Mathematics and Its Applications

Michiel Hazewinkel
Nadiya Gubareni and
V.V. Kirichenko

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by

Michiel Hazewinkel

CWI,
Amsterdam, The Netherlands

Nadiya Gubareni

Technical University of Czestochowa,
Poland

and

V.V. Kirichenko

Kiev Taras Shevchenko University,
Kiev, Ukraine

Springer

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Preface

This book is the natural continuation of "Algebras, rings and modules. vol.I". The main part of it consists of the study of special classes of algebras and rings. Topics covered include groups, algebras, quivers, partially ordered sets and their representations, as well as such special rings as quasi-Frobenius and right serial rings, tiled orders and Gorenstein matrices.

Representation theory is a fundamental tool for studying groups, algebras and rings by means of linear algebra. Its origins are mostly in the work of F.G.Frobenius, H.Weil, I.Schur, A.Young, T.Molien about century ago. The results of the representation theory of finite groups and finite dimensional algebras play a fundamental role in many recent developments of mathematics and theoretical physics. The physical aspects of this theory concern accounting for and using the concepts of symmetry which appear in various physical processes.

We start this book with the main results of the theory of groups. For the convenience of a reader in the beginning of this chapter we recall some basic concepts and results of group theory which will be necessary for the next chapters of the book.

Groups are a central object of algebra. The concept of a group is historically one of the first examples of an abstract algebraic system. Finite groups, in particular permutation groups, are an increasingly important tool in many areas of applied mathematics. Examples include coding theory, cryptography, design theory, combinatorial optimization, quantum computing, and statistics.

In chapter I we give a short introduction to the theory of groups and their representations. We consider the representation theory of groups from the module-theoretical point of view using the main results about rings and modules as recorded in volume I of this book. This theoretical approach was first used by E.Noether who established a close connection between the theory of algebras and the theory of representations. From that point of view the study of the representation theory of groups becomes a special case of the study of modules over rings. In the theory of representations of group a special role is played by the famous Maschke theorem. Taking into account its great importance we give three different proofs of this theorem following J.-P.Serre, I.N.Herstein and M.Hall. As a consequence of the Maschke theorem, the representation theory of groups splits into two different cases depending on the characteristic of a field k: classical and modular (following L.E.Dickson). In "classical" representation theory one assumes that the characteristic of k does not divide the group order $|G|$ (e.g. k can be the field of complex numbers). In "modular" representation theory one assumes that the characteristic of k is a prime, dividing $|G|$. In this case the theory is almost completely different from the classical case.

In this book we consider the results belonging to the classical representation theory of finite groups, such as the characters of groups. We give the basic properties of irreducible characters and their connection with the ring structure of the corresponding group algebras.

A central role in the theory of representations of finite dimensional algebras and rings is played by quivers, which were introduced by P.Gabriel in connection with problems of representations of finite dimensional algebras in 1972. The main notions and result concerning the theory of quivers and their representations are given in chapter 2.

A most remarkable result in the theory of representations of quivers is the theorem classifying the quivers of finite representation type, which was obtained by P.Gabriel in 1972. This theorem says that a quiver is of finite representation type over an algebraically closed field if and only if the underlying diagram obtained from the quiver by forgetting the orientations of all arrows is a disjoint union of simple Dynkin diagrams. P.Gabriel also proved that there is a bijection between the isomorphism classes of indecomposable representations of a quiver Q and the set of positive roots of the Tits form corresponding to this quiver. A proof of this theorem is given in section 2.6.

Another proof of this theorem in the general case, for an arbitrary field, using reflection functors and Coxeter functors has been obtained by I.N.Berstein, I.M.Gel'fand, and V.A.Ponomarev in 1973. In their work the connection between indecomposable representations of a quiver of finite type and properties of its Tits quadratic form is elucidated.

Representations of finite partially ordered sets (posets, in short) play an important role in representation theory. They were first introduced by L.A.Nazarova and A.V.Roiter. The first two sections of chapter 3 are devoted to partially ordered sets and their representations. Here are given the main results of M.M.Kleiner on representations of posets of finite type and results of L.A.Nazarova on representations of posets of infinite type. The most important result in this theory was been obtained by Yu.A.Drozd who showed that there is a trichotomy between finite, tame and wild representation types for finite posets over an algebraically closed field.

One of the main problems of representation theory is to obtain information about the possible structure of indecomposable modules and to describe the isomorphism classes of all indecomposable modules. By the famous theorem on trichotomy for finite dimensional algebras over an algebraically closed field, obtained by Yu.A.Drozd, all such algebras divide into three disjoint classes.

The main results on representations of finitely dimensional algebras are given in section 3.4. Here we give structure theorems for some special classes of finite dimensional algebras of finite type, such as hereditary algebras and algebras with zero square radical, obtained by P.Gabriel in terms of Dynkin diagrams. Section 3.5 is devoted to the first Brauer-Thrall conjecture, of which a proof has been obtained by A.V.Roiter for the case of finite dimensional algebra over an arbitrary field.

Chapter 4 is devoted to study of Frobenius algebras and quasi-Frobenius rings. The class of quasi-Frobenius rings was introduced by T.Nakayama in 1939 as a generalization of Frobenius algebras. It is one of the most interesting and intensively studied classes of Artinian rings. Frobenius algebras are determined by the requirement that right and left regular modules are equivalent. And quasi-Frobenius algebras are defined as algebras for which regular modules are injective.

We start this chapter with a short study of duality properties for finite dimensional algebras. In section 4.2 there are given equivalent definitions of Frobenius algebras in terms of bilinear forms and linear functions. There is also a discussion of symmetric algebras which are a special class of Frobenius algebras. The main properties of quasi-Frobenius algebras are given in section 4.4.

The starting point in studying quasi-Frobenius rings in this chapter is the Nakayama definition of them. The key concept in this definition is a permutation of indecomposable projective modules, which is naturally called **Nakayama permutation**.

Quasi-Frobenius rings are also of interest because of the presence of a duality between the categories of left and right finitely generated modules over them. The main properties of duality in Noetherian rings are considered in section 4.10. Semiperfect rings with duality for simple modules are studied in section 4.11. The equivalent definitions of quasi-Frobenius rings in terms of duality and semi-injective rings are given 4.12. Quasi-Frobenius rings have many interesting equivalent definitions, in particular, an Artinian ring A is quasi-Frobenius if and only if A is a ring with duality for simple modules.

One of the most significant results in quasi-Frobenius ring theory is the theorem of C.Faith and E.A.Walker. This theorem says that a ring A is quasi-Frobenius if and only if every projective right A-module is injective and conversely.

Quivers of quasi-Frobenius rings are studied in section 4.13. The most important result of this section is the Green theorem: the quiver of any quasi-Frobenius ring is strongly connected. Conversely, for a given strongly connected quiver Q there is a symmetric algebra A such that $Q(A) = Q$. Symmetric algebras with given quivers are studied in section 4.14.

Chapter 5 is devoted to the study of the properties and structure of right serial rings. Note that a module is called serial if it decomposes into a direct sum of uniserial submodules, i.e., submodules with linear lattice of submodules. A ring is called right serial if its right regular module is serial.

We start this chapter with a study of right Noetherian rings from the point of view of some main properties of their homological dimensions.

In further sections we give the structure of right Artinian right serial rings in terms of their quivers. We also describe the structure of particular classes of right serial rings, suchas quasi-Frobenius rings, right hereditary rings, and semiprime rings. In section 5.6 we introduce right serial quivers and trees and give their description.

The last section of this chapter is devoted to the Cartan determinant conjecture for right Artinian right serial rings. The main result of this section says that a right Artinian right serial ring A has its Cartan determinant equal to 1 if and only if the global dimension of A is finite.

In chapters 6 and 7 the theory of semiprime Noetherian semiperfect semidistributive rings is developed ($SPSD$-rings). In view of the decomposition theorem (see theorem 14.5.1, vol.I) it is sufficient to consider prime Noetherian $SPSD$-rings, which are called tiled orders.

With any tiled order we can associate a reduced exponent matrix and its quiver. This quiver Q is called the quiver of that tiled order. It is proved that Q is a simply laced and strongly connected quiver. In chapter 6 a construction is given which allows to form a countable set of Frobenius semidistributive rings from a tiled order. Relations between finite posets and exponent (0,1)-matrices are described and discussed. In particular, a finite ergodic Markov chain is associated with a finite poset.

Chapter 7 is devoted to the study of Gorenstein matrices. We say that a tiled order A is Gorenstein if $r.inj.dim_A A = 1$. In this case $r.inj.dim_A A = l.inj.dim_A A = 1$. Moreover, a tiled order is Gorenstein if and only if it is Morita equivalent to a reduced tiled order with a Gorenstein exponent matrix.

Each chapter ends with a number of notes and references, some of which have a bibliographical character and others are of a historical nature.

At the end of the book we give a literature list which can be considered as suggestions for further reading to obtain fuller information concerning other aspects of the theory of rings and algebras.

In closing, we would like to express our cordial thanks to a number of friends and colleagues for reading preliminary versions of this text and offering valuable suggestions which were taken into account in preparing the final version. We are especially greatly indebted to Yu.A.Drozd, V.M.Bondarenko, S.A.Ovsienko, M.Dokuchaev, V.Futorny, V.N.Zhuravlev, who made a large number of valuable comments, suggestions and corrections which have considerably improved the book. Of course, any remaining errors are the sole responsibility of the authors.

Finally, we are most grateful to Marina Khibina for help in preparing the manuscript. Her assistance has been extremely valuable to us.

1. Groups and group representations

Groups are a central subject in algebra. They embody the easiest concept of symmetry. There are others: Lie algebras (for infinitesimal symmetry) and Hopf algebras (quantum groups) who combine the two and more (see volume III). Finite groups, in particular permutation groups, are an increasingly important tool in many areas of applied mathematics. Examples include coding theory, cryptography, design theory, combinatorial optimization, quantum computing.

Representation theory, the art of realizing a group in a concrete way, usually as a collection of matrices, is a fundamental tool for studying groups by means of linear algebra. Its origins are mostly in the work of F.G.Frobenius, H.Weil, I.Schur, A.Young, T.Molien about century ago. The results of the theory of representations of finite groups play a fundamental role in many recent developments of mathematics and theoretical physics. The physical aspects of this theory consist in accounting for and using the concept of symmetry as present in various physical processes – though not always obviously so. As understood at present, symmetry rules physics and an elementary particle is the same thing as an irreducible representation. This includes quantum physics. There is a seeming mystery here which is explained by the fact that the representation theory of quantum groups is virtually the same as that of their classical (Lie group) counterparts.

In this chapter we shall give a short introduction to the theory of groups and their representations. We shall consider the representation theory of groups from the module-theoretic point of view using the main results about rings and modules as described in volume I of this book. This theoretical approach was first used by E.Noether who established a close connection between the theory of algebras and the theory of representations. From this point of view the study of the representation theory of groups becomes a special case of the study of modules over rings. At the end of this chapter we shall consider the characters of groups. We shall give the basic properties of irreducible characters and their connection with the ring structure of group algebras.

For the convenience of a reader in the beginning of this chapter we recall some basic concepts and results of group theory which will be necessary for the next chapters of the book.

1.1 GROUPS AND SUBGROUPS. DEFINITIONS AND EXAMPLES

The notion of an abstract group was first formulated by A.L.Cayley (1821-1895) who used this to identify matrices and quaternions as groups. The first formal definition of an abstract group in the modern form appeared in 1882. Before, a group was exclusively a group of permutations of some set (or a group of matrices). The famous book by Burnside (1905) illustrates this well.

Definition. A **group** is a nonempty set G together with a given binary operation $*$ on G satisfying the following axioms:

(1) $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$; (associativity)

(2) there exists an element $e \in G$, called an **identity** of G, such that $a * e = e * a = a$ for every $a \in G$;

(3) for each $a \in G$ there exists an element $a^{-1} \in G$, called an **inverse** of a, such that $a * a^{-1} = a^{-1} * a = e$.

From the axioms for a group G one can easily obtain the following properties:

(1) the identity element in G is unique;

(2) for each $a \in G$ the element a^{-1} is uniquely determined;

(3) $(a^{-1})^{-1} = a$ for every $a \in G$;

(4) $(a * b)^{-1} = b^{-1} * a^{-1}$.

A group G is called **Abelian** (or **commutative**) if $a*b = b*a$ for all $a, b \in G$. For some commutative groups it is often convenient to use the additive symbol $+$ for the operation in a group and write $x + y$ instead of $x * y$. In this case we call this group **additive**. The identity of an additive group G is called the **zero** and denoted by 0, and the inverse element of x is called its **negative element** and denoted by $-x$. In this case we write $x - y$ instead of $x + (-y)$. Note that this notation is almost never used for non-commutative groups.

For writing an operation of a group G we usually use the multiplicative symbol $\cdot$ and write xy rather that $x \cdot y$. In this case we say that the group G is **multiplicative** and denote the identity of G by 1.

If G is a finite set G is called a **finite group**. The number of elements of a finite group G is called the **order** of G and denoted by $|G|$ or $o(G)$ or $\#G$.

Examples 1.1.1.

1. The sets $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ are groups under the operation of addition $+$ with $e = 0$ and $a^{-1} = -a$ for all a. They are additive Abelian groups.

2. The sets $\mathbf{Q} \setminus \{0\}$, $\mathbf{R} \setminus \{0\}$ and $\mathbf{C} \setminus \{0\}$ are groups under the operation of multiplication $\cdot$ with $e = 1$ and $a^{-1} = 1/a$ for all a. They are multiplicative Abelian groups. The set $\mathbf{Z}\setminus\{0\}$ with the operation of multiplication $\cdot$ is not a group because the inverse to n is $1/n$, which is not integer if $n \neq 1$. The set $\mathbf{R}^+$ of all positive rational numbers is a multiplicative Abelian group under multiplication.

3. The set of all invertible $n \times n$ matrices with entries from a field k forms a group under matrix multiplication. This group is denoted by $\mathrm{GL}_n(k)$ and called

the **general linear group of order** n (in dimension n). This group is finite if and only if k is a finite field.

4. The set of all invertible linear transformations of a vector space V over a field k forms a group under the operation of composition. This group is denoted by $\mathrm{GL}(V, k)$. If V is an n-dimensional vector space over a field k, i.e., $V \simeq k^n$, then there is a one-to-one correspondence between invertible matrices of order n and invertible linear transformations of the vector space k^n. Thus the group $\mathrm{GL}(k^n, k)$ is isomorphic to the group $\mathrm{GL}_n(k)$.

5. Suppose G is the set of all functions $f : [0, 1] \to \mathbf{R}$. Define an addition on G by $(f + g)(t) = f(t) + g(t)$ for all $t \in \mathbf{R}$. Then G is an Abelian group under (pointwise) addition.

Definition. A non-empty subset H of a group G which itself is a group with respect to the operation defined on G is called a **subgroup**.

The following simple statement may be considered as an equivalent definition of the notion of a subgroup.

Proposition 1.1.1. *A subset H of a group G is a subgroup of G if and only if:*

1) *H contains the product of any two elements from H;*
2) *H contains together with any element h the inverse h^{-1}.*

The subset of a group G consisting of the identity element only is clearly a subgroup; it is called the **unit subgroup** of G and usually denoted by E. Also, G is a subgroup of itself. The group G itself and the subgroup E are called **improper subgroups** of G, while all others are called **proper** ones.

One of the central problems in group theory is to determine all proper subgroups of a given group.

Examples 1.1.2.

1. $\mathbf{Z}$ is a proper subgroup of $\mathbf{Q}$ and $\mathbf{Q}$ is a proper subgroup of $\mathbf{R}$ with the operation of addition.

2. The set of all even integers is a subgroup of $\mathbf{Z}$ under addition.

3. If $G = \mathbf{Z}$ under addition, and $n \in \mathbf{Z}$, then $H = n\mathbf{Z}$ is a subgroup of $\mathbf{Z}$. Moreover, every subgroup of $\mathbf{Z}$ is of this form.

4. Let k be a field. Define

$$\mathrm{SL}_n(k) = \{\mathbf{A} \in \mathrm{GL}_n(k) \ : \ \det(\mathbf{A}) = 1\},$$

which is called the **special linear group** or the **unimodular group**. This group is a proper subgroup of $\mathrm{GL}_n(k)$.

For finite groups of not to large order it can be convenient to represent the operation on a group by means of a **multiplication table**, which is often called its **Cayley table**. Such a table is a square array with the rows and columns labelled by the elements of the group. In this table at the intersection of the i-th row and the j-th column we write the product of the elements, which are in the i-th row and the j-th column respectively. It is obvious, that this table is symmetric with respect to the main diagonal if and only if the group is Abelian. For example, consider for a group $G = \{e, a, b, c\}$ the group table:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

This group is called the **Klein 4-group**.

In the general case for a group G one can write down a set of generators S with the property that every element of G can be written as a finite product of elements of S. Any equation in a group G that the generators satisfy is called a **relation** in G. For example, in the previous example the Klein group G has the relations

$$a^2 = b^2 = c^2 = e, \quad ab = c, \quad ac = b, \quad bc = a.$$

Very important examples of non-Abelian groups are groups of transformations of a set, i.e., bijections from a given set to itself. It is interesting that groups first arose in mathematics as groups of transformations. And only later groups were considered as abstract objects independently of groups of transformations. See also above.

Example 1.1.3.

Symmetric groups. Let A be a nonempty set and let S_A be the set of all bijections from A to itself. If $x, y \in S_A$, then their multiplication $z = xy$ is defined by $z(a) = x(y(a))$ for an arbitrary $a \in A$. It is easy to see that $z \in S_A$, and that the operation of multiplication of transformations is associative. The identity of this operation is the identity transformation e of the set A, which is defined by $e(a) = a$ for all $a \in A$.

Obviously, $ex = xe = x$ for all $x \in S_A$. The inverse element to x is defined as the transformation x^{-1} for which $x^{-1}(x(a)) = a$ for all $a \in A$. Clearly, $x^{-1}x = xx^{-1} = e$. Therefore S_A is a group which is called the **symmetric group on the set** A.

In the special case, when $A = \{1, 2, ..., n\}$, each transformation of A is called a **permutation** and the symmetric group on A is called the **permutation group** of A. It is also denoted by S_n and called the **symmetric group of degree** n. The order of the group S_n is $n!$ The group S_n is non-Abelian for all $n \geq 3$.

Example 1.1.4.

Alternating group. Let S_n be a symmetric group, i.e., the group of all permutations of $\{1, 2, ..., n\}$. Let $x_1, x_2, ..., x_n$ be independent variables. Consider the polynomial

$$\Delta = \prod_{i<k} (x_i - x_k), \quad (i, k = 1, 2, ..., n).$$

For each $\sigma \in S_n$ let σ act on Δ by permuting the variables in the same way; i.e., it permutes their indices:

$$\sigma(\Delta) = \prod_{i<k} (x_{\sigma(i)} - x_{\sigma(k)}), \quad (i, k = 1, 2, ..., n).$$

Then

$$\sigma(\Delta) = \pm\Delta, \text{ for all } \sigma \in S_n.$$

For each $\sigma \in S_n$ let $\mathrm{sign}(\sigma) = \varepsilon(\sigma)$ be defined by

$$\varepsilon(\sigma) = \begin{cases} +1 & \text{for } \sigma(\Delta) = \Delta \\ -1 & \text{for } \sigma(\Delta) = -\Delta. \end{cases}$$

A permutation $\sigma \in S_n$ is called an **even permutation** if $\varepsilon(\sigma) = 1$ and an **odd permutation** if $\varepsilon(\sigma) = -1$. A permutation which changes only two indices is called a **transposition** and obviously it is odd. Any permutation is a product of some transpositions. The product of any two even or any two odd permutations is an even permutation. The product of an even permutation and an odd permutation is odd.

The inverse to an even permutation is even, and the inverse to an odd permutation is odd. The identity of S_n is an even permutation. Therefore, the set of all even permutations is a subgroup of S_n. It is called the **alternating group** and denoted by A_n. Note, the set of all odd permutation does not form a group (because the product of any two odd permutation is even). It is easy to show that the number of all even permutations is equal to the number of all odd permutations and so $o(A_n) = \frac{1}{2}n!$

Example 1.1.5.

Let f be any polynomial in n independent variables $x_1, x_2, ..., x_n$. Then

$$\mathrm{Sym}f = \{\sigma \in S_n \ : \ f(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}) = f(x_1, x_2, ..., x_n)\}$$

is a subgroup of the group S_n. In particular, the polynomial f is symmetric if and only if $\mathrm{Sym}f = S_n$.

Example 1.1.6.

Groups in geometry. F.Klein was the first who wrote down the connection between permutation groups and symmetries of convex polygons. He also posed the idea that the background of all different geometries is the notion of a group of transformations. In his famous lecture in 1872 he gave the definition of geometry as the science that studies the properties of figures invariant under a given group of transformations.

Let X be a set and let G be a group of transformations of it. A figure $F_1 \subset X$ is said to be equivalent (or equal) to a figure $F_2 \subset X$ with respect to the group G and will be written $F_1 \sim F_2$ if there is a transformation $\varphi \in G$ such that $F_2 = \varphi(F_1)$. It is easy to verify that this is really an equivalence relation and the three axioms of this equivalence relation amount to the same as the axioms of a group of transformations. Using different kinds of groups of transformations we can build different geometries, such as Euclidean geometry, affine geometry, projective geometry, Lobachevskian geometry (or hyperbolic geometry) and others.

For example, affine geometry is the geometry in which properties are preserved by parallel projections from one plane to another. This geometry may be defined by means of the **affine group** of any affine space over a field k, which is a set of all invertible affine transformations from the space into itself. In particular, an invertible affine transformation of the real space $\mathbf{R}^n$ is a map $F : \mathbf{R}^n \to \mathbf{R}^n$ of the form $F(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{A} \in \mathrm{GL}(n, \mathbf{R})$, $\mathbf{b} \in \mathbf{R}^n$. The affine group contains the full linear group and the group of translations as subgroups.

Example 1.1.7.

Galois groups. In many examples groups appear in the form of automorphism groups of various mathematical structures. This is one of the most important ways of their appearance in algebra. In such a way we can consider Galois groups. Let K be a finite, separable and normal extension of a field k. The automorphisms of K leaving the elements of k fixed form a group $\mathrm{Gal}(K/k)$ with respect to composition, called the **Galois group** of the extension K/k. Let f be a polynomial in x over k and K be the splitting field of f. The group $\mathrm{Gal}(K/k)$ is called the Galois group of f. One of the main applications of Galois theory is connected with the problem of the solvability of equations in radicals. Indeed, the main theorem says that the equation $f(x) = 0$ is solvable in radicals if and only if the group $\mathrm{Gal}(K/k)$ is solvable (see section 1.5). This is where the terminology "solvable" (for groups) comes from.

Example 1.1.8.

Homology groups. This kind of groups, considered in section 6.1 (vol.I), occurs in many areas of mathematics and allow us to study non-algebraic objects by means of algebraic methods. This is a fundamental method in algebraic topology. To each topological space X there is associated a family of Abelian groups $H_0(X)$, $H_1(X)$,..., called the homology groups, while each continuous mapping $f : X \to Y$ defines a family of homomorphisms $f_n : H_n(X) \to H_n(Y)$, $n = 0, 1, 2, \ldots$.[1]

[1] A homomorphism of groups $f : G \to H$ is a map that preserves the unit element and the multiplication, i.e. $f(e_G) = e_H$ and $f(xy) = f(x)f(y)$. It then also preserves inverses, i.e., $f(x^{-1}) = f(x)^{-1}$. See section 1.3.

1.2 SYMMETRY. SYMMETRY GROUPS

Groups were invented as a tool for studying symmetric objects. These can be objects of any kind at all. One can define a **symmetry** of an object as a transformation of that object which preserves its essential structure. Then the set of all symmetries of the object forms a group. The study of symmetry is actually equivalent to the study of automorphisms of systems, and for this reason group theory is indispensable in solving such problems.

An important family of examples of groups is the class of groups whose elements are symmetries of geometric figures. Let E^3 be three dimensional Euclidean space, that is, the vector space $\mathbf{R}^3$ together with the scalar product $(x, y) = x_1y_2 + x_2y_2 + x_3y_3$ for all $x, y \in \mathbf{R}^3$. The distance between x and y in $\mathbf{R}^3$ is $\sqrt{(x-y, x-y)}$. All transformations of E^3 that preserve distance form a group of transformations under composition, which is denoted by $\mathrm{Isom}E^3$. Let F be any geometrical figure in E^3. Then the set

$$\mathrm{Sym}F = \{\varphi \in \mathrm{Isom}E^3 \ : \ \varphi(F) = F\}$$

forms a subgroup in $\mathrm{Isom}E^3$. This group is called the **symmetry group** of the figure F. If this subgroup is not trivial, the figure F is said to be **symmetric**, or to have **symmetry**. In this case there is a special transformation, such as a rotation or a reflection such that the figure looks the same after the transformation as it did before the transformation. These transformations are said to be symmetry transformations of the corresponding geometrical figure.

This was in fact the approach of E.S.Fedorov (see [Fedorov, 1891], [Fedorov, 1949]) for the problem of classification of regular spatial systems of points, which is one of the basic problems in crystallography. Crystals possess a great degree of symmetry and therefore the symmetry group of a crystal is an important characteristic of this crystal. The study of crystallographic groups was started by E.S.Fedorov, and continued by A.Schoenflies at the end of the 19-th century [Schoenflies, 1891]. There are only 17 plane crystallographic groups, which were found directly; there are 230 three-dimensional crystallographic groups, which could be exhaustively classified only by the use of group theory. This is historically the first example of the application of group theory to natural science.

Example 1.2.1.

The symmetry group of an equilateral triangle is isomorphic to S_3. The structure of this group is completely determined by the relations $\sigma^3 = \tau^2 = 1$ and $\sigma\tau = \tau\sigma^{-1}$, where σ is the cyclic permutation $(1, 2, 3)$ and τ is the reflection $(1, 3)$.

Labeling the vertices of the triangle as 1, 2 and 3 permits us to identify the symmetries with permutations of the vertices, and we see that there are three rotation symmetries (through angles of 0, $2\pi/3$ and $4\pi/3$) corresponding to the identity permutation, the cycles $(1, 2, 3)$ and $(1, 3, 2)$, and three reflection symmetries corresponding to the other three elements of S_3.

Example 1.2.2.

Dihedral groups. For each $n \in \mathbf{Z}^+$, $n \geq 3$, let D_n be a set of all symmetries of an n-sided regular polygon. There are n rotation symmetries, through angles $2k\pi/n$, where $k \in \{0, 1, 2, ..., n-1\}$, and there are n reflection symmetries, in the n lines which are bisectors of the internal angles and/or perpendicular bisectors of the sides. Therefore $|D_n| = 2n$. The binary operation on D_n is associative since composition of functions is associative. The identity of D_n is the identity symmetry, denoted by 1, and the inverse of $s \in D_n$ is the symmetry which reverses all rigid motions of s.

In D_n we have the relations: $\sigma^n = 1$, $\tau^2 = 1$ and $\sigma\tau = \tau\sigma^{-1}$, where σ is a clockwise rotation through $\frac{2\pi}{n}$ and τ is any reflection. Moreover, one can show that any other relation between elements of the group D_n may be derived from these three relations. Thus there is the following presentation of the group D_n:

$$D_n = \{\sigma, \tau \ : \ \sigma^n = \tau^2 = 1, \ \ \sigma\tau = \tau\sigma^{-1} \ \}.$$

D_n is called the **dihedral group of order** n. Some authors denote this group by D_{2n}.

The rotation symmetries in the group D_n form a subgroup in it and this group is called the **rotation group** of a given n-sided regular polygon. It is immediate that this subgroup is isomorphic to $\mathbf{Z}_n$.

For $n = 2$ a degenerate "2-sided regular polygon" would be a line segment and in this case we have the simplest dihedral group

$$D_2 = \{\sigma, \tau \ : \ \sigma^2 = \tau^2 = 1, \ \ \sigma\tau = \tau\sigma^{-1} \ \},$$

which is generated by a rotation σ of 180 degrees and a reflection τ across the y-axis. D_2 is isomorphic to the Klein four-group. For $n > 2$ the operations of rotation and reflection in general do not commute and D_n is not Abelian.

Example 1.2.3.

Quasidihedral groups. Quasidihedral groups are groups with similar properties as dihedral groups. In particular, they often arise as symmetry groups of regular polygons, such as an octagon. For each $n \in \mathbf{Z}^+$, $n \geq 3$, the group Q_{2n} has the following presentation:

$$Q_n = \{\sigma, \tau \ : \ \sigma^{2^n} = \tau^2 = 1, \ \ \sigma\tau = \tau\sigma^{2^{n-1}-1} \ \}.$$

This group is called the **quasidihedral group of order** n.

Example 1.2.4.

Generalized quaternion groups. A group is is called the **generalized quaternion group of order** n if it has the following presentation

$$H_n = \{\sigma, \tau \ : \ \sigma^{2^n} = 1, \ \sigma^{2^{n-1}} = \tau^2, \ \ \sigma\tau = \tau\sigma^{-1} \ \}$$

for some integer $n \geq 2$. For $n = 2$ we obtain

$$H_2 = \{\sigma, \tau \ : \ \sigma^4 = 1, \ \sigma^2 = \tau^2, \ \ \sigma\tau = \tau\sigma^{-1} \ \}$$

which is the usual quaternion group $H_2 = \{1, i, j, k, -1, -i, -j, -k\}$ if one takes, for instance, $\sigma = i$ and $\tau = j$ (see example 1.1.12, vol.I).

Example 1.2.5.

Orthogonal groups. Let E_n be a Euclidean space, that is, a real n-dimensional vector space $\mathbf{R}^n$ together with the scalar product $(x, y) = x_1y_1 + x_2y_2 + ... + x_ny_n$ in a given orthonormal basis $e_1, e_2, ..., e_n$. The linear transformations of E_n, which preserve the scalar product, are called **orthogonal**. They form a group $O(n)$ which is called the **orthogonal group** of E_n. The elements of $O(n)$ are orthogonal matrices, i.e.,

$$O(n) = \{\mathbf{X} \in M_n(\mathbf{R}) \ : \ \mathbf{X}\mathbf{X}^T = \mathbf{I}_n\},$$

where $\mathbf{I}_n$ is the identity matrix of degree n. The transformations of $O(n)$ which preserve the orientation of E_n are called the **rotations**. They form the group $SO(n)$. The rotations of the plane E_2 are given by the matrices

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix},$$

where $0 \leq \varphi < 2\pi$.

Example 1.2.6

Symmetry in physical laws. Group theory plays a similar role in physics. The groups of transformations in physics describe symmetries of physical laws, in particular, symmetry of space-time. Thus, the state of a physical system is represented in quantum mechanics by a point in an infinite-dimensional vector space. If the physical system passes from one state into another, its representative point undergoes some linear transformation. The ideas of symmetry and the theory of group representations are of prime importance here.

The laws of physics and invariants in mechanics must be preserved under transformations from one inertial coordinate system to another. The corresponding Galilean transformation of space-time coordinates in Newtonian mechanics has the following form for (uniform) motion along the x-axis with velocity v:

$$x' = x - vt, \quad y' = y; \quad z' = z; \quad t' = t,$$

and in the Einstein's special theory of relativity the Lorentz transformation has the form for motion along the x-axis with velocity v:

$$x' = \frac{x - vt}{\sqrt{1 - (v/t)^2}}, \quad y' = y; \quad z' = z; \quad t' = \frac{t - vx/c^2}{\sqrt{1 - (v/t)^2}},$$

where c is the speed of light.

All Galilean transformations form a group which is called the **Galilean group**, and all Lorentz transformations form the **Lorentz group**. The Lorentz transformations, named after its discoverer, the Dutch physicist and mathematician Henrik Anton Lorentz (1853-1928), form the basis for the special theory of relativity, which has been introduced to remove contradictions between the theory of electromagnetism and classical mechanics. The Lorentz group is the subgroup of the Poincaré group consisting of all isometries that leave the origin fixed. This group was been described in the work of H.A.Lorentz and H.Poincaré as the symmetry group of the Maxwell equations:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

1.3 QUOTIENT GROUPS, HOMOMORPHISMS AND NORMAL SUBGROUPS

Suppose H is a subgroup of a group G with identity e, and $a, b \in G$. We introduce a binary relation on G. The relation $a \sim b$ holds if and only if $ab^{-1} \in H$. This relation is symmetric, reflexive and transitive. Indeed, $a \sim a$, because $aa^{-1} = e \in H$. If $a \sim b$, i.e., $ab^{-1} \in H$, then $(ab^{-1})^{-1} = ba^{-1} \in H$, i.e., $b \sim a$. If $a \sim b$ and $b \sim c$, i.e., $ab^{-1} \in H$ and $bc^{-1} \in H$, then $ac^{-1} = ab^{-1}bc^{-1} \in H$, i.e., $a \sim c$. Therefore we have an equivalence relation and $G = \bigcup\limits_i E_i$ is the union of the equivalence classes E_i with respect to this relation. Each such equivalence class E_i is called a **right coset** or a **right adjacent class** of G by H. Suppose E_i is a right coset and $a \in E_i$. We shall show that $E_i = Ha$. Indeed, let $x \in E_i$, then $xa^{-1} \in H$, and so $x \in Ha$, i.e., $E_i \subseteq Ha$. If $y \in Ha$, then $ya^{-1} \in H$, and so $y \in E_i$. Therefore $E_i = Ha$. Now we shall show that each set of the form Hb is a right coset. Indeed, since $G = \bigcup\limits_i E_i$, $b \in E_j$ for some j, i.e., $b \in E_j$. And as proved above we have that $Hb = E_j$. Since $H = He$, the subgroup H is also a right coset. Therefore any element $a \in G$ can be considered as a representative of the right coset Ha.

Suppose G is a finite group of order n and $H = \{h_1 = e, h_2, ..., h_m\}$ is a subgroup of G. Let $a \in G$. Then all elements of a set $Ha = \{h_1a = a, h_2a, ..., h_ma\}$ are distinct, because $h_ia = h_ja$ implies $h_i = h_j$. Therefore all right cosets contain the same number of elements which is equal to m.

From the decomposition of the group G into a union of right cosets we obtain that $n = km$, where k is a number of right cosets of H in G. Therefore we have proved the following theorem:

Theorem 1.3.1 (Lagrange theorem). *If G is a finite group and H is a subgroup of G, then the order of H divides the order of G, and $|G| = k \cdot |H|$, where k is a number of right cosets of H in G.*

For the proof of Lagrange's theorem we can also introduce another relation defined by $a \sim b$ if and only if $b^{-1}a \in H$. The resulting equivalence classes are called **left cosets**. We can show that in this case each equivalence class has the form aH for some $a \in G$ and each set of the form bH is a left coset. The number of left cosets is also equal to $\frac{m}{n}$, i.e., the number of all right cosets in G. This common number is called the **index** of H in G and denoted by $|G : H|$.

In the case of finite groups from the Lagrange theorem it follows that the index of H in G is equal to $\dfrac{|G|}{|H|}$, that is,

$$|G| = |G : H| \cdot |H|.$$

If a group G is infinite and the number of left (or right) adjacent classes is infinite, then we say that the index of H in G is infinite.

In general, the sets of right cosets and left cosets may be different. It is interesting to know when these sets are the same. Suppose E is a right coset and a left coset simultaneously. Then $E = Ha = aH$ for all $a \in E$. If every right coset is a left coset, then $Ha = aH$ for all $a \in G$. Multiplying the last equality on a^{-1} we obtain $a^{-1}Ha = H$ for all $a \in G$. Subgroups with this property deserve special attention.

Definition. A subgroup H of a group G is called a **normal subgroup** (or **invariant subgroup**) of G if $axa^{-1} \in H$ for every $x \in H$ and every $a \in G$. In this case we write $H \lhd G$ (or $G \rhd H$). A group with no normal proper subgroups is called **simple**.

It is easy to show that H is a normal subgroup of G if and only if $aHa^{-1} = H$ for every $a \in G$ or $aH = Ha$ for every $a \in G$. This last equation yields another definition of a normal subgroup as one whose left and right cosets are equal.

Examples 1.3.1.

1. For any group G the group G itself and the unit subgroup are normal subgroups.

2. If G is an Abelian group, then every subgroup of G is normal.

3. Let $G = \mathrm{GL}_n(k)$ be the set of all square invertible matrices of order n over a field k and let $H = \mathrm{SL}_n(k)$ be the subset of elements from $\mathrm{GL}_n(k)$ with determinant equal to 1. Then H is a normal group in G.

Suppose G is a group, N is a normal subgroup, and G/N is the collection of all left cosets aN, $a \in G$. Then $(aN) \cdot (bN) = (a \cdot b)N$ is a well-defined multiplication on G/N, and with this operation, G/N is a group, which is called the **quotient group** (also called the **factor group**) of the group G by the normal subgroup N. Its identity is N and $(aN)^{-1} = a^{-1}N$. Taking into account that for a normal subgroup N the set of all left cosets aN coincides with the set of all right cosets Na, we can also consider G/N as a set of all right cosets Na, $a \in G$ with operation $(Na) \cdot (Nb) = N(a \cdot b)$. The order of the quotient group G/N is equal to the index of the normal subgroup N.

If G and H are groups, then a map $f : G \to H$ such that $f(ab) = f(a)f(b)$, for all $a, b \in G$, is called a **group homomorphism**. The **kernel** of f is defined by $\mathrm{Ker} f = \{a \in G : f(a) = \bar{e}\}$, where $\bar{e}$ is the identity of H. The **image** of f is a set of elements of H of the form $f(a)$, where $a \in G$, that is, $\mathrm{Im}(f) = \{h \in H : \exists a \in G, h = f(a)\}$. It is easy to show that $\mathrm{Ker} f$ is a subgroup of G and $\mathrm{Im} f$ is a subgroup of H. If f is injective, i.e., $\mathrm{Ker} f = 1$, f is called a **monomorphism**. If f is surjective, i.e., $\mathrm{Im} f = H$, f is called an **epimorphism**. If f is a bijection, then f is called an **isomorphism**. In the case $G = H$, f is called an **automorphism**.

Quotient groups play an especially important role in the theory of groups owing to their connection with homomorphisms of groups. Namely, for any normal subgroup N the quotient group G/N is an image of the group G. And conversely, if G' is a homomorphic image of a group G, then G' is isomorphic to some quotient group of G.

The mapping $\pi : g \to Ng$ is a group epimorphism of G onto G/N, called the **canonical epimorphism** or the **natural projection**.

If $\varphi : G \to G_1$ is an arbitrary epimorphism of G onto a group G_1, then there is an isomorphism ψ of $G/\mathrm{Ker}(\varphi)$ onto G_1 such that the diagram

$$\begin{tikzcd} G \arrow[rr, "\pi"] \arrow[dr, "\varphi"'] & & G/\mathrm{Ker}(\varphi) \arrow[dl, "\psi"] \\ & G_1 & \end{tikzcd} \tag{1.3.1}$$

is commutative, i.e., $\psi\pi = \varphi$, where π is the natural projection.

At one time groups of permutations were the only groups studied by mathematicians. They are incredibly rich and complex, and they are especially important because in fact they give all possible structures of finite groups as shown by the famous Cayley theorem. This theorem establishes a relationship between the subgroups of the symmetric group S_n and every finite group of order n.

Theorem 1.3.2 (A.Cayley). *Let G be a finite group of order n and let S_n be the group of all permutations on the set G. Then G is isomorphic to a subgroup of S_n.*

Proof. For any $a \in G$ we define $f_a : G \to G$ by setting $f_a(g) = g \cdot a$. Let $G = \{g_1, g_2, ..., g_n\}$. From the cancellation law we have that for a given $a \in G$ all n elements $f_a(g_i) = g_i a = g_{\gamma_i}$, $i = 1, ..., n$, are different. This shows that f_a is a bijection from G to G and

$$f_a = \begin{pmatrix} 1 & 2 & \dots & n \\ \gamma_1 & \gamma_2 & \dots & \gamma_n \end{pmatrix} \in S_n.$$

Define a mapping $f : G \to S_n$ by setting $f(a) = f_a$ for any $a \in G$. From the associative law in G we have

$$f_{ab}(g) = (ab)g = a(bg) = f_a(bg) = f_a f_b(g)$$

for any $g \in G$. This shows that $f_{ab} = f_a f_b$. Therefore f is a group homomorphism, because $f(ab) = f_{ab} = f_a f_b = f(a)f(b)$.

Since $ax = bx$ implies $a = b$ for any $x \in G$, $f_a = f_b$ if and only if $a = b$, i.e., f is injective and thus $G \simeq \mathrm{Im}(f) \subset S_n$ is a subgroup of S_n.

As a consequence of this theorem we obtain that the number of all non-isomorphic groups of given order n is finite, because all these groups are isomorphic to subgroups of the finite group S_n, which obviously has only a finite number of subgroups.

Corollary 1.3.3. *Any finite group G is a subgroup of $\mathrm{GL}_n(k)$, where $n = |G|$ and k is a field.*

Proof. This follows from the injection of S_n into $\mathrm{GL}_n(k)$ given by the following rule: $\sigma \mapsto \mathbf{A}_\sigma$, where $(\mathbf{A}_\sigma)_{ij} = 1$ if $\sigma(j) = i$ and $(\mathbf{A}_\sigma)_{ij} = 0$ if $\sigma(j) \neq i$ for any $\sigma \in S_n$.

Definition. For a group G and an element $x \in G$ define the **order**[2] of x to be the smallest positive integer n such that $x^n = 1$, and denote this integer by $|x|$. In this case x is said to be of order n. If no positive power of x is the identity, the order of x is defined to be infinity and x is said to be of infinite order.

The set-theoretic intersection of any two (or any set of) subgroups of a group G is a subgroup of G. The intersection of all subgroups of G containing all elements of a certain non-empty set $M \subset G$ is called the **subgroup generated by the set** M and is denoted by $\{M\}$. If M consists of one element $x \in G$, then $H = \{x\} = \{x^i : i \in \mathbf{Z}\}$ is called the **cyclic subgroup** generated by the element x. It is obvious that the order of this subgroup is equal to the order of the element x. A group that coincides with one of its cyclic subgroups is called a **cyclic group**. The cyclic group C_n of order n consists of n elements $\{1, g, g^2, ..., g^{n-1}\}$, where $g^n = 1$. It is easy to show that any two cyclic groups of the same order n are isomorphic to each other. If a group G is cyclic and H is a subgroup of G, then H is also cyclic.

The following statements are simple corollaries from the Lagrange theorem.

Corollary 1.3.4. *The order of any element of a group G divides the order of G.*

The proof follows from the Lagrange theorem and the fact that the order of an element is equal to the order of the cyclic subgroup generated by this element.

[2]Mathematics has a dearth of words; so a word like 'order' is used in many different meanings

Corollary 1.3.5 (Fermat theorem). *Let a group G be of order n and let $x \in G$, then $x^n = 1$.*

Proof. Suppose the order of an element $x \in G$ is equal to m. Then, by the Lagrange theorem, $n = mk$, hence $x^n = x^{mk} = 1$.

Corollary 1.3.6. *Any group G of order p, where p is a prime integer, is cyclic. Therefore, there is exactly one (up to isomorphism) group of order p.*

Proof. By corollary 1.3.4, the orders of all elements of a finite group G must be divisors of the order of this group. Therefore, for any non-trivial element $g \in G$ the cyclic subgroup $G_p = \{1, g, g^2, ..., g^{p-1}\}$ coincides with G.

Examples 1.3.2.

1. An element of a group has order 1 if and only if it is the identity.

2. In the additive groups $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ every nonzero element has infinite order.

3. The group G_n of all rotations of the plane which carry a regular n-sided polygon to itself, is cyclic of order n with (as one possible) generator g which is a rotation through $2\pi/n$.

4. The additive group $\mathbf{Z}_n$, the ring of integers modulo n, is also a cyclic group, with generator $1 \in \mathbf{Z}$ and the identity element $0 \in \mathbf{Z}$.

5. The set $\mathbf{C}_n$ of all complex numbers satisfying the equality $x^n = 1$ with respect to the usual multiplication is a cyclic group.

6. Let p be a prime number. The set $\mathbf{C}_p^\infty$ of all complex roots of the equality $x^{p^n} = 1$ for some $n \in \mathbf{N}$ forms an infinite Abelian group. This group is isomorphic to the quotient group $\mathbf{Q}/\mathbf{Z}_p$, where $\mathbf{Z}_p = \{\frac{m}{n} \in \mathbf{Q} \mid (n,p) = 1\}$.

In terms of generators and relations this group can be defined by the the countable set of generators $a_1, a_2, \dots, a_n$ and relations:

$$a_1^p = 1, \quad a_{n+1}^p = a_n, \quad n = 2, 3, \dots$$

Note that all proper subgroups of $\mathbf{C}_{p^\infty}$ are finite.

7. Let F be a finite field. Then the multiplicative group $F^* = F \setminus \{0\}$ is cyclic. In general, the finite subgroups of the multiplicative group of a field are cyclic.

1.4 SYLOW THEOREMS

In this section we prove the famous Sylow theorems, named after the Norwegian mathematician Ludwig Sylow (1833-1918). These theorems form a partial converse to the Lagrange theorem, which states that if H is a subgroup of a finite group G, then the order of H divides the order of G. The Sylow theorems guarantee, for certain divisors of the order of G, the existence of corresponding subgroups, and give information about the number of these subgroups.

The main method of proving the Sylow theorems is to use the idea of a group action on a set. It is an important idea in mathematics which allows to study the structure of an algebraic object by seeing how it can act on other structures.

Definition. A **left group action** of a group G on a set X is a mapping $(g, x) \mapsto g \cdot x$ from $G \times X$ to X such that

(1) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for all $x \in X$ and $g_1, g_2 \in G$;

(2) $e \cdot x = x$ for all $x \in X$, where e is the identity element of G.

The expression $g \cdot x$ will usually be written simply as gx when there is no danger of confusing this map with the group operation. Observe that a group action of G on X can be viewed as a rule for multiplying elements of X by elements of G, so that the result is another element of X.

Analogously we can define a **right group action** of a group G on a set X as a mapping $(x, g) \mapsto x \cdot g$ from $X \times G$ to X such that

(1) $x \cdot (g_1 g_2) = (x \cdot g_1) \cdot g_2$ for all $x \in X$ and $g_1, g_2 \in G$;

(2) $x \cdot e = x$ for all $x \in X$, where e is the identity element of G.

Obviously, there is a correspondence between left actions and right actions, given by associating the right action $x \cdot g$ with the left action $g \cdot x$ by the relation: $g \cdot x := x \cdot g^{-1}$. In our context we shall identify right actions with their corresponding left actions, and shall speak only of a left action (or for short, actions omitting the word left).

Let G be a group acting on a set X. Then it is easy to show that the relation $\sim$ on X defined by

$$x \sim y \quad \text{if and only if} \quad x = gy \quad \text{for some } g \in G$$

is an equivalence relation. To each element $x \in X$ we can let correspond the set

$$Gx = \{y \in X \ : \ y = g \cdot x, \ g \in G\},$$

which is the equivalence class of x under the relation $\sim$ and it is called the **orbit** of x under the action of G.

Obviously, the orbits of any two elements, being equivalence classes, either coincide or are disjoint. So we have a partition of the set X into disjoint orbits.

If there is only one orbit, which then is the set X, one says that the group G acts **transitively** on X. In other words, the group G acts transitively on the set X if for each two elements $x_1, x_2 \in X$ there is an element $g \in G$ such that $g \cdot x_1 = x_2$.

Example 1.4.1.

1. For any nonempty set X the symmetric group S_X acts on X by $\sigma \cdot x = \sigma(x)$, for all $\sigma \in S_X$, $x \in X$.

2. Let G be any group and let $X = G$. Define a map from $G \times X$ to X by $g \cdot x = gx$, for each $g \in G$ and $x \in X$, where gx on the right hand side is the product of g and x in the group G. This gives a group action of G on itself. This action is called the **left regular action** of G on itself.

3. The additive group $\mathbf{Z}$ acts on itself by $z \cdot a = z + a$, for all $z, a \in \mathbf{Z}$.

The set

$$\mathrm{St}_G(x) = \{g \in G \ : \ g \cdot x = x\}$$

is called the **stabilizer** of an element $x \in X$. It is easy to show that $\mathrm{St}_G(x)$ is a subgroup of G.

The set

$$\mathrm{Fix}(g) = \{x \in X \ : \ g \cdot x = x\}$$

is called the **set of fixed points** of an element $g \in G$.

Proposition 1.4.1. *If G is a finite group, then the number of elements of Gx is equal to the index of $\mathrm{St}_G(x)$ in the group G, that is,*

$$|Gx| = |G : \mathrm{St}_G(x)|.$$

Proof. Let A be a set of all left cosets of G by $\mathrm{St}_G(x)$. Consider the map $\varphi : Gx \to A$ defined by $\varphi(g \cdot x) = g\mathrm{St}_G(x)$. It is easy to see that φ is a bijection and so we obtain the statement of the proposition.

Let G be a group. Now we consider a particular example of a group action of G on itself. We say that G **acts on itself by conjugation** if $g \cdot x = gxg^{-1}$ for all $g \in G$, $x \in G$. It is easy to show that this definition satisfies the two axioms for a group action.

Definition. Two elements g and h of a group G is called **conjugate** in G if there is an element $x \in G$, called the **conjugating element**, such that $g = xhx^{-1}$.

In other words, two elements of G are conjugate in G if and only if they are in the same orbit of G acting on itself by conjugation. An orbit of G under conjugation is called a **conjugacy class** C and (hence) defined as the set of elements of a group G which can be obtained from a given element of the group G by conjugation. Obviously, distinct conjugacy classes are disjoint. Let $C_1, C_2, ..., C_k$ be all distinct conjugacy classes of a group G. Then we have a partition of the group G:

$$G = C_1 \cup C_2 \cup ... \cup C_k$$

where $C_i \cap C_j = \emptyset$ if $i \neq j$.

The idea of a group action of a group G on itself by conjugation can be generalized. If S is any subset of G, for any $g \in G$ we define

$$gSg^{-1} = \{gsg^{-1} \ : \ s \in S\}.$$

A group G acts on the set $\mathcal{P}(G)$ of all subsets of itself by defining $g \cdot S = gSg^{-1}$ for any $g \in G$ and $S \in \mathcal{P}(G)$. As above, this defines a group action of G on $\mathcal{P}(G)$. Note that if S is the one element set $\{s\}$ then $g \cdot S$ is the one element set $\{gsg^{-1}\}$

and so this action of G on all subsets of G may be considered as an extension of the action of G on itself by conjugation.

Definition. Two subsets S and T of G are said to be **conjugate in** G if there is some $g \in G$ such that $T = gSg^{-1}$.

Now we introduce some important classes of subgroups of an arbitrary group G.

Definition. Let A be any nonempty subset of a group G. Define

$$C_G(A) = \{g \in G \ : \ gag^{-1} = a \ \text{ for all } \ a \in A\},$$

which is called the **centralizer** of A in G. Since $gag^{-1} = a$ if and only if $ag = ga$, $C_G(A)$ is a set of elements of G which commute with every element of A. It is easy to show that $C_G(A)$ is a subgroup of G. In the special case $A = G$ we obtain $C_G(G) = \{g \in G \ : \ gx = xg \ \text{ for all } \ x \in G\}$, which is the **center** of G and denoted by $\mathrm{Cen}(G)$ or $Z(G)$. Obviously, $Z(G)$ is a subgroup of G and $(g) \lhd Z(C_G(g))$. If G is an Abelian group, then $Z(G) = G$. In the special case when $A = \{a\}$ we shall write simply $C_G(a)$ instead of $C_G(\{a\})$. In this case $a^n \in C_G(a)$ for all $n \in \mathbf{Z}$. In an Abelian group G, obviously, $C_G(A) = G$, for all subsets A.

Now define another special subgroup of G. Let A be any nonempty subset of a group G. The set

$$N_G(A) = \{g \in G \ : \ gAg^{-1} = A\}$$

is called the **normalizer** of A in G. It is easy to see that $N_G(A)$ is a subgroup of G and $C_G(A)$ is a subgroup of $N_G(A)$. Note that if $g \in C_G(A)$, then $gag^{-1} = a \in A$ for all $a \in A$ so $C_G(A) \unlhd N_G(A)$. If $A = H$ is a subgroup in G, then $H \lhd N_G(H)$. If $A = g \in G$, then $C_G(g) = N_G(g)$.

Theorem 1.4.2.

1. *The number of subgroups of a finite group G which are conjugate to a given subgroup H is equal to $|G : N_G(H)|$.*
2. *The number of elements of a group G which are conjugate to a given element $g \in G$ is equal to $|G : C_G(g)|$.*

Proof.

1. A group action of a group G on the set $M = \{xHx^{-1} \ : \ x \in G\}$ is given by conjugation: $g \cdot xHx^{-1} = gxHx^{-1}g^{-1}$ for any $g \in G$. This action is transitive and $St_G(H) = N_G(H)$. Then, by proposition 1.4.1, $|M| = |G : N_G(H)|$.
2. This assertion of the proposition follows from the observation that $N_G(\{g\}) = C_G(g)$.

Theorem 1.4.3 (The class equation). *Let G be a finite group and let $g_1, g_2, ..., g_k$ be representatives of the distinct conjugacy classes of G not contained* *in the center $Z(G)$ of G. Then*

$$|G| = |Z(G)| + \sum_{i=1}^{k} |G : C_G(g_i)|.$$

Proof. Let G be a finite group and $x \in G$. Notice that the element $\{x\}$ is a conjugacy class of size 1 if and only if $x \in Z(G)$. Let $Z(G) = \{z_1 = 1, z_2, ..., z_m\}$, and let $C_1, C_2, ..., C_k$ be all distinct conjugacy classes of G not contained in the center. Let g_i be a representative of C_i for each i. Then the full set of conjugacy classes of G is given by

$$\{z_1\}, \{z_2\}, ..., \{z_m\}, C_1, C_2, ..., C_k.$$

Since these classes form a partition of G we have

$$|G| = m + \sum_{i=1}^{k} |C_i|.$$

Then applying theorem 1.4.2 we obtain

$$|G| = |Z(G)| + \sum_{i=1}^{k} |G : C_G(g_i)|.$$

Definition. Let G be a group and let p be a prime. A group of order p^n for some $n \geq 1$ is called a finite p-**group**. Subgroups of G which are finite p-groups are called p-**subgroups**.

Theorem 1.4.4. *The center of a finite p-group G is nontrivial.*

Proof. Let G be a finite group. Then, by theorem 1.4.3 (the class equation),

$$|G| = |Z(G)| + \sum_{i=1}^{k} |G : C_G(g_i)|,$$

where $g_1, g_2, ..., g_k$ are representatives of the distinct non-central conjugacy classes of G. By definition, $C_G(g_i) \neq G$ so, by the Lagrange theorem, p divides $|G : C_G(g_i)|$ for each i. Since $|G| = p^n$, p divides $|Z(G)|$, hence $Z(G)$ must be nontrivial.

Corollary 1.4.5. *If G is a group of order p^2, where p is prime, then G is Abelian.*

Proof. Let G be a group of order p^2, where p is prime, and let $Z(G)$ be its center. Suppose G is not Abelian, then $Z(G) \neq G$. From theorem 1.4.4 it follows that $|Z(G)| = p$ and $|G/Z(G)| = p$. Therefore the group $G/Z(G)$ is cyclic. Let $x \in G/Z(G)$. Then $xZ(G)$ is a generator of $G/Z(G)$. So any element $g \in G$ can be written in the form $g = x^k z$, where $z \in Z(G)$. But any two elements of this form commute, so we have a contradiction.

Definition. Let G be a group and let p be a prime.

1. If G is a group of order $p^n m$, where $(p, m) = 1$, then a subgroup of order p^n is called a **Sylow p-subgroup** of G.

2. The set of Sylow p-subgroups of G will be denoted by $\mathrm{Syl}_p(G)$ and the number of Sylow p-subgroups of G will be denoted by $n_p(G)$ (or just n_p when G is clear from the context).

Theorem 1.4.6 (The first Sylow theorem). *Let G be a group of order $p^n m$, where p is a prime and $(p, m) = 1$. Then there exists a Sylow p-subgroup of G, i.e.,* $\mathrm{Syl}_p(G) \neq \emptyset$.

Proof. We shall prove this theorem by induction on the order of the group G. If $|G| = 1$ then there is no p which divides its order, so the condition is trivial. If G is an Abelian group, then this theorem immediately follows from the structure theorem for finite Abelian groups (see, vol.I, theorem 7.8.6).

Suppose G is not Abelian, $|G| = p^n m > 1$, where p is prime, $(p, m) = 1$, and suppose the proposition holds for all groups of smaller order. Let $Z(G)$ be the center of G. Suppose the order of $Z(G)$ is divisible by p. Let $|Z(G)| = p^r t$, where $(p, t) = 1$. Since G is not Abelian, $|Z(G)| < |G|$ and, by the inductive hypothesis, $Z(G)$ has a subgroup $H \subset Z(G)$ such that $|H| = p^r$. As a subgroup of the centre H is normal (or the fact that $Z(G)$ is Abelian), so the quotient group G/H is well-defined and of order $p^{n-r}m$. By the induction hypothesis, G/H contains a Sylow p-group $K = P/H$ of order p^{n-k}. Then the inverse image $P = \pi^{-1}(K) \subset G$ under the natural projection $\pi : G \to G/H$ is a subgroup of order $|P| = |P : H| \cdot |H| = p^n$, that is, P is a Sylow p-subgroup in G.

Now suppose $s = |Z(G)|$ is not divisible by p. Let $C_1, C_2, ..., C_k$ be all distinct conjugacy classes of G not contained in the center, and let n_i be a number of elements of C_i, $i = 1, 2, .., k$. Then $|G| = p^n m = s + n_1 + n_2 + ... + n_k$. Therefore there exists j such that n_j is not divisible by p and $|G : C_G(g_j)| = n_j$, where g_j is a representative of the class C_j, by theorem 1.4.3. Then $|H| = p^n t < |G|$, where $H = C_G(g_j)$. By the induction hypothesis, arguing as before, there is a subgroup $K \subset H \subset G$ such that $|K| = p^n$, that is, $K \in \mathrm{Syl}_p(G)$.

Proposition 1.4.7. *Let P, Q be Sylow p-subgroups of G. The intersection of the normalizer of P with Q is equal to the intersection of these two Sylow p-subgroups, that is, $Q \cap N_G(P) = Q \cap P$.*

Proof. Let G be a group of order $p^n m$, where p is a prime and $(p, m) = 1$. Let P and Q be Sylow p-subgroups of G, that is, $|P| = |Q| = p^n$. Consider $R = Q \cap N_G(P)$. Obviously, $Q \cap P \subseteq R$. In addition, since $R \subseteq N_G(P)$, RP

is a group and $|RP| = \dfrac{|R| \cdot |P|}{|R \cap P|}$, by the first isomorphism theorem (see theorem 1.3.3, vol.I). Since P is a subgroup of RP, p^n divides its order $|RP|$. But R is a subgroup of Q, and $|P| = p^n$, so $|R| \cdot |P|$ is a power of p. Then it must be that $|RP| = p^n$ because $RP \supset P$, and therefore $P = RP$, and so $R \subseteq P$. Obviously, $R \subset Q$, so $R \subseteq Q \cap P$. Thus $R = Q \cap P$.

The following construction will be used in the proof of the second and the third Sylow theorem.

Given any Sylow p-subgroup P, consider the set of its conjugates Ω. Then $X \in \Omega$ if and only if $X = xPx^{-1}$ for some $x \in G$. Obviously, each $X \in \Omega$ is a Sylow p-subgroup of G. Let Q be an arbitrary Sylow p-subgroup of G. We define a group action of Q on Ω by:

$$g \cdot X = g \cdot xPx^{-1} = gxPx^{-1}g^{-1} = (gx)P(gx)^{-1},$$

for all $g \in Q$. Then we can write Ω as a disjoint union of orbits under the group action of Q on the set Ω:

$$\Omega = \Omega_1 \cup \Omega_2 \cup ... \cup \Omega_r,$$

where $|\Omega| = |\Omega_1| + |\Omega_2| + ... + |\Omega_k|$. Let P_i be a representative of the orbit Ω_i, for $i = 1, 2, ..., r$. By proposition 1.4.1, $|\Omega_i| = |Q : \mathrm{St}_Q(P_i)| = |Q : N_Q(P_i)|$. Using proposition 1.4.7, we have

$$|\Omega_i| = |Q : Q \cap P_i|.$$

If $Q = P_i$, then we obtain that $|\Omega_i| = |P_i : P_i \cap P_i| = 1$. If $Q \neq P_i$, then we obtain that $|\Omega_i| = |Q : Q \cap P_i| > 1$, and since the index of any subgroup of Q divides Q, p divides $|\Omega_i|$.

Proposition 1.4.8. *The number of conjugates of any Sylow p-subgroup of a finite group G is congruent to 1 modulo p.*

Proof. In the construction considered above we take $Q = P_1$. Then $|\Omega_1| = 1$ and p divides $|\Omega_i|$ for $i \neq 1$. Let t be a number of conjugates of P_1. Since $t = |\Omega| = |\Omega_1| + |\Omega_2| + ... + |\Omega_k|$, we have $t = 1 + pk_2 + pk_3 + ... + pk_s \equiv 1(\mathrm{mod} p)$.

Theorem 1.4.9 (The second Sylow theorem). *Any two Sylow p-subgroups of a finite group G are conjugate.*

Proof. Given a Sylow p-subgroup P and any other Sylow p-subgroup Q, consider again the construction considered above. Suppose Q is not conjugate to P, Then $Q \neq P_i$ for each $i = 1, 2, ..., s$. Therefore p divides $|\Omega_i|$ for every orbit. If t is the number of conjugates of P, then $t \equiv 0(\mathrm{mod} p)$, which contradicts proposition 1.4.8.

Theorem 1.4.10 (The third Sylow theorem). *Let G be a group of order $p^n m$, where p is a prime and $(p, m) = 1$. The number n_p of all Sylow p-subgroups* *of G is of the form $1+kp$, i.e., $n_p \equiv 1(\mathrm{mod} p)$. Further, $n_p = |G : N_G(P)|$ for any Sylow p-subgroup P, hence $n_p|m$.*

Proof. Consider again the construction considered above. Since all Sylow p-subgroups are conjugate, $|\Omega|$ is equal to the number n_p of all Sylow p-subgroups of G. By proposition 1.4.8, $n_p \equiv 1(\mathrm{mod} p)$.

Finally, since all Sylow p-subgroups are conjugate, theorem 1.4.2 shows that $n_p = |G : N_G(P)|$ for any $P \in \mathrm{Syl}_p(G)$. Since P is a subgroup of $N_G(P)$, p^n divides $|N_G(P)|$, hence $n_p|m$.

1.5 SOLVABLE AND NILPOTENT GROUPS

Abelian groups are the simplest class of groups in terms of structure. Two broader classes than the class of Abelian groups are the classes of nilpotent groups and solvable groups, the theory of which has also reached a fairly advanced stage.

Recall that a group G is called **simple** if $|G| > 1$ and the only normal subgroups of G are 1 and G.

A **normal series** of a group G is a chain of subgroups

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq ... \trianglelefteq H_s = G$$

such that H_i is a normal subgroup in H_{i+1} for every $i = 0, 1, ..., s-1$. The number s is called the **length** of the normal series and quotient groups H_{i+1}/H_i are called its **factors**.

If all factors in a normal series of a group G are simple, the series is called a **composition series**.

Since any ring is a group we have the following formulations of the Jordan-Hölder theorem for groups (see vol.I, theorem 3.2.1):

Theorem 1.5.1 (Jordan-Hölder). *If a group G has a composition series, then every two composition series of G are isomorphic.*

If a group G has a composition series, then every normal series of it can be refined to a composition series.

Definition. A group G is called **solvable** if it has a normal series with all factors Abelian.

Example 1.5.1.

The subgroup H of all upper triangular matrices of the group $\mathrm{GL}(n, \mathbf{C})$, where $\mathbf{C}$ is the field of complex numbers, is solvable.

Remark 1.5.1. Note that the term 'solvable' arose in Galois theory and is connected with the problem of solvability of algebraic equations in radicals. Let f be a polynomial in x over a field k and K be the (minimal) splitting field of f. The group $\mathrm{Gal}(K/k)$ is called the Galois group of f. The main result of Galois theory says that the equation $f(x) = 0$ is solvable in radicals if and only if the group $\mathrm{Gal}(K/k)$ is solvable.

We now give another characterization of solvable groups.

Let G be a group, $x, y \in G$, and let A, B be nonempty subgroups of G. Recall that the **commutator** of $x, y \in G$ is defined as

$$[x, y] = x^{-1}y^{-1}xy,$$

and the commutator of A, B is

$$[A, B] = \{[x, y] \ : \ x \in A, \ y \in B\}.$$

Define also

$$G' = \{[x, y] \ : \ x, y \in G\},$$

which is the subgroup of G generated by commutators of elements from G and called the **commutator subgroup of** G.

The basic properties of commutators and the commutator subgroup are given by the following statement.

Proposition 1.5.2. *Let G be a group, $x, y \in G$ and let $H \subseteq G$ be a subgroup. Then*

(1) $xy = yx[x, y]$.

(2) *$H \trianglelefteq G$ if and only if $[H, G] \subseteq H$.*

(3) *The group G/G' is Abelian.*

(4) *G/G' is the largest Abelian quotient group of G in the sense that if $H \trianglelefteq G$ and G/H is Abelian, then $G' \subseteq H$. Conversely, if $G' \subseteq H$, then $H \trianglelefteq G$ and G/H is Abelian.*

Proof.

(1) This is immediate from the definition of $[x, y]$.

(2) By definition, $H \trianglelefteq G$ if and only if $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$. For $h \in H$, we have that $g^{-1}hg \in H$ if and only if $h^{-1}g^{-1}hg \in H$, so that $H \trianglelefteq G$ if and only if $[h, g] \in H$ for all $h \in H$ and all $g \in G$. Thus, $H \trianglelefteq G$ if and only if $[H, G] \subseteq H$.

(3) Let xG' and yG' be arbitrary elements of G/G'. By the definition of the group operation in G/G' and since $[x, y] \in G'$ we have

$$(xG')(yG') = (xy)G' = (yx[x, y])G' = (yx)G' = (yG')(xG').$$

(4) Suppose $H \trianglelefteq G$ and G/H is Abelian. Then for all $x, y \in G$ we have $(xH)(yH) = (yH)(xH)$, so

$$H = (xH)^{-1}(yH)^{-1}(xH)(yH) = x^{-1}y^{-1}xyH = [x, y]H.$$

Thus $[x, y] \in H$ for all $x, y \in G$, so that $G' \subseteq H$.

Conversely, if $G' \subseteq H$, then since G/G' is Abelian by (3), every subgroup of G/G' is normal. In particular, $H/G' \trianglelefteq G/G'$. Then, by lemma 1.3.4 and theorem 1.3.5 (see vol.I), it follows that $H \trianglelefteq G$ and

$$G/H \simeq (G/G')/(H/G')$$

so that G/H is Abelian.

Definition. For any group G define the following sequence of subgroups inductively:

$$G^{(0)} = G, \quad G^{(1)} = [G, G] \ \text{ and } \ G^{(i+1)} = [G^{(i)}, G^{(i)}] \ \text{ for all } \ i \geq 1.$$

This series of subgroups is called the **derived** or **commutator series** of G.

Theorem 1.5.3. *A group G is solvable if and only if $G^{(n)} = 1$ for some $n \geq 0$.*

Proof. Assume first that G is solvable and so possesses a normal series

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq ... \trianglelefteq H_s = G$$

such that each quotient group H_{i+1}/H_i is Abelian. We prove by induction that $G^{(i)} \subseteq H_{s-i}$. This is true for $i = 0$, so assume $G^{(i)} \subseteq H_{s-i}$. Then

$$G^{(i+1)} = [G^{(i)}, G^{(i)}] \subseteq [H_{s-i}, H_{s-i}].$$

Since H_{s-i}/H_{s-i-1} is Abelian, by proposition 1.5.2(4), $[H_{s-i}, H_{s-i}] \subseteq H_{s-i-1}$. Thus $G^{(i+1)} \subseteq H_{s-i-1}$, which completes the induction. Since $H_0 = 1$, we have $G^{(s)} = 1$.

Conversely, if $G^{(n)} = 1$ for some $n \geq 0$, proposition 1.5.2(4) shows that if we take H_i to be $G^{(n-i)}$ then H_i is a normal subgroup of H_{i+1} with Abelian quotient, so the derived series itself satisfies the defining condition for solvability of G. This completes the proof.

If a group G is solvable, the smallest nonnegative n for which $G^{(n)} = 1$ is called the **solvable length** of G.

Theorem 1.5.4. *Let G and K be groups, let H be a subgroup of G and let $\varphi : G \to K$ be a surjective homomorphism.*

1. *$H^{(i)} \subseteq G^{(i)}$ for all $i \geq 0$. In particular, if G is solvable, then so is H, i.e., subgroups of solvable groups are solvable.*

2. *$\varphi(G^{(i)}) = K^{(i)}$. In particular, homomorphic images and quotient groups (which are the same thing) of solvable groups are solvable.*

3. *Let N be a normal subgroup of G. If both N and G/N are solvable, then so is G.*

Proof.

1. This follows from the observation that since $H \subseteq G$, by definition of commutator subgroups, $[H, H] \subseteq [G, G]$, that is, $H^{(1)} \subseteq G^{(1)}$. Then, by induction, $H^{(i)} \subseteq G^{(i)}$ for all $i \geq 0$. In particular, if $G^{(n)} = 1$ for some n, then also $H^{(n)} = 1$.

2. Note that, by definition of commutators, $\varphi([x,y]) = [\varphi(x),\varphi(y)]$, so, by induction, $\varphi(G^{(i)}) \subseteq K^{(i)}$. Since φ is surjective, every commutator in K is the image of a commutator in G. Hence again, by induction, we obtain equality for all i. Again, if $G^{(n)} = 1$ for some n then $K^{(n)} = 1$.

3. If G/N and N are solvable of lengths n and m respectively then, by statement 2 of this theorem applied to the natural projection $\varphi : G \to G/N$, we obtain $\varphi(G^{(n)}) = (G/N)^{(n)} = N$, i.e., $G^{(n)} \subseteq N$. Thus $G^{(n+m)} = (G^n)^{(m)} \subseteq N^{(m)} = 1$. Theorem 1.5.3 now implies that G is solvable.

Theorem 1.5.5. *Every finite group of order p^n, where p is prime, is solvable.*

Proof. If G is a finite p-group, then, by theorem 1.4.4, its center $Z(G)$ is not trivial. Then the quotient group $G/Z(G)$ is again a p-group, whose order is less then the order of G. We prove this theorem by induction on the order of a group. Assume that theorem is true for all p-groups with order less then p^n. Then, by induction hypothesis, $Z(G)$ and $G/Z(G)$ are solvable groups. Then, by theorem 1.5.4(3), G is also solvable.

Definition. For any group G define the following subgroups inductively:

$$Z_0(G) = 1, \quad Z_1(G) = Z(G)$$

and $Z_{i+1}(G)$ is the subgroup of G containing $Z_i(G)$ such that

$$Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G)).$$

The chain of subgroups

$$Z_0(G) \subseteq Z_1(G) \subseteq Z_2(G) \subseteq ...$$

is called the **upper central series** of G.

A group G is called **nilpotent** if $Z_m(G) = G$ for some $m \in \mathbf{Z}$. The smallest such m is called the **nilpotency class** of G.

Example 1.5.2.

The subgroup H of all upper triangular matrices of the group $\mathrm{GL}(n, \mathbf{C})$, where $\mathbf{C}$ is the field of complex numbers, is not nilpotent. But the subgroup N of all elements of H with 1 on the main diagonal is nilpotent.

Proposition 1.5.6. *Let p be a prime and let G be a group of order p^m. Then G is nilpotent of nilpotency class at most $m-1$.*

Proof. For each $i \geq 0$, $G/Z_i(G)$ is a p-group, so, by theorem 1.4.4, if $|G/Z_i(G)| > 1$ then $Z(G/Z_i(G))$ is not trivial. Thus if $Z_i(G) \neq 1$ then $|Z_{i+1}(G)| \geq p|Z_i(G)|$ and so $|Z_{i+1}(G)| \geq p^{i+1}$. In particular, $|Z_m(G)| \geq p^m$, so $G = Z_m(G)$. Thus G is nilpotent of nilpotency class $\leq m$. The only way G could be of nilpotence class exactly equal to m would be if $|Z_i(G)| = p^i$ for all i.

In this case, however, Z_{m-2} would have index p^2 in G, so $G/Z_{m-2}(G)$ would be Abelian, by corollary 1.4.5. But then $G/Z_{m-2}(G)$ would equal its center and so $Z_{m-1}(G)$ would equal G, a contradiction. This proves that the nilpotency class of G is $\leq m-1$.

We now give another equivalent definition of a nilpotent group using the notion of a lower central series.

Recall that the commutator of two elements x, y in a group G is defined as

$$[x, y] = x^{-1}y^{-1}xy,$$

and the commutator of two subgroups H and K of G is

$$[H, K] = \{[x, y] \ : \ x \in H, y \in K\}.$$

Definition. For any group G define the following subgroups inductively:

$$G^0 = G, \quad G^1 = [G, G] \text{ and } G^{i+1} = [G, G^i].$$

The chain of groups

$$G^0 \supseteq G^1 \supseteq G^2 \supseteq ...$$

is called the **lower central series** of G.

It is important to note that although $G^{(0)} = G^0$ and $G^{(1)} = G^1$, in general it is not true that $G^{(i)} = G^i$. The difference is that the definition of the $(i+1)$-st term in the lower central series is the commutator of the i-th term with the whole group G whereas the $(i+1)$-st term in the derived series is the commutator of the i-th term with itself. Hence $G^{(i)} \subseteq G^i$ for all i and the containment can be proper. For example, in $G = S_3$ we have $G^1 = A_3$ and $G^2 = [S_3, A_3] = A_3$, whereas $G^{(2)} = [A_3, A_3] = 1$.

Theorem 1.5.7. *A group G is nilpotent if and only if $G^n = 1$ for some $n \geq 0$. More precisely, G is nilpotent of nilpotency class m if and only if m is the smallest nonnegative integer such that $G^m = 1$. If G is nilpotent of nilpotency class m then*

$$Z_i(G) \subseteq G^{m-i-1} \subseteq Z_{i+1}(G) \quad \text{for all} \quad i \in \{0, 1, 2, ..., m-1\}.$$

Proof. This is proved by a straightforward induction on the length of the lower central series.

Corollary 1.5.8. *Each nilpotent group is solvable.*

Proof. This follows immediately from theorem 1.5.7 taking into account that $G^{(i)} \subseteq G^i$ for all i.

Thus, we can summarize the results obtained in this section as the following chain of classes of groups:

$$(cyclic\ \ groups) \subset (Abelian\ \ groups) \subset (nilpotent\ \ groups) \subset$$
$$\subset (solvable\ \ groups) \subset (all\ groups)$$

Proposition 1.5.9. *Let G and K be groups, let H be a subgroup of G and let $\varphi : G \to K$ be a surjective homomorphism.*

1. *$H^i \subseteq G^i$ for all $i \geq 0$. In particular, if G is nilpotent, then so is H, i.e., subgroups of nilpotent groups are nilpotent.*
2. *$\varphi(G^i) = K^i$. In particular, homomorphic images and quotient groups of nilpotent groups are nilpotent.*

Proof.
1. This follows from the observation that since $H \subseteq G$, by definition of commutator subgroups, $[H, H] \subseteq [G, G]$, that is, $H^1 \subseteq G^1$. Then, by induction, $H^i \subseteq G^i$ for all $i \geq 0$. In particular, if $G^n = 1$ for some n, then also $H^n = 1$. And from theorem 1.5.7 it follows that H is nilpotent.
2. Note that, by the definition of commutators, $\varphi([x, y]) = [\varphi(x), \varphi(y)]$, so, by induction, $\varphi(G^i) \subseteq K^i$. Since φ is surjective, every commutator in K is the image of a commutator in G. Hence again, by induction, we obtain equality for all i. Again, if $G^n = 1$ for some n then $K^n = 1$.

1.6 GROUP RINGS AND GROUP REPRESENTATIONS. MASCHKE THEOREM

The **group algebra** of a group G over a field k is the associative algebra over k whose elements are all possible finite sums of the form $\sum\limits_{g\in G} \alpha_g g$, $g \in G$, $\alpha_g \in k$, the operations being defined by the formulas:

$$\sum_{g\in G} \alpha_g g + \sum_{g\in G} \beta_g g = \sum_{g\in G} (\alpha_g + \beta_g) g,$$

$$(\sum_{g\in G} \alpha_g g)(\sum_{g\in G} \beta_g g) = \sum_{h\in G} (\sum_{xy=h, x,y\in G} (\alpha_x \beta_y) h).$$

(All sums in these formulas are finite.) This algebra is denoted by kG; the elements of G form a basis of this algebra; multiplication of basis elements in the group algebra is induced by the group multiplication. The algebra kG is isomorphic to the algebra of functions defined on G with values in k which assume only a finite number of non-zero values. The function associated to $\sum\limits_{g\in G} \alpha_g g$ is $f : g \mapsto \alpha_g$. In this algebra multiplication is the convolution of such functions. Indeed if f_1, f_2 are two functions $G \to k$ with finite support their product is given by

$$f_1 f_2(g) = \sum_{h\in G} f_1(h) f_2(h^{-1} g).$$

The same construction can also be considered for the case when k is an associative ring. One thus arrives at the concept of the group ring of a group G over a ring k; if k is commutative and has a unit element, the group algebra is often called the group algebra of the group over the ring as well.

Note that, by definition of the multiplication, kG is a commutative ring if and only if G is an Abelian group.

Examples 1.6.1.

1. If $G = (g)$ is a cyclic group of order n and k is a field, then the elements of kG are of the form

$$\sum_{i=0}^{n-1} \alpha_i g^i.$$

The map $k[x] \to kG$ which sends x^k to g^k for all $k \geq 0$ extends by k-linearity to a surjective ring homomorphism with kernel equal to the ideal generated by $x^n - 1$. Thus $kG \simeq k[x]/(x^n - 1)$. This is an isomorphism of k-algebras.

Definition. A **k-representation of a group** G on a vector space V over a field k is a group homomorphism $T : G \to \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ is a group of all invertible linear transformations of V over k.

In other words, to define a representation T is to assign to every element $g \in G$ an invertible linear operator $T(g)$ in such a way that $T(g_1g_2) = T(g_1)T(g_2)$ for all $g_1, g_2 \in G$. If the k-vectorspace V is finite dimensional, then its dimension $[V : k]$ is called the **dimension** (or **degree**) of the representation T, and the representation T is called **finite dimensional** over k.

If T is a monomorphism, the representation is said to be **faithful**.

We say that two k-representations $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ of a group are **equivalent** (or **similar**) if there is a k-vector space isomorphism $\theta : V \to W$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\theta"] \arrow[d, "\varphi(g)"'] & W \arrow[d, "\psi(g)"] \\ V \arrow[r, "\theta"] & W \end{tikzcd}$$

is commutative for all $g \in G$, that is, $\theta\varphi(g) = \psi(g)\theta$, for all $g \in G$, which is equivalent to

$$\psi(g) = \theta\varphi(g)\theta^{-1}.$$

In the case where V is of finite dimension n it is common to choose a basis for V and assign to each operator $T(g)$ its matrix $\mathbf{T}_g$ in this basis. The correspondence $g \mapsto \mathbf{T}_g$ defines a homomorphism of the group G into $\mathrm{GL}(n, k)$, the general linear group of invertible $n \times n$ matrices over k, which is called the **matrix representation** of the group G corresponding to the representation T. Thus we can define a matrix representation of a group.

Definition. A **matrix representation** of degree n of a group G over a field k is a group homomorphism $\mathbf{T} : G \to \mathrm{GL}_n(k)$, where $GL_n(k)$ is the general linear group of invertible $n \times n$ matrices over k.

Example 1.6.2.
Consider the cyclic group $C_3 = \{1, u, u^2\}$, where $u^3 = 1$. This group has a two-dimensional representation φ over the field of complex numbers $\mathbf{C}$:

$$\varphi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi(u) = \begin{pmatrix} 1 & 0 \\ 0 & \xi_3 \end{pmatrix}, \quad \varphi(u^2) = \begin{pmatrix} 1 & 0 \\ 0 & (\xi_3)^2, \end{pmatrix}$$

where $\xi_3 = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$ is a primitive 3-rd root of unity. This representation is faithful because φ is a one-to-one map.

If we choose a new basis of a vector space V, then every matrix $\mathbf{T}_g$ transforms into a new matrix of the form $\mathbf{P}\mathbf{T}_g\mathbf{P}^{-1}$, where $\mathbf{P}$ is the matrix of a transformation which does not depend on the element $g \in G$. So in matrix terminology we have the following definition:

Definition. Two matrix representations $\mathbf{T} : G \to \mathrm{GL}_n(k)$ and $\mathbf{S} : G \to \mathrm{GL}_n(k)$ are said to be **equivalent** (or **similar**) if there is a fixed invertible matrix $\mathbf{P} \in \mathrm{GL}_n(k)$ such that $\mathbf{S}_g = \mathbf{P}\mathbf{T}_g\mathbf{P}^{-1}$ for all $g \in G$.

Example 1.6.3.
The cyclic group C_3 also has the representation ψ given by the matrices:

$$\varphi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi(u) = \begin{pmatrix} \xi_3 & & 0 & \\ 0 & 1, & \varphi(u^2) = (\xi_3)^2 & 0 \\ 0 & & 1, & \end{pmatrix}$$

which is equivalent to the representation φ shown in example 1.6.2.

Remark 1.6.1. The representations considered above are also called **linear representations**. Other kinds of representations are permutation representations. A **permutation representation** of a group G on a set S is a homomorphism from G to the group of all permutations of S. In this book "representation" usually means "linear representation".

In this chapter we restrict our attention to finite groups and finite dimensional representations over a field k.

Examples 1.6.4.
1. Let V be a one-dimensional vector space over a field k. Make V into a kG-module by letting $g \cdot v = v$ for all $g \in G$ and $v \in V$. This module corresponds to the representation $\varphi : G \to \mathrm{GL}(V)$ defined by $\varphi(g) = I$, for all $g \in G$, where I is the identity linear transformation. The corresponding matrix representation is defined by $\varphi(g) = 1$. This representation of the group G is called the **trivial representation.** Thus, the trivial representation has degree 1 and if $|G| > 1$, it is not faithful.

2. Consider the representation τ of kG defined by $\tau(a) = T_a$, where $T_a(x) = ax$ for all $x \in G$ and $a \in G$. This representation is called the **left regular representation** of kG. In other words, kG is considered as a left module over itself. If we take the elements of G as a basis of kG, then each $g \in G$ permutes these basis elements: $T_g(g_i) = g \cdot g_i = gg_i = g_j$. With respect to this basis of kG the matrix representation of the group element g has 1 in the intersection of the i-th row and the j-th column, and has zeroes in all other positions. Note that each nonidentity element of G induces a nonidentity permutation on the basis of kG. So the left regular representation is always faithful. Analogously one can define the **right regular representation** of kG.

3. Consider the symmetric group S_3 which has the following matrix two-dimensional representation based on the correspondence with planar symmetry operations of an equilateral triangle:

$$\varphi(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi(a) = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad \varphi(b) = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1, \end{pmatrix}$$

$$\varphi(c) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi(d) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad \varphi(f) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1. \end{pmatrix}$$

The matrices $\varphi(a)$, $\varphi(b)$ and $\varphi(c)$ correspond to reflections, while the matrices $\varphi(d)$ and $\varphi(f)$ correspond to rotations. These matrices form a faithful two-dimensional representation of S_3. Take for example the equilateral triangle with corner points $(-1, -\frac{1}{3}\sqrt{3})$, $(1, -\frac{1}{3}\sqrt{3})$, $(0, \frac{2}{3}\sqrt{3})$ in E^2.

The group S_3 also has the following one-dimensional representations $\psi(e) = \psi(d) = \psi(f) = 1$ and $\psi(a) = \psi(b) = \psi(c) = -1$, which is the mapping from the elements of S_3 to the determinants of the matrix representation discussed above.

Finally, we have the trivial representation of S_3 which is given by $\sigma(e) = \sigma(a) = \sigma(b) = \sigma(c) = \sigma(d) = \sigma(f) = 1$, which is also one-dimensional.

4. Consider the dihedral group D_n which has the presentation:

$$D_n = \{\sigma, \tau \ : \sigma^n = \tau^2 = 1, \quad \sigma\tau = \tau\sigma^{-1} \}.$$

If $\mathbf{S}$ and $\mathbf{T}$ are matrices satisfying the relations $\mathbf{S}^n = \mathbf{T}^2 = \mathbf{E}$ and $\mathbf{ST} = \mathbf{TS}^{-1}$, then the map $\sigma \mapsto \mathbf{S}$ and $\tau \mapsto \mathbf{T}$ extends uniquely to a homomorphism from D_n to the matrix group generated by $\mathbf{S}$ and $\mathbf{T}$, and hence gives a representation of D_n. An explicit example of matrices $\mathbf{S}, \mathbf{T} \in \mathrm{GL}_2(\mathbf{R})$ may be obtained as follows. Take a regular n-polygon drawn on the x, y plane centered at the origin with the line $y = x$ as one of its lines of symmetry then the matrix $\mathbf{S}$ that rotates the plane through $\frac{2\pi}{n}$ radians and the matrix $\mathbf{T}$ that reflects the plane about the line $y = x$ both send this n-polygon onto itself. It follows that these matrices act as symmetries of the n-polygon and so satisfy the above relations. It is easy to compute that

$$\mathbf{S} = \begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix} \quad \text{and} \quad \mathbf{T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

And so the map $\sigma \mapsto \begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix}$ and $\tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ extends uniquely to a representation of D_n into $\mathrm{GL}_2(\mathbf{R})$. The matrices $\mathbf{S}$ and $\mathbf{T}$ have order n and 2 respectively. It is not difficult to check that S and T generate a matrix group of order $2n$ so that this representation is faithful.

5. Consider the quaternion group which has the following presentation:

$$H_2 = \{i, j \ : \ i^4 = 1, \ i^2 = j^2, \ ij = ji^{-1}\}.$$

Then we have a representation $\varphi : H_2 \to \mathrm{GL}_2(\mathbf{C})$ defined by

$$\varphi(i) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \quad \text{and} \quad \varphi(j) = \begin{pmatrix} 0 & -1 \\ 1 & 0. \end{pmatrix}$$

This representation of H_2 is faithful.

As was mentioned above the study of k-representations of a group G is equivalent to the study of modules over the group ring kG. Let G be a group and let V be an n-dimensional vector space over a field k. There is an obvious bijection between kG-modules and pairs (V, φ), where $\varphi : G \to \mathrm{GL}(V)$ is a group representation.

Suppose first that $\varphi : G \to \mathrm{GL}(V)$ is a group representation. For each $g \in G$, $\varphi(g)$ is a linear transformation from V to itself. Make V into a kG-module by defining the action of a ring element on an element of V as follows:

$$(\sum_{g\in G} \alpha_g g) \cdot v = \sum_{g\in G} \alpha_g \varphi(g)(v)$$

for all $\sum_{g\in G} \alpha_g g \in kG$, $v \in V$. Then it is easy to show that V becomes a kG-module.

Conversely, suppose that we have a kG-module V such that $\dim_k V = n$. Since V is a kG-module, it is a k-module, i.e., it is a vector space over k. Also, for each $g \in G$ we obtain a map $\varphi(g) : V \to V$, defined by $\varphi(g)(v) = g \cdot v$ for all $v \in V$, where $g \cdot v$ is the given action of the ring element g on the element v of V. Since the elements of k commute with each $g \in G$, it follows, by the axioms for a module, that for all $u, v \in V$ and all $\alpha, \beta \in k$ we have $\varphi(g)(\alpha u + \beta v) = \alpha\varphi(g)(u) + \beta\varphi(g)(v)$, that is, for each $g \in G$, $\varphi(g)$ is a linear transformation. Furthermore, again by the axioms for a module, it follows that $\varphi(g_1 g_2)(v) = (\varphi(g_1)\varphi(g_2))(v)$. This proves that φ is a group homomorphism.

Therefore to give a representation $\varphi : G \to \mathrm{GL}(V)$ on a vector space V over k is the same as to give a kG-module V. Under this correspondence we shall say that the kG-module V **affords** the representation φ of G.

Definition. A k-representation T of a group G is called **irreducible** if the kG-module V affording it is irreducible, otherwise it is called **reducible**. A k-representation T is called **completely reducible** if the kG-module V affording it is completely reducible. (See vol.I for irreducibility and complete reducibility of modules.)

In matrix terminology, this is equivalent to the following definition.

A matrix representation $\mathbf{T}$ of degree n of a group G is **reducible** if and only if it is equivalent to a matrix representation $\mathbf{S}$ of the form:

$$\mathbf{S}_g = \begin{pmatrix} \mathbf{S}_g^{(1)} & \mathbf{U}_g \\ \mathbf{0} & \mathbf{S}_g^{(2)} \end{pmatrix}$$

for all $g \in G$, where $\mathbf{S}_g^{(i)}$ are matrix representations of degree $n_i < n$ of G; otherwise it is called **reducible**.

A matrix representation $\mathbf{T}$ of degree n of a group G is **completely reducible** if and only if it is equivalent to a matrix representation $\mathbf{S}$ of the form:

$$\mathbf{S}_g = \begin{pmatrix} \mathbf{S}_g^{(1)} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_g^{(2)} & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{S}_g^{(m)} \end{pmatrix}$$

for all $g \in G$, where the $\mathbf{S}_g^{(i)}$ are irreducible matrix representations of degree $n_i < n$ of G $(i = 1, 2, ..., m)$.

Example 1.6.5.

The representations φ and ψ of the group C_3 considered in examples 1.6.2 and 1.6.3 are completely reducible and are direct sums of two one-dimensional representations.

We shall prove below a very important classical theorem. Taking into account its great importance we give three different proofs of it.

Theorem 1.6.1 (H.Maschke). *If G is a finite group and the order of G is not divisible by the characteristic of a field k, then the group algebra kG is semisimple.*

Proof.

1. (A proof according to J.-P.Serre.[3] Let M be any kG-module and let X be an arbitrary kG-submodule of M. Since X is a vector subspace of M, there is a vector subspace Y_0 such that M is a direct sum of these vector subspaces:

$$M = X \oplus Y_0.$$

[3] See [Serre, 1967] .

We need only to show that X is also a direct summand of M as a kG-submodule.

Let $\pi_0 : M \to X$ be the vector space projection of M onto X along Y_0, that is, π_0 is defined by $\pi_0(x + y) = x$ for all $x \in X$ and $y \in Y_0$.

For each $g \in G$ define the map $g^{-1}\pi_0 g : M \to X$ by the formula $g^{-1}\pi_0 g(m) = g^{-1} \cdot \pi_0(g \cdot m)$ for all $m \in M$.

Since X is a kG-submodule of M and π_0 maps M onto X, we have that $g^{-1}\pi_0 g$ maps M onto X. Moreover, $g^{-1}\pi_0 g$ is a k-linear transformation, because g and g^{-1} act as k-linear transformations. Since X is a kG-submodule of M, for each $x \in X$ we have

$$g^{-1}\pi_0 g(x) = g^{-1} \cdot \pi_0(gx) = g^{-1} \cdot gx = x,$$

i.e., $g^{-1}\pi_0 g$ is also a vector space projection of M onto X.

Let $n = |G|$. Consider n as an element of the field k. It is not zero in k by hypothesis. Define

$$\pi = \frac{1}{n} \sum_{g \in G} g^{-1}\pi_0 g.$$

Since π is a scalar multiple of a sum of linear transformations from M to X, it is also a linear transformation from M to X. If $x \in X$, then

$$\pi(x) = \frac{1}{n} \sum_{g \in G} g^{-1}\pi_0 g(x) = \frac{1}{n}(x + ... + x) = x,$$

i.e., π is also a vector space projection of M onto X. We now show that π is a kG-module homomorphism. For any $h \in G$ we have

$$\pi(hm) = \frac{1}{n} \sum_{g \in G} g^{-1}\pi_0 g(hm) = \frac{1}{n} \sum_{g \in G} g^{-1} \cdot \pi_0(g \cdot hm) =$$

$$= \frac{1}{n} \sum_{g \in G} h(h^{-1}g^{-1}) \cdot \pi_0((gh)m) = \frac{1}{n} \sum_{r = gh, g \in G} h(r^{-1} \cdot \pi_0(rm)) =$$

$$= \frac{1}{n} \sum_{r = gh, g \in G} h(r^{-1}\pi_0 r(m)) = h(\frac{1}{n} \sum_{r \in G} r^{-1}\pi_0 r(m)) = h\pi(m),$$

because as g runs over all elements of G, so does $r = gh$.

This establishes the existence of a kG-module projection π of M onto X. Consider the set of elements $Y = \mathrm{Ker}\pi \subset M$. Since π is a kG-module homomorphism, Y is a kG-submodule. If $m \in X \cap Y$, then $m = \pi(m)$ whereas by definition of Y, $\pi(m) = 0$. This shows that $X \cap Y = 0$. On the other hand, an arbitrary element $m \in M$ can be written in the form: $m = \pi(m) + (m - \pi(m))$. By definition, $\pi(m) \in X$. And $\pi(m - \pi(m)) = \pi(m) - \pi(\pi(m)) = \pi(m) - \pi(m) = 0$, i.e., $m - \pi(m) \in \mathrm{Ker}\pi = Y$. This shows that $M = X + Y$ and hence $M = X \oplus Y$. Therefore, any submodule of a kG-module M is a direct summand of it. Then, by proposition 2.2.4 (vol. I), M is a semisimple kG- module and kG is a semisimple ring.

2. (A proof according to I.N.Herstein. [4] Let $a \in kG$, and consider the right regular representation $T_a : kG \to kG$ defined by the formula $T_a(x) = xa$ for any $x \in kG$. It is easy to verify that T_a is a k-linear transformation of the space kG. Moreover the map $\varphi : a \to T_a$ is an isomorphism from the k-algebra kG into the k-algebra $\mathrm{End}_k(kG)$.

We write the transformation T_a by means of its matrix $\mathbf{T}_a$ with respect to the basis consisting of the elements of the group G. Let $\mathrm{Sp}(\mathbf{T}_g)$ be the trace of the matrix $\mathbf{T}_g$. Note that

$$\mathrm{Sp}(\mathbf{T}_g) = \begin{cases} m & \text{for } g = 1; \\ 0 & \text{for } g \neq 1 \end{cases}$$

where $m = |G|$.

Let $R = \mathrm{rad}(kG)$ be the Jacobson radical of kG. Since $[kG : k] = m < \infty$, i.e., kG is a finite dimensional k-algebra, kG is an Artinian algebra. Therefore R is nilpotent, by proposition 3.5.1 (vol. I). Suppose $R \neq 0$, then there is an element $0 \neq x \in R$. Let $x = \alpha_1 g_1 + \alpha_2 g_2 + ... + \alpha_m g_m$.

Without lost of generality, we can assume that $\alpha_1 \neq 0$. Since R is an ideal in kG, $y = xg_1^{-1} \in R$ and

$$y = \alpha_1 \cdot 1 + \alpha_2 h_2 + ... + \alpha_m h_m, \tag{1.6.1}$$

where $h_i \in G$, $y \neq 0$, $\alpha_1 \neq 0$.

Since $y \in R$ and R is nilpotent, the element y is nilpotent as well.

Therefore from any course in linear algebra it is well known that the corresponding linear transformation T_y is nilpotent and $\mathrm{Sp}(\mathbf{T}_y) = 0$.

On the other hand from (1.6.1), taking into account the linear properties of trace, we have

$$\mathrm{Sp}(\mathbf{T}_y) = \alpha_1 \mathrm{Sp}(\mathbf{T}_1) + \alpha_m \mathrm{Sp}(\mathbf{T}_{h_2}) + ... + \alpha_m \mathrm{Sp}(\mathbf{T}_{h_m}) = \alpha_1 \cdot m \neq 0.$$

The obtained contradiction shows that $R = 0$. Thus kG is an Artinian algebra with its Jacobson radical equal to zero. Therefore, by theorem 3.5.5 (vol. I), kG is semisimple.

3. (A proof according to M.Hall.[5] Let $\mathbf{T} : G \to \mathrm{GL}(n, k)$ be a reducible matrix representation of degree n of a group G. Then it is equivalent to a matrix representation of the form

$$\mathbf{S}_g = \begin{pmatrix} \mathbf{S}_g^{(1)} & \mathbf{U}_g \\ \mathbf{0} & \mathbf{S}_g^{(2)} \end{pmatrix}$$

for all $g \in G$, where the $\mathbf{S}_g^{(i)}$ form matrix representations of degree $n_i < n$ of G. We want to show that this matrix representation is completely reducible. Since

[4] See [Herstein, 1968].
[5] See [Hall, 1959].

$\mathbf{S}_{ab} = \mathbf{S}_a\mathbf{S}_b$ for all $a, b \in G$, we have $\mathbf{U}_{ab} = \mathbf{S}_a^{(1)}\mathbf{U}_b + \mathbf{U}_a\mathbf{S}_b^{(2)}$ and hence

$$\mathbf{U}_a = \mathbf{U}_{ab}[\mathbf{S}_b^{(2)}]^{-1} - \mathbf{S}_a^{(1)}\mathbf{U}_b[\mathbf{S}_b^{(2)}]^{-1}.$$

Then

$$|G| \cdot \mathbf{U}_a = \sum_{b\in G}\mathbf{U}_a = \sum_{b\in G}[\mathbf{U}_{ab}[\mathbf{S}_b^{(2)}]^{-1} - \mathbf{S}_a^{(1)}\mathbf{U}_b[\mathbf{S}_b^{(2)}]^{-1}] =$$

$$= \sum_{b\in G}[\mathbf{U}_{ab}[\mathbf{S}_{ab}^{(2)}]^{-1}\mathbf{S}_a^{(2)} - \mathbf{S}_a^{(1)}\mathbf{U}_b[\mathbf{S}_b^{(2)}]^{-1}] = \mathbf{Q}\mathbf{S}_a^{(2)} - \mathbf{S}_a^{(1)}\mathbf{Q},$$

where $\mathbf{Q} = \sum\limits_{g\in G}\mathbf{U}_g[\mathbf{S}_g^{(2)}]^{-1}$. We denote $\mathbf{Q}_1 = \frac{1}{|G|}\mathbf{Q}$, which does not depend on an element $g \in G$. Then $\mathbf{U}_a = \mathbf{Q}_1\mathbf{S}_a^{(2)} - \mathbf{S}_a^{(1)}\mathbf{Q}_1$. Now consider the matrix $\mathbf{C}$ given by:

$$\mathbf{C} = \begin{pmatrix} \mathbf{E}_1 & \mathbf{Q}_1 \\ \mathbf{0} & \mathbf{E}_2 \end{pmatrix},$$

where $\mathbf{E}_i$ is the identity matrix of degree n_i. Then

$$\mathbf{C}^{-1} = \begin{pmatrix} \mathbf{E}_1 & -\mathbf{Q}_1 \\ \mathbf{0} & \mathbf{E}_2 \end{pmatrix},$$

and

$$\mathbf{C}^{-1}\mathbf{S}_g\mathbf{C} = \begin{pmatrix} \mathbf{S}_g^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_g^{(2)} \end{pmatrix},$$

that is, $\mathbf{S}$ is a completely reducible representation.

Remark 1.6.2. This theorem was proved by H.Maschke in 1898 for finite groups when the field k has characteristic 0. For fields whose characteristic does not divide the order of a group G the result was pointed out by L.E.Dickson. Note, that this theorem is also valid in a more general case, namely when k is a commutative ring and $|G| \cdot 1$ is a unit in k, which can be seen from the third proof of this theorem.

Remark 1.6.3. The converse of the Maschke theorem is also true. Namely, if the characteristic of a field k does divide $|G|$, then G possesses finitely generated kG-modules which are not completely reducible. Specifically, the module kG itself is not completely reducible. Indeed, let $e = \sum\limits_{g\in G} g$. Since $ge = eg = e$ for each $g \in G$, e spans a one-dimensional ideal $\mathcal{I}$ in kG. Since $e^2 = 0$, this ideal is nilpotent. Since kG is an Artinian ring (as a finite dimensional algebra), its radical $\mathrm{rad}(kG)$ is nilpotent and contains all nilpotent ideals. So $\mathcal{I} \subset \mathrm{rad}(kG)$. Hence $\mathrm{rad}(kG) \neq 0$ and kG is not semisimple.

As a consequence of the Maschke theorem, the representation theory of groups splits into two different cases depending on the characteristic of the field k: classical and modular (following L.E.Dickson). In "classical" representation theory one assumes that the characteristic of k does not divide the group order $|G|$ (take e.g. k as the field of complex numbers). In "modular" representation theory one assumes that the characteristic of k is a prime, dividing $|G|$. In this case the theory is almost completely different from the classical case.

In this book we shall generally restrict our attention to finite groups in the classical case, as this simplifies things and provides a more complete theory.

1.7 PROPERTIES OF IRREDUCIBLE REPRESENTATIONS

As was shown in the previous section any group algebra kG of a finite group G over a field k with characteristic 0 or characteristic p which does not divide the order of G is semisimple, by the Maschke theorem. So the theorems giving the structure of semisimple rings, proved in volume I, give also full descriptions of the structure of the group algebras kG. We shall state these theorems for this concrete case explicitly and obtain some corollaries and applications from them.

Definition. A representation is called **irreducible, reducible, indecomposable or decomposable** according to whether the kG-module affording it has the corresponding property.

The Wedderburn-Artin theorem for an arbitrary semisimple k-algebra A can be formulated as follows:

Theorem 1.7.1 (Wedderburn-Artin). *A k-algebra A is semisimple if and only if it is isomorphic to a direct product of matrix algebras over division algebras, i.e.,*

$$A \simeq M_{n_1}(D_1) \times M_{n_2}(D_2) \times ... \times M_{n_s}(D_s), \tag{1.7.1}$$

where the D_i are division algebras over the field k.

If k is an algebraically closed field, then we obtain the following corollary:

Corollary 1.7.2 (Molien). *If k is an algebraically closed field, then every semisimple k-algebra A is isomorphic to a direct product of matrix algebras over k, i.e.,*

$$A \simeq M_{n_1}(k) \times M_{n_2}(k) \times ... \times M_{n_s}(k). \tag{1.7.2}$$

If the algebra is commutative we obtain the following statements.

Theorem 1.7.3 (Weierstrass-Dedekind). *A commutative algebra is semisimple if and only if it is isomorphic to a direct product of fields.*

Corollary 1.7.4. *If k is an algebraically closed field, then every commutative semisimple k-algebra A is isomorphic to k^s, where s is the number of simple components of A.*

From the Maschke theorem and the theory of semisimple algebras and modules one can easily obtain a number of important results describing the irreducible representations of a finite group G over an algebraically closed field k, whose characteristic does not divide $|G|$.

Corollary 1.7.5. *There are only a finite number of irreducible non-equivalent representations of a finite group G over an algebraically closed field k whose characteristic does not divide $|G|$, and this number is equal to the dimension of the center of kG over k.*

Theorem 1.7.6. *If $n_1, n_2, ..., n_s$ are the dimensions of all pairwise non-isomorphic irreducible representations of the group G over an algebraically closed field k whose characteristic does not divide $|G|$, then*

$$n_1^2 + n_2^2 + ... + n_s^2 = n, \tag{1.7.3}$$

where $n = |G|$.

Proof. By corollary 1.7.2, $kG \simeq M_{n_1}(k) \times M_{n_2}(k) \times ... \times M_{n_s}(k)$, where $n_1, n_2, ..., n_s$ are the dimensions of all the irreducible representations of the algebra kG, and thus $n = |G| = [kG : k] = n_1^2 + n_2^2 + ... + n_s^2$.

Example 1.7.1.
For the symmetric group $G = S_3$ we have that $|G| = 6$ and we have two one-dimensional irreducible representations and one two-dimensional irreducible representation, as described in example 1.6.2 (3). Thus, using (1.7.3), we have

$$6 = \sum_i n_i^2 = 1^2 + 1^2 + 2^2.$$

So, theorem 1.7.6 tells us that there are no additional distinct irreducible representations of S_3 over an algebraically closed field k whose characteristic does not divide $|G|$.

Theorem 1.7.7. *Each irreducible representation of a finite group G over an algebraically closed field k whose characteristic does not divide $|G|$ appears in the right regular representation of kG with multiplicity equal to the degree of that irreducible representation.*

Proof. Let k be an algebraically closed field, and let G be a finite group. Suppose the order of G is not divisible by the characteristic of the field k. Then, by the Maschke theorem, kG is semisimple, and, by corollary 1.7.2,

$$kG \simeq M_{n_1}(k) \times M_{n_2}(k) \times ... \times M_{n_s}(k).$$

Consider the regular representation of kG, i.e., consider kG as a right kG-module. Since each $M_{n_i}(k)$ decomposes further as a direct sum of n_i isomorphic simple right ideals M_i, and these right ideals give a complete set of isomorphism classes of irreducible kG-modules, we have the corresponding decomposition of the regular kG-module kG over an algebraically closed field k :

$$kG \simeq n_1M_1 \oplus n_2M_2 \oplus ... \oplus n_sM_s, \tag{1.7.4}$$

where $n_i = [M_i : k]$.

Theorem 1.7.8. *The number of irreducible non-equivalent representations of a finite group G over an algebraically closed field k whose characteristic does not divide $|G|$ is finite and is equal to the number of conjugacy classes of the group G.*

Proof. By corollary 1.7.5, the number of non-equivalent irreducible representations of a group G is equal to the dimension of the center of kG. We compute this dimension in another way using the fact that a group algebra kG has a special basis.

Let $C_1, C_2, ..., C_s$ be the distinct conjugacy classes of the group G. Then the group G is partitioned into these pairwise disjoint conjugacy classes. Let $c_i = \sum\limits_{g\in C_i} g$, $i = 1, 2, ..., s$. Note that elements c_i, c_j have no common terms for $i \neq j$. Hence they are linearly independent elements of G. Since conjugation by a group element permutes the elements of each class, $hc_ih^{-1} = c_i$, i.e., c_i belongs to the center of kG. We show that the elements $c_1, c_2, ..., c_s$ form a basis of the center of kG.

Let $x = \sum\limits_{g\in G} \alpha_g g$, where $\alpha_g \in k$, be an element of the center $Z(kG)$. Since $hx = xh$, i.e., $hxh^{-1} = x$ for each $h \in G$, there holds

$$\sum_{g\in G} \alpha_g g = \sum_{g\in G} \alpha_g hgh^{-1}.$$

This means that the coefficients of g and hgh^{-1} in the element x are equal. Since h is arbitrary, every element in the same conjugacy class of a fixed group element g has the same coefficient in x, hence x can be written as a linear combination of the c_i's. Since the c_i's are linearly independent, they form a basis of the center $Z(kG)$, that is, $\dim_k Z(kG) = s$.

Example 1.7.2.

For the group S_3 there are three conjugacy classes: $\{e\}$, $\{a, b, c\}$ and $\{d, f\}$. (See examples 1.6.4(3)). Thus, by theorem 1.7.8, there are three irreducible representations which, as we have seen in example 1.7.1, consist of two one-dimensional representations and one two-dimensional representation.

Corollary 1.7.9. *A finite group G is Abelian if and only if all irreducible representations of G over an algebraically closed field whose characteristic does not divide $|G|$, are one-dimensional.*

Proof. Indeed, it is sufficient to remark that a group is Abelian if and only if every conjugacy class consists of a single element and thus that the number of irreducible non-equivalent representations equals, by theorem 1.7.7, the group order. Applying theorem 1.7.6, we can see immediately that this is possible only when all irreducible representations are one-dimensional.

Corollary 1.7.10. *If G and H are Abelian groups of the same order and k is an algebraically closed field whose characteristic does not divide $|G|$, the group algebras kG and kH are isomorphic.*

1.8 CHARACTERS OF GROUPS. ORTHOGONALITY RELATIONS AND THEIR APPLICATIONS

This section is an introduction to the theory of characters of groups which is one of the important methods for the study of groups and their representations. We shall consider the main properties of characters and their applications to obtain some important results.

Let V be a vector space over a field k with a basis $v_1, v_2, ..., v_n$, and let $\varphi \in \mathrm{GL}(V)$ be a linear transformation with corresponding matrix $\mathbf{A} = (a_{ij})$ on this basis. The **trace** of the transformation φ is the **trace** of the matrix $\mathbf{A}$:

$$\mathrm{Sp}(\varphi) = \mathrm{Sp}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}.$$

From any course on linear algebra it is well known that the trace of the matrix $\mathbf{A}$ does not depend on the choice of a basis $v_1, v_2, ..., v_n$. Indeed, if $\mathbf{B} = \mathbf{PAP}^{-1}$ then $\mathrm{Sp}(\mathbf{B}) = \mathrm{Sp}(\mathbf{PAP}^{-1}) = \mathrm{Sp}(\mathbf{A})$. From the definition it follows immediately, that a trace is a linear function, i.e.,

$$\mathrm{Sp}(\varphi + \psi) = \mathrm{Sp}(\varphi) + \mathrm{Sp}(\psi)$$

$$\mathrm{Sp}(\alpha\varphi) = \alpha\mathrm{Sp}(\varphi)$$

for $\varphi, \psi \in \mathrm{GL}(V)$ and $\alpha \in k$.

Definition. Let $\sigma : G \to \mathrm{GL}(V)$ be a linear representation of a finite group G. The function $\chi_\sigma : G \to k$, which is defined by the formula

$$\chi_\sigma(x) = \mathrm{Sp}[\sigma(x)].$$

for each $x \in G$, is called the **character** of the representation σ.

If there is no chance of misunderstanding we shall write simply χ instead of χ_σ.

Example 1.8.1.

The character of the representation φ of the group C_3 considered in example 1.6.2 is given by: $\chi(1) = 2$, $\chi(u) = 1 + \xi_3$, $\chi(u^2) = 1 + (\xi_3)^2$.

Definition. A character is called **irreducible** or **reducible** according to whether the representation is irreducible or reducible, respectively.

Notice that, by corollary 1.7.5, there is only a finite number of irreducible characters of a finite group G over an algebraically closed field k whose characteristic does not divide $|G|$.

The character of the regular representation is called the **regular character** and denoted by χ_{reg}.

Examples 1.8.2.

1. The character of the trivial representation is the function $\chi(g) = 1$ for all $g \in G$. This character is called the **principal** character of G.

2. For representations of degree 1, the character and the representation are usually identified. Thus for Abelian groups, irreducible representations over an algebraically closed field and their characters are the same.

Proposition 1.8.1. *Let G be a group, and let χ_φ be the character of a representation φ of degree n. Then*

1. $\chi_\varphi(1) = n$;

2. *Equivalent representations have the same characters, and $\chi_\varphi(gxg^{-1}) = \chi_\varphi(x)$ for every $g, x \in G$.*

Proof.

1. Since $\dim_k V = n$, $(\varphi(1)) = \mathbf{E}$ is the $n \times n$ identity matrix. Thus $\mathrm{Sp}(\varphi(1)) = \mathrm{Sp}(\mathbf{E}) = n$, hence $\chi_\varphi(1) = n$.

2. It is well known that $\mathrm{Sp}(ab) = \mathrm{Sp}(ba)$ for any $a, b \in \mathrm{GL}(V)$. Then setting $a = v^{-1}$, $b = vu$, we obtain that $\mathrm{Sp}(u) = \mathrm{Sp}(vuv^{-1})$.

So equivalent representations have the same characters. Therefore $\chi_\varphi(gxg^{-1}) = \mathrm{Sp}[\varphi(gxg^{-1})] = \mathrm{Sp}[\varphi(g)\varphi(x)\varphi(g^{-1})] = \mathrm{Sp}[\varphi(x)] = \chi_\varphi(x)$ for all $g, x \in G$.

Proposition 1.8.2. *Let χ_{reg} be the regular character of a finite group G of order n. Then for any $g \in G$*

$$\chi_{reg}(g) = \begin{cases} n & \text{for } g = 1 \\ 0 & \text{for } g \neq 1 \end{cases}$$

The proof follows immediately from proposition 1.8.1 and example 1.6.2(2).

Examples 1.8.3.

1. Let $\varphi : D_{2n} \to \mathrm{GL}_2(\mathbf{R})$ be the explicit matrix representation described in example 1.6.2(3). If χ is the character of φ, then $\chi(\sigma) = 2\cos(\frac{2\pi}{n})$ and $\chi(\tau) = 0$. Since φ takes the identity of D_{2n} to the 2×2 identity matrix, $\chi(1) = 2$.

2. Let $\varphi : H_2 = Q_8 \to \mathrm{GL}_2(\mathbf{C})$ be the explicit matrix representation described in example 1.6.4(5). If χ is the character of φ, then $\chi(i) = 0$ and $\chi(j) = 0$. Since φ takes the identity of Q_8 to the 2×2 identity matrix, $\chi(1) = 2$.

Proposition 1.8.3. *The character of a direct sum of representations is the sum of the characters of the constituents of the direct sum.*

Proof. Let $\varphi = \varphi_1 \oplus \varphi_2 \oplus ... \oplus \varphi_m$ be a direct sum of representations of a group G. Then the corresponding matrix representation of φ is equivalent to the matrix representation $\mathbf{S}$ of the form:

$$\mathbf{S}_g = \begin{pmatrix} \mathbf{S}_g^{(1)} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_g^{(2)} & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{S}_g^{(m)} \end{pmatrix}$$

for all $g \in G$, where $\mathbf{S}_g^{(i)}$ are the matrix representations corresponding to representations φ_i $(i = 1, 2, ..., m)$.

Therefore

$$\chi_\varphi(g) = \mathrm{Sp}[\mathbf{T}_g] = \mathrm{Sp}[\mathbf{S}_g] = \mathrm{Sp}[\mathbf{S}_g^{(1)}] + \mathrm{Sp}[\mathbf{S}_g^{(2)}] + ... + \mathrm{Sp}[\mathbf{S}_g^{(m)}] =$$
$$= \chi_{\varphi_1} + \chi_{\varphi_2} + ... + \chi_{\varphi_m}.$$

Proposition 1.8.4. *Let k be an algebraically closed field whose characteristic does not divide the order of a finite group G. Then any character χ of the group G has a unique representation in the form:*

$$\chi = \sum_{i=1}^{s} a_i \chi_i,$$

where χ_i is the character afforded by an irreducible kG-module M_i. Moreover, two representations of the group G are equivalent if and only if they have the same characters.

Proof. Since kG is a semisimple k-algebra, $kG = A_1 \times A_2 \times ... \times A_s$, where the A_i are simple k-algebras. According to this decomposition we have a decomposition $1 = e_1 + e_2 + ... + e_s$ of the identity of the algebra kG into a sum of primitive central idempotents.

Let M be a kG-module. Let M_i be a simple A_i-module, and let χ_i be the character afforded by M_i, $i = 1, 2, ..., s$. Then we have a corresponding decompositionof the module M

$$M \simeq a_1M_1 \oplus a_2M_2 \oplus ... \oplus a_sM_s, \tag{1.8.1}$$

where the a_i are nonnegative integers indicating the multiplicity of the irreducible module M_i in the direct sum of the decomposition of M. If χ_M is the character afforded by the module M, then, by proposition 1.8.3,

$$\chi_M = a_1\chi_1 + a_2\chi_2 + ... + a_s\chi_s \tag{1.8.2}$$

and this decomposition is unique, because the decomposition (1.8.1) is unique, by the Krull-Schmidt theorem.

Suppose that two representations T and R are equivalent, then the corresponding matrices $\mathbf{T}_g$ and $\mathbf{R}_g$ are similar for each $g \in G$. Therefore they have the same traces, hence they have the same characters.

Conversely, let two representations T and R have the same characters. Suppose the kG-modules M and N correspond to the representations T and R. If χ_M, χ_N are the characters afforded by the modules M and N, respectively, then, by the proof above, we have decompositions

$$\chi_M = a_1\chi_1 + a_2\chi_2 + ... + a_s\chi_s$$

and

$$\chi_N = b_1\chi_1 + b_2\chi_2 + ... + b_s\chi_s,$$

where the a_i, b_i are nonnegative integers indicating the multiplicity of the irreducible module M_i in the decompositions of the modules M and N.

If $i \neq j$, then $e_jM_i = 0$, i.e., e_j acts on M_i trivially, hence $\chi_i(e_j) = 0$. If $i = j$, then $e_iM_i = M_i$, i.e., e_i acts on M_i as an identity, hence $\chi_i(e_i) = n_i$, where $n_i = \dim_k M_i$. Therefore $\chi_M(e_i) = a_in_i$ and $\chi_N(e_i) = b_in_i$ for all i. Since the characters χ_M and χ_N are equal, $a_i = b_i$ for all i, i.e., $M \simeq N$, and the representations T and R are equivalent.

Corollary 1.8.5. *Let k be an algebraically closed field whose characteristic does not divide the order of a finite group G. Then*

$$\chi_{reg} = \sum_{i=1}^{s} n_i\chi_i, \tag{1.8.3}$$

where χ_i is a character afforded by an irreducible kG-module M_i and $n_i = [M_i : k]$.

Proof. This follows immediately from proposition 1.8.4 and theorem 1.7.7 (see equality 1.7.4).

Note that the center $Z(kG)$ has two different natural bases. Let $1 = e_1 + e_2 + ... + e_s$ be a decomposition of the identity of the algebra kG into a sum of primitive central idempotents. Since the idempotents $e_1, e_2, ..., e_s$ are orthogonal and central, they form a basis of the center $Z(kG)$.

On the other hand, as we have already pointed out in section 1.7, the elements $c_i = \sum\limits_{g\in C_i} g$, for $i = 1, 2, ..., s$, also form a basis of $Z(k)$. Consequently, there are elements $\alpha_{ij}, \beta_{ij} \in k$ such that $c_i = \sum\limits_{j=1}^{s} \alpha_{ij} e_j$ and $e_i = \sum\limits_{j=1}^{s} \beta_{ij} c_j$; and the matrices $\mathbf{A} = (\alpha_{ij})$ and $\mathbf{B} = (\beta_{ij})$ are reciprocal (inverses of each other).

Proposition 1.8.6. *Denote by n_i the dimension of the irreducible representation M_i with character χ_i and denote by h_j the number of elements in the conjugacy class C_j. Then*

$$\alpha_{ij} = \frac{h_i}{n_j}\chi_j(g_i), \quad \beta_{ij} = \frac{n_i}{n}\chi_i(g_j^{-1}), \tag{1.8.4}$$

where $g_j \in C_j$.

Proof. If $i \neq j$, then $e_j M_i = 0$, i.e., e_j acts trivially on M_i, hence $\chi_i(e_j) = 0$. If $i = j$, then $e_i M_i = M_i$, i.e., e_i acts on M_i as an identity, hence $\chi_i(e_i) = n_i$, where $n_i = [M_i : k]$. Therefore

$$\chi_j(c_i) = \chi_j(\sum_{k=1}^{s} \alpha_{ik} e_k) = \sum_{k=1}^{s} \alpha_{ik}\chi_j(e_k) = n_j\alpha_{ij}.$$

On the other hand, $\chi_j(c_i) = h_i\chi_j(g_i)$, and the formula for the α_{ij} follows.

In order to compute β_{ij} we use corollary 1.8.5, which says that $\chi_{reg} = \sum\limits_{i=1}^{n} n_i\chi_i$. From proposition 1.8.1 it follows that $\chi_{reg}(c_k g) = 0$ if $g^{-1} \notin C_k$ and $\chi_{reg}(c_k g) = n$ if $g^{-1} \in C_k$, where $n = |G|$. Therefore, if $g_j \in C_j$, then

$$\chi_{reg}(e_i g_j^{-1}) = \chi_{reg}(\sum_{k=1}^{s} \beta_{ik} c_k g_j^{-1}) = n\beta_{ij}.$$

On the other hand,

$$\chi_{reg}(e_i g_j^{-1}) = \sum_{k=1}^{s} n_i\chi_k(e_i g_j^{-1}) = n_i\chi_i(g_j^{-1})$$

because $\chi_k(e_i g_j^{-1}) = 0$ for $k \neq i$ and $\chi_i(e_i g_j^{-1}) = \chi_i(g_j^{-1})$. The formula for the β_{ij} follows.

Taking into account that the matrices $\mathbf{A}$ and $\mathbf{B}$ are reciprocal, we obtain immediately the following "orthogonal relations" for characters, obtained by F.G.Frobenius.

Theorem 1.8.7 (orthogonality relations). *Let k be an algebraically closed field whose characteristic does not divide the order n of a finite group G. Suppose* *that χ_i is a character of the group G afforded by an irreducible kG-module M_i and $\{g_1, g_2, ..., g_s\}$ is a set of representatives of all conjugacy classes of G. Then*

$$\frac{1}{n}\sum_{k=1}^{s} h_k\chi_i(g_k)\chi_j(g_k^{-1}) = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j; \end{cases}$$

$$\frac{1}{n}\sum_{k=1}^{s} \chi_k(g_i)\chi_k(g_j^{-1}) = \begin{cases} 0 & \text{for } i \neq j, \\ 1/h_j & \text{for } i = j. \end{cases}$$

Corollary 1.8.8. *A representation T of a finite group G over an algebraically closed field whose characteristic does not divide the order of G is irreducible if and only if its character χ satisfies the following equality*

$$\frac{1}{n}\sum_{k=1}^{s} h_k\chi(g_k)\chi(g_k^{-1}) = 1. \tag{1.8.5}$$

Proof. Decompose the representation T into a sum of irreducible representations. Correspondingly, the character χ can be expressed as $\chi = \sum\limits_{i=1}^{s} m_i\chi_i$, where $\chi_1, \chi_2, ..., \chi_s$ are irreducible characters. But then

$$\frac{1}{n}\sum_{k=1}^{s} h_k\chi(g_k)\chi(g_k^{-1}) = \frac{1}{n}\sum_{i,j} m_i m_j \sum_{k=1}^{s} h_k\chi_i(g_k)\chi_j(g_k^{-1}) = \sum_{k=1}^{s} m_i^2,$$

and this sum is equal to 1 if and only if $\chi = \chi_i$ for some i, i.e., in view of proposition 1.8.4, if and only if T is an irreducible representation.

The theory of characters has the most applications in the case when $k = \mathbf{C}$ is the field of complex numbers. Therefore we restate the most important results in this case.

Proposition 1.8.9. *If χ is any character of an m-dimensional representation T of a group G over the field of complex numbers $\mathbf{C}$, then for any $g \in G$*

1. *$\chi(g)$ is a sum of roots of 1 in $\mathbf{C}$.*
2. *$\chi(g^{-1}) = \overline{\chi(g)}$, where $\overline{z}$ is the complex conjugate of the number z.*

Proof. Since G is a finite group, any element of G is of a finite order. So if $|g| = n$ for a given element $g \in G$, then $g^n = 1$ and $[\mathbf{T}_g]^n = \mathbf{E}$, where $\mathbf{T}$ is the matrix representation corresponding to a representation T, and $\mathbf{E}$ is the identity matrix. Since the minimal polynomial of $\mathbf{T}_g$ divides the polynomial $x^n - 1$, which has distinct roots, it follows that the matrix $\mathbf{T}_g$ is similar to the diagonal matrix $\text{diag}\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_m\}$ with m-th roots of 1 on the diagonal, i.e., $\varepsilon_i^m = 1$. Hence

$$\chi(g) = \text{Sp}(\mathbf{T}_g) = \varepsilon_1 + \varepsilon_2 + ... + \varepsilon_m$$

and $\mathbf{T}_{g^{-1}} \simeq \mathrm{diag}\{\varepsilon_1^{-1}, \varepsilon_2^{-1}, ..., \varepsilon_m^{-1}\}$. Moreover, if ε is a root of 1, then $\varepsilon^{-1} = \overline{\varepsilon}$. Therefore

$$\chi(g^{-1}) = \mathrm{Sp}(\mathbf{T}_{g^{-1}}) = \varepsilon_1^{-1} + \varepsilon_2^{-1} + ... + \varepsilon_m^{-1} = \overline{\varepsilon_1} + \overline{\varepsilon_2} + ... + \overline{\varepsilon_m} = \overline{\chi(g)}.$$

Corollary 1.8.10. *A representation T of a finite group G over the field of complex numbers is irreducible if and only if its character χ satisfies the following equality:*

$$\frac{1}{n}\sum_{k=1}^{s} h_k \chi(g_k) \cdot \overline{\chi(g_k)} = 1. \tag{1.8.6}$$

Example 1.8.4.

Consider the representation φ of the group S_3 considered in example 1.6.2(3). Since we have three conjugacy classes of this group: $\{e\}$, $\{a, b, c\}$ and $\{d, f\}$, so we have $h_1 = 1$, $h_2 = 3$ and $h_3 = 2$, respectively. The corresponding charactervalues are $\chi(e) = 2$, $\chi(a) = 0$, $\chi(d) = -1$.

Forming the sum in (1.8.6) and using the fact that $|S_3| = 6$, we obtain

$$\frac{1}{6}\sum_{k=1}^{3} h_k \chi(g_k) \cdot \overline{\chi(g_k)} = 1 \times 4 + 3 \times 0 + 2 \times 1 = 1,$$

which shows that this representation is irreducible by corollary 1.8.10.

Let k be an algebraically closed field and let $\chi_1, \chi_2, ..., \chi_s$ be all the irreducible characters of a finite group G over the field k. Let $C_1, C_2, ..., C_s$ be all the conjugacy classes of the group G. Write $\chi_{ij} = \chi_i(g_j)$, where $g_j \in C_j$. The square matrix $\mathbf{X} = (\chi_{ij})$ is called the **character table** of G over k with conjugacy classes of elements as the columns and characters as the rows. Character tables are central to many applications of group theory to physical problems.

If $g_j \in C_j$, then $\chi_i(g_j^{-1}) = \overline{\chi_{ij}}$. So using proposition 1.8.9 we can rewrite theorem 1.8.7 for the case of the field of complex numbers in the following form:

Theorem 1.8.11 (orthogonality relations). *Let G be a finite group and let (χ_{ij}) be the character table of the group G over the field of complex numbers* $\mathbf{C}$. *Then*

$$\frac{1}{n}\sum_{k=1}^{s} h_k \chi_{ik}\overline{\chi_{jk}} = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j; \end{cases}$$

$$\frac{1}{n}\sum_{k=1}^{s} \chi_{ki}\overline{\chi_{kj}} = \begin{cases} 0 & \text{for } i \neq j, \\ 1/h_j & \text{for } i = j. \end{cases}$$

Let φ and ψ be complex valued functions on a group G. Set

$$(\varphi, \psi) = \frac{1}{|G|}\sum_{g \in G} \varphi(g)\overline{\psi(g)}.$$

It is easy to verify that for all $\alpha, \beta \in \mathbf{C}$

1. $(\alpha\varphi_1 + \beta\varphi_2, \psi) = \alpha(\varphi_1, \psi) + \beta(\varphi_2, \psi)$;
2. $(\varphi, \alpha\psi_1 + \beta\psi_2) = \overline{\alpha}(\varphi, \psi_1) + \overline{\beta}(\varphi, \psi_2)$;
3. $(\varphi, \psi) = \overline{(\psi, \varphi)}$;
4. $(\varphi, \varphi) > 0$ for every $\varphi \neq 0$.

This product is called the **Hermitian inner product** of complex functions on the group G.

Using this definition, theorem 1.8.11 can be rewritten in the following form:

Theorem 1.8.12. *Let G be a finite group, and let χ_i, $i = 1, 2, ..., n$, be the distinct irreducible characters of the group G over the field of complex numbers $\mathbf{C}$. Then*

$$(\chi_i, \chi_j) = \delta_{ij}, \tag{1.8.7}$$

where δ_{ij} is the Kronecker delta.

The orthogonality relations (1.8.7) show that the rows and columns of a character table over the field $\mathbf{C}$ are orthogonal with respect to the Hermitian inner products, which allows one to compute character tables more easily. Note, that the first row of the character table always consists of 1's, and corresponds to the trivial representation. Certain properties of a finite group G can be deduced immediately from its character table:

1. The order of G is given by the sum of $(\chi(1))^2$ over the characters in the table.
2. G is Abelian if and only if $\chi(1) = 1$ for all characters in the table.
3. G is not a simple group if and only if $\chi(1) = \chi(g)$ for some non-trivial character χ in the character table and some non-identity element $g \in G$.

Examples 1.8.5.

1. Consider the cyclic group $C_n = (g)$, where $g^n = 1$. This group may be realized as the rotation group over the angles $2k\pi/n$ over some line. This group is commutative, therefore, by theorem 1.7.3, each irreducible representation over the field of complex numbers is one-dimensional. Each such representation has a character χ such that $\chi(g) = w \in \mathbf{C}$ and $\chi(g^k) = w^k$. Since $g^n = 1$, $w^n = 1$, that is, $w = e^{2\pi k/n}$, where $k = 0, 1, ..., n-1$. Therefore, the n irreducible representations of the group C_n have the characters $\chi_0, \chi_1, \chi_2, ..., \chi_{n-1}$, defined by the formula

$$\chi_h(g^k) = e^{2\pi i hk/n}.$$

For instance, if $n = 3$, we have the following character table (with $w = e^{2\pi i/3}$):

C_3	1	g	g^2
χ_0	1	1	1
χ_1	1	w	w^2
χ_2	1	w^2	w

2. Consider the representations of the group S_3 over the field of complex numbers. Since $|S_3| = 6$ and there are three conjugacy classes of S_3, there are three irreducible representations whose dimensions, by (1.7.3), must satisfy

$$n_1^2 + n_2^2 + n_3^2 = 6.$$

The unique solution of this equation, consisting of positive integers, is $n_1 = 1$, $n_2 = 1$ and $n_3 = 2$, so there are two one-dimensional and one two-dimensional irreducible representations.

Then the character table for S_3 is:

S_3	e	$\{a,b,c\}$	$\{d,f\}$
χ_1	1	1	1
χ_2	1	α	β
χ_3	2	γ	δ

where α, β, γ and δ are quantities that are still to be determined.

