering sublattice of $\underline{2}^X$; for $D, D' \in \mathcal{D}(P)$, $D' \prec D$ holds if and only if $D' = D \setminus x$, with x a maximal element of D.*
2. *The join-irreducible (respectively, meet-irreducible) elements of $\mathcal{D}(P)$ are the n principal ideals $(x]$ (respectively, the complementary sets $X \setminus [x)$ of the n principal filters $[x)$), for $x \in P$. In particular, X (respectively, $\emptyset$) is a join-irreducible (respectively, a meet-irreducible) of $\mathcal{D}(P)$ if and only if P has a greatest element (respectively, a least element).*
3. *The map $(x] \longmapsto m(x) = X \setminus [x)$ is an isomorphism between the ordered set of the join-irreducibles and the ordered set of the meet-irreducibles of $\mathcal{D}(P)$.*
4. *In $\mathcal{D}(P)$, a join-irreducible $(x]$ is less than or equal to a meet-irreducible $X \setminus [y)$ if and only if x is less than or incomparable to y in P.*

Proof (1) We have already observed in the previous section that $\mathcal{D}(P)$ is a topology, sublattice of $\underline{2}^X$ and thus a distributive lattice. If x is a maximal element (for the order of P) of a downset D of P, it is immediate to check that $D \setminus x$ is still a downset, necessarily covered by D in $\mathcal{D}(P)$. Conversely, let $D, D' \in \mathcal{D}(P)$ with $D' \prec D$. There exists at least one element $y \in D \setminus D'$ and one element x, maximal in D, such that $y \leq x$; $x \in D'$ is impossible, since then D' would not be a downset. Thus, $D' \subseteq D \setminus x \prec D$, whence $y = x$ and $D' = D \setminus x$. Therefore, we have shown that the covering relation of $\underline{2}^X$ is preserved in $\mathcal{D}(P)$, which implies in particular that the maximal chains of these two lattices have the same length and thus that the height of $\mathcal{D}(P)$ is n.

(2) Since a principal ideal $(x]$ has a unique maximal element (for the order of P), it covers only the downset $(x[$ in $\mathcal{D}(P)$, which shows that $(x]$ is a join-irreducible of $\mathcal{D}(P)$. It is clear that for any downset D, $D = \bigcup_{x \in D}(x]$ holds. Thus the set of principal ideals is a join-generating set of $\mathcal{D}(P)$ and is the set of all its join-irreducibles. In particular, X is a join-irreducible of $\mathcal{D}(P)$ if and only if there exists $x \in P$ such that $X = (x]$ and thus if and only if P has a greatest element. Write $m(x) = \{y \in X : x \not\leq y\} = X \setminus [x)$ the complementary downset of the principal filter $[x)$, and assume that $m(x) = D_1 \cap D_2$, with D_1 and D_2 two incomparable downsets; $m(x) \subset D_1$ implies that there exists $z \in D_1$ such that $x \leq z$ and thus $x \in D_1$; similarly, $m(x) \subset D_2$ implies $x \in D_2$. Therefore $x \in m(x)$ holds, which is impossible; thus $m(x)$ is a meet-irreducible of $\mathcal{D}(P)$. Check moreover that $D = \bigcap\{m(x) : x \notin D\}$ holds for any downset D; indeed, $x \notin D$ implies $D \subseteq m(x)$ (why?) and thus $D \subseteq \bigcap\{m(x) : x \notin D\}$; on the other hand, $y \in m(x)$ for every $x \notin D$ implies $y \ngeq x$ for every $x \notin D$, so $y \in D$. Thus the sets $m(x)$ are the meet-irreducibles of $\mathcal{D}(P)$. In particular, $\emptyset$ is a meet-irreducible of $\mathcal{D}(P)$ if and only if there exists $x \in P$ such that $\emptyset = X \setminus [x)$, that is, if and only if P has a least element.

(3) It is immediate to check that the map $(x] \longmapsto m(x) = X \setminus [x)$ from the set $J_{\mathcal{D}(P)}$ to the set $M_{\mathcal{D}(P)}$ is a one-to-one correspondence. On the other hand, $(x] \subseteq (y]$ if and only if $x \leq y$, if and only if $[x) \supseteq [y)$, and if and only if $m(x) \subseteq m(y)$. Therefore this map is an isomorphism as required.

(4) This amounts to showing that $(x] \subseteq X \setminus [y)$ if and only if $x \not\geq y$, which is clear. □

The ordered set P is obviously isomorphic to the ordered (by inclusion) set of its principal ideals $(x]$ which itself, according to the above theorem, is isomorphic to the ordered set of the downsets of the form $m(x)$. Thus:

Corollary 5.7 *Every ordered set P is isomorphic to the ordered set of the join-irreducible (respectively, meet-irreducible) elements of a distributive lattice.*

Below we study the behavior of the lattice of downsets of an ordered set with regard to the operations of disjoint union and of linear sum of ordered sets. Let us recall that, if P_1 has a greatest element u_1 and P_2 has a least element o_2, the glued linear sum $P_1 \oplus' P_2$ denotes the ordered set obtained from the linear sum $P_1 \oplus P_2$ by identifying the elements u_1 and o_2 (see Remark 1.43 in Chapter 1).

Proposition 5.8 *Let $P_1 = (X_1, \leq_1)$ and $P_2 = (X_2, \leq_2)$ be two ordered sets with $X_1 \cap X_2 = \emptyset$.*

1. *$\mathcal{D}(P_1 + P_2)$ is isomorphic to $\mathcal{D}(P_1) \times \mathcal{D}(P_2)$.*
2. *$\mathcal{D}(P_1 \oplus P_2)$ is isomorphic to $\mathcal{D}(P_1) \oplus' \mathcal{D}(P_2)$.*

Proof (1) Consider the map associating with any downset D of $P_1 + P_2$ the ordered pair (D_1, D_2) where $D_1 = D \cap X_1$ and $D_2 = D \cap X_2$. It is clear that D_1 (respectively, D_2) is a downset of P_1 (respectively, of P_2), and thus that the latter map goes from $\mathcal{D}(P_1 + P_2)$ to $\mathcal{D}(P_1) \times \mathcal{D}(P_2)$. This map is obviously injective and is also surjective since, if (D_1, D_2) is such that D_1 (respectively, D_2) is a downset of P_1 (respectively, of P_2), $D = D_1 + D_2$ is a downset of $P_1 + P_2$ (indeed, if $x \in D$, then, for example, $x \in D_1$, and $y \leq x$ in $P_1 + P_2$ implies $y \in D_1$ and thus $y \in D$). Finally $D = D_1 + D_2 \subseteq D' = D'_1 + D'_2$ if and only if $D_1 \subseteq D'_1$ and $D_2 \subseteq D'_2$, which shows that this map is an isomorphism, as required.

(2) Let D be a downset of $P_1 \oplus P_2$. It is either such that $D = D_1$ with D_1 a downset of P_1, or such that $D = X_1 + D_2$ with D_2 a non-empty downset of P_2. Then consider the map f associating with D the downset D_1 in the first case and the downset D_2 in the second. This map is a one-to-one correspondence between, $\mathcal{D}(P_1 \oplus P_2)$ and $\mathcal{D}(P_1) \oplus' \mathcal{D}(P_2)$ since, in $\mathcal{D}(P_1) \oplus' \mathcal{D}(P_2)$, the greatest element X_1 of $\mathcal{D}(P_1)$ is identified with $\emptyset$, the least element of $\mathcal{D}(P_2)$. Examining the four possible cases we easily check that $D \subseteq D'$ in $\mathcal{D}(P_1 \oplus P_2)$ if and only if $f(D) \subseteq f(D')$ in $\mathcal{D}(P_1) \oplus' \mathcal{D}(P_2)$ (for example, if $D = D_1$ and $D' = X_1 + D_2$, it is obvious that $D \subseteq D'$ if and only if $f(D) = D_1 \subseteq f(D') = D_2$). Therefore we have shown that f is an isomorphism between $\mathcal{D}(P_1 \oplus P_2)$ and $\mathcal{D}(P_1) \oplus' \mathcal{D}(P_2)$, as required. □

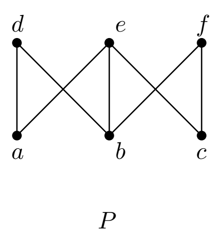
Several other distributive lattices are naturally associated with an ordered set $P = (X, \leq)$. First, instead of considering the lattice $\mathcal{D}(P)$ of downsets of P, we may consider the lattice $\mathcal{U}(P)$ of its upsets. It is immediate to check that the complementation map $D \longmapsto X \setminus D$ is a dual isomorphism between $\mathcal{D}(P)$ and $\mathcal{U}(P)$, which are thus two dual lattices. It is also immediate to check that the two maps of down closure ($Y \longmapsto (Y]$) and of up closure ($Y \longmapsto [Y)$) defined on $\underline{2}^X$ (see Example 3.33) induce two one-to-one correspondences between, on the one hand, the set $\mathcal{A}(P)$ of antichains of P and, on the other hand, $\mathcal{D}(P)$ or $\mathcal{U}(P)$. The set $\mathcal{A}(P)$ may thus be endowed with two dual structures of distributive lattice, respectively isomorphic to $\mathcal{D}(P)$ and to $\mathcal{U}(P)$. For example, let us write for two antichains A, B of P:

$$A \leq_{\mathcal{A}} B \iff (A] \subseteq (B]$$

(i.e., for any x of A, there exists y in B such that $x \leq y$).

So the ordered set $(\mathcal{A}(P), \leq_{\mathcal{A}})$ is a distributive lattice isomorphic to $\mathcal{D}(P)$, where $A \vee B = Max(A \cup B)$ and $A \wedge B = Max((A] \cap (B])$ hold. One may also show that the set of maximum size antichains of P is a sublattice of the lattice $(\mathcal{A}(P), \leq_{\mathcal{A}})$ (Exercise 4.3).

Figure 5.2 shows an ordered set P and the lattice of its downsets. The maximal elements of a downset D are underlined; these elements form the antichain corresponding to D in the one-to-one correspondence between $\mathcal{D}(P)$ and $\mathcal{A}(P)$. When we take into account only these underlined elements, the second ordered set in Figure 5.2 then represents $\mathcal{A}(P)$ endowed with the distributive lattice structure isomorphic



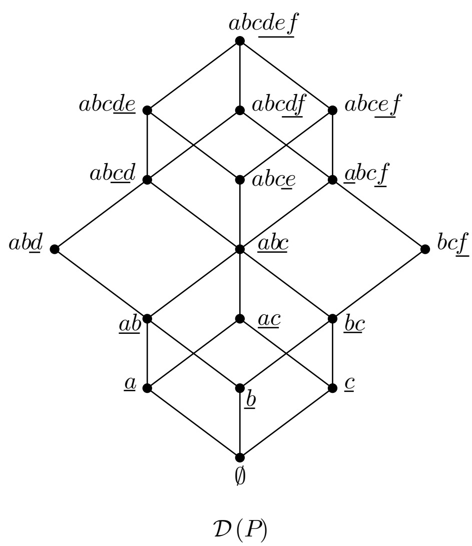


Figure 5.2 An ordered set P and the lattice $\mathcal{D}(P)$ of its downsets.

to $\mathcal{D}(P)$. We obtain the dual lattice $\mathcal{U}(P)$ of upsets of P by taking the complementary sets of the downsets. It is immediate that this lattice is also the lattice of the downsets of the dual of P, i.e., that $\mathcal{U}(P) = \mathcal{D}(P^d)$ holds.

5.3 Representations of a distributive lattice

We now state the fundamental Birkhoff Theorem on the representation of distributive lattices. Let us recall that for any element x of a lattice L, $J_x = \{j \in J_L : j \leq x\}$.

Theorem 5.9 (Birkhoff, 1933) *Let L be a distributive lattice. The map $x \longmapsto d(x) = J_x$ is an isomorphism between L and the lattice $\mathcal{D}(J_L)$ of downsets of the ordered set J_L of the join-irreducible elements of L. The converse isomorphism between $\mathcal{D}(J_L)$ and L is the map $D \longmapsto \bigvee D$.*

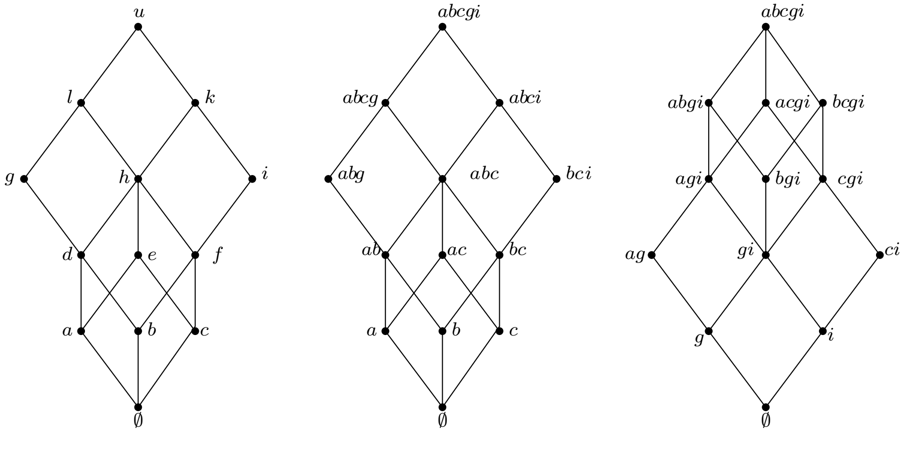
Proof Consider the map d defined in the theorem. It is clear that J_x is a downset of J_L and we know (Proposition 3.11) that $x = \bigvee J_x$. This equality immediately implies that d is injective and that $x \leq y$ if and only if $J_x \subseteq J_y$.

For a downset D of J_L, write $x = \bigvee D$. Thus $D \subseteq J_x$ holds. Let j be a join-irreducible of L such that $j \in J_x$. Since $j \leq x = \bigvee D$, Characterization (4) of distributive lattices (Theorem 5.1) implies that j is less than or equal to an element of the downset D. Therefore $j \in D$, $D = J_x$, and d is surjective, whence an isomorphism. Finally the above two equalities $x = \bigvee J_x$ and $D = d(\bigvee D)$ prove the last assertion. □

The following corollary is an immediate consequence of Theorem 5.9 and of the characterization of the join-irreducibles and the meet-irreducibles of the lattice $\mathcal{D}(J_L)$ given in Theorem 5.6, Item (2).

Corollary 5.10 *Let L be a distributive lattice. The map $j \longmapsto (j]$ is an isomorphism between J_L and $J_{\mathcal{D}(J_L)}$ and the map $J_L \setminus [j) \longmapsto \bigvee(J_L \setminus [j))$ is an isomorphism between $M_{\mathcal{D}(J_L)}$ and M_L.*

Remark 5.11 We have seen in Corollary 3.12 that the map which, with any element x of L, associates the set J_x of join-irreducibles less than or equal to x is a meet-morphism. The previous theorem shows that, if L is a distributive lattice, it is also a join-morphism, i.e., that for all $x, y \in L$, $J_{x \vee y} = J_x \cup J_x$. This equality also immediately results from Characterization (4) of distributive lattices in Theorem 5.1, since $j \leq x \vee y$ implies $j \leq x$ or $j \leq y$. We have already observed that the lattice $\mathcal{D}(P)$ of downsets of an ordered set P is dual of the lattice $\mathcal{U}(P)$ of its upsets (by the complementation map). Then, we deduce from Theorem 5.9 that a distributive lattice L is dual of the lattice $\mathcal{U}(J_L)$ of the upsets of the ordered set of its join-irreducible elements (this dual isomorphism is given by $x \longmapsto J_L \setminus J_x = \{j \in J_L : j \not\leq x\}$). Figure 5.3 shows an example of the representation of a distributive lattice L by $\mathcal{D}(J_L)$ and of its dual L^d by $\mathcal{U}(J_L)$.



(a) L (b) $\mathcal{D}(J_L)$ (c) $\mathcal{U}(J_L)$

Figure 5.3 (a) A lattice L, (b) the lattice $\mathcal{D}(J_L)$, and (c) the lattice $\mathcal{U}(J_L)$.

Proposition 5.12 *The ordered set J_L of join-irreducible elements of a distributive lattice L is isomorphic to the ordered set M_L of its meet-irreducible elements. If we denote this isomorphism ι, then, for any $j \in J_L$:*

1. $\iota(j) = \bigvee\{k \in J_L : j \not\leq k\} = \bigvee\{x \in L : j \not\leq x\}$.
2. $L = [j) + (\iota(j)]$ *with* $j \updownarrow \iota(j)$ *(i.e.,* $j \vee \iota(j) = \iota(j)^+$ *and* $j \wedge \iota(j) = j^-$*).*

Proof (1) From Corollary 5.10 we have the two isomorphisms $j \longmapsto (j]$ between J_L and $J_{\mathcal{D}(J_L)}$ and $J_L \setminus [j) \longmapsto \bigvee(J_L \setminus [j))$ between $M_{\mathcal{D}(J_L)}$ and M_L. On the other hand, we have shown in Theorem 5.6 (Item (3)) that $J_{\mathcal{D}(J_L)}$ is isomorphic to $M_{\mathcal{D}(J_L)}$, by the map $(j] \longmapsto m(j) = J_L \setminus [j)$. Therefore J_L is isomorphic to M_L by the isomorphism ι, composition of the previous three isomorphisms: so $\iota(j) = \bigvee\{k \in J_L : j \not\leq k\}$.

Given $j \in J_L$, write $z = \bigvee\{x \in L : j \not\leq x\} \geq \iota(j)$. Let x be such that $j \not\leq x$ and write $x = j_1 \vee ... \vee j_i \vee ... \vee j_p$ (where the j_i's are join-irreducibles). For any $i, j \not\leq j_i$ holds, whence $j_i \leq \iota(j)$ and thus $x = j_1 \vee ... \vee j_i \vee ... \vee j_p \leq \iota(j)$. Thus $z \leq \iota(j)$ whence $z = \iota(j)$ and (1) is proved.

In order to show (2), consider $j \in J(L)$ and an element $x \in L \setminus [j)$, i.e., such that $j \not\leq x$. By the same argument as above, $x \leq \iota(j)$ holds, whence $L \setminus [j) \subseteq (\iota(j)]$. Conversely from Characterization (4) in Theorem 5.1, $j \not\leq \iota(j)$. Therefore, $x \leq \iota(j)$ implies $j \not\leq x$, i.e., $(\iota(j)] \subseteq L \setminus [j)$, whence finally $L = [j) + (\iota(j)]$. We have obtained a cleavage of L which, as already observed, implies $j \updownarrow \iota(j)$. □

Thus, the isomorphism between J_L and M_L given in this proposition associates with a join-irreducible j of a distributive lattice L the greatest meet-irreducible not greater than or equal to j and dually, it associates with a meet-irreducible m of L the least

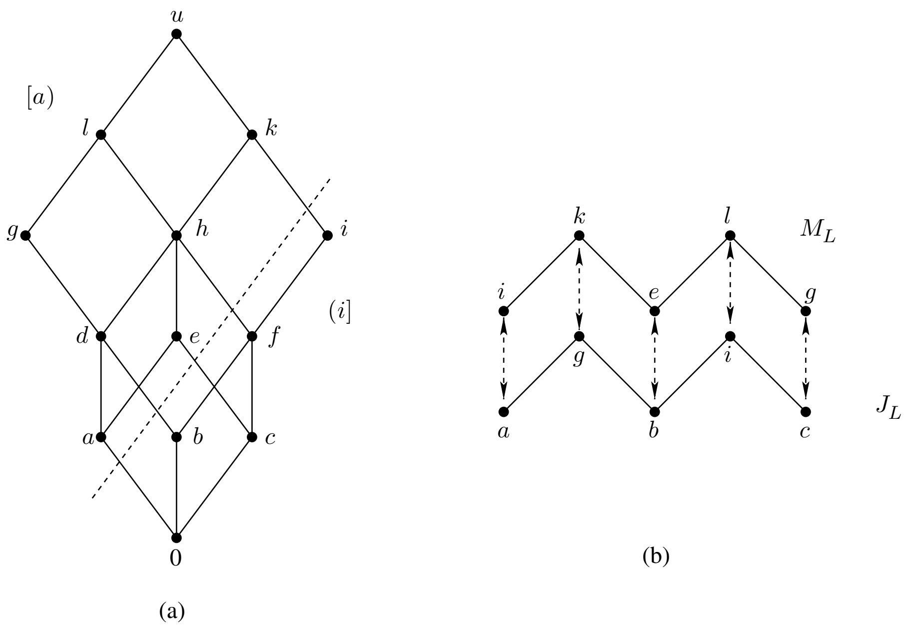


Figure 5.4 (a) $L = [a) + (i]$ and (b) the isomorphism between J_L and M_L.

join-irreducible not less than or equal to m. This is made explicit in Figure 5.4(b) for the distributive lattice L represented in Figure 5.4(a).

Remark 5.13 The isomorphism between J_L and M_L has, among other things, the following consequence: *a distributive lattice L is isomorphic to the lattice $\mathcal{D}(M_L)$ of downsets of M_L and dually isomorphic to the lattice $\mathcal{U}(M_L)$ of upsets of M_L.*

More precisely, this isomorphism between L and $\mathcal{D}(M_L)$ is given by $x \longmapsto (M_L \setminus M^x) = \{m \in M_L : x \not\leq m\}$ and the dual isomorphism by $x \longmapsto M^x = \{m \in M_L : x \leq m\}$. Figure 5.5 shows the distributive lattice L already given in Figure 5.3(a) and its representation by the lattice $\mathcal{D}(M_L)$, as well as the representation of its dual L^d by $\mathcal{U}(M_L)$.

Using Theorem 5.9 on the representation of distributive lattices, the results on the lattice of downsets of an ordered set in Section 5.2, and the duality principle for distributive lattices, it is easy to show the other properties of distributive lattices stated in the following proposition (the proofs are left to the reader).

Proposition 5.14 *Let L be a distributive lattice.*

1. *L is ranked; the rank of an element x is the number of join-irreducibles of L less than or equal to x and is also the number of meet-irreducibles of L not greater than or equal to x: $r(x) = |J_x| = |M_L \setminus M^x|$. In particular, $r(L) = |J_L| = |M_L|$.*

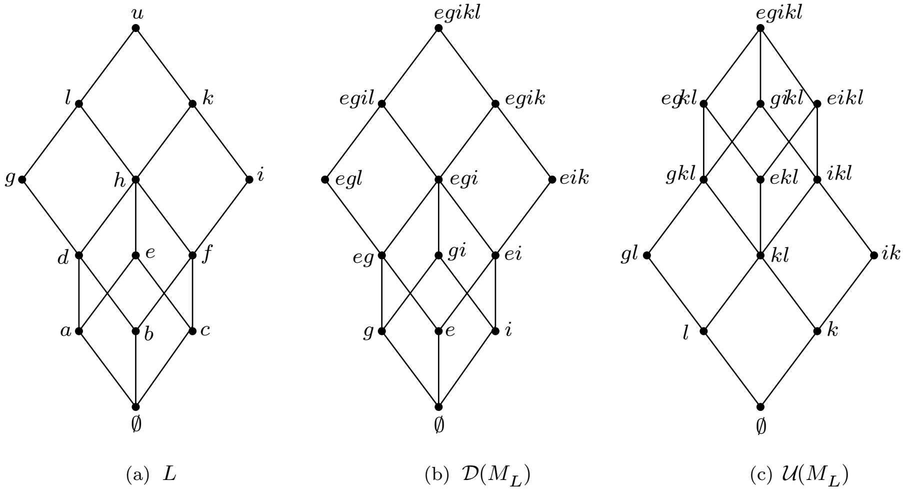


Figure 5.5 (a) A lattice L, (b) the lattice $\mathcal{D}(M_L)$, and (c) the lattice $\mathcal{U}(M_L)$.

2. *For all $x,y \in L$, $x \prec y$ if and only if $J_x = J_y \setminus j$ for a maximal element j of J_y, or if and only if $M^x = M^y + m$ for a minimal element m of M^x.*
3. *The number $|L^- x|$ of elements covered by x in L is equal to the number of maximal elements of J_x.*
4. *The number $|xL^+|$ of elements covering x in L is equal to the number of minimal elements of M^x.*
5. $\max\{|L^- x|, x \in L\} = \max\{|xL^+|, x \in L\} = \alpha(J_L) = \alpha(M_L)$.
6. *For any integer k, $|\{x \in L : |L^- x| = k\}| = |\{x \in L : |xL^+| = k\}| = |\{$antichains of size k of $J_L\}| = |\{$antichains of size k of $M_L\}|$.*

We give another very useful property of distributive lattices. In an arbitrary lattice an element may have several minimal representations as the join of join-irreducible elements (and as the meet of meet-irreducible elements). For example, in the lattice M_3 represented in Figure 5.1, the greatest element has three minimal representations by join-irreducible elements. This is not possible in a distributive lattice, as is specified in the following proposition.

Proposition 5.15 *Every element of a distributive lattice has a unique minimal representation as the join of join-irreducible elements and as the meet of meet-irreducible elements.*

Proof Let x be an element of a distributive lattice having two minimal representations as the join of join-irreducibles: $x = j_1 \vee ... \vee j_k \vee ... \vee j_p = j'_1 \vee ... \vee j'_i \vee ... \vee j'_q$. According to Characterization (4) in Theorem 5.1, there exist i and k such that $j_1 \le j'_i \le j_k$. Since $\{j_1, ..., j_p\}$ is an antichain, $j_1 = j_k = j'_i$ holds. Iterating this result gives the equality of the two representations. We dually obtain the uniqueness of the minimal representation by meet-irreducible elements. □

Remark 5.16 Several of the properties given in Proposition 5.14 and the property given in Proposition 5.15 characterize distributive lattices. For instance, a lattice L is distributive if and only if L is ranked and satisfies $r(L) = |J_L| = |M_L|$, or if and only if any element of L has a unique minimal representation as the join of join-irreducible elements and a unique representation as the meet of meet-irreducible elements.

The remainder of this section is devoted to the study of the generating sets of a distributive lattice, a study useful for the computation of the Boolean dimension of an ordered set in Chapter 6 (see Corollary 6.4).

We first consider the family of all sublattices of a lattice L. Since it is closed under intersection and contains L, it is a Moore family (Definition 3.29). Then, we may associate with this family a closure operator ϕ on $\underline{2}^L$; it associates with every subset A of L the least sublattice $\phi(A)$ of L including A ($\phi(A)$ is the intersection of all the sublattices of L including A). We now set the following definitions:

Definition 5.17 A subset G of a lattice L is said to be a *generating set* (of L) if $\phi(G) = L$, i.e., if the least sublattice of L including G is L. We denote $g(L)$ the minimum size of a generating set of a lattice L.

Let us recall that the meet (respectively, the join) of the empty set is the greatest (respectively, the least) element of lattice L (see Example 3.2 in Chapter 3). Then, any generating set G of L includes a generating set obtained from the former by deleting any of these two elements contained in G. On the other hand, any generating set of L necessarily contains its doubly irreducible elements. Thus, we obtain $g(L) \geq |\{\text{doubly irreducible elements of } L\}|$. We observe that this lower bound of $g(L)$ is obtained for any distributive lattice L generated by its doubly irreducible elements and, in particular, by a chain ($g(\underline{n}) = n - 2$).

We need the following notions and notations:

Definition 5.18 A subset G of a set E is called a *transversal* of a family of subsets $A_1, ..., A_k, ..., A_p$ of E if, for every $k \leq p$, $A_k \cap G \neq \emptyset$.

Definition 5.19 Let L be a lattice.

- We denote $MJ(L)$ the set of join-irreducibles of L which are meet-irreducible in the ordered set J_L.
- We denote $JM(L)$ the set of meet-irreducibles of L which are join-irreducible in the ordered set M_L.

The reader can show that an element of J_L is the meet of other elements in the ordered set J_L if and only if it is the meet of these elements in the lattice L (only one of these implications is not obvious). Thus $j \in MJ(L)$ if and only if ($j \in J_L$ and $j < \bigwedge\{j' \in J_L : j < j'\}$), where $\bigwedge$ is the meet in the lattice L.

For example, for the lattice L in Figure 5.4(a), J_L and M_L are represented in Figure 5.4(b), and $MJ(L) = \{a,g,i,c\}$ and $JM(L) = \{i,e,g\}$ hold.

Using the distributivity of L, the following lemma can be proved by induction on the number of occurrences of elements of G in the expression of x. It will be used to prove Theorem 5.21.

Lemma 5.20 *If G is a generating set of a distributive lattice L, every element x of L may be written as $x = \bigvee_{H\in\mathcal{H}}(\bigwedge H)$, where $\mathcal{H}$ is a family of subsets of G.*

Theorem 5.21 *A subset G of a distributive lattice L is a generating set of L if and only if G is a transversal of the family of intervals $\{[j,m], j \in MJ(L), m \in JM(L)\}$.*

Proof We first consider a generating set G of L and an interval $[j,m]$ of L with $j \in J_L$ and $m \in M_L$. Since j is join-irreducible and generated by G, Lemma 5.20 implies that j is a meet of elements of G. Then $j = \bigwedge_{1\leq k\leq p} g_k \leq m$ holds. Since m is meet-irreducible, there exists k such that $j \leq g_k \leq m$ (Characterization (7) in Remark 5.3). Thus, G is a transversal of the family of intervals $\{[j,m], j \in J_L, m \in M_L\}$ and, in particular, of the family of intervals $\{[j,m], j \in MJ(L), m \in JM(L)\}$.

Conversely, let G be a transversal of the latter family and let $j \in MJ(L)$. Consider a representation of j as a meet of meet-irreducibles of L. Replacing every meet-irreducible not in $JM(L)$ by a join of meet-irreducibles belonging to $JM(L)$ and using distributivity, we write j as a join of meets of subsets of $JM(L)$. Since j is join-irreducible, $j = \bigwedge_{1\leq k\leq p}\{m_k,\ m_k \in JM(L)\}$ holds. Every interval $[j,m_k]$ containing an element g_k of G, $j = \bigwedge_{1\leq k\leq p} g_k$ holds. Therefore, G generates $MJ(L)$, so all join-irreducibles of L and finally, L itself. □

Since a transversal of a family of sets is minimal if (and only if) it is a minimal transversal of the minimal sets of this family, we obtain the following corollary.

Corollary 5.22 *Let L be a distributive lattice. The minimum size $g(L)$ of a generating set of L is equal to* $\min\{|G|, G$ *transversal of the family of intervals* $[j,m]$ *such that* $j \in MJ(L), m \in JM(L)$ *and* $[j,m]$ *minimal for set inclusion*$\}$.

Example 5.23 We illustrate these results by computing the minimal generating sets of the lattice $L = \mathcal{D}(P)$ in Figure 5.2, and so $g(L)$. We first determine $MJ(L) = \{a,c,abd,bcf,abce\}$ and $JM(L) = \{ac,abd,bcf,abcde,abcef\}$. Next we must determine all the minimal intervals $[j,m]$, with $j \in MJ(L)$ and $m \in JM(L)$. There are six such intervals: $[abd,abd], [bcf,bcf], [a,ac], [c,ac], [abce,abcde]$, and $[abce,abcef]$. The first two are given by the two doubly irreducible elements of L and therefore must be contained in any transversal of the family of minimal intervals. Then L has four minimal generating sets:

$$\{ac, abd, bcf, abce\}, \{a, c, abd, bcf, abce\}$$
$$\{ac, abd, bcf, abcef, abcde\}, \{a, c, abd, bcf, abcef, abcde\}$$

and $g(L) = 4$.

In this example, we have easily computed $g(L)$. Yet in fact, Corollary 5.22 shows that the computation of $g(L)$ amounts to solving a very difficult (classic) problem: the search for a minimum transversal of a family of sets, a problem which itself is equivalent to several other problems (see, for instance, Berge (1970), Gondran and Minoux (2009) or Mazbic-Kulma and Sep (2007)). Nevertheless, $g(L)$ may easily be computed in some particular cases (see Remark 6.34 in Chapter 6).

5.4 Dualities: preorders–topologies, orders–distributive lattices

In this section, we consider several dualities coming from a fundamental duality between preorders and topologies defined on the same set X. This duality is obtained from a Galois connection (Definition 3.40) between binary relations on X and families of subsets of X.

Let us begin by specifying the ordinal structures brought into play. We consider the set $P(X^2)$ of all binary relations defined on X and the set $P[P(X)]$, denoted by $P^2(X)$, of all families of subsets of X. These two sets are ordered by inclusion: $R \subseteq R'$, i.e., xRy implies $xR'y$ (we then say that R is *included* in R' or that R' is *compatible with* R), and $\mathcal{F} \subseteq \mathcal{F}'$, i.e., $A \in \mathcal{F}$ implies $A \in \mathcal{F}'$.

Endowed with these orders, $P(X^2)$ and $P^2(X)$ are two Boolean lattices, respectively denoted by $\underline{2}^{X^2}$ and $\underline{2}^{P(X)}$, and where the join and the meet operations are respectively the union and intersection operations.

We now define a map t from $\underline{2}^{X^2}$ to $\underline{2}^{P(X)}$ and a map p from $\underline{2}^{P(X)}$ to $\underline{2}^{X^2}$ and we are going to show that they form a Galois connection between these two lattices. We need some definitions.

First, let R be a binary relation on X. We say that a subset D of X is an *R-downset*, or simply a *downset* if, for every $x \in X$ and every $y \in D$, xRy implies $x \in D$.

Observe that this notion generalizes the notion of a downset of an ordered set. The empty set and the set X are downsets of any binary relation on X; a downset is called *proper* if it is different from X and from the empty set. We denote by $\mathcal{D}(R)$ the set of downsets of R and by $\mathcal{D}^*(R)$ the set of proper downsets of R. Notice that the union and the intersection of two downsets being downsets, $\mathcal{D}(R)$ is a topology on X (Definition 5.5).

Now let $\mathcal{F}$ be a family of subsets of X. We write $\mathcal{F}(x) = \{A \in \mathcal{F} : x \in A\}$ and define a binary relation $R(\mathcal{F})$ on X by writing $xR(\mathcal{F})y$ if $\mathcal{F}(y) \subseteq \mathcal{F}(x)$, i.e., if for every $A \in \mathcal{F}, y \in A$ implies $x \in A$ (which may also be written $x \in \bigcap \mathcal{F}(y)$).

It is obvious that $R(\mathcal{F})$ is a preorder on X (Example 1.20) and that two elements x and y of X are in the same class of this preorder if and only if $\mathcal{F}(x) = \mathcal{F}(y)$.

We have thus associated with every binary relation R on X the family $\mathcal{D}(R)$ of its downsets, and with every family $\mathcal{F}$ of subsets of X the binary relation $R(\mathcal{F})$, and therefore defined the following two maps t and p:

$$\begin{array}{llll} t: & \underline{2}^{X^2} & \longrightarrow & \underline{2}^{P(X)} \\ & R & \longmapsto & t(R) = \mathcal{D}(R) \end{array} \qquad \begin{array}{llll} p: & \underline{2}^{P(X)} & \longrightarrow & \underline{2}^{X^2} \\ & \mathcal{F} & \longmapsto & p(\mathcal{F}) = R(\mathcal{F}) \end{array}$$

We now may state the fundamental result:

Theorem 5.24 *The ordered pair of maps (t,p) is a Galois connection between the lattices $\underline{2}^{X^2}$ and $\underline{2}^{P(X)}$. The relation $pt(R)$, denoted $\pi(R)$, is the least preorder including R. The family $tp(\mathcal{F})$, denoted $\tau(\mathcal{F})$, is the least topology including $\mathcal{F}$.*

Proof In order to show that (t,p) is a Galois connection, it is enough to show (Definition 3.40) that p and t are antitone and that their compositions pt and tp are extensive, that is: (1) $R \subseteq R' \Longrightarrow t(R) \supseteq t(R')$, (2) $\mathcal{F} \subseteq \mathcal{F}' \Longrightarrow p(\mathcal{F}) \supseteq p(\mathcal{F}')$, (3) $R \subseteq pt(R)$, and (4) $\mathcal{F} \subseteq tp(\mathcal{F})$. Let us show Properties (1) and (3) on t and pt. Let R be included in R' and D a downset of R'. Since yRx implies $yR'x$, $x \in D$ and yRx imply $y \in D$ which proves that D is a downset of R, and thus that $t(R')(= \mathcal{D}(R')) \subseteq t(R)(= \mathcal{D}(R))$.

Let x,y be in X with xRy; since any downset of R containing y contains x, $[t(R)](y) = \{D \in \mathcal{D}(R) : y \in D\} \subseteq [t(R)](x)$ holds, i.e., $x[pt(R)]y$. Thus, $R \subseteq pt(R)$. We would similarly prove Properties (2) and (4) on p and tp; thus (t,p) is a Galois connection between the lattices $\underline{2}^{X^2}$ and $\underline{2}^{P(X)}$ as required. It follows from the properties of such a connection between two lattices (Theorem 3.41) that the maps $\pi = pt$ and $\tau = tp$ are two closure operators and that their images $\pi[\underline{2}^{X^2}] = p[\underline{2}^{P(X)}]$ and $\tau[\underline{2}^{P(X)}] = t[\underline{2}^{X^2}]$ are two dual lattices. On the other hand, we have already observed that $\pi(R)$ is a preorder and that $\tau(\mathcal{F})$ is a topology.

Now, we must show that $\pi(R)$ (respectively, $\tau(\mathcal{F})$) is the least preorder including R (respectively, the least topology including $\mathcal{F}$).

Let R be a relation defined on X. We must show that for any preorder Q including R, the preorder $\pi(R)$ is included in Q, i.e., that $y[\pi(R)]x$ implies yQx. Since $R \subseteq Q$, Property (1) of a Galois connection gives $t(Q) = \mathcal{D}(Q) \subseteq \mathcal{D}(R) = t(R)$. For any $x \in X$, the subset $Qx = \{z \in X : zQx\}$ is a Q-downset (since Q is transitive) and so is also an R-downset which contains x (since Q is reflexive). On the other hand, $y[\pi(R)]x = y[p(\mathcal{D}(R)]x$ means that any R-downset containing x also contains y. Since Qx is an R-downset containing x, it contains y and we have shown that $y[\pi(R)]x$ implies yQx, as required.

Now let $\mathcal{F}$ be a family of subsets of X. We must show that, for any topology $\mathcal{T}$ including $\mathcal{F}$, $\tau(\mathcal{F})$ is included in $\mathcal{T}$, i.e., that $T \in \tau(\mathcal{F})$ implies $T \in \mathcal{T}$. Let $T \in \tau(\mathcal{F})$; since τ is a closure operator, it is isotone, so $T \in \tau(\mathcal{T}) = tp(\mathcal{T})$ holds and T is thus a downset of $p(\mathcal{T})$. Write $D(x)$ the principal ideal of $p(\mathcal{T})$ generated by

$x : D(x) = \{y \in X : y[p(\mathcal{T})]x\} = \{y \in X : \mathcal{T}(x) \subseteq \mathcal{T}(y)\}$ (where $\mathcal{T}(x) = \{A \in \mathcal{T} : x \in A\}$). Now $\mathcal{T}(x) \subseteq \mathcal{T}(y)$ is equivalent to $y \in \bigcap \mathcal{T}(x)$. Thus $D(x) = \bigcap \mathcal{T}(x)$ and $D(x) \in \mathcal{T}$ since $\mathcal{T}$ is $\cap$-stable. On the other hand, $T = \bigcup\{D(x), x \in T\}$ obviously holds. Since the $D(x)$'s belong to $\mathcal{T}$ which is $\cup$-stable, their union T also belongs to $\mathcal{T}$, as required. □

Corollary 5.25 *The set $\mathcal{Q}_X$ of preorders and the set $\mathcal{T}_X$ of topologies on X are two dual lattices. Let R_1 and R_2 be two preorders on X and $\mathcal{T}_1$ and $\mathcal{T}_2$ the associated topologies. Their meets and joins are given by the following formulas:*

$$R_1 \wedge R_2 = R_1 \cap R_2 \qquad R_1 \vee R_2 = \pi(R_1 \cup R_2) = p(\mathcal{T}_1 \cap \mathcal{T}_2)$$

and

$$\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_1 \cap \mathcal{T}_2 \qquad \mathcal{T}_1 \vee \mathcal{T}_2 = \tau(\mathcal{T}_1 \cup \mathcal{T}_2) = t(R_1 \cap R_2)$$

Proof This result immediately derives from the characterization of the closure operators π and τ given in Theorem 5.24. Indeed, if R is a preorder (respectively, $\mathcal{F}$ a topology), $\pi(R) = R$ (respectively, $\tau(\mathcal{F}) = \mathcal{F}$) holds and then $\pi(\underline{2}^{X^2}) = \mathcal{Q}_X$ (respectively, $\tau(\underline{2}^{P(X)}) = \mathcal{T}_X$) holds. Since $\mathcal{Q}_X$ and $\mathcal{T}_X$ are Moore families, their meet operation is the set intersection and their join operation is obtained by applying the general formulas on the two dual lattices associated with a Galois connection (formulas found just before Example 3.44). □

Theorem 5.24 characterizes the closure operators π and τ associated with the Galois connection (t,p). In particular, π is nothing else but the reflexo-transitive closure of a binary relation already defined in Example 1.20. If $\mathcal{T}$ is a topology and if the family $\mathcal{F}$ satisfies $\tau(\mathcal{F}) = \mathcal{T}$, we say that $\mathcal{F}$ *generates* $\mathcal{T}$ or that $\mathcal{F}$ is a *generating set* of $\mathcal{T}$. We use the same terms for the closure operator π.

Remark 5.26 We may also define the dual notion of that of a downset of a binary relation, namely the notion of an *upset* of a relation R defined on X: a subset U of X is an *upset* if $x \in X, y \in U$ and yRx imply $x \in U$. We denote by $\mathcal{U}(R)$ the set of upsets of R and by $\mathcal{U}^*(R)$ the set of its proper upsets (i.e., different from X and $\emptyset$). Since the union and the intersection of two upsets are upsets, $\mathcal{U}(R)$ is a topology on X. Observe that $\mathcal{U}(R) = \{X \setminus D, D \in \mathcal{D}(R)\}$.

We are now going to consider some particular classes of preorders and specify the dual classes of associated topologies; we need the following definitions.

Definition 5.27 Let $\mathcal{T}$ be a topology on X.

1. $\mathcal{T}$ is *complemented* if, for every $A \subseteq X$, $A \in \mathcal{T}$ if and only if $X \setminus A \in \mathcal{T}$.
2. $\mathcal{T}$ is *quasi-separated* (or is called a T_0*-topology*) if, for any pair of elements of X, there exists $A \in \mathcal{T}$ containing one and only one of these two elements.
3. $\mathcal{T}$ is *linear* if, for all $A, B \in \mathcal{T}$, $A \subseteq B$ or $B \subseteq A$ holds.
4. $\mathcal{T}$ is *saturated linear* if $\mathcal{T}$ is linear with size $|X| + 1$.

The set denoted by $\mathcal{T}_X^0$ of quasi-separated topologies (or T_0-topologies) defined on X will be shown to be the dual of the set $\mathcal{O}_X$ of orders on X (Proposition 5.29 and Corollary 5.30).

Linear (respectively, saturated linear) topologies are nothing else but extended chains – since they contain $\emptyset$ and X – (respectively, maximal chains) of the lattice $\underline{2}^X$ of subsets of X. In particular, saturated linear topologies are the linear topologies containing the maximum number of possible subsets for such a topology.

We may now state the following result, the easy proof of which is left to the reader.

Proposition 5.28 *In the duality between the set of preorders and the set of topologies defined on X, the set of equivalences (respectively, of orders, of total preorders, of linear orders) corresponds to the set of complemented (respectively, of quasi-separated, of linear, of saturated linear) topologies.*

Figure 5.6 exemplifies the correspondence between equivalences (respectively, orders, total preorders) and complemented (respectively, quasi-separated, linear) topologies.

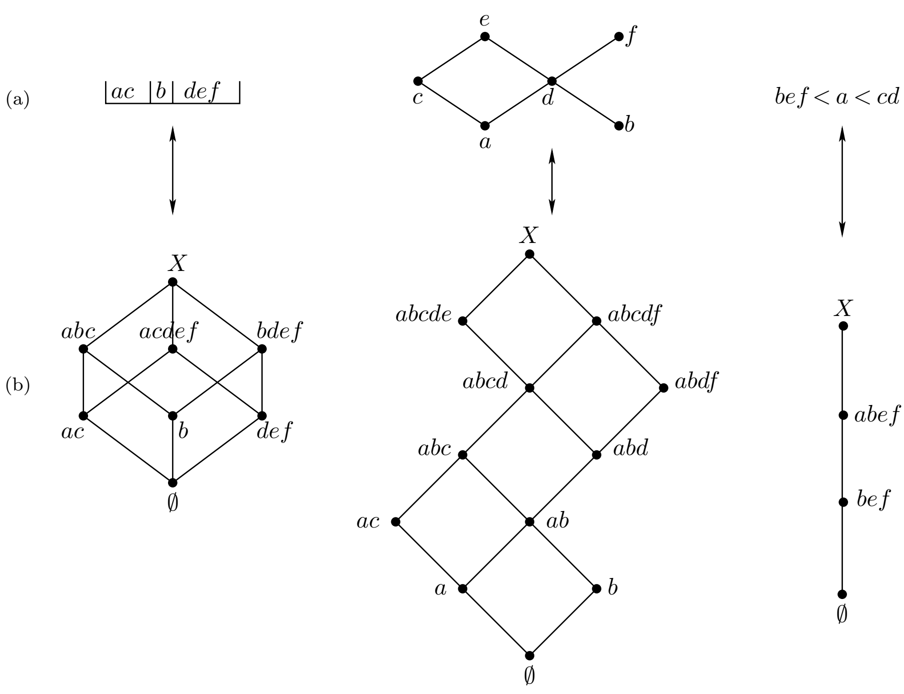


Figure 5.6 (a) An equivalence (defined by its classes), an order, and a total preorder on $X = \{a,b,c,d,e,f\}$, and (b) their associated topologies.

The duality between equivalence relations and complemented topologies is well-known in the form of the classic correspondence between equivalences and partitions: the unions of classes of the equivalence associated with a partition form a complemented topology, i.e., a Boolean lattice the atoms of which are these classes.

Let us somewhat specify the duality between total preorders and extended chains, which will be used in the following chapter for the proof of the fundamental coding theorem (Theorem 6.29). If R is a total preorder, the set of its classes is linearly ordered (Example 1.20). If this total preorder has k classes then, numbering them with respect to this order, we may write it as:

$$X_1 < .... < X_i < .... < X_k \tag{1}$$

We can check that the topology $\tau(R)$ associated with the total preorder R is the extended chain:

$$\emptyset \subset F_1 .... \subset F_i \subset .... \subset F_k = X$$

with, for every $i = 1,...,k$, $F_i = \bigcup_{1 \leq h \leq i} X_h$.

Conversely, if $\mathcal{C} = \emptyset \subset F_1 \subset .... \subset F_i \subset .... \subset F_k = X$ is an extended chain of $\underline{2}^X$ including k non-empty subsets, the corresponding total preorder $\pi(\mathcal{C})$ is obtained in the form (1) by writing $F_0 = \emptyset$ and, for every $i = 1,...,k$, $X_i = F_i \setminus F_{i-1}$.

Thus, the total preorders with k classes correspond to the linear topologies of size $k+1$, i.e., to the extended chains of length k of $\underline{2}^X$. Two cases are particularly interesting:

- If $k = 2$, the total preorders of the form $X_1 < X_2$ correspond to the topologies of the form $\{\emptyset, X_1, X\}$ (observe that the former are the coatoms of the lattice of preorders and that the latter are the atoms of the lattice of topologies).
- If $k = |X| = n$, we obtain the linear orders $x_1 \prec ... \prec x_i \prec x_{i+1} ... \prec x_n$ corresponding to the saturated linear topologies, i.e., the maximal chains of the lattice $\underline{2}^X$:

$$\emptyset \subset \{x_1\} ... \subset \{x_1, x_2, ... x_i\} \subset \{x_1, x_2, ..., x_i, x_{i+1}\} ... \subset \{x_1, x_2, ..., x_n\} = X$$

We may then specify one of the results of Proposition 5.28.

Proposition 5.29 *In the duality between the set of total preorders and the set of linear topologies defined on X, the total preorders with k classes correspond to the extended chains of length k of $\underline{2}^X$. In particular, the total preorders with two classes correspond to the topologies containing a unique element different from $\emptyset$ and X, and the linear orders correspond to the maximal chains of $\underline{2}^X$.*

In Proposition 5.28, the one-to-one correspondences between the different sets are obviously dualities for the (inclusion) orders on these sets. In particular, since the set of orders defined on X is a meet-semilattice (for the intersection operation), the set $\mathcal{T}_X^0$ of quasi-separated topologies is a join-semilattice and we obtain:

Corollary 5.30 (Birkhoff, 1937b) *The meet-semilattice $\mathcal{O}_X$ of orders defined on a set X is dual of the join-semilattice $\mathcal{T}_X^0$ of quasi-separated topologies defined on X.*

In this duality, to an order O corresponds the quasi-separated topology $t(O) = \mathcal{D}(O)$ formed of the downsets of O. To the meet $O_1 \cap O_2$ of two orders O_1 and O_2 corresponds the join $\mathcal{D}(O_1) \vee \mathcal{D}(O_2) = \mathcal{D}(O_1 \cap O_2)$ of the two associated quasi-separated topologies, etc. The reader can specify the other order dualities resulting from Proposition 5.28.

In Corollary 5.30, all orders are defined on the same set X of size (say) n. On the other hand, the quasi-separated topologies defined on X correspond to the distributive lattices of height n (according to Theorem 5.9 and Proposition 5.14). Thus, we obtain a duality between the ordered sets of size n and the distributive lattices of height n. Since this result holds for any n, we may say that there is a duality between the class of ordered sets and the class of distributive lattices. However, the latter duality is of a more general type than that induced by a Galois connection. It could be suitably defined in category theory as a duality between the category of ordered sets and the category of distributive lattices. This would involve defining these categories with their "morphisms," which is beyond the scope of this book (one may refer to Davey and Priestley's book (2001) for a more categorial approach to this duality).

The consequences of the duality between the class of ordered sets and the class of distributive lattices is that any result, construction or question on one of these classes may be translated into a result, construction or question on the other class (and conversely). Depending on the case, it may be easier to realize a construction or to solve a question on one of them or on the other. For example, we may translate the first item of Proposition 5.8 by saying that the direct product of two distributive lattices may be obtained from the disjoint union of their ordered sets of join-irreducible elements. Before ending this section with a second example, let us point out that we will see other examples in the following chapter, where problems on the dimension of ordered sets are translated into problems on the generating sets of a distributive lattice (Corollaries 6.4 and 6.11 in Chapter 6).

From Theorem 5.24 and from the above correspondence between extended chains and total preorders, we also deduce the following result:

Proposition 5.31 *Let $P = (X, O)$ be an ordered set of size n and $\mathcal{D}(P)$ the lattice of downsets of P. The total preorders with k classes including the order O are in a one-to-one correspondence with the extended chains of length k of $\mathcal{D}(P)$. In particular, the linear extensions of P are in a one-to-one correspondence with the maximal chains of $\mathcal{D}(P)$.*

Proof Let $\mathcal{C}$ be an extended chain of length k included in $\mathcal{D}(P) = t(O)$. From the properties of the Galois connection (t,p) (Theorem 5.24), $\mathcal{C} \subseteq t(O)$ is equivalent to $pt(O) = \pi(O) = O \supseteq p(\mathcal{C})$, where $p(\mathcal{C})$ is a total preorder with k classes (Proposition 5.28). Conversely, by a similar reasoning, if R is a total preorder with k classes including O, its image $t(R)$ is an extended chain of length k included in $\mathcal{D}(P)$. If $k = |X|$, we obtain the case of the linear extensions of P. □

Remark 5.32 Proposition 5.31 may be transferred to distributive lattices by means of the representation theorem of these lattices. In particular, the following holds: *the linear extensions of the ordered set J_L of the join-irreducible elements of a distributive lattice L are in a one-to-one correspondence with the maximal chains of this lattice.* If $j_1 \prec j_2 \prec ... \prec j_n$ is a linear extension of J_L, then

$$0_T \prec j_1 \prec j_1 \vee j_2 \prec j_1 \vee j_2 \vee j_3 \prec ... \prec \bigvee J_L = 1_L$$

is a maximal chain of L.

In Exercise 5.14, Proposition 5.31 and the above statement are generalized.

Remark 5.33 Exercise 7.4 shows the existence of a one-to-one correspondence between total preorders and strict weak orders (recall that the asymmetric part of a total preorder is a strict weak order, see Example 1.21). Therefore, we may replace in Propositions 5.29 and 5.31 "total preorder" with "strict weak order." For instance, the strict weak orders which are extensions of a strict order O and have range k are in a one-to-one correspondence with the extended chains of length k of the lattice of downsets of O.

5.5 Duality between orders and spindles of linear orders

We are now going to establish another significant duality between the set $\mathcal{O}_X$ of orders defined on X and the set $\mathcal{S}_X$ of "spindles" (or "convex" subsets or "geodesically convex" subsets) of the set $\mathcal{L}_X$ of linear orders on X (Proposition 5.42). Let us begin with the definition of these notions and the proof of their equivalence.

We recall that $\mathcal{L}(O)$ denotes the set of linear extensions of the order O and that $O = \bigcap\{L : L \in \mathcal{L}(O)\}$ (see Theorem 2.29 in Chapter 2).

Definition 5.34 A set $\mathcal{E} = \{L_1, L_2, ..., L_r\}$ of linear orders on a set X is a *spindle* (*of linear orders*) on X if there exists an order O on X such that $\mathcal{E} = \mathcal{L}(O)$.

The reader can show that this definition is equivalent to writing that, for any linear order L on X, $L \supseteq L_1 \cap L_2 ... \cap L_r$ implies $L \in \mathcal{E}$.

Example 5.35 On the set $X = \{1,2,3,4\}$ consider the set

$$\mathcal{E} = \{1342, 1432, 3142, 3412, 4132, 4312\}$$

of linear orders (here written as permutations of X). The reader can check that $\mathcal{E}$ is the spindle $\mathcal{L}(O)$ with

$$O = \{(1,2), (3,2), (4,2), (1,1), (2,2), (3,3), (4,4)\}$$

However, if we delete from $\mathcal{E}$ any of its orders, it is no longer a spindle.

Definition 5.36 A set $\mathcal{E} = \{L_1,...,L_i,...,L_r\}$ of linear orders on a set X is a *convex subset* of the set $\mathcal{L}_X$ of all linear orders on X if, for all i,j $(1 \leq i,j \leq r)$ and for every linear order L on X, $L_i \cap L_j \subseteq L \subseteq L_i \cup L_j$ implies $L \in \mathcal{E}$.

Let us associate with two linear orders L and L' on X an interval, denoted $[L,L']$, defined as the set of linear orders on X included in $[L \cap L', L \cup L']$. For instance, on $X = \{1,2,3,4\}$, $[1342, 3412] = \{1342, 3142, 3412\}$. A set $\mathcal{E}$ of linear orders is then a convex subset of $\mathcal{L}_X$ if, as soon as it contains the linear orders L_i and L_j, it also includes the interval $[L_i, L_j]$. This is nothing else but the classic definition of a convex subset, once defined a notion of an interval. The reader can check that the set $\mathcal{E}$ given in Example 5.35 is a convex subset of $\mathcal{L}_X$. He will also easily show that, in the above definition, the condition $L_i \cap L_j \subseteq L \subseteq L_i \cup L_j$ may be replaced with any of the two conditions $L \subseteq L_i \cup L_j$ or $L_i \cap L_j \subseteq L$.

We are now going to define the notion of a geodesically convex subset of $\mathcal{L}_X$. With this aim, we endow $\mathcal{L}_X$ with an (undirected) graph structure which comes down to defining a symmetric relation between linear orders. Let L be a linear order on X written as a permutation of X. We say that one carries out a *commutation* on L if one exchanges two consecutive elements in the permutation representing L; one then obtains another linear order L' on X. For instance, if $L = 24135$ the four possible commutations on L generate the four linear orders 42135, 21435, 24315, and 24153. We observe that, if L' is obtained from L by the commutation of two consecutive elements x and y, a unique ordered pair (namely (x,y)) of L is changed and so we may equivalently write $L \cap L'^d = \{(x,y)\}$.

Definition 5.37 The *permutograph* on a set X is the graph, denoted by Σ_X, whose set of vertices is the set $\mathcal{L}_X$ of linear orders on X and whose edges are defined by the following adjacency relation, noted *Adj*, between two linear orders: for $L, L' \in \mathcal{L}_X$, $L Adj L'$ if $|L \cap L'^d| = 1$.

Then, to say that L and L' are adjacent in Σ_X comes down to saying that L' is obtained from L (or L from L') by a commutation.

The permutograph on $X = \{1,2,3,4\}$ is represented in Figure 5.7. Observe that this graph is the neighborhood graph of the permutoedre order presented in Example 1.17.

In a (undirected) graph the *geodesic distance* between two vertices is the length of a shortest path between them. Since it clearly satisfies the axioms of a distance,[1] this graph endowed with it is a metric space. A shortest path between two vertices is called a *geodesic* between these vertices.

[1] A *distance* (also called a *metric*) on a set E is a function d from E^2 to the set $\mathbb{R}^+$ of non-negative real numbers satisfying the following properties for all $e, e', e'' \in E$: $d(e,e') = 0$ if and only if $e = e'$, $d(e,e') = d(e',e)$, and $d(e,e'') \leq d(e,e') + d(e',e'')$.

For the permutograph Σ_X, the geodesic distance between two linear orders L and L' is (by definition) the minimum number of commutations to carry out in order to go from one to the other. We denote it by $\delta(L,L')$.

For instance, the reader can check in Figure 5.7 that $\delta(1432,3412) = 3$ in $\Sigma_{\{1,2,3,4\}}$.

The computation of the geodesic distance between two vertices of a graph with many vertices may be difficult; however, the following Proposition 5.38 shows that it is easy for Σ_X despite its $|X|!$ vertices.

Let L and L' be two linear orders on X; we write

$$d(L,L') = |L \cap L'^{d}| = |L \setminus L'|$$

In other words, $d(L,L')$ is the number of ordered pairs (x,y) with xLy and $yL'x$. If these linear orders represent preferences, such an ordered pair is interpreted as a *disagreement* on the preferences between x and y. Then we will say that $d(L,L')$ is the number of disagreements between these two orders.

Since $|L \setminus L'| = |L' \setminus L|$, $d(L,L') = (|L \setminus L'| + |L' \setminus L|)/2$ also holds, which shows that d is nothing else but half the classic symmetric difference distance between L and L' (recall that the *symmetric difference distance* between two subsets A and B of

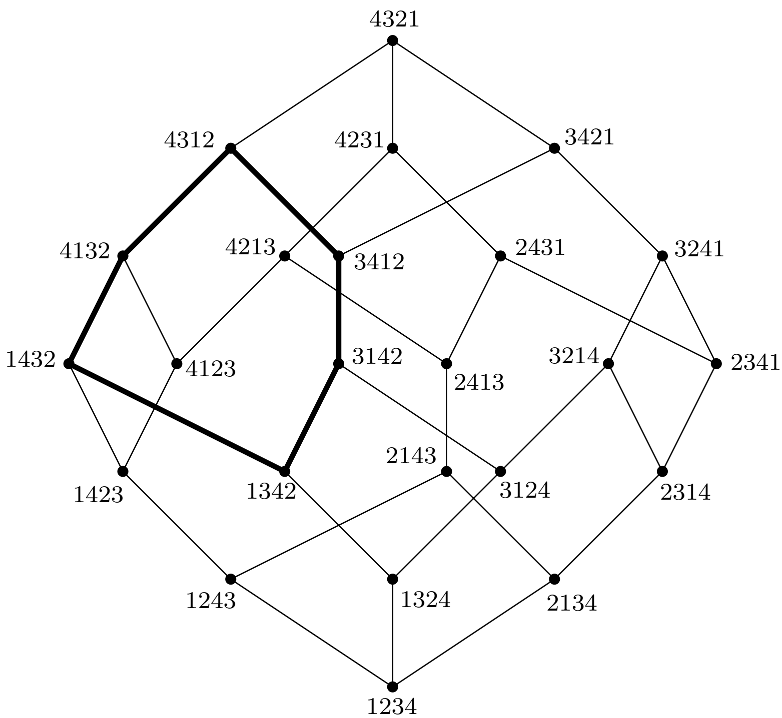


Figure 5.7 The permutograph on $\{1,2,3,4\}$ and the geodesic interval $[3142,4132]_g$.

a set is the size of their symmetric difference ($|A \setminus B| + |B \setminus A|$), see Exercise 5.12). For instance, $d(1432,3412) = |\{(1,4),(1,3),(4,3)\}| = 3$.

