COMPLEX ANALYTIC METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

AN INTRODUCTORY TEXT

Heinrich G. W. Begehr

World Scientific

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In memory

of my parents and my sister Edda Jaeger

Preface

For more than one century complex analysis has fascinated mathematicians since Cauchy, Weierstraß and Riemann had built up the field from their different points of view. Richness, beauty and fascination originate from the coincidence of analytic, algebraic and geometric methods. While the theory of meromorphic functions culminated in value distribution theory, for several decades geometrical function theory flourhished, and complex approximation theory was developed. For recent books in these topics see [Gaie80], [Laga86], [Pomm92] and also the survey article [Gaie90]. An excellent presentation of classical complex analysis including many historical facts and remarks are the two volumes [Remm92]. There are many other facets of complex analysis e.g. analytic number theory and complex ordinary differential equations. And there is the wide field of complex analysis of several variables.
But complex analysis is not any more in the center of mathematical interest neither in one nor in several variables. Nevertheless there is a branch which just recently became quite active: complex analysis in partial differential equations. In the last ten years more than 20 monographs appeared in this area as well as several collections of articles, see the list of references, part a. Although already classical via the theory of harmonic functions this area became very lifely through the investigations of I.N. VEKUA, N.I. MUSKHELISHVILI, L. BERS, F.D. GAKHOV, W. HAACK, R.P. GILBERT and others. While some of these considerations develop the theory of boundary value problems for analytic functions others are concerned with building up some theories for classes of complex partial differential equations and systems.
The present book gives some introduction in complex methods for partial differential equations and systems mainly of first and second order. Classical natural boundary value problems are considered, which in general are reduced to singular integral equations by utilizing proper integral representation formulas. The basic boundary conditions are of RIEMANN and RIEMANN–HILBERT type. Several generalizations and extensions are presented as e.g the POINCARÉ problem and discontinuous boundary value problems. But in general we stay with stronger assumptions in order to keep the introductory character of the book. After the reader has become acquainted with the material he can pass to other secondary literature or even to original research papers.
On the basis of a first course in complex analysis chapter 1 introduces the necessary background of function and potential theory. Properties of CAUCHY integrals, GREEN and NEUMANN functions and SCHWARZ operators, fundamental boundary value problems for analytic functions are extensively discussed. The RIEMANN mapping theorem often used to reduce boundary value problems for simply connected domains to the case of the unit disc is presented with proof. It also serves to motivate the introduction of the GREEN function. For multiply connected domains the BERGMAN kernel serves to find operators of SCHWARZ type. These representation formulas were recently developed by A. DZHURAEV. In chapter 2 beyond analytic functions solutions to nonlinear CAUCHY RIEMANN systems and later on in chapter 3 to generalized BELTRAMI equations are studied. The classical GAUSS theorem leads to a generalization of the CAUCHY representation for analytic functions, the

so-called CAUCHY–POMPEIU formula. The area integral added here is the celebrated T–operator in the VEKUA theory of generalized analytic functions. Its z–derivative is a singular integral operator of CALDERON–ZYGMUND type which serves to transform boundary value problems for the functions just mentioned to singular integral equations. They can be solved and lead to a priori estimates of the solutions as well. This T-operator and its first order z-derivative are just two particular operators of a whole bunch of integral operators useful for first and higher order complex partial differential equations. They were only recently systematically worked out by G.N. HILE and the author after one and the other have occasionally been used before. The section on discontiunuous boundary value problems is technically involved and might be skipped at a first reading of the chapter. For nonlinear BELTRAMI equations the related integral equations become nonlinear. In chapter 4 as an example entire solutions are studied. In principle this leads to the solution of the RIEMANN boundary value problem for these nonlinear equations. Here as well pseudoparabolic equations as first order composite type systems are considered, where the methods developed before turn out to be useful, too. In the final chapter 5 some special boundary value problems for elliptic second and higher order equations are discussed in multiply connected domains and the unit disc, respectively. The singular integral operators involved are expressed by the BERGMAN and related kernel functions. They transform the problem to a singular integral equation to which the FREDHOLM alternative applies.
Boundary value problems in the theory of analytic functions of several variables are difficult in principle because of the complicated structure of the integral representation formulas. In case when analytic functions satisfy some partial differential equations it is possible to solve boundary value problems. Some results of A. DZHURAEV and the author on first order systems in two complex variables with analytic coefficients are presented. Here the DOUGLIS algebra of hypercomplex variables proves to be useful. The SCHWARZ–POISSON formula which turns out as essential for the DIRICHLET problem for analytic functions in chapter 1 can be extended to several variables. This formula was just recently published by A. KUMAR and the author. Its deduction is included here, too.
The main parts of the first three chapters were distributed as Lecture Notes at the University of Assiut, Egypt in 1991 during a short time visitorship granted by the German Academic Exchange Service (DAAD Kurzzeitdozentur). Moreover, the material was presented in special courses at the Freie Universität Berlin. I appreciated the support through DAAD and the scientific discussions with the colleagues and students in Assuit, in Qena and in Berlin very much.
I am much indebted to my secretary Barbara M. Wengel for her careful preparation of the camera-ready copy of the manuscript. The figures were prepared by Ute Fuchs. Thanks to her as well as to the staff of World Scientific for their patience and cooperation.