The orthogonality relations for the rows of the character table yield:

$$1 + 3\alpha + 2\beta = 0,$$
$$1 + 3\alpha^2 + 2\beta^2 = 6.$$

Using the orthogonality relations between the columns of the character table we obtain:

$$1 + \alpha + 2\gamma = 0$$
$$1 + \beta + 2\delta = 0.$$

Using the relations between the elements of the group S_3

$$a^2 = e, \quad b^2 = e, \quad c^2 = e, \quad d^2 = f,$$

we obtain, that $\alpha^2 = 1$ and $\beta^2 = \beta$. Therefore taking into account all these relations it follows that $\alpha = -1$, $\beta = 0$, $\gamma = 0$ and $\delta = -1$. Thus the complete character table for S_3 is given by

S_3	e	$\{a,b,c\}$	$\{d,f\}$
χ_0	1	1	1
χ_1	1	-1	1
χ_2	2	0	-1

Theorem 1.8.13. *Let G be a finite group, and let the χ_i be all the irreducible characters of the group G over the field of complex numbers* **C** *corresponding to irreducible representations M_i, $i = 1, 2, ..., s$. Let M be a linear representation of G with a character χ. Then*

$$\chi = \sum_{i=1}^{n} (\chi, \chi_i)\chi_i, \tag{1.8.8}$$

$$(\chi, \chi) = \sum_{i=1}^{n} a_i^2, \tag{1.8.9.}$$

where a_i is the number of representations M_i appearing in the direct sum of the decomposition of M and $a_i = (\chi, \chi_i)$.

Proof. Let χ_i be the irreducible character afforded by a module M_i, $i = 1, 2, ..., s$ and let M be a linear representation of G with a character χ. Then we have the decomposition of the module M:

$$M \simeq a_1 M_1 \oplus a_2 M_2 \oplus ... \oplus a_s M_s,$$

and the corresponding decomposition of the character χ:

$$\chi = \sum_{i=1}^{n} a_i \chi_i.$$

Using properties 1 and 2 of the inner product and theorem 1.8.12, we obtain that $a_i = (\chi, \chi_i)$. Now computing the inner product (χ, χ) and using the orthogonality relations (1.8.7) we obtain (1.8.9).

Definition. For any complex function χ on a finite group G the **norm** of χ is $(\chi, \chi)^{1/2}$ and will be denoted by $||\chi||$. From equality (1.8.2) we obtain that

$$||\chi|| = (\sum_{i=1}^{n} (a_i)^2)^{1/2}.$$

And hence we have:

Corollary 1.8.14. *A character has norm 1 if and only if it is irreducible.*

1.9 MODULAR GROUP REPRESENTATIONS

The theory of modular representations of finite groups was developed by Richard Brauer starting in the 1930's. In the modular theory one fixes a prime p which divides the order of a group G and studies homomorphisms of the group G into the group of matrices over a field k of characteristic p. The problem of the description of modular representations of a finite group G over a field k is the same as the analogous problem for the Sylow p-subgroups of G.

Theorem 1.9.1 (D.G.Higman).[6] *Let G be a finite group whose order is divisible by the characteristic p of a field k. Then the group algebra kG is of finite representation type if and only if the Sylow p-subgroups of G are cyclic.*

The first result on the classification for representations of non-cyclic p-groups was obtained in 1961. It was the classification for the $(2, 2)$-group. This problem is trivially reduced to the well-known problem of a pencil of matrices up to

[6] see [Higman, 1954].

similarity. Later it became clear that the problem of the description contains, in turn, the problem of the classification of a pair linear operators which act on a finite dimensional vector space (these problems are called wild and the others are called tame). In particular, it was proved that the problem of a description of representations of a group (p,p) for $p \neq 2$ is wild. So, this problem is wild for arbitrary non-cyclic groups. If $p = 2$, the problem is wild for the $(2,4)$ and $(2,2,2)$ groups. From these results it follows that one can hope to obtain a classification of modular representations only for $p = 2$ and for some groups G such that its quotient groups do not contain the groups $(2,4)$ and $(2,2,2)$. These groups are exhausted by the following three infinite sets of 2-groups:

1. Dihedral groups
$$D_m = \{x, y \ : \ x^2 = y^{2^m} = 1, \ \ yx = xy^{-1}\} \ \ (m \geqslant 1);$$
2. Quasidihedral groups
$$Q_m = \{x, y \ : \ x^2 = y^{2^m} = 1, \ \ yx = xy^{2^{m-1}-1}\} \ \ (m \geqslant 3);$$
3. Generalized quaternion groups
$$H_m = \{x, y \ : \ y^{2^m} = 1, \ \ x^2 = y^{2^{m-1}}, \ \ yx = xy^{-1}\} \ \ (m \geqslant 2).$$

We say that a group G is **tame** (**wild**) over a field k if the group algebra kG is of tame (wild) representation type.[7]

V.Bondarenko and Yu.A.Drozd proved that a subgroup of finite index of a tame group (over a field k) is tame (over k). It is easy to see that a generalized quaternion group H_m is isomorphic to the subgroup of the quasidihedral group Q_{m+1}, which is generated by xy and y^2, so that generalized quaternion groups are tame over a field with characteristic 2.

Denote by G' the commutator subgroup of a group G. The above results can be formulated in the following invariant form.

Theorem 1.9.2 (V.M.Bondarenko - Yu.A.Drozd). *A noncyclic infinite p-group G is tame over a field k of a characteristic p if and only if $(G : G') \leq 4$.*

V.M.Bondarenko and Yu.A.Drozd also proved that a finite group is tame over a field with a characteristic $p > 0$ if and only if its p-Sylow subgroup is tame. This result can be formulated in the following invariant form.

Theorem 1.9.3 (V.M.Bondarenko - Yu.A.Drozd). *A finite group G is tame over a field k of characteristic p if and only if an any Abelian p-subgroup in G of order more then 4 is cyclic.*

[7] For more exact definitions see chapter 3.

1.10 NOTES AND REFERENCES

The concept of a group is historically one of the first examples of an abstract algebraic system. The origins of the idea of a group are encountered in a number of disciplines, the principal one being the theory of solving algebraic equations by radicals. Permutations were first employed to satisfy the needs of this theory by J.L.Lagrange (1771) [Lagrange, 1771] and by A.Vandermonde (1771). The former paper is of special importance in group theory, since it gives, in terms of polynomials, what is really a decomposition of a symmetric permutation group into (right) cosets with respect to subgroups. The deep connections between the properties of permutation groups and those of equations were pointed out by N.H.Abel (1824) and by E.Galois (1830). E.Galois must be credited with concrete advances in group theory: the discovery of the role played by normal subgroups in problems of solvability of equations by radicals, the discovery that the alternating groups of order $n \geq 5$ are simple, etc. Also C.Jordan's treatise (1870) on permutation groups played an important role in the systematization and development of this branch of algebra.

The idea of a group arose in geometry, in an independent manner, at the time when the then only existing antique geometry had been augmented in the middle of the nineteenth century by numerous other 'geometries', and finding relations between them had become an urgent problem. This question was solved by studies in projective geometry, which dealt with the behaviour of geometric figures under various transformations. The stress in these studies gradually shifted to the study of the transformations themselves and their classification. Such a 'study of geometric mappings' was extensively conducted by A.Möbius, who investigated congruence, similarity, affinity, collineation, and, finally, 'elementary types of mappings' of geometric figures, that is, actually, their topological equivalence. A.L.Cayley (1854 and later) and other representatives of the English school of the theory of invariants gave a more systematic classification of geometries. A.L.Cayley explicitly used the term 'group', made systematic use of the multiplication table which now carries his name, proved that any finite group can be represented by permutations, and conceived a group as a system which is defined by its generating elements and defining relations. The final stage in this development was the Erlangen program of F.Klein (1872), who based the classification of geometries on the concept of a transformation group.

Number theory is the third source of the concept of a group. As early as 1761 L.Euler, in his study of residues, actually used congruences and their division into residue classes, which in group-theoretic language means the decomposition of groups into cosets of subgroups.

The main notions of abstract group theory arose in the 19-th century. Thus, W.Burnside, writing in 1897, quotes A.Cayley as saying that 'a group is defined by means of the laws of combination of its symbols', and goes on to explain why he, in his own book [Burnside, 1955], does, on the whole, not take that point of view. L.Kronecker discussed axioms for abstract finite groups in 1870 [Kronecker, 1870], and the notion of abstract groups was introduced by A.Cayley in three papers starting in 1849 [Cayley, 1849], [Cayley, 1854a], [Cayley, 1854b], though these papers received little attention at the time. This had certainly changed by the 1890's and a discussion of the basic definitions and some basic properties of abstract groups can be found in H.Weber's influential treatise [Weber, 1899].

Sylow subgroups were introduced by Peter Ludwig Mejdeli Sylow (1833-1918), a Norwegian mathematician. The Sylow theorems were proved also by P.L.M.Sylow in 1872 [Sylow, 1872].

The founder of group representation theory was Ferdinand Georg Frobenius (1849-1917). He discovered the amazing basic properties of irreducible group characters and published them in the 1890's. His student, Issai Schur, was another who made many significant early contributions to the subject.

Group algebras were considered by F.G.Frobenius and I.Schur [Schur, 1905] in connection with the study of group representations, since studying the representations of G over a field k is equivalent to studying modules over the group algebra kG.

H.Maschke is best known today for the Maschke theorem, which he published in 1899. A special case of his theorem H.Maschke proved in the paper [Maschke, 1898]. The general result appeared in the following year [Maschke, 1899].

In its modern form the theory of group representations owes much to the contributions of Emmy Noether (in the 1920's), whose work forms the basis of what is now called "modern algebra".

Modular representation theory was developed by Richard Brauer.

The problem of dividing the class of group algebras into tame and wild ones has been completely solved by V.M.Bondarenko and Yu.A.Drozd [Bondarenko, Drozd, 1977].

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2. Quivers and their representations

This chapter is devoted to the study of quivers and their representations. Quivers were introduced by P.Gabriel in connection with problems of representations of finite dimensional algebras in 1972. And from that time on the theory connected with representations of quivers has developed enormously.

We start this chapter by considering some important algebras, including the Grassmann algebra and the tensor algebra of a bimodule.

In sections 2.3 and 2.4 the main concepts are given pertaining to quivers, their representations and path algebras of quivers. The main theorem in this section says that the path algebra of a quiver over an arbitrary field is hereditary.

Dynkin diagrams and Euclidean diagrams (more often called extended Dynkin diagrams) and the quadratic forms connected with them are studied in section 2.5. The most important theorem, due to I.N.Berstein, I.M.Gel'fand, V.A.Ponomarev, gives the classifications of all graphs from the point of view of their quadratic forms.

The most remarkable result in the theory of representations of quivers is the theorem classifying the quivers of finite representation type, which was obtained by P.Gabriel in 1972. This theorem says that a quiver is of finite representation type over an algebraically closed field if and only if the underlying diagram obtained from the quiver by forgetting the orientations of all arrows is a disjoint union of simple Dynkin diagrams. P.Gabriel also proved that there is a bijection between the isomorphism classes of indecomposable representations of a quiver Q and the set of positive roots of the Tits form corresponding to this quiver. The proof of this theorem is given in section 2.6.

We end this chapter by studying representations of species, also a concept introduced by P.Gabriel. The main results of this theory are stated here for completeness without proofs.

2.1 CERTAIN IMPORTANT ALGEBRAS

Many rings and algebras which occur in the various parts of mathematics and physics are naturally graded. The most important examples of such algebras are tensor algebras and their quotient algebras such as the symmetric and exterior algebras. There are many interesting applications in differential geometry and in physics of these algebras.

Definition. Let A be a ring. An A-module M is said to be **graded** if it can be represented as a direct sum of submodules

$$M = \bigoplus_{n=\infty}^{\infty} M_n. \tag{2.1.1}$$

Of course a module M can admit many different gradings and so a graded module is really a module together with a specified decomposition (2.1.1).

In such a graded module M any element $x \in M$ can be uniquely written in the form $x = \sum_n m_n$, where $m_n \in M_n$, and only a finite number of these summands is not equal to zero. An element $m \in M_n$ is **homogeneous of degree** n, and we write $\deg(m) = n$. The element 0 is called homogeneous of degree n for any n. A graded module M is called **positively graded** if $M_n = 0$ for all $n < 0$, and it is called **negatively graded** if $M_n = 0$ for all $n > 0$.

A submodule B of a graded module M is called **homogeneous**, if $B = \overset{\infty}{\underset{n=\infty}{\oplus}} B_n$, where $B_n = B \cap M_n$ for all n. So, B is a graded module itself. If M is a graded module and B is homogeneous submodule of it, then we can consider the quotient module $\bar{M} = M/B$ which is also a graded module: $\bar{M} = \overset{\infty}{\underset{n=\infty}{\oplus}} \bar{M}_n$, where

$$\bar{M}_n \simeq M_n/B_n.$$

The tensor product of two graded A-modules L and M is again a graded A-module where the grading is given by the formula:

$$(L \otimes M)_n = \sum_{p+q=n} L_p \otimes M_q.$$

Therefore $\deg(l \otimes m) = \deg(l) + \deg(m)$ for all homogeneous $l \in L$ and $m \in M$.

Definition. A ring A is called **graded** if it can be represented as a direct sum of Abelian additive subgroups A_n:

$$A = \overset{\infty}{\underset{n=0}{\oplus}} A_n, \tag{2.1.2}$$

such that $A_i A_j \subseteq A_{i+j}$ for all $i, j \geq 0$.

Let $\mathcal{I} = \overset{\infty}{\underset{n=1}{\oplus}} A_n$. Then $\mathcal{I}$ is obviously a two-sided ideal of A. Let $1 = a_0 + x$ be a decomposition of the identity of A, where $a_0 \in A_0$ and $x \in \mathcal{I}$. Then for all $b \in A_n$ we have $b = 1 \cdot b = a_0 b + xb$. Comparing the degrees of the elements involved it follows that $b = a_0 b$ (and $xb = 0$). Similarly $b = ba_0$, that is, $a_0 \in A_0$ acts like the identity on all homogeneous elements of A. In other words, $1 = a_0 \in A_0$. Since $A_0 A_0 \subseteq A_0$, we obtain that A_0 is a subring of A with the same identity.

Examples 2.1.1.

1. The polynomial ring $A = K[x]$ over a field K is graded with $A_n = Kx^n$. The multiplication in A is defined by $x^p x^q = x^{p+q}$, so that indeed $\deg(x^p x^q) = \deg(x^p) + \deg(x^q)$.

2. Let K be a ring, and let $\{x_1, x_2, \ldots, x_n\}$ be a set of symbols. Consider the free left K-module $\Lambda = K(x_1, x_2, \ldots, x_n)$ with as basis the elements 1 and

all the elements (words) x_i, $x_{i_1}x_{i_2}$, $x_{i_1}x_{i_2}x_{i_3}$, ..., $x_{i_1}x_{i_2}\dots x_{i_n}$, ..., where each $i_j \in \{1, 2, \dots, n\}$. Set $\Lambda_n = \{x_{i_1}x_{i_2}\dots x_{i_n}\}$, the K-vectorspace spanned by all words of length n, then $\Lambda = \overset{\infty}{\underset{n=0}{\oplus}} \Lambda_n$ is a graded K-module. In this module we introduce a multiplication of basis elements by the rule:

$$(\alpha x_{i_1}x_{i_2}\dots x_{i_n})(\beta x_{j_1}x_{j_2}\dots x_{j_m}) = \alpha\beta x_{i_1}x_{i_2}\dots x_{i_n}x_{j_1}x_{j_2}\dots x_{j_m}.$$

Obviously, if $x, y \in \Lambda$, then $\deg(x \cdot y) = \deg x + \deg y$. As a result we obtain a graded ring $\Lambda = K\langle x_1, x_2, \dots, x_n\rangle$ which is called the **free associative K-algebra** over K in the indeterminates $x_1, \dots, x_n$. Consider the ideal $\mathcal{I}$ of the ring Λ which is generated by all elements of the form $x_ix_j - x_jx_i$. This ideal is homogeneous and the quotient ring $\Lambda/\mathcal{I}$ is also a graded ring. This ring is denoted by $K[x_1, x_2, \dots, x_n]$ and it is called the **polynomial ring** (**ring of polynomials**) in n variables over the ring K.

3. If in the previous example we consider the homogeneous ideal $\mathcal{I}$ which is generated by all elements of the form x_ix_i and $x_ix_j + x_jx_i$, for all i, j, then we obtain a quotient ring $\Lambda/\mathcal{I}$ which is denoted by $E(x_1, x_2, \dots, x_n)$ and called the **exterior K-ring** in n variables over the ring K.

Definition. An algebra (associative or non-associative) A over a field K is called **graded** if it is a graded ring, that is, if it is represented in the form of a direct sum of subspaces A_i:

$$A = \overset{\infty}{\underset{i=0}{\oplus}} A_i, \tag{2.1.3}$$

such that $A_iA_j \subseteq A_{i+j}$, and $KA_i \subseteq A_i$ for all i, j

Such a representation of A as a sum (2.1.3) is called a **graduation** (**grading**). The elements of subspaces A_i are called **homogeneous of order** i and we write $\deg a = i$ for $a \in A_i$.

Examples 2.1.2.

Grassmann algebra. An important example of a graded associative algebra is the Grassmann algebra. An associative algebra with 1 over a field K is called a **Grassmann algebra** if it has a system of generators $a_1, a_2, \dots, a_n$ satisfying the following equalities:

$$a_i^2 = 0, \quad a_ia_j + a_ja_i = 0, \quad i, j = 1, 2, \dots, n, \tag{2.1.4}$$

and, moreover, any other identity for these elements is a corollary of the identities (2.1.2). This algebra is denoted by Γ_n or $\Gamma(a_1, a_2, \dots, a_n)$. Note that the relations (2.1.4) are homogeneous and thus generate a homogeneous ideal so that Γ_n inherits a grading from $K(x_1, \dots, x_n)$, see example 2.1.1(2) above. If the characteristic of the field K is unequal to 2 the second group of the equalities (2.1.4) implies the first froup. The equalities (2.1.4) are the **anticommutators** of the elements a_i and a_j.

From the definition of the Grassmann algebra it also follows that Γ_n has a basis consisting of 1 and the following elements:

$$a_i, \quad i = 1, 2, \dots, n,$$

$$a_i a_j, \quad i, j = 1, 2, \dots, n, \ i < j,$$

$$a_i a_j a_k, \quad i, j, k = 1, 2, \dots, n, \quad i < j < k,$$

....

$$a_1 a_2 \dots a_n.$$

Thus, any element $a \in \Gamma_n$ can be written in the form

$$a = \sum_{k \geq 0} \sum_{i_1 < i_2 < \dots < i_k} \gamma_{i_1 i_2 \dots i_k} a_{i_1} a_{i_2} \dots a_{i_k}, \tag{2.1.5}$$

where $\gamma_{i_1 i_2 \dots i_k} \in K$.

An element (2.1.5) of the algebra Γ_n is called **even** (respectively **odd**) if for each nonzero coefficient $\gamma_{i_1 i_2 \dots i_k}$ the number k is even (resp. odd). The set of all even (resp. odd) elements of Γ_n is denoted by Γ_n^0 (resp. Γ_n^1). The set Γ_n^1 is a vector subspace of Γ_n and the set Γ_n^0 is a subalgebra in Γ_n. The following properties hold:

1) if $a, b \in \Gamma_n^1$, then ab, ba and $ab - ba \in \Gamma_n^0$;

2) if $a \in \Gamma_n^1$, $b \in \Gamma_n^0$, then $ab \in \Gamma_n^1$;

3) $\Gamma = \Gamma_n^0 \oplus \Gamma_n^1$.

Grassmann algebras have some important applications in physics where they are used to model various concepts related to fermions and supersymmetry.

Tensor algebra. Another important example of graded algebras are the tensor algebras. Let V be a fixed vector space over a field k, and let $T^0V = k$, $T^1V = V$, $T^2V = V \otimes V$, ..., $T^mV = V \otimes \dots \otimes V$ (m copies). Define $\mathfrak{T}(V) = \bigoplus\limits_{i=0}^{\infty} T^iV$. Introduce an associative product on $\mathfrak{T}(V)$ defined on homogeneous generators of $\mathfrak{T}(V)$ by the following rule: $(v_1 \otimes \dots \otimes v_n)(w_1 \otimes \dots \otimes w_m) = v_1 \otimes \dots \otimes v_n \otimes w_1 \otimes \dots \otimes w_m \in T^{n+m}V$. This makes $\mathfrak{T}(V)$ an associative graded algebra with 1. If V is a finite dimensional algebra, then $\mathfrak{T}(V)$ is generated by 1 together with any basis of V.

The algebra $\mathfrak{T}(V)$ is the universal associative algebra in the following sense.

Proposition 2.1.1 (Universal property.) *For any associative k-algebra $\mathcal{U}$ with 1 and for any k-linear map $\varphi : V \to \mathcal{U}$, there exists a unique homomorphism of k-algebras $\psi : \mathfrak{T}(V) \to \mathcal{U}$ such that $\psi(1) = 1$ and such that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[dr, "\varphi"'] & \mathfrak{T}(V) \arrow[d, "\psi"] \\ & \mathcal{U} \end{tikzcd}$$

where α is the (obvious) inclusion $V = T^1V \subset \mathfrak{T}(V)$.

We shall prove this proposition in a more general situation in the next section.

This proposition gives the motivation to introduce the following formal definition:

Definition. Let V be a vector space over a field k. A **tensor algebra** on V is a pair (T, θ), where T is an associative k-algebra with 1, and $\alpha : V \to T$ is a monomorphism of k-vector spaces such that for any associative k-algebra $\mathcal{U}$ with 1 and for any k-linear map $\varphi : V \to \mathcal{U}$, there exists a unique homomorphism of k-algebras $\psi : T \to \mathcal{U}$ such that $\psi(1) = 1$ and the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\theta"] \arrow[dr, "\varphi"'] & T \arrow[d, "\psi"] \\ & \mathcal{U} \end{tikzcd}$$

Then we obtain the following proposition.

Proposition 2.1.2. *Let V be a vector space over a field k. Suppose that $(\mathfrak{T}(V), \alpha)$ is as before, then*

1. *$(\mathfrak{T}(V), \alpha)$ is a tensor algebra.*

2. *If (T, θ) is any other tensor algebra on V, then there is a unique k-algebra isomorphism $\sigma : \mathfrak{T}(V) \to T$ such that $\sigma\alpha = \theta$, i.e., the diagram*

$$\begin{tikzcd} V \arrow[rr, "\alpha"] \arrow[dr, "\theta"'] & & \mathfrak{T}(V) \arrow[dl, "\sigma"] \\ & T & \end{tikzcd}$$

is commutative.

Proof.

1. This follows immediately from proposition 2.1.1.

2. This uniqueness property of a tensor algebra is of the same type as proved in proposition 4.5.2, vol.I, for the uniqueness of tensor products and therefore its proof is left to the reader as a simple exercise.

Remark 2.1.1. Note that a tensor algebra over V is the same as a free algebra on V, i.e., $\mathfrak{T}(V) \simeq k\langle V\rangle$. If V is a finite dimensional vector space with basis $(x_1, \dots, x_n)$, then $\mathfrak{T}(V) \simeq k\langle x_1, \dots, x_n\rangle$.

Symmetric algebra. Many quotient algebras of the tensor algebra $\mathfrak{T}(V)$ are of interest. First, there is the symmetric algebra on a given vector space V. Let $\mathcal{I}$ be the two-sided ideal in $\mathfrak{T}(V)$ generated by all $x \otimes y - y \otimes x$, where $x, y \in V$. The quotient algebra $\mathfrak{S}(V) = \mathfrak{T}(V)/\mathcal{I}$ is called the **symmetric algebra** on V. We write $\sigma : \mathfrak{T}(V) \to \mathfrak{S}(V)$ for the corresponding canonical map of k-algebras. Since the generators of $\mathcal{I}$ lie in T^2V, it obvious that $\mathcal{I} = (\mathcal{I} \cap T^2V) \oplus (\mathcal{I} \cap T^3V) \oplus \dots$. Therefore σ is injective on $T^0V = k$ and $T^1V = V$. Moreover, $\mathfrak{S}(V)$ inherits a grading from $\mathfrak{T}(V)$: $\mathfrak{S}(V) = \bigoplus\limits_{i=0}^{\infty} S^iV$. The effect of factoring out $\mathcal{I}$ is universal (in the sense of proposition 2.1.1) for linear maps from V into commutative associative k-algebras with 1. Moreover, if $\dim_k V = n$ and $(x_1, \dots, x_n)$ is any fixed basis of V, then $\mathfrak{S}(V)$ is canonically isomorphic to the polynomial algebra $k[x_1, x_2, \dots, x_n]$ over the field k in n variables.

Exterior algebra. Second, we have the exterior algebra of a given vector space V. Let $\mathcal{I}$ be the two-sided ideal in $\mathfrak{T}(V)$ generated by all elements of the form $x \otimes x$, where $x \in V$. The quotient algebra $\Lambda(V) = \mathfrak{T}(V)/\mathcal{I}$ is called the **exterior algebra** on V. We denote by $\sigma : \mathfrak{T}(V) \to \Lambda(V)$ the corresponding canonical map of k-algebras. Since the generators of $\mathcal{I}$ lie in T^2V, it obvious that $\mathcal{I} = (\mathcal{I} \cap T^2V) \oplus (\mathcal{I} \cap T^3V) \oplus \dots$. Therefore σ is injective on $T^0V = k$ and $T^1V = V$. Moreover, $\Lambda(V)$ inherits a grading from $\mathfrak{T}(V)$: $\Lambda(V) = \bigoplus\limits_{i=0}^{\infty} \Lambda^iV$. We use $\wedge$ as the symbol for multiplication in this algebra. The product in $\Lambda(V)$ is often called the **exterior product** (also known as the Grassmann product, or the wedge product). It is a generalization of the cross product in the three-dimensional vector algebra. The exterior product is associative and bilinear. Since $\sigma(T^0V) \simeq \Lambda^0V \simeq k$, and $\sigma(T^1V) \simeq \Lambda^1V \simeq V$, we can identify Λ^0V with k and Λ^1V with V. Then elements of the exterior algebra $\Lambda(V)$ have the following essential property which is alternating on V:

$$v \wedge v = 0$$

for all vectors $v \in V$ which entails

$$u \wedge v = -v \wedge u$$

for all $u, v \in V$ and

$$v_1 \wedge v_2 \wedge \cdots \wedge v_p = 0$$

whenever $v_1, v_2, \dots, v_p \in V$ are linearly dependent.

The exterior algebra is in fact the "most general" algebra with these properties. This means that all equations that hold in the exterior algebra follow from the above properties alone. This generality of $\Lambda(V)$ is formally expressed by a certain universal property given below.

Proposition 2.1.3 (Universal property). *For any associative k-algebra A with 1 and any k-linear map $\varphi : V \to A$ such that $\varphi(v)\varphi(v) = 0$ for all $v \in V$, there exists a unique k-algebra homomorphism $f : \Lambda(V) \to A$ such that $f(v) = \varphi(v)$ for all $v \in V$.*

Elements of the form $v_1 \wedge v_2 \wedge \cdots \wedge v_p$ with $v_1, v_2, \ldots, v_p \in V$ are called **p-vectors**. The subspace $\Lambda^p(V)$ is generated by all p-vectors, and sometimes $\Lambda^p(V)$ is called the p-th **exterior power** of V.

The exterior product has the important property that the product of a p-vector and a q-vector is a $p+q$-vector. Thus the exterior algebra forms a graded algebra where the grading is given by $\Lambda(V) = \oplus\Lambda^p(V)$. These p-vectors have geometric interpretations: the 2-vector $u \wedge v$ represents the oriented parallelogram with sides u and v, while the 3-vector $u \wedge v \wedge w$ represents the oriented parallelepiped with edges u, v, and w.

Let V be an n-dimensional vector space over k with a basis $\{e_1, e_2, \ldots, e_n\}$. Then the set

$$\{e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p} \ : \ 1 \leq i_1 < i_2 < \ldots < i_p \leq n\}$$

is a basis for the p-th exterior power $\Lambda^p(V)$. The reason is the following: given any wedge product of the form $v_1 \wedge v_2 \wedge \ldots \wedge v_p$, every vector v_j can be written as a linear combination of the basis vectors e_j; using the bilinearity of the wedge product, this can be expanded to a linear combination of wedge products of those basis vectors. Any wedge product of elements from $\{e_1, \ldots, e_n\}$ in which the same basis vector appears more than once is zero; any wedge product in which the basis vectors don't appear in the proper order can be reordered, changing the sign whenever two basis vectors change places. The resulting coefficients of the basis k-vectors in a wedge product of $v_1, \ldots, v_m$ can be computed as the minors of the matrix that describes the vectors v_j in terms of the basis e_j.

Counting the basis elements, we see that the dimension of $\Lambda^p(V)$ is equal to $\dfrac{n!}{p!(n-p)!}$. In particular, $\Lambda^p(V) = \{0\}$ for $p > n$.

So in this case the exterior algebra can be written in the following form:

$$\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus \cdots \oplus \Lambda^n(V),$$

where $\Lambda^0(V) = k$ and $\Lambda^1(V) = V$. Thus in this case $\Lambda(V)$ is a finite dimensional algebra with $\dim_k \Lambda(V) = 2^n$.

It is easy to see that $\Lambda(V)$ is isomorphic to the Grassmann algebra $\Gamma_n(k)$.

Denote $\mathcal{J} = \Lambda^1(V) \oplus \ldots \oplus \Lambda^n(V)$. Since $\mathcal{J}^{n+1} = 0$, $\mathcal{J} \subseteq \mathrm{rad}A$. From $\Lambda(V)/\mathcal{J} \simeq k$ it follows that $\mathcal{J} = \mathrm{rad}A$. Therefore $\Lambda(V)$ is a noncommutative local ring.

Example 2.1.3.
Let V be a vector space with basis $\{e_1, e_2, e_3, e_4\}$. Then

$$\Lambda^0(V) = \{1\}$$

$$\Lambda^1(V) = \{e_1, e_2, e_3, e_4\}$$

$$\Lambda^2(V) = \{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$$

$$\Lambda^3(V) = \{e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\}$$

$$\Lambda^4(V) = \{e_1 \wedge e_2 \wedge e_3 \wedge e_4\}$$

and $\Lambda^k(V) = \{0\}$, where $k > \dim V$.

2.2 TENSOR ALGEBRA OF A BIMODULE

In this section we consider a generalization of the construction of a tensor algebra on a vector space, the tensor algebras of bimodules.

Definition. Let B be a ring, and let V be a bimodule over B. The **tensor algebra** of the B-bimodule V is the graded ring $\mathfrak{T}(V) = \mathfrak{T}_B(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$, where $V^{\otimes 0} = B$, and $V^{\otimes k} = V \otimes_B V^{\otimes (k-1)}$ for $k > 0$.

The multiplication in $\mathfrak{T}(V)$ is induced by the natural isomorphisms $V^{\otimes k} \otimes_B V^{\otimes m} \simeq V^{\otimes (k+m)}$. So that we shall identify $V^{\otimes k} \otimes_B V^{\otimes m}$ with $V^{\otimes (k+m)}$. In future we shall also identify B with $V^{\otimes 0}$ and V with $V^{\otimes 1}$.

By construction, $\mathfrak{T}(V)$ contains the subring $B = V^{\otimes 0}$ and the B-bimodule $V = V^{\otimes 1}$. Moreover, as the following theorem shows, $\mathfrak{T}(V)$ is the universal ring with this property.

Theorem 2.2.1. *Let $\varphi : B \to A$ be a ring homomorphism, and let $f : V \to A$ be a homomorphism of B-bimodules, where A is considered as a B-bimodule by means of the homomorphism φ. Then there exists a unique ring homomorphism $F : \mathfrak{T}(V) \to A$ such that the restrictions of F to B and V coincide with φ and f, respectively.*

Proof. The homomorphism f induces B-bimodule homomorphisms

$$f^{\otimes n} : V^{\otimes n} \to A^{\otimes n}.$$

Moreover, the multiplication in the ring A induces a bimodule homomorphism $A^{\otimes n} \to A$ such that the image of $a_1 \otimes_B a_2 \otimes_B \ldots \otimes_B a_n$ is $a_1 a_2 \ldots a_n$. Thus, we obtain a family of homomorphisms $f_n : V^{\otimes n} \to A$ such that

$$f(v_1 \otimes_B v_2 \otimes_B \ldots \otimes_B v_n) = f(v_1) f(v_2) \ldots f(v_n).$$

It is obvious, that in this way we obtain a ring homomorphism $F : \mathfrak{T}(V) \to A$ (where $f_0 = \varphi$ and $f_1 = f$). Moreover, F is unique. This follows from the fact that $\mathfrak{T}(V)$ is generated by the elements of B and V.

Definition. The graded ideal $\mathcal{J} = \mathcal{J}(V) = \bigoplus_{n=1}^{\infty} V^{\otimes n}$ is called the **fundamental ideal** of $\mathfrak{T}(V)$. An ideal $\mathcal{I} \subset \mathfrak{T}(V)$ (not necessarily graded) is called **essential** if $\mathcal{J}^2 \supset \mathcal{I} \supset \mathcal{J}^m$ for some $m > 2$.

Definition. A right Artinian ring A is called a **Wedderburn ring** if it has a subring $B \subset A$ such that $A = B \oplus R$ (a direct sum of additive groups), and $R = V \oplus R^2$ (a direct sum of bimodules), where $R = \mathrm{rad}A$. In this case $B \simeq A/R$ is a semisimple ring and $V \simeq R/R^2$ is a finitely generated right B-bimodule.

Theorem 2.2.2. *Any Wedderburn ring A with radical R is isomorphic to a quotient ring of the form $\mathfrak{T}_B(V)/\mathcal{I}$, where $B = A/R$, $V = R/R^2$, and $\mathcal{I}$ is an essential ideal in $\mathfrak{T}_B(V)$.*

Conversely, if B is a semisimple Artinian ring, V is a finitely generated right B-module, then any quotient ring $A = \mathfrak{T}_B(V)/\mathcal{I}$, where $\mathcal{I}$ is an essential ideal in $\mathfrak{T}_B(V)$, is Wedderburn, moreover, $B \simeq A/R$ and $V \simeq R/R^2$, where $R = \mathcal{J}/\mathcal{I} = \mathrm{rad}A$.

Proof. If A is a Wedderburn ring, then $A = B \oplus R$, so A has a subring $A_1 \simeq B$. Hence, it is possible to define a ring monomorphism $\varphi : B \to A$ and to consider A as a B-bimodule. Since we have a direct sum of B-bimodules $R = V \oplus R^2$, we have a B-bimodule monomorphism $f : V \to A$. Therefore, by proposition 2.2.1, there is a unique homomorphism $F : \mathfrak{T}_B(V) \to A$ which is the identity on B and V. Let $\mathcal{I} = \mathrm{Ker}(F)$. Since F induces an isomorphism $\mathfrak{T}(V)/\mathcal{J}^2 \simeq A/R^2$, we obtain that $\mathcal{I} \subset \mathcal{J}^2$. On the other hand, $F(\mathcal{J}) \subset R$, so $F(\mathcal{J}^n) \subset R^n$ for any $n > 1$. Since A is a right Artinian ring, R is nilpotent. Therefore $F(\mathcal{J}^m) = 0$ for some m, and so $\mathcal{J}^m \subset \mathcal{I}$. Thus $\mathcal{I}$ is an essential ideal in $T(V)$. Since $F(B) = \varphi(B)$ and $F(V) = f(V)$, any element $r \in R$ has the form $F(x) + r'$, where $x \in \mathcal{J}$, $r' \in R^2$. Then any element $r \in R^m$ is also of the form: $F(x) + r'$, with $x \in \mathcal{J}^m$ and $r' \in R^{m+1}$. Since R is nilpotent, it follows immediately that $F(\mathcal{J}) = R$, i.e., F is an epimorphism, and so $A \simeq \mathfrak{T}(V)/\mathcal{I}$.

The second part of the theorem is obvious.

The next part of this section is devoted to studying the global dimension of a tensor algebra $\mathfrak{T}_B(V)$ of a B-bimodule V when V is a projective right (or left) B-module.

We shall need some additional statements concerning projectivity and injectivity of modules.

Proposition 2.2.3. *Let A and B be rings. Assume that M is a right A-module, N is a right B-module and a left A-module. If M is A-projective and N is B-projective, then $M \otimes_A N$ is B-projective.*

Proof. Let X be a right B-module. Since M and N are projective, the functor $\mathrm{Hom}_B(N, X)$ is exact on X and $\mathrm{Hom}_A(M, \mathrm{Hom}_B(N, X))$ is exact on X, by proposition 5.1.1, vol.I. Then, by the adjoint isomorphism (proposition 4.6.3, vol.I), the functor $\mathrm{Hom}_B(M \otimes_A N, X)$ is also exact. So, again by proposition 5.1.1, vol.I, $M \otimes_A N$ is B-projective.

Proposition 2.2.4. *Let A and B be rings. Assume that M is a right A-module and a left B-module, and that N is a left B-module. If M is A-projective and N is B-injective, then $Hom_B(M, N)$ is A-injective.*

Proof. Let X be a right B-module. Since M is A-projective, $M \otimes_A X$ is exact in X, by proposition 6.3.5, vol.I. Since N is B-injective, $\mathrm{Hom}_B(M \otimes_A X, N)$ is also exact in X, by proposition 5.2.1, vol.I. Therefore, by the adjoint isomorphism, the functor $\mathrm{Hom}_A(X, \mathrm{Hom}_B(M, N))$ is exact on X. So again, by proposition 5.2.1, vol.I, $\mathrm{Hom}_B(M, N)$ is A-injective.

Corollary 2.2.5. *If $S = \mathfrak{T}_B(V)$ is the tensor algebra of a bimodule V over a ring B and V is a projective right B-module, then S is also a projective right B-module.*

Proof. Applying proposition 2.2.3 repeatedly we obtain that $V^{\otimes m}$ is projective for any $m \geq 1$. Therefore $S = \mathfrak{T}_B(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$ is also projective, by proposition 5.1.4, vol.I.

Corollary 2.2.6. *Let $S = \mathfrak{T}_B(V)$ be the tensor algebra of a bimodule V over a ring B and let V be a projective right B-module. If Q is an injective left S-module, then Q is also an injective left B-module.*

Proof. This follows from proposition 2.2.4 and corollary 2.2.5 taking into account that ${}_BQ \simeq \mathrm{Hom}_S({}_SS_B, {}_SQ)$.

Corollary 2.2.7. *Let $S = \mathfrak{T}_B(V)$ be the tensor algebra of a bimodule V over a ring B and let V be a projective right B-module. If P is a right projective S-module, then P is also a projective right B-module.*

Proof. This follows from proposition 2.2.3 and corollary 2.2.5 taking into account that $P_B \simeq P \otimes_S S_B$.

Corollary 2.2.8. *Let $S = \mathfrak{T}_B(V)$ be the tensor algebra of a bimodule V over a ring B and let V be a projective right B-module. Then*

(1) $\text{proj. dim}_S (S \otimes_B M) \leq \text{proj. dim}_B M \leq \text{l.gl. dim}\, B$ *for any left B-module M;*

(2) inj. $\dim_S \operatorname{Hom}_B(S, M) \leq$ inj. $\dim_B M \leq$ r.gl. dim B *for any right B-module M.*

Proof.

(1) Let M be a left B-module. Applying the functor $S \otimes_B *$ to a projective resolution of M and taking into account proposition 2.2.3, we obtain a projective resolution of the S-module $S \otimes_B M$.

(2) Let M be a right B-module. Applying the functor $\operatorname{Hom}_B(S, *)$ to an injective resolution of M and taking into account proposition 2.2.4, we obtain an injective resolution of the S-module $\operatorname{Hom}_B(S, M)$.

Taking into account corollaries 2.2.5, 2.2.6, in an analogously way we obtain the following corollary:

Corollary 2.2.9. *For any left S-module N and for any right S-module M we have*:

(1) inj. $\dim_B N \leq$ inj. $\dim_S N$;

(2) proj. $\dim_B M \leq$ proj. $\dim_S M$.

Lemma 2.2.10. *If $S = \mathfrak{T}_B(V)$ and V is a projective right B-module then there is a canonical exact sequence*

$$0 \to S \otimes_B V \otimes_B S \xrightarrow{f} S \otimes_B S \xrightarrow{g} S \to 0 \tag{2.2.1}$$

where $f(s \otimes b \otimes s_1) = sb \otimes s_1 - s \otimes bs_1$ and $g(s \otimes s_1) = ss_1$.

Proof. Suppose $g\big(\sum\limits_{i=1}^{n} x_i \otimes y_i\big) = 0$, where $x_i, y_i \in S$. Then $\sum\limits_{i=1}^{n} x_i y_i = 0$, and therefore

$$\sum_{i=1}^{n} x_i \otimes y_i = \sum_{i=1}^{n} x_i \otimes y_i - \sum_{i=1}^{n} x_i y_i \otimes 1 = \sum_{i=1}^{n} (x_i \otimes 1)(1 \otimes y_i - y_i \otimes 1).$$

So the set of elements of the form $1 \otimes x - x \otimes 1$, where $x \in S$ generate $\operatorname{Ker} g$. Since the map $d : S \to S \otimes_B S$ given by the formula $d(x) = 1 \otimes x - x \otimes 1$ is a differentiation, $\operatorname{Im} d$ is generated by $d(x)$. So $\operatorname{Ker} g = \operatorname{Im} f$.

If $u = \sum\limits_{i=1}^{m} x_i \otimes v_i \otimes y_i \in \operatorname{Ker} f$, where $x_i, y_i \in S; v_i \in V$, then $\sum\limits_{i=1}^{m} x_i v_i \otimes y_i = \sum\limits_{i=1}^{m} x_i \otimes v_i y_i$. We shall show that the last equality holds only in the case that $\sum\limits_{i=1}^{m} x_i \otimes v_i \otimes y_i = 0$.

Indeed, let $\max\limits_{i=1,\ldots,n} \{\deg(x_i)\} = p$ and $\max\limits_{i=1,\ldots,n} \{\deg(y_i)\} = q$ which are the degrees of the homogeneous elements $\overline{x}$ and $\overline{y}$, i.e., $\overline{x} \in V^p$ and $\overline{y} \in V^q$. Then the homogeneous component of $\sum\limits_{i=1}^{m} x_i v_i \otimes y_i$ of maximal degree has the form

$\sum\limits_{i=1}^{m}\overline{x}v_i\otimes\overline{y}\in V^{p+1}\otimes V^q$ and the homogeneous component of $\sum\limits_{i=1}^{m}x_i\otimes v_iy_i$ of maximal degree has the form $\sum\limits_{i=1}^{m}\overline{x}\otimes v_i\overline{y}\in V^p\otimes V^{q+1}$. Since $(V^{p+1}\otimes V^q)\cap(V^p\otimes V^{q+1})=0$, we have $\sum\limits_{i=1}^{m}\overline{x}_iv_i\otimes\overline{y}_i=0=\sum\limits_{i=1}^{m}\overline{x}_i\otimes v_i\overline{y}_i$, i.e., $\sum\limits_{i=1}^{m}x_iv_i\otimes y_i=0=\sum\limits_{i=1}^{m}x_i\otimes v_iy_i$. But the last equality is equivalent to the equality $0=\sum\limits_{i=1}^{m}x_i\otimes v_i\otimes y_i\in S\otimes_B V\otimes_B S$. Therefore $\mathrm{Ker} f=0$ and the lemma is proved.

Theorem 2.2.11. *Let $S=\mathfrak{T}_B(V)$ be the tensor algebra of a bimodule V over a ring B. If V is a projective right (or left) B-module, then*:

$$\text{l.gl. dim } B\le \text{l.gl. dim } S\le \text{l.gl. dim } B+1 \tag{2.2.2}$$

$$\text{r.gl. dim } B\le \text{r.gl. dim } S\le \text{r.gl. dim } B+1 \tag{2.2.3}$$

Proof. Let M be a left S-module. Applying the functor $*\otimes_S M$ to the exact sequence (2.2.1) and taking into account that $\mathrm{Tor}_1^S(S,M)=0$, we obtain the exact sequence:

$$0\to S\otimes_B V\otimes_B M\longrightarrow S\otimes_B M\longrightarrow M\to 0 \tag{2.2.4}$$

Corollary 2.2.9 implies the following inequalities:

$$\text{proj. dim}_S\,(S\otimes_B(V\otimes_B M))\le \text{l.gl. dim } B \tag{2.2.5}$$

$$\text{proj. dim}_S\,(S\otimes_B M)\le \text{l.gl. dim } B \tag{2.2.6}$$

Applying the functor Ext to the exact sequence (2.2.4) and taking into account the inequalities (2.2.5)-(2.2.6), we obtain the inequality proj. dim $_S M\le$ l.gl. dim $B+1$, so that,

$$\text{l.gl. dim } S\le \text{l.gl. dim } B\,+\,1$$

Let N be a right S-module. Applying the functor $\mathrm{Hom}_S(*,N)$ to the exact sequence (2.2.1) and taking into account that $\mathrm{Ext}_1^S(S,M)=0$, we obtain the exact sequence:

$$0\to \mathrm{Hom}_S(S,N)\to \mathrm{Hom}_S(S\otimes_B S,N)\to \mathrm{Hom}_S(S\otimes_B V\otimes_B S,N)\to 0 \tag{2.2.7}$$

Taking into account that

$$\mathrm{Hom}_S(S,N)\simeq N$$

$$\mathrm{Hom}_S(S\otimes_B S,N)\simeq \mathrm{Hom}_B(S,\mathrm{Hom}_S(S,N))\simeq \mathrm{Hom}_B(S,N)$$

$$\mathrm{Hom}_S(S\otimes_B V\otimes_B S,N)\simeq \mathrm{Hom}_B(S\otimes_B V,\mathrm{Hom}_S(S,N))\simeq$$

$$\simeq \mathrm{Hom}_B(S\otimes_B V,N)\simeq \mathrm{Hom}_B(S,\mathrm{Hom}_B(V,N))$$

the sequence (2.2.7) has the form:

$$0 \to N \longrightarrow \mathrm{Hom}_B(S, N) \longrightarrow \mathrm{Hom}_B(S, \mathrm{Hom}_B(V, N)) \to 0 \tag{2.2.8}$$

Corollary 2.2.9 implies the following inequalities:

$$\text{inj. dim}_S\ \mathrm{Hom}_B(S, \mathrm{Hom}_B(V, N)) \leq \text{l.gl. dim } B \tag{2.2.9}$$

$$\text{inj. dim}_S\ \mathrm{Hom}_B(S, N) \leq \text{l.gl. dim } B \tag{2.2.10}$$

Applying the functor Ext to the exact sequence (2.2.8) and taking into account the inequalities (2.2.9)-(2.2.10), we obtain the inequality inj. $\dim_S N \leq$ r.gl. dim $B+1$, i.e.,

$$\text{r.gl. dim } S \leq \text{r.gl. dim } B + 1$$

We shall now prove the right sides of the inequalities (2.2.2)-(2.2.3). To this end consider the exact split sequence of B-bimodules:

$$0 \to \mathcal{J} \to S \to B \to 0 \tag{2.2.11}$$

Applying to this sequence the functors $* \otimes_B M$ and $N \otimes_B *$ we obtain the exact split sequences of B-modules:

$$0 \to S \otimes_B V \otimes_B M \longrightarrow S \otimes_B M \longrightarrow M \to 0 \tag{2.2.12}$$

$$0 \to N \otimes_B V \otimes_B S \longrightarrow N \otimes_B S \longrightarrow N \to 0 \tag{2.2.13}$$

Applying to the sequence (2.2.12) the functor $\mathrm{Ext}^1_B(X, *)$ and to the sequence (2.2.13) the functor $\mathrm{Ext}^1_B(*, X)$ and taking into account corollaries 2.2.8 and 2.2.9 we obtain that

$$\text{inj. dim}_B\ M \leq \text{inj. dim}_B\ S \otimes_B M \leq \text{l.gl. dim } S$$

$$\text{proj. dim}_B\ N \leq \text{proj. dim}_B N \otimes_B S \leq \text{r.gl. dim } S$$

These inequalities mean that

$$\text{l.gl. dim } B \leq \text{l.gl. dim } S$$

$$\text{r.gl. dim } B \leq \text{r.gl. dim } S$$

The theorem is proved.

Remark 2.2.1. If B is a semisimple ring and V is a B-bimodule, then in this case the tensor algebra $\mathfrak{T}_B(V)$ is a **maximal ring** in the sense of G.Hochschild (see [Hochschild, 1947], [Hochschild, 1950]). Since in this case any B-module is projective, from theorem 2.2.11 we obtain immediately the following statement:

Corollary 2.2.12. *A maximal ring is hereditary.*

Remark 2.2.2. Let I be a finite set and let K_i for $i \in I$ be a set of skew fields. Let $B = \prod_{i\in I} K_i$ and and let V be a B-bimodule. In this case $\mathfrak{T}_B(V)$ is called a **special tensor algebra**.

Example 2.2.1.

Let k be a field and $B = \prod_{i=1}^{m} k$. For each $n \geq 1$ let

$$T_n = \begin{pmatrix} B & 0 & 0 & \dots & 0 & 0 \\ V & B & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ V^{\otimes n-2} & V^{\otimes n-3} & \dots & & B & 0 \\ V^{\otimes n-1} & V^{\otimes n-2} & \dots & \dots & V & B \end{pmatrix}$$

be the ring with addition and multiplication given by the matrix operations. Then T_n is a special tensor algebra.

Since for a special tensor algebra $\mathfrak{T}_B(V)$ of a bimodule V over a ring B we have that $B = \prod_{i\in I} K_i$ is always a semisimple ring and V is a projective B-module, from theorem 2.2.11 immediately follows:

Corollary 2.2.13. *A special tensor algebra is hereditary.*

Remark 2.2.3. If the special tensor algebra $\mathfrak{T}_B(V)$ is finite dimensional over a field k, then J.P.Jans and T.Nakayama have shown that $\mathfrak{T}_B(V)$ is a hereditary Artinian k-algebra [Jans, Nakayama, 1957].

Example 2.2.2.

The special tensor algebra T_n from example 2.2.1 is a finite dimensional hereditary Artinian k-algebra.

Remark 2.2.4. If B is a finite dimensional algebra over an algebraically closed field k, then

$$B = B_1 \times B_2 \times \ldots \times B_s,$$

where $B_i \simeq M_{n_i}(k)$ and there is a unique simple (B_i, B_j)-bimodule U_{ij}; moreover, $V = \oplus U_{ij}^{t_{ij}}$. Therefore the tensor algebra $\mathfrak{T}_B(V)$ is uniquely defined by the quiver $Q(B)$ and the ranks n_i. In the important particular case, when all $n_i = 1$, i.e., the algebra B is basic, $\mathfrak{T}_B(V)$ coincides with the path algebra kQ of the quiver $Q(B)$. (See also section 2.3 below.)

Examples 2.2.3.

1. Let k be a field. The algebras

$$A = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} k & 0 & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$$

are finite dimensional special tensor algebras which correspond to the quivers $\bullet \longrightarrow \bullet \longrightarrow \bullet$ and

$$\begin{tikzcd} \bullet & & \bullet \\ & \bullet \arrow[ul] \arrow[ur] & \end{tikzcd}$$

respectively.

2. The algebra $A = k[[\alpha]]$, the ring of formal power series in one variable α over a field k, is an infinite dimensional special tensor algebra which corresponds to the quiver

$$\begin{tikzcd} \bullet \arrow[loop above, "\alpha"] \end{tikzcd}$$

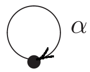
Remark 2.2.5. Using the tensor algebra of a bimodule we can also define the exterior algebra of a bimodule. Let B be a commutative ring with 1 and let V be a B-bimodule. Then we can consider the two-sided ideal $\mathcal{I}$ which is spanned by all elements of the form $x_1 \otimes x_2 \otimes \ldots \otimes x_r$, where $x_i \in V$ and $x_i = x_j$ for some $i \neq j$. Then $\Lambda(V) = \mathfrak{T}(V)/\mathcal{I}$ is called the **exterior algebra** of the bimodule V.

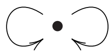
2.3 QUIVERS AND PATH ALGEBRAS

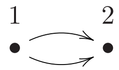
The notion of a quiver of a finite dimensional algebra over an algebraically closed field was introduced by P.Gabriel in 1972 in connection with problems of the representation theory of finite dimensional algebras.

In this section we shall consider quivers in sense of P.Gabriel. And we shall assume that k is an algebraically closed field.

Definition. A **quiver** $Q = (VQ, AQ, s, e)$ is a finite directed graph which consists of finite sets VQ, AQ and two mappings $s, e : AQ \to VQ$. The elements of VQ are called **vertices** (or **points**), and those of AQ are called **arrows**. Usually, the set of vertices VQ will be a set $\{1, 2, ..., n\}$. We say that each arrow $\sigma \in AQ$ starts at the vertex $s(\sigma)$ and ends at the vertex $e(\sigma)$. The vertex $s(\sigma)$ is called the **start** (or **initial**, or **source**) **vertex** and the vertex $e(\sigma)$ is called the **end** (or **target**) **vertex** of σ. Some examples of quivers are:







A quiver can be given by its **adjacency** (or **incidence**) **matrix**

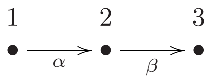
$$[Q] = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ t_{21} & t_{22} & \dots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & \dots & t_{nn} \end{pmatrix},$$

where t_{ij} is the number of arrows from the vertex i to the vertex j.

Two quivers Q_1 and Q_2 are called **isomorphic** if there is a bijective correspondence between their vertices and arrows such that starts and ends of corresponding arrows map into one other. It is not difficult to see that $Q_1 \simeq Q_2$ if and only if the adjacency matrix $[Q_1]$ can be transformed into the adjacency matrix $[Q_2]$ by a simultaneous permutation of rows and columns.

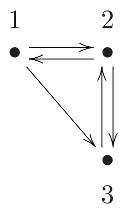
Examples 2.3.1.

1. For the quiver

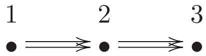


we have $VQ = \{1, 2, 3\}$ and $AQ = \{\alpha, \beta\}$. We also have $s(\alpha) = 1$, $s(\beta) = 2$, $e(\alpha) = 2$ and $e(\beta) = 3$.

2. A quiver may have several arrows in the same or in opposite direction. For example:



and



3. A quiver may also have loops. For example:

For a quiver $Q = (VQ, AQ, s, e)$ and a field k one defines the path algebra kQ of Q over k. Recall that a path p of the quiver Q from the vertex i to the vertex j is a sequence of r arrows $\sigma_1\sigma_2 ... \sigma_r$ such that the start vertex of each arrow σ_m coincides with the end vertex of the previous one σ_{m-1} for $1 < m \le r$, and moreover, the vertex i is the start vertex of σ_1, while the vertex j is the end vertex of σ_r. The number r of arrows is called the **length of the path** p. For such a path p we define $s(p) = s(\sigma_1) = i$ and $e(p) = e(\sigma_k) = j$. By convention we also include into the set of all paths the trivial path ε_i of length zero which connects the vertex i with itself without any arrow and we set $s(\varepsilon_i) = e(\varepsilon_i) = i$ for each $i \in VQ$, and, also, for any arrow $\sigma \in AQ$ with start at i and end at j we set $\varepsilon_i\sigma = \sigma\varepsilon_j = \sigma$. A path, connecting a vertex of a quiver with itself and of length not equal to zero, is called an **oriented cycle**.

Definition. The **path algebra** kQ of a quiver Q over a field k is the (free) vector space with a k-basis consisting of all paths of Q. Multiplication in kQ is defined in the obvious way: the product of two paths is given by composition when possible, and is defined to be 0 otherwise.

Therefore if the path $\sigma_1 \dots \sigma_m$ connects i and j and the path $\sigma_{m+1} \dots \sigma_n$ connects j and k, then the product $\sigma_1 \dots \sigma_m \sigma_{m+1} \dots \sigma_n$ connects i with k. Otherwise, the product of these paths equals 0. Extending the multiplication by distributivity, we obtain a k-algebra kQ (not necessarily finite dimensional), which is obviously associative.

Remark 2.3.1. Note that if a quiver Q has an infinitely many vertices, then kQ has no an identity element. If Q has infinitely many arrows, then kQ is not finitely generated, and so it is not finite dimensional over k. In future we shall always assume that VQ is finite and $VQ = \{1, 2, \dots, n\}$.

In the algebra kQ the set of trivial paths forms a set of pairwise orthogonal idempotents, i.e.,

$$\varepsilon_i^2 = \varepsilon_i \quad \text{for} \quad \text{all} \quad i \in VQ$$

$$\varepsilon_i\varepsilon_j = 0 \quad \text{for} \quad \text{all} \quad i, j \in VQ \quad \text{such} \quad \text{that} \quad i \neq j.$$

If $VQ = \{1, 2, \dots, n\}$, the identity of kQ is the element which is equal to the sum of all the trivial paths ε_i of length zero, that is, $1 = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n$. The elements

ε_1, ε_2, …, ε_n together with the paths of length one generate Q as an algebra. So kQ is a finitely generated algebra.

The subspace $\varepsilon_i A$ has as basis all paths starting at i, and the subspace $A\varepsilon_j$ has as basis all paths ending at j. The subspace $\varepsilon_i A\varepsilon_j$ has as basis all paths starting at i and ending at j.

Since $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ is a set of pairwise orthogonal idempotents for $A = kQ$ with sum equal to 1, we have the following decomposition of A into a direct sum:

$$A = \varepsilon_1 A \oplus \varepsilon_2 A \oplus \dots \oplus \varepsilon_n A.$$

So each $\varepsilon_i A$ is a projective right A-module. Analogously, each $A\varepsilon_i$ is a projective left A-module.

Lemma 2.3.1. *Each ε_i, for $i \in VQ$, is a primitive idempotent, and $\varepsilon_i A$ is an indecomposable projective right A-module.*

Proof. Note that $\varepsilon_i A\varepsilon_i$ is spanned by all paths that start and end at vertex i. In fact they form a basis. Also observe that if p is such a path of length $r > 0$ and $r_1 + r_2 = r$, $r_i \in \mathbf{N} \cup \{0\}$ then there are unique paths x, y of lengths r_1 and r_2 such that $p = xy$. They are of course not necessarily in $\varepsilon_i A\varepsilon_i$. Let f be a nontrivial idempotent of $\varepsilon_i A\varepsilon_i$ and $g = e - f$ where e is the identity of $\mathrm{End}_A(\varepsilon_i A) = \varepsilon_i A\varepsilon_i$ and suppose $fg = gf = 0$. Take paths of maximal length x, y occurring with nonzero coefficient in respectively f and g. Let at least one of them have length > 0. Then we would have $xy \neq 0$ (because x ends at i and y starts at i) and $xy \neq x'y'$ for any other pair (x', y') of such paths. This contradicts $fg = 0$ by the uniqueness of factorization remark above. Hence f or g is a multiple of ε_i and the result follows.

Lemma 2.3.2. *$\varepsilon_i A \not\simeq \varepsilon_j A$, for $i, j \in VQ$ and $i \neq j$.*

Proof. By theorem 2.1.2, vol.I, $Hom_A(\varepsilon_i A, \varepsilon_j A) \simeq \varepsilon_j A\varepsilon_i$ and $Hom_A(\varepsilon_j A, \varepsilon_i A) \simeq \varepsilon_i A\varepsilon_j$. Suppose $\varepsilon_i A \simeq \varepsilon_j A$. Then the inverse isomorphisms give elements $f \in \varepsilon_j A\varepsilon_i$ and $g \in \varepsilon_i A\varepsilon_j$ with $fg = \varepsilon_j$ and $gf = \varepsilon_i$. So $\varepsilon_i \in \varepsilon_i A\varepsilon_j A\varepsilon_i$. Since the subspace $\varepsilon_i A\varepsilon_j A\varepsilon_i$ has as basis all the paths passing through the vertex j and starting and ending at i, and ε_i is the trivial path for the vertex i, we have a contradiction.

Examples 2.3.2.

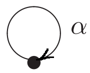
1. Let Q be the quiver

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, "\sigma_1"'] & \overset{2}{\bullet} \arrow[r, "\sigma_2"'] & \overset{3}{\bullet} \end{tikzcd},$$

i.e., $VQ = \{1, 2, 3\}$, $AQ = \{\sigma_1, \sigma_2\}$.

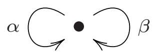
Then kQ has a basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \sigma_1, \sigma_2, \sigma_1\sigma_2\}$ and $kQ \simeq T_3(k) = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix} \subset M_3(k)$. So the algebra kQ is finite dimensional over k.

2. Let Q be the quiver with one vertex and one loop:



Then kQ has a basis $\{\varepsilon, \alpha, \alpha^2, \ldots, \alpha^n, \ldots\}$. Therefore $kQ \simeq k[x]$, the polynomial algebra in one variable x. Obviously, this algebra is finitely generated but it is not finite dimensional.

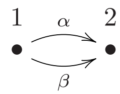
3. Let Q be the quiver with one vertex and two loops:



Then kQ has two generators α, β, and a path in kQ is any word in α, β. Therefore $kQ \simeq k\langle \alpha, \beta \rangle$, the free associative algebra generated by α, β, which is non-commutative and infinite dimensional over k.

If Q is a quiver with one vertex and $n \geq 2$ loops $\alpha_1, \alpha_2, \ldots, \alpha_n$, then $kQ \simeq k\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$, the free associative algebra generated by $\alpha_1, \alpha_2, \ldots, \alpha_n$, which is also non-commutative and infinite dimensional over k.

4. Let Q be the quiver with two vertices and two arrows:



i.e., $VQ = \{1, 2\}$ and $AQ = \{\alpha, \beta\}$. The algebra kQ has a basis $\{\varepsilon_1, \varepsilon_2, \alpha, \beta\}$. This algebra is isomorphic to the Kronecker algebra[1] $A = \begin{pmatrix} k & k \oplus k \\ 0 & k \end{pmatrix}$, which is four-dimensional over k.

Proposition 2.3.3. *The path algebra kQ is finite dimensional over k if and only if VQ is finite and Q has no oriented cycles.*

Proof. If Q contains oriented cycles, then we can construct infinite number of different paths by walking around that cycle n times, for any n. So in this case kQ is not finite dimensional (see, examples 2.3.2(2) and 2.3.2(3)). If Q does not

[1] See example 2.4.4 below for the reason why this is called the Kronecker algebra

contain oriented cycles, then we have only a finite number[2] of paths in Q which form a basis of kQ over k, so it is finite dimensional.

Remark 2.3.2. A path algebra kQ is a special tensor algebra, with B the commutative semisimple algebra $B = \prod\limits_{i \in VQ} k$ and $V = \bigoplus\limits_{\sigma \in AQ} k$, considered as a k-bimodule via $a\sigma a' = (a_i \sigma a'_j)$, where σ is a path stating at i and ending at j, that is, $s(\sigma) = i$ and $e(\sigma) = j$ and $a, a' \in B$ with components a_i, a'_j. Therefore corollary 2.2.13 gives the same statement for any path algebra:

Theorem 2.3.4. *The path algebra kQ of a quiver Q over a field k is hereditary.*

If $A = kQ = T_B(V)$, where $B = \prod\limits_{i \in VQ} k$ and $V = \bigoplus\limits_{\sigma \in AQ} k$, then, by lemma 2.2.10, there is a canonical exact sequence

$$0 \to A \otimes_B V \otimes_B A \xrightarrow{f} A \otimes_B A \xrightarrow{g} A \to 0$$

where $f(a \otimes v \otimes a_1) = av \otimes a_1 - a \otimes va_1$ and $g(a \otimes a_1) = aa_1$ for any $a, a_1 \in A$ and $v \in V$. If X is an A-module, we can apply the functor $* \otimes X$ to this exact sequence. As a result we obtain a new exact sequence which we shall also need in further applications.

Proposition 2.3.5. *Let $A = kQ$. Then for any A-module X there is an exact sequence:*

$$0 \to \bigoplus_{\sigma \in AQ} X_{s(\sigma)} \otimes \varepsilon_{e(\sigma)} A \xrightarrow{f} \bigoplus_{i \in VQ} X_i \otimes \varepsilon_i A \xrightarrow{g} X \to 0 \tag{2.3.1}$$

where $g(x \otimes a) = xa$ for $a \in \varepsilon_i A$, $x \in X_i = X\varepsilon_i$ and $f(x \otimes a) = x \otimes \sigma a - x\sigma \otimes a$ for $a \in \varepsilon_{e(\sigma)} A$ and $x \in X_{s(\sigma)}$.

Proof. Let $A = kQ$ and let X be a right A-module.

Clearly, $gf = 0$ and g is surjective. So we need to show only that $\mathrm{Ker} f = 0$ and $\mathrm{Ker} g \subseteq \mathrm{Im} f$.

1. We shall show that $\mathrm{Ker} f = 0$. Let $u \in \mathrm{Ker} f$. We can write it in the form:

$$u = \sum_{\sigma \in AQ} \sum_a x_{\sigma,a} \otimes a$$

where the second sum is over all paths a with $s(a) = e(\sigma)$ and the elements $x_{\sigma,a} \in X\varepsilon_{s(\sigma)}$ are almost all equal to 0. Then,

$$f(u) = \sum_\sigma \sum_a (x_{\sigma,a} \otimes \sigma a - x_{\sigma,a}\sigma \otimes a).$$

[2]Because if there are no oriented cycles all vertices in a nontrivial path $v_{i_1} \dots v_{i_r}$ must be different.

Let a be a longest path in u with $x_{\sigma,a} \neq 0$ for some σ. Then $f(u)$ involves $x_{\sigma,a} \otimes \sigma a$ and nothing can cancel this, so $f(u) \neq 0$. A contradiction (as in the proof of lemma 2.3.1).

2. We shall show that $\mathrm{Ker} g \subseteq \mathrm{Im} f$. Let $u \in \bigoplus_{i \in VQ} X_i \otimes \varepsilon_i A$. Then one can represent it in the form

$$u = \sum_{i=1}^{n} \sum_{a} x_a \otimes a,$$

where the second sum is over all paths a starting at i and almost all the $x_a \in X_{s(a)}$ are zero. Define $\deg(u)$ to be the length of the longest path a with $x_a \neq 0$.

Now if a is a nontrivial path with $s(a) = i$, we can write it as a product $a = \sigma a'$, where σ is an arrow starting at i and a' is some other path. Then

$$f(x_a \otimes a') = x_a \otimes a - x_a \sigma \otimes a',$$

viewing $x_a \otimes a'$ as an element of the σ-th component.

We now claim that $u + \mathrm{Im} f$ always contains an element of degree 0. Let $\deg(u) = d > 0$. Consider the element

$$\xi = u - f(\sum_{i=1}^{n} \sum_{a} x_a \otimes a'),$$

where the second sum is over all paths a staring at i, and having length equal to d. Then $\deg(\xi) < d$. Now the claim follows by induction.

We are now ready to prove that $\mathrm{Ker} g \subseteq \mathrm{Im} f$. Let $u \in \mathrm{Ker} g$, and let $u' = u + Imf$ with $\deg(u') = 0$. Then

$$0 = g(u) = g(u') = g(\sum_{i=1}^{n} x'_{\varepsilon_i} \otimes \varepsilon_i) = \sum_{i=1}^{n} x'_{\varepsilon_i},$$

which belongs to $\bigoplus_{i=1}^{n} X_i$. So each term in the final sum must be zero. Thus $u' = 0$. Hence $u \in \mathrm{Im} f$.

Corollary 2.3.6. *If X is an arbitrary kQ-module, then* $\mathrm{Ext}^i(X, Y) = 0$ *for all Y and $i \geq 2$.*

Proof. Consider the sequence (2.3.1) constructed in the previous proposition. The maps f and g are A-modules homomorphisms. By lemma 2.3.1, $\varepsilon_i A$ is a projective right A-module. Since $V \otimes \varepsilon_i A$ is isomorphic to the direct sum of $\dim V$ copies of $\varepsilon_i A$, it is also a projective right A-module, by proposition 5.1.4, vol.I. Therefore the exact sequence (2.3.1) is a projective resolution of the kQ-module X.

Remark 2.3.3. The projective resolution (2.3.1) is often called the **Ringel resolution** (see [Ringel, 1976]).

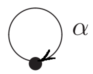
Let Q be a quiver and k be a field. An **admissible relation** is an element of kQ of the form

$$\sum_{i=1}^{m} c_i p_i,$$

where $c_i \in k$ for all i and $p_1,\ p_2\ \dots\ p_m$ are paths in Q with a common start vertex and a common end vertex. If $m = 1$ and $c_1 = 1$ then p is called a **zero-relation**. A relation p with $m = 2$, $c_1 = 1$, $c_2 = -1$ is called a **commutative relation**. An **admissible ideal** is a two-sided ideal in kQ generated by non-zero admissible relations. If $\mathcal{I}$ is an admissible ideal then $kQ/\mathcal{I}$ is called an **algebra of a quiver with relations** (or **bound quiver algebra**).

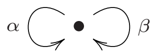
Examples 2.3.3.

1. Let G be the finite group consisting of two elements, and let k be a field. Then we have $kG \simeq kQ/(\alpha^2)$, where Q is the quiver:



and $\mathcal{I} = (\alpha^2)$ is an admissible ideal in kQ.

2. Let $G = \{x, y\ :\ x^2 = y^2 = 1, xy = yx\}$. Then $kG \simeq kQ/(\alpha^2, \beta^2, \beta\alpha - \alpha\beta)$, where Q is the quiver:



and $\mathcal{I} = (\alpha^2, \beta^2, \alpha\beta - \beta\alpha)$ is an admissible ideal[3].

3. Let $A = T_3(k) = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix} \subset M_3(k)$. Then $A \simeq kQ$, where Q is the quiver:

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet}$$

2.4 REPRESENTATIONS OF QUIVERS

Definition. Let $Q = (VQ, AQ, s, e)$ be a quiver and let k be a field. A **representation** $V = (V_x, V_\sigma)$ of Q over k is a family of vector spaces V_x $(x \in VQ)$ together with a family of linear mappings $V_\sigma : V_{s(\sigma)} \to V_{e(\sigma)}$ $(\sigma \in AQ)$.

We assume that Q is a finite quiver and $VQ = \{1, 2, \dots, n\}$. We shall always consider finite dimensional representations of Q, that is, for every such representation V we assume that V_i is a finite dimensional vector space over a field k for all i.

[3] G is of course the Klein four group (see page 4)

Example 2.4.1.
Consider the quiver

$$\overset{1}{\bullet} \xrightarrow{\sigma} \overset{2}{\bullet}$$

For every matrix $\mathbf{X} \in M_{n\times m}(k)$ we can define a representation $V_{\mathbf{X}}$ by $V_{\mathbf{X}}(1) = k^m$, $V_{\mathbf{X}}(2) = k^n$ and $V_{\mathbf{X}}(\sigma) = \mathbf{X}$.

Definition. Given two representations V, W of a quiver Q, a **morphism** $f = (f_x) : V \to W$ is given by a family of linear mappings $f_x : V_x \to W_x$ such that for each $\sigma \in AQ$ the diagram

$$\begin{tikzcd} V_{s(\sigma)} \arrow[r, "f_{s(\sigma)}"] \arrow[d, "V_\sigma"'] & W_{s(\sigma)} \arrow[d, "W_\sigma"] \\ V_{e(\sigma)} \arrow[r, "f_{e(\sigma)}"] & W_{e(\sigma)} \end{tikzcd}$$

commutes, i.e., $f_{s(\sigma)}W_\sigma = V_\sigma f_{e(\sigma)}$. If f_x is invertible for every $x \in VQ$, then f is called an **isomorphism**. We denote the linear space of morphisms from V to W by $\mathrm{Hom}_Q(V, W)$. For two morphisms $f : V \to W$ and $g : W \to W_1$ one can define the composition of morphisms $gf : V \to W_1$ as follows $(gf)_i = g_i f_i$. Obviously all finite dimensional representations of a quiver Q over a field k form a category denoted by $\mathrm{Rep}_k(Q)$, whose objects are finite dimensional representations V and morphisms are defined as above.

Examples 2.4.2.
1. Consider the quiver considered in example 2.4.1. Suppose we have two matrices $\mathbf{X}$ and $\mathbf{Y}$. When are the quiver representations $V_{\mathbf{X}}$ and $V_{\mathbf{Y}}$ isomorphic? According to the definition, we must have invertible linear maps $f_1 : k^n \to k^n$ and $f_2 : k^m \to k^m$ such that $f_2\mathbf{X} = \mathbf{Y}f_1$ or equivalently $f_2\mathbf{X}f_1^{-1} = \mathbf{Y}$. In other words, $V_{\mathbf{X}}$ and $V_{\mathbf{Y}}$ are isomorphic if and only if $\mathbf{Y}$ can be obtained from $\mathbf{X}$ by changing the basis in k^n and changing the basis in k^m.

This is a well known matter and $\mathbf{X}, \mathbf{Y} \in M_{n\times m}(k)$ are equivalent under this equivalence relation if and only if they have the same rank.

2. Consider the one loop quiver

$$\begin{tikzcd} \bullet \arrow[loop above, "\alpha"] \end{tikzcd}$$

For every square matrix $\mathbf{X} \in M_n(k)$ we can define a quiver representation $V_{\mathbf{X}}$ with $V_{\mathbf{X}}(1) = k^n$ and $V_{\mathbf{X}}(\alpha) = \mathbf{X}$. If $\mathbf{X}$ and $\mathbf{Y}$ are two $n \times n$ matrices, then $V_{\mathbf{X}}$ and $V_{\mathbf{Y}}$ are isomorphic if and only if there exists a linear map $f_1 : k^n \to k^n$ such that $f_1\mathbf{X} = \mathbf{Y}f_1$, or equivalently $f_1\mathbf{X}f_1^{-1} = \mathbf{Y}$. In other words, $V_{\mathbf{X}}$ and $V_{\mathbf{Y}}$ are isomorphic if and only if the matrices $\mathbf{X}$ and $\mathbf{Y}$ are conjugate (similar). In this case the category $\mathrm{Rep}_k(Q)$ can be identified with the category of endomorphisms of k-vector spaces.

Definition. Let V and W be representations of Q. A representation W is called a **subrepresentation** of V if

1. W_i is a subspace of V_i, for every $i \in VQ$.
2. The restriction of $V_\sigma : V_{s(\sigma)} \to V_{e(\sigma)}$ to $W_{e(\sigma)}$ is equal to $W_\sigma : W_{s(\sigma)} \to W_{e(\sigma)}$.

In an obvious way we can also introduce the notion of a quotient representation.

Note that every quiver has a representation T with $T_i = 0$ and $T_\sigma = 0$, for all $i \in VQ$ and all $\sigma \in AQ$. This representation is called the **zero representation**. The zero representation and a given representation V itself are called the **trivial subrepresentations** of V.

Definition. A nonzero representation V is called **irreducible** or **simple** if it has only trivial subrepresentations.

Example 2.4.3.

Let Q be the quiver:

$$\overset{1}{\bullet} \xrightarrow{\ \sigma\ } \overset{2}{\bullet}$$

We have three irreducible representations: E_1: $k \longrightarrow 0$, E_2: $0 \longrightarrow k$ and I: $k \xrightarrow{id} k$, which correspond to the indecomposable modules: $(k, 0)$, $(0, k)$ and (k, k). (Here $E_i(j) = k$ if $i = j$ and $E_i(j) = 0$ otherwise, $E_i(\sigma) = 0$.)

Definition. If V and W are representations of a quiver Q, the **direct sum representation** $V \oplus W$ is defined by

$$(V \oplus W)_i = V_i \oplus W_i$$

for each $i \in VQ$ and

$$(V \oplus W)_\sigma = (V_\sigma \oplus W_\sigma),$$

in other words, $(V \oplus W)_\sigma : V_{s(\sigma)} \oplus W_{s(\sigma)} \to V_{e(\sigma)} \oplus W_{e(\sigma)}$ can be represented by the matrix:

$$\left(\begin{array}{c|c} V_\sigma & \mathbf{O} \\ \hline \mathbf{O} & W_\sigma \end{array}\right)$$

A representation V of Q is called **decomposable** if it is isomorphic to a direct sum of non-trivial representations. Otherwise it is called **indecomposable**.

So the category $\mathrm{Rep}_k(Q)$ contains the zero element, direct sums of elements, subrepresentations and quotient representations. Thus, $\mathrm{Rep}_k(Q)$ is an additive category.

Example 2.4.4.
Let Q be the quiver:

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & \overset{2}{\bullet} \end{tikzcd}$$

Then a representation is given by two (not necessarily square) matrices $\mathbf{M}, \mathbf{N}$ and two representations $(\mathbf{M}, \mathbf{N}), (\mathbf{M}', \mathbf{N}')$ are isomorphic if and only if there exist invertible matrices $\mathbf{S}$ and $\mathbf{T}$ such that $\mathbf{SM} = \mathbf{M}'\mathbf{T}$, $\mathbf{SN} = \mathbf{N}'\mathbf{T}$. Thus the theory of the representations of this quiver is the theory of Kronecker pencils of matrices.

Theorem 2.4.1. *If Q is a finite quiver, then the category kQ-mod of right kQ-modules is equivalent to the category $\mathrm{Rep}_k(Q)$ of representations of Q.*

Proof. Let V be a representation of Q. Then we can make $\overline{V} = \bigoplus_{i \in VQ} V_i$ into a kQ-module as follows. For every $i \in VQ$, ε_i acts as the projection onto V_i. For every $\sigma \in AQ$, $\sigma|_{V_i} = 0$ if $i \neq e(\sigma)$ and $\sigma|_{V_{e(\sigma)}} = V_\sigma$. Since kQ is generated by all arrows and the ε_i for all $i \in VQ$, the action for arbitrary path can only be defined in one way. It is obvious that this indeed defines a kQ-module.

On the other hand, if $\overline{V}$ is a kQ-module, then we can define a representation V of Q by $V_i = \varepsilon_i \overline{V}$ for all $i \in VQ$, and since σ maps $\varepsilon_{e(\sigma)}\overline{V}$ into $\varepsilon_{s(\sigma)}\overline{V}$, we define V_σ as the restriction $\sigma : \varepsilon_{e(\sigma)}\overline{V} \to \varepsilon_{s(\sigma)}\overline{V}$.

Corollary 2.4.2 (Krull-Remak-Schmidt theorem). *Every finite dimensional representation V of a quiver Q is isomorphic to a direct sum of indecomposable representations. This decomposition is unique up to isomorphism and order of summands. More precisely, if*

$$V \simeq V_1 \oplus V_2 \oplus \ldots \oplus V_m \simeq W_1 \oplus W_2 \oplus \ldots \oplus W_n$$

with V_i and W_j nonzero and indecomposable, then $n = m$ and there exists a permutation σ such that for each i, $1 \leq i \leq n$ we have $V_i \simeq W_{\sigma(i)}$.

Example 2.4.5.
Let Q be the quiver:

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, "\sigma"] & \overset{2}{\bullet} \end{tikzcd}$$

Then we have $kQ = k\varepsilon_1 \oplus k\varepsilon_2 \oplus k\sigma$ and we have relations: $\sigma\varepsilon_1 = \varepsilon_2\sigma = \sigma$, $\varepsilon_1^2 = \varepsilon_1$, $\varepsilon_2^2 = \varepsilon_2$, $\varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1 = 0$, $\varepsilon_1\sigma = \sigma\varepsilon_2 = 0$.

Let V be a finite dimensional representation of a quiver Q with $VQ = \{1, 2, \ldots, n\}$. In this case we can define the **dimension vector** of a representation V of Q as the vector $\underline{\dim}\, V = (\dim\, V_i)_{i \in VQ} \in \mathbf{N}^n$. Since $V_i = V\varepsilon_i = \mathrm{Hom}\,(V, \varepsilon_i A)$, we have $\dim\, V_i = \dim\, \mathrm{Hom}\,(V, \varepsilon_i A)$.

For a simply laced quiver Q (without loops and multiply arrows) we define a bilinear form by

$$< \alpha, \beta > = \sum_{i=1}^{n} \alpha_i \beta_i - \sum_{\sigma \in AQ} \alpha_{s(\sigma)} \beta_{e(\sigma)}. \tag{2.4.1}$$

And we define the **Tits quadratic form** by

$$q(\alpha) \; = < \alpha, \alpha > . \tag{2.4.2}$$

This is a quadratic form on $\mathbf{Z}^n$. We can also define the corresponding **symmetric bilinear form** by

$$(\alpha, \beta) \; = < \alpha, \beta > + < \beta, \alpha > .$$

Lemma 2.4.3. *Let V, U be finite dimensional representations of a simply laced quiver Q, $A = kQ$, and let q be the corresponding quadratic form of Q. Then*

$$\dim \operatorname{Hom}_A(V, U) - \dim \operatorname{Ext}^1_A(V, U) \; = < \underline{\dim}\, V, \underline{\dim}\, U > .$$

In particular,

$$\dim \operatorname{End}_A(V) - \dim \operatorname{Ext}^1_A(V, V) \; = \; q(\underline{\dim}\, V).$$

Proof. Note that if

$$0 \to V_1 \to V_2 \to \ldots \to V_s \to 0$$

is an exact sequence of vector spaces, then

$$\sum_{i=1}^{s} (-1)^i \dim V_i \; = \; 0. \tag{2.4.1}$$

Let V, U be two finite dimensional representations of a quiver Q. Consider the Ringel resolution (2.3.1) in the form:

$$0 \to P_2 \to P_1 \to V \to 0.$$

We apply to this exact sequence the functor $\operatorname{Hom}_A(*, U)$. Since P_2, P_1 are projective, we obtain the following exact sequence:

$$0 \to \operatorname{Hom}_A(V, U) \to \operatorname{Hom}_A(P_1, U) \to \operatorname{Hom}_A(P_2, U) \to \operatorname{Ext}^1_A(V, U) \to 0.$$

Now applying the equality (2.4.1) to this exact sequence, we obtain:

$$\dim \operatorname{Hom}(V, U) - \dim \operatorname{Ext}^1(V, U) \; = \; \dim \operatorname{Hom}(P_1, U) - \dim \operatorname{Hom}(P_2, U).$$

Since $P_1 = \underset{i \in VQ}{\oplus} V_i \otimes \varepsilon_i A$ and $P_2 = \underset{\sigma \in AQ}{\oplus} V_{s(\sigma)} \otimes \varepsilon_{e(\sigma)} A$, we have, by proposition 4.3.4 and proposition 4.6.3, vol.I, $\mathrm{Hom}(P_1, U) = \underset{i \in VQ}{\oplus} \mathrm{Hom}(V_i \otimes \varepsilon_i A, U) = \underset{i \in VQ}{\oplus} \mathrm{Hom}(V_i, \mathrm{Hom}(\varepsilon_i A, U)) = \underset{i \in VQ}{\oplus} \mathrm{Hom}(V_i, U_i)$. Analogously, $\mathrm{Hom}(P_2, U) = \underset{\sigma \in AQ}{\oplus} \mathrm{Hom}(V_{s(\sigma)}, U_{e(\sigma)})$. Considering $\alpha = \underline{\dim}\, V$ and $\beta = \underline{\dim}\, U$, we obtain:

$$< \alpha, \beta > = \sum_{i=1}^{n} \alpha_i \beta_i - \sum_{\sigma \in AQ} \alpha_{s(\sigma)} \beta_{e(\sigma)} =$$

$$= \dim \big(\overset{n}{\underset{i=1}{\oplus}} \mathrm{Hom}(V_i, U_i)\big) - \dim \big(\underset{\sigma \in AQ}{\oplus} \mathrm{Hom}(V_{s(\sigma)}, U_{e(\sigma)})\big) =$$

$$= \dim \mathrm{Hom}(V, U) - \dim \mathrm{Ext}^1(V, U).$$

So it turns out that the difference $\dim \mathrm{Hom}(V, U) - \dim \mathrm{Ext}^1(V, U)$ depends only on the entries of the dimension vectors of V and U.

Definition. A finite quiver Q is said to be of **finite representation type** (or **finite type**, in short) if, up to an isomorphism, there are only a finite number of indecomposable representations of Q. Otherwise it is called of **infinite type**. A quiver Q is said to be of **tame representation type** if there are infinitely many isomorphism classes of indecomposable representations but these can be parametrized by a finite set of integers together with a polynomial irreducible over k; the quiver Q is said to be of **wild representation type** if for every finite dimensional algebra E over k there are infinitely many pair-wise non-isomorphic representations of Q which have E as their endomorphism algebra. These three classes of quivers are clearly exclusive. They are, as it turns out, also exhaustive.

The main result in the theory of quiver representations is the famous Gabriel theorem classifying the quivers of finite representation type. It turns out that such quivers are closely connected with Dynkin diagrams. The next two sections are devoted to a proof of the Gabriel theorem.

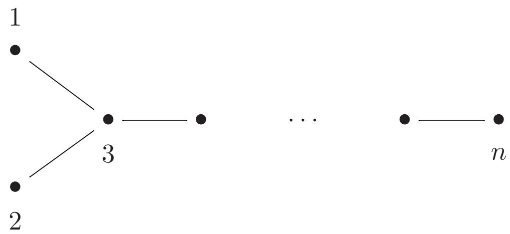
2.5 DYNKIN AND EUCLIDEAN DIAGRAMS. QUADRATIC FORMS AND ROOTS

In this chapter we consider some of the main properties of the special classes of graphs which are called Dynkin and Euclidean diagrams (= extended Dynkin diagrams). They appear in many different parts in mathematics. These diagrams plays an important role not only in the representation theory of algebras and quivers but also in classifications of simple Lie algebras, of finite crystallographic root systems, Coxeter groups, Cohen-Macaulay modules over certain commutative rings, and others.

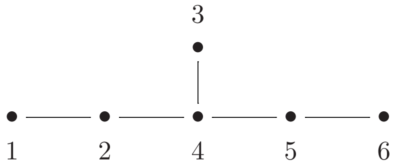
The graphs in the following list are the (simply laced) **Dynkin diagrams** which are also called **simple Dynkin diagrams**:

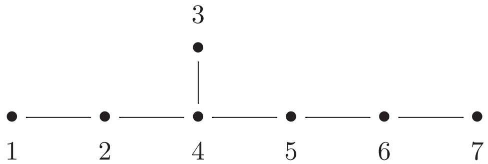
A_n :

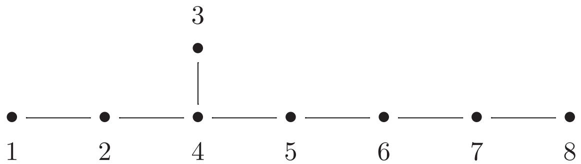

$n \geq 1$

D_n :


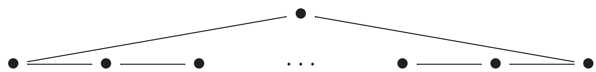
$n \geq 4$

E_6 :


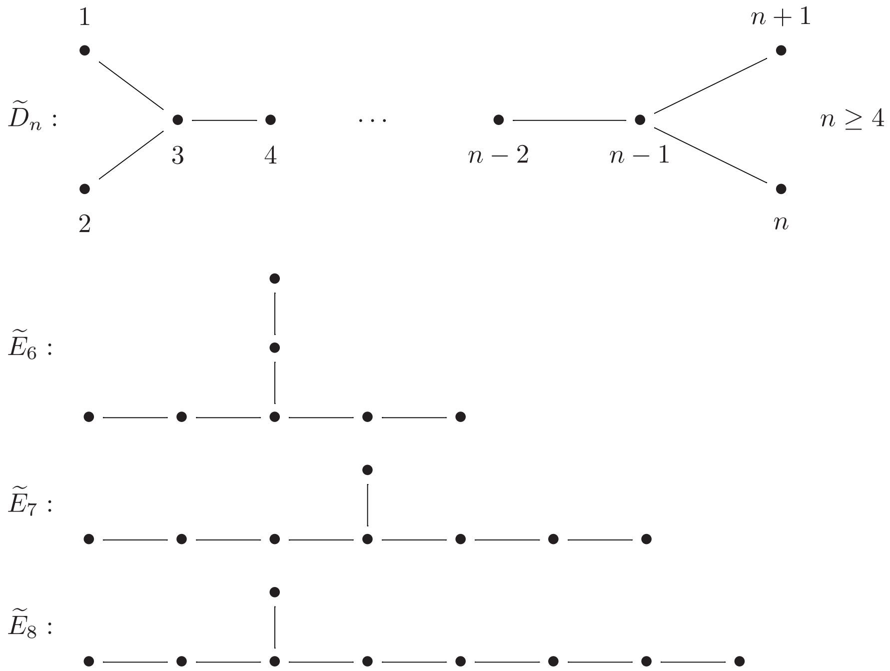
E_7 :


E_8 :


And the following list contains the simple **Euclidean diagrams** which are simply laced graphs. These diagrams are also called the **extended simple Dynkin diagrams**. The corresponding simple Dynkin diagram is obtained from the extended Dynkin diagram by dropping the added vertex and associated edges.

$\widetilde{A}_n$:


$n \geq 1$



Note that the diagram $\widetilde{A}_0$ has one vertex and one loop, and $\widetilde{A}_1$ has two vertices joined by two edges.

Let Γ be a finite connected graph without loops with a set of vertices $\Gamma_0 = \{1, 2, \ldots, n\}$ and a set of natural numbers $\{t_{ij}\}$, where $t_{ij} = t_{ji}$ is a number of edges between the vertex i and the vertex j. Note that Γ may contain multiple edges.

Define the quadratic form $q : \mathbf{Z}^n \to \mathbf{Z}$ by

$$q(\alpha) = \sum_{i=1}^{n} \alpha_i^2 - \sum t_{ij}\alpha_i\alpha_j \tag{2.5.1}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbf{Z}^n$ and the last summation is over all edges. It is easy to see that in the case of a simply laced graph this quadratic form coincides with the Tits form (2.4.2).

Let $(\cdot, \cdot)$ be the symmetric bilinear form on $\mathbf{Z}^n$ defined by

$$(\varepsilon_i, \varepsilon_j) = \begin{cases} -t_{ij} & \text{for } i \neq j, \\ 2 - 2t_{ii} & \text{for } i = j, \end{cases}$$

where ε_i is the i-th coordinate vector.

It is easy to see that $q(\alpha) = \dfrac{1}{2}(\alpha, \alpha)$, and $(\alpha, \beta) = q(\alpha + \beta) - q(\alpha) - q(\beta)$.

If Q is a quiver and Γ is its underlying graph, then $(\cdot,\cdot)$ and q are the same as before. However, the Euler bilinear form $< \cdot,\cdot >$ depends on the orientation of Q. We say that a quadratic form q is **positive definite** if $q(\alpha) > 0$ for all

$0 \neq \alpha \in \mathbf{Z}^n$.

We say that a quadratic form q is **positive semi-definite** (or **nonnegative definite**) if $q(\alpha) \geq 0$ for all $0 \neq \alpha \in \mathbf{Z}^n$.

Examples 2.5.1.

1. Let Γ be the graph: $\bullet \!-\!\!\!-\!\!\!-\!\!\!-\! \bullet$

Then we have

$$q(\alpha) = q(\alpha_1,\alpha_2) = \alpha_1^2 + \alpha_2^2 - \alpha_1\alpha_2 = (\alpha_1 - \frac{1}{2}\alpha_2)^2 + \frac{3}{4}\alpha_2^2 > 0$$

for all $\alpha \neq 0$. Hence q is positive definite.

2. Let Γ be the graph: $\bullet = \bullet$

Then we have

$$q(\alpha) = q(\alpha_1,\alpha_2) = \alpha_1^2 + \alpha_2^2 - 2\alpha_1\alpha_2 = (\alpha_1 - \alpha_2)^2 \geq 0$$

for all α. Hence q is positive semi-definite, but it is not positive definite, since $q(\alpha_1,\alpha_2) = 0$ if $\alpha_1 = \alpha_2$.

3. Let Γ be the graph: $\bullet \equiv \bullet$

Then we have

$$q(\alpha) = q(\alpha_1,\alpha_2) = \alpha_1^2 + \alpha_2^2 - 3\alpha_1\alpha_2 = (\alpha_1 - \alpha_2)^2 - \alpha_2^2$$

Since $q(1,1) = -1$ and $q(1,-1) = 1$, q is not positive semi-definite (nor negative semi-definite).

The set $\mathrm{rad}(q) = \{\alpha \in \mathbf{Z}^n \ : \ (\alpha,\beta) = 0 \ \text{ for all } \ \beta \in \mathbf{Z}^n\}$ is called the **radical** of q. It is easy to see that $\alpha \in \mathrm{rad}(q)$ if and only if

$$(2 - t_{ii})\alpha_i = \sum_{j\neq i} t_{ij}\alpha_j$$

for all i. We say that $\alpha \in \mathbf{Z}^n$ is **strict** if none of its components is zero.

Let $\leq$ be the partial ordering on $\mathbf{Z}^n$ defined by $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for each $i = 1,2,\ldots,n$. Correspondingly we write that $\beta \geq \alpha$ if $\beta_i - \alpha_i \geq 0$ for each i.

Lemma 2.5.1. *Suppose Γ is a connected graph and let $\beta \in \mathbf{Z}^n$ be a nonzero vector such that $\beta \geq 0$ and $\beta \in \mathrm{rad}(q)$. Then β is strict and the form q is positive semi-definite. For any $\alpha \in \mathbf{Z}^n$ the following conditions are equivalent:*

1) $q(\alpha) = 0$;

2) $\alpha \in \mathbf{Q}\,\beta$;

3) $\alpha \in \mathrm{rad}(q)$.

Proof. Since $\beta \in \mathrm{rad}(q)$, we have

$$0 = (\varepsilon_i, \beta) = (2 - 2t_{ii})\beta_i - \sum_{j \neq i} t_{ij}\beta_j.$$

Suppose there exists an i with $\beta_i = 0$, then $\sum_{j \neq i} t_{ij}\beta_j = 0$, and since each term in this sum is ≥ 0, we have $\beta_j = 0$ whenever there is an edge connected the vertex i with the vertex j. Since Γ is connected, it follows that $\beta = 0$, a contradiction. Thus, β is strict. Now,

$$\sum_{i<j} t_{ij}\frac{\beta_i\beta_j}{2}\Big(\frac{\alpha_i}{\beta_i} - \frac{\alpha_j}{\beta_j}\Big)^2 = \sum_{i<j} t_{ij}\frac{\beta_j}{2\beta_i}\alpha_i^2 - \sum_{i<j} t_{ij}(\alpha_i\alpha_j) + \sum_{i<j} t_{ij}\frac{\beta_i}{2\beta_j}\alpha_j^2 =$$

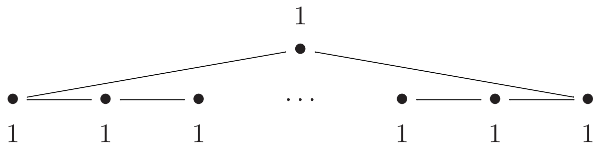
$$= \sum_{i \neq j} t_{ij}\frac{\beta_j}{2\beta_i}\alpha_i^2 - \sum_{i<j} t_{ij}(\alpha_i\alpha_j) =$$

$$= \sum_i (2 - 2t_{ii})\beta_i\frac{1}{2\beta_i}\alpha_i^2 - \sum_{i<j} t_{ij}(\alpha_i\alpha_j) = \sum_i \alpha_i^2 - \sum_{i \leq j} t_{ij}(\alpha_i\alpha_j) = q(\alpha).$$

Hence q is positive semi-definite. If $q(\alpha) = 0$, then $\frac{\alpha_i}{\beta_i} = \frac{\alpha_j}{\beta_j}$ whenever there is an edge connected the vertex i with the vertex j. Since Γ is connected, it follows that $\alpha \in \mathbf{Q}\,\beta$. But that implies that $\alpha \in \mathrm{rad}(q)$, since $\beta \in \mathrm{rad}(q)$, by assumption. Finally, if $\alpha \in \mathrm{rad}(q)$, then $q(\alpha) = 0$. This completes the proof.

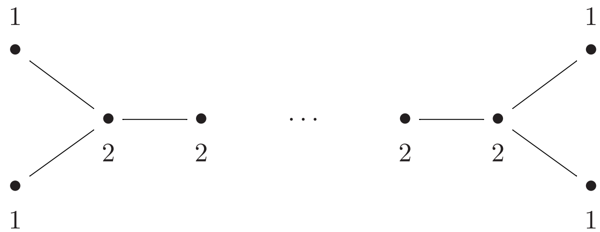
For all extended Dynkin diagrams we can give the corresponding graphs where each vertex i is marked by the value δ_i, where $\delta = (\delta_1, \dots, \delta_n) > 0$ is strict. Here δ is the unique vector such that $\mathrm{rad}Q = \mathbf{Z}\delta$, see lemma 2.5.1.

$\widetilde{A}_n$:



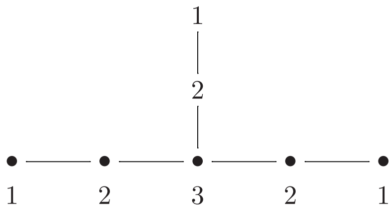
$n \geq 1$

$\widetilde{D}_n$:

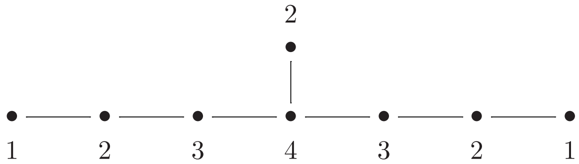


$n \geq 4$

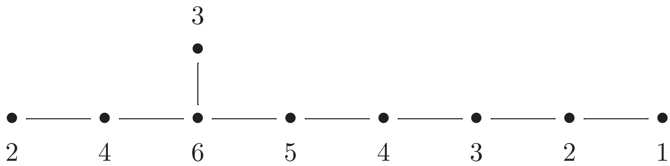
$\widetilde{E}_6$:



$\widetilde{E}_7$:



$\widetilde{E}_8$:



The following main theorem gives the classifications of all graphs from the point of view of their quadratic forms.

Theorem 2.5.2. *Suppose* Γ *is a connected simply laced graph (i.e. a graph without loops and multiply edges) and* q *is its quadratic form.*

(1) q *is positive definite if and only if* Γ *is one of the simple Dynkin diagrams* A_n, D_n, E_6, E_7 *or* E_8.

(2) q *is positive semi-definite if and only if* Γ *is a one of the extended Dynkin diagram* $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ *or* $\tilde{E}_8$*; moreover,* $\mathrm{rad}(q) = \mathbf{Z}\delta$, *where* δ *is the vector indicated by the numbers in the graphs above.*

(3) *Otherwise, there is a vector* $\alpha \geq 0$ *with* $q(\alpha) < 0$ *and* $(\alpha, \varepsilon_i) \leq 0$ *for all* i.

Proof. The full proof of this theorem can be found in [Bourbaki, 1968]. We give here only the proof of the necessary part of this theorem.

(2) Suppose that Γ is a simple Euclidean diagram. First we need to verify that the vector $\delta = \sum\limits_{i=1}^{n} \delta_i \varepsilon_i$ belongs to $\mathrm{rad}(q)$. We have $(\delta, \varepsilon_i) = \sum\limits_{i \neq j} \delta_j t_{ij} + 2\delta_i - 2\delta_i t_{ii} = 0$ for each i. Then q is positive semi-definite and $\mathrm{rad}(q) = \mathbf{Q}\,\delta \cap \mathbf{Z}^n$, by lemma 2.5.1. Since one of the δ_j's is 1, we have $\mathrm{rad}(q) = \mathbf{Q}\,\delta \cap \mathbf{Z}^n = \mathbf{Z}\delta$.

(1) Suppose Γ is a simple Dynkin diagram. We can add one more vertex to this graph to obtain a simple Euclidean diagram $\tilde{\Gamma}$. Note that $q|_{\Gamma}$ is the restriction to $\mathbf{Z}^n$ of the quadratic form $q|_{\tilde{\Gamma}}$. By lemma 2.5.1, if $q(\alpha) = 0$ for a nonzero $\alpha \in \mathbf{Z}^m$, then $\alpha \in \mathbf{Z}\delta$, hence α is strict. So q is strictly positive on all the α's in $\mathbf{Z}^n$ coming from Γ, so q is positive definite on Γ.

(3) Suppose that Γ is neither a simple Euclidean diagram nor a simple Dynkin diagram. Then Γ has a subgraph Γ' which is Euclidean, with a radical vector δ. If all vertices of Γ are in Γ', we can take $\alpha = \delta$. If there is a vertex i of Γ which is not in Γ' and which is connected with Γ' by any edge, we can take $\alpha = 2\delta + \varepsilon_i$. It is now easy to check that $q(\alpha) < 0$ and $(\alpha, \varepsilon_i) \leq 0$ for all i.

Example 2.5.2.
Let Q be a quiver with underlying Dynkin diagram A_n. For all $r, s \in \mathbf{N}$ with $1 \leqslant r < s \leqslant n$ let $V_{r,s}(i) = k$ for $r \leqslant i \leqslant s$ and $V_{r,s}(j) = 0$ for $j < r$ or $j > s$. For $\sigma \in AQ$ we set $V_{r,s}(\sigma) = I$ if σ joints two points in $\{i \ : \ r \leqslant i \leqslant s\}$ and $V_{r,s}(\sigma) = 0$ otherwise. Then $V_{r,s}$ is an indecomposable representation of Q and all indecomposable representations of Q are isomorphic to one of these (see [Gabriel, 1972], [Dlab, Ringel, 1976]).

Example 2.5.3.
Consider the quiver Q_5 :

$$\begin{tikzcd}
\overset{1}{\bullet} \arrow[drr] & \overset{2}{\bullet} \arrow[dr] & \overset{3}{\bullet} \arrow[d] & \overset{4}{\bullet} \arrow[dl] & \overset{5}{\bullet} \arrow[dll] \\
& & \underset{0}{\bullet} & &
\end{tikzcd}$$

with the vertices numbered as indicated. This quiver is wild. We show that any finite dimensional k-algebra A is the algebra of all endomorphisms of some finite dimensional representation V of the quiver Q_5. First consider the quiver Q_4, which is obtained from Q_5, by removing the vertex 5 and the arrow incident with it. We now construct a representation U of Q_4 with $\dim(U) = 2n+1, n = 1, 2, \ldots$ such that the endomorphism algebra of U is k. Let E be an $n+1$-dimensional k-vector space with a basis $e_1, \ldots, e_{n+1}$ and let F be an n-dimensional vector space with a basis $f_1, \ldots, f_n$. We set $U(0) = E \oplus F$, $U(1) = E \oplus 0$, $U(2) = 0 \oplus F$, $U(3) = \{(\lambda(f), f) \,:\, f \in F\}$, $U(4) = \{(\delta(f), f) \,:\, f \in F\}$, where $\lambda, \delta : F \to E$ are defined by $\lambda(f_i) = e_i$, $\delta(f_i) = e_{i+1}$. The maps associated to the arrows are the natural inclusions. Then an endomorphism of U is given by an endomorphism α of $U(0) = E \oplus F$, which preserves the subspaces $U(1), \ldots, U(4)$. One can easily check that this means that α is a multiplication by an element of k, i.e., one finds $\mathrm{End}(U) = k$. Now let A be any finite dimensional algebra over k and let $a_1, \ldots, a_m$ be a set of generators of A (as a k-module). Let $a_0 = 1$ and assume that m is even, $m \geqslant 2$. Let U be the representation of Q_4 constructed above with $\dim(U) = m+1$. We now define a representation V of Q_5 by $V(0) = A \otimes_k U(0)$, $V(i) = A \otimes_k U(i)$ for $i = 1, 2, 3, 4$, $V(5) = \{\sum\limits_{i=0}^{m} aa_i \otimes e_i \ : \ a \in A\}$, where $e_0, \ldots, e_m$ is a basis of $U(0)$. An endomorphism of V is an endomorphism of $V(0)$ which preserves the five subspaces $V(j)$ for $j = 1, \ldots, 5$. Since $\mathrm{End}(U) = k$, an endomorphism of $V(0)$ which preserves $V(1)$,…,$V(4)$ is necessarily of the form $\phi \otimes 1$ where ϕ is a

k-vector space endomorphism of A. Now $(\phi\otimes 1)(\sum_{i=0}^{m} aa_i\otimes e_i) = \sum_{i=0}^{m}\phi(aa_i)\otimes e_i$. Therefore if $\phi\otimes 1$ also preserves $V(5)$, then for all $a\in A$ there exists $b(a)$ such that $\sum_{i=0}^{m}\phi(aa_i)\otimes e_i = \sum_{i=0}^{m} b(a)a_i$. Now $1\otimes e_0,\dots,1\otimes e_m$ is a basis for $A\otimes_k U(0)$ as a module over A, hence $\phi(aa_i)=b(a)a_i$ for all i. Taking $i=0$ we find $\phi(a)=b(a)$. Hence we have that $\phi(aa_i)=\phi(a)a_i$ for all $a\in A$ and all i .

Let $c=\phi(1)$, then $\phi(a_i)=ca_i$ for all i. So ϕ is given by multiplication by $c\in A$. This shows that $\mathrm{End}(V)=A$.

Definition. Suppose Γ is either a simple Euclidean or a simple Dynkin diagram. The following set

$$\Delta=\{0\neq\alpha\in\mathbf{Z}^n\ :\ q(\alpha)\leq 1\}$$

is called the **set of roots** of the quadratic form q of Γ. A root α is called **real** if $q(\alpha)=1$ and it is called **imaginary** if $q(\alpha)=0$.

Note that each ε_i is a root. These roots are called the **simple roots**. If Γ is a simple Dynkin diagram, then there are no imaginary roots for this graph. If Γ is a Euclidean diagram, then its imaginary roots are the integer vectors which are multiples of δ, by lemma 2.5.1.

Lemma 2.5.3.

1. *If $\alpha\in\Delta$ and $\beta\in\mathrm{rad}(q)$, then $-\alpha$ and $\alpha+\beta$ are roots.*
2. *Every root is positive or negative.*
3. *If Γ is a simple Euclidean diagram, then $(\Delta\cup\{0\})/\mathbf{Z}\delta$ is a finite subset of $\mathbf{Z}^m/\mathbf{Z}\delta$.*
4. *If Γ is a simple Dynkin diagram, then Δ is finite.*

Proof.

1. We have $q(\beta\pm\alpha)=q(\beta)+q(\alpha)\pm(\beta,\alpha)=q(\alpha)\leq 1$.

2. Write $\alpha=\alpha^+-\alpha^-$, where $\alpha^+,\alpha^-\geq 0$ and have disjoint support. Then obviously, $(\alpha^+,\alpha^-)\leq 0$, so

$$1\geq q(\alpha)=q(\alpha^+)+q(\alpha^-)-(\alpha^+,\alpha^-)\geq q(\alpha^+)+q(\alpha^-).$$

Since q is positive semi-definite, one of $q(\alpha^+)$ or $q(\alpha^-)$ must be zero, i.e., one of them is an imaginary root. Since each imaginary root is strict, the other of these roots must be zero.

3. Choose an i with $\delta_i=1$. If α is a root with $\alpha_i=0$, then $\delta-\alpha$ and $\delta+\alpha$ are also roots by the condition (1). Since their i-th component is 1, they are positive roots, so $-\delta\leq\alpha\leq\delta$. Therefore

$$S=\{\alpha\in\Delta\ :\ \alpha_i=0\}\subseteq\{\alpha\in\mathbf{Z}^n\ :\ -\delta\leq\alpha\leq\delta\}$$

which is finite. Now if $\beta\in\Delta\cup\{0\}$, then $\beta-\beta_i\delta$ belongs to this finite set S.

4. Embed Γ into the corresponding Euclidean diagram $\tilde{\Gamma}$ by adding a vertex i with $\delta_i = 1$. We can now view a root α of Γ as a root of $\tilde{\Gamma}$ with $\alpha_i = 0$. So the result follows from (3).

In the general case when Q is not necessarily a simply laced quiver the right corresponding quadratic form associated to the quiver is defined differently.

Let Γ be a finite graph without orientation with a set of vertices $\Gamma_0 = \{1, 2, \ldots, n\}$ and a finite set of edges Γ_1. We say that (Γ, d) is a **valued graph** if Γ_1 is provided with non-negative integers d_{ij} for all i, j which are the endpoints of an edge. These integers define a function $d : \Gamma_0 \times \Gamma_0 \to \mathbf{N}$, where we write $d(i, j) = d_{ij}$, and are required to satisfy the following properties:

1) $d_{ii} = 0$ for all $i \in \Gamma_0$;
2) $d_{ij} = 0$ if and only if $d_{ji} = 0$;
3) there exists natural numbers $f_i \in \mathbf{N}$ such that $d_{ij} f_j = d_{ji} f_i$[4].

If $d_{ij} \neq 0$ then this is depicted

$$\underset{i}{\bullet}\overset{(d_{ij},d_{ji})}{\text{———}}\underset{j}{\bullet}$$

If $d_{ij} = d_{ji} = 1$ we simply write

$$\underset{i}{\bullet}\text{———}\underset{j}{\bullet}$$

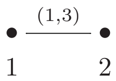
All **Dynkin diagrams** (in addition to the simple Dynkin diagrams drawn above) are represented in the form of valued graphs by the following list:

$$B_n: \quad \underset{1}{\bullet}\overset{(1,2)}{\text{———}}\underset{2}{\bullet}\text{———}\underset{3}{\bullet} \quad \cdots \quad \underset{n-2}{\bullet}\text{————}\underset{n-1}{\bullet}\text{————}\underset{n}{\bullet} \qquad n \geq 2$$

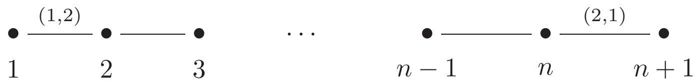
$$C_n: \quad \underset{1}{\bullet}\overset{(2,1)}{\text{———}}\underset{2}{\bullet}\text{———}\underset{3}{\bullet} \quad \cdots \quad \underset{n-2}{\bullet}\text{————}\underset{n-1}{\bullet}\text{————}\underset{n}{\bullet} \qquad n \geq 3$$

$$F_4: \quad \underset{1}{\bullet}\text{———}\underset{2}{\bullet}\overset{(1,2)}{\text{———}}\underset{3}{\bullet}\text{———}\underset{4}{\bullet}$$

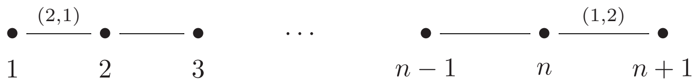
[4]Associate to the valued graph (Γ, d) the generalized Cartan matrix $A = (a_{ij})$, $a_{ii} = 2$, $a_{ij} = -d_{ij}$ if i and j are connected by an edge (and are different) and $a_{ij} = 0$ if there is no edge between i and j. Then condition 3 is equivalent to symmetrizability (on the right) of the GCM A meaning that there is a diagonal integer matrix F such that AF is symmetric. As is easily proved this is equivalent to symmetrizability on the left, meaning that there is a diagonal integer matrix F' such that $F'A$ is symmetric.

G_2 :


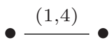
And the valued graphs in the following list are all the **Euclidean diagrams** (or extended Dynkin diagrams)[5], in addition to the simple Euclidean diagrams which were presented above:

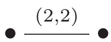
$\widetilde{B}_n$:


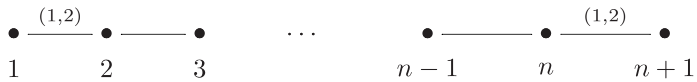
$n \geq 2$

$\widetilde{C}_n$:


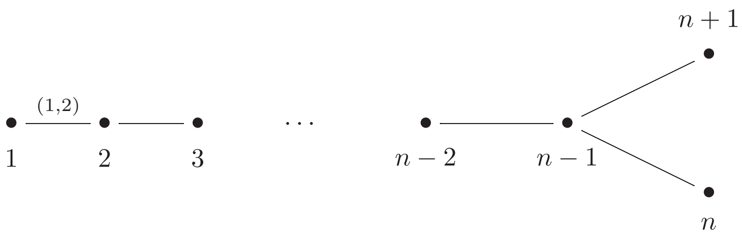
$n \geq 3$

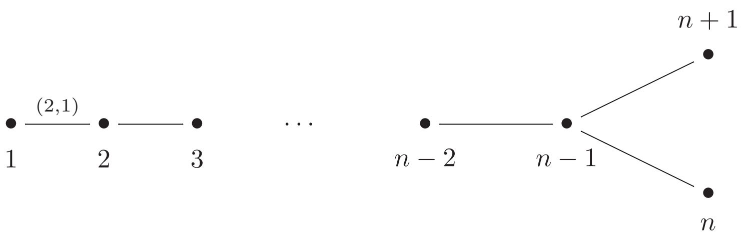
$\widetilde{A}_{11}$:


$\widetilde{A}_{12}$:


$\widetilde{BC}_n$:


$n \geq 2$

$\widetilde{BD}_n$:


$\widetilde{CD}_n$:


[5]It is worth noting that the extra vertex in an extended Dynkin diagram is so placed that omitting any vertex (and the edges involving this vertex) leaves as residue a disjoint union of Dynkin diagrams; see e.g. [Wan, 1991] page 35.

$\widetilde{F}_{41}$: 

$\widetilde{F}_{42}$: 

$\widetilde{G}_{21}$: 

$\widetilde{G}_{22}$: 

These are the notations from [Dlab-Ringel, 1976]. In books on Kac-Moody algebras and quantum groups such as [Kac, 1985], [Wan, 1991], [Hong-Kang, 2002] a different notation is used; e.g. $\widetilde{A}_l = A_l^{(1)}$, $\widetilde{BC}_{2l} = A_{2l}^{(2)}$, $\widetilde{CD}_{2l-1} = A_{2l-1}^{(2)}$

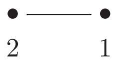
For all these graphs here are the corresponding graphs where each vertex i is marked by the value δ_i, where the strict vector $\delta = (\delta_1, \dots, \delta_n) > 0$ generates the radical.

$\widetilde{B}_n$: 

$n \geq 2$

$\widetilde{C}_n$: 

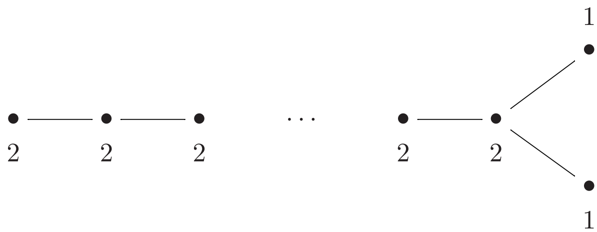
$n \geq 3$

$\widetilde{A}_{11}$: 

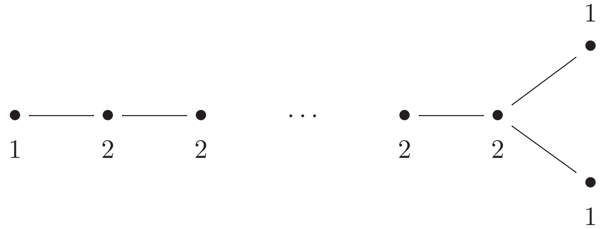
$\widetilde{A}_{12}$: 

$\widetilde{BC}_n$: 

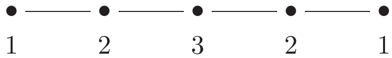
$n \geq 2$

$\widetilde{B}_n$: 

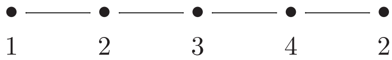
$\widetilde{CD}_n$:



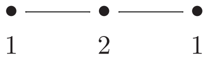
$\widetilde{F}_{41}$:



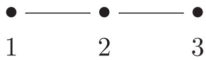
$\widetilde{F}_{42}$:



$\widetilde{G}_{21}$:



$\widetilde{G}_{22}$:



If (Γ, d) is a valued graph and $x, y \in \mathbf{Z}^n$ then the **Dlab-Ringel-Ovsienko-Roiter quadratic form** of (Γ, d) is defined in the following way[6][7]:

$$q(x) = \sum_{i \in \Gamma_0} f_i x_i^2 - \sum_{i<j} d_{ij} f_j x_i x_j \tag{2.5.2}$$

If Q is a quiver and $x, y \in \mathbf{Z}^n$ then the non-symmetric bilinear form of Q is defined as follows:

$$\tilde{q}(x, y) = \sum_{i \in Q_0} f_i x_i y_i - \sum_{i \to j} d_{ij} f_j x_i y_j \tag{2.5.3}$$

This bilinear form was independently introduced by C.M.Ringel (see [Ringel, 1976] and [Dlab-Ringel, 1976]) and S.A.Ovsienko and A.V.Roiter (see [Ovsienko, 1977a]).

In the case when Γ is a simply laced graph these forms coincide with the Tits quadratic form (2.4.2) and the bilinear form (2.4.1).

The notions of positive definiteness and nonnegative definiteness are of course as above.

[6]Here it is customary to take the f_i minimal, i.e. so that their greatest common divisor is 1.

[7]Note that this is the quadratic form associated to the symmetrized generalized Cartan matrix AF.

Examples 2.5.4.

1. Let (Γ, d) be the valued graph:

$$B_3: \quad \underset{1}{\bullet} \overset{(1,2)}{\text{———}} \underset{2}{\bullet} \text{———} \underset{3}{\bullet}$$

For this graph $d_{12} = 1$, $d_{21} = 2$, $d_{23} = d_{32} = 1$. Therefore from the requirement $d_{ij}f_j = d_{ji}f_i$ we obtain that $f_1 = 1$, $f_2 = f_3 = 2$. Then the quadratic form has the following form:

$$q(x) = x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 = (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2$$

and so it is positive definite.

2. Let (Γ, d) be the valued graph:

$$C_3: \quad \underset{1}{\bullet} \overset{(2,1)}{\text{———}} \underset{2}{\bullet} \text{———} \underset{3}{\bullet}$$

For this graph $d_{12} = 2$, $d_{21} = 1$, $d_{23} = d_{32} = 1$. Therefore from requirement $d_{ij}f_j = d_{ji}f_i$ we obtain that $f_1 = 2$, $f_2 = f_3 = 1$. Then the quadratic form has the following form:

$$q(x) = 2x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - x_2x_3 = 2(x_1 - \frac{1}{2}x_2)^2 + (\frac{1}{2}x_2 - x_3)^2 + \frac{1}{4}x_2^2$$

and so it is positive definite.

3. Let (Γ, d) be the valued graph: $\widetilde{A}_{11}: \quad \bullet \overset{(1,4)}{\text{———}} \bullet$

For this graph $d_{12} = 1$, $d_{21} = 4$. Therefore from the requirement $d_{ij}f_j = d_{ji}f_i$ we obtain that $f_1 = 4$, $f_2 = 1$. Then the quadratic form has the following form:

$$q(x) = 4x_1^2 + x_2^2 - 4x_1x_2 = (2x_1 - x_2)^2 \geq 0$$

and so it is nonnegative definite.

4. Let (Γ, d) be the valued graph: $\widetilde{A}_{12}: \quad \bullet \overset{(2,2)}{\text{———}} \bullet$

For this graph $d_{12} = d_{21} = 2$. Therefore from the requirement $d_{ij}f_j = d_{ji}f_i$ we obtain that $f_1 = f_2 = 1$. Then the quadratic form has the following form:

$$q(x) = x_1^2 + x_2^2 - 2x_1x_2 = (x_1 - x_2)^2 \geq 0$$

and so it is nonnegative definite.

The following theorem gives the characterization of all valued graphs from the point of view their quadratic forms:

Theorem 2.5.4. *Suppose Γ is a connected valued graph with quadratic form q. Then*

(1) Γ *is a Dynkin diagram if and only if q is positive definite.*
(2) Γ *is an extended Dynkin diagram if and only if q is positive semi-definite.*

The full proof of this theorem can be found in [Dlab, Ringel, 1976].

Remark 2.5.1. The non-symmetric bilinear form $\tilde{q}$ given by (2.5.3) has the following important property for the representations of a quiver Q (in the case of Dynkin diagrams):

$$\tilde{q}(\underline{\dim}\, V, \underline{\dim}\, U) = \dim \operatorname{Hom}_A(V, U) - \dim \operatorname{Ext}^1_A(V, U) \tag{2.5.4}$$

where V, U are indecomposable representations of Q, and $\underline{\dim}\, V$, $\underline{\dim}\, U$ means the vector dimensions.

Remark 2.5.2. A valued graph on the plane (which is a Dynkin diagram or a Euclidean diagram) is often drawn in some another way. Instead of

$$\underset{i}{\bullet} \overset{(d_{ij},d_{ji})}{\text{——————}} \underset{j}{\bullet}$$

we connect the vertex i with the vertex j by t_{ij} edges, where t_{ij} satisfies the following conditions. If $d_{ij}d_{ji} \leq 4$ then $t_{ij} = \max\{d_{ij}, d_{ji}\}$. In this case and when $d_{ij} < d_{ji}$, all these t_{ij} edges are marked together by one an arrow in the direction from j to i[8]. For example,

$$\underset{1}{\bullet} \overset{(3,1)}{\text{——————}} \underset{2}{\bullet} \quad \text{is drawn as} \quad \underset{1}{\bullet} \Rrightarrow \underset{2}{\bullet}$$

and

$$B_3: \quad \underset{1}{\bullet} \Longleftarrow \underset{2}{\bullet} \text{——————} \underset{3}{\bullet}$$

If $d_{ij} = d_{ji} = 2$, we draw this graph as follows: $\tilde{A}_{12}: \quad \bullet \Longleftrightarrow \bullet$

Finally edges labeled by $(1, 1)$ we draw by one edge: $\bullet \text{——————} \bullet$

Conversely, each such a graph we can transform into a valued graph in the following way. If we have a marked multiply arrow with t_{ij} edges in the direction from j to i then we put $d_{ij} = 1$ and $d_{ji} = t_{ij}$. Otherwise we put $d_{ij} = d_{ji} = t_{ij}$.

[8]This is the convention that makes things come out right when one starts with the lists from [Dlab-Ringel, 1976]. In [Wan, 1991], [Kac, 1985] and [Hong-Kang, 2002] and other books on Kac-Moody algebras and quantum groups another convention is used; viz. there is an arrow towards i iff $d_{ij} \geq 2$ Applied to the list above this switches the B and C diagrams.

Remark 2.5.3. It is often convenient to describe a finite valued graph (Γ, d), where $\Gamma_0 = \{1, 2, \ldots, n\}$ in the terms of an associated matrix $C = (c_{ij})$, called the **Cartan matrix**. By definition, $c_{ii} = 2$, and $c_{ij} = -d_{ij}$. For example, if (Γ, d) is

$$F_4 : \quad \begin{array}{ccccccc} \bullet & \text{———} & \bullet & \overset{(1,2)}{\text{———}} & \bullet & \text{———} & \bullet \\ 1 & & 2 & & 3 & & 4 \end{array}$$

then the Cartan matrix of F_4 is of the form

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

2.6 GABRIEL THEOREM

A basic result in the theory of quiver representations is the famous Gabriel theorem classifying the quivers Q of finite representation type. The proof of this theorem given here is due to J.Tits, P.Gabriel, C.M.Ringel and W.Crawley-Boevey (see [Ringel, 1982], [Crawley-Boevey]).

Theorem 2.6.1 (P.Gabriel).[9] *A connected quiver Q is of a finite type if and only if the underlying undirected graph $\overline{Q}$ of Q (obtained from Q by deleting the orientation of the arrows) is a Dynkin diagram of the form A_n, D_n, E_6, E_7, E_8. The number of isomorphism classes of indecomposable representations of Q is finite if and only if the corresponding quadratic form q_Q is positive definite. In this case the map*

$$V \mapsto \underline{\dim}\, V$$

sets up a bijection between the isomorphism classes of indecomposable representations V of Q and the set of positive roots of the underlying Dynkin diagram of V.

We shall prove this theorem for the case when k is an algebraically closed field. To prove this theorem we need a little bit of algebraic geometry.

Let $\mathbb{A}^N$ be affine N-space, whose points are just the N-tuples from the field k. The coordinate ring $k[\mathbb{A}^N]$ of functions on $\mathbb{A}^N$ is the polynomial ring $k[x_1, x_2, \ldots, x_n]$, where x_i is the i-th coordinate function. This space is endowed with the **Zariski topology**, whose set of closed subsets coincides with the sets of common zeros of the ideals in $k[x_1, x_2, \ldots, x_n]$. A closed set in $\mathbb{A}^N$ is called an **affine variety**. A subset U in $\mathbb{A}^N$ is called **locally closed** if it is open in its closure $\overline{U}$. A non-empty locally closed subset U is called **irreducible** if every non-empty open subset of U is dense in U. The affine space $\mathbb{A}^N$ itself is irreducible.

[9] See [Gabriel, 1972], [Berstein, Gel'fand, Ponomarev, 1973].

The **dimension** of a non-empty locally closed subset U is

$$\sup\{n : \text{there exist irreducible subsets} : Z_0 \subset Z_1 \subset \ldots \subset Z_n \text{ closed in } U\}.$$

We have the following simple properties of dimensions:

1) $\dim U = \dim \overline{U}$;
2) if $W = U \cup V$ then $\dim W = \max\{\dim U, \dim V\}$;
3) $\dim \mathbb{A}^N = N$.

An **algebraic group** is an affine variety G endowed with a multiplication $\nu : G \times G \to G$ and an inverse $i : G \to G$ which are morphisms of affine varieties and which satisfy the usual group axioms, see chapter 1.

Example 2.6.1.
The linear group $\mathrm{GL}_n(k)$ of all $n \times n$ invertible matrices over k is an algebraic group. This group is defined by the non-vanishing of the determinant and so it is open in $M_n(k) \simeq \mathbb{A}^{n^2}$ and is an irreducible affine variety of dimension n^2. To see this consider $\mathbb{A}^{n^2+1}$ with coordinates x_{ij}, $i, j = 1, \ldots, n$ and z and consider the zeros of the polynomial $z\det(x_{ij}) - 1$. These are in obvious bijection with $\mathrm{GL}_n(k)$.

An action of an algebraic group G on an affine variety X is defined in the usual sense. There are the following properties connected with this notion:

1) the orbits of G in X are irreducible and locally closed;
2) if $\mathcal{O}$ is an orbit, its boundary $\overline{\mathcal{O}} \setminus \mathcal{O}$ is a union of orbits of dimensions strictly smaller than $\dim \mathcal{O}$;
3) the stabilizer G_x of a point $x \in \mathcal{O}$ is a closed subgroup in G, and $\dim \mathcal{O} = \dim G - \dim G_x$.

Note that $\mathrm{Hom}_k(k^s, k^t)$ is just the space of all $t \times s$ matrices over k. So we can identify it with the affine variety $\mathbb{A}^{ts}$. Thus it is an irreducible affine variety of dimension ts.

Let Q be a quiver and $A = kQ$. Suppose $\alpha \in \mathbf{N}^n$ is a fixed dimension vector. Define

$$\mathrm{Rep}(\alpha) = \prod_{\sigma \in AQ} \mathrm{Hom}_k(k^{\alpha_{s(\sigma)}}, k^{\alpha_{e(\sigma)}}).$$

This is isomorphic to the irreducible affine variety $\mathbb{A}^r$ of dimension $r = \sum\limits_{\sigma \in AQ} \alpha_{s(\sigma)}\alpha_{e(\sigma)}$.

Given a point $x \in \mathrm{Rep}\,(\alpha)$, define a representation $R(x)$ of Q in the following way: $R(x)_i = k^{\alpha_i}$ for $i = 1, 2, \ldots, n$ and $R(x)_\sigma$ is a linear map with matrix x_σ for $\sigma \in AQ$.

Define $\mathrm{GL}(\alpha) = \prod\limits_{i=1}^{n} \mathrm{GL}(\alpha_i, k)$. This is an algebraic group of dimension $s = \sum\limits_{i=1}^{n} \alpha_i^2$ which is open in $\mathbb{A}^s$.

Define an action of $\mathrm{GL}(\alpha)$ on $\mathrm{Rep}\,(\alpha)$ by means of conjugation:

$$(gx)_\sigma = g_{e(\sigma)} x_\sigma g_{s(\sigma)}$$

for $g \in \mathrm{GL}(\alpha)$ and $x \in \mathrm{Rep}\,(\alpha)$.

Lemma 2.6.2.
1. *There is a bijection $V \mapsto \mathcal{O}_V$ between the set of isomorphism classes of indecomposable representations V of dimension α and the set of* $\mathrm{GL}(\alpha)$*-orbits on* $\mathrm{Rep}\,(\alpha)$*, where $\mathcal{O}_V = \{x \in \mathrm{Rep}\,(\alpha) \,:\, R(x) \simeq V\}$.*
2. $\mathrm{GL}(\alpha)_x \simeq \mathrm{Aut}_A(R(x))$.

Proof. Let V be a representation of dimension α. Then choosing a basis in each V_i allows us to identify each V_i with the vector space k^{α_i} and each V_σ with a map from $k^{\alpha_{s(\sigma)}}$ to $k^{\alpha_{e(\sigma)}}$. The action of $\mathrm{GL}(\alpha)$ on $\mathrm{Rep}(\alpha)$ corresponds to a different choice of bases.

Lemma 2.6.3. *Let V be a representation of dimension α. Then*

$$\dim \mathrm{Rep}\,(\alpha) - \dim \mathcal{O}_V = \dim \mathrm{End}_A(V) - q(\alpha) = \dim \mathrm{Ext}^1_A(V,V).$$

Proof. Let $V \in R(x)$, then

$$\dim \mathrm{GL}(\alpha) - \dim \mathcal{O}_V = \dim \mathrm{GL}(\alpha)_x.$$

By lemma 2.6.2, $\mathrm{GL}(\alpha)_x \simeq \mathrm{Aut}_A(V)$. Since $\mathrm{Aut}_A(V)$ is an open subset in $\mathrm{End}_A(V)$, it has the same dimension. Hence, by lemma 2.4.3, we obtain:

$$\dim \mathrm{Rep}\,(\alpha) - \dim \mathcal{O}_V = \sum_{\sigma \in AQ} \alpha_{s(\sigma)}\alpha_{e(\sigma)} - \sum_{i=1}^{n} \alpha_i^2 + \dim \mathrm{End}_A(V) =$$

$$= \dim \mathrm{End}_A(V) - q(\alpha) = \dim \mathrm{Ext}^1_A(V,V).$$

Corollary 2.6.4. *If $\alpha \neq 0$ and $q(\alpha) < 0$, then there are infinitely many orbits in* $\mathrm{Rep}\,(\alpha)$*, i.e., infinitely many non-isomorphic representations of dimension α.*

Corollary 2.6.5. *Let V be a representation of dimension α. Then $\mathcal{O}_V$ is open in* $\mathrm{Rep}\,(\alpha)$ *if and only if* $\mathrm{Ext}^1_A(V,V) = 0$*, i.e., V has no self-extensions.*

Proof. A proper closed subvariety of an irreducible variety has strictly smaller dimension. Since orbits are open in their closure and $\mathrm{Rep}(\alpha)$ is irreducible, $\mathcal{O}_V$ is open if and only if $\overline{\mathcal{O}_V} = \mathrm{Rep}(\alpha)$ which is the case if and only if $\dim \mathcal{O}_V = \dim \mathrm{Rep}(\alpha)$ which holds if and only if $\dim \mathrm{Ext}^1_A(V,V) = 0$.