The equality $\delta(1432,3412) = d(1432,3412)$ obtained in this example is not fortuitous, since we have the following result:

Proposition 5.38 *The geodesic distance $\delta(L,L')$ between two linear orders L and L', vertices of the permutograph Σ_X, is equal to the number $d(L,L')$ of their disagreements.*

Proof First observe that the inequality $d(L,L') \leq \delta(L,L')$ holds. Indeed, any commutation carried out on L deleting at most one disagreement between L and L', we must make at least $d(L,L')$ commutations to go from L to L'. Now we show the converse inequality $d(L,L') \geq \delta(L,L')$ by induction on $d(L,L')$. If $d(L,L') = 1$, we easily check that $\delta(L,L') = 1$. Assume the property satisfied for $d(L,L') = k > 1$ and consider two linear orders L and L' satisfying $d(L,L') = k+1$. Carrying out a commutation on two consecutive elements x and y of L such that $(x,y) \in L \setminus L'$ (such elements exist since, if not, $L = L'$), we obtain a linear order L'' (different from L') and $d(L,L') = 1 + k = d(L,L'') + d(L'',L')$ holds. By the induction hypothesis, this implies $d(L,L') \geq \delta(L,L'') + \delta(L'',L')$ and, by the triangular inequality applied to the distance δ, $d(L,L') \geq \delta(L,L')$ as required. □

Like in every (undirected) graph, we may now define the notion of a geodesic interval of the permutograph.

Definition 5.39 We call a *geodesic interval* between two linear orders L and L' defined on X – and we denote $[L,L']_g$ – the set of the linear orders which belong to the geodesics linking L and L' in the permutograph Σ_X.

A subset $\mathcal{E}$ of $\mathcal{L}_X$ is *geodesically convex* if it is a geodesically convex subset of the permutograph Σ_X, i.e, if for all $L,L' \in \mathcal{E}$, $[L,L']_g \subseteq \mathcal{E}$.

For instance, in Figure 5.7, the geodesic interval

$$[3142,4132]_g = \{1342,1432,3142,3412,4132,4312\}$$

is visualized.

We have defined three particular subsets of the set $\mathcal{L}_X$ of linear orders on X, namely the spindles of linear orders, the convex subsets, and the geodesically convex subsets. We are now going to show that these three notions are the same by proving Theorem 5.41. We begin with a lemma which is an immediate application of a classic result on the symmetric difference distance to the distance d (half the symmetric difference distance between linear orders). The proof is left to the reader, see Exercise 5.12.

Lemma 5.40 *Let L,L', and M be three linear orders on X. Then:*

$$L \cap L' \subseteq M \subseteq L \cup L' \iff d(L,M) + d(M,L') = d(L,L')$$

Theorem 5.41 *Let $\mathcal{E}$ be a set of linear orders on X. The following properties are equivalent:*

1. *$\mathcal{E}$ is convex.*
2. *$\mathcal{E}$ is geodesically convex.*
3. *$\mathcal{E}$ is a spindle.*

Proof (1) $\Longleftrightarrow$ (2): since $\mathcal{E}$ is convex (respectively, geodesically convex) if and only if it includes $[L,L']$ (respectively, $[L,L']_g$) for all $L,L' \in \mathcal{E}$, we obtain this equivalence by showing that these two intervals are equal. Now since $d = \delta$, $d(L,M) + d(M,L') = d(L,L')$ is equivalent to $\delta(L,M) + \delta(M,L') = \delta(L,L')$. In the permutograph (like in any graph) the latter equality is equivalent to the fact that M is on a geodesic linking L and L'. Then the geodesic interval $[L,L']_g$ is equal to the interval $[L,L']$ – defined above as the set of linear orders included in $[L \cap L', L \cup L']$ – which itself, according to Lemma 5.40, is equal to the set of the linear orders M such that $d(L,M) + d(M,L') = d(L,L')$.

(3) $\Longrightarrow$ (1): every spindle $\mathcal{E}$ is a convex subset since, if $L_i, L_j \in \mathcal{E}$, $M \supseteq L_i \cap L_j$ implies $M \supseteq \bigcap\{L : L \in \mathcal{E}\}$ (see the sentence following Definition 5.34).

(1) $\Longrightarrow$ (3): we show that a convex subset $\mathcal{E}$ is the spindle $\mathcal{L}(O)$ where $O = \bigcap\{L : L \in \mathcal{E}\}$. By definition of O, $\mathcal{E} \subseteq \mathcal{L}(O)$ holds. Assume $\mathcal{E} \subset \mathcal{L}(O)$, and so the existence of $L \supset O$ with $L \notin \mathcal{E}$. Let $L' \in \mathcal{E} \subset \mathcal{L}(O)$ and let $L' = L_0, L_1, ..., L_p = L$ be a geodesic from L' to L in the permutograph. The spindle $\mathcal{L}(O)$ being convex (according to the previous implication (3) $\Longrightarrow$ (1)), and so geodesically convex, this geodesic is included in $\mathcal{L}(O)$. Consider the least i such that $L_i \in \mathcal{E}$ and $L_{i+1} \in \mathcal{L}(O) \setminus \mathcal{E}$. We have $\delta(L_i, L_{i+1}) = 1$, i.e., there exist x and y in X such that $L_i \cap L^d_{i+1} = (x,y)$. Since L_{i+1} does not belong to the convex subset $\mathcal{E}$, then for every M of $\mathcal{E}$, $M \cap L_i \not\subseteq L_{i+1}$ holds. Therefore $M \cap L_i \cap L^d_{i+1}$ is non-empty and is equal to (x,y) and then, for every M of $\mathcal{E}$, (x,y) belongs to M and thus to O, a contradiction with $(x,y) \notin L_{i+1}$. □

Since the notions of a spindle, a convex subset, and a geodesically convex subset are the same, we will indifferently use each of them in the sequel. We will denote by $\mathcal{S}_X$ the set of all spindles of $\mathcal{L}_X$. This set is ordered by set inclusion. In fact, the following proposition shows that $\mathcal{S}_X$ is a join-semilattice for this order:

Proposition 5.42 *The set $\mathcal{O}_X$ of orders and the set $\mathcal{S}_X$ of spindles of linear orders defined on X are two dual semilattices.*

Proof We define two maps between the semilattice $\mathcal{O}_X$ and the lattice $\underline{2}^{\mathcal{L}_X}$ of subsets of $\mathcal{L}_X$ as follows: to an order O corresponds the set of linear orders containing it, i.e., the spindle $\mathcal{L}(O)$; to a set of linear orders corresponds the order obtained as their intersection. It is easy to check that these two maps define a Galois connection between $\mathcal{O}_X$ and $\underline{2}^{\mathcal{L}_X}$ and thus two closure operators on these ordered sets. Since every order is the intersection of the linear orders including it, the set of closed sets of $\mathcal{O}_X$ is the set $\mathcal{O}_X$ itself. As for the closed sets of $\underline{2}^{\mathcal{L}_X}$, they are the images of orders, thus the spindles of linear orders. By the properties of Galois connections (Chapter 3, Theorem 3.41) it follows that $\mathcal{S}_X$ is a join-semilattice dual of the meet-semilattice $\mathcal{O}_X$. □

In this duality, the meet (i.e., the intersection) of two orders in $\mathcal{O}_X$ corresponds to the join in $\mathcal{S}_X$, that is, to the "convex closure" of the union of two spindles (i.e., the least spindle including this union). Combining Corollary 5.30 and Proposition 5.42, we obtain the following result:

Theorem 5.43 *The meet-semilattice $\mathcal{O}_X$ of orders defined on X is dual of the join-semilattice $\mathcal{T}_X^0$ of quasi-separated topologies and of the join-semilattice $\mathcal{S}_X$ of the spindles of linear orders on X; therefore these two join-semilattices are isomorphic.*

We let the reader check that the isomorphism stated in this theorem associates with a quasi-separated topology (so, with a distributive lattice of height $|X|$) the set of linear orders on X corresponding with the maximal chains of this lattice (more precisely they are the images of these chains by the map p of the Galois connection in Theorem 5.24).

5.6 Further topics and references

Some results on distributive lattices appear in the prehistory of lattice theory; indeed as early as 1897, Dedekind proved basic properties of these lattices and in 1900, he raised the problem of computing the size of the free distributive lattice on n generators, a problem solved up to now only for n less than 9 (see Exercise 5.7). Then one must wait for Birkhoff (1933, 1937) to obtain fundamental results like the representation theorem (Theorem 5.9), the isomorphism between the ordered sets of join-irreducibles and of meet-irreducibles (Proposition 5.12) or the uniqueness property of representations by irreducibles (Proposition 5.15). The precisions brought to the structure of distributive lattices (for example in Proposition 5.14) are due, often independently, to different authors like Schützenberger (1949), Avann (1958, 1961a), Bonnet and Pouzet (1969), Stanley (1972), Monjardet (1974), etc. As already said in Remark 5.16, several of these properties allow us to characterize distributive lattices. In addition to the properties mentioned in this remark, we may add the property that "the size of every maximal chain of L is $|J_L| + 1 = |M_L| + 1$" (from Proposition 5.14) or the property that "the number of maximal chains of L is equal to the number of linear extensions of the ordered set of its join-irreducible elements" (from Remark 5.32), two characterizations due to Rival (1976). Let us also mention that the one-to-one correspondence between the maximal chains of a distributive lattice L and the linear extensions of J_L is extended into a one-to-one correspondence between the covering sublattices of L and the order extensions of J_L (Baldy *et al*., 1999). One can find in the literature many other characterizations of distributive lattices, in particular by means of either "projectivity" relations (Avann, 1958, 1961b), or join-prime elements

(Ky Fan, 1972), or also of inequality properties between probability functions or measures defined on the lattice (see Daykin (1977) and Winkler (1986)).

We have described Characterization (5) of distributive lattices given in Theorem 5.1 by means of the notion of a *cleavage* (a term due to Schützenberger (1949)), i.e., of a bipartition of a distributive lattice into a principal filter $[j)$ and a principal ideal $(m]$ (with $j \not\leq m$). Item (1) in Exercise 5.10 shows that such a cleavage induces the existence of two isomorphic intervals of L. Item (2) in the same exercise shows that it is then possible to split L into two distributive lattices which are two disjoint intervals and that, iterating this operation from a linear extension of M_L, one may thus obtain a partition $C_1, C_2, ..., C_{n+1}$ of L into intervals of L (with $C_{n+1} = \{1_T\}$). Moreover, every element x of C_i is covered by a unique element y of $C_{i+1} \cup ... \cup C_{n+1}$ and the set of such ordered pairs (x,y) is a covering tree of L. This tree defined (with the name of "ideal tree") by Habib and Nourine (1996) is used to obtain efficient algorithms on distributive lattices (see A.2.2 in Appendix A).

Finally, Item (3) in Exercise 5.10 describes the converse operation of the previous decomposition. It allows us to go from a distributive lattice L to a distributive lattice L^x obtained by doubling a principal ideal $(x]$ of L. This gives a procedure generating all distributive lattices from the chain $\underline{1}$ (Blair, 1984). Yet, this procedure is in fact a particular case of a procedure allowing us to generate iteratively from the chain $\underline{1}$ the class of the so-called *bounded* lattices,[2] a class containing the distributive lattices. The doubling method is the same but it applies to an arbitrary interval of the lattice instead of an interval defined by a principal ideal (see, for instance, Bertet and Caspard (2002)).

The characterization of the generating sets of a distributive lattice given in Theorem 5.21 appears in Bouchet's thesis (1971); we will go back to this thesis in the last section of the next chapter. A distributive lattice L may have a unique minimal generating set, which is then necessarily the set of its doubly irreducible elements; in this case the computation of the minimum size $g(L)$ of a generating set of L comes down to the simpler computation of the number of its doubly irreducible elements. The distributive lattices satisfying this condition are characterized by the following property: the Dedekind–MacNeille completion of the ordered set of their join-irreducible elements is a distributive lattice (Monjardet and Wille, 1988–89). For instance, this is the case of the lattice $\mathcal{D}(L)$ of downsets of a distributive lattice L (why?).

The Galois connection between binary relations and families of subsets presented in Section 5.4 is independently due to Chacron (1966) and Lorrain (1969) (see also Chapter 6 in Barbut and Monjardet (1970)). It generalizes the duality between orders and quasi-separated topologies obtained by Birkhoff as early as 1933. Feldman-Högaasen (1969) was the first to show the existence of another duality between orders and geodesically convex subsets of the permutograph, a duality resulting from the fact that these subsets are nothing else but the spindles of linear orders. Some applications

[2] Here, the meaning of the term "bounded" is different from the one given just after Definition 1.38.

of these results are in Monjardet (1970). The term "spindle" is the translation of the French term "fuseau" introduced by Frey concerning the problem of obtaining a chronology of Jesus' life from the Synoptic Gospels; more generally it is the problem of reconstructing a linear order from several of its suborders (see Frey and Barbut (1970)). There also exist several generalizations of this Galois connection between binary relations and families of subsets, in particular that providing a duality between closure operators and complete implication systems (see page 233 in Chapter 7 and, for instance, Caspard and Monjardet (2003)).

As already mentioned, the permutograph is the neighborhood graph of the weak Bruhat order defined on the set of linear orders (or of permutations). This ordered set (defined in Example 1.17) is in fact a lattice, called the *permutoedre lattice* (Guilbaud and Rosenstiehl, 1963) and studied by several authors (Barbut and Monjardet, 1970; Le Conte de Poly-Barbut, 1990a,b; Duquenne and Cherfouh, 1994; Markowsky, 1994). Since this lattice is bounded (Caspard, 2000), it may be obtained by the doubling method described above. More generally, the latter property remains true for the lattices defined by finite Coxeter groups endowed with the weak Bruhat order (Caspard *et al.*, 2004).

In the correspondence between ordered sets and distributive lattices, the ordered sets of width at most 2 correspond to planar distributive lattices (see Section 2.5 in Chapter 2). These lattices have many characterizations and one may show that they satisfy the Sperner property of Definition 4.14 (Monjardet, 1976a).

In addition to the fundamental Birkhoff Theorem on the representation of a distributive lattice by some subsets of an ordered set, there exist many results of representation of an arbitrary "abstract" distributive lattice by a "concrete" distributive lattice. Thus, any distributive lattice is isomorphic to the lattice of congruences of a finite lattice (see Grätzer and Schmidt (1962)), which may be taken planar and "small" (Grätzer and Lakser, 1989), to the lattice of *cutsets* or of *strict cutsets* (subsets which meet every maximal chain, only once for the strict ones) of an ordered set (Escalante, 1972; Higgs, 1986), to the lattice of normal subgroups of a solvable finite group (Silcock, 1977; Pálfy, 1987), to the lattice of ideals of an infinite regular ring (Kim and Roush, 1980) and non-necessarily finite (Pálfy, 1987), to the lattice of congruences of an infinite complemented and modular lattice (Schmidt, 1984), to the lattice of maximum size antichains of an ordered set (Koh, 1983), to the lattice of submodules of a finite module (Pálfy, 1987).

Last, every distributive lattice is also isomorphic to a lattice of "stable marriages." Let us specify what the problem of "stable marriages" is, a representative of many allocation problems of individuals to positions on which they have preferences (like, for instance, the allocation of interns in hospitals, or the allocation of roommates in a university college). One considers two disjoint sets H and F of the same size n such that, with every $h \in H$ (respectively, $f \in F$) is associated a linear order $\geq_h$ on F (respectively, $\geq_f$ on H). A one-to-one correspondence β from H to F is called an *unstable marriage* (or a *matching*) if there exist $h \in H$ and $f \in F$ such that $f >_h$

$\beta(h)$ and $h >_f \beta^{-1}(f)$ hold (indeed in this case, to guarantee the couple's peace, the matching associating h and f is preferable to the matching β). A one-to-one correspondence which is not an unstable marriage is called a *stable* marriage. Gale and Shapley (1962) have shown that there always exists at least one stable marriage. One defines a relation on the set of stable marriages by writing $\beta \geq \beta'$ if, for every $h \in H$, $\beta(h) \geq_h \beta'(h)$. Knuth (1976) has proved that this relation is an order endowing this set with a distributive lattice structure (in a similar way one may define an order $\geq_f$ which makes the set of stable marriages a lattice dual of the previous one). In 1984 Blair showed that any finite distributive lattice is isomorphic to a lattice of stable marriages. In order to prove this result, he considered the distributive lattice L of stable marriages between two sets of size n, and showed that, for every $x \in L$, the lattice L^x (defined in Exercise 5.10) is isomorphic to the lattice of stable marriages between two sets of size $2n$. He then deduced the result (another proof and other results may be found in Gusfield and Irving's book (1989) devoted to this subject).

Several other interesting classes of "concrete" distributive lattices linked to combinatorial problems have been studied by Stanley (see especially Stanley, 1975, 1986a).

An especially interesting generalization of the class of distributive lattices is the class of (*lower* or *upper*) *locally distributive* lattices. Initially, (upper) locally distributive lattices were defined by Dilworth (1940) as the lattices satisfying the uniqueness property of minimal representation by (meet-)irreducible elements (stated in Proposition 5.15 for distributive lattices). These lattices have been characterized by himself and other authors (especially Avann, 1961a, 1968) in many ways. The one using the arrow relations is particularly elegant: a lattice L is upper locally distributive if and only if, for every $j \in J_L$, there exists a unique $m \in M_L$ such that $j \downarrow m$. Such lattices appear in many contexts (see Monjardet, 1990a) and, in particular, in the theory of "convex geometries" (Edelman and Jamison, 1985). Indeed convex geometries, whose properties generalize that of convex sets of $\mathbb{R}^n$, are the Moore families which are set representations of lower locally distributive lattices (see Examples 3.31 and 3.32). Moreover, there is a duality between convex geometries and "path-independent" choice functions (see Section 2.5 in Chapter 2 and Monjardet and Raderanirina (2001)).

The semilattice $\mathcal{O}_X$ considered in Proposition 5.42 becomes a lower locally distributive lattice by adding a maximum (see Edelman and Jamison (1985) and Leclerc (2003)). Brualdi *et al.* (1994) study combinatorial properties of this lattice.

5.7 Exercises

Exercise 5.1 Show that in a lattice L, ($[x \wedge y = x \wedge z$ and $x \vee y = x \vee z] \Longrightarrow y = z$) if and only if L is distributive. *Hint*: use Characterization (10) of distributive lattices given in Remark 5.3.

Exercise 5.2 Prove Propositions 2.20 and 2.21 in Chapter 2: the direct product of distributive lattices and every sublattice of a distributive lattice are distributive.

Show that the image of a distributive lattice by a lattice morphism (Chapter 3, Definition 3.3) is distributive. Show that, if L is a distributive lattice and P an ordered set, the set L^P of isotone maps from P to L is a distributive lattice.

Exercise 5.3 [Inclusion-Exclusion Principle] Let L be a distributive lattice and r the rank function of this lattice. Show the following property (a):

for all $x, y \in L$,

$$r(x \vee y) = r(x) + r(y) - r(x \wedge y)$$

(use Proposition 5.14 and Theorem 5.1).

Deduce by induction on n that L satisfies the following property (b):

for all $x_1, ..., x_i, ..., x_n \in L$ with $n \geq 2$,

$$r(\bigvee_{1 \leq i \leq n} x_i) = \Sigma_{1 \leq i \leq n} r(x_i) + ... + (-1)^{k+1} \Sigma_{A \in P_k} (\bigwedge_{i \in A} x_i) + ... + (-1)^{n+1} r(\bigwedge_{1 \leq i \leq n} x_i)$$

where P_k is the set of all subsets of size k of $\{1, ..., i, ..., n\}$.

Write what the formula in (b) becomes when L is the lattice $\underline{2}^X$ of subsets of a set X (this formula is then well-known as the *Inclusion–Exclusion Principle*).

Show that a ranked lattice satisfying Property (b) for $n \leq 3$ is distributive. *Hint*: show that $x \wedge (y \vee z)$ and $(x \wedge y) \vee (x \wedge z)$ are two comparable elements with the same rank.

Note The ranked lattices satisfying Property (a) are called *modular*; they are also characterized by Property (M) of modularity used in the proof of the implication (3) $\Longrightarrow$ (1) in Theorem 5.1. A distributive lattice is thus modular, but the lattice in Figure 5.1(a) shows that the converse is false.

Exercise 5.4 Construct the distributive lattice $\mathcal{D}(P)$ of downsets of the ordered set P when P is:

1. the antichain A_n of size n;
2. the chain C_n of size n.

Draw the lattice $\mathcal{D}(P)$ where P is one of the ordered sets in Appendix B and verify the assertions of Theorem 5.6.

Exercise 5.5 Characterize the distributive lattices isomorphic to the lattice of downsets of P when P is:

1. a weak order;
2. a disjoint union of chains.

Exercise 5.6 Show that a distributive lattice L is isomorphic to the ordered set of the antitone maps from the ordered set J_L of its join-irreducibles to $\underline{2} = \{0 < 1\}$ and dually isomorphic to the ordered set $\underline{2}^{J_L}$ of the isotone maps from J_L to $\underline{2}$ (use Birkhoff's Theorem 5.9).

Exercise 5.7 [Free distributive lattice] An *isotone Boolean map* on a set X is an isotone map f from $\underline{2}^X$ to $\{0 < 1\}$, a *cohereditary family* on X is a family $\mathcal{F}$ of subsets of X satisfying $[A \in \mathcal{F}, B \supseteq A$ imply $B \in \mathcal{F}]$, and a *Sperner family* on X is a family $\mathcal{F}$ of subsets of X satisfying $[A, B \in \mathcal{F}$ and $A \subseteq B$ imply $A = B]$ (thus $\mathcal{F}$ is an antichain of the lattice $\underline{2}^X$). Show that the sets of:

1. all isotone Boolean maps,
2. all cohereditary families,
3. all Sperner families,

defined on the same set X may be endowed with structures of isomorphic distributive lattices.

Draw the diagram of one of these lattices for $|X| = 2$.

Note If we delete from one of these distributive lattices its least and its greatest elements, we obtain a distributive lattice isomorphic to the "free distributive lattice" on $|X|$ generators; this distributive lattice, denoted $FDL(n)$, is generated by a set X of n generators and is such that, for any distributive lattice L generated by $A \subseteq X$, there exists a lattice morphism from $FDL(n)$ onto L the generators of which are fixed points (see Birkhoff (1967) or Barbut and Monjardet (1970)). Therefore the number $f(n) = |FDL(n)| + 2$ is the number of antichains (including the empty chain, to be distinguished from the antichain $\{\emptyset\}$) of the Boolean lattice $\underline{2}^X$. "Dedekind's problem" consists of computing this number (that Dedekind has computed for $n \leq 4$). It is known that $f(1) = 3, f(2) = 6, f(3) = 20, f(4) = 168, f(5) = 7581$, $f(6) = 7\,828\,354$, $f(7) = 2\,414\,682\,040\,998$, $f(8) = 56\,130\,437\,228\,687\,557\,907\,788$, and there exists an asymptotic estimation of $f(n)$ (see Quackenbush (1986), Wiedemann (1991), and http://en.wikipedia.org/wiki/Dedekind_number).

The deletion of the element cg from the lattice represented in Figure 3.9 gives a lattice isomorphic to $FDL(3)$.

Exercise 5.8 Show that the minimum size of a generating set of the lattice L drawn in Figure 5.3(a) is $g(L) = 3$ (for instance by using Corollary 5.22).

Exercise 5.9 Let L and L' be two lattices with least elements 0 and $0'$ and greatest elements 1 and $1'$. We denote ω (respectively, ν) the number of least (respectively, greatest) elements of these two lattices which are meet-irreducible (respectively, join-irreducible). Show the inequality $g(L \times L') \leq g(L) + g(L') + \min(\omega, \nu)$.

Exercise 5.10 [Doubling in a distributive lattice; Habib and Nourine (1996)] Let L be a distributive lattice and J_L (respectively, M_L) the set of its join-irreducible (respectively, meet-irreducible) elements; we use the notations in Proposition 5.12.

1. Describe the isomorphism ι^{-1} between M_L and J_L. If $m \in M_L$ and $j = \iota^{-1}(m)$, show that the intervals $[j^-, m]$ and $[j, m^+]$ are isomorphic (consider the map $x \longmapsto x \vee j$). Show that the interval $[j^-, m^+]$ is isomorphic to the direct product $\underline{2} \times [j^-, m]$.

2. If m is a minimal meet-irreducible of L, show that $(m]$ is isomorphic to $[j, m^+]$ (where $j = \iota^{-1}(m)$) and that $L \setminus (m]$ is a distributive lattice whose set of meet-irreducibles is $M_L \setminus \{m\}$. Deduce that, with any linear extension $m_1 < m_2 < ... < m_n$ of M_L, with $n = |M_L|$, it is possible to associate a partition $C_1, C_2, ..., C_{n+1}$ of L into $n+1$ sublattices of L, with $C_{n+1} = \{1_L\}$.

3. Let x be an element of L; consider the set $\mathcal{I}_x = (x] \times \{1\}$ (every element of which is thus written $z' = z1$ with $z \in (x]$). Consider the set $L + \mathcal{I}_x$ and define a relation $\leq'$ on it as follows:

$$z' \leq' t' \iff \begin{cases} z', t' \in L \text{ and } z' \leq t' \\ \text{or} \\ z' \in L,\ t' = t1 \in \mathcal{I}_x \text{ and } z' \leq t \\ \text{or} \\ z' = z1 \in \mathcal{I}_x,\ t' \in L \text{ and } z \leq t' \\ \text{or} \\ z' = z1, t' = t1 \in \mathcal{I}_x \text{ and } z \leq t \end{cases}$$

Show that $L^x = (L + \mathcal{I}_x, \leq')$ is an ordered set which moreover is a distributive lattice. Determine its join-irreducible and meet-irreducible elements.

Note The construction in the above Item (3) may be done by replacing the principal ideal $(x]$ with an arbitrary interval of an arbitrary lattice L. Starting from the chain $\underline{1}$, one thus obtains all *bounded lattices*, a class of lattices containing the distributive lattices (see, for instance, Bertet and Caspard (2002)).

Exercise 5.11 [Minimum size of a generating set of a Boolean lattice] Consider the Boolean lattice $\underline{2}^E$ where $E = \{1, 2, ..., n\}$. Let $\mathcal{F}$ be a family of subsets of E of size t. We say that $\mathcal{F}$ is *separating* if, for all different elements i and j of E, there exist $A, B \in \mathcal{F}$ such that $i \in A \setminus B$ and $j \in B \setminus A$. For every $i \in E$, we write $\mathcal{F}(i) = \{A \in \mathcal{F} : i \in A\}$ and $\mathcal{F}^* = \{\mathcal{F}(i), i \in E\}$. Show that the following properties are equivalent:

1. $\mathcal{F}$ is separating.
2. For every $i \in E$, $\bigcap \mathcal{F}(i) = \{i\}$.
3. $\mathcal{F}^*$ is a Sperner family of size n.
4. $\mathcal{F}$ is a generating set of the lattice $\underline{2}^E$.

Conclude that $g(\underline{2}^X) = t(n)$, where $t(n) = \min\{t \in \mathbb{N} : n \leq \binom{t}{\frac{t}{2}}\}$. *Hint*: observe that $\mathcal{F}^*$ is a Sperner family on $\mathcal{F}$ and use Sperner's Theorem (Chapter 4, Theorem 4.20).

Exercise 5.12 [Symmetric difference distance] The symmetric difference distance of two subsets A and B of a set E is $|A \Delta B| = |(A \setminus B) \cup (B \setminus A)|$. We write $\delta(A, B) = |A \Delta B|$. Show that:

(a) $\delta(A,B) = |A| + |B| - 2|A \cap B| = 2|A \cup B| - |A| - |B|$.
(b) $\delta(A,B) + \delta(B,C) = \delta(A,C) + 2|(A \cap C) \setminus B| + 2|B \setminus (A \cup C)| = \delta(B, A \cup C) + \delta(B, A \cap C)$.

Deduce that δ is a distance on the set of subsets of E and that $\delta(A,B) + \delta(B,C) = \delta(A,C)$ holds if and only if $A \cap C \subseteq B \subseteq A \cup C$ holds.

Exercise 5.13 [About complementation; Barbut and Monjardet (1970)] In a lattice L where 0 and 1 are the least and the greatest elements, a *complement* of an element x is any element x' such that $x \wedge x' = 0$ and $x \vee x' = 1$. A lattice L is said to be *complemented* if every element x has at least one complement (Chapter 2, Definition 2.19).

1. Show that, if L is distributive, every element has at most one complement.
2. A complemented distributive lattice is called *Boolean*. A lattice every interval of which is a complemented lattice is called *relatively complemented*. Show that a Boolean lattice L is relatively complemented. *Hint*: the complement of x in the interval $[a,b]$ is obtained from the complement of x in L.
3. Show that a relatively complemented lattice is atomistic and coatomistic (Chapter 3, Definitions 3.13 and 3.20).
4. Show that a Boolean lattice is isomorphic to the lattice of subsets of the set of its atoms.

Exercise 5.14 [Generalizing Proposition 5.31] Let $P = (X, O)$ be an ordered set of size n. Show that the preorders with k classes (respectively, the orders) including the order O are in a one-to-one correspondence with the topologies of height k (respectively, of height n) included in $\mathcal{D}(P)$. Transfer this result to distributive lattices.

Exercise 5.15 [Distributive ordered sets] An ordered set P is called *distributive* (or *dissective*) if all its join-irreducibles are *dissective* (see Exercise 3.18). Show that the following three properties are equivalent:

1. P is distributive.
2. The MacNeille completion $Gal(P)$ of P is a distributive lattice.
3. The MacNeille completion $Gal(P)$ of P is isomorphic to $\mathcal{D}(J(P))$.

Hint: Use Corollary 3.59 ($J(P)$ is isomorphic to $J(Gal(P))$), the fact that $Dis(P)$ is isomorphic to $Dis(Gal(P))$ – whose proof is one of the purposes of Exercise 3.18 – and Property (4) in Theorem 5.1 (every join-irreducible of a distributive lattice is join-prime).

Note This result can be found in Reading (2002). In Monjardet and Wille (1988–89) it is also shown that P is distributive if and only if the distributive lattice $\mathcal{D}(P)$ is generated by its doubly irreducible elements. The set of such distributive lattices generated by their doubly irreducible – the latter ordered set being isomorphic to P – is a Boolean interval studied in Berman and Bordalo (1998).

6

Order codings and dimensions

We have several times considered codings (Definition 3.1) from an ordered set to another one. Indeed when an ordered set has a complex structure, it is natural to try to represent it in a simpler ordinal structure. In this chapter, we choose direct products of chains as the simple ordered structures (see Section 1.5.2). A coding of the ordered set P will then be a map sending P to an ordered subset, isomorphic to P, of such a product. This notion is particularized when adding conditions on the sizes of the chains. Thus, when all chains have size k (with k a fixed integer), as always assumed in the sequel, we will speak of a *k-coding*. The minimum number of chains required for the existence of a k-coding of P will be called the *k-dimension* of P.

In the first section, we study the 2-codings of an ordered set P, and the associated 2-dimension – also called *Boolean dimension*. Indeed, a 2-coding of P, i.e., a coding from P to a direct product of p chains of size 2, is equivalent to a coding from P to the Boolean lattice of subsets of a set of size p (such a coding may also be called *Boolean*). The Boolean dimension of an ordered set P is thus the minimum size of a set a family of subsets of which, ordered by set inclusion, reproduces exactly (i.e., is isomorphic to) P. A result stated in Chapter 3 (Proposition 3.6) shows that an ordered set always has a Boolean coding (in the lattice of subsets of any of its join-generating sets). As for the problem of computing the Boolean dimension, it was already encountered in Chapter 1 (Example 1.18) in the particular case where one searches for an optimal coding of a type hierarchy in the lattice $\underline{2}^S$ of subsets of a set S. More generally, this problem has been raised in algorithmics where one seeks the best Boolean coding of an ordered set.

Another important particular case of a k-coding of an ordered set P occurs when k is equal to the size of P or, equivalently, when P is coded in a direct product $\mathbb{N}^q$ (with q an arbitrary integer). The minimum integer q such that there exists such a coding is then called the *dimension* of P and Sections 6.2 and 6.3 are devoted to its study. In Section 6.2, we first give some general results, especially on the dimension behavior with regard to the operations on ordered sets considered in Chapter 1 (Section 1.5). Next, we prove Hiraguchi's Theorem, which states that the dimension of P is upper bounded by $\frac{|P|}{2}$ (for $|P| \geq 4$); the proof uses two other upper bounds including the width of P. In Section 6.3, we study 2-dimensional ordered sets – that is, ordered sets of dimension 2 – and we give several characterizations.

Instead of speaking of the dimension of an ordered set $P = (X, \leq)$, we will also speak of the dimension of the order $\leq$. On the other hand, it will sometimes be convenient to replace the notation $\leq$ with the literal notation O. Indeed, we will encounter other definitions of the notion of the dimension of an order using total preorders, the intersection of which is this order. Thus, the Boolean dimension of $P = (X, O)$ is also the minimum number of total preorders with two classes, the intersection of which is the order O, whereas its dimension is also the minimum number of linear orders, the intersection of which is O. These equivalent definitions are the consequence of a general result on the k-dimension presented in Section 6.4, which itself comes from the fundamental duality between preorders and topologies expounded in Chapter 5.

The fact that the dimension of an order O is the minimum number of linear orders whose intersection is O accounts for the use of this notion in many modelings, for instance in social sciences. For example, if O represents the preference order of a consumer, the dimension of O represents the minimum number of (linearly) ordered criteria accounting for his preference: the consumer prefers a bundle of goods to another one if and only if he prefers the former with respect to all criteria. We will go back to such uses of the notion of the dimension in Section 6.5.

We will use some general notations for a map from an ordered set to a direct product of chains of the same size. We denote by $\underline{k_1} \times ...\underline{k_i} \times ...\underline{k_r}$ the direct product of r chains $\underline{k_1}, ..., \underline{k_i}, ..., \underline{k_r}$, all of size k. Denoting by $\leq_i$ the order of the chain $\underline{k_i}$ and by $\leq$ the order of $\underline{k_1} \times ...\underline{k_i} \times ...\underline{k_r}$, the latter is thus given by $(x_1, ..., x_i, ..., x_r) \leq (x'_1, ..., x'_i, ..., x'_r)$ if and only if $x_i \leq_i x'_i$ for $i = 1, ..., r$. A map c from an ordered set P to $\underline{k_1} \times ...\underline{k_i} \times ...\underline{k_r}$ is given by the r "ith coordinate" maps $c_1, ..., c_i, ..., c_r$ from P to, respectively, $k_1, ..., k_i, ..., k_r$ and we write $c(x) = (c_1(x), ..., c_i(x), ..., c_r(x))$ and $c = (c_1, ..., c_i, ..., c_r)$.

6.1 Boolean codings and Boolean dimension of an ordered set

Here we study the codings from an ordered set to the direct product of chains $\underline{k_1} \times ...\underline{k_i} \times ...\underline{k_r}$, where all chains $\underline{k_i}$ are isomorphic to the chain $\underline{2} = \{0 < 1\}$.

Definition 6.1 1. A 2-*coding*, also called *Boolean coding*, of an ordered set $P = (X, \leq)$ is a map $c = (c_1, ..., c_i, ..., c_r)$ from P to a direct product of r 2-element chains such that:

$$x \leq y \iff c_i(x) \leq_i c_i(y) \text{ for } i = 1, ..., r$$

2. A Boolean coding $c = (c_1, ..., c_i, ..., c_r)$ of P is *strict* if, for $i = 1, ..., r$, $c_i(P) = \underline{2}$.
3. The *Boolean dimension* (or 2-*dimension*) of P, denoted by $\dim_2 P$, is the minimum number of 2-element chains such that there is a Boolean coding from P to their direct product.

The above term “Boolean” is easy to explain. Indeed, the direct product $\underline{2}^r$ of r 2-element chains being isomorphic to the ordered set of subsets of a set E of size r, we may also say that a Boolean (respectively, strict Boolean) coding of P is a map c from P to the Boolean lattice $\underline{2}^E$ of subsets of a set E such that:

$$x \leq y \iff c(x) \subseteq c(y)$$

(respectively, such that: $x \leq y \iff [c(x) \subseteq c(y)$ and, for any $i, c_i(P) \neq \emptyset, E])$.

Likewise, the Boolean dimension of P is the minimum size of a set E such that there is a Boolean coding from P to $\underline{2}^E$.

The fact that every ordered set $P = (X, \leq)$ has a coding in a Boolean lattice $\underline{2}^E$ has already been proved in Chapter 3 since taking any join-generating set of P as the set E is enough (Proposition 3.6). In particular, with $E = X$, we obtain the coding of P by its principal downsets $(x]$ ($x \leq y$ if and only if $(x] \subseteq (y]$). On the other hand, since the minimal join-generating set of P is the set $J(P)$ of its join-irreducible elements, the following result holds:

Proposition 6.2 *Let P be an ordered set. Then* $\dim_2 P \leq |J(P)|$.

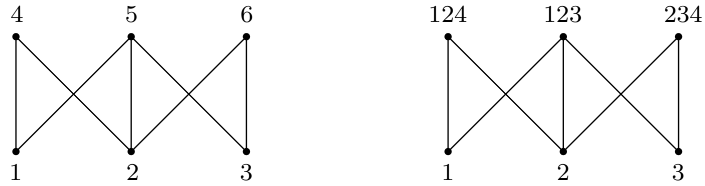
Figure 6.1(b) represents a Boolean coding in $\underline{2}^4$ of the ordered set P of size 6 given in (a). Then, in this example, $\dim_2 P \leq 4 < |J(P)| = 5$, which shows that the Boolean dimension of an ordered set may be less than the number of its join-irreducibles (we shall see later that the Boolean dimension of this ordered set is 4).

Let us observe that a Boolean coding from P to $\underline{2}^E$ may be represented as a “0/1 array”; its lines correspond to the elements x of P and its columns to the elements e of E; the entry $t(x,e)$ is 1 if $e \in c(x)$ and 0 if not; so $c(x) = \{e \in E : t(x,e) = 1\}$. For the example in Figure 6.1(a) and (b), this array is represented in (c).

Conversely, any 0/1 array with lines indexed by a set X and columns by a set E and without identical lines may be considered as representing a Boolean coding of the set of the line labels, ordered by $x \leq y$ if and only if $t(x,e) \leq t(y,e)$ for all columns e of the array.

A strict Boolean coding of P is easily recognized on the associated array, since the coding is strict if and only if this array does not contain columns with only 0’s or only 1’s (which is the case for the coding in Figure 6.1). Non-strict Boolean codings have little interest since they are obtained from strict codings by adding columns of 0’s or of 1’s to the corresponding arrays. In particular, it is clear that the Boolean codings in $\underline{2}^{\dim_2 P}$ are strict. Thus below, we limit ourselves to the case of strict codings when stating the fundamental theorem for Boolean codings.

In this theorem it will be more convenient to denote O the order relation of the ordered set P, thus written $P = (X, O)$. Recall that $\mathcal{D}(P)$ denotes the set of downsets of P and that a downset is proper if it is different from P and the empty set.



(a) P (b)

$X \backslash E$	1	2	3	4
1	1	0	0	0
2	0	1	0	0
3	0	0	1	0
4	1	1	0	1
5	1	1	1	0
6	0	1	1	1

(c)

Figure 6.1 (b) A Boolean coding of the ordered set P and (c) the representation of this coding by a 0/1 array.

Theorem 6.3 *Let $P = (X, O)$ be an ordered set and E a set of size r. There are one-to-one correspondences between the following three sets:*

1. *The set of strict Boolean codings from P to $\underline{2}^E$.*
2. *The set of families $(R_1, ..., R_r)$ of r total preorders with two classes defined on X and the intersection of which is O.*
3. *The set of families $(D_1, ..., D_r)$ of r proper downsets of P generating (by union and intersection) all downsets of P.*

This theorem is obtained by applying to the case where $k = 2$ the fundamental Theorem 6.29 on the codings from P to the chain product $\underline{k}^r$. Therefore, we will give its proof after the proof of Theorem 6.29 (more precisely, just after Proposition 6.33) in the fourth section.

Corollary 6.4 *The Boolean dimension of an ordered set P is the minimum size of a generating set of $\mathcal{D}(P)$ – i.e., the minimum number of proper downsets of P generating by union and intersection all downsets of P:*

$$\dim_2 P = g(\mathcal{D}(P))$$

This corollary is an immediate consequence of Item (3) in Theorem 6.3, since seeking the Boolean dimension of P comes down to seeking a generating set of $\mathcal{D}(P)$

of minimum size. Do not forget when using this result that to generate all downsets of $P = (X, O)$ comes down to generating all its proper downsets (since the empty set and X are always trivially generated).

A consequence of the above corollary is that computing the Boolean dimension of an ordered set comes down to computing the minimum size of a generating set of a distributive lattice, a problem studied in Chapter 5. In Example 5.23, we computed the minimal generating sets of the lattice $\mathcal{D}(P)$ (see Figure 5.2) for an ordered set P isomorphic to the ordered set in Figure 6.1(a). Since we obtained $g(\mathcal{D}(P)) = 4$, the Boolean dimension of P is 4.

Another immediate application of Corollary 6.4 is the computation of the Boolean dimension of a chain $\underline{k}$. Indeed, the lattice of downsets of such a chain is isomorphic to the chain $\underline{k+1}$, whose elements, except for the least and the greatest ones, are doubly irreducible. Hence $\dim_2 \underline{k} = k - 1$ and, here, the bound given in Proposition 6.2 is reached.

The transition from a strict Boolean coding of P to a family of (proper) downsets of P generating $\mathcal{D}(P)$, or the converse transition, is very easily obtained from the 0/1 array defined after Proposition 6.2 (see Figure 6.1(c)). Let t be such an array associated with a coding from P to $\underline{2}^E$; then the corresponding generating family of $\mathcal{D}(P)$ is $\{D_e, e \in E\}$ with $D_e = \{x \in P : t(x,e) = 0\}$.

Thus, in the example of the ordered set P in Figure 6.1, the obtained generating family of $\mathcal{D}(P)$ is $\{236, 13, 124, 1235\}$, which may be checked on the lattice $\mathcal{D}(P)$ in Figure 5.2.

Conversely, let r downsets of P form a generating family of $\mathcal{D}(P)$; we define a 0/1 array with $|P|$ lines and r columns by associating with each downset D_e the column e where $t(x,e) = 0$ if $x \in D_e$ and $t(x,e) = 1$ if not. Then, the lines of this array induce the corresponding Boolean coding (we let the reader check that these assertions follow from Theorem 6.3).

We now give some general results on the Boolean dimension, beginning with its behavior with regard to some operations on ordered sets (see Section 1.5 in Chapter 1).

Proposition 6.5 *Let P, Q, and P_i, $i = 1,...,h$ be ordered sets. Then:*

1. $Q \sqsubseteq P$ *implies* $\dim_2 Q \leq \dim_2 P$.
2. $\dim_2 P^d = \dim_2 P$.
3. $\dim_2(\Sigma_{1 \leq i \leq h} P_i) \leq \min\{\omega, \nu\} + \Sigma_{1 \leq i \leq h} \dim_2 P_i$, *where* $\omega = |\{i : P_i \text{ has a minimum}\}|$ *and* $\nu = |\{i : P_i \text{ has a maximum}\}|$.
4. $\dim_2(\bigoplus_{1 \leq i \leq h} P_i) = t + \Sigma_{1 \leq i \leq h} \dim_2 P_i$, *where* $t = |\{i : P_i \text{ has a maximum and } P_{i+1} \text{ has a minimum}\}|$.
5. $\dim_2(\Pi_{1 \leq i \leq h} P_i) \leq \Sigma_{1 \leq i \leq h} \dim_2 P_i$, *with the equality if all P_i's have a minimum and a maximum.*

Proof (1) Immediate since, if Q is an ordered subset of P, we obtain a Boolean coding of Q by restricting the Boolean coding of P to this subset.

(2) Immediate since, if c is a coding from P to $\underline{2}^E$, we obtain a coding c' from P^d to $\underline{2}^E$ by writing $c'(x) = E \setminus c(x)$.

We show (3), (4), and (5) for two ordered sets $P_1 = (X_1, \leq_1)$ and $P_2 = (X_2, \leq_2)$ and let the reader generalize the proofs to more than two ordered sets.

(3) The Boolean dimension of $P_1 + P_2$ is the minimum size of a generating set of the lattice $\mathcal{D}(P_1 + P_2)$, which is isomorphic to the lattice $\mathcal{D}(P_1) \times \mathcal{D}(P_2)$ (Proposition 5.8). From the result in Exercise 5.9, the minimum size of a generating set of this product lattice is bounded by the sum of the minimum size of a generating set of $\mathcal{D}(P_1)$ and $\mathcal{D}(P_2)$ (i.e., $\dim_2 P_1 + \dim_2 P_2$) and of the minimum of the numbers ω and ν, where ω (respectively, ν) is the number of P_i $(i = 1, 2)$ such that, in $\mathcal{D}(P_i)$, the empty set $\emptyset$ (respectively, the set X_i) is a meet-irreducible (respectively, a join-irreducible). Item (2) in Theorem 5.6 then gives the result.

(4) The Boolean dimension of $P_1 \oplus P_2$ is the minimum size of a generating set of the lattice $\mathcal{D}(P_1 \oplus P_2)$, which is isomorphic to the lattice $\mathcal{D}(P_1) \oplus' \mathcal{D}(P_2)$ (Proposition 5.8). On the other hand, according to Item (2) in Theorem 5.6, the element obtained by identifying the maximum of $\mathcal{D}(P_1)$ and the minimum of $\mathcal{D}(P_2)$ is doubly irreducible (and thus must belong to any generating set of $\mathcal{D}(P_1) \oplus' \mathcal{D}(P_2)$) only if P_1 has a maximum and P_2 a minimum. Now it is easy to see that, if L_1 and L_2 are two lattices, $g(L_1 \oplus' L_2) = g(L_1) + g(L_2) + t$, with $t = 1$ if $u_1 \in J(L_1)$ and $0_2 \in M(L_2)$, and $t = 0$ if not. The required equality follows.

(5) Let c_1 (respectively, c_2) be a coding from P_1 (respectively, P_2) to $\underline{2}^{E_1}$ (respectively, $\underline{2}^{E_2}$) with $|E_1| = \dim_2 P_1$ (respectively, $|E_2| = \dim_2 P_2$). We immediately check that the map c from $P_1 \times P_2$ to $\underline{2}^{E_1 + E_2}$ defined by $c(x_1, x_2) = c_1(x_1) + c_2(x_2)$ – where here $+$ denotes the disjoint union – is a coding, whence the inequality in (5).

Assume that P_i $(i = 1, 2)$ has a minimum 0_i and a maximum u_i. We must show $\dim_2(P_1 \times P_2) = \dim_2 P_1 + \dim_2 P_2$; that is, considering the inequality in (5), $\dim_2(P_1 \times P_2) \geq \dim_2 P_1 + \dim_2 P_2$. Let $c = (c_1, ..., c_i, ..., c_r)$ be a coding from $P_1 \times P_2$ to $\underline{2}^E$, with $E = \{1, ..., i, ..., r\}$ and $r = \dim_2(P_1 \times P_2)$. Write $A = \{i \in E : c_i(0_1, u_2) = 0 < c_i(u_1, 0_2) = 1\}$ and $B = \{i \in E : c_i(0_1, u_2) = 1 > c_i(u_1, 0_2) = 0\}$. Since $(0_1, u_2)$ and $(u_1, 0_2)$ are incomparable, A and B are not empty. We are going to show that the restrictions of the c_i's, for $i \in A$, to $P_1 \times \{0_2\}$ induce a Boolean coding of this ordered set. To do so, it is enough to show that, if two elements x and y are incomparable in P_1, so are the images of $(x, 0_2)$ and $(y, 0_2)$ by these restrictions. Since $(y, 0_2) < (u_1, 0_2)$ and $(0_1, u_2) < (x, u_2)$ in the product order, for any $i \in B$, $c_i(y, 0_2) = 0 < c_i(x, u_2) = 1$ holds; since $(y, 0_2)$ and (x, u_2) are incomparable, there exists $i \in A$ such that $c_i(y, 0_2) = 1 > c_i(x, u_2) = c_i(x, 0_2) = 0$. We similarly show that there exists $k \in A$ with $c_k(y, 0_2) = 0 < c_k(x, 0_2) = 1$, which proves the announced incomparability result. The restrictions of the c_i's, for $i \in A$, to $P_1 \times \{0_2\}$ inducing a Boolean coding of this ordered set and thus of P_1, we deduce $|A| \geq \dim_2 P_1$. We similarly show that $|B| \geq \dim_2 P_2$, whence $\dim_2(P_1 \times P_2) \geq |A| + |B| \geq \dim_2 P_1 + \dim_2 P_2$ and the announced equality. □

Since the Boolean dimension of a chain $\underline{k}$ is $k-1$ (for $k \geq 2$), Item (5) in the above proposition gives the Boolean dimension of a product of chains of arbitrary lengths.

Corollary 6.6 *If $\underline{k_1}, ..., \underline{k_i}, ..., \underline{k_h}$ are h chains all of size greater than 1, then* $\dim_2$ $(\underline{k_1} \times ...\underline{k_i} \times ...\underline{k_h}) = (\Sigma_{1 \leq i \leq h} k_i) - h$.

Computing the Boolean dimension of an ordered set is generally a difficult problem (see Appendix A). It is thus interesting to know easily computable bounds on this dimension, like the ones below. Recall that $\lambda(P)$ and $\alpha(P)$ respectively denote the height and the width of the ordered set P (both are defined on page 19).

Proposition 6.7 *For any ordered set P,*

$$\max\{\log_2|P|, \lambda(P), t(\alpha(P))\} \leq \dim_2 P \leq |P|$$

where $t(n)$ is the least integer t such that $n \leq \binom{t}{\lceil \frac{t}{2} \rceil}$.

Proof Let P be coded in $\underline{2}^d$ with $d = \dim_2 P$; then $2^d \geq |P|$, $\lambda(\underline{2}^d) = d \geq \lambda(P)$, and $\alpha(\underline{2}^d) = \binom{d}{\lceil \frac{d}{2} \rceil}$ (Theorem 4.20) $\geq \alpha(P)$, hence the lower bound. The upper bound is immediate using the fact that the map associating with any element x its principal downset $(x]$ is a coding from P to $\underline{2}^{|P|}$. $\square$

We observe that all the bounds given in this proposition are sharp. For example, Proposition 6.35 shows that the lower bound $\lambda(P)$ is reached by distributive lattices (since the height of such a lattice is equal to the number of its join-irreducibles). The ordered sets P satisfying $\dim_2 P = |P|$ have been characterized (see Section 6.5). The reader can search for examples for the other bounds.

6.2 Dimension of an ordered set

We now study the codings from an ordered set $P = (X, \leq)$ of size n to the direct product of chains $\underline{n}_i$, all isomorphic to the chain $\underline{n} = \{0 < 1 < ... < n-1\}$. Recall that, if c is such a coding, $x \leq y$ holds if and only if $c(x) \leq c(y)$ holds in the product order (Definition 3.1).

Definition 6.8 Let $P = (X, \leq)$ be an ordered set of size n. An *n-coding* of P is a coding c from P to a direct product of chains all of size n. When there are r chains (i.e., if this product is $\underline{n}_1 \times ...\underline{n}_i \times ...\underline{n}_r$, with $n_i = n$ for any $i \leq r$), we write $c = (c_1, ..., c_i, ..., c_r)$ and then:

$$x \leq y \iff c_i(x) \leq_i c_i(y), \textit{ for } i = 1, ..., r$$

An n-coding $c = (c_1, ..., c_i, ..., c_r)$ from P to $\underline{n}^r$ is *strict* if, for each $i \leq r$, $c_i(P) = \underline{n}_i$.

The *dimension* of P, denoted $\dim P$, is the minimum number of chains of size n such that there exists an n-coding from P to their direct product.

In Chapter 1 (Definition 1.33), we gave another definition of the dimension of an ordered set, recalled below:

Definition 6.9 A set of linear extensions of an ordered set P is a *realization* of P (we also say that these linear extensions *realize* P) if P is the intersection of these extensions. A realization of P is *minimal* if the intersection of any of its (strict) subsets strictly includes P. A *basis* of P is a (minimal) realization of P of minimum size (i.e., containing the least possible number of linear extensions of P). The *dimension* of P is the minimum size of a basis of P, i.e., the minimum number of linear extensions of P whose intersection is P.

The next results prove that the above two definitions of the dimension are equivalent between them and with some others. For these results, it will be convenient to use the literal notation $P = (X,O)$ of the ordered set P.

Theorem 6.10 *Let $P = (X,O)$ be an ordered set of size n and let r be a fixed integer. There are one-to-one correspondences between the following three sets:*

1. *The set of strict n-codings from P to $\underline{n}^r$.*
2. *The set of families $(L_1,...,L_i,...,L_r)$ of linear extensions of O such that $O = \bigcap_{1 \le i \le r} L_i$ (i.e., realizing O).*
3. *The set of families $(\mathcal{C}_1,...,\mathcal{C}_i,...,\mathcal{C}_r)$ where, for $i = 1,...,r$, $\mathcal{C}_i$ is a chain of length $n-2$ of proper downsets of P, and such that $\bigcup_{1 \le i \le r} \mathcal{C}_i$ is a generating set of the lattice $\mathcal{D}(P)$ of downsets of P.*

This theorem is the application to the case where $k = n$ of the fundamental Theorem 6.29 on the codings from P to the chain product $\underline{k}^r$ (for an integer $k \ge 2$). Therefore, we will give its proof after the proof of Theorem 6.29 (more precisely, just after Proposition 6.33) in the fourth section.

Corollary 6.11 *The dimension of an ordered set $P = (X,O)$ of size n is given by any of the following expressions:*

1. *The least integer r such that there is a strict n-coding from P to $\underline{n}^r$.*
2. *The least integer r such that there is an n-coding from P to $\mathbb{N}^r$.*
3. *The minimum number of linear extensions of O, the intersection of which is O.*
4. *The minimum number of chains, the elements of which generate $\mathcal{D}(P)$.*
5. *The minimum width of a generating set of $\mathcal{D}(P)$.*
6. *The convex dimension of the set $\mathcal{L}(O)$ of linear extensions of O.*

Proof (1) To prove that, in the definition of the dimension of P, we may only consider strict codings, it is enough to show that the existence of a coding from P to $\underline{n}^r$ implies that of a strict coding from P to $\underline{n}^r$. Let $c = (c_1,...,c_i,...,c_r)$ be an n-coding from $P = (X,O)$ to $\underline{n}^r$. For each $i \in E = \{1,...,r\}$, we define a total preorder R_i on X by xR_iy if and only if $c_i(x) \le c_i(y)$. Since c is a coding, we obtain a family

$(R_1,...,R_i,...,R_r)$ of r total preorders, the intersection of which is O. The total preorder R_i including the order O includes (at least) one linear extension L_i of O (Theorem 2.29). Since, for any $i \in E$, $O \subseteq L_i \subseteq R_i$, then $O = \bigcap_{1 \le i \le r} L_i$ holds. So, the existence of a strict n-coding from P to $\underline{n}^r$ results from Theorem 6.10.

(2) We show this expression by proving that it is equivalent to the expression in (3). The same argument as in (1) proves that, if $P = (X, O)$ is coded in $\mathbb{N}^r$, O is the intersection of r linear orders. Conversely, if $O = \bigcap_{i=1}^{r} L_i$ is the intersection of r linear orders, we obtain a coding from P to $\underline{n}^r$ (and thus in $\mathbb{N}^r$) by writing for any x in P, $c(x) = (r_1(x),...,r_i(x),...,r_r(x))$, where $r_i(x)$ is the (normalized) rank of x in the linear order L_i. These two assertions prove the result.

(3) Immediate from Item (2) in Theorem 6.10.

(4) Since any chain of $\mathcal{D}(P)$ formed of proper downsets of P may be extended into a chain of length $n-2$, this immediately results from Item (3) in Theorem 6.10.

(5) Immediate from Item (4) and from Dilworth's Theorem (this expression also results from the formula given for the k-dimension in Proposition 6.33).

(6) We first have to define what the convex dimension is. In Chapter 5, we have seen that spindles of (linear) orders – i.e., the sets $\mathcal{L}(O)$ formed by all linear extensions of an order O defined on X – are the convex subsets of the set $\mathcal{L}_X$ of all linear orders on X (Theorem 5.41 and Definition 5.36). Since the intersection of convex subsets is convex and since $\mathcal{L}_X$ is convex, the convex subsets form a Moore family with an associated closure operator Φ (Definition 3.29) called the convex closure: the *convex closure* $\Phi(\mathcal{A})$ of $\mathcal{A} \subseteq \mathcal{L}_X$ is the least convex subset including $\mathcal{A}$ and is equal to $\mathcal{L}(O_{\mathcal{A}})$, where $O_{\mathcal{A}} = \bigcap\{L \in \mathcal{A}\}$ is the order intersection of the linear orders in $\mathcal{A}$. We then call a *basis* of a convex subset $\mathcal{L}(O)$ any subset $\mathcal{B}$ of $\mathcal{L}(O)$, minimal with the property that $\Phi(\mathcal{B}) = \mathcal{L}(O)$. The *convex dimension* of $\mathcal{L}(O)$ is the minimum size of a basis. Yet, since $\Phi(\mathcal{B}) = \mathcal{L}(O)$ is equivalent to $O = \bigcap\{L \in \mathcal{B}\}$, a basis $\mathcal{B}$ of $\mathcal{L}(O)$ with regard to the convex closure is nothing else but a basis of O (as defined in Definition 6.9). Then, as required, the convex dimension of $\mathcal{L}(O)$ is the dimension of $P = (X, O)$. $\square$

Corollary 6.11 allows us, for instance, to compute the dimension of the ordered set represented in Figure 6.1(a). Indeed, the expression in (5) of the dimension given in this corollary shows that it is obtained from the minimal generating sets of $\mathcal{D}(P)$. Now, in the previous chapter (Example 5.23 on page 143) we determined the minimal generating sets of the lattice $\mathcal{D}(P)$ for an ordered set P isomorphic to the one in Figure 6.1(a). There are four such subsets, among which three have width 3 and one has width 2. Thus $\dim P = 2$. This example also shows that the minimum width of a generating set of $\mathcal{D}(P)$ is not necessarily the width of a generating set of minimum size: indeed, here it is the generating set of maximum size which has minimum width.

The fact that the order dimension has several expressions (those in Corollary 6.11 or others, such as those given in Exercise 6.7) does not generally make its computation easier (indeed this computation is a "difficult" problem, see Appendix A). Then one

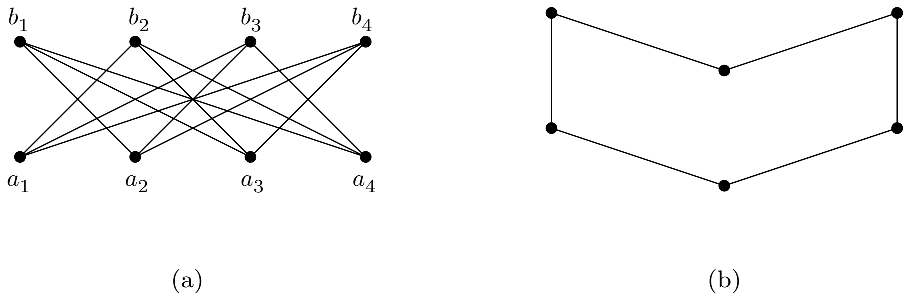


Figure 6.2 (a) The ordered set S_4 and (b) an ordered set P such that $\dim P = \frac{|P|}{2} = 3$.

searches for bounds for the dimension, which is the subject of the end of this section. A proof technique to obtain an upper bound, i.e., to get $\dim P \leq r$ for an integer r, is to provide a realization $\mathcal{E} = \{L_1, ..., L_i, ..., L_r\}$ of P containing r linear orders. From the definitions, this comes down to showing, on the one hand, that the L_i's are linear extensions of P (which will often be obvious) and, on the other hand, that for any pair x, y of incomparable elements of P, there is a pair $L_i, L_j \in \mathcal{E}$ with xL_iy and yL_jx. Observe that in such a case, if moreover one shows $\dim P \geq r$, then $\dim P = r$. We use this technique in the following example, which shows that there are ordered sets of arbitrarily high dimension.