Berlin, May 1994 Heinrich Begehr

Table of Contents

1. Function theoretical tools

1.1 CAUCHY integrals

In this section the behaviour of CAUCHY integrals along smooth curves or systems of curves in the complex plane $\mathbb{C}$ will be reported on.
A smooth curve Γ is a closed or open JORDAN arc with continuously varying tangent. It thus may be represented as

$$\Gamma = \{z : z = z(\tau), 0 \leq \tau \leq 1\}$$

by a continuously differentiable function z mapping the segment $[0,1]$ of the real line $\mathbb{R}$ injectively into $\mathbb{C}$ and satisfying $z'(\tau) \neq 0$ on $[0,1]$. This kind of curves has some properties which will be important in the future (see [Musk53], Chapter 1).

Lemma 1. *Let Γ be a smooth curve. Then for any $\alpha_0 < \frac{\pi}{2}$ there exists a positive $R_0 = R_0(\alpha_0)$ depending only on α_0 such that for any $t \in \Gamma$*

i. $\Gamma \cap \{\zeta : |\zeta - t| < R, R \leq R_0\}$ *consists of a single open arc ab,*

ii. the non-obtuse angle α between the tangents of Γ at two points from the arc ab is less than or equal to α_0.

Remark. R_0 is called the *standard radius* of Γ.

Proof. Let for $\zeta_1, \zeta_2 \in \Gamma$ the length of the arc on Γ between ζ_1 and ζ_2 be denoted by $\sigma = \sigma(\zeta_1, \zeta_2)$. If Γ is a closed curve σ is understood to be the shorter of the two arc lengths. The distance between ζ_1 and ζ_2 is

$$r = r(\zeta_1, \zeta_2) = |\zeta_1 - \zeta_2|.$$

L is the total length of Γ, s the arc length parameter , $\zeta_k = \zeta(s_k), k = 1, 2$.
For any $\sigma_0, 0 < 2\sigma_0 < L$ the subset

$$M := \{(\zeta_1, \zeta_2) \in \Gamma \times \Gamma : \sigma_0 \leq \sigma(\zeta_1, \zeta_2)\}$$

of $\Gamma \times \Gamma$ is compact. On this set the continuous function $r(\zeta_1, \zeta_2)$ attains its minimum, i.e. there exists a pair $(\zeta_1^0, \zeta_2^0) \in M$ such that

$$\varrho = \varrho(\sigma_0) := \min_{(\zeta_1, \zeta_2) \in M} |\zeta_1 - \zeta_2| = \left|\zeta_1^0 - \zeta_2^0\right|.$$

This minimum is positive; for if ϱ were 0 then $\zeta_1^0 = \zeta_2^0$ while $\sigma_0 \leq \sigma(\zeta_1, \zeta_2)$. Hence $\zeta_1^0 = \zeta_2^0$ is a double point of Γ. But Γ is a JORDAN curve, i.e. without multiple points. For any ϱ' with $0 < \varrho' < \varrho$ and $\zeta_0 \in \Gamma$ then

$$\{\zeta : |\zeta - \zeta_0| < \varrho'\} \cap \{\zeta : \zeta \in \Gamma\, , \sigma_0 \leq \sigma(\zeta_0, \zeta)\} = \emptyset.$$

(1) Proof of ii. There exists a $\sigma_0 = \sigma_0(\alpha_0) > 0$ such that the angle α between the tangents at $\zeta_1, \zeta_2 \in \Gamma$ with $\sigma(\zeta_1, \zeta_2) \leq \sigma_0$ satisfies

$$|\alpha| \leq \alpha_0.$$

This follows from the smoothness of Γ by a continuity argument. We may assume $2\sigma_0 \leq L$.

Remark. Let $\zeta_1, \zeta_2 \in \Gamma$ satisfy $\sigma(\zeta_1, \zeta_2) \leq \sigma_0$ and $t_1, t_2 \in \Gamma$ be two points between ζ_1 and ζ_2 (on the shorter arc). Then the absolute value of the non-obtuse angle between the straight line through t_1 and t_2 and the tangent of Γ at ζ_1 (and at ζ_2) is less than or equal to σ_0. This is true because there is always a tangent of Γ parallel to the line through t_1 and t_2 touching Γ at a point lying between t_1 and t_2.