Corollary 2.6.6. *There is at most one representation of Q of dimension α (up to isomorphism) without self-extensions.*

Proof. It is known that any two non-empty open subvarieties inside an irreducible variety intersect. Suppose $\mathcal{O}_X \neq \mathcal{O}_Y$ are open. Since orbits are disjoint, $\mathcal{O}_X \subset \operatorname{Rep}(\alpha) \backslash \mathcal{O}_Y$, and so $\overline{\mathcal{O}_X} \subset \operatorname{Rep}(\alpha) \backslash \mathcal{O}_Y$, which contradicts the irreducibility of $\operatorname{Rep}(\alpha)$.

Lemma 2.6.7. *If*

$$0 \to U \to V \to W \to 0 \tag{2.6.1}$$

is a non-split exact sequence, then $\mathcal{O}_{U \oplus W} \subset \overline{\mathcal{O}}_V \setminus \mathcal{O}_V$.

Proof. For each vertex i we identify U_i with a subspace of V_i. Choose a basis of U_i and extend it to a basis of V_i. Then $V \simeq R(x)$ with a corresponding matrix of the form:

$$x_\sigma = \begin{pmatrix} u_\sigma & s_\sigma 0 & w_\sigma , \end{pmatrix}$$

where $U \simeq R(u)$, $W \simeq R(w)$. For $0 \neq \lambda \in k$ define $g_\lambda \in \mathrm{GL}(\alpha)$ by $(g_\lambda)_\sigma = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$(g_\lambda x)_\sigma = \begin{pmatrix} \lambda u_\sigma & \lambda s_\sigma \\ 0 & w_\sigma \end{pmatrix}.$$

So the closure of $\mathcal{O}_V$ contains the point with matrices $\begin{pmatrix} u_\sigma & 0 \\ 0 & w_\sigma \end{pmatrix}$, which corresponds to $U \oplus W$. Hence $\mathcal{O}_{U \oplus W} \subset \overline{\mathcal{O}}$.

Applying the functor $\operatorname{Hom}(*, U)$ to (2.6.2), we obtain an exact sequence:

$$0 \to \operatorname{Hom}(W, U) \to \operatorname{Hom}(V, U) \to \operatorname{Hom}(U, U) \xrightarrow{f} \operatorname{Ext}^1(W, U).$$

So applying the equality (2.4.1) to this exact sequence, one obtains

$$\dim \operatorname{Hom}(W, U) - \dim \operatorname{Hom}(V, U) + \dim \operatorname{Hom}(U, U) - \dim \operatorname{Im}(f) = 0.$$

Since $f(1) \neq 0$, $\dim \operatorname{Hom}(V, U) \neq \dim \operatorname{Hom}(U \oplus W, U)$, and hence $X \not\simeq U \oplus W$. Therefore, $\mathcal{O}_{U \oplus W} \subset \overline{\mathcal{O}}_V \setminus \mathcal{O}_V$.

Corollary 2.6.8. *If $\mathcal{O}_X$ is an orbit of maximal dimension and $X = U \oplus V$, then* $\operatorname{Ext}^1(V, U) = 0$.

Proof. Suppose there is a non-split exact extension $0 \to U \to V \to W \to 0$, then, by lemma 2.6.7, $\mathcal{O}_X \subset \overline{\mathcal{O}_V} \backslash \mathcal{O}_V$. But that means $\dim \mathcal{O}_X < \dim \mathcal{O}_V$, which contradicts the maximality of $\dim \mathcal{O}_X$.

Corollary 2.6.9. *If $\mathcal{O}_V$ is closed, then V is semisimple.*

Proof. Let U be a submodule of V, and $W = V/U$. We need to show that the extension is split. If not, then, by lemma 2.6.7, $\mathcal{O}_{U \oplus W} \subset \overline{\mathcal{O}}_V \setminus \mathcal{O}_V$. But $\mathcal{O}_V$ is closed, so its boundary is empty. A contradiction.

To prove the sufficiency part of the Gabriel theorem we need the following lemma.

Lemma 2.6.10. *Let V be an indecomposable module and* $\dim \mathrm{End}(V) > 1$. *Then there is a proper indecomposable submodule $U \subset V$ with* $\dim \mathrm{Ext}^1(U,U) > 0$.

Proof. Since V is indecomposable, $E = \mathrm{End}(V)$ is a local ring by the Fitting lemma. Therefore $E/\mathrm{rad}\, E$ is a simple module. Since k is an algebraically closed field, the Schur lemma shows that $E/\mathrm{rad}\, E \simeq \mathrm{End}_E(E/\mathrm{rad}\, E)$ is one dimensional. Hence the assumption that $\dim \mathrm{End}(V) > 1$ means exactly that $\mathrm{rad}\, E \neq 0$. Let $\theta \in E$ be an element such that $\mathrm{Im}\, \theta$ has minimal dimension. Since E is finite dimensional, $\mathrm{rad}\, E$ is nilpotent. Therefore $\theta^2 = 0$.

Write $\mathcal{I} = \mathrm{Im}\, \theta$. Since E is a local ring we have a decomposition $K = \mathrm{Ker}\, \theta = K_1 \oplus K_2 \oplus \ldots \oplus K_r$ into a direct sum of indecomposable modules. Since $\theta^2 = 0$, $\mathcal{I} \subset K$. Choose j so that the composite α of the inclusion $\mathcal{I} \to K$ and the projection $\pi_j : K \to K_j$ are nonzero. Note that $\alpha : \mathcal{I} \to K_j$ is injective. Indeed, the composite

$$X \to \mathcal{I} \to K_j \to X$$

has image $\mathrm{Im}\, \alpha$, hence $\dim(\mathrm{Im}\, \alpha) > \dim(\mathcal{I})$, by the minimality assumption.

Now we claim that $\mathrm{Ext}^1(\mathcal{I}, K_j) \neq 0$. Once we have established that, the lemma follows by applying $\mathrm{Hom}(*, K_j)$ to $0 \to \mathcal{I} \to K_j \to Q \to 0$, and using the fact that Ext^2 vanishes gives $\mathrm{Ext}^1(\mathcal{I}, K_j) \to \mathrm{Ext}^1(K_j, K_j) \to 0$, hence $\mathrm{Ext}^1(K_j, K_j) \neq 0$ and we can take $U = K_j$.

To prove the claim, suppose on the contrary that $\mathrm{Ext}^1(\mathcal{I}, K_j) = 0$. Consider the pushout of the short exact sequence

$$0 \to K \to V \to \mathcal{I} \to 0$$

along π_j, which gives an exact sequence

$$0 \to K_j \to Y \to \mathcal{I} \to 0.$$

If this splits, then K_j has a complement C in Y. But then the inverse image of C is a complement to K_j in V, hence K_j is a summand of V. This contradicts the assumption that V is indecomposable.

Corollary 2.6.11. *Let V be an indecomposable module and* $\dim \mathrm{End}(V) > 1$. *Then there is an indecomposable submodule $U \subset V$ with* $\dim \mathrm{End}(U) = 1$ *and* $\dim \mathrm{Ext}^1(U,U) > 0$, *hence* $q(\underline{\dim}\, U) \leq 0$.

Proof. Apply lemma 2.6.10. If $\mathrm{End} U \neq k$, repeat. Since all modules are finite dimensional, the process must terminate.

Proof of theorem 2.6.1.

1. Necessity. Let Q be a quiver of finite type. Then, by the Krull-Remak-Schmidt theorem, there are a finite number of orbits in $\mathrm{Rep}\,(\alpha)$ for each

dimension α. Then, by corollary 2.6.4, the associated quadratic form q is positive definite. Thus, by theorem 2.5.2, the underlying graph of Q is a Dynkin diagram.

2. Sufficiency. Suppose Q is a quiver whose underlying graph is a Dynkin diagram. So, by theorem 2.5.2, the associated quadratic form q is positive definite. Therefore, by corollary 2.6.11, every indecomposable module V satisfies $\dim \mathrm{End}(V) = 1$ and $\dim \mathrm{Ext}^1(V,V) = 0$. Therefore, if V is an indecomposable module with dimension vector α, then $q(\alpha) = q(\underline{\dim}\, V) = \dim \mathrm{End}(V) - \dim \mathrm{Ext}^1(V,V) = 1$, i.e., α is a positive root of the root system Δ. If V, U are two indecomposable modules with dimension vector α, then, by lemma 2.6.3, $V \simeq U$.

We need to show that for any positive root α of the root system Δ there is at least one indecomposable module of dimension α. Let $\mathcal{O}_V$ be an orbit of maximal dimension in $\mathrm{Rep}\,(\alpha)$. If V is indecomposable, then we are done. Otherwise, $V = U \oplus W$. Then $\mathrm{Ext}^1(W,U) = \mathrm{Ext}^1(U,V) = 0$, by corollary 2.6.8, hence $(\underline{\dim}\, W, \underline{\dim}\, U) \geq 0$. In this case,

$$1 = q(\underline{\dim}\, V) = q(\underline{\dim}\, W + \underline{\dim}\, U) = q(\underline{\dim}\, W) + q(\underline{\dim}\, U) + (\underline{\dim}\, U, \underline{\dim}\, W) \geq 2,$$

a contradiction.

Remark 2.6.1. The Gabriel theorem has also been proved in the more general case when k is an arbitrary field. In this case I.N.Bernstein, I.M.Gel'fand, V.A.Ponomarev proved that if Q is a Dynkin diagram then it has finite representation type (see, [Berstein, Gel'fand, Ponomarev, 1973]). Their proof uses the Weyl group and Coxeter functors.

Theorem 2.6.12. *A quiver Q is of finite (resp. tame) type if and only its quadratic form is positive definite (resp. semipositive definite).*

Remark 2.6.2. The Gabriel theorem was generalized to arbitrary quivers by V.Kac [Kac, 1980a]. V.Kac proved that indecomposable representations occur precisely in those dimension vectors that are roots of the so-called Kac-Moody Lie algebra associated to a given graph. In particular, these dimension vectors do not depend on the orientation of the arrows in a quiver Q.

Theorem 2.6.13 (V.Kac). *Let Q be a quiver and $\alpha > 0$ be a dimension vector. Then there is an indecomposable representation of dimension α if and only if α is a root. If α is a real root, there is a unique indecomposable representation of dimension α; if α is an imaginary root, there are infinitely many indecomposable representations of dimension α.*

Remark 2.6.3. L.A.Nazarova and P.Donovan, M.R.Freislich described independently the quivers of tame representation type over an algebraically closed field (see, [Nazarova, 1973], [Donovan, Freislich, 1973]):

Theorem 2.6.14. *Let k be an algebraically closed field and let Q be a connected quiver without oriented cycles. Then Q is of a tame type if and only if the* *underlying undirected graph $\overline{Q}$ of Q (obtained from Q by deleting the orientation of the arrows) is a Euclidean diagram of the form $\bar{A}_n, \bar{D}_n, \bar{E}_6, \bar{E}_7, \bar{E}_8$.*

In the general case, if k is an arbitrary field, the result was treated by V.Dlab and C.M.Ringel (see [Dlab, Ringel, 1976]).

2.7 K-SPECIES

In this section we consider the notion of k-species introduced by P.Gabriel (see [Gabriel, 1973]). Let k be a fixed field and let I be a finite set. A **species** $\mathcal{L} = (K_i, {}_iM_j)_{i,j\in I}$ is a finite family $(K_i)_{i\in I}$ of skew fields together with a family $({}_iM_j)_{i,j\in I}$ of (K_i, K_j)-bimodules. We say that $(K_i, {}_iM_j)_{i,j\in I}$ is a k-**species** if all K_i are finite dimensional and central over a common commutative subfield k which acts centrally on ${}_iM_j$, i.e., $\lambda m = m\lambda$ for all $\lambda \in k$ and all $m \in {}_iM_j$. We also assume that each bimodule ${}_iM_j$ is a finite dimensional vector space over k. It is a k**-quiver** if moreover $K_i = k$ for each i. We shall consider species without oriented cycles and loops, i.e., we shall consider the case where ${}_iM_i = 0$, and if ${}_iM_j \neq 0$, then ${}_jM_i = 0$.

The **diagram of a species** $(K_i, {}_iM_j)_{i,j\in I}$ is defined in the following way:

1) the set of vertices is the finite set I;

2) the vertex j connects with the vertex i by t_{ij} arrows, where

$$t_{ij} = \dim_{K_i}({}_iM_j) \times \dim({}_iM_j)_{K_j}.$$

If ${}_iM_j = 0$ and $\dim_{K_j}({}_jM_i) > \dim({}_jM_i)_{K_i}$, then we denote this by the following arrow: $\underset{j}{\bullet} \Rrightarrow \underset{i}{\bullet}$ (here $t_{ij} = 3$). (The number of horizontal lines in the “arrows” is t_{ij}.)

Definition. An $\mathcal{L}$**-representation** $(V_i, {}_j\varphi_i)$ of a species $\mathcal{L} = (K_i, {}_iM_j)_{i,j\in I}$ is a family of right K_i-modules V_i and K_j-morphisms:

$${}_j\varphi_i : V_i \otimes_{K_i} {}_iM_j \longrightarrow V_j$$

for each $i, j \in I$. The **category of** $\mathcal{L}$**-representations** $\operatorname{Rep}\mathcal{L}$, is the category whose objects are $\mathcal{L}$-representations and whose morphisms are defined as follows. Let $V = (V_i, {}_j\varphi_i)$ and $W = (W_i, {}_j\psi_i)$ be two $\mathcal{L}$-representations. An $\mathcal{L}$-morphism $\Psi : V \to W$ from V to W is a set of K_i-linear maps $\alpha_i : V_i \to W_i$ such that the following diagram commutes

$$\begin{tikzcd} V_i \otimes_{K_i} {}_iM_j \arrow[r, "{}_j\varphi_i"] \arrow[d, "\alpha_i\otimes 1"'] & V_j \arrow[d, "\alpha_j"] \\ W_i \otimes_{K_i} {}_iM_j \arrow[r, "{}_j\psi_i"] & W_j \end{tikzcd} \tag{2.7.1}$$

Two representations $(V_i, {}_j\varphi_i)$ and $(W_i, {}_j\psi_i)$ are called **equivalent** if there is a set of isomorphisms (α_i) from K_i-module V_i to K_i-module W_i such that for each $i \in I$ the diagram (2.7.1) is commutative.

A representation $(V_i, {}_j\varphi_i)$ is called **indecomposable**, if there are no nonzero sets of subspaces (V_i') and (V_i'') such that $V_i = V_i' \oplus V_i''$ and ${}_j\varphi_i = {}_j\varphi_i' \oplus {}_j\varphi_i''$, where

$$
\begin{aligned}
{}_j\varphi_i &: V_i' \otimes_{K_i} {}_iM_j \longrightarrow V_j' \\
{}_j\varphi_i &: V_i'' \otimes_{K_i} {}_iM_j \longrightarrow V_j''
\end{aligned}
$$

We can define the direct sum of $\mathcal{L}$-representations in the obvious way.

Let $d_i = \dim(V_i)_{K_i}$ be a dimension of V_i as a vector space over K_i. By the dimension of a representation $(V_i, {}_j\varphi_i)$ we shall mean the vector $d = (d_1, d_2, \ldots, d_n)$. We set $d_0 = \sum\limits_{i\in I} \dim(V_i)_{K_i}$. The representation $(V_i, {}_j\varphi_i)$ is called **finite dimensional** if $d_0 < \infty$.

A species $(K_i, {}_iM_j)_{i,j\in I}$ is called of **finite type**, if the number of indecomposable non-isomorphic finite dimensional representations is finite.

In the case when all $K_i = F$, where F is a fixed skew field, and ${}_F({}_iM_j)_F = ({}_FF_F)^{t_{ij}}$, P.Gabriel has characterized k-species of finite type (see [Gabriel, 1972]). This result was extended by V.Dlab and C.M.Ringel to the case where $\mathcal{L}$ is an arbitrary k-species (see [Dlab, Ringel, 1973], [Dlab, Ringel, 1974]), [Dlab, Ringel, 1975]).

Theorem 2.7.1 (V.Dlab, C.M.Ringel). *A k-species is of finite type if and only if its diagram is a finite disjoint union of Dynkin diagrams.*

2.8 NOTES AND REFERENCES

The exterior algebra was invented by Hermann Grassmann (1809 - 1877) in his most important work [Grassmann, 1844]. In this book he developed the idea of an algebra in which the symbols representing geometric entities such as points, lines and planes, are manipulated using certain rules. He represented subspaces of a space by coordinates leading to points in an algebraic manifold now called the Grassmannian. Unfortunately, Grassmann's methods were not understood and adopted at the time. It is only because of the work of Elie Cartan that they became used in studying differential forms and their application to analysis, geometry and physics. In physics differential forms are elements of the Grassmann algebra over the dual vector space. They are used everywhere from elementary mechanics to Hamiltonian mechanics and field theory.

The notion of a tensor algebra as a "maximal ring" was first considered by G.Hochschild (see [Hochschild, 1947], [Hochschild, 1950]). In 1957 in the paper [Eilenberg, 1957], S.Eilenberg, A.Rosenberg and D.Zelinsky showed that for any ring R, the global dimension of the polynomial ring $R[x_1, x_2, \ldots, x_n]$ is equal to

n+gl.dim$R$. In the paper [Hochschild, 1958] G.Hochschild proved that the global dimension of a free ring over R on any set of letters is equal to 1+gl.dim$R$. This theorem was generalized by Yu.V.Roganov for a tensor algebra of a bimodule which is one-sided projective (see [Roganov, 1975]). In the proof of theorem 2.2.11 we follow this paper.

In 1972 in the paper [Gabriel, 1972] P.Gabriel introduced quivers in connection with the classification of finite dimensional algebras of finite type. In this paper he gave a full description of quivers of finite representation type over an algebraically closed field (theorem 2.6.1). P.Gabriel also proved that there is a bijection between the isomorphism classes of indecomposable representations of a quiver Q and the set of positive roots of the Tits form corresponding to this quiver.

Another proof of this theorem in the general case, for an arbitrary field, using reflection functors and Coxeter functors has been given in the paper [Berstein, Gel'fand, Ponomarev, 1973]. This paper also contains the connection between indecomposable representations of a quiver of finite type and properties of the Tits quadratic form (theorem 2.5.2).

The Gabriel theorem was generalized by V.Kac (see [Kac, 1980a], [Kac, 1980b]), who proved that indecomposable representations occur in dimension vectors that are roots of the so-called Kac-Moody Lie algebra associated to a given graph. In particular, these dimension vectors do not depend on the orientation of the arrows in the quiver Q.

The terms "tame type" and "wild type" were introduced by P.Donovan and M.R.Freislich in their paper [Donovan, Freislich, 1972] in analogy with the separation of animals into tame and wild ones. In the same paper they first conjectured that any finite dimensional algebra is either tame or wild.

Tame quivers in terms of extended Dynkin diagrams were classified by L.A.Nazarova in [Nazarova, 1973] and by P.Donovan and M.R.Freislich in [Donovan, Freislich, 1973].

The representation type of factor algebras of path algebras has been discussed by E.Green, who gave some useful algorithms for studying representations of path algebras (see [Green, 1975]).

The theory of species was first considered by P.Gabriel in [Gabriel, 1973]. Note that in fact species and their connections with the representations of algebras were already considered in the papers of T.Yoshii (see [Yoshii, 1956]; [Yoshii, 1957a]; [Yoshii, 1957b]. Later the results of P.Gabriel on representations of species were generalized by V.Dlab and C.M.Ringel (see [Dlab, Ringel, 1974]; [Dlab, Ringel, 1975]; [Dlab, Ringel, 1976]; [Ringel, 1976].

Representations of quivers with relations were considered by P.Donovan and M.Freislich [Donovan, Freislich, 1979], S.Ovsienko [Ovsienko, 1977], C-M.Ringel [Ringel, 1975], [Ringel, 1980], V.Yu.Romanovskij, A.S.Shkabara, and A.G.Zavadskij, (see [Romanovskij, Shkabara, 1976]; [Shkabara, 1978a], [Shkabara, 1978b]; [Zavadskij, Shkabara, 1976].

More references on the history of algebras and their representations can be found in [Gustafson, 1982].

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APPENDIX TO SECTION 2.5. MORE ABOUT DYNKIN AND EXTENDED DYNKIN (= EYCLIDEAN) DIAGRAMS

Recall (following [1]) that a valued graph is a graph with each edge marked by an ordered pair of positive integers. In loc.cit. no loops are allowed, but actually for the tame and finite representation type cases this makes little difference: just the inclusion of one tame case, viz. the one vertex one loop graph.

By convention for an edge with value (1,1) the value indicator is omitted. Thus for example one has the valued diagram

$$B_4 : \begin{array}{ccccccc} \bullet & \overset{(1,2)}{\text{———}} & \bullet & \text{———} & \bullet & \text{———} & \bullet \\ 1 & & 2 & & 3 & & 4 \end{array}$$

which is in fact the diagram denoted by B_4 in [1] and, using a convention explained below, C_4 in [12], [18] and [19] (and most other recent books and papers treating of these matters (especially those dealing with Kac-Moody algebras and quantum groups).

There is a further convention, induced, we feel, by both a sense of historical continuity and nostalgia, that replaces labels with low numbers by an arrow notation. It runs as follows. If the label (d_{ij}, d_{ji}) of the edge between vertex i and vertex j is such that $d_{ij}d_{ji} \leq 4$ then there are $\max\{(d_{ij}, d_{ji})\}$ edges between i and j and there is an arrow towards i (resp. j) if and only if $d_{ij} > 1$ (resp. $d_{ji} > 1$). Using this convention the valued graph above is denoted[1]

$$C_4 : \begin{array}{ccccccc} \bullet & \Longrightarrow & \bullet & \text{———} & \bullet & \text{———} & \bullet \\ 1 & & 2 & & 3 & & 4 \end{array}$$

Here are some further examples

$$\bullet \overset{(2,2)}{\text{———}} \bullet \quad = \quad \bullet \Longleftrightarrow \bullet$$

$$\bullet \overset{(3,1)}{\text{———}} \bullet \quad = \quad \bullet \Lleftarrow \bullet$$

Edges with a label (d_{ij}, d_{ji}) with $d_{ij}d_{ji} > 4$ are left as is. Note that from a diagram with arrow notation the corresponding labeled graph is uniquely recoverable.

Given an arbitrary complex matrix $A = (a_{ij})$ there is an associated interesting (usually infinite dimensional) Lie algebra $g(A)$, see e.g. [12, 19]. Two such matrices

[1] Recall that this convention is different from the one in the main text (section 2.5); whence the switch between B and C.

are said to be equivalent if one can be obtained from the other by simultaneous permutation of rows and columns.

The direct sum of two matrices A_1 and A_2 is the matrix

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

A matrix is called **indecomposable** if it is not equivalent to a direct sum with both A_1, A_2 nonzero.

In the real case, i.e. all a_{ij} are real, and assuming

(KM1) A is indecomposable

(KM2) $a_{ij} = 0$ if and only if $a_{ji} = 0$

(KM3) $a_{ij} \leq 0$ for all $i \neq j$

there is a trichotomy due to E.B.Vinberg [17]:

- *finite*: $\det(A) \neq 0$; there exists a vector $u > 0$ such that $Au > 0$; $Au \geq 0$ implies $u > 0$ or $u = 0$.

- *affine*: corank(A) =1; there exists a $u > 0$ such that $Au = 0$; $Au \geq 0$ implies $u = 0$.

-*indeterminate*: there exists a $u > 0$ such that $Au < 0$; $Av \geq 0$ and $v \geq 0$ imply $v = 0$.

Here for a vector $u = (u_1, u_2, \dots, u_n)$, $u > 0$ means $u_i > 0$ for all i and $u \geq 0$ means $u_i \geq 0$ for all i.

A really deep and fascinating theory with applications to both other areas of mathematics (e.g. the monstrous moonshine conjectures, algebraic combinatorics, ...) and physics (e.g. quantum field theory, string theory) has only been developed (so far) for the integral case with moreover $a_{ii} = 2$ for all i.

Definition. A **generalized Cartan matrix** is an integer entry square matrix satisfying (KM1)-(KM3) and

(KM4) $a_{ii} = 2$ for all i.

In this case the trichotomy mentioned above works out as

- finite: A is positive definite, i.e. all principal minors are positive.

- affine: A is semi-positive definite, i.e. $\det(A) = 0$ and all principal minors are positive.

- indeterminate: all other cases.

There is a natural and rather obvious bijection between valued graphs and generalized Cartan matrices. To a valued graph with edge labels (d_{ij}, d_{ji}) associate the generalized Cartan matrix $A = a_{ij}$ with $a_{ii} = 2$, $a_{ij} = -d_{ij}$ if i and j are connected and $a_{ij} = 0$ if i and j are not connected. In this correspondence connectedness of the valued graph is the same as indecomposability of the matrix.

As it turns out the finite type generalized Cartan matrices A have as associated Lie algebras the classical simple finite dimensional Lie algebras with the Dynkin diagrams A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2 as depicted just below lemma 2.5.1. Nothing much new here. The affine case gives the infinite dimensional affine (or Eyclidean) Lie algebras introduced independently and practically simultaneously by Kac and Moody [11, 14, 15]. These are extraordinary rich in theory and applications.

The corresponding extended Dynkin diagrams, also called Eyclidean diagrams or affine diagrams, are listed in section 2.5 above just below the list of Dynkin diagrams.

It may seem at first sight remarkable that finiteness of the Lie algebra associated to a generalized Cartan matrix should correspond exactly to finite representation type of the corresponding quiver, and that affineness of the Lie algebra associated to a generalized Cartan matrix should correspond precisely to tame representation type of the associated quiver [2]

However, admitting some fairly deep theory in the two cases this is entirely clear.

Whether a connected valued graph (quiver) Γ is finite, tame, wild representation type is ruled by the **Dlab-Ringel-Ovsienko-Roiter quadratic form** of it. This quadratic form is defined as follows. It is assumed that the collections of integers d_{ij} from the labels is (right) symmetrizable; meaning that there are positive integers f_i such that $d_{ij}f_j = d_{ji}f_i$. This is the same as saying that the generalized Cartan matrix associated to the valued graph is (right) symmetrizable (which in turn is the same as being left symmetrizable (as is easy to prove)). This symmetrizability condition is not much of a restriction because a generalized Cartan matrix of finite or affine type is always symmetrizable, see [12, 18].

Then the Dlab-Ringel-Ovsienko-Roiter quadratic form of Γ is

$$q_\Gamma(x) = \sum_{i=1}^{n} 2f_i x_i^2 - \sum_{i \neq j} d_{ij} f_j x_i x_j$$

i.e. it is the quadratic form associated to the symmetric matrix $C(\Gamma)F$, where $C(\Gamma)$ is the generalized Cartan matrix associated to Γ and F is the diagonal matrix $F = \operatorname{diag}(f_1, f_2, \ldots, f_n)$. In terms of this quadratic form a quiver Γ is of finite, tame, or wild representation type according to whether the quadratic form is positive definite, positive semi-definite, or indefinite [3] [4].

[2]The diagrams in the lists correspond bijectively. However, the labels assigned to them in [1] and [12, 18, 19] differ. For instance, $\widetilde{BC}_n$, $\widetilde{F}_{42}$, $\widetilde{G}_{22}$ in [1] correspond respectively to $A^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$ in [12, 18, 19]. It is the latter notation that is now mostly used.

[3]It cannot be negative semi-definite because the diagonal elements are positive.

[4]Recall that the quadratic form of a symmetric matrix is positive definite (resp. positive semi-definite) if and only if all principal minors of the matrix are positive (resp. all proper principal minors are positive and matrix is singular). This does not necessarily hold for matrices

On the other hand the finiteness, affineness, or indefiniteness of the generalized Cartan matrix $C(\Gamma)$ is determined by whether the principal minors of it are positive, zero or negative. But if F is a diagonal matrix with positive diagonal elements then a principal minor of $C(\Gamma)$ is positive, negative, or zero, if and only if the same is true for the corresponding principal minor of $C(\Gamma)F$.

So things fit. In fact the relations between quiver and species representations and Lie algebras go much deeper. See [4,5,6].

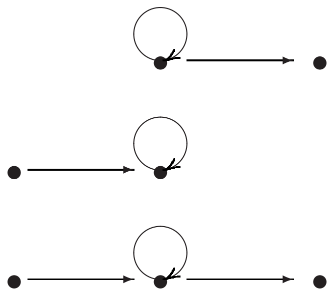
It is an almost trivial matter to incorporate loops in the Dlab-Ringel analysis of [1]. Let (Γ, d) be a connected valued graph with d_{ii} loops at vertex i. The appropriate quadratic form is now

$$q_\Gamma(x) = \sum_{i=1}^{n}(2-2d_{ii})f_i x_i^2 - \sum_{i\neq j} d_{ij} f_j x_i x_j$$

and this adds just one more semi-positive definite case, viz the graph of one vertex with one loop which can be appropriately denoted $\widetilde{A}_0$ (in the kind of notation from [1]).

There is, or has been, a sort of rueful pessimism about doing something in the case of wild quivers and wild representation type in general. This is not quite justified as is evidenced for example by the remarkable results of Kac on dimensions vectors, very nicely explained in [13].

Also, on a much more modest level, a great deal can sometimes be said in specific cases. For instance the quivers



and their representations are of great importance in linear control and systems theory and a great deal is known about (the moduli space of) their representations, see [8,10].

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that are not symmetric. Consider for instance the matrix $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}$ which is the semi-positive definite Cartan matrix of $C_3^{(1)}$ (in the notation of [12]). The quadratic form of this matrix is $2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 - 3x_1x_2 - 2x_2x_3 - 3x_2x_4$ which is indefinite.

Let S be a finite set and let $M = (m_{st})_{s,t\in S}$ be a symmetric matrix such that $m_{ss} = 1$ and $m_{st} \in \{2,3,4,\ldots\}\cup\{\infty\}$ for $s \neq t$. Such a matrix is called a **Coxeter matrix**. Now let W be a group with S as a subset. Then the pair (W, S) is called a **Coxeter system** and W is called a **Coxeter group** if W has a presentation with elements from S as generators and the relations

$$(st)^{m_{st}} = 1 \quad \text{for} \quad \text{all} \quad s, t \in S \quad \text{with} \quad m_{st} \neq \infty \tag{1}$$

In particular $s^2 = 1$ making W a group generated reflections [5]. The Coxeter graph associated to a Coxeter system has $|S|$ vertices and two vertices are joined by an edge iff $m_{st} \geq 3$ and labeled with that number with the number 3 usually omitted by convention. Then the finite Coxeter groups are classified by "Dynkin diagrams"

$$A_n, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4, I_2(m) \tag{2}$$

Most of these groups [6] turn up as the Weyl groups of the simple Lie algebras with the corresponding labels. The correspondence between the Cartan matrix defining the simple Lie algebra and the Coxeter matrix is according to the following table

$a_{st}a_{ts}$	0	1	2	3
m_{st}	2	3	4	6

Note that $G_2 = I_2(6)$. Note also that the simple Lie algebras B_n, C_n give the same (abstract) Coxeter group explaining that C_n is missing from the list (2) above.

Similarly the extended Dynkin diagram give rise to affine Weyl groups which are infinite Coxeter groups. This time there are no extra diagrams like the $I_2(m)$, $m \neq 6$ and H_3, H_4 in the list (2).

See [5] and references quoted there for a great deal more on all this. The subject of reflection groups is a very large one with applications to and/or links with many parts of mathematics.

There are many more areas where Dynkin diagrams turn up, e.g. singularity theory. This happens so often that V.I.Arnol'd posed it as a Hilbert type problem (problem VIII in [16]). Here is the precise formulation:

The A-D-E classifications. The Coxeter-Dynkin graphs A_k, D_k, E_k appear in many independent classification theorems. For instance

(a) classification of Platonic solids (or finite orthogonal groups) in Eyclidean 3-space.

[5] As stated they are just elements of order 2 (involutions). It turns out it is indeed possible to realize W as a group of geometric reflections in some real vector space.

[6] Precisely those that satisfy the socalled "crystallographic condition".

(b) classification of the categories of linear spaces and maps[7]; Gabriel, Gel'fand-Ponomarev, Roiter-Nazarova, see Séminaire Bourbaki, exposé 444, 1974.

(c) classification of the singularities of algebraic hypersurfaces with a definite intersection form of the neighboring smooth fibre, Tjurina.

(d) classification of the critical points of functions having no moduli (see Séminaire Bourbaki, exposé 443, 1974).

(e) classification of the Coxeter groups generated by reflections, or of Weyl groups with roots of equal length.

The problem is to find the common origin of all the $A - D - E$ classification theorems and to substitute a priori proofs for the a posteriori verification of the parallelism of the classifications.

For an introductioon to the ADE problem see [9].

A fair selection of papers related to the ADE problem is mentioned in <http://math.ucr.edu/home/baez/week230.html>. It is also worth the trouble in this connection to look at week62, week63, week64, week65.

The list of Arnol'd by no means exhaust the areas of mathematics where Dynkin diagrams turn up. Some others areas are represented by the papers [2,3,20]. Here one also finds some more different notations for Dynkin diagrams, both symbolically and pictorially. Perhaps the most surprising of these papers (to us) is [3]. The numbers game is a one person game played on a finite simple graph with certain "amplitudes" assigned to its edges and an initial assignment of real numbers to its nodes. The moves use the amplitudes to modify the node numbers. The paper shows that if certain finiteness requirements are to be met one again finds the Dynkin diagrams.

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3. Representations of posets and of finite dimensional algebras

In the theory of representations one tries to study a given object by means of homomorphisms to another object which is in some way more concrete and easier to understand. Such objects in the theory of finite dimensional algebras are endomorphisms of some finite dimensional vector space over a field k. Another, slightly different, but related, point of view is that one attempts to realize an abstract object in terms of more concrete things such as matrices. Whence the terminology "representation".

There exist different approaches to the representation theory of algebras. In this chapter we consider only the part of representation theory connected with the representations of partially ordered sets and Gabriel quivers. This chapter is more of an informative character and it may be considered as a brief survey of those well-known results of this theory which will be needed in this book. Therefore most statements are made here for completeness without proofs.

Representations of finite partially ordered sets (posets, in short) play an important role in representation theory. They were first introduced by L.A.Nazarova and A.V.Roiter. The first two sections of this chapter are devoted to partially ordered sets and their representations. Here there are given the main results of M.M.Kleiner on representations of posets of finite type and the results of L.A.Nazarova on representations of posets of infinite type. The most important result in this theory was been obtained by Yu.A.Drozd who showed that there is a trichotomy between finite, tame and wild representation types for finite posets over an algebraically closed field.

One of the main problems of representation theory is to obtain information about the possible structure of indecomposable modules and to describe the isomorphism classes of all indecomposable modules. By the famous trichotomy theorem for finite dimensional algebras over an algebraically closed field, obtained by Yu.A.Drozd, all such algebras are divided into three disjoint classes.

The main results on representations of finitely dimensional algebras are given in section 3.4. Here we give the structure theorems for some special classes of finite dimensional algebras of finite type, such as hereditary algebras and algebras with zero square radical, obtained by P.Gabriel in terms of Dynkin diagrams. Section 3.5 is devoted to the first Brauer-Thrall conjecture, settled by A.V.Roiter for the case of a finite dimensional algebra over an arbitrary field.

Unless otherwise stated, in this chapter we always suppose that all algebras considered are associative finite dimensional with 1.

3.1 REPRESENTATIONS OF POSETS

The representation theory of partially ordered sets plays a prominent role in the representation theory of finite-dimensional algebras.

Let $\mathcal{P} = (S, \preceq)$ be a finite poset. Since every partial ordering on a finite set can be extended to a total ordering, we can suppose that $S = \{1, 2, ..., m\}$ and

$$i \prec j \quad \Rightarrow i < j.$$

Definition. Let $\mathcal{P} = (S, \preceq)$, where $S = \{1, 2, ..., m\}$, be a finite partially ordered set (or poset, in short). A **representation** of $\mathcal{P}$ over a field k or a **$\mathcal{P}$-space** is a set of finite dimensional k-spaces $V = (V_0; V_i : \ i \in S)$ such that $V_i \subset V_0$ for all $i \in S$ and $V_i \subset V_j$ precisely when $i \preceq j$ in $\mathcal{P}$. Such an object is called a $\mathcal{P}$-space. Let $V = (V_0; V_i : \ i \in S)$ and $W = (W_0; W_i : \ i \in S)$ be two $\mathcal{P}$-spaces. A **morphism** $f : V \to W$ is a k-linear transformation $V_0 \to W_0$ such that $f(V_i) \subset W_i$ for all $i \in S$. The direct sum of V and W is $V \oplus W$ with $(V \oplus W)_0 = V_0 \oplus W_0$ and $(V \oplus W)_i = V_i \oplus W_i$ for all $i \in S$. A nonzero $\mathcal{P}$-space is said to be **indecomposable** if it cannot be written as a direct sum of two nonzero $\mathcal{P}$-spaces.

Therefore $\mathcal{P}$-spaces over a fixed field k form an additive category, which is denoted by $\text{Rep}\,(\mathcal{P}, k)$. Every $\mathcal{P}$-space is a uniquely determined as a direct sum of indecomposable $\mathcal{P}$-spaces, by a corollary of the Krull-Schmidt theorem (see, vol.I, p. 242). This theorem can be written in the following form:

Theorem 3.1.1. *Every representation of a poset decomposes into a direct sum of indecomposable representations uniquely, up to isomorphism and order of summands.*

From this theorem it follows that all representations in $\text{Rep}\,(\mathcal{P}, k)$ are uniquely determined by the indecomposable ones. The set of all isomorphism classes of indecomposable representations in $\text{Rep}\,(\mathcal{P}, k)$ shall be written as $\text{Ind}\,(\mathcal{P}, k)$.

Definition. A partially ordered set $\mathcal{P}$ is called **subspace-finite** (or of **finite representation type**) over a field k if $\text{Ind}\,(\mathcal{P}, k)$ is finite, i.e., there are only a finite number of non-isomorphic indecomposable $\mathcal{P}$-spaces. And $\mathcal{P}$ is called **subspace-infinite** (or of **infinite representation type**) over a field k if $\text{Ind}\,(\mathcal{P}, k)$ is infinite.

Representations of posets were first introduced and considered by L.A.Nazarova and A.V.Roiter [Nazarova, Roiter, 1972]. Every representation of $\text{Rep}\,(\mathcal{P}, k)$ can be given in terms of the language of matrices.

Definition. Let $\mathcal{P} = (S, \preceq)$ be a finite poset, and let $S = \{1, 2, ..., m\}$. A **representation** of $\mathcal{P}$ over a field k is an arbitrary matrix

$$\mathbf{A} = \boxed{\begin{array}{c|c|c|c} \mathbf{A}_1 & \mathbf{A}_2 & \dots & \mathbf{A}_m \end{array}}$$

with entries in k partitioned horizontally into m (vertical) blocks (also called strips).

Here the columns of a block $\mathbf{A}_i$ are formed by the coordinates (with respect to a chosen basis in V_0) of any minimal system of generators of V_i modulo its subspace $\bar{V}_i = \sum\limits_{j \prec i} V_i$.

Let $\mathbf{B}$ be another (matrix) representation of a poset $\mathcal{P} = (S, \preceq)$:

$$\mathbf{B} = \boxed{\begin{array}{c|c|c|c} \mathbf{B}_1 & \mathbf{B}_2 & \dots & \mathbf{B}_m \end{array}}$$

Definition. A representation $\mathbf{A}$ is **isomorphic** to a representation $\mathbf{B}$ of a poset $\mathcal{P} = (S, \preceq)$ if $\mathbf{A}$ can be reduced to $\mathbf{B}$ by the following transformations:

(a) elementary transformations of rows of the whole matrix $\mathbf{A}$;

(b) elementary transformations of columns within each vertical strip $\mathbf{A}_i$;

(c) additions of columns of a strip $\mathbf{A}_i$ to columns of a strip $\mathbf{A}_j$ if $i \prec j$ in $\mathcal{P}$.

Notice that by a sequence of transformations (b) and (c) we can add an arbitrary linear combination of columns of $\mathbf{A}_i$ to a column of $\mathbf{A}_j$ if $i \prec j$ in $\mathcal{P}$.

Definition. The **direct sum** of two representations $\mathbf{A}$ and $\mathbf{B}$ is the representation $\mathbf{A} \oplus \mathbf{B}$ which is equal to:

$$\mathbf{A} \oplus \mathbf{B} = \boxed{\begin{array}{cc|cc|c|cc} \mathbf{A}_1 & \mathbf{O} & \mathbf{A}_2 & \mathbf{O} & \ddots & \mathbf{A}_m & \mathbf{O} \\ \mathbf{O} & \mathbf{B}_1 & \mathbf{O} & \mathbf{B}_2 & \ddots & \mathbf{O} & \mathbf{B}_m \end{array}}$$

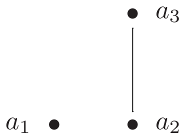
For a representation V of a poset $\mathcal{P} = (S, \preceq)$, where $S = \{1, 2, \dots, m\}$, we define its dimension vector as $d = \underline{\dim} V = (d_0, d_1, \dots, d_m)$ with coordinates $d_0 = \dim V_0$, $d_i = \dim V_i / \bar{V}_i$. For a matrix representation $\mathbf{A}$ there is analogously the **coordinate vector**

$$\mathrm{cdn}(\mathbf{A}) = (s_1, s_2, \dots, s_m, s_{m+1}) \in \mathbf{N}^{m+1},$$

where s_i is the number of nonzero columns in the strip $\mathbf{A}_i$ for $i \in S$, and s_{m+1} is the number of nonzero rows in $\mathbf{A}$.

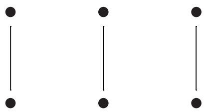
A matrix representation $\mathbf{A}$ is called **exact** (following the terminology of M.Kleiner) if it is indecomposable and all coordinates of the vector $\mathrm{cdn}(\mathbf{A})$ are nonzero.

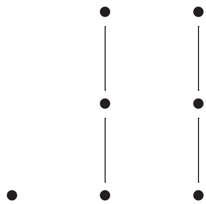
In order to visualize a poset S we shall use its Hasse diagram: the elements of S will be represented by points on the plane and the relation $\prec$ is always thought of as going upwards along the edges drawn. Moreover, we join the point i with the point j if and only if $i \prec j$ and there is no $t \in S$ such that $i \prec t \prec j$. For example, for the poset $\mathcal{P} = (S, \preceq)$, with $S = \{a_1, a_2, a_3\}$ and the sole relation $a_2 \prec a_3$ we have the picture:

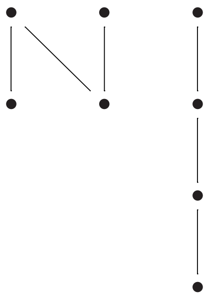


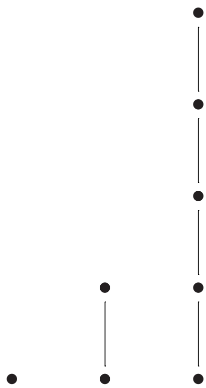
Theorem 3.1.2 (M.Kleiner). *A finite partially ordered set $\mathcal{P}$ is of finite representation type if and only if $\mathcal{P}$ does not contain as a full subposet any poset from the following list:*

$(1, 1, 1, 1)$:

$(2, 2, 2)$: 

$(1, 3, 3)$: 

$(\mathrm{N}, 4)$: 

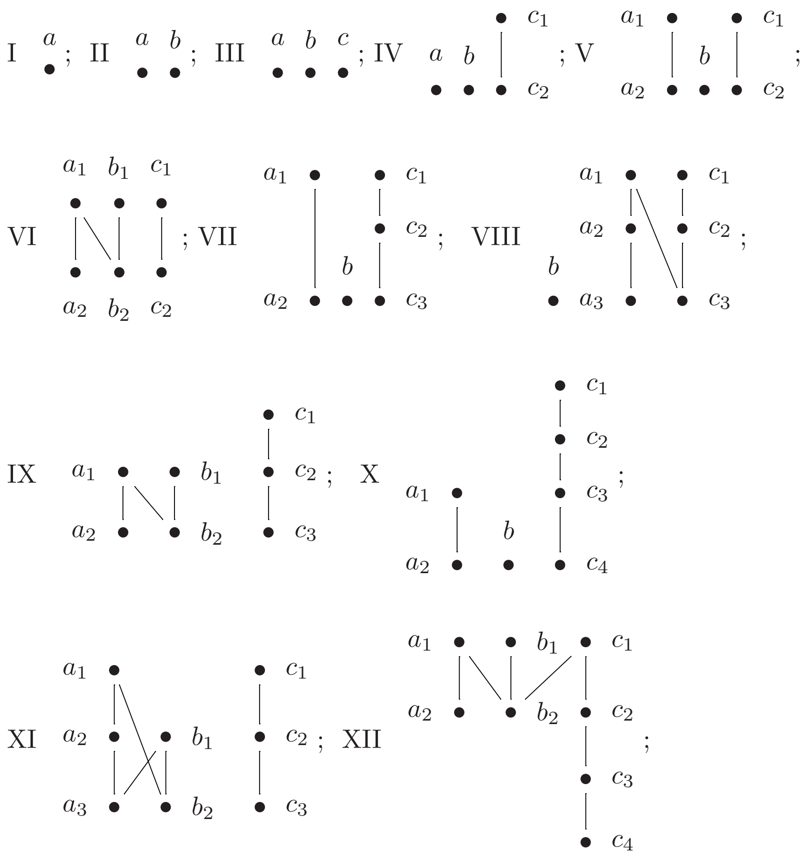
$(1, 2, 5)$: 

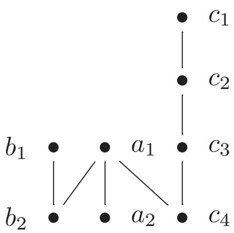
Here the n-element chain $(1 \prec 2 \prec \ldots \prec n)$ is denoted (n), and the disjoint union (direct sum) of $(n_1), \ldots, (n_s)$ is written $(n_1, \ldots, n_s)$. The symbol N denotes the following 4-element poset $\{a_1 \prec a_2 \succ a_3 \prec a_4\}$, and (N, n) stands for the disjoint union of N and the chain (n).

The posets in the list of the Kleiner theorem will be called the **critical subposets**.

A characterization of all exact representations of a finite-representation poset was obtained by M.M.Kleiner [Kleiner, 1972b]. We present here a complete list of all 41 indecomposable exact matrix representations of Kleiner's list with some corrections. As was mentioned in the paper [Arnold, Richman, 1992] five representations from Kleiner's list: (IX_3), (IX_9), (X_2), (X_8), and (XI) are decomposable. We replaced these representations by the right indecomposable ones, and the numeration remains unchanged.

Theorem 3.1.3 (M.Kleiner). *A finite partially ordered set $\mathcal{P}$ of finite type is exact if and only if it has one of the following forms:*



XIII  .

All distinct (up to similarity) exact indecomposable matrix representations of the posets listed above are the following. (Here they are written in the matrix form $(A_i,\ B_i,\ C_i)$, where A_i, B_i, C_i are the matrix blocks in the matrix representations of the posets corresponding to the points a_i, b_i, c_i, respectively.)

If $\mathcal{P} = \text{I}$, then $A = (1)$. (I)
If $\mathcal{P} = \text{II}$, then $A = B = (1)$. (II)
If $\mathcal{P} = \text{III}$, then $A = B = C = (1)$; (III_1)

$$A = \begin{pmatrix} & 1 & \\ 0, & B = & \end{pmatrix} 11,\ C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \qquad (\text{III}_2)$$

If $\mathcal{P} = \text{IV}$, then

$$A = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ C_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ C_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \qquad (\text{IV})$$

If $\mathcal{P} = \text{V}$, then

$$A_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 \\ 0 \\ 0, \end{pmatrix} B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ C_1 = \begin{pmatrix} 0 \\ 1 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 0 \\ 1; \end{pmatrix} \qquad (\text{V}_1)$$

$$A_1 = \begin{pmatrix} 0 \\ 1 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 1 \\ 0 \\ 0, \end{pmatrix} B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0, \end{pmatrix} C_1 = \begin{pmatrix} 0 \\ 1 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 0 \\ 1; \end{pmatrix} \qquad (\text{V}_2)$$

$$A_1 = \begin{pmatrix} 0 \\ 1, \end{pmatrix} A_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 1, \end{pmatrix} C_1 = \begin{pmatrix} 1 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1. \end{pmatrix} \qquad (\text{V}_3)$$

If $\mathcal{P} = \text{VI}$, then

$$A_1 = \begin{pmatrix} 0 \\ 1 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 1 \\ 1 \\ 1, \end{pmatrix} B_1 = \begin{pmatrix} 0 \\ 1 \\ 0, \end{pmatrix} \qquad (\text{VI})$$

$$B_2 = \begin{pmatrix} 1 \\ 0 \\ 0, \end{pmatrix} C_1 = \begin{pmatrix} 0 \\ 1 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 0 \\ 1. \end{pmatrix}$$

If $\mathcal{P} = \mathrm{VII}$, then

$$A_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1, \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0, \end{pmatrix} \tag{VII$_1$}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0, \end{pmatrix} A_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1, \end{pmatrix} B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{VII$_2$}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0\ , \end{pmatrix} A_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1, \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{VII$_3$}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 \\ 1 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 1 \\ 0 \\ 0, \end{pmatrix} B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{VII$_4$}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 \\ 1 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 1 \\ 0 \\ 0, \end{pmatrix} B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0, \end{pmatrix} \tag{VII$_5$}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1. \end{pmatrix}$$

If $\mathcal{P} = \text{VIII}$, then

$$A_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} A_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1, \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0, \end{pmatrix} \qquad \text{(VIII)}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0. \end{pmatrix}$$

If $\mathcal{P} = \text{IX}$, then

$$A_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0, \end{pmatrix} B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0, \end{pmatrix} B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1, \end{pmatrix} \qquad (\text{IX}_1)$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0, \end{pmatrix} B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0, \end{pmatrix} B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1, \end{pmatrix} \qquad (\text{IX}_2)$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1, \end{pmatrix} B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 0, \end{pmatrix} B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1, \end{pmatrix} \qquad (\text{IX}_3)$$

$$C_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0, \end{pmatrix} B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0, \end{pmatrix} B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1, \end{pmatrix} \tag{IX$_4$}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ , \end{pmatrix} A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ , & \end{pmatrix} B_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0, \end{pmatrix} B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1, \end{pmatrix} \tag{IX$_5$}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0, \end{pmatrix} B_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0, \end{pmatrix} B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1, \end{pmatrix} \quad \text{(IX}_6\text{)}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1, \end{pmatrix} B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1, \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0; \end{pmatrix} \tag{IX$_7$}$$

$$A_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1, \end{pmatrix} B_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0, \end{pmatrix} B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{IX$_8$}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 10 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 0 \\ 0 \\ 1, \end{pmatrix} B_1 = \begin{pmatrix} 0 \\ 1 \\ 0, \end{pmatrix} B_2 = \begin{pmatrix} 1 \\ 1 \\ 1, \end{pmatrix} \tag{IX$_9$}$$

$$C_1 = \begin{pmatrix} 0 \\ 0 \\ 1, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 1 \\ 0 \\ 0. \end{pmatrix}$$

If $\mathcal{P} = \mathrm{X}$, then

$$A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0, \end{pmatrix} A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1, \end{pmatrix} B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{X$_1$}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1, \end{pmatrix} B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{X$_2$}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0, \end{pmatrix} C_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0, \end{pmatrix} A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1, \end{pmatrix} B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{X$_3$}$$

$$C_1 = \begin{pmatrix} 0\\0\\0\\0\\1\\0, \end{pmatrix} C_2 = \begin{pmatrix} 0 & 0\\0 & 0\\1 & 0\\0 & 1\\0 & 0\\0 & 0, \end{pmatrix} C_3 = \begin{pmatrix} 0\\1\\0\\0\\0\\0, \end{pmatrix} C_4 = \begin{pmatrix} 1\\0\\0\\0\\0\\0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 & 0\\1 & 0\\0 & 1\\0 & 0\\0 & 0\\0 & 0, \end{pmatrix} A_2 = \begin{pmatrix} 1 & 0\\0 & 0\\0 & 0\\0 & 0\\0 & 0\\0 & 1, \end{pmatrix} B = \begin{pmatrix} 0 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\\1 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0, \end{pmatrix} \qquad (\mathrm{X}_4)$$

$$C_1 = \begin{pmatrix} 1 & 0\\0 & 1\\0 & 0\\0 & 0\\0 & 0\\0 & 0, \end{pmatrix} C_2 = \begin{pmatrix} 0\\0\\1\\0\\0\\0, \end{pmatrix} C_3 = \begin{pmatrix} 0\\0\\0\\1\\0\\0, \end{pmatrix} C_4 = \begin{pmatrix} 0\\0\\0\\0\\1\\0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 & 0\\0 & 1\\0 & 0\\0 & 0\\0 & 0\\0 & 0, \end{pmatrix} A_2 = \begin{pmatrix} 0 & 0\\0 & 0\\0 & 0\\0 & 0\\1 & 0\\0 & 1, \end{pmatrix} B = \begin{pmatrix} 1 & 0 & 1\\0 & 1 & 1\\0 & 0 & 1\\0 & 1 & 0\\0 & 1 & 0\\1 & 0 & 0 \end{pmatrix}, \qquad (\mathrm{X}_5)$$

$$C_1 = \begin{pmatrix} 1\\0\\0\\0\\0\\0, \end{pmatrix} C_2 = \begin{pmatrix} 0\\1\\0\\0\\0\\0, \end{pmatrix} C_3 = \begin{pmatrix} 0\\0\\1\\0\\0\\0, \end{pmatrix} C_4 = \begin{pmatrix} 0\\0\\0\\1\\0\\0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 & 0\\1 & 0\\0 & 1\\0 & 0\\0 & 0, \end{pmatrix} A_2 = \begin{pmatrix} 1 & 0\\0 & 0\\0 & 0\\0 & 0\\0 & 1, \end{pmatrix} B = \begin{pmatrix} 0 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\\1 & 0 & 1\\1 & 0 & 0 \end{pmatrix}, \qquad (\mathrm{X}_6)$$

$$C_1 = \begin{pmatrix} 1\\0\\0\\0\\0, \end{pmatrix} C_2 = \begin{pmatrix} 0\\1\\0\\0\\0, \end{pmatrix} C_3 = \begin{pmatrix} 0\\0\\1\\0\\0, \end{pmatrix} C_4 = \begin{pmatrix} 0\\0\\0\\1\\0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1, \end{pmatrix} B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{X$_7$}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} C_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0, \end{pmatrix} A_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1, \end{pmatrix} B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{X$_8$}$$

$$C_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0, \end{pmatrix} A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1, \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{X$_9$}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} C_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1, \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \tag{X$_{10}$}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} C_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0, \end{pmatrix} A_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1, \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{X_{11}}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} C_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0, \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}, \tag{X_{12}}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0, \end{pmatrix} C_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0, \end{pmatrix} A_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \tag{X_{13}}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0, \end{pmatrix} C_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1; \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}, \tag{X_{14}}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0, \end{pmatrix} C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0, \end{pmatrix} C_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1. \end{pmatrix}$$

If $\mathcal{P} = \mathrm{XI}$, then

$$A_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0, \end{pmatrix} A_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0, \end{pmatrix} A_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1, \end{pmatrix} B_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0, \end{pmatrix} B_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \tag{XI}$$

$$C_1 = \begin{pmatrix}0\\0\\1\\0,\end{pmatrix} C_2 = \begin{pmatrix}0\\1\\0\\0,\end{pmatrix} C_3 = \begin{pmatrix}1\\0\\0\\0.\end{pmatrix}$$

If $\mathcal{P} = \mathrm{XII}$, then

$$A_1 = \begin{pmatrix}1\\0\\0\\0\\0,\end{pmatrix} A_2 = \begin{pmatrix}1 & 1\\0 & 1\\1 & 0\\0 & 0\\0 & 1,\end{pmatrix} B_1 = \begin{pmatrix}0 & 1\\1 & 0\\0 & 0\\1 & 0\\0 & 0,\end{pmatrix} B_2 = \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}, \tag{XII}$$

$$C_1 = \begin{pmatrix}1\\0\\0\\0\\0,\end{pmatrix} C_2 = \begin{pmatrix}0\\1\\0\\0\\0,\end{pmatrix} C_3 = \begin{pmatrix}0\\0\\1\\0\\0,\end{pmatrix} C_3 = \begin{pmatrix}0\\0\\0\\1\\0.\end{pmatrix}$$

If $\mathcal{P} = \mathrm{XIII}$, then

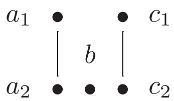
$$A_1 = \begin{pmatrix}0\\0\\1\\0\\0,\end{pmatrix} A_2 = \begin{pmatrix}0 & 1\\1 & 0\\0 & 0\\0 & 0\\1 & 0,\end{pmatrix} B_1 = \begin{pmatrix}1 & 0\\0 & 1\\1 & 1\\0 & 1\\0 & 0,\end{pmatrix} B_2 = \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}, \tag{XIII}$$

$$C_1 = \begin{pmatrix}1\\0\\0\\0\\0,\end{pmatrix} C_2 = \begin{pmatrix}0\\1\\0\\0\\0,\end{pmatrix} C_3 = \begin{pmatrix}0\\0\\1\\0\\0,\end{pmatrix} C_3 = \begin{pmatrix}0\\0\\0\\1\\0.\end{pmatrix}$$

Example 3.1.1.

In this example we show in what way one can obtain the vector representations of posets from the list above.

Consider the exact indecomposable matrix representation (V_2) of the poset

$\mathcal{P} = V$:  from the list above:

$$A_1 = \begin{pmatrix}0\\1\\0,\end{pmatrix} A_2 = \begin{pmatrix}1\\0\\0\\,\end{pmatrix} B = \begin{pmatrix}1 & 1\\0 & 1\\1 & 0,\end{pmatrix} C_1 = \begin{pmatrix}0\\1\\0,\end{pmatrix} C_2 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Then the corresponding matrix representation can be written as the following matrix partitioned horizontally into 5 vertical blocks:

$$\mathbf{A}=\begin{array}{|c|c|c c|c|c|}\hline 0 & 1 & 1 & 1 & 0 & 0\\ 1 & 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 1\\ \hline\end{array}$$

The corresponding vector representation $V = \{V_0, V_{a_1}, V_{a_2}, V_b, V_{c_1}, V_{c_2}\}$ over a field k can be obtained in the following way. Since the number of rows of the matrix A is equal to 3, $\dim_k V_0 = 3$. Therefore $V_0 = \{e_1, e_2, e_3\}$ is a vector space spanned by the basis elements $e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$. Since $a_2 \preceq a_1$ in $\mathcal{P}$, $V_{a_2} \subset V_{a_1}$ and so $V_{a_2} = \{e_1\}$, $V_{a_1} = \{e_1, e_2\}$. Since $c_2 \preceq c_1$ in $\mathcal{P}$ and a_1, c_2 are incomparable in $\mathcal{P}$, $V_{c_2} \subset V_{c_1}$, and so $V_{c_2} = \{e_3\}$, $V_{c_1} = \{e_2, e_3\}$, $V_b = \{e_1 + e_3, e_1 + e_2\} = k\begin{pmatrix}1\\0\\1\end{pmatrix} + k\begin{pmatrix}1\\1\\0\end{pmatrix}$.

For a finite poset $\mathcal{P} = (S, \preceq)$ of size m one introduces the rational **Tits quadratic form** $q_{\mathcal{P}} : \mathbf{Q}^{m+1} \to \mathbf{Q}$ defined by:

$$q_{\mathcal{P}} = \sum_{i=1}^{m+1} x_i^2 + \sum_{i \prec j \leq m} x_i x_j - (\sum_{i=1}^{m} x_i) x_{m+1}.$$

The rational Tits quadratic form $q_{\mathcal{P}}$ is called **weakly positive** if $q_{\mathcal{P}}(x) > 0$ for any nonzero vector $x = (x_1, x_2, \ldots, x_{m+1}) \in \mathbf{Q}^{m+1}$ with $x_1, x_2, \ldots, x_{m+1} \geq 0$. And this form is called **weakly non-negative**, if $q_{\mathcal{P}}(x) \geq 0$ for any nonzero vector $x = (x_1, x_2, \ldots, x_{m+1}) \in \mathbf{Q}^{m+1}$ with $x_1, x_2, \ldots, x_{m+1} \geq 0$.

Examples 3.1.2.

1. Let $\mathcal{P}$ be the poset $\left\{ \begin{array}{l} \bullet\ 2 \\ \mid \\ \bullet\ 1 \end{array} \right\}$. The corresponding rational Tits quadratic form

$$q_{\mathcal{P}} = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 - (x_1 + x_2) x_3$$

is weakly positive since

$$q_{\mathcal{P}} = \frac{1}{2}(x_1 + x_2)^2 + \frac{1}{2}(x_1 - x_3)^2 + \frac{1}{2}(x_2 - x_3)^2 > 0$$

for any nonzero vector (x_1, x_2, x_3).

2. Let $\mathcal{P}$ be the poset $\{\bullet \quad \bullet \quad \bullet \quad \bullet\}$. The corresponding rational Tits quadratic form

$$q_{\mathcal{P}} = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - (x_1 + x_2 + x_3 + x_4) x_5$$

is weakly non-negative since

$$q_{\mathcal{P}} = (x_1 - \frac{1}{2}x_5)^2 + (x_2 - \frac{1}{2}x_5)^2 + (x_3 - \frac{1}{2}x_5)^2 + (x_4 - \frac{1}{2}x_5)^2 \geq 0$$

for any nonzero vector $(x_1, x_2, x_3, x_4, x_5)$.

The following result of Yu.A.Drozd gives a characterization of finite posets of finite type using the Tits quadratic form:

Theorem 3.1.4 (Yu.A.Drozd). *Let $\mathcal{P} = (S, \preceq)$ be a finite poset over a field k with rational Tits quadratic form $q_{\mathcal{P}}$. Then*

1. *The poset $\mathcal{P}$ is of finite representation type if and only if $q_{\mathcal{P}}$ is weakly positive.*

2. *If $\mathcal{P}$ is a finite poset of finite representation type and $\mathbf{V}$ is (a matrix of) an indecomposable $\mathcal{P}$-space with* $\mathrm{cdn}(\mathbf{V}) = (s_1, s_2, \ldots, s_m, s_{m+1})$, *then* $q_{\mathcal{P}}(\mathrm{cdn}(\mathbf{V})) = 1$, $s_j \leq 6$ *for* $j = 1, 2, \ldots, m+1$ *and* $\mathrm{End}(\mathbf{V}) = k$.

3. *If $\mathcal{P}$ is a finite poset of finite representation type, and $\mathbf{V}$ and $\mathbf{W}$ are indecomposable $\mathcal{P}$-spaces, then $\mathbf{V} \simeq \mathbf{W}$ if and only if* $\mathrm{cdn}(\mathbf{V}) = \mathrm{cdn}(\mathbf{W})$.

As in the case of representations of quivers, for finite posets one can also introduce the notions of tame representation type and wild representation type.

Let $\mathcal{P} = (S, \preceq)$, where $S = \{1, 2, ..., m\}$, be a finite poset. And let $V = (V_0; V_i : i \in S)$ be a representation of $\mathcal{P}$ over a field k. As already said, the dimension vector $d = \underline{\dim} V = (d_0, d_1, \ldots, d_m)$ with coordinates $d_0 = \dim V_0$, $d_i = \dim V_i/\bar{V}_i$ is called the **dimension of the representation** V. Write $\omega(V) = d_0 + \sum\limits_{i=1}^{m} d_i$.

Definition. We say that a poset $\mathcal{P}$ is of **tame representation type** over a field k, if for any natural number n there exists a finite set of representations $M_n \subset \mathrm{Rep}(\mathcal{P}, k[x])$ such that any $V \in \mathrm{Ind}(\mathcal{P}, k)$ with $\omega(V) \leq n$ has the form $S \otimes B$, where $S \in M_n$ and B is a finite dimensional $k[x]$-module.

Definition. We say that a poset $\mathcal{P}$ is of **wild representation type** over a field k, if there exists a representation S of $\mathcal{P}$ over the free algebra $k\langle x, y\rangle$ such that for any two non-isomorphic and indecomposable finite dimensional $k\langle x, y\rangle$-modules B_1 and B_2 the representations $S \otimes B_1$ and $S \otimes B_2$ are indecomposable and non-isomorphic over k.

The fundamental result due to Yu.A.Drozd says that there is a trichotomy between finite, tame and wild representation type for finite posets over an algebraically closed field k:

Theorem 3.1.5 (Yu.A.Drozd). *If k is an algebraically closed field and $\mathcal{P}$ is a finite poset it is of finite, tame or wild representation type and these types are mutually exclusive.*

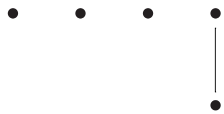
The following famous theorem gives a characterization of a wild partially ordered sets.

Theorem 3.1.6 (L.A.Nazarova). *Let $\mathcal{P}$ be a poset of infinite representation type over an algebraically closed field k with the rational Tits quadratic form $q_{\mathcal{P}}$. Then the following conditions are equivalent*:

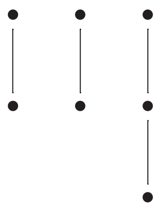
1. *The poset $\mathcal{P}$ is of tame representation type.*
2. *The poset $\mathcal{P}$ is not of wild representation type.*
3. *The quadratic form $q_{\mathcal{P}}$ is weakly non-negative.*
4. *The poset $\mathcal{P}$ does not contain as a full subposet any poset from the following list:*

$\mathcal{N}_1 = (1,1,1,1,1):$

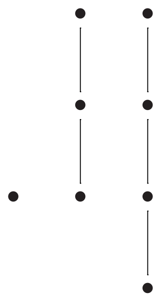
$\mathcal{N}_2 = (1,1,1,2):$



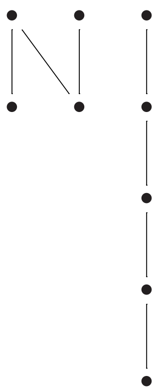
$\mathcal{N}_3 = (2,2,3):$



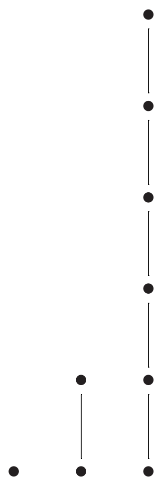
$\mathcal{N}_4 = (1,3,4):$



$\mathcal{N}_5 = (N,5):$



$\mathcal{N}_6 = (1,2,6):$



Remark 3.1.1. This theorem was proved by L.A.Nazarova [Nazarova, 1975] and independently by P.Donovan, M.R.Freislich [Donovan, M.R.Freislich, 1974].

3.2 DIFFERENTIATION ALGORITHMS FOR POSETS

The results characterizing posets of finite representation type and posets of wild type were obtained by M.M.Kleiner and L.A.Nazarova using methods developed by L.A.Nazarova and V.A.Roiter in their paper [Nazarova, Roiter, 1972]. An important role in these methods is played by a differentiation algorithm for posets. In 1977 A.G.Zavadskij in his paper [Zavadskij, 1977] introduced another differentiation algorithm for computing representations of posets. Since both these algorithms are of great importance and are used in various applications we shall give them, but without proof.

Definition. The **width** $w(\mathcal{P})$ of a poset $\mathcal{P} = (S, \preceq)$ is a maximal number of pairwise incomparable elements in $\mathcal{P}$, i.e. the maximal length of an antichain.

A poset $\mathcal{P}$ is called a **chain** if any two of its elements are comparible.

Examples 3.2.1.

1. The posets (2,2,2), (1,3,3), (N,4), (1,2,5) in theorem 3.1.2 have width equal to 3, whereas the poset (1,1,1,1) has width equal to 4.
2. A poset $\mathcal{P}$ is of width one if and only if $\mathcal{P}$ is a chain.

Lemma 3.2.1. *If the width $w(\mathcal{P})$ of a poset $\mathcal{P}$ is greater than or equal to four, then $\mathcal{P}$ is of infinite representation type.*

Proof. Suppose that $w(\mathcal{P}) \geq 4$. Then $\mathcal{P}$ contains a full subposet $\mathcal{R}$ consisting of four incomparable elements. It is sufficient to show that $\mathcal{R}$ is a poset of infinite type over a field k. For this purpose we note that for any $\lambda \in k$ the matrix representation

$$\mathbf{A}^{(n,\lambda)} = \begin{array}{|c|c|c|c|} \hline J(n,\lambda) & \mathbf{E} & \mathbf{E} & \mathbf{O} \\ \mathbf{E} & \mathbf{E} & \mathbf{O} & \mathbf{E} \\ \hline \end{array}$$

with $cdn(\mathbf{A}^{(n,\lambda)}) = (n, n, n, n, 2n)$ is an indecomposable representation of $\mathcal{R}$, where $\mathbf{E}$ is the identity matrix in $M_n(k)$ and $J(n, \lambda)$ is a Jordan block of size $n \times n$.

From this lemma it follows that a poset of finite representation type has width $\leq$ 3.

For a poset $\mathcal{P} = (S, \preceq)$ and $x \in S$ we define the **upper** and **lower cone** of x:

$$x^{\Delta} = \{y \in S : \ x \preceq y\}$$

$$x^{\nabla} = \{y \in S \ : \ y \preceq x\}$$

Definition. Let $\mathcal{P} = (S, \preceq)$ and $\mathcal{T} = (T, \preceq)$ be any two (disjoint) posets. The **cardinal sum** $\mathcal{P} \cup \mathcal{T}$ or $(\mathcal{P}, \mathcal{T})$ of $\mathcal{P}$ and $\mathcal{T}$ is the set of all $s \in S$ and $t \in T$ with a relation $\preceq$ such that $s \preceq s_1$ and $t \preceq t_1$ $(s, s_1 \in S;\ t, t_1 \in T)$ have unchanged meanings and there are no other relations in $\mathcal{P} \cup \mathcal{T}$.

Definition. Let $\mathcal{P} = (S, \preceq)$ be a poset and let $a \in S$ be a maximal element such that

$$w(P \setminus a^{\nabla}) \leq 2.$$

We define a new poset

$$\partial_a \mathcal{P} \;=\; \{\mathcal{P} \setminus a\} \cup \{(p, q) \,:\, p, q, a \text{ are incomparable in } \mathcal{P}\}.$$

The order relation $\preceq$ in $\partial_a P$ is defined by keeping the relation between elements of P and by the following formulas:

(1) $i \preceq (p, q)$ if and only if $i \preceq p$ or $i \preceq q$;

(2) $(p, q) \preceq i$ if and only if $p \preceq i$ and $q \preceq i$;

(3) $(p, q) \preceq (p_1, q_1)$ if and only if for any $x \in \{p, q\}$ there is $y \in \{p_1, q_1\}$ such that $x \preceq y$.

It is clear that $\partial_a \mathcal{P}$ becomes a poset which is called the **differential of** $\mathcal{P}$ **with respect to the maximal element** $a \in \mathcal{P}$.

Remark 3.2.1. Note that a maximal element with property $w(P \setminus a^{\nabla}) \leq 2$ does not always exist for any poset. For example, the poset $\mathcal{P} :\ \{\bullet\ \bullet\ \bullet\ \bullet\}$ has no such maximal element. However for any poset $\mathcal{P}$ of width ≤ 3 each maximal element $a \in \mathcal{P}$ has this property, i.e., $w(P \setminus a^{\nabla}) \leq 2$.

Theorem 3.2.2. *Suppose that $\mathcal{P}$ is a finite poset, $w(\mathcal{P}) \leq 3$ and $a \in \mathcal{P}$ is a maximal element. Then the posets $\mathcal{P}$ and $\partial_a \mathcal{P}$ have the same representation type (i.e., both are either of finite representation type or of infinite representation type).*

The following result shows that the differentiation procedure is an algorithm for determining matrix representations of posets.

Theorem 3.2.3. *A poset $\mathcal{P}$ is of finite representation type if and only if $w(\mathcal{P}) \leq 3$ and the differentiation procedure reduces $\mathcal{P}$ in finitely many steps to the empty poset.*

Remark 3.2.2. Note that the differentiation procedure described above does not depend on the choice of a sequence of maximal elements (see [Gabriel, 1972/1973]).

The differentiation procedure with respect to a maximal element, which is often called the **reduction algorithm of Nazarova-Roiter**, is always applicable to posets of width at most three.

We shall now describe the differentiation procedure for posets due to V.Zavadskij. This two-point differentiation algorithm can be applied to posets in more general situations in comparison with the reduction algorithm of Nazarova-Roiter.

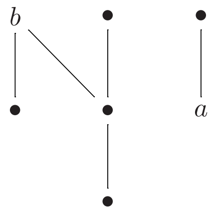
Definition. Let $\mathcal{P} = (S, \preceq)$ be a poset, and $a, b \in S$. A pair (a, b) is called **suitable** if $a \not\prec b$, $a \neq b$ and the subposet

$$\mathcal{P}_b^a = \mathcal{P} \setminus (a^\Delta \cup b^\nabla)$$

of $\mathcal{P}$ is either empty or is a chain. We call (a, b) **irreducible** if there is no relation $b \prec j$ and $i \prec a$ with $i, j \in \mathcal{P}_b^a$.

Remark 3.2.3. A suitable pair of elements does not always exist for any poset. For example, the poset $\mathcal{P}: \ \{\bullet \ \ \bullet \ \ \bullet \ \ \bullet\}$ has no such suitable pair of elements. If a poset contains no suitable pair of elements it is called **non-differential**.

Example 3.2.2.
Consider the poset $\mathcal{P}$ of the following form:



which width equal to 3. Here b is a maximal element of $\mathcal{P}$ and a is a minimal element in $\mathcal{P} \setminus b^\nabla$. Here (a, b) is a suitable pair.

Definition. Let (a, b) be a suitable pair of elements in a finite poset $\mathcal{P} = (S, \preceq)$, and let $\mathcal{P}_b^a = C = \{c_1 \prec c_2 \prec \ldots \prec c_n\}$ be a chain or the empty set. We define a new poset

$$\delta_{(a,b)}\mathcal{P} = b^\nabla + C^- + C^+ + a^\Delta$$

according to the following rules:

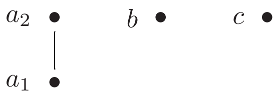
(1) If C is not empty replace C by two chains $C^+ = \{c_1^+ \prec c_2^+ \prec \ldots \prec c_n^+\}$ and $C^- = \{c_1^- \prec c_2^- \prec \ldots \prec c_n^-\}$ such that c_i^+ and c_i^- satisfy the same relations as c_i in $\mathcal{P}$ for $i = 1, 2, \ldots, n$. If C is empty $C^+ = C^-$ is empty as well. The partial ordering on a^Δ and b^∇ is as in $\mathcal{P}$.

(2) Add new relations, namely $a \prec b$, $a \prec c_1^+$, $c_n^- \prec b$ and $c_i^- \prec c_i^+$ for $i = 1, 2, ..., n$.

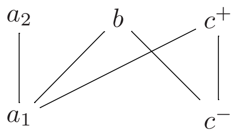
(3) If during the second step we get elements x, y with $x \prec y$ and $y \prec x$, then we identify them in $\delta_{(a,b)}\mathcal{P}$.

The poset $\delta_{(a,b)}\mathcal{P}$ will be called the **differential of $\mathcal{P}$ with respect to a suitable pair** (a, b) in $\mathcal{P}$.

Example 3.2.3.
Let $\mathcal{P}$ be of the following form:



Then $\delta_{(a_1,b)}\mathcal{P}$ has the form:



Remark 3.2.4. If C is the empty set, then $\delta_{(a,b)}\mathcal{P} = (S_1, \preceq_1)$, where $\preceq_1$ is the same as $\preceq$ in $\mathcal{P}$ with a new added relation $a \prec_1 b$. If there is a relation $b \prec a$ then we identify a and b in S, otherwise $S_1 = S$.

Remark 3.2.5. The poset $\delta_{(a,b)}\mathcal{P}$ was defined by A.G.Zavadskij and V.V.Kirichenko in the particular case when $|C| \leq 1$ [Zavadskij, Kirichenko, 1977]. In the general case this definition was introduced by A.G.Zavadskij in his paper [Zavadskij, 1977].

Definition. We say that a poset $\mathcal{P}$ is **$\mathcal{N}_*$-free** if $\mathcal{P}$ contains as a full subposet any poset of the form $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{N}_4$, $\mathcal{N}_5$ and $\mathcal{N}_6$ from the list of theorem 3.1.6.

Any poset which is a cardinal sum of one-pointed posets and posets of the form (1,1) is called a **garland**.

A disjoint union decomposition $\mathcal{P} = \mathcal{P}' + C + \mathcal{P}''$ is called a **splitting decomposition** of $\mathcal{P}$ if

(1) The subset C is either empty or a chain.

(2) $x \prec y$ for all $x \in \mathcal{P}'$ and all $x \in \mathcal{P}''$.

For the study of indecomposable representations of posets of tame type by using the differentiation procedure with respect to a suitable pair the most important statements are the following (see [Nazarova, Zavadskij, 1977], [Bondarenko, Nazarova, Zavadskij, 1979]):

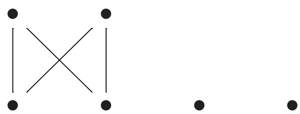
Proposition 3.2.4. *If $(a.b)$ is an irreducible suitable pair in the poset $\mathcal{P}$ and $\mathcal{P}$ is $\mathcal{N}_*$-free, then the derived poset $\delta_{(a,b)}\mathcal{P}$ is also $\mathcal{N}_*$-free.*

Proposition 3.2.5. *Any exact non-differential poset which does not contain as a full subposet any poset of the form $\mathcal{N}_1 = (1,1,1,1,1)$ and $\mathcal{N}_2 = (1,1,1,2)$ is a sum of two garlands.*

Theorem 3.2.6. *Suppose $\mathcal{P}$ is an $\mathcal{N}_*$-free poset and $w(\mathcal{P}) \geq 2$.*

(1) *If $\mathcal{P}$ has no nontrivial splitting decomposition and $\mathcal{P}$ does not contain as a full subposet any poset of the following form:*

$\mathcal{NZ}$:



then either $\mathcal{P} = (1,1,1,1)$ or there is an irreducible suitable pair (a,b) in $\mathcal{P}$.

(2) *If (a,b) is a suitable pair in $\mathcal{P}$ and $\mathcal{P}$ does not contain as a full subposet any poset of the form $\mathcal{NZ}$ then $\delta_{(a,b)}\mathcal{P}$ also does not contain a subposet of the form $\mathcal{NZ}$.*

Remarks 3.2.5. From results of the paper [Nazarova, Roiter, 1973] it follows that all posets which are a sum of two garlands are of tame type.

For a given suitable pair (a,b) of elements of a poset $\mathcal{P}$ define a map

$$\delta_{(a,b)} : \mathrm{Rep}(\mathcal{P}, k) \to \mathrm{Rep}(\delta_{(a,b)}\mathcal{P}, k)$$

as follows. Let $C = \mathcal{P}_b^a = \{c_1 \prec c_2 \prec \ldots \prec c_n\}$, and let $V = (V_0; V_i \ : \ i \in \mathcal{P})$ be a $\mathcal{P}$-space. Choose a k-subspace $U \subseteq V_a$ such that $V_a + V_b = U \oplus V_b$ and set $\delta_{(a,b)}(V) = W = (W_0; W_i \ : \ i \in \delta_{(a,b)}\mathcal{P})$, where

(i) $W_0 = V_0/U$;

(ii) $W_{c_i^+} = (V_a + V_{c_i})/U$ for $i = 1, 2, \ldots, n$;

(iii) $W_{c_i^-} = ((V_b \cap V_{c_i}) + U)/U$ for $i = 1, 2, \ldots, n$;

(iv) $W_j = (V_j + U)/U$ for $j \in (a^{\Delta} \cup b^{\nabla})$.

We call $\delta_{(a,b)}$ a **differentiation map with respect to the suitable pair** (a,b).

It can be shown that the definition of $\delta_{(a,b)}(V)$ does not depend of the choice of U and so we have a well defined map.

The following fundamental result gives the main tool in the study of indecomposable representations of posets of finite type and posets of tame type.

Theorem 3.2.7 (A.G.Zavadskij). *Let $\mathcal{P}$ be a finite poset with a suitable pair (a,b), and let $\mathcal{P}_b^a = C = \{c_1 \prec c_2 \prec \ldots \prec c_n\}$ be a chain or an empty set. Then the differentiation map $\delta_{(a,b)}$, defined above, has the following properties*:

(i) $\delta_{(a,b)}(V \oplus V') \simeq \delta_{(a,b)}(V) \oplus \delta_{(a,b)}(V')$;

(ii) *Suppose V is an indecomposable representation of $\mathcal{P}$. Then $\delta_{(a,b)}(V) = 0$ if and only if V is isomorphic to one of the $\mathcal{P}$-spaces P_a, $P_{(a,c_i)}$, $i = 1, 2, \ldots, n$, where P_a is the induced representation of the one-element subset $\{a\}$, $P_{(a,c_i)}$ is the induced representation of the two-element subset $\{a, c_i\}$ so that $P_{(a,c_i)} = P_a + P_{c_i}$. If $\delta_{(a,b)}(V) \neq 0$ then it is indecomposable.*

(iii) *The mapping* $\delta_{(a,b)} : \mathrm{Rep}(\mathcal{P}, k) \to \mathrm{Rep}(\delta_{(a,b)}\mathcal{P}, k)$ *yields a mapping correspondence* $\delta_{(a,b)} : \mathrm{Ind}(\mathcal{P}, k) \to \mathrm{Ind}(\delta_{(a,b)}\mathcal{P}, k)$.

(iv) $|\mathrm{Ind}(\mathcal{P}, k)| = |\mathrm{Ind}(\delta_{(a,b)}\mathcal{P}, k)| + 1 + n$.

As an immediate consequence of this theorem we have the following result:

Corollary 3.2.7. *A poset* $\mathcal{P}$ *is of finite representation type if and only if the derived poset* $\delta_{(a,b)}\mathcal{P}$ *is of finite type.*

3.3 REPRESENTATIONS AND MODULES. THE REGULAR REPRESENTATIONS

Let k be a field, and let A be an associative finite dimensional k-algebra with 1, where k is an arbitrary field.

Definition. A **representation** of a k-algebra A is an algebra homomorphism $T : A \to \mathrm{End}_k(V)$, where V is a k-vector space.

In other words, to define a representation T is to assign to every element $a \in A$ a linear operator $T(a)$ in such a way that

$$T(a + b) = T(a) + T(b)$$

$$T(\alpha a) = \alpha T(a)$$

$$T(ab) = T(a)T(b)$$

$$T(1) = E \quad \text{(the identity operator)}$$

for arbitrary $a, b \in A$, $\alpha \in k$.

The action of the operators $T(a)$ on V is written on the right, i.e. $T(a) : V \to V$, $v \mapsto vT(a)$.

If the vector space V is finite dimensional over k, then its dimension is called the **dimension** (or **degree**) of the representation T. Obviously, the image of a representation T forms a subalgebra in $\mathrm{End}_k(V)$. If T is a monomorphism, then this subalgebra is isomorphic to the algebra A. In this case the representation T is called **faithful**.

Let $T : A \to \mathrm{End}_k(V)$ and $S : A \to \mathrm{End}_k(W)$ be two representations of a k-algebra A. A **morphism** from the representation T to the representation S is a linear k-vector map $\varphi : V \to W$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "T(x)"'] & W \arrow[d, "S(x)"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd}$$

is commutative for all $x \in A$, that is, $\varphi T(x) = S(x)\varphi$, for all $x \in A$.

If φ is an invertible morphism, then it is called an **isomorphism** of representations. Two representations T and S are called **isomorphic** if there is an isomorphism φ from the representation T to the representation S, and in this case we have

$$S(x) = \varphi T(x)\varphi^{-1}. \tag{3.3.1}$$

Theorem 3.3.1 (A.L.Cayley). *Every finite dimensional algebra admits a faithful representation. In other words, every algebra is isomorphic to a subalgebra of an algebra of linear operators.*

Proof. From the axioms for algebras it follows that for every $a \in A$ the map $T(a) : x \mapsto xa$, $x \in A$, is a linear operator on the space A. Moreover, $T(a+b) = T(a) + T(b)$, $T(\alpha a) = \alpha T(a)$, $T(ab) = T(a)T(b)$ and $T(1) = E$ (the identity operator). Thus, T is a representation of the algebra A. If $a \neq b$, then $1 \cdot a \neq 1 \cdot b$. This shows that the operators $T(a)$ and $T(b)$ are distinct and T is a faithful representation, as required.

The representation constructed in the proof of Cayley's theorem is called (right) **regular**. The dimension of the regular representation equals the dimension of the algebra.

If the dimension of a representation T is equal to n, then one may choose a basis $(e_1, e_2, ..., e_n)$ in the space V and assign to each operator $T(a) \in \mathrm{End}_k(V)$ its matrix $\mathbf{T}_a = (a_{ij})$ in this basis so that

$$Te_j = \sum_{i=1}^{n} a_{ij}e_i, \quad a_{ij} \in k.$$

Obviously, the correspondence $a \mapsto \mathbf{T}_a$ is a homomorphism from the algebra A to the matrix algebra $M_n(k)$. Such a homomorphism is called a **matrix representation** of A.

If one chooses bases in the vector spaces V and W, and $\mathbf{T}_x$, $\mathbf{S}_x$ are the matrices of the linear operators T_x and S_x in these bases, the condition (3.3.1) can be rewritten in the following equivalent form:

$$\mathbf{S}_x = \mathbf{P}\mathbf{T}_x\mathbf{P}^{-1}, \tag{3.3.2}$$

where $\mathbf{P} \in \mathrm{GL}(n, k)$ does not depend on the element $x \in A$.

A **subrepresentation** of a representation T is given by a subspace W of V which is $T(a)$-invariant for all $a \in A$. In this case one can construct a representation on V/W, called the **quotient representation**.

Given a representation $T : A \to \mathrm{End}_k(V)$ of A, there is the **dual** (or **contragredient**) **representation** $T^* : A^{op} \to \mathrm{End}_k(V^*)$ of the opposite algebra A^{op} of A (this is the algebra on the underlying vector space of A with multiplication $*$

defined by $a * b = ba$). By definition, $vT^*(a)(\varphi) = \varphi(vT(a))$ for $a \in A^{op} = A$, $v \in V$, $\varphi \in V^* = \mathrm{Hom}_k(V, k)$.

There is a close connection between the representations of an algebra A and its modules. For any representation of A we can construct a right module over this algebra, and, vice versa, for any right module we can construct a representation.

Let $T : A \to \mathrm{End}_k(V)$ be a representation of an algebra A. Define $va = vT(a)$ for $v \in V$, $a \in A$. From the definition of a representation it follows immediately that, in this way, V becomes a right A-module. We say that this module corresponds to the representation T. On the other hand, for any right A-module we can construct a representation of A. Indeed, if M is a right A-module, then for a fixed $a \in A$, the map $T(a) : m \mapsto ma$ is a linear transformation in M. Assigning to every $a \in A$ the operator $T(a)$ we obtain a representation of the algebra A corresponding to the module M.

Given two representations $T_1 : A \to \mathrm{End}_k(V_1)$ and $T_2 : A \to \mathrm{End}_k(V_2)$, a mapping $f : T_1 \to T_2$ is a linear transformation $f : V_1 \to V_2$ satisfying $f(vT_1(a)) = f(v)T_2(a)$ for $v \in V$, $a \in A$, or, rewritten, $f(va) = f(v)a$; hence it is an A-module homomorphism. Thus, the category of all representations of A is equivalent to the category of all right A-modules.

To left A-modules there correspond the dual representations of the algebra A. In particular, by considering the algebra A as a left module over itself, we obtain the concept of the regular left module and the **regular dual representation**.

If $T_i : A \to \mathrm{End}_k(V_i)$ is a family of representations, their direct sum is the representation $T : A \to \mathrm{End}_k(V)$, where $V = \bigoplus_i V_i$ is a direct sum of vector spaces and $T(A)|_{V_i} = T_i(x)$ for all $x \in A$. The category of all representations is an additive Abelian category.

Definition. A representation of A is said to be **simple** (or **irreducible**) provided it is nonzero and the only proper subrepresentation is the zero representation.

By choosing a suitable basis of a vector space V all matrices of a reducible representation T in this basis have the form:

$$\mathbf{T}_x = \begin{array}{c|c} \mathbf{T}_x^{(1)} & \mathbf{U}_x \\ \hline \mathbf{0} & \mathbf{T}_x^{(2)} \end{array}$$

for all $x \in A$, where the $\mathbf{T}_x^{(i)}$ are square matrices of degree $n_i < n$, where $n = \dim_k V$.

The Schur lemma (see proposition 2.2.1, vol.I) says that the endomorphism ring of a simple representation is a division ring.

Definition. A representation T of a k-algebra A is said to be **indecomposable** if its corresponding right A-module is indecomposable. In other words, a representation T is indecomposable if it cannot be written as a direct sum of nonzero representations. Otherwise it is called **decomposable**.

A simple module is obviously indecomposable. But an arbitrary indecomposable module may have proper submodules.

Definition. A representation T of a k-algebra A is said to be **completely reducible** if it is a direct sum of irreducible representations.

By choosing a suitable basis of a vector space V all matrices of a completely reducible representation T in this basis have the form:

$$\mathbf{T}_x = \begin{pmatrix} \mathbf{T}_x^{(1)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_x^{(2)} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{T}_x^{(m)} \end{pmatrix}$$

for all $g \in G$, where the $\mathbf{T}_x^{(i)}$ form irreducible square matrix representations of degree $n_i < n$, where $n = \dim_k V$, $i = 1, 2, ..., m$.

Any finite dimensional module over an algebra A can be uniquely written (up to isomorphism) in the form of a direct sum of indecomposable modules by the Krull-Remak-Schmidt theorem. This means that for many questions one can restrict attention to the consideration of indecomposable modules.

Let A be a finite dimensional k-algebra, and let V be a finite dimensional right A-module. We construct a category $C(V)$ whose objects are the submodules of tensor products of the form $U \otimes_k V$, the U being finite dimensional vector spaces; a morphism from an object $X \subset U \otimes_k V$ to an object $Y \subset W \otimes_k V$ is a linear mapping $\phi : U \to W$ such that $(\phi \otimes 1)(X) \subset Y$. The problem of classifying the objects of $C(V)$ up to isomorphism is, by definition, a **linear matrix problem**.

The following considerations and constructions serve to reformulate such a linear matrix problem as a representation theoretic problem. Clearly, the module V may be assumed faithful (by replacing, if needed, the algebra A with its quotient algebra $A/(\mathrm{Ker}(A \to \mathrm{End}_k(V)))$). Thus A is identified with a subalgebra of $E = \mathrm{End}_k V$. Let $\mathcal{O}$ be a discrete valuation ring with field of residues k, and let π be a prime element of $\mathcal{O}$. Consider an $\mathcal{O}$-lattice L (i.e., a free $\mathcal{O}$-module) of rank $n = \dim V$. If $\Gamma = \mathrm{End}_{\mathcal{O}} L$, then $\Gamma/\pi\Gamma \simeq E$, and $L/\pi L \simeq V$ as an E-module. Let Λ stand for the preimage of the subalgebra $A \subset E$ in Γ, and consider the category $\mathrm{Rep}(\Lambda)$ of representations of Λ (over $\mathcal{O}$); that is, of Λ-modules that are $\mathcal{O}$-lattices. Every such module M can be naturally embedded into the Γ-module $M\Gamma$ which is a representation of Γ. But every Γ-module is of the form $F \otimes_{\mathcal{O}} L$, where F is some $\mathcal{O}$-lattice, and every Γ-homomorphism $F \otimes_{\mathcal{O}} L \to G \otimes_{\mathcal{O}} L$ is of the form $f \otimes 1$, where $f : F \to G$ is a homomorphism of $\mathcal{O}$-modules, and f is an isomorphism if and only if $\bar{f} : U \to W$ is an isomorphism, where $U = F/\pi F$ and $W = G/\pi G$.

Since $\Lambda \supset \pi\Gamma$, it follows that $M \supset \pi M\Gamma$. Set $\bar{M} = M/\pi M\Gamma$. This is an A-submodule of $\overline{M\Gamma} = M\Gamma/\pi M\Gamma$. Let $g : M \to N$ be a homomorphism of Λ-representations. This can be extended to a Γ-homomorphism $g\Gamma : M\Gamma \to N\Gamma$, where if $M\Gamma = F \otimes_{\mathcal{O}} L$, $N\Gamma = G \otimes_{\mathcal{O}} L$ and $g\Gamma = f \otimes 1$. Then g, $g\Gamma$ and f are all isomorphisms if one of them is. Moreover, if $U = F/\pi F$, $W = G/\pi G$ and $\bar{f} : U \to W$ is the homomorphism induced by f, then $\overline{M\Gamma} = U \otimes_k V$, $\overline{N\Gamma} = W \otimes_k V$ and $(\bar{f} \otimes 1)(\bar{M}) \subset \bar{N}$.

We note that an A-submodule $X \subset U \otimes_k V$ is of the form $\bar{M}$ if and only if $XE = U \otimes_k V$. However, for every $X \subset U \otimes_k V$, XE is a direct summand of $U \otimes_k V$; that is, $U = U_1 \oplus U_2$, where $X \subset U_1 \otimes_k V$ and $XE = U_1 \otimes_k V$. Therefore an object $X \subset U \otimes_k V$ in $C(V)$ is the direct sum of the object $X \subset U_1 \otimes_k V$ and an object $Y \subset U_2 \otimes_k V$. The second summand is isomorphic with O^m, where O is the object in $C(V)$ determined by the zero submodule of V, and $m = \dim U_2$. These arguments yield the following statement:

Theorem 3.3.2. *Let M and N be representations of Λ, let $g : M \to N$ be a homomorphism, and suppose that $M\Gamma = F \otimes_{\mathcal{O}} L$, $N\Gamma = G \otimes_{\mathcal{O}} L$, $U = F/\pi F$, $W = G/\pi G$ and $g\Gamma = f \otimes 1$, where $f : F \to G$ and $\bar{f} : U \to W$ is the mapping induced by f. Setting $\Psi(M) = \bar{M} \subset U \otimes_k V$ and $\Psi(g) = \bar{f}$, we obtain a functor $\Psi : \mathrm{Rep}(\Lambda) \to C(V)$. Here $\Psi(M) \simeq \Psi(N)$ if and only if $M \simeq N$. Every object of $C(V)$ is isomorphic to a $\Psi(M) \oplus O^m$ for some $M \in \mathrm{Rep}(\Lambda)$.*

Thus the classification of representations of Λ is equivalent to the given linear matrix problem.

In conclusion we show that this scheme includes the representations of a poset S; that is, homomorphisms of S into the lattice of subspaces of a finite-dimensional space U. For this purpose construct the algebra $A = A(S)$ with basis $\{a_{ij} \ : \ i, j \in S;\ i \preceq j\}$ and multiplication table $a_{ij}a_{kl} = \delta_{jk}a_{il}$, and the A-module $V = V(S)$ with basis $\{v_i \ : \ i \in S\}$ and the operator action given by $\nu_i a_{kj} = \delta_{ij}\nu_k$, where δ_{ij} is the Kronecker delta symbol. If X is an A-submodule of $U \otimes_k V$, then associating $i \in S$ with the subspace $X_i = \{u \in U \ : \ u \otimes \nu_i \in X\}$ gives a representation of S in U. Conversely, if $i \mapsto X_i$ is a representation of S in U, then $X = \sum_i x_i \otimes \nu_i$ is a submodule of $U \otimes_k V$. If $S = \{1, \ldots, n\}$, then A can be identified with the subalgebra of $M_n(k)$ spanned by the basis $\{e_{ij} \ : \ i, j \in S;\ i \preceq j\}$ (here the e_{ij} are the matrix units). Then the corresponding ring $\Lambda = \Lambda(S)$ is a subring of $M_n(\mathcal{O})$ with $\mathcal{O}$-basis $\{d_{ij}e_{ij} \ : \ i, j \in S\}$, where $d_{ij} = 1$ for $i \preceq j$ and $d_{ij} = \pi$ otherwise. The representations of $\Lambda(S)$ over $\mathcal{O}$ are classified up to isomorphism by the representations of the poset S.

Remarks 3.3.1.

1) The classification of the representations of an arbitrary $\mathcal{O}$-order Λ such that $\Gamma \supset \Lambda \supset \pi\Gamma$ for some maximal order Γ can be reduced in a similar fashion to a

matrix problem (though not necessarily to a linear one). If $\Lambda \not\supset \pi\Gamma$ we obtain such a problem but not over k this time, but over the quotient ring $\mathcal{O}/\pi^k\mathcal{O}$.

2) Replacing the submodule $X \subset U \otimes_k V$ by its projective cover P, or more exactly by the corresponding homomorphism $P \to U \otimes_k V$, we obtain an interpretation of the linear matrix problem in terms of V-matrices (see [Drozd, 1972]). For posets this gives essentially the original matrix treatment of L.A.Nazarova and A.V.Roiter (see [Nazarova, Roiter, 1972]).

3.4 ALGEBRAS OF FINITE REPRESENTATION TYPE

One of the main problems in the theory of representations is to get information about the structure of indecomposable modules. And, if possible, to obtain the complete description of all indecomposable modules up to isomorphism. All algebras are divided in different types of representation classes. As was conjectured by P.Donovan and M.Freislich and established by Yu.A.Drozd, for the case of an algebraically closed field k, there are only three different representation classes: finite, tame and wild representation type.

Definition. A k-algebra A is said to be of **finite representation type** (or **finite type**, in short) if A has only a finite number of non-isomorphic finite dimensional indecomposable representations up to isomorphism. Otherwise A is said to be of **infinite representation type**.

Examples 3.4.1.

1. Every indecomposable representation of a finite dimensional semisimple algebra is equivalent to a direct summand of the regular representation, by the Wedderburn theorem. Hence, every finite dimensional semisimple algebra is an algebra of finite representation type.
2. A k-algebra $k[x]/(x^n)$ is an algebra of finite representation type.
3. The algebra $A = \{1, r, s \ : \ r^2 = s^2 = rs = sr = 0\}$ is an algebra of infinite representation type.
4. The k-algebra $k[x,y]/(x^n, y^m)$, for $n, m \geq 2$, is an algebra of infinite representation type.
5. The group algebra KG of a finite group G over a field K of characteristic $p > 0$ has finite type if and only if the p-Sylow subgroup of G is cyclic.
6. Any serial algebra is of finite type.

A main invariant in the theory of representations is the **length** of a module which is defined as the length of its composition series. Recall that a module has finite length if and only if it is both Artinian and Noetherian. Such modules are called finite length modules. It is obvious that any finite length module is a direct sum of indecomposable modules of finite length. And by the Fitting lemma the endomorphism ring of any indecomposable module of finite length is a local ring. Then, by the Krull-Schmidt theorem (see theorem 10.4.11, vol.I), the decomposition of a finite length module into indecomposable modules is unique up to isomorphism.

Definition. We say that a ring A is of **bounded representation type** if there is a bound on the lengths of indecomposable A-modules. Otherwise it is said to be of **unbounded representation type**.

We say that a ring A has a **strongly unbounded representation type**, if there is an infinite sequence $d_1 < d_2 < \ldots$ such that A has infinitely many indecomposable modules for each length d_i.

Theorem 3.4.1 (The first Brauer-Thrall conjecture). *If the number of indecomposable modules of length 1 is finite, then bounded representation type implies finite representation type.*

This conjecture was proved by A.V.Roiter in 1968 [Roiter, 1968] for a finite dimensional algebra A over an arbitrary field.

And it was generalized by M.Auslander for Artinian algebras [Auslander, 1971], [Auslander, 1974a], [Auslander, 1974b].

Theorem 3.4.2 (M.Auslander). *An Artinian algebra A is of finite representation type if and only if there is a bound on the lengths of the indecomposable A-modules.*

A stronger version of the Brauer-Thrall conjecture is the following statement:

(The second Brauer-Thrall conjecture). *If a finite dimensional algebra A has infinite representation type, then it is of strongly unbounded representation type.*

So far this stronger conjecture has been proved for a finite dimensional algebra over an algebraically closed field k by R.Bautista [Bautista, 1985] and K.Bongartz [Bongartz, 1985]:

Theorem 3.4.3. *If a finite dimensional algebra A over an algebraically closed field k is not of finite representation type, there are infinitely many dimensions for which the number of isomorphism classes of indecomposable finite dimensional A-modules is infinite.*

All k-algebras of infinite type are further divided into **algebras of wild representation type** and **algebras of tame representation type**.

Definition. An algebra A said to be of **tame representation type** (or a **tame algebra**, in short) if it is of infinite type but all families of indecomposable representations are 1-parametric. In other words, for any r there are $(A, k[x])$-bimodules $M_1, \ldots, M_n$ (where the natural number n may depend on r), which are finitely generated and free over $k[x]$ such that any indecomposable A-module of dimension r is isomorphic to some $M_i \otimes k[x]/(x-\lambda)$.

An algebra A is said to be of **wild representation type** (or a **wild algebra**, in short) if there is an $(A, k\langle x, y\rangle)$-bimodule M which is finitely generated and free over $k\langle x,y\rangle$ and such that the functor $M\otimes_{k\langle x,y\rangle} *$ sends non-isomorphic finite dimensional $k\langle x,y\rangle$-modules to non-isomorphic A-modules. In this case the category of all finite-dimensional A-modules includes the classification problem for pairs of square matrices up to simultaneous equivalence.

Yu.A.Drozd proved that for an algebraically closed field k there is a trichotomy between finite, tame and wild representation type for finite dimensional algebras [Drozd, 1980]:

Theorem 3.4.4 (Yu.A.Drozd). *Let A be a finite dimensional algebra over an algebraically closed field. Then A is of finite, tame or wild representation type.*

Examples 3.4.2.
1. The Kronecker algebra

$$A = \begin{pmatrix} k & k\oplus k \\ 0 & k \end{pmatrix}$$

where k is a field, is a four-dimensional algebra, which is of infinite representation type. The problem of the classification of all indecomposable modules over this algebra is equivalent to the classification of the indecomposable matrix pencils. This problem was considered by K.Weierstrass and then solved by L.Kronecker in 1890. This algebra is of tame type.
2. Let $G = \{x, y \ : \ x^2 = y^2 = 1, xy = yx\}$ be the Klein 4-group, and let k be a field. Then the group algebra kG is of wild representation type.

In the representation theory of associative algebras it is important to obtain necessary and sufficient conditions for a given algebra to be of finite, tame or wild type. Another important problem is to describe all indecomposable representations in the finite and tame cases. These problems have still not been solved in the general case.

A full description of algebras of finite or tame type and their representations has been obtained only for some particular classes of algebras, for example, for hereditary algebras and algebras in which the square of the radical equals zero.

Taking into account theorems 2.4.1 and 2.3.4 we have the following statement.

Theorem 3.4.5. *If Q is a quiver of finite representation type, then the path algebra kQ is an Artinian hereditary algebra of finite type.*

With any finite dimensional algebra A over a field k one can associate its Gabriel quiver $Q(A)$ (see section 11.1, vol.I).

Let $P_1, ..., P_s$ be all pairwise nonisomorphic principal right A-modules. Write $R_i = P_iR$ $(i = 1, ..., s)$ and $V_i = R_i/R_iR$ where R is the radical. Since V_i is a semisimple module, $V_i = \mathop{\oplus}\limits_{j=1}^{s} U_j^{t_{ij}}$, where the $U_j = P_j/R_j$ are simple modules. This is equivalent to the isomorphism $P(R_i) \simeq \bigoplus_{j=1}^{s} P_j^{t_{ij}}$. To each module P_i assign a vertex i and join the vertex i with the vertex j by t_{ij} arrows. The thus constructed graph is called the **quiver** of A in the sense of P.Gabriel and denoted by $Q(A)$.

For any finite quiver $Q = (VQ, AQ, s, e)$ we can construct a bipartite quiver $Q^b = (VQ^b, AQ^b, s_1, e_1)$ in the following way. Let $VQ = \{1, 2, ..., s\}$, $AQ = \{\sigma_1, \sigma_2, ..., \sigma_k\}$. Then $VQ^b = \{1, 2, ..., s, b(1), b(2), ..., b(s)\}$ and $AQ^b = \{\tau_1, \tau_2, ..., \tau_k\}$, such that for any $\sigma_j \in AQ$ we have $s_1(\tau_j) = s(\sigma_j)$ and $e_1(\tau_j) = b(e(\sigma_j))$. In other words, in the quiver Q^b from the vertex i to the vertex $b(j)$ go t_{ij} arrows if and only if in the quiver Q from the vertex i to the vertex j go t_{ij} arrows. As before, denote by $\overline{Q}$ the undirected graph which is obtained from Q by deleting the orientation of all arrows.

Example 3.4.3.

Let k be a field and

$$A = \begin{pmatrix} k & 0 & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$$

Then

$$R = rad\, A = \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & k \\ 0 & 0 & 0 \end{pmatrix}$$

and so $R^2 = 0$. The right principal modules are $P_1 = \begin{pmatrix} k & 0 & k \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 & k & k \end{pmatrix}$, $P_3 = \begin{pmatrix} 0 & 0 & k \end{pmatrix}$, while the simple right modules are the $U_i \simeq P_i/P_iR$ for $i = 1, 2, 3$. Therefore $P_1R = \begin{pmatrix} 0 & 0 & k \end{pmatrix} \simeq U_3$, $P_2R = \begin{pmatrix} 0 & 0 & k \end{pmatrix} \simeq U_3$, $P_3R = 0$. Thus the quiver $Q^b(A)$ has the following form:

$$\begin{tikzcd}
\overset{1}{\bullet} \arrow[drr] & \overset{2}{\bullet} \arrow[dr] & \overset{3}{\bullet} \\
\underset{b_1}{\bullet} & \underset{b_2}{\bullet} & \underset{b_3}{\bullet}
\end{tikzcd}$$

If A is a finite dimensional algebra over an algebraically closed field k with zero square radical and with associated quiver Q, then an explicit connection between the category $\mathrm{mod}_r(A)$ and the category $\mathrm{Rep}\, Q^b(A)$ was established by P.Gabriel. He has proved the following theorem:

Theorem 3.4.6 (P.Gabriel). *Let A be a finite dimensional algebra over an algebraically closed field k with zero square radical and quiver Q. Then A is of finite type if and only if the quiver $Q^b(A)$ is of finite type.*

Proof. We shall give only a short sketch of the proof. Let A be a finite dimensional algebra over an algebraically closed field k with Jacobson radical R

and $R^2 = 0$. First we construct the Grassmann category $G(A)$ of the algebra A. The objects of $G(A)$ are triples (X, Y, f), where X, Y are A/R-modules, and $f : X \otimes_{A/R} R \to Y$ is a A/R-epimorphism. A morphism $(X, Y, f) \to (X_1, Y_1, f_1)$ is a pair (φ, ψ), where $\varphi : X \to X_1$, $\psi : Y \to Y_1$ are A/R-homomorphisms such that $f_1(\varphi \otimes 1_R) = \psi f$. Next construct the functor $F : \mathrm{mod}_r A \to G(A)$ given by $F(M) = (M/MR, MR, f)$, where $f : (M/MR) \otimes_{A/R} R \to MR$ is induced by $m \otimes r \mapsto mr$ for any right finite dimensional A-module M. Then it can be proved that this functor induces a one-to-one correspondence between the isomorphism classes of indecomposable objects in $mod_r\, A$ and $G(A)$.

Let $P_1, ..., P_s$ be all pairwise nonisomorphic principal right A-modules, $R_i = P_i R$ $(i = 1, ..., s)$, $V_i = R_i/R_iR$, and let the $U_j = P_j/R_j$ be the simple modules. Since V_i is a semisimple module, $V_i = \mathop{\oplus}\limits_{j=1}^{s} U_j^{t_{ij}}$. Since $R^2 = 0$, $R_i \simeq V_i$ and so $R \simeq \mathop{\oplus}\limits_{i,j=1}^{s} U_j^{t_{ij}}$.

Let $T = (X, Y, f) \in G(A)$. Since X, Y are A/R-modules, we can write $X = U_1^{m_1} \oplus \ldots \oplus U_s^{m_s}$ and $Y = U_1^{k_1} \oplus \ldots \oplus U_s^{k_s}$. Then the map $f : X \otimes_{A/R} R \to Y$ induces t_{ij} k-homomorphisms $U_i^{m_i} \to U_j^{k_j}$.

Therefore this construction yields a fully faithful functor F from the category $G(A)$ to the category of k-representations of A. It is easy to see that indecomposable objects of $G(A)$ turn into (again) indecomposable representations of A.

Taking into account theorem 3.4.6 and theorem 2.6.1, we obtain the following theorem:

Theorem 3.4.7 (P.Gabriel). *Let A be a finite dimensional algebra over an algebraically closed field k with zero square radical and quiver Q. Then A is of finite type if and only if $\overline{Q^b}$ is a finite disjoint union of Dynkin diagrams of the form A_n, D_n, E_6, E_7, E_8.*

This theorem has been generalized to the case of arbitrary fields.

With any finite dimensional algebra k-algebra A we can associate a k-species. Let B be the basic algebra of A. Then $B/\mathrm{rad}B \simeq K_1 \oplus K_2 \oplus \ldots \oplus K_n$, where the K_i are skew fields for $i = 1, \ldots n$ which are finite dimensional over k. We can write $\mathrm{rad}B/(\mathrm{rad}B)^2 = \mathop{\oplus}\limits_{i,j=1}^{n} {}_iM_j$, where the ${}_iM_j$ are K_i-K_j-bimodules. This yields the k-species $\mathcal{L}_A = (K_i, {}_iM_j)_{i,j \in I}$.

Given a k-species $(K_i, {}_iM_j)_{i,j \in I}$, define its **separated diagram** as follows. The finite set $I \times \{0, 1\}$ is the set of all vertices, and there are $t_{ij} = \dim_{K_i}({}_iM_j) \times \dim({}_iM_j)_{K_j}$ edges between $(i, 0)$ and $(j, 1)$.

Moreover, there is a (multiple) arrow $\Longrightarrow$ provided $\dim_{K_i}({}_iM_j) < \dim({}_iM_j)_{K_j}$, which connects $(i,0)$ with $(j,1)$ and contains t_{ij} edges. Note that there are no edges between $(i,0)$ and $(j,0)$, nor between $(i,1)$ and $(j,1)$.

Theorem 3.4.8. *Let A be a finite dimensional k-algebra A over an arbitrary field k with $(\mathrm{rad}A)^2 = 0$. Then A is of finite type if and only if the diagram of its k-species is a finite disjoint union of Dynkin diagrams.*

Remark 3.4.1. This theorem was proved by P.Gabriel in the case when the k-species $\mathcal{L}_A = (K_i, {}_iM_j)_{i,j\in I}$ has the property that all K_i are equal to a fixed skew field F and ${}_F({}_iM_j)_F = ({}_FF_F)^{n_{ij}}$ for some natural number n_{ij}. Also P.Gabriel has shown that the structure of a k-algebra A of finite type with $(\mathrm{rad}A)^2 = 0$ can be recovered from the known results in the case when k is a perfect field [Gabriel, 1972], [Gabriel, 1973]. In its general form for arbitrary fields k theorem 3.4.8 was proved by V.Dlab and C.M.Ringel [Dlab, Ringel, 1973], [Dlab, Ringel, 1975].

With any species $\mathcal{L} = (K_i, {}_iM_j)_{i,j\in I}$ one can associate the special tensor algebra of a bimodule of the following form $\mathfrak{T}(\mathcal{L}) = \mathfrak{T}_B(M) = B \oplus M \oplus M\otimes_B M \oplus \ldots$, where $B = \prod\limits_{i\in I} K_i$ and $M = \bigoplus\limits_{i,j\in I} {}_iM_j$.

Theorem 3.4.9 (V.Dlab, C.M.Ringel). *Let $\mathcal{L}$ be a k-species. Then the category $\mathrm{Rep}\,\mathcal{L}$ of all representations of $\mathcal{L}$ and the category $\mathrm{mod}_r\,\mathfrak{T}(\mathcal{L})$ of all right $\mathfrak{T}(\mathcal{L})$-modules are equivalent.*

Proof. Let X be a right $\mathfrak{T}(\mathcal{L})$-module. Then it can be considered as a right B-module, because B is a subring of $\mathfrak{T}(\mathcal{L})$. Let $e_1, \ldots, e_n$ be the set of all primitive idempotents of B and $1 = e_1 + \ldots + e_n$. Then X decomposes uniquely into a direct sum $X = \bigoplus\limits_{i=1}^{n} X_i$, where $X_i = Xe_i$ are right K_i-modules and $X_iK_j = 0$ for $i \neq j$. Since M is a B-bimodule the multiplication of X_B gives rise to a right B-map $\varphi : X \otimes_B M \to X$. Since $X_i \otimes_{K_j} {}_jM_s = 0$ for $i \neq j$, the map φ given by $\varphi : \bigoplus\limits_{i=1}^{n} (X_i \otimes_B {}_iM_j) \to \bigoplus\limits_{i=1} X_i$ is determined by the family of K_j-linear maps ${}_j\varphi_i : X_i \otimes_B {}_iM_j = X_i \otimes_{K_i} {}_iM_j \to X_j$.

Now a functor $R : \mathrm{mod}_r\,\mathfrak{T}(\mathcal{L}) \to \mathrm{Rep}\,\mathcal{L}$ is defined as follows: for any right $\mathfrak{T}(\mathcal{L})$-module X we set $R(X) = (X_i, {}_j\varphi_i)$. Remark, that the right $\mathfrak{T}(\mathcal{L})$-module structure on X is uniquely determined by the map φ and so by the family of morphisms $\{{}_j\varphi_i\}$.

Let X, Y be two right $\mathfrak{T}(\mathcal{L})$-modules and let $\alpha : X \to Y$ be a $\mathfrak{T}(\mathcal{L})$-homomorphism. Let $R(X) = (X_i, {}_j\varphi_i)$ and $R(Y) = (Y_i, {}_j\psi_i)$. Since α is also a B-homomorphism, $\alpha(X_i) \subseteq Y_i$ and it is determined by the family of restrictions $\alpha_i : X_i \to Y_i$. Set $R(\alpha) = \{\alpha_i\}$. Since α is a $\mathfrak{T}(\mathcal{L})$-homomorphism, we have that ${}_j\psi_i(\alpha_i \otimes 1) = \alpha_j \cdot {}_j\varphi_i$, i.e., $\{\alpha_i\}$ is a map in $\mathrm{Rep}\,\mathcal{L}$.

Conversely, let $(X_i, {}_j\varphi_i) \in \mathrm{Rep}\,\mathcal{L}$. Define a functor $P : \mathrm{Rep}\,\mathcal{L} \to \mathrm{mod}_r\,\mathfrak{T}(\mathcal{L})$ as follows: $P(X_i, {}_j\varphi_i) = X$, where $X = \bigoplus\limits_{i}^{n} X_i$ and the ring $B = \bigoplus\limits_{i} K_i$ operates on

X via the projection $B \to K_i$. Then the scalar multiplication by $M^{(n)}$ on X is defined inductively by $\varphi^{(n)} : X \otimes_B M^{(n)} \to X$ with

$$\varphi^{(1)} = \bigoplus_{i,j=1}^{n} {}_j\varphi_i : X \otimes_B M = \bigoplus_{i,j=1}^{n} (X_i \otimes_B {}_iM_j) =$$
$$= \bigoplus_{i,j=1}^{n} (X_i \otimes_{K_i} {}_iM_j) \to \bigoplus_{i=1}^{n} X_i = X$$

and

$$\varphi^{(n+1)} = \varphi(\varphi^{(n)} \otimes 1) : X \otimes_B M^{(n+1)} = (X \otimes_B M^{(n)}) \otimes_B M \xrightarrow{\varphi^{(n)}\otimes 1} X \otimes_B M \xrightarrow{\varphi} X$$

If $\{\alpha_i\} : (X_i, {}_j\varphi_i) \to (Y_i, {}_j\psi_i)$ is in $\operatorname{Rep}\mathcal{L}$, then $\alpha = \bigoplus_{i=1}^{n} \alpha_i : X = \bigoplus_{i=1}^{n} X_i \to Y = \bigoplus_{i=1}^{n} Y_n$ is a right $\mathfrak{T}(\mathcal{L})$-map and so we can set $P(\{\alpha_i\}) = \alpha$.

It is easy to verify that R and P are mutually inverse equivalences of categories, as required.

Remark 3.4.2. Theorem 3.4.9 was proved by V.Dlab and C.M.Ringel (see [Dlab, Ringel, 1973], [Dlab, Ringel, 1975]. A different version of this theorem was obtained by E.L.Green [Green, 1975].

From theorem 3.4.9 the following theorem follows immediately.

Theorem 3.4.10.[1] *Let $\mathcal{L}$ be a k-species. The special tensor algebra $\mathfrak{T}(\mathcal{L})$ is of finite representation type is and only if the k-species $\mathcal{L}$ is of finite type.*

Let $\mathcal{L}$ be a k-species. From corollary 2.2.13 it follows that the special tensor algebra $\mathfrak{T}(\mathcal{L})$ is hereditary. It was shown that the converse is also true for the case of finite type.

Theorem 3.4.11. *A finite dimensional k-algebra A is a hereditary algebra of finite type if and only if A is Morita equivalent to a tensor algebra $\mathfrak{T}(\mathcal{L})$, where $\mathcal{L}$ is a k-species of finite type.*

Remark 3.4.3. This theorem first was proved by P.Gabriel[2] for the case of an algebraically closed field k and then was proved by V.Dlab, C.M.Ringel for an arbitrary field k.[3]

Theorem 3.4.12.[4] *Let A be a finite dimensional hereditary k-algebra with associated k-species $\mathcal{L}$. Then A is of finite type if and only if the diagram of $\mathcal{L}$ is a finite disjoint union of Dynkin diagrams.*

These theorems have been generalized to the case of Artinian algebras.

[1] see [Dlab, Ringel, 1975].
[2] see [Gabriel, 1972], [Gabriel, 1973]
[3] see [Dlab, Ringel, 1975].
[4] see [Dlab, Ringel, 1976].

Definition. Let R be a commutative Artinian ring. An R-algebra A is a ring together with a ring morphism $\varphi : R \to A$ such that $\mathrm{Im}(\varphi) \subseteq \mathrm{Cen}(A)$. In other words we can assume that $ra = ar$ for each $r \in R$ and $a \in A$.

We say that A is an **Artinian R-algebra**, or **Artinian algebra**, in short, if A is finitely generated as an R-module.

Important examples of Artinian algebras are finite dimensional algebras over a field.

Applying theorem 2.7.1, V.Dlab and C.M.Ringel proved the following theorems which give a full description of hereditary Artinian algebras of finite type.

Theorem 3.4.13. *A hereditary Artinian algebra A is of finite type if and only if A is Morita equivalent to a tensor algebra $\mathcal{T}(\Omega)$, where Ω is a k-species of finite type.*

Theorem 3.4.14. *Let A be a hereditary Artinian algebra with quiver Q. Then A is of finite type if and only if the underlying graph $\overline{Q}$ is a finite disjoint union of Dynkin diagrams.*

More generally questions concerning finite dimensional algebras over algebraically closed fields are treated by considering quivers with relations.

3.5 ROITER THEOREM

In this section we shall give the proof of the first Brauer-Thrall conjecture for finite dimensional algebras over arbitrary fields following A.V.Roiter.

Theorem 3.5.1 (The first Brauer-Thrall conjecture). *If a finite dimensional k-algebra A (where k is an arbitrary field) is of infinite representation type, then it has indecomposable modules of arbitrary large dimensions.*

Throughout this section Λ denotes a fixed finite dimensional algebra over a field k.

Definition. Let A and B be Λ-modules. We say that A **divides**[5] B, and write $A|B$, if $A \cdot \mathrm{Hom}\,(A, B) = B$, where

$$A \cdot \mathrm{Hom}\,(A, B) = \sum_{\varphi \in \mathrm{Hom}\,(A,B)} \mathrm{Im}\,\varphi.$$

It is easy to see that if A and B are Noetherian modules then $A|B$ if and only if there is an integer n such that there is an exact sequence $A^{(n)} \to B \to 0$.

Definition. A decomposition $A = A_1 \oplus A_2 \oplus \ldots \oplus A_n$ is called **normal** if $A_i|A_j$

[5] The relation "|" defined here is reflexive and transitive but it is not symmetric just as in the case of the usual relation of divisibility of integers.

for $i < j$. A module A which cannot be decomposed into a nontrivial normal direct sum is called **normally indecomposable**.

It is obvious that any Noetherian module A can be decomposed into a direct sum $A = A' \oplus A''$, where A' is normally indecomposable and $A'|A$.

Proposition 3.5.2. *Let Λ be a finite dimensional algebra over a field k, let B be a finitely generated Λ-module, and let A be a normally indecomposable quotient module of it. If $A|B$, then the exact sequence $B \xrightarrow{\varphi} A \to 0$ splits (where φ is the quotient map).*

Proof. Let $U = \mathrm{Hom}_\Lambda(A, A)$, which is a finite dimensional k-algebra. Set $T = \mathrm{Hom}_\Lambda(A, B)\varphi$. Obviously, T is a left ideal of U and $AT = A$. Let $R = \mathrm{rad}\,\Lambda$ be the radical of Λ. Write $\overline{U} = U/R$ and $\overline{T} = (T + R)/R$. Then $\overline{T}$ is a left ideal in $\overline{U}$ and $\overline{T} = \overline{e}\overline{U}$, where $\overline{e}$ is an idempotent of $\overline{T}$. Let $e \in T$ be an idempotent corresponding to the idempotent $\overline{e}$. Then $T \subseteq eU + R$, $A = AT = AeU + AR$. Therefore, by the Nakayama lemma, $A = AeU$. Since $A = \mathrm{Im}e \oplus \mathrm{Ker}e$, $\mathrm{Im}e|A$. By assumption, A is normally indecomposable, so $\mathrm{Im}e = A$, i.e., e is an identity of $U = \mathrm{Hom}(A, A)$. Since $e \in T = \mathrm{Hom}(A, B)\varphi$, $e = \psi\varphi$, where $\psi \in \mathrm{Hom}(A, B)$. So, by proposition 4.2.1, vol.I, the exact sequence $B \xrightarrow{\varphi} A \to 0$ splits.

The following statement is obvious:

Lemma 3.5.3. *Let*

$$B \xrightarrow{\varphi} A \to 0 \tag{3.5.1}$$

be an exact sequence, and let X be an arbitrary module. Consider

$$B \oplus X \xrightarrow{\overline{\varphi}} A \oplus X \to 0 \tag{3.5.2}$$

which is also an exact sequence, where $(b, x)\overline{\varphi} = (b\varphi, x)$. Then the sequence (3.5.1) splits if and only the sequence (3.5.2) splits.

In what follows in this section we shall assume that all modules are finitely generated over a finite dimensional k-algebra Λ. Each such module A obviously has finite length which is denoted by $l(A)$.

Denote by $\mathfrak{M}$ the set of all indecomposable Λ-modules. We shall construct in $\mathfrak{M}$ a sequence of subsets $\mathfrak{M}_1$, $\mathfrak{M}'_1$, ..., $\mathfrak{M}_i$,$\mathfrak{M}'_i$, ... in the following way. $\mathfrak{M}_1 = \{A \in \mathfrak{M} \ : \ l(A) = 1\}$. Assuming that $\mathfrak{M}_1$, ...,$\mathfrak{M}_k$ have been constructed, define $\mathfrak{M}'_k$ by setting $A \in \mathfrak{M}'_k$ if and only if

1) $A \in \mathfrak{M}$;

2) $A \notin \bigcup\limits_{i=1}^{k} \mathfrak{M}_i$;

3) any proper indecomposable quotient module of A is contained in $\bigcup\limits_{i=1}^{k} \mathfrak{M}_i$.

Then we set

$$\mathfrak{M}_{k+1} = \{A \in \mathfrak{M}'_k \ : \ l(A) = \max_{B \in \mathfrak{M}'_k} l(B)\}.$$

Since $\mathfrak{M}'_k \cap \overset{k}{\underset{i=1}{\cup}} \mathfrak{M}_i = \emptyset$, and $\mathfrak{M}_{k+1} \subset \mathfrak{M}'_k$, it is obvious that $\mathfrak{M}_i \cap \mathfrak{M}_j = \emptyset$ for $i \neq j$.

Lemma 3.5.4. *$\mathfrak{M}_i \neq \emptyset$ for only finitely many i.*

Proof. Since the lengths of all modules from $\mathfrak{M}$ are bounded it is sufficient to show that modules with a fixed length k can be in only a finite number of $\mathfrak{M}_i$. We shall prove this by induction on k. For $k = 1$ this statement is trivial because all modules of length 1 are in $\mathfrak{M}_1$ and only in $\mathfrak{M}_1$. Set $\mathfrak{R} = \overset{\infty}{\underset{i=1}{\cup}} \mathfrak{M}_i$, $\mathfrak{R}_s = \overset{s}{\underset{i=1}{\cup}} \mathfrak{M}_i$. Assume that for $j \leq k$ the modules of length j belong only to a finite number of $\mathfrak{M}_i$. Then there is a number s such that for any $A \in \mathfrak{R}_s$ its length $l(A) \geq k$. Suppose that $A \in \mathfrak{R}_s$ and $l(A) = k+1$. If C is a proper indecomposable quotient module of A, then $C \in \mathfrak{R}$, since otherwise $A \notin \mathfrak{M}'_i$ for any i, and so $A \notin \mathfrak{R}$. Since $l(C) < l(A) = k+1$, $C \in \overset{s}{\underset{i=1}{\cup}} \mathfrak{M}_i$. Therefore $A \in \mathfrak{M}'_s$, and it is obvious that $A \in \mathfrak{M}'_i$ for all $i \geq s$. So if B is any other module such that $B \in \mathfrak{M}_t$ for $t > s$ and $l(b) = k+1$ then also $A \in \mathfrak{M}_t$, i.e., all modules from $\mathfrak{R}_s$, whose length is equal to $k+1$, belong to not more than one set $\mathfrak{M}_i$.

Lemma 3.5.5. *If $A \in \mathfrak{M}$ and $A \notin \overset{k}{\underset{i=1}{\cup}} \mathfrak{M}_i$, then there is a quotient module B of A which is contained in $\mathfrak{M}'_k$.*

Proof. If all indecomposable quotient modules of A belong to $\overset{k}{\underset{i=1}{\cup}} \mathfrak{M}_i$, then we can take $B = A$. Otherwise we choose a quotient module $A_1 \in \mathfrak{M}$ such that $A_1 \notin \overset{k}{\underset{i=1}{\cup}} \mathfrak{M}_i$. If $A_1 \in \mathfrak{M}'_k$ then take $A_1 = B$. If $A_1 \notin \mathfrak{M}'_k$, consider a quotient module $A_2 \in \mathfrak{M}$ such that $A_2 \notin \overset{k}{\underset{i=1}{\cup}} \mathfrak{M}_i$, and so on. From the ascending chain condition it now follows that in a finite number of steps we find a quotient module which belongs to $\mathfrak{M}'_k$.

Lemma 3.5.6. *There is an integer n such that $\overset{n}{\underset{i=1}{\cup}} \mathfrak{M}_i = \mathfrak{M}$.*

Proof. By lemma 3.5.4, there exists an integer n such that $\mathfrak{M}_i = \emptyset$ for $i > n$. If $\overset{n}{\underset{i=1}{\cup}} \mathfrak{M}_i \neq \mathfrak{M}$, then, by lemma 3.5.5, $\mathfrak{M}'_n \neq \emptyset$, and so $\mathfrak{M}_{n+1} \neq \emptyset$.

Proposition 3.5.7. *If Λ is a finite dimensional algebra of bounded representation type, then there is a function $f\colon$ mod-$\Lambda \to \mathbf{N}$ such that:*

1) *from the existence of a non-split exact sequence*

$$\overset{s}{\underset{i=1}{\oplus}} B_i \to A \to 0,$$

where the $B_1, \ldots, B_s, A$ *are indecomposable modules, it follows that* $f(A) < \max\limits_{i=1,2,\ldots,s} f(B_i)$;

2) $f(A) = f(B)$ *implies* $l(A) = l(B)$;

3) *there is an integer* n *such that* $f(A) \leq n$ *for any indecomposable module* A.

Proof. From the construction of the sets $\mathfrak{M}_i$ and lemma 3.5.6 it follows that for any indecomposable module $A \in \mathfrak{M}$ there exists a unique integer i such that $A \in \mathfrak{M}_i$. We set $f(A) = i$. From lemma 3.5.6 it follows that the function f satisfies condition 3), the fulfillment of condition 2) follows from the construction of the sets $\mathfrak{M}_{k+1}$. So it remains to verify condition 1).

Let there be a non-split exit sequence

$$\overset{s}{\underset{i=1}{\oplus}} B_i \to A \to 0, \tag{3.5.3}$$

where the B_1, ..., B_s, A are indecomposable modules. Let $k = \max\limits_{i=1,2,\ldots,s} f(B_i)$. We need to show that $f(A) < k$. We shall this prove by induction on k. For $k = 1$ the statement is obvious, because in this case all B_i are irreducible and the sequence 3.5.3 can not be non-split.