Example 6.12 We denote by S_n ($n \geq 2$) the ordered set defined on $X = \{a_1, ..., a_n, b_1, ..., b_n\}$ as follows: the ordered pairs of S_n are all the ordered pairs (a_i, b_j) with $i \neq j$ (S_4 is represented in Figure 6.2(a)). We are going to show that $\dim S_n = n$.

We first show that $\dim S_n \leq n$ by giving a realization of S_n with n linear extensions. The reader can check that it is the case taking, for $i = 1, ..., i, ..., n$, $L_i = a_i b_{i+1} a_{i+1} b_{i+2} a_{i+2} ... a_{i-2} b_{i-1} a_{i-1} b_i$ (with $a_{n+1} = a_1$ and $b_{n+1} = b_1$).

Let us now show that $\dim S_n \geq n$. Indeed, let (b_i, a_i) and (b_j, a_j) be two ordered pairs with i different from j. Since they are formed of incomparable elements of S_n and $a_i L_i b_i$ and $a_j L_j b_j$ hold, a realization of S_n must contain two linear extensions L_k and L_h of S_n such that $b_i L_k a_i$ and $b_j L_h a_j$. Yet, $L_k = L_h = L$ is impossible, since it would imply $a_j L b_i L a_i L b_j L a_j$. Thus, a realization of S_n must have at least as many linear orders as ordered pairs (b_i, a_i) and then $\dim S_n \geq n$. Finally $\dim S_n = n$ holds, as announced.

We now study the behavior of the dimension with regard to some usual operations on ordered sets (see Section 1.5 in Chapter 1). In the following proposition, we consider some operations on h ordered sets P_i (in the formulas (4) to (7) below, the index i varies from 1 to h).

Proposition 6.13 *Let P, Q, and P_i with $i = 1,...,h$ ($h \geq 2$) be ordered sets. Then:*

1. $Q \sqsubseteq P$ *implies* $\dim Q \leq \dim P$.
2. $\dim P^d = \dim P$.
3. $\dim P - 1 \leq \dim(P \setminus x) \leq \dim P$, *for any* $x \in P$.
4. $\dim(\Sigma_{1 \leq i \leq h} P_i) = \max\{2, \max_{1 \leq i \leq h}\{\dim P_i\}\}$.
5. $\dim(\bigoplus_{1 \leq i \leq h} P_i) = \max_{1 \leq i \leq h}\{\dim P_i\}$.
6. $\dim(Q_{y_1,...,y_h}^{P_1,...,P_h}) = \max\{\dim Q, \max_{1 \leq i \leq h}\{\dim P_i\}\}$.
7. $\max_{1 \leq i \leq h}\{\dim P_i\} \leq \dim(\Pi_{1 \leq i \leq h} P_i) \leq \Sigma_{1 \leq i \leq h} \dim P_i$, *with* $\dim(\Pi_{1 \leq i \leq h} P_i) = \Sigma_{1 \leq i \leq h} \dim P_i$ *if the P_i's all have a maximum and a minimum and are of size at least 2.*

Proof (1) Immediate since, if Q is an ordered subset of P, the restriction to Q of a coding of P is a coding of Q.

(2) Immediate since, if $\{L_1,...,L_r\}$ is a realization of P, it is clear that $\{L_1^d,...,L_r^d\}$ is a realization of P^d.

(3) From (1), removing an element x from P decreases or does not change its dimension. We must show that, if it decreases, it becomes $\dim P - 1$, i.e., that $\dim P \leq \dim(P \setminus x) + 1$. To do so, we show that, if $\{L_1,...,L_r\}$ is a basis of $P \setminus x$, there is a realization of P of size $r+1$. With that aim, we build from the linear extension L_1 of $P \setminus x$ two linear extensions M_1 and M_2 of P. In their definitions given below, the index 1 denotes the restriction of the order L_1 to the indexed subset (and $I(x)$ is the set of elements of P incomparable with x):

$$M_1 = [(x[\cup I(x))]_1 \oplus \{x\} \oplus]x)_1 \qquad\qquad M_2 = (x[_1 \oplus \{x\} \oplus [I(x) \cup]x)]_1$$

It is easy to check that M_1 and M_2 are two linear extensions of P and that, if y is an element incomparable with x, yM_1x and xM_2y hold. Let us show that, if y and z are two incomparable elements of P different from x, the ordered pair (y,z) – for instance – cannot belong to all the linear extensions $M_1, M_2, L_2, ..., L_r$. We only have to consider the case where yL_iz holds for every $i \geq 2$ and so where zL_1y holds. In this case, from the definition of M_1 (respectively, of M_2), if yM_1z (respectively, yM_2z) then $y \in (x[\cup I(x)$ and $z \in [x)$ (respectively, $y \in (x]$ and $z \in I(x) \cup [x)$); thus (yM_1z and yM_2z) is impossible since, then, we would obtain $y <_P x <_P z$, a contradiction with the assumption. We deduce that $\{M_1, M_2, L'_2, ..., L'_r\}$, where the L'_i are the linear extensions of P obtained from the L_i by inserting the element x between $(x[_i$ and $]x)_i$, is a realization of P.

(4) and (5) are particular cases of (6) where, in the first case, Q is an antichain and a chain in the second (see Section 1.5.1).

(6) We consider the case where $h = 2$ (the reader can generalize). Let $\{L_1, ..., L_{p_1}\}$ be a basis of P_1, $\{M_1, ..., M_{p_2}\}$ a basis of P_2, and $\{N_1, ..., N_q\}$ a basis of Q with, for example, $p_1 \geq p_2 \geq q$. A – somewhat tedious – checking shows that the

p_1 linear orders

$$(N_1)_{y_1y_2}^{L_1M_1}, (N_2)_{y_1y_2}^{L_2M_2}, ..., (N_q)_{y_1y_2}^{L_qM_q}, (N_q)_{y_1y_2}^{L_{q+1}M_{q+1}}, ..., (N_q)_{y_1y_2}^{L_{p_2}M_{p_2}},$$
$$(N_q)_{y_1y_2}^{L_{p_2+1}M_{p_2+1}}, ..., (N_q)_{y_1y_2}^{L_{p_1}M_{p_1}}$$

form a realization of $Q_{y_1y_2}^{P_1P_2}$. Since $Q_{y_1y_2}^{P_1P_2}$ includes P_1 as an ordered subset, its dimension is (according to (1)) greater than or equal to the dimension p_1 of P_1 and thus the previous realization of $Q_{y_1y_2}^{P_1P_2}$ is in fact a basis.

(7) Like above, we consider the case where $h = 2$. Clearly, if we may embed P_1 in $\mathbb{N}^{p_1}$ and P_2 in $\mathbb{N}^{p_2}$, we may embed $P_1 \times P_2$ in $\mathbb{N}^{p_1} \times \mathbb{N}^{p_2} = \mathbb{N}^{p_1+p_2}$. Whence $\dim(P_1 \times P_2) \leq \dim P_1 + \dim P_2$.

Since P_i $(i = 1,2)$ is isomorphic to an ordered subset of $P_1 \times P_2$, the first inequality comes from (1).

Assume that P_i $(i = 1,2)$ has a minimum 0_i and a maximum u_i. We must show $\dim(P_1 \times P_2) = \dim P_1 + \dim P_2$, i.e., considering the above inequality, $\dim(P_1 \times P_2) \geq \dim P_1 + \dim P_2$. Since $(0_1,u_2)$ and $(u_1,0_2)$ are incomparable in $P_1 \times P_2$, we may find a basis $\{L_1,...,L_{p+q}\}$ of $P_1 \times P_2$, with $(0_1,u_2)L_i(u_1,0_2)$, for $1 \leq i \leq p$, and $(u_1,0_2)L_i(0_1,u_2)$ for $p+1 \leq i \leq p+q$. Let x and y be incomparable in P_1; since $(y,0_2) < (u_1,0_2)$ and $(0_1,u_2) < (x,u_2)$ in the product order then $(y,0_2)L_i(x,u_2)$ holds for every $i > p$ and thus there exists $i \leq p$ with $(x,u_2)L_i(y,0_2)$. Then, $(x,0_2)L_i(y,0_2)$ holds. A similar argument shows that there exists $k \leq p$ with $(y,0_2)L_k(x,0_2)$. This implies that, for $1 \leq i \leq p$, the restrictions of L_i to $P_1 \times \{0_2\}$ form a realization of $P_1 \times \{0_2\}$ and so induce a realization of P_1. Thus, $p \geq \dim P_1$ holds. We similarly show that $q \geq \dim P_2$ whence $\dim(P_1 \times P_2) \geq \dim P_1 + \dim P_2$ and, finally, the required equality. □

Proposition 6.13 allows us to compute some dimensions. Thus, since the dimension of a chain is 1, Item (7) in this proposition gives the dimension of a product of chains of arbitrary (non-zero) lengths.

Corollary 6.14 *Let $\underline{k_1},...,\underline{k_i},...,\underline{k_h}$ be h chains all of size greater than or equal to 1. Then* $\dim(\underline{k_1} \times ...\underline{k_i} \times ...\underline{k_h}) = h$.

As another example of an application of Proposition 6.13, we may observe that its Item (6) allows us to show that the N-free ordered sets (see Section 2.2, Chapter 2), and that are different from a chain, are 2-dimensional. Indeed, these ordered sets are the series–parallel ordered sets obtained by substitution from ordered sets whose dimension is less than or equal to 2 and where at least one is 2-dimensional.

The remainder of the section is devoted to obtaining bounds on the dimension of an arbitrary ordered set. We are especially going to show the fundamental Hiraguchi's result (Theorem 6.21) stating that the dimension of an ordered set P is at most $\frac{|P|}{2}$. Our proof method allows us to obtain two other bounds using the width of the ordered set (Proposition 6.15 and Corollary 6.20).

Proposition 6.15 *The dimension of an ordered set P is less than or equal to its width:* $\dim P \leq \alpha(P)$.

To obtain this result, it is enough to show that $\dim P \leq \theta(P)$ since, according to Dilworth's Theorem (Theorem 4.2), $\theta(P) = \alpha(P)$. To do so, we first prove the following lemma, then the proposition.

Lemma 6.16 *Let C be a chain of an ordered set P. There exists a linear extension L of P such that:*

$$\forall x \in P \setminus C, \forall y \in C \textit{ with } x||_P y,\ xLy \textit{ holds}$$

In other words, in such a linear extension every element x not belonging to the chain is less than any element of the chain incomparable with x. Such an extension is called a *lower linear extension* of P (a dual definition associates with a chain an *upper linear extension*).

Proofs We first prove the lemma. Let C be a chain of the ordered set $P = (X, O)$. Write $A_C = \{(x,c) : x \in X \setminus C, c \in C \text{ and } x||_P c\}$ and show that the relation $O \cup A_C$ defined on X is cycle-free. It is clear that O and A_C (which is a bipartite ordered set) are cycle-free. Assume that $O \cup A_C$ contains a cycle $(a_1 ... a_i ... a_k a_1)$ which we choose of minimal length. If $(a_i, a_{i+1}) \in O$, $(a_{i+1}, a_{i+2}) \notin O$ (by minimality of the cycle). If $(a_i, a_{i+1}) \in A_C$, $(a_{i+1}, a_{i+2}) \notin A_C$ (by definition of A_C). Then $(a_1 a_2 ... a_k a_1)$ may be written without loss of generality $(c_1 x_1 ... c_i x_i ... c_r x_r c_1)$ with, for each $i \leq r$, $c_i \in C$, $x_i \in X \setminus C$, $(c_i, x_i) \in O$, and $(x_i, c_{i+1}) \in A_C$. Let then c_h be the greatest element in the subchain $\{c_1, ..., c_i, ..., c_r\}$ of C. So $(c_h, x_h) \in O$ and $(c_{h+1}, c_h) \in O$ hold, whence $(c_{h+1}, x_h) \in O$, a contradiction with x_h incomparable with c_{h+1}. Since $O \cup A_C$ is cycle-free, we may extend this relation into a linear order which is a linear extension of P (Chapter 2, Theorem 2.23) and which, by definition, satisfies the required condition.

Let us now prove Proposition 6.15. To do so, we consider a partition $\{C_1, ..., C_i, ..., C_{\alpha(P)}\}$ of $P = (X, O)$ into $\alpha(P)$ chains. Lemma 6.16 allows us to associate with each of these chains C_i a lower linear extension L_i. Thus, $O \subseteq \bigcap\{L_i, i = 1, ..., \alpha(P)\}$ holds. Assume that there exists $(x,y) \in \bigcap\{L_i, i = 1, ..., \alpha(P)\} \setminus O$. Then $x||_P y$ holds and x and y belong to two different chains C_i and C_k. From the definition of a lower linear extension, we obtain $xL_i y$ and $yL_k x$, a contradiction. So the L_i's form a realization of P by $\alpha(P)$ linear extensions. □

To obtain the bounds in Proposition 6.19 and Corollary 6.20, we need two other lemmas using the following notations: if x and y are two distinct elements of the ordered set $P = (X, \leq)$, we write $x \sim y$ if $(x[= (y[$ and $]x) =]y)$. It is then clear that x and y are incomparable. On the other hand, we write $x <_\sim y$ if $x < y$ and if, for every z different from x and y, $z < x$ holds if and only if $z < y$ holds and $z > x$ holds if and only if $z > y$ holds (which implies $x \prec y$ with x a meet-irreducible and y a join-irreducible).

Lemma 6.17 *Let x,y be two distinct elements of an ordered set P. Then:*

1. *$x \sim y$ implies $\dim(P \setminus x) = \dim P$, unless $P \setminus x$ is a chain (and in this case, $\dim(P \setminus x) = \dim P - 1$).*
2. *$x <_{\sim} y$ implies $\dim(P \setminus x) = \dim(P \setminus y) = \dim P$.*

Proof These two results come from Item (6) in Proposition 6.13. For (1) we substitute the antichain $\{x,y\}$ for y in $P \setminus x$. For (2), we substitute the chain $\{x < y\}$ for y in $P \setminus x$ (or for x in $P \setminus y$). □

We observe that, in order to compute the dimension of an ordered set, Item (1) (respectively, Item (2)) in the lemma allows us to come back to the case where two elements x and y of the ordered set never satisfy $x \sim y$ (respectively, $x <_{\sim} y$).

Lemma 6.18 *If an ordered set P has an antichain A such that $|P \setminus A| = 2$, then $\dim P \leq 2$.*

Proof If P is an antichain then $\dim P = 2$.

Otherwise, let A be an antichain of P such that $P \setminus A = \{x,y\}$ and first assume $x||y$. Without loss of generality we may assume that there exists $a \in A$ with $a < x$. Then for every $a' \in A$, $a' < x$ or $a'||x$.

If there exists $b \in A$ with $b < y$, then $a' < y$ or $a'||y$, for every $a' \in A$. So, we may obtain a partition of A into four sets: $A_1 = \{a \in A : a < x \text{ and } a||y\}$, $A_2 = \{a \in A : a < y \text{ and } a||x\}$, $A_3 = \{a \in A : a < x \text{ and } a < y\}$, $A_4 = \{a \in A : a||x \text{ and } a||y\}$. If two elements z and t belong to the same A_i then – with the notation introduced before Lemma 6.17 – $z \sim t$ holds; thus, using this lemma, we may bring this back to the case where $|A_i| \leq 1$ (with $|A_1 \cup A_2 \cup A_3| \geq 1$). Then, P is isomorphic to an ordered subset of the ordered set in Figure 6.3(a).

If there exists $b \in A$ with $y < b$ or if there exists $a \in A$ with $x < a$, we similarly show that P is isomorphic to an ordered subset of the ordered set in Figure 6.3(b) or to an ordered subset of the two dual ordered sets of those in the previous figures.

If x and y are comparable with, for example, $y < x$, we similarly show that P is isomorphic to an ordered subset of the ordered set in Figures 6.3(c), (d) or (e).

Since it is easy to show that the ordered sets in Figure 6.3 are 2-dimensional (the reader can search for a basis of two linear orders for each of them), we obtain that $\dim P \leq 2$ always holds (and $\dim P = 2$ if $|A| \geq 2$). □

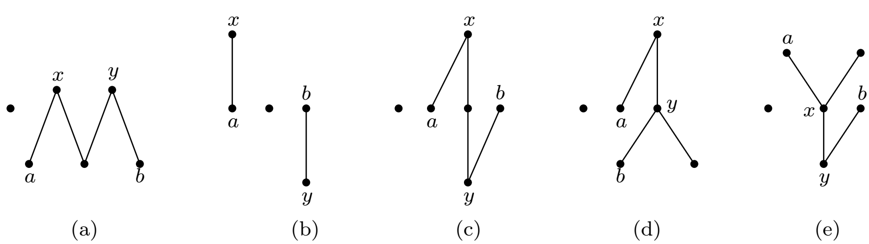


Figure 6.3 Illustration of Lemma 6.18.

Proposition 6.19 *If A is an antichain of an ordered set P satisfying $|P \setminus A| \geq 2$, then* $\dim P \leq |P \setminus A|$ *holds.*

Proof This proposition is proved by induction on the integer $k = |P \setminus A|$. Lemma 6.18 shows that it is true for $k = 2$. Assume that it is true for an integer $k \geq 2$ and let A be an antichain of an ordered set P satisfying $|P \setminus A| = k+1$. Then, for $x \in P \setminus A$, $|P \setminus (x+A)| = k \geq 2$ holds and thus $\dim(P \setminus x) \leq k$; but since the removal of an element from an ordered set decreases its dimension by at most 1 (Item (3) in Proposition 6.13), $\dim P \leq k+1$ holds. □

Let us now consider an antichain A of size $\alpha(P)$ of P. Using Proposition 6.19 if $|P \setminus A| \geq 2$ or trivially if not, we obtain the following result:

Corollary 6.20 *The dimension of an ordered set P satisfies the inequality* $\dim P \leq \max\{2, |P| - \alpha(P)\}$.

Let P be an ordered set such that $|P| - \alpha(P) \geq 2$. Since $\dim P \leq \alpha(P)$ (Proposition 6.15) and $\dim P \leq \max\{2, |P| - \alpha(P)\}$ (Corollary 6.20), we deduce $\dim P \leq \frac{|P|}{2}$. Moreover, if $|P| - \alpha(P) \leq 1$, it is easy to check that the same inequality holds, unless $|P| = 1$ or P is an ordered set of size 2 or 3 and different from a chain. Then we have proved Hiraguchi's result:

Theorem 6.21 (Hiraguchi, 1951) *If P is an ordered set of size $|P| \geq 4$, then* $\dim P \leq \frac{|P|}{2}$.

Remark 6.22 In Example 6.12, we considered the ordered set S_n of size $2n \geq 4$ and showed that its dimension is n. On the other hand, it is clear that $\alpha(S_n) = n$. Thus, $\dim S_n = \alpha(S_n) = |S_n| - \alpha(S_n) = \frac{|S_n|}{2}$, which shows that the above three bounds for the dimension are sharp.

6.3 2-dimensional ordered sets

The ordered sets of dimension 2 are called 2-dimensional ordered sets and are especially interesting since they allow a simple interpretation of an order as the intersection of two linear orders. For example, consider an ordered set modeling the preference of an individual; if it is 2-dimensional, we may presume that this preference is obtained from two linearly ordered criteria, the individual preferring x to y if and only if he/she prefers the former to the latter on the two criteria. In fact, situations where such ordered sets appear are rather frequent. Moreover, problems on 2-dimensional ordered sets may algorithmically become easier (it is the case for their recognition or the computation of the number of their downsets, see Appendix A). We obtain some first characterizations of 2-dimensional ordered sets by rewriting Corollary 6.11 in this particular case:

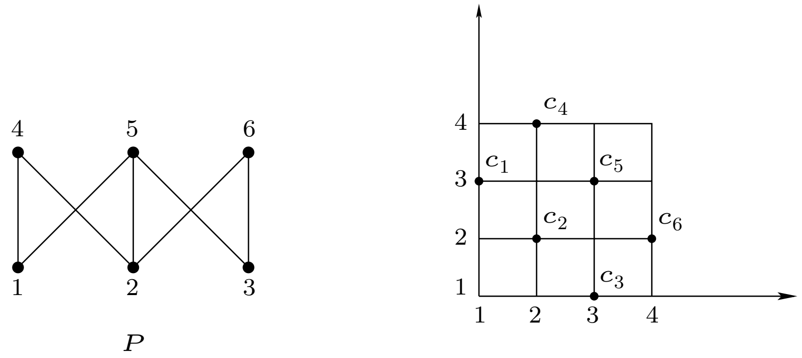


Figure 6.4 A coding from the ordered set P to $\mathbb{N}^2$.

Proposition 6.23 *Let P be an ordered set different from a chain. Then the following are equivalent:*

1. *P is 2-dimensional (that is,* $\dim P = 2$*).*
2. *There exists a coding from P to* $\mathbb{N}^2$*.*
3. *P is the intersection of two linearly ordered sets.*
4. *The lattice* $\mathcal{D}(P)$ *of downsets of P is generated (by union and intersection) by the downsets belonging to two chains.*
5. *The minimum width of a generating set of* $\mathcal{D}(P)$ *is 2.*
6. *The convex subset* $\mathcal{L}(P) = \{$*linear extensions of P*$\}$ *of the permutoedre graph is the convex closure of two linear orders.*

Thus, in order to show that an ordered set P (different from a chain) is 2-dimensional, it is enough for instance to produce an ordered subset of $\mathbb{N}^2$ isomorphic to P, which is done in Figure 6.4. Observe that the coding c from P to $\mathbb{N}^2$ shown in Figure 6.4 induces a strict coding from P to $\underline{4}^2$.

In order to give other characterizations of 2-dimensional ordered sets, we introduce some definitions.

Definition 6.24 Two ordered sets $P = (X, \leq_P)$ and $Q = (X, \leq_Q)$ are said to be *conjugate* if, for any pair $\{x,y\}$ of elements of X, x and y are comparable in one and only one of these two ordered sets: ($x <_P y$ or $x >_P y$) if and only if $x||_Q y$. In this case, we also say that Q (respectively, P) is a *conjugate* of P (respectively, of Q).

A linear extension L of an ordered set P is called *non-separating* if two comparable elements of P are never "separated" in L by an element incomparable to them both: $x <_P y$ and $x <_L z <_L y$ imply $x <_P z$ or $z <_P y$.

Now we may state the further characterizations of 2-dimensional ordered sets.

Theorem 6.25 *Let P be an ordered set. Then the following are equivalent:*

1. *P is 2-dimensional.*
2. *P may be coded in the ordered – by inclusion – set of intervals of a linearly ordered set.*
3. *The incomparability graph of P is a comparability graph.*
4. *P has a conjugate.*
5. *P has a non-separating linear extension.*

Proof (1) $\Longrightarrow$ (2): let L and M be two linear extensions of $P=(X,\leq_P)$ realizing P. Write $X=\{x_1,...,x_i,...,x_n\}$ and let $X'=\{x'_1,...,x'_i,...,x'_n\}$ be a set such that $X\cap X'=\emptyset$. Let $M'=(X',\leq_{M'})$ be the linearly ordered set obtained by writing $x'_j\leq_{M'}x'_i$ if and only if $x_i\leq_M x_j$. Writing $f(x_i)=[x'_i,x_i]$, we define a map f from P to the intervals of the linearly ordered set $N=M'\oplus L$ (the linear sum of M' and L). Then we have $x_i\leq_P x_j$ if and only if $[x_i\leq_L x_j$ and $x_i\leq_M x_j]$, if and only if $[x_i\leq_L x_j$ and $x'_j\leq_{M'}x'_i]$, if and only if $[x'_j\leq_N x'_i\leq_N x_i\leq_N x_j]$, if and only if $f(x_i)\subseteq f(x_j)$. Since f is injective, it is the required coding.

(2) $\Longrightarrow$ (3): let I_x be the interval image of x in the isomorphism between P and a set of intervals of a linearly ordered set L, and denote by i_x the origin of this interval. If x and y are two incomparable elements of P, then i_x and i_y are different. We write $x<y$ if $i_x<_L i_y$. Clearly, we then obtain a transitive orientation of the edges of the incomparability graph of P, which is thus a comparability graph.

(3) $\Longrightarrow$ (4): since there exists an ordered set Q whose comparability graph is equal to the incomparability graph of P, Q is a conjugate of P.

(4) $\Longrightarrow$ (5): let Q be a conjugate ordered set of $P=(X,\leq_P)$. We write $x\leq_L y$ if and only if $[x\leq_P y$ or $x\leq_Q y]$ holds; from the definition of a conjugate, $\leq_L$ is a total and antisymmetric relation. We show that it is transitive, hence a linear order. Let $x<_P y, y<_Q z$ and assume $z<_L x$ whence, for example, $z<_P x$. This implies $z<_P y$, a contradiction with $y<_Q z$ (since Q is conjugate of P). Therefore $L=(X,\leq_L)$ is a linear extension of P; it remains to show that it is non-separating. If $x<_P y$ and $x<_L z<_L y$ also hold, then $x<_P z$ or $x<_Q z$, and $z<_P y$ or $z<_Q y$ hold. Since $x<_Q z$ and $z<_Q y$ would imply $x<_Q y$, which is impossible, $x<_P z$ or $z<_P y$ hold and L is non-separating.

(5) $\Longrightarrow$ (1): let L be a non-separating linear extension of $P=(X,\leq_P)$. We write $Q=(X,<_Q)$ with $x<_Q y$ if ($x<_L y$ and $x||_P y$); we first show that $<_Q$ is a strict order by proving – since it is asymmetric – that it is transitive. Let $x<_Q y$ and $y<_Q z$; then $x<_L y<_L z$, $x||_P y$ and $y||_P z$ hold. If $x<_P z$, since L is non-separating, $x<_P y$ or $y<_P z$ would hold, which is impossible. Since $z<_P x$ is also impossible, $x||_P z$ holds. Thus, $x<_Q z$ and we have shown that $<_Q$ is a strict order. Then write $M=(X,\leq_M)$ with $x\leq_M y$ if ($x\leq_P y$ or $y<_Q x$); the relation $\leq_M$ is antisymmetric and total and we are going to show that it is also transitive. Indeed, consider $x<_M y$ and $y<_M z$ with, for example, $x<_P y$ and $z<_Q y$ (so $y||_P z$). If $z<_M x$, either $z<_P x$

or $x <_Q z$ hold and, using the transitivity of P and Q, we have a contradiction in both cases. Thus $\leq_M$ is a linear order, M is a linear extension of P, and clearly L and M realize P. □

In particular, the proof of this theorem makes it appear that, if P is a 2-dimensional ordered set and Q is its conjugate, then $\{P \cup Q, P \cup Q^d\}$ (respectively, $\{P \cup Q, P^d \cup Q\}$) is a basis of P (respectively, of Q).

Here is an example where we show that an ordered set is 2-dimensional by using the above characterization (5). Recall that a tree-ordered set is a meet-semilattice where $x \wedge y$ is the greatest element of the intersection of the two chains from 0 to x on the one hand and to y on the other hand (see Exercise 2.5). Let P be the tree-ordered set in Figure 6.5(a). We consider the linear extension L of P defined by the following listing of elements: 0 1 3 5 6 7 8 9 4 2. We say that it is defined by a "left" traversing path of the diagram of P: from the element 0, we choose as a successor of x in the list the element y the most "on the left" (with regard to the diagram) among the elements not yet listed and covering the greatest element which satisfies the following two conditions:

- it is already listed,
- it is covered in P by at least one element not yet listed.

We are going to show that L is a non-separating linear extension of P. To do so let $i <_L j <_L k$ with $i <_P k$; we must show that $i <_P j$ holds. Since $i <_L j$, either $i <_P j$ or $i||_P j$ hold. In the latter case, $i \wedge j$ is less than i. If j was on a chain on "the left" of $i \wedge j$, $j <_L i$ would hold, which is impossible; and if j was on a chain on "the right" of $i \wedge j$, $k <_L j$ would hold, which is also impossible. Therefore, L is non-separating and P is 2-dimensional. We obtain another (non-separating) linear extension of P forming

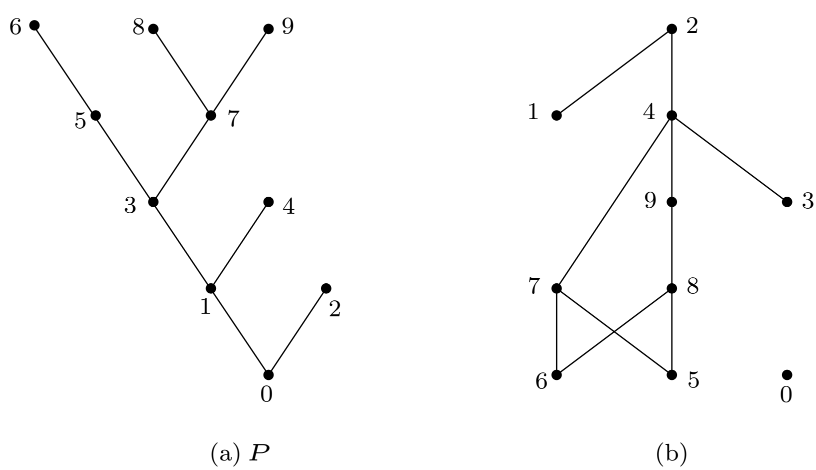


Figure 6.5 (a) A tree-ordered set P and (b) its conjugate.

with L a realization of P when we list the elements according to a "right" traversing path (in the example, we obtain 0 2 1 4 3 7 9 8 5 6).

Observe that the 2-dimensionality of (arbitrary) tree-ordered sets may be obtained more simply. Indeed, these ordered sets are N-free and so we may apply a result given in Exercise 6.4. The aim of the above proof was only to provide examples of non-separating linear extensions.

6.4 *k*-dimension of an ordered set

In this section, we define the notions of a k-coding and the k-dimension of an ordered set (for an integer $k \geq 2$) and we prove the fundamental results on these notions. As particular cases, we obtain the proofs of results on the Boolean dimension and the dimension of an ordered set given in the previous sections.

Definition 6.26 A *k-coding* of an ordered set $P = (X, \leq)$ is a map $c = (c_1, ..., c_i, ..., c_r)$ from P to a direct product of r k-element chains such that:

$$x \leq y \iff c_i(x) \leq_i c_i(y) \text{ for } i = 1, ..., r$$

A k-coding $c = (c_1, ..., c_i, ..., c_r)$ of P is *strict* if, for $i = 1, ..., r$, $c_i(P) = \underline{k}$.

The *k-dimension* of P is the integer denoted by $\dim_k P$ and defined by:

$$\dim_k P = \min\{r \in \mathbb{N} : \text{there exists a coding from } P \text{ to } \underline{k}^r\}$$

The k-dimension trivially satisfies the following:

$$P \sqsubseteq Q \Longrightarrow \dim_k P \leq \dim_k Q \qquad\qquad \dim_k P = \dim_k P^d$$

The following lemma is useful for the proof of Proposition 6.28 giving the relations between the different k-dimensions (with $k \geq 2$ an arbitrary integer).

Lemma 6.27 *Let P be an ordered set of size n. For any integer $k \geq 2$, $\dim_n P \leq \dim_k P$ holds.*

Proof To show this result it is enough to prove that a coding c from $P = (X, O)$ to $\underline{k}^r$ induces a coding from P to $\underline{n}^r$ (with $n = |P|$). For $i \in E = \{1, ..., r\}$, we denote c_i the ith coordinate map associated with a k-coding c: $c(x) = (c_1(x), ..., c_i(x), ..., c_r(x))$. For each $i \in E$, we define a total preorder R_i on X by xR_iy if and only if $c_i(x) \leq c_i(y)$. Since c is a coding, we then obtain a family $(R_1, ..., R_i, ..., R_r)$ of r total preorders, the intersection of which is O. Since the total preorder R_i includes the order O, it also includes a linear extension L_i of O (see Theorem 2.29). Since, for any $i \in E$, $O \subseteq L_i \subseteq R_i$ holds, so does $O = \bigcap_{i \in E} L_i$. Then for any $i \in E$, we define a map c'_i from P to the chain $\{0 < 1 < ... < n-1\}$ of size n by writing $c'_i(x) = r_{L_i}(x)$ (the rank of x in the linear order L_i, see Theorem 2.27). Finally, since xL_iy if and only if $r_{L_i}(x) \leq r_{L_i}(y)$, $c' = (c'_1, ..., c'_i, ..., c'_r)$ is a coding from P to $\underline{n}^r$. □

Proposition 6.28 *For any ordered set P of size n and for $2 \leq k \leq n$,* $\dim P = \dim_n P \leq ... \leq \dim_k P \leq ... \leq \dim_2 P$ *and, for $k \geq n$,* $\dim_k P = \dim P$.

Proof By definition, $\dim_n P = \dim P$. If $k \geq k'$, a coding from P to $\underline{k'}^r$ induces a coding from P to $\underline{k}^r$ and so $\dim_k P \leq \dim_{k'} P$. In particular, $k \geq n$ implies $\dim_k P \leq \dim_n P$. But in this case, Lemma 6.27 gives the converse inequality. □

Before stating the fundamental results on k-codings, recall that a downset of an ordered set $P = (X, \leq)$ is proper if it is different from $\emptyset$ and from X.

Theorem 6.29 *Let $P = (X, O)$ be an ordered set. There are one-to-one correspondences between the following three sets:*

1. *The set of strict codings from P to the direct product $\underline{k}^r$ of r chains $\underline{k}$.*
2. *The set of families $(R_1, ..., R_i, ..., R_r)$ of r total preorders with k classes defined on X and the intersection of which is O.*
3. *The set of families $(\mathcal{C}_1, ..., \mathcal{C}_i, ..., \mathcal{C}_r)$ where, for $i = 1, ..., r$, $\mathcal{C}_i$ is a chain of length $k-2$ of proper downsets of P, and such that $\bigcup_{i \leq r} \mathcal{C}_i$ is a generating set of the lattice $\mathcal{D}(P)$ of downsets of P.*

Proof Let c be a strict coding from $P = (X, O)$ to $\underline{k}^r$ and $c_1, ..., c_i, ..., c_r$ the r associated coordinate maps : $c(x) = (c_1(x), ..., c_i(x), ..., c_r(x))$. For any $i \leq r$, we define a total preorder R_i on X by xR_iy if $c_i(x) \leq c_i(y)$. Thus, we obtain a family $(R_1, ..., R_i, ..., R_r)$ of r total preorders with k classes and – since c is a coding – the intersection of which is O.

We denote $\mathcal{T}_i = t(R_i)$ the extended chain of length k of downsets of P associated with the total preorder R_i in the Galois connection (t,p) defined in Chapter 5 (Theorem 5.24 and Proposition 5.29). By definition of the map t, $t(O) = \mathcal{D}(O)$ $(= \mathcal{D}(P))$ holds. On the other hand, $t(O) = t(\bigcap_{i \leq r} R_i) = \tau(\bigcup_{i \leq r} \mathcal{T}_i)$ (see the properties of this Galois connection stated in Corollary 5.25). Therefore, $\mathcal{D}(P) = \tau(\bigcup_{i \leq r} \mathcal{T}_i)$, which means that $\bigcup_{i \leq r} \mathcal{T}_i$ is a generating set of $\mathcal{D}(P)$. Writing $\mathcal{C}_i = \mathcal{T}_i \setminus \{X, \emptyset\}$, we thus obtain a family of r chains of proper downsets of P, all of length $k-2$ and which generate all proper downsets of P and thus all downsets of P (since the downsets $\emptyset$ and X are always trivially generated). In other words, $\bigcup_{i \leq r} \mathcal{C}_i$ is a generating set of $\mathcal{D}(P)$.

Now, consider a family $(\mathcal{C}_1, ..., \mathcal{C}_i, ..., \mathcal{C}_r)$ of r chains of proper downsets of P, all of length $k-2$ and such that $\bigcup_{i \leq r} \mathcal{C}_i$ is a generating set of $\mathcal{D}(P)$. Adding to every $\mathcal{C}_i$ the empty set and X, we obtain r extended chains $\mathcal{T}_1, ..., \mathcal{T}_i, ..., \mathcal{T}_r$ (all of length k) and which generate all downsets of P. For every $i \leq r$, denote R_i the total preorder with k classes $p(\mathcal{T}_i)$ associated with the extended chain $\mathcal{T}_i$ (Proposition 5.29). Since $t(O) = \mathcal{D}(O) = \tau(\bigcup_{i \leq r} \mathcal{T}_i) = t(\bigcap_{i \leq r} R_i)$ (Corollary 5.25) and since the map t is injective on the set of preorders, we obtain that O is intersection of the r total preorders R_i.

Let us finally assume that the order O of P is the intersection of r total preorders $R_i = (X_1^{(i)} < ...X_h^{(i)} < ...X_k^{(i)})$, each with k classes. We define, for $i = 1,...,r$, the map c_i from X to $\underline{k} = \{1 < ...h < ...k\}$ by $c_i(x) = h$ if $x \in X_h^{(i)}$. This map is surjective and satisfies xR_iy if and only if $c_i(x) \le c_i(y)$. Thus, $c = (c_1,...,c_i,...,c_r)$ is a strict coding from P to $\underline{k}^r$.

The previous constructions allow us to define maps between any two of the three sets considered in the theorem. To end the proof it remains to check that all pairwise compositions of these maps are equal to the identity map, which is left to the reader. □

Above we have considered strict codings from an ordered set P to $\underline{k}^r$. If we consider arbitrary codings, we obtain – with almost identical proofs – the following result.

Theorem 6.30 *Let $P = (X,O)$ be an ordered set. There are one-to-one correspondences between the following three sets:*

1. *The set of codings from P to the direct product $\underline{k}^r$ of r chains $\underline{k}$.*
2. *The set of families $(R_1,...,R_i,...,R_r)$ of r total preorders with at most k classes defined on X and the intersection of which is O.*
3. *The set of families $(\mathcal{C}_1,...,\mathcal{C}_i,...,\mathcal{C}_r)$ where, for $i = 1,...,r$, $\mathcal{C}_i$ is a chain of length at most $k-2$ of proper downsets of P, and such that $\bigcup_{i\le r}\mathcal{C}_i$ is a generating set of the lattice $\mathcal{D}(P)$ of downsets of P.*

The latter theorem immediately gives the following:

Corollary 6.31 *The k-dimension of an ordered set $P = (X,O)$ is the minimum number of total preorders with at most k classes, the intersection of which is O, as well as the minimum number of chains of proper downsets of P, the lengths of which are at most $k-2$ and which generate (by union and intersection) all downsets of P.*

We now introduce a definition to obtain a concise expression of the k-dimension.

Definition 6.32 A *k-chain covering* of an ordered set $P = (X,O)$ is a covering of X by chains of length at most k. We write:

$$\theta_k(P) = \text{minimum number of chains in a } k\text{-chain covering of } P$$

We observe that $\theta_0(P) = |P|$ and that, for $k \ge \lambda(P)$ (where $\lambda(P)$ is the height of P), $\theta_k(P) = \theta(P) = \alpha(P)$ (the latter equality is Dilworth's Theorem 4.2).

A subset $\mathcal{G}$ of $\mathcal{D}(P)$ is a *proper* generating set of $\mathcal{D}(P)$ if it contains neither $\emptyset$ nor X. Then we may write the second expression of the k-dimension in Corollary 6.31 as follows:

Proposition 6.33 *For any ordered set P,*

$$\dim_k P = \min\{\theta_{k-2}((\mathcal{G},\subseteq)),\ \mathcal{G} \textit{ a proper generating set of } \mathcal{D}(P)\}$$

We may now give the proofs of Theorem 6.3 on the Boolean dimension and of Theorem 6.10 on the dimension, together with an alternative proof of Item (5) in Corollary 6.11.

Proofs For $k = 2$, the k-dimension of an ordered set $P = (X, O)$ is its Boolean dimension. Then, Items (1) and (2) in Theorem 6.29 become Items (1) and (2) in Theorem 6.3. It is the same for Item (3), since a chain of length 0 of proper downsets of P is nothing but a proper downset of P.

Observe that Corollary 6.4 – a consequence of Theorem 6.3 – is also a direct consequence of Proposition 6.33 since, for $k = 2$, we obtain $\dim_2 P = \min\{\theta_0((\mathcal{G}, \subseteq))\} = \min\{|\mathcal{G}|\}$ (for $\mathcal{G}$ a proper generating set of $\mathcal{D}(P)\}$.

Similarly, for $k = n$ $(= |P|)$, Items (1), (2) and (3) in Theorem 6.29 become the same items in Theorem 6.10 (since a total preorder with $n = |X|$ classes including the order O is a linear extension of this order).

Proposition 6.33 becomes $\dim_n P = \min\{\theta_{n-2}((\mathcal{G}, \subseteq)) : \mathcal{G}$ a proper generating set of $\mathcal{D}(P)\}$. Since $(\mathcal{G}, \subseteq)$ is an ordered subset of the lattice $\mathcal{D}(P)$ of height n (Theorem 5.6) and contains neither $\emptyset$ nor X, its height is at most $n - 2$. Thus, $\theta_{n-2}((\mathcal{G}, \subseteq)) = \theta((\mathcal{G}, \subseteq)) = \alpha((\mathcal{G}, \subseteq))$. We then obtain that the dimension of P is equal to the minimum width of an (arbitrary) generating set of $\mathcal{D}(P)$, that is to say Item (5) in Corollary 6.11. □

Remark 6.34 The computation of the k-dimension of an ordered set P requires the computation of the minimal generating sets of the lattice $\mathcal{D}(P)$ of its downsets. When $\mathcal{D}(P)$ has a unique minimal generating set, this computation is much simpler. For example, it has been shown (Monjardet and Wille, 1988–89) that, if L is a distributive lattice, the set $DIR(\mathcal{D}(L))$ of its doubly irreducible elements generates $\mathcal{D}(L)$ and thus is its unique minimal generating set (see also Exercise 5.15). On the other hand, it is not difficult to show that, in this case, $DIR(\mathcal{D}(L))$ ordered by inclusion is isomorphic to the ordered set J_L of join-irreducibles of L. Then we obtain:

Proposition 6.35 *For any distributive lattice L,* $\dim_k L = \theta_{k-2}(J_L)$. *In particular, the Boolean dimension (respectively, the dimension) of a distributive lattice is equal to the size (respectively, the width) of the ordered set of its join-irreducibles.*

Moreover, Exercise 6.12 shows that any distributive lattice may be embedded in a direct product of chains by a coding satisfying good properties.

6.5 Further topics and references

The notion of the dimension of an ordered set first appears in Dushnik and Miller's paper (1941) as the minimum number of linear extensions realizing this ordered set. Indeed, these authors observe that the proof of Szpilrajn's Theorem (1930) (showing that any order has a linear extension) immediately enables us to show that any order is the intersection of its linear extensions (see Theorem 2.29). They also give the dimension of S_n (Example 6.12) and the characterizations of 2-dimensional ordered sets stated in Theorem 6.25. In the latter, the fine formulation of Characterization (3) comes from Baker *et al*. (1972). It is equivalent to say that a graph G is the comparability graph of a 2-dimensional ordered set if and only if G and its complementary graph are both comparability graphs; such graphs have also been called *permutation graphs*. Using the known characterizations of comparability graphs, one obtains "efficient" recognition algorithms of these permutation graphs or of 2-dimensional ordered sets (see Appendix A). Another characterization of 2-dimensional ordered sets in Theorem 6.25 states that they are isomorphic to the inclusion order on a family of intervals of a linear order. More generally, it has been shown in Leclerc (1976) that orders of dimension at most r are characterized as isomorphic to the inclusion order on some subtrees of a tree with r leaves.[1]

The equivalence between Dushnik and Miller's definition of the dimension and the one (found for example in Öre's book (1962)) using the notion of a coding is in Hiraguchi (1955). The notion of a Boolean coding of an ordered set P is already – implicitly – in Dushnik and Miller's paper, since they use the isomorphism between P and the set ordered by inclusion of its principal downsets. As for the notion of the k-dimension, it may be dated from Novák (1963); indeed, his '"α-pseudodimension" is the k-dimension when the (possibly infinite) chain α is the chain $\underline{k}$ (Novák shows that the k-dimension of $\underline{k}^r$ is r, and he gives the Boolean dimensions of C_n and A_n). But these notions of codings in a direct product of chains of size k and of k-dimension are independently taken again and considerably developed in Bouchet's thesis (1971). Bouchet's motivations were algorithmic (the Boolean coding of an ordered set enables computer processing). Especially, Bouchet obtains the fundamental results stated in Theorems 6.29, 6.30 and in Corollary 6.31 on the equivalence of several definitions of the k-dimension. In fact, his results (unfortunately not published except in Bouchet (1984)) are more general, since they concern codings of an ordered set in a direct product of chains of different sizes. Such "optimal" codings are also studied in Habib *et al*. (1995).

In his 1950 paper, Dilworth proves his famous decomposition theorem (Chapter 4, Theorem 4.2). Yet in fact, his motivation is then to prove a result equivalent to the following (see Proposition 6.35): the dimension of a distributive lattice is the width of the ordered set of its join-irreducibles. This proposition on the k-dimension of a distributive lattice, a significant generalization of Dilworth's result, is due to Trotter (1975a), as well as the expression of the k-dimension by means of chain coverings of the ordered set given in Proposition 6.33. It is easy to see that, for any ordered set P and any $k > 2$, $\dim_k P \leq \theta_{k-2}(P)$, but it is much more difficult to prove that $\dim_3 P \leq \lceil \frac{|P|}{2} \rceil$ if $|P| \geq 5$ and $\dim_4 P \leq \lfloor \frac{|P|}{2} \rfloor$ if $|P| \geq 6$ (Trotter, 1976a). For the case where $k = 2$,

[1] Recall that a tree is a connected and cycle-free undirected graph and that a leaf is a vertex adjacent to a unique other vertex.

Trotter (1975b) characterizes the ordered sets P, the Boolean dimension of which is the upper bound $|P|$. The other results on the Boolean dimension in Section 6.1 are due to Bouchet (Theorem 6.3) and Trotter (Proposition 6.5). Concerning algorithmics, Stahl (see West, 1985) shows that computing the k-dimension is $\mathcal{NP}$-hard (see Appendix A). We may mention that the number of Boolean codings from an ordered set P to $\underline{2}^p$ (with p an arbitrary integer) has been studied by Hillman (1955) who gives formulas for $|P| \leq 4$ and for $P = \underline{k}$ and by Markowsky (1980). The fact that any 0/1 array induces two Boolean codings of ordered sets may be useful in ordinal data analysis, since such an array thus induces two orders, one on its lines and the other on its columns. This is used from a more theoretical point of view in Monjardet and Netchine-Grynberg (1988) concerning works in child developmental psychology.

After Dushnik and Miller's founder paper (1941), the first significant works on the dimension were carried out by Hiraguchi (1951, 1955) and Baker (1961). In particular, Hiraguchi proves several of the results in Proposition 6.13 (dimension of $P \setminus x$, of the lexicographic sum, and of the product of arbitrary ordered sets), defines the notion of a d-irreducible ordered set (see below and Exercise 6.5), and obtains the upper bounds $\alpha(P)$ and $\frac{|P|}{2}$ for the dimension of P. For the latter fundamental result, we have given Trotter's proof (1975c) (based on a Bogart idea). The ordered sets P, the dimension of which is equal to the bound $\frac{|P|}{2}$, are characterized (Bogart and Trotter, 1973; Kimble, 1973). But it is not the case for those, the dimension of which is equal to the bounds $\alpha(P)$ or $|P| - \alpha(P)$. Baker proves the formula on the dimension of the product of bounded ordered sets (Item (7) in Proposition 6.13) and the result stating that the dimension of an ordered set is equal to the dimension of its completion (see Chapter 3, Section 3.5.3). Exercise 6.8 provides a proof of the latter. Baker also shows that a lattice is planar (see Section 2.5, Further topics and references in Chapter 2) if and only if its dimension is less than or equal to 2. This result will enable the reader to give another proof of the 2-dimensionality of a tree-ordered set (different from a chain). Let us observe the following consequence of Baker's results: the dimension of an ordered set P is less than or equal to 2 if and only if its completion $Gal(P)$ is a planar lattice.

In fact, it is mostly from the 1970s that works on dimension develop considerably, especially with Trotter and Kelly. Their papers (Kelly and Trotter, 1982; Trotter, 1983, 1994) and Trotter's book (1992) review the many obtained results, so we will just mention the most significant and/or latest ones.

The most remarkable result concerns the notion of an ordered set called *d-irreducible*. An ordered set P is d-irreducible if $\dim P = d \geq 2$ and if the dimension of every ordered subset of P (different from P) is less than d or – equivalently to the latter condition (why?) – if, for every x of P, $\dim(P \setminus x) = d - 1$. An ordered set of dimension d always includes a d-irreducible ordered subset. Exercise 6.5 enables us to deduce that the dimension of an ordered set is at most d (with d greater than or equal to 2) if and only if it includes no $(d+1)$-irreducible ordered subset. In particular, any ordered set P different from a chain is 2-dimensional if and only if it does not include any 3-irreducible ordered subset. These 3-irreducible ordered sets have (independently) been determined by Trotter and Moore (1976) on the one hand, and by Kelly (1977) on the other. They consist (up to isomorphism) of nine infinite families of ordered sets and of eighteen particular ordered sets of size $n = 6$ (2) or $n = 7$ (16). Trotter and Moore's proof is based on Characterization (3) of 2-dimensional ordered sets given in Theorem 6.25 (i.e., by the fact that their incomparability graph is a comparability graph). Therefore, an ordered set is 3-irreducible if and only if its incomparability graph is one of the forbidden subgraphs characterizing comparability graphs (these subgraphs were determined by Gallaï (1967)). Kelly's proof exploits Baker's result (mentioned above) that the dimension of an ordered set P is less than or equal to 2 if and only if its completion $Gal(P)$ is a planar lattice, combined with the characterization of planar lattices by forbidden sublattices in Kelly and Rival (1975). For arbitrary d-irreducible ordered sets, Trotter and Ross (1983) show that any d-irreducible ordered set ($d \geq 3$) is an ordered subset of a $(d+1)$-irreducible ordered set.

Exercise 6.7 gives another characterization of the dimension of an ordered set based on its critical ordered pairs. This characterization induces another characterization where the dimension of an ordered set P is the chromatic number of a hypergraph[2] $H(P)$ associated with P. To define $H(P)$, we first define the notion of an IP-cycle: an *IP-cycle* (or *alternating cycle*) of $P = (X, \leq)$ is a sequence $(x_1,y_1),...,(x_i,y_i),...,(x_p,y_p)$ $(p \geq 2)$ of P-critical ordered pairs satisfying the conditions $y_1 \leq x_2,...,y_i \leq x_{i+1},...,y_p \leq x_1$; an IP-cycle is said to be *strong* if $y_i \not\leq x_j$ for every $j \neq i+1$ (modulo p). Then the hypergraph $H(P)$ associated with an ordered set P is defined as follows: its vertices are the P-critical ordered pairs and its edges are the *strong IP-cycles* contained in the set $Crit(P)$ of the P-critical ordered pairs. Now, Trotter and Moore (1977) have shown that, if K is a subset of incomparability ordered pairs of P, there exists a linear extension of P including K if and only if K has no strong IP-cycle. Using this result and those in Exercise 6.7, one then easily obtains $\dim P = \chi(H(P))$ (for P different from a chain). For instance, let us consider the ordered set S_n (Example 6.12): $Crit(S_n)$ is the set of the n ordered pairs (b_j, a_j) and a strong IP-cycle is formed of any two such ordered pairs. Thus, the hypergraph $H(S_n)$ is isomorphic to the complete graph on n vertices, the chromatic number of which is n, another proof that $\dim S_n = n$. For dimension 2, a result from Cogis' thesis on the Ferrers dimension (1980, see further on) shows that the hypergraph $H(P)$ may be replaced with the graph $G(H(P))$ whose edges are those of $H(P)$ containing two vertices: an ordered set P is 2-dimensional if and only if the chromatic number of the graph $G(H(P))$ is 2. Yet, this reduction to the chromatic number of a graph is no longer true in dimension greater than 2, as shown by an example in Trotter (1983).

[2] A *hypergraph* is a pair $H = (X, \mathcal{E})$, where X is a set and $\mathcal{E}$ a family of subsets of X. The elements of $\mathcal{E}$ are called the *(hyper)edges* of H. Let $H = (X, \mathcal{E})$ be a hypergraph. A subset of X is a *stable* (or an *independent*) of H if it contains no edge of $\mathcal{E}$ of size greater than 1. The *chromatic number* $\chi(H)$ of H is the minimum number of stables partitioning X.

Several results concern the dimension of particular classes of orders. For instance, Baker's result stating the 2-dimensionality of a planar lattice (different from a chain) was first completed by Trotter and Moore (1977), who proved that a planar ordered set with a maximum or a minimum may be 3-dimensional, then by Kelly (1981), who provided planar ordered sets of arbitrary dimension. The dimension of interval orders is bounded by their range, whereas the dimension of semiorders is less than or equal to 3 (Rabinovitch, 1978a,b). Brightwell and Trotter (1993) showed that the set ordered by inclusion of vertices, edges, and faces of any planar map (with loops and multiple edges) has dimension at most 4. This result generalizes Schnyder's result (1989) showing that an (undirected) graph G is planar if and only if the dimension of an ordered set P_G associated with G is less than or equal to 3 (the elements of P_G are the vertices and the edges of G and they are ordered by $x < u$ if the vertex x is an extremity of the edge u).

To compute the dimension of an ordered set is a "difficult" problem, unless its dimension is at most 2 (see Appendix A). Then, from the 1980s on, two main research directions have been followed. On the one hand, some classes of orders have been obtained, where the computation of the dimension becomes "easy" (see Appendix A). On the other hand, other – more easily computable – notions of dimension have been defined. We describe the most significant one, called the *greedy dimension*. Informally, one obtains a *greedy linear extension* of an ordered set P by applying the rule "always go as high as possible." So, the algorithm is the following: choose any minimal element x in P and then always choose y in P which covers x and such that all elements covered by y have already been chosen; if no such y exists, take any element which is minimal in the not chosen elements.

One proves that P is the intersection of all its greedy linear extensions, which enables us to define the greedy dimension of P as the minimum number of greedy linear extensions realizing P (Bouchitté *et al*., 1985). In the same research direction one may, for instance, define the notions of *super greedy* linear extension and of *super greedy* dimension of an ordered set (see Trotter's book (1992) for results on such dimensions).

We end this section with the notion of the *Ferrers dimension* of an arbitrary relation, a notion inducing another approach of order dimension. This notion dates back to Bouchet (1971), who intended to define the notions of codings and dimensions for an arbitrary binary relation R (possibly defined between two different sets A and B). To do so, Bouchet considered *Ferrers relations* (Definition 7.8 given in Chapter 7 for $A = B$ may be applied to the general case) that he called *step-type relations* ("relations en escalier"). It is easy to check that any relation R is the intersection of the Ferrers relations including R. Then, Bouchet studied the minimum number of Ferrers relations, the intersection of which is R, a number now called the *Ferrers dimension* (or sometimes the *bidimension*) of R. Since Bouchet showed that the Ferrers dimension of an order is equal to its (order) dimension, he obtained a generalization of the latter.

The Ferrers dimension has been studied particularly by Cogis (1982a,b), Doignon *et al.* (1984), and Koppen (1987, 1989). Their studies situated in the context of generalizations of Guttman scale, such as Coombs and Kao conjunctive–disjunctive models. Indeed, the Ferrers dimension may be used to provide models explaining the answers of a subject to a yes/no questionnaire: these answers may, for instance, be assumed to result from the state of his knowledge on several linearly ordered dimensions, and thus as the intersection of Guttman scales. Concerning "theoretical" results, Bouchet has shown that the Ferrers dimension of a relation is equal to the dimension of the Galois lattice associated with this relation (see Section 3.5.1). When this relation is an order (and so when its Ferrers dimension is its dimension), one obtains the result in Exercise 6.8, namely $\dim P = \dim Gal(P)$. Bouchet has also shown that if an ordered set P may be coded in a (complete) lattice, so may $Gal(P)$; then one obtains the more general result $\dim_k P = \dim_k Gal(P)$. So, the dimension of an ordered set P is equal to the Ferrers dimension of the relation $(J(P), M(P), \leq)$. The Ferrers dimension has been used by Reuter (1989) to obtain results on the difficult problem of determining the dimension of a product of ordered sets, and by Flath (1993) to obtain the dimension of lattices of "multipermutations" and, in particular, to show that the dimension of the permutoedre lattice Σ_n (Example 1.17, Chapter 1) is $n-1$. Finally, as we will see in Section 7.1, irreflexive Ferrers relations are nothing but strict interval orders. Restricting oneself to these particular Ferrers relations, one obtains another notion of dimension called the *interval dimension* and studied by Trotter and others (see West, 1985).

6.6 Exercises

Exercise 6.1 Compute the Boolean dimension of the antichain A_n (see Sperner's Theorem 4.20).

Exercise 6.2 Compute the Boolean dimensions of the ordered sets of size $n \leq 4$. Show that, for $n = 2$ (respectively, 3, 4), there exists one (respectively, two, four) ordered sets of Boolean dimension n.

Exercise 6.3 What is the Boolean dimension of the 2-chain $\underline{n-1} + \underline{1}$? Find again this result by considering the lattice of downsets of this 2-chain (use Proposition 5.8 and Corollary 6.4). Show that this lattice has 2^{n-2} generating sets of minimum size.

Exercise 6.4 [Dimension of an N-free ordered set] Let the P_i's be the ordered subsets formed from the connected components of an ordered set $P = (X, O)$. Why does $\dim P = \max\{2, \max\{\dim P_i\}\}$ hold?

Show that the dimension of an N-free ordered set (Definition 2.12) is at most 2 (read again Section 2.2 and Proposition 6.13).

Exercise 6.5 [d-irreducible ordered sets] An ordered set P is said to be *d-irreducible* if $\dim P = d \geq 2$ and if the dimension of every ordered subset of P (different from P) is less than d or – equivalently for the latter condition – if for every x of P, $\dim(P \setminus x) = d-1$. Show that the dimension of an ordered set is less than or equal to d (with $d \geq 2$) if and only if it has no $(d+1)$-irreducible ordered subset. What are the 2-irreducible ordered sets?

Show that, for $d \geq 2$, every d-irreducible ordered set is indecomposable for the substitution operation (find the relevant item in Proposition 6.13). Show that the ordered set S_3 (in Example 6.12) is 3-irreducible.

Exercise 6.6 Let P be a bipartite ordered set with neither minimum nor maximum. Why does $\dim P \leq \min\{|MinP|, |MaxP|\}$ hold?

Exercise 6.7 [Dimension and P-critical ordered pairs] Show that a set $\{L_1, ..., L_i, ..., L_k\}$ of linear extensions of an ordered set $P = (X, O)$ realizes P (i.e., $O = \bigcap_{1 \leq i \leq k} L_i$) if and only if any P-critical ordered pair (Definition 1.34) belongs to one of these extensions.

Denoting $Crit(P,L)$ the set $L \cap Crit(P)$ of P-critical ordered pairs contained in the linear extension L of P, deduce that the dimension of P is the minimum number of its linear extensions L_i such that $Crit(P) = \bigcup_{1 \leq i \leq k} Crit(P, L_i)$.

Draw the diagram of the forcing order (Definition 1.35) for the ordered set S_3 (Example 6.12). Deduce $Crit(S_3)$ and the dimension of S_3.

Exercise 6.8 [Dimension of the lattice $Gal(P)$] Recall that $IR(P) = J(P) \cup M(P)$ is the set of irreducible elements of an ordered set P. Show that the forcing order (Definition 1.35) on the incomparability ordered pairs of $IR(P)$ is the restriction to these ordered pairs of the forcing order on the incomparability ordered pairs of P (use Exercise 3.15).

Deduce that $Crit(P) = Crit(IR(P))$. Let $Gal(P)$ be the completion of P. Show that $Crit(Gal(P)) = Crit(P)$.

Deduce from these results and from the expression of the dimension of P that $\dim P = \dim IR(P) = \dim Gal(P)$.

Note More generally, it has been shown that $\dim_k P = \dim_k Gal(P)$ (see Bouchet (1971) and Ritzert (1977)). It has also been shown that the interval dimensions of P and of $Gal(P)$ are the same (Habib *et al.*, 1993a).