(2) Proof of i. Consider for fixed $\zeta_0 \in \Gamma$ the subarc

$$\Gamma_0 := \{\zeta : \zeta \in \Gamma, \sigma(\zeta, \zeta_0) \leq \sigma_0\}.$$

We at first assume that none of the end points of Γ belongs to Γ_0 if Γ is an open curve. Then ζ_0 splits Γ_0 into two parts corresponding to $s \leq s_0$ and $s_0 \leq s$, where $\zeta_0 = \zeta(s_0)$. Introducing polar coordinates in ζ_0 we have

$$\zeta = \zeta_0 + re^{i\varphi}, \qquad ds = |d\zeta| = |dr + ird\varphi|.$$

Hence, $dr = \pm ds \cos\alpha$ where $|\alpha| \leq \alpha_0$ according to the preceding remark where the + sign holds for $s \geq s_0$ and the − sign for $s \leq s_0$.

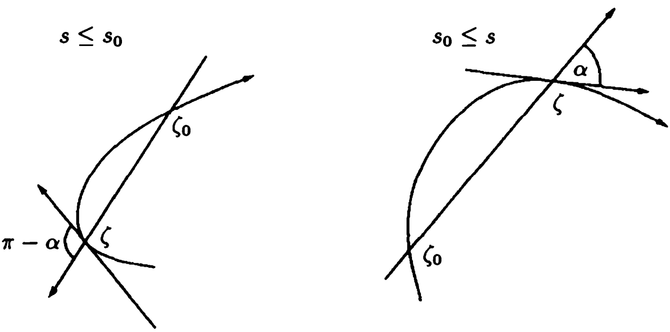

Figure 1.

Because $1 \geq \cos\alpha \geq \cos\alpha_0 =: k_0 > 0$ this means that r is monotone increasing as well on the right as on the left-hand side of ζ_0 on Γ_0, when moving away from ζ_0 on Γ. Moreover,

$$k_0|s - s_0| \leq |\zeta - \zeta_0| \leq |s - s_0|$$

holds for $\zeta = \zeta(s) \in \Gamma_0$. Let

$$R_0 := \min\{\varrho(\sigma_0), k_0\sigma_0\}.$$

Then Γ intersects the circle $|\zeta - \zeta_0| = R$ for $0 < R \leq R_0$ in exactly two points, one on the right and one on the left of ζ_0 on Γ_0. This is true because when s monotonically varies from s_0 to $s_0 \pm \sigma_0$ then $|\zeta - \zeta_0|$ monotonically increases from 0 to some $r_1 \geq k_0\sigma_0 \geq R_0 \geq R$.
If an end point on Γ belongs to Γ_0 there might be only one intersection point of Γ with the circle $|\zeta - \zeta_0| = R$ which e.g. will happen if ζ_0 coincides with this end point.

Definition 1. *A function f of a real or complex variable z is said to satisfy a* HÖLDER *condition or to be* HÖLDER *continuous on a set D if there exists $0 < H$ and $0 < \alpha < 1$ such that*

$$|f(z_1) - f(z_2)| \leq H|z_1 - z_2|^\alpha$$

for all $z_1, z_2 \in D$. $H = H_\alpha(f) = H(f; D, \alpha)$ is called the HÖLDER *constant, α the* HÖLDER *exponent. In case $\alpha = 1$ the condition is called* LIPSCHITZ *condition. The set of* HÖLDER *continuous functions on D is denoted by $C^\alpha(D)$. $C^\alpha(D; \mathbb{C})$ means the complex valued* HÖLDER *continuous functions in $D, C^\alpha(D; \mathbb{R})$ the real valued ones.*

Obviously, if D is bounded set then a HÖLDER condition with exponent α implies HÖLDER continuity with any $\beta \leq \alpha$.
Let Γ be a rectifiable curve in the complex plane and φ be integrable along Γ. Then

$$\phi(z) := \frac{1}{2\pi i} \int_\Gamma \varphi(\zeta) \frac{d\zeta}{\zeta - z} \tag{1.1.1}$$

is an analytic function in $\hat{\mathbb{C}} \setminus \Gamma$ - where $\hat{\mathbb{C}}$ denotes the RIEMANN sphere $\mathbb{C} \cup \{\infty\}$ - vanishing at ∞. In general ϕ does not exist for points of Γ. Consider e.g. a real segment $[a, b]$ and let $a < c < b$. Then the improper integral