Suppose that for all $j < k$ the statement is proved. We represent the module $B = B_1 \oplus B_2 \oplus \ldots \oplus B_s$ in the form $B = B' \oplus B''$, where B' is normally indecomposable and $B'|B$. Renumbering the B_i if needed and taking into account the Krull-Schmidt theorem we may assume that $B' = \overset{r}{\underset{i=1}{\oplus}} B_i$, where $r \leq s$. Since $B'|B$ and $B|A$, we obtain that $B'|A$. So there exists an exact sequence

$$B'^{(m)} \xrightarrow{\varphi} A \to 0. \tag{3.5.4}$$

If this sequence would be split, then, by the Krull-Schmidt theorem, $A \simeq B_i$ for some i. Consider the sequence

$$B \oplus A \longrightarrow A \oplus X \to 0,$$

where $X = B_1 \oplus \ldots \oplus B_{i-1} \oplus B_{i+1} \oplus \ldots \oplus B_r$, which is split, by proposition 3.5.2. Then, by lemma 3.5.3, the sequence 3.5.3 is also split. If $f(B_i) < k$ for all $i = 1, 2, \ldots, r$, then the required inequality $f(A) < k$ follows from the inductive assumption. We can assume that $f(B_i) = k$ for $i \leq q \leq r$ and $f(B_i) < k$ for $q < i \leq r$. Also $B'|B$. Renumbering the B_i and taking into account the Krull-Schmidt theorem we find that there exists a (proper or improper) quotient module C of the module A that is in $\mathfrak{M}'_{k-1}$. Since $B'|A$ and $A|C$, it follows that $B'|C$, i.e., there is a set of homomorphisms $\varphi_{ij} : B_i \to C$, $i = 1, \ldots, q$ such that

$\sum\limits_{i,j} \mathrm{Im}\varphi_{ij} = C$. Since $C \in \mathfrak{M}'_{k-1}$, $l(B_i) > l(C)$ for $i = 1, \ldots, q$. Therefore for $1 \leq i \leq q$ and for any φ_{ij} we have that either 1) $\mathrm{Im}\varphi_{ij}$ is a proper quotient module or 2) $\mathrm{Im}\varphi_{ij} = C \simeq B_i$.

If for all i ($1 \leq i \leq q$) we have the first case, then from the construction of f it follows that the values of f on all direct summands of $\mathrm{Im}\varphi_{ij}$ are not more than $k-1$. Since $f(B_i) \leq k-1$ for $q < i \leq r$, we then obtain that $D|C$, where $D = \big(\sum\limits_{i,j} \oplus \mathrm{Im}\varphi_{ij}\big) \oplus \big(\sum\limits_{i=q+1}^{r} \oplus B_i\big)$ and the values of f on all indecomposable direct summands of D are not more than $k-1$. Taking an exact sequence $D^{(n)} \to C \to 0$, we see that either this sequence splits and then $f(C) \leq k-1$, by the Krull-Schmidt theorem, or this sequence is not split and then $f(C) < k-1$, by the induction assumption. In the both cases we have a contradiction with $C \in \mathfrak{M}'_{k-1}$.

Now consider the second case when $C \simeq B_i$ for some i, where $1 \leq i \leq r$. In this case we consider the exact sequence:

$$A \longrightarrow C \to 0. \tag{3.5.5}$$

We also construct the sequence

$$A \oplus X \longrightarrow C \oplus X \to 0,$$

where $X = B_1 \oplus \ldots \oplus B_{i-1} \oplus B_{i+1} \oplus \ldots \oplus B_r$, which is split, by proposition 3.5.2. Then, by lemma 3.5.3, the sequence 3.5.5 is also split. A contradiction.

Lemma 3.5.8. *Let $M_1, \ldots, M_t, B$ be a set of modules. There exists an integer N such that for any exact sequence*

$$0 \to B \longrightarrow X \longrightarrow M_1^{n_1} \oplus \ldots \oplus M_t^{(n_t)} \to 0 \quad (n_i \geq 0)$$

it follows from $l(X) > N$ that $X = Y \oplus M_i$ for some i and some Y.

Proof. The group $\mathrm{Ext}(M_i, B)$ is a finite dimensional vector space over k. Suppose its dimension is equal to s_i. If $n_i > s_i$ then there is a module M_i which is a direct summand of X. Therefore it is sufficient to take

$$N = \sum_{i=1}^{t} s_i l(M_i) + l(B).$$

Proof of theorem 3.5.1. Assume the contrary. Consider the function f on the set of all indecomposable representations of an algebra A, which satisfies the conditions of proposition 3.5.7. We set $\mathfrak{M}_i = \{A \in \mathfrak{M} \ : \ f(A) = i\}$. The set $\mathfrak{M}_1$ is finite, because any finite dimensional algebra has a finite number of modules of length 1. On the other hand, $\bigcup\limits_{i=1}^{n} \mathfrak{M}_i = \mathfrak{M}$ (for some n). Therefore there is an integer k such that the $\mathfrak{M}_i$ are finite for $i < k$, and $\mathfrak{M}_k$ is infinite. We denote by $\overline{\mathfrak{M}}$ the set of modules of whose all indecomposable direct summands are contained in $\overset{k-1}{\underset{i=1}{\cup}} \mathfrak{M}_i$. We now show that if A, B are submodules in C, and $A, B \in \overline{\mathfrak{M}}$, then the submodule $D = A + B \in \overline{\mathfrak{M}}$ as well. Indeed, in this case $(A \oplus B) \,|\, D$. Decomposing A, B, D into direct sums we obtain exact sequences of the form $X \to D_i \to 0$, where all indecomposable direct summands of the module X are contained in $\overset{k-1}{\underset{i=1}{\cup}} \mathfrak{M}_i$, where the D_i are the indecomposable direct summands of D. For those i, for which this sequence is split, from the Krull-Schmidt theorem we obtain that $f(D_i) \le k - 1$; and for those i, for which this sequence is not split, from property 1) of the function f it follows that $f(D_i) < k - 1$. Thus, $D \in \overline{\mathfrak{M}}$, and so any module A has a submodule $U(A) \in \overline{\mathfrak{M}}$ which contains any other submodule belonging to $\overline{\mathfrak{M}}$.

We decompose the set $\mathfrak{M}_k$ into the classes by considering $A, B \in \mathfrak{M}_k$ to belong to the same class if $U(A) \simeq U(B)$. Taking into account that the lengths of all modules from $\mathfrak{M}_k$ are equal, the lengths of all possible $U(A)$ are bounded, and as $\overline{\mathfrak{M}}$ contains only a finite number of indecomposable modules, we obtain that $\mathfrak{M}_k$ decomposes into a finite number of classes. Therefore there is a class $\mathfrak{T}$ containing an infinite number of modules.

Let $A_1, \dots, A_m$ be a distinct modules from $\mathfrak{T}$. Consider $A = \sum\limits_{i=1}^{m} \oplus A_i$ and write $U_i = U(A_i)$. By definition, $U_i \simeq U_j$ for $i, j = 1, \dots, m$. Let $\varphi_i : U_m \to U_i$ be one of these isomorphisms. Consider the submodule $U = \{(u_1, \dots, u_m) : \ u_i \in U_i, \text{ and } u_i = \varphi(u_m)\}$ in A. Let $V = A/U$. We shall show that there is an integer m_0 such that V is indecomposable for any $m > m_0$.

Let $V = V_1 \oplus \dots \oplus V_s$. It is obvious that $A \,|\, V \,|\, V_j$ for $1 \le j \le s$. Therefore taking into account property 1) of the function f and the Krull-Schmidt theorem, we conclude that either there is a $V_j \simeq A_i$ or $f(V_j) < k$ for all j. In the second case, since $\overset{k-1}{\underset{i=1}{\cup}} \mathfrak{M}_i$ is a finite set, we can choose an integer m_0 such that V_i is a direct summand of A for $m > m_0$, by lemma 3.5.8 (where for $M_1, \dots, M_t$ one takes $\overset{k-1}{\underset{i=1}{\cup}} \mathfrak{M}_i$, for B one takes U, and for X one takes A). This contradicts the Krull-Schmidt theorem.

This takes care of the first case, i.e., there is a $V_j \simeq A_i$ for $m > m_0$. Suppose $V = A_1 \oplus Y$. Denote by φ the epimorphism $A \to V$, by ψ the projection $V \to A_1$, and set $\alpha = \varphi\psi$, which is the homomorphism $A \to A_1$, and denote by α_i the restriction of α to $A_i \subset A$. We shall show that α_1 is an isomorphism, Indeed, otherwise all $\operatorname{Im} \alpha_i$ are proper quotient modules of A_i, and so all direct summands of $\operatorname{Im} \alpha_i$ belong to $\overset{k-1}{\underset{i=1}{\cup}} \mathfrak{M}_i$ and their direct sum cannot divide A_1.

Thus, α_1 is an isomorphism. We set $T = A_2 \oplus \dots \oplus A_m$, $\beta = \alpha_2 + \dots + \alpha_m :$ $T \to A_1$. Consider the submodule $T' = \{t - t\beta\alpha_1^{-1} \ : \ t \in T\}$ in A. We now show that T' contains U. Indeed, to start with, obviously $A = A_1 \oplus T$. On the other hand, $T'\alpha = 0$. Take an arbitrary element $u \in U$. Let $u = u_1 + u_2$, where

$u_1 \in A_1$, $u_2 \in T'$. Applying the homomorphism $\alpha = \varphi\psi$ to the element u we see that $u\alpha = 0$, $u_2\alpha = 0$, and so $u_1\alpha = 0$. Since $u_1\alpha = u_1\alpha_1$ and α_1 is an isomorphism, $u_1 = 0$, i.e., $u \in T'$. Taking into account that $T' = \{t - t\beta_1^{-1}\}$ and $U = \{(u_1, \dots, u_m) : \ u_i \in U_i, u_i = \varphi_i(u_m)\} \subset T'$, where $U_i = U(A_i)$ is a submodule of A_i ($U_1 \simeq U_2 \simeq \dots \simeq U_m$, and $\varphi_i : U_i \to U_m$ is an isomorphism), we conclude that $U_1 \subseteq \operatorname{Im}\beta \subseteq A_1$, and the homomorphism $\beta : A_2 \oplus \dots \oplus A_m \to A_1$ maps the module $U' = \{(u_2, \dots, u_m) \mid u_i \in U_i, u_i = \varphi_i(u_m)\}$ onto U_1. We shall show that $\operatorname{Im}\beta = U_1$. Indeed, $\operatorname{Im}\beta = \sum\limits_{i=2}^{m} \operatorname{Im}\alpha_i$. Each $\operatorname{Im}\alpha_i$ $(i \neq q)$ is a proper quotient module of A_i. Therefore $\operatorname{Im}\beta$ is a module belonging to $\mathfrak{M}$, and, by construction, U_1 is a maximal submodule of the module A_1 belonging to $\overline{\mathfrak{M}}$.

Denote by γ the natural mapping of U onto $U' \subset T$, which we can consider as a homomorphism $\gamma : U \to T$. Then we obtain two homomorphisms $\beta : T \to U$ and $\gamma : U \to T$, where β is an epimorphism and γ is a monomorphism, and $\gamma\beta \in \operatorname{Hom}(U, U)$ is an isomorphism (since β maps U' onto U_1). Therefore U is a direct summand of $T = A_2 \oplus \dots \oplus A_m$, which contradicts the Krull-Schmidt theorem.

The theorem is proved.

3.6 NOTES AND REFERENCES

The basic problems studied in the representation theory of associative algebras are that of obtaining necessary and sufficient conditions for an algebra to belong to one of the types: finite, tame or wild, as well as that of classifying the indecomposable representations in the finite and tame cases. In the general case these problems have not been solved.

Representations of partially ordered sets were first introduced by L.A.Nazarova and A.V.Roiter in 1972 [Nazarova, Roiter, 1973]. In this paper they gave the algorithm which allows to check whether a given poset is of finite type. Using this algorithm M.M.Kleiner in 1972 characterized posets of finite type [Kleiner, 1972]. Moreover M.M.Kleiner classified all the indecomposable P-spaces of finite type. He also found that the dimensions of all such indecomposable P-spaces are bounded by 6 [Kleiner, 1972b]. In 1975 L.A.Nazarova characterized posets of infinite type (see [Nazarova, 1975], [Nazarova, 1974]). These results were independently also obtained by P.Donovan, M.R.Freislich (see [Donovan, Freislich, 1974]). The algorithm of Nazarova-Roiter works only for posets of width at most three. In 1977 A.G.Zavadskij has proposed the new differentiation algorithm for computing representations of posets [Zavadskij, 1977]. This algorithm has been used to give a new proof for the characterizations of poset of tame type (see [Nazarova, Zavadskij, 1977]). O.Kerner showed that this algorithm is quite useful also in the case of finite representation type. He has used this algorithm to give a new proof of Kleiner's theorem (see [Kerner, 1981]).

Theorem 3.1.4 was proved by Yu.A.Drozd [Drozd, 1974].

More results on posets of tame type one can found also in the following papers:

[Nazarova, Roiter, 1983]; [Bondarenko, Zavadskij, 1991]; [Bondarenko, Zavadskij, 1992]; [Kleiner, 1988].

The fundamental monograph of D.Simson [Simson, 1992] is devoted to the theory of representations of posets.

The connections between Abelian groups and representations of finite partially ordered sets were studied by D.M.Arnold, S.Brenner, M.C.R.Butler, M.Dugas, E.L.Lady, H.Krause, C.Ringel, C.Vinsonhaler and others. D.M.Arnold and M.Dugas obtained interesting results concerning the representations of posets over discrete valuation rings and their connections with Butler groups (see [Arnold, Dugas, 1997]; [Arnold, Dugas, 1999]. Most of these results are presented in the monograph of A.D.Arnold [Arnold, 2000].

Theorem 3.3.2 was proved by Yu.A.Drozd, A.G.Zavadskij and V.V.Kirichenko in the paper [Drozd, Zavadskij, Kirichenko, 1974].

In 1940 T.Nakayama in the paper [Nakayama, 1940] first posed the problem of finding algebras of unbounded representation type. In 1941 R.Brauer in his abstract [Brauer, 1941] asserted that he had found some sufficient conditions for an algebra to have infinite representation type. Several years later, in 1947, R.Thrall in his note [Thrall, 1941] also asserted that he had found sufficient conditions for an algebra to have infinite representation type formulated in terms of the Cartan matrices of factors of the algebra by powers of its radical. Unfortunately these results have never been published in detail.

In 1954, D.G.Higman showed that a group algebra over a field of characteristic $p > 0$ is of finite type if and only if a p-Sylow subgroup is cyclic, and of unbounded type otherwise (see [Higman, 1954]).

In 1957 J.Jans in the paper [Jans, 1957] has given conditions under which a finite dimensional algebra has infinitely many indecomposable representations of degree d for infinitely many d (i.e., strongly unbounded type). He also gave the first published announcement of the **Brauer-Thrall conjectures**:

(1) *if the degrees of the indecomposable representations of A are bounded (bounded type) then the number of inequivalent indecomposable representations is finite (finite type)*;

(2) *over an infinite field, the lack of a bound for the degrees of the indecomposable representations (unbounded type) implies that the algebra has strongly unbounded type.*

That the first conjecture is true for finite dimensional algebra with zero square radical over an algebraically closed field was shown by T.Yoshii [Yoshii, 1956].

In 1956 T.Yoshii attempted to give necessary and sufficient assumption for an algebra with zero square radical to be of unbounded type (see [Yoshii, 1956], [Yoshii, 1957]). Unfortunately these results of T.Yoshii had an error, which was discovered independently by P.Gabriel and S.A.Krugljak. They also independently published a correct solution of this problem (see [Gabriel, 1972]; [Krugljak, 1972]).

In 1968 the first Brauer-Thrall conjecture (theorem 3.5.1) was proved by A.V.Roiter for finite dimensional algebras over an arbitrary field and for Artin algebras (see [Roiter, 1968]) using a remarkably simple argument. The proof of this theorem "marks the beginning of the new representation theory of finite dimensional algebras", as C.M.Ringel wrote in his paper [Ringel, 2004].

The first Brauer-Thrall conjecture in a more general form (in particular, for Artinian algebras and one-sided Artinian rings) was proved by M.Auslander (see [Auslander, 1974a], [Auslander, 1974b]). In these papers M.Auslander proved that if C is a skeletally small Abelian category with only a finite number of non-isomorphic simple objects and such that each object has finite composition length, then C has only a finite number of indecomposable objects if and only if C satisfies the following conditions:

(1) C has a.c.c. on chains of indecomposable objects;

(2) if $\{M_i \xrightarrow{f_i} M_{i-1}\}_{i\in N}$ is a sequence of epimorphisms, then there is an n such that for $i \geq n$, f_i is an isomorphism.

M.Auslander also constructed a one-to-one correspondence between isomorphism classes of Artin algebras of finite type and isomorphism classes of Artin algebras of global dimension at most two and dominant dimension at least two (see [Auslander, 1971]). The generalizations of these results to Artinian rings were given by M.Auslander and H.Tachikawa (see [Auslander, 1974b], [Auslander, 1975] and [Tachikawa, 1973]).

The first Brauer-Thrall conjecture for arbitrary algebras (not necessarily finite dimensional) was studied by A.D.Bell and K.R.Goodearl. We say that a k-algebra A is of **right bounded finite dimensional representation type** if it has an upper bound on the k-dimensions of the finite dimensional indecomposable right A-modules. They showed that if a k-algebra A is either finitely generated as a k-algebra, or Noetherian as a ring, then bounded finite dimensional type implies that A has only finitely many isomorphism classes of finite dimensional indecomposable modules (see [Bell, Goodearl, 1995]).

The classification of hereditary finite dimensional k-algebras and algebras with zero square radical of finite representation type were obtained by P.Gabriel in the case when the corresponding k-species has the property that all K_i are equal to a fixed skew field F and $F({}_iM_j)_F = ({}_FF_F)^{n_{ij}}$ for some natural n_{ij} (see [Gabriel, 1972]; [Gabriel, 1972/1973], [Gabriel, 1973], [Gabriel, 1974]). In the general case these theorems were proved by V.Dlab, C.M.Ringel (see [Dlab, Ringel, 1975] and [Dlab, Ringel, 1976]).

The description of hereditary finite dimension algebras of tame type and wild type was obtained by V.Dlab and C.M.Ringel (see [Dlab, Ringel, 1976], [Ringel, 1976], [Ringel, 1978]).

In 1980 Yu.A.Drozd proved his famous theorem which says that for an algebraically closed field k there is a trichotomy between finite, tame and wild representation type for finite dimensional algebras (see [Drozd, 1980]).

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4. Frobenius algebras and quasi-Frobenius rings

This chapter is devoted to the study of Frobenius algebras and quasi-Frobenius rings. The class of quasi-Frobenius rings was introduced by T.Nakayama in 1939 as a generalization of Frobenius algebras. It is one of the most interesting and intensively studied classes of Artinian rings. Frobenius algebras are determined by the requirement that the right and left regular modules are equivalent. And quasi-Frobenius algebras are defined as algebras whose regular modules are injective.

We start this chapter with a short study of duality properties for finite dimensional algebras. In section 4.2 there are given equivalent definitions of Frobenius algebras in terms of bilinear forms and linear functions. There also symmetric algebras are studied who are a special class of Frobenius algebras. The main properties of quasi-Frobenius algebras are given in section 4.4.

The starting point in studying quasi-Frobenius rings in this chapter is the Nakayama definition of them. The key concept in this definition is a permutation of indecomposable projective modules. It is natural to call such a permutation a **Nakayama permutation**.

Quasi-Frobenius rings are also of interest because of the presence of a duality between the categories of left and right finitely generated modules over them. The main properties of duality in Noetherian rings are considered in section 4.10. Semiperfect rings with duality for simple modules are studied in section 4.11. The equivalent definitions of quasi-Frobenius rings in terms of duality and semi-injective rings are given in section 4.12. Quasi-Frobenius rings have many interesting equivalent definitions, in particular, an Artinian ring A is quasi-Frobenius if and only if A is a ring with duality for simple modules.

One of the most significant results of quasi-Frobenius rings is the theorem of C.Faith and E.A.Walker. This theorem says that a ring A is quasi-Frobenius if and only if every projective right A-module is injective and conversely.

Quivers of quasi-Frobenius rings are studied in section 4.13. The most important result of this section is the Green theorem: the quiver of any quasi-Frobenius ring is strongly connected. Conversely, for a given strongly connected quiver Q there is a symmetric algebra A such that $Q(A) = Q$. Symmetric algebras with given quivers are studied in section 4.14.

4.1 DUALITY PROPERTIES

Let A be a finite dimensional algebra over a field k. We shall establish a duality between the category $\mathrm{mod}_r A$ of right finite dimensional A-modules and the category $\mathrm{mod}_l\, A$ of left finite dimensional A-modules.

Let $M \in \mathrm{mod}_r A$. Denote by $M^* = \mathrm{Hom}_k(M, k)$ the conjugate (linear dual) space which is the vector space of linear functionals on M. Then M^* is naturally a left A-module. Indeed, for $\varphi \in M^*$ and $a \in A$ we define $a\varphi$ by the formula $(a\varphi)(m) = \varphi(ma)$, where $m \in M$. It is easy to verify that M^* is a finite dimensional left A-module, i.e., $M^* \in \mathrm{mod}_l A$. The module M^* is called the **dual** of M.

Analogously, if $M \in \mathrm{mod}_l A$, then the conjugate space M^* is a finite dimensional right A-module, i.e., $M^* \in \mathrm{mod}_r A$.

Obviously, $\dim_k M^* = \dim_k M$ and $M^{**} \simeq M$ as A-modules. Every linear map $\varphi : M \to N$ induces a conjugate map $\varphi^* : N^* \to M^{**}$ defined by $(\varphi^* f)(m) = f(\varphi m)$.

We can check readily that if φ is an A-homomorphism, so is φ^*. Moreover, $(\varphi\psi)^* = \psi^*\varphi^*$ and $1^* = 1$.

Hence, assigning to every right A-module M the left A-module M^* and to every homomorphism φ the homomorphism φ^* we obtain a contravariant exact functor $*$ from $\mathrm{mod}_r A$ to $\mathrm{mod}_l A$. Analogously, we have a contravariant exact functor $*$ from $\mathrm{mod}_l A$ to $\mathrm{mod}_r A$. We shall call these functors the **duality functors**.

Proposition 4.1.1. *There is a bijective correspondence between submodules of M and those of M^*, reversing the inclusion.*

Proof. Let $N \subset M$ be a submodule of M. Then it defines a natural epimorphism $\pi : M \to M/N$, and thus a monomorphism $\pi^* : (M/N)^* \to M^*$, that is, a left submodule of M^*. This submodule has a simple interpretation: it coincides with the "orthogonal complement" $N^\perp = \{\varphi \in M^* : \varphi(N) = 0\}$. Moreover, $M^*/N^\perp \simeq N^*$.

It is easy to verify that the correspondence $N \to N^\perp$ satisfies the following conditions:

(1) for $N_1 \subset N_2$ we have $N_2^\perp \subset N_1^\perp$;
(2) $(N_1 + N_2)^\perp = N_1^\perp \cap N_2^\perp$ and $(N_1 \cap N_2)^\perp = N_1^\perp + N_2^\perp$;
(3) $N^{\perp\perp} = N$.

A correspondence satisfying these conditions is called an **anti-isomorphism of lattices**.

Proposition 4.1.2. *A right A-module U is simple if and only if the left A-module U^* is simple.*

Proof. Since $U \simeq U^{**}$, it is sufficient to show that if U is not simple, then U^* is not simple as well. Let $N \subset U$ be a non-trivial submodule of U. Consider the exact sequence:

$$0 \to N \to U \to U/N \to 0.$$

Applying the exact duality functor $* = \mathrm{Hom}(-, k)$ to this sequence, we obtain the exact sequence

$$0 \to (U/N)^* \to U^* \to N^* \to 0,$$

where both $(U/N)^*$ and N^* are nonzero modules. So U^* is not simple. Consequently, the modules U and U^* are simple simultaneously.

Remark 4.1.1. Taking into account the dual definitions of projective and injective modules, it is easy to see that any indecomposable left injective A-module Q is equal to a $P^* = \mathrm{Hom}_k(P, k)$, where P is a principal right A-module.

Remark 4.1.2. By the annihilation lemma (see vol.I, p.265), for any simple right A-module U there exists a unique canonical idempotent $f \in A$ such that $Uf = U$. By the definition of U^*, we have $Uf = U$ if and only if $fU^* = U^*$.

Remark 4.1.3. Let M be a finite dimensional module over a finite dimensional algebra A with radical R. Then $\mathrm{rad}\, M = MR$, by proposition 5.1.8, vol.I.

Recall that the **socle** of a right A-module M, which is denoted by $\mathrm{soc}M$, is the sum of all simple right submodules of M. If there are no such submodules, then $\mathrm{soc}M = 0$.

Proposition 4.1.3. *For any finitely generated right A-module M we have that $(\mathrm{rad}M)^{\perp}$ is the socle of M^* and $soc\, M^* \simeq (M/\mathrm{rad}\, M)^*$.*

Proof. Since, by definition, $\mathrm{rad}M$ is the intersection of all maximal submodules of M, the statement follows from proposition 4.1.1.

Obviously, a finite dimensional algebra is an Artinian ring and consequently it is a semiperfect ring. Therefore applying the duality functor $* = \mathrm{Hom}\,(-, k)$ to theorem 10.4.10, vol.I, i.e., by "inverting all arrows", we immediately obtain the following statement:

Theorem 4.1.4. *Any indecomposable injective module Q over a finite dimensional algebra A is finite dimensional; it is the injective hull of a simple A-module and has exactly one simple submodule* $\mathrm{soc}\, Q$. *There is a one to one correspondence between the mutually nonisomorphic indecomposable injective A-modules $Q_1, \ldots, Q_s$ and the mutually nonisomorphic simple A-modules $U_1, \ldots, U_s$ which is given by the following correspondences*:

$$Q_i \mapsto \mathrm{soc}\, Q_i = U_i$$

and

$$U_i \mapsto E(U_i),$$

where $E(U_i)$ is the injective hull of U_i.

4.2 FROBENIUS AND SYMMETRIC ALGEBRAS

Definition. A finite dimensional k-algebra A is called **Frobenius** if the right modules A_A and $({}_AA)^*$ are isomorphic.

If an algebra A is Frobenius then the left modules ${}_AA$ and $(A_A)^*$ are also isomorphic.

Taking remark 4.1.1 into account, it follows that if A is a Frobenius algebra, then the right module $({}_AA)^*$ is injective. Analogously, the left module $(A_A)^*$ is injective.

Theorem 4.2.1. *Let A be a finite dimensional k-algebra. The following statements are equivalent*:

(1) *A is a Frobenius algebra*;

(2) *there exists a non-degenerate bilinear form $f : A \times A \to k$ which is associative, i.e., $f(ab,c) = f(a,bc)$ for all $a,b,c \in A$;*

(3) *there exists a linear function $\sigma : A \to k$ such that the kernel of σ contains neither left nor right ideals.*

Proof.

(1) $\Rightarrow$ (2). Let the left A-modules ${}_AA$ and $(A_A)^*$ be isomorphic and let

$$\Theta : {}_AA \to (A_A)^*$$

be an isomorphism. Then $\Theta(ab) = a\Theta(b)$ for all $a,b \in A$. So $\Theta(ab)x = (a\Theta(b))x = \Theta(b)xa$, where $x \in A$.

Define a bilinear form $f : A \times A \to k$ by the formula $f(x,y) = \Theta(y)x$.

We shall show that $f(x,y)$ is non-degenerate. Assume $f(x,y) = 0$ for all $x \in A$. Then $\Theta(y) = 0$, and so $y = 0$ since Θ is an isomorphism.

Let $e_1, \ldots, e_n$ be a basis of A, $x = x_1e_1 + \ldots x_ne_n$ and $y = y_1e_1 + \ldots y_ne_n$, $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, $\mathbf{G} = (f(e_i,e_j))$. Obviously,

$$f(x,y) = \mathbf{x}\mathbf{G}\mathbf{y}^T,$$

where

$$\mathbf{y}^T = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

The implication

$$f(x,y) = 0 \ \text{ for all } x \in A \quad \Rightarrow \quad y = 0$$

is equivalent to the implication

$$\mathbf{G}\,\mathbf{y}^T = \mathbf{0} \Rightarrow \mathbf{y}^T = \mathbf{0}.$$

So $\det \mathbf{G} \neq 0$ and $\mathbf{xG} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. Consequently, if $f(x, y) = 0$ for all $y \in A$, then $x = 0$. Therefore the bilinear form $f(x, y)$ is non-degenerate.

We have $f(xy, z) = \Theta(z)xy$ and $f(x, yz) = \Theta(yz)x = \Theta(z)xy$. Consequently,

$$f(xy, z) = f(x, yz).$$

and (1) $\Rightarrow$ (2) is proved.

(2) $\Rightarrow$ (1). Let $f : A \times A \to k$ be an associative non-degenerate bilinear form. Define a map $\Theta : {}_AA \to (A_A)^*$ by the formula $\Theta(y)x = f(x, y)$, where $x, y \in A$.

Since $f(x, y)$ is an associative non-degenerate form, Θ is an A-isomorphism.

(2) $\Rightarrow$ (3). Define a linear function $\sigma : A \to k$ by the formula $\sigma(x) = f(x, 1)$, where $x \in A$.

Let $\mathcal{I} = Aa_1 + \ldots + Aa_n$ be a left ideal in A and $\sigma(\mathcal{I}) = 0$. Consequently, $\sigma(Aa_i) = 0$ for $i = 1, \ldots, n$ and there is a nonzero left ideal $\mathcal{I}$ contained in $\operatorname{Ker} \sigma$ if and only if there exists an $a \neq 0$, $a \in A$ such that $\sigma(Aa) = 0$. If $\sigma(Aa) = 0$, then $f(Aa, 1) = f(A, a) = 0$ and $a = 0$. Analogously, $f(aA, 1) = 0$ implies $a = 0$. So $\operatorname{Ker} \sigma$ contains neither left nor right ideals.

(3) $\Rightarrow$ (2). Let σ be a linear function as in condition (3). It is easy to verify that the bilinear form $f(x, y) = \sigma(xy)$ satisfies condition (2). The theorem is proved.

Definition. A finite dimensional k-algebra A is called **symmetric** if there exists a non-degenerate associative bilinear form $f : A \times A \to k$ which is symmetric, i.e.,

$$f(a, b) = f(b, a)$$

for all $a, b, c \in A$.

From theorem 4.2.1 there immediately follows the following equivalent definition of a symmetric algebra.

Proposition 4.2.2. *A finite dimensional k-algebra A is symmetric if and only if there exists a linear function $\sigma : A \to k$ such that* $\operatorname{Ker} \sigma$ *contains neither left nor right ideals and $ab - ba \in \operatorname{Ker} \sigma$ for all $a, b \in A$.*

Obviously, each symmetric algebra is Frobenius. And any commutative Frobenius k-algebra is a symmetric algebra. One of the most important examples of symmetric algebras is given by the following statement.

Theorem 4.2.3. *Let $A = kG$ be the group algebra of a finite group G over a field k. Then A is a symmetric algebra.*

Proof. Define the linear function $\sigma : A \to k$ on the k-algebra $A = kG$ by the formula

$$\sigma(\sum_{g \in G} \alpha_g g) = \alpha_1,$$

where 1 is the identity of G. Assume Kerσ contains a right ideal $\mathcal{I}$. Then for any $a \in \mathcal{I}$ we have $\sigma(aA) = 0$ and $a = \sum\limits_{g \in G} \alpha_g g$. Obviously, $\sigma(ag_1^{-1}) = \alpha_{g_1} = 0$ for all $g_1 \in G$. So $a = 0$, i.e., $\mathcal{I} = 0$. Analogously, Ker σ does not contain any nonzero left ideal.

Let $a = \sum\limits_{g \in G} \alpha_g g$ and $b = \sum\limits_{g \in G} \beta_g g$, then $\sigma(ab) = \sum\limits_{gh=1} \alpha_g \beta_h = 1$ and $\sigma(ba) = \sum\limits_{gh=1} \beta_h \alpha_g$. So $\sigma(ab - ba) = 0$. Thus A is a symmetric algebra, by proposition 4.2.2.

4.3 MONOMIAL IDEALS AND NAKAYAMA PERMUTATIONS OF SEMIPERFECT RINGS

Let $1 = f_1 + \ldots + f_s$ be a canonical decomposition of the identity of a semiperfect ring A with Jacobson radical R. Then $A_A = f_1 A \oplus \ldots \oplus f_s A$ (resp. ${}_A A = A f_1 \oplus \ldots \oplus A f_s$), where $f_i A = P_i^{n_i}$ (resp. $A f_i = Q_i^{n_i}$), for $i = 1, \ldots, s$, is called the **canonical decomposition** of a ring A into a direct sum of its principal right (left) A-modules. Let $A_{ij} = f_i A f_j$ $(i, j = 1, \ldots, s)$. The two-sided Peirce decomposition of the ring A

$$A = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1s} \\ A_{21} & A_{22} & \ldots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \ldots & A_{ss} \end{pmatrix} \tag{4.3.1}$$

is called a **canonical two-sided Peirce decomposition** of A.

From theorem 11.1.7, vol.I, it follows that every other canonical Peirce decomposition of A can be obtained from (4.3.1) by a simultaneous permutation of rows and columns and a replacement of all the Peirce components A_{ij} by $aA_{ij}\, a^{-1}$ for some invertible element $a \in A$.

Let $1 = e_1 + \ldots + e_n$ be a decomposition of 1 into a sum of pairwise orthogonal idempotents. By an ideal we shall mean here a two-sided ideal. For any ideal $\mathcal{I}$ of A the Abelian group $e_i \mathcal{I} e_j$ $(i, j = 1, \ldots, n)$ obviously lies in $\mathcal{I}$, and $\mathcal{I} = \bigoplus\limits_{i,j=1}^{n} \mathcal{I}_{ij}$ is a decomposition of $\mathcal{I}$ into a direct sum of Abelian subgroups. Such a decomposition is called the **two-sided Peirce decomposition** of $\mathcal{I}$ corresponding to $1 = e_1 + \ldots + e_n$. It has the following matrix form:

$$\mathcal{I} = \begin{pmatrix} \mathcal{I}_{11} & \mathcal{I}_{12} & \cdots & \mathcal{I}_{1n} \\ \mathcal{I}_{21} & \mathcal{I}_{22} & \cdots & \mathcal{I}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{I}_{n1} & \mathcal{I}_{n2} & \cdots & \mathcal{I}_{nn} \end{pmatrix}$$

If $\mathcal{J} = \bigoplus\limits_{i,j=1}^{n} \mathcal{J}_{ij}$ is also an ideal, then

$$\mathcal{I}+\mathcal{J}=\begin{pmatrix}\mathcal{I}_{11}+\mathcal{J}_{11} & \mathcal{I}_{12}+\mathcal{J}_{12} & \cdots & \mathcal{I}_{1n}+\mathcal{J}_{1n}\\ \mathcal{I}_{21}+\mathcal{J}_{21} & \mathcal{I}_{22}+\mathcal{J}_{22} & \cdots & \mathcal{I}_{2n}+\mathcal{J}_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \mathcal{I}_{n1}+\mathcal{J}_{n1} & \mathcal{I}_{n2}+\mathcal{J}_{n2} & \cdots & \mathcal{I}_{nn}+\mathcal{J}_{nn}\end{pmatrix},$$

and each Peirce component $(\mathcal{I}\mathcal{J})_{ij}$ of the product $\mathcal{I}\mathcal{J}$ is given by

$$(\mathcal{I}\mathcal{J})_{ij}=\sum_{k=1}^{n}\mathcal{I}_{ik}\mathcal{J}_{kj}\quad (i,j=1,\ldots,n),$$

so that addition and multiplication of elements from $\mathcal{I}$ and $\mathcal{J}$ can be done by addition and multiplication of the corresponding matrices.

Let A be a semiperfect ring and let $1=f_1+\ldots+f_s$ be a canonical decomposition of $1\in A$ into a sum of pairwise orthogonal idempotents (see vol.I, p.265). Then $\mathcal{I}=\bigoplus\limits_{i,j=1}^{s}\mathcal{I}_{ij}$ with $\mathcal{I}_{ij}=f_i\mathcal{I}f_j$ $(i,j=1,\ldots,s)$ is called the **canonical two-sided Peirce decomposition** of $\mathcal{I}$. From theorem 11.1.7, vol.I, it follows that one canonical Peirce decomposition of $\mathcal{I}$ can be obtained from another one by a simultaneous permutation of rows and columns and the replacement of each Peirce component $\mathcal{I}_{ij}$ by $a\mathcal{I}_{ij}a^{-1}$, where $a\in A$.

Definition. An ideal $\mathcal{I}$ of a semiperfect ring A will be called **monomial** if each row and each column of a canonical two-sided Peirce decomposition of $\mathcal{I}$ contains exactly one nonzero Peirce component.

If $\mathcal{I}$ is a monomial ideal, then there exists a permutation ν of $\{1,\ldots,s\}$ such that $\mathcal{I}_{i\nu(i)}\neq 0$. Clearly, ν is uniquely determined up to conjugation by an element from the symmetric group on s letters. We denote such a permutation by $\nu(\mathcal{I})$.

Lemma 4.3.1. *Let A be a semiperfect ring. If $\mathcal{I}$ is a monomial ideal of A then each canonical two-sided Peirce component of $\mathcal{I}$ is an ideal of A.*

Proof. Let $1=f_1+\ldots+f_s$ be a canonical decomposition of $1\in A$ into a sum of pairwise orthogonal idempotents. Write $\nu=\nu(\mathcal{I})$, then $\mathcal{I}=\bigoplus\limits_{i=1}^{s}f_i\mathcal{I}f_{\nu(i)}$. Obviously, $f_i\mathcal{I}f_{\nu(i)}f_kAf_l=0$ if $k\neq\nu(i)$. Moreover, $f_i\mathcal{I}f_{\nu(i)}f_{\nu(i)}Af_l\subseteq f_i\mathcal{I}f_l$ which is nonzero if and only if $l=\nu(i)$, since $\mathcal{I}$ is monomial. Similarly, $f_kAf_lf_i\mathcal{I}f_{\nu(i)}\neq 0$ if and only if $k=l=i$. It follows that $f_i\mathcal{I}f_{\nu(i)}$ is an ideal of A for each $i=1,\ldots,n$.

Let $A_A=P_1^{n_1}\oplus\ldots\oplus P_s^{n_s}$ (resp. ${}_AA=Q_1^{n_1}\oplus\ldots\oplus Q_s^{n_s}$) be the canonical decomposition of a semiperfect ring A into a direct sum of right (left) principal modules.

Let M be a right A-module and N be a left A-module. We set $\operatorname{top}M=M/MR$ and $\operatorname{top}N=N/RN$. Since P_i is a principle right A-module, $\operatorname{top}P_i$is a simple right module, and analogously, $\operatorname{top} Q_i$ is a simple left module for $i = 1, \dots, s$. We denote by $\operatorname{soc} M$ (resp. $\operatorname{soc} N$) the largest semisimple right (resp. left) submodule of M (resp. N).

Example 4.3.1.

Let $A = T_n(D)$ be a ring of all upper-triangular matrices over a division ring D. Let $P_1 = e_{11}A, \dots, P_n = e_{nn}A$ and ${}_AP_1 = Ae_{11}, \dots, {}_AP_n = Ae_{nn}$. Let $Z_r = \operatorname{soc} A_A$ and $Z_l = \operatorname{soc}_A A$. Obviously $Z_r = {}_AP_n = \underbrace{U_n \oplus \ldots \oplus U_n}_{n \ times}$, where $U_n = P_n$ is a simple right A-module and $Z_l = P_1 = \underbrace{V_1 \oplus \ldots \oplus V_1}_{n \ times}$, where $V_1 = {}_AP_1$ is a simple left A-module. Thus $Z_r \cap Z_l = e_{11}Ae_{nn}$ is a two-sided ideal. Obviously, $(Z_l \cap Z_r)_A = U_n$ and ${}_A(Z_l \cap Z_r) = V_1$. Moreover, $\dim_D Z_r = \dim_D Z_l = n$.

Definition. We say that a semiperfect ring A admits a **Nakayama permutation** $\nu(A) : i \to \nu(i)$ of $\{1, \dots, s\}$ if the following conditions are satisfied:

(np1) $\operatorname{soc} P_k = \operatorname{top} P_{\nu(k)}$,

(np2) $\operatorname{soc} Q_{\nu(k)} = \operatorname{top} Q_k$.

Let A be a semiperfect ring, which admits a Nakayama permutation $\nu(A)$. By condition (np1), the socle of every principal module is simple and, moreover, two principal modules with isomorphic socles have to be isomorphic, by theorem 10.4.10, vol.I. By condition (np2), the socles of the principal left modules are also simple.

Theorem 4.3.2. *Let A be a semiperfect ring such that the socles of all principal right A-modules and of all principal left A-modules are simple. Suppose furthermore, that if the socles of two principal right A-modules P and P' are isomorphic then $P \cong P'$. Then A satisfies the following conditions*:

(i) $\operatorname{soc} A_A = \operatorname{soc} {}_AA = \mathcal{Z}$ *and $\mathcal{Z}$ is a monomial ideal*;

(ii) *the ring A admits a Nakayama permutation $\nu = \nu(A)$ with $\nu(A) = \nu(\mathcal{Z})$.*

Proof. Let $1 = f_1 + \ldots + f_s$ be a canonical decomposition of $1 \in A$ into a sum of pairwise orthogonal idempotents and $f_iA = P_i^{n_i}$ $(i = 1, \dots, s)$. Set $\mathcal{Z}_r = \operatorname{soc} A_A$ and $\mathcal{Z}_l = \operatorname{soc} {}_AA$. The equality ${}_AA = Af_1 \oplus \ldots \oplus Af_s$ implies that $\mathcal{Z}_l = \operatorname{soc} Af_1 \oplus \ldots \oplus \operatorname{soc} Af_s$ and $\mathcal{Z}_l f_i = \operatorname{soc} Af_i$ for all $i = 1, \dots s$. Similarly, $\mathcal{Z}_r = \operatorname{soc} f_1A \oplus \ldots \oplus \operatorname{soc} f_sA$ and $f_j\mathcal{Z}_r = \operatorname{soc} f_jA$ $(j = 1, \dots s)$. From our hypothesis it follows that $\operatorname{soc} P_1, \dots, \operatorname{soc} P_s$ is a permutation of the simple modules $U_1 = \operatorname{top} P_1, \dots, U_s = \operatorname{top} P_s$.

We write $e \in f_i$ if there is a decomposition $f_i = e_1 + \ldots + e_{n_i}$ into a sum of pairwise orthogonal local idempotents and $e = e_j$ for some $j \in \{1, \dots, n_i\}$. For a fixed $i = 1, \dots, s$ and each local idempotent $e \in f_i$, we obtain, by the annihilation lemma, that $\mathcal{Z}_r e \neq 0$. Then $\mathcal{Z}_r e$ must contain $\operatorname{soc} Ae$, since $\operatorname{soc} Ae$ is simple. Hence $\mathcal{Z}_l f_i = \operatorname{soc} Af_i \subseteq \mathcal{Z}_r f_i$ for all $i = 1, \dots, s$, which yields $\mathcal{Z}_l \subseteq \mathcal{Z}_r$.

Similarly, $f_j\mathcal{Z}_r \subseteq f_j\mathcal{Z}_l$ $(j = 1, \dots, s)$, which implies that $\mathcal{Z}_r \subseteq \mathcal{Z}_l$. Consequently, $\mathcal{Z}_r = \mathcal{Z}_l$.

Since $\mathcal{Z}_r \cong U_{\nu(1)}^{n_1} \oplus \dots \oplus U_{\nu(s)}^{n_s}$, then, by the annihilation lemma of simple modules, $\mathcal{Z}_r$ has the following two-sided Peirce decomposition: $\mathcal{Z}_r = \bigoplus_{i=1}^{s} f_i\mathcal{Z}_r f_{\nu(i)}$. Thus $\mathcal{Z} = \mathcal{Z}_r = \mathcal{Z}_l$ is a monomial ideal and $\nu(\mathcal{Z})$ is a Nakayama permutation of A.

Recall the notions of right and left annihilators (see vol.I, p.219). Let S be a subset in a ring A. Then $\mathrm{r.ann}_A(S) = \{x \in A : Sx = 0\}$ and $\mathrm{l.ann}_A(S) = \{x \in A : xS = 0\}$. We shall write $r(S)$ instead of $\mathrm{r.ann}_A(S)$ and $l(S)$ instead of $\mathrm{l.ann}_A(S)$.

Proposition 4.3.3. *Let A be a semiperfect ring. Then $\mathrm{soc}(A_A)$ coincides with the left annihilator $l(R)$ of $R = \mathrm{rad}\,A$, whereas $\mathrm{soc}({}_AA)$ coincides with the right annihilator $r(R)$. In particular, $\mathrm{soc}({}_AA)$ and $\mathrm{soc}(A_A)$ are two-sided ideals.*

Proof. If U is a simple right A-module, then, obviously, $UR = 0$ and, consequently, $\mathrm{soc}(A_A) \subseteq l(R)$. On the other hand, the equality $l(R)R = 0$ implies that $l(R)$ is a semisimple right A-module, so it must be contained in the right socle of A, hence, $l(R) = \mathrm{soc}(A_A)$. Similarly, $r(R) = \mathrm{soc}({}_AA)$. The statement is proved.

4.4 QUASI-FROBENIUS ALGEBRAS

In this section we shall consider an important class of algebras first introduced by T.Nakayama. These algebras are a generalization of the Frobenius algebras considered in section 4.2.

Definition. A finite dimensional algebra A over a field k is called **quasi-Frobenius** if the regular right module A_A is injective. For short, we shall call it a **QF-algebra**.

This condition is equivalent to the fact that any projective right A-module is injective.

Remark 4.4.1. From the duality properties it is easy to see that A_A is injective if and only if ${}_AA$ is injective. So, the definition of quasi-Frobenius algebras is right and left symmetric.

Examples 4.4.1.

1. Any Frobenius algebra is quasi-Frobenius, since in this case $A_A \simeq ({}_AA)^*$ is injective.
2. Any group algebra kG of a finite group G over a field k is quasi-Frobenius, by theorem 4.2.3.

Let $A_A = P_1^{n_1} \oplus \dots \oplus P_s^{n_s}$ $({}_AA = Q_1^{n_1} \oplus \dots \oplus Q_s^{n_s})$ be a decomposition of the right (left) regular A-module into a direct sum of non-isomorphic principal right (left) A-modules.

By remark 4.1.1, any indecomposable right injective A-module has the form $Q_i^* = \mathrm{Hom}_k(Q_i, k)$ for some $i = 1, \dots, s$. By theorem 4.1.4, the modules

$Q_1^*, \dots, Q_s^*$ are all pairwise non-isomorphic indecomposable injective A-modules.

If A is a Frobenius algebra, then $A_A^* \simeq P_1^{n_1} \oplus \dots \oplus P_s^{n_s}$. If A is a quasi-Frobenius algebra, then $A_A^* \simeq P_1^{m_1} \oplus \dots \oplus P_s^{m_s}$, but in general $n_i \neq m_i$ for $i = 1, \dots, s$.

Recall that a finite dimensional algebra A over a field k is called **basic** if the quotient algebra $\bar{A} = A/R$ is a direct product of division algebras. It is equivalent to the fact that if $A_A = P_1 \oplus P_2 \oplus \dots \oplus P_s$ is a decomposition of A_A into a direct sum of principal modules, then $P_i \not\simeq P_j$ for $i \neq j$ $(i, j = 1, \dots, s)$.

The fact that the definition of quasi-Frobenius algebra has been given in terms of module categories implies the following proposition.

Proposition 4.4.1. *Every finite dimensional algebra Morita equivalent to a quasi-Frobenius algebra is quasi-Frobenius.*

In particular, every quasi-Frobenius algebra is Morita-equivalent to a Frobenius algebra (namely, to its basic algebra).

Theorem 4.4.2. *The following conditions for a finite dimensional algebra A are equivalent*:

(1) *A is quasi-Frobenius*;

(2) *A admits a Nakayama permutation.*

Proof. Without loss of generality, we may assume that A is basic. Let $A_A = P_1 \oplus P_2 \oplus \dots \oplus P_s$ $({}_AA = Q_1 \oplus \dots \oplus Q_s)$ be a canonical decomposition of the right (resp. left) regular A-module A_A (resp. ${}_AA$) into a direct sum of non-isomorphic principal right (left) A-modules, and let $1 = f_1 + f_2 + \dots + f_n$ be the corresponding decomposition of the identity of A into a sum of pairwise orthogonal local idempotents.

Any indecomposable right injective A-module has the form $Q_i^* = \mathrm{Hom}_k(Q_i, k)$ for some $i = 1, \dots, s$. By theorem 4.1.4, the modules $Q_1^*, \dots, Q_s^*$ are all pairwise non-isomorphic indecomposable injective A-modules.

(1) $\Rightarrow$ (2). Suppose that A is quasi-Frobenius, then there exists a permutation ν on $\{1, 2, \dots, s\}$ such that

$$P_i \simeq Q_{\nu(i)}^*, \dots, P_s \simeq Q_{\nu(s)}^* \tag{4.4.1}$$

Then, taking into account proposition 4.1.3, we have

$$\mathrm{soc}\, P_i \simeq \mathrm{soc}\, Q_{\nu(i)}^* \simeq \left(Q_{\nu(i)}/RQ_{\nu(i)}\right)^*.$$

We shall show that

$$\left(Q_{\nu(i)}/RQ_{\nu(i)}\right)^* \simeq U_{\nu(i)},$$

where $U_i = P_i/P_iR$ and $V_i = Q_i/RQ_i$. Really that, by the annihilation lemma, $f_iV_i \neq 0$. Then, by remark 4.1.2, $V_if_i \neq 0$, as well. So, by the annihilation lemma $V_i^* \simeq U_i$. Thus,

$$\mathrm{soc}\, P_i \simeq U_{\nu(i)} = \mathrm{top}\, P_{\nu(i)}.$$

Since ${}_AA$ is also injective by remark 4.4.1, all principal left A-modules $Q_1, \dots, Q_s$ have pairwise non-isomorphic simple socles, as well. Then, by theorem 2.3.2, we obtain that a quasi-Frobenius algebra A admits the Nakayama permutation $\nu(A) : i \to \nu(i)$ of $\{1, \dots, s\}$.

(2) $\Rightarrow$ (1). Suppose that a finite dimensional algebra A admits a Nakayama permutation. Let $E(P_1), \dots, E(P_s)$ be injective hulls of the principal right A-modules $P_1, \dots, P_s$ respectively. Then, $E(P_1), \dots, E(P_s)$ are indecomposable and pairwise non-isomorphic. Obviously, $\dim E(P_i) \geq \dim P_i$ for $i = 1, \dots, s$. By remark 4.1.1, we have $E(P_i) = Q^*_{\nu(i)}$ for $i = 1, \dots, s$.

Suppose there exists a number j $(a \leq j \leq s)$ such that $\dim E(P_j) > \dim P_j$. Since $\dim Q_i = \dim Q_i^*$,

$$\dim A = \sum_{i=1}^{s} \dim Q_i = \sum_{i=1}^{s} \dim E(P_i) > \sum_{i=1}^{s} \dim P_i = \dim A.$$

This contradiction proves that $P_i = E(P_i)$ for $i = 1, \dots, s$ and A is quasi-Frobenius.

Remark 4.4.2. It is easy to see that a quasi-Frobenius algebra is Frobenius if and only if $n_{\nu(i)} = n_i$ for all $i = 1, \dots, s$.

Theorem 4.4.3. *An algebra A is quasi-Frobenius if and only if the socle of each principal A-module is simple, and for any non-isomorphic principal A-modules P_1 and P_2,* $\operatorname{soc} P_1 \not\simeq \operatorname{soc} P_2$.

Proof. If A is a quasi-Frobenius algebra then, by theorem 4.4.2, A admits a Nakayama permutation. So it is enough to prove the sufficiency of the theorem. Without loss of generality, we may assume that A is basic. Then $A_A = P_1 \oplus \dots \oplus P_s$ and ${}_AA = Q_1 \oplus \dots \oplus Q_s$, where the $P_1, \dots, P_s$ are pairwise non-isomorphic right principal modules and the $Q_1, \dots, Q_s$ are pairwise non-isomorphic left principal modules. Let $A^* = Q_1^* \oplus \dots \oplus Q_s^*$. By remarks 4.1.1, 4.1.2 and theorem 4.1.4, all the $\operatorname{soc} Q_1^*, \dots, \operatorname{soc} Q_s^*$ are the pairwise non-isomorphic simple A-modules $U_1, \dots, U_s$. By condition of the theorem, the map $\nu(A) : i \mapsto \nu(i)$ of the set $\{1, \dots, s\}$ such that $\operatorname{soc} P_k = \operatorname{top} P_{\nu(k)}$ is a transposition. By proposition 5.3.7, vol.I injective hulls $E(U_{\nu(k)})$ and $E(P_k)$ of $U_{\nu(k)}$ and U_k, respectively, coincide. As above, if there exists a number j $(1 \leq j \leq s)$, such that $\dim E(P_j) > \dim P_j$, then

$$\dim A = \sum_{i=1}^{s} \dim E(P_i) > \sum_{i=1}^{s} \dim P_i = \dim A.$$

This contradiction proves that $P_i = E(P_i)$ for $i = 1, \dots, s$ and therefore A is Frobenius.

Theorem 4.4.4. *A Nakayama permutation $\nu(A)$ of a symmetric k-algebra A is the identity permutation.*

Proof. Let $A_A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be a canonical decomposition of A into a direct sum of right principal modules and let $1 = f_1 + \ldots + f_s$ be a canonical decomposition of the identity of A. By theorem 4.3.2, $\operatorname{soc} A_A = \operatorname{soc}_A A = \mathcal{Z}$ and $\mathcal{Z}$ is a monomial ideal with $\nu(\mathcal{Z}) = \nu(A)$. Suppose that $\nu(A)$ is not the identity permutation. Then there are $i \neq j$ such that $f_i \mathcal{Z} f_j \neq 0$ and $f_i \mathcal{Z} f_j$ is a two-sided ideal.

Since A is a symmetric algebra, by proposition 4.2.2 there exists a linear function $\sigma : A \to k$ such that $\mathrm{Ker}\sigma$ contains neither left nor right ideals and $ab - ba \in \mathrm{Ker}\,\sigma$ for all $a, b \in A$.

Consider $f_i \mathcal{Z} f_j$, which is a two-sided ideal by lemma 4.3.1. Let $z \in \mathcal{Z}$ and $z \neq 0$. We have $\sigma(f_i z f_j) = \sigma(f_i z f_j \cdot f_j) = \sigma(f_j \cdot f_i z f_j) = \sigma(0) = 0$. So $f_i \mathcal{Z} f_j = 0$ and we have a contradiction. Consequently, for any $i = 1, \ldots, s$ it must be the case that $f_i \mathcal{Z} f_i \neq 0$ and $\nu(\mathcal{Z}) = \nu(A) = E$ is the identity permutation.

Definition. A quasi-Frobenius algebra A is called **weakly symmetric** if the Nakayama permutation $\nu(A)$ of A is the identity permutation.

Theorem 4.4.5. *Let A be a weakly symmetric algebra. Then A is Frobenius and every algebra C Morita equivalent to A is also Frobenius. Conversely, if every finite dimensional algebra C Morita equivalent to a Frobenius algebra A is Frobenius, then A is a weakly symmetric algebra.*

Proof. By theorem 4.4.2 and remark 4.4.2, every QF-algebra with identity Nakayama permutation is automatically Frobenius. Clearly, every algebra which is Morita equivalent to a Frobenius algebra with identity Nakayama permutation is Frobenius.

Let A be a Frobenius algebra and let $\nu(A)$ be not the identity. Then we can assume that $\operatorname{soc} P_1 = \operatorname{top} P_2$. Let $A = P_1^{n_1} \oplus P_2^{n_2} \oplus \ldots \oplus P_s^{n_s}$ be a canonical decomposition of A into a direct sum of non-isomorphic principal A-modules. From the definition of a Frobenius algebra it follows that $n_2 = n_1$. Set $P = P_1^2 \oplus P_2 \oplus \ldots \oplus P_s$. Then $C = \mathrm{End}_A P$ is a QF-algebra, $\nu(A) = \nu(C)$, the multiplicity of the first principal C-module is 2 and does not coincide with the multiplicity of the second principal C-module. Therefore, C is not Frobenius.

Definition. A local serial (=uniserial) algebra is called a **Köthe algebra.**

Proposition 4.4.6. *A Köthe algebra is Frobenius.*

Proof. This immediately follows from theorem 4.4.2.

Examples 4.4.2.

(a) Let $G = \{ g \}$ be a cyclic group of order 4, and let $k = F_2$ be the field of two elements, $A = kG$.

Set $r = 1 + g$. Then $R = rA$ is the Jacobson radical of A, and the elements $1, r, r^2, r^3$ form a basis of A, i.e., A is a Köthe algebra of length 4.

(b) Let $G = \{a\} \times \{b\}$ be the Klein four-group, i.e., $a^2 = 1$ and $b^2 = 1$; $k = F_2$, $A = kG$. Set $r_1 = 1 + a$, $r_2 = 1 + b$. Then $r_1 r_2 = 1 + a + b + ab$, and $R = \{\alpha r_1 + \beta r_2 + \gamma r_1 r_2 \ : \ \alpha, \beta, \gamma \in k\}$ is the Jacobson radical of A, $R^2 = \{\delta r_1 r_2 \ : \ \delta \in k\}$. For every $r \in R$ we have $r^2 = 0$.

(c) We set $A^* = \mathrm{Hom}_k(A, k)$ for a finite dimensional algebra A over a field k. Let A be the four-dimensional algebra over $k = F_2$ with basis $1, \sigma_1, \sigma_2, \sigma_1^2$ such that $\sigma_1^2 = \sigma_2^2$ and $\sigma_1\sigma_2 = \sigma_2\sigma_1 = 0$. Let $\psi \in A^*$, $\psi(\sigma_1^2) = 1$ and $\psi(1) = \psi(\sigma_1) = \psi(\sigma_2) = 0$. Then the multiplication table for the basis elements $1, \sigma_1, \sigma_2, \sigma_1^2$ is the following

	1	σ_1	σ_2	σ_1^2
1	1	σ_1	σ_2	σ_1^2
σ_1	σ_1	σ_1^2	0	0
σ_2	σ_2	0	σ_1^2	0
σ_1^2	σ_1^2	0	0	0

Therefore the matrix $\mathbf{B}$ of the bilinear form $f(x, y) = \psi(x \cdot y)$ in the basis $1, \sigma_1, \sigma_2, \sigma_1^2$ is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and A is a symmetric algebra.

(d) Let $H_s^{(m)}(\mathcal{O}) = H_s(\mathcal{O})/R^m$, where $\mathcal{O} = k[[x]]$ is the ring of formal power series over a field k. A k-algebra $H_s^{(m)}(\mathcal{O})$ is a Frobenius algebra, which is not symmetric for $m \not\equiv 1(mod\, s)$.[1]

It is easy to see that the algebras given in examples 4.4.2(b) and 4.4.2(c) are non-isomorphic. Indeed, in case (b) $x^2 = 0$ for all $x \in R$, on the other hand, in case (c) this does not hold.

We have the following strict inclusions:[2]

$$\text{(group algebras)} \subset \text{(symmetric algebras)} \subset$$
$$\subset \text{(Frobenius algebras)} \subset \text{(quasi-Frobenius algebras)}.$$

The algebra from example 4.4.2(c) is an example of a symmetric algebra which is not a group algebra, and so the first inclusion is strict.

Example 4.4.2(d) shows that the second inclusion is strict. From theorem 4.4.4 it follows that the third inclusion is also strict.

Example 4.4.3.

In conclusion we give an example of a semidistributive weakly symmetric algebra A over the field $k = F_2 = \{0, 1\}$. This algebra is a quotient algebra of the

[1] see example 4.5.1.

[2] see *C.W.Curtis, I.Reiner, Methods of Representation Theory I,II. John Wiley and Sons, New York, 1990,* §*66.*

path algebra kQ of the quiver with adjacency matrix

$$[Q] = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

An admissible ideal $\mathcal{J}$ of kQ is generated by the following paths: $\sigma_{21}\sigma_{12} = 0$, $\sigma_{12}r_2 = 0$, $r_2\sigma_{21} = 0$, $r_2^m = 0$, $m \geq 2$. Here σ_{12} is the arrow from 1 to 2 and r_2 is the loop at 2 and paths are read from left to right (as before).

Let $A = kQ/\mathcal{I}$. A basis of the first principal module P_1 is: e_1, σ_{12}, $\sigma_{12}\sigma_{21}$.

A basis of the second principal module P_2 is: $e_2, \sigma_{21}, r_2, r_2^2, \dots, r_2^{m-1}$, for $m \geq 2$. So, dim $A = m+4$ and this finite dimensional algebra contains 2^{m+4} elements. Obviously, $\operatorname{soc} P_1 = U_1$ and $\operatorname{soc} P_2 = U_2$, i.e., the algebra A is weakly symmetric.

Suppose that A is a symmetric algebra. By the definition of a symmetric algebra, there exists a linear function $\sigma : A \to k$ such that $\operatorname{Ker} \sigma$ contains neither left nor right ideals and $\sigma(ab) = \sigma(ba)$ for all $a, b \in A$.

Obviously, $\mathcal{J} = \{0, \sigma_{12}\sigma_{21}\}$ is a right ideal in A consisting of two elements. Then $\sigma(\sigma_{21}\sigma_{12}) = \sigma(\sigma_{12}\sigma_{21}) = 0$ and $\sigma(\mathcal{J}) = 0$. This contradiction proves that A is not symmetric. Note that $l(P_1) = 3$ and $l(P_2) = m + 1$. So, $l(P_2)$ may be arbitrarily large. (Here $l(M)$ denotes the length of a module M, i.e., the length of a composition series.)

4.5 QUASI-FROBENIUS RINGS

Let $P_1, \dots, P_s$ be the non-isomorphic principal right A-modules and let $Q_1, \dots, Q_s$ be the non-isomorphic principal left A-modules of a two-sided Artinian ring A. And let $A_A = P_1^{n_1} \oplus \dots \oplus P_s^{n_s}$ (resp. ${}_AA = Q_1^{n_1} \oplus \dots \oplus Q_s^{n_s}$) be a decomposition of the right (left) regular A-module into a direct sum of principal right (left) A-modules.

Definition. A two-sided Artinian ring A is called **quasi-Frobenius** (abbreviated, a QF**-ring**), if A admits a Nakayama permutation $\nu(A)$ of $\{1, 2, \dots, s\}$.

A quasi-Frobenius ring A is called **Frobenius**, if $n_{\nu(i)} = n_i$ for all $i = 1, \dots, s$.

Clearly, ν is determined up to conjugation in the symmetric group on s letters, and conjugations correspond to renumberings of the principal modules $P_1, \dots, P_s$.

We now construct some examples of quasi-Frobenius rings. Recall that a local ring $\mathcal{O}$ with unique nonzero maximal right ideal $\mathcal{M}$ is called a discrete valuation ring, if it has no zero divisors, the right ideals of $\mathcal{O}$ form the unique chain:

$$\mathcal{O} \supset \mathcal{M} \supset \mathcal{M}^2 \supset \dots \supset \mathcal{M}^n \supset \dots,$$

and, moreover, this chain is also the unique chain of left ideals of A. Then, obviously, $\mathcal{O}$ is Noetherian, but not Artinian, all powers of $\mathcal{M}$ are distinct and $\bigcap_{k=1}^{\infty} \mathcal{M}^k = 0$. Moreover, $\mathcal{M}$ is principal as a right (left) ideal.

Example 4.5.1.
Denote by $H_s(\mathcal{O})$ the ring of all $s \times s$ matrices of the following form:

$$H = H_s(\mathcal{O}) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{M} & \mathcal{O} & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \mathcal{O} & \mathcal{O} \\ \mathcal{M} & \mathcal{M} & \dots & \mathcal{M} & \mathcal{O} \end{pmatrix}.$$

It is easy to see that the radical R of $H_s(\mathcal{O})$ is

$$R = \begin{pmatrix} \mathcal{M} & \mathcal{O} & \dots & \dots & \mathcal{O} \\ \mathcal{M} & \mathcal{M} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \mathcal{O} \\ \mathcal{M} & \mathcal{M} & \dots & \mathcal{M} & \mathcal{M} \end{pmatrix}.$$

and

$$R^2 = \begin{pmatrix} \mathcal{M} & \mathcal{M} & \mathcal{O} & \dots & \dots & \mathcal{O} \\ \mathcal{M} & \mathcal{M} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \mathcal{O} \\ \mathcal{M} & \mathcal{M} & \dots & & \mathcal{M} & \mathcal{M} \\ \mathcal{M}^2 & \mathcal{M} & \dots & \dots & \mathcal{M} & \mathcal{M} \end{pmatrix}.$$

The principal right modules of H are the "row-ideals" of H and the submodules of each of them form a chain. In particular, the submodules of the "first-row-ideal" form the following chain:

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \supset \begin{pmatrix} \mathcal{M} & \mathcal{O} & \cdots & \mathcal{O} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \supset \quad \cdots \quad \supset$$

$$\supset \begin{pmatrix} \mathcal{M} & \cdots & \mathcal{M} & \mathcal{O} \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} \supset \begin{pmatrix} \mathcal{M} & \mathcal{M} & \cdots & \mathcal{M} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

It is easy to see that each other row-ideal of H is isomorphic to a submodule of the above module. In a similar way, the principal left H-modules are the column-ideals, whose submodules form corresponding chains. Thus, H is a serial ring. Let $P_1, \dots, P_s$ be the principal right modules of the quotient ring $A = H_s(\mathcal{O})/R^2$ and let $Q_1, \dots, Q_s$ be the principal left A-modules numbered in a such way that $P_i = e_{ii}A, Q_i = Ae_{ii}, (i = 1, \dots, s)$, where e_{ij} denotes the elementary $s \times s$ matrix whose (i, j)-th entry is 1 while all other entries are zero. Then the submodules of every P_i and Q_i form finite chains, and a direct verification shows that

$$\text{soc } P_1 \cong \text{top } P_2, \text{soc } P_2 \cong \text{top } P_3, \dots, \text{soc } P_s \cong \text{top } P_1$$

and

$$\text{top } Q_1 \cong \text{soc } Q_2, \text{top } Q_2 \cong \text{soc } Q_3, \dots, \text{top } Q_s \cong \text{soc } Q_1.$$

Moreover, each of these modules is a one-dimensional vector space over $\mathcal{O}/\mathcal{M}$. Hence, A is a quasi-Frobenius ring with Nakayama permutation $(1, 2, \dots, s)$.

More generally, the quotient ring $A = H_s(\mathcal{O})/R^m$ $(m \geq 2)$ is a quasi-Frobenius ring with Nakayama permutation $(1, 2, \dots, s)^{m-1}$. It follows, in particular, that A has the identity permutation as Nakayama permutation if and only if $m \equiv 1 (mod\ s)$.

Example 4.5.2.
Let $B_{2,s}$ be the ring of $2s \times 2s$ matrices of the form:

$$B_{2,s} = \begin{pmatrix} H & R \\ R & H \end{pmatrix},$$

where $H = H_s(\mathcal{O})$ as in the previous example.

It is easy to see that, the Jacobson radical of $B_{2,s}$ is $\begin{pmatrix} R & R \\ R & R \end{pmatrix}$.

Consider the ideal

$$\mathcal{J} = \begin{pmatrix} R^s & R^{s+1} \\ R^{s+1} & R^s \end{pmatrix}.$$

A direct verification shows that the quotient ring $A = B_{2,s}/\mathcal{J}$ is a semidistributive quasi-Frobenius ring with Nakayama permutation

$$\nu = (1,\ s+1)(2,\ s+2) \dots (s,\ 2s).$$

Remark 4.5.1. It can be verified that $B_{2,s}/\mathcal{J}$ is semidistributive.

The following theorem follows immediately from theorem 4.3.2.

Theorem 4.5.1. *Let A be a quasi-Frobenius ring. Then* $\mathrm{soc}({}_AA) = \mathrm{soc}(A_A)$. *Moreover,* $\mathcal{Z} = \mathrm{soc}({}_AA)$ *is a monomial ideal and* $\nu(\mathcal{Z})$ *coincides with the Nakayama permutation* $\nu(A)$ *of* A.

The definition of quasi-Frobenius rings in terms of permutations, given in this section, is due to T.Nakayama [Nakayama, 1939]. There are numerous other equivalent definitions of a quasi-Frobenius ring. One of the equivalent definition of a QF-ring is given in the following theorem, which is a generalization of theorem 4.4.3 for quasi-Frobenius algebras.

Theorem 4.5.2. *Let A be a two-sided Artinian ring. Then A is a QF-ring if and only if the following conditions hold:*

1) *the socle of any principal A-module is simple.*
2) *if P_1 and P_2 are non-isomorphic principal A-modules then* $\mathrm{soc}\, P_1 \not\simeq \mathrm{soc}\, P_2$.

Proof. Let A be a quasi-Frobenius ring. Then A admits a Nakayama permutation, and so, by condition (np1) of the definition, the socle of any principal A-module P is simple. Moreover, if $P_1 \not\simeq P_2$ then $\mathrm{soc}\, P_1 \not\simeq \mathrm{soc}\, P_2$.

Conversely, suppose all conditions of the theorem hold. From these conditions it follows that $\mathrm{soc}\, P_1, \dots, \mathrm{soc}\, P_s$ describe a permutation ν of the simple modules $U_1 = \mathrm{top}\, P_1, \dots, U_s = \mathrm{top}\, P_s$. So, we have $\mathrm{soc}\, P_k = \mathrm{top}\, P_{\nu(k)}$, that is, condition (np1) holds. By theorem 4.3.2, $\mathrm{soc}\, A_A = \mathrm{soc}_A A = \mathcal{Z}$ is a monomial ideal with a permutation ν such that $\mathcal{Z} = \bigoplus\limits_{k=1}^{s} f_k \mathcal{Z} f_{\nu(k)}$. Therefore, by the annihilation lemma, we have $\mathrm{soc}\, Q_{\nu(k)}^{n_k} = \mathrm{soc}\, Af_{\nu(k)} = V_k^{n_k}$, where the V_k are left simple A-modules. So we have $\mathrm{soc}\, Q_{\nu(k)} = \mathrm{top}\, Q_k$, that is, the condition (np2) holds as well. Thus, A is a quasi-Frobenius ring. The theorem is proved.

Some other equivalent definitions of QF-rings in terms of self-injective rings and duality will be given at the end of this chapter.

4.6 THE SOCLE OF A MODULE AND A RING

In this section we shall study in detail the main properties of socles of modules and rings.

Recall the basic definitions (see vol. I, p. 129).

Definition. Let M be a right A-module. The **socle** of M, denoted by *soc* (M), is the sum of all simple right submodules of M. If there are no such submodules, then $\mathrm{soc}\,(M) = 0$.

If $M = A_A$, then $\mathrm{soc}\,(A_A)$ is the sum of all minimal right ideals of A and it is a right ideal of A. Analogously, $\mathrm{soc}\,({}_AA)$ is a left ideal in A.

For a semisimple module M we have $\mathrm{soc}\,(M) = M$.

Since a homomorphic image of a simple module is a simple module or zero, for any A-homomorphism $\varphi : M \to N$ of A-modules M, N we have that $\varphi(\mathrm{soc}\,(M)) \subseteq \mathrm{soc}\,(N)$.

Let A be a ring with Jacobson radical R.

Proposition 4.6.1. *If M is a semisimple A-module, then $MR = 0$.*

Proof. Let M be a simple nonzero right A-module. Obviously, $M = mA$ for all nonzero $m \in M$. If $MR = M$, then, by the Nakayama lemma, $M = 0$. This contradiction shows that $MR = 0$. The general case follows from proposition 3.4.3, vol.I. The proposition is proved.

Recall that a ring A is called **semilocal** if $\bar{A} = A/R$ is a right Artinian ring (see vol. I, p.228).

By theorem 3.5.5, vol.I, we obtain that $\bar{A}$ is semisimple (i.e., A_A and ${}_AA$ are right and left semisimple modules respectively).

The next proposition is a generalization of proposition 4.3.3.

Proposition 4.6.2. *Let A be a semilocal ring with Jacobson radical R. Then* $\mathrm{soc}(A_A)$ *coincides with the left annihilator $l(R)$, whereas* $\mathrm{soc}({}_AA)$ *coincides with the right annihilator $r(R)$. In particular,* $\mathrm{soc}({}_AA)$ *and* $\mathrm{soc}(A_A)$ *are two-sided ideals.*

Proof. If M is a semisimple right A-module, then, $MR = 0$, by proposition 4.6.1, and consequently $\mathrm{soc}(A_A) \subseteq l(R)$. On the other hand, the equality $l(R)R = 0$ implies that $l(R)$ is a right $\bar{A}$-module and, by theorem 2.2.5, vol.I, $l(R)$ is a semisimple module, so it must be contained in the right socle of A, hence, $l(R) = \mathrm{soc}(A_A)$. Similarly, $r(R) = \mathrm{soc}({}_AA)$. The proposition is proved.

Example 4.6.1.

Let $A = T_n(D)$ be the ring of upper triangular matrices of degree n over a division ring D. Obviously, $\mathrm{soc}\,{}_AA$ is the "first row ideal" of $T_n(D)$ and $\mathrm{soc}\,A_A$ is the "last column ideal" of $T_n(D)$. Therefore $\mathrm{soc}\,{}_AA = V_1^n$ and $\mathrm{soc}\,A_A = U_n^n$, where U_i (resp. V_i) is a right (resp. left) simple A-module ($i = 1, 2, \ldots, n$). For $n \geq 2$ $\mathrm{soc}\,{}_AA \neq \mathrm{soc}\,A_A$. Obviously, $\dim_D(\mathrm{soc}\,{}_AA) = \dim_D(\mathrm{soc}\,A_A) = n$.

Example 4.6.2.

Let k be a field and $B = B_n(k)$ be the subalgebra of $M_n(k)$ of matrices of the following form:

$$\begin{pmatrix} a_{11} & 0 & 0 & \ldots & 0 & 0 \\ a_{21} & a_{22} & 0 & \ldots & 0 & 0 \\ a_{31} & 0 & a_{33} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & 0 & 0 & \ldots & a_{n-1,n-1} & 0 \\ a_{n1} & 0 & 0 & \ldots & 0 & a_{nn} \end{pmatrix}.$$

Obviously,

$$R = \mathrm{rad}(B) = \left\{ \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ a_{21} & 0 & 0 & \dots & 0 \\ a_{31} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & 0 & 0 & \dots & 0 \end{pmatrix} \right\}.$$

It is easy to see that

$$l(R) = \mathrm{soc}\, B_B = \left\{ \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & 0 & \dots & 0 \end{pmatrix} \right\}$$

and

$$r(R) = \mathrm{soc}\, {}_B B = \left\{ \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & 0 & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & 0 & 0 & \dots & a_{nn} \end{pmatrix} \right\}.$$

Consequently, $\mathrm{soc}\, B_B = U_1^n$ and $\mathrm{soc}\, {}_B B = V_2^2 \oplus \ldots \oplus V_n^2$. We have $\dim {}_k(\mathrm{soc}\, B_B) = n$ and $\dim {}_k(\mathrm{soc}\, {}_B B) = 2n - 2$, where U_i (resp. V_i) is a right (resp. left) simple B-module ($i = 1, 2, \dots, n$).

Example 4.6.3.
Here we give an example of a right Noetherian serial ring A for which $\mathrm{soc}\, A_A = U_2 \oplus U_2$, and $\mathrm{soc}\, {}_A A = 0$.

Let $\mathbf{Q}$ be the field of rational numbers, p be a prime integer, $\mathbf{Z}_{(p)} = \{m/n \in \mathbf{Q} \ : \ (n, p) = 1\}$. Set

$$A = \begin{pmatrix} \mathbf{Z}_{(p)} & \mathbf{Q} \\ 0 & \mathbf{Q} \end{pmatrix}.$$

It is clear that $R = \mathrm{rad} A = \begin{pmatrix} p\mathbf{Z}_{(p)} & \mathbf{Q} \\ 0 & 0 \end{pmatrix}$ and $P_1 = (\mathbf{Z}_{(p)}, \ \mathbf{Q}), P_2 = (0, \ \mathbf{Q})$. The left principal A-modules are:

$$Q_1 = \begin{pmatrix} \mathbf{Z}_{(p)} \\ 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} \mathbf{Q} \\ \mathbf{Q} \end{pmatrix}.$$

Since A is serial, if the socle of a principal left (or right) module P is nonzero, then $\mathrm{soc}\, P$ has to be simple.

Obviously, $P_1R = (p\mathbf{Z}_{(p)},\ \mathbf{Q})$ and $P_2R = 0$. The submodules of P_1 are P_1R^j $(j = 1, 2, \ldots)$ and $(0,\ \mathbf{Q})$. The last module is isomorphic to U_2. Hence $\operatorname{soc} A_A = U_2 \oplus U_2 = U_2^2$.

In the left case $RQ_1 = \begin{pmatrix} p\mathbf{Z}_{(p)} \\ 0 \end{pmatrix}$ and $RQ_2 = \begin{pmatrix} \mathbf{Q} \\ 0 \end{pmatrix}$. It is clear that the socles of these modules are zero. Thus, $\operatorname{soc}{}_AA = 0$.

Example 4.6.4.

Let Q_∞ be the following countable directed graph:

$$\begin{array}{ccccccccccc} 1 & & 2 & & 3 & & \cdots & & n & & n+1 & & \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & & \cdots & & \bullet & \longrightarrow & \bullet & & \cdots \end{array}$$

So, the set of vertices VQ_∞ is the set of all natural integers $\mathbf{N}$ and the set of arrows $AQ_\infty = \{\sigma_{k,k+1}\ :\ k \in \mathbf{N}\}$. For all $i < j$, where $(i, j) \in \mathbf{N} \times \mathbf{N}$, there is a path $p_{ij} = \sigma_{i,i+1} \ldots \sigma_{j-1,j}$. Let k be a field and consider kQ_∞. This is an infinite dimensional k-algebra with the following basis:

$$\{\varepsilon_i, i \in \mathbf{N};\ p_{ij},\ i < j,\ i, j \in \mathbf{N}\}.$$

Much related is the infinite dimensional k-algebra Ω with basis

$$\{\ 1; p_{ij},\ i, j \in \mathbf{N},\ i < j\}$$

with 1 as the unit element and $p_{ij}p_{rs} = \delta_{jr}p_{is}$, where δ_{jr} is the Kroneker delta. Obviously, a basis of the Jacobson radical $R = \operatorname{rad}(\Omega)$ is $\{p_{ij}\ :\ i < j\}$.

The product $\sigma_{12}\sigma_{23} \ldots \sigma_{n-1,n}$ is nonzero for any $n \in \mathbf{N}$.

Let $w \in \Omega$ and $w = \sum\limits_{i,j} \alpha_{ij}p_{ij}$. Let $s(w) = \min\limits_{\exists j\ with\ \alpha_{ij} \neq 0} \{i\}$. If $a, b \in R$ and $ab \neq 0$, then $s(ab) < s(b)$. So, if $s(a_1) = m$ then $a_{m+1}a_m \ldots a_2a_1 = 0$ for any $a_{m+1}, \ldots, a_2 \in R$. Therefore, Ω is right perfect, but not left perfect.

Obviously, $\operatorname{soc}\Omega_\Omega = 0$ and $\operatorname{soc}{}_\Omega\Omega = \{\sum\limits_{k=1}^{n} \alpha_{1k}p_{1k} :$ for any $n \in \mathbf{N}\}$. Thus, the set $\{p_{12}, p_{13}, \ldots, p_{1n}, \ldots\}$ is a basis of $\operatorname{soc}{}_\Omega\Omega$.

Example 4.6.5.

Here we give an example of a commutative local semiprimary ring A whose socle is simple, but A is not Artinian.

Let $k[x_1, x_2, \ldots, x_n, \ldots]$ be the polynomial ring over a field k in a countable number of variables and let J be the ideal of $k[x_1, x_2, \ldots, x_n, \ldots]$ generated by the elements: $x_1^2, x_2^2, \ldots, x_n^2, \ldots$ and $x_1x_2 - x_ix_j$ for $i \neq j$. Consider the quotient ring $A = k[x_1, x_2, \ldots, x_n, \ldots]/J$. Let $\pi : k[x_1, x_2, \ldots, x_n, \ldots] \to A$ be the natural epimorphism. Denote $\pi(x_1x_2)$ by a_0. The images of $1, x_1, x_2, \ldots, \ldots, x_n, \ldots$ in A will be denoted by the same symbols.

Now $\pi(x_ix_jx_k) = 0$ for all i, j, k. Indeed if $i = j$, then $\pi(x_i^2x_k) = \pi(x_i^2)\pi(x_k) = 0$. This follows from $x_i^2 = 0$. And if $i \neq j$ then $\pi(x_ix_jx_k) = \pi(x_1x_2x_k)$ which is zero if $k \in \{1, 2\}$ and which is equal to $\pi(x_1x_1x_2) = 0$ for $k \geq 3$.

Obviously the square (resp. third power) of any element of the form $r = \alpha_1 x_1 + \ldots + \alpha_m x_m + \alpha_0 a_0$ is a sum of monomials of degree 2 (resp. 3). So $r^3 = 0$ in A for all such r. It immediately follows that the Jacobson radical R of A is the infinite dimensional vector space with basis $a_0, x_1, x_2, \ldots, x_n, \ldots$. Therefore the Loewy series of A is:

$$A \supset R \supset R^2 \supset 0,$$

where $R^2 = \operatorname{soc} A$ is the simple A-module generated by a_0.

4.7 OSOFSKY THEOREM FOR PERFECT RINGS

The following lemma is a generalization of the Nakayama lemma for perfect rings.

Lemma 4.7.1. *Let R be the Jacobson radical of a right (or left) perfect ring A. Suppose, that $X + R^2 = R$ for some right (or left) ideal X in R. Then $X = R$.*

The proof follows from theorem 10.5.1, vol.I.

Remark 4.7.1. Let R be as above. If $R^m \neq 0$, then $R^{m+1} \neq R^m$.

Let $\{A_n : n = 0, 1, \ldots\}$ be a countable family of finite sets, and let F be a family of functions $\{f_n : A_n \to \mathcal{P}(A_{n+1})\}$ (here $\mathcal{P}(X)$ is the power set of X). Consider the pair $(\{A_n\}, F)$. A path in this pair is a set of elements $\{a_m\}$ such that $a_0 \in A_0$, and $a_m \in f_{m-1}(a_{m-1})$ for $m \geq 1$.

The following lemma is known as the **König graph theorem**.

Lemma 4.7.2. *If the pair $((\{A_n\}), F)$ has arbitrarily long paths, then it has a path of infinite length.*

Proof. We call $a_n \in A_n$ "good" if there are paths of arbitrary length containing a_n. Obviously, there is a good element $a_0 \in A_0$. If every element of the finite set $f_0(a_0)$ is not good, then there is an upper bound on the lengths of paths through each of these elements, and hence the maximum of these upper bounds plus 1 is an upper bound on the length of a path through a_0. This contradicts the assumption that a_0 is good. We conclude that some element of $f_0(a_0)$ is good.

Now assume a_n is good. As above we conclude that some element of $f_n(a_n)$ is good. We then obtain an infinite path by selecting $a_0 \in A_0$ as a good element in A_0 and a_n as a good element in $f_{n-1}(a_{n-1})$, by induction.

Theorem 4.7.3 (B.Osofsky). *Let A be a right (or left) perfect ring with Jacobson radical R. Then A is a right Artinian if and only if the right quiver $Q(A)$ of A is defined, i.e., the quotient ring A/R^2 is right Artinian.*

Proof. Obviously, A/R^2 is right Artinian if and only if $W = R/R^2$ is a finitely generated right A-module.

Obviously, if A is a right Artinian ring, then the right quiver $Q(A)$ is defined.

Conversely, suppose that A/R^2 is a right Artinian ring. Let $W = R/R^2$. We have $W = \sum_{i=1}^{n} \bar{r}_i A$, where $\bar{r}_i = f(r_i)$ for some $r_i \in R$ and $f : R \to W$ is the natural epimorphism. By lemma 4.7.2, we have:

$$R = \sum_{i=1}^{n} r_i A. \tag{4.7.1}$$

Obviously, any element from R^2 has the following form: $\sum_{i,j=1}^{n} r_i r_j a_{ij}$ and

$$R^2 = \sum_{i,j=1}^{n} r_i r_j A. \tag{4.7.2}$$

Consequently,

$$R^k = \sum r_{i_1} r_{i_2} \ldots r_{i_k} A. \tag{4.7.3}$$

Write $S = (r_1, \ldots, r_n)$. Let A_n consist of all products $r_{i_0} \cdots r_{i_n} \neq 0$. Consider

$$f_n(r_{i_0} \cdots r_{i_n}) = \{r_k r_{i_0} \cdots r_{i_n} \neq 0;\ r_{i_0} \cdots r_{i_n} r_t \neq 0 \ \ : \ \ r_k \in S \text{ and } r_t \in S\},$$

which is a subset of A_{n+1}.

By the König graph theorem, there is an integer m such that each product of m r_i's is zero. Consequently, every product $r_{i_1} \cdots r_{i_m}$ equals zero. From formula (4.7.3) it follows that $R^m = 0$. Let t be the smallest integer such that $R^t = 0$. We have the following Loewy series for A:

$$A \supset R \supset R^2 \supset \ldots \supset R^{t-1} \supset 0.$$

By remark 4.7.1, all inclusions in this Loewy series are strict and all R^k are finitely generated A-modules. Obviously, the R^k/R^{k+1} are semisimple and finitely generated for $k = 1, \ldots, t-1$. Therefore A has a composition series and A is a right Artinian ring. The theorem is proved.

Corollary 4.7.4. *A semidistributive right (or left) perfect ring A is right (or left) Artinian.*

The proof follows from theorem 14.1.6, vol.I, and the Osofsky theorem.

Proposition 4.7.5. *Let R be the Jacobson radical and let $Pr(A)$ be the prime radical of a ring A. If A is a one-sided perfect ring then $Pr(A) = R$.*

The proof follows from the following assertion: any element $r \in R$ of a one-sided perfect ring is strongly nilpotent.

4.8 SOCLES OF PERFECT RINGS

Definition. A ring A is called **left** (resp. **right**) **socular** if every nonzero left (resp. right) A-module has a nonzero socle. A ring which is right and left socular is called **socular**.

Let A be a right socular ring, and let M be a right A-module. We define inductively: $M_0 = 0$, $M_1 = \operatorname{soc} M$, if α is not a limit ordinal $M_{\alpha+1}$ is the right A-module such that $M_{\alpha+1}/M_\alpha$ is the socle of the right module M/M_α, and, if α is a limiting ordinal, $M_\alpha = \bigcup\limits_{\beta<\alpha} M_\beta$. Since A is right socular there is an ordinal γ such that $M = M_\gamma$.

The resulting ascending chain of submodules of M

$$M_0 \subset M_1 \subset \ldots \subset M_\alpha \subset \ldots$$

is called the **transfinite ascending Loewy series**.

Proposition 4.8.1. *The following conditions are equivalent for a ring A*:
(1) *A is right socular;*
(2) *for any right A-module M there exists a transfinite ascending Loewy series.*

The proof is obvious.

Theorem 4.8.2 (H.Bass). *The following conditions are equivalent*:
(1) *a ring A is right (resp. left) perfect;*
(2) *a ring A is semilocal and left (resp. right) socular.*

Proof.
(1) $\Rightarrow$ (2). Let A be a right (or left) perfect ring with Jacobson radical R. Then A is semilocal, by theorem 10.5.3, lemma 10.4.9 and theorem 10.4.8, vol.I.

Let M be a left A-module and suppose that $\operatorname{soc} M = 0$. If $RM = 0$ then M is semisimple, by theorem 2.2.5, vol.I. Therefore, there exists $r_1 \in R$ such that $r_1 M \neq 0$. Let $r_1 m \neq 0$. Consider $Am \supset Ar_1 m$. If $Ar_1 m$ is not semisimple, then there exists r_2 such that $r_2 r_1 \cdot m \neq 0$. Continuing this process we obtain a sequence $\{r_k\}$ of elements of R such that $r_n \cdot r_{n-1} \ldots r_1 \neq 0$ for any $n \in \mathbf{N}$. This contradiction proves (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). Let $0 \supset R_1 \supset \ldots \supset R_\alpha \supset \ldots \supset R_\delta = R$ be the transfinite ascending Loewy series of R. Therefore, if $r \in R$, we can define $h(r)$ to be the least α such that $r \in R_\alpha$. Note that $h(r)$ can never be a limiting ordinal since if $r \in \bigcup\limits_{\beta<\alpha} R_\beta$, then $r \in R_\beta$ for some $\beta < \alpha$. Thus, if $r \in R$ we can write $h(r) = \beta + 1$ for some β. Since $R_\beta \supset R \cdot R_{\beta+1}$, we have $h(b \cdot r) < h(r)$ for any $b \in R$ and $h(r) \neq 0$ for $r \neq 0$.

Suppose we have a sequence $\{r_n\}$ of elements of R. Then if $r_n r_{n-1} \ldots r_1 \neq 0$ for all n the sequence $\{h(r_n r_{n-1} \ldots r_1)\}$ is a countable strongly decreasing chain of ordinals from a totally ordered ascending set of ordinals with a unique least element. This is impossible. The theorem is proved.

The following proposition follows immediately from theorem 4.8.2.

Proposition 4.8.3. *For a semilocal ring A the following conditions are equivalent:*
(i) A is socular;
(ii) A is perfect.

In particular, if A is perfect, then soc ${}_AA \neq 0$ and soc $A_A \neq 0$.

4.9 SEMIPERFECT PIECEWISE DOMAINS

Let $A_A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be a decomposition of a semiperfect ring A into a direct sum of principal right A-modules.

Recall that a semiperfect ring A is called a **piecewise domain** if every nonzero homomorphism between indecomposable projective A-modules is a monomorphism.

Piecewise domains coincide with l-**hereditary rings** A, that is, rings such that every local one-sided ideal of A is projective. From theorem 10.7.9, vol.I, we immediately obtain the following lemma:

Lemma 4.9.1. *A semihereditary and semiperfect ring A is a piecewise domain.*

Proposition 4.9.2. *A local piecewise domain A is a domain. In particular, if A is a piecewise domain, then for any local idempotent $e \in A$, eAe is a domain.*

The proof is obvious.

Corollary 4.9.3. *The prime radical $Pr(A)$ of a piecewise domain A is nilpotent.*

Proof. Let e be a local idempotent. By proposition 4.9.2, the ring eAe is a domain. By proposition 11.2.9 and corollary 11.2.7, vol.I, we have that $Pr(eAe) = 0$. By theorem 11.4.1, vol.I, $Pr(A)$ is nilpotent.

From corollary 11.8.7, vol.I, and corollary 4.9.3 we obtain the following theorem.

Theorem 4.9.4. *A semiperfect piecewise domain can be uniquely decomposed into a finite direct product of indecomposable rings $A_1, \ldots, A_m$ with connected prime quivers $PQ(A_i)$, $i = 1, \ldots, m$.*

Theorem 4.9.5. *The prime quiver $PQ(A)$ of a semiperfect piecewise domain A is an acyclic simply laced quiver. Conversely, for every acyclic simply laced* *quiver* Γ *there is a finite dimensional hereditary* k*-algebra* A *such that* $PQ(A) = Q(A) = \Gamma$ *(for any field* k*).*

Proof. We may assume that the piecewise domain A is reduced. Let

$$A = \begin{pmatrix} A_1 & A_{12} & \dots & A_{1t} \\ A_{21} & A_2 & \dots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ A_{t1} & A_{t2} & \dots & A_t \end{pmatrix}$$

be a two-sided Peirce decomposition so that

$$Pr(A) = \begin{pmatrix} Pr(A_1) & A_{12} & \dots & A_{1t} \\ A_{21} & Pr(A_2) & \dots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ A_{t1} & A_{t2} & \dots & Pr(A_t) \end{pmatrix}$$

and $\bar{A} = A/Pr(A) = \bar{A}_1 \times \ldots \times \bar{A}_t$, where $\bar{A}_k = A_k/Pr(A_k)$ for $k = 1, \ldots, t$. Such a Peirce decomposition exists by proposition 11.7.1, vol.I. By proposition 4.9.2, for every local idempotent $e \in A$ the ring eAe is a domain and therefore a prime ring. Consider the set $\bar{e}\bar{A}\bar{e}$. This set coincides with the set of cosets $\{eae + Pr(A) \ : \ a \in A\}$ which is identified with the set $eae + Pr(A)/Pr(A)$. By theorem 1.3.3, vol.I, we have

$$(eAe + Pr(A))/Pr(A) \simeq eAe/(eAe \cap Pr(A)) = eAe/ePr(A)e.$$

The set eAe coincides with the prime radical of the ring eAe, by proposition 11.2.9, vol.I. But eAe is a domain, so $Pr(eAe) = 0$ and $\bar{e}\bar{A}\bar{e} = eAe$. By theorem 14.4.6, vol.I, the ring $\bar{A}$ is a direct product of prime rings. So, all the rings $\bar{A}_i$ are prime. We shall show that $Pr(A_1) = 0, \ldots, Pr(A_t) = 0$.

First we assume that $\bar{A} = A/Pr(A)$ is prime. Let $1 = e_1 + \ldots + e_s$ be a decomposition of $1 \in A$ into a sum of mutually orthogonal local idempotents and

$$Pr(A) = I = \bigoplus_{p,q=1}^{s} e_p I e_q.$$

As above $e_p I e_p = 0$ for $p = 1, \ldots, s$. If $e_p I e_q \neq 0$ then $e_p I e_q \cdot e_q A e_p \subset e_p I e_p = 0$. Consequently, $e_q A e_p = 0$. By proposition 9.2.13, vol.I, the ring $B_{pq} = (\bar{e}_p + \bar{e}_q \bar{A}(\bar{e}_p + \bar{e}_q)$ must be prime, but if $\bar{e}_p\bar{A}\bar{e}_q \neq 0$ then $\bar{e}_p\bar{A}\bar{e}_q$ is a nilpotent ideal in B_{pq}. Therefore, $\bar{e}_p\bar{A}\bar{e}_q = 0$ for $p \neq q$; $p, q = 1, \ldots, s$. This contradicts the fact that $\bar{A}$ is prime. So, $Pr(A_1) = 0, \ldots, Pr(A_t) = 0$ and the rings $A_1, \ldots, A_t$ are prime.

Returning to the general case. Let $1 = g_1 + \ldots + g_t$ be a decomposition of $1 \in A$ into a sum of mutually orthogonal idempotents such that $A_{ij} = g_i A g_j$ for $i \neq j$ and $A_i = g_i A g_i$ $(i, j = 1, \ldots, t)$. We may assume that $A_{12} \neq 0$. Let $g_1 = e_1 + \ldots + e_m$ and $g_2 = h_{m+1} + \ldots h_{m+k}$ be the decompositions of g_1 and g_2 into a sum of pairwise orthogonal local idempotents. We shall show that $e_iAh_j = 0$ for $i = 1,\dots,m$ and $j = 1,\dots,k$. Since the rings A_1 and A_2 are prime, there are sets of elements $a_{\alpha\beta} \in e_\alpha A e_\beta$ ($a_{\alpha\beta} \neq 0$ for $\alpha,\beta = 1,\dots,m$) and $b_{\gamma\delta} \in h_\gamma A h_\delta$ ($b_{\gamma\delta} \neq 0$ for $\gamma,\delta = 1,\dots,k$). From the inequality $A_{12} \neq 0$ it follows that there is a nonzero element $a_{12} \in A_{12}$ such that $a_{12} = e_{p_0}a_{12}h_{q_0}$. The product $a_{pp_0}a_{12}b_{q_0q}$ is nonzero in the piecewise domain A. Consequently, $e_iAh_j \neq 0$ for $i = 1,\dots,m$ and $j = 1,\dots,k$.

We shall show that $PQ(A)$ is an acyclic quiver. From the equalities $Pr(A_1) = 0,\dots,Pr(A_t) = 0$ it follows that $PQ(A)$ has no loops. Suppose that $PQ(A)$ contains an oriented cycle

$$\begin{tikzcd}[row sep=tiny] 1 & 2 & & r-1 & r \\ \bullet \arrow[r] & \bullet & \cdots & \bullet \arrow[r] & \bullet \arrow[llll, bend left] \end{tikzcd}$$

where $r > 1$. Obviously, $A_{12} \neq 0,\dots,A_{r-1r} \neq 0$ and $A_{r1} \neq 0$. Therefore, the product $A_{12}\dots A_{r-1r}A_{r1} \neq 0$. This is a contradiction with $Pr(A_1) = 0$.

Now we shall prove the converse assertion. Let Γ be a cyclic simply laced quiver and k be a field. By proposition 2.3.3 and theorem 2.3.4, the path algebra $A = k\Gamma$ is hereditary and finite dimensional. Moreover, $Q(A) = PQ(A) = \Gamma$. The theorem is proved.

It follows from theorem 11.8.6, vol.I, that a semiperfect piecewise domain A has the following kind of two-sided Peirce decomposition:

$$A = \begin{pmatrix} A_1 & A_{12} & \dots & A_{1t} \\ 0 & A_2 & \dots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_t \end{pmatrix}, \tag{4.9.1}$$

where the $A_1,\dots,A_t$ are prime rings and the prime radical has the following form:

$$Pr(A) = \begin{pmatrix} 0 & A_{12} & \dots & A_{1t} \\ 0 & 0 & \dots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}. \tag{4.9.2}$$

Consequently, $\bar{A} = A/Pr(A) = A_1 \times \dots \times A_t$.

Corollary 4.9.6. *A semiperfect piecewise domain A considered as an Abelian group is a direct sum of the ring $A_0 \simeq \bar{A}$ and $Pr(A)$: $A = A_0 \oplus Pr(A)$.*

Corollary 4.9.7. *Let t be the number of vertices in the prime quiver $PQ(A)$ of a semiperfect piecewise domain A. Then $(Pr(A))^t = 0$.*

The proof follows from formula (4.9.2).

In conclusion there is the following theorem.

Theorem 4.9.8. *Let A be a right hereditary semiperfect ring. Then the following conditions are equivalent:*

(a) *A contains a monomial ideal;*

(b) *A is isomorphic to a finite direct product of rings of the form $M_{n_k}(B)$, where all the B_k are local right hereditary domains.*

Moreover, if the ring A is semidistributive, then all the rings B_i are either division rings or discrete valuation rings.

Proof.

$(b) \Rightarrow (a)$. Obvious.

$(a) \Rightarrow (b)$. Let A be an indecomposable ring. From lemma 4.9.1 it follows that A is a piecewise domain. Every semiperfect piecewise domain has a triangular Peirce decomposition (see formula (4.9.1)). So A is a prime right hereditary semiperfect ring. Since A contains a monomial ideal, $A = P^n$ and $A \simeq M_n(\mathrm{End}_A P)$, where $B = \mathrm{End}_A P$ is a local right hereditary domain.

If a ring A is semidistributive, then B is a right Noetherian right hereditary local semidistributive domain. By proposition 14.4.10 and corollary 14.4.11, vol.I, B is either a division ring or a discrete valuation ring.

Corollary 4.9.9. *Let A be a right Artinian and right hereditary ring with a monomial ideal. Then A is semisimple.*

The proof is obvious.

4.10 DUALITY IN NOETHERIAN RINGS

The theme of this section is again duality theory, as in section 4.1. But there is an important difference. In section 4.1 the duality function was $\mathrm{Hom}_k(*, k)$ where k is the underlying field of a k-algebra A; here the duality functors are $\mathrm{Hom}_A(*, {}_AA)$ and $\mathrm{Hom}_A(*, A_A)$.

Let M be a right A-module and let

$$M^* = \mathrm{Hom}_A(M, A_A). \tag{4.10.1}$$

Obviously, this is an additive group and it can be considered as a left A-module if we define $a\varphi$ by the formula $(a\varphi)(m) = \varphi(ma)$, where $a \in A$, $\varphi \in M^*$, $m \in M$. This left A-module is called **dual** to the right A-module M. Analogously, for any left A-module N we can define the dual module

$$N^* = Hom_A(N, {}_AA),$$

which is a right A-module, if we set $(\psi a)(x) = \psi(x)a$ for $a \in A$, $\psi \in N^*$, $x \in N$. Obviously, isomorphic modules have isomorphic duals.

Let $f : N \to M$ be a homomorphism of right A-modules. Then we may define a map $f^* : M^* \to N^*$ by the formula $f^*(\varphi) = \varphi f$ for $\varphi \in M^*$. It is easy to show that f^* is an A-homomorphism of left A-modules. This homomorphism f^* is called **dual** to f.

Let F be a free A-module with a finite free basis $f_1, f_2, \ldots, f_n$. Define an A-homomorphism $\varphi_i : F \to A$ by $\varphi_i(f_j) = \delta_{ij}$ for $i, j = 1, 2, \ldots, n$, where δ_{ij} is the Kronecker symbol. Then $\varphi_i \in F^*$. It is easy to show that F^* is a free A-module with a free basis $\varphi_1, \ldots, \varphi_n$. This basis is called **dual** to the basis $f_1, f_2, \ldots, f_n$.

Lemma 4.10.1. *Let P be a finitely generated projective module. Then the dual module P^* is also a finitely generated projective A-module.*

Proof. Let P be a finitely generated module generated by elements $x_1, \ldots x_n$ and let F be a free module with a free basis $f_1, f_2, \ldots, f_n$. Then there is an epimorphism $\pi : F \to P$ with $\pi(f_i) = x_i$ for $i = 1, \ldots, n$. Since P is projective, there is a homomorphism $\sigma : P \to F$ such that $\pi\sigma = 1_P$. Consequently, $\sigma^*\pi^* = (\pi\sigma)^* = 1_{P^*}$. Therefore P^* is a direct summand of a free module F^* which is free with a finite basis of n elements. So P^* is a finitely generated projective module.

Lemma 4.10.2. *Let A be a right Noetherian ring. Then the dual to any finitely generated left A-module is also finitely generated.*

Proof. Let M be a finitely generated left A-module. Then there is an exact sequence $0 \to N \to F \to M \to 0$ with F a free module with a finite basis. Applying the duality functor $\mathrm{Hom}_A(*, A)$ we obtain that M^* is a submodule of F^*. Since F^* is a free right A-module with a finite basis and A is a right Noetherian ring, M^* is a finitely generated A-module, by corollary 3.1.13, vol.I.

Let M be a right A-module with dual module M^*. Then M^* itself has a dual module M^{**}. Suppose $m \in M$ and $f \in M^* = \mathrm{Hom}_A(M, A)$. Define a mapping

$$\varphi_m : M^* \to A$$

by $\varphi_m(f) = f(m)$. Obviously,

$$\varphi_m(f_1 + f_2) = \varphi_m(f_1) + \varphi_m(f_2).$$

For any $a \in A$ we have $\varphi_m(af) = af(m) = a\varphi_m(f)$. Thus φ_m is an A-homomorphism, i.e., $\varphi_m \in M^{**}$. Consider the mapping

$$\delta_M : M \to M^{**} \tag{4.10.2}$$

defined by $\delta_M(m)(f) = f(m)$ for $m \in M$ and $f \in M^*$. It is easy to verify that δ_M is an A-homomorphism.

Definition. A module M is called **reflexive** if δ_M is an isomorphism. And it is called **semi-reflexive** if δ_M is a monomorphism.

Note that any finite dimensional vector space as a module over its field k is reflexive; on the other hand an infinite dimensional vector space over a field k is never reflexive.[3]

Lemma 4.10.3. *Any submodule of semi-reflexive module is semi-reflexive and any direct summand of a reflexive module is reflexive.*

Proof. Suppose M is a semi-reflexive A-module and N is a submodule of M. Let $i : N \to M$ be the inclusion mapping. Then the following diagram

$$\begin{array}{ccc} N & \xrightarrow{\delta_N} & N^{**} \\ \downarrow i & & \downarrow i^{**} \\ M & \xrightarrow{\delta_M} & M^{**} \end{array}$$

is commutative. Since i and δ_M are monomorphisms, δ_N is also a monomorphism. Therefore N is semi-reflexive.

Let N be a direct summand of a reflexive module M. Then there are an inclusion mapping $i : N \to M$ and an epimorphism $\pi : M \to N$ such that $\pi i = 1_N$. Then $\pi^{**}i^{**} = (\pi i)^{**} = 1_{N^{**}}$ is an epimorphism. The following diagram

$$\begin{array}{ccc} M & \xrightarrow{\delta_M} & M^{**} \\ \downarrow \pi & & \downarrow \pi^{**} \\ M & \xrightarrow{\delta_N} & N^{**} \end{array}$$

is commutative. Since δ_M is an isomorphism and π^{**} is an epimorphism, δ_N is also an epimorphism. But from the first part of this lemma δ_N is a monomorphism. So δ_N is an isomorphism, i.e., N is reflexive.

Proposition 4.10.4. *Each finitely generated projective module is reflexive. In particular, a free module with finite free basis is reflexive.*

Proof. Let F be a free module with finite free basis $f_1, f_2, \dots, f_n$ and let $\varphi_1, \varphi_2, \dots, \varphi_n$ be the free basis of F^* dual to $f_1, f_2, \dots, f_n$. Let $\psi_1, \psi_2, \dots, \psi_n$ be a basis of F^{**} dual to the basis $\varphi_1, \varphi_2, \dots, \varphi_n$. Then $\delta_F(f_i)$ and ψ_i both belong to $\mathrm{Hom}_A(F^*, A)$ and

$$\delta_F(f_i)(\varphi_j) = \varphi_j(f_i) = \delta_{ji} = \psi_i(\varphi_j).$$

This implies that $\delta_F(f_i) = \psi_i$ and that δ_F is an isomorphism, i.e., F is reflexive.

Let P be a finitely generated projective module. Then P is a direct summand of a free module with a finite basis. Hence P is reflexive by lemma 4.10.3.

[3]In general the matter of reflexivity of a module is a delicate matter involving higher set theory. For free Abelian groups (as modules over $\mathbf{Z}$) the answer is as follows: A free Abelian group is reflexive if and only if its cardinality is non-ω-measurable. In particular a countable free Abelian group is reflexive. See P.C.Eklof, A.H.Mekler, Almost free modules. Set-theoretic methods, North Holland, 1990 for this and much more.

Lemma 4.10.5. *The dual of an arbitrary module is semi-reflexive. The dual of a reflexive module is reflexive.*

Proof. Let M be A-module. If we apply the duality functor to the A-homomorphism $\delta_M : M \to M^{**}$ we obtain an A-homomorphism $\delta_M^* : M^{***} \to M^*$. Then it is easy to show that

$$\delta_M^* \delta_{M^*} = 1_{M^*}. \tag{4.10.3}$$

From this equality it follows that δ_{M^*} is a monomorphism, so M^* is semi-reflexive. If M is reflexive, then δ_M is an isomorphism, and so is δ_M^*. But then from (4.10.3) $\delta_{M^*} = (\delta_M^*)^{-1}$ is also an isomorphism, i.e., M^* is reflexive.

Lemma 4.10.6. *Let A be a right Noetherian ring. Then any finitely generated submodule of a free module with a finite basis is semi-reflexive.*

Proof. This follows from proposition 4.10.4 and lemma 4.10.3.

4.11 SEMIPERFECT RINGS WITH DUALITY FOR SIMPLE MODULES

Lemma 4.11.1. *Let U (resp. V) be a simple right (resp. left) A-module. Then A has a right (resp. left) ideal isomorphic to U (resp. V) if and only if $U^* \neq 0$ (resp. $V^* \neq 0$).*

Proof. This follows immediately from the Schur lemma.

Definition. We say that A is a **ring with duality for simple modules** (shortly, a DSM**-ring**) if for each simple right A-module U the dual module U^* is simple and the same also holds for simple left A-modules.

Theorem 4.11.2. *Let A be a semiperfect semiprime ring, then* $\operatorname{soc} A_A$ *(*$\operatorname{soc}{}_A A$*) is nonzero if and only if $A \simeq M_{n_1}(D_1) \times \ldots \times M_{n_s}(D_s)$, where the $D_1, \ldots, D_s$ are division rings, i.e., A is a semisimple Artinian ring.*

Proof. Let A be a semiperfect semiprime ring with Jacobson radical R. If A is a semisimple ring, then $R = 0$. Then, by proposition 4.3.3, $\operatorname{soc} A_A = l(R) = A \neq 0$.

Conversely, suppose $\operatorname{soc} A_A \neq 0$. If the Jacobson radical R of a ring A equals zero, then, by lemma 10.4.9 and theorem 10.4.8, vol.I, A is a semisimple Artinian ring.

Suppose $R \neq 0$. Let $A_A = P_1^{n_1} \oplus P_2^{n_2} \oplus \ldots \oplus P_s^{n_s}$ be a decomposition of A into a direct sum of principal right A-modules and let $1 = f_1 + f_2 + \ldots + f_s$ be the corresponding canonical decomposition of $1 \in A$, i.e., $f_i A = P_i^{n_i}$ $(i = 1, 2, \ldots, s)$. Suppose A contains a simple projective module, say $P_1 = eA$, where $f_1 = e + g$ and $e^2 = e$, $eg = ge = 0$. Denote $h = f_2 + \ldots + f_s$ and $X = f_1 A h$. Obviously, X is a nonzero two-sided ideal and $X^2 = 0$. Since A is a semiprime ring, this contradiction shows that A doesn't contain simple projective modules and so $\operatorname{soc} A_A \subset R$. By proposition 4.12.1, $\operatorname{soc} A_A = l(R)$ is a two-sided ideal in A. Let $1 = e_1 + \ldots + e_n$ be a decomposition of the identity of A into a sum of pairwise orthogonal local idempotents. Then for any e_i we have $e_il(R) \subseteq e_iA$, where e_iA is a principal right A-module. If there exists a number i such that $e_il(R) = e_iA$, then e_iA is a simple projective module, and this case has been considered above. So we can assume that $e_il(R) \subseteq e_iR$ for all $e_i \in A$, and then $l(R) \subseteq R$. Since $l(R)R = 0$, we have $l(R)^2 = 0$ and so $l(R) = 0 = \operatorname{soc} A_A$, since A is semiprime. This contradiction shows that $R = 0$.

For $\operatorname{soc}{}_AA$ the statement is proved analogously. The theorem is proved.

Proposition 4.11.3. *Let A be a semiperfect ring with duality for simple modules, and let P be a simple projective A-module. Then $A = M_{n_1}(\mathcal{D}) \times A_2$, where $\mathcal{D} = \operatorname{End}_A P$. Conversely, if $A = M_{n_1}(\mathcal{D})$, where $\mathcal{D}$ is a division ring, then $U^* = V$ and $V^* = U$, where U is the unique simple right A-module and V is the unique simple left A-module.*

Proof. Let $P_1 = eA$, where e is a local idempotent of A. Then obviously $Ae \subset P_1^*$. Therefore the left simple A-module $P_1^* = V_1$ coincides with Ae. Let $A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be a canonical decomposition of A into a sum of principal A-modules and let $1 = f_1 + f_2 + \ldots + f_s$ be a corresponding canonical decomposition of $1 \in A$, i.e., $f_iA = P_i^{n_i}$ $(i = 1, 2, \ldots, s)$.

Set $P' = P_2^{n_2} \oplus \ldots \oplus P_s^{n_s}$. Clearly $\operatorname{Hom}_A(P', P_1^{n_1}) = 0$. Suppose, that $\operatorname{Hom}_A(P_1^{n_1}, P') \neq 0$. Then there exists a canonical idempotent f_i such that $f_iaf_1 \neq 0$ and $i \neq 1$. Obviously, $f_iaf_1 \subset (f_1A)^* = V_1^{n_1}$. Since $f_1Af_1 \neq 0$, for the simple left A-module V_1 we have $f_1V_1 = V_1$ and $f_iV_1 = V_1$ which contradicts the annihilation lemma. Hence $\operatorname{Hom}_A(P_1^{n_1}, P') = 0$ and $A = M_{n_1}(\operatorname{End} P_1) \times A_2$.

The converse statement is obvious.

Definition. The following conditions will be called the **Nakayama conditions** for a semiperfect ring A:

(α) the socles of the principal right and left A-modules are simple;

(β) principal modules with isomorphic socles are isomorphic.

Lemma 4.11.4. *Let A be a semiperfect ring A with Jacobson radical R, let $P = eA$ be an indecomposable projective A-module, and let $U = P/PR$ be a simple module. Then the dual A-module U^* is isomorphic to $l(R)e$, i.e., $U^* \simeq \operatorname{soc} A_A \cdot e$.*

Proof. Let $U = eA/eR$ and $\varphi \in U^*$. Write $z = \varphi(e + eR)$. Obviously, $z = ze$ and for all $p \in eA$ we have

$$\varphi(p + eR) = \varphi(ep + eR) = zp.$$

Moreover, for $r \in R$ we have $zr = \varphi(e+eR)r = \varphi(er+eR) = 0$. Consequently, $z \in l(R) \cap Ae = l(R)e$ and every $\varphi \in U^*$ has the form: $\varphi = \varphi_z$, where $\varphi_z(p+eR) = zp$.

Conversely, for every $z \in l(R)e$ define $\varphi_z \in U^*$ by the formula: $\varphi_z(p+eR) = zp$. Obviously, the map: $z \mapsto \varphi_z$ is an isomorphism of the left A-module $l(R)e$ to U^*. The lemma is proved.

Observe that the dual of a right A-module $eAeJ$ is isomorphic to the left A-module $l(J)e$.

Theorem 4.11.5. *Let A be a semiperfect ring. Then the following are equivalent*:

(1) *A is a ring with duality for simple modules;*

(2) *A admits a Nakayama permutation $\nu(A)$;*

(3) *A satisfies the Nakayama conditions.*

Moreover, from (1), (2), (3) it follows that $\mathrm{soc}\,(A_A) = \mathrm{soc}\,({}_AA) = \mathcal{Z}$ *and $\mathcal{Z}$ is a monomial ideal.*

Proof. Obviously, we can suppose that a ring A is reduced and indecomposable. Therefore, every local idempotent is canonical.

Then $A_A = P_1 \oplus \ldots \oplus P_s$ and ${}_AA = Q_1 \oplus \ldots \oplus Q_s$, where $P_1, \ldots, P_s$ are pairwise non-isomorphic right principal modules and $Q_1, \ldots, Q_s$ are pairwise non-isomorphic left principal modules. Let R be the Jacobson radical of A. We set $U_i = P_i/P_iR$ and $V_i = Q_i/RQ_i$ $(i = 1, \ldots, s)$. By proposition 11.1.1, vol.I, $U_1, \ldots, U_s$ (resp. $V_1, \ldots, V_s$) are all right (resp. left) simple A-modules.

(1) $\Rightarrow$ (2). Let $\mathcal{Z}_r = \mathrm{soc}\ A_A$ and $\mathcal{Z}_l = \mathrm{soc}\ {}_AA$. Since A is semiperfect, $\mathcal{Z}_r = l(R)$ and $\mathcal{Z}_l = r(R)$. By lemma 4.11.1, $\mathcal{Z}_r$ contains at least one copy of each simple A-module $U_i, i = 1, \ldots, s$ and $\mathcal{Z}_l$ contains at least one copy of each simple left A-module $V_i, i = 1, \ldots, s$.

Therefore, by the annihilation lemma for simple modules, for each canonical idempotent $e_k \in A$ we have $\mathcal{Z}_r e_k \neq 0$ and $e_k \mathcal{Z}_l \neq 0$.

By lemma 4.11.4, the module V_i^*, which is dual to $V_i = Ae_i/Re_i$, has the form: $V_i^* = e_i r(R) = e_i \mathcal{Z}_l$. By assumption, V_i^* is simple.

Thus $e_i \mathcal{Z}_l \subset e_i \mathcal{Z}_r$ for $i = 1, \ldots, s$. Therefore $\mathcal{Z}_l \subset \mathcal{Z}_r$. By symmetry, $\mathcal{Z}_r \subset \mathcal{Z}_l$ which implies $\mathcal{Z}_l = \mathcal{Z}_r = \mathcal{Z}$.

Suppose now that the right module $e_i\mathcal{Z}$ is simple and isomorphic to $U_{\nu(i)}$. Then $e_i\mathcal{Z} = e_i\mathcal{Z}e_{\nu(i)}$. Similarly $\mathcal{Z}_{\nu(i)}$ is a simple left A-module, and, consequently, it coincides with V_i, as $e_i\mathcal{Z}e_{\nu(i)} \neq 0$.

Therefore $\mathcal{Z} = \overset{s}{\underset{i=1}{\oplus}} e_i\mathcal{Z}e_{\nu(i)}$ is a monomial ideal with $\nu(\mathcal{Z}) = \nu(A)$ and $\mathcal{Z}$ is determined by $V_i^* = U_{\nu(i)}$.

The equivalence of (2) and (3) is obvious.

(2) $\Rightarrow$ (1). We show that $\mathcal{Z}_r = \mathcal{Z}_l$. For each local idempotent e_i we have $e_i\mathcal{Z}_l \neq 0$ and $e_i\mathcal{Z}_l \supset e_i\mathcal{Z}_r$. Therefore $\mathcal{Z}_l \supset \mathcal{Z}_r$. Symmetrically $\mathcal{Z}_r \supset \mathcal{Z}_l$. Hence, $\mathcal{Z}_r = \mathcal{Z}_l = \mathcal{Z}$, and $e_i\mathcal{Z} = \mathcal{Z}e_{\nu(i)} = e_i\mathcal{Z}e_{\nu(i)}$, and, by theorem 4.3.2, $V_i^* = e_i\mathcal{Z} = U_{\nu(i)}$, and $U_{\nu(i)}^* = \mathcal{Z}e_{\nu(i)} = V_i$ for $i = 1, \ldots, s$.

As a corollary of this theorem we obtain another equivalent definition of a quasi-Frobenius ring.

Theorem 4.11.6. *The following conditions are equivalent for an Artinian ring A:*

(1) *A is a quasi-Frobenius ring;*

(2) *A is a ring with duality for simple modules.*

4.12 SELF-INJECTIVE RINGS

For any ring A the right regular module A_A is free and so it is a projective module. But in the general case A_A is not injective.

Definition. A ring A is called **right self-injective** if the right regular module A_A is injective. A **left self-injective ring** is defined analogously.

Examples 4.12.1.

1. Each Frobenius algebra and each quasi-Frobenius algebra is a self-injective ring.
2. The ring of integers $\mathbf{Z}$ is not self-injective.
3. The ring $\mathbf{Z}/n\mathbf{Z}$, for $n > 0$ is self-injective.
4. A semisimple Artinian ring is right and left self-injective.

In general, a ring may be right self-injective without being left self-injective. One example of such a ring has been constructed by B.Osofsky[4].

Examples 4.12.2.

Let $T = \{\sum\limits_{i\in\alpha} a_i x^i \ : \ a_i \in K\}$ where $\alpha \subset Q_{\geq 0}$ and the exponent set α is such that each lower limit point is in α (i.e., every subset of α has a smallest element (in α)). There is a (nondiscrete) valuation on T obtained by

$$v(\sum_{i\in\alpha} a_i x^i) = \text{ smallest } j \text{ in } \alpha \text{ with } a_j \neq 0.$$

Let $R = T/U_1$, $Y = T/V_1$, where $U_1 = \{f \in T \ : \ v(f) > 1\}$, $V_1 = \{f \in T \ : \ v(f) \geq 1\}$. And let $J = \{x \in R \ : \ v(x) > 0\}$, $S = \{x \in R \ : \ v(x) = 1\}$. Then R is a self-injective ring and Y is an injective R-module as well as a self-injective ring. And the ring

$$A = \begin{pmatrix} R & J \\ Y & Y \end{pmatrix}$$

is a right self-injective and not a left self-injective ring with $\mathrm{soc}({}_AA) = \begin{pmatrix} S & S \\ 0 & 0 \end{pmatrix}$ which is essential in ${}_AA$.

[4] see [Osofsky, 1984]

The next statement gives connections between a self-injective ring and annihilators of its ideals.

Proposition 4.12.1. *If a ring A is right self-injective, then it satisfies the following conditions*

(1) *for any right ideals H_1, H_2*

$$l(H_1 \cap H_2) = l(H_1) + l(H_2) \tag{4.12.1}$$

(2) *for any finitely generated left ideal H*

$$l(r(H)) = H. \tag{4.12.2}$$

Proof. For all right ideals H_1, H_2 of A, obviously, always $l(H_1) + l(H_2) \subset l(H_1 \cap H_2)$. Let $x \in l(H_1 \cap H_2)$. Consider the map $\varphi : H_1 + H_2 \to A$ defined by $\varphi(a+b) = xb$ for $a \in H_1$ and $b \in H_2$. This is well-defined because $x \in l(H_1 \cap H_2)$. It is easy to show that φ is an A-homomorphism. Since A_A is injective, by the Baer criterion (see proposition 5.2.4, vol.I), there is a $y \in A$ such that $\varphi(a+b) = y(a+b) = xb$ for all $a \in H_1$ and $b \in H_2$. In particular, $0 = \varphi(a) = ya$ for all $a \in H_1$, that is, $y \in l(H_1)$. For all $b \in H_2$ we have $\varphi(b) = yb = xb$, so $z = x - y \in l(H_2)$. Therefore $x = y + z \in l(H_1) + l(H_2)$. Thus, $l(H_1 \cap H_2) \subset l(H_1) + l(H_2)$, and hence $l(H_1 \cap H_2) = l(H_1) + l(H_2)$.

Let H be a finitely generated left ideal of A. Then there are elements $h_1, h_2, \ldots, h_n$ such that $H = Ah_1 + Ah_2 + \ldots + Ah_n$. It is easy to see that

$$r(H) = r\Big(\sum_{i=1}^{n} Ah_i\Big) = \bigcap_{i=1}^{n} r(Ah_i).$$

Applying (4.12.1), we obtain that

$$l(r(H)) = \sum_{i=1}^{n} l(r(Ah_i)).$$

Therefore it needs only to be shown that $l(r(Ax)) = Ax$ for any $x \in A$.

Obviously, $Ax \subset l(r(Ax))$. Let $y \in l(r(Ax))$. Then $r(x) \subset r(y)$ and therefore the map $\psi : xA \to A$, which is given by $\psi(xa) = ya$, is an A-homomorphism. Since A_A is injective, by the Baer criterion, there is an element $z \in A$ such that $\psi(xa) = zxa$. Therefore $zx = y$, i.e., $y \in Ax$. Thus, $l(r(Ax)) \subset Ax$, and so $l(r(Ax)) = Ax$.

Recall that a ring A is called an **FDI-ring** if there exists a decomposition of the identity $1 \in A$ into a finite sum $1 = e_1 + \ldots + e_n$ of pairwise orthogonal primitive idempotents e_i (see vol.I, p.56). Typical examples of FDI-rings are one-sided Noetherian rings and semiperfect rings.

Lemma 4.12.2. *The endomorphism ring of every indecomposable injective module is local.*

Proof. Let Q be an indecomposable injective right A-module, and let $\varphi \in \mathrm{End}_A Q$. Observe that $\mathrm{Ker}\,\varphi \cap \mathrm{Ker}\,(1-\varphi) = 0$. Indeed, let $a \in \mathrm{Ker}\,\varphi \cap \mathrm{Ker}\,(1-\varphi)$. Then $\varphi(a) = 0$ and $a - \varphi(a) = 0$, so $a = 0$. Suppose $\mathrm{Ker}\,\varphi \neq 0$ and $\mathrm{Ker}\,(1-\varphi) \neq 0$. Let $E(\mathrm{Ker}\,\varphi)$ be an injective hull of $\mathrm{Ker}\,\varphi$ and let $E(\mathrm{Ker}\,(1-\varphi))$ be an injective hull of $\mathrm{Ker}\,(1-\varphi)$. There is an exact sequence

$$0 \to \mathrm{Ker}\,\varphi \oplus \mathrm{Ker}\,(1-\varphi) \to Q$$

Since Q is injective, $Q = Q_1 \oplus Q_2$, where $Q_1 \simeq E(\mathrm{Ker}\,\varphi \oplus \mathrm{Ker}\,(1-\varphi))$, by proposition 5.3.6, vol.I. Since Q is indecomposable, $Q \simeq Q_1$. Thus $Q \simeq E(\mathrm{Ker}\,\varphi \oplus \mathrm{Ker}\,(1-\varphi)) \simeq E(\mathrm{Ker}\,\varphi) \oplus E(\mathrm{Ker}\,(1-\varphi))$, by the same proposition. Since $E(\mathrm{Ker}\,\varphi) \neq 0$ and $E(\mathrm{Ker}\,(1-\varphi)) \neq 0$, we obtain a contradiction. Therefore $\mathrm{Ker}\,\varphi = 0$ or $\mathrm{Ker}\,(1-\varphi) = 0$. We may assume that $\mathrm{Ker}\,\varphi = 0$. Consider $\mathrm{Im}\,\varphi$. If $\mathrm{Im}\,\varphi \neq Q$, then, by proposition 5.3.6, vol.I, $Q = Q_1 \oplus Q_2$, where $Q_1 \simeq E(\mathrm{Im}\,\varphi)$, i.e., Q is decomposable. Therefore $\mathrm{Im}\,\varphi = Q$, and consequently, φ is an isomorphism. If $\mathrm{Ker}\,(1-\varphi) = 0$, then analogously we can show that $1-\varphi$ is an isomorphism. So either φ or $1-\varphi$ is invertible and $\mathrm{End}_A Q$ is local.

Theorem 4.12.3. *A right (left) self-injective FDI-ring A is semiperfect.*

Proof. Let e be a primitive idempotent of A. The right projective module eA is indecomposable and injective. So $\mathrm{End}_A\, eA \simeq eAe$ is local, by lemma 4.12.2, and $A \simeq \mathrm{End} A_A$ is semiperfect by theorem 10.3.8, vol. I.

Corollary 4.12.4. *If a ring A is right (resp. left) Noetherian and right (resp. left) self-injective then* $\mathrm{soc}\,({}_AA) \neq 0$ *(resp.* $\mathrm{soc}\,(A_A) \neq 0$*).*

Proof. By theorem 4.12.3, A is semiperfect. Let R be the Jacobson radical of A. By proposition 4.3.3, $r(R) = \mathrm{soc}\,({}_AA)$. If $r(R) = 0$ then $l(r(R)) = A$. But, by the property (2) of proposition 4.12.1, $l(r(R)) = R$. Therefore, $\mathrm{soc}\,({}_AA) \neq 0$.

Proposition 4.12.5. *Suppose a ring A satisfies properties (1) and (2) of proposition 4.12.1. Then for any finitely generated right ideal $\mathcal{I}$ of A and each $f \in \mathrm{Hom}_A(\mathcal{I}, A)$ there is an element $x \in A$ such that $f(a) = xa$ for all $a \in A$.*

Proof. We shall prove this statement by induction on the number n of generators of a finitely generated right ideal $\mathcal{I}$ of A.

Suppose $n = 1$. Then $\mathcal{I} = xA$. Take a homomorphism $\varphi : \mathcal{I} \to A$. Since $xa = 0$ implies $\varphi(xa) = 0 = \varphi(x)a$, we have $r(x) \subset r(\varphi(x))$. Hence $r(Ax) \subset r(A\varphi(x))$. Then, by (4.12.2), we obtain that $A\varphi(x) = l(r(A\varphi(x))) \subset l(r(Ax)) = Ax$. Therefore there is an element $b \in A$ such that $\varphi(x) = bx$ and so $\varphi(xa) = \varphi(x)a = bxa$, as required. .

Suppose the statement is true for all finitely generated right ideals with n generators. Let a right ideal $\mathcal{I}$ have $n+1$ generators, i.e., $\mathcal{I} = \sum\limits_{i=1}^{n+1} x_i A$. Take a homomorphism $\varphi : \mathcal{I} \to A$. By the induction hypothesis, there are elements $b_1, b_2 \in A$ such that

$$\varphi\Big(\sum_{i=1}^{n} x_i a_i\Big) = b_1 \sum_{i=1}^{n} x_i$$

and

$$\varphi(x_{n+1}a_{n+1}) = b_2 x_{n+1}a_{n+1}.$$

Taking into account (4.12.1) we have

$$b_1 - b_2 \in l\Big(\sum_{i=1}^{n} x_i A \cap x_{n+1}A\Big) = l\Big(\sum_{i=1}^{n} x_i A\Big) + l(x_{n+1}A).$$

So there are elements $y \in l(\sum\limits_{i=1}^{n} x_i A)$ and $z \in l(x_{n+1}A)$ such that $b_1 - b_2 = y - z$. Set $b = b_1 - y = b_2 - z$. Then

$$\varphi\Big(\sum_{i=1}^{n+1} x_i a_i\Big) = \varphi\Big(\sum_{i=1}^{n} x_i a_i\Big) + \varphi(x_{n+1}a_{n+1}) =$$

$$= (b_1 - y)\sum_{i=1}^{n} x_i a_i + (b_2 - z)x_{n+1}a_{n+1} = b\sum_{i=1}^{n+1} x_i a_i,$$

as required.

Theorem 4.12.6. *Let A be a right Noetherian ring. Then A is right self-injective if and only if it satisfies properties (1) and (2) of proposition 4.12.1.*

Proof. Since A is right Noetherian, each of its right ideals is finitely generated. Taking into account the Baer criterion we obtain the statement as a corollary of propositions 4.12.1 and 4.12.5.

Theorem 4.12.7 (J.Levitzki). *If A is a right Noetherian ring, then each of its one-sided nil-ideals is nilpotent.*

Proof. Since A is a right Noetherian ring, A has a maximal two-sided nilpotent ideal N. Let $B = A/N$. Then 0 is the only nilpotent ideal in B. We shall show that 0 is also the only left nil-ideal in B.