Exercise 6.9 [Dimensions of a 0/1 array] Let t be an array with n lines and p columns such that the entries $t(i,j)$ are 0 or 1. Order the set of its lines (respectively, of its columns) by writing $i \leq i'$ if $t(i,j) \leq t(i',j)$ for every column j (respectively, $j \leq j'$ if $t(i,j) \leq t(i,j')$ for every line i). Draw the obtained ordered sets for the 0/1 array, the six lines of which are 110111, 101111, 011111, 001111, 010110, and 100100.

Show that the dimensions of these two ordered sets are respectively 3 and 2.

Generalize this example to obtain a 0/1 array with $2n$ lines and $2n$ columns whose order on the lines (respectively, on the columns) has dimension n (respectively, 2).

Note This example is due to Trotter (personal communication).

Exercise 6.10 [Dimension 2: a graphical test] Let x be an element of an ordered set P. Associate with x the point $p(x)$ of the plane and the portion $Q(x)$ of the plane formed from the set of points above and on the right of $p(x)$. Show that P is 2-dimensional if and only if the following are equivalent:

- $x < y$,
- y may be placed in $Q(x)$.

Exercise 6.11 [Dimension of $\underline{2}^n$] Show that the dimension of the ordered set $\underline{2}^n$ is n (one of the numerous possible proofs consists of showing that $\underline{2}^n$ includes an ordered subset isomorphic to S_n – see Example 6.12). Deduce that $\dim_k \underline{k}^n = n$.

Note Using a result in Section 6.2 provides an immediate proof.

Exercise 6.12 [A coding of a distributive lattice] Let L be a distributive lattice and $(C_1, ..., C_i, ..., C_p)$ be a chain covering (see Definition 6.32) of the ordered set J_L of its join-irrreducibles satisfying, for all $i \neq j$, $C_i \cap C_j = \{0_L\}$ (the minimum of L). Define a map c from L to the direct product $\Pi_{1 \leq i \leq p} C_i$ by writing, for all $x \in L$ and $i = 1, 2, ..., p$, $c(x) = (x_1, ..., x_i, ..., x_p)$ where $x_i = \max\{(x] \cap C_i\}$. Show that the map c is a strict join- and meet-coding in this direct product and that it preserves the covering relation of L.

Note This result may easily be generalized to obtain the inequality $\dim_k L \leq \theta_{k-2}(J_L)$ (in fact, an equality stated in Proposition 6.35).

Exercise 6.13 Show that the maximum size of a chain of the ordered set $\underline{k}^r$ is $r(k-1)+1$. Deduce that $\dim_k \underline{n} = \lceil \frac{n-1}{k-1} \rceil$, for any $n \geq 2$.

Exercise 6.14 Show that, if $|P| = 4$, $\dim_3 P = 2$, except for $P = A_4$ or $P = A_2 + C_2$ (in both cases, $\dim_3 P = 3$).

Exercise 6.15 [Permutation graph] Let π be a permutation of $X = \{1, 2, ..., n\}$. The *inversion graph* associated with π is the graph $G = (X, U)$, where $(i,j) \in U$ if i and j are reversed in π (i.e., if $\pi^{-1}(i) > \pi^{-1}(j)$). A graph $G = (X, U)$ is a *permutation graph* if there exists a permutation of X whose inversion graph is G. Show that G is a permutation graph if and only if G is the comparability graph of a 2-dimensional order.

Note Permutation graphs have been studied extensively (see, for example, Spinrad (1985)).

7
Some uses

7.1 Models of preferences

In Chapter 1 (Example 1.21), we mentioned that the classic utility function of economists that represents the preferences of a consumer on a set of commodity bundles (bundle x is preferred to bundle y if $u(y) < u(x)$) defines a particular (strict) order, called a *weak order*. In this modeling of preferences by a utility function, two bundles with the same utility are indifferent for the consumer. Then his indifference relation is transitive. Yet, it was observed long ago that this assumption is not necessarily satisfied. This observation has led us to define other preference ordinal models allowing a numerical representation of the preference along with a non-transitive indifference relation, namely *interval orders* and *semiorders*. The orders of these two classes have been studied extensively. In this section, we concentrate on their basic properties and their numerical representations obtained in the frameworks of psychophysics and preference modeling. First, let us observe or specify several points.

The order relations studied in this section are in particular used in the many areas where one needs to modelize preferences, i.e., not only in microeconomics but more generally in the normative or descriptive decision theories (preferences of a decision-maker over alternatives, preferences of a player on lotteries) or in voting theory (preferences of a voter over candidates).

In these models, one can modelize either the so-called *strict* preference (interpreted as "object x is better than object y") or the so-called *weak* preference (interpreted as "object x is at least as good as object y"). In this section, we choose the former alternative where preference is modelized by a strict order relation, and it will be more convenient to adopt, for a strict order, the literal notation O rather than the notation "$<$." Then, yOx is interpreted as "y is less good than x" (or "x is better than y"). The strict preference O being given, we can define the weak preference R by $(x,y) \in R$ if and only if $(y,x) \notin O$. This weak preference relation is then total and its symmetric part (xRy and yRx) – which modelizes indifference – is equal to the incomparability relation ($(x,y) \notin O$ and $(y,x) \notin O$) of the strict order O. So in this section, we consider that the incomparability relation of the strict order O modelizes the indifference relation. We shall go back to the equivalence – in fact the duality – between these two classes of models at the end of this section.

We shall successively consider the preference models defined by the following classes of orders: strict weak orders, strict interval orders, and strict semiorders. Yet in this section, we shall simplify the statements concerning these orders by systematically omitting the term "strict." Thus, for example, the term "weak order" which in Chapter 2 is defined as a reflexive order will stand here for the corresponding irreflexive order. For each above-mentioned class of orders, we shall first give several characterizations of the orders of the class, then their numerical representation properties. These characterizations use notions that hold for any binary relation, like for instance the notion of a *tableau* of a binary relation. We shall not always immediately give the proofs of the stated results. Indeed, it will be simpler to define a special class of relations called *Ferrers relations*, or *biorders*. These relations are generally not orders but all the above-mentioned orders are particular biorders. So, we will apply to these particular biorders the general (and easy to prove) results on the characterizations and the numerical representation of arbitrary biorders. There is nevertheless an exception to this strategy in the case of the (not so easy to prove) result on the constant threshold numerical representation of semiorders (Scott–Suppes Theorem 7.16). We will use a direct but specific type of proof to show it.

In Chapter 2 (Definition 2.12) we have defined (reflexive) weak orders, interval orders, and semiorders by forbidden suborders properties. Once again, we make use of these definitions at Items (1) in Propositions 7.4, 7.5, and 7.7 (now by forbidding strict suborders). The latter provide several characterizations of these orders, based on notions on arbitrary binary relations defined below.

Definition 7.1 Let R and R' be two binary relations defined on a set X. Their *composition*, denoted RR', is the relation defined by $xRR'y$ if there exists $t \in X$ such that xRt and $tR'y$.

Definition 7.2 A *tableau* of a binary relation R on a set X is a triple (R, L_1, L_2) where L_1 and L_2 are two linear orders on X.

A tableau (R, L_1, L_2) is *step-type* if $L_1RL_2 \subseteq R$, i.e., if for all x, y, z, t in X, xL_1y, yRz, and zL_2t imply xRt.

A tableau where $L_1 = L_2 = L$ is denoted (R, L). A step-type tableau (R, L) is *upperdiagonal* if $R \subseteq L$.

Note that, in Definition 7.2, the orders L_1 and L_2 are reflexive. It follows that the condition $L_1RL_2 \subseteq R$ also implies $L_1R \cup RL_2 \subseteq R$ (why?).

The incidence matrix of a relation R on a set X of size n is the 0/1 matrix M with n lines and n columns, where $m_{x,y} = 1$ if $(x, y) \in R$ and $m_{x,y} = 0$ if not. When a relation R has a step-type tableau we can give a step-type representation of its incidence matrix. It is enough to rank the lines and the columns of this matrix according to the orders L_1 and L_2. Figure 7.2(a) is an example of an order having a step-type tableau with the same order $L = cabedh$ on lines and columns (Table 7.2). Since, moreover, all the 1's of the corresponding 0/1 incidence matrix are above the main diagonal, this tableau is upperdiagonal. The order in Figure 7.1(a) also has a step-type tableau represented in Table 7.1. In spite of appearances, it is not an upperdiagonal tableau: indeed, the order *bacde* on its lines is different from the order *abcde* on its columns.

Let R be a binary relation on a set X. Generalizing the notations used for order relations, we denote R^c the *complementary relation* of R (xR^cy if $(x,y) \notin R$) and R^d the *dual relation* of R (xR^dy if yRx). The relation $(R^c)^d$ (= $(R^d)^c$) is denoted R^{cd} and is equal to $\{(x,y) : (y,x) \notin R\}$; it is called the *codual relation* of R.

In this section, we work with strict orders that can be characterized by forbidden suborders, all of which are disjoint unions of strict linear orders. Then we take again the notation given in Definition 1.3 for these orders. So $\underline{k}_s$ denotes the strict linear order defined on a set of size k and, for example, the notation $\underline{2}_s + \underline{2}_s$ represents the strict bipartite order formed by two ordered pairs (x,y) and (z,t) of distinct elements (its diagram is the same as that of the corresponding reflexive order $C_2 + C_2$ shown in Figure 2.5).

We begin with Proposition 7.4 characterizing weak orders by means of several equivalent conditions. The second condition uses the negative transitivity property defined in Definition 7.3. The fifth condition shows that weak ordered sets are nothing else but linear sums (see Section 1.5.1) of antichains (an antichain being here irreflexive, i.e., without any ordered pair, see Item (2) in Remark 1.44). The sixth condition shows that weak orders can be "represented" by a numerical function, the notion of a representation being precisely defined in the latter condition. Let us develop this point: a (strict) order is naturally associated with any numerical function defined on a set X, namely the order induced by the strictly increasing values of the function; this order is a weak order and, conversely, any weak order is induced by a numerical function (in fact, by an infinity of such functions). This rather obvious representation property of weak orders will be very useful for obtaining other types of numerical representation.

Definition 7.3 A binary relation R defined on a set X is *negatively transitive* if, for all x,y,z in X, xR^cy and yR^cz imply xR^cz.

In other words, R is negatively transitive if R^c is transitive.

When O is a binary relation, we write $I = O^c \cap O^{cd}$. So, when O is a strict order, I is the incomparability relation Inc_O of this order. In this section, we shall use the notation I instead of Inc_O.

Proposition 7.4 *Let O be a binary relation defined on a set X and let $I = O^c \cap O^{cd}$. The following conditions are equivalent:*

1. *O is a weak order, i.e., a strict order including no $\underline{1}_s + \underline{2}_s$.*
2. *O is asymmetric and negatively transitive.*

3. *O is a strict order and $O = OI = IO$.*
4. *O is a strict order and I is an equivalence relation.*
5. *(X,O) is a linear sum of antichains.*
6. *There exists a real-valued function $u : X \longmapsto \mathbb{R}$ such that*

$$xOy \iff u(x) < u(y)$$

Proof (1) $\Longrightarrow$ (2): by definition, O is asymmetric. If O is not negatively transitive, there exist x,y, and z with xO^cy, yO^cz, and xOz. Yet, yOx is impossible (because it would imply yOz, a contradiction). Similarly, zO^cy holds. Then the restriction of O to $\{x,y,z\}$ is isomorphic to $\underline{1}_s + \underline{2}_s$, which is impossible.

(2) $\Longrightarrow$ (3): since O is asymmetric, it is irreflexive. Then, I is reflexive and $O \subseteq OI$. If $OI \not\subseteq O$, there exist x,y, and z such that xOy, yIz, and xO^cz. But xO^cz, zO^cy, and O negatively transitive imply xO^cy, a contradiction. So $O = OI$ and we likewise show $O = IO$.

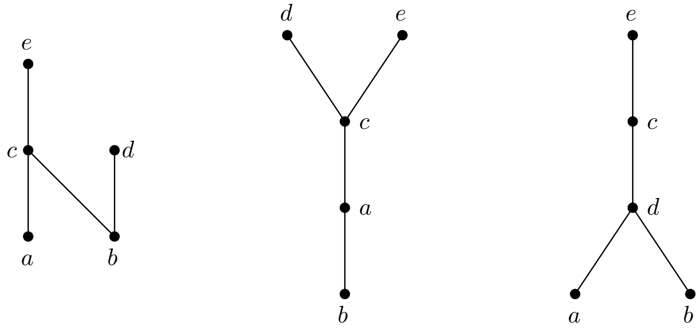
(3) $\Longrightarrow$ (4): as above I is reflexive. Since $(O^c \cap O^{cd})^d = O^{cd} \cap O^c$, I is symmetric. Let x,y,z with xIy and yIz. If xI^cz then either xOz or zOx. In the first case, since $IO = O$, then yIx and xOz imply yOz, a contradiction. The second case implies a similar contradiction. Then I is transitive and so an equivalence.

(4) $\Longrightarrow$ (5): let X/I be the quotient set of X by the equivalence I. Its elements, the incomparability classes of O, are the maximal antichains of (X,O). We define a relation $<$ on this quotient set by writing, for any pair $\{A,A'\}$ of such antichains, $A < A'$ if there exist $x \in A$ and $y \in A'$ with xOy. The reader can check that this relation is equivalently defined by the condition "for every $x \in A$ and every $y \in A'$, xOy holds," and that $<$ is a strict order on X/I. Moreover, this order is a (strict) linear order. Indeed, if for two different (maximal) antichains A and A' of X/I, there were $x \in A$ and $y \in A'$ with xO^cy and yO^cx, then xIy would hold; a contradiction. Then, this linear order $<$ can be written $A_1 \prec \ldots \prec A_p$. It is then clear that, in the ordered set (X,O), x is covered by y if and only if there exists $i < p$ such that $x \in A_i$ and $y \in A_{i+1}$, which means that (X,O) is the linear sum of the antichains A_i.

(5) $\Longrightarrow$ (6): let $(X,O) = \bigoplus_{i=1}^{p} A_i$ be a linear sum of p antichains A_i. Define a map r from X to $\mathbb{N}$ by writing $r(x) = i$ if $x \in A_i$. Since xOy means ($x \in A_i$ and $y \in A_j$, with $i < j$), then xOy if and only if $r(x) < r(y)$, as required (observe that (X,O) is ranked with r as a rank function).

(6) $\Longrightarrow$ (1): let u be a map from X to $\mathbb{R}$ and O the relation defined by xOy if and only if $u(x) < u(y)$. It is immediate to check that O is irreflexive and transitive, i.e., a strict order. On the other hand, xIy if and only if $u(x) = u(y)$. Then it is impossible to have three elements x,y,z with xOy, xIz, and yIz (since, in this case, $u(x) = u(z) = u(y)$ and $u(x) < u(y)$ would hold). $\square$

In microeconomics, one generally defines the preference of a consumer over a set X of commodity bundles by means of a utility function defined on X. This implies that this (strict) preference is a weak order and so that the indifference relation of a



(a) O (b) OI (c) IO

Figure 7.1 (b, c) The two weak orders OI and IO associated with the interval order O

Table 7.1 A step-type tableau of the interval order O in Figure 7.1(a)

	a	b	d	c	e
b	0	0	1	1	1
a	0	0	0	1	1
c	0	0	0	0	1
d	0	0	0	0	0
e	0	0	0	0	0

consumer is transitive: in fact, x is indifferent to y if and only if $u(x) = u(y)$. The notions of an interval order and of a semiorder allow us to obtain a modeling of preferences that no longer assumes a transitive indifference but preserves a property of numerical representation that will be specified in Theorems 7.14 and 7.16. We begin with several ordinal or combinatorial characterizations of interval orders and semiorders.

Proposition 7.5 *Let O be a binary relation defined on a set X and $I = O^c \cap O^{cd}$. The following conditions are equivalent:*

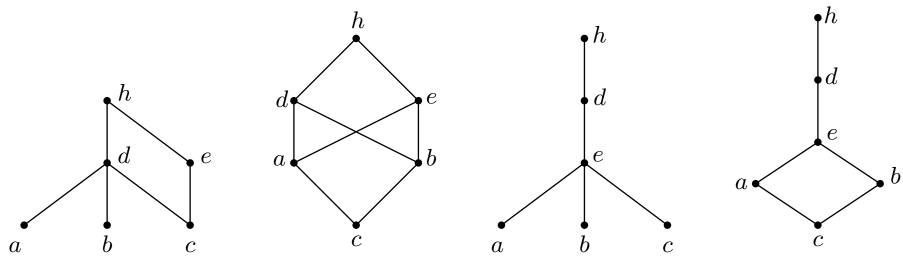
1. *O is an interval order, i.e., a strict order including no $\underline{2}_s + \underline{2}_s$.*
2. *O is irreflexive and, for all elements x,y,z,t of X, $[xOy$ and $zOt]$ implies $[xOt$ or $zOy]$.*
3. *O is a strict order and OI is a weak order.*
4. *O is irreflexive and has a step-type tableau (O,L_1,L_2).*

As said above, we shall prove this proposition after the proof of the more general Proposition 7.9 (and, precisely, after Lemma 7.12).

Note In Condition (3) above, one could replace "OI is a weak order" with "IO is a weak order."

Table 7.2 An upperdiagonal step-type tableau of the semiorder O in Figure 7.2(a).

	c	a	b	e	d	h
c	0	0	0	1	1	1
a	0	0	0	0	1	1
b	0	0	0	0	1	1
e	0	0	0	0	0	1
d	0	0	0	0	0	1
h	0	0	0	0	0	0



(a) O (b) OI (c) IO (d) $OI \cup IO$

Figure 7.2 (b–d) The weak orders OI, IO, and $OI \cup OI$ associated with the semiorder O in (a).

Example 7.6 We illustrate the above characterizations of interval orders with the (strict) order O given in Figure 7.1(a). It is an interval order since it includes no $\underline{2}_s + \underline{2}_s$. The two associated weak orders OI and IO (Condition (3) in Proposition 7.5) are represented in Figures 7.1(b) and 7.1(c). A step-type tableau of O (Condition (4) in Proposition 7.5) is given in Table 7.1.

Proposition 7.7 *Let O be a binary relation defined on a set X and $I = O^c \cap O^{cd}$. The following conditions are equivalent:*

1. *O is a semiorder, i.e., a strict order including neither $\underline{2}_s + \underline{2}_s$ nor $\underline{1}_s + \underline{3}_s$.*
2. *O is irreflexive and, for all elements x, y, z, t of X, $[xOy$ and $zOt]$ implies $[xOt$ or $zOy]$, and $[xOy$ and $yOz]$ implies $[xOt$ or $tOz]$.*
3. *O is a strict order and $OI \cup IO$ is a weak order.*
4. *O has an upperdiagonal step-type tableau (O, L).*

As said above, we shall prove this proposition after the proof of the more general Proposition 7.9 (and, precisely, after Lemma 7.12).

We illustrate the above characterizations of a semiorder on the example of the semiorder O given by its diagram in Figure 7.2(a). It is easy to check that this order includes neither $\underline{2}_s + \underline{2}_s$ nor $\underline{1}_s + \underline{3}_s$. The associated weak orders OI, IO, and $OI \cup IO$ are represented in Figures 7.2(b), (c), and (d), respectively. An upperdiagonal step-type tableau of O is given in Table 7.2.

In order to provide easy proofs of Propositions 7.5 and 7.7, we now define the class of Ferrers relations (also called biorders) and we give several characterizations of such relations.

Definition 7.8 A binary relation R defined on a set X is *a Ferrers relation* (or a *biorder*) if, for all $x,y,z,t \in X$, xRy and zRt imply xRt or zRy.

Observe that, if $x = z$ or $y = t$, the condition of this definition is trivially satisfied, but that it is not so if some other elements are equal. For example, with $x = y$ and $z = t$, it appears that a biorder R cannot include the sub-relation xRx, zRz, $xR^c z$, and $zR^c x$.

Proposition 7.9 *Let R be a binary relation defined on a set X. The following conditions are equivalent:*

1. *R is a Ferrers relation.*
2. *For all pairwise distinct x,y,z,t in X, xRy and zRt imply [xRt or zRy].*
3. *RR^{cd} is a weak order.*
4. *R has a step-type tableau (R, L_1, L_2).*

We begin with the proof of the following lemma on an arbitrary binary relation R defined on X. For $x \in X$, $xR = \{y \in X : xRy\}$ and $Rx = \{y \in X : yRx\}$.

Lemma 7.10 *Let R be a binary relation defined on a set X. Then:*

1. *For all $x,y \in X$:*
   - $xRR^{cd}y \iff xR \not\subseteq yR$.
   - $xR^{cd}Ry \iff Ry \not\subseteq Rx$.
   - $x[(RR^{cd})^{cd}]y \iff xR \supseteq yR$.
   - $x[(R^{cd}R)^{cd}]y \iff Rx \subseteq Ry$.
2. *The relations RR^{cd} and $R^{cd}R$ are negatively transitive.*
3. *The following properties are equivalent:*
   - *RR^{cd} is asymmetric.*
   - *$R^{cd}R$ is asymmetric.*
   - *RR^{cd} is a weak order.*
   - *$R^{cd}R$ is a weak order.*
4. *If R is a strict order, denoted by O, and $I = O^c \cap O^{cd}$, then $O^{cd} = O + I$, $OI = OO^{cd}$ and $IO = O^{cd}O$.*

Proof (1) $xRR^{cd}y$ means that there exists an element t of X such that xRt and yR^ct, which is equivalent to $xR \not\subseteq yR$. Then $x[(RR^{cd})^{cd}]y$ is equivalent to $xR \supseteq yR$. We similarly prove the other two equivalences.

(2) According to (1), $x[(RR^{cd})^c]y$ is equivalent to $yR \supseteq xR$, and $x[(R^{cd}R)^c]y$ is equivalent to $Ry \subseteq Rx$, so the negative transitivity of RR^{cd} and of $R^{cd}R$ is obvious.

(3) Assume RR^{cd} is asymmetric and $R^{cd}R$ is not. Thus, there exist two distinct elements x and y with $xR^{cd}Ry$ and $yR^{cd}Rx$, which implies that there exist two distinct elements z and t with zRy, zR^cx, tRx, and tR^cy. Since tRx and zR^cx, then $tRR^{cd}z$ holds, whereas zRy and tR^cy imply $zRR^{cd}t$, a contradiction with the asymmetry of RR^{cd}. We similarly show that, if $R^{cd}R$ is asymmetric, so is RR^{cd}. At last since, by (2), RR^{cd} is also negatively transitive, this relation is a weak order (Proposition 7.4). It is the same for $R^{cd}R$.

(4) A binary relation O is asymmetric if $O \subseteq O^{cd}$, which implies $O^{cd} = O^{cd} \cap (O+O^c) = O+I$. For any binary relation O, $OI \subseteq OO^{cd}$ obviously holds. If O is a strict order and $xOO^{cd}y$, then there exists t with xOt, and yO^ct holds. If tO^cy, then tIy and so $xOIy$ holds. If tOy, the transitivity of O implies xOy and the reflexivity of I implies yIy, then $xOIy$ holds. So we have proved $OI = OO^{cd}$. We similarly prove $IO = O^{cd}O$. □

Remark 7.11 It results immediately from Item (1) in the previous lemma that $(RR^{cd})^{cd}$ and $(R^{cd}R)^{cd}$ are total preorders (the weak orders RR^{cd} and $R^{cd}R$ of which are the asymmetric parts). These preorders, already considered in Exercise 1.11, are often called (*right* and *left*) *trace preorders*. Exercise 7.3 allows us to show some additional results on these two preorders (respectively denoted T_r and T_l) or their intersection (denoted T).

We now prove Proposition 7.9.

Proof of Proposition 7.9 (1) $\Longrightarrow$ (2): obvious.

(2) $\Longrightarrow$ (3): from Item (3) in Lemma 7.10, to show that RR^{cd} is a weak order amounts to showing that this relation is asymmetric. If RR^{cd} is not asymmetric, there exist $x,y \in X$ with $xRR^{cd}y$ and $yRR^{cd}x$. This implies that there exist t and z with xRt, yR^ct, yRz, and xR^cz, a contradiction with (2).

(3) $\Longrightarrow$ (4): from Lemma 7.10, RR^{cd} being a weak order, so is $R^{cd}R$. From Item (1) in Theorem 2.29 (applied to strict orders), there exist two strictly linear orders $L_{1,s}$ and $L_{2,s}$ such that $RR^{cd} \subseteq L_{1,s}$ and $R^{cd}R \subseteq L_{2,s}$. On the other hand, observe that the codual relation of a strictly linear order L_s is the associated linear order $(L_s)^{cd} = L_s + \{(x,x), x \in X\}$. Denote $L_1 = (L_{1,s})^{cd}$ and $L_2 = (L_{2,s})^{cd}$ the two linear orders associated with $L_{1,s}$ and $L_{2,s}$. We are going to show that (R, L_1, L_2) is a step-type tableau, i.e., that xL_1y, yRz, and zL_2t imply xRt. Begin with an obvious observation: if $R \subseteq R'$ then $R^{cd} \supseteq R'^{cd}$. In particular, $RR^{cd} \subseteq L_{1,s}$ implies $L_1 = (L_{1,s})^{cd} \subseteq (RR^{cd})^{cd}$. So xL_1y implies $x[(RR^{cd})^{cd}]y$, that is to say, $xR \supseteq yR$ (Lemma 7.10). We similarly

show that zL_2t implies $Rz \subseteq Rt$. Since yRz holds, the first implication leads to xRz, and with the second implication, xRt holds.

(4) $\Longrightarrow$ (1): let (R, L_1, L_2) be a step-type tableau of R and let x, y, z, t such that xRy and zRt. Since L_1 is a linear order, then xL_1z or zL_1x hold. By definition of (R, L_1, L_2), $L_1R \subseteq R$ and, in the first case, xRt holds whereas, in the second case, zRy holds, whence (1). □

The following lemma characterizing interval orders and semiorders as particular Ferrers relations will lead immediately to easy proofs for Propositions 7.5 and 7.7.

Lemma 7.12 *Let O be a binary relation defined on a set X.*

1. *O is an interval order if and only if O is an irreflexive Ferrers relation.*
2. *O is a semiorder if and only if O is an irreflexive Ferrers relation including no $\underline{1}_s + \underline{3}_s$.*

Proof (1) If O is an interval order, it is irreflexive. Moreover, since it includes no $\underline{2}_s + \underline{2}_s$, it satisfies Characterization (2) of a Ferrers relation (Proposition 7.9). Conversely, if O is an irreflexive Ferrers relation, it is transitive since xOy, yO^cy, and yOt imply xOt (Item (2) in Proposition 7.9). Thus, O is a strict order which includes no $\underline{2}_s + \underline{2}_s$ since O is a Ferrers relation. So it is an interval order.

(2) This results immediately from (1) and from the fact that, by definition, a semiorder is an interval order including no $\underline{1}_s + \underline{3}_s$. □

We now prove Propositions 7.5 and 7.7.

Proof of Proposition 7.5 Apply Proposition 7.9 when R is an irreflexive Ferrers relation, i.e., an interval order denoted O, and show that the conditions in this proposition become the four conditions in Proposition 7.5, which proves their equivalence. It is obvious for Items (2) and (4). For Item (1), this results from the above lemma stating that irreflexive Ferrers relations are interval orders. For Item (3), if OO^{cd} is a weak order, O is a Ferrers relation (Proposition 7.9) which, being irreflexive, is a strict (interval) order, according to what has just been said. From Item (4) in Lemma 7.10, $OO^{cd} = OI$ holds. Conversely, if O is a (strict) order, the same item implies $OI = OO^{cd}$. □

Proof of Proposition 7.7 (1) $\Longrightarrow$ (2): a semiorder being an interval order including no $\underline{1}_s + \underline{3}_s$, it satisfies the first implication of (2) (by Proposition 7.5). For the second one, it is easy to see that if it was not satisfied, the restriction of O to $\{x, y, z, t\}$ would be isomorphic to $\underline{1}_s + \underline{3}_s$, which is impossible.

(2) $\Longrightarrow$ (3): by Proposition 7.5, O is an interval order and OI is a weak order; it is the same for IO (see the comment following Proposition 7.5, on page 196). Since it is clear that the union of two negatively transitive relations is negatively transitive, so is $OI \cup IO$. To show that this relation is a weak order, it therefore suffices to show that it is asymmetric. Assume on the contrary that there exist x, y with $x(OI \cup IO)y$

and $y(OI \cup IO)x$; OI and IO being asymmetric, one must have, for example, $xOIy$ and $yIOx$. This implies that there exist z and t with xOz, zIy, yIt, and tOx. These four elements are necessarily different (for example, $t \neq y$ since, if not, the transitivity of O implies yOz; a contradiction). But then the restriction of O to $\{y,t,x,z\}$ must be isomorphic to $\underline{1}_s + \underline{3}_s$, which is impossible.

(3) $\Longrightarrow$ (4): by Items (4) and (2) in Lemma 7.10, the relations OI and IO are negatively transitive. But since they are included in the weak order $OI \cup IO$, they are also asymmetric and so weak orders (Proposition 7.4). From the implication of (4) by (3) shown at Proposition 7.5, O has a step-type tableau (O, L_1, L_2), where L_1 and L_2 are two linear orders. Moreover, these orders respectively include OI and IO. Indeed, it results from the proof of the implication of (4) by (3) in Proposition 7.9 and from the fact that, O being an order, $OO^{cd} = OI$ and $O^{cd}O = IO$ hold (see Item (4) in Lemma 7.10). Yet, since the weak orders OI and IO are included in the weak order $OI \cup IO$, we can take $L_1 = L_2 = L$ as a linear order including $OI \cup IO$. The relation I being reflexive, then $O \subseteq OI \cup IO \subseteq L$ and so the tableau (O, L) is upperdiagonal.

(4) $\Longrightarrow$ (1): since O has a step-type tableau (O, L), it is an interval order (Proposition 7.5). Let x, y, z, t with xOy, yOz (and so xOz). The tableau (O, L) being upperdiagonal, then $O \subseteq L$ and so xLy, yLz, and xLz. Since $LO \subseteq O$ and yOz, if tLy holds, so does tOz. Similarly, yLt implies xOt since $OL \subseteq O$ and xOy. Therefore the restriction of O to $\{x,y,z,t\}$ cannot be isomorphic to $\underline{1}_s + \underline{3}_s$, which shows that O is a semiorder. □

We now come to the results on the numerical representation of interval orders and semiorders. The result on interval orders (Theorem 7.14) will be an immediate consequence of the result on the numerical representation of Ferrers relations (Proposition 7.13) whereas that on semiorders (Theorem 7.16) needs a specific proof. So we begin with the case of Ferrers relations.

Proposition 7.13 (Numerical representation of Ferrers relations) *Let R be a binary relation defined on a set X. The following conditions are equivalent:*

1. *R is a Ferrers relation.*
2. *There exist two functions f and g from X to $\mathbb{R}$ such that*

$$xRy \iff f(x) < g(y)$$

3. *There exist two functions u and s from X to $\mathbb{R}$ such that*

$$xRy \iff u(x) + s(x) < u(y)$$

Proof (1) $\Longrightarrow$ (2): By Item (3) in Proposition 7.9, RR^{cd} is a weak order. Then, by Proposition 7.4, there exists a function $f : X \longmapsto \mathbb{R}$ such that $xRR^{cd}z$ if and only if

$f(x) < f(z)$. We write for every y in X:

$$g(y) = \begin{cases} \max\{f(z), z \in X\} + 1 & \text{if } yR^{cd} = \emptyset \text{ (i.e., if there does not exist } z \\ & \text{such that } yR^{cd}z) \\ \min\{f(z) : yR^{cd}z\} & \text{if not} \end{cases}$$

We first show that xRy implies $f(x) < g(y)$. Let x, y with xRy. If $yR^{cd} = \emptyset$, $f(x) \leq \max\{f(z), z \in X\} < g(y)$.

If $yR^{cd} \neq \emptyset$, let z be such that $yR^{cd}z$. Since xRy and $yR^{cd}z$, $xRR^{cd}z$ holds. So $f(x) < f(z)$ for every $z \in yR^{cd}$, whence $f(x) < \min\{f(z) : yR^{cd}z\} = g(y)$.

For the converse, let x and y be such that $f(x) < g(y)$. If $g(y)$ is equal to $\max\{f(z), z \in X\} + 1$ or $\min\{f(z) : yR^{cd}z\}$, we obtain $y(R^{cd})^c x$, i.e., xRy.

(2) $\Longrightarrow$ (3): write for every $t \in X$, $u(t) = g(t)$ and $s(t) = f(t) - u(t)$. Then xRy if and only if $f(x) = u(x) + s(x) < u(y)$.

(3) $\Longrightarrow$ (1): assume there exist x, y, z, t such that xRy, zR^cy, zRt and xR^ct. Then $u(x) + s(x) < u(y) \leq u(z) + s(z) < u(t) \leq u(x) + s(x)$, which is impossible. □

We can now state and easily prove the result on the numerical representation of interval orders:

Theorem 7.14 (Numerical representation of interval orders) *Let O be a binary relation defined on a set X and $I = O^c \cap O^{cd}$. The following conditions are equivalent:*

1. *O is an interval order, i.e., a strict order including no $\underline{2}_s + \underline{2}_s$.*
2. *There exist two functions u from X to $\mathbb{R}$ and s from X to $\mathbb{R}^+$ such that*

$$xOy \iff u(x) + s(x) < u(y)$$

3. *There exists a function F from X to the set of intervals of $\mathbb{R}$ such that*

$$xOy \iff F(x) < F(y)$$

(where the order $<$ on the intervals of $\mathbb{R}$ is defined by $F(x) < F(y)$ if, for all $a \in F(x), b \in F(y)$, $a < b$).

Proof To prove the equivalence of Conditions (1) and (2), it is sufficient to apply Proposition 7.13 when $R = O$ is an irreflexive Ferrers relation, i.e., an interval order, and to observe that, O being irreflexive, $u(x) + s(x) \geq u(x)$ and thus $s(x) \geq 0$. Now, Condition (3) is obviously implied by Condition (2): it suffices to set $F(x) = [u(x), u(x) + s(x)]$. Conversely, we show that, if (3) is satisfied, O is an interval order, by showing that O satisfies Condition (2) in Proposition 7.13. First, $F(x) \not< F(x)$ implies that O is irreflexive. Let x, y, z, t with xOy, zOt, and xO^ct; we must show zOy. The first two relations imply $u(x) + s(x) < u(y)$ and $u(z) + s(z) < u(t)$

and the third relation gives $u(x) + s(x) \geq u(t)$. We deduce $u(z) + s(z) < u(y)$, i.e., zOy. $\square$

Condition (2) implies that in this interval order model x is indifferent to y if and only if $[u(x) \leq u(y) + s(y)$ and $u(y) \leq u(x) + s(x)]$. Condition (3) implies that x is indifferent to y if and only if the two intervals $F(x)$ and $F(y)$ have a non-empty intersection (see the further topics in Section 7.6 for the so-called *intersection graphs* defined in a similar manner).

Example 7.15 We illustrate the above characterizations of interval orders with the (strict) order O given in Figure 7.1(a).

In order to obtain the numerical representation in Condition (2), we use the method given in the proof of Proposition 7.13 to obtain the numerical representation (2) of a Ferrers relation. It first allows us to find two functions f and g such that xOy if and only if $f(x) < g(y)$. For f we can, for example, take a rank function of the order OI (by Item (4) in Lemma 7.10, OO^{cd} is equal to OI). Thus, we obtain $f(b) = 1, f(a) = 2$, $f(c) = 3$, and $f(d) = f(e) = 4$. Then since the set $\{y \in X : xO^{cd}y\} = xO^{cd} = x(O+I)$ is never empty, we obtain $g(x) = \min\{f(z) : xO^{cd}z\}$, which implies $g(a) = g(b) = 1$, $g(c) = 3, g(d) = 2$, and $g(e) = 4$. Now, taking $u = g$ and $s = f - g$, we obtain $s(a) = 1$, $s(b) = s(c) = s(e) = 0$, and $s(d) = 2$. The reader can check that, then, xOy if and only if $u(x) + s(x) < u(y)$. In order to obtain the function F in Condition (3), we write $F(x) = [u(x), u(x) + s(x)]$, whence $F(a) = [1,2], F(b) = [1,1], F(c) = [3,3], F(d) = [2,4]$, and $F(e) = [4,4]$. Observe that some intervals obtained in this representation of the order O reduce to a number (but it is easy to deduce from this representation another one without such trivial intervals).

We now give the theorem on the numerical representation of semiorders; that is, the Scott–Suppes Theorem.

Theorem 7.16 (Scott–Suppes, 1958) *Let O be a binary relation defined on a set X and $I = O^c \cap O^{cd}$. The following conditions are equivalent:*

1. *O is a semiorder.*
2. *There exist a function u from X to $\mathbb{R}$ and a real number s such that*

$$xOy \iff u(x) + s < u(y).$$

3. *There exists a function F from X to the set of unit-length intervals of $\mathbb{R}$ such that*

$$xOy \iff F(x) < F(y)$$

(where the order $<$ on the intervals of $\mathbb{R}$ is defined by $F(x) < F(y)$ if, for all $a \in F(x), b \in F(y)$, $a < b$).

Proof

(1) $\Longrightarrow$ (3): The proof is made by induction on the size of X but we begin with an observation. Since O is a semiorder and thus an interval order, there exists a function F from X to the set of intervals of $\mathbb{R}$ such that xOy if and only if $F(x) < F(y)$ (Theorem 7.14). We write $F(x) = I_x = [l_x, r_x]$ and consider the element z such that $l_z = \max\{l_t, t \in X\}$. The set A of elements incomparable to z is an antichain. Indeed, if there existed $u, v \in A$ with vOu, one would have $l_z < r_v < l_u$, a contradiction with the choice of z.

Write now $Y = X \setminus z$, and assume inductively that there exists a function F from Y to the set of unit-length intervals of $\mathbb{R}$ such that xOy if and only if $F(x) < F(y)$, and such that no intervals share the same endpoint. We are going to show that we can represent z by a unit-length interval. Since the set A is an antichain, the union of the intervals representing the elements of A has length less than two. Let w and $w' \in A$ such that $l_w = \min\{l_s, s \in A\}$ and $r_{w'} = \max\{r_s, s \in A\}$. Then $J = [l_{w'}, r_w] = \bigcap\{I_s, s \in A\}$.

First assume that, for any $x \in X \setminus z$ with xOz, $J \not\subset I_x$ holds; that is, $r_x < r_w$. Then we can represent z by the interval $F(z) = [l_z, l_z + 1]$, with l_z in the interval $[\max\{r_x, xOz\}, r_w]$.

On the contrary, assume that there exists $x \in X \setminus z$ with xOz and $J \subset I_x$. Then the left endpoint of the required interval $F(z)$ cannot be in J, a contradiction with the fact that z is incomparable to any element of A. We claim that, in this case, we can shift I_x leftward so that $r_x < r_w$. Indeed, it would be impossible if there existed yOx with y incomparable to w. Because if we do such a shift then we must also shift I_y leftward so that $r_y < l_x$ and then, since I_x and I_w have unit length, we obtain $r_y < l_x < l_w$, a contradiction with the fact that y is incomparable to w. Now, we observe that it is impossible to have yOx with y incomparable to w. Indeed in this case, we would have $yOxOz$ and z, y, and x incomparable to w and so a restriction of O isomorphic to $\underline{1}_s + \underline{3}_s$, a contradiction with O semiorder.

Thus, we can shift all such I_x leftward so that the maximum of the corresponding r_x is less than $r_w - \epsilon$, for some $\epsilon > 0$. Finally, taking $F(z) = I_z = [r_w - \frac{\epsilon}{2}, r_w - \frac{\epsilon}{2} + 1]$, we get the required unit-length interval representation.

(3) $\Longrightarrow$ (2): Obvious. We take $u(x) = l_x$ (with $F(x) = [l_x, r_x]$) and $s = 1$.

(2) $\Longrightarrow$ (1): We already know (Proposition 7.14) that O is an interval order. We show that it cannot include a restriction isomorphic to $\underline{1}_s + \underline{3}_s$. Let $x, y, z, t \in X$ be such that $xOyOz$ and x is incomparable to t. Then, $u(x) + s < u(y)$, $u(y) + s < u(z)$, and $u(x) - s \leq u(t) \leq u(x) + s$. Then $u(t) \leq u(y)$, $u(t) + s < u(y) + s < u(z)$ and so tOz. □

To end this section, we go back to the duality between the modelings of the strict and the weak preferences of an individual. This duality is a consequence of the next elementary result:

Proposition 7.17 *The sets of asymmetric binary relations and of total binary relations defined on a set X and ordered by inclusion are two dual semilattices by the coduality map $R \longmapsto R^{cd}$.*

Proof The intersection (respectively, the union) of two asymmetric (respectively, total) relations defined on X is obviously asymmetric (respectively, total). The set of asymmetric (respectively, total) relations defined on X is therefore a meet-semilattice (respectively, a join-semilattice). Since an asymmetric (respectively, total) relation R is characterized by $R \subseteq R^{cd}$ (respectively, $R \supseteq R^{cd}$), $(R^{cd})^{cd} = R$, and $R \subseteq R'$ if and only if $R^{cd} \supseteq R'^{cd}$, it is easy to deduce that the coduality map is a dual isomorphism between these two semilattices. □

So, each time a particular class of asymmetric relations is defined, the coduality map transforms it into a dual class of total relations. The dual class of weak orders is the class of total preorders (see Exercise 7.4). The dual class of orders is the class of total relations, the asymmetric part of which is transitive (in the economics literature, such relations are often called *quasi-transitive relations*). While the codual relation of an interval order did not receive any particular name, the codual relation of a semiorder has been called a *total semiorder*, but also a *semiorder* (indeed, in the literature, the term semiorder is used in the two senses of an asymmetric or of a total relation).

The duality between these two classes of asymmetric and of total relations allows us to immediately translate any result obtained on one class to a result on the other one. Thus, it is the same to modelize the notion of a preference as a strict preference – i.e., by a strict order – or as a weak preference – i.e., by a quasi-transitive relation. If, in microeconomics, it is more common to opt for the second choice, the first one has at least two advantages: on the one hand, it conforms more to the parsimony principle (the codual relation of an asymmetric relation contains much more ordered pairs); on the other hand, and above all, it situates the study of the mathematical models of preference in the field of ordered set theory.

Remark 7.18 If O is an asymmetric relation and if R denotes the codual total relation of O, then $R = O^{cd} = O + I$, with $I = O^c \cap O^{cd} = R \cap R^d$. Thus, if O is an order modeling a strict preference, one finds again that the incomparability relation of this order is the symmetric part of the relation R modeling a weak preference, namely the relation which, in the classic model, represents the indifference relation.

7.2 Preference aggregation: Arrowian theorems for orders

Since Antiquity, citizens' votes have been used to choose representatives or leaders, for instance the Senate members or the consuls of the Roman Republic. In the eighteenth century, the French Academy of Science used to elect its new members by means of a simple voting rule, still often used (for instance, for the election of the members of the British Parliament): the chosen candidate is the one supported by the greatest number of votes in a ballot where each voter gives his preferred candidate. The reader can easily construct examples where this type of ballot leads to electing the candidate who is ranked last by a majority of the electorate. This fact led Borda to propose the method bearing his name. In Borda's method each voter gives his preference order (assumed to be a linear order) on the candidates; then the latter are ranked according to the sum of their ranks obtained in these different orders. Soon afterwards Condorcet observed that a majority of voters could prefer a candidate different from that elected by Borda's method (an example will be given later). He then proposed his own method, the majority rule, as a remedy. In the majority rule one considers all pairs of candidates and for each pair, the majority ordered pair is kept; the collective preference is then the union of all these majority ordered pairs. Yet, Condorcet realized that his method also had a defect: the collective preference obtained in this way may contain cycles, i.e., may not be an order. This fact has been called "l'effet Condorcet" or the "voting paradox." The simplest example of such an effect is obtained with the following preferences of three voters on three candidates a, b and c: abc, bca, and cab. Indeed in this case the collective preference obtained by Condorcet's majority rule is the cycle where a is preferred to c, c to b, and b to a. Condorcet will seek to correct this defect, the explanation of which will be provided much later when Arrow, formalizing the difficulties encountered by economists who try to build a collective utility from individual utilities, will state his famous impossibility theorem. Arrow's Theorem fostered a considerable development of the so-called social choice theory. Here we shall simply present some initial results of the latter field showing the difficulties of attempting to aggregate satisfactorily individual preferences modelized by linear orders into a collective preference modelized by an order.

We shall begin by specifying the aggregation rules proposed by Borda and Condorcet and by showing some of their defects. Afterwards we will describe a large class of rules generalizing Condorcet's rule and using "generalized majorities" (or "winning coalitions"). The search for those rules where the collective preference is a linear order will lead us to a first impossibility result. Next, we shall present the "axiomatic" approach introduced by Arrow and consisting of the search for aggregation rules satisfying "good" properties. The previously mentioned impossibility result will then lead us to several Arrowian theorems showing the difficulties of aggregating linear orders into an order. In particular, Arrow's Theorem – proved here in the case of linear orders – will provide a "dictatorial" result or, in more mathematical terms, a characterization of the projections defined on some product space.

We first describe the mathematical model used for preference aggregation.

$X = \{x, y, z, ...\}$ is a finite set of size m. Throughout this section, the elements of X will be called the "candidates" but, according to the context, they may be commodity bundles, options, etc.

$N = \{1, 2, ..., n\}$ is a finite set of size n. Throughout this section, the elements of N will be called the "voters" but, according to the context, they may be consumers, decision-makers, etc.

Afterwards, we always assume that there are at least three candidates and three voters, so $m, n \geq 3$ (this assumption, necessary for some of the presented results, will not be repeated).

We also assume that each voter has a preference linear order on the set of candidates. Contrary to Section 7.1, the orders considered in this section are reflexive.

The consideration of the voters' preferences then defines a map from N to the set $\mathcal{L} = \mathcal{L}_X$ of the $m!$ linear orders defined on X. This map is called a *preference profile* (or simply a *profile*) π and is denoted:

$$\pi = (L_1, ..., L_n)$$

where L_i is the preference linear order of voter i on the candidates and where yL_ix is interpreted as "voter i prefers candidate x to candidate y." The set of all preference profiles of the N voters is denoted $\mathcal{L}^N$.

Let $\pi = (L_1, ..., L_i, ..., L_n)$ be a profile. For all $x, y \in X$, we write

$$N_\pi(y,x) = \{i \in N : yL_ix\}, \quad n_\pi(y,x) = |\{i \in N : yL_ix\}|$$

$N_\pi(y,x)$ is then the set of voters preferring candidate x to candidate y and $n_\pi(y,x)$ is the number of these voters.

Example 7.19 $X = \{a, b, c, d\}$ is a set of four candidates and N a set of seven voters. The preference profile of the voters on the candidates is $\pi = (cadb : 2, bcad : 2, dbca : 3)$ where, for example, $cadb : 2$ means that 2 voters have the order $c < a < d < b$ as their preference on the candidates. Table 7.3 gives the array of the $n_\pi(x,y)$'s for this profile. For example, the value 2 of $n_\pi(a,b)$ is at the intersection of line a and column b in this array.

Table 7.3 Array of the $n_\pi(y,x)$ for the profile π

$n_\pi(y,x)$	a	b	c	d
a	7	2	0	4
b	5	7	5	2
c	7	2	7	4
d	3	5	3	7

Finally, the rank of the element x in the linear order L_i, where the rank is defined by $r_{L_i}(x) = |L_ix| = |\{y \in X : yL_ix\}|$, will be more simply denoted by $r_i(x)$. So the candidates have ranks from 1 (for the least appreciated) to n (for the most appreciated).

The preference aggregation problem consists of defining functions that satisfy "good" properties and that associate with each possible preference profile a relation on X representing at best the collective preference of the voters of this profile. It would be *a priori* desirable that the collective preference be a linear order like the individual preferences. But the difficulty (or even the impossibility) of obtaining such good aggregation functions leads us to relax this objective. Thus one may only require that the collective preference be an order or even be cycle-free since, then, the collective preference can be extended to a linear order (Theorem 2.23, for the latter). However, it was seen above that an aggregation rule as "natural" as Condorcet's majority rule may generate collective preferences containing cycles. So we give a definition of an aggregation function allowing us to include these different cases.

Definition 7.20 Let $\mathcal{M}$ be a set of binary relations on the set X. An *$\mathcal{M}$-preference aggregation function* ($\mathcal{M} - PAF$) is a map F from the set $\mathcal{L}^N$ of all profiles of linear orders to $\mathcal{M}$: for each $\pi \in \mathcal{L}^N$, $F(\pi) \in \mathcal{M}$.

Let us begin by specifying the $\mathcal{M}$-preference aggregation functions first proposed by Borda then by Condorcet.

Let $\pi = (L_1, ..., L_n) \in \mathcal{L}^N$. For each $x \in X$, we write $R(x,\pi) = \sum_{i=1}^{n} r_i(x)$ for the sum of the ranks of x in the orders of profile π.

We define a binary relation $R_B(\pi)$ on X by $yR_B(\pi)x$ if $R(y,\pi) < R(x,\pi)$.

It is obvious that, for each $\pi \in \mathcal{L}^N$, $R_B(\pi)$ is a weak order on X (Condition (6) in Proposition 7.4). The preference aggregation function defined in this way is thus a $\mathcal{W} - PAF$ – where $\mathcal{W}$ is the set of (strict) weak orders defined on X – called *Borda's aggregation function* (or *Borda's rule*).

To show that this aggregation function may contradict the majority principle, consider the following example. There are seven voters with the following preferences on three candidates a, b and c: cba (i.e., $c < b < a$) for three voters, bac for two others, and acb for the last two, i.e., the associated preference profile is $\pi = (cba : 3, bac : 2, acb : 2)$. The reader can check that Borda's weak order is then the linear order cba, thus with a as the winner. Yet c is last for Borda's rule, although he is preferred to a by four voters against three.

Exercise 7.10 shows some other types of "voting paradoxes" likely to be obtained with Borda's rule: for instance, a change of voters' preferences on a single candidate may completely reverse the collective preference.

We now define the two forms of Condorcet's majority rule.

Definition 7.21 *Condorcet's preference aggregation function* (respectively, *weak function*) maps each profile $\pi \in \mathcal{L}^N$ to its *majority relation* $R_{MAJ}(\pi)$ (respectively, to its *weak majority relation* $R_{WMAJ}(\pi)$). These two relations are defined as follows: for all $x, y \in X$,

$$yR_{MAJ}(\pi)x \ \text{ if } \ n_\pi(y,x) > \frac{n}{2}$$

and

$$yR_{WMAJ}(\pi)x \text{ if } n_\pi(y,x) \geq \frac{n}{2}$$

Then, for instance, x is preferred to y in the majority relation if the number of voters who prefer x to y is strictly greater than $\frac{n}{2}$.

For the sequel, it is useful to give another formulation of these definitions by making explicit the majority notions that they use: a subset S of the set N of voters is a *(strict) majority* if $|S| > \frac{n}{2}$, and a *weak majority* if $|S| \geq \frac{n}{2}$. With $\mathcal{F}_{MAJ} = \{$majorities of $N\}$ and $\mathcal{F}_{WMAJ} = \{$weak majorities of $N\}$, we can then write

$$yR_{MAJ}(\pi)x \text{ if } N_\pi(y,x) \in \mathcal{F}_{MAJ} \quad \text{and} \quad yR_{WMAJ}(\pi)x \text{ if } N_\pi(y,x) \in \mathcal{F}_{WMAJ}$$

The following proposition, the proof of which is the purpose of Exercise 7.11, gives the codomains $\mathcal{M}$ of the two Condorcet's preference aggregation functions.

Proposition 7.22 *1. For each preference profile $\pi \in \mathcal{L}^N$, $R_{MAJ}(\pi)$ is a reflexive and antisymmetric binary relation, and $R_{WMAJ}(\pi)$ is a total relation.*

2. *For each reflexive and antisymmetric (respectively, total) binary relation R defined on X, there exists a set N of voters of even size and a profile π of $\mathcal{L}^N$ such that $R_{MAJ}(\pi) = R$ (respectively, $R_{WMAJ}(\pi) = R$).*
3. *For each set N of odd size and each profile $\pi \in \mathcal{L}^N$, $R_{MAJ}(\pi) = R_{WMAJ}(\pi)$ is a tournament relation and, for each tournament R defined on X, there exists a set N of voters of even size and a profile $\pi \in \mathcal{L}^N$ such that $R_{MAJ}(\pi) = R$.*

Thus, it appears that the use of Condorcet's majority rules may generate any reflexive and antisymmetric, or total, binary relation as the collective preference, which contradicts the wish for a good aggregation rule. This is illustrated in Figure 7.3, where the preference profile is that in Example 7.19. The majority ordered pairs are then those supported by at least four voters and, according to Table 7.3, the obtained majority relation is the non-transitive tournament given in Figure 7.3. This example will be continued in Exercise 7.10.

In order to try to palliate the problems raised by Condorcet's majority rule, we define a much broader class of majority functions, associated with the families called *federations*.

Definition 7.23 1. A *federation* on the set N is a family $\mathcal{F}$ of subsets of N such that, for all $T \subseteq N$ and $S \in \mathcal{F}$, $S \subseteq T$ implies $T \in \mathcal{F}$.

2. Let $\mathcal{F}$ be a federation on N and $\pi \in \mathcal{L}^N$. The *collective preference relation* $R_{\mathcal{F}}(\pi)$ associated with $\mathcal{F}$ and π is defined by $yR_{\mathcal{F}}(\pi)x$ if $N_\pi(y,x) \in \mathcal{F}$.
3. The *(generalized) majority function* associated with the federation $\mathcal{F}$ on N is the preference aggregation function associating with each profile π of $\mathcal{L}^N$ the relation $R_{\mathcal{F}}(\pi)$; it is denoted by $F_{\mathcal{F}}$.

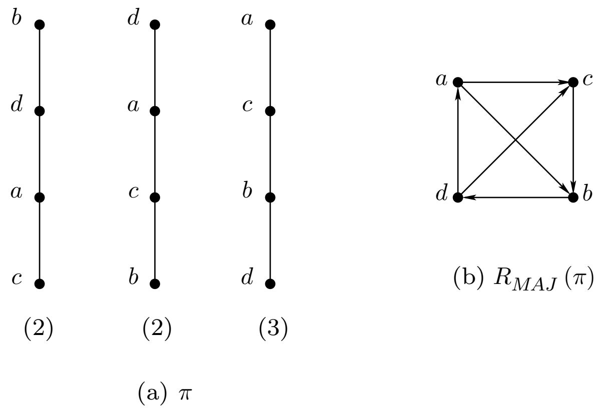


Figure 7.3 (a) A profile of seven voters on four candidates, the majority tournament (b) of which is not transitive.

Observe that a federation is nothing but an upset of the ordered set $\underline{2}^N$ of subsets of N ordered by inclusion. In the literature a federation has also been called a "family of generalized majorities" or a "simple game" and the members of the latter are often called "winning coalitions." Then, Item 2 above is written: x is preferred to y if the voters of profile π preferring x to y form a winning coalition.

We give examples of (generalized) majority functions where the federations used are the so-called filters and ultrafilters. These examples may seem very particular but we will see that they are those which appear in the Arrowian theorems given further in the section.

Definition 7.24 A family $\mathcal{F}$ of subsets of N is a *filter* on N if it is a federation which is $\cap$-stable (i.e., such that $S, T \in \mathcal{F}$ implies $S \cap T \in \mathcal{F}$) and different from the set $P(N)$ of subsets of N. A filter is an *ultrafilter* if it is a maximal filter for the inclusion order on filters.

We observe immediately that a filter may contain neither the empty set nor two disjoint subsets. Exercise 7.12 allows us to show other well-known results (on a finite set): each filter has the form $\mathcal{F}_V = \{S \subseteq N : V \subseteq S\}$, where V is a non-empty subset of N. We say that $\mathcal{F}_V$ is the *filter of basis* V (observe that $\mathcal{F}_V$ is the principal upset $[V)$ in $\underline{2}^N$). Ultrafilters on N are nothing but the n filters of basis i, for $i \in N$; such a filter, denoted by $\mathcal{F}_i$, is thus the family of all subsets of N containing the voter i. These notions are illustrated in Figure 7.4.

In the proof of the further Proposition 7.29 which is a preliminary result to Arrow's Theorem for linear orders (Theorem 7.32), it will be profitable to use a concise characterization of ultrafilters requiring the below notion of the Nakamura number. We keep the following notation: for a family $\mathcal{F}$ of subsets of N, $\bigcap\mathcal{F}$ denotes the intersection $\bigcap\{S \subseteq N : S \in \mathcal{F}\}$ of all subsets belonging to $\mathcal{F}$.

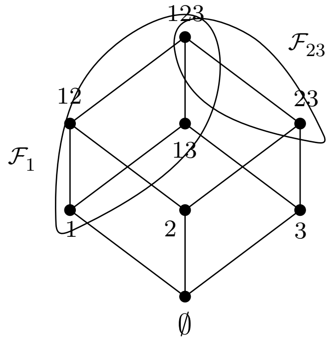


Figure 7.4 The filter $\mathcal{F}_{23}$ and the ultrafilter $\mathcal{F}_1$ on $N = \{1,2,3\}$, represented in the lattice $\underline{2}^N$.

Definition 7.25 Let $\mathcal{F}$ be a family of subsets of N. The *Nakamura number* of $\mathcal{F}$, denoted by $\nu(\mathcal{F})$, is the number defined by:

$$\nu(\mathcal{F}) = \begin{cases} \min\{|\mathcal{F}'|, \mathcal{F}' \subseteq \mathcal{F} \textit{ and } \bigcap \mathcal{F}' = \emptyset\} & \textit{if } \bigcap \mathcal{F} = \emptyset, \\ +\infty & \textit{if not}. \end{cases}$$

When the intersection $\bigcap \mathcal{F}$ of the family $\mathcal{F}$ is empty, $\nu(\mathcal{F})$ is then the minimum size of a sub-family of $\mathcal{F}$ the intersection of which is empty. So for example, if $\nu(\mathcal{F}) > 3$, $\mathcal{F}$ does not contain three subsets with an empty intersection. Observe that $\nu(\mathcal{F}) = 1$ means that $\emptyset \in \mathcal{F}$.

Using the Nakamura number we obtain the following characterization of ultrafilters, allowing a simple proof of Proposition 7.29.

Lemma 7.26 *Let $\mathcal{F}$ be a family of subsets of N. The following conditions are equivalent:*

1. *$\mathcal{F}$ is an ultrafilter.*
2. *$\nu(\mathcal{F}) > 3$ and, for each $S \subseteq N$, $S \notin \mathcal{F}$ implies $N \setminus S \in \mathcal{F}$.*

Proof If $\mathcal{F}$ is an ultrafilter, it is equal to $\mathcal{F}_i$ for some $i \in N$. Then $\nu(\mathcal{F}) = +\infty$ and the second property is immediate.

For the converse, we first show that $\mathcal{F}$ is a filter. Indeed, $\nu(\mathcal{F}) > 3$ first implies that $\mathcal{F}$ does not contain the empty set. If there exist $S \in \mathcal{F}$ and $T \supset S$ with $T \notin \mathcal{F}$, we have $N \setminus T \in \mathcal{F}$, whence $S \cap (N \setminus T) = \emptyset$, a contradiction with $\nu(\mathcal{F}) > 3$. Finally if $S, T \in \mathcal{F}$ and $S \cap T \notin \mathcal{F}$, then $N \setminus (S \cap T) \in \mathcal{F}$, whence $S \cap T \cap (N \setminus (S \cap T)) = \emptyset$, and the same contradiction. The family $\mathcal{F}$ is thus a filter. Assume that it is not a maximal filter, i.e., that there exists a filter $\mathcal{G}$ on N such that $\mathcal{G} \supset \mathcal{F}$. So, there exists $S \subseteq N$ such that $S \in \mathcal{G}$ and $S \notin \mathcal{F}$. But then we have $N \setminus S \in \mathcal{F}$, whence S and $N \setminus S$ are in $\mathcal{G}$, which is impossible. □

We now define the preference aggregation functions associated with a filter or an ultrafilter.

Definition 7.27 A preference aggregation function F is an $\cap$-*projection* (respectively, a *projection*) if there exists a filter (respectively, an ultrafilter) $\mathcal{F}$ on N such that $F = F_{\mathcal{F}}$.

When F is an $\cap$-projection, $\mathcal{F} = \mathcal{F}_V$ is a filter of basis V. We then obtain $F_{\mathcal{F}}(\pi) = \bigcap\{L_i : i \in V\}$ for each $\pi \in \mathcal{L}^N$, and the codomain of this *PAF* is the set $\mathcal{O} = \mathcal{O}_X$ of all orders on X.