$$\int_a^b \frac{dx}{x - c} \tag{1.1.2}$$

does not exist because

$$\lim_{\varepsilon_1 \to 0, \varepsilon_2 \to 0} \left[-\int_a^{c-\varepsilon_1} \frac{dx}{c - x} + \int_{c+\varepsilon_2}^b \frac{dx}{x - c} \right] = \lim_{\varepsilon_1 \to 0, \varepsilon_2 \to 0} \left[\log \frac{b - c}{c - a} + \log \frac{\varepsilon_1}{\varepsilon_2} \right]$$

in general does not exist. But if the limits are taken symmetrically, i.e. $\varepsilon_1 = \varepsilon_2 \to 0$ then

$$\lim_{\varepsilon\to 0}\left[\int_a^{c-\varepsilon}\frac{dx}{x-c}+\int_{c+\varepsilon}^{b}\frac{dx}{x-c}\right]=\log\frac{b-c}{c-a}.$$

This value is called the CAUCHY principal value of the singular integral (1.1.2).

Definition 2. *Let Γ be a smooth curve in $\mathbb{C}$. For a fixed $c \in \Gamma$ let $\varphi(\cdot\,; c)$ be an integrable function on $\Gamma \backslash \{c\}$ having a singularity at $\zeta = c$. Denote for $0 < \varepsilon$*

$$\Gamma_\varepsilon := \Gamma\backslash\{\zeta : |\zeta - c| < \varepsilon\}.$$

If

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon}\varphi(\zeta; c)d\zeta$$

exists this value is denoted as the CAUCHY *principal value of the singular integral, written as*

$$C.P.\int_\Gamma \varphi(\zeta; c)d\zeta$$

or shortly as

$$\int_\Gamma \varphi(\zeta; c)d\zeta.$$

Similarly, if $D \subset \mathbb{C}$ is a domain, say and $\varphi(\cdot\,; c)$ a function in $D\backslash\{c\}$ for some point $c \in D$ such that

$$\int_{D_\varepsilon}\varphi(z; c)dxdy, \quad D_\varepsilon := D\backslash\{z : |z - c| < \varepsilon\},$$

exists for any small enough $\varepsilon > 0$ then

$$\int_D \varphi(z; c)dxdy := \lim_{\varepsilon\to 0}\int_{D_\varepsilon}\varphi(z; c)dxdy$$

is called CAUCHY *principal value of the singular integral if the limit exists.*

Theorem 1. *Let Γ be a simply closed (piecewise) smooth curve in $\mathbb{C}$ and $\varphi \in \mathbb{C}^\alpha(\Gamma)$, then ϕ given by (1.1.1) exists as* CAUCHY *principle integral on Γ.*

Proof. $\tau \in \Gamma, 0 < \varepsilon, \Gamma_\varepsilon = \Gamma\backslash\{\zeta : |\zeta - \tau| < \varepsilon\}$.

$$\int_{\Gamma_\varepsilon}\varphi(\zeta)\frac{d\zeta}{\zeta-\tau}=\int_{\Gamma_\varepsilon}\frac{\varphi(\zeta)-\varphi(\tau)}{\zeta-\tau}d\zeta+\varphi(\tau)\int_{\Gamma_\varepsilon}\frac{d\zeta}{\zeta-\tau}.$$

From $|\varphi(\zeta)-\varphi(\tau)| \leq H|\zeta-\tau|^{\alpha}$ we have with $\zeta=\zeta(s), \tau=\zeta(s_0)$

$$\left|\int_{\Gamma_\varepsilon} \frac{\varphi(\zeta)-\varphi(\tau)}{\zeta-\tau} d\zeta\right| \leq H \int_0^L \left|\frac{\zeta(s)-\zeta(s_0)}{s-s_0}\right|^{\alpha-1} \frac{ds}{|s-s_0|^{1-\alpha}}$$

$$\leq 2H\left(\min_{0\leq s\leq L}|\zeta'(s)|\right)^{\alpha-1} \int_0^L \frac{ds}{s^{1-\alpha}} = \frac{2}{\alpha} H L^{\alpha}\left(\min_{0\leq s\leq L}|\zeta'(s)|\right)^{\alpha-1}.$$

Moreover, if Γ has a tangent in z then

$$\lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} \frac{d\zeta}{\zeta-\tau} = \lim_{\varepsilon\to 0} \log \frac{\zeta_2(\varepsilon)-\tau}{\zeta_1(\varepsilon)-\tau} = i\pi ,$$

where $\zeta_1(\varepsilon)$ is the first intersection point of Γ with $|\zeta-\tau|=\varepsilon$ starting from τ in the positive direction of Γ and $\zeta_2(\varepsilon)$ is the second. Here we may take any branch of the log-function. In case of a piecewise smooth Γ there might be a corner in τ with inner angle $\rho\pi, 0<\rho<2$.
Then

$$\int_{\Gamma} \frac{d\zeta}{\zeta-\tau} = \rho\pi i .$$

Hence, we have for the CAUCHY principle integral ($\rho=1$)

$$\frac{1}{2\pi i}\int_{\Gamma} \varphi(\zeta)\frac{d\zeta}{\zeta-\tau} = \frac{1}{2\pi i}\int_{\Gamma} \frac{\varphi(\zeta)-\varphi(\tau)}{\zeta-\tau} d\zeta + \frac{1}{2}\varphi(\tau) , \tag{1.1.3}$$

where the integral on the right-hand side exists as an improper integral.