Suppose there is a nonzero left nil-ideal $\mathcal{I}$ in B. Since B is a right Noetherian ring, the set of right annihilators $r_B(x)$, where $0 \neq x \in \mathcal{I}$, has a maximal element, say $r_B(y)$. Let $b \in B$ with $by \neq 0$. Since $\mathcal{I}$ is a nil-ideal, there is $n > 0$ such that $(by)^{n+1} = 0$ and $(by)^n \neq 0$. Obviously, $r_B(y) \subset r_B(by) \subset r_B((by)^n)$, so by maximality $r_B(by) = r_B((by)^n)$. Thus $yby = 0$. Therefore $(ByB)^2 = 0$ and $y = 0$. This contradiction shows that 0 is the only left nil-ideal in B.

Thus if L is a left nil-ideal in A, then $(L+N)/N = 0 \in A/N$ and $L \subseteq N$, i.e., N contains every right nil-ideal of A. Let $x \in A$, and let Ax be a left nil-ideal. Then for any $a \in A$ we have $(ax)^n = 0$. Therefore $(xa)^{n+1} = 0$ and so the right ideal xA is also a nil-ideal. Thus N also contains every right nil-ideal of A. Since N is nilpotent, every one-sided nil-ideal is nilpotent.

Corollary 4.12.8. *Let A be a right Noetherian ring. If A/R is semisimple and R is a nil-ideal, then A is right Artinian.*

Proof. From theorem 4.12.7, R is nilpotent. So A is a right Noetherian semiprimary ring. By corollary 3.7.2, vol.I, A is right Artinian.

Corollary 4.12.9. *If A is a right Noetherian and right perfect ring, then A is right Artinian.*

Proof. Since A is a right perfect ring, A/R is semisimple and R is right T-nilpotent, and so R is a right nil-ideal. By corollary 4.12.8, A is right Artinian.

Lemma 4.12.10. *If A is a right Noetherian and right self-injective ring, then it is a two-sided Artinian ring.*

Proof. Since A is right self-injective ring, $l(r(L)) = L$ for all left finitely generated ideals L of A, by proposition 4.12.1. This condition implies that if we have some infinite descending sequence of left finitely generated ideals

$$L_1 \supset L_2 \supset \ldots \supset L_n \supset \ldots$$

then we have also an infinite ascending sequence of right ideals

$$r(L_1) \subset r(L_2) \subset \ldots \subset r(L_n) \subset \ldots$$

Since A is right Noetherian the last sequence must stabilize and so A satisfies the d.c.c. for all finitely generated left ideals and, in particular, for all principal left ideals. From theorem 10.5.5, vol.I, it follows that A is a right perfect ring. Therefore A/R is semisimple and $R = \mathrm{rad}A$ is right T-nilpotent. By corollary 4.12.9 and theorem 4.12.7, A is a right Artinian ring and R is nilpotent. Now we shall show that A is also a left Noetherian ring. As we have shown above A satisfies the d.c.c. for all finitely generated left ideals. Suppose there is a left ideal L in A which is not finitely generated. Then for every finitely generated left ideal $H \subset L$ there is a finitely generated ideal H_1 such that $H \subset H_1 \subset L$. Then we can build by induction an infinite strictly ascending chain of finitely generated ideals of A. This contradiction shows that A is a left Noetherian ring. Since A/R is semisimple and R is nilpotent, A is left Artinian, by corollary 4.12.8. Thus, A is a two-sided Artinian.

Lemma 4.12.11. *If A is a right Noetherian ring which satisfies the conditions (1) and (2) of proposition 4.12.1, then it is a two-sided Artinian and right self-injective ring.*

Proof. Since A is a right Noetherian ring, any of its right ideals is finitely generated. Then from proposition 4.12.5 and the Baer criterion it follows that A is a right self-injective ring. And so, by lemma 4.12.10, A is two-sided Artinian.

Lemma 4.12.12. *If A is a right (resp. left) Noetherian ring and $r(l(\mathcal{I})) = \mathcal{I}$ (resp. $l(r(\mathcal{I})) = \mathcal{I}$) for all two-sided ideals $\mathcal{I}$, then* $\mathrm{rad}(A)$ *is nilpotent.*

Proof. Suppose A is a right Noetherian ring and $r(l(\mathcal{I})) = \mathcal{I}$ for all two-sided ideals $\mathcal{I}$ of A. Let $R = \mathrm{rad}A$. Consider the descending chain of ideals

$$R \supseteq R^2 \supseteq R^3 \supseteq \dots$$

and the corresponding ascending chain of annihilators:

$$l(R) \subseteq l(R^2) \subseteq l(R^3) \subseteq \dots$$

Since A is right Noetherian, the last chain is finite, i.e., there is an integer n such that $l(R^n) = l(R^{n+1})$. Hence $R^n = r(l(R^n)) = r(l(R^{n+1})) = R^{n+1}$. Since A is right Noetherian, R^n is a right finitely generated ideal, and, by the Nakayama lemma (lemma 3.4.11, vol.I), $R^n = 0$.

Lemma 4.12.13. *Let A be a right Noetherian and right self-injective ring. Then any indecomposable right A-module can be embedded in the right regular module A_A.*

Proof. By lemma 4.12.10, A is a two-sided Artinian ring. Therefore any A-module M contains a simple A-module. Suppose M contains a simple A-module U. By theorem 4.8.2, A is a socular ring. In particular, $\mathrm{soc}\,({}_AA) \neq 0$ and $\mathrm{soc}\,(A_A) \neq 0$. Then, by lemma 4.11.4, $U^* \neq 0$, as well. Therefore, by lemma 4.11.1, A has a right ideal isomorphic to U, i.e., we have a monomorphism $\varphi : U \to A_A$. This means that we have the diagram of the form:

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "\psi"] \arrow[d, "\varphi"] & M \\ & A_A & \end{tikzcd}$$

with the top row exact. Since A is self-injective, this diagram can be completed to a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "\psi"] \arrow[d, "\varphi"] & M \arrow[dl, "h"'] \\ & A_A & \end{tikzcd}$$

Since M is indecomposable, h is a monomorphism.

Theorem 4.12.14. *For any ring A the following conditions are equivalent:*

(1) *A is a right Noetherian and right self-injective ring.*

(2) *A is two-sided Artinian and satisfies the following double annihilator conditions:*

$$(2a) \quad r(l(H)) = H \quad \textit{for any right ideal } H$$

$$(2b) \quad l(r(L)) = L \quad \textit{for any left ideal } L.$$

Proof.

2) $\Rightarrow$ 1) Since A is a two-sided Artinian ring, it is also two-sided Noetherian by theorem 3.5.6, vol.I.

Let H_1, H_2 be right ideals of A. Then from condition (2a) it follows that

$$r(l(H_1) + l(H_2)) = r(l(H_1)) \cap r(l(H_2)) = H_1 \cap H_2.$$

Taking left annihilators we obtain that

$$l(H_1) + l(H_2) = l(H_1 \cap H_2).$$

Therefore, by theorem 4.12.6, A is right self-injective.

1) $\Rightarrow$ 2) Suppose A is a right Noetherian and right self-injective ring. Then any of its right ideals of is finitely generated, and so, by proposition 4.12.1, $l(r(L)) = L$ for any left ideal L and $r(H_1 \cap H_2) = r(H_1 + H_2)$ for any right ideals H_1, H_2 of A. We must only show that $r(l(H)) = H$ for any right ideal of A. Obviously, $r(l(H)) \subseteq H$. Suppose $r(l(H)) \neq H$. Let $M = r(l(H))/H$ be a right A-module and consider $f \in \mathrm{Hom}_A(M, A_A)$. We may view f as a homomorphism $r(l(H)) \to A_A$ vanishing on H. Since A_A is injective, this homomorphism, by the Baer criterion, is given by left multiplication by some $y \in A$. But $yH = 0$ implies $yx = 0$ for any $x \in r(l(H))$, so $f = 0$. Therefore $\mathrm{Hom}_A(M, A_A) = 0$. We shall show that in this case $M = 0$.

By lemma 4.12.10, A is a two-sided Artinian ring. Suppose M is a decomposable A-module and $M = \overset{n}{\underset{i}{\oplus}} M_i$, then $\mathrm{Hom}_A(M, A_A) = \overset{n}{\underset{i}{\oplus}} \mathrm{Hom}_A(M_i, A_A)$, by proposition 4.3.4, vol.I. Therefore we can assume that M is an indecomposable module. But in this case, by lemma 4.12.13, from $\mathrm{Hom}_A(M, A_A) = 0$ it follows that $M = 0$. Thus, $r(l(H)) = H$ for any right ideal H.

Remark 4.12.1. Since condition 2) of theorem 4.12.14 is left-right symmetric, the theorem implies the same for condition 1). Therefore we can rewrite this theorem in the following symmetric form:

Theorem 4.12.14*. *For any ring A the following conditions are equivalent:*

1) *A is right Noetherian and right self-injective.*

2) *A is left Noetherian and left self-injective.*

3) *A is Artinian and satisfies the following double annihilator conditions:*

$$(3a) \quad r(l(H)) = H \quad \textit{for any right ideal } H$$

$$(3b) \quad l(r(L)) = L \quad \textit{for any left ideal } L.$$

Remark 4.12.2. Note that in the case if A is neither left Noetherian nor right Noetherian self-injectivity is not necessarily left-right symmetric. For example, the endomorphism ring of any infinitely generated free left module over a quasi-Frobenius k-algebra is left self-injective but not right self-injective (see [Osofsky, 1966], [Sandomerski, 1970]. On the other hand there are right and left self-injective ring which are not Artinian (see [Goodearl, 1974]).

In section 4.11 we have introduced rings with duality for simple modules (DSM-rings). By theorem 4.11.5, any quasi-Frobenius ring is a DSM-ring. Now we shall show that any two-sided Artinian DSM-ring satisfies the two double annihilator conditions.

Proposition 4.12.15. *Let A be a two-sided Artinian DSM-ring. Then $r(l(H)) = H$ for any right ideal H and $l(r(L)) = L$ for any left ideal L.*

Lemma 4.12.16. *Let $K \subseteq L$ be right ideals in a ring A and let $(L/K)^*$ be a simple left A-module. Then $l(K)/l(L)$ is either zero or isomorphic to $(L/K)^*$.*

Proof. We can define a map $f : l(K) \to (L/K)^*$ by the formula $f(x)(y+K) = xy$ for $x \in l(K)$ and $y \in L$. This is well-defined because $x \in l(K)$. It is easy to see that f is a homomorphism of left modules and that $\mathrm{Ker} f = l(L)$. Therefore $l(K)/l(L)$ is isomorphic to a submodule of the left simple module $(L/K)^*$. The lemma is proved.

Proof of proposition 4.12.15. Consider any composition series of A_A:

$$0 = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_n = A \tag{4.12.3}$$

By assumption, each $(K_{i+1}/K_i)^*$ is simple. So, by lemma 4.12.16, $l(K_i)/l(K_{i+1})$ is either zero or simple.

Thus

$$0 = l(K_n) \subseteq \ldots \subseteq l(K_i) \subseteq l(K_0) = A \tag{4.12.4}$$

is a series of submodules with either simple or zero factors.

Consequently, $\mathrm{length}({}_AA) \leq \mathrm{length}(A_A)$. By symmetry we have $\mathrm{length}(A_A) \leq \mathrm{length}({}_AA)$. Therefore $l(K_i) \neq l(K_{i+1})$ for $i = 0, \ldots, n$. Applying the right annihilator to (4.12.4) we obtain that $0 = r(l(K_0)) \subseteq \ldots \subseteq r(l(K_i)) \subseteq \ldots \subseteq r(l(K_n)) = A$ is also a composition series for A_A. Obviously, $K_i \subseteq r(l(K_i))$ and $K_i = r(l(K_i))$. Any right ideal H may be a member of a composition series so we have proved the double annihilator property for right ideals. By symmetry the same hold for left ideals. The proposition is proved.

The following statement gives equivalent definitions of QF-rings.

Theorem 4.12.17 (S.Eilenberg, T.Nakayama). *For each ring A the following conditions are equivalent:*

1) *A is a quasi-Frobenius ring;*

2) *A is two-sided Artinian and satisfies the following double annihilator conditions:*

$$(2a) \quad r(l(H)) = H \quad \textit{for any right ideal } H$$

$$(2b) \quad l(r(L)) = L \quad \textit{for any left ideal } L.$$

3) *A is right Noetherian and right self-injective.*

4) *A is left Noetherian and left self-injective.*

Proof. The conditions (2), (3) and (4) are equivalent by theorem 4.12.14*. So assume that A is two-sided Artinian and right and left self-injective. Let P be a principal right A-module. Then it contains a simple right A-module U. Since P is a direct summand of A_A and A is self-injective, P is injective as well. So P is an indecomposable injective A-module, and then from proposition 5.3.6, vol.I, it follows that $P \simeq E(U)$. So U is an essential simple module in P, therefore $U = \operatorname{soc} P$. If P and P' are principal indecomposable A-modules then their injectivity implies that $\operatorname{soc} P \simeq \operatorname{soc} P'$ if and only if $P \simeq P'$. Thus, every simple right A-module is isomorphic to the socle of some right indecomposable injective module. Analogously, if Q and Q' are left principal indecomposable then $\operatorname{soc} Q$ and $\operatorname{soc} Q'$ are simple and A is quasi-Frobenius, by theorem 4.5.2.

Let A be a quasi-Frobenius ring. Then A is a DSM-ring and, by proposition 4.12.15, A satisfies the two double annihilator conditions, i.e., we proved (1) $\Rightarrow$ (2). Therefore the theorem is proved.

Lemma 4.12.18. *Let A be a right Noetherian ring. Then any injective right A-module M is a direct sum of indecomposable injective submodules.*

Proof. Let A be a right Noetherian ring. We first show that any injective right A-module Q contains an indecomposable injective submodule. Let $x \in Q$. Then it suffices to consider the case when Q is an injective hull of xA, i.e., $Q = E(xA)$. Since A is right Noetherian, xA cannot contain an infinite direct sum. Since Q is an essential extension of xA, Q also cannot contain an infinite direct sum as well. Hence it follows that Q contains an indecomposable injective submodule.

Let M be an injective right A-module. Consider a set of all indecomposable injective submodule of M whose sum is direct. By Zorn's lemma, there exists such a set $\{M_i : i \in I\}$ which is maximal. From theorem 5.2.12, vol.I, it follows that $\bigoplus\limits_i M_i$ is an injective module, and so $M = X \oplus \bigoplus\limits_i M_i$, where X is some submodule of M, by propositions 5.2.2 and 5.3.6, vol.I. Since M is injective, X is injective as well, and it contains an indecomposable injective submodule. From the maximality of $\bigoplus\limits_i M_i$ it follows that $X = 0$. So $M = \bigoplus\limits_i M_i$ as required.

Theorem 4.12.19 (C.Faith, E.A.Walker). *The following conditions are equivalent:*

1) *A is a quasi-Frobenius ring;*

2) *a right (or left) A-module M is projective if and only if it is injective.*

Proof.

1) $\Rightarrow$ 2) Let A be a quasi-Frobenius ring, then, by theorem 4.12.16, A is right and left self-injective. So every free module is injective. Hence every projective module is injective as well.

Conversely, suppose M is an injective module. Since A is a Noetherian ring, by lemma 4.12.17, $M \simeq \mathop{\oplus}\limits_i M_i$, where the M_i are indecomposable injective modules. By lemma 4.12.13, M_i may be embedded in the injective module A_A. Therefore M_i is projective, and so is M.

2) $\Rightarrow$ 1) Conversely, since A itself is projective, we obtain that A_A is injective. Since any direct sum of projective modules is projective, by proposition 5.1.4, vol.I, and any projective module is injective, any direct sum of injective module is injective. Then, by theorem 5.2.12, vol.I, A is right Noetherian. Therefore, by theorem 4.12.17, A is quasi-Frobenius.

Corollary 4.12.20 (M.Auslander). *Let A be a quasi-Frobenius ring. Then* $\operatorname{gl.dim} A = 0$ *or* $\operatorname{gl.dim} A = \infty$

Proof. Suppose that $\operatorname{l.gl.dim} A = n < \infty$. Let M be a left A-module such that $\operatorname{l.gl.dim}_A(M) = n$. Then there exists a left A-module X such that $\operatorname{Ext}^n_A(M, X) \neq 0$ and $\operatorname{Ext}^{n+1}_A(M, N) = 0$ for any left A-module N. Consider an exact sequence

$$0 \to Y \to F \to X \to 0$$

where F is a free A-module. Then $\operatorname{Ext}^n_A(M, F) = 0$ since F is a left free module and A is quasi-Frobenius, and hence by theorem 4.12.19 F is injective. Since $\operatorname{Hom}_A(*, Q)$ is an exact functor for any injective A-module Q, we conclude that $n = 0$, and so M is projective. In this case A is a semisimple ring, by proposition 6.6.6, vol.I.

In section 4.10 we have studied reflexive modules. The following theorem gives a characterization of finitely generated modules over quasi-Frobenius rings.

Theorem 4.12.21. *Let A be a quasi-Frobenius ring. Then*

1) *all finitely generated A-modules are reflexive.*

2) *a right (left) A-module M is finitely generated if and only if M^* is finitely generated.*

To prove this theorem we need the following lemma.

Lemma 4.12.22. *Let A be a quasi-Frobenius ring. Then any finitely generated semi-reflexive A-module is reflexive.*

Proof. Let M be a finitely generated semi-reflexive A-module. Consider an exact sequence

$$0 \to N \to F \xrightarrow{\varphi} M \to 0$$

where F is a free A-module with a finite free basis. Since A is right and left self-injective, by applying two times the duality functor we obtain an exact sequence:

$$0 \to N^{**} \to F^{**} \xrightarrow{\varphi^{**}} M^{**} \to 0$$

Consider the commutative diagram:

$$\begin{array}{ccc} F & \xrightarrow{\delta_F} & F^{**} \\ \downarrow \varphi & & \downarrow \varphi^{**} \\ M & \xrightarrow{\delta_M} & M^{**} \end{array}$$

Here δ_F is an isomorphism by proposition 4.10.4, and δ_M is a monomorphism, by assumption. Then $M \simeq \delta_M(M) = \operatorname{Im} \varphi^{**} = M^{**}$ and $\operatorname{Coker} \delta_M = 0$. Therefore δ_M is an isomorphism, i.e., M is reflexive.

Proof of theorem 4.12.21. (1) Let M be a right finitely generated A-module. Then there exists an exact sequence

$$0 \to N \to F \xrightarrow{\varphi} M \to 0$$

where F is a free A-module with a finite free basis, i.e., $F = A^n$, and N is a submodule of F. Then F is a reflexive modules, by proposition 4.10.4. Since A is Noetherian, N is a finitely generated A-module, as well. Then N is a semi-reflexive module, by lemma 4.10.6. And, by lemma 4.12.22, N is also reflexive.

Since A is right self-injective, the duality functor $\operatorname{Hom}_A(*, A_A)$ is exact from $mod_r A$ to $mod_l A$. And analogously, the duality functor $\operatorname{Hom}_A(*, {}_A A)$ is also exact. Consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow \delta_N & & \downarrow \delta_F & & \downarrow \delta_M & & \\ 0 & \longrightarrow & N^{**} & \longrightarrow & F^{**} & \longrightarrow & M^{**} & \longrightarrow & 0 \end{array}$$

Since δ_N and δ_F are isomorphisms, δ_M is also an isomorphism, by corollary 4.2.6, vol.I, i.e., M is reflexive.

(2) Let M be a finitely generated right A-module. Since A is a two-sided Noetherian ring, M^* is also a finitely generated module by lemma 4.10.2. Conversely, let M^* be a finitely generated left A-module. Then by the above, M^{**} is also a finitely generated module. Consider an injective hull $E(M)$. By theorem 14.12.19, $E(M)$ is projective, so it embeds into a free module F. So we have a chain of inclusions $M \subseteq E(M) \subseteq F$, which says that $\operatorname{Hom}_A(M, A_A) \neq 0$ for any $M \neq 0$. This means that $\delta_M : M \to M^{**}$ is a monomorphism. Since M^{**}

is finitely generated and A is a Noetherian ring, M is also a finitely generated A-module.

4.13 QUIVERS OF QUASI-FROBENIUS RINGS

Let Q be a quiver, let i and j be two points of Q and let σ_{ij} be an arrow from i to j. A path x_{ij} is called **simple** if all its points $i_1, i_2, \dots, i_r$ are different except maybe the first and last one. If in addition $i = j$ then x_{ij} is called a **simple cycle**.

Denote $Q(i) = \{j \in Q : \text{ there exists a path } x_{ij} \text{ from } i \text{ to } j\}$.

Let $Q = Q(A)$ be the quiver of a two-sided Artinian ring A. We introduce a notion of a "maximal path" in Q connected with properties of the Jacobson radical R of A. We can assume that A is basic, since $Q(A) = Q(B)$ for any ring B which is Morita equivalent to A. Let $A = P_1 \oplus \dots \oplus P_s$ and let $1 = e_1 + \dots + e_s$ be a corresponding decomposition of $1 \in A$, i.e., $e_iA = P_i$ $(1 = 1, \dots, s)$. Let σ_{ij} be an arrow of the quiver $Q(A)$. This means that there is a direct summand $P_j^{t_{ij}} \neq 0$ in the decomposition $P(P_iR) = \overset{s}{\underset{j=1}{\oplus}} P_j^{t_{ij}}$ and that there is a nonzero homomorphism $\varphi_{ji} : P_j \to P_i$ which is given by the restriction of a homomorphism $P(P_iR) \to P_iR$ onto one of the direct summands $P_j^{t_{ij}}$.

So each arrow σ_{ij} of the quiver $Q(A)$ of the ring A naturally corresponds to a homomorphism $\varphi_{ji} : P_j \to P_i$ and every path $x_{ij} = \sigma_{ii_1}\sigma_{i_1i_2}\dots\sigma_{i_rj}$ naturally corresponds to a homomorphism $\Phi_{ji} : P_j \to P_i$ where $\Phi_{ji} = \varphi_{ji_r}\varphi_{i_ri_{r-1}}\dots\varphi_{i_1i}$.

A path x_{ij} is called **maximal** if $\Phi_{ji} \neq 0$ but $\varphi_{kj}\Phi_{ji} = 0$, where φ_{kj} corresponds to an arrow σ_{jk}. In this case j is called the **end** of the maximal path x_{ij} with start at i.

Since $\mathrm{Hom}_A(P_j, P_i) \simeq e_iAe_j$, an arrow σ_{ij} naturally corresponds to an element $a_{ij} = e_iae_j \neq 0$ and $x_{ij} = \sigma_{ii_1}\sigma_{i_1i_2}\dots\sigma_{i_rj}$ naturally corresponds to an element $c_{ij} = a_{ii_1}a_{i_1i_2}\dots a_{i_rj}$. Suppose that the path x_{ij} is maximal. Consider the right ideal $c_{ij}A$. Since x_{ij} is maximal, $c_{ij}AR = 0$, i.e., $c_{ij}Ae_j \neq 0$ and $c_{ij}Ae_k \neq 0$ for $k \neq j$. By the annihilation lemma, $c_{ij}A \simeq U_j^{m_j}$, where $m_j > 0$.

Let A be a QF-ring. From theorem 4.5.2 it follows that the map $i \mapsto \pi(i)$ which sends every vertex i of $Q(A)$ to the end of a maximal path with start at i is a permutation. This permutation coincides with a Nakayama permutation of A.

Clearly, the permutation $i \mapsto \pi(i)$ satisfies the following conditions:

(α): for any σ_{ij} either $\pi(i) = j$ or $\pi(i) \in Q(j)$;

(β): for any vertex k of Q and any vertex $i \in Q(k)$, we have $\pi(i) \in Q(k)$.

Lemma 4.13.1. *Suppose that there is a permutation on the set of vertices of a connected quiver Q satisfying the conditions (α) and (β). Then the quiver Q is strongly connected.*

Proof. Assume that Q is not strongly connected. Then there exist vertices k and l such that there is no path from k to l. Hence $l \notin Q(k)$. Let T be the set of vertices of Q that do not belong to $Q(k)$. Since Q is connected, there exists $i \in T$ and an arrow σ_{ij} such that $j \in Q(k)$. Clearly, $Q(j) \subset Q(k)$. Let $Q(k) = \{1, 2, \ldots, m\}$. Then from the property (β) it follows that $Q(k) = \{\pi(1), \pi(2), \ldots, \pi(m)\}$ and (α) implies that $\pi(i) \in Q(k)$. But $i \in T$ and so $i \notin Q(k)$ and some vertex from $Q(k)$ is mapped to $\pi(i)$. The obtained contradiction completes the proof of the lemma.

The main result of this section is the following theorem.

Theorem 4.13.2 (E.L.Green). *The quiver of an indecomposable quasi-Frobenius ring is strongly connected.*

Proof. Since all rings Morita equivalent to a QF-ring are QF-rings, we can assume that the quasi-Frobenius ring A is basic. Since a QF-ring is a two-sided Artinian ring, we can consider its quiver Q. Then the proof of this theorem follows from lemma 4.13.1.

4.14 SYMMETRIC ALGEBRAS WITH GIVEN QUIVERS

It is easy to prove the following lemma. The proof is left to the reader.

Lemma 4.14.1. *Let Q be a connected quiver with at least one arrow. The following statements are equivalent:*

(i) *Q is strongly connected;*

(ii) *there is a cycle $c = (\sigma_1, \ldots, \sigma_m)$ such that every arrow σ of Q occurs as some σ_i.*

Recall the definition of a path algebra (see vol. I, section 11.3).

Given a quiver $Q = (VQ, AQ, s, e)$ and a field k, the **path algebra** kQ of Q over k is the (free) vector space with a k-basis consisting of all paths of Q. Multiplication in kQ is defined in an obviously way: if the path $\sigma_1 \ldots \sigma_m$ connects the vertex $i \in VQ$ with the vertex $j \in VQ$ and the path $\sigma_{m+1} \ldots \sigma_n$ connects the vertex j with the vertex $k \in VQ$, then the product $\sigma_1 \ldots \sigma_m \sigma_{m+1} \ldots \sigma_n$ connects the vertex i with the vertex k. Otherwise, the product of these paths equals 0.

By convention, we shall consider that the path ε_i of length zero connects the vertex $i \in VQ$ with itself without any arrow.

The identity of this algebra kQ is the sum of all paths ε_i for $i \in VQ$ of length zero. Extending the multiplication by the distributivity, we obtain a k-algebra (non necessarily finite dimensional).

Note that kQ is finite dimensional if and only if Q is finite and has no cyclic path. Moreover, in this case kQ is a basic split algebra.

Let $Q = (VQ, AQ, s, e)$ be a quiver, where $VQ = \{1, 2, \ldots, n\}$. Let $B_0 = \{\varepsilon_1, \ldots, \varepsilon_n\}$, $B_1 = AQ$, $B_2 = \{$all paths of Q of length 2$\}$, …, $B_m = \{$all paths of Q of length $m\}$, …, $B = \bigcup\limits_{i=0}^{\infty} B_i$. Obviously, B is a k-basis of $\Omega = kQ$. Denote by $\mathcal{J}$ the two-sided ideal in $\Omega = kQ$ with k-basis $B \setminus B_0$. This ideal is called the **fundamental ideal** of a path algebra kQ (see section 2.2).

For a given k-linear map $\mu : \Omega \to k$ we set $S = \{b \in B : \mu(b) \neq 0\}$ and $\mathcal{I}(\mu) = \{w \in \Omega : \mu(\Omega w \Omega) = 0\}$.

Lemma 4.14.2. *Let $\mu : \Omega \to k$ be a k-linear map. Then*
(a) *$\mathcal{I}(\mu)$ is the largest two-sided ideal contained in* $\operatorname{Ker}\mu$*;*
(b) *S is a finite set if and only if $\mathcal{J}^m \subset \mathcal{I}(\mu)$ for some m.*

Proof. (a) Obviously, $\mathcal{I}(\mu)$ is a two-sided ideal. Let L be a two-sided ideal in Ω and $L \subset \operatorname{Ker}\mu$. Then $\mu(\Omega\, x\, \Omega) = 0$ for all $x \in L$. Consequently, $x \in I(\mu)$ and property (a) is proved.

(b) Let S be a finite set. Therefore, there exists an m such that $\mu(b) = 0$ for all $b \in \bigcup\limits_{i=m}^{\infty} B_i$. Hence, if $w \in \mathcal{J}^m$ then $\mu(w) = 0$ and $\mathcal{J}^m \subseteq \operatorname{Ker}\mu$. Moreover, $\Omega w \Omega \subseteq \mathcal{J}^m$ for all $w \in \mathcal{J}^m$ and $\mathcal{J}^m \subseteq \mathcal{I}(\mu)$. Conversely, if $\mathcal{J}^m \subset \mathcal{I}(\mu)$, then $\mu(b) = 0$ for all $b \in \bigcup\limits_{i=m}^{\infty} B_i$. Consequently, if $\mu(b) \neq 0$ then $b \in \bigcup\limits_{i=0}^{m-1} B_i$ and as $\bigcup\limits_{i=0}^{m-1} B_i$ is finite it follows that S is finite.

Definition. Let μ be as above (with corresponding S). We say that (S, μ) is a **Frobenius system** if the following conditions hold:
(fs1) S is a finite set;
(fs2) for all $w \in \Omega$, $w\Omega \subset \operatorname{Ker}\mu$ if and only if $\Omega w \subset \operatorname{Ker}\mu$.

Theorem 4.14.3. *If $\Omega = kQ$ is the path algebra of a quiver Q over a field k and (S, μ) is a Frobenius system then $\Omega/\mathcal{I}(\mu)$ is a Frobenius algebra.*

Proof. Suppose that (S, μ) is a Frobenius system. Since S is a finite set, by lemma 4.14.2, we have $\mathcal{J}^m \subseteq \mathcal{I}(\mu)$ for some integer m. Thus $A = \Omega/\mathcal{I}(\mu)$ is a finite-dimensional k-algebra. Since $\mathcal{I}(\mu) \subset \operatorname{Ker}\mu$, the linear map $\mu : \Omega \to k$ induces a linear map $\bar{\mu} : A \to k$. By theorem 4.2.1, it suffices to show that $\operatorname{Ker}\bar{\mu}$ contains neither left nor right ideals.

Let $\mathfrak{b}$ be a left ideal of A and $\mathfrak{b} \subseteq \operatorname{Ker}\bar{\mu}$. Consider the commutative diagram

$$\begin{tikzcd} \Omega \arrow[r, "\mu"] \arrow[d, "\varphi"'] & k \\ A \arrow[ur, "\bar{\mu}"'] & \end{tikzcd}$$

where φ is the canonical ring surjection. Let $\mathfrak{C} = \varphi^{-1}(\mathfrak{b})$. Then $\mathfrak{C}$ is a left ideal in Ω and $\mathfrak{C} \subseteq \operatorname{Ker}\mu$. Obviously, $\mathfrak{C} \supset \mathcal{I}(\mu)$. We shall show that $\mathfrak{C}\Omega \subseteq \mathcal{I}(\mu)$.

Let $w \in \mathfrak{C}$. Then $\Omega w \subseteq \mathfrak{C} \subseteq \operatorname{Ker}\mu$. By definition of the Frobenius system (S, μ), we have $w\Omega \subseteq \operatorname{Ker}\mu$. Therefore, $\mathfrak{C}\Omega \subseteq \mathcal{I}(\mu)$ and $\varphi(\mathfrak{C}) = 0 = \varphi\varphi^{-1}(\mathfrak{b}) = \mathfrak{b}$. A similar proof shows that if $\mathfrak{b}$ is a right ideal in A and $\mathfrak{b} \subseteq \operatorname{Ker}\bar{\mu}$ then $\mathfrak{b} = 0$. Therefore, A is a Frobenius algebra, by theorem 4.2.1.

Definition. We say that a subset S of B is **cyclic** if there is a cycle $c = (\sigma_1, \ldots, \sigma_m)$ such that

(1) every arrow in AQ occurs as some σ_s;

(2) if $f \in B$ then

$$f \in S \Leftrightarrow f = \begin{cases} c, & \text{or} \\ (\sigma_j, \ldots, \sigma_m, \sigma_1, \ldots, \sigma_{j-1}) & \text{for some } j = 2, \ldots, m. \end{cases}$$

Note that S is a finite set.

Define a set map: $\mu : B \to k$ by the formula

$$\mu(f) = \begin{cases} 1 & \text{if and only if } f \in S \\ 0 & \text{otherwise} \end{cases} \tag{4.14.1}$$

Extend this map to a k-linear map $\Omega \to k$ by "linearity", i.e., for $w \in \Omega$, $w = \alpha_1 b_1 + \ldots + \alpha_n b_n$ we set $\mu(w) = \sum\limits_{i=1}^{n} \alpha_i \mu(b_i)$.

Proposition 4.14.4. *Let Q be a quiver with basis B of its path algebra. If S is a cyclic subset of B and μ is the k-linear map defined by (4.14.1), then (S, μ) is a Frobenius system.*

Proof. Clearly, S is a finite set, so condition (fs1) holds.

Since S is cyclic, it follows that $f \cdot g \in S \Leftrightarrow g \cdot f \in S$ for all $f, g \in B$. This implies

$$\mu(f \cdot g) = 0 \Leftrightarrow \mu(g \cdot f) = 0 \tag{4.14.2}$$

for all $f, g \in B$.

Since μ is the identity on S and 0 on $B \setminus S$, we see that the linearity of μ and (4.14.2) imply

$$\mu(w \cdot w') = \mu(w' \cdot w) \tag{4.14.3}$$

for all $w, w' \in \Omega$.

Property (fs2) easily follows from (4.14.3) and we conclude that (S, μ) is a Frobenius system.

Theorem 4.14.5. *Let Q be a strongly connected quiver. Then for any field k there is a symmetric k-algebra A such that $Q = Q(A)$.*

Proof. If Q is a point, then k is a symmetric algebra and $Q(k)$ is a point. We may assume that Q is a strongly connected quiver with at least one arrow. By lemma 4.14.1, there is a cyclic subset S of B and a k-linear map: $\mu : kQ \to k$, such that (S, μ) is a Frobenius system. By theorem 4.14.3, the quotient k-algebra $A = \Omega/\mathcal{I}(\mu)$, where $\Omega = kQ$, is Frobenius. Let $\bar{\mu} : A \to k$ be the linear map induced by μ. From equality (4.14.3) we have $\bar{\mu}(a \cdot a') = \bar{\mu}(a' \cdot a)$ for all $a, a' \in A$. By the definition of a symmetric algebra, it follows that A is a symmetric k-algebra. Obviously, $\mathcal{J}^{m+1} \subset \mathcal{I}(\mu)$.

It remains to show that $Q(A) = Q$. It is sufficiently to show that $\mathcal{I}(\mu) \subset \mathcal{J}^2$. Let $t \in \Omega \setminus \mathcal{J}^2$. We shall show that $t \notin \mathcal{I}(\mu)$.

As $t \in \Omega \setminus \mathcal{J}^2$, it is of the form

$$t = \sum_{i=1}^{n} \alpha_i \varepsilon_i + \sum_{\sigma \in AQ} \alpha_\sigma \sigma + \sum_{i \in VQ} \beta_i h_i,$$

where $h_i \in B \cap \mathcal{J}^2$ and α_i, α_σ, $\beta_i \in k$ with at least one α_i or α_σ unequal to zero.

First assume there exists $i \in VQ$ such that $\alpha_i \neq 0$. We may assume that $\alpha_1 \neq 0$. There is an $f \in S$ such that $\varepsilon_1 f = f$. It is easy to see that

$$\mu(tf - \alpha_1 \varepsilon_1 f) = 0.$$

So, $\mu(tf) = \alpha_1 \neq 0$ and $t \notin \mathcal{I}(\mu)$. Consequently we can assume that $t = \sum\limits_{\sigma \in AQ} \alpha_\sigma \sigma + \sum\limits_{i \in VQ} \beta_i h_i$. Let $\alpha_\sigma \neq 0$ for some σ. We can assume that $c = (\sigma, \sigma_2, \ldots, \sigma_m)$ is a cycle. Let $f^* = \sigma_2 \ldots \sigma_m$ and $f = \sigma_1 \ldots \sigma_m$. Obviously,

$$\mu(tf^*) = \alpha_\sigma + \sum_{\tau \neq \sigma} \alpha_\tau \mu(\tau \sigma_2 \ldots \sigma_m),$$

where $\tau \sigma_2 \ldots \sigma_m \in B$. The number of times which σ occurs in $(\tau, \sigma_2, \ldots, \sigma_m)$ is one less than the number of times which σ occurs in c. Thus $(\tau, \sigma_2, \ldots, \sigma_m)$ is not a reordering of c and we conclude that $\mu(\tau \sigma_2 \ldots \sigma_m) = 0$ for $\tau \neq \sigma$.

So, $\mu(t \cdot f^*) = \alpha_\sigma \neq 0$ and $t \notin \mathcal{I}(\mu)$. Consequently, $\mathcal{I}(\mu) \subseteq \mathcal{J}^2$ and $Q(A) = Q$. The theorem is proved.

4.15 REJECTION LEMMA

Through this section A denotes a two-sided Artinian ring.

Suppose A is an Artinian ring and M a right A-module. Let $\operatorname{End}(M)$ denote the ring of endomorphisms of M. It is natural to view M as a left $\operatorname{End}(M)$-module. Denote the ring of endomorphisms of the $\operatorname{End}(M)$-module M by $A(M)$. The ring $A_M = A/\operatorname{ann} M$ is a subring of $A(M)$ where the embedding is induced by $A \to \operatorname{End}_{\operatorname{End}(M)}(M)$, $a \mapsto (m \mapsto ma)$.

Proposition 4.15.1. *For any A-module M there exists a monomorphism $A_M \to M^n$ for some n.*

Proof. Let us view M as a left $\operatorname{End}(M)$-module. there exists an exact sequence

$$\operatorname{End}(M)^J \longrightarrow \operatorname{End}(M)^I \longrightarrow M \longrightarrow 0\,.$$

Apply to this sequence the left exact functor $\operatorname{Hom}_{\operatorname{End}(M)}(*, M)$. We obtain

$$0 \longrightarrow A(M) \longrightarrow M^I\ ;$$

consequently,

$$0 \longrightarrow A_M \xrightarrow{\varphi} M^I .$$

We now show that A_M can be embedded in a direct sum of a finite number of copies of M. Let φ_i, $i \in I$, denote the projection of φ on the i-th coordinate of M^I. Since φ is a monomorphism, $\bigcap\limits_{i\in I} \operatorname{Ker}\varphi_i = 0$. Since the ring A_M is Artinian, there exist $i_1, \dots, i_n$ such that $\bigcap\limits_{i\in I}^{n} \operatorname{Ker}\varphi_{i_k} = 0$. Therefore, A_M can be embedded in M^n.

Definition. A module over an Artinian ring is called **bijective** if it is both projective and injective. We shall prove the following lemma.

Lemma 4.15.2 (Rejection Lemma). *Suppose M is an indecomposable bijective module over an Artinian ring A. There exists a proper quotient ring A_1 of A such that every indecomposable A-module except M is an A_1-module.*

Proof. Let $\mathcal{I} = \bigcap \operatorname{ann} N$, where N ranges over all indecomposable A-modules except M. Since A is an Artinian ring, there exists a finite set of indecomposable modules $N_1, \dots, N_t$ such that $\mathcal{I} = \bigcap\limits_{i=1}^{t} \operatorname{ann} N_i$, where $N_i \not\simeq M$ $(i = 1, \dots, t)$. Let $Y = \bigoplus\limits_{i=1}^{t} N_i$. Obviously, $\operatorname{ann} Y = \mathcal{I}$. Let $A_1 = A/\mathcal{I}$. Clearly, every indecomposable A-module except M is an A_1-module. If M is also an A_1-module, then $A_1 = A_Y = A$. By proposition 4.15.1, we have in this case a monomorphism $A \to Y^m$, from which we obtain a monomorphism $\psi : M \to Y^m$. Since M is an indecomposable bijective module, it has exactly one minimal submodule (see proposition 5.3.6, vol.I). Therefore, if each projection ψ_i of M on N_i has nonzero kernel, then ψ is not monomorphism. Consequently, there exists a monomorphism $M \to N_i$, from which we obtain that $M \simeq N_i$, since M is injective and N_i is indecomposable. This contradiction proves the lemma.

We denote the quotient ring A_1 by $A - (M)$. We say that A_1 is obtained from A by rejecting the indecomposable bijective module M.

Lemma 4.15.3. *Let M be an indecomposable bijective A-module, let M_1 be its unique maximal submodule, and let M_2 be its unique minimal submodule; let $A_1 = A - (M)$. Then M/M_2 is a projective A_1-module and M_1 is an injective A_1-module.*

Proof. We shall show that any diagram of A_1-modules

$$\begin{tikzcd} 0 \arrow[r] & N' \arrow[r, "i"] \arrow[d, "\alpha"'] & N \\ & M_1 & \end{tikzcd}$$

where i is a monomorphism, can be extended to a commutative diagram by a suitable morphism $N \to M_1$. Let us view this diagram as a diagram of A-modules. Then there exists a homomorphism $\beta : N \to M$ such that $\alpha = \beta i$. We shall show that $\operatorname{Im}\beta \subset M_1$. Suppose this is not so. Then $\operatorname{Im}\beta = M$. Since M is projective, we obtain $N \simeq M \oplus X$, which contradicts the fact that N is an A_1-module.

Write $\bar{M} = M/M_2$. We shall show that any diagram of A_1-modules

$$\begin{tikzcd} & \bar{M} \arrow[d, "\psi"] & \\ N \arrow[r, "\pi"] & N'' \arrow[r] & 0 \end{tikzcd}$$

where π is an epimorphism, can be extended to a commutative diagram. Consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & M_2 \arrow[r] & M \arrow[r, "\pi_1"] & \bar{M} \arrow[r] \arrow[d, "\psi"] & 0 \\ & & N \arrow[r, "\pi"] & N'' \arrow[r] & 0 \end{tikzcd}$$

By the projectivity of M we obtain that there exists $h : M \to N$ such that $\pi h = \psi\pi_1$. If $\operatorname{Ker} h = 0$, then $N = M \oplus X$. This contradicts the fact that N is an A_1-module. So $\operatorname{Ker} h \supseteq M_2$ and h induces a homomorphism $\bar{h} : \bar{M} \to N$ such that $\pi\bar{h} = \psi$. The lemma is proved.

Proposition 4.15.4. *Let A be an Artinian ring and P a simple bijective A-module. Then $A \simeq A_1 \times A_2$, where A_1 is a simple Artinian ring and $A-(P) \simeq A_2$.*

Proof. Suppose M is an arbitrary A-module and that $\varphi : M \to P$ is a nonzero homomorphism. Since P is simple, it follows that φ is an epimorphism; hence $M = P \oplus X$. If $\psi : P \to M$ is a nonzero homomorphism, then, since P is simple, it is a monomorphism; and since P is injective, $M = P \oplus Y$.

Let $A_A = P^s \oplus P'$, where P does not occur in a direct decomposition of P'. Then it follows from what was said above that $\operatorname{Hom}(P^s, P') = \operatorname{Hom}(P', P^s) = 0$ and $A \simeq A_1 \times A_2$, where $A_1 \simeq \operatorname{End}(P^s) \simeq M_s(\operatorname{End}(P))$, where $\operatorname{End}(P)$ is a division ring, and $A_2 \simeq \operatorname{End}(P')$. Obviously, $A - (P) \simeq A_2$.

Now we shall give a characterization of primary indecomposable serial Artinian rings. For the definitions and other results see vol.I, p. 316.

Theorem 4.15.5. *For a two-sided indecomposable Artinian ring A the following two conditions are equivalent*:

a) $A \simeq M_n(B)$, *where B is a local uniserial ring*;

b) *A is quasi-Frobenius and A has a minimal (by inclusion) two-sided ideal $\mathcal{I}$ such that $A/\mathcal{I}$ is also quasi-Frobenius.*

Proof. (a)$\Rightarrow$(b). Denote By $\mathcal{M}$ the Jacobson radical of B and by R the Jacobson radical of $A = M_n(B)$. Let $\mathcal{M}^t \neq o$ and $\mathcal{M}^{t+1} = 0$. Then $A \supset R \supset \ldots \supset R^t \supset 0$ is the unique series of two-sided ideals in A. Consequently, by definition of quasi-Frobenius and Frobenius rings, we obtain that A is a Frobenius ring and R^t is a minimal (by inclusion) two-sided ideal such that A/R^t is also Frobenius.

To prove the reverse implication, we shall need the following simple lemma.

Lemma 4.15.6. *Let $\mathcal{I}$ be a minimal two-sided ideal of a quasi-Frobenius ring A. Then there is a principal A-module P such that every indecomposable A-module different from P is an $A/\mathcal{I}$-module.*

Proof. Clearly there is a principal A-module P which is not an $A/\mathcal{I}$-module. It follows from lemma 4.15.2 that there is a proper quotient ring $A_1 = A - (P)$ of A such that every indecomposable A-module, except P itself, is an A_1-module. Since $A/\mathcal{I}$ is clearly an A_1-module, it follows from the minimality of $\mathcal{I}$ that $A_1 = A/\mathcal{I}$.

Proof. b)$\Rightarrow$ a). Let $A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be a decomposition of A into a direct sum of principal A-modules. We can assume without loss of generality that $A/\mathcal{I} = A_1 = A - (P_1)$. Let U denote the unique minimal submodule of P_1. By lemma 14.5.3, the A_1-modules $\bar{P}_1 = P_1/U$ and P_1R are both projective.

It is clear that P_1R and $\bar{P}_1$ are indecomposable. Since A_1 is quasi-Frobenius, it follows that P_1R is a principal A_1-module. By the definition of quasi-Frobenius rings, U cannot be minimal submodule of any of the P_i $(i = 2, \ldots, s)$, and so $P_1R/P_1R^2 \simeq P_1/P_1R \simeq U_1$. Since P_1 has a finite length, it follows easily from this that P_1 has a unique composition series whose simple factors are all isomorphic to U_1.

Writing $P' = P_2^{n_2} \oplus \ldots \oplus P_s^{n_s}$, we claim that

$$\mathrm{Hom}_A(P_1^{n_1}, P') = \mathrm{Hom}_A(P', P_1^{n_1}) = 0$$

for which it is sufficient to prove that

$$\mathrm{Hom}_A(P_1, P_j) = \mathrm{Hom}_A(P_j, P_1) = 0$$

for $j = 2, \ldots, s$. If $\mathrm{Hom}_A(P_1, P_j) \neq 0$, then P_j has a minimal submodule isomorphic to U_1, contrary to the definition of quasi-Frobenius rings. If

$\mathrm{Hom}_A(P_j, P_1) \neq 0$, then the composition series of P_1 has a simple factor isomorphic to U_j, and this is a contradiction. Since A is indecomposable as a ring, it follows that $A = P_1^{n_1}$. Arguing in just the same way for left modules, we find that A is uniserial. This completes the proof of theorem 4.15.5.

Proposition 4.15.7. *If A is a quasi-Frobenius ring with zero square radical, then A is a generalized uniserial ring, i.e., a serial Artinian ring.*

The proof is obvious.

In conclusion we note without proof the following theorem.

Theorem 4.15.8.[5] *The following conditions are equivalent for an Artinian ring A:*

(a) *A is serial;*

(b) *any quotient ring of A (including A itself) has a nontrivial bijective module.*

4.16 NOTES AND REFERENCES

Quasi-Frobenius rings and algebras were introduced by Tadasi Nakayama as a generalization of Frobenius algebras (see [Nakayama, 1939], [Nakayama, 1941]). F.G.Frobenius in his paper [Frobenius, 1903] studied the special class of algebras for which the left and right regular representations are equivalent. He also gave necessary and sufficient condition for this equivalence. A most important example of these algebras are group algebras. R.Brauer and C.Nesbitt (see [Brauer, Nesbitt, 1937], [Nesbitt, 1938]) pointed out the importance of these algebras and named them Frobenius algebras. They proved, in particular, the equivalence of conditions (1), (3) of theorem 4.2.1.

The main properties and the structure of quasi-Frobenius rings and algebras were established by T.Nakayama and M.Ikeda (see [Nakayama, Ikeda, 1950], [Ikeda, Nakayama, 1954], [Ikeda, 1952]).

The key concept in the classical definition of quasi-Frobenius rings, given by T. Nakayama, is a permutation of indecomposable projective modules, which is naturally called a **Nakayama permutation** (see [Müller, 1974], [Oshiro, Rim, 1997], [Yousif, 1997]).

Since T.Nakayama introduced the notion of a QF-ring, quasi-Frobenius rings have been extensively studied. One of the most significant results on quasi-Frobenius rings was obtained by C. Faith and E. A. Walker in the paper [Faith, Walker, 1967]. It establishes the equivalence of the following conditions on ring A: (1) A is quasi-Frobenius; (2) each injective right A-module is projective; (3) each injective left A-module is projective. Another of their main results, obtained also in this paper, says that a ring A is quasi-Frobenius if and only if left (or right) A-modules always embed in free A-modules.

Still another equivalent definition of a quasi-Frobenius ring which says that an

[5] see [Kirichenko, 1976].

Artinian ring A is quasi-Frobenius if and only if A is a ring with duality for simple modules was proved by C.Curtis, J.Reiner [Curtis, Reiner, 1962].

B.Osofsky proved in the paper [Osofsky, 1966a] the following equivalent definition for QF-rings: A is a QF-ring if and only if A is right (left) perfect and right and left semi-injective.

It is well known that if A is a QF-ring, then the ring $M_n(A)$, the ring of $n \times n$ matrices over A, and AG, the group ring over A for any finite group G, are both QF-rings. The construction of QF-rings in the general case has been studied by T.A.Hannula in the paper [Hannula, 1973].

In section 4.5, which is devoted to duality in Noetherian rings, we follow to a considerable extent the book by D.G.Northcott [Northcott, 1973].

Rings with duality for simple modules were considered in [Dieudonné, 1958] and in [Morita, Tachikawa, 1956]. These rings were also studied in [Curtis, Reiner, 1962]. M.A.Dokuchaev and V.V.Kirichenko studied semiperfect rings with duality for simple modules (see [Dokuchaev, Kirichenko, 2002]. For an equivalent formulation of the Osofsky theorem see the paper [Osofsky, 1966a]. In this paper she also gave the following equivalent definition for QF-rings: a ring A is quasi-Frobenius if and only if A is right (left) perfect and right and left semi-injective.

The notion of a symmetric algebra was introduced by R.Brauer and C.Nesbitt (see [Brauer, Nesbitt, 1937], [Nesbitt, 1938]). The properties of symmetric algebras were studied by T.Nakayama [Nakayama, 1939]. The structure of symmetric algebras was studied by H.Kupisch (see [Kupisch, 1965] and [Kupisch, 1970]).

Cohomological dimension of Frobenius algebras and quasi-Frobenius rings has been considered in the paper of S.Eilenberg and T.Nakayama [Eilenberg, Nakayama, 1955]. In this paper they proved that for Frobenius algebras the cohomological dimension is either 0 or ∞. They also proved that for right or left Noetherian rings the notions "quasi-Frobenius ring" and "left self-injective ring" are equivalent. In our proof of this theorem (theorem 4.12.17 in this chapter) we follow to a considerable extent F.Kasch [Kasch, 1982]. For left Noetherian rings which are left self-injective Eilenberg and Nakayama proved that their left global dimension is either 0 or ∞. Corollary 4.12.20 in this chapter, which states that the global dimension of a quasi-Frobenius ring is equal to 0 or ∞, was proved by M.Auslander in his paper [Auslander, 1955].

Piecewise domains were first considered by R.Gordon, L.W.Small in [Gordon, Small, 1972].

Theorem 4.9.5 was proved in [Kirichenko, 1993].

Theorem 4.13.2 and theorem 4.14.5 first were proved by E.Green in the paper [Green, 1978].

The classification of quasi-Frobenius algebras of finite representation type in terms of Dynkin diagrams was obtained by C.Riedtmann (see [Riedtmann, 1980a]):

Theorem (C.Riedtmann). *Let A be a quasi-Frobenius algebra of finite representation type over an algebraically closed field. Then the quiver $\overline{Q^b}$ of A is a disjoint union of Dynkin diagrams.*

Quasi-Frobenius algebras of finite representation type were also studied by H.Kupisch (see [Kupisch, 1975], [Kupisch, 1965] and [Kupisch, 1970]. All self-injective algebras of finite representation type were been classified in the papers: [Bretscher, 1982], [Hughes, Waschbüsch, 1983], [Riedtmann, 1980b], [Riedtmann, 1983], [Waschbüsch, 1981].

And quasi-Frobenius algebras of infinite representation type were studied by I.Assen, A.Skowroński, J.Nehring, (see [Assen, 1988], [Nehring, 1989], [Nehring, Skowroński, 1989], [Skowroński, 1989] and K.Erdmann (see [Erdmann, 1990],[Erdmann, 1992]).

Generalizations of quasi-Frobenius algebras were proposed by R.M.Thrall [Thrall, 1948] who introduced three kinds of algebras called QF-1, QF-2 and QF-3 algebras. These algebras were intensively studied by K.Morita (see, for example, [Morita, 1958], [Morita, 1969]), by B.J.Müller (see, for example, [Müller, 1974]) and by others.

Quasi-Frobenius rings appear in different branches of mathematics, for example in number theory, algebraic geometry and combinatorics, in topology and geometry. Finite Frobenius rings have many important applications in coding theory (see, for example, [Dinh, 2004a], [Dinh, 2004b], [Greferath, 2004], [Greferath, O'Sullivan, 2004], [Wood, 1999], [Wood, 1999]).

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5. Right serial rings

This chapter is devoted to the study of properties and structures of right serial rings. Note that a module is called **serial** if it decomposes into a direct sum of uniserial submodules, i.e., submodules whose lattice of submodules is linear. A ring is called **right serial** if its right regular module is serial.

This chapter starts with the study of right Noetherian rings from the point of view of some main properties of their homological dimensions.

In the following sections we give the structure of right Artinian right serial rings in terms of their quivers. We also describe the structure of particular classes of right serial rings, such as quasi-Frobenius rings, right hereditary rings, and semiprime rings. In section 5.6 we introduce right serial quivers and trees and give their description.

The last section of this chapter is devoted to the Cartan determinant conjecture for right Artinian right serial rings. The main result of this section says that a right Artinian right serial ring A has Cartan determinant equal to 1 if and only if the global dimension of A is finite.

5.1 HOMOLOGICAL DIMENSIONS OF RIGHT NOETHERIAN RINGS

In section 6, vol.I, we introduced the main notions of homological dimensions and considered some of their properties for various kinds of rings. In this section we shall introduce the notion of flat dimension and consider some properties of homological dimensions for right Noetherian rings.

Proposition 5.1.1. *Let A be a right Noetherian ring, and let X be a finitely generated right A-module. Then* $\text{proj.dim}_A X \le n$ *if and only if* $\text{Ext}_A^{n+1}(X, Y) = 0$ *for any right finitely generated A-module Y.*

Proof. The necessity of this statement follows from proposition 6.5.4, vol.I.

To prove sufficiency we consider an exact sequence

$$0 \to M \to P \to X, \tag{5.1.1}$$

where P is a projective finitely generated A-module. Since A is a right Noetherian ring, M is a finitely generated A-module. Suppose $n = 0$. Then $\text{Ext}_A^1(X, M) = 0$, and therefore the map $\text{Hom}_A(X, P) \to \text{Hom}_A(X, X)$ is surjective. So the exact sequence (5.1.1) is split, and thus X is projective, i.e., $\text{proj.dim}_A X = 0$.

Now assume that $n > 0$ and the statement holds for $n-1$. Then $\text{proj.dim}_A M \le n-1$, since $\text{Ext}_A^n(M, Y) \simeq \text{Ext}_A^{n+1}(X, Y) = 0$. Then, by proposition 6.5.1, vol.I, $\text{proj.dim}_A X = \text{proj.dim}_A M + 1 \le n$.

Lemma 5.1.2 (O.Villamayor). *Let* $0 \to X \to F \to P \to 0$ *be an exact sequence of right* A*-modules, where* F *is free with a basis* $\{e_i : i \in I\}$*. If* P *is a flat module, then for every* $x \in X$ *there is a* $\theta \in \mathrm{Hom}_A(F, X)$ *with* $\theta(x) = x$.

Proof. F is a free module with a basis $\{e_i : i \in I\}$, and P is flat. For a given $x \in X$ we have $x = a_1 e_{i_1} + a_2 e_{i_2} + \ldots + a_m e_{i_m}$, where $a_i \in A$. Let $\mathcal{I} = a_1 A + a_2 A + \ldots + a_m A$. Since P is flat, we have $x \in X \cap F\mathcal{I} = X\mathcal{I}$, by proposition 5.4.12, vol.I. Therefore $x = \sum x_j c_j$, where $x_j \in X$ and $c_j \in \mathcal{I}$. Now each $c_j = \sum\limits_i a_i b_{ij}$, so that $x = \sum\limits_i a_i x_i'$, where $x_i' = \sum\limits_j x_j b_{ij}$. Define $\theta : F \to X$ by $\theta(e_{i_k}) = x_k'$, while θ sends all the other basis elements of F into 0. Then

$$\theta(x) = \theta(\sum_k a_k e_{i_k}) = \sum_k a_k \theta(e_{i_k}) = \sum_k a_k x_k' = x.$$

As was shown earlier (see corollary 5.4.5, vol.I), every projective module is flat. The converse to this statement is not true in general. The following statement gives an example of a case where a flat module is always projective.

Proposition 5.1.3. *If* A *is a right Noetherian ring, then every finitely generated flat right* A*-module is projective.*

Proof. Suppose P is a finitely generated flat right A-module. Then there is an exact sequence:

$$0 \to X \xrightarrow{\alpha} F \to P \longrightarrow 0,$$

where F is a finitely generated free A-module. Since A is a right Noetherian ring, X is a finitely generated submodule of the free module F. By lemma 5.1.2, there is a map $\theta : F \to X$ with $\theta\alpha = 1_X$, so the sequence splits and P is projective since it is a direct summand of the free module F.

In chapter 4, vol. I, we considered the projective and injective dimensions of modules. There are also other dimensions associated to modules.

Definition. A right A-module X has **flat dimension** n and we write $\mathrm{w.dim}_A X = n$ if there is an exact sequence

$$0 \longrightarrow F_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0, \tag{5.1.2}$$

where F_n is flat, all P_i are projective and there is no shorter such sequence.

We set $\mathrm{w.dim}_A M = \infty$ if there is no n with $\mathrm{w.dim}_A X \leq n$.

Lemma 5.1.4. *Let*

$$\ldots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} X \longrightarrow 0 \tag{5.1.3}$$

be a projective resolution of an A*-module* X*. Then for all left* A*-modules* Y *and all* n $\mathrm{Tor}_{n+1}^A(X, Y) \simeq \mathrm{Tor}_1^A(\mathrm{Ker}\, d_{n-1}, Y)$.

Proof. Since $\mathrm{Tor}_{n+1}^A(X,Y)$ can be computed by using the projective resolution (5.1.3) of X and $\mathrm{Tor}_n^A(\mathrm{Ker}\, d_0, Y)$ can be computed by using the associated projective resolution of $\mathrm{Ker}\, d_0$ (where $d_0 = \pi$):

$$\ldots \longrightarrow P_k \longrightarrow P_{k-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow \mathrm{Ker}\, d_0 \longrightarrow 0,$$

it follows that $\mathrm{Tor}_{n+1}^A(X,Y) \simeq \mathrm{Tor}_n^A(\mathrm{Ker}\, d_0, Y)$. Iterating this argument we obtain:

$$\mathrm{Tor}_{n+1}^A(X,Y) \simeq \mathrm{Tor}_n^A(\mathrm{Ker}\, d_0, Y) \simeq \mathrm{Tor}_{n-1}^A(\mathrm{Ker}\, d_1, Y) \simeq ... \simeq \mathrm{Tor}_1^A(\mathrm{Ker}\, d_{n-1}, Y)$$

Proposition 5.1.5. *The following conditions are equivalent for a right A-module X:*

1) $\mathrm{w.dim}_A X \le n$;
2) $\mathrm{Tor}_k^A(X,Y) = 0$ *for all left A-modules Y and all $k \ge n+1$;*
3) $\mathrm{Tor}_{n+1}^A(X,Y) = 0$ *for all left A-modules Y;*
4) $\mathrm{Ker}\, d_{n-1}$ *is a flat A-module for any projective resolution of X.*

Proof.
1) $\Rightarrow$ 2) and 2) $\Rightarrow$ 3) are trivial.
3) $\Rightarrow$ 4). Consider a projective resolution of X

$$\ldots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} X \longrightarrow 0$$

By lemma 5.1.4, $\mathrm{Tor}_{n+1}^A(X,Y) \simeq \mathrm{Tor}_1^A(\mathrm{Ker}\, d_{n-1}, Y)$, therefore $\mathrm{Tor}_1^A(\mathrm{Ker}\, d_{n-1}, Y) = 0$ for all Y. Then, by proposition 6.3.7, vol.I, $\mathrm{Ker}\, d_{n-1}$ is flat.

4) $\Rightarrow$ 1). Consider a projective resolution of X. Then we have an exact sequence

$$0 \longrightarrow \mathrm{Ker}\, d_{n-1} \longrightarrow P_{n-1} \longrightarrow ... \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

with projective modules P_0, P_1,..., P_{n-1}, and a flat module $\mathrm{Ker}\, d_{n-1}$. Hence $\mathrm{w.dim}_A X \le n$.

Definition. If A is a ring, then its **right weak global dimension**, abbreviated as r.w.gl.dim, is defined as follows:

$$\mathrm{r.w.gl.dim}\, A = \sup\{\mathrm{w.dim}_A M : M \in mod_r\, A\}$$

Analogously we can introduce the **left weak global dimension** of A:

$$\mathrm{l.w.gl.dim}\, A = \sup\{\mathrm{w.dim}_A M : M \in mod_l\, A\}$$

Theorem 5.1.5 immediately implies:

Corollary 5.1.6. r.w.gl.dim $A \le n$ *if and only if* $\mathrm{Tor}^A_{n+1}(X,Y) = 0$ *for all right A-modules X and all left A-modules Y.*

Theorem 5.1.7. *For any ring A* r.w.gl.dim $A =$ l.w.gl.dim A.

Proof. This follows immediately from corollary 5.1.6 and its analog for l.w.gl.dim A.

Definition. The common value of r.w.gl.dim A and l.w.gl.dim A is called the **weak dimension** of a ring A and it is written as w.gl.dim A.

Theorem 5.1.8. *For any ring A* w.gl.dim $A \le \min\{$r.gl.dim A, l.gl.dim $A\}$.

Proof. Let X be a right A-module, then any projective resolution of A has a projective $(n-1)$-st kernel, which is obviously flat. So w.gl.dim$_A X \le$ r.proj.dim$_A X$ and w.gl.dim $A \le$ r.gl.dim A. Analogously we obtain that w.gl.dim $A \le$ r.gl.dim A.

Proposition 5.1.9. *For any ring A,*

$$\text{w.gl.dim}\, A = \sup\{\text{r.w.dim}_A\,(A/\mathcal{I}) \ : \ \mathcal{I} \ \textit{ is a right ideal of } A\} =$$

$$= \sup\{\text{l.w.dim}_A\,(A/\mathcal{I}) \ : \ \mathcal{I} \textit{ is a left ideal of } A\}$$

Proof. First observe that a right A-module B is flat if and only if $\mathrm{Tor}^A_1(A/\mathcal{I}, B) = 0$ for all right ideals $\mathcal{I}$ (see proposition 6.3.9, vol.I). If $\sup\{\text{r.w.dim}_A\,(A/\mathcal{I})\} = \infty$, we are done. Therefore we can assume that r. w.dim$_A\,(A/\mathcal{I}) \le n$ for all right ideals $\mathcal{I}$. We shall prove that r.w.dim$_A\,(A/\mathcal{I}) \le n$ for each left A-module B. Thus assuming $\mathrm{Tor}^A_{n+1}(A/\mathcal{I}, B) = 0$ for every right ideal $\mathcal{I}$, we must show that $\mathrm{Tor}^A_{n+1}(X, B) = 0$ for every right A-module X. Consider a projective resolution of B with n-th kernel $\operatorname{Ker} d_{n-1}$. If $\mathrm{Tor}^A_{n+1}(A/\mathcal{I}, B) = 0$, then $\mathrm{Tor}^A_1(A/\mathcal{I}, \operatorname{Ker} d_{n-1}) = 0$, by lemma 5.1.4. So $\operatorname{Ker} d_{n-1}$ is flat, by proposition 6.3.9, vol.I. Hence, for every module X, $\mathrm{Tor}^A_1(X, \operatorname{Ker} d_{n-1}) = 0$. Using lemma 5.1.4 again, we obtain that $\mathrm{Tor}^A_{n+1}(X, B) = 0$.

In the same way we can prove the second equality.

We shall also show that there holds an analogous statement for projective global dimension for an arbitrary ring.

The following statements are dual to corresponding statements on projective modules over arbitrary rings as given in section 6.5, vol.I.

Proposition 5.1.10. *Let M be a right A-module.*

(1) *M is injective if and only if* $\mathrm{Ext}^i_A(N, M) = 0$ *for all $i > 0$ and all right A-modules N.*

(2) *Let*

$$0 \to M \to Q_0 \xrightarrow{d_0} Q_1 \xrightarrow{d_1} Q_2 \longrightarrow \dots$$

be an injective resolution of the right A-module M. Then for any right A-module N

$$\mathrm{Ext}^{n+1}_A(N, M) \simeq \mathrm{Ext}^1_A(N, \mathrm{Im}\, d_{n-1})$$

(3) $\mathrm{inj.dim}_A M \leq n$ *if and only if* $\mathrm{Ext}^i_A(N, M) = 0$ *for all* $i > n$ *and all right A-modules N.*

Theorem 5.1.11 (M.Auslander). *Let A be an arbitrary ring and M be a right A-module. Then* $\mathrm{inj.dim}\,_A M \leq n$ *if and only if* $\mathrm{Ext}^{n+1}_A(A/\mathcal{I}, M) = 0$ *for any right ideal $\mathcal{I}$ of A.*

Proof. Let $\mathrm{Ext}^{n+1}_A(A/\mathcal{I}, M) = 0$ for any right ideal $\mathcal{I}$ in A. Consider an exact sequence

$$0 \to M \to Q_0 \to Q_1 \to \ldots \to Q_{n-1} \to N \to 0$$

with the Q_i injective ($i = 1, 2, \ldots, n-1$). By proposition 5.1.10(2), $\mathrm{Ext}^{n+1}_A(A/\mathcal{I}, M) \simeq \mathrm{Ext}^1_A(A/\mathcal{I}, N)$. Therefore $\mathrm{Ext}^1_A(A/\mathcal{I}, N) = 0$ and N is an injective A-module, by the Baer criterion. Therefore $\mathrm{inj.dim}\,_A M \leq n$.

Conversely, if $\mathrm{inj.dim}_A M \leq n$ then, by proposition 5.1.10(3), $\mathrm{Ext}^{n+1}_A(A/\mathcal{I}, M) = 0$. The theorem is proved.

Corollary 5.1.12. *A right A-module B is injective if and only if* $\mathrm{Ext}^1_A(A/\mathcal{I}, B) = 0$ *for all right ideals of A.*

Theorem 5.1.13 (M.Auslander). *For any ring A,*

$$\mathrm{r.gl.dim}\, A = \sup\{\mathrm{r.proj.dim}_A\,(A/\mathcal{I}) : \mathcal{I} \textit{ is a right ideal of } A\}.$$

Proof. If $\sup\{\mathrm{r.proj.dim}_A\,(A/\mathcal{I})\} = \infty$, we are done. Therefore we can assume that $r.\mathrm{proj.dim}_A\,(A/\mathcal{I}) \leq n$ for all right ideals $\mathcal{I}$. We shall prove that $\mathrm{r.inj.dim}_A\,(A/\mathcal{I}) \leq n$ for each right A-module B. Then the theorem will follow from theorem 6.5.5, vol.I. Thus assuming $\mathrm{Ext}^{n+1}_A(A/\mathcal{I}, B) = 0$ for every right ideal $\mathcal{I}$, we must show that $\mathrm{Ext}^{n+1}_A(X, B) = 0$ for every right A-module X. Consider an injective resolution of B with n-th image $\mathrm{Im}\, d_{n-1}$.

If $\mathrm{Ext}^{n+1}_A(A/\mathcal{I}, B) = 0$, then $\mathrm{Ext}^1_A(A/\mathcal{I}, \mathrm{Im}\, d_{n-1}) = 0$, by lemma 6.5.7, vol.I. So $\mathrm{Im}\, d_{n-1}$ is injective, by lemma 5.1.12. Hence, for every module X, $\mathrm{Ext}^1_A(X, Im\, d_{n-1}) = 0$. Using lemma 6.5.7, vol.I, again, we obtain that $\mathrm{Ext}^{n+1}_A(X, B) = 0$.

Remark 5.1.1. Thus, propositions 5.1.9 and 5.1.13 say that to compute the global dimension and weak dimension of a ring, it suffices to know the dimensions of the cyclic modules, and, in particular, the dimensions of finitely generated modules. In the general case the weak dimension and global dimension of a ring are distinct things, but for a right Noetherian ring they are the same, as we shall show below.

Proposition 5.1.14. *If A is a right Noetherian ring, then* $\mathrm{w.dim}_A X = \mathrm{r.proj.dim}_A X$ *for any finitely generated right A-module X.*

Proof. Let X be a finitely generated right A-module, where A is a right Noetherian ring. Since $\mathrm{w.dim}_A X \le \mathrm{r.proj.dim}_A X$ always holds, we need only show that $\mathrm{w.dim}_A X \ge \mathrm{r.proj.dim}_A X$, i.e., if $\mathrm{Tor}^A_{n+1}(X,Y) = 0$ for all left A-modules Y, then $\mathrm{Ext}^{n+1}_A(X,B) = 0$ for all right A-modules B.

Since A is right Noetherian, there is a projective resolution of X in which every term is a finitely generated A-module. Indeed, since X is finitely generated, there is an exact sequence

$$0 \to K_0 \to P_0 \to X \to 0,$$

where P_0 is a finitely generated free module. Since A is right Noetherian, K_0 is finitely generated as well. So we can also form an exact sequence

$$0 \to K_1 \to P_1 \to K_0 \to 0,$$

where P_1 a finitely generated free module. The usual iteration gives a projective resolution in which each P_i is a finitely generated free module. Since $\mathrm{Tor}^A_{n+1}(X,Y) = 0$ for all left A-modules Y, we have $\mathrm{Tor}_1(\mathrm{Ker}\, d_{n-1}, Y) = \mathrm{Tor}^A_{n+1}(X,Y) = 0$ for all B. Therefore, by proposition 6.3.7, vol.I, $\mathrm{Ker}\, d_{n-1}$ is flat. Thus $\mathrm{Ker}\, d_{n-1}$ is a finitely generated flat right A-module. Then, by proposition 5.1.3, $\mathrm{Ker}\, d_{n-1}$ is projective. Therefore $\mathrm{Ext}^1_A(\mathrm{Ker}\, d_{n-1}, B) = 0$ for all right A-modules B, hence $\mathrm{Ext}^{n+1}_A(X,B) = 0$, by lemma 6.5.3, vol.I.

Theorem 5.1.15. *If A is a right Noetherian ring, then*

$$\mathrm{w.gl.dim}\, A = \mathrm{r.gl.dim}\, A.$$

Similarly, if A is a left Noetherian ring, then

$$\mathrm{w.gl.dim}\, A = \mathrm{l.gl.dim}\, A.$$

Proof. This follows from propositions 5.1.9, 5.1.13 and 5.1.14.

As a simple corollary from theorem 5.1.15 we obtain a famous result which was proved by M.Auslander (see [Auslander, 1955]).

Theorem 5.1.16 (M.Auslander). *If A is a two-sided Noetherian ring, then*

$$\mathrm{r.gl.dim}\, A = \mathrm{l.gl.dim}\, A.$$

5.2 STRUCTURE OF RIGHT ARTINIAN RIGHT SERIAL RINGS

Recall that a right A-module is called **serial** if it is decomposes into a direct sum of uniserial modules, that is, modules possessing a linear lattice of submodules. A ring A is said to be **right serial** if it is a right serial module over itself. Analogously one can define left serial rings. A ring which is both a right and left serial ring is called a **serial ring**.

Remark 5.2.1. T.Nakayama [Nakayama, 1941] has studied Artinian serial rings and he called them generalized uniserial rings. I.Murase proved a number of structure theorem for these rings and described most of them in terms of quasi-matrix rings over division rings (see [Murase, 1963a], [Murase, 1963b], [Murase, 1964]). Serial non-Artinian rings were studied and described by R.B.Warfield and V.V.Kirichenko. In particular, they gave a full description of the structure of serial Noetherian rings. Most of these results are described in chapters 12, 13 of vol.I of this book.

Let A be a right Artinian ring with Jacobson radical R. Then for this ring we can define the right quiver $Q(A)$ of A (see chapter 11, vol.I). Recall the definition. Let $P_1, P_2, \dots, P_n$ be all pairwise non-isomorphic principal right A-modules. Consider the projective cover of $R_i = P_iR$ $(i = 1, ..., s)$, which we shall denote by $P(R_i)$. Let $P(R_i) = \bigoplus_{j=1}^{s} P_j^{t_{ij}}$. We assign to the principal modules $P_1, \dots, P_s$ the vertices $1, \dots, s$ in the plane and join the vertex i with the vertex j by t_{ij} arrows. The so constructed graph is called the **right quiver of the ring** A and denoted by $Q(A)$. From the definition of a projective cover it follows that $Q(A) = Q(A/R^2)$.

Theorem 11.1.9, vol.I, gives us that if A is a right Artinian ring then the following conditions are equivalent:

1) A is an indecomposable ring;
2) A/R^2 is an indecomposable ring;
3) the quiver of A is connected.

By theorem 10.3.7, vol.I, a right serial ring is semiperfect. And, by proposition 12.1.1, vol.I, a right serial ring with nilpotent Jacobson radical is right Artinian. So we can define the right quiver $Q(A)$ of a right serial ring A with Jacobson radical R by the formula $Q(A) = Q(A/R^2)$. For short, we shall call it the **quiver** of A. A vertex i of a quiver Q will be called an **end vertex** (or **target vertex**), if there are no arrow that starts at this vertex.

Remark 5.2.2. If a ring A is left Artinian we can analogously construct the left quiver $Q'(A)$. Note that there is a bijection between the right principal A-modules and the left principal A-modules which is given by a mapping $\varphi : P_i \to P_i^*$, where $\varphi(P_i) = \mathrm{Hom}_A(P_i, A)$, and, moreover, $\mathrm{Hom}_A(P_j, P_i) \simeq \mathrm{Hom}_A(P_i^*, P_j^*)$. Therefore if we have an arrow from the vertex i to the vertex j in the quiver $Q(A)$, then there is an arrow from the vertex j to the vertex i in the quiver $Q'(A)$. In particular, both quivers are connected or disconnected simultaneously.

If A is a finite dimensional algebra over an algebraically closed field k, then the number of arrows from the vertex i to the vertex j in the quiver $Q(A)$ is equal to the number of arrows from the vertex j to the vertex i in the quiver $Q'(A)$. However, this is not true in general.

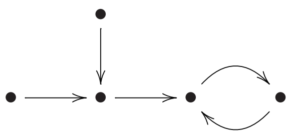
In this section we shall study the structure of right Artinian right serial rings in terms of quivers. If a right Artinian ring A is right serial, then every one of its right principal module is uniserial, i.e., the lattice of its submodules is linear.

Theorem 5.2.1. *A ring A is right serial if and only if each vertex of the quiver $Q(A)$ is the start of at most one arrow.*

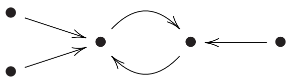
Proof. Let A be a right serial ring with Jacobson radical R, and let $P_1, P_2, \dots, P_n$ be all pairwise nonisomorphic principal right A-modules. Then $R_i = P_i R = 0$ or $R_i R$ is a unique maximal submodule in R_i. In the first case there is no arrow which starts at the vertex i, and in the second case $R_i / R_i R$ is a simple module, i.e., there is exactly one arrow which starts at the vertex i.

Conversely, suppose that each vertex of a quiver $Q(A)$ is the start of at most one arrow. Then for any $i \in VQ$ we have either $R_i = 0$ or $R_i R$ is the unique maximal submodule in R_i. We shall show by induction on k that if $P_i R^k \neq 0$, then $P_i R^{k+1}$ is a unique maximal submodule in $P_i R^k$. For $k = 1$ the statement is true. Suppose the statement is true for $k-1$, i.e., $P_i R^k$ is the unique maximal submodule in $P_i R^{k-1}$. Then $P_i R^k$ is a quotient module of R_i and it has exactly one maximal submodule which is $P_i R^{k+1}$. This equivalent to the fact that P_i is a uniserial module.

Example 5.2.1.
A ring with quiver



or

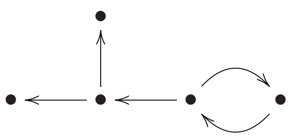


is right serial.

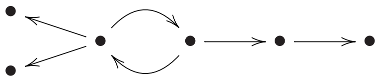
Taking into account remark 5.2.2 we have the analogous theorem for left serial rings:

Theorem 5.2.1*. *A ring A is left serial if and only if each vertex of the quiver $Q(A)$ is the end of at most one arrow.*

Example 5.2.2.
A ring with the quiver



or

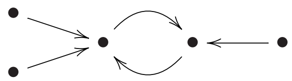


is left serial.

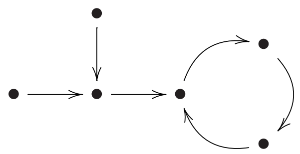
Corollary 5.2.2. *A ring A is right (left) serial if and only if A/R^2 is right (left) serial.*

Corollary 5.2.3. *An indecomposable ring A is right serial if and only if the quiver $Q(A)$ is a tree with a single end vertex or it contains only one cycle, and each arrow that does not form part of the cycle is oriented towards it.*

Example 5.2.3.
A ring with quiver



or



is an indecomposable right serial ring.

Corollary 5.2.3*. *An indecomposable ring A is left serial if and only if the quiver $Q(A)$ is a tree with a single start vertex or it contains only one cycle, and each arrow not part of the cycle is oriented away from it.*

Remark 5.2.3. From corollary 5.2.3 and corollary 5.2.3* it follows that the quiver of an indecomposable serial ring is a cycle or a chain. In this case all division rings entering into the Wedderburn-Artin decomposition of the ring A/R are isomorphic. If A is a finite dimensional algebra over a field k these conditions are also sufficient for A to be a serial ring. The following example shows that this is not true in general.

Example 5.2.2.
Consider the set A of all matrices of the form

$$\begin{pmatrix} f(x^2) & g(x) \\ 0 & h(x), \end{pmatrix}$$

where the f, g, h are arbitrary rational functions over a field k. Then A forms an infinite dimensional algebra for which the conditions above hold, but this algebra is not serial.

Definition. Let B be a semisimple Artinian ring. A B-bimodule V is called **right serial** if eV is a simple B-bimodule for any minimal idempotent $e \in B$.

(Note that it is sufficient to verify this for minimal idempotents from a fixed decomposition of the identity of B.)

Corollary 5.2.4. *Let $B = A/R$ and $V = R/R^2$. Then the ring A is right serial if and only if the B-bimodule V is right serial.*

Theorem 5.2.5. *If V is a right bimodule over a semisimple Artinian ring B, and $\mathcal{I}$ is an essential ideal in the tensor algebra $\mathfrak{T}_B(V)$, then the quotient ring $\mathfrak{T}_B(V)/\mathcal{I}$ is right serial. Moreover, any Wedderburn right serial ring is of this form.*

Proof. The proof follows immediately from corollary 5.2.4 and theorem 2.2.2.

Fix a semisimple Artinian ring B and a right serial B-bimodule V. We shall describe the essential ideals in $\mathfrak{T} = \mathfrak{T}_B(V)$. Let $U_1, \dots, U_s$ be all pairwise non-isomorphic simple B-modules; and let $e_1, ..., e_s$ be idempotents in B such that $U_i \simeq e_i B$. Let $T_i = e_i\mathfrak{T}$, let $\mathcal{J}$ be the fundamental ideal of $\mathfrak{T}_B(V)$ and let $l(X)$ denote the length of a B-module X. For any ideal $\mathcal{I} \subset \mathfrak{T}$, $Q_i = T_i\mathcal{I}$ is a submodule in T_i; moreover, $\mathcal{I}$ is an essential ideal if and only if $T_i\mathcal{J}^2 \supset Q_i \supset T_i\mathcal{J}^n$ for some n, i.e., $l(P_i/Q_i) = l_i < \infty$, and if $T_i\mathcal{J} \neq 0$ (which is equivalent to the inequality $V^{\otimes l_i} \neq 0$), then $l_i \geq 2$. From theorem 5.2.5 it follows that $T_i/T_i\mathcal{J}^n$ is a uniserial module, and therefore Q_i is uniquely defined by the number l_i such that $Q_i = T_i\mathcal{J}^{l_i}$.

Let $B \simeq U_1^{n_1} \oplus \dots \oplus U_s^{n_s}$, then $\mathfrak{T} \simeq T_1^{n_1} \oplus \dots \oplus T_s^{n_s}$. Therefore the right ideal $\mathcal{I} = Q_1^{n_1} \oplus \dots \oplus Q_s^{n_s}$ is defined by giving Q_i (or what is the same the number l_i). We shall determine for which conditions on l_i the ideal $\mathcal{I}$ is a two-sided ideal. Let $V_i = V^{\otimes l_i}$. If $l_i \neq 1$, then V_i is a simple B-module, that is, there is a unique number j such that $V_i \simeq U_j$. This is the unique number such that in the quiver $Q = Q(\mathfrak{T}/\mathcal{J}^2)$ there is an arrow from i to j.

Since $\mathcal{J} = V\mathfrak{T}$, we have $T_i\mathcal{I} = V_i\mathfrak{T} \simeq P_j$ and $T_i\mathcal{I}^2 = V_iV_j\mathfrak{T}$. Continuing this process we have that

$$Q_i = V_{i_1}V_{i_2}\dots V_{i_l}\mathfrak{T},$$

where $l = l_i$, and $i = i_1, i_2, \dots, i_l$ is the unique set of indexes such that there is an arrow from i_k to i_{k+1} in the quiver Q. Hence it follows that $B\mathcal{I} = \mathcal{I}$ and we only need to know when $V\mathcal{I} \subset \mathcal{I}$. Obviously, this is true if and only if $V_{i_0}V_{i_1}\dots V_{i_l} \subset Q_j$ for any index i_0 such that $V_{i_0} \simeq U_i$, that is, there is an arrow from i_0 to i. Since $Q_j = V_{i_0}V_{i_1}\dots V_{i_m-1}\mathfrak{T}$, where $m = l_{i_0}$, our inclusion is equivalent to the inequality $l_{i_0} - 1 \geq l_i$. Thus, we have proved the following statement.

Proposition 5.2.6. *Let B be a semisimple Artinian ring, let V be a right serial B-bimodule, and let $Q = Q(\mathfrak{T}(V)/\mathcal{J}^2)$.*

An essential ideal $\mathcal{I} \subset \mathfrak{T}(V)$ is uniquely defined by a set of numbers $\{l_1, \dots, l_s\}$ such that $l_i = 1$ if the vertex i is an end vertex in Q, and $2 \le l_i \le l_j + 1$ if there is an arrow from the vertex i to the vertex j where s is the number of vertices of the quiver Q, that is, the number of simple summands of B.

Suppose that $A = \mathfrak{T}(V)/\mathcal{I} \simeq \mathfrak{T}(V_1)/\mathcal{I}_1$, where V_1 is a right serial bimodule over a semisimple Artinian ring B_1, and $\mathcal{I}_1$ is an essential ideal in $\mathfrak{T}(V_1)$. Then $B \simeq A/R \simeq B_1$ and $V \simeq R/R^2 \simeq V_1$ (where $R = \mathrm{rad}A$). Moreover, if $\pi : \mathfrak{T}(V) \to A$ and $\pi_1 : \mathfrak{T}(V_1) \to A$ are the natural epimorphisms onto the quotient rings, then there are unique isomorphisms $\varphi : B \to B_1$ and $f : V \to V_1$ such that $\pi(b) \equiv \pi_1\varphi(b)(\mathrm{mod}\, R)$ and $\pi(v) \equiv \pi_1 f(v)(\mathrm{mod}\, R^2)$. Hence it follows that φ and f are compatible, that is, $f(bv) = \varphi(b)f(v)$, $f(vb) = f(v)\varphi(b)$ for any $b \in B$ and $v \in V$. Besides, if $U_1, \dots, U_s$ are the simple B-modules, and $U'_1, \dots, U'_s$ are the simple B_1-modules, while $T_1, ..., T_s$ and $T'_1, \dots, T'_s$ are the corresponding components of $\mathfrak{T}(V)$ and $\mathfrak{T}(V_1)$; $l_i = l(T_i/T_i\mathcal{I})$; $l'_i = l(T'_i/T'_i\mathcal{I})$; then $l_i = l'_i$ if $\varphi(U_i) \simeq U'_i$. (Obviously, $l_i = l'_i = l(P_i)$, where $P_i = \pi(T_i) \simeq \pi_1(T'_i)$).

Conversely, if $\varphi : B \to B_1$ and $f : V \to V_1$ are compatible isomorphisms, they induce a ring isomorphism $F : \mathfrak{T}(V) \to \mathfrak{T}(V_1)$ and an isomorphism of quivers: $\sigma : Q \to Q_1$, where $Q = Q(\mathfrak{T}(V)/\mathcal{J}^2)$, $Q_1 = Q(\mathfrak{T}(V_1)/\mathcal{J}_1^2)$. Let $\mathcal{I}$ be an essential ideal in $\mathfrak{T}(V)$ defined by the set of numbers $\{l_1, \dots, l_s\}$. Then $\mathcal{I}_1 = F(\mathcal{I})$ is an essential ideal in $T(V_1)$ defined by the set of numbers $\{l'_1, \dots, l'_s\}$, where $l'_i = l_j$ if $\sigma(j) = i$. This analysis together with theorem 5.2.5 gives a full description of Wedderburn right serial rings.

Theorem 5.2.7. *A right serial Wedderburn ring A is uniquely defined by a tuple $(B; V; l_1, \dots, l_s)$, where $B = A/R$ is a semisimple Artinian ring; $V = R/R^2$ is a right serial B-bimodule; $l_i = l(P_i)$ is a integer function on the quiver $Q(A) = Q(\mathfrak{T}(V)/\mathcal{J}^2)$, and, moreover, $l_i = 1$ if the vertex i is an end vertex of $Q(A)$, and $2 \le l_i \le l_j + 1$, if there is an arrow from the vertex i to the vertex j.*

The tuples $(B; V; l_1, \dots, l_s)$ and $(B_1; V_1; l'_1, \dots, l'_s)$ define isomorphic rings if and only if there is a pair of compatible isomorphisms $\varphi : B \to B_1$ and $f : V \to V_1$ such that $l'_i = l_j$ if $i = \sigma(j)$, where σ is the mapping of quivers $\sigma : Q(\mathfrak{T}(V)/\mathcal{J}^2) \to Q_1(\mathfrak{T}(V_1)/\mathcal{J}_1^2)$ induced by a pair of isomorphisms (φ, f).

Remark 5.2.4. Let A be a finite dimensional algebra over an algebraically closed field k. Then $B = B_1 \times B_2 \times \ldots \times B_s$, where $B_i \simeq M_{n_i}(k)$, and there is a unique simple B_i-B_j-bimodule U_{ij}, moreover, $V = \oplus U_{ij}^{t_{ij}}$. Thus B and V are defined by a quiver $Q(A)$ and a set of multipliers $\{n_1, \dots, n_s\}$. Therefore a finite dimensional right serial k-algebra is defined by a tuple $\{Q; n_1, \dots, n_s; l_1, \dots, l_s\}$, where Q is a quiver whose connected components satisfy corollary 5.2.3; $l_i = 1$ if the vertex i is an end vertex of $Q(A)$, and $2 \le l_i \le l_j + 1$, if there is an arrow from the vertex i to the vertex j.

Two tuples $\{Q; n_1, \ldots, n_s; l_1, \ldots, l_s\}$ and $\{Q_1; n'_1, \ldots, n'_s; l'_1, \ldots, l'_s\}$ define isomorphic algebras if and only if there is an isomorphism of quivers $\sigma : Q \to Q_1$ such that if $\sigma(j) = i$ then $l'_i = l_j$ and $n_i = n_j$.

5.3 QUASI-FROBENIUS RIGHT SERIAL RINGS

Recall that a ring A is quasi-Frobenius if A is an injective module over itself by theorem 4.12.17.

Proposition 5.3.1. *A right serial quasi-Frobenius ring is left serial, and therefore it is serial.*

Proof. This follows from the fact that in this case the functor $\mathrm{Hom}_A(*, A)$ is exact and it establishes a duality of categories of right and left finite generated A-modules by theorem 4.12.21.

Theorem 5.3.2. *An Artinian right serial ring A is quasi-Frobenius if and only if*

1. $Q(A) \simeq Q'(A)$ *and both are a disjoint union of cycles;*
2. $l_1 = \ldots = l_s$, *where* $l_i = l(P_i)$.

Proof. Let A be an Artinian right serial ring A. Without loss of generality we can assume that A is an indecomposable ring. Suppose A is a quasi-Frobenius ring, then, by proposition 5.3.1, it is a serial ring. Therefore, by theorem 12.1.12, vol.I, the quiver $Q(A)$ of A is a cycle or a chain. Thus, by theorem 4.13.2, $Q(A)$ is a cycle. Since A is quasi-Frobenius, all principal modules P_i are projective and injective simultaneously. Therefore any epimorphism $\varphi : P_j \to R_i$, where $R_i = \mathrm{rad}P_i$, is not a monomorphism. Thus,

$$l_j = l(P_j) > l(R_i) = l(P_i) - 1,$$

i.e., $l_j \geq l_i$. Taking into account that $Q(A)$ is a cycle, we obtain

$$l_1 \leq l_2 \leq \ldots \leq l_s \leq l_1,$$

so all l_i are equal.

Conversely, let A be a right Artinian and right serial ring, and suppose that conditions 1 and 2 of the theorem both hold. Let $P = P_i$ be a principal right A-module, and let M_k be its unique submodule such that $l(P/M_k) = k$ (clearly, $M_k = PR^k$, where $R = \mathrm{rad}A$). For convenience, we write $P_{s+1} = P_1$, $P_{s+2} = P_2$, etc. Then $P(M_1) \simeq P_{i+1}$ and thus M_2 is an epimorphic image of R_{i+1}; whence $P(M_2) \simeq P_{i+2}$. In general, $P(M_k) \simeq P_{i+k}$ for $M_k \neq 0$. In particular, $\mathrm{soc}P = M_{l-1}$ and thus $P(\mathrm{soc}P_i) = P_{i+l-1}$. It is clear that the modules P_{i+l-1} are non-isomorphic for $i = 1, 2, \ldots, s$. Therefore the socle of a principal right A-module P_i is simple and for $P_i \not\simeq P_j$, $\mathrm{soc}P_i \not\simeq \mathrm{soc}P_j$. By theorem 4.5.2, A is a quasi-Frobenius ring, as required.

Remark 5.3.1. For finite dimensional algebras instead of an isomorphism $Q(A) \simeq Q'(A)$ it is sufficient to require that $A/R \simeq M_{n_1}(D_1) \times \ldots \times M_{n_s}(D_s)$. In general this property follows from condition 1, but not conversely.

Remark 5.3.2. Let k be an algebraically closed field. Then a finite dimensional quasi-Frobenius right serial k-algebra A is defined by a tuple

$$\{Q; m; n_1, \ldots, n_s\},$$

where Q is a disjoint union of cycles. Moreover, if V is the bimodule defined by $\{Q; m; n_1, \ldots, n_s\}$, then $A \simeq \mathfrak{T}(V)/\mathcal{J}^m$.

5.4 RIGHT HEREDITARY RIGHT SERIAL RINGS

Recall that a ring A is called **right** (resp. **left**) **hereditary** if each right (resp. left) ideal is projective. If a ring A is both right and left hereditary, we say that A is a **hereditary ring**. In section 5.5, vol.I, we have studied some properties of right hereditary rings. The main properties of right hereditary rings that follow from theorem 5.5.6, vol.I and propositions 6.5.5 and 6.6.6, vol.I, can be formulated as the following two statements:

Theorem 5.4.1. *Let A be a right hereditary ring. Then*
1. $\text{proj.dim}_A M \leq 1$ *for any right A-module M.*
2. $\text{Ext}^n_A(X, Y) = 0$ *for all right A-modules X, Y and all $n \geq 2$.*

Theorem 5.4.2. *The following conditions are equivalent for a ring A*
1. *A is is a right hereditary ring.*
2. *Any submodule of a right projective A-module is projective.*
3. $\text{r.gl.dim}\, A \leq 1$.

Proposition 5.4.3. *Let A be a quasi-Frobenius ring. Then A is right hereditary if and only if A is semisimple.*

Proof. Since A is a right hereditary ring, $\text{r.gl.dim}\, A \leq 1$, by proposition 6.6.6, vol.I. Since A is a quasi-Frobenius ring, $\text{gl.dim}\, A = 0$ or $\text{gl.dim}\, A = \infty$, by corollary 4.12.20. Since each quasi-Frobenius ring is a two-sided Artinian ring, we have $\text{r.gl.dim}\, A = \text{l.gl.dim}\, A$, by the Auslander theorem 5.1.16. Therefore, $\text{gl.dim}\, A = 0$, i.e., A is semisimple.

The converse statement is trivial.

Recall that an indecomposable projective right module over a semiperfect ring A is called a **principal right module**. Note that any right Artinian ring is right Noetherian and semiperfect.

Proposition 5.4.4. *Let A be a right Artinian ring. Then the following conditions are equivalent for the ring A:*

1) *A is a right hereditary ring;*
2) *every submodule of a projective right A-module is projective;*
3) *every submodule of a principal right A-module is projective;*
4) rad A *is a projective right A-module.*

Proof.
1) $\Longleftrightarrow$ 2). This follows from theorem 5.5.6, vol.I.
2) $\Rightarrow$ 3) is trivial.
3) $\Rightarrow$ 4). Let $A = P_1 \oplus P_2 \oplus \ldots \oplus P_n$ be the decomposition of the right regular module A_A into a direct sum of principal right A-modules. Then, by proposition 3.4.3, vol.I, $\operatorname{rad} A = R_1 \oplus R_2 \oplus \ldots \oplus R_n$, where R_i is the radical of P_i. Since all the P_i are projective, $\operatorname{rad} A$ is projective as well.
4) $\Rightarrow$ 1). Let $R = \operatorname{rad} A$ be a projective right A-module, i.e., $\text{r.proj.dim}_A\, R = 0$. Consider the exact sequence $0 \to R \to A \to A/R \to 0$. Then, by proposition 6.5.1, vol.I, $\text{r.proj.dim}_A\,(A/R) = \text{r.proj.dim}_A\,(R) + 1 = 1$. We shall show that $\text{r.gl.dim}\, A \leq \text{r.proj.dim}_A\,(A/R) = 1$, which means, by proposition 6.6.6, vol.I, that A is right hereditary. Since A/R is a semisimple A-module, it is a direct sum of simple modules. So,

$$1 = \text{r.proj.dim}_A\,(A/R) = \max\{\text{r.proj.dim}_A\,(S) \ : \ S \ \text{ is a simple A-module}\}$$

Therefore $\text{r.proj.dim}_A\, S \leq 1$ for each simple A-module S. We shall prove by induction on $l(X)$ that for any finitely generated right A -module X $\text{r.proj.dim}_A\, X \leq 1$. If $l(X) = 0$ we are done. Suppose that $l(X) = m > 0$ and $\text{r.proj.dim}_A\, Y \leq 1$ for any finitely generated A-module Y with $l(Y) < m$. Since X is finitely generated, there is an exact sequence $0 \to S \to X \to X' \to 0$, where S is a simple A-module and $X' \simeq X/S$ is finitely generated. Then $l(X') < m$ and so $\text{r.proj.dim}_A\, X' \leq 1$. Then $\text{r.proj.dim}_A\, X \leq \max\{\text{r.proj.dim}_A S, \text{r.proj.dim}_A X'\} \leq 1$. Therefore, by theorem 5.1.13, $\text{r.gl.dim}\, A \leq 1$, hence A is right hereditary.

Proposition 5.4.5. *If a right serial ring A is right hereditary, then the quiver $Q(A)$ does not contain oriented cycles.*

Proof. Let $Q = Q(A)$ be the quiver of a right hereditary and right serial ring A. If there is an arrow $\sigma \in AQ$ with $s(\sigma) = i$ and $e(\sigma) = j$, then there is a non-zero homomorphism $f_\sigma : P_i \to P_j$ and $\operatorname{Im}(f_\sigma) \subset \operatorname{rad}(P_j)$. Suppose $Q(A)$ contains an oriented cycle and let $p = \sigma_1\sigma_2 \ldots \sigma_m$ be a path of Q with start and end at the vertex i. Then $f = f_{\sigma_1} f_{\sigma_2} \ldots f_{\sigma_m}$ is a homomorphism from P_i to P_i. Since all P_i are indecomposable and projective, by lemma 5.5.8, vol.I, all homomorphisms f_{σ_k} are monomorphisms. So f is also a monomorphism, that is, $P_i \simeq \operatorname{Im}(f)$. But $\operatorname{Im}(f) \subset \operatorname{rad}(P_i)$ and $\operatorname{rad}(P_i) \neq P_i$, by proposition 5.1.8, vol.I. This contradiction proves the proposition.

Let Q be an arbitrary quiver without cycles with an adjacency matrix (t_{ij}). We define on Q an integral function $d_i = 1$ for any end vertex and otherwise

$d_i = \sum t_{ij} d_j + 1$. This function is well-defined. Denote by $l(M)$ the length of a module M.

Theorem 5.4.6. *A right Artinian right serial ring A is right hereditary if and only if the quiver $Q(A)$ does not contain oriented cycles and $l_i = l(P_i) = d_i$ for each principal right A-module P_i.*

Proof. Let A be a right Artinian right serial ring. If A is a hereditary ring, then, by proposition 5.4.5, $Q(A)$ does not contain oriented cycles. For any principal right A-module P_i, $R_i = \operatorname{rad}(P_i)$ is a projective A-module, as well. Therefore $R_i \simeq P(R_i) = \bigoplus_j P_j^{t_{ij}}$ and $l_i = \sum t_{ij} l_j + 1$. If $i \in Q(A)$ is an end vertex, then P_i is a simple module, and so $l_i = 1$. Hence it follows that $l_i = d_i$.

Conversely, let A be a right Artinian right serial ring and let the quiver $Q(A)$ have no oriented cycles and $l_i = d_i$ for all i. Then $l(R_i) = l_i - 1 = \sum t_{ij} l_j + 1 = l(P(R_i))$, therefore $R_i \simeq P(R_i)$ is a projective module. Therefore $R = \operatorname{rad} A$ is also a projective module, since it is a direct sum of projective modules P_i. Thus A is a right hereditary ring, by proposition 5.4.4.

Corollary 5.4.7. *A right Artinian right serial ring A is right hereditary if and only if $Q(A)$ is a disjoint union of trees with only one end vertex and $l_i = l(P_i)$ is one more than the length of the (unique) path from the vertex i to the end vertex (for all i).*

Theorem 5.4.6 and corollary 5.4.7 give a full description of right Artinian right serial hereditary rings. Moreover, theorem 5.4.6 and a simple calculation of the length of the principal $\mathfrak{T}(V)$-modules give a description of all Wedderburn hereditary rings.

Corollary 5.4.8. *A right hereditary Wedderburn ring A is isomorphic to the tensor algebra $\mathfrak{T}_B(V)$, where $B = A/R$; $V = R/R^2$. Conversely, if the tensor algebra $\mathfrak{T}_B(V)$ is a right Artinian ring, then it is right hereditary.*

Remark 5.4.1. Let k be an algebraically closed field. A hereditary finite dimensional right serial k-algebra is defined by a tuple $\{Q; n_1, \dots, n_s\}$, where Q is a disjoint union of trees with only one end vertex. It is isomorphic to the tensor algebra $\mathfrak{T}(V)$, where V is the associated bimodule.

5.5 SEMIPRIME RIGHT SERIAL RINGS

In this section all ideals will be two-sided ideals.

Lemma 5.5.1. *A semiprime local right serial ring A is prime.*

Proof. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two nonzero ideals from the ring A such that $\mathcal{I}_1\mathcal{I}_2 = 0$. Since A is right serial ring, we can assume that $\mathcal{I}_1 \supset \mathcal{I}_2$. But then $\mathcal{I}_2^2 = 0$ and from the semiprimality of A it follows that $\mathcal{I}_2 = 0$. A contradiction.

Theorem 5.5.2. *A semiprime right serial ring is a direct product of prime right serial rings.*

Proof. Since a right serial ring is semiperfect, the proof immediately follows from lemma 5.5.1, proposition 9.2.13, vol.I, and theorem 14.4.6, vol.I.

Recall that a module M is called **distributive** if

$$K \cap (L + N) = K \cap L + K \cap N$$

for all submodules K, L, N. A module is called **semidistributive** if it is a direct sum of distributive modules. A ring A is called **right** (**left**) **semidistributive** if the right (left) regular module A_A (${}_AA$) is semidistributive. A right and left semidistributive ring is called **semidistributive**. Obviously, every uniserial module is a distributive module and every serial module is a semidistributive module. We write SPSDR-ring for a semiperfect right semidistributive ring and SPSD-ring for a semiperfect semidistributive ring.

Theorem 5.5.3. *A prime hereditary SPSDR-ring A is right serial.*

Proof. One can assume that A is a reduced ring. Let $1 = e_1 + e_2 + \ldots + e_n$ be a decomposition of the identity of A into a sum of pairwise orthogonal local idempotents, and let $P_i = e_iA$ be a right principal A-module ($i = 1, 2, \ldots, n$). Suppose A is not right serial. Let $P = eA$ be an indecomposable projective right A-module which is not uniserial (where e is a local idempotent of A). Since A is a hereditary ring, there is a submodule $K \subseteq P$ such that $K = P_i \oplus P_j$ ($i \neq j$). Since A is right semidistributive, $K = eae_iA \oplus ebe_jA$, where $a, b \in A$. Obviously, $eae_iA = (eae_iAe_1, eae_iAe_2, \ldots, eae_iAe_n)$ and $ebe_jA = (ebe_jAe_1, ebe_jAe_2, \ldots, ebe_jAe_n)$. Since we have a direct sum of submodules, $eae_iA \cap ebe_jA = 0$. By theorem 14.2.1, vol.I, we have $eae_iAe_k = 0$ or $ebe_jAe_k = 0$ This is a contradiction with the assumption that A is prime. This proves that A is right serial.

Let $\mathcal{O}$ be a discrete valuation ring with radical $R = \pi\mathcal{O} = \mathcal{O}\pi$, where $\pi \in \mathcal{O}$ is a prime element, and skew field of fractions D.

Consider a subring $\Lambda \subset M_n(D)$ of the following form

$$\Lambda = \begin{pmatrix} \mathcal{O} & \pi^{\alpha_{12}}\mathcal{O} & \ldots & \pi^{\alpha_{1n}}\mathcal{O} \\ \pi^{\alpha_{21}}\mathcal{O} & \mathcal{O} & \ldots & \pi^{\alpha_{2n}}\mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \pi^{\alpha_{n1}}\mathcal{O} & \pi^{\alpha_{n2}}\mathcal{O} & \ldots & \mathcal{O} \end{pmatrix},$$

where the α_{ij} are integers such that $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ for all i, j, k and where $\alpha_{ii} = 0$ for each i. Such a ring is two-sided Noetherian semiperfect semidistributive prime ring, and $M_n(D)$ is its classical ring of fractions.

We shall use the following notation: $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$, where $\mathcal{E}(\Lambda) = (\alpha_{ij})$ is the exponent matrix of the ring Λ.

Theorem 5.5.4. *The following conditions for a right Noetherian semiperfect semiprime and semidistributive ring A are equivalent:*

(a) *the ring A is right serial;*

(b) *the ring A is two-sided hereditary;*

(c) *the ring A is serial;*

(d) *the ring A is Morita-equivalent to a direct product of skew-fields and rings of the form $H_{s_j}(\mathcal{O}_j)$ $(j = 1, 2, \ldots, n)$.*

Proof. It is obvious, that in all cases we can consider that the ring A is reduced and indecomposable.

$(a) \Rightarrow (b)$. By theorem 14.5.1, vol.I, the ring A can be considered to be a division ring or a ring of the form $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$. Obviously, a division ring is a hereditary ring. Since Λ is a reduced ring, the matrix $\mathcal{E}(\Lambda)$ has no symmetric zeros, therefore the first row corresponding to the first indecomposable projective Λ-module P_1 can be made zero. The module $P_1 R$ is projective (it contains a unique maximal submodule since the ring Λ is right serial) and so it coincides with $P_2 = (\alpha_{21}, 0, \ldots, 0) = (1, 0, \ldots, 0)$; the module $P_2 R = P_3 = (1, 2, 0, \ldots, 0)$ and continuing this process we obtain that $P_{s-1} R = P_s = (1, 1, \ldots, 1, 0)$. Therefore the matrix $\mathcal{E}(\Lambda)$ coincides with a matrix of the following form:

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & 0 & 0 \\ 1 & 1 & 1 & \ldots & 1 & 0 \end{pmatrix}$$

Thus, A is two-sided hereditary, by corollary 12.3.7, vol.I.

$(b) \Rightarrow (c)$ follows from theorem 14.5.1, vol.I, and theorem 5.5.3.

$(c) \Rightarrow (d)$ follows from theorem 14.5.1, vol.I, and corollary 12.3.7, vol.I.

$(d) \Rightarrow (a)$ follows from theorem 14.2.1, vol.I, and corollary 12.3.7, vol.I.

Example 5.5.1.

Let $\mathbf{Z}_{(p)}$ be the ring of p-integral numbers (p is a prime number), i.e., $\mathbf{Z}_{(p)} = \{\frac{m}{n} \in \mathbf{Q} : (n, p) = 1\}$, and let $\mathbf{F}_p = \mathbf{Z}_{(p)}/p\mathbf{Z}_{(p)}$ be the field consisting of p elements. Consider the ring A of 2×2-matrices of the following form:

$$A = \begin{pmatrix} \mathbf{F}_p & \mathbf{F}_p \\ 0 & \mathbf{Z}_{(p)} \end{pmatrix}$$

(where $\mathbf{F}_p$ is considered as an $\mathbf{Z}_p$-module via the canonical quotient map $\mathbf{Z}_p \to \mathbf{F}_p$).

It is easy to see that

$$R = \operatorname{rad} \mathrm{A} = \begin{pmatrix} 0 & \mathbf{F}_p \\ 0 & p\mathbf{Z}_{(p)} \end{pmatrix}$$

and

$$R^2 = \begin{pmatrix} 0 & 0 \\ 0 & p^2\mathbf{Z}_{(p)}. \end{pmatrix}$$

So

$$R/R^2 = \begin{pmatrix} 0 & \mathbf{F}_p \\ 0 & p\mathbf{Z}_{(p)}/p^2\mathbf{Z}_{(p)} \end{pmatrix}$$

and, by the Q-Lemma, the quiver $Q(A)$ of A is the two-pointed quiver with the adjacency matrix

$$[Q(A)] = \begin{pmatrix} 0 & 1 \\ 0 & 1. \end{pmatrix}$$

Therefore, by theorem 14.2.1, vol.I, A is a right serial and left semidistributive ring. So there is no analogue of the decomposition theorem for serial Noetherian rings (see theorem 12.3.8, vol.I) even for Noetherian right serial and left semidistributive rings with a two-pointed quiver.

5.6 RIGHT SERIAL QUIVERS AND TREES

Definition. A quiver Q is called **right serial** if each of its vertices is the start of at most one arrow.

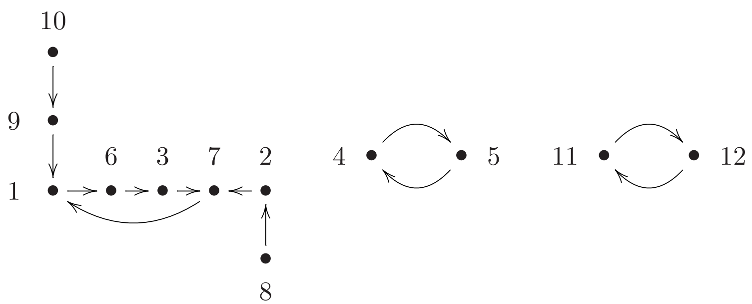
Examples 5.6.1.

1. The quiver of a right serial ring is right serial.

2. Let $N_n = \{1, 2, \dots, n\}$ and consider a map $\varphi : N_n \to N_n$. We represent φ as a right serial quiver Q_φ by drawing arrows from i to $\varphi(i)$. For example, the map

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 6 & 7 & 7 & 5 & 4 & 3 & 1 & 2 & 1 & 9 & 12 & 11 \end{pmatrix}$$

is represented by the following right serial quiver Q_φ:

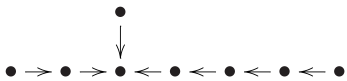


Quivers which come from maps in this way have the property that each vertex is the start of precisely one arrow. They are therefore right serial.

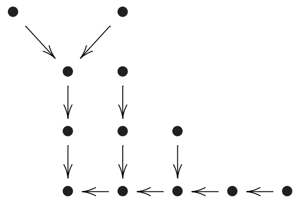
3. The following quivers are right serial:

(a)

(b)



(c)



In order to describe all right serial quivers we introduce the following definitions.

A **circuit of a quiver** Q is a sequence of pairwise distinct vertices $\{i_1, i_2, \dots, i_t\}$ and a sequence of arrows $\{\sigma_1, \sigma_2, \dots, \sigma_t\}$ such that each σ_k goes from i_k to i_{k+1} or from i_{k+1} to i_k (where $i_{t+1} = i_1$).

Note that it is possible that $t = 1$. Of course, every cycle is a circuit, but the converse is not true. A connected quiver without circuits we shall call a **tree**.

A **root** of a tree is a vertex which is not a start vertex of any arrow in the given tree. It is unique, see below.

Theorem 5.6.1. *A connected right serial quiver Q is either a tree with a single root, or a quiver with a unique circuit which is a cycle such that after deleting all arrows of this cycle, the remaining quiver is a disconnected union of trees whose roots are the vertices of the cycle. Conversely, every such a quiver is right serial.*

Proof. Assume that Q is a right serial quiver. Let $\{i_1, i_2, \dots, i_t\}$, $\{\sigma_1, \dots, \sigma_t\}$ be a circuit of Q. Assume that $\sigma_1 : i_2 \to i_1$. Then $\sigma_2 : i_2 \to i_3$ is impossible and thus $\sigma_2 : i_3 \to i_2$. Similarly, $\sigma_3 : i_4 \to i_3$, ..., $\sigma_t : i_1 \to i_t$, and thus the sequence $\{\sigma_t, \sigma_{t-1}, \dots, \sigma_1\}$ is a cycle (if $\sigma_1 : i_1 \to i_2$ then $\{\sigma_1, \sigma_2, \dots, \sigma_t\}$ is a cycle).

If there is no circuit in Q, then Q is a tree. Let i be a root of Q and let VQ_1 be the non-empty set of all vertices of Q from which there is a path to i. Similarly, denote by VQ_2 the set of all vertices from which there is a path to any other root of Q. Obviously, $VQ_1 \cap VQ_2 = \varnothing$ and $VQ = VQ_1 \cup VQ_2$. Therefore, since Q is connected, $VQ_2 = \varnothing$. So Q has a unique root.

Now let $\{\sigma_1, \sigma_2, \dots, \sigma_t\}$ be the cycle of Q. Observe that $\sigma_k : i_k \to i_{k+1}$ (with $i_{t+1} = i_1$) is the single arrow starting at i_k. Denote by VQ_k the set of all vertices of Q from which there is a path to i_k not containing any arrow of the cycle. In particular, each $i_k \in VQ_k$ (the trivial path ε_k). Let $VQ_0 = VQ \setminus \bigcup\limits_{k=1}^{t} VQ_k$. Obviously, all VQ_k ($k = 1, \dots, t$) are pairwise disjoint. Denote by Q_k the subquiver of Q, whose vertices are in VQ_k. Since every $i \in VQ_k$, $i \neq i_k$, is a start vertex for a unique arrow σ whose end vertex is again in VQ_k, there are no cycles in Q_k. Obviously, there is no arrow with a start vertex in VQ_0 and an end vertex in one of the VQ_k. Since Q is connected, $VQ_0 = \varnothing$. Thus, deleting all arrows $\sigma_1, \sigma_2, \dots, \sigma_t$, we obtain a disjoint union of the subquivers Q_k each of which is a tree with a single root i_k.

Conversely, let Q be a quiver described above. First, if Q is a tree with a single root, then there is no vertex which is a start vertex of more than one arrow because there are no circuits in Q. It is easy to see that the same conclusion holds if there is a single cycle in Q and its complement is a disjoint union of trees with unique roots which are vertices of the cycle.

Remark 5.6.1. Let Q_φ be a right serial quiver constructed by means of a map $\varphi : N_n \to N_n$. Then Q_φ contains at least one cycle.

We shall use right serial quivers to prove the Cayley formula for the number of different trees on a vertex set $\{1, 2, \dots, n\}$.

We shall use the main notions of graph theory (see [Harary, 1969]). In what follows we consider undirected graphs without loops and multiple edges.

Definition. A connected graph without circuits is called a **tree**.

The following proposition is well-known and we state it without proof (see [Harary, 1969]. [Harary, 1973]).

Proposition 5.6.2. *Let T_n be an undirected graph with n vertices and m edges. Then the following conditions are equivalent:*

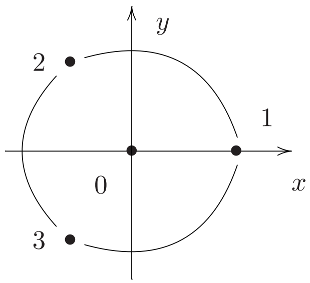
1. *T_n is a tree.*
2. *T_n is a connected graph and $m = n - 1$.*
3. *There is only one path in T_n from p to q for any vertices $p \neq q$.*

Let the vertices of a tree T_n be numbered by the numbers $1, \dots, n$. It will be suitable for us to set these vertices in the points corresponding to the vertices of n-sided regular polygon which are situated on a circumference with radius 1 on the complex plane. The first vertex is situated on the axis Ox and the next vertices are situated in the direction opposite to the clock hand.

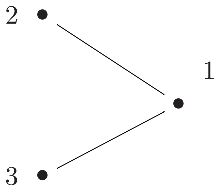
We shall assume that this numbering is fixed and say that T_n is a tree with vertex set $VT_n = \{1, \dots, n\}$.

Examples 5.6.1.

1. $n = 3$.



In this case we can construct 3 trees:



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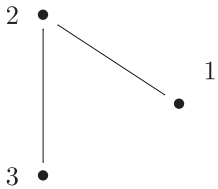

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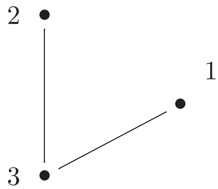
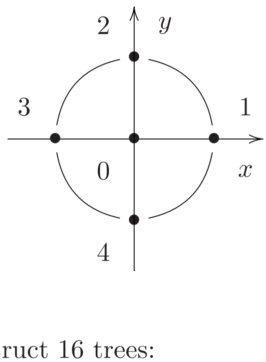
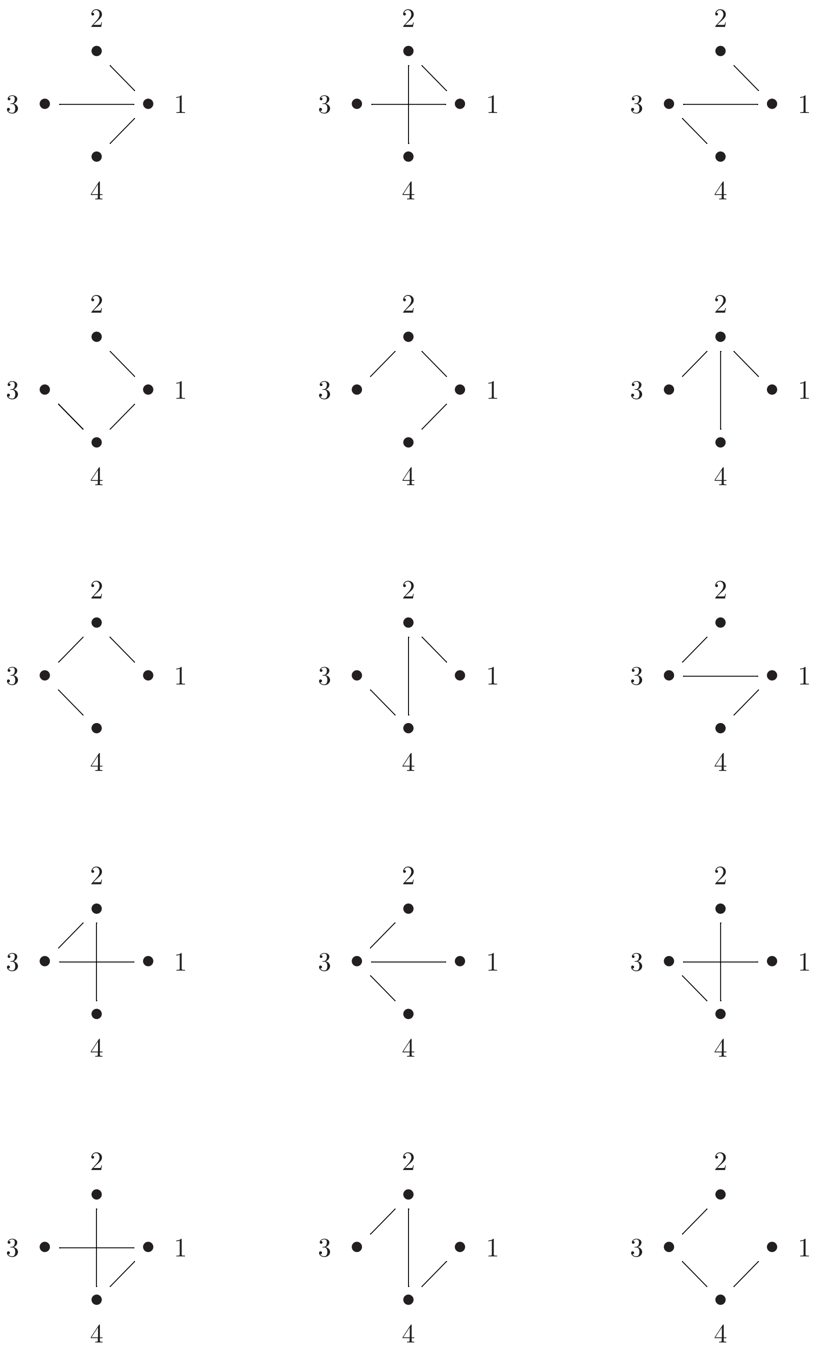


2. $n = 4$.



In this case we can construct 16 trees: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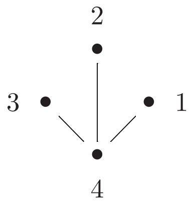


Theorem 5.6.3. (The Cayley formula). *For any $n \geq 2$ there are n^{n-2} different trees with vertex set $\{1, \dots, n\}$.*

Proof. Let T be a tree with vertex set $VT = \{1, \dots, n\}$. Fix two vertices x and y in T.

Case I. $x = y = k$.
For any vertex $i \in VT$ there is a single path that starts at i and ends at the vertex k. Let $\overset{i}{\bullet}\text{——}\overset{m}{\bullet}$ be the first edge in this path. Define $\varphi(T, k, k) = \varphi_k$ in the following way: $\varphi_k(k) = k$ and $\varphi_k(i) = m$ for each $i \neq k$. Obviously, φ_k is a well-defined map.

Case II. $x \neq y$ ($x = p$, $y = q$).
Let $M = \{p, i_1, \dots, i_{k-2}, q\}$ be the sequence of vertices of the unique path P starting at the vertex p and ending at the vertex q, where i_1 is the end of the edge $\overset{p}{\bullet}\text{——}\overset{i_1}{\bullet}$, i_2 is the end of the edge $\overset{i_1}{\bullet}\text{——}\overset{i_2}{\bullet}$, etc., and q is the end of the edge $\overset{i_{k-1}}{\bullet}\text{——}\overset{q}{\bullet}$. Write

$$\varphi(T, x, y)|_M = \begin{pmatrix} a_1 & a_2 & \dots & a_{k-1} & a_k \\ p & i_1 & \dots & i_{k-2} & q \end{pmatrix}$$

such that the numbers $a_1, a_2, \dots, a_{k-1}, a_k$ in the first row are the numbers $p, i_1, \dots, i_{k-2}, q$ in their natural order. Let $\varphi = \varphi(T, x, y)$. Consequently, write $\varphi(a_1) = p$, $\varphi(a_2) = i_1$, …, $\varphi(a_{k-1}) = i_{k-2}$, $\varphi(a_k) = q$. And for all remaining vertices we define $\varphi(i) = m$, where $\overset{i}{\bullet}\text{——}\overset{m}{\bullet}$ is the first edge in the unique path starting at the vertex i and ending at the vertex p.

Therefore, a tree T with vertex set $\{1, \dots, n\}$ defines n^2 maps $\varphi : N_n \to N_n$.

Conversely, let $f : N_n \to N_n$ be any map. We shall represent f as a right serial quiver Q_f by drawing arrows from i to $f(i)$.

Let $Q_f^{(k)}$ be a connected component of Q_f. By remark 5.6.1, there is a unique cycle in $Q_f^{(k)}$. Let $M \subset N_n$ be a union of the vertex sets of these cycles and

$M = \{a_1, \ldots, a_k : a_1 < a_2 < \ldots < a_k\}$. Obviously, M is a unique maximal subset of N_n such that the restriction of f onto M acts as a bijection on M. The tree with vertex set $\{1, \ldots, n\}$ corresponding to the map f is now constructed as follows: write

$$f\,|_M = \begin{pmatrix} a_1 & a_2 & \ldots & a_{k-1} & a_k \\ f(a_1) & f(a_2) & \ldots & f(a_{k-1}) & f(a_k) \end{pmatrix}$$

and draw

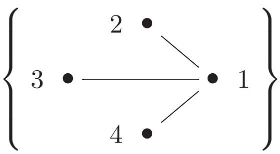
$$f(a_1), f(a_2), \ldots, f(a_{k-1}), f(a_k)$$

in this order as a path beginning at the vertex $f(a_1)$ and ending at the vertex $f(a_k)$. If the cycle is of length 1 (i.e., $a_k = a_1$), no edges are drawn. The remaining vertices are connected as they were connected in Q_f without orientation.

Now the proof follows from the fact that the set of all mappings from N_n into N_n contains n^n elements and the fact that a tree with vertex set $\{1, \ldots, n\}$ defines n^2 different mappings from N_n to N_n.

The algorithms used in the proof of theorem 5.6.3 are illustrated by the following examples.

Example 5.6.2.

For $n = 4$ consider the tree $T = T_4 =$ 

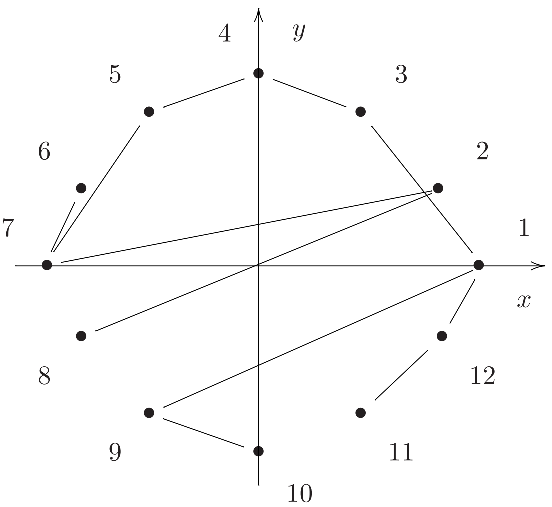
.

Construct the 16 mappings $N_4 = \{1, 2, 3, 4\}$ onto itself. It is easy to see that these mappings are given by the following tables (where on the intersection of the i-th row and the j-th column there is $\varphi(T, i, j)$).

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 4 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 4 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 3 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 1 & 3 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 1 & 4 \end{pmatrix}$

Example 5.6.3.

Consider the mapping $\varphi : N_{12} \to N_{12}$ for $N_{12} = \{1, 2, \dots, 12\}$ as in example 5.6.1(2). The tree corresponding to this map φ has the following form:



Note that this tree has leafs 6, 8, 10, 11 (where a leaf is defined as an end vertex of a tree).

5.7 CARTAN MATRIX FOR A RIGHT ARTINIAN RIGHT SERIAL RING

Throughout in this section A will denote a right Artinian ring with Jacobson radical R. Let $P_1, P_2, \ldots, P_n$ be all pairwise nonisomorphic principal right A-modules and let $U_1, U_2, \ldots, U_n$ be the set of simple right A/R-modules given by $P_i/P_iR \simeq U_i$. Suppose that $e_1, e_2, \ldots, e_n$ are pairwise orthogonal primitive idempotents corresponding to the principal right A-modules $P_1, P_2, \ldots, P_n$ so that $P_i = e_iA$ for $i = 1, \ldots, n$.

Let $\mathrm{mod}_r A$ be the category of finitely generated right A-modules. If $M \in \mathrm{mod}_r A$ then $l(M)$ denotes the composition length of M, and $c_i(M)$ denotes the number of factors in a composition series for M that are isomorphic to U_i.

Thus $c_i(M) = l((Me_i)_{e_iAe_i})$, the composition length of the right e_iAe_i-module Me_i.

Let $0 \to N \to M \to L \to 0$ be an exact sequence of A-modules. If M has a composition series then $l(M) = l(N) + l(L)$, by proposition 3.2.3, vol.I. Using the Jordan-Hölder theorem (see theorem 3.2.1, vol.I), we also obtain the following simple corollary:

Corollary 5.7.1. *Let $0 \to N \to M \to L \to 0$ be an exact sequence of A-modules. If N and L have finite length then M has also finite length and $l(M) = l(N) + l(L)$. And moreover $c_i(M) = c_i(L) + c_i(N)$*

An important role in the studying modules of finite length is played by a special group. Denote by $|\mathrm{mod}_r A|$ the set of all isomorphism classes of modules in $\mathrm{mod}_r A$ (including P_i and U_i, $i = 1, \ldots, n$). Denote by $i(M)$ the isomorphism class of a module $M \in \mathrm{mod}_r A$.

Definition. The **Grothendick group** of $\mathrm{mod}_r A$ is

$$K_0(\mathrm{mod}_r A) = \mathcal{F}/\mathcal{R},$$

where $\mathcal{F}$ is the free Abelian group with basis $|\mathrm{mod}_r A|$ and $\mathcal{R}$ is the subgroup of $\mathcal{F}$ generated by the expressions $i(M) - i(M') - i(M'')$ for all exact sequences

$$0 \to M' \to M \to M'' \to 0$$

in $\mathrm{mod}_r A$.

We denote by $[M]$ the coset of $i(M)$ with respect to $\mathcal{R}$, therefore

$$[M] = [M'] + [M'']. \tag{5.7.1}$$

Theorem 5.7.2. *Let $K_0(\mathrm{mod}_r A)$ be the Grothendick group of $\mathrm{mod}_r A$. Then it is a free Abelian group with basis $[U_1], \ldots, [U_n]$ and the map $[M] \to (c_1(M), \ldots, c_n(M))$ defines an isomorphism $K_0(\mathrm{mod}_r A) \simeq Z^{(n)}$, and for each $M \in \mathrm{mod}_r A$ we have*

$$[M] = \sum_{i=1}^{n} c_i(M)[U_i]. \tag{5.7.2}$$

Proof. Denote by $S = \{i(U_1), \ldots, i(U_n)\}$ the set of all isomorphism classes of simple A-modules. Let $\mathcal{SF}$ be a subgroup of $\mathcal{F}$ generated by S. Define a map $\alpha : \mathcal{SF} \to K_0(\mathrm{mod}_r A)$ by the formula $\alpha\left(i(U_i)\right) = [U_i]$. For any exact sequence

$$0 \to M' \to M \to M'' \to 0$$

in $mod_r A$ we have $c_i(M) = c_i(M') + c_i(M'')$ for any simple module U_i. Define a map $\beta : K_0(\mathrm{mod}_r A) \to \mathcal{SF}$ by the formula $\beta([M]) = \sum\limits_{i=1}^{n} c_i(M)[U_i]$ for any $M \in \mathrm{mod}_r A$. Then it is easy to check that $\beta\alpha = 1_{\mathcal{SF}}$ and $\alpha\beta = 1_{K_0(\mathrm{mod}_r A)}$. So α is an isomorphism. The theorem is proved.

Definition. The (**right**) **Cartan matrix** of A is the $n \times n$ matrix $C(A) = (c_{ij}) \in M_n(\mathbf{Z})$, where $c_{ij} = c_i(P_j) = c_i(e_j A)$ is the number of composition factors of P_j that are isomorphic to U_i. The integers c_{ij} are called the **Cartan invariants** of A. The left Cartan matrix of A is defined similarly.

Applying formula (5.7.2) to a principal right A-module P_j we obtain the following simple formula:

$$[P_j] = \sum_{i=1}^{n} c_{ij}[U_i]. \tag{5.7.3}$$

Clearly, the ring A has the same Cartan matrix as its basic ring. There is also the following obvious proposition:

Proposition 5.7.3. *If A is a decomposable right Artinian ring $A = A_1 \times A_2$, then its Cartan matrix is block diagonal*:

$$C(A) = \begin{pmatrix} C(A_1) & 0 \\ 0 & C(A_2) \end{pmatrix}$$

Therefore further on in this section we shall assume that A is basic and indecomposable.

Examples 5.7.1.