When F is a projection, $\mathcal{F} = \mathcal{F}_i$ is an ultrafilter of basis i. We then obtain $F_{\mathcal{F}}(\pi) = L_i$ for each $\pi \in \mathcal{L}^N$. In other words, F is the ith projection of $\mathcal{L}^N$ and its codomain is the set $\mathcal{L} = \mathcal{L}_X$ of all linear orders on X.

Remark 7.28 In social choice theory, an $\cap$-projection is called an "oligarchic" function since, then, the collective preference is determined only by the voters belonging to the "oligarchy" V. The particular case where $V = \{i\}$ – and where F is a projection – is called a "dictatorial" function (since, then, the collective preference is that of the "dictator" i).

The proof of the next results uses preference profiles for which the voters' preferences are fixed on some candidates and arbitrary on others. We adopt the following type of notation for such profiles: $\pi = (xyz : S, yzx : T, zyx : U)$, where (S,T,U) is a partition of N and where the preferences of all voters of S (respectively, of T, of U) on the three candidates x, y, and z are xyz (respectively, yzx, zyx), the preferences on the other candidates being arbitrary.

Considering the *PAFs* $F_{\mathcal{F}}$ for which the collective preference is always a linear order – i.e., the $\mathcal{L}$-*PAFs* $F_{\mathcal{F}}$ – we obtain the following result:

Proposition 7.29 *The PAF $F_{\mathcal{F}}$ associated with the federation $\mathcal{F}$ on N is an $\mathcal{L}$-PAF if and only if $\mathcal{F}$ is an ultrafilter, i.e., if and only if $F_{\mathcal{F}}$ is a projection.*

Proof Let us show the necessary condition. To prove that $\mathcal{F}$ is an ultrafilter it is sufficient to show that $\mathcal{F}$ satisfies the two conditions in Lemma 7.26. We begin by proving that $\nu(\mathcal{F}) > 3$; if not, $\nu(\mathcal{F}) \leq 3$. First assume $\nu(\mathcal{F}) = 3$. So there exist $S,T,U \in \mathcal{F}$ such that $S \cap T \cap U = \emptyset$. Then we consider the following profile: $\pi = (xyz : S \cap T, yzx : T \setminus S, zyx : N \setminus T)$. For this profile $N_\pi(x,y) \supseteq S$, $N_\pi(y,z) = T$, and $N_\pi(z,x) \supseteq U$. So (since $\mathcal{F}$ is a federation) $xR_{\mathcal{F}}(\pi)yR_{\mathcal{F}}(\pi)zR_{\mathcal{F}}(\pi)x$ holds, a contradiction with $R_{\mathcal{F}}(\pi)$ linear order. If $\nu(\mathcal{F}) = 2$ there exist $S,T \in \mathcal{F}$ such that $S \cap T = \emptyset$ and, for the profile $\pi = (xy : N \setminus T, yx : T)$, $xR_{\mathcal{F}}(\pi)yR_{\mathcal{F}}(\pi)x$ holds, another contradiction. Finally observe that $\nu(\mathcal{F}) \neq 1$ since $\nu(\mathcal{F}) = 1$ means that $\emptyset \in \mathcal{F}$ and so we would obtain that $R_{\mathcal{F}}(\pi)$ is equal to the complete relation X^2 for any profile π.

Then we prove that $\mathcal{F}$ satisfies the second condition of Lemma 7.26. If not, there exists $S \subseteq N$ such that $S \notin \mathcal{F}$ and $N \setminus S \notin \mathcal{F}$. Then consider $x,y \in X$ and the profile

$\pi = (xy : S, yx : N \setminus S)$. Then $(x,y) \notin R_{\mathcal{F}}(\pi)$ and $(y,x) \notin R_{\mathcal{F}}(\pi)$, which is impossible since $R_{\mathcal{F}}(\pi) \in \mathcal{L}$.

The sufficient condition is obvious. □

Corollary 7.30 *There do not exist non-dictatorial generalized majority functions that are $\mathcal{L}$-PAF.*

The above results allow us to give simple proofs of Arrowian theorems. Arrow's approach consisted of axiomatically defining the "good" preference aggregation functions, then seeking the functions satisfying these axioms. We follow this approach in the case where the individual preferences are linear orders (see Section 7.6.2, Further topics and references, for the case of total preorders considered by Arrow). Afterwards we consider $\mathcal{M}$-*PAFs* from $\mathcal{L}^N$ to $\mathcal{M}$, where the set $\mathcal{M}$ of possible collective preference relations will be specified in each case. We require that the aggregation function satisfies the two properties of Pareto and independence defined below:

Definition 7.31 Let F be an aggregation function from $\mathcal{L}^N$ to a set $\mathcal{M}$ of relations.

- F satisfies *Pareto's property* (or simply *Pareto*) if, for each $\pi \in \mathcal{L}^N$ and all $x,y \in X$, $N_\pi(y,x) = N$ implies $yF(\pi)x$. It is also said that F is *Paretian* or that it satisfies the *unanimity property*.
- F satisfies the *independence property* if, for all $\pi, \pi' \in \mathcal{L}^N$ and all $x,y \in X$, $N_\pi(y,x) = N_{\pi'}(y,x)$ implies $[yF(\pi)x \iff yF(\pi')x]$. It is also said that F is *independent*.

The first of these properties simply means that, if all voters prefer candidate x to candidate y, this unanimous preference must be a collective preference, which is the least one can require. The aggregation rules defined above by a federation as well as Borda's rule satisfy this Paretian property. As for the independence property, it means that the collective preference between x and y depends only on the individual preferences between these two candidates and not on the preferences on other candidates, which seems a reasonable demand. We observe that this requirement is clearly satisfied by the *PAFs* defined by a federation. However, it is not satisfied by Borda's rule, as shown in Exercise 7.10. Yet, if we add to these two apparently justified requests the demand that the collective preference be either a linear order or simply an order, we obtain dictatorial or oligarchic results.

Theorem 7.32 (Arrow for linear orders) *Let N be a set of n voters, $\mathcal{L}^N$ the set of their preference profiles, and $F : \mathcal{L}^N \mapsto \mathcal{L}$ an $\mathcal{L}$-preference aggregation function. Then F is independent and Paretian if and only if F is a projection.*

Proof The sufficient condition is obvious.

The proof of the necessary condition is obtained by showing that, if F is independent and Paretian, there exists a federation $\mathcal{F}$ on N such that $F = F_{\mathcal{F}}$. Proposition 7.29

allows us to conclude, since $\mathcal{F}$ is then an ultrafilter. To simplify the notations, we denote by L the linear order $F(\pi)$.

To obtain the federation $\mathcal{F}$, we introduce the notion of a decisive set. Let (y,x) be an ordered pair of different elements of X. A subset S of N is a (y,x)-*decisive set* (for the function F) if, for each $\pi \in \mathcal{L}^N$ such that $N_\pi(y,x) = S$, yLx holds. Observe that, F being independent, it is enough to have a profile π with $N_\pi(y,x) = S$ and yLx in order that S be (y,x)-decisive. Denoting by $\mathcal{F}(y,x)$ the family of (y,x)-decisive sets (for F), we then have yLx if and only if $N_\pi(y,x) \in \mathcal{F}(y,x)$. Observe that $\mathcal{F}(y,x)$ is never empty since, by Pareto's property, it contains N.

We say that a subset S of N is a *decisive set* (*for* F) if S is (y,x)-decisive for each ordered pair (y,x) of different elements of X.

We are going to show that, if S is a (y,x)-decisive set for at least one ordered pair (y,x), it is a decisive set. First show that, if S is (y,x)-decisive, it is (z,x)-decisive for each $z \neq x,y$, i.e., that $\mathcal{F}(y,x) \subseteq \mathcal{F}(z,x)$. Let $S \subset N$ with $S \in \mathcal{F}(y,x)$, and z different from x and y. Consider the profile $\pi = (zyx : S, xzy : N \setminus S)$. The set S is (y,x)-decisive and F is Paretian so yLx and zLy hold. By transitivity of the linear order L, zLx holds. But since $N_\pi(z,x) = S$, we obtain $S \in \mathcal{F}(z,x)$. Likewise, the reader can show that, for each $z \neq x,y$, $\mathcal{F}(y,x) \subseteq \mathcal{F}(y,z)$, and then the identity of all families of (y,x)-decisive sets for all the ordered pairs (y,x) of different elements of X; they are thus all equal to the family $\mathcal{F}$ of decisive sets.

So we may write that yLx if and only if $N_\pi(y,x) \in \mathcal{F}$, which means that $F = F_{\mathcal{F}}$. To use Proposition 7.29, it only remains to prove that $\mathcal{F}$ is a federation. Let $S \in \mathcal{F}$, $N \supseteq T \supset S$, and $\pi = (zyx : S, yzx : T \setminus S, yxz : N \setminus T)$. Since F is Paretian and the set S is decisive, yLx and zLy hold. Since L is transitive, zLx holds and, since $N_\pi(z,x) = T$, $T \in \mathcal{F}$ follows. □

We now broaden the domain of the possible collective preferences by assuming that it is equal to the set $\mathcal{O} = \mathcal{O}_X$ of all orders on X. The theorem below shows that the "dictatorship" then becomes an "oligarchy"; that is, an $\cap$-projection (Definition 7.27).

Theorem 7.33 *Let N be a set of n voters, $\mathcal{O}^N$ the set of their preference profiles, and $F : \mathcal{O}^N \mapsto \mathcal{O}$ an $\mathcal{O}$-preference aggregation function. F is independent and Paretian if and only if F is an $\cap$-projection.*

Proof The sufficient condition is obvious.

The proof of the necessary condition is obtained by showing that, in this case, there exists a filter $\mathcal{F}$ on N such that $F = F_{\mathcal{F}}$ and thus a subset V of N such that, for every profile $\pi, F(\pi) = \bigcap\{L_i : i \in V\}$.

We define the notions of decisive sets as in Theorem 7.32 and we first observe that the proof given there of the identity of all the families of (y,x)-decisive sets only uses the transitivity of $F(\pi)$. So it remains valid here and we may write $F = F_{\mathcal{F}}$. It is the same for the proof that $\mathcal{F}$ is a federation. The empty set cannot belong to $\mathcal{F}$ (since, by Nakamura's Theorem, $\nu(\mathcal{F}) > |N| \geq 3$). To show that the federation $\mathcal{F}$ is a filter, it remains to show that it is $\cap$-stable. Let $S,T \in \mathcal{F}$ with $S \not\subseteq T$ and $S \not\supseteq T$; $\nu(\mathcal{F}) > 3$ implies $S \cap T \neq \emptyset$. Let $\pi = (zyx : S \cap T, yxz : S \setminus T, xzy : T \setminus S, xyz : N \setminus (S \cup T))$. The sets S and T being decisive, yOx (with $O = F(\pi)$) and zOy hold. The transitivity of O implies zOx. Since $N_\pi(z,x) = S \cap T$, we obtain $S \cap T \in \mathcal{F}$ as required. □

Remark 7.34 The above two theorems can be stated as impossibility results by adding non-dictatorship or non-oligarchy to the conditions on the aggregation function. One may wonder what an independent and Paretian aggregation function becomes when the admissible domain of collective preferences is a particular class of orders containing the linear orders (for instance the class of weak orders, of semiorders or of interval orders). The answer is given for these three classes in Exercise 7.14.

To end this section we go back to the *PAF*s which are majority functions $F_{\mathcal{F}}$ and introduce a minimal hypothesis in order that a such majority function be considered as satisfactory, namely that the collective preference relations obtained by this function be cycle-free.

Nakamura's Theorem uses the Nakamura number $\nu(\mathcal{F})$ (Definition 7.25) to characterize the families for which $F_{\mathcal{F}}(\pi)$ is always cycle-free. Denoting by $\mathcal{A}$ the set of all cycle-free relations defined on X, these aggregation functions are then $\mathcal{A}$-*PAFs*.

Theorem 7.35 (Nakamura, 1975) *Let $\mathcal{F}$ be a federation on N, different from $\underline{2}^N$, and $F_{\mathcal{F}}$ the PAF which associates the relation $R_{\mathcal{F}}(\pi)$ with each profile π of $\mathcal{L}^N$. Then $F_{\mathcal{F}}$ is an $\mathcal{A}$-PAF if and only if $\nu(\mathcal{F}) > |N|$.*

In other terms, $F_{\mathcal{F}}$ associates a cycle-free relation with each preference profile π if and only if the Nakamura number of $\mathcal{F}$ is strictly greater than the size of N. The not too difficult proof of this result is the purpose of Exercise 7.13. Observe that the first part of the proof of Proposition 7.29 is a corollary of Nakamura's Theorem (since in this section, we have assumed $|N| \geq 3$).

7.3 The roles of orders in cluster analysis

Cluster analysis (or classification) is a branch of data analysis the concern of which is the determination of *classes* (or *clusters*) in a set E of objects to be classified. A cluster $C \subseteq E$ must be understood as a set of objects grouped with respect to common properties or to some kind of proximity between them. For instance, Example 3.48 illustrates *Galois classification*, where each concept (F,G) corresponds to a class F of elements of E sharing the attributes in G. We will go back to this approach in Section 7.4, which is especially devoted to relational databases and knowledge extraction. Nevertheless, the Galois approach is not the most usual one in cluster analysis. A more frequently considered tool consists of constructing a function d on E, called a *dissimilarity*, measuring the degree of dissemblance between the elements to be classified. Such functions are more general than metrics as may be observed in the definition below.

Definition 7.36 A *dissimilarity* on a set E is a function d from E^2 to the set $\mathbb{R}^+$ of non-negative real numbers satisfying the following properties for all $e, e' \in E$:

- $d(e,e') = 0$ if and only if $e = e'$.
- $d(e,e') = d(e',e)$.

As said above, the meaning of a dissimilarity is that the more similar the objects e and e' the smaller the value $d(e,e')$. From a dissimilarity (or sometimes from data of another kind) a classification system on E – that is, a family $\mathcal{C}$ of subsets of E the elements of which are the wanted clusters – can be derived, w.r.t. the two following goals:

1. The objects inside a cluster have to be more similar to one another than to those outside the cluster.
2. The family $\mathcal{C}$ belongs to a particular type of classification model which guarantees simplicity and clarity of the classification process output.

Simultaneously reaching these aims can be done only approximately. The presentation of the numerous available methods (and algorithms) is not the purpose of this section. The reader can refer to various manuals such as those of Arabie *et al*. (1996), Mirkin (1997) or Everitt *et al*. (2011).

Here, we tackle some ordinal aspects about the description of several types of usual classification model on the one hand, and the structure of the set $\mathcal{M}$ of all the solutions of a classification problem on the other hand. Indeed the ordinal approach is an important tool for "derived" problems such as classification comparison and aggregation. So, we first present several types of model frequently considered in classification: partitions, equivalences, hierarchies, valued hierarchies, dendrograms, and ultrametrics, but also Moore families. Observing that the corresponding sets of models are meet-semilattices, and frequently lattices, we then tackle comparison and aggregation problems in such structures. A meet-semilattice L is endowed with a particular metric related to the representation of any element x of L by the set J_x of the join-irreducibles less than or equal to x (Chapter 3, Proposition 3.11 and Corollary 3.12). The metric aggregation method consists of searching for an element, called a *median*, minimizing the sum of its distances to a k-tuple of some given elements of L (or in searching for all medians when there is more than one). Properties of medians linked with (the lattice form of) the majority rule defined in the previous section are obtained and we define the so-called *median semilattices*, where medians are actually obtained by this rule. The section ends with a presentation of how these rules apply to classification models.

Let us begin with the latter models. Recall that a partition $\mathbf{P} = \{C_1, C_2, ..., C_p\}$ of E (Chapter 1, Example 1.14) is a family of pairwise disjoint subsets of E (called the

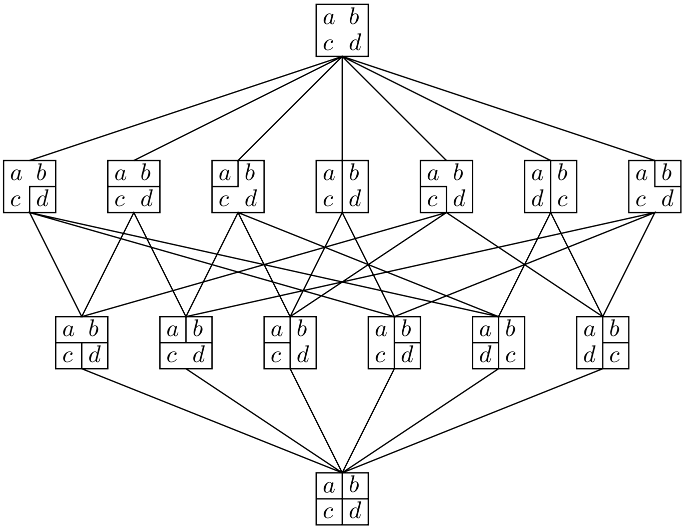


Figure 7.5 The lattice $\mathcal{P}_{\{a,b,c,d\}}$ of partitions of $\{a,b,c,d\}$ endowed with the refinement order.

classes of **P**), the union of which is E: so each element of E belongs to a unique class of **P**. It is well-known that a partition **P** of E is equivalent to an equivalence (that is, a reflexive, symmetric, and transitive) relation $R_{\mathbf{P}}$ on E. The one-to-one correspondence between **P** and $R_{\mathbf{P}}$ is given by $(e,e') \in R_{\mathbf{P}}$ if and only if e and e' belong to the same class of the partition **P**. The set $\mathcal{E}_E$ of all equivalences on E contains the total equivalence E^2 and is intersection-stable (the reader can check that). So, it is a Moore family (Definition 3.29) on E^2 and a lattice for the inclusion order. The corresponding closure is nothing but the reflexo-transitive closure which associates with any symmetric relation on E the minimum equivalence including it. Let us consider the refinement order on the set $\mathcal{P}_E$ of partitions of E which has been defined in Example 1.14. It is easy to see that, for all $\mathbf{P},\mathbf{P}' \in \mathcal{P}_E$, **P** is finer than $\mathbf{P}'$ (that is, every class of **P** is included in a class of $\mathbf{P}'$) if and only if $R_{\mathbf{P}} \subseteq R_{\mathbf{P}'}$. So the set $\mathcal{P}_E$, endowed with the refinement order, is a lattice. Its minimum is the finest partition $\mathbf{P_0}$ (so with $|E|$ classes) and its maximum is the coarsest partition $\mathbf{P_1}$ the unique class of which is E (see Exercises 7.15 and 7.16 for some properties of this lattice). Figure 7.5 shows the diagram of the lattice $\mathcal{P}_{\{a,b,c,d\}}$ endowed with the refinement order.

The other classic models presented here correspond with several types of *classification trees*. In *hierarchical* classification, the program outputs are of the type represented in Figure 7.6; the information on the set $E = \{1,2,3,4,5,6,7,8,9,10\}$ provided by such a tree may be read at several levels, corresponding with several classification models that we are going to precisely define.

The minimal elements of the tree in Figure 7.6(a) are the "singletons" (1-element subsets) of E. Each of its horizontal solid lines corresponds with the subset H of E

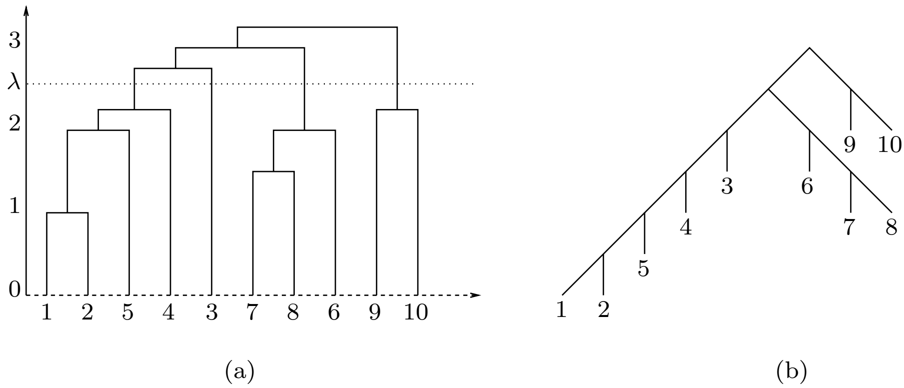


Figure 7.6 Classification trees corresponding to a hierarchy.

containing all the elements of E placed “below” this line. Thus, the tree in this figure represents the family of subsets (classes)

$$\{\{1\},\{1,2\},\{1,2,5\},\{1,2,4,5\},\{1,2,3,4,5\},\{1,2,3,4,5,6,7,8\},\{2\},\{3\},\{4\},\{5\},$$

$$\{6\},\{6,7,8\},\{7\},\{7,8\},\{8\},\{9\},\{9,10\},\{10\},E\}.$$

Observe that two classes are either disjoint or ordered by inclusion. Indeed, such a tree is equivalent to a family H of subsets of E of the type defined as follows:

Definition 7.37 A *hierarchy* on a set E is a family $\mathcal{H}$ of subsets of E satisfying the following properties:

1. $E \in \mathcal{H}$, $\emptyset \notin \mathcal{H}$, $\{e\} \in \mathcal{H}$ for any $e \in E$.
2. For all $H, H' \in \mathcal{H}$, $H \cap H' \in \{\emptyset, H, H'\}$.

The set of all hierarchies on E is denoted by $\mathbb{H}$.

Two hierarchies on E can differ only on classes of size between 2 and $n-1$; the other classes (that is, E and the singletons) are said to be *trivial*. Property (2) above implies that, for any non-empty subset A of E, the set of the classes of $\mathcal{H}$ including A is linearly ordered by inclusion (the reader can check that). So it has a minimum class, denoted by H_A. The following properties then follow:

- The ordered set $(\mathcal{H}, \subseteq)$ is a join-semilattice with, for all $H', H“ \in \mathcal{H}$, $H' \vee H'' = H_{(H' \cup H“)}$.
- The ordered set $(\mathcal{H}, \subseteq)$ is a tree-ordered set (Chapter 2, Definition 2.12).

So, hierarchies are tree join-semilattices and they form the first and basic model of classification trees. Exercise 2.5 in Chapter 2 enumerates a number of properties which dually (since the exercise concerns tree meet-semilattices) apply to hierarchies. For instance observing that, for all $e, e' \in E$, $\{e\} \vee \{e'\} = H_{\{e,e'\}}$ we obtain:

Proposition 7.38 *Let $\mathcal{H}$ be a hierarchy on E. For all $e, e', e'' \in E$, the inequality $|\{H_{\{e,e'\}}, H_{\{e,e''\}}, H_{\{e',e''\}}\}| < 3$ holds.*

Figure 7.6(b) gives the diagram of the hierarchy in Figure 7.6(a). It is enough to give labels to the singletons since any other vertex corresponds with the set of singletons below it. Hierarchical classification methods frequently provide, in addition to a hierarchy $\mathcal{H}$, a height function[1] ι which is understood as measuring the class cohesion: the lower the height of a class, the more coherent the class.

Definition 7.39 A *valued hierarchy* on a set E is a pair $(\mathcal{H}, \iota)$ where:

- $\mathcal{H}$ is a hierarchy on E.
- ι is a height function, i.e., a strictly isotone map from $\mathcal{H}$ to $\mathbb{R}^+$ satisfying $\iota(\{e\}) = 0$ for any $e \in E$.

Although they are a standard classification model, these trees have many denominations in the literature (ranked trees, valued trees, dendrograms, numerically stratified clusterings, etc.). With the scale on its left, Figure 7.6(a) gives in fact a representation of a valued hierarchy. Cutting the tree with a horizontal line of ordinate λ as shown with a dotted line in this figure, the classes of the hierarchy lying just below or on this line form a partition $f(\lambda)$ of E. In the figure, $f(\lambda) = \{\{1,2,4,5\},\{3\},\{6,7,8\},\{9,10\}\}$. Since $\lambda \leq \lambda'$ implies $f(\lambda) \leq f(\lambda')$ – the second inequality using the refinement order on partitions of E – we have in fact defined an isotone map f from $\mathbb{R}^+$ to the lattice $\mathcal{P}_E$ of partitions of E. The image by f of $\mathbb{R}^+$ in $\mathcal{P}_E$ is an extended chain; that is, containing the finest (with n classes) and the coarsest (with one class) partitions. Moreover, for all $e, e' \in E$, $\iota(H_{\{e,e'\}})$ is the minimum of the values $\lambda \in \mathbb{R}^+$ such that e and e' are in the same class of $f(\lambda)$. These observations lead to the following definitions:

Definition 7.40 Let E be a set and L a subset of $\mathbb{R}^+$ containing the 0 number. An *L-dendrogram* on E is an isotone map f from L to the lattice $\mathcal{P}_E$ of partitions of E satisfying the following property (DG):

(DG) for all $e, e' \in E$, the converse image

$$f^{-1}(\{\mathbf{P} \in \mathcal{P}_E : e \textit{ and } e' \textit{ are in the same class of } \mathbf{P}\})$$

has a minimum in L.

Especially, an $\mathbb{R}^+$-dendrogram on E is called a *dendrogram* on E.

There are correspondences between valued hierarchies, dendrograms, and also ultrametrics, that we define below:

[1] *Note* The height function ι mentioned here must be distinguished from the height of an element defined on any ordered set on page 44.

Definition 7.41 An *ultrametric* on E is a dissimilarity u on E satisfying the following inequality (U):

$$\text{For all } e,e',e'' \in E, u(e,e') \leq \max(u(e,e''),u(e',e'')) \qquad (U)$$

Note that an ultrametric is obviously a distance.

We derive a dissimilarity u on E from a valued hierarchy $(\mathcal{H},\iota)$ by setting $u(e,e') = \iota(H_{\{e,e'\}})$ for all $e,e' \in E$. As a consequence of Proposition 7.38, the dissimilarity u is an ultrametric on E.

In Exercises 7.18 to 7.20, we consider the finite case by taking the chain $\underline{k} = \{0 < 1 < ... < k-1\}$ as L. Then Condition (DG) is satisfied by any isotone map from $\underline{k}$ to $\mathcal{P}_E$. These exercises consist of proving, on the one hand, that $\underline{k}$-dendrograms on E are exactly residual maps (Definition 3.35) from $\underline{k}$ to $\mathcal{P}_E$ and, on the other hand, that the following three sets are in pairwise one-to-one correspondence:

- the set of valued hierarchies on E where the height values belong to the chain $\underline{k}$,
- the set of $\underline{k}$-dendrograms on E,
- the set of ultrametrics on E taking their values in $\underline{k}$.

Moreover, the last two of these sets are dual ordered sets of maps for the exponentiation order (Chapter 3, Definition 3.4).

Though the above-described classification models are the most usual ones, many others have been proposed in the literature. It is not possible to review all of them here, but we may however observe that the following conditions may be significant for a classification system $\mathcal{F} \subseteq \mathcal{P}(E)$:

(C1) $E \in \mathcal{F}$.
(C2) $C, C' \in \mathcal{F}$ implies $C \cap C' \in \mathcal{F}$.
(C3) For any $e \in E, \{e\} \in \mathcal{F}$.

Condition (C1) assumes the existence of a "universal" class and (C2) means that, as soon as two classes C and C' are obtained, it is natural to also consider the class $C \cap C'$ of the elements common to C and C'. Put together, Conditions (C1) and (C2) imply that $\mathcal{F}$ is a Moore family on E (Definition 3.29). Adding Condition (C3), which corresponds to the possibility of distinguishing elements of E from one another in $\mathcal{F}$, we also have $\emptyset \in \mathcal{F}$. We observe that neither partitions nor hierarchies directly constitute such families. Nevertheless in Chapter 3 (Example 3.32) we have described how a unique Moore family $\mathcal{F} = \mathbf{m}(\mathcal{A}) = \{\bigcap \mathcal{B} : \mathcal{B} \subseteq \mathcal{A}\}$ can be associated with any family $\mathcal{A}$ of subsets of E. If $\mathbf{P} = \{C_1, C_2, ..., C_p\}$ is a partition of E, then $\mathbf{m}(\mathbf{P}) = \mathbf{P} \cup \{E, \emptyset\}$. If $\mathcal{H}$ is a hierarchy on E, then $\mathbf{m}(\mathcal{H}) = \mathcal{H} \cup \{\emptyset\}$.

Still in Chapter 3 (Proposition 3.46 and Example 3.48) we have considered the case of a set E of objects described with a set E' of binary attributes; that is, the data consisting of a relation $R \subseteq E \times E'$. We have then shown that the Galois lattice of the relation R is isomorphic to a Moore family $\mathcal{F}$ on E and dual of a Moore family $\mathcal{G}$ on E'.

Such examples account for the addition of Moore families to the previous model sets. On the one hand, the former are a natural generalization of the latter. On the other hand, they are the classification systems obtained in Galois classification.

Finally, the sets $\mathcal{M}$ of models that we are going to consider are always ordered sets. The orders are:

- the refinement order for $\mathcal{M} = \mathcal{P}_E$, the set of partitions of E,
- the inclusion for $\mathcal{M} = \mathbb{H}$, the set of hierarchies on E, or $\mathcal{M} = \mathbb{F}$, the set of Moore families on E,
- the exponentiation order if $\mathcal{M}$ is the set of L-dendrograms, or that of ultrametrics on E.

Moreover, the orders on these model sets are lattices or at least meet-semilattices: we have encountered the partition lattice and we will point out in this section that the ordered set $(\mathbb{H}, \subseteq)$ is a meet-semilattice of a particular type. Exercises 7.18 and 7.20 will show how to define lattice orders on the sets of $\underline{k}$-dendrograms and of ultrametrics on E. Finally, the lattice structure of the set $\mathbb{F}$ will be made explicit in Section 7.4. We now use the ordinal structure of $\mathcal{M}$ to tackle two types of problem:

- *Comparison problem*. This is about defining a metric on $\mathcal{M}$ which is easy to compute, even if the size of the set $\mathcal{M}$ exponentially increases with the size of E. For instance, what easily computable distance can be taken to compare two partitions on E?
- *Aggregation problem*. This is about transposing to classification the problems and approaches considered in Section 7.2 in the case of preference aggregation. For instance, what method can be chosen to find the *consensus* of several hierarchies on E?

In the latter case, we consider a profile $\pi = (M_1, M_2, ..., M_v) \in \mathcal{M}^v$ (we say that π is a *profile of* $\mathcal{M}$) and aggregation functions on $\mathcal{M}^v$ which associate with such a profile one or several classifications. This consensus problem appeared in classification in the 1960s about data analysis and phylogenetic reconstruction (see Section 7.6).

In what follows, we make the link between the previous two problems and mainly consider them within the abstract framework of *metric* aggregation in semilattices. The obtained results will then be applied to the various above-mentioned classification lattices or semilattices. So we now consider a meet-semilattice $(L, \leq)$ endowed with a metric (or distance) d.

Definition 7.42 Let $(L, \leq)$ be a meet-semilattice, d a distance on L, $\pi = (x_1, ..., x_i, ..., x_v)$ a profile of L, and x an element of L. The *remoteness* of x from π is the sum $R_d(\pi, x) = \Sigma_{1 \leq i \leq v} d(x_i, x)$. An element x of L is called a (*metric*) *median* of π if its remoteness from π is minimum. The aggregation procedure which associates with any profile $\pi \in L^v$ the set $Med_d(\pi)$ of medians of π for the distance d is called the *median procedure*.

Remark 7.43 The term *median* was used in the different meaning of a particular lattice polynomial after Corollary 5.2 on page 132. Generally, this *lattice* median must be distinguished from the *metric* medians as defined above. In fact, one of the purposes of this section is the recognition of a wide domain where these two notions coincide.

For the choice of an appropriate distance on L we use the meet-coding $x \longmapsto J_x$ from L to $\underline{2}^{J_L}$ defined in Chapter 3 (Proposition 3.6 and Corollary 3.12), where J_L is the set of join-irreducible elements of L and $J_x = \{j \in J_L : j \leq x\}$. Using the results in Exercise 5.12 about the symmetric difference distance on the subsets of a set, we define a distance on L which generalizes the latter and corresponds to usual metrics in various particular cases:

Definition 7.44 Let L be a meet-semilattice. The *symmetric difference distance* on L is the function δ from L^2 to $\mathbb{R}^+$ defined by: for all $x,x' \in L$,

$$\begin{aligned}\delta(x,x') &= |J_x \Delta J_{x'}| \\ &= |\{j \in J : [j \in J_x \textit{ and } j \notin J_{x'}] \textit{ or } [j \notin J_x \textit{ and } j \in J_{x'}]\}| \\ &= |J_x \setminus J_{x'}| + |J_{x'} \setminus J_x| \\ &= |J_x \cup J_{x'}| - |J_x \cap J_{x'}|.\end{aligned}$$

For instance, in the lattice in Figure 7.7, $\delta(g,i) = |\{a,b,g\}|\Delta|\{b,c,i\}| = |\{a,g\}| + |\{c,i\}| = 4$.

By extension of the lattice case, a meet-semilattice L is said to be *distributive* if the lattice $(x]$ is distributive for any $x \in L$. We now show that such a semilattice is ranked and that its distance δ has expressions in terms of ranks or path lengths in the neighborhood graph $Neigh(L)$ of L (Chapter 1, Section 1.1.2):

Proposition 7.45 *Let L be a distributive meet-semilattice, δ the symmetric difference distance on L, and $x,x' \in L$. Then the following properties hold:*

1. *L is ranked with $r(x) = |J_x|$.*
2. *$\delta(x,x') = r(x) + r(x') - 2r(x \wedge x')$.*
3. *$\delta(x,x')$ is the minimum path length between x and x' in the graph $Neigh(L)$.*
4. *If $x \vee x'$ exists, then $\delta(x,x') = r(x \vee x') - r(x \wedge x')$.*

Proof It immediately follows from Item (1) in Proposition 5.14 that such a semilattice is ranked and that the rank $r(x)$ of an element x is $|J_x|$. We then obtain the formula in Item (2) by observing that $|J_x \setminus J_{x'}| = |J_x| - |J_x \cap J_{x'}|$ and, similarly,

$|J_{x'} \setminus J_x| = |J_{x'}| - |J_x \cap J_{x'}|$. In a distributive lattice, the map $x \longmapsto J_x$ is also a join-coding (Item (6) in Theorem 5.1). The equalities $r(x \vee x') = |J_{x \vee x'}| = |J_x \cup J_{x'}|$ and $r(x \wedge x') = |J_{x \wedge x'}| = |J_x \cap J_{x'}|$ then provide (4) when $x \vee x'$ exists. If x and x' are adjacent in the graph *Neigh*(L), for instance if $x \prec x'$ holds, then the sets J_x and $J_{x'}$ differ from exactly one element. So, for arbitrary x and x' the quantity $|J_x \Delta J_{x'}|$ is a lower bound for the length of a path in *Neigh*(L) between x and x'. This bound is sharp since the formula in Item (2) corresponds to the existence of a path with this length going through $x \wedge x'$, and (3) is satisfied. □

Going back to medians, their computation requires us to determine the remoteness $R_\delta(\pi,x)$ of x from π. In Proposition 7.48, we are going to give a simple formula for this remoteness, which involves some new definitions:

Definition 7.46 Let L be a meet-semilattice, $\pi = (x_1,...,x_i,...,x_v) \in L^v$ a profile of L, and write $V = \{1,...,v\}$. With any join-irreducible $j \in J_L$, we associate the following parameters:

- $v_\pi(j) = |\{i \in V : j \leq x_i\}|$.
- $v'_\pi(j) = |\{i \in V : j \not\leq x_i\}|$.
- $w_\pi(j) = v_\pi(j) - v'_\pi(j)$.

The element j is said to be:

- *majority* if $2v_\pi(j) > v$,
- *balanced* if $2v_\pi(j) = v$.

So, when j is majority, it belongs to a strict majority of the join-irreducible representations of the elements of the profile.

When no ambiguity may occur – that is, most of the time – the index π is omitted in the above notations.

Observe that the equalities $v(j) + v'(j) = v$ and $w(j) = 2v(j) - v$ are immediate. They imply that j is majority (respectively, balanced) if $w(j) > 0$ (respectively, $w(j) = 0$).

Example 7.47 Consider the (distributive) lattice L in Figure 7.7 and the profile $\pi = (a,a,e,f,g,l)$ of L. We have $J_L = \{a,b,c,g,i\}$. The values of functions v and w are given in Table 7.4. The unique majority join-irreducible is a, whereas b and c are balanced.

When L is an arbitrary meet-semilattice, there is a simple formula for the remoteness $R_\delta(\pi,x)$ of an element x of L from a profile π of L.

Proposition 7.48 *Let L be a meet-semilattice. For any profile $\pi = (x_1,...,x_i,...,x_v) \in L^v$ and for any $x \in L$, the following holds:*

$$R_\delta(\pi,x) = \Sigma_{1 \leq i \leq v} |J_{x_i}| - \Sigma_{j \in J_x} w(j).$$

Table 7.4 Functions v and w on the join-irreducible elements of the distributive lattice in Figure 7.7

$j \in J_L$	a	b	c	g	i
$v(j)$	5	3	3	1	1
$w(j)$	4	0	0	−4	−4

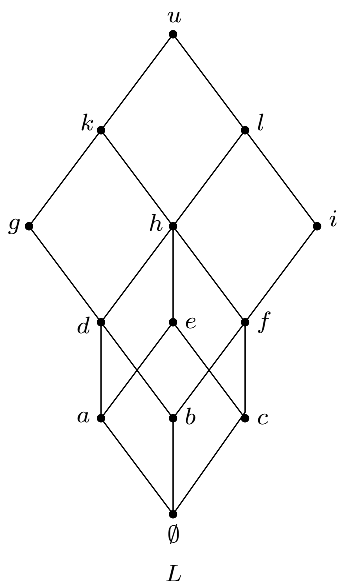


Figure 7.7 Example 7.47.

Proof From the definition of $R_\delta(\pi,x)$ we have $R_\delta(\pi,x) = \Sigma_{1\leq i\leq v}\delta(x_i,x) = \Sigma_{1\leq i\leq v}|J_{x_i}\Delta J_x|$. Let c_i be the characteristic function of $J_{x_i}\Delta J_x$ defined by:

for any $j \in J_L$,

$$c_i(j) = \begin{cases} 1 & \text{if } j \in J_{x_i}\Delta J_x, \\ 0 & \text{if not.} \end{cases}$$

We then have $R_\delta(\pi,x) = \Sigma_{1\leq i\leq v}\Sigma_{j\in J_L}c_i(j) = \Sigma_{j\in J_L}\Sigma_{1\leq i\leq v}c_i(j)$. Partitioning J_L into J_x and its complementary set, we find:

$$R_\delta(\pi,x) = \Sigma_{j\notin J_x}\Sigma_{1\leq i\leq v}c_i(j) + \Sigma_{j\in J_x}\Sigma_{1\leq i\leq v}c_i(j) = \Sigma_{j\notin J_x}v(j) + \Sigma_{j\in J_x}(v - v(j))$$

In the latter expression, we then add the quantity $\Sigma_{j\in J_x}v(j)$ to the first sum and substract it from the second to obtain the required formula:

$$R_\delta(\pi,x) = \Sigma_{j\in J_L}v(j) + \Sigma_{j\in J_x}(v - 2v(j)) = \Sigma_{1\leq i\leq v}|J_{x_i}| - \Sigma_{j\in J_x}w(j)$$

□

In the latter formula, the first sum is a constant which depends only on the given profile π whereas in the second sum the quantity $-w(j)$ happens to be the contribution of the join-irreducible $j \in J_x$ to the remoteness of x. This contribution is negative if j is a majority join-irreducible, null if it is balanced, and positive otherwise. In order to obtain a remoteness as small as possible, the best is to search for an element x of L whose representation J_x would contain all majority join-irreducibles, possibly some balanced ones, and no others. In the sequel, the purpose is to present the class of semilattices where such elements exist for any profile and to give the algebraic formulas then available for the metric median. In particular, we will see that this class generalizes the class of distributive lattices.

We first introduce a further notation:

Definition 7.49 Let $\pi \in L^v$ be a profile of L and σ an integer. We write $J(\pi,\sigma) = \{j \in J_L : v_\pi(j) \geq \sigma\}$.

This set will generally be simply denoted by $J(\sigma)$. The following result will be useful later (in Proposition 7.53).

Proposition 7.50 *Let L be a meet-semilattice, $\pi = (x_1,...,x_i,...,x_v) \in L^v$ a profile of L, and σ an integer. The set $J(\sigma)$ is a downset of the ordered subset $(J_L,\leq)$ of L.*

Proof If $j \in J(\sigma)$, then there exists a subset $W \subseteq V = \{1,...,v\}$ such that $|W| \geq \sigma$ and $j \leq x_i$ for any $i \in W$. Then $j' \leq j$ implies $j' \leq x_i$ for any $i \in W$ and so $j' \in J(\sigma)$. □

We observe that:

- For $\alpha = \frac{v+1}{2}$, $J(\alpha)$ is the set of majority join-irreducibles.
- For $\beta = \frac{v}{2}$, $J(\beta) \setminus J(\alpha)$ is the set (empty for odd v) of balanced ones.

It follows from Proposition 7.48 that any element x of L satisfying $J(\alpha) \subseteq J_x \subseteq J(\beta)$ minimizes the remoteness $R_\delta(\pi,x)$, i.e., is a median metric.

We now associate with any profile $\pi = (x_1,...,x_i,...,x_v) \in L^v$ and any integer σ some elements of L which may – or not – exist. In some meet-semilattices, some of these elements will provide algebraic formulas for metric medians.

Definition 7.51 For a profile $\pi = (x_1,...,x_i,...,x_v) \in L^v$ and subject to the existence of such elements, we write:

$$x(\sigma) = \bigvee J(\sigma) \text{ and } x'(\sigma) = \bigvee\{\bigwedge_{i \in W} x_i : W \subseteq V, |W| \geq \sigma\}.$$

The second formula is a "lattice polynomial."

We observed in Chapter 2 (Proposition 2.16) that any upper bounded subset L' of a meet-semilattice L has a join, which is the meet of the upper bounds of L'. Especially, for any $x \in L$, the ordered subset $\{x' \in L : x' \leq x\}$ is a lattice.

Proposition 7.52 *Let L be a meet-semilattice, π a profile of L, and σ an integer. If any of the elements $x(\sigma)$ and $x'(\sigma)$ exist, so does the other and the equality $x(\sigma) = x'(\sigma)$ holds.*

Proof Assume that $x'(\sigma)$ exists and consider some $j \in J(\sigma)$. Then there exists a subset W of V such that $|W| \geq \sigma$ and $j \leq x_i$ for any $i \in W$. So $j \leq \bigwedge_{i \in W} x_i \leq x'(\sigma)$. Then $x'(\sigma)$ is an upper bound of $J(\sigma)$, whence the existence of $x(\sigma)$ with $x(\sigma) \leq x'(\sigma)$.

On the other hand, since the join-irreducible representation is a meet-coding, we have $J_{\bigwedge_{i \in W} x_i} = \bigcap_{i \in W} J_{x_i} = \{j \in J_L : j \leq x_i$ for any $i \in W\}$. For $|W| \geq \sigma$ this set of join-irreducibles is a subset of $J(\sigma)$. It follows that the element $x(\sigma) = \bigvee J(\sigma)$, whose existence is ascertained by the first part of this proof, is an upper bound of $\bigwedge_{i \in W} x_i = \bigvee J_{\bigwedge_{i \in W} x_i}$. So, $x'(\sigma)$ is a join of elements all upper bounded by $x(\sigma)$, which implies $x(\sigma) \geq x'(\sigma)$ and the equality.

Assume that $x(\sigma)$ exists. It is then an upper bound of any meet $\bigwedge_{i \in W} x_i$ with $|W| \geq \sigma$. So, the element $x'(\sigma)$ exists and the previous result applies. □

In the sequel, $x'(\alpha)$ will be called the *lattice median* of π since it generalizes the notion of a median in a distributive lattice (see page 132).

In the case where π is a profile $(L_1, ..., L_i, ..., L_n)$ of linear orders, its weak majority relation $R_{WMAJ}(\pi)$ was defined in Section 7.2 on page 208. It is immediate to check that $R_{WMAJ}(\pi) = \bigcup\{(y,x) : n_\pi(y,x) \geq \frac{n}{2}\} = \bigcup\{\bigcap_{i \in W} L_i, W \subseteq N$ and $|W| \geq \frac{n}{2}\}$, where the $\cup$ and $\cap$ operations are those of the lattice $P(X^2)$ of binary relations on X. Thus the lattice median $x'(\alpha)$ happens to be a lattice formalization of the majority rule.

From Proposition 7.52, the element $x(\alpha)$, when it exists, has the lattice polynomial expression $x(\alpha) = \bigvee\{\bigwedge_{i \in W} x_i : W \subseteq V, |W| = \frac{v+1}{2}\}$ on the one hand and a join-irreducible representation $J_{x(\alpha)}$ containing all majority join-irreducibles on the other hand. Yet in general, the set $J_{x(\alpha)}$ also contains some join-irreducibles which are neither majority nor balanced.

Proposition 7.53 *Let L be a distributive meet-semilattice and π a profile of L. If $x(\alpha)$ exists, then $J_{x(\alpha)} = J(\alpha)$; that is, for any $j \in J_L$, $j \leq \bigvee J(\alpha)$ if and only if $v(j) \geq \alpha$.*

Proof By definition, $v(j) \geq \alpha$ implies $j \in J(\alpha)$ and so $j \leq \bigvee J(\alpha) = x(\alpha)$. We now show the converse implication. In a distributive lattice and, by extension, in a distributive meet-semilattice, it follows from Item (4) in Theorem 5.1 that, for any join-irreducible $j \leq x(\alpha) = \bigvee J(\alpha)$, there exists an element j' of $J(\alpha)$ (thus a majority join-irreducible) such that $j \leq j'$. Then Proposition 7.50 implies that j is also majority and, finally, so are all elements of $J_{x(\alpha)}$. □

We now define a class of distributive meet-semilattices where the element $x(\alpha)$ always exists.

Definition 7.54 A *median semilattice* is a distributive meet-semilattice L such that, for all $x_1, x_2, x_3 \in L$, the element $x_1 \vee x_2 \vee x_3$ exists as soon as all the three elements $x_1 \vee x_2$, $x_1 \vee x_3$, and $x_2 \vee x_3$ do.

Table 7.5 Remoteness of elements of L from the profile π (Example 7.56)

$x \in L$	$\emptyset$	a	b	c	d	e	f	g	h	i	k	l	u
J_x	$\emptyset$	a	b	c	a,b	a,c	b,c	a,b,g	a,b,c	b,c,i	a,b,c,g	a,b,c,i	J_L
$R_\delta(\pi,x)$	13	9	13	13	9	9	13	13	9	17	13	13	17

The existence of $x'(\alpha) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3)$ holds for any profile $\pi = (x_1,x_2,x_3)$ of length 3 of such a semilattice (why?). A straightforward algebraic computation extends the existence of the lattice median $x'(\alpha)$ (but not that of $x'(\beta)$) to any profile of arbitrary finite length. The next characterization of medians for the metric δ in such semilattices follows. It especially applies to distributive lattices (which obviously are median semilattices).

Theorem 7.55 *Let L be a median semilattice and $\pi \in L^v$ a profile of L. If v is odd, then the lattice median $x(\alpha)(=x'(\alpha))$ is the unique (metric) median of π; if v is even, then the set of medians of π is $Med_\delta(\pi) = \{\bigvee J' : J(\alpha) \subseteq J' \subseteq J(\beta)$ and $\bigvee J'$ exists$\}$.*

Especially, in a distributive lattice, $Med_\delta(\pi) = [x(\alpha), x(\beta)]$.

Proof We already observed that, as a consequence of Proposition 7.48, any element x satisfying $J(\alpha) \subseteq J_x \subseteq J(\beta)$ minimizes $R_\delta(\pi,x)$. From the above considerations and Proposition 7.52, the element $x(\alpha) = \bigvee J(\alpha)$ exists for any profile of a median semilattice. Let $j \in J$ such that $j \leq x(\alpha)$. From Proposition 7.53, the equality $J_{x(\alpha)} = J(\alpha)$ holds, which proves that $x(\alpha)$ is a median. As we did in the latter proposition we show that $j \leq \bigvee J'$ and $J' \subseteq J(\beta)$ imply $j \in J(\beta)$. So the form of the elements with the same remoteness as $x(\alpha)$ is $\bigvee J'$ with $J(\alpha) \subseteq J' \subseteq J(\beta)$. If v is odd, then $J(\alpha) = J(\beta)$ and $x(\alpha)$ is the unique median.

In particular if $x(\beta)$ exists – as is always the case in a distributive lattice – we have $Med_\delta(\pi) = [x(\alpha), x(\beta)]$. □

The essential fact in this result is that, in a median semilattice, the lattice median is also a metric median. Possible other metric medians are derived from the former by adding some "neutral" (that is, balanced) join-irreducibles.

Example 7.56 (continuing Example 7.47, on page 223) Table 7.5 gives, for each element x of the lattice L, its representation J_x, then its remoteness $R_\delta(\pi,x)$. The latter is determined from Proposition 7.48 with $\Sigma_{1\leq i\leq v}|J_{x_i}| = 13$ and the values $w(j)$ given in Table 7.4. In this example we have $x(\alpha) = a$ and $x(\beta) = h$. Observe that the set $\{a,d,e,h\}$ of the medians of $\pi = (a,a,e,f,g,k)$ is the interval $[a,h]$ of the lattice.

We have defined the median procedure for profiles of fixed length but, in fact, it may be defined for profiles of non-fixed length. A profile may just be assumed to have a finite length v and to belong to the set $L^* = \bigcup_{v\in\mathbb{N}} L^v$. Its length, denoted by $v(\pi)$, is now a parameter of the profile, with $\beta = \frac{v(\pi)}{2}$ whereas α is the least integer strictly greater than β. An axiomatic characterization of this more general procedure has been obtained in median semilattices. Given two profiles $\pi = (x_1,...,x_{v(\pi)})$ and $\pi' = (x'_1,...,x'_{v(\pi')}) \in L^*$, the *concatenation* of π and π' is the profile $\pi\pi' = (x_1,...,x_{v(\pi)},x'_1,...,x'_{v(\pi')})$ of length $v(\pi\pi') = v(\pi)+v(\pi')$. Then, the theorem below holds (its proof may be found in McMorris *et al.* (2000) – recall that j^- is the unique element covered by j):

Theorem 7.57 *Let L be a median semilattice and $F : L^* \longmapsto (P(L) \setminus \{\emptyset\})$ an aggregation procedure. Then F is the median procedure if and only if it satisfies the properties:*

CONDORCET: *$\pi \in L^v$ with even v, $j \in J(\beta) \setminus J(\alpha)$, $x \in L$ and $x \vee j$ exists imply $(x \vee j^- \in F(\pi) \iff x \vee j \in F(\pi))$.*

CONSISTENCY: *$\pi, \pi' \in L^*$ and $F(\pi) \cap F(\pi') \neq \emptyset$ imply $F(\pi\pi') = F(\pi) \cap F(\pi')$.*

FAITHFULNESS: *$\pi = (x) \in L^1$ implies $F(\pi) = \{x\}$.*

So, median semilattices constitute a type of structure where medians have a simple characterization. Moreover, the determination of the algebraic median $x(\alpha)$ is easy, as soon as that of the set J_x for any $x \in J_L$ and the computation of the join are so. However, the search for medians for the symmetric difference distance in a lattice or a semilattice of another type generally becomes difficult.

In a distributive meet-semilattice L which is not median, the conclusions in Theorem 7.55 still apply to any profile such that $x(\alpha)$ exists. If $x(\alpha)$ does not exist, one has to find the elements x of L of the form $x = (\bigvee J_1) \vee (\bigvee J_2)$ where:

- J_1 is a set of majority join-irreducibles such that $\bigvee J_1$ exists and maximizing $\Sigma_{j \in J_1} w(j)$ with this condition, and
- J_2 is a set of balanced join-irreducibles such that x exists in L.

Such a search may be difficult.

When L is not distributive, it is no longer sure that $x(\alpha)$ is a median, even if this element exists. Nevertheless, there remain some relations between medians and the majority rule (Leclerc, 1994a):

Theorem 7.58 *Let L be a meet-semilattice. For any profile π of L and for any median x of π, the following properties hold:*

1. *If $x(\beta)$ exists, then $x \leq x(\beta)$.*
2. *If $x(\alpha)$ exists, then there exists a median x' such that $x' \leq x \wedge x(\alpha)$ and such that any element x'' satisfying $x' \leq x'' \leq x$ is a median.*

Proof Let x be a median of π such that $x \not\leq x(\beta)$. Write $x' = x \wedge x(\beta) < x$. The set $J_x \setminus J_{x'}$ is not empty and its elements are join-irreducibles j which are neither majority nor balanced; that is, which satisfy $w(j) < 0$. The inequality $R_\delta(\pi, x') < R_\delta(\pi, x)$ then follows from Proposition 7.48, a contradiction with the assumption that x is a median.

For the second part, consider a median x such that $x \not\leq x(\alpha)$ and the element $x'' = x \wedge x(\alpha)$. As above, observe that the elements of $J_x \setminus J_{x''}$ are not majority. So they are balanced since x is a median. The equality $R_\delta(\pi, x'') = R_\delta(\pi, x)$ follows, which extends to any element intermediate between x'' and x. □

In order to go back to problems of classification aggregation, let us examine the consequences of Theorems 7.55 and 7.57. They apply to any distributive lattice, in particular to chains and direct products of chains such as, for example, Boolean lattices and so to lattices $\underline{2}^{E^2}$ of binary relations on E. The interest in the formulation of these results in the more general framework of median semilattices comes from the observation that such semilattices, that are not lattices, are frequently encountered. Tree-ordered sets (see Exercise 2.5) provide a first class of such semilattices: they satisfy the condition in Definition 7.54 since, in a tree-ordered set, the three elements $x \vee x'$, $x \vee x''$, and $x' \vee x''$ exist if and only if the subset $\{x, x', x''\}$ is a chain.

Another class of median semilattices, neither including nor included in the previous ones, appears when considering a set E endowed with a symmetric relation A modeling the notion of a "compatibility" of some type. The subsets C of pairwise compatible elements of E then correspond with the cliques of the graph $G = (X, A)$. We first observe that the set of all these cliques is a downset of the Boolean lattice $\underline{2}^E$ and so a distributive meet-semilattice (which is a lattice only in the very particular case where G is a complete graph). Moreover, given three cliques C, C', and C'' of G, the set $(C \cap C') \cup (C \cap C'') \cup (C' \cap C'')$ is still a clique of G. It follows that the ordered set of cliques of G ordered by inclusion is a median semilattice (which moreover is atomistic). When for instance A is the comparability or the incomparability relation of an ordered set P, we obtain:

Proposition 7.59 *Let P be an ordered set. The sets of chains and of antichains of P, ordered by inclusion, are median semilattices.*

The ordered set $(\mathbb{H}, \subseteq)$ of hierarchies on E is another example of a clique semilattice, as shown in the proof of the following result:

Proposition 7.60 *The set $(\mathbb{H}, \subseteq)$ of all hierarchies on a set E, ordered by inclusion, is a median semilattice.*

Proof Consider the graph $G = (P^*(E), A)$, with $P^*(E) = \{E' \subseteq E : 2 \leq |E'| < n\}$ and, for $E', E'' \in P^*(E)$, $(E', E'') \in A$ if and only if $E' \cap E'' \in \{\emptyset, E', E''\}$. It is easy to see that any hierarchy $\mathcal{H}$ on E corresponds to a clique of the graph G, with the addition of the trivial classes. So the ordered set $(\mathbb{H}, \subseteq)$ is a median semilattice. □

We then check the following points:

- The atoms of the semilattice $\mathbb{H}$ are the hierarchies with a unique non-trivial class H (and they are the only join-irreducibles of $\mathbb{H}$).
- The symmetric difference distance between two hierarchies $\mathcal{H}$ and $\mathcal{H}'$ is the number $\delta(\mathcal{H},\mathcal{H}') = |\mathcal{H}\Delta\mathcal{H}'|$ of classes belonging either to $\mathcal{H}$ or to $\mathcal{H}'$ but not to both.

Consider a profile $\pi = (\mathcal{H}_1, ..., \mathcal{H}_v)$ of $\mathbb{H}$. Then, it follows from Theorem 7.55 and Proposition 7.60 that the set $\mathcal{H}(\alpha)$ of all the classes belonging to more than half the elements of π is a median hierarchy of π (for the previous distance). This median is unique for odd v; otherwise the other medians are obtained by adding balanced classes to $\mathcal{H}(\alpha)$ (that is, classes belonging to half the elements of π), provided that these additions do not contradict Condition (2) in Definition 7.37.

Theorem 7.58 applies to the other classification ordered sets described in this section (partitions, dendrograms, ultrametrics, Moore families) and also, for instance, to the lattice of preorders or the meet-semilattice of orders.

7.4 Implicational systems, Moore families and Galois data analysis

In many situations, the knowledge on a set E under study takes the form of a family $\mathcal{D}$ of subsets of E. It is for instance the case for the clusters obtained by the use of a classification method (as mentioned in the previous section) or for the Moore families obtained by a Galois classification as described in Chapter 3. Other similar cases occur in the framework of databases or knowledge representation. For instance a databasis is frequently identified with the family $\mathcal{D}$ of the subsets (then often called *transactions*) of the elements (or *items*) satisfying each possible request. In Doignon and Falmagne's theory of "knowledge (or learning) spaces" (1999, 2011, www/aleks.com/) the basic set contains "knowledge units," for instance mathematical problems of fourth-year primary school. A "knowledge state" is the set of the problems that a pupil is able to solve and a "knowledge structure" is a set of knowledge states; that is, a family of subsets of the set of knowledge units. (Formally, a "knowledge space" is a dual closure system; that is, up to a duality, a Moore family.)

In these various contexts, one frequently searches for *implications* (or *association rules*, or *functional dependencies*). For instance, there exist such implications between the above-mentioned mathematical problems; indeed any pupil able to solve some subset A of these problems is also able to solve some other subset B (which can be written as $A \longrightarrow B$). This can be expressed as the fact that any element of the family $\mathcal{D}$ of knowledge states which includes A also includes B and we will say that $A \longrightarrow B$ is a $\mathcal{D}$-*implication*. In Definition 7.62 we will give another definition of a $\mathcal{D}$-*implication* and, in Proposition 7.64, we will prove the equivalence of these two definitions.

A standard example is that of a marketing office concerned with observations such as the following: in some "customer segment," any coffee and filter buyer is also a sugar buyer, which then expresses as $\{coffee, filter\} \longrightarrow \{sugar\}$ (a famous example associates *beer* with *man* and *diaper*). In fact, rather than exact implications, those considered in the latter example are often approximate with, for instance, a condition on a minimal proportion of cases where they are satisfied. We consider approximate implications only in the Further topics section. As for exact implications, a significant observation is that they may be very numerous and so an essential purpose consists of selecting a small number of them forming a *basis*; that is, allowing us to recover all others.

Let $\mathcal{D} = \{A_1, ..., A_m\}$ be a family of m subsets of a set E of size n. This family is equivalent to a binary relation R on $\mathcal{D} \times E$ which may be represented by a 0/1 array t with m lines and n columns (where $t[i,j] = 1$ if $j \in A_i$). Conversely, given a binary relation R on $E' \times E$ (or the associated 0/1 array), one can associate two families of subsets : the first $\{Re : e \in E\}$ on the set E' and the second $\{e'R : e' \in E'\}$ on the set E. In Galois data analysis the data is given in the form of a 0/1 array between a set E' of objects and a set E of attributes. Beware that in Chapter 3, the set of attributes was denoted by E' and that of objects by E but here it is more convenient to permute these notations since we want to focus on the implications between attributes. So we consider the implications determined by the family $\{e'R : e' \in E'\}$. Such an implication $A \longrightarrow B$ (with $A, B \subseteq E$) means that:

$$\forall e' \in E', (\forall e \in A, e'Re) \Longrightarrow (\forall e \in B, e'Re)$$

We illustrate this with the following example, which will be continued in the sequel.