Remark. (1.1.1) can be considered for open curves Γ, too when z is different from the end points of Γ.

Theorem 2. *Take Γ and φ as before. Then*

$$\psi(z) := \int_{\Gamma} \frac{\varphi(\zeta)-\varphi(\tau)}{\zeta-z} d\zeta, \quad \tau\in\Gamma , \tag{1.1.4}$$

satisfies

$$\lim_{z\to\tau} \psi(z) = \psi(\tau) , \tag{1.1.5}$$

if the limit is taken non-tangentially from any of the two sides of Γ.

Proof. Decompose the integral

$$\psi(z) - \psi(\tau) = \int_{\Gamma} \frac{(z-\tau)(\varphi(\zeta) - \varphi(\tau))}{(\zeta - z)(\zeta - \tau)} d\zeta$$

into the sum of two integrals along the curves Γ_ε and $\Gamma \backslash \Gamma_\varepsilon$ where

$$\Gamma_\varepsilon := \Gamma \backslash \{\zeta : |\zeta - \tau| < \varepsilon\} .$$

According to

$$\left|\frac{ds}{dr}\right| \leq \frac{1}{k_0}$$

(see the proof of Lemma 1) and taking a non-tangential limit so that $0 < \omega_0 \leq \omega$,

$$\left|\frac{z-\tau}{\zeta - z}\right| = \frac{\sin\beta}{\sin\omega} \leq \frac{1}{\sin\omega_0} =: K$$

we have

$$\left|\int_{\Gamma\backslash\Gamma_\varepsilon} \frac{(z-\tau)(\varphi(\zeta) - \varphi(\tau))}{(\zeta - z)(\zeta - \tau)} d\zeta\right| \leq HK \int_{\Gamma\backslash\Gamma_\varepsilon} \frac{|d\zeta|}{|\zeta - \tau|^{1-\alpha}}$$

$$\leq 2\frac{HK}{k_0} \int_0^\varepsilon \frac{dr}{r^{1-\alpha}} = \frac{2HK}{\alpha k_0} \varepsilon^\alpha .$$

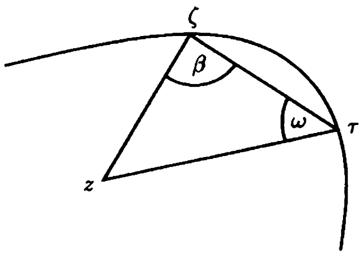

Figure 2.

Because on Γ_ε we have $|\zeta - \tau| \geq \varepsilon$ for the second integral taking $|z - \tau| \leq \frac{1}{2}\varepsilon$ then

$$\left|\int_{\Gamma_\varepsilon} \frac{(z-\tau)(\varphi(\zeta) - \varphi(\tau))}{(\zeta - z)(\zeta - \tau)} d\zeta\right| \leq \frac{2HL}{\varepsilon^{2-\alpha}} |z - \tau| .$$

Remark. The above estimations do not depend on $\tau \in \Gamma$. Hence, $\psi(z)$ tends uniformly with respect to the position of τ on Γ to $\psi(\tau)$ if z tends to τ non-tangentially. Thus, ψ - and hence ϕ - is continuous on Γ, too: for $\tau_1, \tau_2 \in \Gamma$ close to one another choose $z \notin \Gamma$ close to τ_1 and τ_2. Then

$$|\psi(\tau_1) - \psi(\tau_2)| \leq |\psi(\tau_1) - \psi(z)| + |\psi(z) - \psi(\tau_2)|$$

shows the smallness of $\psi(\tau_1) - \psi(\tau_2)$.

With this in mind one can show that (1.1.5) also holds for taking the limit tangentially. Because if z tends tangentially to $\tau \in \Gamma$ choose some $\tau_1 \in \Gamma$ arbitrarily close to τ and a non-tangential curve through z and τ_1. As $\psi(z) - \psi(\tau_1)$ and $\psi(\tau) - \psi(\tau_1)$ are small this is true also for $\psi(z) - \psi(\tau)$ (see [Musk53],§14 and §16).