1. Let A be a local right Artinian ring. Then $C(A) = (m)$, where $m = l({}_A A)$.

2. The Cartan matrix of a right Artinian ring is the identity matrix if and only if A is semisimple.

3. Let A be the ring of upper triangular $n \times n$ matrices over a field k. Then $\det C(A) = 1$.

It is easy to check the following simple properties of the Cartan matrix:

Lemma 5.7.4. *Let A be a right Artinian ring with a set of pairwise orthogonal primitive idempotents $e_1, e_2, \ldots, e_n$ and Cartan matrix $C(A)$. Then*

1. $c_{ij} = l({}_{e_iAe_i}(e_iAe_j))$;
2. $l(e_iA) = \sum\limits_{j=1}^{n} c_{ij}$.

An important result in the theory of Cartan matrices has been obtained by S.Eilenberg, who in 1954 showed that the determinant of the Cartan matrix of an Artinian ring A of finite global dimension must be $+1$ or -1. We shall prove this interesting statement following K.R.Fuller.[1] This proof uses the Grothendick group of the category $\text{mod}_r A$. We shall call the determinant of the Cartan matrix the **Cartan determinant**.

Theorem 5.7.5. **(S.Eilenberg).** *Let A be a right Artinian ring. If* $\text{gl.dim}\, A < \infty$ *then* $\det C(A) = \pm 1$.

Proof. If P is a projective module, then there are non-negative integers m_k such that $P \simeq \bigoplus\limits_{k=1}^{n} P_k^{m_k}$, so $[P] = \sum\limits_{k=1}^{n} m_k[P_k]$. Consider projective resolutions

$$0 \to P_{jt} \to \ldots \to P_{j1} \to P_{j0} \to U_j \to 0$$

of the U_j, with P_{ji} projective, $j = 1, \ldots, n$, $i = 0, 1, \ldots, t$. Applying to these resolutions the formulas (5.7.1) and (5.7.3) we obtain integers h_{kj} such that

$$[U_j] = \sum_l (-1)^l [P_{jl}]$$

$$= \sum_{k=1}^{n} h_{kj}(-1)^l [P_k]$$

$$= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} c_{ik} h_{kj} \right) [U_i].$$

Thus, writing $H = (h_{ij}) \in M_n(\mathbf{Z})$, we see that $C(A)^{-1} = H \in M_n(\mathbf{Z})$, the ring of $n \times n$ integral matrices. Therefore $\det C(R) = \pm 1$.

Remark 5.7.1. The determinant of the right Cartan matrix is not necessarily equal to the determinant of the left Cartan matrix even in the case when a ring is two-sided Artinian (see [Fuller, 1992]). However they are equal to one-another when A is an Artin algebra (see [Nakayama, 1938]).

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[1] see [Fuller, 1992]

Proposition 5.7.6. *If A is an Artin algebra then* $\det C({}_AA) = \det C(A_A)$.

Proof. We shall write

$$C({}_AA) = [d_{ij}] \text{ and } C(A_A) = [c_{ij}]$$

to distinguish between the two Cartan matrices. Thus

$$d_{ij} = l({}_{e_iAe_i}(e_iAe_j)) \quad \text{and} \quad c_{ij} = l((e_jAe_i)_{e_iAe_i}).$$

Now if A is an Artin algebra over K and $t_i = l({}_K(e_iAe_i/e_iRe_i))$ then

$$t_i d_{ij} = l({}_K(e_iAe_j)) = t_j c_{ji},$$

so that

$$t_1 \dots t_n \det C({}_AA) = \det[t_i d_{ij}] = \det[t_j c_{ij}] = t_1 \dots t_n \det C(A_A)^T.$$

This proves the proposition.

In 1957 J.Jans and T.Nakayama[2] proved that if a ring A has finite global dimension and $R^2 = 0$, where $R = \mathrm{rad}A$, or A is a quotient ring of a hereditary ring, then the Cartan determinant is equal to 1. Since there are known examples of Artinian rings of finite global dimension with Cartan determinant equal -1, the problem was posed to settle what is now known as the **Cartan determinant conjecture**:

If A is an Artinian ring of finite global dimension then its Cartan determinant is equal to 1.

At the end of this section we shall prove this conjecture for right serial rings. This conjecture was first proved in 1985 by W.D.Burgess, K.R.Fuller, E.R.Voss and B.Zimmermann-Huisgen in their paper [Burgess, 1985].

It is easy to prove the following lemma.

Lemma 5.7.7. *Let A be a right serial right Artinian ring with a set of primitive pairwise orthogonal idempotents $e_1, e_2, \dots, e_n$ and corresponding Cartan matrix $C(A) = (c_{ij})$. Then*

$$l(e_iA) = \sum_{j=1}^{n} c_{ij}.$$

Definition. The sequence $e_1A, e_2A, \dots, e_nA$ is called a (**right**) **Kupisch series** if e_iA is a projective cover of $e_{i+1}R$ for $1 \le i \le n-1$, and e_nA is a projective cover of e_1R or $e_1R = 0$.

[2]see [Jans, Nakayama, 1957].

Lemma 5.7.8. *Let A be a right serial ring, and suppose that A has a Kupisch series $e_1A, \dots, e_nA$, all of whose members have the same composition length m. Write $m = an + r$ with $0 \le r < n$. Then C has the following form:*

$$c_{ij} = \begin{cases} a+1 & \textit{if } \ 0 \le j-i < r \ \textit{ or } \ n-r < i-j \le n \\ a & \textit{otherwise} \end{cases}$$

(in other words, for $r > 0$, the matrix C has entries a on its first $n-r$ subdiagonals and the last $n-r$ superdiagonals, and $a+1$ on the remaining diagonals.)

Proof. Denote by $[k]$ the least positive remainder of k modulo n. Then the sequence of composition factors of e_jA is U_j, $U_{[j-1]}$, $\dots$, $U_{[j-(n-1)]}$, U_j,$U_{[j-1]}$, $\dots$; it continues for m terms. Thus there are $a+1$ copies of the first r candidates in the list and a copies of the others.

Remark 5.7.2. Since for a quasi-Frobenius serial ring A all e_iA have the same composition length by theorem 5.3.2, the previous lemma gives the structure of Cartan matrices for quasi-Frobenius serial rings.

Remark 5.7.3. The matrices that occur in lemma 5.7.8 are a special type of circulant matrices.[3] By a **circulant matrix** of order n is meant a square matrix of the form

$$C = \mathrm{circ}(c_1, c_2, \dots, c_n) =$$
$$= \begin{pmatrix} c_1 & c_2 & \dots & c_n \\ c_n & c_1 & \dots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_2 & c_3 & \dots & c_1 \end{pmatrix}$$

The elements of each row of C are identical to those of the previous row, but are moved one position to the right and wrapped around. A circulant matrix can be also written in the form:

$$C = (c_{jk}) = (c_{[k-j+1]}),$$

where $[t]$ is the least positive remainder of t modulo n. Denote by (m, n) the greatest common divisors of two integers m and n. We shall use the following statement:

Proposition 5.7.9.[4] *Let $C = \mathrm{circ}(a, a, \dots, a; b, b, \dots, b)$ be a circulant matrix of degree $m + n$, where the first m elements of the first row are a and the last n elements are b. Then*

$$\det C = \begin{cases} (ma + nb)(a-b)^{m+n-1} & \textit{if } \ (m, n) = 1, \\ 0 & \textit{if } \ (m, n) \ne 1. \end{cases}$$

[3] see [Davis, 1979]

[4] see [Davis, 1979], Problem 27, p.81.

Taking this lemma into account there results the following as an immediately corollary of lemma 5.7.8.

Lemma 5.7.10. *Let C be a matrix as in lemma 5.7.8. Then*

$$\det C = \begin{cases} m & \text{if } (m,n) = 1, \\ 0 & \text{if } (m,n) \neq 1. \end{cases}$$

Lemma 5.7.11. *If A is a right serial right Artinian ring with a simple module of finite projective dimension, then A has a simple module of projective dimension ≤ 1.*

Proof. If A has a simple projective module, then everything is done. Otherwise A has a simple module U of finite projective dimension and so it has a finite projective resolution of the form:

$$0 \to e_i A \xrightarrow{\alpha} e_j A \to \ldots \to U \to 0$$

Let $\operatorname{Im} \alpha = e_j R^m$. Suppose we have a projective cover of $e_j R^{m-1}$:

$$e_k A \xrightarrow{g} e_j R^{m-1} \to 0$$

then g induces a split epimorphism $e_k R \xrightarrow{f} e_j R^m \simeq e_i A$. Since $e_k R$ is indecomposable, f is an isomorphism. Therefore $U_k = e_k A / e_k R$ has projective dimension 1.

Lemma 5.7.12. *Let A be a right serial right Artinian ring and $e = 1 - e_1$. Suppose A has a simple module $U_1 = e_1 A / e_1 R$ with* $\text{proj.dim}\, U_1 \leq 1$. *Then the right global dimension of A is finite if and only if the right global dimension of eAe is finite, and* $\text{r.gl.dim}\, A \leq \text{r.gl.dim}\, eAe + 2$.

Proof. First suppose that $\text{r.gl.dim}\, eAe$ is finite and $\text{proj.dim}\, U_1 \leq 1$. If $\text{proj.dim}\, U_1 = 0$, we have $e_1 R = 0$. If $\text{proj.dim}\, U_1 = 1$, then $e_1 R / e_1 R^2 \not\simeq U_1$. In both cases $\text{Ext}^1(U_1, U_1) = 0$.

Now let $i \neq 1$ and $\text{proj.dim}_{eAe}\, eU_i = m$. In order to verify that $\text{proj.dim}\, U_i \leq m + 2$, consider a projective resolution

$$\ldots \to P_m \xrightarrow{f_m} P_{m-1} \to \ldots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} U_i \to 0$$

of U_i, where all P_m are indecomposable projective. Since $\text{proj.dim}\, U_1 \leq 1$, we have

$$e_1 R \simeq \bigoplus_{i \neq 1} e_i A^{m_i}$$

Therefore $e_1 A e = e_1 R e$ is projective over eAe and so the sequence

$$\ldots \to eP_m \longrightarrow eP_{m-1} \to \ldots \to eP_1 \longrightarrow eP_0 \longrightarrow eU_i \to 0$$

is an eAe-projective resolution of the eAe-module eU_i. In particular, $f_m(eP_m)$ is projective and nonzero. Since eP_m is indecomposable, $f_m|_{eP_m}$ is a monomorphism. Set $T = \operatorname{soc} P_m$.

If $T \not\simeq U_1$, we have $0 \neq f_m(eT) \subset f_m(T)$, whence $f_m : P_m \to P_{m-1}$ is also a monomorphism and $\text{proj.dim}\, U_i \leq m$.

Let $T \simeq U_1$ and $L = \operatorname{soc} P_m \subset P_m$. Since $\operatorname{Ext}^1(U_1, U_1) = 0$, we have that $L/T \not\simeq U_1$. From $0 \neq f_m(eL) \subseteq f_m(L)$, it follows that $L \not\subseteq \operatorname{Ker} f_m$ and, consequently, $\operatorname{Ker} f_m \subseteq L$. From $\operatorname{Ker} f_m \neq L$ we conclude further that $\operatorname{Ker} f_m$ equals either T or 0. In the latter case we have $\text{proj.dim}\, U_i \leq m$ as above. In the former we obtain an exact sequence

$$\ldots \to e_1R \to e_1A \to P_m \xrightarrow{f_m} \ldots \to P_0 \to U_i \to 0,$$

which, in view of the projectivity of e_1R, yields $\text{proj.dim}\, U_i \leq m + 2$.

Conversely, finiteness of $\text{r.gl.dim}\, A$ implies finiteness of $\text{r.gl.dim}\, eAe$, since, as we have already remarked, multiplication of an A-projective resolution of a simple module U_i $(i \geq 2)$ by e results in an eAe-projective resolution of eU_i.

Lemma 5.7.13. *Suppose that the projective dimension of the simple right A-module $U_1 = e_1A/e_1R$ is ≤ 1 and set $e = 1 - e_1$. Then* $\det C(eAe) = \det C(A)$.

Proof. By hypothesis, e_1R is projective, and, since A is a semiperfect ring,

$$e_1R = (e_2A)^{m_2} \oplus \ldots \oplus (e_nA)^{m_n}$$

with $m_i \geq 0$.

Denoting the j-th column of the Cartan matrix C of A by

$$c_j = \begin{pmatrix} c_{1j} \\ \vdots \\ c_{nj}, \end{pmatrix}$$

we obtain

$$c_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sum_{j=2}^{n} m_j c_j.$$

Thus, subtraction of m_j times the j-th column from the first column for $j = 2, \ldots, n$ yields the matrix

$$\begin{pmatrix} 1 & c_{12} & \cdots & c_{1n} \\ 0 & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_{n2} & \cdots & c_{nn}, \end{pmatrix}$$

and consequently,

$$\det C = \det \begin{pmatrix} c_{22} & \cdots & c_{2n} \\ \vdots & \ddots & \vdots \\ c_{n2} & \cdots & c_{nn}. \end{pmatrix}$$

But the latter matrix is the Cartan matrix of eAe.

Theorem 5.7.14. *Given a right serial ring A, the determinant of its Cartan matrix is 1 if and only if its right global dimension is finite. In any case the determinant is nonnegative.*

Proof. We can assume that A is basic and indecomposable.

If $\text{r.gl.dim}\, A < \infty$, then the combination of lemmas 5.7.11-5.7.13 allows successive elimination of primitive idempotents corresponding to simple modules of projective dimension ≤ 1 until we are left with one idempotent. But, in this situation, from the finite dimension property of A it follows that A is semisimple, and hence $\det C(A) = 1$.

Assume that $\text{r.gl.dim}\, A = \infty$. By induction on the number n of primitive idempotents, we shall show that either $\det C(A) = 0$ or $\det C(A) > 1$. The case $n = 1$ is trivial. For the induction step we may start with $n > 1$ primitive idempotents e_i, none of which gives rise to a simple module e_iA/e_iR of projective dimension ≤ 1 (otherwise lemmas 5.7.12 and 5.7.13 would permit us to discard one idempotent and invoke the induction hypothesis). Since A is right serial, its quiver $Q(A)$ can be either a tree with a single end vertex or it contains a cycle or a loop, by corollary 5.2.3. And in any case each vertex of $Q(A)$ is a start of at most one arrow, by theorem 5.2.1. If the quiver of A is a tree then any root vertex corresponds to a projective module of length 1 which is impossible by induction assumption. Therefore the quiver contains a cycle or loop with vertices corresponding to idempotents $e_1, \dots, e_k$ $(k \geq 1)$, say. Since the composition series of $e_1A, \dots, e_kA$ contains only the simple modules $U_1, \dots, U_k$, the Cartan matrix of A has the block form

$$\begin{pmatrix} C_1 & X \\ 0 & C_2, \end{pmatrix}$$

where C_1 is a $k \times k$ matrix as treated in lemmas 5.7.8 and 5.7.10. Hence $\det C_1 = 0$ or $\det C_1 > 1$. Note that C_2 is the Cartan matrix of eAe, where $e = 1 - e_1 - e_2 - \ldots - e_k$. If $\text{r.gl.dim}\, eAe < \infty$, we have $\det C_2 = 1$ by the first part of the proof; otherwise the induction hypothesis yields that $\det C_2 \geq 0$. In either case, $\det C = (\det C_1)(\det C_2)$ is either zero or greater than 1.

Corollary 5.7.15. *Let A be an Artin algebra which is left or right serial, and let C be its right Cartan matrix. Then* $\det C = 1$ *if and only if* $\text{gl.dim}\, A < \infty$. *In any case* $\det C \geq 0$.

Proof. It is sufficient to note that in this case the left and right Cartan matrices have the same determinant by proposition 5.7.6. So the right and left global dimensions are equal.

5.8 NOTES AND REFERENCES

Most of the results included in section 5.1 were obtained by M.Auslander [Auslander, 1955].

Left serial algebras were studied in the papers: [Janusz, 1972], [Tachikawa, 1974]. The structure of two-sided Pierce decompositions of left Artinian left generalized uniserial rings was studied by G.Ivanov in his paper [Ivanov, 1974]

The main results of sections 5.2, 5.3 and 5.4 were obtained by N.Gubareni, Yu.A.Drozd and V.V.Kirichenko in the paper [Gubareni, 1976]. The results from section 5.5 are due to M.Khibina [Khibina, 2001].

By means of a slight modification of the statement given by S.Eilenberg in 1954 (see [Eilenberg, 1954]) there follows the main result which says that the Cartan matrix of an Artinian ring A of finite global dimension must be $+1$ or -1. The proof of theorem 5.7.5 in section 5.7 above follows K.R.Fuller [Fuller, 1972]. D. Zacharia has proved in the article [Zacharia, 1983] that if A is an Artinian algebra of global dimension 2, then the determinant of its Cartan matrix equals $+1$. This result is a generalization of previous results of P.Donovan, A.M.Freislich, and K.Igusa, G.Todorov, and G.Wilson. P.Donovan, A.M.Freislich have shown that if $A = kG$ is a group algebra of finite representation type then its Cartan determinant equals $+1$ (see [Donovan, Freislich, 1974]). G.Wilson has proved the same result for a finite dimensional algebra of finite representation type over an algebraically closed field (see [Wilson, 1982]). The same result was proved by K.Igusa, G.Todorov without making any assumption on the ground field.

The Cartan determinant conjecture asserts that if A is a right Artinian ring with finite global dimension then the determinant of its right Cartan matrix is equal to $+1$. The Cartan determinant conjecture for right Artinian right serial rings was first proved in 1985 by W.D.Burgess, K.R.Fuller, E.R.Voss and B.Zimmermann-Huisgen in the paper [Burgess, 1985].

S.Singh has studied modules over right serial rings. He proved that for an Artinian ring A with Jacobson radical R such that A/R is a direct product of matrix rings over finite-dimensional divisions rings the following statements are equivalent: 1) A is right serial; 2) every indecomposable injective left A-module is uniserial (see [Singh, 1997]).

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6. Tiled orders over discrete valuation rings

6.1 TILED ORDERS OVER DISCRETE VALUATION RINGS AND EXPONENT MATRICES

Exponent matrices appear in the theory of tiled orders over a discrete valuation ring. Many properties of such an order and its quiver are completely determined by its exponent matrix. We prove that an arbitrary strongly connected simply laced quiver with a loop in every vertex is realized as the quiver of a reduced exponent matrix.

Recall that a **tiled order** over a discrete valuation ring is a Noetherian prime semiperfect semidistributive ring A with nonzero Jacobson radical. By theorem 5.1.1, vol.I, any tiled order A is of the form

$$A = \begin{pmatrix} \mathcal{O} & \pi^{\alpha_{12}}\mathcal{O} & \dots & \pi^{\alpha_{1n}}\mathcal{O} \\ \pi^{\alpha_{21}}\mathcal{O} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \pi^{\alpha_{n1}}\mathcal{O} \\ \pi^{\alpha_{n1}}\mathcal{O} & \dots & \pi^{\alpha_{n2}}\mathcal{O} & \mathcal{O} \end{pmatrix}, \tag{6.1.1}$$

where $n \geq 1$, $\mathcal{O}$ is a discrete valuation ring with a prime element π, and where the α_{ij} are integers such that $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ for all i, j, k, and $\alpha_{ii} = 0$ for any $i = 1, \dots, n$.

Therefore $A = \sum\limits_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O}$, where the e_{ij} are the matrix units. If a tiled order A is reduced, then $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, \dots, n$, $i \neq j$.

The ring $\mathcal{O}$ is embedded into its classical division ring of fractions $\mathcal{D}$, and so that the tiled order A is the subset of all matrices $(a_{ij}) \in M_n(\mathcal{D})$ such that

$$a_{ij} \in \pi^{\alpha_{ij}}\mathcal{O} = e_{ii}Ae_{jj},$$

where the $e_{11}, \dots, e_{nn}$ are the matrix units of $M_n(\mathcal{D})$. It is clear that $Q = M_n(\mathcal{D})$ is the classical ring of fractions of A.

Since $eAe = \mathcal{O}$ is a discrete valuation ring for any primitive idempotent e of a tiled order A, we have the following simple lemma.

Lemma 6.1.1. *All principal endomorphism rings of a tiled order are isomorphic.*

Throughout this section, unless specifically noted, A denotes a tiled order with nonzero Jacobson radical R and classical ring of fractions Q.

We now recall the important notions of a band and a semilattice.

Let E be the set of idempotent elements of a semigroup S. For $e, f \in E$ we define $e \preceq f$ iff $ef = fe = e$. In this case we say that e is **under** f and f is **over** e. To see that $\preceq$ is a partial ordering of E, let $e, f, g \in E$. Then

(1) $e^2 = e$, and hence $e \preceq e$.

(2) If $e \preceq f$ and $f \preceq e$ then $ef = fe = e$ and $fe = ef = f$, whence $e = f$.

(3) If $e \preceq f$ and $f \preceq g$ then $ef = fe = e$ and $fg = gf = f$, whence $eg = (ef)g = e(fg) = ef = e$, and $ge = g(fe) = (gf)e = fe = e$. Hence $e \preceq g$.

We shall call $\preceq$ the **natural partial ordering** of E.

An element b of a partially ordered set X is called an **upper bound** of a subset Y of X if $y \preceq b$ for every y in Y. An upper bound b of Y is called a **least upper bound** (or **join**) of Y if $b \preceq c$ for every upper bound c of Y. If Y has a join in X, it is clearly unique. **Lower bound** and **greatest lower bound** (or **meet**) are defined dually. A partially ordered set X is called an **upper** (resp. **lower**) **semilattice** if every two-element subset $\{a, b\}$ of X has a join (resp. meet) in X; it follows that every finite subset of X has a join (or meet). The join (resp. meet) of $\{a, b\}$ will be denoted by $a \cup b$ (resp. $a \cap b$). A partially ordered set which is both an upper and lower semilattice is called a **lattice**. A lattice X is said to be **complete** if every subset of X has a join and a meet.

We recall that a **band** is a semigroup S in which every element is an idempotent. Thus $S = E$ if S is a band, and so the natural partial ordering ($a \preceq b$ if and only if $ab = ba = a$) applies to all of S.

Theorem 6.1.2. *A commutative band S is a lower semilattice with respect to the natural partial ordering of S. The meet $a \cap b$ of two elements a and b of S is just their product ab. Conversely, a lower semilattice is a commutative band with respect to the meet operation.*

Proof. That $\preceq$ is a partial ordering of S $(= E)$ was shown above. We must show that the product ab $(= ba)$ of two elements a and b of S is in fact the meet of $\{a, b\}$. From $(ba)a = ba^2 = ba$ and $(ab)b = ab^2 = ab$, we see that $ab \preceq a$ and $ab \preceq b$. Suppose $c \preceq a$ and $c \preceq b$. Then $(ab)c = a(bc) = ac = c$, and similarly $c(ab) = c$, whence $c \preceq ab$.

The converse is evident.

Definition. Let Q be the classical ring of fractions of both rings A and C. If $A \subset C$, the ring C is called an **overring** of A.

The following lemma is obvious:

Lemma 6.1.3. *Let $\mathcal{O}$ be a discrete valuation ring and D be its classical ring of fractions, which is a division ring. Then D is a uniserial right and left $\mathcal{O}$-module. If $X \subset D$ is a right $\mathcal{O}$-module ($X \neq D$), then $X = \pi^t\mathcal{O} = \mathcal{O}\pi^t$.*

Corollary 6.1.4. *If $\Delta \neq D$ is an overring of $\mathcal{O}$ then $\Delta = \mathcal{O}$.*

Proof. The ring Δ is an $\mathcal{O}$-bimodule. Since $\Delta \neq D$, we obtain $\Delta = \pi^t\mathcal{O} = \mathcal{O}\pi^t$. Consider $\Delta = \Delta \cdot \Delta = \pi^{2t}\mathcal{O} = \mathcal{O}\pi^{2t}$. By the Nakayama lemma $t = 0$.

Definition. Let A be a tiled order. A **right (left) A-lattice** is a right (left) A-module which is a finitely generated free $\mathcal{O}$-module.

In particular, all finitely generated projective A-modules are A-lattices.

Among all A-lattices we single out the so-called **irreducible** A-lattices. These are the A-lattices contained in the simple right (resp. left) Q-module U (resp. V). These lattices form a poset $S_r(A)$ (resp. $S_l(A)$) with respect to inclusion. As was shown in section 14.5, vol.I, any right (resp. left) irreducible A-lattice M (resp. N) lying in U (resp. in V) is an A-module with $\mathcal{O}$-basis $\pi^{\alpha_1}e_1, \dots, \pi^{\alpha_n}e_n$ with

$$\begin{cases} \alpha_i + \alpha_{ij} \geq \alpha_j, & \text{if } (\alpha_1, \dots, \alpha_n) \in S_r(A) \\ \alpha_j + \alpha_{ij} \geq \alpha_i, & \text{if } (\alpha_1, \dots, \alpha_n)^T \in S_l(A). \end{cases} \tag{6.1.2}$$

where the letter T stands for the transposition operation.

Such right (resp. left) modules always exist. For instance for a fixed chosen k take $\alpha_i = \alpha_{ki}$. Then

$$\alpha_i + \alpha_{ij} = \alpha_{ki} + \alpha_{ij} \geq \alpha_{kj} = \alpha_j.$$

Further if $(\alpha_1, \dots, \alpha_n)$ characterizes an irreducible right A-lattice where A is a tiled order as in (6.1.1), then

$$\begin{pmatrix} \pi^{\alpha_1} & 0 & \dots & \dots & 0 \\ 0 & \pi^{\alpha_2} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \pi^{\alpha_n} \end{pmatrix} A \begin{pmatrix} \pi^{-\alpha_1} & 0 & \dots & \dots & 0 \\ 0 & \pi^{-\alpha_2} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \pi^{-\alpha_n} \end{pmatrix}$$

is an isomorphic tiled order with all exponents ≥ 0. Thus, if desired, this additional property can always be assumed.

For our purposes, it suffices to consider a reduced tiled order A. In this case, the elements of $S_r(A)$ ($S_l(A)$) are in a bijective correspondence with integer-valued row vectors $\vec{a} = (\alpha_1, \dots, \alpha_n)$ (column vectors $\vec{a}^T = (\alpha_1, \dots, \alpha_n)^T$), where $\vec{a}$ and

$\vec{a}^T$satisfy the conditions (6.1.2). We shall write $M = (\alpha_1, \dots, \alpha_n)$, if $M \in S_r(A)$.

Let $\vec{b} = (\beta_1, \dots, \beta_n)$. The order relation $\vec{a} \preceq \vec{b}$ in $S_r(A)$ is defined as follows:

$$\vec{a} \preceq \vec{b} \iff \alpha_i \geq \beta_i \text{ for } i = 1, \dots, n.$$

Since A is a semidistributive ring, $S_r(A)$ and $S_l(A)$ are distributive lattices with respect to addition and intersection.

Proposition 6.1.5. *There exists only a finite number of irreducible A-lattices up to isomorphism.*

Proof. Let $A = \{\mathcal{O}, \mathcal{E}(A)\}$ be a tiled order with exponent matrix $\mathcal{E}(A) = (\alpha_{ij})$. Let $M = (\alpha_1, \dots, \alpha_n) \in S_r(A)$. Let $a = \min(\alpha_1, \dots, \alpha_n)$. Then $M_1 = (\alpha_1 - a, \dots, \alpha_n - a)$ is an irreducible A-lattice and $M_1 \simeq M$. Suppose that $\alpha_i = a$. Then $M_1 = (\beta_1, \dots, \beta_n)$, where the $\beta_1, \dots, \beta_n$ are non-negative and $\beta_i = 0$. Consequently, every irreducible A-lattice M is isomorphic to a lattice M_1 with at least one zero coordinate. We obtain from (6.1.2) that $0 \leq \beta_j \leq \alpha_{ij}$. So, the number of irreducible A-lattices of the form M_1 is finite. The proposition is proved.

Proposition 6.1.6. *All overrings of a tiled order A are tiled orders. They form a finite lower semilattice.*

Proof. Let $C \supseteq A$ be an overring of the tiled order A. If there is k such that $e_{kk}Ce_{kk} = D$ then $C = M_n(D)$. Therefore $\mathcal{O} \subseteq e_{ii}Ce_{ii} \neq D$ for $i = 1, \dots, n$ and, by lemma 6.1.3, $e_{ii}Ce_{ii} = \mathcal{O}$ and C is a tiled order. If C_1 and C_2 are overrings of A then $C_1 \cap C_2$ is an overring A. So all overrings of A form a lower semilattice $OR(A)$.

To show that $OR(A)$ is finite take any $C \in OR(A)$. Let $\mathcal{E}(C) = (c_{ij})$. Then every row $(c_{i1}, \dots, 0, \dots, c_{in})$ defines an irreducible A-lattice. By proposition 6.1.5 (or rather a variant of its proof), the number of such rows is finite. Therefore the semilattice $OR(A)$ is finite. The proposition is proved.

Using the properties of projective covers of finitely generated modules over semiperfect rings, one can characterize projective modules of the lattice $S_r(A)$ (resp. $S_l(A)$) in the following way:

Proposition 6.1.7. *An irreducible A-lattice is projective if and only if it contains exactly one maximal submodule.*

Proof. Let M be an irreducible A-lattice and suppose that M contains exactly one maximal submodule. Then $M/MR = U$, where U is a simple A-module and $P = P(U) = P(M)$ is an indecomposable principal A-module (where $P(M)$ stands for is a projective cover of the module M). Let $\varphi : P(M) \to M$ be the associated projection. Since M is an A-lattice, $\operatorname{Ker} \varphi = 0$. Therefore, $M \simeq P(M)$

is projective. Conversely, every indecomposable projective A-module (where A is a tiled order) is an irreducible A-lattice with exactly one maximal submodule.

We denote by $\mathcal{M}_r(A)$ (resp. $\mathcal{M}_l(A)$) the partially ordered subset of $S_r(A)$ (resp. $S_l(A)$), which is formed by all irreducible projective right (left) A-lattices.

If it is not necessary to distinguish the partially ordered sets (or lattices) formed by left or right modules, then we shall write them without the indices l and r.

Definition. Let $\mathcal{P}$ be an arbitrary poset. A subset of $\mathcal{P}$ is called a **chain** if any two of its elements are comparable. A subset of $\mathcal{P}$ is called an **antichain** if no two distinct elements of the subset are comparable. We shall denote a chain of n elements by CH_n and an antichain of n elements by ACH_n.

Recall that the maximal number of pairwise incomparable elements in $\mathcal{P}$ is called the **width** of $\mathcal{P}$ and denoted by $w(\mathcal{P})$. Obviously, if $w(\mathcal{P})$ is a finite number then it is equal to the number of elements in a maximal antichain of $\mathcal{P}$.

We introduce the notion of a strongly dependent subset C of an infinite countable poset $\mathcal{P}$ of finite width $w = w(\mathcal{P})$.

Definition. A subset $C \subset \mathcal{P}$ is **strongly dependent** if for any finite poset $S \subset \mathcal{P}$ there exists a representation $S = \bigcup\limits_{i=1}^{k} L_i$, where $L_1, \dots, L_k$ are pairwise disjoint chains ($k \le w$) such that the intersection $S \cap C$ is in one of chains $L_1, \dots, L_k$.

Remark that every strongly dependent set C is a chain. Indeed, let $x, y \in C$ be incomparable. Consider $S = \{x, y\}$. Obviously, $S \cap C = S$ is not a chain.

Theorem 6.1.8 (R.P.Dilworth).[1] *For a poset P with finite width $w(P)$ the minimal number of disjoint chains that together contain all elements of P is equal to $w(P)$.*

Remark 6.1.1. This theorem holds for any poset $\mathcal{P}$ of finite width. Here is a proof of this theorem for the countable case. To prove this theorem we use the following lemma.

Lemma 6.1.9. *Let $\mathcal{P}$ be a countable partially ordered set of finite width $w = w(P)$. Suppose there are m disjoint chains that together contain all elements of $\mathcal{P}$. Then $m \le w$.*

Proof (finite case). We use induction on the number of elements of $\mathcal{P}$. The start of the induction, $|\mathcal{P}| = 1$, is obvious.

Let L be a maximal chain in $\mathcal{P}$. Consider the poset $\mathcal{P} \setminus L$. Obviously, $w(\mathcal{P} \setminus L)$ can only be w or $w - 1$. If $w(\mathcal{P} \setminus L) = w - 1$, then, by the induction hypothesis, $\mathcal{P} \setminus L$ is a disjoint union of $s \le w - 1$ chains: $\mathcal{P} \setminus L = \bigcup\limits_{i=1}^{s} C_i$ and $\mathcal{P} = L \bigcup (\bigcup\limits_{i=1}^{w-1} C_i)$

[1] see [Dilworth, 1950]

is a representation of $\mathcal{P}$ as a disjoint union of $(s+1) \leq w$ chains. Now suppose that $w(\mathcal{P} \setminus L) = w$. Let $a_1, \dots, a_w$ be an antichain in $\mathcal{P} \setminus L$, and let c (d) be a maximal (minimal) element of L.

Define $A = \{x \in \mathcal{P} \ : \ x \geq a_i \text{ for some } i = 1, \dots, w\}$ and $B = \{y \in \mathcal{P} \ : \ y \leq a_i \text{ for some } i = 1, \dots, w\}$. Let $p \in \mathcal{P}$, then $p, a_1, \dots, a_w$ is not an antichain. So, $p \in A$ or $p \in B$ and $\mathcal{P} = A \cup B$. Now $|A| < |\mathcal{P}|$ and $|B| < |\mathcal{P}|$. Indeed, if $A = \mathcal{P}$ then $d \in A$ and there exists an i such that $d \geq a_i$. This contradicts the maximality of L. Analogously, if $B = \mathcal{P}$, then $c \in B$ and there exists a_i such that $c \leq a_i$. Again a contradiction. So, $|A| < |\mathcal{P}|$ and $|B| < |\mathcal{P}|$. Obviously, the $a_1, \dots, a_w$ are minimal elements in A and maximal elements in B. By induction, there exist disjoint chains $T_1, \dots, T_w$ and $S_1, \dots, S_w$ such that $A = \bigcup\limits_{i=1}^{w} T_i$ and $B = \bigcup\limits_{i=1}^{w} S_i$ It is possible to number $T_1, \dots, T_w$ and $S_1, \dots, S_w$ in such a way that $a_i \in T_i \cap S_i$ for $i = 1, \dots, w$. Obviously, $C_k = T_k \cup S_k$ is a chain for all $i = 1, \dots, w$ and $\mathcal{P} = \bigcup_{k=1}^{w} C_k$ is a disjoint union of w chains.

Proof (infinite countable case). We use induction on $w = w(\mathcal{P})$. The start of the induction, $w(\mathcal{P}) = 1$, is obvious.

Claim: there exists a maximal strongly dependent set. By assumption, $\mathcal{P}$ is a countable set. So, $\mathcal{P} = \bigcup\limits_{i=0}^{\infty} \{a_i\}$. Denote $C_0 = \{a_0\}$ and let $S \subset \mathcal{P}$ be finite. If $S \cap C_0 = \emptyset$, then $\emptyset \subset L$, where L is an arbitrary chain. Let $S \cap C_0 = a_0$ and $S = \bigcup\limits_{i=1}^{k} L_i$. Obviously, $a_0 \in L_i$ for some i. For every $k \geq 0$ let $C_{k+1} = C_k \cup \{a_k\}$, if $C_k \cup \{a_k\}$ is strongly dependent and $C_{k+1} = C_k$ otherwise. Then $C = \bigcup\limits_{i=0}^{\infty} C_i$ is a maximal strongly dependent set.

If $w(\mathcal{P} \setminus C) = w - 1$, then, according to the induction hypothesis, we can represent $\mathcal{P} \setminus C$ as a union of the $w - 1$ disjoint chains

$$\mathcal{P} \setminus C = \bigcup_{i=0}^{w-1} C_i$$

and $\mathcal{P} = (\bigcup\limits_{i=0}^{w-1} C_i) \cup C$ is a union of the w disjoint chains.

Let $w(\mathcal{P} \setminus C) = w$ and let $a_1, \dots, a_w$ be a maximal antichain. Then every set $C \cup \{a_i\}$ is not strongly dependent for $i = 1, \dots, w$. It means that for every $i = 1, \dots, w$ there exists a finite subset $S_i \subset \mathcal{P}$ such that $w(S_i \cap (C \cup \{a_i\})) = 2$. Consequently, $a_i \in S_i$ for $i = 1, \dots, w$.

Consider the finite set $S = \bigcup\limits_{i=1}^{w} S_i$. The set S may be presented as a union of w disjoint chains $S = \bigcup\limits_{i-1}^{w} K_i$ by the lemma in the finite case (which has been proved above). Renumbering the K_i's if needed we can assume that $S \cap C \subset K_1$. We can also assume that $a_1 \in K_1$. So, $w(S \cap (C \cup a_1)) = 1$ and we have a contradiction. The lemma is proved.

Proof of theorem 6.1.8. Let m be the minimal number of disjoint chains that together contain all elements of P, and let $w = w(P)$. So, there exist pairwise incomparable elements $x_1, \dots, x_w$. Obviously, any chain cannot contain two of these elements together. Consequently, $m \geq w$.

From lemma 6.1.9 it follows that $m \leq w$. Therefore $m = w$.

Definition. The width of $\mathcal{M}_r(A)$ is called the **width of a tiled order** A and it is denoted by $w(A)$. Obviously, $1 \leq w(A) \leq m$, where m is the number of elements of $\mathcal{M}_r(A)$.

Let $\mathcal{P}$ be an arbitrary partially ordered set. Then one can construct a new partially ordered set $\widetilde{\mathcal{P}}$, whose elements are the nonempty subsets of $\mathcal{P}$ that consist of pairwise incomparable elements, including the subsets consisting of only one element of $\mathcal{P}$. We shall introduce an order in $\widetilde{\mathcal{P}}$ in the following way. If $A, B \in \widetilde{\mathcal{P}}$, then $A \preceq B$ in $\widetilde{\mathcal{P}}$ if and only if for any $a \in A$ there exists $b \in B$ such that $a \preceq b$ in P. The poset $\mathcal{P}$ is naturally embedded in $\widetilde{\mathcal{P}}$: an element $a \in \mathcal{P}$ is mapped into the singleton set $\{a\}$.

Example 6.1.1.

If $\mathcal{P}$ is an antichain then $\widetilde{\mathcal{P}}$ is the poset of all non-empty subsets of $\mathcal{P}$ partially ordered by inclusion.

In particular if $\mathcal{P}$: $\overset{1}{\bullet} \qquad \overset{2}{\bullet}$ then

$$\widetilde{\mathcal{P}} : \begin{tikzcd} \{1,2\} \arrow[d] \arrow[dr] & \\ \{1\} & \{2\} \end{tikzcd}$$

;

If $\mathcal{P}$: $\overset{1}{\bullet} \quad \overset{2}{\bullet} \quad \overset{3}{\bullet}$ then

$$\widetilde{\mathcal{P}}: \quad \begin{tikzcd}
 & \{1,2,3\} \arrow[dl] \arrow[d] \arrow[dr] & \\
\{1,2\} \arrow[d] \arrow[dr] & \{1,3\} \arrow[dl] \arrow[dr] & \{2,3\} \arrow[dl] \arrow[d] \\
\{1\} & \{2\} & \{3\}
\end{tikzcd}$$

Proposition 6.1.10. *The set $\widetilde{\mathcal{M}_r}(A)$ is a lattice. There is a natural isomorphism of lattices $\widetilde{\mathcal{M}}_r(A)$ (resp. $\widetilde{\mathcal{M}}_l(A)$) and $S_r(A)$ (resp. $S_l(A)$), which is the identity on $\mathcal{M}_r(A)$ (resp. $\mathcal{M}_l(A)$).*

Proof. Let $M \in S_r(A)$. Since M is an irreducible A-lattice, there exists a projective cover of M:

$$P(M) \xrightarrow{\varphi} M \to 0$$

Let $P(M) = \bigoplus\limits_{i=1}^{m} P_i$, where the P_i are principal modules. The restriction of the homomorphism φ to P_i is either the zero map or a monomorphism, since P_i is an irreducible A-lattice. Consequently, any irreducible module M admits a representation $M = P_1 + \ldots + P_m$, where $P_i \in \mathcal{M}_r(A)$ and $P_i \not\subset P_j$ for $i \neq j$. Thus, the correspondence

$$\Theta : (P_1, \ldots, P_m) \to P_1 + \ldots + P_m$$

defines an epimorphism of the partially ordered set $\widetilde{M}_r(A)$ onto $S_r(A)$. Let $M = P_1 + \ldots + P_m = P_{i_1} + \ldots + P_{i_t}$. Then $P_1 \subseteq P_{i_1} + \ldots + P_{i_t}$. We may assume that $P_1 \simeq e_{11}A$, i.e., $P_1 = (a, \alpha_{12} + a, \ldots, \alpha_m + a)$ for some $a \in \mathbf{Z}$.

Obviously, if $L_1 = (\alpha_1, \ldots, \alpha_n)$ and $L_2 = (\beta_1, \ldots, \beta_n)$ are irreducible A-lattices from $S_r(A)$ then $L_1 + L_2 = (\min(\alpha_1, \beta_1), \ldots, \min(\alpha_n, \beta_n))$.

Let $P_{i_k} = (a_1^{(i_k)}, \ldots, a_n^{(i_k)})$. Then

$$P_{i_1} + \ldots + P_{i_t} = (\min_{1 \leq k \leq t} a_1^{(i_k)}, \ldots, \min_{1 \leq k \leq t} a_n^{(i_k)}).$$

From the ordering relation in $S_r(A)$ we obtain that $\min\limits_{1 \leq k \leq t} a_1^{(i_k)} \leq a$, i.e., $a_1^{(i_k)} \leq a$ for some k. Let $P_{i_k} \simeq e_{i_k i_k}A$ and $P_{i_1} = (\alpha_{i1} + b, \ldots, \alpha_{in} + b)$ for some $b \in \mathbf{Z}$.

We obtain $\alpha_{ij} + b \leq \alpha_{i1} + \alpha_{1j} + b \leq \alpha_{1j} + a$ for $j = 1, \ldots, n$ and $P_1 \subseteq P_{i_k}$. So for every $j = 1, \ldots, m$ there exists $\tau(j) \in \{i_1, \ldots, i_t\}$ and $P_j \subseteq P_{\tau(j)}$. Therefore $M = P_1 + \ldots + P_m = P_{\tau(1)} + \ldots + P_{\tau(m)}$.

If some inclusion $P_j \subseteq P_{\tau(j)}$ is strong, we have $M \subset M$ and so $M \neq M$, a contradiction. Therefore we obtain $P_j = P_{\tau(j)}$ and $m = t$. Consequently Θ is an injective map.

Moreover, if $P_1 + \ldots + P_m \subseteq P_{i_1} + \ldots + P_{i_t}$ then it follows from the above that $(P_1, \ldots, P_m) \preceq (P_{i_1}, \ldots, P_{i_t})$ in $\widetilde{M}_r(A)$.

Note that if $\Theta : \mathcal{L}_1 \to \mathcal{L}_2$ is an isomorphism of partially ordered sets and as $\mathcal{L}_2$ is (of course) a lattice, then $\mathcal{L}_1$ is (of course) a lattice as well, and Θ is an isomorphism of lattices. So the proposition is proved.

Remark 6.1.2. The notions of a lattice and an A-lattice used in this proof are different; the one is a partially ordered set, the other is a special kind of module.

Note that $\mathcal{M}_r(A) = S_r(A)$ if and only if the width of the set $\mathcal{M}_r(A)$ is equal to 1.

Let a tiled order A be reduced. Then $\alpha_{ij} + \alpha_{ji} > 0$ for $i \neq j$, and the set $\mathcal{M}_r(A)$ can be simply constructed from the matrix (α_{ij}): let $p_i^k = (\alpha_{i1} + k, \ldots, \alpha_{in} + k)$, where $i = 1, \ldots, n$ and $k \in \mathbf{Z}$. Suppose $p_i^k = p_j^l$. This means that $\alpha_{im} + k = \alpha_{jm} + l$ for $m = 1, \ldots, n$. In particular, $\alpha_{ii} + k = \alpha_{ji} + l$, i.e., $k = \alpha_{ji} + l$. Analogously, $\alpha_{ij} + k = \alpha_{jj} + l$, i.e., $l = \alpha_{ij} + k$. Therefore, $k + l = \alpha_{ji} + \alpha_{ij} + k + l$ and we obtain the contradiction: $\alpha_{ji} + \alpha_{ij} = 0$. Obviously, the partial ordering on $\mathcal{M}_r(A)$ can be defined as follows: $p_i^k \preceq p_j^l$ if and only if $\alpha_{jm} + l \leq \alpha_{im} + k$ for $m = 1, \ldots, n$. Therefore, $\alpha_{ji} + l \leq k$.

Conversely, let $\alpha_{ji} + l \leq k$. Adding α_{im} to the both sides of this equality we obtain $\alpha_{ji} + \alpha_{im} + l \leq k + \alpha_{im}$. But $\alpha_{ji} + \alpha_{im} \geq \alpha_{jm}$ and $\alpha_{ji} + \alpha_{im} + l \geq k + \alpha_{jm} + l$. Thus, $\alpha_{jm} + l \leq k + \alpha_{im}$, i.e., $p_i^k \preceq p_j^l$.

Analogously, we can construct $\mathcal{M}_l(A)$. The element p_i^k is in $\mathcal{M}_l(A)$ if $p_i^k = (\alpha_{1i} + k, \ldots, \alpha_{ni} + k)^T$ and, as above, all elements p_i^k are different and $p_i^k \preceq p_j^l$ if and only if $k \geq l + \alpha_{ij}$.

Proposition 6.1.11. *The sets $\mathcal{M}_l(A)$ and $\mathcal{M}_r(A)$ are anti-isomorphic.*

Proof. We use the correspondence $p_i^k \mapsto p_i^{-k}$, where $p_i^k = (\alpha_{i1} + k, \ldots, \alpha_{in} + k)$ and $p_i^{-k} = (\alpha_{1i} - k, \ldots, \alpha_{ni} - k)^T$. Then $p_i^k \leq p_j^l$ if and only if $k \geq l + \alpha_{ji}$, and $p_j^{-l} \leq p_i^{-k}$ if and only if $-l \geq -k + \alpha_{ji}$. This proves the statement.

Remark 6.1.3. One should note that this anti-isomorphism cannot be extended to an anti-isomorphism of the lattices $S_l(A)$ and $S_r(A)$, since the anti-isomorphism of lattices is given by the correspondence:

$$(\alpha_1, \ldots, \alpha_n) \to (-\alpha_1, \ldots, -\alpha_n)^T$$

Proposition 6.1.12. *The following properties of a tiled order A are equivalent:*

(a) *the width of the set $\mathcal{M}_r(A)$ does not exceed m;*

(b) *each right irreducible A-module has not more than m maximal submodules;*

(c) *for any $M \in S_r(A)$ the length of the A/R-module M/MR does not exceed m, where R is the Jacobson radical of A.*

Proof. The equivalence of (a) and (b) follows from the fact that any irreducible A-lattice $M \in S_r(A) = \widetilde{\mathcal{M}}_r(A)$, is identified with a collection $T = \{t_1, \ldots, t_k\}$ of pairwise incomparable elements of the set $\mathcal{M}(A)$, has exactly k maximal submodules identified by the sets $T_1, \ldots, T_k$, where T_i is the union of the sets $T \setminus \{t_i\}$ with the set of points strictly less than t_i, but not comparable with any point of $T \setminus \{t_i\}$. On the other hand, the number k is the number of indecomposable summands of a projective cover $P(M)$ of M, which is equal to the length of the module M/MR and this proves the equivalence of (a) and (c).

Remark 6.1.4. In proposition 6.1.12 right modules can be replaced by left ones.

We consider now the connection between the semilattice of overrings of a tiled order A and $S_r(A)$.

Definition. Let $M \in S_r(A)$, i.e., $M \subset U$, where U is a simple Q-module. By the **ring of multipliers** $A(M)$ of an irreducible lattice M will be meant the ring $A_r(M) = \{x \in Q : Mx \subset M\}$, if M is a right module, and $A_l(M) = \{x \in Q : xM \subset M\}$, if M is a left module.

Obviously, $A_r(M)$ and $A_l(M)$ are overrings of A. From proposition 6.1.6 it follows that rings of multipliers are again tiled orders.

Let C be an overring of a tiled order A. Then, obviously, $S_r(C)$ is a sublattice of the lattice $S_r(A)$.

Proposition 6.1.13. *The set of overrings C of a tiled order A is in one-to-one correspondence with the set of nonempty sublattices of the lattice $S_r(A)$ which contain along with each module all its isomorphs. More precisely, if S_1 is a sublattice of $S_r(A)$, then $S_1 = S_r(C)$, where $C = \bigcap\limits_{M \in S_1} A_r(M)$.*

Proof. Let S_1 be a nonempty sublattice of the lattice $S_r(A)$, which contains along with each module all its isomorphs, and let $C = \bigcap\limits_{M \in S_1} A_r(M)$. It is clear that $C \supset A$ and $S_r(C) \supset S_1$. Moreover, there exists a finite collection of modules $M_1, \ldots, M_t \in S_1$, such that $C = \bigcap\limits_{k=1}^{t} A_r(M_k)$. If $M_k = (a_{k1}, \ldots, a_{kn})$, then $\mathcal{E}(A(M_k)) = (\alpha_{ij}^k)$, where $\alpha_{ij}^k = a_{kj} - a_{ki}$, and $C = (c_{ij})$, where $c_{ij} = \max\limits_k(\alpha_{ij}^k)$. Whence it follows that any module from $\mathcal{M}_r(C)$ is the intersection of modules isomorphic to $M_1, \dots, M_t$. Since S_1 is a lattice, this implies $\mathcal{M}_r(C) \subset S_1$, whence $S_r(C) \subset S_1$, that is, $S_r(C) = S_1$.

Corollary 6.1.14. *The number of maximal overrings of a tiled order A is equal to the number of its irreducible A-lattices.*

Definition. An integer matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbf{Z})$ is called

- an **exponent matrix** if $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ and $\alpha_{ii} = 0$ for all i, j, k;
- a **reduced exponent matrix** if $\alpha_{ij} + \alpha_{ji} > 0$ for all i, j, $i \neq j$.

We shall use the following notation: $A = \{\mathcal{O}, \mathcal{E}(A)\}$, where $\mathcal{E}(A) = (\alpha_{ij})$ is the exponent matrix of the tiled order A, i.e.,

$$A = \sum_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O},$$

where the e_{ij} are the matrix units. If a tiled order is **reduced**, i.e., A/R is a direct product of division rings, then $\alpha_{ij} + \alpha_{ji} > 0$ if $i \neq j$, i.e., $\mathcal{E}(A)$ is a reduced exponent matrix.

Let $\mathcal{E} = (\alpha_{ij})$ be a $n \times n$ reduced exponent matrix. Define $n \times n$ matrices

$$\mathcal{E}^{(1)} = (\beta_{ij}), \qquad \text{where} \qquad \beta_{ij} = \begin{cases} \alpha_{ij} & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

and

$$\mathcal{E}^{(2)} = (\gamma_{ij}), \qquad \text{where} \qquad \gamma_{ij} = \min_{1 \leq k \leq n} (\beta_{ik} + \beta_{kj}).$$

Obviously, $[Q] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ is a $(0, 1)$-matrix.

The following theorem is the same as theorem 14.7.1, vol.I, where it was proved using the fact that a tiled order is a prime ring. Here we shall give a direct proof of this theorem using only the matrix definitions.

Theorem 6.1.15. *The matrix $[Q] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ is the adjacency matrix of the strongly connected simply laced quiver $Q = Q(\mathcal{E})$.*

Proof. Since $[Q]$ is a $(0, 1)$-matrix, it is the adjacency matrix of a simply laced quiver.

We shall show that $[Q]$ is a strongly connected quiver. Suppose the contrary. This means that there is no path from the vertex i to the vertex j in Q for some i, j. Denote by $VQ(i) = V_1$ a set of all vertices $k \in Q$ such that there exists a path beginning at the vertex i and ending at the vertex k. Then, by assumption, $V_2 = VQ \backslash VQ(i) \neq \emptyset$ (because $j \in V(Q) \backslash V(Q)(i)$). Consequently, $VQ = V_1 \cup V_2$

and $V_1 \cap V_2 = \emptyset$. It is clear, that there are no arrows from V_1 to V_2. One can assume, that $V_1 = \{1, \ldots, m\}$ and $V_2 = \{m+1, \ldots, n\}$. Conjugation with a diagonal matrix of the form $\mathrm{diag}(\pi^{b_1}, \ldots, \pi^{b_n})$ gives an isomorphic tiled order with exponent matrix

$$\alpha'_{ij} = \alpha_{ij} + b_i - b_j.$$

This new tiled order is also reduced if the original one was reduced. This does not change the matrix $[Q]$ as is easily checked. Thus taking $b_1 = 0$ and $b_j = \alpha_{1j}$ it can be assumed that the exponent matrix $\mathcal{E}$ has its first row zero

$$\alpha_{1p} = 0 \quad p = 1, \ldots, n.$$

It follows that $\alpha_{pq} \geq 0$ for $p, q = 1, \ldots, n$, because $\alpha_{1p} + \alpha_{pq} \geq \alpha_{1q}$.

By assumption $[Q]$ is of the form

$$[Q] = \left(\begin{array}{c|c} * & 0 \\ \hline * & * \end{array}\right).$$

Partition $\mathcal{E}$ in the same way

$$\mathcal{E} = \left(\begin{array}{c|c} \mathcal{E}_1 & * \\ \hline * & \mathcal{E}_2 \end{array}\right),$$

with, hence, $\mathcal{E}_1 \in M_m(\mathbf{Z})$, $\mathcal{E}_2 \in M_{n-m}(\mathbf{Z})$. Associate to the matrix $\mathcal{E}_2$ a partially ordered set by defining

$$i \preceq j \ \text{ if and only if } \ \alpha_{ij} = 0.$$

Note that this is transitive because $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik} \geq 0$. We can assume that $m+1$ is a minimal element. It follows that

$$\alpha_{i,m+1} > 0, \quad i > m+1.$$

Indeed, as $m+1$ is minimal, either $i \geq m+1$ and so $\alpha_{m+1,i} = 0$ so that $\alpha_{i,m+1} > 0$ because $\alpha_{i,m+1} + \alpha_{m+1,i} > 0$ (as A is reduced), or i and $m+1$ are incomparable which means that both $\alpha_{i,m+1}$ and $\alpha_{m+1,i}$ are > 0.

Since $q_{1,m+1}$, the $(1, m+1)$-th entry of $[Q]$, is zero, there is a k such that

$$0 = q_{1,m+1} = \min_k(\beta_{1,k} + \beta_{k,m+1}) - \beta_{1,m+1}.$$

This k cannot be 1 or $m+1$ because $\beta_{1,1} = 1 = \beta_{m+1,m+1}$. Hence there is a $k \in \{1, \ldots, n\} \setminus \{1, m+1\}$ such that

$$\alpha_{1,k} + \alpha_{k,m+1} = \alpha_{1,m+1}.$$

Also $k \leq m$ because $\alpha_{1,k} = 0 = \alpha_{1,m+1}$ while $\alpha_{k,m+1} > 0$ for $k > m+1$. Thus there is a k, $2 \leq k \leq m+1$ with $\alpha_{k,m+1} = 0$. Interchanging the 2-nd and k-th columns and rows simultaneously we can assume that $\alpha_{2,m+1} = 0$. Further $q_{2,m+1} = 0$, and so (arguing as before)

$$0 = \alpha_{2,m+1} = \alpha_{k,m+1} + \alpha_{2,k}$$

for some $k \neq 2, m+1$. Again $k \leq m$ is not possible because $\alpha_{k,m+1} > 0$ for $k > m+1$. Also $k = 1$ is not possible as $\alpha_{2,1} + \alpha_{1,2} > 0$ and $\alpha_{1,2} = 0$. Thus $3 \leq k \leq m$ and we can assume that $k = 3$ giving $\alpha_{2,3} = 0 = \alpha_{3,m+1}$. Now use $q_{3,m+1} = 0$ so that

$$0 = \alpha_{3,m+1} = \alpha_{k,m+1} + \alpha_{3,k}$$

for some $k \neq 3, m+1$ and $k \leq m$. Again $k = 1$ is not possible because $\alpha_{3,1} > 0$ and $k = 2$ is also not possible as $\alpha_{2,3} = 0$ so that $\alpha_{3,2} > 0$. So $4 \leq k \leq m$ and we can take $k = 4$ giving that $\alpha_{4,m+1} = 0 = \alpha_{3,4}$. Continuing this way $\alpha_{12} = \alpha_{23} = \ldots = \alpha_{m-1,m} = 0$, and $\alpha_{k,m+1} = 0$, $k = 1, \ldots, m$. Thus the first superdiagonal of $\mathcal{E}_1$ is zero. But $0 \leq \alpha_{13} \leq \alpha_{12} + \alpha_{23}$ and so also $\alpha_{13} = 0$ (which we knew anyway). But quite generally, $0 \leq \alpha_{i,i+2} \leq \alpha_{i,i+1} + \alpha_{i+1,i+2} = 0$ and so the second superdiagonal is also zero. Continuing one finds that $\alpha_{k,l} = 0$ for all $k \leq l$ (making the matrix $\mathcal{E}_1$ lower triangular). In particular

$$q_{m,m+1} = \min_k (\beta_{m,k} + \beta_{k,m+1}) - \beta_{m,m+1}.$$

Now $\beta_{m,m+1} = \alpha_{m,m+1} = 0$. Further for $k \geq m+1$, $\beta_{k,m+1} \geq 1$ as $\alpha_{k,m+1} > 0$ for $k > m+1$; $\beta_{m,m} = 1$ and for $k < m$ $\beta_{m,k} = \alpha_{m,k} > 0$ because $\alpha_{m,k} + \alpha_{k,m} > 0$ and $\alpha_{k.\,m} = 0$. Thus $q_{m,m+1} = 1$, a contradiction that proves the theorem. (NB. The argument in the case $|V_1| = 1$ has to be changed slightly; this is left to the reader.)

Recall that the quiver of a reduced exponent matrix $\mathcal{E}$ is the quiver $Q(\mathcal{E})$ with adjacency matrix $[Q]$. A strongly connected simply laced quiver is called **admissible** if it is a quiver of a reduced exponent matrix.

A reduced exponent matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbf{Z})$ is **Gorenstein** if there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for $i, k = 1, \ldots, n$.

Theorem 6.1.16. *An arbitrary strongly connected simply laced quiver Q with a loop in each vertex is admissible.*

Proof. Consider the matrix $\mathcal{E} = (\alpha_{ij})$, where $\alpha_{ii} = 0$ and α_{ij} is equal to the minimum length of a path from the vertex i to the vertex j for $i \neq j$. (Note that a path of minimum length always exists because Q is a strongly connected quiver. There may be more than one path of minimum length.)

Let us show that $\mathcal{E}$ is a reduced exponent matrix. Since the minimum length of a path from i to k is less than or equal to the minimum length of a path from

i to k which passes through the vertex j, we have $\alpha_{ik} \leq \alpha_{ij} + \alpha_{jk}$. By definition, $\alpha_{ij} \geq 1$ if $i \neq j$, and so $\alpha_{ij} + \alpha_{ji} > 0$.

Let us show that $Q(\mathcal{E}) = Q$. Since $\alpha_{ij} + \alpha_{ji} \geq 2$ for $j \neq i$, $q_{ii} = \min_k(\beta_{ik} + \beta_{ki} - \beta_{ii} \geq 1$, and so there exists a loop at each vertex of $Q(\mathcal{E})$. Assume there exists an arrow from i to j. Then $\alpha_{ij} = 1$. Since $\alpha_{tk} \geq 1$ if $t \neq k$, there is no $k \neq i, j$ such that $\alpha_{ik} + \alpha_{kj} = 1 = \alpha_{ij}$. Therefore $\alpha_{ik} + \alpha_{kj} > \alpha_{ij}$ for all $k \neq i, j$, and

$$\gamma_{ij} - \beta_{ij} = \min_k(\beta_{ik} + \beta_{kj}) - \beta_{ij} = \min\{1, \min_k(\alpha_{ik} + \alpha_{kj} - \alpha_{ij})\} = 1,$$

i.e., there exists an arrow from i to j in $Q(\mathcal{E})$.

Assume there is no arrow from i to j in Q. Then a path of minimal length from i to j passes through a vertex $t \neq i, j$. Let

$$\sigma_1 \ldots \sigma_u \sigma_{u+1} \ldots \sigma_v : i \to j$$

be a path of minimum length from i to j, where

$$\sigma_1 \ldots \sigma_u : i \to t, \qquad \sigma_{u+1} \ldots \sigma_v : t \to j,$$

and $\alpha_{ij} \geq 2$. Then $\sigma_1 \ldots \sigma_u$ and $\sigma_{u+1} \ldots \sigma_v$ are paths of minimum length from i to t and from t to j, respectively. Hence, $\alpha_{it} + \alpha_{tj} = \alpha_{ij}$. Thus

$$\gamma_{ij} - \beta_{ij} = \min_k(\beta_{ik} + \beta_{kj}) - \beta_{ij} = 0,$$

i.e., there is no arrow in $Q(\mathcal{E})$ as well.

Example 6.1.2.
It is easy to see that the quiver Q with the adjacency matrix

$$[Q] = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

is not admissible.

Definition. Two exponents matrices $\mathcal{E} = (\alpha_{ij})$ and $\Theta = (\theta_{ij})$ are called **equivalent** if they can be obtained from each other by transformations of the following two types :

(1) subtracting an integer α from the entries of the l-th row with simultaneous adding of that integer α to the entries of the l-th column;

(2) simultaneous interchanging of two rows and the same numbered columns.[2]

Proposition 6.1.17. *Suppose $\mathcal{E} = (\alpha_{ij})$, $\Theta = (\theta_{ij})$ are exponent matrices, and Θ is obtained from $\mathcal{E}$ by a transformation of type (1). Then $[Q(\mathcal{E})] = [Q(\Theta)]$.*

[2]Both these transformations were used in the proof of theorem 6.1.15.

If $\mathcal{E}$ is a reduced Gorenstein exponent matrix with a permutation $\sigma(\mathcal{E})$, then Θ is also reduced Gorenstein with $\sigma(\Theta) = \sigma(\mathcal{E})$.

Proof. We have

$$\theta_{ij} = \begin{cases} \alpha_{ij}, & \text{if } i \neq l, j \neq l, \\ 0, & \text{if } i = l, j = l, \\ \alpha_{lj} - t, & \text{if } i = l, j \neq l, \\ \alpha_{il} + t, & \text{if } i \neq l, j = l, \end{cases}$$

where t is an integer. One can directly check that if $\alpha_{ij} + \alpha_{jk} = \alpha_{ik}$ for some i, j, k, then $\theta_{ij} + \theta_{jk} = \theta_{ik}$. Since these transformations are invertible, the converse transformations have a similar form. So the equality $\theta_{ij} + \theta_{jk} = \theta_{ik}$ implies $\alpha_{ij} + \alpha_{jk} = \alpha_{ik}$. Therefore, $\theta_{ij} + \theta_{jk} = \theta_{ik}$ if and only if $\alpha_{ij} + \alpha_{jk} = \alpha_{ik}$.

Denote $\Theta^{(1)} = (\mu_{ij})$ and $\Theta^{(2)} = (\nu_{ij})$.

The equalities $\gamma_{ij} = \beta_{ij}$, $\nu_{ij} = \mu_{ij}$ or inequalities $\gamma_{ij} > \beta_{ij}$, $\nu_{ij} > \mu_{ij}$ hold simultaneously for the entries of the matrices $(\beta_{ij}) = \mathcal{E}_1$, $(\mu_{ij}) = \Theta^{(1)}$, $(\gamma_{ij}) = \mathcal{E}^{(2)}$, $(\nu_{ij}) = \Theta^{(2)}$. Therefore, $\mathcal{E}^{(2)} - \mathcal{E}^{(1)} = \Theta^{(2)} - \Theta^{(1)}$ and $[Q(\mathcal{E})] = [Q(\Theta)]$.

Suppose that $\mathcal{E}$ is a reduced Gorenstein exponent matrix with a permutation $\sigma(\mathcal{E})$, i.e., $\alpha_{ij} + \alpha_{j\sigma(i)} = \alpha_{i\sigma(i)}$. Whence, $\theta_{ij} + \theta_{j\sigma(i)} = \theta_{i\sigma(i)}$. This means that the matrix Θ is also Gorenstein with the same permutation $\sigma(\mathcal{E})$.

Let τ be a permutation which determines simultaneous transpositions of rows and columns of the reduced exponent matrix $\mathcal{E}$ under a transformation of type (2). Then $\theta_{ij} = \alpha_{\tau(i)\tau(j)}$ and $\Theta = P_\tau^T \mathcal{E} P_\tau$, where $P_\tau = \sum\limits_{i=1}^{n} e_{i\tau(i)}$ is the permutation matrix, and P_τ^T stands for the transposed matrix of P_τ. Since $\alpha_{ij} + \alpha_{j\sigma(i)} = \alpha_{i\sigma(i)}$ and $\alpha_{ij} = \theta_{\tau^{-1}(i)\tau^{-1}(j)}$, we have $\theta_{\tau^{-1}(i)k} + \theta_{k\tau^{-1}(\sigma(i))} = \theta_{\tau^{-1}(i)\tau^{-1}(\sigma(i))}$. Hence the permutation π of Θ satisfies $\pi(\tau^{-1}(i)) = \tau^{-1}(\sigma(i))$ for all i. Whence, $\pi = \tau^{-1}\sigma\tau$.

Since

$$\mu_{ij} = \beta_{\tau(i)\tau(j)}, \quad \nu_{ij} = \min_k(\mu_{ik} + \mu_{kj}) = \min_l(\beta_{\tau(i)l} + \beta_{l\tau(j)}) = \gamma_{\tau(i)\tau(j)},$$

it follows that,

$$\widetilde{q}_{ij} = \nu_{ij} - \mu_{ij} = \gamma_{\tau(i)\tau(j)} - \beta_{\tau(i)\tau(j)} = q_{\tau(i)\tau(j)},$$

where $[\widetilde{Q}] = (\widetilde{q}_{ij})$ is the adjacency matrix of the quiver $\widetilde{Q}$ of Θ. So we have proved the following statement.

Proposition 6.1.18. *Under transformations of the second type the adjacency matrix $[\widetilde{Q}]$ of $Q(\Theta)$ is changed according to the formula: $[\widetilde{Q}] = P_\tau^T[Q]P_\tau$, where $[Q] = [Q(\mathcal{E})]$. If $\mathcal{E}$ is Gorenstein, then Θ is also Gorenstein, and for the new permutation π we have: $\pi = \tau^{-1}\sigma\tau$, i.e., $\sigma(\Theta) = \tau^{-1}\sigma(\mathcal{E})\tau$.*

Note that the type (= conjugacy class) of a permutation is not changed under transformations of the second type. Therefore, in order to describe reduced

Gorenstein exponent matrices, one needs to examine matrices with different types of permutations. Further, to simplify calculations we can assume that some row or some column of $\mathcal{E}$ is zero. This can always be assured by transformations of the first type and then the entries of a new exponent matrix will be non-negative integers (see the proof of theorem 6.1.15).

Indeed, let $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbf{Z})$ be an exponent matrix. Subtracting α_{1i} from the entries of the i-th column and adding this number to the entries of the i-th row, we obtain a new exponent matrix

$$\Theta = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \theta_{21} & 0 & \theta_{23} & \dots & \theta_{2n} \\ \theta_{31} & \theta_{32} & 0 & \dots & \theta_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta_{n1} & \theta_{n2} & \theta_{n3} & \dots & 0 \end{pmatrix}$$

The first row of Θ equals zero. Consequently, $\theta_{1i} + \theta_{ij} \geq \theta_{1j} = 0$ and $\theta_{ij} \geq 0$ for $i, j = 1, \dots, n$.

6.2 DUALITY IN TILED ORDERS

In this section we shall introduce a duality for tiled orders and study its properties. Throughout in this section, unless specifically noted, A denotes a tiled order with nonzero Jacobson radical R and classical ring of fractions Q.

Proposition 6.2.1. *Let A be a tiled order with classical ring of fractions Q. Then Q is a flat and injective two-sided A-module.*

Proof. The classical ring of fractions Q is the direct limit of flat submodules $\pi^k A = A\pi^k$ of A, for $k \in \mathbf{Z}$ (as $k \to -\infty$). Thus Q is flat, by proposition 5.4.6, vol.I.

To prove the injectiveness of Q we use the Baer criterion (see proposition 5.2.4, vol.I). Let $\mathcal{I}$ be a right ideal in A. Since A is a Noetherian ring, $\mathcal{I}$ is a finitely generated ideal. Consider a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}_A \arrow[r, "i"] \arrow[d, "\varphi"'] & A_A \, , \\ & Q & \end{tikzcd}$$

where i is a monomorphism. Since Q is flat, the sequence

$$0 \longrightarrow \mathcal{I}_A \otimes Q \xrightarrow{i \otimes 1_Q} A_A \otimes Q$$

is exact. Then, by proposition 5.4.11, vol. I, we obtain the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{\mathcal{I}} \arrow[r, "\widetilde{i}"] \arrow[d, "\widetilde{\varphi}"'] & Q , \\ & Q & \end{tikzcd}$$

where $\widetilde{\mathcal{I}} \simeq \mathcal{I}Q$ and $\widetilde{\varphi} = \varphi \otimes 1_Q$. Since $Q = M_n(D)$ is a simple Artinian ring, Q is a two-sided injective Q-module. Therefore, by the Baer criterion, there is a homomorphism $\widetilde{\psi} : Q \to Q$ such that $\widetilde{\varphi} = \widetilde{\psi i}$. Restricting $\widetilde{i}$ and $\widetilde{\varphi}$ to $\mathcal{I}_A$, and $\widetilde{\psi}$ to A_A we obtain $\varphi = \psi i$. Thus Q is an injective A-module.

Now consider finitely generated semi-reflexive A-modules.

Proposition 6.2.2. *A finitely generated A-module M is semi-reflexive if and only if M is isomorphic to a submodule of a free A-module of finite rank A^m.*

Proof. If $M \subset A^m$, then M is semi-reflexive, by lemma 4.10.3.

Conversely, let M be a finitely generated semi-reflexive A-module. We shall write $X^* = \mathrm{Hom}_A(X, A)$ for any A-module X. An epimorphism $A^m \to M \to 0$ induces a monomorphism $0 \to M^* \to (A^m)^*$. But $A^* = \mathrm{Hom}_A(A, A) \simeq A$ and so M^* is isomorphic to a submodule of A^m. Since A is a Noetherian ring, M^* is a finitely generated A-module and therefore there is an exact sequence $A^r \to M^* \to 0$. Then $0 \to M^{**} \to A^r$ is a monomorphism. Since M is semi-reflexive $\delta_M : M \to M^{**}$ is a monomorphism. Therefore, M is isomorphic to a submodule of a free A-module of a finite rank.

Let A be a tiled order of the form (6.1.1). Recall that an A module M is called an A-lattice if it is a finitely generated free $\mathcal{O}$-module, where $\mathcal{O}$ is the discrete valuation ring of A (see vol.I, p.353). We shall denote by $\mathrm{Lat}_r(A)$ (resp. $\mathrm{Lat}_l(A)$) the category of right (resp. left) A-lattices.

Proposition 6.2.3. *Let A be a tiled order. Then an A-module M is finitely generated semi-reflexive if and only if M is an A-lattice.*

Proof. Let $A = \sum\limits_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O} \subset \sum\limits_{i,j=1}^{n} e_{ij}D = Q = M_n(D)$. Denote by E_n the identity matrix of $M_n(D)$. Obviously, $E_n = \sum\limits_{i=1}^{n} e_{ii}$, where the e_{ii} are the local matrix idempotents of A. Let $X = \{x \in M_n(D) : xe_{ij} = e_{ij}x \text{ for } i, j = 1, \dots, n\}$ and $Y = \{y \in A : ye_{ij} = e_{ij}y \text{ for } i, j = 1, \dots, n\}$. Obviously, $X = \{dE_n\}$, where $d \in D$ and $Y = \{\alpha E_n\}$, where $\alpha \in \mathcal{O}$. So we can view D as a subring of $M_n(D)$ and $\mathcal{O}$ as a subring of A (where D coincides with X and $\mathcal{O}$ coincides with Y). Therefore, A is a free $\mathcal{O}$-module of rank n^2, i.e., an A-lattice. By proposition 6.2.2, an A-module M is finitely generated semi-reflexive if and only if M is an A-lattice. Obviously, $A \otimes_{\mathcal{O}} D = M_n(D) = Q$ and $M \otimes_A Q = M \otimes_A (A \otimes_{\mathcal{O}} D) = M \otimes_{\mathcal{O}} D$, by proposition 4.5.3, vol.I. In this case $\widetilde{M} = M \otimes_{\mathcal{O}} D$ is a finite dimensional vector space over D and M is a complete right A-lattice in $\widetilde{M}$, where $\mathrm{rank}_{\mathcal{O}} M = \dim_D \widetilde{M}$.

Proposition 6.2.4. *Let*

$$0 \longrightarrow L \xrightarrow{i} M \xrightarrow{p} N \longrightarrow 0$$

be an exact sequence of right A-modules. If $L, N \in \mathrm{Lat}_r(A)$ then $M \in \mathrm{Lat}_r(A)$ as well.

Proof. Let $m \neq 0$, $m \in M$ and $m\pi^t E_n = 0$ for some positive $t \in \mathbf{Z}$. Then $p(m)\pi^t E_n = 0$ and $p(m) = 0$. Therefore $m \in \mathrm{Ker}\, p = \mathrm{Im}\, i$, i.e., $m = i(l)$, where $l \in L$ and $m\pi^t E_n = i(l\pi^t E_n) = 0$. Thus $l\pi^t E_n = 0$. Since $L \in \mathrm{Lat}_r(A)$ we obtain $l = 0$ and $m = 0$.

We shall establish now the duality between the category $\mathrm{Lat}_r(A)$ and $\mathrm{Lat}_l(A)$. Let $M \in \mathrm{Lat}_r(A)$ and let $M^{\#} = \mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$. For any $f \in M^{\#}$ and $a \in A$ we can define af by the formula $(af)(m) = f(ma)$ where $m \in M$. Then it is easy to verify that $M^{\#}$ is a left A-module.

Since $M \in \mathrm{Lat}_r(A)$, it is a free $\mathcal{O}$-module with a finite $\mathcal{O}$-basis $e_1, e_2, \ldots, e_n$. As in section 4.10, we can define an $\mathcal{O}$-homomorphism $\varphi_i : M \to \mathcal{O}$ by the formula $\varphi_i(e_j) = \delta_{ij}$ for $i, j = 1, \ldots, n$, where δ_{ij} is the Kronecker symbol. Then $\varphi_i \in M^{\#}$. It is easy to see that $M^{\#}$ is a free $\mathcal{O}$-module with $\mathcal{O}$-basis $\varphi_1, \ldots, \varphi_n$. This $\mathcal{O}$-basis is called the **dual** $\mathcal{O}$-basis of $M^{\#}$. Thus, $M^{\#} \in \mathrm{Lat}_l(A)$. If $M \in \mathrm{Lat}_l(A)$, then $M^{\#} \in \mathrm{Lat}_r(A)$.

Let $\varphi : M \to N$ be a homomorphism of $M, N \in \mathrm{Lat}_r(A)$, i.e., $\varphi \in \mathrm{Hom}_A(M, N)$. Then $\varphi^{\#} : N^{\#} \to M^{\#}$ can be defined by the formula $(\varphi^{\#} f)(m) = f\varphi(m)$, where $f \in N^{\#}$, is a homomorphism from $N^{\#}$ to $M^{\#}$, i.e., $\varphi^{\#} \in \mathrm{Hom}_A(N^{\#}, M^{\#})$. Obviously, if we have homomorphisms $\psi : L \to M$ and $\varphi : M \to N$, then $(\varphi\psi)^{\#} = \psi^{\#}\varphi^{\#}$ and $1_M^{\#} = 1_{M^{\#}}$. Moreover, for any $M \in \mathrm{Lat}_r(A)$ we have $M^{\#\#} = M$ and for any $N \in \mathrm{Lat}_l(A)$ it is true $N^{\#\#} = N$. Besides, for any $\varphi : M \to N$ we have $\varphi^{\#\#} = \varphi$. It is also obvious that $(M \oplus N)^{\#} = M^{\#} \oplus N^{\#}$.

Proposition 6.2.5. *Let L be a submodule of M and $L, M/L \in \mathrm{Lat}_r(A)$. Let $p : M \to M/L$ be the natural projection. Then $M \in \mathrm{Lat}_r(A)$ and M has the following $\mathcal{O}$-basis: $e_1, \ldots, e_s, p^{-1}(n_1), \ldots, p^{-1}(n_t)$, where $e_1, \ldots, e_s$ is a $\mathcal{O}$-basis of L and $n_1, \ldots, n_t$ is a $\mathcal{O}$-basis of M/L.*

Proof. By proposition 6.2.4, $M \in \mathrm{Lat}_r(A)$. Denote $N = M/L$. Let $e_1\alpha_1 + \ldots + e_s\alpha_s + p^{-1}(n_1)\beta_1 + \ldots + p^{-1}(n_t)\beta_t = 0$. Then $e_1\alpha_1 + \ldots + e_s\alpha_s + p^{-1}(n_1\beta_1 + \ldots + n_t\beta_t) = 0$. Obviously, $p(e_1\alpha_1 + \ldots + e_s\alpha_s + p^{-1}(n_1\beta_1 + \ldots + n_t\beta_t)) = n_1\beta_1 + \ldots + n_t\beta_t = 0$. Thus $\beta_1 = \ldots = \beta_t = 0$ and $e_1\alpha_1 + \ldots + e_s\alpha_s = 0$. We obtain $\alpha_1 = \ldots = \alpha_s$. Let $m \in M$. Then $p(m) = n_1\beta_1 + \ldots + n_t\beta_t$ and $m - p^{-1}(n_1\beta_1 + \ldots + n_t\beta_t) \in \mathrm{Ker}\, p$. We obtain $m - p^{-1}(n_1\beta_1 + \ldots + n_t\beta_t) =$

$e_1\alpha_1 + \ldots + e_s\alpha_s$ and $m = e_1\alpha_1 + \ldots + e_s\alpha_s + p^{-1}(n_1)\beta_1 + \ldots + p^{-1}(n_t)\beta_t$. The proposition is proved.

Proposition 6.2.6. *Let $L, M, N = M/L$ be as in the previous proposition. Let*

$$0 \longrightarrow L \longrightarrow M \xrightarrow{p} N \longrightarrow 0$$

be the corresponding exact sequence. Then there is a dual $\mathcal{O}$-basis $\varphi_1, \ldots, \varphi_s, p^{\#}\Theta_1, \ldots, p^{\#}\Theta_t$ of $M^{\#}$, where $\varphi_1, \ldots, \varphi_s$ is a dual $\mathcal{O}$-basis of $L^{\#}$ and $\Theta_1, \ldots, \Theta_t$ is a dual basis of $N^{\#}$.

Proof. By proposition 6.2.5, M has an $\mathcal{O}$-basis $e_1, \ldots, e_s$, $p^{-1}(n_1), \ldots, p^{-1}(n_t)$, where $e_1, \ldots, e_s$ is an $\mathcal{O}$-basis of L and $n_1, \ldots, n_t$ is an $\mathcal{O}$-basis of N. It is now easy to see that $\varphi_1, \ldots, \varphi_s, p^{\#}\Theta_1, \ldots, p^{\#}\Theta_t$ is a dual $\mathcal{O}$-basis to the $\mathcal{O}$-basis $e_1, \ldots, e_s, p^{-1}(n_1), \ldots, p^{-1}(n_s)$. By definition, $\varphi_i(e_j) = \delta_{ij}$ for $i, j = 1, \ldots, s$. Consider $p^{\#}\Theta_i(p^{-1}(n_j)) = \Theta_i p(p^{-1}(n_j)) = \Theta_i(n_j) = \delta_{ij}$ for $i, j = 1, \ldots, t$.

Corollary 6.2.7. *Let*

$$0 \longrightarrow L \longrightarrow M \xrightarrow{p} M \longrightarrow 0$$

be an exact sequence as above. Then the sequence

$$0 \longrightarrow N^{\#} \xrightarrow{p^{\#}} M^{\#} \longrightarrow L^{\#} \longrightarrow 0$$

is exact.

Corollary 6.2.8. $\mathrm{Ext}^1_A\,(N,\ {}_AA^{\#}) = 0$ *for any* $N \in \mathrm{Lat}_r(A)$.

Proof. Let

$$0 \longrightarrow {}_AA^{\#} \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence. By corollary 6.2.7, we obtain that

$$0 \longrightarrow N^{\#} \longrightarrow M^{\#} \longrightarrow {}_AA \longrightarrow 0$$

is an exact sequence of left A-lattices. Then from the projectivity of ${}_AA$ we have $M^{\#} \simeq A \oplus N^{\#}$. Therefore $M^{\#\#} = M \simeq_A A \oplus N$, i.e., $\mathrm{Ext}^1_A(N,\ {}_AA^{\#}) = 0$.

It is simple to establish the duality of irreducible and duality of completely decomposable A-lattices.

Let $M \in S_r(A)$ and $M = \sum_{i=1}^n e_i\pi^{\alpha_i}\mathcal{O}$. If $\varphi_1, \ldots, \varphi_n$ is the dual $\mathcal{O}$-basis for $e_1, \ldots, e_n$, then $\pi^{-\alpha_1}\varphi_1, \ldots, \pi^{-\alpha_n}\varphi_n$ is the dual $\mathcal{O}$-basis for the $\mathcal{O}$-basis $e_1\pi^{\alpha_1}, \ldots, e_n\pi^{\alpha_n}$. Consequently, if $M = (\alpha_1, \ldots, \alpha_n)$, then $M^{\#} =$

$(-\alpha_1, \ldots, -\alpha_n)$. Using the same formula for $N = (\beta_1, \ldots, \beta_n)^T$, we obtain $N^{\#} = (-\beta_1, \ldots, -\beta_n)$. It is easy to see that

$$(M_1 + M_2)^{\#} = M_1^{\#} \cap M_2^{\#} \text{ and } (M_1 \cap M_2)^{\#} = M_1^{\#} + M_2^{\#}$$

for any $M_1, M_2 \in S_r(A)$. Further, if $M_1 \subset M_2$ are two irreducible A-lattices then $M_2^{\#} \subset M_1^{\#}$. (In this case the lattice M_2 is called an **overmodule** of M_1).

Definition. An A-lattice M is said to be **relatively injective** if $M \simeq {}_AP^{\#}$, where ${}_AP$ is a finitely generated projective left A-module.

Definition. An A-lattice M is called **completely decomposable** if it is a direct sum of irreducible A-lattices.

Corollary 6.2.9. *A relatively injective A-lattice M is completely decomposable and any relatively injective indecomposable M has the following form: $M = {}_AP^{\#}$, where ${}_AP$ is an indecomposable projective left A-module.*

Proof. A tiled order

$$A = \sum_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O}$$

is a completely decomposable right A-lattice

$$A_A = e_{11}A \oplus \ldots \oplus e_{nn}A$$

and also a completely decomposable left A-lattice

$${}_AA = Ae_{11} \oplus \ldots \oplus Ae_{nn}.$$

Every finite generated left projective A-module ${}_AP$ has the following form: ${}_AP = (Ae_{11})^{m_1} \oplus \ldots \oplus (Ae_{nn})^{m_n}$. Obviously, ${}_AP \in Lat_l(A)$ and

$${}_AP^{\#} = (Ae_{11})^{\# m_1} \oplus \ldots \oplus (Ae_{nn})^{\# m_n}.$$

So, ${}_AP^{\#}$ is a completely decomposable right A-lattice. In particular, M is indecomposable if and only if $M = (Ae_{ii})^{\#}$ for some $i = 1, \ldots, n$. The corollary is proved.

In what follows we assume that the tiled order A is reduced. In this case $\mathcal{E}(A)$ is reduced, i.e., $\alpha_{ij} + \alpha_{ji} > 0$ for $i \neq j$. An A-lattice $N \subset M_n(D)$ is said to be **complete** if $N \simeq (\mathcal{O}_{\mathcal{O}})^{n^2}$ as a right $\mathcal{O}$-module. If a complete A-lattice N is a left A-module then $e_{ii}Ne_{jj} \subset N$. So $e_{ii}Ne_{jj} = \pi^{\gamma_{ij}}\mathcal{O}$ and

$$N = \sum_{i,j=1}^{n} e_{ij}\pi^{\gamma_{ij}}\mathcal{O}.$$

Note that N is a right and left A-module if and only if $\gamma_{ij} + \alpha_{ik} \geq \gamma_{ik}$ and $\alpha_{ik}+\gamma_{kj} \geq \gamma_{ij}$ for all i, j, k. In this case the matrix (γ_{ij}) is said to be the **exponent matrix** of the A-lattice N and we write it as $\mathcal{E}(N)$. Complete A-lattices which are left A-modules are said to be **fractional ideals** of the order A. Denote by Δ the completely decomposable lattice $A^{\#}_A$.

Lemma 6.2.10. *A completely decomposable left A-lattice Δ is a complete right A-lattice, and*

$$\mathcal{E}(\Delta) = \begin{pmatrix} 0 & -\alpha_{21} & \dots & -\alpha_{n1} \\ -\alpha_{12} & 0 & \dots & -\alpha_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{1n} & -\alpha_{2n} & \dots & 0 \end{pmatrix}.$$

Proof. Let us show that the k-th row $(-\alpha_{1k}, -\alpha_{2k}, \dots, -\alpha_{nk})$ of the matrix $\mathcal{E}(\Delta)$ defines an irreducible right A-lattice. Write $\beta_i = -\alpha_{ik}$. We can rewrite the inequality $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ in the form $-\alpha_{ik} + \alpha_{ij} \geq -\alpha_{jk}$, i.e., $\beta_i + \alpha_{ij} \geq \beta_j$, which implies the assertion of the lemma.

Corollary 6.2.11. *A fractional ideal Δ is a relatively injective right and a relatively injective left A-lattice.*

Proof. The proof follows from the relation ${}_AA^{\#} = A_A$.

Let A be a reduced tiled order and $R = \mathrm{rad} A$. Then

$$\mathcal{E}(R) = \begin{pmatrix} 1 & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & 1 & \dots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & 1 \end{pmatrix}$$

and

$$\mathcal{E}({}_AR^{\#}) = \mathcal{E}(R^{\#}_A) = \begin{pmatrix} -1 & -\alpha_{21} & \dots & -\alpha_{n1} \\ -\alpha_{12} & -1 & \dots & -\alpha_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{1n} & -\alpha_{2n} & \dots & -1 \end{pmatrix}$$

Write $X = {}_AR^{\#}$, $\Delta = (A_A)^{\#}$.

Lemma 6.2.12. *For $i = 1, \dots, n$ we have that $e_{ii}X$ (Xe_{ii}) is the unique minimal overmodule of $e_{ii}\Delta$ (Δe_{ii}) and $e_{ii}X/e_{ii}\Delta = U_i$, $Xe_{ii}/\Delta e_{ii} = V_i$, where U_i is a simple right A-module and V_i is a simple left A-module.*

Proof. The proof for the left case follows from the fact that $e_{ii}R$ is the only maximal submodule of $e_{ii}A$ and from the duality properties and the annihilation lemma. The proof for the right case is just the same.

Note once more, that the $e_{ii}\Delta$ (Δe_{ii}) are all indecomposable relatively right (left) injective A-lattices (up to isomorphism) and the $e_{ii}X$ (Xe_{ii}) are unique minimal overmodules of $e_{ii}\Delta$ (Δe_{ii}). Moreover, the notions of indecomposable relatively injective A-lattice and irreducible relatively injective A-lattice coincide.

Let A_1 and A_2 be Morita equivalent tiled orders. Then the relatively injective indecomposable A_1-lattices correspond to relatively injective indecomposable A_2-lattices. Thus, from lemma 6.2.12 there follows the following lemma

Lemma 6.2.13. *Every relatively injective irreducible A-lattice Q has only one minimal overmodule. Let Q_1 and Q_2 be relatively injective irreducible A-lattices, and let $X_1 \supset Q_1$ and $X_2 \supset Q_2$ be the unique minimal overmodules of Q_1 and Q_2, respectively. Then the simple A-modules X_1/Q_1 and X_2/Q_2 are isomorphic if and only if $Q_1 \simeq Q_2$.*

The dual statement to proposition 6.1.7 is the following proposition, the proof of which can be simply obtained from duality properties:

Proposition 6.2.14. *An irreducible A-lattice is relatively injective if and only if it has exactly one minimal overmodule.*

The following proposition states some interesting fact about the injective dimension of the lattice ${}_AA^{\#}$.

Proposition 6.2.15. *Let A be a tiled order. Then* $\text{inj.dim}_A\,({}_AA^{\#}) = 1$.

Proof. Let $\mathcal{I}$ be a right ideal of A. Consider the exact sequence $0 \to \mathcal{I} \to A \to A/\mathcal{I} \to 0$. We shall show that $\text{Ext}^2_A(A/\mathcal{I},\ {}_AA^{\#}) = 0$. Indeed, by proposition 5.1.10(2), we obtain $\text{Ext}^2_A(A/\mathcal{I},\ {}_AA^{\#}) = \text{Ext}^1_A(\mathcal{I},\ {}_AA^{\#})$. But $\text{Ext}^1_A(\mathcal{I},\ {}_AA^{\#}) = 0$, by corollary 6.2.8. Consequently, $\text{inj.dim}_A\,({}_AA^{\#}) \le 1$. Since $\text{inj.dim}_A\,({}_AA^{\#}) \ne 0$, we obtain that $\text{inj.dim}_A\,({}_AA^{\#}) = 1$, as required.

Consider the quotient module $Q_1 = M_n(D)/{}_AA^{\#}$. We have an exact sequence $0 \to {}_AA^{\#} \to Q_0 = M_n(D) \to Q_1 \to 0$. By corollary 6.2.17, we obtain that Q_1 is an injective A-module. Assume that the tiled order A is reduced. Then by lemma 6.2.12 the injective hulls of the simple A-modules $U_1, \dots, U_s$ may be written in the following form: $E(U_i) = e_{ii}M_n(D)/e_{ii}\Delta$, where the e_{ii} are the matrix idempotents, $i = 1, 2, \dots, n$.

6.3 TILED ORDERS AND FROBENIUS RINGS

In this section we shall construct a countable set of pairwise non-isomorphic Frobenius quotient rings with identity Nakayama permutation for any reduced tiled order over a given discrete valuation ring. In particular, for any finite poset $\mathcal{P} = (p_1, \dots, p_n)$, we shall construct Frobenius rings $F_m(\mathcal{P})$, all different, such that the quivers $Q(F_m(\mathcal{P}))$ of all the rings $F_m(\mathcal{P})$ coincide. Denote by $\mathcal{P}_{\max}$ the set of all maximal elements of $\mathcal{P}$, and denote by $\mathcal{P}_{\min}$ the set of all minimal elements of $\mathcal{P}$, and denote by $\mathcal{P}_{\max} \times \mathcal{P}_{\min}$ their Cartesian product.

To state the relationship between the quiver $Q(F_m(\mathcal{P}))$ of one of these rings and the poset $\mathcal{P}$ we recall the definition of the diagram of a poset $\mathcal{P}$.

The **diagram** of a poset $\mathcal{P} = (p_1, \ldots, p_n)$ is the quiver $Q(\mathcal{P})$ with as set of vertices $VQ(\mathcal{P}) = \{1, \ldots, n\}$ and the set of arrows $AQ(\mathcal{P})$ is given by: there is an arrow from a vertex i to a vertex j if and only if $p_i \prec p_j$, and moreover, if $p_i \preceq p_k \preceq p_j$ then either $k = i$ or $k = j$.

The quiver $Q(F_m(\mathcal{P}))$ is obtained from the diagram of $Q(\mathcal{P})$ by adding arrows σ_{ij} for any $(p_i, p_j) \in \mathcal{P}_{\max} \times \mathcal{P}_{\min}$ (see theorem 14.6.3, vol. I).

Therefore, if $\mathcal{P}$ is a totally ordered set of n elements, then $Q(F_m(\mathcal{P}))$ is a simple cycle C_n, and hence all rings $F_m(\mathcal{P})$ are serial in this case.

For any finite poset $\mathcal{P} = \{p_1, \ldots, p_n\}$ we can construct a reduced tiled $(0,1)$-order $A(\mathcal{P})$ by setting

$$\mathcal{E}(A(\mathcal{P})) = (\alpha_{ij}),$$

where $\alpha_{ij} = 0 \Longleftrightarrow p_i \preceq p_j$ and $\alpha_{ij} = 1$, otherwise.

Then $A(\mathcal{P}) = \{\mathcal{O}, \mathcal{E}(A(\mathcal{P}))\}$ is a reduced $(0,1)$-order (see vol.I, §14.6).

Theorem 6.3.1. *For any finite poset $\mathcal{P}$ there is a countable set of Frobenius rings $F_m(\mathcal{P})$ with identity Nakayama permutation such that $Q(F_m(\mathcal{P})) = Q(A(\mathcal{P}))$.*

Proof. Denote $A = A(\mathcal{P})$, $R = \mathrm{rad}A$, and $X = {}_AR^{\#}$. Let $\Delta = A^{\#}_A$ be the fractional ideal, as above. Then there exists a least positive integer t such that $\pi^t\Delta \subset R^2$. It is clear that $J = \pi^t\Delta$ is a two-sided ideal of the $(0,1)$-order $A(\mathcal{P})$. Write

$$F_m(\mathcal{P}) = A(\mathcal{P})/\pi^m J.$$

Since $\pi^m J \subset R^2$, it follows that $Q(F_m(\mathcal{P})) = Q(A(\mathcal{P}))$. The description of the quiver $Q(A(\mathcal{P}))$ is given by theorem 14.6.3, vol.I. The Artinian ring $F_m(\mathcal{P})$ is a Frobenius ring. Indeed, we have the following chain of inclusions:

$$A \supset R \supset R^2 \supset \pi^{m+t}X \supset \pi^m J.$$

Every indecomposable projective $F_m(\mathcal{P})$-module is of the form $\bar{P}_i = e_{ii}A/e_{ii}\pi^m J$. Therefore, top $\bar{P}_i = U_i$, and from lemma 6.2.12 it follows that

$$\operatorname{soc} \bar{P}_i = e_{ii}\pi^{m+t}X/e_{ii}\pi^{m+t}\Delta = U_i \quad \text{for } i = 1, \ldots, n.$$

The same relation holds for the left modules. Therefore, the Nakayama permutation of the ring $F_m(\mathcal{P})$ is the identity permutation.

Theorem 6.3.2. *For every reduced tiled order A over a discrete valuation ring, there is a countable set of Frobenius rings $F_m(A)$ with identity Nakayama permutation such that $Q(F_m(A)) = Q(A)$.*

Proof. For the fractional ideal Δ, there is the least positive integer t such that $\pi^t\Delta \subset R^2$. Then the quotient ring $Q(F_m(A)) = A/\pi^{m+t}\Delta$ is a Frobenius ring with identity Nakayama permutation.

Example 6.3.1.
Let k be a field, $\mathcal{O} = k[[x]]$ and $\pi = x$. Let

$$A = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \pi^\alpha\mathcal{O} & \mathcal{O} \end{pmatrix},$$

where $\alpha \geq 2$. Obviously,

$$\mathcal{E}(A) = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} \text{ and } [Q(A)] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

In this case

$$\mathcal{E}(\Delta) = \begin{pmatrix} 0 & -\alpha \\ 0 & 0 \end{pmatrix}.$$

We have

$$\mathcal{E}(R^2) = \begin{pmatrix} 2 & 1 \\ \alpha+1 & 2 \end{pmatrix}.$$

Consequently, $t = \alpha + 1$ and

$$\mathcal{E}(\pi^{\alpha+1}\Delta) = \begin{pmatrix} \alpha+1 & 1 \\ \alpha+1 & \alpha+1 \end{pmatrix}$$

and the quotient ring $F_m(A) = A/\pi^{m+t}\Delta$ is a Frobenius ring with the identity Nakayama permutation.

Note that

$$\mathcal{E}(\pi^{m+t}\Delta) = \begin{pmatrix} m+\alpha+1 & m+1 \\ m+\alpha+1 & m+\alpha+1 \end{pmatrix}.$$

Let k be a finite field with q elements. Then $F_m(A)$ is a finite Frobenius ring and $|F_m(A)| = q^{4m+3\alpha+4}$.

Theorem 6.3.3. *For any permutation $\sigma \in S_n$ there exists a countable set of Frobenius semidistributive algebras A_m such that $\nu(A_m) = \sigma$.*

Proof. Indeed, let $\mathcal{O}$ be a discrete valuation ring with unique maximal ideal $\mathcal{M}$, and let

$$K_n(\mathcal{O}) = \begin{pmatrix} \mathcal{O} & \mathcal{M} & \dots & \dots & \mathcal{M} \\ \mathcal{M} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \mathcal{M} \\ \mathcal{M} & \dots & \dots & \mathcal{M} & \mathcal{O} \end{pmatrix}$$

be a tiled order.

Let $\sigma : i \to \sigma(i)$ be a permutation of $\{1, \ldots, n\}$ and let $\mathcal{I}_m = (\mathcal{M}^{w_{ij}})$ be the two-sided ideal of $K_n(\mathcal{O})$, where $w_{i\sigma(i)} = m + 1$, $w_{ij} = m$ for $j \neq \sigma(i)$ $(i, j = 1, \ldots, n)$.

It is easy to see that $F_m(\mathcal{O}) = K_n(\mathcal{O})/\mathcal{I}_m$ is a Frobenius ring with Nakayama permutation σ.

Let $\mathcal{O} = k[[t]]$ be the ring of formal power series over a field k, then $F_m(k[[t]]) = K_n(k[[t]])/I_m$ is a countable set of Frobenius semidistributive algebras $A_m = F_m(k[[t]])$ such that $\nu(A_m) = \sigma$. If k is finite, then all the algebras A_m are finite.

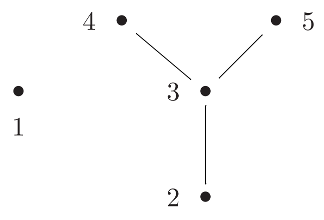
Remark 6.3.1. Recall that QF-algebras with identity Nakayama permutation are called **weakly symmetric algebras**. If for $\mathcal{O}$ we take the ring of formal power series $k[[t]]$ over a field k, then we obtain a countable series of weakly symmetric algebras for every reduced tiled order over $k[[t]]$.

6.4 Q-EQUIVALENT PARTIALLY ORDERED SETS

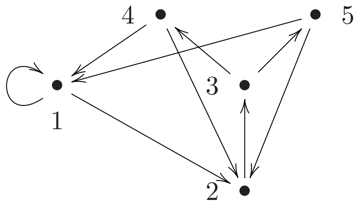
To any finite poset $\mathcal{P} = \{1, \ldots, n\}$, as before, assign a reduced $(0,1)$-matrix $\mathcal{E}_{\mathcal{P}} = (a_{ij})$ in the following way: $a_{ij} = 0 \Leftrightarrow i \preceq j$, and otherwise $a_{ij} = 1$. Then $A(\mathcal{P}) = \{\mathcal{O}, \mathcal{E}_{\mathcal{P}}\}$ is a reduced $(0,1)$-order.

The original poset $\mathcal{P}$ and the $[Q]$-matrix of the exponent matrix $\mathcal{E}_{\mathcal{P}}$ have a great deal to do with one another. Recall that the diagram of $\mathcal{P}$ is the quiver $Q(\mathcal{P})$ which has an arrow $i \to j$ if and only if $i \prec j$ and there is no $k \in \mathcal{P}$ such that $i \prec k \prec j$; see the beginning of section 6.3 above. Let $\widetilde{Q}(\mathcal{P})$ (the "extended diagram" of $\mathcal{P}$) be obtained from $Q(\mathcal{P})$ by adding an arrow $i \to j$ to each pair (i, j) such that i is maximal in $\mathcal{P}$ and j is minimal in $\mathcal{P}$.

For example, if $\mathcal{P}$ is the two component poset



then $\widetilde{Q}(\mathcal{P})$ is



Such extended diagrams have played a role before, see theorem 14.6.3, vol.I.

There is now the following relation between $\widetilde{Q}(\mathcal{P})$ and the "$[Q]$-matrix" of the exponent matrix $\mathcal{E}_{\mathcal{P}}$. Let (p_{ij}) be the adjacency matrix of $\widetilde{Q}(\mathcal{P})$ and $(q_{ij}) = [Q(\mathcal{E}_{\mathcal{P}})]$, then $p_{ij} = q_{ij}$, $i, j = 1, \ldots, n$.

The proof of this is a rather simple matter of checking various cases, as follows. There are 4 cases: $i = j$, $i \prec j$, $j \prec i$, i and j are incomparable. Let $\mathcal{E}_{\mathcal{P}} = (\alpha_{ij})$, where

$$\alpha_{ij} = \begin{cases} 0 & \text{if } i \preceq j \\ 1 & \text{otherwise} \end{cases}$$

Then

$$q_{ij} = \min_k(\beta_{ik} + \beta_{kj}) - \beta_{ij}$$

where

$$\beta_{ij} = \begin{cases} 1 & \text{if } i = j \\ \alpha_{ij} & \text{if } i \neq j \end{cases}$$

Note that for all i, j always $\beta_{ii} + \beta_{ij} - \beta_{ij} = \beta_{ii} = 1$ and $\beta_{ik} + \beta_{kj} \geq \beta_{ij}$ so that $0 \leq q_{ij} \leq 1$. Also, always $\beta_{ij} + \beta_{jj} - \beta_{ij} = \beta_{jj} = 1$. So to calculate q_{ij} it only remains to deal with the $k \notin \{i, j\}$.

Case 1: $i = j$. There are two subcases.

Case 1.1: $i = j$ and i is not comparable to any other element of $\mathcal{P}$ so that $\{i\}$ is a one-element component of $\mathcal{P}$. Then i is both maximal and minimal and there is a loop at i in $\widetilde{Q}(\mathcal{P})$. Thus $p_{ii} = 1$. On the other hand, $\beta_{ii} + \beta_{ii} - \beta_{ii} = \beta_{ii} = 1$ and for all $k \neq i$ (if any) $\beta_{ik} + \beta_{ki} - \beta_{ii} = 1 + 1 - 1 = 1$ because k and i are incomparable. Thus $q_{ii} = 1$.

Case 1.2: $i = j$ and i is comparable to some $k \neq i$. Then $1 = \alpha_{ik} + \alpha_{ki} = \beta_{ik} + \beta_{ki}$. And so $\beta_{ik} + \beta_{ki} - \beta_{ii} = 0$ and $q_{ii} = 0$. On the other hand, $p_{ii} = 0$ because i is not maximal and of course not $i \prec i$.

Case 2: $i \prec j$, so that $\alpha_{ij} = 0$. There are again two subcases.

Case 2.1: $i \prec j$ and there is no k such that $i \prec k \prec j$, i.e. j covers i. Then $p_{ij} = 1$. On the other hand, $\alpha_{ij} = 0$. Note that always $\beta_{ii} + \beta_{ij} - \beta_{ij} = 1$, $\beta_{ij} + \beta_{jj} - \beta_{ij} = 1$ so that to calculate q_{ij} it remains to look at the $k \notin \{i, j\}$. For the location of k vis à vis i and j there are 4 subsubcases.

$i \prec k$ and $k \prec j$. This is not possible under case 2.1.

$i \prec k$ and not $(k \prec j)$. Then $\beta_{ik} + \beta_{kj} - \beta_{ij} = \alpha_{ik} + \beta_{kj} - \alpha_{ij} = 0 + 1 - 0 = 1$.

not $(i \prec k)$ and $k \prec j$. Then $\beta_{ik} + \beta_{kj} - \beta_{ij} = \alpha_{ik} + \alpha_{kj} - \alpha_{ij} = 1 + 0 - 0 = 1$.

not $(i \prec k)$ and not $k \prec j$. Then $\beta_{ik} + \beta_{kj} - \beta_{ij} = \alpha_{ik} + \alpha_{kj} - \alpha_{ij} = 1 + 1 - 0 = 2$. Thus $q_{ij} = 1$.

Case 2.2. There is a k such that $i \prec k \prec j$. Then $p_{ij} = 0$ and for this particular k, $\beta_{ik} + \beta_{kj} - \beta_{ij} = \alpha_{ik} + \alpha_{kj} - \alpha_{ij} = 0 + 0 - 0 = 0$, so that also $q_{ij} = 0$.

Case 3: $j \prec i$. In this case always $\alpha_{ij} = 1$. There three subcases.

Case 3.1: i is maximal and j is minimal. Then $p_{ij} = 1$. On the other hand, for $k \neq i, j$ it cannot be that $i \prec k$ or $k \prec j$ (because i is maximal and j is minimal). So for all $k \neq \{i, j\}$, $\alpha_{ik} = 1$, $\alpha_{kj} = 1$. Also not $(i \prec j)$ so that $\alpha_{ij} = 1$. Thus $\beta_{ik} + \beta_{kj} - \beta_{ij} = \alpha_{ik} + \alpha_{kj} - \alpha_{ij} = 1 + 1 - 1 = 1$, and $q_{ij} = 1$.

Case 3.2: i is not maximal. Then $p_{ij} = 0$. On the other hand, there is a $k \succ i$. And so, for this particular k, $\alpha_{ik} = 0$, $\alpha_{kj} = 1$ and so $\beta_{ik} + \beta_{kj} - \beta_{ij} = \alpha_{ik} + \alpha_{kj} - \alpha_{ij} = 0 + 1 - 1 = 0$, and so $q_{ij} = 0$.

Case 3.3: j is not minimal. Then $p_{ij} = 0$. On the other hand, there is a $k \prec j$. So for this particular k, $i \succ j \succ k$ so that $\alpha_{ik} = 1$, $\alpha_{kj} = 0$, $\alpha_{ij} = 1$ and so $\beta_{ik} + \beta_{kj} - \beta_{ij} = \alpha_{ik} + \alpha_{kj} - \alpha_{ij} = 1 + 0 - 1 = 0$, and $q_{ij} = 0$.

Case 4: i and j are incomparable. Then $\alpha_{ij} = 1$. There again three subcases.

Case 4.1: i is maximal and j is minimal. Then $p_{ij} = 1$. Let $k \notin \{i, j\}$. As always it suffices to look at these k. As i is maximal it cannot be that $i \prec k$ and so $\alpha_{ik} = 1$. Also as j is minimal it cannot be that $k \prec j$ and so $\alpha_{kj} = 1$. Thus $\beta_{ik} + \beta_{kj} - \beta_{ij} = 1 + 1 - 1 = 1$ for all k, and $q_{ij} = 1$.

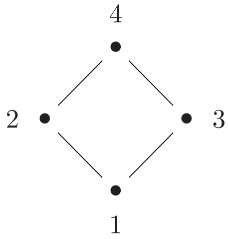
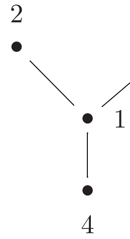
Case 4.2: i is not maximal. Then $p_{ij} = 0$. As i is not maximal there is a $k \succ i$ and so for this particular k, $\alpha_{ik} = 0$. Also $\alpha_{kj} = 1$ because otherwise we would have $i \prec k \preceq j$ which would make i and j comparable. Thus $\alpha_{ik} + \alpha_{kj} - \alpha_{ij} = 0 + 1 - 1 = 0$ for this particular k, so that $q_{ij} = 0$.

Case 4.3: j is not minimal. Then there is a $k \prec j$ and so $\alpha_{kj} = 0$. But it cannot be that $i \preceq k$ because then $i \preceq k \preceq j$ making i and j comparable. Thus $\alpha_{ik} = 1$ and $\alpha_{ik} + \alpha_{kj} - \alpha_{ij} = 1 + 0 - 1 = 0$ for this particular k and $q_{ij} = 0$.

This concludes the proof that the adjacency matrix of $\widetilde{Q}(\mathcal{P})$ is equal to $[Q(\mathcal{E}_{\mathcal{P}})]$.

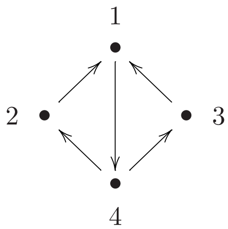
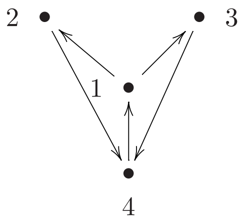
Definition. Two finite partially ordered sets S and T are called **Q-equivalent** if the reduced exponent $(0, 1)$-matrices $\mathcal{E}_S$ and $\mathcal{E}_T$ are equivalent (meaning that $\mathcal{E}_T$ can be obtained from $\mathcal{E}_S$ by repeated use of the transformations (1) and (2) in the definition just above proposition 6.1.17 above).

Example 6.4.1. The following posets are Q-equivalent:

$S =$  and $T =$ 

Obviously,

$$\mathcal{E}_S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \text{ and } \mathcal{E}_T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\widetilde{Q}(S) =$  $\widetilde{Q}(T) =$ 

$$[\widetilde{Q}(S)] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} = [Q(\mathcal{E}_S)]$$

$$[\widetilde{Q}(T)] = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = [Q(\mathcal{E}_T)]$$

Note that the two adjacency matrices go into one another by a simultaneous interchange of the first row and fourth row and first column and fourth column, making $\widetilde{Q}(S)$ and $\widetilde{Q}(T)$ isomorphic quivers.

The matrix $\mathcal{E}_T$ is obtained from $\mathcal{E}_S$ by subtracting 1 from the last row of $\mathcal{E}_S$ with simultaneous adding 1 to the last column.

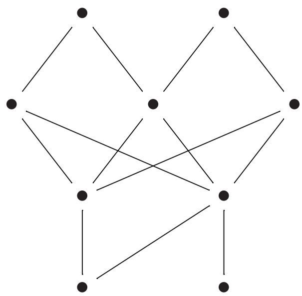
A finite poset with a connected diagram will be called **connected**.

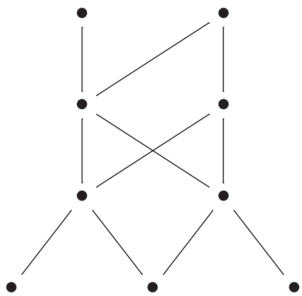
Proposition 6.4.1. *For any two posets S and T, if the exponent matrices $\mathcal{E}_S$ and $\mathcal{E}_T$ are equivalent then $Q(\mathcal{E}_S)$ and $Q(\mathcal{E}_T)$ are isomorphic.*

The proof follows from propositions 6.1.17 and 6.1.18.

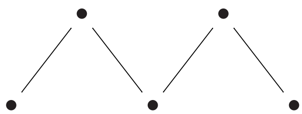
Theorem 6.4.2. *Two finite connected posets $\mathcal{P}_1$ and $\mathcal{P}_2$ are Q-equivalent if and only if these posets are either isomorphic or there exist partitions of these posets $\mathcal{P}_1 = \mathcal{P}_1' \bigcup \mathcal{P}_1''$ and $\mathcal{P}_2 = \mathcal{P}_2' \bigcup \mathcal{P}_2''$ such that each element of $\mathcal{P}_1''$ is not greater then any element from $\mathcal{P}_1'$, and each element of $\mathcal{P}_2'$ is not greater then any element from $\mathcal{P}_2''$, and $\mathcal{P}_1' \simeq \mathcal{P}_2'$, $\mathcal{P}_1'' \simeq \mathcal{P}_2''$.*

Example 6.4.2.
The following two posets satisfy the conditions of theorem 6.4.2.

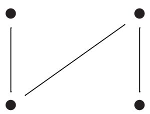
(a) 

(b) 

Both posets contain the partially ordered subsets



and



and moreover in these posets each element of the one subset is not greater than any element of the other subset.

Proof of theorem 6.4.2. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be finite numbered connected Q-equivalent posets. Then $\widetilde{Q}(\mathcal{P}_1) \simeq \widetilde{Q}(\mathcal{P}_2)$. Here $\widetilde{Q}(\mathcal{P}_i)$ is the quiver with adjacency matrix $[Q(\mathcal{E}_{\mathcal{P}_i})]$, see the definition just above theorem 6.1.15. Renumber the elements of the set $\mathcal{P}_2$ (a renumbering of the vertices of a quiver does not affect the quiver) in such a way that $\widetilde{Q}(\mathcal{P}_1) = \widetilde{Q}(\mathcal{P}_2)$ (inclusive the numbering). Let $\mathcal{P}_1 = \{\alpha_1, \alpha_2, ..., \alpha_n\}$, $\mathcal{P}_2 = \{\gamma_1, \gamma_2, ..., \gamma_n\}$ and $[\widetilde{Q}(\mathcal{P}_1)] = [\widetilde{Q}(\mathcal{P}_2)] = (q_{ij})$ (where $[\widetilde{Q}(\mathcal{P}_i)]$ is the adjacency matrix of $\widetilde{Q}(\mathcal{P}_i)$.

If $q_{ij} = 0$, then the element α_j does not cover the element α_i and either α_i isn't maximal or α_j isn't minimal (or both); and the element γ_j does not cover the element γ_i and either γ_i isn't maximal or γ_j isn't minimal (or both).

If $q_{ij} = 1$, then either the element α_j covers the element α_i or α_i is maximal and α_j is minimal; and either the element γ_j covers the element γ_i or γ_i is maximal and γ_j is minimal.

Assume that $Q(\mathcal{P}_1) \neq Q(\mathcal{P}_2)$. Then (because $\widetilde{Q}(\mathcal{P}_1) = \widetilde{Q}(\mathcal{P}_2)$) there exist an i and j such that α_j covers α_i, but γ_i is maximal and γ_j is minimal (or vice versa).

Let $\alpha_j = \alpha_{j_1}$ cover the elements $\alpha_i = \alpha_{i_1}, \dots, \alpha_{i_s}$. Then $\gamma_j = \gamma_{j_1}$ is a minimal element, and $\gamma_{i_1}, \dots, \gamma_{i_s}$ are maximal elements, moreover, there are no other maximal elements in the poset $\mathcal{P}_2$. Indeed, if there was a maximal element $\gamma_{i_0} \notin \{\gamma_{i_1}, \dots, \gamma_{i_s}\}$ in $\mathcal{P}_2$ there would be an arrow from γ_{j_0} to γ_j in $\widetilde{Q}(\mathcal{P}_2)$ and hence an arrow from α_{i_0} to α_j in $\widetilde{Q}(\mathcal{P}_1)$. But this would mean that α_j covers α_{i_0}. Therefore $\alpha_{i_0} \in \{\alpha_{i_1}, \dots, \alpha_{i_s}\}$, this is a contradiction.

If $\mathcal{P}_{2\,\min} = \{\gamma_{j_1}, ..., \gamma_{j_r}\}$, then all the $\alpha_{j_1}, ..., \alpha_{j_r}$ cover every element of $\{\alpha_{i_1}, \dots, \alpha_{i_s}\}$.

Let $\mathcal{P}_{1\,\max} = \{\alpha_{p_1}, ..., \alpha_{p_m}\}$, $\mathcal{P}_{1\,\min} = \{\alpha_{l_1}, ..., \alpha_{l_t}\}$, $\mathcal{P}_{2\,\max} = \{\gamma_{i_1}, ..., \gamma_{i_s}\}$. Then, if $\alpha_p = \alpha_{p_v} \in \mathcal{P}_{1\,\max}$ and $\alpha_l = \alpha_{l_u} \in \mathcal{P}_{1\,\min}$, then γ_{l_u} covers γ_{p_v}.

Denote $\mathcal{P}_1' = \{\alpha_q \in \mathcal{P}_1 : \alpha_q > \alpha_{i_1}\}$, $\mathcal{P}_1'' = \mathcal{P}_1 \backslash \mathcal{P}_1'$; $\mathcal{P}_2'' = \{\gamma_q \in \mathcal{P}_2 : \gamma_q > \alpha_{p_1}\}$, $\mathcal{P}_2' = \mathcal{P}_2 \backslash \mathcal{P}_2''$.

Then $Q(\mathcal{P}_1') = Q(\mathcal{P}_2')$ and $Q(\mathcal{P}_1'') = Q(\mathcal{P}_2'')$. Indeed, since in each set $\mathcal{P}_1', \mathcal{P}_2', \mathcal{P}_1'', \mathcal{P}_2''$ there are no simultaneously elements of $\mathcal{P}_{1\,\max}$ and $\mathcal{P}_{1\,\min}$; $\mathcal{P}_{2\,\max}$ and $\mathcal{P}_{2\,\min}$, from the equality $q_{ij} = 1$ for, e.g., $\alpha_i, \alpha_j \in \mathcal{P}_1'$, it follows that α_j covers α_i and γ_j covers γ_i, where $\gamma_i, \gamma_j \in \mathcal{P}_2''$.

Conversely, let the connected posets $\mathcal{P}_1$ and $\mathcal{P}_2$ be either isomorphic or there exist partitions of these sets $\mathcal{P}_1 = \mathcal{P}_1' \bigcup \mathcal{P}_1''$ and $\mathcal{P}_2 = \mathcal{P}_2' \bigcup \mathcal{P}_2''$ such that each element of $\mathcal{P}_1''$ is not greater than any element of $\mathcal{P}_1'$, each element of $\mathcal{P}_2'$ is not greater than any element of $\mathcal{P}_2''$, and $\mathcal{P}_1' \simeq \mathcal{P}_2'$, $\mathcal{P}_1'' \simeq \mathcal{P}_2''$. If $\mathcal{P}_1 \simeq \mathcal{P}_2$ then, obviously, $\mathcal{E}(\mathcal{P}_1) \sim \mathcal{E}(\mathcal{P}_2)$ (renumbering of elements of the poset $\mathcal{P}_2$ corresponds to the second type equivalent transformations of the matrix $\mathcal{E}(\mathcal{P}_2)$). If $\mathcal{P}_1 \not\simeq \mathcal{P}_2$ and there exists a suitable partition then

$$\mathcal{E}(\mathcal{P}_1) = \begin{pmatrix} \mathcal{E}(\mathcal{P}_1') & U \\ 0 & \mathcal{E}(\mathcal{P}_1'') \end{pmatrix}, \mathcal{E}(\mathcal{P}_2) = \begin{pmatrix} \mathcal{E}(\mathcal{P}_2'') & U \\ 0 & \mathcal{E}(\mathcal{P}_2') \end{pmatrix},$$

where U is a matrix with all entries equal to 1, $\mathcal{E}(\mathcal{P}_1') = \mathcal{E}(\mathcal{P}_2')$, $\mathcal{E}(\mathcal{P}_1'') = \mathcal{E}(\mathcal{P}_2'')$ (therefore $\mathcal{P}_1' \simeq \mathcal{P}_2'$, $\mathcal{P}_1'' \simeq \mathcal{P}_2''$).

Matrices $\mathcal{E}(\mathcal{P}_1)$ and $\mathcal{E}(\mathcal{P}_2)$ are equivalent. Really, at first, by means of transformations of the second type from the matrix $\mathcal{E}(\mathcal{P}_2)$ we obtain a matrix

$$\mathcal{E}(\overline{\mathcal{P}}_2) = \begin{pmatrix} \mathcal{E}(\overline{\mathcal{P}}_2') & 0 \\ U & \mathcal{E}(\overline{\mathcal{P}}_2'') \end{pmatrix},$$

where $\mathcal{E}(\overline{\mathcal{P}}_2'') = \mathcal{E}(\mathcal{P}_2'') = \mathcal{E}(\mathcal{P}_1'')$, $\mathcal{E}(\overline{\mathcal{P}}_2') = \mathcal{E}(\mathcal{P}_2') = \mathcal{E}(\mathcal{P}_1')$. Then, by means of equivalent transformations of the first type from the matrix $\mathcal{E}(\overline{\mathcal{P}}_2)$ we obtain the matrix $\mathcal{E}(\mathcal{P}_1)$. Thus, the posets $\mathcal{P}_1$ and $\mathcal{P}_2$ are Q-equivalent. The theorem is proved.

In the proof of this theorem, we have also proved the following proposition.

Proposition 6.4.3. *For finite connected posets S and T, if $Q(\mathcal{E}_S)$ and $Q(\mathcal{E}_T)$ are isomorphic, then $\mathcal{E}_S$ and $\mathcal{E}_T$ are equivalent.*

Theorem 6.4.4. *Two finite non-connected posets $\mathcal{P}$ and S are Q-equivalent if and only if the diagrams $Q(\mathcal{P})$ and $Q(S)$ are isomorphic.*

Proof. Let $\mathcal{P}$ and S be two non-connected Q-equivalent posets, $\mathcal{P} = \mathcal{P}_1 \bigcup \mathcal{P}_2$, $S = S_1 \bigcup S_2, Q(\mathcal{P}) = Q(\mathcal{P}_1) \bigcup Q(\mathcal{P}_2), Q(S) = Q(S_1) \bigcup Q(S_2)$ and, besides, between $Q(\mathcal{P}_1)$ and $Q(\mathcal{P}_2)$, and between $Q(S_1)$ and $Q(S_2)$ there are no arrows. As the sets $\mathcal{P}$ and S are Q-equivalent, by proposition 6.4.1 the quivers $\widetilde{Q}(\mathcal{P})$ and $\widetilde{Q}(S)$ are isomorphic. Let $\varphi : \widetilde{Q}(\mathcal{P}) \to \widetilde{Q}(S)$ be an isomorphism of quivers. Let $\mathcal{P}_{1\,\max} = \{\alpha_{i_1}, ..., \alpha_{i_m}\}, \mathcal{P}_{1\,\min} = \{\alpha_{j_1}, ..., \alpha_{j_s}\}, \mathcal{P}_{2\,\max} = \{\alpha_{k_1}, ..., \alpha_{k_t}\}, \mathcal{P}_{2\,\min} = \{\alpha_{l_1}, ..., \alpha_{l_r}\}$. From each vertex α_{i_v} of the set $\mathcal{P}_{1\,\max}$ as well as from each vertex α_{k_u} of the set $\mathcal{P}_{2\,\max}$, in the quiver $\widetilde{Q}(\mathcal{P})$ arrows go to each vertex α_{j_p} of the set $\mathcal{P}_{1\,\min}$ as well as to each vertex α_{l_q} of the set $\mathcal{P}_{2\,\min}$. Then from each vertex $\varphi(\alpha_{i_v})$ of the set $\varphi(\mathcal{P}_{1\,\max})$ and from each vertex $\varphi(\alpha_{k_u})$ of the set $\varphi(\mathcal{P}_{2\,\max})$ (resp. in the quiver $\widetilde{Q}(S)$, arrows go to each vertex $\varphi(\alpha_{j_p})$ of the set $\varphi(\mathcal{P}_{1\,\min})$ and to each vertex $\varphi(\alpha_{l_q})$ of the set $\varphi(\mathcal{P}_{2\,\min})$).

Suppose that $\varphi(\alpha_{i_v})$ does not belong to the set $S_{\max}$. Then $\varphi(\alpha_{j_1}), ..., \varphi(\alpha_{j_s}), \varphi(\alpha_{l_1}), ..., \varphi(\alpha_{l_r}) \notin S_{\min}$ and $\varphi(\alpha_{i_1}), ..., \varphi(\alpha_{i_m})$, $\varphi(\alpha_{k_1}), ..., \varphi(\alpha_{k_t}) \notin S_{\max}$. There are arrows in $\widetilde{Q}(S)$ from each of the $\varphi(\alpha_{i_1}), ..., \varphi(\alpha_{i_m})$; $\varphi(\alpha_{k_1}), ..., \varphi(\alpha_{k_t})$ to each of the $\varphi(\alpha_{j_1}), ..., \varphi(\alpha_{j_s})$; $\varphi(\alpha_{l_1}), ..., \varphi(\alpha_{l_r})$. As none of the $\varphi(\alpha_{i_v})$, $v = 1, \dots, m$; $\varphi(\alpha_{k_w})$, $w = 1, \dots, t$ is maximal these arrows must be in $Q(S)$ itself. Thus the $\varphi(\alpha_{j_v})$, $\varphi(\alpha_{k_w})$, $\varphi(\alpha_{j_x})$, $\varphi(\alpha_{l_y})$ form a connected subquiver T of $Q(S)$.

Now suppose that there are no minimal elements in S among the $\varphi(\alpha_{i_1}), ..., \varphi(\alpha_{i_m})$. Take any $\alpha \in \mathcal{P}_1$ that is neither maximal nor minimal (if any). Then there is a path $\alpha \to \ldots \to \alpha_{i_k}$ for some i_k. This gives a path

$$\varphi(\alpha) \to \varphi(\alpha_{n_1}) \to \cdots \to \varphi(\alpha_{n_a}) \to \varphi(\alpha_{i_k}) \tag{$*$}$$

in $\widetilde{Q}(S)$. Now the $\varphi(\alpha_{n_i})$, $i = 1, \dots, a$ are visibly nonminimal and $\varphi(\alpha_{i_k})$ is nonminimal by hypothesis, so all the arrows of (*) are in fact in $Q(S)$ itself. So $\varphi(\alpha)$ is connected to T for every $\alpha \in \mathcal{P}_1$. Similarly if there is no minimal element among the $\varphi(\alpha_{k_1}), ..., \varphi(\alpha_{k_t})$ every $\varphi(\alpha)$ for $\alpha \in \mathcal{P}_2$ is connected to T. Thus if there is no minimal in S among $\{\varphi(\alpha_{i_1}), ..., \varphi(\alpha_{i_m})\} \cup \{\varphi(\alpha_{k_1}), ..., \varphi(\alpha_{k_t})\}$ S would be connected contrary to the hypothesis in the statement of the theorem.

Thus there is a minimal element in S among the

$$S^{(1)} = \{\varphi(\alpha_{i_1}), ..., \varphi(\alpha_{i_m})\};\ S^{(2)} = \{\varphi(\alpha_{k_1}), ..., \varphi(\alpha_{k_t})\}.$$

In the same there is a maximal element in S among the

$$S^{(3)} = \{\varphi(\alpha_{j_1}), ..., \varphi(\alpha_{j_s})\}; \quad S^{(4)} = \{\varphi(\alpha_{l_1}), ..., \varphi(\alpha_{l_r})\}.$$

There are 4 cases to consider.

Case 1. There is a minimal element from S in $S^{(1)}$, say $\varphi(\alpha_{i_b})$, and a maximal element from S in $S^{(4)}$, say $\varphi(\alpha_{l_c})$. Then there is an arrow from $\varphi(\alpha_{l_c})$ to $\varphi(\alpha_{i_b})$ in $\widetilde{Q}(S)$ and hence an arrow $\alpha_{l_c} \to \alpha_{i_b}$ in $\widetilde{Q}(\mathcal{P})$ which cannot be because α_{l_c} is minimal and α_{i_b} is maximal and $\alpha_{l_c} \neq \alpha_{i_b}$ (being from $\mathcal{P}_2$ and $\mathcal{P}_1$ respectively.

Case 2. There is a minimal element from S in $S^{(1)}$, say $\varphi(\alpha_{i_b})$, and a maximal element from S in $S^{(3)}$, say $\varphi(\alpha_{j_c})$. Then there is an arrow from $\varphi(\alpha_{j_c})$ to $\varphi(\alpha_{i_b})$. So there is an arrow $\alpha_{j_c} \to \alpha_{i_b}$ in $\widetilde{Q}(\mathcal{P})$. But α_{i_b} is maximal and α_{j_c} is minimal. So this is only possible if $\alpha_{j_c} = \alpha_{i_b}$ and we have a loop at α_{j_c} and $\{\alpha_{j_c}\}$ and $\{\varphi(\alpha_{j_c})\}$ are one-element components of $\mathcal{P}$ and S. Let α be a maximal element of $\mathcal{P} \setminus \{\alpha_{i_b}\}$. Then in $\widetilde{Q}(\mathcal{P})$ there is an arrow $\alpha \to \alpha_{i_b}$. So there is also an arrow $\varphi(\alpha) \to \varphi(\alpha_{i_b})$ in $\widetilde{Q}(S)$. This arrow cannot be in $Q(S)$ because $\{\varphi(\alpha_{i_b}\}$ is a one vertex component of S. Thus $\varphi(\alpha) \to \varphi(\alpha_{i_b})$ is in $\widetilde{Q}(S) \setminus Q(S)$ which can only be the case if $\varphi(\alpha)$ is maximal. This holds for all maximal α contradicting the hypothesis that there is an α_{i_v} such that $\varphi(\alpha_{i_v})$ is not maximal.

Case 3. There is a minimal element from S in $S^{(2)}$ and a maximal element from S in $S^{(3)}$. This is, mutatis mutandi, treated just as case 1.

Case 4. There is a minimal element from S in $S^{(2)}$ and a maximal element from S in $S^{(4)}$. This is, mutatis mutandi, treated just as case 2.

Thus all possible cases lead to a contradiction under the assumption that there is a α_{i_v} with $\varphi(\alpha_{i_v})$ nonmaximal. In the same way the assumption that there is a α_{k_u} such that $\varphi(\alpha_{i_v})$ is nonminimal leads to a contradiction.

Thus $\varphi(\mathcal{P}_{\max}) \subset S_{\max}$. In a quite similar way one proves that $\varphi(\mathcal{P}_{\min}) \subset S_{\min}$. Working with φ^{-1} instead of φ gives $\varphi^{-1}(S_{\max}) \subset \mathcal{P}_{\max}$, $\varphi^{-1}(S_{\min}) \subset \mathcal{P}_{\min}$. And so $\varphi(\mathcal{P}_{\max}) = S_{\max}$, $\varphi(\mathcal{P}_{\min}) = S_{\min}$.

Since the quivers $\widetilde{Q}(\mathcal{P})$ and $\widetilde{Q}(S)$ are isomorphic, and maximal and minimal elements, under the isomorphism, are mapped into maximal and minimal elements, it follows that the restriction of the isomorphism $\varphi : \widetilde{Q}(\mathcal{P}) \to \widetilde{Q}(S)$ to $Q(\mathcal{P})$ gives an isomorphism $Q(\mathcal{P}) \to Q(S)$.

Conversely, if the diagrams $Q(\mathcal{P})$ and $Q(S)$ are isomorphic then the exponent matrices $\mathcal{E}_{\mathcal{P}}$ and $\mathcal{E}_S$ are equivalent, i.e., the posets $\mathcal{P}$ and S are Q-equivalent. The theorem is proved.