Example 7.61 Using Galois lattices, Duquenne (1995) resumes an anthropological study on 98 Javanese peasants (Schweizer, 1993). The latter are described with 31 binary attributes bearing on their housing, furniture, and livestock. Especially the six attributes on livestock correspond to the following possessions:

H: Bantam chicken	P: pedigree chicken
M: Manila duck	D: (common) duck
U: water buffalo	G: goat

So, we have a Galois lattice $Gal(E', E, R)$ (Chapter 3, Section 3.5.1) where E' is the set of peasants of the study, $E = \{D, G, H, M, P, U\}$ is the set of above-mentioned attributes, and R is the relation from E' to E defined by $(e', e) \in R$ if peasant e' has livestock of type e. As said above, we are concerned with the $\mathcal{D}$-implications where $\mathcal{D} = \{e'R : e' \in E'\}$, i.e., the implications between attributes of the following type: all the peasants who possess some given livestock species also possess some others. The reader could check that these implications are the same as those associated with the Moore family $\psi_R(\underline{2}^E)$ on E, here denoted by $\mathcal{F}$ (see Propositions 7.63 and 7.64). The closure operator ψ_R on $\underline{2}^E$ is that associated with R in Section 3.5.1: a subset F of E

Table 7.6 The arrowed table of the lattice in Figure 7.8

	DMP	*DHM*	*DGH*	*DHU*	*DHP*	*GH*	*HU*	*HP*	*G*	*U*
D	$\times$	$\times$	$\times$	$\times$	$\times$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\downarrow$	$\downarrow$
G	$\updownarrow$	$\updownarrow$	$\times$	$\updownarrow$	$\updownarrow$	$\times$	$\downarrow$	$\downarrow$	$\times$	$\downarrow$
U	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\times$	$\updownarrow$	$\downarrow$	$\times$	$\downarrow$	$\downarrow$	$\times$
P	$\times$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\times$	$\downarrow$	$\downarrow$	$\times$	$\downarrow$	$\downarrow$
H	$\updownarrow$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\updownarrow$	$\updownarrow$
DM	$\times$	$\times$	$\updownarrow$	$\updownarrow$	$\updownarrow$					

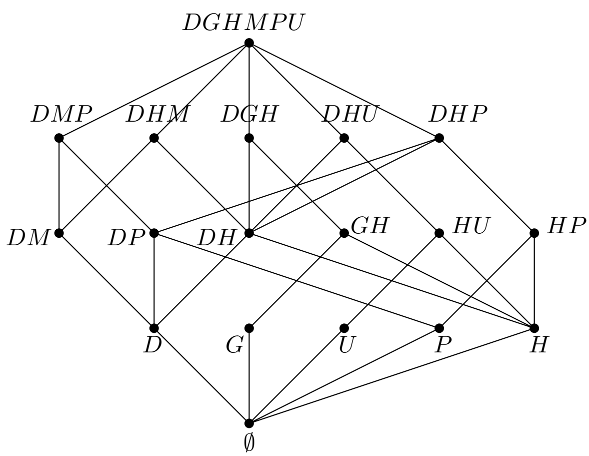


Figure 7.8 The lattice of livestock possessions (Example 7.61).

is closed by ψ_R if and only if there exists a class of peasants $A \subseteq E'$ who all share all the possessions in F and have no others in common.

In the above example, the Moore family $\mathcal{F}$ has 18 elements. Figure 7.8 shows the lattice $(\mathcal{F}, \subseteq)$ and Table 7.6 gives the arrowed table of this lattice. The set of meet-irreducibles of $\mathcal{F}$ is

$$\mathcal{M}_{\mathcal{F}} = \{DMP, DHM, DGH, DHU, DHP, GH, HU, HP, G, U\}$$

where, for instance, DGH is an abbreviated notation for $\{D, G, H\}$. The entire family $\mathcal{F}$ is obtained by making all possible intersections of subsets of $\mathcal{M}_{\mathcal{F}}$.

Since we observe that any closed set including M also includes D, we have the (simple and natural) association rule $M \longrightarrow D$ ("having *Manila ducks* implies having *common ducks*"). We may also observe other rules, all the more interesting for specialists since they are less expected, such as $DU \longrightarrow H$ ("having *common ducks* and *water buffalo* implies having *Bantam chickens*"). In the quoted paper, Duquenne exhibits a set of nine particular implications allowing us to recover all the others in a way that will be made explicit in the sequel.

The purpose of this section is the study of the properties of the implications of the above type together with the obtainment of a reduced subset of implications accounting for the whole data. Let $\mathcal{D}$ be any arbitrary family of subsets of a given set E. We first define the complete implicational system $i(\mathcal{D})$ on $P(E)$; that is, the relation on $P(E)$ formed from all the implications of the type $A \longrightarrow B$ induced by $\mathcal{D}$ (see Definition 7.62). We then characterize these relations in Theorem 7.65. A particular Galois connection between the sets $\underline{2}^{P(E)}$ of families of subsets of E and $\underline{2}^{(P(E))^2}$ of binary relations on $P(E)$, both ordered by inclusion, is defined. Among other consequences, it follows that the sets $\mathbb{F}$ of all Moore families on E and $\mathbb{I}$ of all complete implicational systems on $P(E)$ constitute two dual lattices (Theorem 7.68). We give several properties of these lattices, useful when going back to the search for reduced sets of implications allowing us to recover all the implications of an element of $\mathbb{I}$. The existence of such reduced sets is established, especially the *Guigues–Duquenne canonical implication basis* which is described with its main properties (Theorem 7.79 and Proposition 7.80).

As in Chapter 3 (Example 3.32) and in the previous section, we denote by $\mathbf{m}(\mathcal{D}) = \{\bigcap \mathcal{C} : \mathcal{C} \subseteq \mathcal{D}\}$ the Moore family obtained by completing an arbitrary family $\mathcal{D}$ of subsets of a set E with intersections. Then, in Example 7.61 above, $\mathcal{F} = \mathbf{m}(\mathcal{M}_{\mathcal{F}})$. The definition of the closure operator $\varphi_{\mathcal{F}}$ on $\underline{2}^E$ associated with a Moore family $\mathcal{F}$ (Definition 3.29) then extends and a complete implicational system is associated with any family $\mathcal{D}$ of subsets.

Definition 7.62 Let $\mathcal{D}$ be a family of subsets of a set E.

(1) A map $\varphi_{\mathcal{D}}$ on $P(E)$ is associated with $\mathcal{D}$ by writing $\varphi_{\mathcal{D}}(A) = \bigcap\{D \in \mathcal{D} : A \subseteq D\}$ for any $A \subseteq E$. A subset F of E such that $\varphi_{\mathcal{D}}(A) = F$ for some $A \subseteq E$ is said to be *$\mathcal{D}$-closed* (or simply *closed* when there is no ambiguity).

(2) A map i from $P(P(E))$ to $P((P(E))^2)$ is defined by writing $i(\mathcal{D}) = \{(A,B) \in (P(E))^2 : B \subseteq \varphi_{\mathcal{D}}(A)\}$. The binary relation $i(\mathcal{D})$ on $P(E)$ is called the *complete implicational system* associated with $\mathcal{D}$. The ordered pair $(A,B) \in i(\mathcal{D})$ is called a *$\mathcal{D}$-implication* (or simply an *implication* when there is no ambiguity) and will be frequently written $A \longrightarrow_{\mathcal{D}} B$ (or simply $A \longrightarrow B$).

(3) A binary relation I on $P(E)$ is a *complete implicational system* on $P(E)$ if it is equal to $i(\mathcal{D})$ for some family $\mathcal{D}$ of subsets of E.

The expression "complete implicational system" will be abbreviated as CIS and the set of all CIS on $P(E)$ will be denoted by $\mathbb{I}$.

It is worth noticing that the definition of $\varphi_{\mathcal{D}}$ implies $\varphi_{\mathcal{D}}(A) = E$ as soon as A is included in no element of $\mathcal{D}$ (see Example 3.2).

The denomination of a $\mathcal{D}$-closed set is justified by showing that the map $\varphi_{\mathcal{D}}$ on $P(E)$ is identical to the closure map $\varphi_{\mathbf{m}(\mathcal{D})}$ on $\underline{2}^E$ associated with the Moore family $\mathbf{m}(\mathcal{D})$.

Proposition 7.63 *Let $\mathcal{D}$ be a family of subsets of a set E. With the previous notations, the equality $\varphi_{\mathcal{D}}(A) = \varphi_{\mathbf{m}(\mathcal{D})}(A)$ holds for any $A \subseteq E$.*

Proof We already know that the meet operation of the lattice $\mathbf{m}(\mathcal{D})$ is the intersection whence $\varphi_{\mathbf{m}(\mathcal{D})}(A) = \bigcap\{D \in \mathbf{m}(\mathcal{D}) : A \subseteq D\}$. Now since $\mathcal{D}$ is a meet-generating set of $\mathbf{m}(\mathcal{D})$, $\varphi_{\mathbf{m}(\mathcal{D})}(A) = \bigcap\{D \in \mathcal{D} : A \subseteq D\} = \varphi_{\mathcal{D}}(A)$ (Proposition 3.16). □

Thus, the pairs (A,B) in the relation $i(\mathcal{D})$ may be characterized in several ways, with the consequence that all implications in a CIS are obtained from a Moore family on E, namely $\mathbf{m}(\mathcal{D})$.

Proposition 7.64 *Let $\mathcal{D}$ be a family of subsets of a set E and A and B two subsets of E. The following four conditions are equivalent:*

1. *$A \longrightarrow_{\mathcal{D}} B$; that is, $B \subseteq \varphi_{\mathcal{D}}(A)$.*
2. *For any $D \in \mathcal{D}, A \subseteq D \Longrightarrow B \subseteq D$.*
3. *For any $F \in \mathbf{m}(\mathcal{D}), A \subseteq F \Longrightarrow B \subseteq F$.*
4. *$B \subseteq \varphi_{\mathbf{m}(\mathcal{D})}(A)$; that is, $A \longrightarrow_{\mathbf{m}(\mathcal{D})} B$.*

Proof (1) $\Longrightarrow$ (2) is a direct consequence of the definition of $\varphi_{\mathcal{D}}$.

(2) $\Longrightarrow$ (3): let $A,B \subseteq E$ satisfying Condition (2) and F be an element of $\mathbf{m}(\mathcal{D}) \setminus \mathcal{D}$. If $F = E$, then $A \subseteq E$ and $B \subseteq E$, which matches (3). Otherwise F is the intersection of a non-empty subset $\mathcal{C}$ of $\mathcal{D}$. Then $A \subseteq F$ implies $A \subseteq D$ and so $B \subseteq D$ for any $D \in \mathcal{C}$. Thus $B \subseteq F = \bigcap \mathcal{C}$ as required.

(3) $\Longrightarrow$ (4): from (3) any element of $\mathbf{m}(\mathcal{D})$ including A includes B. This is in particular the case for $\varphi_{\mathbf{m}(\mathcal{D})}(A)$.

(4) $\Longrightarrow$ (1) is an immediate consequence of Proposition 7.63. □

A consequence of this proposition is that, when $\mathcal{D}$ is a Moore family, the CIS $i(\mathcal{D})$ may be derived from the only family $\mathcal{M}_{\mathcal{D}}$ of meet-irreducibles of the lattice $\mathcal{D}$ (since $\mathbf{m}(\mathcal{M}_{\mathcal{D}}) = \mathcal{D}$). Theorem 7.65 below gives a characterization of CIS which is a variant of that by Armstrong (1974). In the literature, a relation on $P(E)$ satisfying the conditions of this theorem – that is, a CIS – is also called a *complete family of functional dependencies*.

Theorem 7.65 *A binary relation I on $P(E)$ is a CIS if and only if it satisfies the following three properties for all $A,B,C,D \subseteq E$:*

1. *$B \subseteq A \Longrightarrow (A,B) \in I$.*
2. *$(A,B) \in I$ and $(B,C) \in I \Longrightarrow (A,C) \in I$.*
3. *$(A,B) \in I$ and $(C,D) \in I \Longrightarrow (A \cup C, B \cup D) \in I$.*

Proof If $I = i(\mathcal{D})$ it results from Proposition 7.64 that there exists a Moore family $\mathcal{F} = \mathbf{m}(\mathcal{D})$ on E such that $I = i(\mathcal{F})$. Then $(A,B) \in I$ is equivalent to $B \subseteq \varphi_{\mathcal{F}}(A)$ with the following consequences:

- $B \subseteq A \Longrightarrow B \subseteq \varphi_{\mathcal{F}}(A)$ (since $A \subseteq \varphi_{\mathcal{F}}(A)$) and Property (1) holds.
- $B \subseteq \varphi_{\mathcal{F}}(A) \Longrightarrow \varphi_{\mathcal{F}}(B) \subseteq \varphi_{\mathcal{F}}(\varphi_{\mathcal{F}}(A)) = \varphi_{\mathcal{F}}(A)$; then $C \subseteq \varphi_{\mathcal{F}}(B) \Longrightarrow C \subseteq \varphi_{\mathcal{F}}(A)$ and Property (2) holds.
- $[B \subseteq \varphi_{\mathcal{F}}(A)$ and $D \subseteq \varphi_{\mathcal{F}}(C)] \Longrightarrow [B \subseteq \varphi_{\mathcal{F}}(A \cup C)$ and $D \subseteq \varphi_{\mathcal{F}}(A \cup C)] \Longrightarrow B \cup D \subseteq \varphi_{\mathcal{F}}(A \cup C)$ and Property (3) holds.

Conversely, let I be a relation on $P(E)$ satisfying Properties (1), (2), and (3). Write $\varphi(A) = \{e \in E : (A,\{e\}) \in I\}$ for any subset A of E. We first show the equivalence $(A,B) \in I \iff B \subseteq \varphi(A)$.

It follows from (3) that $(A,B) \in I$ and $e \in \varphi(A)$ imply $(A, B \cup \{e\}) \in I$. Then, it may be shown by induction on $|B|$ that $B \subseteq \varphi(A)$ implies $(A,B) \in I$.

Now, assume $(A,B) \in I$ and consider an element $e \in B$. From (1), we have $(B,\{e\}) \in I$ and so, from (2), $(A,\{e\}) \in I$; that is, $e \in \varphi(A)$. The inclusion $B \subseteq \varphi(A)$ and the required equivalence follow.

We then show that the map φ is a closure operator on $P(E)$. Indeed, this map is:

- Extensive: from (1), $e \in A$ implies $e \in \varphi(A)$ whence $A \subseteq \varphi(A)$.
- Isotone: let A and B be two subsets of E with $A \subseteq B$, and $e \in E$ such that $(A,\{e\}) \in I$. Then (1) implies $(B,A) \in I$ whence, from (2), $(B,\{e\}) \in I$. So $e \in \varphi(B)$, which implies $\varphi(A) \subseteq \varphi(B)$.
- Idempotent: since $(A,\varphi(A)) \in I$ and $(\varphi(A),\varphi(\varphi(A))) \in I$, we have, from (2), $(A,\varphi(\varphi(A))) \in I$ which implies $\varphi(\varphi(A)) \subseteq \varphi(A)$ and, since φ is extensive, $\varphi(\varphi(A)) = \varphi(A)$.

We then conclude that $(A,B) \in I$ if and only if $B \subseteq \varphi_{\mathcal{F}}(A)$, where $\mathcal{F}$ is the Moore family associated with the closure φ; that is, $I = i(\mathcal{F})$. $\square$

According to Property (1) above, a CIS is reflexive, and, according to Property (2), it is transitive. So, the CIS form a particular class of preorders on $P(E)$, namely those compatible with the dual inclusion order and satisfying Property (3) of union preserving.

It is easy to see that the set $\mathbb{I}$ of all CIS on $P(E)$ is a Moore family, which allows us to give the following definition.

Definition 7.66 Let R be a relation on $P(E)$. We denote by $\mathbf{a}(R)$ the smallest CIS on $P(E)$ including R.

The map $R \longrightarrow \mathbf{a}(R)$ is the closure operator on $\underline{2}^{(P(E))^2}$ associated with the Moore family $\mathbb{I}$.

The map i from $\underline{2}^{P(E)}$ to $\underline{2}^{(P(E))^2}$ was introduced in Definition 7.62. It associates with any family of subsets $\mathcal{D} \subseteq P(E)$ its CIS $i(\mathcal{D}) \subseteq (P(E))^2$. On the other hand, we may associate a family of subsets of E with any binary relation R on $P(E)$.

Definition 7.67 The *saturation operator* s is defined on $(P(E))^2$ by $s(R) = \{F \subseteq E : A \subseteq F$ and $(A,B) \in R$ imply $B \subseteq F\}$ for any binary relation R on $P(E)$. The elements of $s(R)$ are called the *saturated subsets of E for R*.

Observe that saying F is saturated is equivalent to saying that the family $(F]$ is an "upset" of $P(E)$ for the relation R (see Remark 5.26 on page 146).

In the next theorem, we prove that the pair (i,s) is a Galois connection with $\mathbb{I}$ and $\mathbb{F}$ as images.

Theorem 7.68 *Let E be a set. Then, with the previous definitions and notations:*

1. *The pair (i,s) constitutes a Galois connection between the ordered sets $\underline{2}^{P(E)}$ of families of subsets of E and $\underline{2}^{(P(E))^2}$ of binary relations on E.*
2. *The image of $\underline{2}^{(P(E))^2}$ by s is the set $\mathbb{F}$ of Moore families on E and, for any $\mathcal{D} \in \underline{2}^{P(E)}$, the equality $si(\mathcal{D}) = \mathbf{m}(\mathcal{D})$ holds.*
3. *The image of $\underline{2}^{P(E)}$ by i is the set $\mathbb{I}$ of CIS on $P(E)$ and, for any $R \in \underline{2}^{P(E)^2}$, the equality $is(R) = \mathbf{a}(R)$ holds.*

Proof In order to show (1) we first establish Item (2) in Theorem 3.39: for all $\mathcal{D} \in \underline{2}^{P(E)}$, $R \in \underline{2}^{(P(E))^2}$, $\mathcal{D} \subseteq s(R)$ is equivalent to $R \subseteq i(\mathcal{D})$.

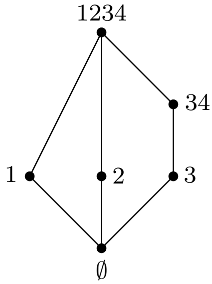
Assume $\mathcal{D} \subseteq s(R)$ and show that any pair $(A,B) \in R$ belongs to $i(\mathcal{D})$; that is, $B \subseteq \varphi_{\mathcal{D}}(A) = \bigcap\{D \in \mathcal{D} : A \subseteq D\}$. We search for the elements D of $\mathcal{D}$ including A. If there is no such element then, by definition, $(A,B) \in i(\mathcal{D})$ since, in the lattice $\underline{2}^E$, the meet of the empty family is E. Otherwise, by assumption, these subsets D belong to $s(R)$, which means that all of them include B. So $B \subseteq \varphi_{\mathcal{D}}(A)$ and $(A,B) \in i(\mathcal{D})$. Finally $\mathcal{D} \subseteq s(R)$ implies $R \subseteq i(\mathcal{D})$.

Assume $R \subseteq i(\mathcal{D})$ and show that any element D of $\mathcal{D}$ belongs to $s(R) = \{F \subseteq E : A \subseteq F$ and $(A,B) \in R$ imply $B \subseteq F\}$. If there is no pair $(A,B) \in R$ with $A \subseteq D$, then $D \in s(R)$ holds. Otherwise any such pair satisfies $(A,B) \in i(\mathcal{D})$ whence, from Definition 7.62, $B \subseteq \varphi_{\mathcal{D}}(A) \subseteq D$. So we obtain $D \in s(R)$ and $R \subseteq i(\mathcal{D})$ implies $\mathcal{D} \subseteq s(R)$ as required.

Let us show (2). Obviously, $E \in s(R)$ for any relation R on $P(E)$. Let $F, F' \in s(R)$ and a pair $(A,B) \in R$ such that $A \subseteq F \cap F'$. We then have $A \subseteq F$ and, by assumption, $B \subseteq F$ and similarly $A \subseteq F'$ and $B \subseteq F'$, whence $B \subseteq F \cap F'$. Thus $F, F' \in s(R)$ implies $F \cap F' \in s(R)$, whence $s(R) \in \mathbb{F}$. So the image of $\underline{2}^{(P(E))^2}$ by s is included in the set $\mathbb{F}$.

Let $\mathcal{F}$ be a Moore family on E. From the properties of Galois connections, si is a closure operator on $\underline{2}^{P(E)}$, with $\mathcal{F} \subseteq si(\mathcal{F}) = \mathcal{G}$. So the set $\mathcal{G}$ is a Moore family on E and is, by definition of the map s, the set of the subsets G of E such that $(A,B) \in i(\mathcal{F})$ and $A \subseteq G$ imply $B \subseteq G$. In other terms $A \subseteq G$ implies $B \subseteq G$ and, if $B \subseteq \varphi_{\mathcal{F}}(A)$ then $B \subseteq \varphi_{\mathcal{G}}(A)$. The inequality $\varphi_{\mathcal{F}} \leq \varphi_{\mathcal{G}}$ for the exponentiation order follows, which, according to Proposition 3.26, implies $\mathcal{G} \subseteq \mathcal{F}$ and so $\mathcal{F} = \mathcal{G}$. Thus any Moore family belongs to the image of $\underline{2}^{(P(E))^2}$ by s and the equalities $s(\underline{2}^{(P(E))^2}) = si(\underline{2}^{P(E)}) = \mathbb{F}$ hold.

If $\mathcal{D}$ is an arbitrary family of subsets of E, then $si(\mathcal{D})$ is the smallest Moore family including $\mathcal{D}$; that is, $\mathbf{m}(\mathcal{D})$.



$$\begin{aligned}
3 &\longrightarrow 4\\
12 &\longrightarrow 1234\\
13 &\longrightarrow 1234\\
14 &\longrightarrow 1234\\
23 &\longrightarrow 1234\\
24 &\longrightarrow 1234\\
123 &\longrightarrow 1234\\
124 &\longrightarrow 1234\\
134 &\longrightarrow 1234\\
234 &\longrightarrow 1234
\end{aligned}$$

Figure 7.9 A Moore family $\mathcal{F}$ and the non-trivial implications of its associated CIS.

Finally, (3) is a direct consequence of Definition 7.62 together with the fact that the relation $\mathbf{a}(R)$ (see Definition 7.66) is the smallest CIS including R. □

From Item (3) in Theorem 3.41, we obtain the following:

Corollary 7.69 *The ordered sets* $(\mathbb{F}, \subseteq)$ *of Moore families on* E *and* $(\mathbb{I}, \subseteq)$ *of CIS on* $P(E)$ *are two dual lattices.*

We will say that an implication $A \longrightarrow B$ is *trivial* if $B \subseteq A$. The example in Figure 7.9 shows the non-trivial implications of the CIS associated with a Moore family $\mathcal{F}$.

According to the above theorem, the set $\mathbb{F}$ of Moore families on E (respectively, the set $\mathbb{I}$ of CIS on $P(E)$) is a Moore family on $P(E)$ (respectively, on $(P(E))^2$). The ordered sets $(\mathbb{F}, \subseteq)$ and $(\mathbb{I}, \subseteq)$ are dual lattices. The minimum of the lattice $\mathbb{F}$ is the Moore family $\{E\}$ and its maximum is $P(E)$. Its atoms are the Moore families $\mathcal{F}_A = \{A, E\}$ for any $A \subset E$. The minimum of the lattice $\mathbb{I}$ is the dual of the inclusion order on $P(E)$ (why?) and its maximum is $i(\{E\}) = (P(E))^2$.

Since $\mathbb{F}$ is the lattice of the families of $\underline{2}^{P(E)}$ closed by $\mathbf{m}$, Proposition 3.28, Definition 3.29, and their comments lead to the first of the following expressions, the second being easy to obtain:

$$\text{for all } \mathcal{F}, \mathcal{F}' \in \mathbb{F}, \mathcal{F} \vee \mathcal{F}' = \mathbf{m}(\mathcal{F} \cup \mathcal{F}') = \{F \cap F' : F \in \mathcal{F}, F' \in \mathcal{F}'\}$$

These expressions of the join in the lattice $\mathbb{F}$ allow a simple proof of the next proposition which implies that any Moore family is a join of atoms $\mathcal{F}_A$. As above, $\mathcal{M}_{\mathcal{F}}$ denotes the set of meet-irreducibles of the lattice $\mathcal{F}$ (that is, the subsets $\mathcal{F}$ which cannot be obtained as the intersection of other elements of $\mathcal{F}$).

Proposition 7.70 *Let* $\mathcal{F}$ *be a Moore family on a set* E *and* $\mathcal{D}$ *a subset of* $\mathcal{F} \setminus \{E\}$. *Then, the equality* $\mathcal{F} = \bigvee\{\mathcal{F}_D : D \in \mathcal{D}\}$ *in the lattice* $\mathcal{F}$ *is equivalent to the inclusion* $\mathcal{M}_{\mathcal{F}} \subseteq \mathcal{D}$.

Proof If $\mathcal{D}$ is a subset of $\mathcal{F} \setminus \{E\}$, then the equalities

$$\mathbf{m}(\mathcal{D}) = \mathbf{m}(\mathcal{D} \cup \{E\}) = \mathbf{m}(\bigcup\{\{D,E\} : D \in \mathcal{D}\}) = \bigvee\{\mathcal{F}_D : D \in \mathcal{D}\}$$

hold, where the latter is a consequence of the first expression of the join in $\mathbb{F}$ given above. If $\mathcal{D}$ includes $\mathcal{M}_{\mathcal{F}}$ then, according to the second expression of the join, the equality $\mathbf{m}(\mathcal{D}) = \bigvee\{\mathcal{F}_D : D \in \mathcal{D}\} = \mathcal{F}$ follows. On the contrary, if there exists a meet-irreducible A of $\mathcal{F}$ which does not belong to $\mathcal{D}$, then A cannot be obtained as the intersection of other elements of $\mathcal{F}$, and so of $\mathcal{D}$, and A does not belong to the Moore family $\bigvee\{\mathcal{F}_D : D \in \mathcal{D}\}$ which cannot be equal to $\mathcal{F}$. □

According to this proposition, every Moore family is a join of atoms; that is, the lattice $\mathbb{F}$ is atomistic (Definition 3.13). Moreover, any Moore family $\mathcal{F}$ has a unique minimal join-irreducible representation corresponding to $\mathcal{D} = \mathcal{M}_{\mathcal{F}}$. As a consequence the lattice $\mathbb{F}$ is lower locally distributive (see page 158). Precisely it is a convex geometry (see the end of Section 5.6). The covering relation of $\mathbb{F}$ is easily characterized:

Proposition 7.71 *Let $\mathcal{F}$ and $\mathcal{G}$ be two Moore families on a set E. Then $\mathcal{G}$ covers $\mathcal{F}$ in the lattice $\mathbb{F}$ of Moore families if and only if there exists a meet-irreducible A of $\mathcal{G}$ such that $\mathcal{G} = \mathcal{F} + \{A\}$.*

Proof Assume that there exists a subset A of E such that $\mathcal{G}$ is equal to $\mathcal{F} + \{A\}$. Then $A \in \mathcal{M}_{\mathcal{G}}$ since otherwise $\mathcal{F}$ is not intersection-stable. And $\mathcal{G}$ covers $\mathcal{F}$ in the lattice $\mathbb{F}$ since the two families differ from only one element.

Conversely, assume $\mathcal{G}$ covers $\mathcal{F}$ in the lattice $\mathbb{F}$ and there exist two subsets A and A' of E such that $\{A, A'\} \subseteq \mathcal{G} \setminus \mathcal{F}$. We then have $\mathcal{G} = \mathcal{F} \vee \mathcal{F}_A = \mathcal{F} \vee \mathcal{F}_{A'}$ in $\mathbb{F}$, which implies the existence of two elements G and G' of $\mathcal{F}$ such that $A = G \cap A'$ and $A' = G' \cap A$, whence $A' = G \cap G' \cap A'$. So $A' \subseteq G$ and $A = G \cap A' = A'$. □

The map i (Definition 7.62) sends the join-irreducibles of the lattice $\mathbb{F}$ – that is, its atoms – to the meet-irreducibles of the dual lattice $\mathbb{I}$ – that is, its coatoms. The latter are characterized in Exercise 7.24. By duality with $\mathbb{F}$, the lattice $\mathbb{I}$ is upper locally distributive, which in particular implies that any CIS has a unique minimal meet-irreducible representation.

Remark 7.72 Let E be a set and R a relation on $P(E)$. The closure map $is(R)$ is the intersection of the CIS including R. So, it is the intersection of all the meet-irreducibles of the lattice $\mathbb{I}$ which include R.

In what follows, we go back to the representations of a CIS $is(R)$ as the join of a set of elements of the lattice $\mathbb{I}$, especially to the representations by bases (with the notion of a basis defined in Definition 7.74). We first need the following lemma.

Lemma 7.73 *Let A and B be two subsets of a set E.*

1. *The family of subsets* $\mathcal{F}_{A,B} = \{F \subseteq E : A \not\subseteq F \text{ or } B \subseteq F\}$ *is a Moore family and the relation* $[A \longrightarrow B] = \{(C,D) \in P(E)^2 : C \not\subseteq A \text{ or } D \subseteq B\}$ *is a CIS. The equalities* $\mathcal{F}_{A,B} = s(\{(A,B)\})$ *and* $[A \longrightarrow B] = i(\mathcal{F}_{A,B}) = is(\{(A,B)\})$ *hold.*
2. *For any relation R on P(E), the equality* $is(R) = \bigvee\{[A \longrightarrow B], (A,B) \in R\}$ *holds.*
3. $\mathbb{G} = \{[A \longrightarrow B] : A,B \subseteq E\}$ *is a join-generating set of the lattice* $\mathbb{I}$ *of CIS.*

Proof (1) is straightforward from the definitions of the maps i and s and Theorem 7.68.

For (2), consider a relation R on $P(E)$. First observe that, since is is a closure map on $\underline{2}^{(P(E))^2}$ (as a consequence of Theorem 7.68), we have $is(\{(A,B)\}) = [A \longrightarrow B] \subseteq is(R)$ for any pair $(A,B) \in R$. The inclusion $\bigvee\{[A \longrightarrow B] : (A,B) \in R\} \subseteq is(R)$ follows, where the join is that of the lattice $\mathbb{I}$. We also have $(A,B) \in is(\{(A,B)\}) = [A \longrightarrow B]$, whence $R \subseteq \bigcup\{[A \longrightarrow B] : (A,B) \in R\} \subseteq \bigvee\{[A \longrightarrow B] : (A,B) \in R\}$ and so, since the latter element is closed by is, $is(R) \subseteq is(\bigvee\{[A \longrightarrow B] : (A,B) \in R\}) = \bigvee\{[A \longrightarrow B] : (A,B) \in R\}$. So the equality in (2) is obtained.

Taking an element $R \in \mathbb{I}$ we have $R = is(R) = \bigvee\{[A \longrightarrow B] : (A,B) \in R\}$. So, any CIS is obtained as a join of elements of the subset $\mathbb{G}$ of $\mathbb{I}$, which is (3).

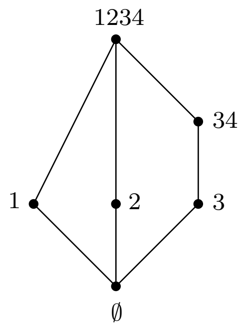
□

Definition 7.74 Given a CIS I, a relation R on $P(E)$ is said to be a *basis* of I if $is(R) = I$ and R is minimal for inclusion with this property.

Given a Moore family $\mathcal{F}$ on E, a relation R on $P(E)$ is said to be an *implication basis* of $\mathcal{F}$ (or of the associated closure $\varphi_{\mathcal{F}}$) if R is a basis of the relation $i(\mathcal{F})$ (or equivalently if $s(R) = \mathcal{F}$).

In other terms, the relation R is a basis of a CIS I if I is the smallest CIS including R and if the closure $is(R')$ of any proper sub-relation R' of R is strictly included in I. Similarly an implication basis of a Moore family $\mathcal{F}$ – or of the associated closure $\varphi_{\mathcal{F}}$ – is a minimal set of implications which generates $\mathcal{F}$ (by the map s). It results from Item (2) in Lemma 7.73 and Corollary 3.12 that the set $\mathbb{G}$ of implications of the type $[A \longrightarrow B]$ contains the join-irreducibles of $\mathbb{I}$ (that Exercise 7.24 proposes to characterize). It is then clear that any minimal join-irreducible representation of a CIS I of $\mathbb{I}$ corresponds to a basis of I. Nevertheless, the important point is that there also exist bases of I (or $\mathcal{F}$) the elements of which are not join-irreducibles of $\mathbb{I}$ and which contain in general a smaller number of implications. This is the case for the so-called *Guigues–Duquenne canonical implication basis*. Such a basis is defined for any CIS I and so for the corresponding Moore family $\mathcal{F} = s(I)$ (it is also defined for any family $\mathcal{D}$ such that $\mathbf{m}(\mathcal{D}) = \mathcal{F}$). It will be denoted by $K_{\mathcal{F}}$ in the sequel.

Obviously, it is possible to delete a closed set F from a Moore family without leaving $\mathbb{F}$ if and only if F is a meet-irreducible of $\mathbb{F}$. The following definitions, which focus on possible additions, will allow us to characterize the Guigues–Duquenne basis in Theorem 7.79 and to explain in Proposition 7.80 in what sense it is "canonical."



$$4 \longrightarrow_{\mathcal{F}} 3$$
$$12 \longrightarrow_{\mathcal{F}} 34$$
$$13 \longrightarrow_{\mathcal{F}} 24$$
$$23 \longrightarrow_{\mathcal{F}} 14$$

Figure 7.10 A Moore family $\mathcal{F}$ and its Guigues–Duquenne canonical basis.

We will go back to the importance of this basis in Section 7.6.4 of Further topics and references.

Definition 7.75 Let $\mathcal{F}$ be a Moore family on a set E and $F \in \mathcal{F}$. A subset Q of E is:

- A *quasi-closed set of* $\mathcal{F}$ if $Q \notin \mathcal{F}$ and $\mathcal{F} + \{Q\} \in \mathbb{F}$.
- An F*-quasi-closed set of* $\mathcal{F}$ if Q is a quasi-closed set of $\mathcal{F}$ and $\varphi_{\mathcal{F}}(Q) = F$.
- An F*-critical set of* $\mathcal{F}$ if Q is an F-quasi-closed set of $\mathcal{F}$ and is minimal with that property.
- A *critical set of* $\mathcal{F}$ if Q is F-critical for some $F \in \mathcal{F}$.

The set of critical sets of $\mathcal{F}$ is denoted by $\mathcal{C}_{\mathcal{F}}$. The *Guigues–Duquenne canonical implication basis* of $\mathcal{F}$ is the relation $K_{\mathcal{F}} = \{(Q, \varphi_{\mathcal{F}}(Q) \setminus Q) : Q \in \mathcal{C}_{\mathcal{F}}\}$, which also writes as the set of implications $K_{\mathcal{F}} = \{Q \longrightarrow_{\mathcal{F}} \varphi_{\mathcal{F}}(Q) \setminus Q : Q \in \mathcal{C}_{\mathcal{F}}\}$.

Observe that a set $Q \notin \mathcal{F}$ is a quasi-closed set of $\mathcal{F}$ if $Q \cap F \in \mathcal{F} + \{Q\}$ for any $F \in \mathcal{F}$, which also writes as $\mathbf{m}(\mathcal{F} + \{Q\}) = \mathcal{F} + \{Q\}$.

The previous definitions are illustrated on the Moore family $\mathcal{F}$ in the following example.

Example 7.76 Figure 7.10 shows a Moore family $\mathcal{F}$ given in its lattice form. Its F-quasi-closed sets are 4 (for $F = 34$) and 12, 13, 134, 23, and 234 (for $F = 1234$). Its critical sets are 4, 12, 13, and 23 and its canonical implication basis is $\{4 \longrightarrow_{\mathcal{F}} 3, 12 \longrightarrow_{\mathcal{F}} 34, 13, \longrightarrow_{\mathcal{F}} 24, 23 \longrightarrow_{\mathcal{F}} 14\}$.

The definitions in Definition 7.75 immediately extend to any family $\mathcal{D}$ of subsets of E by replacing $\mathcal{F}$ with $\mathbf{m}(\mathcal{D})$. Exercise 7.25, where particular bases are associated with any implication basis of $\mathcal{F}$, allows us to possibly replace implications of the type $Q \longrightarrow_{\mathcal{F}} \varphi_{\mathcal{F}}(Q) \setminus Q$ with those of the type $Q \longrightarrow_{\mathcal{F}} \varphi_{\mathcal{F}}(Q)$. Observe that this simpler general formulation in fact corresponds to redundant implications in the sense that the elements of Q are repeated in their right-hand part.

Proposition 7.71 implies the existence of quasi-closed sets – and so of critical sets – for any Moore family $\mathcal{F}$ on E, different from the maximum $P(E)$ of the lattice $\mathbb{F}$. Indeed in this case, $\mathcal{F}$ is covered by $\mathcal{G} = \mathcal{F} + \{Q\}$ (so with Q a quasi-closed set of $\mathcal{F}$). Exercise 7.26 leads to the recognition of some critical sets.

Example 7.77 Consider again the Moore family $\mathcal{F}$ of Example 7.61. The set $\mathcal{Q}$ of quasi-closed sets of $\mathcal{F}$ is given by:

$$\mathcal{Q} = \{M, DG, DU, GP, GU, PU, GHP, GHU, GPU, HPU, DGHM, DGHP, DGHU, DHMP, DHMU, DHPU, GHPU, DGHMP, DGHMU, DGHPU, DGMPU, DHMPU\}$$

For instance, DG is a quasi-closed set of $\mathcal{F}$ since the intersection of any closed set of $\mathcal{F}$ with DG belongs to $\{\emptyset, D, G, DG\}$. It is not the same for GM since its intersection with the closed set DM is M which is not closed. With some patience, the reader may check the above list of quasi-closed sets. On the other hand, he can easily check that $\varphi_{\mathcal{F}}(M) = DM$, $\varphi_{\mathcal{F}}(DG) = DGH$, $\varphi_{\mathcal{F}}(DU) = DHU$, and $\varphi_{\mathcal{F}}(Q) = E$ in all other cases. Applying the above definitions, we observe that M is DM-critical, DG is DGH-critical, DU is DHU-critical, whereas GP, GU, PU, $DGHM$, $DHMP$, and $DHMU$ are E-critical. So the canonical implication basis $K_{\mathcal{F}}$ of $\mathcal{F}$ is formed of the following nine implications:

(1) $M \longrightarrow_{\mathcal{F}} D$	(2) $DG \longrightarrow_{\mathcal{F}} H$	(3) $DU \longrightarrow_{\mathcal{F}} H$
(4) $GP \longrightarrow_{\mathcal{F}} DHMU$	(5) $GU \longrightarrow_{\mathcal{F}} DHMP$	(6) $PU \longrightarrow_{\mathcal{F}} DGHM$
(7) $DHMU \longrightarrow_{\mathcal{F}} GP$	(8) $DHMP \longrightarrow_{\mathcal{F}} GU$	(9) $DGHM \longrightarrow_{\mathcal{F}} PU$

Note that the three pairs (4) and (7), (5) and (8), (6) and (9) are pairs of converse implications: for instance, GP (possession of *goat* and *pedigree chicken*) implies $DHMU$ (possession of *common duck*, *Manila duck*, *Bantam chicken*, and *water buffalo*) and conversely. The six left-hand terms in these implications are the minimal sets the presence of which implies that of all species of the livestock.

The justification of the "canonical basis" denomination has two parts. The following theorem states that the previously defined set $K_{\mathcal{F}}$ of implications is an implication basis – in the sense of Definition 7.74 – of the Moore family $\mathcal{F}$. Then, we specify how this basis is canonical, this time referring to the literature for a proof. We first give a proposition which will be useful in the proof of the theorem. It is related to several results of this section (Proposition 7.71 and Exercise 7.26).

Proposition 7.78 *Let $\mathcal{F}$ and $\mathcal{G}$ be two Moore families on a set E with $\mathcal{F} \subset \mathcal{G}$, and A an element of $\mathcal{G} \setminus \mathcal{F}$.*

1. *If A is minimal w.r.t. inclusion in $\mathcal{G} \setminus \mathcal{F}$, then A is a quasi-closed set of $\mathcal{F}$.*
2. *If A is maximal w.r.t. inclusion in $\mathcal{G} \setminus \mathcal{F}$, then A is a meet-irreducible of the lattice $\mathcal{G}$.*

Proof (1) Let A be a minimal element of $\mathcal{G} \setminus \mathcal{F}$. If A is not a quasi-closed set of $\mathcal{F}$, then there exists a closed set F in $\mathcal{F}$ with $A \cap F \notin \mathcal{F} + \{A\}$. Then we have

$A \cap F \subset A$ and, since $A \cap F \in \mathcal{G} \setminus \mathcal{F}$, we find a contradiction with the assumption that A is minimal.

(2) Let A be a maximal element of $\mathcal{G} \setminus \mathcal{F}$. If A is not a meet-irreducible of $(\mathcal{G}, \subseteq)$, then there exist $G, G' \in \mathcal{G}$ with $A = G \cap G'$, $A \subset G$, and $A \subset G'$. By the maximality assumption on A, this implies $G, G' \in \mathcal{F}$ and so $A \in \mathcal{F}$, a contradiction. □

Theorem 7.79 *Let $\mathcal{F}$ be a Moore family on a set E and $\mathcal{C}_\mathcal{F}$ the set of critical sets of $\mathcal{F}$. Then the relation $K_\mathcal{F} = \{(Q, \varphi_\mathcal{F}(Q) \setminus Q) : Q \in \mathcal{C}_\mathcal{F}\}$ on $P(E)$ is an implication basis of $\mathcal{F}$.*

Proof The relation $K_\mathcal{F}$ is an implication basis of $\mathcal{F}$ if $s(K_\mathcal{F}) = \mathcal{F}$, where s is the saturation operator (given in Definition 7.67), and if $\mathcal{F}$ is minimal for this equality.

We first show the equality; that is, $\{F \subseteq E : (A,B) \in K_\mathcal{F}$ and $A \subseteq F$ imply $B \subseteq F\} = \mathcal{F}$. The inclusion $\mathcal{F} \subseteq s(K_\mathcal{F})$ is obtained when observing that, by Definition 7.62, any implication $Q \longrightarrow \varphi_\mathcal{F}(Q) \setminus Q$ of $K_\mathcal{F}$ belongs to the relation $i(\mathcal{F})$. So $K_\mathcal{F} \subseteq i(\mathcal{F})$, whence $\mathcal{F} = \mathbf{m}(\mathcal{F}) = si(\mathcal{F}) \subseteq s(K_\mathcal{F})$. Indeed, the former equality follows from the fact that $\mathcal{F}$ is a Moore family and so is closed by $\mathbf{m}$, the latter from Item (2) in Theorem 7.68, and the inclusion from the antitony of the map s, a consequence of Item (1) in the same theorem.

Now, we show that this inclusion is in fact an equality; that is, any subset A of E which belongs to $s(K_\mathcal{F})$ also belongs to $\mathcal{F}$. To do so, we first characterize the subsets $A \in s(K_\mathcal{F})$. We deduce $s(K_\mathcal{F}) = sis(K_\mathcal{F})$ from the properties of Galois connections (Item (1) in Theorem 3.41). The previously observed formula $is(K_\mathcal{F}) = \bigvee\{[Q \longrightarrow \varphi_\mathcal{F}(Q) \setminus Q] : Q \in \mathcal{C}_\mathcal{F}\}$ (Lemma 7.73) implies $s(K_\mathcal{F}) = \bigcap\{s([Q \longrightarrow \varphi_\mathcal{F}(Q) \setminus Q]) : Q \in \mathcal{C}_\mathcal{F}\} = \bigcap\{\mathcal{F}_{Q,\varphi_\mathcal{F}(Q)\setminus Q} : Q \in \mathcal{C}_\mathcal{F}\}$, since the restriction of s to $\mathbb{I}$ is a dual isomorphism between the lattices $\mathbb{I}$ and $\mathbb{F}$ and since $s(\{(A,B)\}) = \mathcal{F}_{A,B}$ (Item (1) in Lemma 7.73). In other terms, a subset A of E belongs to $s(K_\mathcal{F})$ if and only if, for any $Q \in \mathcal{C}_\mathcal{F}$, it satisfies the property $[Q \not\subseteq A$ or $\varphi_\mathcal{F}(Q) \setminus Q \subseteq A]$, which also simply expresses as $[Q \not\subseteq A$ or $\varphi_\mathcal{F}(Q) \subseteq A]$.

Assume $\mathcal{F} \subset s(K_\mathcal{F})$ and consider a subset Q, minimal in $s(K_\mathcal{F}) \setminus \mathcal{F}$. From Proposition 7.78, Q is a quasi-closed set of $\mathcal{F}$. Then, there exists a critical set $Q' \in \mathcal{C}_\mathcal{F}$ for which the inclusions $Q' \subseteq Q$ and $Q \subset \varphi_\mathcal{F}(Q') = \varphi_\mathcal{F}(Q)$ hold, a contradiction with our assumption that Q belongs to $s(K_\mathcal{F})$. So we have $\mathcal{F} = s(K_\mathcal{F})$ as required.

Finally, we complete the proof by showing that the relation $K_\mathcal{F}$ is minimal w.r.t. the equality $s(K_\mathcal{F}) = \mathcal{F}$. We have to show that, for any $Q \in \mathcal{C}_\mathcal{F}$, the strict inclusion $\mathcal{F} \subset s(K_\mathcal{F} \setminus \{(Q, \varphi_\mathcal{F}(Q) \setminus Q)\})$ holds. This is satisfied since Q is not an element of $\mathcal{F}$ and we are going to show that $Q \in s(K_\mathcal{F} \setminus \{(Q, \varphi_\mathcal{F}(Q) \setminus Q)\}) = \bigcap\{\mathcal{F}_{Q',\varphi_\mathcal{F}(Q')\setminus Q'} : Q' \in \mathcal{C}_\mathcal{F} \setminus \{Q\}\}$. So we have to show that, for any $Q' \in \mathcal{C}_\mathcal{F} \setminus \{Q\}$, $Q' \not\subseteq Q$ or $\varphi_\mathcal{F}(Q') \setminus Q' \subseteq Q$. Indeed let $Q' \in \mathcal{C}_\mathcal{F} \setminus \{Q\}$. If $Q' \not\subseteq Q$, then $Q \in \mathcal{F}_{Q',\varphi_\mathcal{F}(Q')\setminus Q'}$. Otherwise, we have to show that $\varphi_\mathcal{F}(Q') \setminus Q' \subseteq Q$. Since Q is quasi-closed, $Q \cap \varphi_\mathcal{F}(Q') \in \{Q\} \cup \mathcal{F}$. Now $Q \cap \varphi_\mathcal{F}(Q') = Q$ would imply $Q \subset \varphi_\mathcal{F}(Q')$ whence, since $Q' \subset Q$ is assumed, $\varphi_\mathcal{F}(Q) = \varphi_\mathcal{F}(Q')$, which is impossible (since Q and Q' are critical sets). So $Q' \subset$

$Q \cap \varphi_{\mathcal{F}}(Q') \in \mathcal{F}$ and, as a consequence, $\varphi_{\mathcal{F}}(Q') = Q \cap \varphi_{\mathcal{F}}(Q')$. Now, since Q is a critical set, $\varphi_{\mathcal{F}}(Q') \subset Q$ and $\varphi_{\mathcal{F}}(Q') \setminus Q' \subset Q$ as required. □

To end this section we state a property of the canonical implication basis $K_{\mathcal{F}}$ of a Moore family $\mathcal{F}$ which implies, in particular, that this basis has a minimal size among all implication bases of $\mathcal{F}$. Moreover, any implication basis of the same size is related to $K_{\mathcal{F}}$. In fact, Proposition 7.80 below is a simplified version of Theorem 51 in Caspard and Monjardet (2003). The proof is omitted and we refer the reader to this paper for a more complete characterization of implication bases of $\mathcal{F}$ with its proof. There one also finds further references and a survey of the terminology variants on this topic in the literature.

The following statement uses the – easy to prove – fact that the family obtained by adding all its quasi-closed sets to a Moore family $\mathcal{F}$ is still a Moore family $\mathcal{G}$. The closure operator associated with $\mathcal{G}$ is denoted by σ.

Proposition 7.80 *Let $\mathcal{F}$ be a Moore family on a set E and $R = \{(A_1, B_1), ..., (A_r, B_r)\}$ an implication basis of $\mathcal{F}$. Then, for any critical set $C \in \mathcal{C}_{\mathcal{F}}$, there exists $i \in \{1, ..., r\}$ such that $A_i \subseteq \sigma(A_i) = C$.*

Thus, any relation R on $P(E)$ including an implication basis of a Moore family $\mathcal{F}$ (that is, such that $s(R) = \mathcal{F}$) has a size greater than or equal to that of the canonical implication basis $K_{\mathcal{F}}$. Exercise 7.29 shows that the size of $K_{\mathcal{F}}$ may be large.

7.5 Orders in scheduling

The word "scheduling" has several meanings. Here we use it within the framework of management and operations research. In the most general way, one must temporally assign some resources to a set of interdependent jobs (or tasks), that are required to carry out a given project. More precisely, we will focus here on the problems of the following type: so as to carry out this project, one has a number of resources (CPUs, machines, etc.) used to execute the jobs, some of which have to be completed before some others may get started; then the problem is about finding a temporal allocation of the resources to the jobs, which is optimal with respect to some criterion. That kind of problem has many variants, for which one must search for algorithms to obtain an optimal solution and study their complexity. In the Further topics and references section, as well as in Appendix A, we provide some general references on these scheduling problems and on the complexity of the solving algorithms. Yet, in the present section, we are only concerned with showing the ordinal aspects of some of these problems and, so, we limit ourselves to presenting three of them, simple but classic. The fact that the first two problems have an ordinal dimension is obvious since the precedence constraints between jobs involve a strict (most often partial) order on the latter. The third problem will illustrate a case which, although it is free from precedence constraint, will also lead us to consider an order and its linear extensions.

In the first two problems, the basic assumptions on the jobs and the machines require that the jobs may not be parceled out, that each machine can only carry out one job at a time, and that each job be carried out without interruption.

The criterion to minimize in the first problem is a cost whereas, in the last two, it is the completion time of all the jobs; that is, the total duration of the project (nevertheless, "time is money"!).

In the sequel, all ordered sets are assumed to be strict and the word "order" will always stand for "strict order."

7.5.1 The single-machine scheduling problem

We first consider the easiest variant of the problem, whose mathematical modeling is as follows:

- An ordered set $P = (X,<)$, where $X = \{x,y,z,...\}$ is the set of the n jobs to be carried out and where $<$ is the order on the jobs given by some precedence constraints: $x < y$ if job y may not start before job x is completed.
- A machine to carry out the jobs.

In such a problem, the various possible orders for the execution of the set of jobs are then all linear extensions of P. It is assumed that a cost is associated with each of these linear extensions. The problem then becomes to determine a linear extension of minimal cost.

Suppose for instance that, when x and y are two jobs incomparable for the precedence order on jobs, carrying out y just after x has a cost, namely $c(x,y)$. In a linear extension of P containing the ordered pair (x,y), the latter then constitutes a jump (Chapter 1, Definition 1.32) and the cost of that extension is the sum of the costs of its jumps. If one moreover assumes that the cost of a jump is constant, the obtained problem is to find a linear extension minimizing the number of jumps. This problem is known to be $\mathcal{NP}$-hard but may become polynomial for some classes of orders (see Appendix A). A *fortiori*, the same observations may be made, in the case where the costs of the jumps are no longer constant, on the problem of minimizing the sum of the costs of the jumps.

A more realistic version of the single machine problem is the following: with a job x are associated its execution time $t(x)$ and a weight $p(x)$. Consider a possible scheduling of the jobs; that is, a linear extension $L = x_1...x_i...x_n$ of their precedence order. Thus the job x_i is ended at the time $d(x_i) = \Sigma_{j \leq i} t(j)$. Then, with the linear order L is associated the quantity $D(L) = \Sigma_{i=1}^{n} p(x_i)d(x_i)$, called the *average weighted completion time*. The problem is then to find a linear extension of the precedence order that minimizes that quantity. This optimization problem is in general very difficult ($\mathcal{NP}$-hard in the sense given in Appendix A). It may however be solved for some order classes, such as the class of series–parallel orders. These results will be found in Section A.2.3 of Appendix A.

7.5.2 The m-machine scheduling problem

The considered mathematical model is the following:

- An ordered set $P = (X, <)$, where $X = \{x, y, z, ...\}$ is the set of the n jobs to be carried out and where $<$ is the order on the latter given by some precedence constraints: $x < y$ if job y may not start before job x is completed.
- A set $I = \{1, 2, ..., m\}$ of m identical parallel machines to carry out the jobs.

The problem consists of finding an optimal scheduling of the jobs; that is, a scheduling which minimizes the total completion time.

Saying that the machines are identical, we mean that each of them will carry out a given job with the same duration time. Saying that they are parallel, we mean that they can execute different jobs simultaneously. On the other hand, $x < y$ means that job y may not start on any machine before job x has been completed on any machine. Time is measured by a (positive) integer and, in order to simplify the exposition, we will assume here that executing a job takes a constant time independent of the job and chosen equal to 1 time unit.

The notion of a scheduling in such a model is then formalized by the following definition:

Definition 7.81 Let $P = (X, <)$ be an ordered set (of jobs) and $I = \{1, 2, ..., m\}$ a set (of machines). An *m-scheduling of P* is a map $f = (f_1, f_2)$ from P to the direct product $I \times \mathbb{N}$, such that the map f_2 is strictly isotone (i.e., $x < y$ implies $f_2(x) < f_2(y)$).

This definition must be understood as follows:

- The map f_1 assigns a machine to a job: $f_1(x) = i$ if machine i is assigned to job x.
- The map f_2 measures the execution time: $f_2(x) = p$ if job x is carried out during the pth time unit. The fact that f_2 must be isotone is the expression of the precedence constraints on jobs.

Example 7.82 The set X of jobs is $\{a, b, c, d, e, f, g\}$ and their precedence order is given by the diagram in Figure 7.11(a). Two machines are available. A 2-scheduling is given by the table in Figure 7.11(b): the first machine successively executes the jobs a, c, e, g, the second deals with the jobs b, d, f. Such a table is called a *Gantt chart*.

The criterion chosen to determine an optimal scheduling is that of the total completion time of the n jobs; that is, $\max\{f_2(x), x \in P\}$, denoted by $C_{\max}(f)$. An m-scheduling problem is then defined as follows:

Definition 7.83 For a given ordered set P, the *m-scheduling problem* consists of minimizing the value of $C_{\max}(f) = \max\{f_2(x), x \in P\}$ on all m-schedulings f of P.

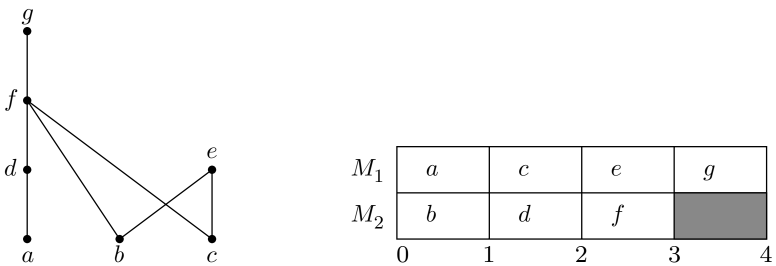


Figure 7.11 (a) An ordered set P of jobs and (b) the Gantt chart of a 2-scheduling of P.

We first give a number of translations of the latter problem by presenting, in an ordinal framework, several equivalent notions of an m-scheduling. To do so, we use strict weak orders (see Proposition 7.4), the width, the range, the extensions, and the lattice of downsets of an ordered set (Definitions 1.30, 1.31, and Theorem 5.6).

Proposition 7.84 *Let $P = (X,<)$ be an ordered set and k an integer. The following properties are equivalent:*

1. *P has an m-scheduling of completion time equal to k.*
2. *P has as an extension a weak ordered set of width at most equal to m and of range k.*
3. *In the lattice $\mathcal{D}(P)$ of downsets of P, there exists an extended chain $\emptyset = D_0 \subset D_1 \subset ... \subset D_k = X$ of length k and such that, for $i = 1,2...,k$, the set $A_i = D_i \setminus D_{i-1}$ contains at most m maximal elements of D_i.*

Proof (1) $\Longrightarrow$ (2): given an m-scheduling $f = (f_1,f_2)$, the map f_2 induces a weak order O including the order of P and defined by xOy if and only if $f_2(x) < f_2(y)$ (see Proposition 7.4); since an antichain of size t of this order corresponds to a set of t jobs incomparable for the precedence order – so to which t distinct machines can be simultaneously assigned – the size of the antichain is at most m; since the order is weak, its range is the number of such antichains and so, is equal to the total completion time of all jobs, i.e., the integer k.

(2) $\Longrightarrow$ (1): let O be a weak order, extension of P, of width at most m and of range k. According to Proposition 7.4, it may be written as a linear sum $A_1 \oplus A_2 \oplus ... \oplus A_k$ of k antichains, each of size at most m. We write $A_i = \{x_{i1},...,x_{iq_i}\}$ with $1 \leq q_i \leq m$. An m-scheduling $f = (f_1,f_2)$ is realized by writing, for $i = 1,2,...,k$ and $j = 1,2...,q$, $f_1(x_{ij}) = j$ and $f_2(x_{ij}) = i$.

(2) $\Longleftrightarrow$ (3): we know (see Remark 5.33) that the weak orders of range k which are extensions of the order of P are in a one-to-one correspondence with the extended chains of length k of the lattice of downsets of P. In this correspondence, $A_i = D_i \setminus D_{i-1}$ $(i = 1,...,k)$ is the antichain of rank i of the weak order, whence the second condition of (3). $\square$

It follows from this proposition that the problem of searching for an m-scheduling of P which minimizes the completion time comes down to searching for an extended chain $\emptyset = D_0 \subset D_1 \subset ... \subset D_k = X$ in the lattice of downsets of P such that, for any $i = 1, 2, ..., k$, the set $A_i = D_i \setminus D_{i-1}$ contains at most m maximal elements of D_i, and whose length k is minimal. So, in the m-scheduling, A_i is the set of the jobs which are executed during the ith time unit and its size is at most m.

One may also state this problem in terms of graphs. Given the ordered set $P = (X, <)$, one builds the directed graph $G_m(P)$ whose vertices are all downsets of P and whose arcs are the ordered pairs (D, D') of downsets such that $D \subset D'$ and where $D' \setminus D$ has at most m maximal elements of D'. Searching for an optimal m-scheduling of P is then reduced to the search for a shortest path from the subset $\emptyset$ to the subset X in the graph $G_m(P)$.

We illustrate this procedure in Example 7.82, where $m = 2$. The lattice of downsets of the ordered set P in this example (Figure 7.11) is represented in Figure 7.12(a). In this lattice one only needs to build the graph $G'_2(P)$, keeping only some of the vertices of $G_2(P)$ defined as follows: starting from a kept vertex D (which in the beginning is the empty set), one keeps all vertices D' and all arcs (D, D') where D' is a downset of the form $D + A$ with A an antichain of size 2 or maximal of size 1. In Figure 7.12(a) the graph $G'_2(P)$ is represented on the diagram of $\mathcal{D}(P)$ as follows: its vertices are the labeled ones and its arcs are represented either by an arrow put on the used covering arc or by a dotted arc when the used arc is not covering. We see in this figure that there exist three paths from the subset $\emptyset$ to the subset X, among which one of length 5 and two of length 4, the latter two thus corresponding to two optimal 2-schedulings of P. Figure 7.11(b) shows one of these two schedulings, which corresponds to the first weak order represented in Figure 7.12(b). As for the second weak order represented in the latter, it corresponds to the path of length 5 in $G'_2(P)$ and so with a non-optimal scheduling.