Theorem 3. (PLEMELJ–SOKHOTZKI). *Under the above conditions the* CAUCHY *integral (1.1.1) has boundary values*

$$\phi^+(\tau) := \lim_{\substack{z\to\tau \\ z\in D^+}} \phi(z), \quad \phi^-(\tau) := \lim_{\substack{z\to\tau \\ z\in D^-}} \phi(z) \,,$$

where D^+ is the bounded domain with $\partial D^+ = \Gamma$ and $D^- = \hat{\mathbb{C}} \setminus (D^+ \cup \Gamma)$. Moreover, for $\tau \in \Gamma$

$$\phi^+(\tau) = \frac{1}{2}\varphi(\tau) + \phi(\tau), \quad \phi^-(\tau) = -\frac{1}{2}\varphi(\tau) + \phi(\tau) \,, \tag{1.1.6}$$

where $\phi(\tau)$ is understood as CAUCHY *principal value.*

Remark. Formulae (1.1.6) are called the PLEMELJ–SOKHOTZKI formulae. They equivalently can be written as

$$\phi^+(\tau) - \phi^-(\tau) = \varphi(\tau), \quad \phi^+(\tau) + \phi^-(\tau) = 2\phi(\tau), \quad \tau \in \Gamma \,. \tag{1.1.7}$$

Proof. Rewriting for $z \notin \Gamma$

$$2\pi i \phi(z) = \psi(z) + \varphi(\tau) \int_\Gamma \frac{d\zeta}{\zeta - z}$$

and observing (Γ is assumed to be smooth)

$$\int_\Gamma \frac{d\zeta}{\zeta - z} = \begin{cases} 2\pi i, & z \in D^+ \\ \pi i, & z \in \Gamma \\ 0, & z \in D^- \end{cases}$$

we get

$$\phi^+(\tau) = \frac{1}{2\pi i}\psi(\tau) + \varphi(\tau) = \phi(\tau) + \frac{1}{2}\varphi(\tau)\,,$$

$$\phi^-(\tau) = \frac{1}{2\pi i}\varphi(\tau) + 0 = \phi(\tau) - \frac{1}{2}\varphi(\tau)\,.$$

Theorem 4. (PLEMELJ–PRIVALOV). $\phi^+, \phi^- \in C^\alpha(\Gamma)$.

Proof. It is enough to show the function $\psi(\tau)$ from Theorem 2 to belong to $C^\alpha(\Gamma)$. For $\tau_1, \tau_2 \in \Gamma$ we have

$$\psi(\tau_2) - \psi(\tau_1) = \int_\Gamma \left\{ \frac{\varphi(\zeta) - \varphi(\tau_2)}{\zeta - \tau_2} - \frac{\varphi(\zeta) - \varphi(\tau_1)}{\zeta - \tau_1} \right\} d\zeta\,.$$

In order to decompose this integral let $|\tau_2 - \tau_1| < \delta$ where $\delta > 0$ is so small that

$$\Gamma \cap \{\zeta : |\zeta - \tau_1| = \delta\} = \{a', b'\}$$

which is possible for Γ is smooth. Let $k > 1$ be defined by $\delta = k|\tau_2 - \tau_1|$. As before $\sigma(\tau_1, \tau_2)$ denotes the length of the shorter of the two subarcs of Γ between τ_1 and τ_2. We have

$$\sigma(\tau_1, \tau_2) \le \frac{1}{k_0}|\tau_1 - \tau_2|$$

(see proof of Lemma 1). Let γ be that arc on Γ consisting of two subarcs intersecting at the point τ_1 and each having the total arc length $2\sigma(\tau_1, \tau_2)$. Let the end points of γ be denoted by a and b. τ_1 is the midpoint of γ and τ_2 lies on γ between but different from a and b. The two points a' and b' are lying on γ and are different from a and b. We now write $\psi(\tau_2) - \psi(\tau_1)$ as the sum of the following four integrals

$$I_1 := \int_\gamma \frac{\varphi(\zeta) - \varphi(\tau_2)}{\zeta - \tau_2} d\zeta, \quad I_2 := -\int_\gamma \frac{\varphi(\zeta) - \varphi(\tau_1)}{\zeta - \tau_1} d\zeta,$$

$$I_3 := \int_{\Gamma\setminus\gamma} \frac{\varphi(\tau_1) - \varphi(\tau_2)}{\zeta - \tau_1} d\zeta, \quad I_4 := \int_{\Gamma\setminus\gamma} \frac{(\varphi(\zeta) - \varphi(\tau_2))(\tau_2 - \tau_1)}{(\zeta - \tau_1)(\zeta - \tau_2)} d\zeta\,.$$