The diagram $Q(\mathcal{P})$ of a finite poset $\mathcal{P}$ is the union of its connected components $Q(\mathcal{P}_1), \ldots, Q(\mathcal{P}_s)$. The subsets $\mathcal{P}_1, \ldots, \mathcal{P}_s$ will be called the **connected components** of the poset $\mathcal{P}$.

Theorem 6.4.5. *Let $\mathcal{P} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_n$ and $S = S_1 \cup \cdots \cup S_m$ be partitions of two finite disconnected posets into connected components. The posets $\mathcal{P}$ and S*

are Q-equivalent if and only if $m = n$ and there exists a permutation $\theta \in S_n$ such that $\mathcal{P}_i$ and $S_{\theta(i)}$ are isomorphic for all $i = 1, \dots, n$.

The proof of this theorem follows from theorem 6.4.4.

Theorem 6.4.6. *For finite posets S and T the following conditions are equivalent:*
(a) $Q(\mathcal{E}_S)$ and $Q(\mathcal{E}_T)$ are isomorphic;
(b) $\mathcal{E}_S$ and $\mathcal{E}_T$ are equivalent.

Proof. (b) $\Rightarrow$ (a) follows from proposition 6.4.1.
(a) $\Rightarrow$ (b) follows from theorem 6.4.4 and theorem 14.6.3, vol.I.

6.5 INDICES OF TILED ORDERS

We recall some facts concerning the relations between square matrices and quivers.

Let $B = (b_{ij})$ be an arbitrary real square $n \times n$-matrix, i.e., $B \in M_n(\mathbf{R})$. Using B one can construct a simply laced quiver $Q(B)$ in the following way: the set of vertices $VQ(B)$ of $Q(B)$ is $\{1, \dots, n\}$. The set of arrows $AQ(B)$ is defined as follows: there is an arrow from i to j if and only if $b_{ij} \neq 0$.

Let τ be a permutation of the set $\{1, 2, \dots, n\}$ and let

$$P_\tau = \sum_{i=1}^{n} e_{i\tau(i)}$$

be the corresponding permutation matrix, where the e_{ij} are the matrix units. Clearly, $P_\tau^T P_\tau = P_\tau P_\tau^T = E_n$ is the identity matrix of $M_n(\mathbf{R})$. In particular,

$$D_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

is a P_σ, where $\sigma = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \end{pmatrix}$, and $D_n^T = D_n$.

Recall that a matrix $B \in M_n(\mathbf{R})$ is called **permutationally reducible** if there exists a permutation matrix P_τ such that

$$P_\tau^T B P_\tau = \begin{pmatrix} B_1 & B_{12} \\ 0 & B_2 \end{pmatrix},$$

where B_1 and B_2 are square matrices of order less that n. Otherwise, the matrix B is called **permutationally irreducible**.

From the equality

$$D_n \begin{pmatrix} B_1 & B_{12} \\ 0 & B_2 \end{pmatrix} D_n = \begin{pmatrix} B_1^{(1)} & 0 \\ B_{21} & B_2^{(2)} \end{pmatrix}$$

it follows that B is permutationally reducible if and only if there exists a permutation matrix P_ν such that

$$P_\nu^T B P_\nu = \begin{pmatrix} B_1^{(1)} & 0 \\ B_{21} & B_2^{(2)} \end{pmatrix},$$

where $B_1^{(1)}$ and $B_2^{(2)}$ are square matrices of order less that n.

By theorem 11.3.2, vol.I, a matrix B is permutationally irreducible if and only if the simply laced quiver $Q(B)$ is strongly connected.

Recall that a matrix $A = (a_{ij}) \in M_n(\mathbf{R})$ is called **positive** if $a_{ij} > 0$ for $i,j = 1,\ldots,n$. If all $a_{ij} \geq 0$, A is called **non-negative**.

In 1907 O.Perron found a remarkable property of the spectra (i.e., characteristic values and characteristic vectors) of positive matrices.

Theorem 6.5.1 (O.Perron).[3] *A positive matrix $A = (a_{ij})$ $(i,j = 1,\ldots,n)$ always has a real and positive characteristic value r which is a simple root of the characteristic equation and which is larger that the absolute values of all other characteristic values. To this maximal characteristic value r there corresponds a characteristic vector $z = (z_1, z_2, \ldots, z_n)$ of A with positive coordinates $z_i > 0$ $(i = 1,\ldots,n)$.*

A positive matrix is a special case of an permutationally irreducible non-negative matrix. G.Frobenius generalized the Perron theorem by investigating the spectral properties of permutationally irreducible non-negative matrices.

Theorem 6.5.2 (Frobenius).[4] *A permutationally irreducible non-negative matrix $A = (a_{ij})$ $i,j = 1,\ldots,n$ always has a positive characteristic value r which is a simple root of the characteristic equation. The absolute values of all the other characteristic values do not exceed r. To the maximal characteristic value r there corresponds a characteristic vector with positive coordinates.*

Moreover, if A has h characteristic values $\lambda_0 = r, \lambda_1, \ldots, \lambda_{h-1}$ of absolute value r, then these numbers are all distinct and are roots of the equation

$$\lambda^h - r^h = 0. \tag{6.5.1}$$

[3] see [Perron, 1907].
[4] see [Frobenius, 1912]

More generally: The whole spectrum $\lambda_0, \lambda_1, \dots, \lambda_{h-1}$ of A, regarded as a system of points in the complex λ-plane, goes over into itself under a rotation of the plane by the angle $2\pi/h$. If $h > 1$, then, by means of a permutation, A can be brought into the following block cyclic form:

$$A = \begin{pmatrix} 0 & A_{12} & 0 & \dots & \dots & 0 \\ 0 & \ddots & A_{23} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \ddots & A_{h-1,h} \\ A_{h1} & 0 & \dots & \dots & 0 & 0 \end{pmatrix}, \tag{6.5.2}$$

where there are square blocks along the main diagonal.

Remark 6.5.1. Let

$$s_i = \sum_{j=1}^{n} a_{ij} \quad (i = 1, 2, \dots, n), \ s = \min_{1 \le i \le n} s_i, \ S = \max_{1 \le i \le n} s_i.$$

Then for a permutationally irreducible matrix $A \geq 0$

$$s \leq r \leq S,$$

and the equality sign on the left or the right of r holds for $s = S$ only; i.e., they hold only when all the row-sums $s_1, s_2, \dots, s_n$ are all equal.

Remark 6.5.2. A permutationally irreducible matrix $A \geq 0$ cannot have two linearly independent non-negative characteristic vectors.

Theorem 6.5.3. *A non-negative matrix $A = (a_{ij})$ $(i, j = 1, \dots, n)$ always has a non-negative characteristic value r such that the moduli (= absolute values) of all the characteristic values of A do not exceed r. To this maximal characteristic value r there corresponds a non-negative characteristic vector*

$$A\vec{y} = r\vec{y} \quad (\vec{y} \geq 0, \vec{y} \neq 0).$$

Let A be a semiperfect ring with Jacobson radical R. Suppose that the quotient ring A/R^2 is right Artinian. In this case the quiver $Q(A)$ of the ring A is defined.

Write $[Q(A)] = (t_{ij})$ for the adjacency matrix of $Q(A)$. Recall that t_{ij} is the number of arrows between i and j, where $VQ(A) = \{1, \dots, n\}$.

By theorem 6.5.3, there exists a non-negative characteristic value r such that the moduli of all the characteristic values of A do not exceed r.

Definition. The maximal characteristic value r of $[Q(A)]$ is called the **index of the ring** A. We shall denote this number by $\operatorname{inx} A$.

Let $A \geq 0$ be a permutationally irreducible matrix. If A has only one eigenvalue of module r $(h = 1)$, then A is called **primitive**, otherwise A is called **imprimitive**.

In the following two remarks it is supposed that all rings are semiperfect and that their quotient rings by the square of the Jacobson radical are right Artinian.

Remark 6.5.3. If a ring A_1 is Morita equivalent to a ring A_2 then $\operatorname{inx} A_1 = \operatorname{inx} A_2$.

Remark 6.5.4. If $A = A_1 \times A_2$ is a direct product of two rings A_1 and A_2 then $\operatorname{inx} A = \max(\operatorname{inx} A_1, \operatorname{inx} A_2)$.

Theorem 6.5.5. *Let A be a Noetherian semiprime and semiperfect ring then A is semisimple Artinian if and only if* $\operatorname{inx} A = 0$.

Proof. If A is a semisimple Artinian then $Q(A) = \{\bullet, \bullet, \dots, \bullet\}$ and $[Q(A)]=0$. So, $\operatorname{inx} A = 0$. Conversely, by remark 6.5.4, we can assume that A is an indecomposable ring. By theorem 14.6.1, vol.I, $Q(A)$ is strongly connected. If $Q(A)$ contains an arrow, then, by remark 6.5.1, $\operatorname{inx} A \geq 1$.

Therefore, $Q(A)$ does not contain an arrow and $[Q(A)] = 0$ and $Q(A) = \{\bullet\}$, by theorem 11.1.9, vol.I. We obtain $A \simeq M_n(D)$, by theorem 11.6.9 vol.I. The theorem is proved.

Let B be a semiperfect ring with Jacobson radical $J = \operatorname{rad} B$. Denote by $H_n(B)$ the following ring:

$$H_n(B) = \begin{pmatrix} B & B & \dots & \dots & B \\ J & B & \ddots & & B \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & B \\ J & \dots & \dots & J & B \end{pmatrix}$$

Let R_n be a Jacobson radical of $H_n(B)$. Obviously,

$$R_n = \begin{pmatrix} J & B & \dots & \dots & B \\ J & J & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & B \\ J & \dots & \dots & J & J \end{pmatrix}, \quad R_n^2 = \begin{pmatrix} J & J & B & \dots & \dots & B \\ J & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & B \\ J & J & \dots & \dots & J & J \\ J^2 & J & \dots & \dots & J & J \end{pmatrix},$$

$$\text{and} \quad R_n/R_n^2 = \begin{pmatrix} 0 & B/J & 0 & \dots & \dots & 0 \\ 0 & 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \ddots & 0 & B/J \\ J/J^2 & 0 & \dots & \dots & 0 & 0 \end{pmatrix}. \tag{6.5.3}$$

Consider the following two chains of inclusions:

$$(i) \qquad B \supset J \supset J^2;$$

$$(ii) \qquad H_n(B) \supset R_n \supset R_n^2.$$

It follows from (6.5.3) that (i) has a composition series if and only if (ii) does. Consequently, the quiver $Q(B)$ is defined if and only if $Q(H_n(B))$ is defined.

Denote by $[Q(B)]$ the adjacency matrix of $Q(B)$. Then (6.5.3) implies

$$[Q(H_n(B))] = \begin{pmatrix} 0 & E_m & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & & 0 & E_m \\ [Q(B)] & 0 & \dots & \dots & 0 & 0 \end{pmatrix}, \tag{6.5.4}$$

where E_m stands for the identity $m \times m$ matrix.

Proposition 6.5.5. *For any Noetherian semiperfect ring A:*

$$\operatorname{inx} H_n(A) = \sqrt[n]{\operatorname{inx} A}$$

Proof. The proof follows from (6.5.4). Indeed, in this case we have $\chi_{[Q(H_n(B))]}(x) = \det(x^n E - [Q(B)])$, where $\chi_M(x)$ is the characteristic polynomial of a square matrix M.

Example 6.5.1.
Let

$$U_n = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}$$

be the square $n \times n$-matrix all of whose entries are 1, and let $E_n \in M_n(\mathbf{R})$ be the identity matrix.

Let $K_n(\mathcal{O})$ be the following tiled order: $K_n(\mathcal{O}) = \{\mathcal{O}, \mathcal{E}(K_n(\mathcal{O}))\}$, where $\mathcal{E}(K_n(\mathcal{O})) = U_n - E_n$. Obviously, the corresponding matrix $(\beta_{ij}) = U_n$ and so $[Q(K_n(\mathcal{O}))] = U_n$ and $\operatorname{inx} K_n(\mathcal{O}) = n$. If $n \to \infty$ then $\operatorname{inx} H_n(K_n(\mathcal{O})) = \sqrt[n]{n}$ goes to 1.

Theorem 6.5.6. *Let A be an indecomposable semiprime and semiperfect Noetherian ring. Then* $\operatorname{inx} A = 1$ *if and only if A is Morita-equivalent to a ring $H_n(\mathcal{O})$, where $\mathcal{O}$ is a discrete valuation ring.*

Proof. We can assume that A is reduced. Suppose that $\operatorname{inx} A = 1$ and in the i-th row of $[Q(A)]$ there are two elements 1. By theorem 14.6.1, vol.I, $Q(A)$ is strongly connected and, by remark 6.5.1, $\operatorname{inx} A > 1$. So in each row of A there is only one 1. The matrices $[Q(A)]^T$ and $[Q(A)]$ are permutationally irreducible simultaneously and their maximal real eigenvalues coincide. Consequently, in each column of A there is only one 1. Thus $[Q(A)] = P_\sigma$ for some $\sigma \in S_n$. From the indecomposability of A and theorem 11.1.9, vol.I, σ is a cycle. By corollary 12.3.7 and theorem 12.3.8, vol.I, A is Morita-equivalent to $H_n(\mathcal{O})$. It is obvious that $\operatorname{inx} \mathcal{O}$ equals 1 for a discrete valuation ring $\mathcal{O}$. It follows from proposition 6.5.6 that $\operatorname{inx} H_n(\mathcal{O}) = 1$. The theorem is proved.

Theorem 6.5.7. *Let A be a tiled order and suppose $Q(A)$ contains n vertices. then* $1 \leq \operatorname{inx} A \leq n$ *and for any integer k* $(1 \leq k \leq n)$ *there exists a tiled order A_k with* $\operatorname{inx} A_k = k$.

Proof. Recall that $[Q(A)]$ is a $(0,1)$-matrix. Therefore the inequalities $1 \leq \operatorname{inx} A \leq n$ follows from remark 6.5.1. We know that $\operatorname{inx} H_n(\mathcal{O}) = 1$ and $\operatorname{inx} K_n(\mathcal{O}) = n$. Let $\sigma \in S_n$ and $\sigma = (12\ldots n)$ be a cyclic permutation. Let $D_k = E_n + P_\sigma + \ldots + P_{\sigma^{k-1}}$, where $2 \leq k \leq n-1$ and let Q_k be the quiver with adjacency matrix D_k. By theorem 6.1.16 the quiver Q_k is admissible. So there exists a tiled order A_k with quiver Q_k. Obviously, $\operatorname{inx}[Q_k] = \operatorname{inx} D_k = k$. The theorem is proved.

6.6 FINITE MARKOV CHAINS AND REDUCED EXPONENT MATRICES

Recall some notions from the theory of Markov chains (see, e.g. [Kemeny, Snell, 1960]).

Let there be n possible states of a certain system

$$S_1, S_2, \ldots, S_n \tag{6.6.1}$$

and a sequence of instants

$$t_0, t_1, t_2, \ldots .$$

Suppose that at each of these instants the system is in one and only one of the states (6.6.1) and that p_{ij} denotes the probability of finding the system in the state S_j at the instant t_k if it is known that at the preceding instant t_{k-1} the system was in the state S_i $(i, j = 1, 2, \ldots, n;\ k = 1, 2, \ldots)$. If the transition probabilities p_{ij} $(i, j = 1, 2, \ldots, n)$ do not depend on the index k (of the instant t_k), then the process is called a **homogeneous Markov chain** with a finite number of states.

The matrix

$$P = (p_{ij}) \in M_n(\mathbf{R})$$

is called the **transition matrix** for the Markov chain. From the above assumptions it is obvious that

$$p_{ij} \geq 0, \qquad \text{and} \qquad \sum_{j=1}^{n} p_{ij} = 1 \ (i, j = 1, 2, \ldots, n). \tag{6.6.2}$$

Definition. A square $n \times n$-matrix $P = (p_{ij}) \in M_n(\mathbf{R})$ is called (row) **stochastic** if P is non-negative and the sum of the elements of each row of P is 1, i.e., if the relations (6.6.2) hold.

Thus, every stochastic matrix can be regarded as the transition matrix for a finite (homogeneous) Markov chain and, conversely, the transition matrix for such a Markov chain is stochastic.

Definition. Let $P = (p_{ij}) \in M_n(\mathbf{R})$ be the transition matrix for a Markov chain MC_n. The **quiver** $Q(MC_n)$ **of the Markov chain** MC_n is the quiver $Q(P)$ of its transition matrix P.

Obviously, $Q(MC_n)$ is a simply laced quiver.

The following definitions are in [Kemeny, Snell, 1960]). A Markov chain is called **ergodic**, if its transition matrix is permutationaly irreducible. An ergodic Markov chain MC_n is called **regular** if its transition matrix is primitive, otherwise MC_n is called **cyclic**.

Definition. A stochastic matrix $P = (p_{ij}) \in M_n(\mathbf{R})$ is called **doubly stochastic** if it also satisfies $\sum\limits_{k=1}^{n} p_{kj} = 1$ for $j = 1, \ldots, n$.

Proposition 6.6.1. *Let S be a doubly stochastic matrix. Then the quiver $Q(S)$ is a disjoint union of strongly connected quivers.*

Proof. Let $S \in M_n(\mathbf{R})$ be a doubly stochastic matrix. Suppose that the quiver $Q(S)$ is connected but non-strongly connected. Then there exists a permutation matrix P_τ such that $P_\tau^T S P_\tau = \begin{pmatrix} S_1 & X \\ 0 & S_2 \end{pmatrix}$.

The matrix $P_\tau^T SP_\tau$ is also doubly stochastic as a product of the doubly stochastic matrices. Therefore S_1^T and S_2 are stochastic matrices (or because, obviously, if S is doubly stochastic so are SP and PS for any permutation matrix P).

Let $S_1 \in M_m(\mathbf{R})$ and $S_2 \in M_{n-m}(\mathbf{R})$, and $m \geq 1$. Denote by $\Sigma(Y)$ the sum of all elements of an arbitrary matrix $Y \in M_n(\mathbf{R})$. Obviously, $\Sigma(P_\tau^T SP_\tau) = \Sigma(S_1) + \Sigma(S_2) + \Sigma(X)$. For any stochastic matrix $S \in M_n(\mathbf{R})$, the equality $\Sigma(S) = \Sigma(S^T) = n$ holds. This sum does not change under a simultaneous transposition of rows and columns. Hence, $\Sigma(P_\tau^T SP_\tau) = n$. Clearly, S_1^T and S_2 are stochastic matrices. Consequently, $n = m + n - m + \Sigma(X)$. Whence, $\Sigma(X) = 0$ and $X = 0$. Thus, the doubly stochastic matrix S is permutationally decomposable. This completes the proof.

Definition. A finite homogeneous Markov chain with transition matrix P is called **ergodic** if the quiver $Q(P)$ is strongly connected. This fits with the definition given on the previous page.

Let Q be a quiver with adjacency matrix $[Q] = (q_{ij})$. We shall refer to the eigenvectors (resp. eigenvalues) of $[Q]$ as the eigenvectors (resp. eigenvalues) of the quiver Q. If Q is strongly connected, then the **index** of Q (written $\operatorname{inx} Q$) is the maximal real eigenvalue of $[Q]$; its eigenvector

$$\vec{f} = (f_1, \ldots, f_n)^T$$

is called its **Frobenius vector**. The numeration of Q is called **standard** if $f_1 \geq f_2 \geq \ldots \geq f_n$

Definition. A quiver Q with $VQ \neq \varnothing$ is called **Frobenius** if it has a positive right eigenvector.

Theorem 6.6.2.[5] *For any Frobenius quiver Q there exists a stochastic matrix P such that $Q(P) = Q$.*

Proof. Suppose $[Q]$ has a positive eigenvector $\vec{z} = (z_1, z_2, \ldots, z_n) > 0$. This means that $z_i > 0$ for $i = 1, \ldots, n$.

Let λ be the eigenvalue corresponding to the eigenvector $\vec{z}$, i.e.,

$$[Q]\vec{z} = \lambda\vec{z} \tag{6.6.3}$$

We shall show that $\lambda > 0$. Since $VQ \neq \varnothing$, $[Q]$ is a nonzero non-negative matrix. Hence, on the left hand side of (6.6.3) we have a nonzero positive vector, and the vector on its right hand side has nonzero coordinates. Consequently, $\lambda\vec{z} > 0$ and $\lambda > 0$. Consider the diagonal matrix $Z = \operatorname{diag}(z_1, z_2, \ldots, z_n)$. Then the matrix $P = (p_{ij}) = \lambda^{-1}Z^{-1}[Q]Z$ is stochastic. Indeed, we have $\sum\limits_{j=1}^{n} q_{ij}z_j = \lambda z_i$ and

[5] Compare this with [Gantmakher, 1960], Ch.13, §6 and [Kostrikin, 2000]

$\sum_{j=1}^{n} p_{ij} = \lambda^{-1} z_i^{-1} \sum_{j=1}^{n} q_{ij} z_j = \lambda^{-1} z_i^{-1} \lambda z_i = 1$. Obviously, $[Q(P)] = [Q]$.

The Markov chain with this stochastic matrix is called the Markov chain of the Frobenius quiver Q.

It follows from the Perron-Frobenius theorem[6] and corollary 11.3.3, vol.I that every strongly connected quiver is Frobenius.

Example 6.6.1.

Let

$$P = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then

$$Q(P) = \begin{tikzcd} \underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} \arrow[loop above] \end{tikzcd}$$

is a Frobenius quiver.

Example 6.6.2.

Let $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$. Then $[Q(P)] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

Obviously, $\chi_{[Q(P)]} = x(x-1)^2(x-2)$ and we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}.$$

Consequently, the quiver of a Markov chain is not necessarily Frobenius.

Recall some facts from section 11.3, vol.I. An arrow $\sigma : i \to j$ of an acyclic quiver Q is called **extra** if there exists also a path from i to j of length greater then 1.

Let Q^* be the condensation[7] of a quiver Q. If we delete from Q^* all extra arrows, then, by proposition 11.3.7, vol.I, we obtain the diagram of a finite partially ordered set, which shall be denoted by $S(Q)$. In particular, with any matrix $B \in M_n(\mathbb{R})$ we associate the finite poset $S(B) = S(Q(B))$.

Definition. Let MC_n be a finite Markov chain. The partially ordered set $SQ(MC_n)$ is called the **associated poset of** MC_n. In particular, if MC_n is ergodic, then $SQ(MC_n)$ contains only one element.

[6]See also [Gantmakher, 1960], Ch.13, §2

[7]See Volume I, Page 277.

6.7 FINITE PARTIALLY ORDERED SETS, (0,1)-ORDERS AND FINITE MARKOV CHAINS

Recall that a tiled order $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$ is called a **(0,1)-order** if $\mathcal{E}(\Lambda)$ is a (0,1)-matrix (see vol.I, section 14.6).

Definition. The **index** (written $\operatorname{inx}\mathcal{P}$) of a finite partially ordered set $\mathcal{P}$ is the maximal real eigenvalue of the adjacency matrix of $\widetilde{Q}(\mathcal{P})$.

Thus, $\operatorname{inx}\mathcal{P} = \operatorname{inx}\Lambda(\mathcal{P})$.

In this case $\widetilde{Q}(\mathcal{P}) = Q(\mathcal{E}_{\mathcal{P}})$, where $\mathcal{E}_{\mathcal{P}}$ is a reduced exponent matrix, corresponding to $\mathcal{P}$ (see section 6.4). By theorem 6.1.15 the quiver $Q(\mathcal{E}_{\mathcal{P}})$ is strongly connected.

Definition. A finite partially ordered set $\mathcal{P}$ is called **regular** if the adjacency matrix $[Q(\mathcal{E}_{\mathcal{P}})]$ of $Q(\mathcal{E}_{\mathcal{P}})$ is primitive, otherwise $\mathcal{P}$ is **cyclic**.

Definition. We say that two finite posets $S = \{s_1, \ldots, s_n\}$ and $T = \{t_1, \ldots, t_n\}$ are Q**-equivalent** if $\widetilde{Q}(S) \simeq \widetilde{Q}(T)$.

This agrees with the definition in section 6.4 above as shown there.

Example 6.7.1.
The index of a finite linearly ordered set CH_n is 1.

Example 6.7.2.
Let

$$ACH_n = \left\{ \begin{matrix} 1 & 2 & 3 & \cdots & n-1 & n \\ \bullet & \bullet & \bullet & \cdots & \bullet & \bullet \end{matrix} \right\}$$

be an antichain of width n. Clearly, $\widetilde{Q}(ACH_n)$ is a complete simply laced quiver with n vertices. Thus $\operatorname{inx} ACH_n = n$.

Example 6.7.3.
Let $\mathcal{P}_{m,n} = (m, m, \ldots, m)$ be a primitive poset formed by n linearly ordered disjoint sets each of length m. It is easy to verify that $\operatorname{inx}\mathcal{P}_{m,n} = \sqrt[m]{n}$.

Example 6.7.4.
Consider

$$\mathcal{P}_4 = \left\{ \begin{tikzcd} \bullet & \bullet \\ \bullet \arrow[u] \arrow[ur] & \bullet \arrow[u] \arrow[ul] \end{tikzcd} \right\}.$$

Denote by

$$U_n = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}$$

the square $n \times n$-matrix whose all entries are 1. Obviously, the adjacency matrix $\widetilde{Q}(\mathcal{P}_4)$ is

$$[\widetilde{Q}(\mathcal{P}_4)] = \begin{pmatrix} 0 & U_2 \\ U_2 & 0 \end{pmatrix}$$

and $\operatorname{inx} \mathcal{P}_4 = 2$

Example 6.7.5.
Let

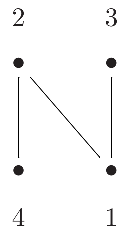
$$\mathcal{P}_{2n} = \left\{ \begin{tikzcd}[row sep=small] 1 & 3 & 5 & & 2n-3 & 2n-1 \\ \bullet \arrow[r] \arrow[dr] & \bullet \arrow[r] \arrow[dr] & \bullet & \cdots & \bullet \arrow[r] \arrow[dr] & \bullet \\ \bullet \arrow[r] \arrow[ur] & \bullet \arrow[r] \arrow[ur] & \bullet & \cdots & \bullet \arrow[r] \arrow[ur] & \bullet \\ 2 & 4 & 6 & & 2n-2 & 2n \end{tikzcd} \right\}.$$

Obviously,

$$[\widetilde{Q}(\mathcal{P}_{2n})] = \begin{bmatrix} 0 & U_2 & 0 & \dots & \dots & 0 \\ 0 & 0 & U_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & \dots & & 0 & U_2 \\ U_2 & 0 & \dots & \dots & 0 & 0 \end{bmatrix}$$

and $\operatorname{inx} \mathcal{P}_{2n} = 2.$

Example 6.7.6.
Let

N :  (6.7.1)

be a partially ordered set with 4 elements. Obviously,

$$[Q(\mathcal{E}_N)] = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and

$$\chi_B(x) = x^2(x^2 - 3).$$

So $\operatorname{inx}(N) = \sqrt{3}$ and the Frobenius eigenvector is

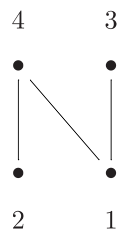
$$\vec{f} = (2,\ \sqrt{3},\ \sqrt{3},\ 1)^T.$$

The numeration (6.7.1) of N is standard with Frobenius eigenvector $(2, \sqrt{3}, \sqrt{3}, 1)^T$. The transition matrix T_N of the Markov chain associated with N is

$$T_N = 1/\sqrt{3} \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 & 0 \\ 0 & 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 2/3 & 0 & 0 & 1/3 \\ 2/3 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Evidently, T_N defines an ergodic cyclic Markov chain. So, the poset N is cyclic. The numeration of N:

 (6.7.2)

is nonstandard and this is in accordance with the Frobenius theorem (see, [Gantmakher, 1998], Section XIII, § 2).

Indeed, let $\mathcal{E}'_N$ be an exponent matrix of N corresponding the numeration (6.7.2). We have

$$[Q(\mathcal{E}'_N)] = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

and the Frobenius eigenvector corresponding the numeration (6.7.2) is $(2, 1, \sqrt{3}, \sqrt{3})^T$. So, the numeration (6.7.2) is not standard.

Below there is the list of indexes and Frobenius vectors of posets with at most four elements.

Remark 6.7.1. Obviously, $\mathrm{inx}\, CH_n = 1$ and $\mathrm{inx}\, ACH_n = n$. The vector $(1, \ldots, 1)^T$ is the Frobenius vector of CH_n and ACH_n.

I. $(1) = \{\, \bullet \,\}$, $\mathrm{inx}\,(I, 1) = 1$.

II. $(1) = \left\{ \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right\}$, $inx\,(II, 1) = 1$; $(2) = \{\, \bullet \quad \bullet \,\}$, $\mathrm{inx}\,(II, 2) = 2$.

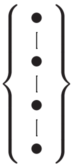
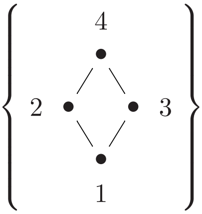
III. $(1) = \left\{ \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \right\}$, $\mathrm{inx}\,(III, 1) = 1$;

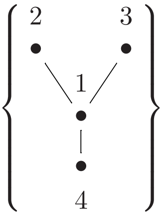
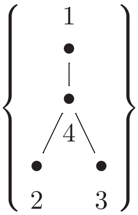
$(2) = \left\{ \begin{array}{ccc} & 3 & \\ & \bullet & \\ / & & \backslash \\ \bullet & & \bullet \\ 1 & & 2 \end{array} \right\}$,

$(3) = \left\{ \begin{array}{ccc} 1 & & 2 \\ \bullet & & \bullet \\ \backslash & & / \\ & \bullet & \\ & 3 & \end{array} \right\}$, $\mathrm{inx}\,(III, 2) = \mathrm{inx}\,(III, 3) = \sqrt{2}$; $\vec{f}(III, 2) = \vec{f}(III, 3) = (1, 1, \sqrt{2})^T$;

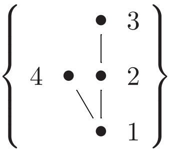
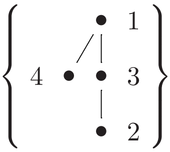
$(4) = \left\{ \begin{array}{cc} & 3 \\ & \bullet \\ & | \\ \bullet & \bullet \\ 1 & 2 \end{array} \right\}$, $\mathrm{inx}\,(III, 4) = \dfrac{1 + \sqrt{5}}{2}$; $(5) = \{\, \bullet \quad \bullet \quad \bullet \,\}$, $\mathrm{inx}\,(III, 5) = 3$;

$\vec{f}(III, 4) = (1, \dfrac{\sqrt{5} - 1}{2}, 1)^T$.

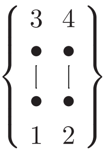
IV. (1) $=$ , $\operatorname{inx}(IV,1) = 1$; (2) $=$ , $\operatorname{inx}(IV,2) = \sqrt[3]{2}$;

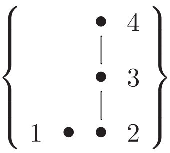
(3) $=$ , (4) $=$ , $\operatorname{inx}(IV,3) = \operatorname{inx}(IV,4) = \sqrt[3]{2}$;

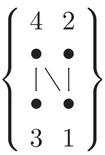
$\vec{f}(IV,2) = \vec{f}(IV,3) = \vec{f}(IV,4) = (\sqrt[3]{4}, 1, 1, \sqrt[3]{2})^T$

(5) $=$ , (6) $=$ ; $\chi_{5,6}(x) = x(x^3 - x - 1)$ and

$1.32 < \operatorname{inx}(IV,5) = \operatorname{inx}(IV,6) < 1.33$; $\vec{f}(IV,5) = \vec{f}(IV,6) = (\lambda^2, 1, \lambda, \lambda)^T$, where $\lambda^3 - \lambda - 1 = 0$.

(7) $=$ , $\operatorname{inx}(IV,7) = \sqrt{2}$; $\vec{f}(IV,7) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1, 1)^T$;

(8) $=$ , $\chi_8(x) = x(x^3 - x^2 - 1)$ and $1.46 < \operatorname{inx}(IV,8) < 1.47$;
$\vec{f}(IV,8) = (1, \lambda - 1, \lambda^2 - \lambda, 1)^T$, where $\lambda^3 - \lambda^2 - 1 = 0$.

(9) $=$ , $\operatorname{inx}(IV,9) = \sqrt{3}$; $\vec{f}(IV,9) = (2, \sqrt{3}, 1, \sqrt{3})^T$.

$$(10) = \left\{ \begin{array}{ccc} & 4 & \\ & \bullet & \\ / & | & \backslash \\ \bullet & \bullet & \bullet \\ 1 & 2 & 3 \end{array} \right\}, \ (11) = \left\{ \begin{array}{ccc} 1 & 2 & 3 \\ \bullet & \bullet & \bullet \\ \backslash & | & / \\ & \bullet & \\ & 4 & \end{array} \right\}, \ \operatorname{inx}(IV,10) = \operatorname{inx}(IV,11) = \sqrt{3};$$

$\vec{f}(IV,10) = \vec{f}(IV,11) = (\sqrt{3}, \sqrt{3}, \sqrt{3}, 3)^T$.

$$(12) = \left\{ \begin{array}{cccc} & & 4 & \\ & & \bullet & \\ & / & & \backslash \\ \bullet & \bullet & & \bullet \\ 1 & 2 & & 3 \end{array} \right\}, \ (13) = \left\{ \begin{array}{cccc} 1 & 2 & & 3 \\ \bullet & \bullet & & \bullet \\ & \backslash & & / \\ & & \bullet & \\ & & 4 & \end{array} \right\},$$

$\operatorname{inx}(IV,12) = \operatorname{inx}(IV,13) = 2$; $\vec{f}(IV,12) = (2,1,1,2)^T$; $\vec{f}(IV,13) = (1,1,1,1)^T$.

$$(14) = \left\{ \begin{array}{cc} 3 & 4 \\ \bullet & \bullet \\ | \times | & \\ \bullet & \bullet \\ 1 & 2 \end{array} \right\}, \ \operatorname{inx}(IV,14) = 2; \ \vec{f}(IV,14) = (1,1,1,1)^T;$$

$$(15) = \left\{ \begin{array}{ccc} & & 4 \\ & & \bullet \\ & & | \\ \bullet & \bullet & \bullet \\ 1 & 2 & 3 \end{array} \right\}, \ \chi_{15}(x) = x^2(x^2 - 2x - 1) \text{ and } \operatorname{inx}(IV,15) = 1 + \sqrt{2};$$

$\vec{f}(IV),15) = (1+\sqrt{2}, 1+\sqrt{2}, 1, 1+\sqrt{2})^T$;

$(16) = \{ \bullet \quad \bullet \quad \bullet \quad \bullet \}$, $\operatorname{inx}(IV,16) = 4$.

Note that the posets $(IV,2)$, $(IV,3)$ and $(IV,4)$ are Q-equivalent. The other non-singleton Q-equivalence classes are $\{(III,2),(III,3)\}$, $\{(IV,10),(IV,11)\}$. For the posets $N = (IV,9)$ and $F_4 = (IV,11)$ we have $\operatorname{inx} N = \operatorname{inx} F_4 = \sqrt{3}$, but N and F_4 are not Q-equivalent.

The posets $(IV,12)$ and $(IV,13)$ are antiisomorphic, and $w(IV,12) = w(IV,13) = 3$ but $(IV,12)$ and $(IV,13)$ are not Q-equivalent, because $\vec{f}(IV,12) = (2,1,1,2)^T$ and $\vec{f}(IV,13) = (1,1,1,1)^T$. The posets $(IV,13)$ and $(IV,14)$ have both index and Frobenius vector equal but they are not Q-equivalent because $\widetilde{Q}(IV,13)$ has a loop and $\widetilde{Q}(IV,14)$ does not.

6.8 ADJACENCY MATRICES OF ADMISSIBLE QUIVERS WITHOUT LOOPS

In [Harary, 1969] (Appendix 2, Digraph diagrams) there is a list of simply laced digraphs without loops for $s \leq 4$ (s is the number of vertices Q). The number of such quivers is 3 for $s = 2$, 16 for $s = 3$, and 218 for $s = 4$. Using this list, it is easy to determine the number of strongly connected quivers among them. This gives 1 for $s = 2$, 5 for $s = 3$, and 83 for $s = 4$.

We shall give the list of admissible quivers without loops for $2 \leq s \leq 4$.

The number of these quivers is 1 for $s = 2$, 2 for $s = 3$ and 11 for $s = 4$.

We use the following notations:

$$H_s = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 0 & 0 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix}, \quad F_s = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix},$$

$\Omega_s = (\omega_{ij})$, where $\omega_{ij} = 0$ for $i \leq j$ and $\omega_{ij} = i-j$ for $i \geq j$; $H_s, F_s, \Omega_n \in M_s(\mathbf{Z})$. Note that notation F_s agrees (up to renumbering) with the notation F_4 for case (IV, 11) at the end of section 6.7 in that $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ is the exponent matrix of the poset $\begin{array}{ccc} 2 & 3 & 4 \\ \bullet & \bullet & \bullet \\ \setminus & | & / \\ & \bullet & \\ & 1 & \end{array}$.

Proposition 6.8.1. *There exists only one admissible quiver without loops for $s = 2$ which is $C_2 = Q(H_2)$ and there are precisely two admissible quivers without loops $C_3 = Q(H_3)$ and $Q(\Omega_3)$ for $s = 3$.*

Proof. In what follows we assume that exponent matrices are reduced and their first rows are zero, which can be done because we are after admissible quivers, which are quivers coming from reduced exponent matrices..

Let $s = 2$. Then $\mathcal{E} = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$, $\mathcal{E}^{(1)} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ and $\mathcal{E}^{(2)} = \begin{pmatrix} (2,\alpha) & (1,\alpha) \\ \alpha+1 & (2,\alpha) \end{pmatrix}$, where $(\alpha_1, \dots, \alpha_t) = \min(\alpha_1, \dots, \alpha_t)$. Here, quite generally, as before, see just below corollary 6.1.14 in section 6.1, the following notation is used. If $\mathcal{E}$ is an (reduced) exponent matrix, $\mathcal{E} = (\alpha_{ij})$, then $\mathcal{E}^{(1)} = (\beta_{ij})$ with $\beta_{ii} = 1$ and $\beta_{ij} = \alpha_{ij}$ if $i \neq j$, $\mathcal{E}^{(2)} = (\gamma_{ij})$, $\gamma_{ij} = \min_K(\beta_{ik} + \beta_{kj})$, so that $[Q(\mathcal{E})] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$. So $[Q(\mathcal{E})] = \begin{pmatrix} (1, \alpha-1) & 1 \\ 1 & (1, \alpha-1) \end{pmatrix}$ and $Q(\mathcal{E})$ is either C_2 for $\alpha = 1$ or $\mathcal{L}C_2$ for $\alpha \geq 2$ (C_n is a simple cycle with n vertices, $[\mathcal{L}C_n] = [C_n] + E_n$, E_n is the identity $n \times n$ matrix).

Let $s = 3$. Then $\mathcal{E} = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & \delta \\ \beta & \gamma & 0 \end{pmatrix}$, $\mathcal{E}^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & \delta \\ \beta & \gamma & 1 \end{pmatrix}$ and $\mathcal{E}^{(2)} =$

$$\begin{pmatrix} (2,\alpha,\beta) & (1,\gamma) & (1,\delta) \\ (\alpha+1,\beta+\delta) & (2,\alpha,\gamma+\delta) & (\alpha,\delta+1) \\ (\beta+1,\alpha+\gamma) & (\beta,\gamma+1) & (2,\beta,\gamma+\delta) \end{pmatrix}.$$

Obviously, one can suppose $1 \le \alpha \le \beta$. Then $\alpha = 1$. Indeed, if $\alpha \ge 2$ we have a loop in the first vertex. If $\beta = 1$ we have either $\mathcal{E} \sim H_3$ or $\mathcal{E} \sim F_3$. Obviously, $\Omega_3 \sim F_3$. If $\beta = 2$ then $\mathcal{E} \sim \Omega_3$.

Let $s = 4$. As above we obtain the admissible quivers without loops, listed below. The notation $\mathcal{E} \sim \Theta$ means equivalence of these matrices by transformations of the first type.

We put $d = d(\mathcal{E}) = \sum_{i,j=1}^{n} \alpha_{ij}$ for an exponent matrix $\mathcal{E} = (\alpha_{ij})$. Obviously, $d(\mathcal{E}) = d(\Theta)$ for equivalent reduced exponent matrices $\mathcal{E}$ and Θ.

It is convenient to place the first six exponent matrices in the following sequence:

(1) $d = \mathbf{6}$, $\mathcal{E}_1 = H_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$, $[Q(\mathcal{E}_1)] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$;

(2) $d = \mathbf{7}$, $\mathcal{E}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{pmatrix}$, $[Q(\mathcal{E}_2)] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

and $\mathcal{E}_2 \sim \Theta_2$, where $\Theta_2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$;

(3) $d = \mathbf{8}$, $\mathcal{E}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{pmatrix}$, $[Q(\mathcal{E}_3)] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

and $\mathcal{E}_3 \sim \Theta_3$, where $\Theta_3 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$;

(4) $d = \mathbf{9}$, $\mathcal{E}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{pmatrix}$, $[Q(\mathcal{E}_4)] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

and $\mathcal{E}_4 \sim \Theta_4$, where $\Theta_4 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$;

(5) $d = \mathbf{10}$, $\mathcal{E}_5 = \Omega_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix}$, $[Q(\Omega_4)] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$;

(6) $d = \mathbf{9}$, $\mathcal{E}_6 = F_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$, $[Q(F_4)] = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$;

(7) $d = \mathbf{8}$, $\mathcal{E}_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{pmatrix}$, $[Q(\mathcal{E}_7)] = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$,

and $\mathcal{E}_7 \sim \Theta_7$, where $\Theta_7 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$;

(8) $d = \mathbf{10}$, $\mathcal{E}_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{pmatrix}$, $[Q(\mathcal{E}_8)] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$;

(9) $d = \mathbf{10}$, $\mathcal{E}_9 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{pmatrix}$, $[Q(\mathcal{E}_9)] = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, $\sigma(\mathcal{E}_9) =$ (1423);

(10) $d = \mathbf{11}$, $\mathcal{E}_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{pmatrix}$, $[Q(\mathcal{E}_{10})] = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$;

(11) $d = \mathbf{11}$, $\mathcal{E}_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{pmatrix}$, $[Q(\mathcal{E}_{11})] = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$.

6.9 TILED ORDERS AND WEAKLY PRIME RINGS

Recall that a ring A is called **semiprime** if it does not have nonzero nilpotent ideals. A ring A is called prime if the product of any two nonzero two-sided ideals of A is nonzero. In what follows in this section any ideal is a two-sided ideal. Let R be the Jacobson radical of a ring A.

Definition. A ring A is called **weakly prime** if the product of any two nonzero ideals not contained in R is nonzero.

Clearly, any prime ring is weakly prime and a weakly prime ring is indecomposable.

Lemma 6.9.1. *If e is a nonzero idempotent of a weakly prime ring A, then the ring eAe is weakly prime.*

Proof. Let $\mathcal{I}$ and $\mathcal{J}$ be two-sided ideals in eAe which are not contained in the Jacobson radical eRe of the ring eAe (see proposition 3.4.8, vol.I), and let $1 = e + f$. Consider the following two-sided ideals

$$\widetilde{\mathcal{I}} = \mathcal{I} + \mathcal{I}eAf + fAe\mathcal{I} + fAe\mathcal{I}eAf$$

and

$$\widetilde{\mathcal{J}} = \mathcal{J} + \mathcal{J}eAf + fAe\mathcal{J} + fAe\mathcal{J}eAf$$

of the ring A. It is clear that $\widetilde{\mathcal{I}} \not\subseteq R$ and $\widetilde{\mathcal{J}} \not\subseteq R$. Thus, $\widetilde{\mathcal{I}}\widetilde{\mathcal{J}} \neq 0$. On the other hand,

$$\widetilde{\mathcal{I}}\widetilde{\mathcal{J}} = \mathcal{I}\mathcal{J} + \mathcal{I}\mathcal{J}eAf + fAe\mathcal{I}\mathcal{J} + fAe\mathcal{I}\mathcal{J}eAf.$$

This implies $\mathcal{I}\mathcal{J} \neq 0$ and so the lemma is proved.

Theorem 6.9.2. *Let $1 = e_1 + e_2 \ldots + e_n$ be a decomposition of the unity of a semiperfect ring A into the sum of mutually orthogonal local idempotents and $A_{ij} = e_iAe_j$ $(i, j = 1, \ldots, n)$. The ring A is weakly prime if and only if $A_{ij} \neq 0$ for all i, j.*

Proof. Assume that the ring A is weakly prime and $A_{pq} = 0$ for some $p \neq q$. Consider the ring $C = A_{pp} + A_{qq} + A_{qp}$. Then, $Z = A_{pp} + A_{qp}$ and $N = A_{qq} + A_{qp}$ are two-sided ideals of the ring C that do not belong to the Jacobson radical of C and such that $ZN = 0$. But this contradicts lemma 6.9.1.

We shall show that, if all $A_{ij} \neq 0$, then A is weakly prime. There exists a decomposition of the identity of the ring A into the sum of mutually orthogonal idempotents $1 = f_1 + \ldots + f_s$ such that $f_iAf_i = M_{n_i}(\mathcal{O}_i)$, with local rings $\mathcal{O}_i$, $i = 1, \ldots, s$, and $f_iAf_j = f_iRf_j$ $(i \neq j,\ i, j = 1, \ldots, s)$ (see proposition 11.1.1, vol.I).

Consider two-sided ideals $\mathcal{I}$ and $\mathcal{J}$ such that $\mathcal{I} \nsubseteq R$, $\mathcal{J} \nsubseteq R$. Let $f_k\mathcal{I}f_l = \mathcal{I}_{kl}$ and $f_p\mathcal{J}f_q = \mathcal{J}_{pq}$. It is clear that there exists an index t_0 such that $\mathcal{I}_{t_0t_0} \nsubseteq R$. Without loss of generality, we can assume that $t_0 = 1$. Similarly, there exists an index t_1 such that $\mathcal{J}_{t_1t_1} \nsubseteq R$. If $t_1 = 1$, then $\mathcal{I}\mathcal{J} \neq 0$. Hence, we can assume that $t_1 = 2$ and $\mathcal{J}_{22} = M_{n_2}(\mathcal{O}_2)$. Since $\mathcal{I}_{11}f_1Af_2 = f_1Af_2 \subseteq \mathcal{I}$ and $f_1Af_2\mathcal{J}_{22} = f_1Af_2 \subseteq \mathcal{J}$, we obtain $f_1\mathcal{I}f_1f_1\mathcal{J}f_2 = f_1Af_1f_1Af_2 = f_1Af_2 \neq 0$. The theorem is proved.

Here is an example of an indecomposable semiprime ring which is not weakly prime. Let $\mathbf{Q}$ be the field of rational numbers and let $\mathbf{Z}_{(p)}$ (p is a prime integer) be the ring of p-integral numbers, i.e., $\mathbf{Z}_{(p)} = \{\frac{m}{n} \in \mathbf{Q} \mid (n,p) = 1\}$ (see example 1.1.9, vol.I). Consider the $\mathbf{Q}$-algebra $M_2(\mathbf{Q}) \times M_2(\mathbf{Q})$. Let $e_{11}, e_{12}, e_{21}, e_{22}$ be the matrix units of the first matrix ring and $f_{11}, f_{12}, f_{21}, f_{22}$ be the matrix units of the second matrix ring. Let A be the $\mathbf{Z}_{(p)}$-order in the $\mathbf{Q}$-algebra $M_2(\mathbf{Q}) \times M_2(\mathbf{Q})$ with $\mathbf{Z}_{(p)}$-basis $e_{11}, pe_{12}, e_{21}, pe_{22}, e_{22} + f_{11}, pf_{12}, f_{21}, f_{22}$. Let $e_1 = e_{11}, e_2 = e_{22} + f_{11}, e_3 = f_{22}$. Then, $1 = e_1 + e_2 + e_3$ is a decomposition of $1 \in A$ into the sum of mutually orthogonal local idempotents. Clearly, $e_1Ae_3 = 0$ and, hence, A is not weakly prime, by theorem 6.9.2. On the other hand, the ring A is semiprime as an order in $M_2(\mathbf{Q}) \times M_2(\mathbf{Q})$.

Theorem 6.9.3. *The quiver $Q(A)$ of a weakly prime semiperfect Noetherian ring A is strongly connected.*

The proof follows from theorem 11.6.3, vol.I and theorem 6.9.2.

Proposition 6.9.4. *If A is a tiled order, then the quotient ring $B = A/\pi A$ is weakly prime Noetherian and semidistributive.*

Proof. Since A is semidistributive and Noetherian, B is the same. Let $1 = e_{11} + \ldots + e_{nn}$ be the decomposition of $1 \in A$ into the sum of mutually orthogonal matrix idempotents. Write $\bar{e}_{ii} = e_{ii} + \pi A$ for $i = 1, \ldots, n$. Obviously, $\bar{e}_{ii}B\bar{e}_{jj} \neq 0$ for $i, j = 1, \ldots, n$. So, by theorem 6.9.2, B is weakly prime.

Proposition 6.9.5. *Let A be a tiled order and $B = A/\pi A$. The quiver $Q(B)$ of the ring B is obtained from the quiver $Q(A)$ of the tiled order A by deleting all loops.*

Proof. We can assume that A is reduced. Obviously, for any $i = 1, \ldots, n$ we have $\bar{e}_{ii}B\bar{e}_{ii} \simeq \mathcal{O}/\pi\mathcal{O} = T$, where T is a division ring. Let $R(B)$ be the Jacobson radical of B. Since A is reduced, B is also reduced and $\bar{e}_{ii}R(B)\bar{e}_{ii} = 0$ for $i = 1, \ldots, n$. By the Q-lemma (vol.I, p.266), $Q(B)$ has no loops.

Recall that $Q(A)$ is simply laced and strongly connected. Let there exist an arrow from i to j in $Q(A)$ $(i \neq j)$. This means that $e_{ii}Re_{jj} = \pi^{\alpha_{ij}}\mathcal{O}$ and $e_{ii}R^2e_{jj} = \pi^{\alpha_{ij}+1}\mathcal{O}$, where R is the Jacobson radical of A. Therefore in $Q(B)$ there exists an arrow from i to j $(i \neq j)$, and so in $Q(A)$ there exists an arrow from i to j.

Let $\mathcal{O} = k[[x]]$ be the ring of formal power series over a field k. We know that $\mathcal{O}$ is a discrete valuation ring with a prime element x. Let $A = \{k[[x]], \mathcal{E}(A) = (\alpha_{ij})\}$ be a reduced tiled order. Then A/xA is an n^2-dimensional weakly prime semidistributive algebra over a field k, where n is the number of vertices in $Q(A)$.

Theorem 6.9.6. *Let Q be an arbitrary simply laced quiver without loops. There exists a weakly prime semidistributive Artinian ring B such that $Q(B) = Q$.*

Proof. Consider the quiver Q_1 with the following adjacency matrix: $[Q_1] = [Q] + E_n$. By theorem 6.1.16, every strongly connected simply laced quiver Q_1 with a loop in each vertex is admissible. So there exists a reduced exponent matrix $\mathcal{E} = (\alpha_{ij})$ such that $Q(\mathcal{E}) = Q_1$. Let $\mathcal{O}$ be a discrete valuation ring with a prime element π and let $A = \{\mathcal{O}, \mathcal{E}\}$ be a tiled order. Obviously, by proposition 6.9.5, the quiver $Q(B)$ coincides with Q, where $B = A/\pi A$.

Corollary 6.9.7. *For any simply laced quiver Q with n vertices and for any field k there exists a weakly prime semidistributive n^2-dimensional algebra B over k such that $Q(B) = Q$.*

Proof. It is sufficient to consider the ring $A = \{\mathcal{O}, \mathcal{E}\}$, where $\mathcal{O} = k[[x]]$.

Remark 6.9.1 H.Fujita in [Fujita, 2003] introduced an interesting class of finite dimensional algebras in the following way:

Let K be a field and n an integer with $n \geq 2$. Let $\mathbb{A} = (A_1, \ldots, A_n)$ be an n-tuple of $n \times n$ matrices $A_k = (a_{ij}^{(k)}) \in M_n(K)$ $(1 \leq k \leq n)$ satisfying the following three conditions.

(A1) $a_{ij}^{(k)} a_{il}^{(j)} = a_{il}^{(k)} a_{kl}^{(j)}$ for all $i, j, k, l \in \{1, \ldots, n\}$,

(A2) $a_{kj}^{(k)} = a_{ik}^{(k)} = 1$ for all $i, j, k \in \{1, \ldots, n\}$, and

(A3) $a_{ii}^{(k)} = 0$ for all $i, k \in \{1, \ldots, n\}$ such that $i \neq k$.

Let $A = \bigoplus\limits_{1 \leq i,j \leq n} Ku_{ij}$ be a K-vector space with basis $\{u_{ij} \,|\, 1 \leq i, j \leq n\}$. Then, using $\mathbb{A}$, define a multiplication of A as follows:

$$u_{ik}u_{lj} = \begin{cases} a_{ij}^{(k)}u_{ij}, & \text{if } k = l \\ 0 & \text{otherwise} \end{cases}.$$

Then A is an associative n^2-dimensional K-algebra with Jacobson radical $R(A) = \bigoplus\limits_{i \neq j;\, 1 \leq i,j \leq n} Ku_{ij}$. So $A/R(A) = \underbrace{K \times K \times \ldots \times K}_{n\,\text{times}}$, i.e., A is a split reduced n^2-dimensional associative K-algebra. In what follows, we assume that $a_{ij}^{(k)} = 0$ or 1 for all $1 \leq i, j, k \leq n$. We call such algebras as **Fujita algebras**.

Recall that a basis $e_1, \ldots, e_t$ of a finite dimensional algebra B is multiplicative (= monomial) if a product e_ie_j is either zero or e_k for some $k = 1, \ldots, t$. It is easy to see that a Fujita algebra has a multiplicative basis.

Obviously, $u_{ii}u_{jj} = \delta_{ij}u_{jj}$, where δ_{ij} is the Kronecker delta. For $1 \in A$ we have $1 = u_{11} + \ldots + u_{nn}$. Therefore for any Fujita algebra A it is the case that: $u_{ii}Au_{jj} = K$, $1 \leq i, j \leq n$. By theorem 6.9.2 and theorem 14.2.1, vol. I, any Frobenius algebra A is weakly prime and semidistributive. By theorem 14.3.1, vol. I and theorem 6.9.3 the quiver $Q(A)$ is a simply laced and strongly connected quiver without loops.

We remark without proof, that for any simply laced strongly connected quiver Q which has no loops, there exists a Fujita algebra A with quiver $Q(A) = Q$. The properties of Fujita algebras are considered in [Fujita, 2006].

As an example of weakly prime semiperfect rings we consider right 2-rings. Recall that a ring A is called a **right 2-ring** if each right ideal in A is two-generated.

Lemma 6.9.8. *If A is a semiperfect Noetherian weakly prime 2-ring and $A = \bigoplus\limits_{i=1}^{s} P_i^{n_i}$ is the decomposition as in lemma 6.5.9, then $n_1 = n_2 = \ldots = n_s$.*

Proof. Assume that the modules $P_1, P_2, \ldots, P_s$, are numbered in such a way that $n_1 \geq n_2 \ldots \geq n_s$. Suppose that $n_1 = n_2 = \ldots = n_k$ but $n_k > n_{k+1}$. We shall show that $e_jAe_i \simeq \mathrm{Hom}_A(P_{k+1}, P_k) = 0$. Indeed, otherwise exists a nonzero homomorphism $\psi : P_{k+1} \to P_k$. Obviously, $P_{k+1}^{n_k}$ is a projective cover of $(\mathrm{Im}\,\psi)^{n_k}$. Therefore a projective cover of the finitely generated right ideal $\mathcal{I} = \mathrm{Im}\,\psi)^{n_k} \oplus P_{k+1}^{n_{k+1}}$ is equal to $P_{k+1}^{n_k+n_{k+1}}$, where $n_k + n_{k+1} > 2n_{k+1}$. Let $\mu_A(\mathcal{I})$ be the minimal number of generators of the right ideal $\mathcal{I}$. Then, by lemma 11.1.8, vol.I, we obtain that $\mu_A(\mathcal{I}) \geq 3$. Let $1 = e_1 + \ldots + e_n$ be a decomposition of $1 \in A$ into a sum of local idempotents such that for some e and f from this decomposition we have $P_k = eA$ and $P_{k+1} = fA$. Consequently, $\mathrm{Hom}_A(P_{k+1}, P_k) \simeq eAf = 0$. We have a contradiction with theorem 6.9.2. The lemma is proved.

Theorem 6.9.9. *The quiver $Q(A)$ of a Noetherian weakly prime semiperfect 2-ring A consists of at most two points. If $Q(A)$ consists of one point, then there are at most two loops at this point. If $Q(A)$ consists of two points, then there is only one arrow from one point to another and each point has at most one loop, i.e., the two-pointed quiver $Q(A)$ is simply laced.*

Proof. By lemma 6.9.8 we can assume that $A = P_1^n \oplus \ldots \oplus P_s^n$. But then $A \simeq M_n(\mathrm{End}\,(P_1 \oplus \ldots \oplus P_s))$ and the ring $B = \mathrm{End}\,_A(P_1 \oplus \ldots \oplus P_s)$ is a basic ring, i.e., the quotient ring of B by its Jacobson radical is a direct product of division rings.

It is clear that the rings A and B are 2-rings simultaneously.

Let A be a basic Noetherian semiperfect weakly prime 2-ring, and let $1 = e_1 + \ldots + e_n$ be a decomposition of $1 \in A$ into a sum of pairwise orthogonal idempotents.

Assume that the quiver $Q(A)$ of the ring A consists of more then two points. Let $P_1 = e_1A, P_2 = e_2A, P_3 = e_3A$ be nonisomorphic indecomposable projective modules. Since $e_1Ae_3 \neq 0$ and $e_2Ae_3 \neq 0$, there exist nonzero homomorphisms $\phi : P_3 \to P_1$ and $\psi : P_3 \to P_2$. Thus, P_3^3 is a projective cover of the right ideal $\operatorname{Im}\phi \oplus \operatorname{Im}\psi \oplus P_3$ which contradicts, by lemma 11.1.8, vol.I, with the fact that A is a 2-ring. Hence, the quiver $Q(A)$ consists of at most two points. If it has only one point, then there are at most two loops at this point by lemma 11.1.8, vol.I.

Assume now that $Q(A)$ consists of two points and $A = P_1 \oplus P_2$. Suppose that the point 1, for example, has more than one loop. Since A is weakly prime, there exists a nontrivial homomorphism $\nu : P_1 \to P_2$. Then P_1^3 is a direct summand of the projective cover of the right ideal $P_1R \oplus \operatorname{Im}\nu$ which gives a contradiction, by lemma 11.1.8, vol.I. Thus, each point of $Q(A)$ has at most one loop. Similarly, one can prove that there is only one arrow from one point to another. The theorem is proved.

From theorem 6.9.9 we immediately obtain the following corollaries.

Corollary 6.9.10 (**Theorem of reduction for Noetherian weakly prime 2-rings**.) *Any Noetherian weakly prime semiperfect right 2-ring A is isomorphic to a ring $M_n(B)$, where B is either a local 2-ring or $B = P_1 \oplus P_2$ (where P_1, P_2 are nonisomorphic, indecomposable projective B-modules), and the quiver $Q(B)$ is a two-pointed simply laced quiver.*

Corollary 6.9.11. *Let A be a Noetherian weakly prime semiperfect and semidistributive right 2-ring. Then A is isomorphic to a ring $M_n(B)$, where B is either a discrete valuation ring or a uniserial Artinian ring. In this case the quiver $Q(A)$ has one vertex. If $Q(A)$ contains two vertices then it is a simply laced quiver.*

Corollary 6.9.12. *If a tiled order A is a right 2-ring, then either $A \simeq M_n(\mathcal{O})$, where $\mathcal{O}$ is a discrete valuation ring or $A = M_n(B)$, where*

$$B = \left\{ \mathcal{O}, \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} \right\}$$

and $\begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} \in M_2(\mathbf{Z})$.

If M is a uniserial A-module such that $\bigcap\limits_{n=1}^{\infty} MR^m = 0$, where $R = \operatorname{rad} A$, and $m \in MR^t \setminus MR^{t+1}$, we write $t = d(m)$. Let M be an arbitrary A-module. An element $m \in M$ is called a **primitive generator** if $m = me$ for some primitive idempotent $e \in A$. In this case there exists a homomorphism $\varphi : eA \to mA$ such that $\varphi(e) = m$.

Proposition 6.9.13. *If B is a reduced Noetherian SPSD-ring with two-pointed quiver $Q(B)$, then B is a right and left 2-ring.*

Proof. Let $1 \in B$ and $1 = e_1 + e_2$ be a decomposition of 1 into a sum of two local idempotents. Write $B_i = e_iBe_i$ and $R_i = \mathrm{rad}\,(B_i)$, $i = 1, 2$. Let $X = e_1Be_2$ and $Y = e_2Be_1$. By theorem 14.2.1, vol. I, B_1 and B_2 are uniserial rings, X is a uniserial right B_2-module and a uniserial left B_1-module; Y is a uniserial right B_1-module and a uniserial left B_2-module. Moreover, by corollary 14.2.4, vol. I, $R_1X = XR_2$ and $YR_1 = R_2Y$.

Let $\mathcal{J}$ be a right ideal in B. There is the decomposition $\mathcal{J} = \mathcal{J}e_1 \oplus \mathcal{J}e_2$. Obviously, all nonzero elements from $\mathcal{J}e_1$ are primitive generators for e_1 and all nonzero elements from $\mathcal{J}e_2$ are primitive generators for e_2. If $b \in \mathcal{J}e_1$, then $b = \begin{pmatrix} a & 0 \\ y & 0 \end{pmatrix}$. Let $t_0 = \min d(a)$, where $\begin{pmatrix} a & 0 \\ y & 0 \end{pmatrix} \in \mathcal{J}e_1$ and $\begin{pmatrix} a_0 & 0 \\ y_0 & 0 \end{pmatrix} \in \mathcal{J}e_1$ with $d(a_0) = t_0$. Consequently, for any $\begin{pmatrix} a & 0 \\ y & 0 \end{pmatrix} \in \mathcal{J}e_1$ we have $a = a_0a_1$, where $a_0, a_1, a \in B_1$. Obviously,

$$\begin{pmatrix} a_0 & 0 \\ y_0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_0a_1 & 0 \\ y_0a_1 & 0 \end{pmatrix} \in \mathcal{J}e_1.$$

Let $t_1 = \min d(y)$, where $\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \in \mathcal{J}e_1$. We obtain that $y = y_1a_1$, where $d(y_1) = t_1$ and $a_1 \in B_1$. Let $\varphi_1 : e_1B \to \mathcal{J}$ be a homomorphism with $\varphi_1(e_1) = \begin{pmatrix} a_0 & 0 \\ y_0 & 0 \end{pmatrix}$ and let $\varphi_2 : e_1B \to \mathcal{J}$ be a homomorphism with $\varphi_2(e_1) = \begin{pmatrix} 0 & 0 \\ y_1 & 0 \end{pmatrix}$. Consider $\varphi : e_1B \oplus e_1B \to \mathcal{J}$, where $\varphi = (\varphi_1, \varphi_2)$ and $\varphi(e_1b_1, e_1b_2) = \varphi_1(e_1b_1) + \varphi_2(e_2b_2)$. Obviously, $\mathrm{Im}\,\varphi \supseteq \mathcal{J}e_1$. Analogously, there exists a homomorphism $\psi : e_2B \oplus e_2B \to \mathcal{J}e_2$ such that $\mathrm{Im}\,\psi \supseteq \mathcal{J}e_2$. By lemma 11.1.8, vol.I, we obtain that $\mu_A(\mathcal{J}) \leq 2$. Analogously, for any left ideal $L \subset A$ we obtain that $\mu_A(L) \leq 2$. The proposition is proved.

Let's consider a Noetherian reduced $SPSD$-ring B with two-pointed quiver in more detail. Let B be weakly prime. Then there are the following possibilities for the quiver of B (up to a renumbering of vertices).

(a) $\begin{tikzcd} \overset{1}{\bullet} \arrow[r, bend left] & \overset{2}{\bullet} \arrow[l, bend left] \end{tikzcd}$

(b) $\begin{tikzcd} \overset{1}{\bullet} \arrow[loop left] \arrow[r, bend left] & \overset{2}{\bullet} \arrow[l, bend left] \end{tikzcd}$

(c) $$\begin{tikzcd} \overset{1}{\bullet} \arrow[loop left] \arrow[r, bend left] & \overset{2}{\bullet} \arrow[loop right] \arrow[l, bend left] \end{tikzcd}$$

Case (a). In this case the ring B is serial, by theorem 12.3.11, vol.I. We shall study in which cases B is not Artinian. Let $1 \in B$ and let $1 = e_1 + e_2$ be a decomposition of 1 into a sum of local pairwise orthogonal idempotents. Write $R = \operatorname{rad} B$, $B_i = e_iBe_i$ and $R_i = \operatorname{rad}(B_i)$, $X = e_1Be_2$, $Y = e_2Be_1$, where B_1 and B_2 are uniserial rings, X is an uniserial right B_2-module and an uniserial left B_1-module; Y is an uniserial right B_1-module and an uniserial left B_2-module, by theorem 14.2.1, vol.I. If B_1 and B_2 are Artinian, then B is Artinian. Let R be the Jacobson radical of B. Then, as usual,

$$R = \begin{pmatrix} R_1 & X \\ Y & R_2 \end{pmatrix} \text{ and } R^2 = \begin{pmatrix} R_1^2 + XY & R_1X + XR_2 \\ YR_1 + R_2Y & YX + R_2^2 \end{pmatrix}.$$

Moreover, by corollary 14.2.4, vol.I, $R_1X = XR_2$ and $YR_1 = R_2Y$.

Case (b). Suppose that $Q(B)$ is the quiver of the ring B in this case. By the Q-lemma (see vol.I, p.266), we obtain $YX + R_2^2 = R_2$. By the Nakayama lemma, $YX = R_2$. So $XYX = XR_2$ and $XYX \subseteq R_1^2X = XR_2^2$. Therefore, $XR_2 \subseteq XR_2^2$ and, again by the Nakayama lemma, $XR_2 = 0$, i.e., $XYX = 0$. Note that $R_1X = XR_2$. Consequently, $R_1X = 0$. Analogously, $YR_1 = R_2Y = 0$. Now note that $R_2^2 = 0$. Indeed, $YXYX = R_2^2 = 0$. If $XY \neq 0$, then $XY = R_1^m$. Therefore, $R_1^{m+1} = XYR_1 = XR_2Y = XYXY = 0$ and B is an Artinian semidistributive ring. We obtain that if B_1 is a discrete valuation ring then $XY = 0$. Suppose that B_1 is a discrete valuation ring. Then there is the following countable descending Loewy series:

$$B \supset R \supset \begin{pmatrix} R_1^2 & 0 \\ 0 & R_2 \end{pmatrix} \supset \begin{pmatrix} R_1^3 & 0 \\ 0 & 0 \end{pmatrix} \supset \ldots \supset \begin{pmatrix} R_1^m & 0 \\ 0 & 0 \end{pmatrix} \supset \ldots$$

and $\bigcap\limits_{m=0}^{\infty} R^m = 0$. Note that e_2B is serial and that $l(e_2B) \leq 3$.

Case (c). If B is not Artinian, then B_1 is a discrete valuation ring (up to a renumbering). Assume that B_2 is Artinian. If $XY \neq 0$, then $XY = R_1^m$. For some t we have $(YX)^t = 0$, since $YX \subseteq R_2^2$ and R_2 is nilpotent. Therefore, $X(YX)^tY = R_1^{m(t+1)} = 0$. This contradiction proves that $XY = 0$.

Let $B_i = \mathcal{O}_i$ $(i = 1, 2)$ be discrete valuation rings. If $XY \neq 0$, but $YX = 0$ we obtain that B_1 is Artinian. Therefore, XY and YX are not equal to zero simultaneously. Let $XY = R_1^m \neq 0$ and $YX = R_2^m \neq 0$. Obviously, $XYX = R_1^mX = XR_2^n$. If $m \neq n$, then B must be Artinian. Consequently, $m = n$. If $R_1^tX = 0$ for some t, then $R_1^tXY = R_1^{t+m} = 0$. It is easy to see that B is a prime ring, i.e., B is a tiled order, and $B \simeq \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \pi^m\mathcal{O} & \mathcal{O} \end{pmatrix}$, where $m \geq 1$.

The last case is the following: $XY = 0$ and $YX = 0$.

In this case there are the following countable descending Loewy series for $P_1 = e_1B$ and $P_2 = e_2B$:

$$P_1 \supset P_1R = (R_1, X) \supset P_1R^2 = (R_1^2, R_1X) \supset \ldots \supset P_1R^m = (R_1^m, R_1^{m-1}X) \supset \ldots$$

and

$$P_2 \supset P_2R = (Y, R_2) \supset P_2R^2 = (R_2Y, R_2^2) \supset \ldots \supset P_2R^m = (R_2^{m-1}Y, R_2^m) \supset \ldots .$$

In the following proposition the ring B is a Noetherian but non-Artinian reduced ring with two-pointed quiver $Q(B)$.

Proposition 6.9.14. *If B is not a tiled order then there exists a two-sided nilpotent ideal $\mathcal{J}$ of B such that the quotient ring $B/\mathcal{J}$ is serial (it is possible $\mathcal{J} = 0$).*

Proof. If $Q(B)$ is strongly connected and B is weakly prime then the proof follows from the consideration of the cases: (a), (b) and (c) above. Let B be not weakly prime. Up to renumbering we may assume that $Y = 0$. If $X = 0$, then $B = B_1 \times B_2$, where B is serial (B_1 and B_2 are uniserial). If $X \neq 0$, then $L = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ is a two-sided ideal of B, $L^2 = 0$ and $B/L \simeq B_1 \times B_2$ is serial.

Let A be a ring. Denote $\mu_r^*(A) = \max\limits_{\mathcal{I} \subseteq A} \mu_A(\mathcal{I})$, where $\mathcal{I}$ is a right ideal A. Analogously, one can define $\mu_l^*(A)$. By definition, A is a right principal ideal ring if and only if $\mu_r^*(A) = 1$.

Example 6.9.1 (I.N.Herstein).[8]

Let $\mathbf{Q}$ be the field of rational numbers and let $\mathbf{Q}(x)$ be the field of rational functions over $\mathbf{Q}$ in an indeterminate x. Write $K = \mathbf{Q}(x)$. Define a monomorphism $\varphi : K \to K$ by $(\varphi(f))(x) = f(x^2)$. Let $K[y, \varphi] = \{\sum y^i \alpha_i \ : \ \alpha_i \in K\}$ be the ring of right polynomials in y over K, where equality and the addition are defined as usual but the multiplication is defined by means of the rule $\alpha y = y\varphi(\alpha)$ for $\alpha \in K$. It is easy to show that every right ideal of $K[y, \varphi]$ is principal, and so $K[y, \varphi]$ is right Noetherian, but $K[y, \varphi]$ is not left Noetherian.

Note that $\mu_r^*(K[y, \varphi]) = 1$ and $\mu_l^*(K[y, \varphi]) = \infty$.

Let $\mathcal{O} = K[[y, \varphi]]$ be a ring of formal power series, where K is as above and $\alpha y = y\varphi(\alpha)$. Denote by $\mathcal{M} = y\mathcal{O}$ the unique right maximal ideal in $\mathcal{O}$. Therefore, $\mathcal{M}$ is the Jacobson radical of $\mathcal{O}$ and $\mathcal{O}/\mathcal{M} = K$. Obviously, $\mathcal{O}$ is right uniserial and every right ideal of $\mathcal{O}$ is principal, but $\mathcal{O}$ is not left Noetherian. So, $\mu_r^*(\mathcal{O}) = 1$ and $\mu_l^*(\mathcal{O}) = \infty$.

Let $\mathcal{O}$ be as above and $\mathcal{M} = y\mathcal{O}$. Consider the ring B of 2×2-matrices of the following form:

[8] see [Herstein, 1968]

$$B = \begin{pmatrix} \mathcal{O} & K \\ K & K \end{pmatrix}.$$

We describe the multiplication and addition in B. Denote by $e_{11}, e_{12}, e_{21}, e_{22}$ the matrix units of B: $e_{12}e_{21} = 0$ and $e_{21}e_{12} = 0$. Let $\varphi : \mathcal{O} \to K$ be the canonical epimorphism. If $\alpha \in \mathcal{O}$, then $(\alpha e_{11})e_{12} = \varphi(\alpha)e_{12} = e_{12}\varphi(\alpha)$ and $e_{21}(\alpha e_{11}) = e_{21}\varphi(\alpha) = \varphi(\alpha)e_{21}$. Further, $\alpha e_{11} = e_{11}\alpha$ for $\alpha \in \mathcal{O}$ and $\beta e_{22} = e_{22}\beta$ for $\beta \in K$. The multiplication in K is defined as multiplication of 2×2-matrices and the addition is defined elementwise. It is easy to see that $\mu_r^*(B) = 2$ and $\mu_l^*(B) = \infty$.

6.10 GLOBAL DIMENSION OF TILED ORDERS

In this section we want to give a short survey on some more fairly recent results of the theory of tiled orders. Most of these results are presented without proofs.

Theorem 2.3.24, vol.I, asserts that the ideals in a semisimple ring

$$A = M_{n_1}(\mathcal{D}_1) \times M_{n_2}(\mathcal{D}_2) \times \ldots \times M_{n_s}(\mathcal{D}_s)$$

form a finite Boolean algebra consisting of 2^s elements. Therefore, any two-sided ideal $\mathcal{I}$ in A is idempotent, i.e., $\mathcal{I}^2 = \mathcal{I}$.

Let A be an associative ring with $1 \neq 0$. Denote by $I(A)$ the set of all idempotent ideals in A.

Proposition 6.10.1. *The set $I(A)$ is a commutative band under addition.*

Proof. Let $\mathcal{J}_1, \mathcal{J}_2 \in I(A)$. Then $(\mathcal{J}_1+\mathcal{J}_2)^2 = \mathcal{J}_1^2+\mathcal{J}_1\mathcal{J}_2+\mathcal{J}_2\mathcal{J}_1+\mathcal{J}_2^2 = \mathcal{J}_1+\mathcal{J}_2$.

Obviously, for any idempotent $e \in A$ the ideal AeA is idempotent. This follows from the following inclusions: $AeAAeA \subset AeA$ and $AeA \subset AeAAeA$.

Recall that an associative ring A with 1 is called **semiprimary** if its Jacobson radical R is nilpotent and A/R is a semisimple ring. An ideal $\mathcal{J}$ of A is said to be **heredity** if $\mathcal{J}^2 = \mathcal{J}$, $\mathcal{J}R\mathcal{J} = 0$ and $\mathcal{J}$, considered as a right A-module $\mathcal{J}_A$, is projective. In fact, this also implies that the left A-module ${}_A\mathcal{J}$ is projective. A semiprimary ring A is called **quasi-hereditary** if there is a chain

$$0 = \mathcal{J}_0 \subset \mathcal{J}_1 \subset \ldots \subset \mathcal{J}_{t-1} \subset \mathcal{J}_t \subset \ldots \subset \mathcal{J}_m = A$$

of ideals of A such that, for any $1 \leq t \leq m$, $\mathcal{J}_t/\mathcal{J}_{t-1}$ is a heredity ideal of $A/\mathcal{J}_{t-1}$. Such a chain of idempotent ideals is called a **heredity chain**.

Let $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$ be a tiled order over a discrete valuation ring $\mathcal{O}$ and let $M_n(\mathcal{D})$ be its classical ring of fractions, where $\mathcal{D}$ is the classical division ring of fractions of $\mathcal{O}$, and write $\mathcal{E}(\Lambda) = (\alpha_{ij})$. Let $\mathcal{E}(\Lambda)^T = (\alpha_{ji})$ and $\Lambda^T = \{\mathcal{O}, \mathcal{E}(\Lambda)^T\}$. Then the following proposition is obvious.

Proposition 6.10.2. *Λ^T is a tiled order and Λ is anti-isomorphic to Λ^T.*

Proposition 6.10.3. gl.dim Λ = gl.dim Λ^T.

Proof. The proof follows from the equality gl.dim Λ^T = l.gl.dim Λ and from Auslander's theorem 5.1.16, which asserts that l.gl.dim Λ = r.gl.dim Λ if Λ is two-sided Noetherian.

We now consider the global dimension of tiled orders having finite global dimension and width at most 2.

Note that if two tiled orders A_1 and A_2 are Morita equivalent, then $\operatorname{gl.dim} A_1 = \operatorname{gl.dim} A_2$. So, we can assume (for global dimension considerations) that a tiled order Λ with finite global dimension is reduced.

Let Λ be a reduced tiled order of finite global dimension with $w(\Lambda) = 2$. The following theorem is stated without proof.

Theorem 6.10.4.[9] *The following conditions are equivalent for a tiled order Λ:*

(a) the endomorphism ring of any indecomposable Λ-lattice is a discrete valuation ring;

(b) every Λ-lattice M is a direct sum of irreducible Λ-lattices;

(c) every irreducible Λ-lattice has no more then two maximal submodules;

(d) $w(\Lambda) \leq 2$.

Since, by this theorem, any tiled order of width at most 2 is a direct sum of irreducible lattices, to determine the global dimension of such a tiled order it is sufficient to check the projective dimensions of irreducible Λ-lattices, by the Auslander theorem (theorem 5.1.13).

It is obvious that the notion of the diagram of a finite poset (see vol. I, p.279 and section 6.3 above) may be extended in the same way to $\mathcal{M}(\Lambda)$, where $\mathcal{M}(\Lambda)$ is the partially ordered set, which is formed by all irreducible projective A-lattices. We shall denote this infinite quiver by $Q(\mathcal{M}(\Lambda))$.

Lemma 6.10.5. *Let Λ be a tiled order of width at most two. If* $\operatorname{gl.dim} \Lambda < \infty$, *then there exists an element $P \in \mathcal{M}(\Lambda)$ such that only one arrow in $Q(\mathcal{M}(\Lambda))$ starts from P.*

Proof. If $w(\Lambda) = 1$, then $\operatorname{gl.dim} \Lambda = 1$, by propositions 6.1.7 and 6.1.10. In this case lemma is true.

Suppose that $w(\Lambda) = 2$. By the Dilworth theorem (theorem 6.1.8), we can assume that $\mathcal{M}(\Lambda) = L_1 \cup L_2$, where L_1, L_2 are chains and $L_1 \cap L_2 = \varnothing$. So, from any $P \in \mathcal{M}(\Lambda)$ in $Q(\mathcal{M}(\Lambda))$ there start at most two arrows. Let $M \in S(\Lambda)$ with maximal $\operatorname{proj.dim}_\Lambda M$. Obviously, $M = P_1 + P_2$, where $P_1 \in L_1$ and $P_2 \in L_2$ and P_1, P_2 are non-comparable. Suppose two arrows $P_1 \to Q_1$ and $P_1 \to Q_2$ start

[9] see [Zavadskij, Kirichenko, 1976]

from P_1, where $Q_1 \in L_1$ and $Q_2 \in L_2$. Analogously, let $P_2 \to R_1$ and $P_2 \to R_2$ be two arrows, where $R_1 \in L_1$ and $R_2 \in L_2$. We have the following (part) diagrams:

$$\begin{tikzcd} Q_1 & & Q_2 & R_1 & & R_2 \\ & P_1 \arrow[ul] \arrow[ur] & & & P_2 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Suppose that $R_1 \subseteq P_1$. Then $P_2 \subseteq R_1$ and $P_2 \subseteq P_1$. So $R_1 \supseteq Q_1 \supset P_1$. Analogously, $Q_2 \supseteq R_2 \supset P_2$. Consider $R_1 \cap Q_2$. We have $R_1 \supseteq P_1$ and $Q_2 \supseteq P_1$, i.e., $P_1 \subseteq R_1 \cap Q_2$. Analogously, $Q_2 \supseteq P_2$ and $R_1 \supseteq P_2$, i.e., $P_2 \subseteq R_1 \cap Q_2$. Consequently, $R_1 \cap Q_2 \supseteq P_1 + P_2$. We now show that $R_1 \cap Q_2 = P_1 + P_2$. Indeed, if $R_1 \cap Q_2 = F_1 + F_2$, where $F_1 \in L_1$ and $F_2 \in L_2$, then $F_1 \subseteq R_1 \cap Q_2$ and therefore $F_1 \subseteq P_1$. Analogously, $F_2 \subseteq P_2$.

Thus $R_1 \cap Q_2 = P_1 + P_2$. Write $M_1 = R_1 + Q_2$. Let $\pi : R_1 \oplus Q_2 \to M_1$ be the canonical epimorphism. Obviously, $\operatorname{Ker} \pi \simeq R_1 \cap Q_2 = M$. So, $\operatorname{proj.dim}_A M_1 = \operatorname{proj.dim}_A M + 1$. We obtain a contradiction. The lemma is proved.

Let $\Lambda = \{\mathcal{O}, \mathcal{E} = (\alpha_{ij})\}$ be a reduced tiled order and let $1 = e_1 + \ldots + e_n$ be a decomposition of $1 \in \Lambda$ into a sum of mutually orthogonal local idempotents, $P_1 = e_1\Lambda, \ldots, P_{n-1} = e_{n-1}\Lambda, P_n = e_n\Lambda$. Write $f = e_1 + \ldots + e_{n-1}$, $e = e_n$, $B = f\Lambda f$, $\mathcal{O} = e\Lambda e$, $X = f\Lambda e$, $Y = e\Lambda f$ and $\{P_n\} = \{(\alpha_{n1} + c, \ldots, \alpha_{nn-1} + c, c),\ \text{where } c \in \mathbf{Z}\}$.

Lemma 6.10.6. *$\mathcal{M}(\Lambda) \setminus \{P_n\} \simeq \mathcal{M}(f\Lambda f)$.*

Proof. Let $P = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) \in \mathcal{M}(\Lambda) \setminus \{P_n\}$. We assert that $\alpha_1, \ldots, \alpha_{n-1} \in \mathcal{M}(f\Lambda f)$.

Indeed, by proposition 6.1.7, P has a unique maximal submodule PR and $P/PR = U_i$ for some $i = 1, \ldots, n-1$. So, $P = (\alpha_{i1} + c, \ldots, \alpha_{in-1} + c, \alpha_{in} + c)$ and $(\alpha_{i1} + c, \ldots, \alpha_{in-1} + c) \in \mathcal{M}(f\Lambda f)$. Conversely, let $P_i f = (\alpha_{i1} + c, \ldots, \alpha_{in-1} + c) \in \mathcal{M}(f\Lambda f)$. Consider $P_i = (\alpha_{i1} + c, \ldots, \alpha_{in-1} + c, \alpha_{in} + c) \in \mathcal{M}(\Lambda) \setminus \{P_n\}$. Obviously, these maps are inverse to each other and preserve the ordering.

If $w(\Lambda) = 2$, then every irreducible Λ-lattice is either projective or $M = P_1 + P_2$, where $P_1 \in L_1$, $P_2 \in L_2$ and P_1, P_2 are non-comparable.

Assume that $\operatorname{gl.dim} \Lambda < \infty$. Let $M \in S(\Lambda)$ and $M = P_1 + P_2$ as above. If $P_1 \cap P_2$ is not projective, then $P_1 \cap P_2 = P_1^{(1)} + P_2^{(1)}$, where $P_1^{(1)} \in L_1$ and $P_2^{(1)} \in L_2$, and $P_1^{(1)}$ and $P_2^{(1)}$ are non-comparable. Obviously, from $P_1^{(1)}$ there start two arrows and from $P_2^{(1)}$ there also start two arrows. So, to a projective resolution of M we can assign a sequence of pairs of non-comparable elements $(P_1, P_2), (P_1^{(1)}, P_2^{(1)}), \ldots, (P_1^{(k)}, P_2^{(k)})$, where $P_i^{(j)} \in L_i$ for $i = 1, 2$, $j = 1, \ldots, k$

and such that from $P_i^{(j)}$ there start two arrows. We can assume that $P_1^{(k)} \cap P_2^{(k)}$ is a projective module and that from $P_1^{(k)} \cap P_2^{(k)}$ there also start two arrows.

Lemma 6.10.7. *Let* $\Lambda = \{\mathcal{O}, \mathcal{E} = (\alpha_{ij})\}$ *be a reduced tiled order of width 2 with* $\mathrm{gl.\,dim}\,\Lambda < \infty$. *Then there exists a decomposition* $1 = e + f$, *where* e *is a local idempotent, such that*

$$\mathrm{gl.\,dim}\,\Lambda - 1 \leq \mathrm{gl.\,dim}\, f\Lambda f \leq \mathrm{gl.\,dim}\,\Lambda.$$

Proof. By lemma 6.10.5, there exists an element $P \in \mathcal{M}(\Lambda)$ such that only one arrow in $Q(\mathcal{M}(\Lambda))$ issues from P. Let $P = e\Lambda$, and $1 = e + f$. If $af, bf \in \mathcal{M}(f\Lambda f)$ is a pair of non-comparable elements in $\mathcal{M}(f\Lambda f)$, then $a, b \in \mathcal{M}(\Lambda)$ is a pair of non-comparable elements in $\mathcal{M}(\Lambda)$. Consider a projective resolution of $a + b$ as a Λ-module. To a projective resolution of $a + b$ we can assign a sequence of non-comparable pairs $(a_1, b_1), \dots, (a_k, b_k)$, where $a_j \in L_1$, $b_j \in L_2$ for $j = 1, \dots, k$ and from a_k and b_k there start two arrows. Let $a_k \cap b_k$ be projective. Obviously, two arrows start from $a_k \cap b_k$. Consequently, $a, b, a_1, b_1, \dots, a_k, b_k \in \mathcal{M}(\Lambda) \setminus \{P_n\}$. By lemma 6.10.6, $(af, bf), (a_1 f, b_1 f), \dots, (a_k f, b_k f)$ corresponds to a finite projective resolution of the $f\Lambda f$-module $af + bf$. So, $\mathrm{gl.\,dim}\, f\Lambda f \leq \mathrm{gl.\,dim}\,\Lambda$. Conversely, in a sequence $(P_1, P_2), (P_1^{(1)}, P_2^{(1)}), \dots, (P_1^{(k)}, P_2^{(k)})$, where $P_1^{(k)} \cap P_2^{(k)}$ is projective, all the modules $P_1^{(1)}, P_2^{(1)}, \dots, P_1^{(k)}, P_2^{(k)}, P_1^{(k)} \cap P_2^{(k)} \in \mathcal{M}(\Lambda) \setminus \{P_n\}$. Therefore, $\mathrm{gl.\,dim}\, f\Lambda f \geq \mathrm{gl.\,dim}\,\Lambda - 1$.

Theorem 6.10.8. *Let* Λ *be a tiled order in* $M_n(\mathcal{D})$ *and* $w(\Lambda) \leq 2$. *If* $\mathrm{gl.dim}\,\Lambda < \infty$ *then* $\mathrm{gl.dim}\,\Lambda \leq n - 1$.

Proof. This will be proved by induction on n. Let $n = 2$. We can assume that

$$\Lambda = \left\{ \mathcal{O}, \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} \right\}.$$

If $\alpha = 1$, then $\Lambda = H_2(\mathcal{O})$ is hereditary and $\mathrm{gl.\,dim}\,\Lambda = 1$. If $\alpha \geq 2$, then $(\alpha - 1, 0) = (\alpha - 1, \alpha - 1) + (\alpha, 0)$ and $(\alpha - 1, \alpha - 1) \cap (\alpha, 0) = (\alpha, \alpha - 1) \simeq (1, 0)$. Obviously, $(1, 0) = (1, 1) + (\alpha, 0)$ and $(1, 1) \cap (\alpha, 0) = (\alpha, 1) \simeq (\alpha - 1, 0)$. So, $\mathrm{proj.\,dim}_\Lambda(\alpha - 1, 0) = \infty$.

In the general case, when $n > 2$, by lemma 6.10.7, there exists $f^2 = f \in \Lambda$ with $\mathrm{gl.\,dim}\, f\Lambda f < \infty$ and $\mathrm{gl.\,dim}\, f\Lambda f \geq \mathrm{gl.\,dim}\,\Lambda - 1$, $w(f\Lambda f) \leq 2$, and f is a sum of $n - 1$ mutually orthogonal local idempotents. By the induction hypothesis, $\mathrm{gl.\,dim}\, f\Lambda f \leq n - 2$. So, $n - 2 \geq \mathrm{gl.\,dim}\,\Lambda - 1$ and $\mathrm{gl.\,dim}\,\Lambda \leq n - 1$. The theorem is proved.

Theorem 6.10.9. *Let* Λ *be a tiled order and* $w(\Lambda) \leq 2$. *If* $\mathrm{gl.dim}\,\Lambda = k < \infty$, *then for any* $m\,(1 \leq m \leq k)$ *there exists an idempotent* $e \in \Lambda$ *such that* $\mathrm{gl.dim}\, e\Lambda e = m$.

The proof of this theorem follows from lemma 6.10.7 and the fact that all tiled orders in $M_2(D)$ of finite global dimension are isomorphic to $H_2(\mathcal{O})$ with $\operatorname{gl.dim} H_2(\mathcal{O}) = 1$.

Remark 6.10.1. Theorems 6.10.8 and 6.10.9 first were proved by Kh.M.Danlyev (see [Danlyev, 1989]).

Example 6.10.1.
The tiled order $\Lambda_n = \{\mathcal{O}, \mathcal{E}(\Lambda_n)\}$, where

$$\mathcal{E}(\Lambda_n) = \begin{pmatrix} 0 & 0 & \dots & \dots & \dots & 0 \\ 1 & \ddots & \ddots & & & \vdots \\ 2 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 2 & \dots & \dots & 2 & 1 & 0 \end{pmatrix}$$

is an $n \times n$-matrix, is a triangular tiled order of width 2 and $\operatorname{gl.dim} \Lambda_n = n-1$.

Example 6.10.2.
The tiled order $\Omega_n = \{\mathcal{O}, \mathcal{E}(\Omega_n)\}$, where

$$\mathcal{E}(\Omega_n) = \begin{pmatrix} 0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 1 & \ddots & \ddots & & & & & \vdots \\ 2 & \ddots & \ddots & \ddots & & & & \vdots \\ 3 & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ n-3 & \dots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ n-2 & n-3 & \dots & \ddots & \ddots & \ddots & \ddots & 0 \\ n-1 & n-2 & n-3 & \dots & 3 & 2 & 1 & 0 \end{pmatrix}$$

is an $n \times n$-matrix, is triangular tiled order and $\operatorname{gl.dim} \Omega_n = 2$.

The following proposition is very useful. The proof of it can be found in the paper [Kirkman, Kuzmanovich, 1989].

Proposition 6.10.10. *Let Λ be an order, and let e be an idempotent of Λ such that $e\Lambda e$ is a hereditary ring and $\mathcal{I} = \Lambda e \Lambda$. Then* $\operatorname{gl.dim}(\Lambda/\mathcal{I}) \leq \operatorname{gl.dim}\Lambda \leq \operatorname{gl.dim}(\Lambda/\mathcal{I}) + 2$.

We use the Dlab-Ringel example of a serial ring A with $\mathrm{gl.dim}\, A = 4$ and Kupisch series 4,4,3 for the construction of a tiled order Λ of width 2 with $\mathrm{gl.dim}\, \Lambda = 4$ and such that the quiver $Q(\Lambda)$ has five vertices.[10].

Let $H_3(\mathcal{O}) = \left\{\mathcal{O}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\right\}$ and let $\mathcal{L} \subset H_3(\mathcal{O})$ be the two-sided ideal with exponent matrix $\mathcal{E}(\mathcal{L}) = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 2 & 1 \end{pmatrix}$. Consider the tiled order Δ_5 with

$$\mathcal{E}(\Delta_5) = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 \end{array}\right).$$

Obviously, $w(\Delta_5) = 2$. We shall show that Δ_5 has global dimension 4. This follows from proposition 6.10.6. Indeed, let $e = e_{44} + e_{55}$ and $\mathcal{I} = \Delta_5 e \Delta_5$.

$$\mathcal{E}(\mathcal{I}) = \left(\begin{array}{ccc|cc} 2 & 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 \end{array}\right)$$

and $\mathrm{gl.dim}\, \Delta_5 \geq \mathrm{gl.dim}\, \Lambda/\mathcal{I} = 4$. From theorem 5.10.4 it follows that $\mathrm{gl.dim}\, \Delta_5 = 4$. Let $f = e_{11} + e_{22} + e_{33}$. Then

$$\mathcal{E}(\mathcal{J}) = \mathcal{E}(\Delta_5 f \Delta_5) = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 2 & 2 \end{array}\right).$$

It is easy to see that $\mathrm{gl.dim}\, \Delta_5/\mathcal{J} = 2$. Consequently, we have $\mathrm{gl.dim}\, \Delta_5 = \mathrm{gl.dim}\, \Delta_5/\mathcal{I}$ and $\mathrm{gl.dim}\, \Delta_5 = \mathrm{gl.dim}\, \Delta_5/\mathcal{J} + 2$.

From this example it follows that both equalities in proposition 6.10.10 may hold.

Theorem 6.10.11.[11] *If Λ is a tiled order and* $\mathrm{gl.dim}\, \Lambda < \infty$, *then $Q(\Lambda)$ has no loops.*

[10] see [Dlab, Ringel, 1989]
[11] see [Weidemann, Roggenkamp, 1983]

Theorem 6.10.12. *If Λ is a tiled order and $Q(\Lambda)$ has at most 3 vertices, then* $\mathrm{gl.dim}\,\Lambda$ *is finite if and only if $Q(\Lambda)$ has no loops. In this case $w(\Lambda) \leq 2$.*

A proof follows from theorem 6.10.11 and proposition 6.7.1. If $Q(\Lambda)$ is a cycle, then Λ is hereditary and $\mathrm{gl.dim}\Lambda = 1$. If $\Omega_3(\mathcal{O}) = \{\mathcal{O}, \Omega_3\}$, then $\mathrm{gl.dim}\,\Omega_3(\mathcal{O}) = 2$.

Remark 6.10.2. This theorem was first proved by R.B.Tarsy (see [Tarsy, 1970]).

The list of the orders Λ with $\mathrm{gl.dim}\,\Lambda < \infty$ and such that $Q(\Lambda)$ has 4 vertices is given in the papers [Fujita, 1990], [Fujita, 1991]. The first six exponent matrices (1)-(6) from section 6.8 exhaust this list.

Recall that $\mathcal{E}(F_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$. Obviously, $w(F_4) = 3$.

Note, that all tiled orders of finite global dimension, whose quivers have at most four vertices, are isomorphic to $(0,1)$-orders, except Ω_4. Now we give a list of the associated posets P_Λ, where $\mathrm{gl.dim}\,\Lambda < \infty$ and Λ is a $(0,1)$-order.

List of posets:

$$n = 1, \quad \mathcal{P}_1 = \{\bullet\}, \quad \mathrm{gl.dim}\,\Lambda_{\mathcal{P}_1} = 1;$$

$$n = 2, \quad \mathcal{P}_2 = \left\{ \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right\}, \quad \mathrm{gl.dim}\,\Lambda_{\mathcal{P}_2} = 1;$$

$$n = 3, \quad \mathcal{P}_3 = \left\{ \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \right\}, \quad \mathrm{gl.dim}\,\Lambda_{\mathcal{P}_3} = 1;$$

$$n = 3, \quad \mathcal{P}_4 = \left\{ \begin{array}{ccccc} \bullet & & & & \bullet \\ & \diagdown & & \diagup & \\ & & \bullet & & \end{array} \right\}, \quad \mathrm{gl.dim}\,\Lambda_{\mathcal{P}_4} = 2;$$

$$n = 4, \quad \mathcal{P}_5 = \left\{ \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \right\}, \quad \mathrm{gl.dim}\,\Lambda_{\mathcal{P}_5} = 1;$$

$$n = 4, \quad \mathcal{P}_6 = \left\{ \begin{array}{ccc} \bullet & & \\ | & & \\ \bullet & & \bullet \\ | & / & \\ \bullet & & \end{array} \right\}, \quad \text{gl.dim}\,\Lambda_{\mathcal{P}_6} = 2;$$

$$n = 4, \quad \mathcal{P}_7 = \left\{ \begin{array}{ccccc} \bullet & & & & \bullet \\ & \diagdown & & / & \\ & & \bullet & & \\ & & | & & \\ & & \bullet & & \end{array} \right\}, \quad \text{gl.dim}\,\Lambda_{\mathcal{P}_7} = 2;$$

$$n = 4, \quad \mathcal{P}_8 = \left\{ \begin{array}{ccc} \bullet & & \bullet \\ | & \diagdown & | \\ \bullet & & \bullet \end{array} \right\}, \quad \text{gl.dim}\,\Lambda_{\mathcal{P}_8} = 3;$$

$$n = 4, \quad \mathcal{P}_9 = \left\{ \begin{array}{ccccc} \bullet & & \bullet & & \bullet \\ & \diagdown & | & / & \\ & & \bullet & & \end{array} \right\}, \quad \text{gl.dim}\,\Lambda_{\mathcal{P}_9} = 2.$$

It follows from proposition 6.10.2 that if the finite posets $\mathcal{P}_{\Lambda_1}$ and $\mathcal{P}_{\Lambda_2}$, which are associated with $(0,1)$-orders Λ_1 and Λ_2, are anti-isomorphic, then $\text{gl.dim}\,\Lambda_1 = \text{gl.dim}\,\Lambda_2$.

Proposition 6.10.13. *If* $\text{gl.dim}\,\Lambda \leq 2$, *then* $\mathcal{M}(\Lambda)$ *is a lower semilattice.*

Proof. By the Michler theorem (see theorem 12.3.4, vol.I), $\text{gl.dim}\,\Lambda = 1$ if and only if $\mathcal{M}(\Lambda)$ is a chain. In this case $\mathcal{M}(\Lambda)$ is a lower semilattice. If $\mathcal{M}(\Lambda)$ is not a chain, let P_i and P_j be non-comparable elements of $\mathcal{M}(\Lambda)$. Then $P_i + P_j = M$ and the projective cover $P(M)$ of M is $P_i \oplus P_j$. Let $\varphi : P(M) \longrightarrow M$. Then $\text{Ker}\,\varphi \simeq P_i \cap P_j$ is projective.

Proposition 6.10.14. *If a poset* $\mathcal{P}_\Lambda$ *associated with a* $(0,1)$*-order* Λ *has a unique maximal element or a unique minimal element, then* $\text{gl.dim}\,\Lambda < \infty$.

Proof. By proposition 6.10.2, we can assume that $\mathcal{P}_\Lambda$ has a unique minimal element and

$$\mathcal{E}(\Lambda) = \left(\begin{array}{c|cccc} 0 & 0 & \dots & 0 & 0 \\ \hline 1 & 0 & & & \\ & & & & * \\ \vdots & & & \ddots & \\ & & * & & \\ 1 & & & & 0 \end{array} \right).$$

Let $e = e_{11}$ and $\mathcal{I} = \Lambda e \Lambda$. In this case the quotient ring $\Lambda/\mathcal{I}$ is, obviously, an Artinian piecewise domain. Thus $\text{gl.dim}\,\Lambda/\mathcal{I}$ is finite and, by proposition 6.10.10, $\text{gl.dim}\,\Lambda$ is finite.

Proposition 6.10.15. *The chain of the ideals*

$$\mathcal{I}_1 \subset \mathcal{I}_2 \ldots \subset \mathcal{I}_{n-1} \subset \Omega_n$$

with

$$\mathcal{E}(\mathcal{I}_1) = \begin{pmatrix} n-1 & n-2 & \ldots & 1 & 0 \\ n-1 & n-2 & \ldots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ n-1 & n-2 & \ldots & 1 & 0 \end{pmatrix},$$

$$\mathcal{E}(\mathcal{I}_2) = \begin{pmatrix} n-2 & n-3 & \ldots & 0 & 0 \\ n-2 & n-3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ n-2 & n-3 & \ldots & 0 & 0 \\ n-1 & n-2 & \ldots & 1 & 0 \end{pmatrix},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathcal{E}(\mathcal{I}_{n-1}) = \begin{pmatrix} 1 & 0 & \ldots & \ldots & 0 \\ 1 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ n-2 & \ddots & \ddots & \ddots & 0 \\ n-1 & n-2 & \ldots & 1 & 0 \end{pmatrix},$$

$$\mathcal{E}(\Omega_n) = \begin{pmatrix} 0 & 0 & \ldots & \ldots & \ldots & 0 \\ 1 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ n-3 & \ldots & \ddots & \ddots & \ddots & \vdots \\ n-2 & n-3 & \ldots & \ddots & \ddots & 0 \\ n-1 & n-2 & n-3 & \ldots & 1 & 0 \end{pmatrix}$$

is a chain of projective idempotent ideals of Ω_n and the quotient ring $\Omega_n/\mathcal{I}_1$ is quasi-hereditary.

The proof is obvious.

Proposition 6.10.16. *The chain of the ideals*

$$\mathcal{J}_1 \subset \mathcal{J}_2 \subset \ldots \subset \mathcal{J}_{n-1} \subset \Lambda_n$$

with

$$\mathcal{E}(\mathcal{J}_1) = \begin{pmatrix} 2 & 2 & \dots & 1 & 0 \\ 2 & 2 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 2 & \dots & 1 & 0 \\ 2 & 2 & \dots & 1 & 0 \end{pmatrix}, \mathcal{E}(\mathcal{J}_2) = \begin{pmatrix} 2 & \dots & \dots & 2 & 1 & 0 & 0 \\ 2 & \dots & \dots & 2 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & \dots & \dots & 2 & 1 & 0 & 0 \\ 2 & \dots & \dots & 2 & 2 & 1 & 0 \end{pmatrix}$$

$$\dots, \ \mathcal{E}(\mathcal{J}_{n-1}) = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & 0 & \ddots & & & \vdots \\ 2 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 2 & \dots & \dots & 2 & 1 & 0 \end{pmatrix} \mathcal{E}(\Lambda_n) = \begin{pmatrix} 0 & 0 & \dots & \dots & \dots & 0 \\ 1 & \ddots & \ddots & & & \vdots \\ 2 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 2 & \dots & \dots & 2 & 1 & 0 \end{pmatrix}$$

is a chain of projective idempotent ideals of Λ_n and the quotient ring $\Omega_n/\mathcal{J}_1$ is quasi-hereditary.

The proof is obvious.

Now we shall compute the quiver $Q(\Omega_n)$ and its transition matrix for the reduced exponent matrix Ω_n. We use the formula $[Q(\Omega)_n] = \Omega_n^{(2)} - \Omega_n^{(1)}$. Obviously,

$$\Omega_n^{(2)} = \begin{pmatrix} 1 & 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 2 & \ddots & \ddots & \ddots & & & & \vdots \\ 2 & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ 3 & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ n-2 & n-3 & \dots & \ddots & \ddots & \ddots & \ddots & 1 \\ n-1 & n-2 & n-3 & \dots & 3 & 2 & 2 & 1 \end{pmatrix}$$

and $[Q(\Omega_n)] = J_n^-(0) + J_n^+(0) = Y_n$, where $J_n^+(0) = e_{12} + e_{23} + \ldots + e_{n-1n}$ and $J_n^-(0) = e_{21} + e_{32} + \ldots + e_{nn-1}$. We have that $\operatorname{inx}\Omega_n = 2\cos\dfrac{\pi}{n+1}$ and $\vec{f} = (a_1, a_2, \ldots, a_n)$ is a positive eigenvector of Y with eigenvalue $\lambda = 2\cos\dfrac{\pi}{n+1}$, $Z = \operatorname{diag}(a_1, \ldots, a_n)$, where $a_i = \sin\dfrac{i\pi}{n+1}$ for $i = 1, 2, \ldots, n$.

Thus the transition matrix S_n for the quiver $Q(\Omega_n)$ is:

$$S_n = \lambda^{-1} Z^{-1} Y_n Z =$$
$$= \frac{1}{2\cos\frac{\pi}{n+1}} \cdot \mathrm{diag}(\frac{1}{a_1}, \ldots, \frac{1}{a_n}) \cdot Y_n \cdot \mathrm{diag}(a_1, \ldots, a_n) = \frac{1}{2\cos\frac{\pi}{n+1}} \cdot C,$$

where $Y_n = (y_{ij})$, $C = (c_{ij})$ and $a_i = \sin\dfrac{i\pi}{n+1}$ for $i = 1, 2, \ldots, n$,

$$y_{ij} = \begin{cases} 1, & \text{if} \quad i = j-1 \ \text{ or } \ i = j+1, \\ 0, & \text{otherwise,} \end{cases}$$

$$c_{ij} = \begin{cases} \dfrac{a_{i+1}}{a_i}, & \text{if} \quad i = j-1, \\[2ex] \dfrac{a_i}{a_{i+1}}, & \text{if} \quad i = j+1, \\[2ex] 0, & \text{otherwise,} \end{cases}$$

for $i, j = 1, 2, \ldots, n$.

The matrix S_n defines a random walk on the set $\{1, 2, \ldots, n\} \subset \mathbf{N}$.

6.11 NOTES AND REFERENCES

The main concepts of this chapter are reduced exponent matrices and their quivers. Note that exponent matrices appeared first in the study of completely decomposable orders (see [Kirichenko, 1967], [Zavadskij, 1973]) and were used for the study of semimaximal rings of finite type (see [Zavadskij, Kirichenko, 1976], [Zavadskij, Kirichenko, 1977]).

Theorem 6.1.8 was proved by R.P.Dilworth in the paper [Dilworth, 1950]. The direct proof of theorem 6.1.12 was given in [Kirichenko, 2004]. Theorem 6.1.13 was first proved in [Kirichenko, 2005].

In section 6.3 we have followed the paper [Dokuchaev, 2002]. In sections 6.6 and 6.7 we have followed the papers [Chernousova, 2002], [Chernousova, 2003]. The list of the section 6.7 was given in [Chernousova, 2003].

The notions of weakly prime and right 2-rings were introduced in [Danlyev, 1997]. Theorem 6.9.1 and 6.9.9 were proved in this article.

Consider the following example, due to H.Fujita [Fujita, 1990]. Let A be the tiled order with the following exponent matrix:

$$\mathcal{E} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 2 & 1 & 0 \end{pmatrix}$$

Then gl. dim$A = 6$ and this is also a counterexample to the Tarsy conjecture (see [Tarsy, 1970]) saying that the maximal possible finite global dimension of a tiled order in $(K)_n$ is $n-1$.

The following example of W.Rump shows that the global dimension of an order is determined not only by its exponent matrix even for the case of (0,1)-orders.

Consider the tiled $\mathcal{O}$-order Λ in the matrix algebra $\mathrm{M}_{14}(K)$, where $\mathcal{O}$ is a discrete valuation domain with quotient field K with exponent matrix:

$$\mathcal{E} = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0
\end{pmatrix}$$

If the characteristic of the residue class field k of $\mathcal{O}$ is 2, then $\mathrm{gl.dim}\,(\Lambda) = 4$, otherwise $\mathrm{gl.dim}\,(\Lambda) = 3$.

Details can be found in example 1 of the paper [Rump, 1996].

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7. Gorenstein matrices

7.1 GORENSTEIN TILED ORDERS. EXAMPLES

In this section we consider a special type of tiled orders which is defined by the equivalent conditions of the following theorem:

Theorem 7.1.1. *The following conditions are equivalent for a tiled order A*:
(i) $\mathrm{inj.dim}_A A_A = 1$;
(ii) $\mathrm{inj.dim}_A {}_AA = 1$;
(iii) A^*_A *is projective left A-module*;
(iv) ${}_AA^*$ *is projective right A-module.*

Proof. (i)$\Rightarrow$ (iv). Write $Q = Q_0 = M_n(D)$, where D is is the classical ring of fractions of $\mathcal{O}$. By proposition 6.2.1, Q is an injective right and left A-module. If $\mathrm{inj.dim}_A A_A = 1$ then there exists an exact sequence

$$0 \to A_A \to Q_0 \to Q_0/A_A \to 0.$$

By proposition 6.5.5, vol. I, the module Q_0/A_A is injective. Obviously, every indecomposable direct summand of Q_0/A_A has the form $e_{ii}Q_0/e_{ii}A$. Since $e_{ii}Q_0/e_{ii}A$ is indecomposable injective $\mathrm{soc}\,(e_{ii}Q_0/e_{ii}A)$ is simple. Therefore every $e_{ii}A$ is a relatively injective irreducible A-lattice by proposition 6.2.14, and A_A is a relatively injective right A-module. By definition, $A_A \simeq_A P^*$. By duality properties, ${}_AP ={}_A P_1 \oplus \ldots \oplus_A P_s \oplus P$, where ${}_AP_1, \ldots {}_A P_s$ are all pairwise non-isomorphic left principal A-modules and every indecomposable direct summand of P is isomorphic to some ${}_AP_i$. Therefore, ${}_AA^*$ is a projective right A-module.

From corollary 6.2.11 we obtain (iii) $\Leftrightarrow$ (iv). In conclusion, we obtain that (iv) $\Rightarrow$ (i), by corollary 6.2.15 and the fact, that ${}_AA^*$ and A_A contain the same indecomposable summands if ${}_AA^*$ is projective. The equivalence (ii) $\Leftrightarrow$ (iii) for left modules is proved just like (i) $\Leftrightarrow$ (iv) for right modules. The theorem is proved.

Definition. A tiled order A, which satisfies the equivalent conditions of theorem 7.1.1, is called a **Gorenstein tiled order**[1]. As follows from theorem 7.1.1 the definition of a Gorenstein tiled order is right-left symmetric.

[1]Recall that a commutative ring is called Gorenstein if its injective dimension is finite. These rings were first considered by H.Bass (see [Bass, 1963])

Proposition 7.1.2. *Let $A = \{\mathcal{O}, \mathcal{E}(A)\}$ be a reduced tiled order with exponent matrix $\mathcal{E}(A) = (\alpha_{ij}) \in M_n(\mathbf{Z})$. A is Gorenstein if and only if the matrix $\mathcal{E}(A)$ is Gorenstein, i.e., there exists a permutation σ of the set $\{1, \ldots, n\}$ such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for $i, k = 1, \ldots, n$.*

Proof. Since A is reduced we have that ${}_AA^* \simeq A_A$. But

$$\mathcal{E}(A_A) = \begin{pmatrix} 0 & \alpha_{12} & \cdots & \cdots & \alpha_{1n} \\ \alpha_{21} & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \alpha_{n-11} \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn-1} & 0 \end{pmatrix}$$

and

$$\mathcal{E}({}_AA^*) = \begin{pmatrix} 0 & -\alpha_{21} & \cdots & \cdots & -\alpha_{n1} \\ -\alpha_{12} & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -\alpha_{nn-1} \\ -\alpha_{1n} & -\alpha_{2n} & \cdots & -\alpha_{n-1n} & 0 \end{pmatrix}.$$

Therefore, there exists a permutation σ of $\{1, \ldots, n\}$ such that

$$(\alpha_{i1}, \ldots, 0, \ldots, \alpha_{in}) = (-\alpha_{1\sigma(i)} + c_i, \ldots, -\alpha_{n\sigma(i)} + c_i),$$

where $c_i \in \mathbf{Z}$ for $i = 1, \ldots, n$. Consequently, $\alpha_{ik} + \alpha_{k\sigma(i)} = c_i$ for $i, k = 1, \ldots, n$. For $i = k$ we obtain $\alpha_{i\sigma(i)} = c_i$ and hence $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ and $\mathcal{E}(A)$ is a Gorenstein matrix. Conversely, if $\mathcal{E}(A)$ is Gorenstein then ${}_AA^* \simeq A_A$ and, by theorem 7.1.1, the tiled order A is Gorenstein. The proposition is proved.

Example 7.1.1.
Let

$$A = \left\{ \mathcal{O}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \right\}$$

and $P_1 = (0,0,0)$; $P_2 = (1,0,1)$; $P_3 = (1,1,0)$ be projective A-modules. Obviously, $P_1^* = (0,0,0)^T$; $P_2^* = (-1,0,-1)^T \simeq (0,1,0)^T$ and $P_3^* = (-1,-1,0)^T \simeq (0,0,1)^T$. Therefore, the modules P_2 and P_3 are relatively injective, but A is not Gorenstein. It is well-known that $\mathrm{gl.dim}\, A = 2$.

In this section we shall assume that the first row of a Gorenstein matrix $\mathcal{E}$ is zero.

Let

$$T_{n,\alpha} = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ \alpha & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & 0 \\ \alpha & \dots & \dots & \alpha & 0 \end{pmatrix},$$

where α is a natural integer, $T_{n,\alpha} \in M_n(\mathbf{Z})$. Obviously, $T_{n,\alpha}$ is a cyclic Gorenstein matrix with $\sigma = \sigma(T_{n,\alpha}) = (n\, n{-}1 \dots 2\, 1)$. Write $J_n^+(0) = e_{12}+e_{23}+\ldots+e_{n-1n}$. It is easy to see that $[Q(T_{n,1})] = J_n^+(0) + e_{n1}$ and $[Q(T_{n,\alpha})] = E_n + J_n^+(0) + e_{n1}$, where E_n is the identity $n \times n$-matrix and $\alpha \geq 2$.

We shall write $T_{n,1} = H_n$. Therefore, $Q(H_n)$ is a simple cycle C_n. For the quiver $Q(T_n, \alpha)$ $(\alpha \geq 2)$ we use the notation $\mathcal{L}C_n$, i.e., $\mathcal{L}C_n$ is a simple cycle with a loop in each vertex.

Obviously, $\operatorname{inx} H_n = 1$ and $\operatorname{inx} T_{n,\alpha} = 2$. In this case $[Q(H_n)] = P_{\sigma^{n-1}}$ and $[Q(T_{n,\alpha})] = P_{id} + P_{\sigma^{n-1}}$.

Definition. A tiled order $A = \{\mathcal{O}, E(A)\}$ is called **triangular** if the matrix $\mathcal{E}(A) = (\alpha_{ij})$ is triangular, i.e., $\alpha_{ij} = 0$ for $i \leq j$.

The following theorem is stated without proof.

Theorem 7.1.3.[2] *A reduced tiled order $A = \{\mathcal{O}, E(A)\}$ is a Gorenstein triangular tiled order if and only if $Q(A)$ is a simple cycle C_s or a quiver $\mathcal{L}C_s$. In this case A is isomorphic to the order $T_{n,\alpha}$.*

Below the Gorenstein tiled orders and hence Gorenstein matrices are described in detail for $n \leq 6$.

To do this efficiently it is important to recall that up to equivalence it can be assumed that the first row of an exponent matrix $\mathcal{E} = (\alpha_{ij})$ consists of zeroes (and then $\alpha_{ij} \geq 0$ for all i, j). Up to equivalence that still leaves the freedom to permute columns and rows simultaneously by a permutation τ which leaves 1 fixed. Such an operation turns a Gorenstein matrix $\mathcal{E}$ with first row zero and permutation σ into a new Gorenstein matrix $\mathcal{E}'$ with first row zero and permutation $\sigma' = \tau^{-1}\sigma\tau$.

A second important point to note is that the permutation of a reduced Gorenstein matrix cannot have a fixed point. Indeed, suppose that $\sigma(i) = i$. Then for all k, $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ and so $\alpha_{ik} + \alpha_{ki} = \alpha_{ii} = 0$ contradicting that $\mathcal{E}$ is reduced.

We shall use the decomposition of a permutation $\sigma \in S_n$ into a product of independent cycles. For example, the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$ is the

[2] see [Roggenkamp, 2001]

product of two cycles $(1\,3\,4)(2\,5)$ of lengths 3 and 2. The permutation of a reduced Gorenstein matrix has no cycles of length 1.

Let $\sigma \in S_n$ be a product of m independent cycles of lengths $d_1, \ldots, d_m$ and $\tau \in S_n$ be a product of m' independent cycles of lengths $d'_1, \ldots, d'_m$. Two permutations σ and τ have the same cyclic type if $m = m'$ and there exists a permutation $\mu \in S_m$ such that $d'_i = d_{\mu(i)}$ for $i = 1, \ldots, m$. It is well-known that two permutations are conjugate if and only if they have the same cyclic type. By proposition 6.1.18, we can consider a permutation $\sigma(\mathcal{E})$ for a Gorenstein matrix up to cyclic type.

Let $n = 2$. Then any reduced exponent matrix is equivalent to $T_{2,\alpha}$.

Let $n = 3$. We may assume that $\sigma(\mathcal{E}) = (3\,2\,1)$. From the Gorenstein condition follows:

$$\begin{cases} \alpha_{12} + \alpha_{23} = \alpha_{13} \\ \alpha_{23} + \alpha_{31} = \alpha_{21} \\ \alpha_{31} + \alpha_{12} = \alpha_{32} \end{cases}.$$

Since $\alpha_{12} = \alpha_{13} = 0$, we obtain $\alpha_{23} = 0$ and $\alpha_{31} = \alpha_{32} = \alpha_{21}$. Set $\alpha_{21} = \alpha$. Therefore,

$$\mathcal{E} = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \alpha & \alpha & 0 \end{pmatrix} = T_{3,\alpha} \quad (\alpha \geq 1).$$

So there is the following statement.

Proposition 7.1.4. *For $n = 2$ every reduced exponent matrix is cyclic Gorenstein, and for $n = 3$ every Gorenstein matrix is cyclic. Obviously,* $\operatorname{inx} H_n = w(H_n) = 1$ *and* $\operatorname{inx} T_{n,\alpha} = w(T_{n,\alpha}) = 2$ *for $\alpha \geq 2$.*

For $n = 4$ there are two possibilities for $\sigma(\mathcal{E})$: a simple cycle and a product of two transpositions. By proposition 4.1.12, we may assume that

$$\sigma(\mathcal{E}) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \text{ and } \sigma(\mathcal{E}) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

Case (a): $\sigma = \sigma(\mathcal{E}) = (4\,3\,2\,1)$.

We shall use notations like: $(\alpha_1, \ldots, \alpha_k) = \min(\alpha_1, \ldots, \alpha_k)$ and $\delta = (2, \alpha, \beta)$. It is easy to see that

$$\mathcal{E} = \mathcal{E}_{\alpha,\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha & 0 & \alpha - \beta & 0 \\ \beta & \beta & 0 & 0 \\ \alpha & \beta & \alpha & 0 \end{pmatrix} \quad (\alpha \geq \beta > 0).$$

We have

$$\mathcal{E}^{(1)}_{\alpha,\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & \alpha - \beta & 0 \\ \beta & \beta & 1 & 0 \\ \alpha & \beta & \alpha & 1 \end{pmatrix}$$

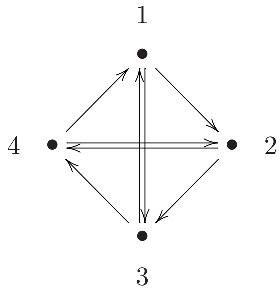
and

$$\mathcal{E}^{(2)}_{\alpha,\beta} = \begin{pmatrix} \delta & 1 & (1,\alpha-\beta) & 0 \\ \alpha & \delta & (\alpha,\alpha-\beta+1) & (1,\alpha-\beta) \\ (\alpha,\beta+1) & \beta & \delta & 1 \\ \alpha+1 & (\alpha,\beta+1) & \alpha & \delta \end{pmatrix}.$$

Therefore, $\mathcal{E}_{1,1} = H_4$ and $Q(\mathcal{E}_{1,1}) = C_4$. If $\alpha \geq 2$, then $\mathcal{E}_{\alpha,\alpha} = T_{4,\alpha}$ and $Q(\mathcal{E}_{\alpha,\alpha}) = \mathcal{L}C_4$.

If $\beta = 1$ we have $\alpha \geq 2$ and

$$[Q(\mathcal{E}_{\alpha,1})] = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$



In this case, $[Q(\mathcal{E}_{\alpha,1})] = P_{\sigma^3} + P_{\sigma^2}$ and $\operatorname{inx}\mathcal{E}_{\alpha,1} = 2$.

If $\alpha > \beta > 1$, then

$$[Q(\mathcal{E}_{\alpha,\beta})] = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} = P_{\sigma^2} + P_{\sigma^3} + P_{\sigma^4}$$

and $\operatorname{inx}\mathcal{E}_{\alpha,\beta} = 3$.

Case (b): $\sigma = \sigma(\mathcal{E}) = (1\,2)(3\,4)$.

In this case

$$\mathcal{E}_{\gamma,\delta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \gamma+\delta & 0 & \gamma & \delta \\ \delta & 0 & 0 & \delta \\ \gamma & 0 & \gamma & 0 \end{pmatrix}, \quad (\gamma > 0, \delta > 0).$$

$$\mathcal{E}^{(1)}_{\gamma,\delta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \gamma+\delta & 1 & \gamma & \delta \\ \delta & 0 & 1 & \delta \\ \gamma & 0 & \gamma & 1 \end{pmatrix} \text{ and } \mathcal{E}^{(2)}_{\gamma,\delta} = \begin{pmatrix} \Delta & 0 & 1 & 1 \\ \gamma+\delta & \Delta & \gamma+1 & \delta+1 \\ \delta+1 & 1 & \Delta & \delta \\ \gamma+1 & 1 & \gamma & \Delta \end{pmatrix},$$

where $\Delta = (2, \gamma, \delta)$.

Therefore,

$$[Q(\mathcal{E}_{\alpha,\delta})] = \begin{pmatrix} \Delta-1 & 0 & 1 & 1 \\ 0 & \Delta-1 & 1 & 1 \\ 1 & 1 & \Delta-1 & 0 \\ 1 & 1 & 0 & \Delta-1 \end{pmatrix}.$$

If $\Delta = 1$ we have

$$B = [Q(\mathcal{E}_{1,\delta})] = [Q(\mathcal{E}_{\gamma,1})] = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

and $\operatorname{inx}\mathcal{E}_{1,\delta} = \operatorname{inx}\mathcal{E}_{\gamma,1} = 2$. In this case, $B = P_\tau + P_w$, where

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

For $\Delta \geq 2$, i.e., $\gamma \geq 2$ and $\delta \geq 2$ we have

$$[Q(\mathcal{E}_{\gamma,\delta})] = E + B = P_\tau + P_{\tau^2} + P_w$$

and $\operatorname{inx}\mathcal{E}_{\gamma,\delta} = 3$.

As above, for $n = 5$ we may assume:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$$

or $\sigma = (1\,2)(3\,4\,5)$.

Case (a):
$n = 5$, $\sigma = (5\,4\,3\,2\,1)$. It is easy to see that

$$\mathcal{E}_{\mu,\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \mu & 0 & \mu-\nu & \mu-\nu & 0 \\ \nu & \nu & 0 & \mu-\nu & 0 \\ \nu & 2\nu-\mu & \nu & 0 & 0 \\ \mu & \nu & \nu & \mu & 0 \end{pmatrix} \quad (\mu \geq \nu > 0)$$

$$\mathcal{E}^{(1)}_{\mu,\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \mu & 1 & \mu-\nu & \mu-\nu & 0 \\ \nu & \nu & 1 & \mu-\nu & 0 \\ \nu & 2\nu-\mu & \nu & 1 & 0 \\ \mu & \nu & \nu & \mu & 1 \end{pmatrix}.$$

Let $\delta = \min(2, \mu, \nu)$. The matrix $\mathcal{E}^{(2)}_{\mu,\nu}$ is:

$$\begin{pmatrix} \delta & (1, 2\nu - \mu) & (1, \mu - \nu) & (1, \mu - \nu) & 0 \\ \mu & \delta & (\nu, \mu - \nu + 1) & (\mu, \mu - \nu + 1) & (1, \mu - \nu) \\ (\mu, \nu + 1) & \nu & \delta & (\nu, \mu - \nu + 1) & (1, \mu - \nu) \\ (\mu, \nu + 1) & (\nu, 2\nu - \mu + 1) & \nu & \delta & (1, 2\nu - \mu) \\ (\mu + 1, 2\nu) & (\mu, \nu + 1) & (\mu, \nu + 1) & \mu & \delta \end{pmatrix}$$

and

$$[Q(\mathcal{E}_{\mu,\nu})] = \mathcal{E}^{(2)}_{\mu,\nu} - \mathcal{E}^{(1)}_{\mu,\nu} =$$

$$= \begin{pmatrix} \delta - 1 & (1, 2\nu - \mu) & (1, \mu - \nu) & (1, \mu - \nu) & 0 \\ 0 & \delta - 1 & (1, 2\nu - \mu) & (1, \mu - \nu) & (1, \mu - \nu) \\ (1, \mu - \nu) & 0 & \delta - 1 & (1, 2\nu - \mu) & (1, \mu - \nu) \\ (1, \mu - \nu) & (1, \mu - \nu) & 0 & \delta - 1 & (1, 2\nu - \mu) \\ (1, 2\nu - \mu) & (1, \mu - \nu) & (1, \mu - \nu) & 0 & \delta - 1 \end{pmatrix}.$$

If $\mu = \nu$ we obtain that $\mathcal{E}_{1,1} = H_5$ and $\mathcal{E}_{\mu,\mu} = T_{5,\mu}$ for $\mu \geq 2$.

If $\mu = 2\nu$ and $\nu = 1$, we have

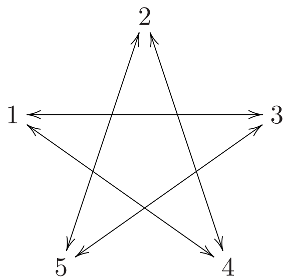
$$C = [Q(\mathcal{E}_{2,1})] = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Obviously, $C = P_{\sigma^2} + P_{\sigma^3}$ and $\operatorname{inx} C = 2$. Denote $\mathcal{E}_{2,1} = \mathcal{E}_5$. We have

$$\mathcal{E}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 2 & 0 \end{pmatrix}$$

and

$Q(\mathcal{E}_5) =$



If $\mu = 2\nu > 2$, then

$$[Q(\mathcal{E}_{2\nu,\nu})] = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} = E + C = P_{id} + P_{\sigma^2} + P_{\sigma^3}$$

and $\operatorname{inx}(E + C) = 3$. If $\nu = 1$, then $2 - \mu \geq 0$ and $\mu = 1, 2$. These cases were considered above.

Let $2\nu = \mu > 1$ and $\mu > \nu = 1$. Obviously, $[Q(\mathcal{E}_{\mu,\nu})] = P_{id} + P_{\sigma^2} + P_{\sigma^3} + P_{\sigma^4}$ and $\operatorname{inx} \mathcal{E}_{\mu,\nu} = 4$.

Case (b): $\sigma = (1\,2)(3\,4\,5)$. We obtain the following Gorenstein matrices:

$$\mathcal{E}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 4\alpha & 0 & 2\alpha & 2\alpha & 2\alpha \\ 2\alpha & 0 & 0 & 2\alpha & \alpha \\ 2\alpha & 0 & \alpha & 0 & 2\alpha \\ 2\alpha & 0 & 2\alpha & \alpha & 0 \end{pmatrix}.$$

$$\mathcal{E}_\alpha^{(2)} = \begin{pmatrix} 2 & 0 & 1 & 1 & 1 \\ 4\alpha & 2 & 2\alpha+1 & 2\alpha+1 & 2\alpha+1 \\ 2\alpha+1 & 1 & 2 & 2\alpha & \alpha+1 \\ 2\alpha+1 & 1 & \alpha+1 & 2 & 2\alpha \\ 2\alpha+1 & 1 & 2\alpha & \alpha+1 & 2 \end{pmatrix}$$

and

$$[Q(\mathcal{E}_\alpha)] = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix} = P_{id} + P_\tau + P_w + P_\theta,$$

where $\tau = (1\,3\,5\,2\,4)$, $w = (1\,4\,3\,2\,5)$ and $\theta = (1\,5\,4\,2\,3)$. Therefore $\operatorname{inx} \mathcal{E}_\alpha = 4$.

The next proposition follows from our considerations.

Proposition 7.1.5. *An adjacency matrix of the quiver of a Gorenstein matrix G for $n = 2, 3, 4, 5$ has the form $[Q(G)] = \lambda S$, where S is a doubly stochastic matrix and $\lambda = 1, 2, 3, 4$.*

Let $n = 6$. In this case there are four types of permutations:

$$\text{(a)}\, \sigma = (6\,5\,4\,3\,2\,1);\ \text{(b)}\, \sigma = (3\,2\,1)(6\,5\,4);$$

$$\text{(c)}\, \sigma = (4\,3\,2\,1)(5\,6);\ \text{(d)}\, \sigma = (1\,2)(3\,4)(5\,6).$$

Case (a): $\sigma = (6\,5\,4\,3\,2\,1)$. In this case we have the following Gorenstein matrix:

$$\mathcal{E}_{\alpha,\beta,\gamma} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha+\beta\gamma & 0 & \alpha & \beta & \alpha & 0 \\ \beta+\gamma & \beta+\gamma & 0 & \beta & \beta & 0 \\ \alpha+\gamma & \gamma & \alpha+\gamma & 0 & \alpha & 0 \\ \beta+\gamma & \gamma & \gamma & \beta+\gamma & 0 & 0 \\ \alpha+\beta+\gamma & \beta+\gamma & \alpha+\gamma & \beta+\gamma & 0 & 0 \end{pmatrix},$$

where $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, $\beta+\gamma \geq 1$, $\alpha+\gamma \geq 1$. Then

$$[Q(\mathcal{E}_{\alpha,\beta,\gamma})] = \begin{pmatrix} \delta & (1,\gamma) & (1,\alpha,\gamma) & (1,\beta) & (1,\alpha,\beta) & 0 \\ 0 & \delta & (1,\gamma) & (1,\alpha,\gamma) & (1,\beta) & (1,\alpha,\beta) \\ (1,\alpha,\beta) & 0 & \delta & (1,\gamma) & (1,\alpha,\gamma) & (1,\beta) \\ (1,\beta) & (1,\alpha,\beta) & 0 & \delta & (1,\gamma) & (1,\alpha,\gamma) \\ (1,\alpha,\gamma) & (1,\beta) & (1,\alpha,\beta) & 0 & \delta & (1,\gamma) \\ (1,\gamma) & (1,\alpha,\gamma) & (1,\beta) & (1,\alpha,\beta) & 0 & \delta \end{pmatrix},$$

where $\delta = (2, \beta+\gamma, \gamma+\alpha) - 1$.