What precedes allows us to obtain complexity results. Since the search for a shortest path in a graph may be done in polynomial time, it will be the same when searching for an optimal m-scheduling of P, as soon as building $G_m(P)$ will itself take a polynomial time. In particular, since the number of antichains – and so of downsets – of an ordered set of size n and of width at most ℓ is bounded by n^{ℓ} (why?), the following result is obtained:

Corollary 7.85 *For some fixed integer ℓ, the m-scheduling problem (with $m \leq \ell$), restricted to the class of ordered sets the width of which is at most ℓ, can be solved in polynomial time.*

7.5.3 The two-step (and two-machine) scheduling problem

In this problem, the jobs to be carried out must be executed in two steps on two different machines. Machine M executes the first part of job x_i in time t_i, and machine M' deals with the second part in time t'_i, which may start only after the end of the execution of

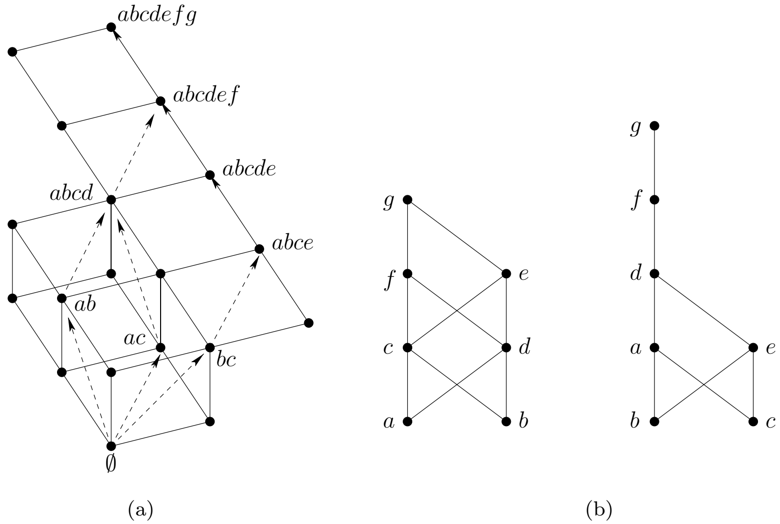


Figure 7.12 (a) The lattice $\mathcal{D}(P)$ and the graph $G'_2(P)$ of the ordered set P in Figure 7.11(a); (b) two weak orders of width 2, extensions of P.

the first part on M. Unlike the previous problems, no precedence constraint between jobs is assumed here but we always suppose that each machine can execute one (part of a) job at a time and that it carries it out without any interruption.

The mathematical model is as follows:

- A set $X = \{x_1, x_2, ..., x_n\}$ of n jobs to be carried out.
- A map f from X to the direct product $\mathbb{R}^+ \times \mathbb{R}^+ : f(x_i) = (t_i, t'_i)$, where t_i (respectively, t'_i) is the execution time of x_i on machine M (respectively, on machine M').

The problem consists of finding an optimal scheduling of the jobs; that is, a scheduling which minimizes the total completion time.

Here a scheduling can be expressed as a pair (L, L') of linear orders on X: machine M executes the first parts of the jobs in the order of L and machine M' executes the second parts in the order of L'.

First notice that an optimal scheduling of the form (L, L) can always be obtained. Indeed, consider on the one hand the linear order L written as a permutation $x_1 x_2 ... x_n$ on X and on the other hand the principal downset $x_1 x_2 ... x_i x_j$ of the linear order L' up to the first element x_j $(j > i+1)$, where L and L' differ (to simplify the notation, x_i in these orders stands either for the first or the second part of the job). This implies that x_j and x_{i+1} have already been made on M. If we change L' by inserting x_{i+1} between x_i and x_j and by executing on M' the remaining jobs as soon as possible, then the total completion time may not be increased (the reader can easily check that with a drawing).

Table 7.7 Minimums of execution times

$\min(t_i,t'_j)$	$t'_1=3$	$t'_2=5$	$t'_3=2$	$t'_4=3$
$t_1=1$		1	1	1
$t_2=4$	3		2	3
$t_3=2$	2	2		2
$t_4=4$	3	4	2	

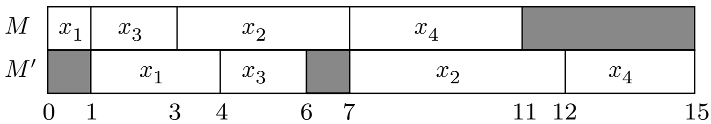


Figure 7.13 Gantt chart of an optimal scheduling on the jobs x_1,x_2,x_3,x_4.

From now on, we search for an optimal scheduling given by a linear order L on X. Machine M will execute the first parts of the jobs without interruption and following the order L, so in a time equal to $\Sigma_{i=1}^{n} t_i$. Machine M' will execute the second part of a job from the moment that the execution of the first part is finished by M (so with possible interruptions).

Example 7.86 Figure 7.13 uses a Gantt chart to illustrate such a scheduling for the following example: $X = \{x_1,x_2,x_3,x_4\}$ and the execution times on M and M' are given in Table 7.7.

Intuitively, one conceives that job x_i should be executed early if t_i is small and late if t'_i is large. More precisely, the following result holds, whose (non-difficult) proof is left as an exercise (Exercise 7.34).

Lemma 7.87 *Let $x_1 \ldots x_{k-1}x_k \ldots x_l \ldots x_n$ be the order L on the jobs. If job x_l ($l > k$) satisfies $t_l \leq \min(t_k,t'_l)$, the order $L' = x_1 \ldots x_{k-1}x_l x_k \ldots x_{l-1}x_{l+1} \ldots x_n$ obtained by inserting x_l between x_{k-1} and x_k has a completion time at most equal to that given by L.*

In this case, one then considers that one had better execute x_l before x_k. The same result holds if $t'_k \leq \min(t_k,t'_l)$. This leads to defining a relation $<$ on the set X of jobs, that is shown to be a strict order:

Lemma 7.88 *The relation $<$ defined on X by $x_i < x_j$ if $\min(t_i,t'_j) < \min(t_j,t'_i)$ is a strict order.*

Proof In a linearly ordered set like $(\mathbb{R}^+,\leq)$ the minimum of two elements u and v is equal to their meet: $\min(u,v) = u \wedge v$. Since $<$ is irreflexive, we just have to prove its transitivity; that is, if $t_i \wedge t'_j < t_j \wedge t'_i$ and $t_j \wedge t'_k < t_k \wedge t'_j$ hold, so does $t_i \wedge t'_k < t_k \wedge t'_i$. Now $t_i \wedge t'_j < t_j \wedge t'_i$ and $t_j \wedge t'_k < t_k \wedge t'_j$ imply $(t_i \wedge t'_j) \wedge (t_j \wedge t'_k) < (t_j \wedge t'_i) \wedge (t_k \wedge t'_j)$. The latter inequality may be rewritten as $(t_i \wedge t'_k) \wedge (t_j \wedge t'_j) < (t_k \wedge t'_i) \wedge (t_j \wedge t'_j)$, which in turn implies (since $\mathbb{R}^+$ is linearly ordered) that $t_i \wedge t'_k < t_k \wedge t'_i$, as required. □

We call *priority order (of the two-step scheduling problem)* the order $<$ defined above. Lemma 7.87 then implies that, if x_l is after x_k in the order L on jobs and before or incomparable to x_k in the priority order, L may be modified by inserting x_l before x_k without increasing the execution time. The following result then holds:

Proposition 7.89 *All linear extensions of the priority order of a two-step scheduling problem are optimal schedulings.*

Proof Consider an arbitrary scheduling L. If it is not a linear extension of the priority order $<$, there exist x_i, x_j such that $x_i < x_j$ and $x_j L x_i$. The transformation given in Lemma 7.87 can then be made by putting x_i between x_{j-1} and x_j in L, which does not increase the completion time. Iterating this operation, we obtain a linear extension of the priority order which does not increase the completion time. In particular, if L is an optimal scheduling we obtain a linear extension of the priority order, which itself is optimal. Moreover, it results from Theorem 5.41 that all linear extensions of an order are obtained from one of them by commutations (swaps between two consecutive elements). Since such commutations made on the obtained optimal linear extension may neither reduce nor increase (Lemma 7.87) the completion time, the latter remains optimal. □

It is worth noticing that the converse is not true. There may exist optimal schedulings that are not linear extensions of the priority order. We provide the following example.

Example 7.90 Consider the set $X = \{x_1, x_2, x_3, x_4\}$ of jobs. Table 7.7 gives the minimum values $\min(t_i, t'_j)$ of their execution times. We deduce the priority order on X, that we represent in Figure 7.14. This order has three linear extensions $x_1x_3x_2x_4$, $x_1x_2x_3x_4$, and $x_1x_2x_4x_3$, with which correspond three optimal schedulings, of total completion time 15. The first one was represented in Figure 7.13.

If, in this example, the time t'_3 becomes 1 instead of 2, we obtain a priority order which is the linear order $x_1x_2x_4x_3$, of total completion time equal to 14. We can check that $x_1x_2x_3x_4$ is also an optimal scheduling.

Exercise 7.35 describes a simple algorithm to obtain a linear extension of the priority order without need to explicitly building the latter order.

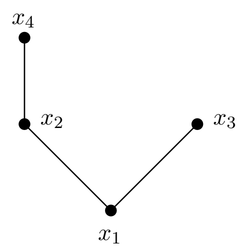


Figure 7.14 The priority order on X.

7.6 Further topics and references

7.6.1 Preference models

As we already mentioned, the ordinal models of preference presented in this section are used in many fields. For operations research, one can for instance see the publications of the "French school" of decision aid (Bouyssou and Roy, 1993; Bouyssou and Vincke 1997, 1998; Bouyssou *et al.*, 2009). For artificial intelligence or databases, one can consult the sites of specialized conferences, for instance the one held in 2006 (Multidisciplinary Workshop on Advances in Preference Handling, www.mycosima.com/ecai2006-preferences; see also Chomicki, 2003).

We now provide more details about the various motivations that have led to many rediscoveries of interval orders. It is a long and discontinuous history beginning in the early twentieth century. Then, in Cambridge, the child prodigy mathematician Norbert Wiener tried to answer a question raised by his mentor Bertrand Russell: how to obtain the notion of an instant in time (or of a point on a line) from the notion of an event in time (or of an interval on a line)? In his 1914 paper, Wiener defines a relation O of "complete succession" by two conditions: O is irreflexive and satisfies $OIO \subseteq O$, which is one of the characterizations of interval orders given in Exercise 7.4. Two further Wiener papers (the main content of which, translated in modern notations, can be found in Fishburn and Monjardet (1992)) contain developments on these orders. They are motivated by the project of finding a valid measurement theory for quantities such as the psychological perception of tonal height. For such quantities one can only define a "just-noticeable difference" (jnd) relation introduced by psychophysicists at the end of the nineteenth century. In this case, as observed by Poincaré, the indiscernibility relation between two quantities is no longer necessarily an equivalence, since it is not necessarily transitive.

In the 1930s, economists like Georgescu-Roegen or Armstrong observed a similar phenomenon for preferences: the indifference relation can be non-transitive. And 25 years later, the desire to obtain an individual preference model taking account of this possible intransitivity of indifference led Luce (1956) to define semiorders. While Wiener's papers have been largely forgotten (an exception is Riguet's 1951 CRAS), interval orders come out again in Fishburn's 1970 paper generalizing Luce's semiorders. In that paper, they are defined as irreflexive relations O satisfying the condition that xOy and zOt imply xOt or zOy.

Previously, for a problem in genetics, Benzer had considered "interval graphs." It was later observed that these (undirected) graphs are exactly the incomparability graphs of interval orders. Indeed the vertices of an interval graph are intervals (on the line) and the edges link two intervals having a non-empty intersection.

Since interval orders, semiorders, interval graphs, and hypergraphs, or indifference graphs (i.e., the incomparability graphs of semiorders) have appeared (or appeared again, since they have often been rediscovered) in multiple contexts in pure or applied mathematics and computer science: numerical and set representations of graphs, statistical estimation, seriation and archeology, data analysis, utility theory in mathematical psychology and microeconomics, decision aid, scheduling, temporal logics etc. Very many results and references on these structures can be found in books by Berge (1970), Golumbic (1980), Mirkin and Rodin (1984), Fishburn (1985), Pirlot and Vincke (1997), Aleskerov *et al.* (2007), in Trotter's survey (1997) or, much earlier, in Monjardet (1978, 1988). Let us only quote an interesting result, in particular for the algorithmic recognition of interval orders. Consider the following strict order $<$ between the maximal antichains of an order O: $A < A'$ if, for each x in $A \setminus A'$, there exists x' in A' such that xOx' (the associated reflexive order $\leq$ is a lattice; see Behrendt (1988)). Then a strict order is an interval order if and only if the order $<$ between its maximal antichains is a strictly linear order (see, for instance, Felsner *et al.* (1998) for more general results linking maximal chains in the lattice (P) with interval reductions or extensions of an arbitrary ordered set P). Moreover the number of maximal antichains of an interval order is exactly the number of "steps" of its step-type tableaux. Let us also mention two basic references for algorithmic aspects concerning these structures, namely the two Möhring's surveys (1984, 1989).

The essential property for various applications of interval orders (respectively, semiorders, weak orders) is that of their numerical representation with variable threshold (respectively, constant threshold, 0 threshold). Theorem 7.14 on the numerical representation of interval orders is due to Fishburn (1970) and Mirkin (1972). Theorem 7.16 on the numerical representation of semiorders was called the Scott–Suppes Representation Theorem, as a reference to the authors of its first proof (1958). But, after their rather complicated constructive proof, many other proofs have been found. A graph-theoretical proof due to Roy and Vincke is sketched in Exercise 7.7, and we have given the particularly simple inductive proof due to Balof and Bogart (2003). Other interesting proofs of the numerical representation results are based on inductive characterizations of interval orders and semiorders by Leblet and Rampon (2009). Let us also add that several classes of orders that generalize interval orders and semiorders by means of properties bearing on numerical representations have been studied; see their description in Fishburn's survey (1997) and in Fishburn and Trotter (1999b) about those called *split semiorders*.

The notion of a Ferrers relation was introduced by Riguet in 1951, the "Ferrers" appellation coming from the link with the Ferrers–Sylvester graph associated with an integer partition. However, the numerical representation is not mentioned and the notion did not give rise to much interest. But almost 20 years later, Ducamp and Falmagne (1969), motivated by the study of "Gutmann scales" in questionnaires analysis, defined the corresponding notion for a binary relation defined between two disjoint (and not necessarily finite) sets X and Y. They gave the following representation result: there exist two functions f from X to $\mathbb{R}$ and g from Y to $\mathbb{R}$ such that xRy if and only if $f(x) < g(y)$. The same result was independently obtained by Bouchet (1971, 1984) in his study of binary relations codings. Sometimes renamed biorders, Ferrers relations and their valued generalizations (allowing, in "probabilistic consistency" theory, to modelize the preferences of a subject at different times) have fostered many works, especially about the "Ferrers dimension" (or "bidimension") of a relation (see for example Monjardet (1976a), Cogis (1982a,b), Ducamp *et al*. (1984) or Doignon *et al*. (1986) as well as Falmagne's book (1985)).

7.6.2 Preference aggregation: Arrowian theorems for orders

As stated in the introduction of Section 7.2, Arrow's Impossibility Theorem has led to a considerable development of social choice theory illustrated, for example, by the books of Fishburn (1973), Kelly (1978), Moulin (1988) or Aizerman and Aleskerov (1995). Before mentioning some of these developments, let us specify the difference between our Theorem 7.32 ("Arrow for linear orders") and Arrow's Theorem itself. The latter deals with preference aggregation functions (called "social welfare functions" by Arrow) whose domain and codomain are respectively the set of preference profiles formed of total preorders and the set of total preorders or, equivalently (see Proposition 7.17 and Exercise 7.4), the set of profiles of weak orders and the set of weak orders. It does not characterize those functions that are independent and Paretian (the latter property concerning strict preferences) functions. It only states the existence of a *dictator*, i.e., a voter imposing his strict preference: if he/she strictly prefers candidate x to candidate y, so does the collective preference. But if the dictator is indifferent between these two candidates, the collective preference is not determined. It becomes so by strengthening some axioms. Then one obtains, for example, a *hierarchy of dictators*: each one can impose his/her strict preference if all his/her hierarchical superiors are indifferent. Whereas the initial version given by Arrow to his theorem was partly wrong (and was corrected in the second edition of his book, 1963), the correct version has now many proofs. We simply mention two proofs related to ordinal considerations. First Leclerc (1991) gave a generalization of Arrow's Theorem when individual and collective preferences are "fuzzy preorders." Next one can get this theorem from Theorem 7.32 that proves it for linear orders (see for example Monjardet (2003)); now the latter theorem characterizing projections can be obtained from results on projective ordered sets (Pouzet, 1998), where an ordered set P is called *projective* if every isotone and idempotent map from P^n to P is a projection. It is interesting to observe that, in Theorem 7.33 characterizing $\cap$-projections, the independency property can be replaced by a purely ordinal property, namely that F be a residual (see Leclerc and Monjardet (2012) for a more general result).

Several works of research motivated by Arrow's Theorem consist of checking its robustness by weakening the conditions making it possible. As seen in Section 7.2 (see also Barthélemy (1982)), to require that the collective preference be an order rather than a linear order is not enough since, then, one obtains oligarchic preference aggregation functions. If one weakens still more the demands on the collective preference by only requiring it to be cycle-free (which can be considered as a minimal level of rationality), the obtained functions are hardly more satisfactory since, for example, they give veto rights to some voters.

Another research direction consists of weakening the hypothesis that the preference aggregation function must provide a collective preference for any possible profile of individual preferences. In fact if the domain of admissible profiles is suitably restricted, Condorcet's majority rule provides a transitive collective preference. Thus this leads to searching for the so-called *Condorcet* (or *acyclic*) *domains*, i.e., for subsets $\mathcal{C}$ of $\mathcal{L}$ such that the majority relation $R_{MAJ}(\pi)$ applied to any profile π of linear orders taken in $\mathcal{C}$ is cycle-free. The first example of such a domain is when the linear orders representing the voters' preferences "respect" an "objective" linear order on the candidates: if a voter prefers x to y and if, in the objective order, y is preferred to z, then he also prefers x to z. It has been shown that this Condorcet domain, defined by Black (1958), is a particular case of a large class of Condorcet domains, all distributive and covering sublattices of the permutoedre lattice – see Section 5.6 in Chapter 5 (Chameni-Nembua, 1989; Galambos and Reiner, 2005). An example of such a Condorcet domain is given in Figure 7.15. It is a distributive covering sublattice of size 45 of the permutoedre lattice Σ_6 and it has been shown that this size is maximum for a Condorcet domain in Σ_6. These Condorcet domains can be obtained from a maximal chain of the permutoedre lattice (Abello, 1991) or by ad hoc procedures such as the "alternating scheme" (Fishburn, 1997). These results on Condorcet domains provide another link between social choice theory and ordered set theory (for a survey, see Monjardet (2009)). Working on a related problem, Fishburn (1974) initiated another link relating these theories. One considers the profile π of linear orders formed by the set $\mathcal{L}(P)$ of linear extensions of an ordered set P. The strict majority relation $R_{MAJ}(\pi)$ applied to this profile may contain a 3-cycle only if the width $\alpha(P)$ of P is greater than or equal to 3 (why?). Fishburn (1974) shows that, as soon as the size n of the ordered set is greater than or equal to 31, there exists an ordered set P of width 3 and height $n - 3$ such that the majority relation of the profile $\mathcal{L}(P)$ contains a 3-cycle. This means that, even applied to almost unanimous preferences, the majority rule may generate cycles. Since then it has been shown that the minimum size of P for which a cycle may appear is 9 (Ewacha *et al.*, 1990), and such ordered sets have been counted up to 13 elements (de Loof *et al.*, 2010).

One of the interests of Arrow's "axiomatic" approach in preference aggregation is to provide a theoretical basis for the fact that any aggregation rule has some undesirable effects (its "paradoxes"), like the Condorcet effect for the majority rule or that shown in Exercise 7.10 for Borda's rule. Then one has sought to specify the characteristics of the possible rules and, in particular, to characterize these rules by some properties. Thus for example there are "axiomatic" characterizations of Borda's rule (Young, 1974) or of the median procedure considered in Example 1.24 (Young and Levenglick, 1978). There we mentioned that the latter procedure coincides with Condorcet majority rule if voters' preferences are not arbitrary. In fact, it is the case

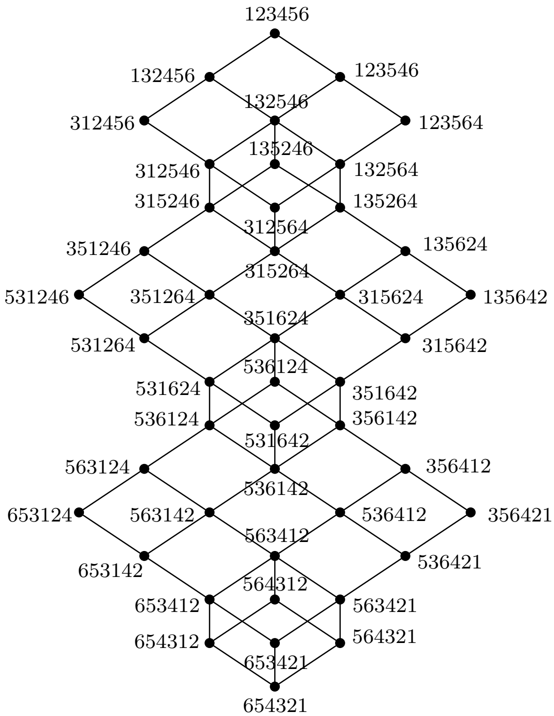


Figure 7.15 A maximal Condorcet domain in S_6.

if these preferences belong to some of the above-mentioned Condorcet domains (for which the majority relation is cycle-free). Indeed a linear order L is a median order for a profile $\pi = (L_1, ..., L_n)$ of linear orders if it maximizes among all possible linear orders the sum of its agreements with the orders of this profile, i.e., the quantity $\sum_{(y,x)\in L} n_\pi(y,x)$. Hence it is easy to see that if the majority relation associated with a profile π is cycle-free, any linear order including it is a median order for this profile. The fact that these Condorcet domains are distributive lattices is connected with the considerations of Section 7.6.3 on the links between majority irreducible elements and median elements in median semilattices (and in particular in distributive lattices). One will find a survey on these questions in Hudry *et al.* (2009).

Let us finally mention that Arrow's "axiomatic" approach in preference aggregation has been used in several other fields like for example data analysis, where one seeks to aggregate several classifications obtained on the same set of objects into a consensus classification. Several results similar to Arrow's Theorem have been shown, and they often bring into play the ordinal structure of the sets of classifications. Since then it has been possible to develop an "axiomatic" ordinal (or lattice) consensus theory allowing us to obtain such results in a unified way (see Monjardet (1990b), Leclerc and Monjardet (1995), and the Day and McMorris survey (2003)). This theory as well as the metric ordinal approach will be mentioned in Section 7.6.3.

7.6.3 The roles of orders in cluster analysis

The applications of ordered sets presented in this section concern the so-called (by Arabie (1982)) "Combinatorial Data Analysis," see for instance the survey by Hubert *et al.* (2001). The significance of the contribution of ordered set theory to cluster analysis appeared with an ordinal axiomatic approach of fitting problems due to Janowitz (1978), then with two surveys (Barthélemy *et al.*, 1984, 1986). In particular, Janowitz pointed out how residuation theory provides formalizations and generalizations to already known fitting methods (see also Leclerc (1994b)), while the papers by Barthélemy *et al.* gave the first results stemming from the ordinal formalization of problems of comparison and consensus (that is, aggregation) of classifications. More recently, Domenach and Leclerc (2001, 2002) proposed a formalization of fitting problems, subject to constraints on data and models, which extends some aspects of those in Janowitz (1978) and Barthélemy *et al.* (1984).

The symmetric difference distance in a distributive semilattice L has been expressed (Definition 7.44 and Proposition 7.45) in several ways: from the counting of join-irreducibles (which amounts to (unit) weighting), or from the rank function of L, or from path lengths in the neighborhood graph of L. Each of these characterizations leads to the definition of a family of metrics in ordered sets. Exercises 7.21 and 7.22 give an idea of some of these generalizations. In lattice theory, the study of valuations and their associated metrics dates back to Glivenko's work (1938). Monjardet's survey (1981) is concerned with the more general case of arbitrary ordered sets. A good illustration of the practical interest of these developments is provided by the case of partitions: various geodesic distances have been proposed for their comparison (see Arabie and Boorman, 1973). For many of these distances the simple problem of computing the distance between two partitions is $\mathcal{NP}$-hard (Day and Wells, 1984). On the contrary, the distances associated with (lower or upper) valuations are easily computable. Barthélemy and Leclerc (1995) surveyed several such valuations in the partition lattice (including Shannon's entropy).

The first works on classification consensus concerned partition aggregation. The latter was especially considered in statistics, where Régnier proposed as soon as 1965 to search for median partitions (that he called "central") in the sense of Definition 7.42 for the symmetric difference distance. Yet linked with the well-known fact that the partition lattice is not distributive, the problem of obtaining these medians turned out to be $\mathcal{NP}$-hard (see Barthélemy and Leclerc (1995) and Appendix A). The study of the aggregation of classification trees was developed in taxonomy in the 1980s, especially about phylogenetic reconstruction problems. Margush and McMorris showed as early as 1981 that using the majority rule on a profile of hierarchies provides a median hierarchy, unique for an odd profile. A similar property was already known for distributive lattices (Birkhoff and Kiss, 1947; Barbut, 1961; Monjardet, 1980) and for undirected trees of graph theory (Jordan, 1869; Zelinka, 1968). Theorem 7.55 unifies these results and extends the field where the majority rule and the median procedure coincide, namely the class of median semilattices (Bandelt and Barthélemy, 1984). This coincidence remains true with any metric associated with a weighting as in Exercise 7.22 (Leclerc, 1994a). The graphs where any triple of vertices has a unique median (for the distance of the minimum path length) are called *median graphs*; for instance, trees are such graphs. Median graphs are exactly the neighborhood graphs of median semilattices (Avann, 1961c). The reader will find in Hudry *et al*. (2009) a general exposition on the relations between metrics and lattice medians. Theorem 7.58 shows that, in other lattices and semilattices, the relations between medians and majorities remain in a weakened form (Leclerc, 1990b, 1993, 1994a).

Other approaches have been used for classification consensus, especially the axiomatic ones mentioned in Section 7.2. For this topic, the reader may refer to Monjardet (1990b), Leclerc and Monjardet (1995), and to Day and McMorris (2003). The characterization of the median procedure in median semilattices given in Theorem 7.57, due to McMorris *et al*. (2000), improves a previous version of Barthélemy and Janowitz (1991). The consistency condition used in this characterization is a general property of metric medians pointed out by Young and Levenglick (1978) in a social choice context. The name of the first condition is a distorted echo of the 1785 use by Condorcet of the majority rule in voting procedures.

7.6.4 Implicational systems, Moore families, and Galois data analysis

A considerable amount of work has been developed about the notion of an implication and an association, the latter being an implication with statistical measures of range (support) and precision (confidence). It was generally motivated by their use in various fields of applications. Several such fields are mentioned at the beginning of this section (knowledge discovery in databases, learning spaces, data analysis, etc.). It is worth adding, among others, artificial intelligence and data mining, where the aim is to extract some relevant information from a large amount of poorly structured data. The multiplicity and variety of these fields entail a terminological multiplicity and it is hard to find one's way through the latter.

For instance, in Ganter and Wille (1999), the Guigues–Duquenne (canonical implication) basis is called the "stem basis" and, in the theory of knowledge spaces mentioned at the beginning of Section 7.4, the complete implication systems are called "entail relations." In Caspard and Monjardet (2003) one will find some of these terminological variations.

As already emphasized, an implication is defined here as the systematic association of some attributes with some others. The term "implication" itself also appears in other logic or algebraic contexts. In particular it is found, in the same or a close meaning, in the Boolean analysis of questionnaires (Degenne, 1972; Flament, 1976; Theuns, 1998), in knowledge spaces theory (Doignon and Falmagne, 1999), and in inductive item tree analysis (Sargin and Ünlü, 2009).

The correspondence between Moore families and complete implicational systems was first observed by Armstrong (1974) and then developed by many authors to lead to the form given in Theorem 7.68 (see for instance Burosch *et al.* (1987) and Doignon and Koppen (1989) who use dual closure systems, Demetrovics *et al.* (1992) or Caspard and Monjardet (2003)).

Another equivalent notion is that of a *nesting relation*. Such a relation $\eth$ is defined on $P(E)$ by $A \eth B$ if $A \subset B$ and $\varphi(A) \subset \varphi(B)$, where φ is a closure operator on $P(E)$. The latter is a strict order, studied and characterized in Domenach and Leclerc (2004). In a more theoretic viewpoint, it is worth noticing that, since any lattice may be represented by a Moore family (Theorem 3.52 in Chapter 3), it may also be represented by a CIS, or a nesting order.

From an applied point of view we consider, as illustrated in Example 3.48, the binary relation between a set of objects and a set E of attributes given in the form of a 0/1 array and, equivalently, by a family $\mathcal{D}$ of subsets (of the set of attributes). The Galois lattice of the latter is isomorphic to the Moore family $\mathcal{F} = \mathbf{m}(\mathcal{D})$. From Theorem 7.68, the latter is in turn determined by the set $i(\mathcal{D})$ of implications associated with $\mathcal{D}$ and also by its closure operator $\varphi_{\mathcal{D}}$.

In practice, the relation $i(\mathcal{D})$ is too large to provide a readable description of the data represented by the family $\mathcal{D}$. It is then interesting to search for exhaustive summarizations and so for implication bases of $\mathcal{D}$ (Definition 7.74). In that direction, a fundamental result is the existence and characterization of the Guigues–Duquenne canonical implication basis (Guigues and Duquenne, 1986). The highlighting of the latter has been preceded, attended, and followed by many works in the same direction, for instance Maier (1983), Duquenne (1987), Luxenburger (1991), Wild (1994), Ganter and Wille (1999). See also the website at: http://dl.kr.org/dl2008/?id=25 (go to "Invited Speakers," then to "Slides of Bernhard Ganter"). Theorem 7.79 and Proposition 7.80 are variants of Guigues and Duquenne's results. The proof given for Theorem 7.79 is based on Caspard and Monjardet (2003). The canonical basis has been characterized by Caspard (1999), whereas the critical sets have been characterized by Diatta (2009) from a new characterization of quasi-closed sets.

Using the canonical basis is not so easy since its determination is a difficult algorithmic problem (in the sense of Appendix A). This is why, among other reasons, several authors have defined alternative implication bases, especially the *direct canonical basis*, studied in Bertet and Monjardet (2005). They show in particular that this basis has been rediscovered many times (in different forms) and they point out its relation with the meet-irreducible representation of a Moore family in the lattice $\mathbb{F}$, as well as the representation of a Horn Boolean function by its prime implicants.

The implications between attributes considered in this section are "exact," in the sense that they correspond, as already said, to systematic associations. The problem then arises that some of these implications may have poor evidence in the family $\mathcal{D}$, i.e., may be satisfied by few objects. The extreme case is that of a subset A of the set E of attributes, which is shared by no object. Then, since $\varphi_{\mathcal{D}}(A) = E$, we obtain the implication $A \longrightarrow_{\mathcal{D}} E \setminus A$ which in fact is purely theoretic in the considered family $\mathcal{D}$. In his 1995 paper (from which Example 7.61 is extracted), Duquenne manages this type of problem by taking into account the numbers of answers (an aspect that, for the sake of brevity, we did not retain here). Another problem is that an implication satisfied in, say, one hundred elements of $\mathcal{D}$ but contradicted in the one hundred and first no longer appears in the above-described strict model. The need to consider all significant implications (in a statistical sense) and to distinguish them from fortuitous ones has prompted the introduction of statistical validations based on more or less sophisticated probabilistic models (for instance in Bernard and Poitrenaud (1999)).

In the same purpose of the selection of significant but not necessarily exact implications, a simple approach has known an important growth in data mining since the 1993 paper of Agrawal *et al.* It consists of the search for implications $A \longrightarrow_{\mathcal{D}} B$ that are "frequently enough" satisfied in the databasis $\mathcal{D}$. This search is done in two steps. First the *frequent* or the *frequent closed itemsets*; that is, the sets of attributes *supported by* (i.e., belonging to) at least p objects (where p is a fixed threshold called the minsupp) are found. Exercise 7.23 is concerned with properties of the set of these frequent itemsets. Then one searches for a partition of each frequent itemset into two parts A and B in such a way that the implication $A \longrightarrow_{\mathcal{D}} B$ has enough occurrences.

The current literature is mainly devoted to the first step. The extraction of frequent itemsets, which is essential in the procedure, has induced many algorithms, for instance: *APRIORI* (Agrawal and Srikant, 1994), *CHARM* (Zaki and Hsiao, 2002), *TITANIC* (Stumme *et al.*, 2002), *LCM* (Uno *et al.*, 2003), *ECLAT Z* (Szathmary *et al.*, 2008). Some of these works develop an approach based on Galois lattices (Ben Yahia and Mephu Nguifo, 2004) or Valtchev *et al.* (2002, 2008).

It is interesting to observe that the frequent closed itemsets ordered by inclusion form a lattice, called the *iceberg lattice* (see Stumme *et al.*, 2002).

A more sophisticated approach initiated in the 2005 paper of Ventos and Soldano to find frequent closed itemsets leads to the notion of *alpha Galois lattice*. In this approach, the set of objects is partitioned into classes and an alpha closed set is formed of attributes locally frequent w.r.t. these classes.

Finally let us mention the approach based on a notion of stability indicating how much the concept intent depends on particular objects of the extent (Roth *et al.*, 2008).

7.6.5 Orders in scheduling

Scheduling problems are one of the most studied classes of problems in operations research. A very complete report is provided in Leung (2004). Some reports stressing the ordinal aspects of these problems can be found in Lenstra and Rinnooy Kan (1984), Möhring (1984, 1989), Jansen (1993) or Poguntke (1986), the presentation of which has been chosen for this section.

It has been shown that the m-machine scheduling problem can be solved in polynomial time as soon as $m = 2$ and the job execution times are all equal (see, for instance, Fujii *et al.* (1969)). Many works exist proving that the latter result remains true or on the contrary that the problem becomes $\mathcal{NP}$-hard, depending on the value of m, the type of precedence order between jobs, and the assumptions made on the execution times of the jobs (see Appendix A and/or Lenstra and Rinnooy Kan (1984)).

In the examples handled in the section, we have only considered the time dimension for the job accomplishment. More complete modelings take into account the resources needed for the job execution; there are different types of resources with the result that a resource vector is associated with any job. Every resource has a cost that depends on its type. Therefore, one searches for minimizing a function which depends, on the one hand, on a global performance measure based on execution times and, on the other hand, on a total measure of the costs of the required resources. In fact, the problem often arises in a different form where the aim is to optimize the temporal performance criterion, the other criterion appearing as a constraint, each resource being available only in a limited quantity. Thus, a classic problem (which includes many variants) consists of minimizing the completion time with resource constraints. Radermacher (1977, 1985–6) and Möhring (1984, 1989) have developed an ordinal model for this problem, which leads to the obtention of branch and bound solving algorithms. In such a model, a job scheduling (which respects the precedence constraints) is given and defined by the job beginnings. The basic idea is then to consider the interval order on X associated with the family $[t_i, t_i + d_i]$ of intervals, where t_i is the beginning time of job x_i and d_i its duration. This order is an extension of the precedence order on jobs and induces another scheduling which can only bring the job beginnings forward (or maintain them). In other words, the newly obtained scheduling is necessarily better than (or equivalent to) the initial one. On the other hand, one can determine the global amount of a given resource used in the latter scheduling. Indeed if one defines the weight – with respect to a resource r – of a job x_i as the cost of r required for the execution of x_i, the global amount of a given resource used in the latter scheduling is equal to the maximum weight of an antichain of the interval order. Then, it can be shown that an optimal scheduling (which moreover respects the resource constraints) is obtained from some interval orders that are extensions of the precedence order and that can be characterized.

7.7 Exercises

7.7.1 Preference models

Exercise 7.1 Let (X, O) be the ordered set where $X = \{a,b,c,d,e,f,g,h,i,j\}$ and where the covering relation of O is

$$(a,b),(c,a),(c,d),(e,g),(g,a),(g,d),(g,f),(h,j),(j,c),(j,e), \text{ and } (i,g)$$

(and where, for instance, (a,b) means that $a \prec b$ in O).

Draw a diagram of this ordered set.

Show that O is an interval order (for instance, by computing the relations IO and OI).

Give a step-type tableau of this interval order and compare the number of "steps" of this tableau with the number of maximal antichains of O (see Section 7.6.1 of further topics).

Exercise 7.2 Let (X,O) be the ordered set where $X=\{a,b,c,d,e,f,g,h,i,j\}$ and where the covering relation of O is

$$(d,a),(e,a),(e,b),(e,c),(f,a),(f,b),(f,c),(g,b),(g,c),(h,d),$$

$$(h,e),(h,f),(i,d),(i,e),(i,f),(h,j),(j,e),(j,g),\text{ and }(j,f)$$

Answer the same questions as in Exercise 7.1 and, in particular, determine which class of orders O belongs to.

Exercise 7.3 ["Trace" preorders T_r and T_l] Let R be a binary relation defined on a set X. For $x \in X$, write $d^+(x)=|xR|=|\{y \in X : xRy\}|$, $d^-(x)=|Rx|=|\{y \in X : yRx\}|$, and $s(x)=d^+(x)-d^-(x)$. Consider the following relations on X:

- xT_ry if $xR \supseteq yR$,
- xT_ly if $Rx \subseteq Ry$,
- $T=T_r \cap T_l$,
- xT_+y if $d^+(x) \geq d^+(y)$,
- xT_-y if $d^-(x) \leq d^-(y)$,
- xT_sy if $s(x) \geq s(y)$

Decomposing these relations into their asymmetric and symmetric parts F and E, we write:

- $T_r=F_r+E_r$
- $T_l=F_l+E_l$
- $T_+=F_++E_+$
- $T_-=F_-+E_-$
- $T=F+E$
- $T_s=F_s+E_s$

Show the following relations:

1. $T_r=(RR^{cd})^{cd} \subseteq T_+$, $T_l=(R^{cd}R)^{cd} \subseteq T_-$ and $T \subseteq T_s$.
2. $F_r=(RR^{cd})^{cd} \cap (RR^{cd})$, $F_l=(R^{cd}R)^{cd} \cap (R^{cd}R)$, and $F=(F_r \cap T_l) \cup (F_l \cap T_r)$.

When R is a strict order O and $I=O^{cd} \cap O^c$, show the following relations:

- $T_r=(OI)^{cd}$,
- $F_r=(OI)^{cd} \cap OI$,
- $T_l=(IO)^{cd}$,
- $F_l=(IO)^{cd} \cap IO$,
- $T=(OI)^{cd} \cap (IO)^{cd}=(OI \cup IO)^{cd}$,
- $F=(OI \cup IO) \cap (OI \cup IO)^{cd}$,
- $E=(OI)^c \cap (OI)^{cd} \cap (IO)^c \cap (IO)^{cd}$,
- $O=OE \cup EO$

Show the equivalences xEy if and only if [$xO=yO$ and $Ox=Oy$], and if and only if $xI=yI$.

Exercise 7.4 [Weak orders and total preorders] Let O be a binary relation defined on a set X. Show that the following three properties are equivalent:

1. O is a weak order.
2. O^{cd} is a total preorder.

3. There exists a numerical function f defined on X such that $xO^{cd}y$ if and only if $f(x) \leq f(y)$.

Exercise 7.5 [Multiple characterizations of interval orders] Let O be a strict order defined on a set X. Using the notations in Exercise 7.3, show that the following properties are equivalent:

1. O is an interval order.
2. $OIO \subseteq O$.
3. OI is cycle-free.
4. OI is asymmetric.
5. $OI = F_r$.
6. T_r is a total preorder.
7. $T_r = T_+$.

Show that, in Characterizations (2), (3), (6), and (7) of an interval order O, the latter does not have to be irreflexive. Find other characterizations of interval orders using IO, T_l, or T_-.

Exercise 7.6 [Multiple characterizations of semiorders] Let O be a strict order defined on a set X. Using the notations in Exercise 7.3, show that the following properties are equivalent:

1. O is a semiorder.
2. $OIO \subseteq O$ and $O^2I \subseteq O$.
3. $OI \cup IO$ is cycle-free.
4. $OI \cup IO$ is asymmetric.
5. $OI \cup IO = F$.
6. $T = OI \cup IO \cup E$.
7. T is a total preorder.
8. $T = T_+$.
9. O is an interval order satisfying $F_r \cap (F_l)^d = \emptyset$.
10. For each pair $\{x,y\}$ of incomparable elements of O, either (x,y) or (y,x) is an O-critical ordered pair.

Show that, if O is a semiorder, then $T = T_+ \cap T_-$. Show that, in Characterizations (2), (5), and (7) of a semiorder O, the latter does not have to be irreflexive.

Exercise 7.7 [Towards another proof of the Scott–Suppes Theorem] Let O be an asymmetric binary relation defined on a set X and $I = O^{cd} \cap O^c$.

Show that O is an interval order if and only if each cycle of the relation O^{cd} $(= O + I)$ contains at least two consecutive ordered pairs belonging to I.

Show that O is a semiorder if and only if the relation O^{cd} is total and each cycle of length 3 or 4 of this relation contains strictly more ordered pairs belonging to I than belonging to O.

Note This characterization of semiorders, used with the existence of a "potential function" for a valued relation, allows us to give a proof of Theorem 7.16 of a constant threshold numerical representation of semiorders (see Pirlot and Vincke (1997), or Aleskerov *et al.* (2007)).

Exercise 7.8 ["S-orders"] Let O be a binary relation defined on a set X. Assume O is irreflexive and satisfies the following condition (S):

For all $x,y,z,t \in X$, xOy and yOz imply xOt or tOz $\qquad (S)$

Show that:

1. O is a strict order (such a strict order is called an *S-order*).
2. A strict order O is an S-order if and only if there does not exist $x,y,z,t \in X$ with xOy, yOz, xIt, yIt, and zIt (where $I = O^{cd} \cap O^c$).

Show that the following three conditions are equivalent for an irreflexive binary relation O defined on a set X:

1. O is an S-order.
2. For all $x,y \in X$, $xO \supseteq yO$ or $Oy \subseteq Ox$.
3. For all $x,y \in X$, $xO = yO$ or $Ox = Oy$ or ($xO \supset yO$ and $Ox \subset Oy$) or ($yO \supset xO$ and $Oy \subset Ox$).

Note Observe that a semiorder is an interval order that is also an S-order. The reader will find a study of S-orders in Monjardet (1978) and in Doble *et al.* (2001), where they are called *almost connected orders*.

7.7.2 Preference aggregation: Arrowian theorems for orders

Exercise 7.9 [Borda and pairwise comparisons] Let N be a set of n voters and X a set of m candidates. Associate with a profile $\pi \in \mathcal{L}^N$ the numbers $n_\pi(y,x)$ giving the results of the pairwise comparisons between candidates x and y. Show that Borda's preorder can be deduced from these numbers by proving the equality $R(x,\pi) = \sum_{y \in X} n_\pi(y,x)$.

Show that $\Sigma_{x \in X} R(x,\pi) = \frac{nm(m+1)}{2}$.

Then prove that a candidate beaten in each pairwise comparison with the other candidates cannot be the first one in Borda's preorder.

Exercise 7.10 [A "paradox" of Borda's rule, Fishburn (1981)] Let N be a set of seven voters having the following preference profile on a set $X = \{x, a_1, a_2, a_3\}$ of four candidates: $\pi = (a_1a_2a_3x : 3; a_2a_3xa_1 : 2; a_3xa_1a_2 : 2)$ (where, for example, $a_1a_2a_3x : 3$ means that three voters have $a_1 < a_2 < a_3 < x$ as their preference order on the four candidates).

1. Compute Borda's preorder $R_B(\pi)$ for this profile.

2. Now assume that the four voters that do not have x as the preferred candidate put him in last position in their ranking (this is a classic example of a strategic voting where a voter downgrades the candidate the most "dangerous" for his preferred one). Compute Borda's preorder $R_B(\pi)$ for this new profile and compare it with that obtained in (1).

3. Generalizing the previous example, show that on a set $X = \{x, a_1, a_2, ..., a_p\}$ of size $p+1 \geq 3$, one can find a profile π of $2p+1$ voters such that $R_B(\pi) = a_p a_{p-1} ... a_1 x$ and such that for the profile π' where all the voters that do not have x as their preferred candidate put him in last position, one obtains $R_B(\pi') = x a_1 ... a_{p-1} a_p$.

Exercise 7.11 [Proof of Proposition 7.22, McGarvey (1953)] Let X and N be two sets.

1. For each $\pi \in \mathcal{L}^N$, show that $R_{MAJ}(\pi)$ is a reflexive and antisymmetric relation and that $R_{WMAJ}(\pi)$ is a total relation.

2. Let R be a reflexive and antisymmetric relation defined on X. With each ordered pair (x,y) of distinct elements of R, one associates the two linear orders $L_1(x,y) = Mxy$ and $L_2(x,y) = xyM^d$ defined on X and where M is an arbitrary linear order on $X \setminus \{x,y\}$. One thus defines a profile π of linear orders on X; compute its majority relation $R_{MAJ}(\pi)$. Deduce Item (2) in Proposition 7.22 from this result and from a similar result for total relations.

3. Show Item (3) in Proposition 7.22.

Exercise 7.12 [The semilattice of filters] Let $\mathcal{F}$ and $\mathcal{F}'$ be two filters defined on a set N.

1. Show that $\mathcal{F} \cap \mathcal{F}'$ is a filter. We write $\mathcal{F} \vee \mathcal{F}' = \{F \cap F', F \in \mathcal{F}, \text{ and } F' \in \mathcal{F}'\}$. Show that $\mathcal{F} \vee \mathcal{F}'$ is a filter if and only if $F \cap F'$ is never empty.

2. Show that $\mathcal{F}$ is an ultrafilter if and only if $\mathcal{F}$ is a filter such that, for every $S \subseteq N$, $S \notin \mathcal{F}$ implies $N \setminus S \in \mathcal{F}$ (clue for the necessary condition: if there exists $S \subseteq N$ such that $S \notin \mathcal{F}$ and $N \setminus S \notin \mathcal{F}$, consider $\mathcal{F} \vee \mathcal{F}_S$, with $\mathcal{F}_S = \{T \subseteq N : S \subseteq T\}$).

3. Show that every filter $\mathcal{F}$ defined on a set N has the form $\mathcal{F} = \mathcal{F}_V$, where the basis V of the filter is a non-empty subset of N. Deduce that, if $V = \{i\}$, $\mathcal{F}_i$ is an ultrafilter.

4. Deduce from the previous results that the ultrafilters on N are the n filters $\mathcal{F}_i$ of basis $i \in N$ (they are thus the maximal elements of the semilattice of filters).

Exercise 7.13 [Proof of Nakamura's Theorem 7.35 (Nakamura, 1975)] Let R be a binary relation defined on a set X.

1. Show that R is cycle-free if and only if it has no cycles of length at most equal to the size of X.

2. Let $\mathcal{F} \subseteq P(N)$ be a federation on N and $F_{\mathcal{F}}$ the *PAF* that associates with each profile π of $\mathcal{L}^N$ the relation $R_{\mathcal{F}}(\pi)$ ($yR_{\mathcal{F}}(\pi)x$ if $N_\pi(y,x) \in \mathcal{F}$). Show that, if $\nu(\mathcal{F}) > p$ (for some integer p), $R_{\mathcal{F}}(\pi)$ has no cycles of length at most equal to p, for each $\pi \in \mathcal{L}^N$

(clue: show that, if not, one can find a sub-family $\mathcal{F}'$ of $\mathcal{F}$ such that $|\mathcal{F}'| \leq p$ and $\bigcap \mathcal{F}' = \emptyset$).

3. With the notations in (2) show that, if $R_{\mathcal{F}}(\pi)$ has no cycles of length at most p, $\nu(\mathcal{F}) > p$ holds. *Clue:* if $\nu(\mathcal{F}) = k < p$, build a profile π of $\mathcal{L}^N$ such that $R_{\mathcal{F}}(\pi)$ contains a cycle $x_1x_2,...,x_kx_1$; to do so consider a sub-family $\mathcal{F}'$ of $\mathcal{F}$ of size k and with an empty intersection; writing $\mathcal{F}' = \{S_1, S_2, ..., S_k\}$, begin by associating with each $i \in \bigcup\{S_h, S_h \in \mathcal{F}'\}$ a relation $R(i)$ formed from all the ordered pairs (x_h, x_{h+1}) for which $i \in S_h$.

4. Deduce Nakamura's Theorem from the previous results.

Note This proof, that simplifies Nakamura's proof, can be found for example in Moulin (1988).

Exercise 7.14 Let $\mathcal{M}$ be a set of orders defined on a set X and including the set $\mathcal{L} = \mathcal{L}_X$ of linear orders on X. $\mathcal{M}$ is said to be $\mathcal{L}$-$\cap$-*stable* if, for all $L, L' \in \mathcal{L}$, $L \cap L' \in \mathcal{M}$ (or, equivalently, if $\mathcal{M}$ contains all 2-dimensional orders). Show that, if $\mathcal{M}$ is not $\mathcal{L}$-$\cap$-stable, an $\mathcal{M}$-*PAF* from $\mathcal{L}^N$ to $\mathcal{M}$ is independent and Paretian if and only if it is a projection.

Deduce that, if $\mathcal{M}$ is the set of weak orders, of semiorders, or of interval orders, an independent and Paretian $\mathcal{M}$-*PAF* from $\mathcal{L}^N$ to $\mathcal{M}$ is dictatorial.

7.7.3 The roles of orders in cluster analysis

Exercise 7.15 [The partition lattice is upper semimodular] Let $\mathcal{P}_E$ be the set of partitions of a set E. Show that, for $\mathbf{P}, \mathbf{P}' \in \mathcal{P}_E$, the inequality $\mathbf{P} \leq \mathbf{P}'$ (for the refinement order) is satisfied if and only if any class of $\mathbf{P}'$ is a union of classes of $\mathbf{P}$. Especially, show that $\mathbf{P}'$ covers $\mathbf{P}$ if and only if $\mathbf{P}'$ is obtained from the union of exactly two classes of $\mathbf{P}$. Deduce that $(\mathcal{P}_E, \leq)$ satisfies the condition of upper semimodularity (Chapter 2, Definition 2.7).

Exercise 7.16 [The partition lattice is atomistic and coatomistic] Let $\mathcal{P}_E$ be the partition lattice of a set E of size n. What are the atoms and the coatoms of $\mathcal{P}_E$ (Chapter 3, Definitions 3.13 and 3.20)? Prove that $\mathcal{P}_E$ is atomistic and coatomistic.

Note So the partition lattice is a *geometric lattice* (that is, upper semimodular and atomistic). To know more on this lattice, see for instance Grätzer (1998).

Exercise 7.17 Let E be a set and d a dissimilarity on E. Given $e \in E$ and $\lambda > 0$, the *ball* $B(e, \lambda)$ is the set $\{e' \in E : d(e, e') \leq \lambda\}$. Show that d is an ultrametric if and only if, for any fixed $\lambda \geq 0$, the set $\{B(e, \lambda) : e \in E\}$ is a partition of E, denoted by $\mathbf{P}_{d,\lambda}$.

Exercise 7.18 [Ultrametrics, valued hierarchies and dendrograms, Barthélemy *et al.* (1984)] Let E be a set and $\underline{k} = \{0 < 1 < ... < k-1\}$ a chain.

1. Let u be an ultrametric on E taking its values in $\underline{k}$ and f_u the map which associates with any $\lambda \in \underline{k}$ the partition $\mathbf{P}_{d,\lambda}$ defined in the previous exercise. Show that f_u is a $\underline{k}$-dendrogram on E and that the map $u \longmapsto f_u$ is injective.

2. Let f be a $\underline{k}$-dendrogram on E and $\mathcal{H}_f = \bigcup\{f(\lambda) : \lambda \in \underline{k}\}$. Write $\iota_f(H) = \min\{\lambda \in \underline{k} : H$ is included in a class of $f(\lambda)\}$ for any $H \in \mathcal{H}_f$. Show that the pair $(\mathcal{H}_f, \iota_f)$ is a valued hierarchy on E and that the map $f \longmapsto (\mathcal{H}_f, \iota_f)$ is injective.

3. Let $(\mathcal{H}, \iota)$ be a valued hierarchy on E taking its height values in $\underline{k}$. Write $u_{(\mathcal{H},\iota)}(e,e') = \iota(H_{\{e,e'\}})$ for all $e, e' \in E$. Show that the so-defined dissimilarity $u_{(\mathcal{H},\iota)}$ on E is an ultrametric (taking its values in $\underline{k}$). Show that the map $(\mathcal{H}, \iota) \longmapsto u_{(\mathcal{H},\iota)}$ is injective.

4. Deduce from the previous results that the sets $\mathbf{U}_k$ of ultrametrics on E with values in $\underline{k}$, $\mathbf{D}_k$ of $\underline{k}$-dendrograms on E, and $\mathbb{H}_k^\iota$ of valued hierarchies on E with height values in $\underline{k}$ are in pairwise one-to-one correspondence. Let $u \in \mathbf{U}_k$, $f \in \mathbf{D}_k$, and $(\mathcal{H}, \iota) \in \mathbb{H}_k^\iota$ be three pairwise corresponding elements. Show the following equalities for all $e, e' \in E$: $u(e,e') = \min\{\lambda \in \underline{k} : e$ and e' are in the same class of $f(\lambda)\} = \iota(H_{\{e,e'\}})$.

Exercise 7.19 [Dendrograms are residual maps, Janowitz (1978) in the version of Barthélemy *et al.* (1984)] Let u be an ultrametric on E with values in $\underline{k} = \{0 < 1 < \ldots < k-1\}$, and f_u the $\underline{k}$-dendrogram associated with u in Item (1) in the previous exercise. Define a map g_u from $\mathcal{P}_E$ to $\underline{k}$ by writing $g_u(\mathbf{P}) = \max\{u(e,e') : e$ and e' are in the same class of $\mathbf{P}\}$ for any partition $\mathbf{P}$ of E (so $g_u(\mathbf{P_0}) = 0$, where $\mathbf{P_0}$ is the finest partition).

Show that, for all $\mathbf{P} \in \mathcal{P}_E$ and $\lambda \in \underline{k}$, $\mathbf{P} \le f_u(\lambda)$ if and only if $g_u(\mathbf{P}) \le \lambda$.

Deduce that the $\underline{k}$-dendrogram f_u is a residual map from $\underline{k}$ to $\mathcal{P}_E$ and that g_u is the associated residuated map (Definition 3.35 in Chapter 3).

Exercise 7.20 [Duality between the dendrogram and the ultrametric lattices, Leclerc (1981)] Consider, on the one hand, the set $\mathbf{D}_k$ of $\underline{k}$-dendrograms on a set E, endowed with the exponentiation order (with the usual order on $\underline{k}$ and the refinement order on $\mathcal{P}_E$) and, on the other hand, the set $\mathbf{U}_k$ of ultrametrics on E with values in $\underline{k}$, endowed with the pointwise order: $u \le u'$ if $u(e,e') \le u'(e,e')$ for all $e, e' \in E$.

1. Show that, for all $f, f' \in \mathbf{D}_k$, the maps $f \vee f'$ and $f \wedge f'$ defined, for any $\lambda \in \underline{k}$, by $(f \vee f')(\lambda) = f(\lambda) \vee f'(\lambda)$ and $(f \wedge f')(\lambda) = f(\lambda) \wedge f'(\lambda)$, are still $\underline{k}$-dendrograms on E.

2. Show that, for all $u, u' \in \mathbf{U}_k$, the dissimilarity $u \vee u'$ defined for all $e, e' \in E$ by $(u \vee u')(e,e') = \max(u(e,e'), u'(e,e'))$ is still an element of $\mathbf{U}_k$ and deduce that $\mathbf{U}_k$ is a lattice.

3. Show that, for the correspondence defined in Item (1) in Exercise 7.18, the inequality $u \le u'$ implies $f_{u'} \le f_u$. Deduce that $\mathbf{U}_k$ and $\mathbf{D}_k$ are dual lattices.

Exercise 7.21 [Lower valuation, Monjardet (1976b)] Let L be a meet-semilattice. A strictly isotone function v from L to $\mathbb{R}^+$ is called a *lower valuation* if, for all x, y in L such that $x \vee y$ exists, the inequality $v(x) + v(y) \le v(x \wedge y) + v(x \vee y)$ holds. Show that, if v is a lower valuation on L, then the function d_v from L^2 to $\mathbb{R}^+$ defined for all $x, y \in L$ by $d_v(x,y) = v(x) + v(y) - 2v(x \wedge y)$, is a metric on L; that is, it satisfies $d_v(x,z) \leq d_v(x,y) + d_v(y,z)$ for all $x,y,z \in L$.

Exercise 7.22 [Valuation, Leclerc (1994a)] Let P be an ordered set and w a real strictly positive weight function on the set J_P of join-irreducibles of P.

1. Show that the function δ_w defined, for all $x,y \in P$, by $\delta_w(x,y) = \Sigma_{j \in J_x \Delta J_y} w(j)$, is a distance on P^2.

2. Assume that P is a meet-semilattice. Using the definitions in the previous exercise, show that the function v defined on P by $v(x) = \Sigma_{j \in J_x} w(j)$ is a lower valuation on P and that $d_v(x,y) = \delta_w(x,y)$ for all $x,y \in L$.

3. Assume that P is a distributive meet-semilattice. Show that the function v satisfies $v(x) + v(y) = v(x \vee y) + v(x \wedge y)$ for all x,y in P such that $x \vee y$ exists (then v is called a *valuation* on P).

4. Show that, conversely, any valuation on a distributive meet-semilattice P may be obtained as above from a function w defined on J_P.

7.7.4 Implicational systems, Moore families, and Galois data analysis

Exercise 7.23 [Frequent itemsets] Let $\mathcal{D}$ be a family of subsets of a set E and $p \leq |\mathcal{D}|$ an integer. A non-empty subset M of E is called a *frequent itemset* if it is included in at least p elements of $\mathcal{D}$. Show that the set $\mathcal{M}$ of frequent itemsets constitutes a downset of the ordered set $\underline{2}^E$. Show that the maximal frequent itemsets are closed by the closure $\varphi_{\mathcal{D}}$ associated with $\mathcal{D}$.

Take $\mathcal{D} = \mathcal{M}_{\mathcal{F}}$, where $\mathcal{M}_{\mathcal{F}}$ is the family of 10 meet-irreducibles of the Moore family $\mathcal{F}$ in Example 7.61 and $p = 3$. Find the corresponding frequent itemsets (there are 6 such itemsets, 4 of which are maximal).

Exercise 7.24 [Meet-irreducibles of the lattice $\mathbb{F}$ and join-irreducibles of the lattice $\mathbb{I}$] Show that the meet-irreducible elements of the lattice $\mathbb{F}$ of Moore families on a set E are the families of the form $\mathcal{F}_{A,\{e\}}$ (Definition 7.74), where A is a non-empty subset of E and $e \in E \setminus A$. Deduce a general expression for the join-irreducibles of the lattice $\mathbb{I}$ of CIS.

From the form of the join-irreducibles of $\mathbb{F}$, show that a meet-irreducible of $\mathbb{I}$ has the form $\{(C,D) \in (P(E))^2 : C \not\subseteq A \text{ or } D \subseteq A\}$.

Exercise 7.25 Let $\mathcal{F}$ be a Moore family on a set E and $R = \{(A_1,B_1),\ldots,(A_r,B_r)\}$ a relation on $P(E)$. Write $R' = \{(A_1,\varphi_{\mathcal{F}}(A_1)),\ldots,(A_r,\varphi_{\mathcal{F}}(A_r))\}$. Use Theorem 7.65 to show that, if R is an implication basis of $\mathcal{F}$, then so is R'.

Assume that, moreover, $\varphi_{\mathcal{F}}(A_i) \setminus A_i \subseteq B_i$ for $i = 1,\ldots,r$. Show that R is then an implication basis of $\mathcal{F}$ if and only if so is R'.

Exercise 7.26 [Easily computable critical sets] Let $\mathcal{F}$ be a Moore family on a set E. Show that any element minimal w.r.t. inclusion in $P(E) \setminus \mathcal{F}$ is a critical set of $\mathcal{F}$. Which elements of E are critical sets? In Example 7.77, which critical sets are of the previous type?

Exercise 7.27 [A characterization of the quasi-closed sets of a Moore family, Caspard and Monjardet (2003)] Let $\mathcal{F}$ be a Moore family on a set E, φ the associated closure, and $Q \in P(E) \setminus \mathcal{F}$. Show that, if Q is a quasi-closed set of $\mathcal{F}$, then, for any $A \subset Q$, $\varphi(A) \subset \varphi(Q)$ implies $\varphi(A) \subset Q$.

Conversely, let $Q \in P(E) \setminus \mathcal{F}$ satisfying $\varphi(A) = \varphi(Q)$ or $\varphi(A) \subset Q$ for any $A \subseteq Q$. Show that $F \cap Q \in \mathcal{F}$ for any $F \in \mathcal{F}$ incomparable with Q w.r.t. inclusion. Deduce that such a Q is a quasi-closed set of $\mathcal{F}$. Conclude that a subset $Q \in P(E) \setminus \mathcal{F}$ is a quasi-closed set of $\mathcal{F}$ if and only if, for any $A \subset Q$, $\varphi(A) \subset \varphi(Q)$ implies $\varphi(A) \subset Q$.