$$|I_1| \le H \int_\gamma \frac{|d\zeta|}{|\zeta - \tau_2|^{1-\alpha}} \le \frac{2H}{k_0^{1-\alpha}} \int_0^{3\sigma(\tau_1,\tau_2)} \frac{ds}{s^{1-\alpha}} \le \frac{2H}{k_0^{1-\alpha}} \int_0^{\frac{3}{k_0}|\tau_1-\tau_2|} \frac{ds}{s^{1-\alpha}} = \frac{2}{\alpha} H \frac{3^\alpha}{k_0} |\tau_1 - \tau_2|^\alpha,$$

$$|I_2| \le H \int_\gamma \frac{|d\zeta|}{|\zeta - \tau_1|^{1-\alpha}} \le \frac{2H}{k_0^{1-\alpha}} \int_0^{\frac{2}{k_0}|\tau_1-\tau_2|} \frac{ds}{s^{1-\alpha}} = \frac{2^{1+\alpha}}{\alpha k_0} H |\tau_1 - \tau_2|^\alpha\,,$$

$$|I_3| \le H|\tau_1 - \tau_2|^\alpha \left| \int_{\Gamma\setminus\gamma} \frac{d\zeta}{\zeta - \tau_1} \right| = H|\tau_1 - \tau_2|^\alpha \left| \log \frac{b - \tau_1}{a - \tau_1} \right| \le 2H\Big(\frac{1}{k_0} + \pi\Big)|\tau_1 - \tau_2|^\alpha\,.$$

For the last estimation we may assume without loss of generality $|a - \tau_1| \geq |b - \tau_1|$. Then

$$\left|\log \frac{b - \tau_1}{a - \tau_1}\right| \leq \log \left|\frac{a - \tau_1}{b - \tau_1}\right| + |\arg(a - \tau_1) - \arg(b - \tau_1)|\,,$$

$$\log \left|\frac{a - \tau_1}{b - \tau_1}\right| \leq \left|\frac{a - \tau_1}{b - \tau_1}\right| - 1 \leq \left|\frac{a - b}{b - \tau_1}\right| \leq \frac{\sigma(a, b)}{k_0\sigma(\tau_1, b)} = \frac{4\sigma(\tau_1, \tau_2)}{2k_0\sigma(\tau_1, \tau_2)}\,,$$

so that

$$\left|\log \frac{b - \tau_1}{a - \tau_1}\right| \leq \frac{2}{k_0} + 2\pi\,.$$

$$|I_4| \leq H|\tau_1 - \tau_2| \int\limits_{\Gamma\setminus\gamma} \frac{|d\zeta|}{|\zeta - \tau_1||\zeta - \tau_2|^{1-\alpha}} = H|\tau_1 - \tau_2| \int\limits_{\Gamma\setminus\gamma} |\zeta - z_1|^{\alpha-2} \left|\frac{\zeta - \tau_1}{\zeta - \tau_2}\right|^{1-\alpha} |d\zeta|\,.$$

To estimate the integrand we observe for $\zeta \in \Gamma\setminus\gamma$

$$|\zeta - \tau_1| - |\tau_1 - \tau_2| \leq |\zeta - \tau_2|\,, \quad k|\tau_1 - \tau_2| = \delta \leq |\zeta - \tau_1|\,,$$

so that

$$\left|\frac{\zeta - \tau_1}{\zeta - \tau_2}\right| \leq \frac{1}{1 - |\frac{\tau_1 - \tau_2}{\zeta - \tau_1}|} \leq \frac{k}{k - 1}\,.$$

Hence,

$$|I_4| \leq H\left(\frac{k}{k - 1}\right)^{1-\alpha} |\tau_1 - \tau_2| \int\limits_{\Gamma\setminus\gamma} |\zeta - \tau_1|^{\alpha-2}|d\zeta|\,,$$

$$\int\limits_{\Gamma\setminus\gamma} |\zeta - \tau_1|^{\alpha-2}|d\zeta| \leq k_0^{\alpha-2} \int\limits_{2\sigma(\tau_1,\tau_2)}^{L} s^{\alpha-2}ds$$

$$\leq k_0^{\alpha-2}\frac{2^{\alpha-1}}{1 - \alpha}\sigma(\tau_1, \tau_2)^{\alpha-1} \leq \frac{2^{\alpha-1}k_0^{\alpha-2}}{1 - \alpha}|\tau_1 - \tau_2|^{\alpha-1}\,,$$

$$|I_4| \leq \frac{H}{1 - \alpha}\frac{1}{k_0}\left(\frac{k}{2k_0(k - 1)}\right)^{1-\alpha} |\tau_1 - \tau_2|^{\alpha}\,.$$

These estimates give

$$|\psi(\tau_2) - \psi(\tau_1)| \leq CH|\tau_2 - \tau_1|^{\alpha}\,,$$

where C is a constant depending on α, k_0 and k.