Let $\alpha = \beta = 0$, then $\mathcal{E}_{0,0,\gamma} = T_{6,\gamma}$. In this case, $\mathcal{E}_{0,0,1} = H_6$ and $\operatorname{inx} \mathcal{E}_{0,0,\gamma} = 2$ for $\gamma \geq 2$. In the case $\alpha = \delta = 0$ we have $\gamma = 1$ and

$$[Q(\mathcal{E}_{0,\beta,1})] = \begin{pmatrix} 0 & 1 & 0 & (1,\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 & (1,\beta) & 0 \\ 0 & 0 & 0 & 1 & 0 & (1,\beta) \\ (1,\beta) & 0 & 0 & 0 & 1 & 0 \\ 0 & (1,\beta) & 0 & 0 & 0 & 1 \\ 1 & 0 & (1,\beta) & 0 & 0 & 0 \end{pmatrix}.$$

If $\beta = 0$, then we obtain the matrix H_6. Let $\alpha = \delta = 0$, $\gamma = 1$ and $\beta \geq 1$. In this case, we obtain $[Q(\mathcal{E}_{0,\beta,1})] = P_{\sigma^3} + P_{\sigma^5}$ and $\operatorname{inx} \mathcal{E}_{0,\beta,1} = 2$.

Now we consider the following case: $\alpha = 0$, $\gamma \geq 2$, $\beta \geq 1$. We obtain $[Q(\mathcal{E}_{0,\beta,\gamma})] = P_{id} + P_{\sigma^3} + P_{\sigma^5}$ and $\operatorname{inx} \mathcal{E}_{0,\beta,\gamma} = 3$.

Let $\alpha \geq 1$, $\beta = \delta = 0$. As above $\gamma = 1$. We obtain the following adjacency matrices:

$$[Q(\mathcal{E}_{\alpha,0,1})] = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

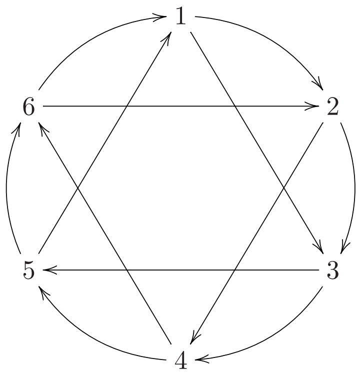
Obviously, $[Q(\mathcal{E}_{\alpha,0,1})] = P_{\sigma^4} + P_{\sigma^5}$ and $\operatorname{inx} \mathcal{E}_{0,\beta,1} = 2$.

Write $\mathcal{E}_{\alpha,0,1} = \Gamma_\alpha$. We obtain, that

$$\Gamma_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha+1 & 0 & \alpha & 0 & \alpha & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \alpha+1 & 1 & \alpha+1 & 0 & \alpha & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ \alpha+1 & 1 & \alpha+1 & 1 & \alpha+1 & 0 \end{pmatrix}$$

and

$Q(\Gamma_\alpha) =$



If $\alpha \geq 1$, $\beta = 0$, $\delta = 1$ and $\gamma \geq 2$, then $[Q(\mathcal{E}_{\alpha,0,\gamma})] = P_{id} + P_{\sigma^4} + P_{\sigma^5}$ and $\operatorname{inx} \mathcal{E}_{\alpha,0,\gamma} = 3$.

Let $\gamma = 0$, $\delta = 0$. We obtain the following adjacency matrices:

$$[Q(\mathcal{E}_{\alpha,\beta,0})] = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} = P_{\sigma^2} + P_{\sigma^3}$$

and $\operatorname{inx} \mathcal{E}_{\alpha,\beta,o} = 2$.

If $\gamma = 0$ and $\delta = 1$, then $[Q(\mathcal{E}_{\alpha,\beta,0})] = P_{id} + P_{\sigma^2} + P_{\sigma^3}$ and $\operatorname{inx} \mathcal{E}_{\alpha,\beta,o} = 3$.

Consequently, we may assume, that $\alpha, \beta, \gamma \geq 1$. Therefore, $\delta = 1$ and $[Q(\mathcal{E}_{\alpha,\beta,\gamma})] = U_6 - P_\sigma$, where

$$U_n = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}$$

for any n and $\operatorname{inx} \mathcal{E}_{\alpha,\beta,\gamma} = 5$. Obviously, $U_6 - P_\sigma = P_{id} + P_{\sigma^2} + P_{\sigma^3} + P_{\sigma^4} + P_{\sigma^5}$.

Remark 7.1.1. For U_n there is the obvious equality:

$$U_n = \sum_{i=1}^{n} P_{\sigma^i},$$

where $\sigma = (n\ n-1\ \dots\ 2\ 1)$.

From the considerations above there results the following proposition.

Proposition 7.1.6. *The adjacency matrix of any cyclic Gorenstein matrix $\mathcal{E}$ with $\sigma = \sigma(\mathcal{E}) = (6\,5\,4\,3\,2\,1)$ is a sum of some powers of the permutation matrix P_σ and for $\operatorname{inx}\mathcal{E}$ there are the possibilities: $\operatorname{inx}\mathcal{E} = 1, 2, 3, 5$.*

Remark 7.1.2. The matrix

$$\Gamma_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 4 & 3 & 3 \\ 4 & 0 & 0 & 4 & 2 & 2 \\ 4 & 0 & 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 2 & 0 & 3 \\ 3 & 0 & 1 & 2 & 3 & 0 \end{pmatrix}$$

is Gorenstein with permutation

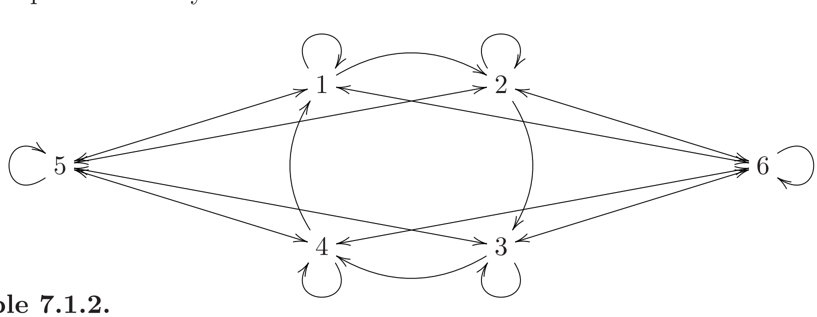
$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 6 & 5 \end{pmatrix}.$$

Note that

$$[Q(\Gamma_6)] = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

is not a multiple of a doubly stochastic matrix. We have that

$Q(\Gamma_6) =$



Example 7.1.2.

To conclude this section, here is an example of a non-Gorenstein reduced exponent matrix Δ such that $[Q(\Delta)] = 3P$, where P is a stochastic matrix, but not doubly stochastic. Let

$$\Delta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ 3 & 3 & 2 & 0 \end{bmatrix}$$

$$\Delta^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 0 \\ 3 & 3 & 2 & 1 \end{bmatrix}, \quad \Delta^{(2)} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 3 & 2 & 2 & 1 \\ 3 & 2 & 2 & 1 \\ 4 & 3 & 3 & 2 \end{bmatrix},$$

$$B = [Q(\Delta)] = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

and $\chi_B(x) = x^2(x-1)(x-3)$, where $\chi_B(x)$ is a characteristic polynomial of the matrix B.

7.2 CYCLIC GORENSTEIN MATRICES

Let $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbf{Z})$ be a Gorenstein (reduced) matrix with a permutation σ.

Lemma 7.2.1. *If $\alpha_{ij} + \alpha_{jk} = \alpha_{ik}$ for some i, j, k, then*

$$\alpha_{\sigma(i)\sigma(j)} + \alpha_{\sigma(j)\sigma(k)} = \alpha_{\sigma(i)\sigma(k)}.$$

Proof. Consider the sum

$$\alpha_{i\sigma(i)} + \alpha_{j\sigma(j)} + \alpha_{k\sigma(k)} \tag{7.2.1}$$

On the one hand using the Gorenstein property repeatedly and using $\alpha_{ij} + \alpha_{jk} = \alpha_{ik}$, this is equal to

$$\alpha_{ik} + \alpha_{k\sigma(i)} + \alpha_{j\sigma(i)} + \alpha_{\sigma(i)\sigma(j)} + \alpha_{k\sigma(j)} + \alpha_{\sigma(j)\sigma(k)} =$$

$$\alpha_{ij} + \alpha_{jk} + \alpha_{k\sigma(i)} + \alpha_{j\sigma(i)} + \alpha_{k\sigma(j)} + \alpha_{\sigma(i)\sigma(j)} + \alpha_{\sigma(j)\sigma(k)} \tag{7.2.2}$$

On the other hand (7.2.1) is also equal to

$$\alpha_{ij} + \alpha_{j\sigma(i)} + \alpha_{jk} + \alpha_{k\sigma(j)} + \alpha_{k\sigma(i)} + \alpha_{\sigma(i)\sigma(k)} \tag{7.2.3}$$

Comparing (7.2.2) and (7.2.3) is follows that $\alpha_{\sigma(i)\sigma(j)} + \alpha_{\sigma(j)\sigma(k)} = \alpha_{\sigma(i)\sigma(k)}$, proving the lemma.

Lemma 7.2.2. *For $i, j = 1, \ldots, s$ the following equalities hold:*

$$\alpha_{ij} + \alpha_{ji} = \alpha_{\sigma(i)\sigma(j)} + \alpha_{\sigma(j)\sigma(i)}.$$

Proof. The proof follows from the equalities:

$$1)\ \alpha_{ij} + \alpha_{j\sigma(i)} = \alpha_{i\sigma(j)} + \alpha_{\sigma(j)\sigma(i)} = \alpha_{i\sigma(i)},$$
$$2)\ \alpha_{ji} + \alpha_{i\sigma(j)} = \alpha_{j\sigma(i)} + \alpha_{\sigma(i)\sigma(j)} = \alpha_{j\sigma(j)}.$$

Adding them gives

$$\alpha_{ij} + \alpha_{ji} + \alpha_{j\sigma(i)} + \alpha_{i\sigma(j)} = \alpha_{i\sigma(j)} + \alpha_{j\sigma(i)} + \alpha_{\sigma(j)\sigma(i)} + \alpha_{\sigma(i)\sigma(j)}$$

proving the lemma.

Corollary 7.2.3. *If $\alpha_{ij} + \alpha_{jk} = \alpha_{ik}$ for some i, j, k, then*

$$\alpha_{\sigma^m(i)\sigma^m(j)} + \alpha_{\sigma^m(j)\sigma^m(k)} = \alpha_{\sigma^m(i)\sigma^m(k)}$$

for all positive integers m. For $i, j = 1, \ldots, n$

$$\alpha_{ij} + \alpha_{ji} = \alpha_{\sigma^m(i)\sigma^m(j)} + \alpha_{\sigma^m(j)\sigma^m(i)}$$

for any positive integer m.

Corollary 7.2.4. *If for any $j \neq i$, $j \neq k$, $\alpha_{ij} + \alpha_{jk} > \alpha_{ik}$ for some $i, k\,(i \neq k)$, then*

$$\alpha_{\sigma^m(i)\sigma^m(j)} + \alpha_{\sigma^m(j)\sigma^m(k)} > \alpha_{\sigma^m(i)\sigma^m(k)}.$$

Proof. Assume that

$$\alpha_{\sigma^m(i)\sigma^m(j)} + \alpha_{\sigma^m(j)\sigma^m(k)} = \alpha_{\sigma^m(i)\sigma^m(k)}.$$

Let σ^n be the identity. By corollary 7.2.3, we have

$$\alpha_{\sigma^{n-m}(\sigma^m(i))\sigma^{n-m}(\sigma^m(j))} + \alpha_{\sigma^{n-m}(\sigma^m(j))\sigma^{n-m}(\sigma^m(k))} = \alpha_{\sigma^{n-m}(\sigma^m(i))\sigma^{n-m}(\sigma^m(k))}.$$

So $\alpha_{\sigma^n(i)\sigma^n(j)} + \alpha_{\sigma^n(j)\sigma^n(k)} = \alpha_{\sigma^n(i)\sigma^n(k)}$, i.e., $\alpha_{ij} + \alpha_{jk} = \alpha_{ik}$. The corollary is proved.

Let $[Q] = (t_{ij})$ be the adjacency matrix of the quiver $Q(\mathcal{E})$ of a Gorenstein matrix $\mathcal{E}$.

Lemma 7.2.5. *Let $i, k = 1, \ldots, n$ and let m be a positive integer. Then $t_{\sigma^m(i)\sigma^m(k)} = t_{ik}$.*

Proof. Let $t_{ij} = 0$ for $i \neq j$. By the definition of $[Q(\mathcal{E})]$, there exists an integer k such that $\beta_{ik} + \beta_{kj} = \beta_{ij}$. If $k = i$ or $k = j$ then $\beta_{ik} + \beta_{kj} > \beta_{ij}$. Consequently, i, j, k are distinct and $\beta_{ik} = \alpha_{ik}$, $\beta_{kj} = \alpha_{kj}$, $\beta_{ij} = \alpha_{ij}$. By corollary 7.2.3, we have

$$\beta_{\sigma^m(i)\sigma^m(k)} + \beta_{\sigma^m(k)\sigma^m(j)} = \beta_{\sigma^m(i)\sigma^m(j)}.$$

Then $t_{\sigma^m(i)\sigma^m(j)} = \min\left(\beta_{\sigma^m(i)k} + \beta_{k\sigma^m(j)}\right) - \beta_{\sigma^m(i)\sigma^m(j)} = 0 = t_{ij}$.

If $t_{ij} = 1$ for $i \neq j$, then $\beta_{ik} + \beta_{kj} > \beta_{ij}$ for all k. We shall prove that

$$\beta_{\sigma^m(i)\sigma^m(k)} + \beta_{\sigma^m(k)\sigma^m(j)} > \beta_{\sigma^m(i)\sigma^m(j)}$$

for $k = 1, \dots, n$.

It is obvious for $i = k$ or $j = k$. Therefore we can consider that i, j, k are distinct. And so the inequality $\beta_{ik} + \beta_{kj} > \beta_{ij}$ is the same thing as the inequality $\alpha_{ik} + \alpha_{kj} > \alpha_{ij}$. By corollary 7.2.4, we have

$$\alpha_{\sigma^m(i)\sigma^m(k)} + \alpha_{\sigma^m(k)\sigma^m(j)} > \alpha_{\sigma^m(i)\sigma^m(j)}$$

and $\beta_{\sigma^m(i)\sigma^m(k)} + \beta_{\sigma^m(k)\sigma^m(j)} > \beta_{\sigma^m(i)\sigma^m(j)}$. As k ranges over $1, \dots, n$ so does $\sigma^m(k)$ and hence $t_{\sigma^m(i)\sigma^m(j)} > 0$ and hence $t_{ij} = 1$ as $[Q(\mathcal{E})]$ is a $(0,1)$-matrix. So, $t_{ij} = t_{\sigma^m(i)\sigma^m(j)}$ for $i \neq j$.

Let $t_{ii} = 0$. Then $\alpha_{ik} + \alpha_{ki} = 1$ for some $k \neq i$. By corollary 7.2.3,

$$\alpha_{\sigma^m(i)\sigma^m(k)} + \alpha_{\sigma^m(k)\sigma^m(i)} = 1.$$

Hence, $\gamma_{\sigma^m(i)\sigma^m(i)} = 1$, and $t_{\sigma^m(i)\sigma^m(i)} = 0$. If $t_{ii} = 1$ then $\gamma_{ii} = 2$. Since $\beta_{ii} + \beta_{ii} = 1 + 1 = 2$ for $i = 1, \dots, n$ it follows that $\beta_{ij} + \beta_{ji} \geq 2$ for $i \neq j$. But $\beta_{ij} = \alpha_{ij}$ for $i \neq j$ and, by corollary 7.2.3, we have

$$\beta_{\sigma^m(i)\sigma^m(j)} + \beta_{\sigma^m(j)\sigma^m(i)} = \beta_{ij} + \beta_{ji} \geq 2.$$

So $\gamma_{\sigma^m(i)\sigma^m(i)} \geq 2$ and $t_{\sigma^m(i)\sigma^m(i)} = 1$. The lemma is proved.

Theorem 7.2.6. *Let $\mathcal{E}$ be a cyclic Gorenstein matrix. Then $[Q(\mathcal{E})] = \lambda P$, where λ is a positive integer and P is a doubly stochastic matrix.*

Proof. Let σ be a cyclic permutation. Then the integers

$$t_{ik}, t_{\sigma(i)\sigma(k)}, \dots, t_{\sigma^{s-1}(i)\sigma^{s-1}(k)}$$

belong to different rows and columns. Let $C_i = \sum\limits_{j=1}^{n} t_{ij}$ and $D_j = \sum\limits_{i=1}^{n} t_{ij}$. Then

$$C_i = \sum_{j=1}^{n} t_{ij} = \sum_{j=1}^{n} t_{\sigma^m(i)\sigma^m(j)} = C_{\sigma^m(i)}$$

and

$$D_j = \sum_{i=1}^{n} t_{ij} = \sum_{i=1}^{n} t_{\sigma^m(i)\sigma^m(j)} = D_{\sigma^m(j)}$$

for any m, i.e., $C_i = C$ and $D_j = D$ for $i, j = 1, \dots, n$. Obviously,

$$\sum_{i=1}^{n} C_i = \sum_{i,j=1}^{n} t_{ij} = \sum_{j=1}^{n} D_j.$$

Then $nC = nD$, i.e., $C = D = \lambda$ and $[Q(\mathcal{E})] = \lambda P$, where P is a doubly stochastic matrix. The theorem is proved.

Theorem 7.2.7 (G.D.Birkhoff). *Every doubly stochastic matrix P is a linear combination of permutation matrices*

$$P = \sum_{\sigma \in S_n} \tau_\sigma P_\sigma,$$

where $\tau_\sigma \geq 0$, $\sum\limits_{\sigma \in S_n} \tau_\sigma = 1$ and the P_σ are permutation matrices. Conversely, $\sum\limits_{\sigma \in S_n} \tau_\sigma P_\sigma$ is a doubly stochastic matrix if $\tau_\sigma \geq 0$ and $\sum\limits_{\sigma \in S_n} \tau_\sigma = 1$.

The proof of this theorem is based on the Frobenius-König lemma for which we shall need to introduce some new notions.

Definition. A $(p \times q)$-submatrix $N = (n_{ks})$ of an $(n \times n)$-matrix $M = (m_{ij}) \in M_n(\mathbf{R})$ is a $(p \times q)$-matrix with entries from M such that $n_{ks} = m_{i_k, j_s}$ for $k = i_1, \dots, i_p$ and $s = j_1, \dots, j_q$. We shall write also

$$N = N_{i_1, \dots, i_p; j_1, \dots, j_q}.$$

Let $T = (t_{ij}) \in M_n(\mathbf{R})$ be a non-negative matrix. A **normal set of elements of** T is a set of n elements $t_{1j_1}, \dots, t_{nj_n}$ of T, where

$$\begin{pmatrix} 1 & 2 & \dots & n \\ j_1 & j_2 & \dots & j_n \end{pmatrix}$$

is an element of the symmetric group S_n of degree n.

Lemma 7.2.8 (Frobenius-König). *Let $T = (t_{ij}) \in M_n(\mathbf{R})$ be a non-negative matrix. If each normal set of T contains a zero element, then there is a zero $(p \times q)$-submatrix of T such that $p + q = n + 1$.*

Proof. We shall prove the lemma by induction on the degree of the matrix. We assume that the statement is true for all matrices of degree $< n$.

The case $n = 1$ is trivial: $t_{11} = 0$ and $1 + 1 = 2$.

Let $n > 1$. We can assume that $T \neq 0$ and $t_{nn} \neq 0$. Therefore by permuting rows and columns (independently) if necessary

$$T = \left(\begin{array}{c|c} T_1 & * \\ \hline * & t_{nn} \end{array}\right)$$

The matrix T_1 satisfies the conditions of the lemma, since otherwise there is a normal set of positive elements of T. So, by the induction hypothesis, there is a zero $(p_1 \times q_1)$-submatrix N of T_1 such that $p_1 + q_1 = n$; $p_1, q_1 \leq n - 1$. We may assume that $N = N_{1,\dots,p_1;1,\dots q_1}$ and T has the following form

$$\begin{array}{c|c|c|} \cline{2-3} p_1 & 0 & T_2 \\ \cline{2-3} q_1 & T_3 & \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{q_1} & \multicolumn{1}{c}{p_1} \end{array}$$

At least one of the square matrices T_2 or T_3 satisfies the conditions of the lemma. Otherwise there would be a normal set of T with all elements nonzero. Let the matrix T_2 satisfies the conditions of the lemma. By the induction hypothesis, there exists a zero $(p_2 \times q_2)$-submatrix of T_2 such that $p_2 + q_2 = p_1 + 1$. We can assume that T_2 has the form

$$\begin{array}{c|c|c|} \cline{2-3} p_2 & 0 & * \\ \cline{2-3} & * & * \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{q_2} & \multicolumn{1}{c}{} \end{array}$$

Then the matrix T has the form:

$$\begin{array}{c|c|c|c|} \cline{2-4} p_2 & 0 & 0 & * \\ \cline{2-4} p_1 - p_2 & 0 & * & * \\ \cline{2-4} & * & * & * \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c}{q_1} & \multicolumn{1}{c}{q_2} & \multicolumn{1}{c}{} \end{array}$$

And thus there is a zero $(p_2 \times (q_1 + q_2))$-submatrix of T. Therefore, $p_2 + q_1 + q_2 = p_1 + q_1 + 1 = n + 1$. The lemma is proved.

Lemma 7.2.9. *Let $B = (b_{ij}) \in M_n(\mathbf{R})$ be a non-negative matrix and*

$$\sum_{i=1}^{n} b_{ij} = \sum_{j=1}^{n} b_{ij} = \omega.$$

Then there exists a normal set $b_{1i_1}, \dots, b_{ni_n}$ of strictly positive elements of B.

Proof. Suppose that the statement is not true. Then, by the previous lemma, we can assume that B has the following form:

$$\begin{array}{r|c|c|} \cline{2-3} p & 0 & \\ \cline{2-3} n-p & & \\ \cline{2-3} \multicolumn{1}{r}{} & \multicolumn{1}{c}{q} & \multicolumn{1}{c}{n-q} \end{array}$$

and $p + q = n + 1$. The sum of the elements of the first p rows is equal to $p\omega$ and the sum of the first q columns is equal to $q\omega$. The sum of all elements of the matrix B is equal to $n\omega$. Therefore $p\omega + q\omega = (n+1)\omega > n\omega$. This contradiction proves the lemma.

Proof of theorem 7.2.7. Let T be a doubly stochastic matrix. By the previous lemma, there is a normal set

$$t_{1j_1}, \dots, t_{nj_n} \tag{7.2.4}$$

of positive elements of T. Let

$$\min_s t_{sj_s} = \tau_1 \qquad (\tau_1 > 0) \tag{7.2.5}$$

and let P_1 be a permutation matrix which has nonzero elements on the places corresponding to the set (7.2.4). Consider the matrix

$$B_1 = T - \tau_1 P_1.$$

The elements of B_1 are non-negative, and the sum of elements in each row and each column of B_1 is equal to $1 - \tau_1 = \omega_1 \geq 0$. The number of zero elements in B_1 is at least one more than the number of zero elements in T. If $1 - \tau_1 = 0$ we are through. Continuing this process after k steps we obtain:

$$B_k = T - \tau_1 P_1 - \tau_2 P_2 - \ldots - \tau_k P_k.$$

As the number of zeros in B_k is at least one more than the number of of zeros in B_{k-1} this procedure terminates, proving that a doubly stochastic matrix is a linear sum of permutation matrices whose coefficients are positive and sum to 1.

Conversely, the right side of the equality $T = \sum\limits_{\sigma \in S_n} \tau_\sigma P_\sigma$, where $\tau_\sigma \geq 0$ and $\sum\limits_{\sigma \in S_n} \tau_\sigma = 1$, is obviously a doubly stochastic matrix. The theorem is proved.

Theorem 7.2.10. *The adjacency matrix of a cyclic Gorenstein matrix with permutation σ is a sum of powers of the permutation matrix P_σ.*

Proof. Let $\mathcal{E}$ be a cyclic Gorenstein matrix with permutation σ. Then the adjacency matrix $[Q(\mathcal{E})]$ of the quiver $Q(\mathcal{E})$ is a multiple of a doubly stochastic matrix. By the Birkhoff theorem every non-negative doubly stochastic matrix is a linear combination of permutation matrices with non-negative coefficients: $P = \sum\limits_{\tau \in S_n} \alpha_\tau P_\tau$, where $a_\tau \geq 0$. Consequently, $[Q(\mathcal{E})]$ is a linear combination of permutation matrices.

By lemma 7.2.5, $t_{ij} = t_{\sigma(i)\sigma(j)} = \ldots = t_{\sigma^{n-1}(i)\sigma^{n-1}(j)}$, where σ is a cyclic permutation. As σ is cyclic $j = \sigma^k(i)$ for some $k \in \{0, 1, \ldots, n-1\}$. Hence, $t_{i\sigma^k(i)} = t_{\sigma(i)\sigma^{k+1}(i)} = \ldots = t_{\sigma^{n-1}(i)\sigma^{n+k-1}(i)}$. Renumbering the columns (and rows) simultaneously of $\mathcal{E}$ so that σ becomes $(1\,2\,\ldots\,n{-}1\,n)$ (which can be done, see the text just below the statement of theorem 7.1.3), this gives for all k, $t_{1\sigma^k(1)} = t_{2\sigma^k(2)} = \ldots = t_{n\sigma^k(n)}$. Since $[Q] = \sum\limits_{i,j=1}^{n} t_{ij}e_{ij}$ and $P_\tau = \sum\limits_{i=1}^{n} e_{i\tau(i)}$, where the e_{ij} are the matrix units, we have

$$[Q] = \sum_{i,j=1}^{n} t_{ij}e_{ij} = \sum_{i,k=1}^{n} t_{i\sigma^k(i)}e_{i\sigma^k(i)} = \sum_{k=1}^{n} t_{1\sigma^k(1)}P_{\sigma^k}.$$

Taking into account that $P_{\tau^m} = (P_\tau)^m$ for any permutation matrix P_τ, we obtain $[Q] = \sum\limits_{k=1}^{n} t_{1\sigma^k(1)}P_\sigma^k$, where $t_{1\sigma^k(1)}$ equals either 0 or 1.

Example 7.2.1.

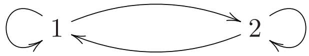
Let $\mathcal{E}_{2m} \in M_{2m}(\mathbf{Z})$ be the following exponent matrix:

$$\mathcal{E}_{2m} = \begin{pmatrix} A & C & \ldots & \ldots & C \\ B & A & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & C \\ B & \ldots & \ldots & B & A \end{pmatrix},$$

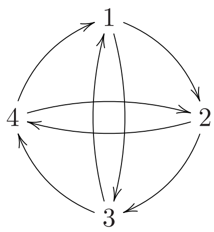
where $A = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

It is easy to see that $\mathcal{E}_{2m}$ is a cyclic Gorenstein matrix and $[Q(\mathcal{E}_{2m})] = P_{\sigma^{n-2}} + P_{\sigma^{n-1}}$, where $n = 2m$ with $\sigma = (n\, n-1\, \ldots\, 2\, 1)$.

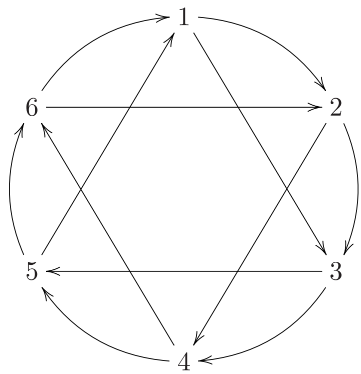
$Q(\mathcal{E}_2) =$



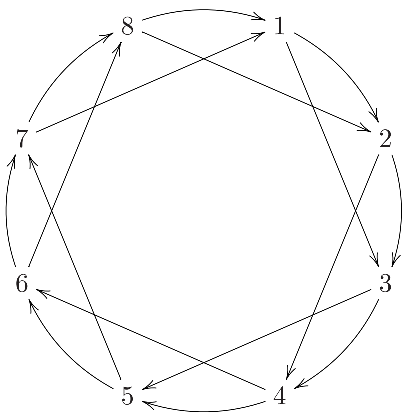
$Q(\mathcal{E}_4) =$



$Q(\mathcal{E}_6) =$



$Q(\mathcal{E}_8) =$



7.3 GORENSTEIN $(0,1)$-MATRICES

Recall that an (exponent) matrix $\mathcal{E} = (\alpha_{ij})$ is called a $(0,1)$-**matrix** if $\alpha_{ij} \in \{0,1\}$.

Consider the following Gorenstein $(0,1)$-matrices:

I. The $(n \times n)$-matrix

$$H_n = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ 1 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 1 & \dots & \dots & 1 & 0 \end{pmatrix}$$

is a Gorenstein cyclic matrix with permutation

$$\sigma = \sigma(H_n) = \begin{pmatrix} 1 & 2 & \dots & n \\ n & 1 & \dots & n-1 \end{pmatrix}.$$

For the adjacency matrix $[Q(H_n)]$ we have that $[Q(H_n)] = P_{\sigma^{n-1}}$.

II. The $(2m \times 2m)$-matrix

$$G_{2m} = \left| \begin{array}{c|c} H_m & H_m^{(1)} \\ \hline H_m^{(1)} & H_m \end{array} \right|$$

is Gorenstein with the permutation

$$\sigma(G_{2m}) = \begin{pmatrix} 1 & 2 & \dots & m & m+1 & m+2 & \dots & 2m \\ m+1 & m+2 & \dots & 2m & 1 & 2 & \dots & m \end{pmatrix}.$$

If $m = 1$ then

$$[Q(G_2)] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = E + P_\tau,$$

where τ is the transposition $(1\,2)$.

In the general case, $[Q(G_{2m})] = P_{\tau^{m-1}} + P_{\tau^{2m-1}}$, where

$$\tau = \begin{pmatrix} 1 & 2 & \dots & 2m \\ 2m & 1 & \dots & 2m-1 \end{pmatrix}$$

is a cycle and $\operatorname{inx} G_{2m} = 2$.

Associate with a reduced exponent $(0,1)$-matrix the poset

$$\mathcal{P}_{\mathcal{E}} = \{1, \dots, n\}$$

with the relation $\preceq$ defined by the formula $i \preceq j \Leftrightarrow \alpha_{ij} = 0$. It is easy to see that $(\preceq, \mathcal{P}_{\mathcal{E}})$ is a poset. Conversely, with any finite poset

$$\mathcal{P} = \{1, \dots, n\}$$

there is associated, as before (see just above theorem 6.3.1), the reduced exponent $(0,1)$-matrix $\mathcal{E}_{\mathcal{P}} = (\alpha_{ij})$ defined by: $\alpha_{ij} = 0$ if and only if $i \preceq j$ in $\mathcal{P}$, otherwise $\alpha_{ij} = 1$.

Definition. The width of the poset $\mathcal{P}_{\mathcal{E}}$ is called the **width of a reduced exponent** $(0,1)$**-matrix** and is denoted by $w(\mathcal{E})$.

Let $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbf{Z})$ be a reduced exponent matrix. We shall use the notations: $P_i = (\alpha_{i1}, \dots, \alpha_{in})$ and $\operatorname{rad} P_i = (\alpha_{i1}, \dots, 1, \dots, \alpha_{in})$.

We shall identify the poset $\mathcal{P}_{\mathcal{E}}$ with the set $\{P_1, \dots, P_n\}$ and $P_i \preceq P_j$ if and only if $\alpha_{ij} = 0$.

Then the diagram of $\mathcal{P}_{H_n}$ is: $P_1 \to P_2 \to \dots \to P_{n-1} \to P_n$ and the diagram of $\mathcal{P}_{G_{2s}}$ is the garland $\mathcal{P}_{2s}$:

$$\mathcal{P}_{2s} = \begin{tikzcd}
P_1 \arrow[r] \arrow[dr] & P_2 \arrow[r] \arrow[dr] & P_3 \arrow[r] \arrow[dr] & \cdots \arrow[r] \arrow[dr] & P_{s-1} \arrow[r] \arrow[dr] & P_s \\
P_{s+1} \arrow[r] \arrow[ur] & P_{s+2} \arrow[r] \arrow[ur] & P_{s+3} \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & P_{2s-1} \arrow[r] \arrow[ur] & P_{2s}
\end{tikzcd} \tag{7.3.1}$$

Lemma 7.3.1. *Let $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbf{Z})$ be a reduced exponent $(0,1)$-matrix and $w(\mathcal{E}) = 1$. Then $\mathcal{E}$ is equivalent to H_n.*

Proof. We may assume that $\alpha_{12} = \alpha_{23} = \dots = \alpha_{n-1n} = 0$ because $P_{\mathcal{E}}$ is a chain. From the exponent matrix inequalities we have $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$. Therefore, $\alpha_{ik} = 0$ for $i \leq k$. Since there are no symmetric zeroes in $\mathcal{E}$, we have that $\alpha_{pq} = 1$ for $p > q$. The lemma is proved.

Lemma 7.3.2. *Let $\mathcal{E} = (\alpha_{ij})$, $i, j = 1, \dots, n$, be a Gorenstein $(0,1)$-matrix with permutation $\sigma = \sigma(\mathcal{E})$ and let there exist an i such that $\alpha_{i\sigma(i)} = 0$. Then $\mathcal{E} \simeq H_n$ and $w(\mathcal{P}_{\mathcal{E}}) = 1$.*

Proof. If $n = 2$, then $\mathcal{E} \simeq H_2$ and $w(P_{\mathcal{E}}) = 1$. Then $n \geq 3$. We may assume that $i = 1$ and $\sigma(1) = n$. From $\alpha_{1j} + \alpha_{j\sigma(1)} = \alpha_{1n} = 0$ follows that $\alpha_{1j} = 0$ and $\alpha_{jn} = 0$ for $j = 1, \dots, n$, i.e., the first row and the last column of $\mathcal{E}$ is zero. This means that $P_1 = (0, 0, \dots, 0)$ is the unique minimal element in $P_{\mathcal{E}}$. Consider $\operatorname{rad} P_1 = (1, 0, \dots, 0)$. The matrix $\mathcal{E}$ does not have symmetric zeroes hence, $\alpha_{j1} = 1$

for $j = 2, \dots, n$. There exists such k that $\sigma(k) = 1$. By proposition 6.1.18 one can assume that $k = 2$. Therefore, $\alpha_{21} = \alpha_{2\sigma(2)} = \alpha_{2j} + \alpha_{j1} = 1$ and $\alpha_{2j} = 0$ for $j = 2, \dots, n$. So $P_2 = \operatorname{rad} P_1 = (1, 0, \dots, 0)$. As above $\alpha_{j2} = 1$ for $j \geq 3$.

Let $\sigma(3) = 2$. We have $\alpha_{32} = 1 = \alpha_{3\sigma(3)} = \alpha_{3j} + \alpha_{j2}$ and $\alpha_{3j} = 0$ for $j = 3, \dots, n$. So $P_3 = (1, 1, 0, \dots, 0) = \operatorname{rad} P_2$ and $\alpha_{j3} = 1$ for $j \geq 4$.

Continuing this process we obtain the following chain of the elements of $P_{\mathcal{E}}$

$$P_1 \preceq \operatorname{rad} P_1 = P_2 \preceq \operatorname{rad} P_2 \preceq \ldots \preceq \operatorname{rad} P_{n-1} = P_n.$$

The exponent matrix has the following form

$$\mathcal{E} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 0 & 0 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix}.$$

So $w(P_{\mathcal{E}}) = 1$ and $\mathcal{E} \simeq H_n$. The lemma is proved.

Theorem 7.3.3. *Any reduced Gorenstein* $(0, 1)$*-matrix* $\mathcal{E}$ *is equivalent either to* H_n *or to* G_{2m}.

Proof. Let $\sigma = \sigma(\mathcal{E})$ be the permutation of an $\mathcal{E}$. First of all we shall prove that the width $w(\mathcal{E})$ of a reduced Gorenstein (0,1)-matrix $\mathcal{E}$ is not greater than 2.

Let $w(\mathcal{E}) \geq 3$. Consequently there exist 3 pairwise non-comparable indecomposable rows

$$P_i = (\alpha_{i1}, \dots, 0, \dots, \alpha_{in});$$
$$P_j = (\alpha_{j1}, \dots, 0, \dots, \alpha_{jn});$$
$$P_k = (\alpha_{k1}, \dots, 0, \dots, \alpha_{kn}).$$

Using the elementary transformations of the second type (see proposition 6.1.18) we can assume that $i = 1$, $j = 2$, $k = 3$. Then

$$\mathcal{E} = \left(\begin{array}{ccc|c} 0 & 1 & 1 & \\ 1 & 0 & 1 & * \\ 1 & 1 & 0 & \\ \hline & * & & * \end{array}\right).$$

Obviously, $\sigma(i) > 3$ for $i = 1, 2, 3$. As above, we can consider that $\sigma(1) = 4$, $\sigma(2) = 5$, $\sigma(3) = 6$. ¿From the Gorenstein conditions it follows that $\alpha_{i4} = 1 - \alpha_{1i}$, $\alpha_{i5} = 1 - \alpha_{2i}$, $\alpha_{i6} = 1 - \alpha_{3i}$. First of all we shall compute the elements of $\mathcal{E}$ for $i = 1, 2, 3$, and after that for $i = 4, 5, 6$. This is done as follows. The Gorenstein conditions and lemma 7.3.2 say that $\alpha_{1i}+\alpha_{i\sigma(1)}+\alpha_{1\sigma(1)}=\alpha_{14}=1$. Then, taking $i=2,3$, we obtain $\alpha_{24}=\alpha_{34}=0$. Similarly $\alpha_{2i}+\alpha_{i\sigma(i)}=\alpha_{25}$. By lemma 7.3.2, $\alpha_{25}=1$. For $i=3$ we have $\alpha_{23}+\alpha_{35}=1$ and $\alpha_{35}=0$. Analogously, $\alpha_{21}+\alpha_{15}=1$ and $\alpha_{15}=0$ (recall that we are dealing with a $(0,1)$-matrix). Similarly $\alpha_{16}=\alpha_{26}=0$. Next the α_{ij} for $4\leq i,j\leq 6$ are calculated. For instance $\alpha_{34}+\alpha_{46}=\alpha_{36}=1$, so that $\alpha_{46}=1$. Further $\alpha_{24}+\alpha_{45}=\alpha_{25}$ and so $\alpha_{45}=1$. As a result $\mathcal{E}$ looks like

$$\mathcal{E}=\begin{pmatrix} \begin{matrix}0&1&1\\1&0&1\\1&1&0\end{matrix} & \begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} & * \\ & & \\ * & \begin{matrix}0&1&1\\1&0&1\\1&1&0\end{matrix} & * \\ & & \\ * & * & * \end{pmatrix}.$$

As $\alpha_{ij}+\alpha_{ji}>0$ $(i\neq j)$ it follows that $\alpha_{51}=\alpha_{61}=\alpha_{42}=\alpha_{62}=\alpha_{43}=\alpha_{53}=1$. Further $\alpha_{24}+\alpha_{41}\geq\alpha_{21}=1$ and as $\alpha_{24}=0$ this gives $\alpha_{41}=1$; $\alpha_{15}+\alpha_{52}\geq\alpha_{12}=1$ and $\alpha_{15}=0$ gives $\alpha_{52}=1$; $\alpha_{16}+\alpha_{63}\geq\alpha_{13}=1$ and $\alpha_{16}=0$ gives $\alpha_{63}=1$. Thus the matrix $\mathcal{E}$ is of the form

$$\mathcal{E}=\begin{pmatrix} \begin{matrix}0&1&1\\1&0&1\\1&1&0\end{matrix} & \begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} & * \\ & & \\ \begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix} & \begin{matrix}0&1&1\\1&0&1\\1&1&0\end{matrix} & * \\ & & \\ * & * & * \end{pmatrix}.$$

Now observe that $\sigma(i)>6$ for $i=4,5,6$. Indeed, if $\sigma(4)=1$, then $\alpha_{4\sigma(4)}=1=\alpha_{42}+\alpha_{21}=2$, a contradiction. Analogously, the cases $\sigma(4)=2$ and $\sigma(4)=3$ are impossible. Recall that $\sigma(1)=4$, $\sigma(2)=5$ and $\sigma(3)=6$. Therefore, $\sigma(i)\notin\{4,5,6\}$ for $i=4,5,6$. So it can be assumed that $\sigma(4)=7$, $\sigma(5)=8$, $\sigma(6)=9$. Again, of course, the Gorenstein condition is used in the form

$$\alpha_{4i}+\alpha_{i7}=\alpha_{47},\ \alpha_{5i}+\alpha_{i8}=\alpha_{58},\ \alpha_{6i}+\alpha_{i9}=\alpha_{69}.$$

As $\alpha_{41}=\alpha_{51}=\alpha_{61}=1$, this gives $\alpha_{47}=\alpha_{58}=\alpha_{69}=1$. Put $i=4,6$ in $\alpha_{5i}+\alpha_{i8}=1$ to find $\alpha_{48}=\alpha_{68}=0$; put $i=5,6$ in $\alpha_{4i}+\alpha_{i7}=1$ to find $\alpha_{57}=\alpha_{67}=0$; put $i=4,5$ in $\alpha_{6i}+\alpha_{i9}=1$ to find $\alpha_{49}=\alpha_{59}=0$.

Further put $i=8,9$ in $\alpha_{4i}+\alpha_{i7}=1$ to find $\alpha_{87}=1$, $\alpha_{97}=1$; put $i=7,9$ in $\alpha_{5i}+\alpha_{i8}=\alpha_{58}=1$ to find $\alpha_{78}=\alpha_{98}=1$, put $i=7,8$ in $\alpha_{6i}+\alpha_{i9}=1$ to find $\alpha_{79}=\alpha_{89}=1$.

Next put $i = 1, 2, 3$ in $\alpha_{4i} + \alpha_{i7} = 1$, $\alpha_{5i} + \alpha_{i8} = 1$, $\alpha_{6i} + \alpha_{i9} = 1$ to find $\alpha_{ij} = 0$ for $i \in \{1, 2, 3\}$, $j \in \{7, 8, 9\}$.

Thus the matrix $\mathcal{E}$ is of the form

$$\mathcal{E} = \left(\begin{array}{ccccccccccccc}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & * \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
 & & & & & & & & & \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & * \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & \\
 & & & & & & & & & \\
 & & & & & & 0 & 1 & 1 & \\
 & * & & & * & & 1 & 0 & 1 & * \\
 & & & & & & 1 & 1 & 0 & \\
 & & & & & & & & & \\
 & * & & & * & & & * & & *
\end{array}\right).$$

Because $\alpha_{ij} + \alpha_{ji} > 0$, $(i \neq j)$, it now follows that $\alpha_{ij} = 1$ for $i \in \{7, 8, 9\}$, $j \in \{1, 2, 3\}$ and $\alpha_{75} = \alpha_{76} = \alpha_{84} = \alpha_{86} = \alpha_{94} = \alpha_{95} = 1$. Next

$$\alpha_{57} + \alpha_{74} \geq \alpha_{54} = 1, \ \alpha_{48} + \alpha_{85} \geq \alpha_{45} = 1, \ \alpha_{49} + \alpha_{96} \geq \alpha_{46} = 1$$

and $\alpha_{57} = \alpha_{48} = \alpha_{49} = 0$, so that $\alpha_{74} = \alpha_{85} = \alpha_{96} = 1$. So the exponent matrix $\mathcal{E}$ looks like

$$\mathcal{E} = \left(\begin{array}{ccccccccccccc}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & * \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
 & & & & & & & & & \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & * \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & \\
 & & & & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & * \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \\
 & & & & & & & & & \\
 & * & & & * & & & * & & *
\end{array}\right).$$

Again, $\sigma(i) > 6$ for $i = 7, 8, 9$. Continuing this process we finally obtain after m

steps that the Gorenstein matrix $\mathcal{E}$ has the following block form:

$$\mathcal{E} = \begin{pmatrix} A & E & O & \dots & O & * \\ U & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & O & \vdots \\ \vdots & & \ddots & \ddots & E & \vdots \\ U & \dots & \dots & U & A & * \\ * & \dots & \dots & * & * & * \end{pmatrix},$$

where $n = 3m + r$, $0 \le r \le 2$, the bottom $*$'s are matrices with r rows. The $*$'s on the right are have r columns, and where

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \ U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \ E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

But then the Gorenstein condition forces (as above) that $\sigma(i) > 3m$ for $i \in \{1, 2, \dots, 3m\}$ for which there is no room. Thus the hypothesis $w(\mathcal{E}) \ge 3$ leads to a contradiction proving that $w(\mathcal{E}) \le 2$.

Consider the case $w(\mathcal{P}_{\mathcal{E}}) = 2$, that means $\mathcal{P}_{\mathcal{E}}$ has two non-comparable elements. Let they be P_1 and P_2. Then $\alpha_{12} = \alpha_{21} = 1$, and the exponent matrix has the following form:

$$\mathcal{E} = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & * \\ * & * \end{pmatrix}.$$

Suppose, $\sigma(1), \sigma(2) > 2$. One may then assume that $\sigma(1) = 3$ and $\sigma(2) = 4$. Then, in view of the Gorenstein condition and lemma 7.3.2, we obtain $\alpha_{1j} + \alpha_{j3} = \alpha_{13} = 1$ and $\alpha_{2j} + \alpha_{j4} = \alpha_{24} = 1$.

As $\alpha_{12} = 1$ it follows that $\alpha_{13} = 1$ and $\alpha_{23} = 0$; and as $\alpha_{21} = 1$ it follows that $\alpha_{24} = 1$ and $\alpha_{14} = 0$.

Further $\alpha_{23} + \alpha_{34} = \alpha_{24} = 1$, so $\alpha_{34} = 1$; $\alpha_{14} + \alpha_{43} = \alpha_{13} = 1$, so $\alpha_{43} = 1$. Thus the exponent matrix $\mathcal{E}$ looks like

$$\mathcal{E} = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & * \\ * & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & * \\ * & * & * \end{pmatrix}.$$

Next as $\alpha_{ij} + \alpha_{ji} > 0$ for $i \neq j$, it follows that $\alpha_{32} = \alpha_{41} = 1$. Also $\alpha_{14} + \alpha_{42} \geq \alpha_{12} = 1$ and as $\alpha_{14} = 0$, $\alpha_{42} = 1$; $\alpha_{23} + \alpha_{31} \geq \alpha_{21} = 1$, $\alpha_{23} = 0$, so that $\alpha_{31} = 1$. So $\mathcal{E}$ is of the form

$$\mathcal{E} = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & * \\ & & \\ \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & * \\ & & \\ * & * & * \end{pmatrix}.$$

Now observe that it must be the case that $\sigma(3), \sigma(4) > 4$. Indeed, suppose $\sigma(3) \in \{1, 2, 4\}$ (a fixed point being impossible). The Gorenstein condition says

$$\alpha_{34} + \alpha_{4\sigma(3)} = \alpha_{3\sigma(3)} = 1.$$

As $\alpha_{34} = 1$ this gives $\alpha_{4\sigma(3)} = 0$, so $\sigma(3) \neq 1, 2$. Also if $\sigma(3) = 4$ it would be the case that

$$\alpha_{3i} + \alpha_{i4} = \alpha_{34} = 1$$

for all i. But it is not true for $i = 2$. Thus, indeed, $\sigma(3) > 4$. A similar argument gives $\sigma(4) > 4$. Thus we can assume $\sigma(3) = 5$, $\sigma(4) = 6$.

Then $\alpha_{3i} + \alpha_{i5} = \alpha_{35}$, $\alpha_{34} = 1$, so $\alpha_{35} = 1$, $\alpha_{45} = 0$; $\alpha_{4i} + \alpha_{i6} = \alpha_{46}$, $\alpha_{43} = 1$, so $\alpha_{46} = 1$, $\alpha_{36} = 0$. Further $\alpha_{31} + \alpha_{15} = \alpha_{35} = 1$, $\alpha_{31} = 1$, so $\alpha_{15} = 0$; $\alpha_{41} + \alpha_{16} = \alpha_{46} = 1$, $\alpha_{41} = 1$, so $\alpha_{16} = 1$, $\alpha_{32} + \alpha_{25} = \alpha_{35} = 1$, $\alpha_{32} = 1$, so $\alpha_{25} = 0$; $\alpha_{42} + \alpha_{26} = \alpha_{46} = 1$, $\alpha_{42} = 1$, so $\alpha_{26} = 0$.

Next $\alpha_{45} + \alpha_{56} = \alpha_{46} = 1$, $\alpha_{45} = 0$, so $\alpha_{56} = 1$; $\alpha_{36} + \alpha_{65} = \alpha_{35} = 1$, $\alpha_{36} = 0$, so $\alpha_{65} = 1$. Therefore the exponent matrix $\mathcal{E}$ is of the form

$$\mathcal{E} = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & * \\ & & & \\ \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & * \\ & & & \\ * & * & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & * \\ & & & \\ * & * & * & * \end{pmatrix}.$$

As $\alpha_{ij} + \alpha_{ji} > 0$ for all $i \neq j$ it now follows that $\alpha_{51} = \alpha_{52} = \alpha_{61} = \alpha_{62} = 1$ and $\alpha_{54} = \alpha_{63} = 1$. Further $\alpha_{36} + \alpha_{64} \geq \alpha_{34} = 1$, $\alpha_{36} = 0$, so $\alpha_{64} = 1$; $\alpha_{15} + \alpha_{53} \geq \alpha_{13} = 1$, $\alpha_{15} = 0$, so $\alpha_{53} = 1$.

Thus the exponent matrix $\mathcal{E}$ looks like

$$\mathcal{E} = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & * \\ & & & \\ \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & * \\ & & & \\ \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} & \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & * \\ * & * & * & * \end{pmatrix}.$$

Now observe that the Gorenstein condition forces $\sigma(i) > 6$ for $i \in \{1, 2, 3, 4, 5, 6\}$ and continue in the same manner. The result, after m steps, is that $\mathcal{E}$ must be of the form

$$\mathcal{E} = \begin{pmatrix} A & E & O & \dots & \dots & O & * \\ U & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & O & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & E & \vdots \\ U & \dots & \dots & \dots & U & A & * \\ * & \dots & \dots & \dots & * & * & * \end{pmatrix},$$

where $n = 2m + r$, $0 \leq r \leq 1$, where the bottom $*$'s are matrices with r rows and the $*$'s on the right are blocks with r columns and where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Gorenstein condition forces $\sigma(i) > 2m$ for $i \in \{1, 2, \dots, 2m\}$ for which there is no room, giving a contradiction with the hypothesis $\sigma(1), \sigma(2) > 2$.

Hence, at least one of the numbers $\sigma(1)$ or $\sigma(2)$ is less than 3. Suppose $\sigma(1) = 2$, but $\sigma(2) \neq 1$. Let $\sigma(2) = 3$ and $\alpha_{2\sigma(2)} = 1 = \alpha_{23} = \alpha_{2i} + \alpha_{i3}$. We assume that $\alpha_{12} = \alpha_{21} = 1$. Then $\alpha_{i3} = 1 - \alpha_{2i}$. Therefore $\alpha_{13} = 0$ and $\alpha_{23} = 1$. Consequently

$$\mathcal{E} = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \begin{matrix} 0 \\ 1 \end{matrix} & * \\ * & * & * \end{pmatrix}.$$

Since $\mathcal{E}$ is reduced, $\alpha_{ij} + \alpha_{ji} > 0$ for $i \neq j$, and $\alpha_{13} + \alpha_{32} \geq \alpha_{12}$. So $\alpha_{31} = \alpha_{32} = 1$. Hence

$$\mathcal{E} = \begin{pmatrix} \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix} & * \\ * & * \end{pmatrix}.$$

Now consider $\sigma(3)$. Since $\sigma(1) = 2$ and $\sigma(2) = 3$, $\sigma(3) = 2$ cannot hold. If $\sigma(3) = 1$ then $\alpha_{32} + \alpha_{21} = 2$. So $\sigma(3) \notin \{1, 2\}$ and $\sigma(3) > 3$. It can be assumed that $\sigma(3) = 4$. The Gorenstein condition gives $\alpha_{3i} + \alpha_{i4} = \alpha_{34}$. As $\alpha_{31} = 1$, $\alpha_{34} = 1$ and $\alpha_{14} = 0$, and then also $\alpha_{24} = 0$ because $\alpha_{32} = 1$. So $\mathcal{E}$ looks like

$$\mathcal{E} = \begin{pmatrix} 0 & 1 & 0 & 0 & \\ 1 & 0 & 1 & 0 & * \\ 1 & 1 & 0 & 1 & \\ & & * & & * \end{pmatrix}.$$

Further $\alpha_{41} = \alpha_{42} = 1$ because $\alpha_{14} = \alpha_{24} = 0$ and $\alpha_{43} = 1$ because $\alpha_{24} + \alpha_{43} = \alpha_{23} = 1$ and $\alpha_{24} = 0$. Thus $\mathcal{E}$ has the form

$$\mathcal{E} = \begin{pmatrix} 0 & 1 & 0 & 0 & \\ 1 & 0 & 1 & 0 & \\ 1 & 1 & 0 & 1 & * \\ 1 & 1 & 1 & 0 & \\ & & * & & * \end{pmatrix}.$$

Continuing this process it follows that $\mathcal{E}$ is equal to

$$\mathcal{E} = \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 1 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 1 \\ 1 & \dots & \dots & \dots & 1 & 0 \end{pmatrix}.$$

and that $\sigma(i) = i + 1$, $i = 1, \dots, n - 1$. But then $\sigma(n)$ has to be 1 and $\alpha_{n2} + \alpha_{21} = \alpha_{n1}$ which is not true because $\alpha_{n2} = 1 = \alpha_{21} = \alpha_{n1}$ (if $n \geq 3$). Thus $\sigma(2) > 2$ cannot hold and we must have $\sigma(1) = 2$, $\sigma(2) = 1$.

Quite generally this means that if P_i and P_j are two incomparable elements then $\sigma(i) = j$, $\sigma(j) = i$. And further, if P_k $(k \neq j)$ would also be incomparable with P_i, $\sigma(i) = k$, $\sigma(k) = i$ which cannot be. So for any P_k, $k \neq i, j$, P_k is comparable with both P_i and P_j. This means that P splits up in a number s of pairs of incomparable elements which after renumbering can be labelled

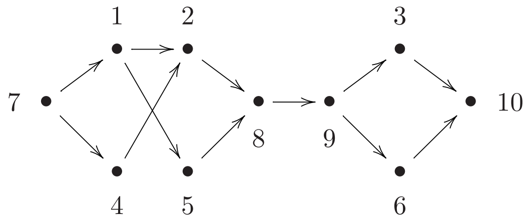
$$P_1, P_{s+1}; P_2, P_{s+2}; \dots; P_s, P_{2s}$$

and a number of singletons $P_{2s+1}, \dots, P_n$ such that for all $t = 1, \dots, s$, P_t is only incomparable with P_{t+s} and P_{2s+r} is comparable with all other P_t for $r = 1, \dots, n - 2s$. A further renumbering, if needed, sees to it that the ordered set looks like

$$P_i,\, P_{s+1} \prec P_j,\, P_{s+j},\ i,j \in \{1,\ldots,s\},\, i < j;$$

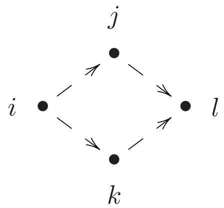
$$P_{2s+i} \prec P_{2s+j},\ i,j \in \{1,2,\ldots,n-2s\},\, i < j.$$

But nothing can be done by renumbering about the pattern odd singletons and pairs in the poset $\mathcal{P}_{\mathcal{E}}$. For instance $\mathcal{P}_{\mathcal{E}}$ could look like



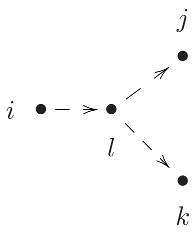
Now we show that if such an $\mathcal{E}$ is to be Gorenstein there can not be both pairs and singletons.

Indeed suppose that there are both pairs and singletons. Now σ takes each pair $\{P_i, P_j\}$ into itself. It follows that σ takes singletons into singletons. Thus there must be at least one pair and 2 singletons. This means that $\mathcal{P}_{\mathcal{E}}$ must have one of the 3 following subposets.



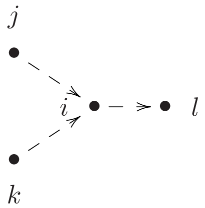
(a)

$\sigma(j) = k$, $\sigma(k) = j$, $\sigma(l) = i$.



(b)

$\sigma(j) = k$, $\sigma(k) = j$, $\sigma(i) = l$.



(c)

$\sigma(j) = k$, $\sigma(k) = j$, $\sigma(i) = l$.

Indeed take for i' the minimal singleton (among the singletons) and for $\{j,k\}$ the minimal pair (among the pairs). Suppose $i \prec j,k$ and $\sigma(i) \prec j,k$ then we have case (b) with $i = i'$ and $l = \sigma(i)$. Next suppose $i \prec j,k$ and $\sigma(i) \succ j,k$. Let $A = \{c : c$ is a singleton and $c \prec j,k\}$, $B = \{c : c$ is a singleton and $c \succ j,k\}$. Then σ takes $A \cup B$ into itself and as $\sigma(i') \in B$, $i' \in A$ there must be an $l \in B$ with $\sigma(l) \in A$. Take $i = \sigma(l)$. This gives subposet (a). Finally if $i' \succ j,k$, $\sigma(i') \succ i'$ (as i' is minimal), so take $i = i'$, $l = \sigma(i)$ to find case (c).

Case (a). The Gorenstein condition says

$$\alpha_{lj} + \alpha_{j\sigma(l)} = \alpha_{lj} + \alpha_{ji} = \alpha_{li}.$$

But $\alpha_{lj} = 1$, $\alpha_{ji} = 1$, $\alpha_{li} = 1$, so this cannot be.

Case (b). Here the Gorenstein condition says that

$$\alpha_{ij} + \alpha_{jl} = \alpha_{il} = 0.$$

But $\alpha_{ij} = 0$, $\alpha_{jl} = 1$, $\alpha_{il} = 0$, so this also cannot be.

Case (c). Finally in this case the Gorenstein condition says that

$$\alpha_{ij} + \alpha_{jl} = \alpha_{il}.$$

But $\alpha_{il} = 0$, $\alpha_{ij} = 1$, so this cannot be.[3]

It follows that there are either only singletons so that $\mathcal{P}_{\mathcal{E}}$ is a chain and $\mathcal{E} = H_n$ or there are only pairs and then $\mathcal{P}_{\mathcal{E}}$ is a garland (see (7.3.1)) and $\mathcal{E} = G_{2m}$.

7.4 INDICES OF GORENSTEIN MATRICES

Let r be the maximal eigenvalue of a permutationally irreducible non-negative matrix $A = (a_{ij})$. Write $s_i = \sum\limits_{k=1}^{n} a_{ik}$ $(i = 1, 2, \ldots, n)$, $s = \min\limits_{1 \le i \le n} s_i$, $S = \max\limits_{1 \le i \le n} s_i$. We recall:

Proposition 7.4.1. *Let A be a permutationally irreducible non-negative matrix. Then $s \le r \le S$ and the equality sign on the left side or on the right side of*

[3] The argument suggest e.g. in case (a) that a chain $\underset{i}{\bullet} \longrightarrow \underset{j}{\bullet} \longrightarrow \underset{l}{\bullet}$ cannot give a Gorenstein exponent matrix. That is not true of course. The reason that cases (a), (b), (c) are not possible is that the specified permutation is the wrong one in each case.

r holds only for $s = S$, i.e., it holds only when all the "row-sums" $s_1, s_2, \ldots, s_n$ are equal.

Corollary 7.4.2. *Let A be a permutationally irreducible $(0,1)$-matrix and $s = k$, $S = k+1$. Then r is not integer.*

The proof is obvious.

Definition. Let X and Y be any two (disjoint) posets. The **ordinal sum** $X \oplus Y$ of X and Y is the set of all $x \in X$ and $y \in Y$ (as a set, i.e. the disjoint union of the sets X and Y). The ordering relation $\preceq$ on it is defined as following: $x \prec y$ for all $x \in X$ and $y \in Y$; the relations $x \ \preceq x_1$ and $y \preceq y_1$ $(x, x_1 \in X;\ y, y_1 \in Y)$ are as before.

For instance

$$(\bullet \longrightarrow \bullet) \quad \oplus \quad (\bullet \longrightarrow \bullet) \quad = \quad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

and

$$(\bullet \qquad \bullet) \quad \oplus \quad (\bullet \longrightarrow \bullet) \quad = \quad \begin{tikzcd}[row sep=small] \bullet \arrow[dr] & & \\ & \bullet \arrow[r] & \bullet \\ \bullet \arrow[ur] & & \end{tikzcd}$$

The ordinal sum is associative (but not commutative), and we can consider the ordinal power $X^{\oplus n} = \underbrace{X \oplus \ldots \oplus X}_{n}$ for any poset X.

In particular, $CH_n = CH_1^{\oplus n}$ and $P_{2n} = ACH_2^{\oplus n}$.

If X and Y are finite posets, then

$$\mathcal{E}_{X \oplus Y} = \begin{pmatrix} \mathcal{E}_X & 0_{m \times n} \\ U_{n \times m} & \mathcal{E}_Y \end{pmatrix},$$

where m (resp. n) is a number of elements in X (resp. in Y); $0_{m \times n}$ is an $m \times n$-matrix, with all entries 0 and $U_{n \times m}$ is an $n \times m$-matrix, all of whose entries are 1. As usual one writes, $U_{n \times n} = U_n$ and $0_{n \times n} = 0_n$.

Remark 7.4.1. $\operatorname{inx} CH_n = w(CH_n) = 1$ and $\operatorname{inx} \mathcal{P}_{2n} = w(\mathcal{P}_{2n}) = 2$.

Recall that the index of a poset $\mathcal{P}$ is the largest eigenvalue of the adjacency matrix $[\widetilde{Q}(P)]$.

Proposition 7.4.3. *If $\operatorname{inx} \mathcal{P} = 1$, then $\mathcal{P}$ is CH_n for some n.*

Proof. The proof follows from proposition 7.4.1.

Proposition 7.4.4. *For any finite poset $\mathcal{P}$ we have:*

$$\operatorname{inx}\mathcal{P} \leq w(\mathcal{P}).$$

Proof. Let $c_1, \ldots, c_m$ form the antichain of all minimal elements of $\mathcal{P}$. There are exactly m arrows from a maximal element a to each c_i, $(i = 1, \ldots, m)$.

The elements $a_1, \ldots, a_k \in \mathcal{P}$ which cover a $b \in \mathcal{P}$ form an antichain. Thus, there are exactly k arrows to b from $a_1, \ldots, a_k$. Obviously, $m \leq w(\mathcal{P})$ and $k \leq w(\mathcal{P})$. Let $[\tilde{Q}(\mathcal{P})] = B = (b_{ij})$. Then, $S = \max\limits_{1 \leq i \leq n} \sum\limits_{j=1}^{n} b_{ij} \leq w(\mathcal{P})$ and, by proposition 7.4.1, we have $\operatorname{inx}\mathcal{P} \leq w(\mathcal{P})$.

Example 7.4.1.

The quiver Q with adjacency matrix

$$[Q] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

is not a quiver associated with any finite poset $\mathcal{P}$.

Theorem 7.4.5. *Let $\mathcal{P}$ be a finite poset. Then* $\operatorname{inx}\mathcal{P} = w(\mathcal{P}) = 2$ *if and only if* $\mathcal{P} = \mathcal{P}_{2n} = ACH_2^{\oplus n}$.

Proof. The equalities $\operatorname{inx}\mathcal{P}_{2n} = w(\mathcal{P}_{2n}) = 2$ follow from proposition 7.4.1.

Let $\mathcal{P} = \{p_1, \ldots, p_n\}$, $n \geq 3$ and $\operatorname{inx}\mathcal{P} = 2$. We shall show first that $\widetilde{Q}(\mathcal{P})$ has no loops when $n \geq 3$. For $n = 2$, $w(\mathcal{P}) = 2$ implies directly that $\mathcal{P} = ACH_2$. Let p_n be an isolated element. Then, as $w(\mathcal{P}) = 2$, $\{p_1, \ldots, p_{n-1}\}$ is a chain CH_{n-1}. One can suppose that

$$p_1 \prec p_2 \prec \ldots \prec p_{n-1}.$$

Thus, for $n \geq 3$

$$[\widetilde{Q}(\mathcal{P})] = \begin{pmatrix} 0 & 1 & 0 & \ldots & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & \ddots & \ddots & 0 \\ 1 & 0 & \ldots & \ldots & 0 & 1 \\ 1 & 0 & \ldots & \ldots & 0 & 1 \end{pmatrix}.$$

We have $s_1 = 1$ and $s_n = 2$. By corollary 7.4.2, $1 < \operatorname{inx}\mathcal{P} < 2$ contradicting $\operatorname{inx}(\mathcal{P}) = 2$. So $\widetilde{Q}(\mathcal{P})$ has no loops as required. Consequently, the $(0,1)$-matrix $[Q(\mathcal{P})]$ with $\operatorname{inx}\mathcal{P} = 2$ has the zero main diagonal and exactly two 1's in each row. Thus, $\mathcal{P}_{min}$ consists of two elements as there is an arrow from each maximal element to all minimal elements.

Denote by $\mathcal{P}^T$ the poset anti-isomorphic to $\mathcal{P}$. Obviously, $\operatorname{inx}\mathcal{P} = \operatorname{inx}\mathcal{P}^T$. Then $\operatorname{inx}\mathcal{P}^T = 2$ and $\mathcal{P}^T$ has exactly two minimal elements. Hence $\mathcal{P}$ also has precisely two maximal elements, say p_{n-1} and p_n. Thus, one can assume that $\mathcal{P}_{min} = \{p_1, p_2\}$, $\mathcal{P}_{max} = \{p_{n-1}, p_n\}$. The $(0,1)$-matrix $[\widetilde{Q}(\mathcal{P})]$ has zero main diagonal and exactly two 1's in each row and in each column.

Every partial order can be refined to a total order. Number the elements of the poset $\mathcal{P} \setminus \{1, 2, n-1, n\}$ according to such a total order with the numbers $3, 4, \dots, n-2$. Then in $\mathcal{P}$

$$p_i \prec p_j \Rightarrow i < j \tag{7.4.1}$$

From every$i$, $1 \le i \le n-2$ there issue precisely two arrows of $Q(\mathcal{P})$ and in every i, $3 \le i \le n$ there arrive precisely 2 arrows (because each column and row of $[\widetilde{Q}(\mathcal{P})]$ has precisely two entries equal to 1 and is otherwise zero).

Now consider the element 3. Two arrows must terminate in p_3 and as there are no loops these must come from p_1 and p_2 because of (7.4.1). Thus

$$\mathcal{P} = \begin{tikzcd} p_1 \arrow[r] & p_3 \\ p_2 \arrow[ur] & \end{tikzcd} \qquad \dots ? \dots$$

Next consider the element p_4. Two arrows must terminate in p_4. These must come from $\{p_1, p_2, p_3\}$. Suppose one of them comes from p_3. Let the other come from p_1. This gives $p_1 \longrightarrow p_3 \longrightarrow p_4$ and $p_1 \longrightarrow p_4$ which is a contradiction as p_4 then does not cover p_1. Similarly, if the second arrow arriving in 4 comes from p_2 we would have $p_2 \longrightarrow p_3 \longrightarrow p_4$, $p_2 \longrightarrow p_4$ also a contradiction. Thus the two arrows arriving in p_4 come from p_1 and p_2 and $\mathcal{P}$ looks like

$$\begin{tikzcd} p_1 \arrow[r] \arrow[dr] & p_3 \\ p_2 \arrow[ur] \arrow[r] & p_4 \end{tikzcd} \qquad \dots$$

Now consider p_5. There are two arrows terminating in p_5. These must come from $\{p_1, p_2, p_3, p_4\}$. But there are already 2 arrows starting in p_1, p_2. Thus these two arrows ending in p_5 must come from p_3, p_4 and $\mathcal{P}$ is of the form

$$\begin{tikzcd} p_1 \arrow[r] \arrow[dr] & p_3 \arrow[r] & p_5 \\ p_2 \arrow[ur] \arrow[r] & p_4 \arrow[ur] & \end{tikzcd} \qquad \dots$$

Now look at p_6. There must be precisely 2 arrows ending in p_6. These must issue from $\{p_3, p_4, p_6\}$. Arguing exactly as for p_4 we see that $p_5 \longrightarrow p_6$ is not possible. And so $\mathcal{P}$ looks like

$$\begin{tikzcd} p_1 \arrow[r] \arrow[dr] & p_3 \arrow[r] \arrow[dr] & p_5 & \cdots \\ p_2 \arrow[r] \arrow[ur] & p_4 \arrow[r] \arrow[ur] & p_6 & \end{tikzcd}$$

Continuing in this way the partially ordered set $\mathcal{P}$ would look like

$$\begin{tikzcd} p_1 \arrow[r] \arrow[dr] & p_3 & \cdots & p_{n-4} \arrow[r] \arrow[dr] & p_{n-2} \arrow[r] & p_n \\ p_2 \arrow[r] \arrow[ur] & p_4 & & p_{n-3} \arrow[r] \arrow[ur] & p_{n-1} \arrow[ur] & \end{tikzcd}$$

if n is odd. But that contradicts that there are two maximal elements. Thus n is even and

$$\mathcal{P} = \begin{tikzcd} p_1 \arrow[r] \arrow[dr] & p_3 & \cdots & p_{n-3} \arrow[r] \arrow[dr] & p_{n-1} \\ p_2 \arrow[r] \arrow[ur] & p_4 & & p_{n-2} \arrow[r] \arrow[ur] & p_n \end{tikzcd} = (ACH_2)^{\oplus n/2}$$

Remark 7.4.2. Similarly, one can show that if $\operatorname{inx} \mathcal{P} = w(\mathcal{P}) = m$ and $\widetilde{Q}(\mathcal{P})$ has no loops, then $\mathcal{P} = ACH_m^{\oplus n}$.

The description of reduced Gorenstein $(0, 1)$-orders is given by theorem 7.3.3. In view of theorem 7.4.5 and the definition of an ordinal power, we have the following statements.

Theorem 7.4.6. *A reduced* $(0, 1)$*-matrix* $\mathcal{E}$ *is Gorenstein if and only if* $\mathcal{P}_{\mathcal{E}}$ *is an ordinal power of either a singleton or an antichain with two elements.*

Theorem 7.4.7. *A reduced* $(0, 1)$*-matrix* $\mathcal{E}$ *is Gorenstein if and only if either* $\operatorname{inx} \mathcal{P}_{\mathcal{E}} = w(\mathcal{P}_{\mathcal{E}}) = 1$ *or* $\operatorname{inx} \mathcal{P}_{\mathcal{E}} = w(\mathcal{P}_{\mathcal{E}}) = 2$. *In the first case, the reduced tiled* (0,1)*-order with the exponent matrix* $\mathcal{E}$ *is hereditary.*

Denote by $M_n(\mathbf{Z})$ the ring of all square $n \times n$ matrices over the integers $\mathbf{Z}$. Let $A = (a_{ij}) \in M_n(\mathbf{Z})$.

Definition. A matrix $A = (a_{ij})$ is called a $(0,1,2)$**-matrix** if $a_{ij} \in \{0,1,2\}$.

Theorem 7.4.8. *For any permutation σ on $\{1,\ldots,n\}$ without fixed elements there exists a Gorenstein $(0,1,2)$-matrix $\mathcal{E}_\sigma$.*

Proof. Let $\sigma : i \mapsto \sigma(i)$ be a permutation on $\{1,\ldots,n\}$ without fixed elements and let $\mathcal{E}_\sigma = (\alpha_{ij})$ be the following $(0,1,2)$-matrix:

- $\alpha_{ii} = 0$ and $\alpha_{i\sigma(i)} = 2$ for $i = 1,\ldots,n$;
- $\alpha_{ij} = 1$ for $i \neq j$ and $j \neq \sigma(i)$ $(i,j = 1,\ldots,n)$.

Obviously, $\mathcal{E}_\sigma$ is a Gorenstein matrix with permutation σ.

Let σ be an arbitrary permutation on $\{1,\ldots,n\}$ without fixed elements and let $\mathcal{E}_\sigma$ be the Gorenstein $(0,1,2)$-matrix as in theorem 7.4.8. Let $P_\sigma = \sum\limits_{i=1}^{n} e_{i\sigma(i)}$ be the permutation matrix of σ. It is easy to see that $[Q(\mathcal{E}_\sigma)] = U_n - P_\sigma$.

Here is how one can represent the matrix $[Q(\mathcal{E}_\sigma)]$ as a sum of permutation matrices.

Let $\sigma_1,\ldots,\sigma_{n-1}$ be the permutations: $\sigma_k(i) = \sigma(i) + k \ (mod\, n)$. Obviously, $\sigma_k(i) \neq \sigma_m(i)$ for $k \neq m$ and $[Q(\mathcal{E}_\sigma)] = \sum\limits_{k=1}^{n-1} P_{\sigma_k}$.

Examples 7.4.2.
I. Let

$$\mathcal{E}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 0 \end{pmatrix}$$

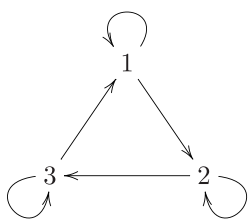
be the Gorenstein matrix with the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Straight forward calculation gives

$$\mathcal{E}_3^{(2)} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \text{ and } [Q(\mathcal{E}_3)] = E + P_{\sigma^2}.$$

Thus, $Q(\mathcal{E}_3)$ has the following form:



II. Let

$$\mathcal{E}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 \end{pmatrix}$$

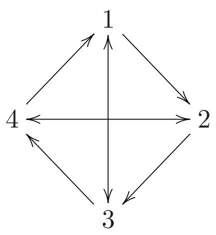
be the Gorenstein matrix with permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}.$$

One calculates

$$\mathcal{E}_4^{(2)} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 2 & 1 \end{pmatrix} \text{ and } [Q(\mathcal{E}_4)] = P_{\sigma^2} + P_{\sigma^3} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Hence, $Q(\mathcal{E}_4)$ has the following form:



III. Let

$$\mathcal{E}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 2 & 0 \end{pmatrix}$$

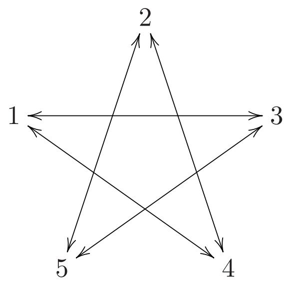
be the Gorenstein matrix with permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}.$$

By straight forward calculation

$$\mathcal{E}_5^{(2)} = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 2 & 1 \end{pmatrix} \text{ and } [Q(\mathcal{E}_5)] = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

So, $Q(\mathcal{E}_5)$ has the following form



and $[Q(\mathcal{E}_5)] = P_{\sigma^2} + P_{\sigma^3}$.

IV. Let

$$\mathcal{E}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 2 & 0 \end{pmatrix}$$

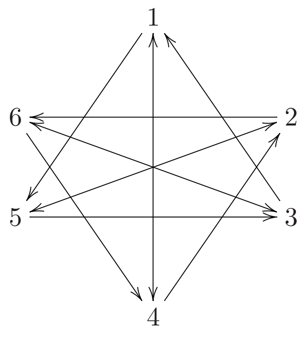
be the Gorenstein matrix with permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

Then

$$\mathcal{E}_6^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 2 & 2 & 1 \end{pmatrix} \quad [Q(\mathcal{E}_6)] = P_{\sigma^2} + P_{\sigma^3} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

And $Q(\mathcal{E}_6)$ looks as follows



In the general case we have that

$$\mathcal{E}_n = \begin{pmatrix} 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 \\ 2 & 0 & 1 & \dots & \dots & \dots & 1 & 0 \\ 1 & 1 & \ddots & \ddots & & & \vdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & 1 & 0 \\ 1 & 0 & \dots & \dots & 0 & 1 & 0 & 0 \\ 2 & 1 & \dots & \dots & \dots & 1 & 2 & 0 \end{pmatrix}$$

is a Gorenstein matrix with permutation

$$\sigma(\mathcal{E}_n) = \sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ n & 1 & \dots & n-1 \end{pmatrix}.$$

It is easy to show that $[Q(\mathcal{E}_n)] = P_{\sigma^2} + P_{\sigma^3}$.

Proposition 7.4.9. *For every positive integer n there exists a Gorenstein cyclic $(0,1,2)$-matrix $\mathcal{E}_n$ such that $\operatorname{inx}\mathcal{E}_n = w(\mathcal{E}_n) = 2$.*

7.5 D-MATRICES

Definition. Let $A \in M_{m\times n}(\mathbf{R})$ and $A \geq 0$, i.e., if $A = (a_{ij})$, then $a_{ij} \geq 0$. We say that A is a d**-matrix** for some $d > 0$, if $\sum\limits_{j=1}^{n} a_{ij} = d$ and $\sum\limits_{i=1}^{m} a_{ij} = d$ for all i, j.

Lemma 7.5.1. *If A is a d-matrix, then $m = n$.*

Proof. Obviously, $\sum\limits_{i,j} a_{ij} = md = nd$. So, $m = n$.

Lemma 7.5.2. *If A is a d-matrix and B is a d_1-matrix, then AB is a dd_1-matrix.*

The proof is obvious.

From proposition 6.6.1 it follows that the quiver $Q(A)$ of a d-matrix A is a disjoint union of strongly connected quivers.

Proposition 7.5.3. *The set D_n of all $(n \times n)$ doubly stochastic matrices is a semigroup with identity with respect to multiplication.*

The proof follows immediately from lemma 7.5.2.

Theorem 7.5.4. *If A is a $(0,1)$ d-matrix, then A is a sum of permutation matrices.*

Proof. We shall prove this theorem by induction on the number d. If $d=1$, then A is a permutation matrix. By lemma 6.2.9, in an arbitrary d-matrix ($d \geq 2$) there exists a normal set $a_{1i_1}, \ldots, a_{1i_n}$ of 1's in A. Denote by σ the permutation

$$\sigma = \begin{pmatrix} 1 & \ldots & n \\ i_1 & \ldots & i_n \end{pmatrix}.$$

Consider $A - P_\sigma = A_1$. Obviously, A_1 is a $(0,1)$ $(d-1)$-matrix and, by induction, $A_1 = \sum\limits_{\tau \in S_n} \alpha_\tau P_\tau$, where α_τ is either 0 or 1.

Note that a presentation $A = \sum \alpha_\tau P_\tau$ need not be unique. For example,

$$U_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = P_{id} + P_\sigma + P_{\sigma^2},$$

where $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. Write

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},$$

then also $U_3 = P_{\sigma_1} + P_{\sigma_2} + P_{\sigma_3}$.

Remark 7.5.1. Let A be a d-matrix and let there exists a permutation matrix P_τ such that

$$B = P_\tau^T A P_\tau = \begin{pmatrix} 0 & A_{12} & 0 & \ldots & 0 \\ 0 & 0 & A_{23} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & A_{h-1h} \\ A_{h1} & 0 & 0 & \ldots & 0 \end{pmatrix}$$

By lemma 7.5.2, $P_\tau^T A P_\tau = B$ is a d-matrix. Therefore the matrices $A_{12}, A_{23}, \ldots, A_{h1}$ are d-matrices. By lemma 7.5.1, all these matrices are square.

Example 7.5.1.

By theorem 7.3.3, any Gorenstein $(0,1)$-matrix $\mathcal{E}$ is equivalent to either H_s or G_{2s}. In the case H_s we obtain the adjacency matrix $[Q(H_s)]$ of $Q(H_s)$ in the following form

$$[Q(H_s)] = P_\sigma,$$

where $\sigma = (12\ldots s)$. The exponent matrix G_{2s} is equivalent to the matrix $\mathcal{E}_{2s}$

$$\mathcal{E}_{2s} = \begin{pmatrix} A & O & O & \dots & O & O \\ U & A & O & \dots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ U & U & U & \dots & A & O \\ U & U & U & \dots & U & A \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Obviously,

$$[Q(\mathcal{E}_{2s})] = \begin{bmatrix} 0 & U_2 & 0 & \dots & 0 \\ 0 & 0 & U_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & U_2 \\ U_2 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

We recall some facts which can be found in the book [Gantmakher, 1959] (Chapter III, §5 Primitive and imprimitive matrices).

Definition. Let $A \geq 0$ be a permutationally irreducible matrix, and let the maximal characteristic root be r. Suppose there are exactly h characteristic numbers of modulus r, i.e., $\lambda_1 = \ldots = |\lambda_h| = r$. If h is 1, the matrix is called **primitive**; if $h > 1$, the matrix is called **imprimitive**, and the number h is called the **index of imprimitivity**.

Theorem 7.5.5. *A nonnegative matrix A is primitive if and only if there is a power of A which is positive:*

$$A^p > 0 \ \ (\textit{for some } p \geq 1).$$

Remark 7.5.2. An algebraic proof of this theorem was given by I.N. Herstein in the paper [Herstein, 1954].

Let $A_1, \dots, A_t$ be the square matrices of orders $m_1, \dots, m_t$. Denote by $A = \operatorname{diag}(A_1, \dots, A_t)$ the following block diagonal matrix of order $m = m_1 + \ldots + m_t$:

$$A = \begin{pmatrix} A_1 & 0 & \dots & 0 & 0 \\ 0 & A_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A_{t-1} & 0 \\ 0 & 0 & \dots & 0 & A_t \end{pmatrix}.$$

Theorem 7.5.6. *Let A be a nonnegative permutationally irreducible matrix, and let some power A^q of A be reducible. Then A^q is completely reducible, i.e., there is a permutation of indices such that A^q can be written in the form*

$$A^q = \operatorname{diag}(A_1, A_2, \dots, A_d), \quad (d > 1)$$

after the permutation. Moreover, $A_1, A_2, \dots, A_d$ are permutationally irreducible matrices. The maximal characteristic numbers of these matrices are equal. Hence, d is the greatest common divisor of the numbers q, h, where h is the index of imprimitivity of the matrix A.

Corollary 7.5.7. *Every power a of primitive matrix $A \geq 0$ is permutationally irreducible, and therefore primitive.*

In the formulation of the next corollary we use the notation of theorem 6.5.2.

Corollary 7.5.8. *If $A \geq 0$ is an imprimitive matrix, the index of imprimitivity being h, then the matrix A^h is (after permutation of indices) a diagonal block matrix*

$$A^h = \operatorname{diag}\{A_{12} \dots A_{h-1n} A_{h1}, A_{23} \dots A_{h1} A_{12}, \dots, A_{h1} A_{12} \dots A_{h-1h}\},$$

where each block is permutationally irreducible, and each block has the same maximal characteristic number.

Lemma 7.5.9. *If Q is a quiver and $[Q]^m = (t_{ij})$, then t_{ij} is the number of all paths from a vertex i to a vertex j of length m.*

The proof goes by induction on m.

In accordance with the terminology of Markov chains, we shall call a strongly connected quiver Q **regular** if its adjacency matrix $[Q]$ is primitive. Otherwise, Q is called **cyclic**.

Theorem 7.5.10. *A quiver Q is regular if and only if it is strongly connected and the lengths of all its cycles have greatest common divisor 1.*

Proof. Let Q be regular. By theorem 7.5.5, there exists an integer m such that $[Q]^m > 0$. Therefore $[Q]^{m+1} > 0$ and there exist two cycles from 1 to 1. The first cycle has the length m, the second $m + 1$. So, the greatest common divisor of all cycles equals 1.

If Q is not regular then, by theorem 6.5.2, there exists a permutation matrix

P_τ such that

$$P_\tau^T[Q]P_\tau = \begin{pmatrix} 0 & A_{12} & 0 & \dots & 0 \\ 0 & 0 & A_{23} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_{h-1,h} \\ A_{h1} & 0 & 0 & \dots & 0 \end{pmatrix},$$

where there are square zero blocks along the main diagonal.

Denote by m_i the order of the i-th zero square block. The set VQ of vertices of Q may be numerated in such a way that

$$VQ = V_1 \cup V_2 \cup \ldots \cup V_h, \quad V_i \cap V_j = 0,$$

for $i \neq j$ and $|\,V_i\,| = m_i$ for $i = 1, \ldots, h$. Obviously, in Q there exists an arrow $k \to l$ only in the case if $k \in V_i$ and $l \in V_{i+1}$ for $i = 1, \ldots, h-1$ and $k \in V_h$ for $i = h$, $l \in V_1$. So, in Q the greatest common divisor d of the lengths of all cycles is, at least, $h \geq 2$. We shall show that if $d = 1$, then $[Q]$ is primitive. The matrix $[Q]$ is permutationally irreducible, therefore, by definition, $[Q]$ is either primitive or imprimitive. If $[Q]$ is imprimitive, then $d \geq h \geq 2$ and we obtain a contradiction. Therefore, $[Q]$ is primitive and Q is regular. The theorem is proved.

This theorem was first proved in the book [Dulmage, Mendelsohn, 1967].

Remark 7.5.3. Let Q be the quiver with adjacency matrix

$$[Q] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Obviously, Q is the simple cycle

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[dr] & \overset{2}{\bullet} \arrow[dl] \\ \underset{3}{\bullet} \arrow[u] & \underset{4}{\bullet} \arrow[u] \end{tikzcd}$$

and $h = 2$ is not the greatest common divisor of all cycles.

Assume we have a homogeneous Markov chain with a finite number of states and transition matrix $P = (p_{ij}) \in M_n(\mathbf{R})$. Let $Q(P)$ be the simply laced quiver associated with P (see vol.I, pp. 275-276 and section 6.6 above).

Corollary 7.5.11. *A Markov chain is regular if and only if the quiver $Q(P)$ of its transition matrix is regular, i.e., the greatest common divisor of all cycles of $Q(P)$ equals 1.*

The following theorem is stated without proof (see [Menon, 1967]).

Theorem 7.5.12. *To a given square matrix A with strictly positive elements there corresponds exactly one doubly stochastic matrix T which can be expressed in the form $T = D_1AD_2$, where D_1 and D_2 are diagonal matrices with strictly positive diagonal elements. The matrices D_1 and D_2 themselves are unique up to a scalar factor.*

7.6 CAYLEY TABLES OF ELEMENTARY ABELIAN 2-GROUPS

We introduce the following notations:

$$\Gamma_0 = (0), \qquad \Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix},$$

$$U_n = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \in M_n(\mathbf{Z}), \qquad X_{k-1} = 2^{k-1}U_{2^{k-1}};$$

$$\Gamma_k = \begin{pmatrix} \Gamma_{k-1} & \Gamma_{k-1} + X_{k-1} \\ \Gamma_{k-1} + X_{k-1} & \Gamma_{k-1} \end{pmatrix} \text{ for } k = 1, 2, \dots .$$

The matrix $\Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the Cayley table of the cyclic group G_1 of order 2 and is a Gorenstein matrix with permutation $\sigma(\Gamma_1) = (12)$.

Clearly, the Cayley table of the Klein four-group $\mathbf{Z}/(2) \times \mathbf{Z}/(2) = V_4 = (2) \times (2)$ can be written as

$$\Gamma_2 = \begin{pmatrix} \Gamma_1 & \Gamma_1 + 2U_2 \\ \Gamma_1 + 2U_2 & \Gamma_1 \end{pmatrix}.$$

Here the elements of $\mathbf{Z}/(2) \times \mathbf{Z}/(2)$ are numbered as follows: $(0, 0)$ has the label 0, and $(1, 0), (0, 1), (1, 1)$ have respectively the labels 1, 2, 3.

Consider

$$\Gamma_k = \begin{pmatrix} \Gamma_{k-1} & \Gamma_{k-1} + X_{k-1} \\ \Gamma_{k-1} + X_{k-1} & \Gamma_{k-1} \end{pmatrix}.$$

Proposition 7.6.1. *The matrix Γ_k is an exponent matrix for any natural number k.*

Proof. The proof is obvious.

Let $G = H\times < g >$ be a finite Abelian group, $H = \{h_1, \dots, h_n\}$, $g^2 = e$. We shall consider the Cayley table of H as the matrix $C(H) = (h_{ij})$ with entries in H, where $h_{ij} = h_i h_j$. The following proposition is obvious.

Proposition 7.6.2. *The Cayley table of G is*

$$C(G) = \begin{pmatrix} C(H) & gC(H) \\ gC(H) & C(H) \end{pmatrix}.$$

Proposition 7.6.3. *The matrix Γ_k is the Cayley table of the elementary Abelian group G_k of order 2^k.*

Proof. The proof goes by induction on k. The basis of the induction has already been done. If Γ_{k-1} is the Cayley table of G_{k-1}, then, by proposition 7.6.2, Γ_k is the Cayley table of G_k.

Proposition 7.6.4. *The matrix Γ_k is Gorenstein with permutation*

$$\sigma(\Gamma_k) = \begin{pmatrix} 1 & 2 & 3 & \dots & 2^k-1 & 2^k \\ 2^k & 2^k-1 & 2^k-2 & \dots & 2 & 1 \end{pmatrix}.$$

Proof. This is obvious for $k = 1$. Suppose that $\Gamma_k = (\alpha_{ij}^k)(\ i, j = 1, 2, \dots, 2^k)$ is Gorenstein and $\sigma(\Gamma_k) = \sigma_k$, where $\sigma_k(i) = 2^k + 1 - i$. Now $\alpha_{ij}^k + \alpha_{j\sigma_k(i)}^k = \alpha_{i\sigma_k(i)}^k$ for all $i, j = 1, 2, \dots, 2^k$. Since

$$\alpha_{2^k+i,j}^{k+1} = \alpha_{i,2^k+j}^{k+1} = \alpha_{ij}^k + 2^k, \ \ \alpha_{2^k+i,2^k+j}^{k+1} = \alpha_{ij}^{k+1} = \alpha_{ij}^k \quad \text{for all } i, j = 1, 2, \dots, 2^k$$

and $(\alpha_{ij}^k + 2^k) + \alpha_{j\sigma_k(i)}^k = (\alpha_{ij}^k + \alpha_{j\sigma_k(i)}^k) + 2^k = \alpha_{i\sigma_k(i)}^k + 2^k$, we obtain that

$$\alpha_{ij}^{k+1} + \alpha_{j,2^k+\sigma_k(i)}^{k+1} = \alpha_{i,2^k+\sigma_k(i)}^{k+1}, \quad \alpha_{i,2^k+j}^{k+1} + \alpha_{2^k+j,2^k+\sigma_k(i)}^{k+1} = \alpha_{i,2^k+\sigma_k(i)}^{k+1},$$

$$\alpha_{2^k+i,2^k+j}^{k+1} + \alpha_{2^k+j,\sigma_k(i)}^{k+1} = \alpha_{2^k+i,\sigma_k(i)}^{k+1}, \quad \alpha_{2^k+i,j}^{k+1} + \alpha_{j\sigma_k(i)}^{k+1} = \alpha_{2^k+i,\sigma_k(i)}^{k+1},$$

$i, j = 1, 2, \dots, 2^k$. Putting $\sigma_{k+1}(i) = 2^k + \sigma_k(i)$, $\sigma_{k+1}(2^k + i) = \sigma_k(i)$, we have $\alpha_{pq}^{k+1} + \alpha_{q\sigma_{k+1}(p)}^{k+1} = \alpha_{p\sigma_{k+1}(p)}^{k+1}$ for all $p, q = 1, 2, \dots, 2^{k+1}$, i.e., Γ_{k+1} is Gorenstein with the permutation $\sigma(\Gamma_{k+1}) = \sigma_{k+1}$, where $\sigma_{k+1}(i) = 2^{k+1} + 1 - i$.

Now we compute the adjacency matrix of the quiver $Q(\Gamma_k)$.

Let $\Gamma_k^{(1)} = (\beta_{ij}^k)$ and $\Gamma_k^{(2)} = (\gamma_{ij}^k)$. We have

$$\Gamma_k^{(1)} = \begin{pmatrix} \Gamma_{k-1}^{(1)} & \Gamma_{k-1} + X_{k-1} \\ \Gamma_{k-1} + X_{k-1} & \Gamma_{k-1}^{(1)} \end{pmatrix},$$

$$\Gamma_k^{(2)} = \begin{pmatrix} \Gamma_{k-1}^{(2)} & \Gamma_{k-1}^{(1)} + X_{k-1} \\ \Gamma_{k-1}^{(1)} + X_{k-1} & \Gamma_{k-1}^{(2)} \end{pmatrix}.$$

Hence,

$$[Q(\Gamma_k)] = \left[\begin{array}{cc} [Q(\Gamma_{k-1})] & E \\ E & [Q(\Gamma_{k-1})] \end{array} \right].$$

Here is the characteristic polynomial $\chi_{k+1}(x) = \chi_{[Q(\Gamma_{k+1}]}(x)$.

$$\chi_{k+1}(x) = |xE - [Q(\Gamma_{k+1})]| = \left| \begin{array}{cc} xE - [Q(\Gamma_k)] & -E \\ -E & xE - [Q(\Gamma_k)] \end{array} \right| =$$

$$= \left| \begin{array}{cc} xE - [Q(\Gamma_k)] - E & 0 \\ -E & xE - [Q(\Gamma_k)] + E \end{array} \right| =$$

$$= |(x-1)E - [Q(\Gamma_k)]| \cdot |(x+1)E - [Q(\Gamma_k)]|$$

Therefore,

$$\chi_{k+1}(x) = \chi_k(x-1) \cdot \chi_k(x+1). \tag{7.6.1}$$

Since

$$\chi_1(x) = \left| \begin{array}{cc} x-1 & -1 \\ -1 & x-1 \end{array} \right| = x(x-2),$$

we obtain $\chi_2(x) = (x-3)(x-1)(x-1)(x+1) = (x-3)(x-1)^2(x+1)$, $\chi_3(x) = (x-4)(x-2)^2 x(x-2)x^2(x+2) = (x-4)(x-2)^3 x^3 (x+2)$.

Proposition 7.6.5. $\chi_m(x) = \prod\limits_{i=0}^{m} (x-m-1+2i)^{C_m^i}$, *where* $C_m^i = \dfrac{m!}{(m-i)!i!}$.

Proof. This proposition is proved by induction on m. The basis of induction is clear. Suppose that the formula is true for $m = k$. Then, by formula (7.6.1), we obtain

$$\chi_{k+1}(x) = \prod_{i=0}^{k} (x-k-2+2i)^{C_k^i} \cdot \prod_{j=0}^{k} (x-k+2j)^{C_k^j} =$$

$$= (x-k-2) \prod_{i=1}^{k} (x-k-2+2i)^{C_k^i} \cdot \prod_{j=0}^{k-1} (x-k+2j)^{C_k^j} (x+k) =$$

$$= (x-k-2) \prod_{i=0}^{k-1} (x-k+2i)^{C_k^{i+1}} \cdot \prod_{j=0}^{k-1} (x-k+2j)^{C_k^j} (x+k) =$$

$$= (x-k-2) \prod_{i=0}^{k-1} (x-k+2i)^{C_k^i + C_k^{i+1}} (x+k).$$

Since $C_k^i + C_k^{i+1} = C_{k+1}^{i+1}$, we obtain $\chi_{k+1}(x) = (x-k-2)\prod_{i=0}^{k-1}(x-k+2i)^{C_{k+1}^{i+1}}(x+k) = (x-k-2)\prod_{j=1}^{k}(x-k+2(j-1))^{C_{k+1}^{j}}(x+k) = \prod_{j=0}^{k+1}(x-(k+1)-1+2j)^{C_{k+1}^{j}}$.

By induction on k, it is easy to prove that $\sum_{i=1}^{2^k} q_{ij}(\Gamma_k) = k+1$, $\sum_{j=1}^{2^k} q_{ij}(\Gamma_k) = k+1$. Thus, $[Q(\Gamma_k)] = (k+1)P_k$, where P_k is a doubly stochastic matrix.

Examples 7.6.1.

I. The Latin square

$$\mathcal{L}'_4 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 3 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{pmatrix}$$

is a Gorenstein matrix with permutation

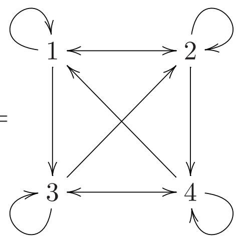
$$\sigma = \sigma(\mathcal{L}'_4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

and

$$[Q(\mathcal{L}'_4)] = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

So $[Q(\mathcal{L}'_4)] = E + P_{\sigma^2} + P_{\sigma^3}$ and

$$Q(\mathcal{L}'_4) =$$



II. The Latin square

$$\Gamma_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

is the Cayley table of the Klein four-group and it is a Gorenstein matrix with permutation $\sigma(\Gamma_2) = (14)(23)$. By propositions 4.1.11 and 4.1.12, the matrices Γ_2 and $\mathcal{L}'_4$ are non-equivalent.

$$\Gamma_2^{(1)} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 \\ 2 & 3 & 1 & 1 \\ 3 & 2 & 1 & 1 \end{pmatrix}; \quad \Gamma_2^{(2)} = \begin{pmatrix} 2 & 2 & 3 & 3 \\ 2 & 2 & 3 & 3 \\ 3 & 3 & 2 & 2 \\ 3 & 3 & 2 & 2 \end{pmatrix}.$$

$$[Q(\Gamma_2)] = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$Q(\Gamma_2) = \begin{tikzcd} 1 \arrow[loop left] \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & 2 \arrow[loop right] \arrow[d, leftrightarrow] \\ 3 \arrow[loop left] \arrow[r, leftrightarrow] & 4 \arrow[loop right] \end{tikzcd}$$

Theorem 7.6.6. *Suppose that a Latin square $\mathcal{L}_n$ with first row and first column $(0\,1\,\ldots\,n-1)$ is an exponent matrix. Then $n = 2^m$ and $\mathcal{L}_n = \Gamma_m$ is the Cayley table of the direct product of m copies of the cyclic group of order 2.*

Conversely, the Cayley table Γ_m of the elementary Abelian group $G_m = \mathbf{Z}/(2) \times \ldots \times \mathbf{Z}/(2) = (2) \times \ldots \times (2)$ (m factors) of order 2^m is a Latin square and a Gorenstein symmetric matrix with first row $(0, 1, \ldots, 2^m - 1)$ and permutation

$$\sigma(\Gamma_m) = \begin{pmatrix} 1 & 2 & 3 & \ldots & 2^m-1 & 2^m \\ 2^m & 2^m-1 & 2^m-2 & \ldots & 2 & 1 \end{pmatrix}.$$

The second part of this theorem follows from proposition 7.6.3.

Lemma 7.6.7. *Let $\mathcal{L}_n = (\alpha_{ij})$ be defined as above. Then*

$$|\, i-j \,| \le \alpha_{ij} \le i+j-2.$$

Proof. Obviously, $\alpha_{1i} + \alpha_{ij} \ge \alpha_{1j}$ and so $\alpha_{ij} \ge j-1-(i-1) = j-i$. Analogously, $\alpha_{ij} + \alpha_{j1} \ge \alpha_{i1}$ and $\alpha_{ij} \ge i-1-(j-1) = i-j$, i.e., $\alpha_{ij} \ge |\, i-j \,|$. Also $\alpha_{i1} + \alpha_{1j} \ge \alpha_{ij}$ so that $\alpha_{ij} \le i+j-2$.

Lemma 7.6.8. *The last row of $\mathcal{L}_n$ is $(n-1, n-2, \ldots, 1)$.*

Proof. We have $\alpha_{n1} = n-1$, by the definition of $\mathcal{L}_n$. By lemma 7.6.7, we have $\alpha_{ni} \geq n-i$. So, $\alpha_{n2} = n-2$, $\alpha_{n3} = n-3$ and $\alpha_{nn} = 0$ because otherwise the row sum of the last row would be larger than that of the 1-st row.

Corollary 7.6.9. *The last column of $\mathcal{L}_n$ is $(n-1, n-2, \dots, 1)^T$, where T is the transpose.*

Lemma 7.6.10. *Let $\mathcal{L}_n = (\alpha_{ij})$ be defined as above. Then*

$$|i+j-(n+1)| \leq |n-1-\alpha_{ij}|.$$

Proof. By lemma 7.6.8 and corollary 7.6.9, we have $\alpha_{ij} \leq \alpha_{in} + \alpha_{nj} = (n-i)+(n-j) = 2n-(i+j)$. By lemma 7.6.7, $\alpha_{ij} \leq i+j-2$. From the first inequality we have

$$\alpha_{ij} - (n-1) \leq n+1-(i+j).$$

From the second inequality we have

$$\alpha_{ij} - (n-1) \leq (i+j)-(n+1).$$

So

$$|(i+j)-(n+1)| \leq |\alpha_{ij}-(n-1)|.$$

Corollary 7.6.11. *The integer n in theorem 7.6.6 is even.*

Proof. By lemma 7.6.7, if the integer $n-1$ appears in position (i,j) then $|(n-1)-(n-1)| \geq |i+j-(n+1)|$, i.e., $i+j = n+1$. Hence, the secondary diagonal, $(a_{1n}, a_{2n-1}, \dots, a_{n1})$, has the following form: $(n-1, \dots, n-1)$ because, $\mathcal{L}_n$ being a Latin square, $n-1$ must appear once in each row. If n is odd then for $i = j = \dfrac{n+1}{2}$ we have $\alpha_{ii} = n-1$. This is a contradiction. So $n = 2n_1$.

Proof of theorem 7.6.6. If $\alpha_{ij} = 1$, then $|i-j| = 1$ by lemma 7.6.7. By assumptions in the statement of the theorem $\alpha_{12} = 1 = \alpha_{21}$. Now look at the 3-rd row. The number 1 must occur somewhere. This can only be at $(3,2)$ or $(3,4)$. But at $(3,2)$ is not possible because that would give two 1's at the 2-nd column. Thus $\alpha_{34} = 1$. Next look at the 3-rd column. There must be a 1 somewhere in this column. This can only occur at $(2,3)$ or $(4,3)$. But $(2,3)$ is impossible because that would give two 1's in the second row and so $\alpha_{43} = 1$. Further look at the fifth row. There must be a 1 somewhere. This can only be at $(5,4)$ or $(5,6)$. But $(5,4)$ is not possible as that would give two 1's in the 4-th column. So $\alpha_{56} = 1$. Next look at the 5-th column which must have a 1 somewhere. This can only be at $(4,5)$ or $(6,5)$, but at $(4,5)$ is not possible because that would give two 1's in the fourth row. So $\alpha_{65} = 1$. Continuing this argument it follows that $\mathcal{L}_n$ look like

$$\mathcal{L}_n = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & * & \dots & * \\ * & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{pmatrix} = \begin{pmatrix} \Gamma_1 & * & \dots & * \\ * & \Gamma_1 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & \Gamma_1 \end{pmatrix}.$$

If $\alpha_{ij} = 2$ then $|\, i - j \,| \leq 2$, by lemma 7.6.7. By the assumption of the theorem $\alpha_{13} = \alpha_{31} = 2$. There must be a 2 in the second row. This can only happen at $(2,1)$, $(2,2)$, $(2,3)$, $(2,4)$. But $(2,1)$ and $(2,2)$ are already occupied (by a 1 and a 0) and $(2,3)$ is impossible because that would give two 2's in the 3-rd column. So $\alpha_{24} = 2$. Similarly, looking at the 2-nd column and the 3-rd row, one obtains $\alpha_{42} = 2$. Now look at the 5-th row. Columns 3 and 4 already have a 2 and places $(5,5)$, $(5,6)$ are already occupied, so $\alpha_{57} = 2$. Next look at row 6. Column 4 already has a 2, places $(6,5)$ and $(6,6)$ are occupied, column 7 already has a 2 and it follows that $\alpha_{68} = 2$. Next look at column 5. Rows 3, 4 already have a 2, places $(5,5)$ and $(6,5)$ are already occupied and so $\alpha_{75} = 2$. Now examine column 6. Rows 4, 5 already have a 2, places $(5,6)$, $(6,6)$ are already occupied, row 7 already has a 2 and so $\alpha_{86} = 2$. Continuing in this way it follows that $\mathcal{L}_n$ looks like

$$\mathcal{L}_n = \begin{pmatrix} \begin{array}{|cccc|cccc|} \hline 0 & 1 & 2 & & & & & \\ 1 & 0 & & 2 & & & & \\ 2 & & 0 & 1 & & * & & \\ & 2 & 1 & 0 & & & & \\ \hline & & & & 0 & 1 & 2 & \\ & & & & 1 & 0 & & 2 \\ & * & & & 2 & & 0 & 1 \\ & & & & & 2 & 1 & 0 \\ \hline \end{array} & & * \\ & \ddots & \\ * & & \begin{array}{|cccc|} \hline 0 & 1 & 2 & \\ 1 & 0 & & 2 \\ 2 & & 0 & 1 \\ & 2 & 1 & 0 \\ \hline \end{array} \end{pmatrix}$$

It also follows that n is divisible by 4. Otherwise we would obtain a "cutoff version" like illustrated below

$$\left(\begin{array}{cccc|cc|cc}
0 & 1 & 2 & & & & & \\
1 & 0 & & 2 & & * & & \\
2 & & 0 & 1 & & & & \\
 & 2 & 1 & 0 & & & & \\
\hline
 & * & & & 0 & 1 & 2 & \\
 & & & & 1 & 0 & & 2 \\
\hline
 & & & & 2 & & 0 & 1 \\
 & & & & & 2 & 1 & 0
\end{array}\right)$$

and the last row would have no 2 in it.

Obviously the next step is to figure out where the 3's must be located. If $\alpha_{ij} = 3$, then $|\,i - j\,| \leq 3$. By the assumption of the theorem $\alpha_{14} = 3 = \alpha_{41}$. Consider places $(4t+2, 4t+3)$, $t = 0, 1, \ldots, (n/4 - 1)$. Now $\alpha_{4t+2,4t+3}$ must be ≥ 3 because all the 0, 1, 2 have been placed. On the other hand

$$\alpha_{4t+2,4t+3} \leq \alpha_{4t+2,4t+1} + \alpha_{4t+1,4t+3} = 1 + 2 = 3$$

and so $\alpha_{4t+2,4t+3} = 3$. Similarly $\alpha_{4t+3,4t+2} \geq 3$, but

$$\alpha_{4t+3,4t+2} \leq \alpha_{4t+3,4t+1} + \alpha_{4t+1,4t+2} = 2 + 1 = 3$$

and so $\alpha_{4t+3,4t+2} = 3$.

Also

$$\alpha_{4t+1,4t+4} \leq \alpha_{4t+1,4t+2} + \alpha_{4t+2,4t+4} = 1 + 2 = 3$$

$$\alpha_{4t+4,4t+1} \leq \alpha_{4t+4,4t+2} + \alpha_{4t+2,4t+1} = 2 + 1 = 3$$

and the same argument gives $\alpha_{4t+1,4t+4} = 3 = \alpha_{4t+4,4t+1}$. Thus the matrix $\mathcal{L}_n$ is of the following form

$$\mathcal{L}_n = \begin{pmatrix}
\begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{matrix} & * & \cdots & * \\
* & \begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{matrix} & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & \begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{matrix}
\end{pmatrix} = \begin{pmatrix}
\Gamma_2 & * & \ldots & * \\
* & \Gamma_2 & \ldots & * \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & \Gamma_2
\end{pmatrix}.$$

The next and final step is to prove by induction the following two statements.

- for every k there exists a unique Latin square $\mathcal{L}_n$ of order 2^k satisfying theorem 7.6.6, viz. $\mathcal{L}_n = \Gamma_k$;
- for every k the number of blocks Γ_k ($\Gamma_k \neq \mathcal{L}_n$) on the main block diagonal of $\mathcal{L}_k$ is even.

The start of the induction was taken care of above.

Now assume that $\mathcal{L}_n$ has blocks Γ_k, $2^k < n$ on the main block diagonal and let there be n_k of them. Note that this has been shown to be the case for $k = 1$ and $k = 2$. So $\mathcal{L}_n$ looks like

$$\mathcal{L}_n = \begin{pmatrix} \Gamma_k & ** & * & \dots & * \\ * & \Gamma_k & * & \dots & * \\ * & * & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ * & * & \dots & * & \Gamma_k \end{pmatrix}$$

If $\alpha_{ij} = 2^k$ then $|\, i - j \,| \leq 2^k$. Claim

$$\alpha_{i,i+2^k} = 2^k \text{ for } i = 1, \dots, 2^k.$$

It helps to realize that this is a statement about the $2^k \times 2^k$ block indicated by $**$ above. The claim is proved by induction. For $i = 1$ it is true by the assumptions of theorem 7.6.6. Now let $i > 1$. The number 2^k must occur at one of the places $(i, 1), (i, 2), \dots, (i, i-1), (i, i), (i, i+1), \dots, (i, i+2^k)$. The places $(i, 1), \dots, (i, 2^k)$ are already filled. Further the columns $2^k + 1, \dots, 2^k + i - 1$ already contain a 2^k by the induction hypothesis. So it must be the case that $\alpha_{i,i+2^k} = 2^k$.

Similarly, switching rows and columns in the argument, one finds

$$\alpha_{i+2^k,i} = 2^k \text{ for } i = 1, 2, \dots, 2^k.$$

Moreover, this pattern persists in that

$$\alpha_{t2^{k+1}+i,t2^{k+1}+i+2^k} = 2^k = \alpha_{t2^{k+1}+i+2^k,t2^{k+1}+i} \tag{7.6.2}$$

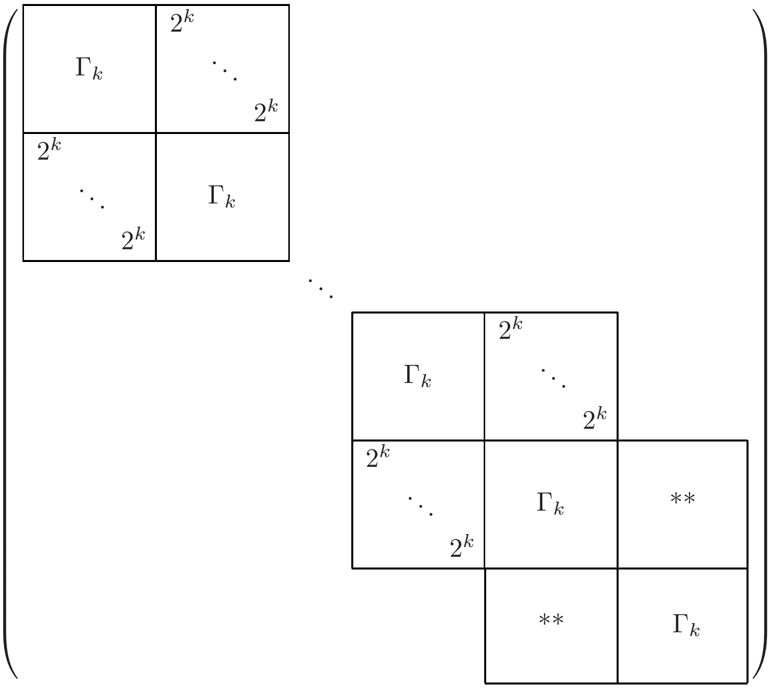
for $t = 0, 1, \dots$; $i = 1, 2, \dots, 2^k$. This is again proved by induction on i. Consider row $t2^{k+1} + i$. The number 2^k must occur at one of the places

$$(t2^{k+1} + i, t2^{k+1} + i - 2^k + j), \quad j = 0, 1, \dots, 2^{k+1}.$$

Now the place $(t2^{k+1}+i, t2^{k+1}+i-2^k+j)$ with $i+j \leq 2^k$ is in the $(2t+1)$-th Γ_k and hence already occupied by one of the numbers $0, 1, \dots, \dots, 2^k - 1$. By induction on i, the columns with with index $t2^{k+1}+1+2^k, \dots, t2^{k+1}+(i-1)+2^k$ already have a

2^k in them. So it must be that 2^k occurs at position $(t2^{k+1}+i, t2^{k+1}+i+2^k)$. This proves the first half of (7.6.2). The second half is handled by the same argument (switching rows and columns).

Now suppose that the number of blocks Γ_k is odd. Then by what has just been proved the matrix $\mathcal{L}_n$ looks like



But then there is no 2^k in the blocks indicated by $**$'s and hence no 2^k in the last 2^k rows and columns because $\alpha_{ij} = 2^k$ implies $|\, i-j \,| \le 2^k$.

Let Y_t denote the t-th $2^{k+1} \times 2^{k+1}$ diagonal block of $\mathcal{L}_n$, $t = 1, 2, \ldots, 2^{-1}n_k$, and write

$$Y_t \;=\; \begin{pmatrix} \Gamma_k & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \Gamma_k \end{pmatrix}.$$

As $0, 1, 2, \ldots, 2^k - 1$ occur in each row (and column) of Γ_k and $\mathcal{L}_n$ is a Latin square all elements of $\mathcal{E}_{12}$ must be $\ge 2^k$. On the other hand for any element of an $\mathcal{E}_{12}$

$$\alpha_{t2^{k+1}+i,t2^{k+1}+2^k+j} \le \alpha_{t2^{k+1}+i,t2^{k+1}+j} + \alpha_{t2^{k+1}+j,t2^{k+1}+j+2^k}$$

By (7.6.2) above, the second element on the right hand side is equal to 2^k and the first element on the right hand side is in a Γ_k and hence $< 2^k$. Hence $\mathcal{E}_{12}$ is a Latin square on the numbers $2^k, 2^k + 1, \ldots, 2^{k+1} - 1$. Similarly for $\mathcal{E}_{21}$.

Since $2^kU_{2^k} \le \mathcal{E}_{12} < 2^{k+1}$, we obtain $0 \le \mathcal{E}_{12} - 2^kU_{2^k} < 2^k$, and $\mathcal{E}_{12} - 2^kU_{2^k}$ is a Latin square on the set $\{0, 1, 2, \dots, 2^k - 1\}$.

Since $2^kU_{2^k} \le \mathcal{E}_{21} < 2^{k+1}$, we obtain $0 \le \mathcal{E}_{21} - 2^kU_{2^k} < 2^k$, and $\mathcal{E}_{21} - 2^kU_{2^k}$ is a Latin square on the set $\{0, 1, 2, \dots, 2^k - 1\}$.

urthermore

$$\alpha_{t\cdot 2^{k+1}+i,t\cdot 2^{k+1}+2^k+1} \le \alpha_{t\cdot 2^{k+1}+i,t\cdot 2^{k+1}+1}+\alpha_{t\cdot 2^{k+1}+1,t\cdot 2^{k+1}+2^k+1} = (i-1)+2^k = 2^k+i-1$$

$$\alpha_{t\cdot 2^{k+1}+2^k+i,t\cdot 2^{k+1}+1} \le \alpha_{t\cdot 2^{k+1}+2^k+i,t\cdot 2^{k+1}+i}+\alpha_{t\cdot 2^{k+1}+i,t\cdot 2^{k+1}+1} = 2^k+(i-1) = 2^k+i-1$$

for all $i = 1, 2, \dots, 2^k$. Thus for $i = 1, 2, \dots, 2^k$ we have

$$\alpha_{t\cdot 2^{k+1}+i,t\cdot 2^{k+1}+2^k+1} = 2^k + i - 1$$

$$\alpha_{t\cdot 2^{k+1}+2^k+i,t\cdot 2^{k+1}+1} = 2^k + i - 1$$

for all $i = 1, 2, \dots, 2^k$. So the first row (resp. column) of the matrices $\mathcal{E}_{12} - 2^kU_{2^k}$ and $\mathcal{E}_{21} - 2^kU_{2^k}$ have the following form $(0, 1, \dots, 2^k - 1)$ (resp. $(0, 1, \dots, 2^k - 1)^T$).

By induction, we have $\mathcal{E}_{12} - 2^kU_{2^k} = \mathcal{E}_{21} - 2^kU_{2^k} = \Gamma_k$ and blocks

$$\Gamma_{k+1} = \begin{pmatrix} \Gamma_k & \Gamma_k + 2^kU_{2^k} \\ \Gamma_k + 2^kU_{2^k} & \Gamma_k \end{pmatrix}$$

are on the main diagonal.

With induction Γ_k is a $2^k \times 2^k$ matrix and hence $\mathcal{L}_n$ is the Γ_k with $n = 2^k$.

$$\Gamma_m = \begin{pmatrix} \Gamma_{m-1} & \Gamma_{m-1} + X_{m-1} \\ \Gamma_{m-1} + X_{m-1} & \Gamma_{m-1} \end{pmatrix}, \quad X_{m-1} = 2^{m-1}U_{2^{m-1}}.$$

By proposition 7.6.3, Γ_m is the Cayley table of the elementary Abelian group G_k of order 2^k. Theorem 7.6.2 is proved.

Remark 7.6.1. Example 7.6.1(I) shows, that for theorem 7.6.2 the condition for a Latin square having the first row and the first column of the form $(0\,1\,\dots\,n-1)$ is essential.

7.7 QUASI-FROBENIUS RINGS AND GORENSTEIN TILED ORDERS

Theorem 7.7.1. *A reduced tiled order A is Gorenstein if and only if the ring $B = A/\pi A$ is Frobenius. In this case $\nu(B) = \sigma(A)$.*

Proof. Let $A = \{\mathcal{O}, \mathcal{E}(A)\}$ be a Gorenstein tiled order. By proposition 7.1.2, A is Gorenstein if and only if $\mathcal{E}(A)$ is Gorenstein. Obviously, $\mathcal{E}(\pi A) = \mathcal{E}(A) + U_n$, where

$$U_n = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}.$$

Let $P_\sigma = P_{\sigma(A)}$ be the permutation matrix of $\sigma = \sigma(A)$. It is easy to see that the set $X \subset A$, which is defined by the matrix $\mathcal{E}(X) = \mathcal{E}(A) + U_n - P_\sigma$, is a two-sided ideal in A. Further, $\bar{X} = X/\pi A$ is a right and left semisimple module: $X = U_{\sigma(1)} \oplus \ldots \oplus U_{\sigma(n)}$ as a right A-module and $X = V_1 \oplus \ldots \oplus V_n$ as a left A-module. Consider $B = A/\pi A$. Obviously, $\bar{X} = \mathrm{soc}\,(B_B) = \mathrm{soc}\,({}_B B)$ and $\bar{X}$ is a monomial ideal. By definition of a quasi-Frobenius ring, the ring B is Frobenius with Nakayama permutation $\sigma(A)$.

Conversely, let $B = A/\pi A$ be Frobenius and $\nu(B)$ be its Nakayama permutation. Obviously, $\nu(B)$ is a fixed point free permutation. So every indecomposable projective πP_i has only one minimal overmodule X_i and $X_i/\pi P_i \simeq U_{\nu(i)}$. Consequently, by lemma 5.2.13, $\pi P_1, \ldots, \pi P_s$ are indecomposable relatively injective pairwise nonisomorphic A-lattices and ${}_A A^{\#}$ is a projective right A-module. It is easy to see that $\nu(B) = \sigma(A)$. The theorem is proved.

Definition. A permutation σ on the set $S = \{1, \ldots, n\}$ is called a **fixed point free permutation** if $\sigma(i) \neq i$ for all $i \in S$.

Corollary 7.7.2. *For any fixed point free permutation σ there exists a semidistributive weakly prime Artinian Frobenius ring B with $\nu(B) = \sigma$.*

The proof follows from theorem 7.4.8 and theorem 7.7.1.

Lemma 7.7.3. *Let e and f be non-zero idempotents of a ring A such that the modules eA and fA are indecomposable. In this case eA and fA are isomorphic if and only if the following equality holds: $f = f\lambda_0 e\lambda_1$, where $\lambda_0, \lambda_1 \in A$.*

Proof. Suppose that modules eA and fA are isomorphic, and $\varphi : eA \to fA$ is this isomorphism. Then $\varphi(e\lambda) = \varphi(e \cdot e\lambda) = \varphi(e)e\lambda = f\lambda_0 e \cdot \lambda$, i.e., φ acts on $e\lambda$ by left multiplication by $f\lambda_0$. Therefore $f = \varphi(e\lambda_1) = f\lambda_0 e\lambda_1$. Conversely, let $f = f\lambda_0 e\lambda_1$. Then $\varphi : eA \to fA$, where $\varphi(eA) = f\lambda_0 e \cdot e\lambda$ is an epimorphism. Since the module fA is projective, we have $eA \simeq fA \oplus X$. Note that $fA \neq 0$, and that the module eA is indecomposable, so $eA \simeq fA$. The lemma is proved.

Theorem 7.7.4. *Let $A = \{\mathcal{O}, \mathcal{E}(A)\}$ be a reduced tiled Gorenstein order with Jacobson radical R and let $\mathcal{J}$ be a two-sided ideal of A such that $A \supset R^2 \supset \mathcal{J} \supset R^n$ $(n \geq 2)$. The quotient ring $A/\mathcal{J}$ is quasi-Frobenius if and only if there exists a $p \in R^2$ such that $\mathcal{J} = pA = Ap$.*

Proof. Let $A = \sum_{i,j=1}^{s} e_{ij}\pi^{\alpha_{ij}}\mathcal{O}$, and let $M_s(\mathcal{D}) = \sum_{i,j=1}^{s} e_{ij}\mathcal{D}$ be the ring of fractions of A, where $\mathcal{D}$ is the division ring of fractions of $\mathcal{O}$, and the e_{ij} are the matrix units $(i, j = 1, \ldots, s)$. Let $\mathcal{J} = pA = Ap$ be a two-sided ideal of A and $A \supset R^2 \supset \mathcal{J} \supset R^n$. Obviously, $M_s(\mathcal{D})\mathcal{J}M_s(\mathcal{D}) = M_s(\mathcal{D})pAM_s(\mathcal{D}) = M_s(\mathcal{D})pM_s(\mathcal{D})$ is a non-zero two-sided ideal of $M_s(\mathcal{D})$. Therefore $p \in M_s(\mathcal{D})$ has an inverse p^{-1} in $M_s(\mathcal{D})$. Since the quotient ring A/R^n is Artinian, the quotient ring $\bar{A} = A/\mathcal{J}$ is also Artinian. We now show that the quotient ring $\bar{A}$ is quasi-Frobenius. Let

$1 = e_{11} + \ldots + e_{ss}$ be the decomposition of $1 \in A$ into a sum of matrix idempotents, $A = \oplus_{i=1}^{s} P_i$, where $P_i = e_{ii}A$. Obviously, every indecomposable projective $\bar{A}$-module is of the form $P_i \simeq e_{ii}\Lambda/e_{ii}p\Lambda$ for some $i = 1, \ldots, s$. The right modules A and pA are isomorphic because p is an invertible element in $M_s(\mathcal{D})$. Since, $\mathcal{J} = pA = \oplus_{i=1}^{s} e_{ii}pA$, we have $e_{ii}A \supset e_{ii}R^2 \supset e_{ii}pA \supset e_{ii}R^n$. Therefore for each module $e_{ii}pA$ there exists a unique minimal overmodule. It follows that $\operatorname{soc} \bar{P}_i$ is a simple module for $i = 1, \ldots, s$. Suppose that $\operatorname{soc} \bar{P}_i \simeq \operatorname{soc} \bar{P}_k$ for $i \neq k$. Then we have $e_{ii}pA \simeq e_{kk}pA$. This implies that $p^{-1}e_{ii}pA \simeq p^{-1}e_{kk}pA$. Since, $pA = Ap$ we have $p^{-1}Ap = A$, therefore $p^{-1}e_{ii}p \in A$ for $i = 1, \ldots, s$. By lemma 7.7.3, the following equality holds: $p^{-1}r_{kk}\lambda_0 e_{ii}p\lambda_1 = p^{-1}e_{kk}p$, so $e_{kk}p\lambda_0 p^{-1}e_{ii}p\lambda_1 p^{-1} = e_{kk}$. Since $p\lambda_0 p^{-1}$ and $p\lambda_1 p^{-1}$ are elements of A, we have $e_{ii}A \simeq e_{kk}A$. This contradiction shows that if $i \neq k$ then $\operatorname{soc} \bar{P}_i \not\simeq \operatorname{soc} \bar{P}_k$. The same holds for left modules, so, by the classical definition of QF-rings by Nakayama, we see that $\bar{A}$ is a QF-ring.

Conversely, let $A \supset R^2 \supset \mathcal{J} \supset R^n$, and let $\bar{A} = A/\mathcal{J}$ be a QF-ring. Since $e_{ii}A \supset e_{ii}R^2 \supset e_{ii}\mathcal{J} \supset e_{ii}R^n$, we see that for every module $e_{ii}J$ there exists a unique minimal overmodule X_i such that $X_i/e_{ii}\mathcal{J} \not\simeq X_j/e_{jj}\mathcal{J}$ for $i \neq j$. Since A is Gorenstein, by lemma 6.2.13, one can see that the modules $e_{ii}\mathcal{J}(i = 1, \ldots, s)$ are all pairwise non-isomorphic indecomposable projective A-modules. It follows that the right A-module $\mathcal{J} = \oplus_{i=1}^{s} e_{ii}\mathcal{J}$ is isomorphic to A_A. Since $End_A A \simeq A$, we can conclude that there exists a monomorphism $\varphi : A \to A$ defined by the formula $\varphi(\lambda) = q\lambda$ and $\mathrm{Im}\varphi = \mathcal{J}$. Therefore $\mathcal{J} = qA$, where q is a regular element. Analogously, $Ap = \mathcal{J}$, where p is a regular element. Then $q = a_0 p$ and $p = qa_1, q = a_0 q a_1$. Obviously, the elements a_0 and a_1 are regular. Let us show that if $b_1 A \subset b_2 A$ and $b_1 A \neq b_2 A$ for some elements $b_1, b_2 \in A$, then $xb_1 A \subset xb_2 A$ and $xb_1 A \neq xb_2 A$ for any regular element $x \in A$. Suppose that $xb_1 A = xb_2 A$. Then $xb_1 y = xb_2$, i.e., $b_2 = b_1 y$ and $b_2 A = b_1 yA \subset b_1 A$. So we obtain a contradiction with our assumption. Thus if $a_1 A \subset A$ and $a_1 A \neq A$ then $qa_1 A \neq qA$. Therefore $a_0 q a_1 A$ is a proper submodule of $a_0 qA$. It follows that qA is a proper submodule of $a_0 qA \subset \mathcal{J}$. This contradiction shows that $a_1 A = A$ and a_1 is an invertible element of the ring A. Finally, we see that $pA = qa_1 A = qA = Ap$, i.e., $\mathcal{J} = pA = Ap$, where $p \in R^2$. The theorem is proved.

7.8 NOTES AND REFERENCES

Gorenstein rings were introduced by D.Gorenstein in the paper [Gorenstein, 1952]. In the article [Bass, 1962] H.Bass wrote (footnote 2 on page 18): "After writing this paper I discovered from Professor Serre that these rings have been encountered by Grothendieck, the latter having christened them “Gorenstein rings”. They are described in this setting by the fact that a certain module of differentials is locally free of rank one”. (See, also [Bass, 1963]).

Let $\mathcal{O}$ be a Dedekind ring with a field of fractions K, and let Λ be an $\mathcal{O}$-order in a finite dimensional separable K-algebra A (see [Curtis, Reiner, 1981]). In this case it is natural to consider Λ-lattices, i.e., finitely generated $\mathcal{O}$-torsion free Λ-modules.

Noncommutative Gorenstein $\mathcal{O}$-orders appeared first in [Drozd, 1967] (see the definition and proposition 6.1). An $\mathcal{O}$-order Λ is left Gorenstein if and only if the injective dimension of Λ as a left Λ-module is 1 ($\mathcal{O} \neq K$). From the definition and proposition 6.1 in the paper mentioned above it follows that Λ is left Gorenstein if and only if it is right Gorenstein.

Given a Λ-lattice M, a sublattice N of M is called **pure** if M/N is $\mathcal{O}$-torsion free.

The following theorem is proved in [Gustafson, 1974]:

An $\mathcal{O}$-order Λ is Gorenstein if and only if each left Λ-lattice is isomorphic to a pure sublattice of a free Λ- lattice.

In [Nishida, 1988] there is the example of the $(0,1)$-order $\Lambda(P_5)$ associated with the finite poset

$P_5 =$ 

which is such that $\operatorname{inj.dim} \Lambda(P_5) = 2$ and $\operatorname{gl.dim}\Lambda(P_5) = \infty$.

Let Λ be a Gorenstein order. If Λ has the additional property that every $\mathcal{O}$-order containing Λ is also Gorenstein, then Λ is called a Bass order. The following inclusions are easily verified:

$$\text{(maximal orders)} \subseteq \text{(hereditary orders)} \subseteq$$

$$\subseteq \text{(Bass orders)} \subseteq \text{(Gorenstein orders)}$$

(see §37 in [Curtis, Reiner, 1981]).

Denote by $\mu_\Lambda(X)$ the minimal number of generators of a finitely generated Λ-module X. The following theorem is proved in [Roiter, 1966] (see also theorem 37.17 in [Curtis, Reiner, 1981]).

Let Λ be an $\mathcal{O}$-order such that $\mu_\Lambda(I) \leq 2$ for each left ideal I of Λ. Then Λ is a Bass order.

Obviously, the $\mathbf{Z}$-order

$$\begin{pmatrix} \mathbf{Z} & 4\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{pmatrix}$$

is a Bass order, because for every left ideal J we have $\mu_\Lambda(I) \leq 2$, (see also [Chatters, Hajaruavis, 2003], [Chatters, 2006]).

Tiled orders over a discrete valuation rings appeared first in [Tarsy, 1970] (see also [Jategaonkar, 1973] and [Jategaonkar, 1974]). The Gorenstein condition for exponent matrices of tiled orders was formulated in [Kirichenko, 1978]. Note that the notion of an exponent matrix appeared first in the English translation of [Zavadskij, Kirichenko, 1977].

Theorem 7.7.4 was proved in [Roggenkamp, 2001].

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