Exercise 7.28 Let $\mathcal{F}$ be a Moore family on a set E and φ the associated closure. Write $xR(\mathcal{F})y$ if $\varphi(x) \subseteq \varphi(y)$. Use the result in Exercise 7.27 to show that, if a subset Q of E is a quasi-closed set of $\mathcal{F}$ of size at least 2, then Q is a downset of the preorder $R(\mathcal{F})$ (see page 144).

Exercise 7.29 [Canonical basis of a downset of $\underline{2}^E$] Let E be a set and $\mathcal{A}$ an antichain of the Boolean lattice $\underline{2}^E$. Show that the canonical basis of the CIS $i((\mathcal{A}[)$ is the set of all implications of the form $A \longrightarrow_{(\mathcal{A}[} E \setminus A$, where $A \in \mathcal{A}$. Use the results in Section 4.3 (Chapter 4) to derive the maximum number of implications that such a basis may contain.

Note The Moore families of the form $\mathcal{F} = (\mathcal{A}[\cup \{E\}$ have various characterizations and it is the same for their CIS of the form $i((\mathcal{A}[)$. Then, in relational databases, such a CIS is said to be "Boyce–Codd normal form" (see Caspard and Monjardet (2003), Proposition 33 (3) and Remark 64 (2)).

7.7.5 Orders in scheduling

Exercise 7.30 [k-jump critical ordered sets] An ordered set P is said to be *k-jump critical* if $s(P) = k$ (see Definition 1.32) and if, for any element x of P, $s(P \setminus x) < k$. Show that the ordered set S_3 (see Example 6.12, on page 172) is 3-jump critical. Let $P = A_1 \oplus A_2 \oplus ... \oplus A_k$ be a weak ordered set, where A_i is an antichain of size n_i $(i = 1, 2, ..., k)$. Prove that P is $[(n_1 + n_2 + ... + n_k) - k]$-jump critical. Consider the case $k = 3$ and find four other 3-jump critical ordered sets.

Exercise 7.31 Prove that the jump number of an ordered set P satisfies the following inequalities: for any $x \in P$, $s(P) - 1 \leq s(P \setminus x) \leq s(P)$.

Exercise 7.32 Let $P = (X, O)$ be an ordered set with $X = \{a, b, c, d, e, f, g, h, i\}$ and where the order O is given by the following covering relation:

$$(a,d),(b,e),(b,d),(c,d),(c,f),(d,g),(f,h),(g,i)$$

Compute an optimal 2-scheduling and an optimal 3-scheduling of P.

Exercise 7.33 [Execution time and range] Let k be the total execution time of an optimal m-scheduling of an ordered set P. Show that the range of P (Definition 1.30) satisfies $k \geq \kappa(P)$. Give an example of an ordered set for which the execution time of an optimal 2-scheduling is strictly greater than its range. More generally, give an example of a connected ordered set P for which the latter time is equal to $\kappa(P) + h$ (for an arbitrary integer h).

Exercise 7.34 Prove the result of Lemma 7.87 (examine the different possible cases).

Exercise 7.35 [Computing an optimal linear extension] Consider the priority order associated with a 2-step 2-machine problem (Lemma 7.88). In order to build a particular linear extension of the priority order, the following algorithm is proposed: consider the minimum of all the execution times; if the latter is some t_i (respectively, some t_i'), x_i is placed as the minimum (respectively, the maximum of the sought linear extension). That process is then iterated (after deletion of x_i, t_i, and t_i'). Show that the obtained result is a linear extension of the priority order, whence an optimal linear extension.

Appendix A

About algorithmic complexity

Using the notions and results presented in this book requires answering some questions about an ordered set modeling such situations. It may, for instance, be about determining a linear extension of the ordered set, or the lattice of its downsets, or its covering graph, or computing its width or its dimension.

The effective resolution of these problems requires the use of an "efficient algorithm" implemented by a program to be run on a computer. But given a problem, is there a solving algorithm and, if so, is it efficient and how could we measure this efficiency? We consider a two-level study of that type of question.

On the one hand there is the computational complexity theory which, from a formalization of the notions of a *problem* and an *algorithm* (for example by means of "languages accepted" by a "Turing machine"), leads to a problem classification with respect to the difficulty of solving them algorithmically – and independently of the algorithm used. The aim of the first part of this appendix is briefly to provide an intuitive idea on this classification.[1]

On the other hand, for a given problem, we will search for the most efficient algorithms (or heuristics[2]), taking into account the numerous factors which in practice may improve their efficiency.[3]

In the second part of this appendix, we will give a list of problems on ordered sets with, for each of them, the mention of the complexity of at least one resolution algorithm (but not necessarily of the "best" algorithm) and associated references.

[1] However, the reader should be warned: the definition of – for instance – the class $\mathcal{P}$ of problems, that will be found further on, makes sense only in the formalized context of computational complexity theory. The goal here is not to present this theory and so we refer the reader to excellent books, such as those written by Garey and Johnson (1979), Harel and Feldman (2004), Lewis and Papadimitriou (1998) or Wegener (2005) and, for French-reading readers, to two brilliant books written by Stern (1990) and by Wolper (1991) as well as the related chapters in Barthélemy *et al.* (1992) or Charon *et al.* (1996).

[2] Unlike an algorithm, the aim of a heuristic is not to provide the exact answer to a problem but rather to approach at best the solution or, at least, to provide an acceptable one.

[3] For the conception and analysis of algorithms and heuristics one may, for instance, refer to the books of Cormen *et al.* (2009), Sedgewick and Wayne (2011), and, obviously, to the fundamental 2011 Knuth treatise.

A.1 Computational complexity theory

Informally speaking, an algorithm is a procedure allowing us to solve a problem and which, from an "input" (the data of the problem) leads by means of a finite number of instructions to an "output" (the solution of the problem). The specification of the data that define the algorithm input is a so-called *instance* of the problem. For example, for the dimension computation (in the sense of Chapter 6), an instance will be some ordered set. On the other hand, solving a problem with an algorithm implemented on a computer requires that the data of the problem be comprehensible by the machine; that is, represented by a sequence of 0 and 1. The length λ of this sequence defines the *size* of the instance of the problem. The *time complexity* of an algorithm supposed to solve the problem is the function $t(\lambda)$ on $\mathbb{N}$ which returns the maximum number of "elementary operations"[4] required by the execution of the algorithm for solving the problem on an instance of size λ of this problem (the worst possible case is then considered).[5]

The algorithm complexity will depend not only on the algorithm itself but also on the type of coding used for the instance. For example, an ordered set may be described by the whole set of its ordered pairs or only by those of its covering relation, and the ordered pairs may be given by an adjacency matrix or by a predecessor or a successor list. However, the definition (given further on) of an efficient algorithm provided in complexity theory will allow us, in return for some precautions, to leave this factor aside. Thus an efficient algorithm for a "reasonable" coding of the instance will still be efficient for any *polynomially related* coding (that is, such that the instance size in the first coding is majored by a polynomial function of the instance size in the second one, and conversely). Therefore, one proves that it is possible to replace the instance size, as defined above as a sequence of 0 and 1, by an "intuitive" size which, in the case of an ordered set, may be the number n of elements of the set, but also the number m of ordered pairs of its order relation, the number $m_{\prec}$ of ordered pairs of its covering relation, or the sums $n+m$ or $n+m_{\prec}$. For the following definitions, we choose n as the size and we then consider the time complexity function $t(n)$ which returns, for a given algorithm, the maximum number of elementary operations required for the algorithm execution for solving the problem on an instance of size n. The exact computation of the complexity will often be difficult. Consequently one searches for a bound of it, using classic notions and notations on orders of magnitude, that we recall below.

Two relations O and Ω are defined on the set of functions on $\mathbb{N}$ as follows: for f and g two functions on $\mathbb{N}$, g is said to *asymptotically upper* (respectively, *lower*)

[4] The notion of an elementary operation will depend on the considered problem; it may be about an arithmetic operation, a comparison operation or an affectation of elements. However, the notions of *classes of complexity* introduced further on are independent of a precise definition of elementary operations.

[5] Here we will not adress the "space complexity," which measures the memory space required for the algorithm.

bound f if there exist a non-negative real number c and a positive integer n_0 such that, for any integer $n > n_0$, $f(n) \leq cg(n)$ (respectively, $f(n) \geq cg(n)$). Rather than the notation fOg (respectively, $f\Omega g$), one writes $f = O(g)$ (respectively, $f = \Omega(g)$) and one also says that f *is* $O(g)$ (respectively, f *is* $\Omega(g)$).

The reader can check that both relations O and Ω define two dual preorders on the set of functions on $\mathbb{N}$. He will also specify the induced equivalence, which is denoted by $f = \Theta(g)$ and is translated into the expression "f and g have (asymptotically) the same order of magnitude."

An algorithm is said to be *polynomial-time* if its time complexity function $t(n)$ is $O(n^k)$ for a given number $k \geq 1$. In this case the algorithm is considered to be *efficient* or "good."[6] In the other cases the algorithm is said to be *exponential-time*[7] and is considered as *inefficient*.[8] It is, for example, the case if its complexity is $\Omega(k^n)$ for some number $k > 1$, i.e., if it is lower bounded by an exponential function.

One easily realizes how relevant this distinction is when comparing the values $t(n)$ obtained for a polynomial and an exponential function. Assuming that $t(n)$ is expressed in seconds, the computation of n^3 and 3^n with, for example, $n = 40$, respectively gives a little less than 18 hours and 3855 centuries! Thus, except for data of very small size, the exponential complexity of an algorithm makes it normally impracticable.

We now go back to the following question: is there for any given problem an efficient algorithm allowing us to solve it? The theory of complexity classes proposes to study this question by building up a typology of the so-called *decision* problems. A *decision problem* is made up of an input and a question. The input specifies a particular instance of the problem and the question has only two possible answers, "yes" or "no." Formally a decision problem is defined as a pair (I,J) where I is the set of all instances of the problem and J the (non-empty) subset of I of all *positive* instances; that is, those instances whose answer to the question is "yes." Here, we are concerned with decision problems on ordered sets. Thus, the input will be a given ordered set P and the question will be to know whether P satisfies a given property (for example, is P an interval ordered set?). The problem of searching for the value of a parameter of P, for example its dimension, is not a decision problem but it may be re-expressed so as to become one (for example, is the dimension of P less than or equal to a given number q?).

We also note that – unlike the so-called *undecidable* problems, for which there is no algorithm allowing us to answer the question – for the decision problems about a finite ordered set, there normally is an algorithm allowing us (in theory) to solve them.

[6] It is clear that, in practice, the efficiency of an algorithm will be even better since k is small. In particular one may consider as very efficient the algorithms said to be *linear-time*, i.e., those whose time complexity function is $\Theta(n)$.

[7] Even though $t(n)$ is not an exponential function, for example, if $t(n) = n!$.

[8] That terminology should not be understood too literally. One knows exponential-time algorithms that are in practice very efficient. It is the case for the simplex algorithm for linear programming problems and for the *APRIORI* algorithm mentioned at the end of Section 7.6.4.

For example, for the above-mentioned problem on the dimension of an ordered set P, it "is enough" to compute all its linear extensions (by going through the $n!$ possible linear orders), then to compute all possible intersections of q among the latter, to make sure that one of these intersections is – or is not – equal to P. Such an algorithm has clearly an exponential complexity and, therefore, is inefficient. Actually the decision problem about the dimension is one of the many examples of problems for which no efficient algorithm is known, even though it is not either proved that an efficient algorithm does not exist. That point will be specified with the definitions below of the two fundamental complexity classes.

Definition A.1 The class $\mathcal{P}$ is the set of all decision problems for which, for any instance of any problem, there is a polynomial-time algorithm allowing us to determine if the answer is "yes" or "no."

The class $\mathcal{NP}$ is the set of all decision problems for which, for any instance of any problem, there is a polynomial-time algorithm applied to a "succinct certificate" (i.e., a certificate of polynomial size with respect to the instance size) allowing us to check that the answer is "yes" if and only if the instance is positive.

Let us consider, for example, the problem whose statement is as follows: "is the dimension of P less than or equal to q?" (with P an ordered set and $q \leq \frac{|P|}{2}$, according to Theorem 6.21). Such a certificate will be the data of $q \leq \frac{|P|}{2}$ linear extensions of P and it may be checked in polynomial time that their intersection is P.

More precisely, the problems of the class $\mathcal{NP}$ are said to be "solved" by a *non deterministic algorithm* in the following sense: in the first step of the algorithm an "oracle" provides a certificate (of polynomial size) allowing for a positive instance of the problem to answer yes. In that case, the second step is the verification one and consists in a polynomial-time algorithm applied to the latter certificate; so it allows us to answer yes in case of a positive instance. If, on the contrary, the instance is negative, there is no such certificate. Such an algorithm does not really solve the problem since, if there was no soothsayer to provide a certificate (if it exists), it might be necessary to test an exponential number of cases (for example the binomial coefficient $\binom{n!}{q}$ of sets of q linear extensions of P) to have the answer.

More formally, in computational complexity theory, one modelizes the computation made by an algorithm by means of a "Turing machine." Problems in the class $\mathcal{P}$ (respectively, in the class $\mathcal{NP}$) are then those which can be solved in polynomial time by a "deterministic" (respectively, "non-deterministic") Turing machine. The reader can refer to the books quoted in footnote 1 on page 270 for the formal exposition of the theory.

We want only to stress the fact that translating "$\mathcal{NP}$" into "non-polynomial" would be wrong, since it means on the contrary "polynomial in a non-deterministic way."

Result: $\mathcal{P}$ is included in $\mathcal{NP}$.

Open question: is this inclusion strict or are these classes equal?

In fact it is considered highly probable that the class $\mathcal{P}$ is different from the class $\mathcal{NP}$ but, since it has not been proved so far, the assertion "$\mathcal{P} \neq \mathcal{NP}$" is still a conjecture. If anyone proves it right (which will allow him/her to win a million dollars, see the site www.claymath.org/millennium/), that will imply all the problems in $\mathcal{NP}$ that are not in $\mathcal{P}$ are "difficult," in the sense that there is no polynomial-time algorithm to solve them.

A decision problem is said to be *$\mathcal{NP}$-complete* if it belongs to the class $\mathcal{NP}$ and is at least as hard as any problem in the class $\mathcal{NP}$, in the sense that the existence of a polynomial-time algorithm to solve a $\mathcal{NP}$-complete problem would imply that of polynomial-time algorithms to solve any problem in the class $\mathcal{NP}$ (one would then have $\mathcal{P} = \mathcal{NP}$).[9]

Consequence: $\mathcal{P} \neq \mathcal{NP}$ holds if and only if the class of $\mathcal{NP}$-complete problems and the class $\mathcal{P}$ are disjoint.

Remark A.2 The previous notions are in fact ordinal. Let us define a relation of *polynomial reduction* between problems of $\mathcal{NP}$ by saying that a problem A *reduces to* a problem B if there is a polynomial-time computable function that transforms any instance of A into an instance of B with the same (positive or negative) nature. This relation is a preorder on $\mathcal{NP}$ and the associated equivalence defines equivalent problem classes. The quotient order between the latter classes has a least element, namely the class $\mathcal{P}$, and a greatest element, namely the class of $\mathcal{NP}$-complete problems. If $\mathcal{P} \neq \mathcal{NP}$, it can also be shown that the previous preorder has an infinity of classes formed of problems whose difficulty is "intermediate" between that of the minimum and the maximum classes, and that it is not total.

The fundamental two classes $\mathcal{P}$ and $\mathcal{NP}$ have been defined in the 1970s. Since then many other classes have been added (see www.complexityzoo.com for the "complexity zoo") and we here give only a few examples.

A problem is said to be *$\mathcal{NP}$-hard* if it is at least as hard (in the same sense as previously) as an $\mathcal{NP}$-complete problem. Any $\mathcal{NP}$-complete problem is then $\mathcal{NP}$-hard and the following relation holds:

$$\{\mathcal{NP}\text{-hard problems}\} \cap \{\text{problems in } \mathcal{NP}\} = \{\mathcal{NP}\text{-complete problems}\}$$

The complement of a problem is the problem where the yes/no answers are reversed. For instance, the problem: "is the ordered set P an interval order?" has for complement the problem "is the ordered set P not an interval order?"

A problem is said to be in *co-$\mathcal{NP}$* if and only if its complement is in $\mathcal{NP}$, i.e., if it can be solved by a non-deterministic algorithm (in the sense given above for the

[9] The notion of $\mathcal{NP}$-complete problems has been introduced by Cook (1971), who proved the existence of such problems by showing that the problem of the *satisfiability* of a Boolean expression (in conjunctive normal form) is $\mathcal{NP}$-complete. Since then many problems have been proved to be $\mathcal{NP}$-complete and we will mention some of them – that deal with ordered sets – in the second part of this appendix.

problems in $\mathcal{NP}$). A problem is said to be *co-$\mathcal{NP}$-complete* if it belongs to the class co-$\mathcal{NP}$ and is at least as hard as any problem in the class co-$\mathcal{NP}$ (in the same sense as that defined above for a $\mathcal{NP}$-complete problem).

It should be observed that here we use the term "problem" without having specified its definition. The latter will depend on the context; for example, optimization problems are those where, with any instance of the problem, is associated a set S of "possible solutions" and where one searches for a solution minimizing (or maximizing) a real-valued function f defined on S. Any optimization problem has an associated decision problem. For example, to search for the jump number $s(P)$ of an ordered set P is an optimization problem consisting of minimizing on the set $\mathcal{L}(P)$ of linear extensions L of P the number $s(P,L)$ of jumps of L. The associated decision problem has the ordered set P and an integer k as the input and asks the question: does $s(P) \leq k$ hold? If the decision problem associated with an optimization problem is $\mathcal{NP}$-complete, the latter is $\mathcal{NP}$-hard.

The classes $\#\mathcal{P}$, *$\#\mathcal{P}$-complete*, and *$\#\mathcal{P}$-hard* of problems have been introduced (by Valiant (1979a,b)) to report the difficulty of enumerative problems for some combinatorial objects; that is, of determining the number of such objects and/or listing them (for example, all linear extensions of an ordered set). Similarly to what is written above, one has:

$$\{\#\mathcal{P}\text{-hard problems}\} \cap \{\text{problems in } \#\mathcal{P}\} = \{\#\mathcal{P}\text{-complete problems}\}$$

The enumeration version of most $\mathcal{NP}$-complete problems is a $\#\mathcal{P}$-complete problem but it may also be the case for problems in the class $\mathcal{P}$ (for example the problem of computing the number of matchings in a bipartite directed graph).

On the other hand, for the problems of enumerating combinatorial objects on a set of size n, and where the number N of these objects may be exponential in n (for example when considering the linear extensions of an ordered set), other notions have been introduced. A listing algorithm of the N objects (that is, which provides a listing of the latter) is in *polynomial time* if its complexity is in $O((n+N)^k)$, with k a constant at least equal to 1. It is in *polynomial* (respectively, *linear*) *time by object* if its complexity is in $O(n^k N)$, with k a constant at least equal to 1 (respectively, in $O(nN)$). In that case the algorithm is also said to have a polynomial (respectively, linear) *amortized time complexity*. When the complexity is in $O(N)$ one says that the algorithm has a *constant amortized time complexity* or also that it is in *constant amortized time* (TCA). If moreover the (time) complexity for going from an object of the list to its successor is upper bounded by a constant, the algorithm is said to be *loopless* (or in *constant worst case time*). At last, the provided list is of the *Gray code type* if the difference between two successive objects of the list is small (for example if, in a list of all linear orders, two successive orders differ only by the inversion of an ordered pair).

A.2 Complexity results

Consider the following problem: is the dimension of an ordered set at most equal to $k > 2$? The difficulty of this problem depends on the class of ordered sets concerned in the question. Trivial when applied on linear ordered sets, the problem is $\mathcal{NP}$-complete when considering the class of all ordered sets but becomes polynomial for some given classes of ordered sets. We shall then start in Section A.2.1 with complexity results for algorithms on "easy" problems (that may or may not be decision problems); that is, for which there always exists a polynomial-time algorithm. Then in Section A.2.2 we will give a list of "difficult" problems ($\mathcal{NP}$-complete, $\mathcal{NP}$-hard or $\#\mathcal{P}$-complete) on the class of all ordered sets. There are several possibilities to solve such problems on a specified ordered set. First, the size of the ordered set may happen to be small enough to allow the problem being solved by an exponential-time algorithm. Then, the ordered set may belong to a particular class of ordered sets for which the problem has been proved to be polynomial; in Section A.2.3 we will give a table presenting some classes of (polynomially recognizable) ordered sets for which some of the previous difficult problems become polynomial. At last, if none of the previous possibilities applies, one may turn to approximation (polynomial) algorithms – when they exist – giving an approximate solution whose quality may sometimes be guaranteed by means of *performance bounds*; we will not give further information on this possibility, which is the purpose of many research works; we refer the interested reader to Vazirani's book (2001) or to the Demange and Paschos text (2010) for example (however and with few exceptions, the approximation results for order problems may only be found in journals or reviews of specialized conferences).

Finally, the results stated in the following three subsections provide a substantial sample of those obtained in the algorithmic literature on orders. The reader will find some of the latter results with, if need be, the description of the corresponding algorithms in the surveys due to Bouchitté and Habib (1989), Möhring (1984, 1989), and Spinrad (1994).

It is worth noticing that the complexity of some problems has so far remained unknown. It is the case for the decision problem about the isomorphism of two ordered sets (like – more generally – that of two graphs). Likewise for the problem of enumerating all minimal transversals of a federation (see Definition 7.23 in Chapter 7).

A.2.1 Easy problems (polynomial-time algorithms)

The problems on ordered sets quoted below may be solved by a polynomial-time algorithm, no matter whether they are decision problems of the class $\mathcal{P}$ or others. The complexity of such an algorithm may depend significantly on the data structures used for the representation of the ordered set. However, the purpose here is not to specify that data structure nor the algorithm used; in fact, the reader can find the (time) complexity of one or two algorithms solving the problem and some references giving the algorithms together with detailed information on their implementation. Let us specify that the references mentioned are not always the original ones – the latter are sometimes not easily accessible – and that the given complexity is not always the "best" one.

As already mentioned, the complexity will be evaluated as a function of a "size" of the ordered set $P = (X, \leq)$. The latter may be the size n of X, the number m of ordered pairs of the order relation $\leq$, the number $m_{\prec}$ of the ordered pairs of its covering relation $\prec$, or still a function of n, m or $m_{\prec}$. On the other hand, the input of these algorithms is sometimes a cycle-free non-directed graph $G = (X, U)$ rather than the order or the covering relation of an ordered set. Considering the transitive closure of this graph as the ordered set, it amounts to taking a sub-relation U of the latter closure such that $\prec$ is included in U which itself is included in $\leq$. The size is then measured by the number of arcs of the graph G, i.e., by the integer m_U (with $m_{\prec} \leq m_U \leq m$). Thus, in the following complexities, the appearance of $m_{\prec}$ (respectively, m_U or m) in an expression means that the input data of the algorithm is the covering relation of an order (respectively, a cycle-free graph or an order relation).

1. Computation of the order relation of P from a sub-relation (including the covering relation of P).

(a) $O(n^3)$: Roy (1959), Warshall (1962), Schröder (2002).
(b) $O(nm_U)$: Goralcikova and Koubek (1979), Mehlhorn (1984).
(c) If P is ranked, $O(n + m_U)$: Goralcikova and Koubek (1979).
(d) If P is an ordered set of dimension 2 (respectively, a distributive lattice), $O(n^2)$: Ma and Spinrad (1991) (respectively, Bordat (1991a)).
(e) If P has a bounded width, $O(n + m)$: Habib *et al.* (1993b).

2. Computation of the covering relation of P from its order relation.

(a) $O(n^3)$: Schröder (2002).
(b) $O(nm + n^2)$: Freese *et al.* (1995).
(c) If P is an ordered set of dimension 2, $O(n^2)$: Ma and Spinrad (1991).
(d) If P is an N-free ordered set (respectively, of bounded width), $O(n + m)$: Ma and Spinrad (1991) (respectively, Habib *et al.* (1993b)).
(e) If P is a lattice: $O(n^2)$, Freese *et al.* (1995).

Note The former problem is that of the reflexo-transitive closure of a graph (here directed and cycle-free). Both problems 1. and 2. are equivalent to that of the computation of the product of two $n \times n$ matrices. The upper bound in n^3 for this problem has been improved but thanks to algorithms which are not necessarily implementable in practice.

3. Downset $(A]$ generated by a subset A of P.

$O(n + m_U)$: Bordat (1985).

4. Computation of a linear extension of P.

$O(n+m_U)$: Kahn (1962), Knuth and Szwarcfiter (1974), Freese *et al.* (1995).

Note If $G=(X,U)$ is an arbitrary directed graph, the algorithm allows us to test if it is cycle-free (and so if the reflexo-transitive closure of U is an order).

5. Decomposition tree of P.

(a) $O(n^2)$: Müller and Spinrad (1984, 1989), McConnell (1995).
(b) $O(n+m)$: Cournier and Habib (1994), McConnell and de Montgolfier (2005).

COMPUTATION OF SOME PARAMETERS OF P

6. Range $\kappa(P)$ of P.

(a) $O(n^2)$: Möhring (1984).
(b) If P is a 2-dimensional ordered set, $O(n\log\log n)$: see Spinrad (1985), in the equivalent form of the search for a clique of maximum size in a permutation graph (see Exercise 6.12).

7. Width $\alpha(P)$ of P.

(a) $O(n^{\frac{5}{2}})$: via Hopfcroft and Karp (1973).
(b) $O(nm)$: via Nemhauser and Wolsey (1988).
(c) If P is an interval ordered set (given by an interval representation), $O(n^2)$: Schröder (2002).
(d) If P is a 2-dimensional ordered set, $O(n\log\log n)$: see Spinrad (1985), in the equivalent form of the search for an independent subset of maximum size in a permutation graph (see Exercise 6.12).

Note 1. The algorithm in (a) (respectively, (b)) first associates with the ordered set the appropriate bipartite graph (see on page 124 and Exercise 4.5) then uses the algorithm of maximum matching in a bipartite graph due to Hopfcroft and Karp (1973) (respectively, Nemhauser and Wolsey (1988)).

2. Some changes in the previous algorithm in (a) allow us, with a complexity in $O(kn^2)$, to decide whether $\alpha(P)\leq k$ (with k an integer) and, if so, to obtain a maximum antichain or a partition of P into $\alpha(P)$ chains: Felsner *et al.* (2003) – see also Gavril (1987).

3. An algorithm due to Bogart and Magagnosc allows us to find $\alpha(P)$ and a partition of P into $\alpha(P)$ chains in $O(n^3)$ (see Freese *et al.*, 1995).

RECOGNITION PROBLEMS

In these problems the question is the following: is P an ordered set of the indicated type?

8. Series–parallel ordered set.

(a) $O(n^3)$: Möhring (1989).
(b) $O(n+m_U)$: Valdes *et al.* (1982).

Note These algorithms provide the decomposition tree of P and, for (b), the order and the covering relation of P.

9. N-free ordered set.

$O(n+m_{\prec})$: Valdes *et al.* (1982) Sysło (1982), Habib and Jégou (1985) – see Möhring (1989).

Note The Habib and Jégou algorithm gives the decomposition tree of P.

10. Interval ordered set.

(a) $O(n+m)$: Papadimitriou and Yannakakis (1979).
(b) $O(n+m_U)$: Baldy and Morvan (1993).

Note These algorithms provide an interval representation of P.

11. Semiordered set.

$O(n+m_U)$: Mitas (1994).

Note The algorithm provides a unit-interval representation of P.

12. 2-dimensional ordered set.

(a) $O(n^2)$: Spinrad and Valdes (1983), Spinrad (1985), Ma and Spinrad (1991).
(b) $O(n+m_U)$: McConnell and Spinrad (1999).

Note The input of the first two algorithms is the order relation of P; in the third one it is a sub-relation of the order including the covering relation of P and, in the fourth one, a comparability graph $G=(X,U)$. These algorithms provide two linear orders forming a basis of P.

13. Ordered set of width at most k.

$O(kn^2)$: Gavril (1987), Felsner *et al.* (2003).

14. Ordered set whose decomposition diameter is at most k.

$O(n^2)$: Müller and Spinrad (1984).

15. Ranked ordered set.

$O(n+m_U)$.

16. Semimodular ordered set.

(a) $O(n^3)$, $O(n(n+m_U))$: Bordat (1985).
(b) If P is a semilattice, $O(n^2)$: Bordat (1985).

17. Semilattice.

(a) $O(n^{\frac{5}{2}})$, $O(n+m_U)$: Goralcik *et al.* (1981), Freese *et al.* (1995).
(b) $O(n(n+m_U))$: Bordat (1985).

Note The input of the algorithm in (a) is the table of a binary operation on X or a cycle-free graph. The input of the algorithm in (b) is a cycle-free graph.

18. Lattice.

$O(n^{\frac{5}{2}})$: Goralcik *et al.* (1981), Freese *et al.* (1995).

Note The input of the algorithm is an arbitrary relation and allows us, if the latter is a lattice order, to obtain the tables of its join and meet operations, as well as its covering relation.

19. Distributive lattice.

(a) $O(n^2)$: Bordat (1985).
(b) $O(n+m_{\prec})$: Bordat (1985).

Note 1. The input of the algorithm in (a) is a directed graph whose transitive closure is a lattice whereas for the algorithm in (b), it is the covering graph of a lattice.

2. The ideal tree associated with a linear extension of the ordered set M_L of a distributive lattice L (see page 156) is obtained in $O(n+m_{\prec})$ from the covering relation of the lattice and conversely allows us, with the same complexity, to obtain the latter; using such an ideal tree, the transitive closure of the relation $\prec$ may be computed – as well as the operations join and meet of L – in $O(|M_L|)$ (Habib and Nourine, 1996; Habib *et al.*, 2001).

The last considered recognition problem is not about an ordered set but about a non-directed graph.

20. Recognition of a comparability graph.

$O(\Delta(G))$, where $\Delta(G)$ is the maximum degree of a vertex of the non-directed graph G, Golumbic (1977) – see Möhring (1984).

Note The transitive orientation of a comparability graph is done in $O(n^2)$ or $O(m \log n)$ if the input is the decomposition tree of the comparability graph (Spinrad, 1985, 1994) and in $O(n+m_U)$ if the input is the latter graph $G=(X,U)$ (McConnell and Spinrad, 1999).

A.2.2 Difficult problems

Difficult problems are the $\mathcal{NP}$-complete, $\mathcal{NP}$-hard, co-$\mathcal{NP}$-complete, co-$\mathcal{NP}$-hard, $\#\mathcal{P}$-complete, $\#\mathcal{P}$-hard problems for which no polynomial-time algorithm is known. In this section we will also mention enumeration problems that are not difficult (in the above sense) but for which the size of the answer is not bounded by a polynomial function of the size of the data (for example, searching for all elements of the Galois lattice of a relation).

The following decision problems are $\mathcal{NP}$-complete.

Input: a non-directed graph G.
Question: is there an ordered set P satisfying $G = Neigh(P)$?
Nešetřil and Rödl (1987), Brightwell (1993b).
Note Remains $\mathcal{NP}$-complete for a 4-colorable graph (Brightwell, 1993b).

Input: an ordered set P and an integer $k \geq 3$.
Question: does $\dim P \leq k$ hold?
Yannakakis (1982).
Note Remains $\mathcal{NP}$-complete for an N-free ordered set, Kierstead and Penrice (1989).

Input: a bipartite ordered set P.
Question: does $\dim P \leq 4$ hold?
Yannakakis (1982).

Input: an ordered set P and an integer k.
Question: does $\dim_2 P \leq k$ hold?
Stahl and Wille (1986), Habib *et al.* (2004).
Note In Proposition 6.7 we have proved that $\dim_2 P \geq \lambda(P)(= \kappa(P) - 1)$. The problem is recognizing whether $\dim_2 P = \lambda(P)$ is $\mathcal{NP}$-complete (Habib *et al.*, 2004). A study of the heuristics for the computation of the 2-dimension is given in Caseau *et al.* (1999).

Input: an ordered set P.
Question: does $s(P) = \alpha(P) - 1$ hold?
Bouchitté and Habib (1987).
Note By definition, the latter question amounts to recognizing whether P is a Dilworth ordered set.

The following optimization problems are $\mathcal{NP}$-hard:

Input: an ordered set P.
Output: a linear extension L of P minimizing the jump number ($s(L) = s(P)$).
Pulleyblank (1981, unpublished), Bouchitté and Habib (1987).
Note The computation of $s(P)$ remains $\mathcal{NP}$-hard for bipartite ordered sets (Pulleyblank, 1981) or interval ordered sets (Mitas, 1991).

Input: an ordered set P of size n and two tuples $(t_1, t_2, ..., t_n), (p_1, p_2, ..., p_n) \in \mathbb{R}^n$.
Output: a linear extension of P minimizing the total weighted completion time.
Lawler (1978).

Note The problem remains $\mathcal{NP}$-hard on N-free ordered sets even when all t_i's (or all p_i's) are equal (Habib and Möhring, 1987). For approximation algorithms, see Woeginger (2001).

Input: a profile $\pi = (L_1, ..., L_i, ..., L_n)$ of linear orders.
Output: a median order of π.
See Hudry (2004) and, for approximation algorithms, Hudry (1997).

The following counting problems are $\#\mathcal{P}$-complete.

Input: an ordered set P.
Output: the number $|\mathcal{D}(P)|$ of downsets of P (or also the number of its antichains).
Provan and Ball (1983).

Note Enumeration of $\mathcal{D}(P)$:

(a) $O(n|\mathcal{D}(P)|)$: Steiner (1986), in Gray code manner (Pruesse and Ruskey, 1994).
(b) $O(\log n|\mathcal{D}(P)|)$: Squire (1995).
(c) $O(\alpha(P)|\mathcal{D}(P)|)$: Bordat (1991b).
(d) $O(\Delta(P)|\mathcal{D}(P)|)$ (where $\Delta(P)$ is the maximum number of elements covered by an element of P): Habib *et al.* (2001) – see also Habib *et al.* (1997).
(e) $O(n|\mathcal{D}(P)|)$ in Gray code manner, Abdo (2009).
(f) If P is an interval ordered set, $O(|\mathcal{D}(P)|)$ (in Gray code manner with a "loop-free" algorithm): Habib *et al.* (1997).

Covering relation of $\mathcal{D}(P)$:

(a) $O(\alpha(P)|\mathcal{D}(P)|)$: Bordat (1991b).
(b) $O(n+m)$ (with the covering tree of $\mathcal{D}(P)$ as the input, see page 156): Habib *et al.* (2001).

Input: an ordered set P.
Output: the number $|\mathcal{L}(P)|$ of linear extensions of P.
Brightwell and Winkler (1991).

Note Enumeration of $\mathcal{L}(P)$:

(a) $O(|\mathcal{L}(P)|)$: Pruesse and Ruskey (1994).
(b) Korsh and LaFollette (2002, with a "loop-free" algorithm).
(c) For approximation algorithms, see Brightwell and Winkler (1991).
(d) For an enumeration algorithm of all extensions of an ordered set, see Corrêa and Szwarcfiter (2005).

Input: (E', E, R), where $R \subseteq E' \times E$ is a binary relation from E' to E.
Output: $|Gal(E', E, R)|$.
Kuznetsov (2001).

Note In particular, computing the number of elements of a lattice given by its table is $\#\mathcal{P}$-complete. Many algorithms exist for enumerating the closed elements of the Galois lattice and/or determining its covering relation. See for example Ganter (1984), Kuznetsov and Obiedkov (2001) where the efficiencies of 10 algorithms are compared, Nourine and Raynaud (2002) or Gely (2005).

The following counting problem is $\#\mathcal{P}$-hard.

Input: (E', E, R), where $R \subseteq E' \times E$ is a binary relation from E' to E.
Output: the number $|\mathcal{C}_{\mathcal{F}}|$ of critical sets of the Moore family $\mathcal{F}$ on E associated with R.
Kuznetsov (2004).

Note Recognition of a critical set.
The following decision problem is co-$\mathcal{NP}$-complete.
Input: (E', E, R), where $R \subseteq E' \times E$ is a binary relation from E' to E, and a set $C \subseteq E$.
Question: is C a critical set of the Moore family $\mathcal{F}$ on E associated with R?
Babin and Kuznetsov (2010).
Enumeration of $|\mathcal{C}_{\mathcal{F}}|$.
See Distel and Sertkaya (2011).

A.2.3 Difficult problems and particular classes of orders

Table A.1 crosses a number of order classes easy to recognize and some problems that are in general difficult (or whose status is unknown, like the isomorphism problem) but that may become polynomial for such classes. The corresponding cell in the table is then labeled "$\mathcal{P}$" (for polynomial) or contains the complexity of a polynomial-time algorithm. The cells filled in with a "?" are those for which we know that the answer is unknown, whereas the empty cells are those for which we have found no information. The "jump number" and "total weighted completion time" problems (the third and fourth columns) are the two one-machine scheduling problems presented in Section 7.5.1.

For each of the order classes in the table, we give below – ordered with respect to the problems given in the columns – the references of the complexity results given in the table cells.

1. Series–parallel order.

- Isomorphism: Lawler (1978), Valdes *et al*. (1982).
- Dimension (it is equal to 2): Valdes *et al*. (1982).
- Jump number: Cogis and Habib (1979).
- Total weighted completion time: Lawler (1978).

Table A.1 Order classes and problem complexities

Order	Isomorphism	Dim_P	Jump number
series–parallel order	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$
N-free order	complete isomorphism[a]	$\mathcal{NP}$	$\mathcal{P}$
2-dimensional order	$O(n^2)$	–	?
interval order	$\mathcal{P}$	?	$\mathcal{NP}$
semiorder	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$
weak order	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$
order of width at most k	?	?	$O(n^k)$
order of decomposition diameter at most k	$O(n^2k^2k!)$	$O(n^2k!)$	$O((2k)!n^2)$

[a] A problem is *complete isomorphism* if it is at least as difficult as the graph isomorphism problem.

Order	total weighted completion time	$\lvert\mathcal{D}(P)\rvert$	$\lvert\mathcal{L}(P)\rvert$
series–parallel order	$O(n\log n)$	$\mathcal{P}$	$\mathcal{P}$
N-free order	$\mathcal{NP}$	?	?
2-dimensional order	?	$\mathcal{P}$	?
interval order	?	$\mathcal{P}$	?
semiorder		$\mathcal{P}$	?
weak order	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$
order of width at most k		$O(n^{2k+1})$	$O(n^{k+1})$
order of decomposition diameter at most k	$O(n^{k^2})$	$\mathcal{P}$	$O(n^{k^2})$

- Number of downsets and of linear extensions: they are obtained by formulas using the decomposition tree of P, Faigle *et al.* (1986), Faigle and Schrader (1986).

2. *N*-free order.

- Isomorphism: Habib and Möhring (1987).
- Dimension: Kierstead and Penrice (1989).
- Jump number: Rival (1982, “greedy" algorithm of construction of an optimal linear extension), Faigle *et al*. (1985), Bouchitté and Habib (1989).
- Total weighted completion time: Habib and Möhring (1987).

3. 2-dimensional order.

- Isomorphism: Spinrad and Valdes (1983).
- Number of downsets: Steiner (1984).

Note Following a 2003 Ceroi result showing that a weighted version of the problem of the jump number on a 2-dimensional ordered set is $\mathcal{NP}$-complete, it may be conjectured that so is the non-weighted version.

4. Interval order.

- Isomorphism: Lueker and Booth (1979).
- Jump number: Mitas (1991); for approximation algorithms, see Faigle and Schrader (1985) and Felsner (1990).
- Number of downsets: it is obtained by recurrence formulas, Faigle *et al*. (1986).

5. Semiorder.

- Jump number: Arnim and de la Higuera (1994).
- Number of downsets: Faigle and Schrader (1986).

6. Weak order.

- Folklore.

7. Order of width at most *k*.

- Recognition: Felsner *et al*. (2003).
- Jump number: Colbourn and Pulleyblank (1985).
- Number of downsets: Faigle and Schrader (1986).
- Number of linear extensions: Steiner (1987).

8. Order of decomposition diameter at most *k*.

- Isomorphism: Habib and Möhring (1987).
- Dimension: Habib and Möhring (1987), Möhring (1989).
- Jump number: Habib and Möhring (1987), Möhring (1989).
- Total weighted completion time: Möhring and Radermacher (1985).
- Number of downsets: Faigle and Schrader (1986).
- Number of linear extensions: Habib and Möhring (1987).

Appendix B

The 58 types of connected ordered sets of size at most 5

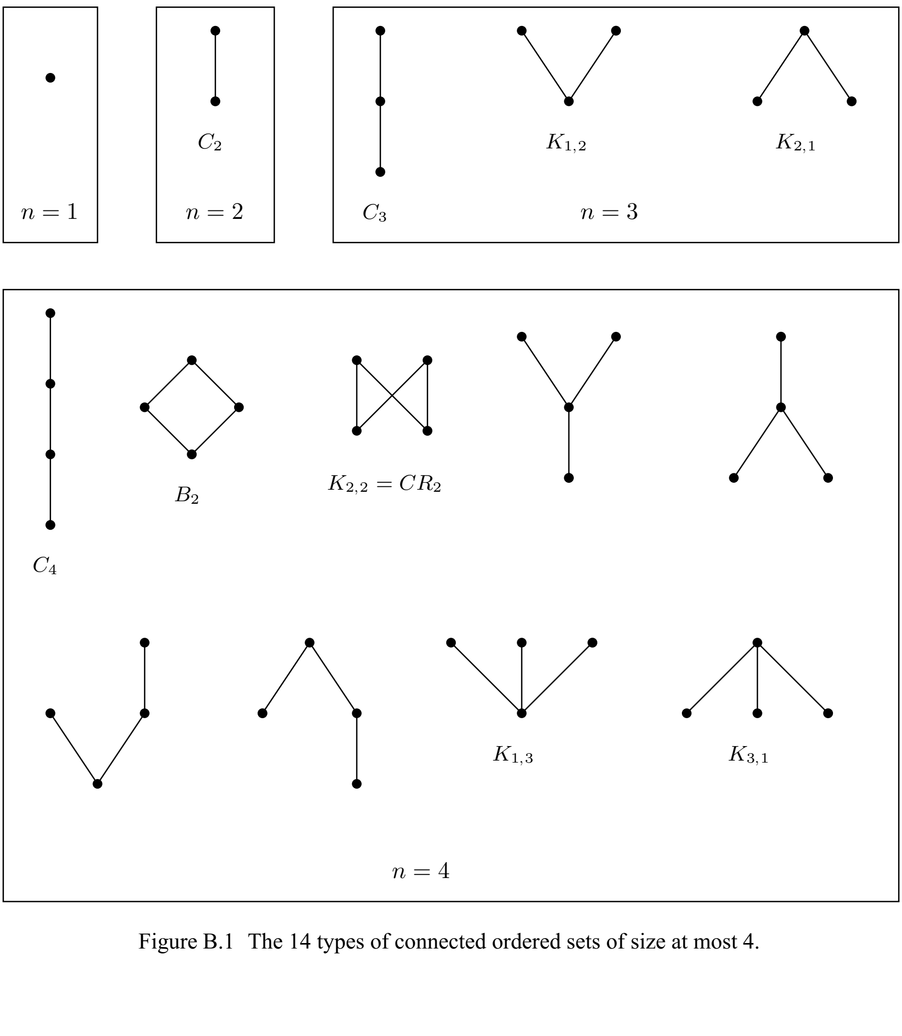
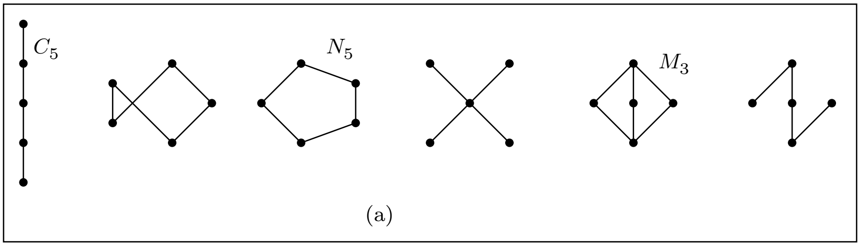
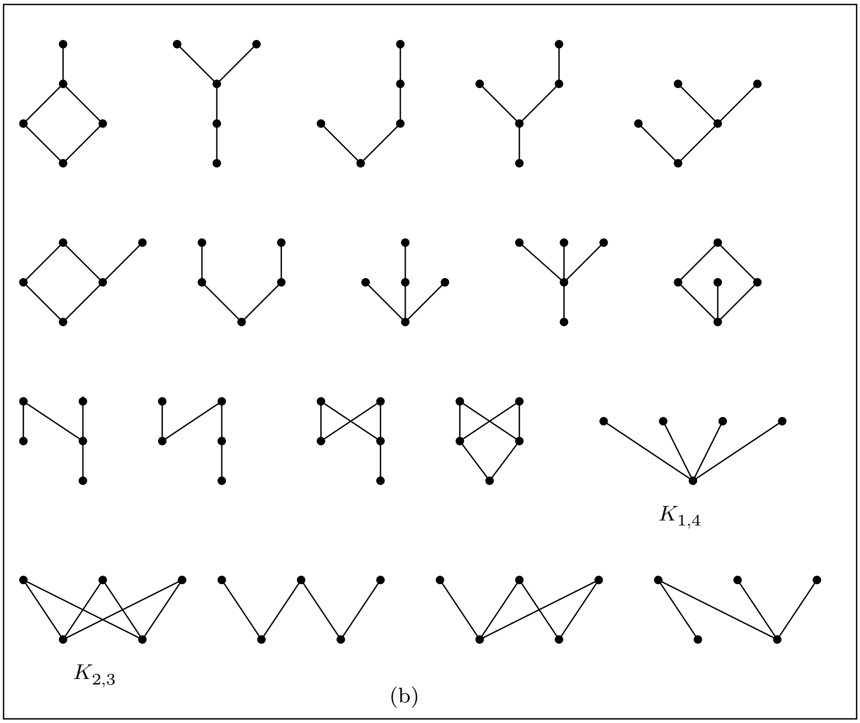


Figure B.1 The 14 types of connected ordered sets of size at most 4.





$n = 5$

Figure B.2 The 44 types of connected ordered sets of size 5. (a) Autodual ones. (b) Others, without the representation of their dual ordered sets; the reader can turn the book upside down in order to see the latter appear.

Appendix C

The numbers of ordered sets and of types of ordered sets

Table C.1 Numbers of orders and of connected orders defined on a set with $n \leq 18$ elements (see Brinkmann and McKay (2002))

n	Orders	Connected orders
1	1	1
2	3	2
3	19	12
4	219	146
5	4231	3060
6	130 023	101 642
7	6 129 859	5 106 612
8	431 723 379	377 403 266
9	44 511 042 511	40 299 722 580
10	6 611 065 248 783	6 138 497 261 882
11	1 396 281 677 105 899	1 320 327 172 853 172
12	414 864 951 055 853 499	397 571 105 288 091 506
13	171 850 728 381 587 059 351	166 330 355 795 371 103 700
14	98 484 324 257 128 207 032 183	96 036 130 723 851 671 469 482
15	77 567 171 020 440 688 353 049 939	76 070 282 980 382 554 147 600 692
16	83 480 529 785 490 157 813 844 256 579	82 226 869 197 428 315 925 408 327 266
17	122 152 541 250 295 322 862 941 281 269 151	120 722 306 604 121 583 767 045 993 825 620
18	241 939 392 597 201 176 602 897 820 148 085 023	239 727 397 782 668 638 856 762 574 296 226 842

Bounds for the number $|\mathcal{O}_n|$ of orders defined on a set with n elements

$$\frac{1}{4}n^2 + \frac{1}{3}n - 3\log_2 n \leq \log_2 |\mathcal{O}_n| \leq \frac{1}{4}n^2 + \frac{3}{2}n + c\log_2 n$$

where c is a constant.

See Kleitman and Rothschild (1975).

Table C.2 Numbers of order types and of types of connected orders defined on a set with $n \leq 16$ elements (see Brinkmann and McKay, 2002).

n	Order types	Types of connected orders
1	1	1
2	2	1
3	5	3
4	16	10
5	63	44
6	318	238
7	2 045	1 650
8	16 999	14 512
9	183 231	163 341
10*	2 567 284	2 360 719
11	46 749 427	43 944 974
12	1 104 891 746	1 055 019 099
13	33 823 827 452	32 664 984 238
14	1 338 193 159 771	1 303 143 553 205
15	68 275 077 901 156	66 900 392 672 168
16	4 483 130 665 195 087	4 413 439 778 321 689

* The number 2 567 284 of order types on a set of size 10 comes from Culberson and Rawlins (1991).

Appendix D

Documentation marks

As already said, the study of ordered sets has long been almost exclusively devoted to that of lattices. Things started to change in the 1970s and, since then, hundreds of papers have been concerned with other ordered sets, in particular with finite ones. It would therefore require several copious books to describe just the principal results obtained in this field (in comparison, the "Handbook" on Boole algebras published by Elsevier in 3 volumes has 1440 pages). However, with regard to books, the situation has not changed much. Whereas there exist dozens of books on lattice theory, the number of books on ordered sets is still very low and the latter are often concerned with particular aspects. That is why we have given, in the last section of each chapter, a number of notions and important results, referring the reader to the (numerous) references allowing us to know more on these subjects. Below, we give some marks and indications to help the reader find his bearings among these references and, more generally, in the documentary resources of the field.

D.1 Internet and the inescapable Google

If we ask that search engine, for instance, for "partial order dimension," we obtain almost 14 million answers, with many appropriate references in the first pages. It is then clear that Google may be a very efficient tool, even though important – but old – references may not be found.

Besides, there exist generalist sites where one may find a number of definitions and results:

- `http://en.wikipedia.org/wiki/Category:Order_theory`
- `http://en.wikipedia.org/wiki/Partially_ordered_set`
- `http://mathworld.wolfram.com/search/?query=poset&x=0&y=0`
- `www.math.niu.edu/ rusin/known-math/index/06-XX.html`

The latter contains some sites with queries/answers on ordered sets.

D.2 Books

D.2.1 Generalist books on ordered sets

So far, only two other books exist in English on arbitrary ordered sets. These two excellent books deal with finite as well as infinite ordered sets, the second being much more devoted to the latter. Neither of them is concerned with applications of ordered sets.

- B.S.W. Schröder (2002), *Ordered Sets. An Introduction*, Birkhaüser, Boston.
- E. Harzheim (2005), *Ordered Sets*, Series: Advances in Mathematics, Vol. 7, Springer-Verlag, New York.

D.2.2 Specialized books on ordered sets

In English, one finds books devoted to a specific topic in the theory of ordered sets:

- B.A. Davey and H.A. Priestley (1990, 2002), *Introduction to Lattices and Order*, Cambridge University Press, Cambridge.

Indeed, in spite of its title, the brilliant introduction of Davey and Priestley mainly concentrates on lattices, as does the last book of the following list:

- P.C. Fishburn (1985), *Interval Orders and Interval Graphs. A Study of Partially Ordered Sets*, John Wiley & Sons, Inc. New York.
- I. Anderson (1987), *Combinatorics of Finite Sets*, Clarendon Press, Oxford.
- W.T. Trotter (1992), *Combinatorics and Partially Ordered Sets: Dimension Theory*, The John Hopkins University Press, Baltimore, MD.
- M. Pirlot and Ph. Vincke (1997), *Semiorders. Properties, Representations, Applications*, Kluwer, Dordrecht.
- K. Engel (1997), *Sperner Theory*, Encyclopedia of Mathematics and its Applications 65, Cambridge University Press, Cambridge.
- S. Roman (2009), *Lattices and Ordered Sets*, Springer, New York.

D.2.3 Book chapters or parts

In many books on discrete mathematics and combinatorics, one finds chapters/parts dedicated to ordered sets. This is true for textbooks as well as for research books. In the former category we can quote for example:

- T.S. Foldes (1994), *Fundamental Structures of Algebra and Discrete Mathematics*, John Wiley & Sons, Inc., New York.
- V. Krishnamurthy (1986), *Combinatorics: Theory and Applications*, Halsted Press, New York.

- R.C. Penner (1999), *Discrete Mathematics: Proof Techniques and Mathematical Structures*, World Scientific Publishing Co. Pte. Ltd, Singapore.
- H.F. Mattson (1993), *Discrete Mathematics with Applications*, John Wiley and Sons, Inc., New York.

For the second category, we can mention the following books:

- M. Aigner (1979), *Combinatorial Theory*, Springer-Verlag, Berlin.
- R.P. Stanley (1986), *Enumerative Combinatorics*, Vol. 1, Wadsworth and Brooks, Monterey.
- D. Stanton and D. White (1986), *Constructive Combinatorics*, Springer-Verlag, New York.

Let us also observe that, as early as 1962, Öre devoted two chapters to ordered sets in his book *Theory of Graphs*.

D.2.4 Other references

One may notice that some books on ordered sets were published earlier in other languages. For instance, one finds in French:

- N. Bourbaki (1956), *Éléments de Mathématiques* – Livre I – *Théorie des ensembles*, Chapter 3, Ensembles ordonnés. N. Hermann & Cie Editeurs, Paris, and (2006) Éléments de Mathématiques – Théorie des ensembles, Springer, Berlin.

The particular Bourbaki style does not allow us to recommend this chapter as an introductory text, certainly, but we have to note that, at that time, it contained – with its exercises – most of what was known on arbitrary ordered sets. On the other hand, one of its merits consisted of the introduction of a terminology for the basic notions which almost became a standard for French-speaking authors.

- M. Barbut and B. Monjardet (1970), *Ordre et Classification, Algèbre et Combinatoire*, tomes I and II, Hachette, Paris.

This book (out of print for a long time but which may be found in good libraries) concentrates on applications of ordinal structures to human sciences. However, lattices occupy a great deal of the book.

More recent books are:

- in German, the marvellous:

  M. Erné (1982), *Einführung in die Ordnungstheorie*, Bibliographisches Institut, Mannheim.
- in French:

N. Caspard, B. Leclerc, and B. Monjardet (2007), *Ensembles ordonnés finis: concepts, résultats et usages*, Springer, Berlin.

In fact, the present book in English is an updated and deeply revised version of the latter (2007).

D.3 Journals

Papers on ordered sets have for long been published in very diverse mathematical reviews. This is still the case but, in 1984, Ivan Rival (at that time author or co-author of more than 60 papers in the field) launched the publication in D. Reidel of the specialized journal *Order*.

The latter journal (now published by Springer) has then become a privileged place to publish excellent papers on ordered sets. After I. Rival and W.T. Trotter, the present main editor is D. Duffus.

The list of published papers may be found on the website of the journal:

`www.springerlink.com/content/100324/`

We shall not forget the two journals of reviews whose ambition is to give a brief comment (sometimes limited to a summary) for any published paper in mathematics:

- *Mathematical Reviews*, published by the American Mathematical Society and which presents the papers with respect to a classification whose last update dates back to 2010; the latter may be found on the site:

  `www.ams.org/mathscinet/msc/msc2010.html`

- *ZentralBlatt Mathematik* (`www.zentralblatt-math.org/zmath/`).

D.4 Reviews and proceedings

Ivan Rival (deceased in 2002) wanted to publish advanced research in *Order* but also encourage diffusion of the results and communication between researchers and their fields of study. With this aim, he was the initiator of several conferences principally dedicated to survey talks, presented by one or two specialists (these conferences also had sessions of open problems). These talks were later published in the following proceedings:

- *Ordered sets*. Proceedings of the NATO Advanced Study Institute held in Banff, Alta, August 28–September 12, 1981. I. Rival (ed.). NATO ASI Series C 83. D. Reidel Publishing Co., Dordrecht-Boston, MA, 1982. xviii+966 pp.

- *Graphs and order*. The role of graphs in the theory of ordered sets and its applications. Proceedings of the NATO Advanced Study Institute held in Banff, Alta, May 18–31, 1984. I. Rival (ed.), NATO ASI Series, 147. D. Reidel Publishing Co., Dordrecht-Boston, MA, 1985. xix+796 pp.
- *Combinatorics and ordered sets*. Proceedings of the AMS–IMS–SIAM joint summer research conference held at Humboldt State University, Arcata, CA, August 11–17, 1985. I. Rival (ed.). Contemporary Mathematics, 57. American Mathematical Society, Providence, RI, 1986. xvi+285 pp.
- *Algorithms and order*. Proceedings of the NATO Advanced Study Institute held in Ottawa, Ontario, June 1–12, 1987. I. Rival (ed.). Kluwer Academic Publishers, Dordrecht-Boston, MA, 1989. x+498 pp.

We should also mention the conference proceedings that may also contain reviews:

- *Orders*: *Descriptions and Roles*, M. Pouzet and D. Richard (eds), *Annals of Discrete Mathematics* **23**, 1984, North-Holland.
- *Orders, Algorithms and Applications*, International Workshop ORDAL'94, Lyon, France, July 4–8, 1994. Proceedings Series: Lecture Notes in Computer Science, Vol. 831, Bouchitté, V., Morvan, M. (eds) 1994, IX, 204 pp.
- *Orders, Algorithms and Applications*, International Workshop ORDAL'96, Ottawa, Ont., Canada, August 5–9, 1996. *Theoretical Computer Science*, **217** (2), I. Rival, N. Zaguia (eds) 1999, Elsevier Science Publishers, B.V., Amsterdam, 1999. pp. i–iv and 173-436.

We may also add to this list two survey texts exclusively devoted to finite ordered sets:

- C. Greene and D. Kleitman (1978), Proof techniques in the theory of finite sets, in *Studies in Combinatorics*, ed. by G.-C.Rota, Mathematical Association of America, 22–79.
- W.T. Trotter (1995), Chapter 8, Partially ordered sets, in R.L. Graham, M. Grötschel, L. Lovász (eds), *Handbook of Combinatorics*, Vol. 1, Elsevier, Amsterdam, 433–480.

D.5 Software

Software called "the posets package" and containing 41 programs written in MAPLE has been developed by John Stembridge (`www.math.lsa.umich.edu/jrs/maple.html`). Among the procedures available for an ordered set, one finds the search for its covering relation, for chains or antichains, the determination of its linear extensions or of its lattice of downsets, some recognition tests of a number of properties (existence of a rank, recognition of a lattice structure). One also finds a test for the isomorphism of two ordered sets and a library of the 19.449 types of ordered set of size at most 8 and of the 7.372 types of lattice of size at most 10.

Another software package called “A Mathematica Package for Studying Posets” has been developed by Erica Greene and Curtis Greene:

`www.haverford.edu/math/cgreene/posets.html`

They write: “It is designed to generate, display, and explore partially ordered sets, with an emphasis on topics of current interest in combinatorics. The package has two distinctive features: (1) a large repertoire of standard examples (2) the ability to generate a poset directly from its formal definition, i.e., from a Mathematica program giving its covering function” (last version: summer 2008).

Another more specialized Mathematica package called “Mathematica package to cope with partially ordered sets” has been developed by P. Codara. In particular, it offers the ability to enumerate, create, and display monotone and regular partitions of partially ordered sets as well as the capability of constructing its lattices of partitions:

`www.cody.it/pietrocodara_files/pub_cod_ugm2010_pre.pdf`

Karell Bertet has incorporated in the free software Graphviz of graph vizualization some functions to handle orders (transitive/symmetric closure/reduction) and a set of Java classes allowing us to instantiate an implicational system (set of implication rules), to test and apply some properties on these rules, and to generate the associated closure system. See:

`http://perso.univ-lr.fr/kbertet/dotty.html`

Besides, let us mention the last chapter, called “Computational aspects of lattice theory,” of the book *Free Lattices* (Freese *et al.*, 1995). It contains numerous algorithms, written in a simple language, on orders and lattices. In particular, it contains the following: transitive closure, transitions between different representations of an order, linear extensions, antichains, chain partitions.

Software for graph handling, such as LEDA – “a Library of Efficient Data types and Algorithms”:

`www.cs.sunysb.edu/ algorith/implement/LEDA/`
`implement.shtml`

contains algorithms of transitive closure and reduction.

At last, let us note that a lot of software has been developed for the use of ordinal tools in data analysis and data mining. We can quote, for instance:

TOCKIT: `http://tockit.sourceforge.net/`

GALICIA: `www.iro.umontreal.ca/galicia`

and CONCEPTS: `www.st.cs.uni-saarland.de/lindig/#`
`colibri`

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List of symbols

Index