Theorem 4 ensures the HÖLDER continuity of the boundary values of the analytic function ϕ from inside and outside Γ. In the next theorem ϕ extended by ϕ^+ and ϕ^-, respectively to the closure $\overline{D^+}$ and $\overline{D^-}$ of D^+ and D^-, respectively will be shown to be HÖLDER continuous, too. For the proof we need the following lemmas.

Lemma 2. *Let $R_0 = R_0(\alpha_0)$ be the standard radius of Γ and $0 < \rho < R_0$, $\tau \in \Gamma, z \notin \Gamma, |z-\tau| \le \rho$. The non-obtuse angle between the straight line $\overrightarrow{\tau z}$ from τ to z and the tangent of Γ at τ is assumed to be not less than $\beta_0 > \alpha_0 > 0$. Then there exists a positive constant $M = M(\alpha, \rho, \Gamma)$ such that for any $\varphi \in C^\alpha(\Gamma)$*

$$|\phi'(z)| \le M|z-\tau|^{\alpha-1}[\max_{z\in\Gamma} |\varphi(z)| + H(\varphi;\Gamma,\alpha)] .$$

Proof. Consider the subarc

$$\gamma := \{\zeta : \zeta \in \Gamma, |\zeta - \tau| < R_0\}$$

having endpoints a and b. In order to estimate

$$\phi'(z) = \frac{1}{2\pi i}\int_\Gamma \varphi(\zeta)\frac{d\zeta}{(\zeta-z)^2}$$

we decompose the integral in a sum of ϕ_1 and ϕ_2 corresponding to γ and $\Gamma\backslash\gamma$, respectively. We have

$$\phi_1(z) = \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\zeta)-\varphi(\tau)}{(\zeta-z)^2}d\zeta + \frac{\varphi(\tau)}{2\pi i}\left[\frac{1}{z-b} - \frac{1}{z-a}\right] ,$$

$$\left|\frac{1}{z-b} - \frac{1}{z-a}\right| = \frac{|b-a|}{|z-b||z-a|} \le \frac{2R_0}{(R_0-\rho)^2} \le \frac{2R_0^{2-\alpha}}{(R_0-\rho)^2}\delta^{\alpha-1} ,$$

$$|\zeta - z|^2 = |\zeta-\tau+\tau-z|^2 = |\zeta-\tau|^2 + 2Re(\zeta-\tau)(\overline{\tau-z}) + |\tau-z|^2$$

$$= r^2 + 2r\delta\cos\omega + \delta^2, \quad \zeta \in \gamma ,$$

with $r := |\zeta-\tau|, \delta := |z-\tau|, \omega := |\arg(\zeta-\tau) - \arg(\tau-z)|$. Because the angle between the line from τ to z and the tangent at τ is greater than or equal to β_0 and the angle between this tangent and the secant from τ to ζ is less than or equal to α_0 we can estimate

$$\omega \ge \beta_0 - \alpha_0 =: \omega_0 > 0 .$$

Thus, for $\zeta \in \gamma$

$$|\zeta - z|^2 \ge r^2 - 2r\delta\cos\omega_0 + \delta^2 = (r-\delta\cos\omega_0)^2 + \delta^2\sin^2\omega_0 ,$$

$$\begin{aligned}
\left|\int_\gamma \frac{\varphi(\zeta)-\varphi(\tau)}{(\zeta-z)^2}d\zeta\right| &\le \frac{2H}{k_0}\int_0^{R_0} \frac{r^\alpha dr}{(r-\delta\cos\omega_0)^2 + \delta^2\sin^2\omega_0} \quad (r = \delta t) \\
&\le \frac{2H}{k_0}\delta^{\alpha-1}\left[\int_0^2 \frac{t^\alpha dt}{\sin^2\omega_0} + \int_2^\infty \frac{t^\alpha dt}{(t-\cos\omega_0)^2}\right] \\
&= \frac{2H}{k_0}\delta^{\alpha-1}\left[\frac{2^{1+\alpha}}{(1+\alpha)\sin^2\omega_0} + \frac{2^{1+\alpha}}{1-\alpha}\right]
\end{aligned}$$

where for $2 \leq t$ we used $t \leq 2(t-1)$. Hence,

$$|\phi_1(z)| \leq \left[\frac{R_0^{2-\alpha}}{\pi(R_0-\rho)^2}\max_{\zeta\in\Gamma}|\varphi(\zeta)| + \frac{2^{1+\alpha}H}{\pi k_0}\left(\frac{1}{(1+\alpha)\sin^2\omega_0} + \frac{1}{1-\alpha}\right)\right]\delta^{\alpha-1} .$$

$$|\phi_2(z)| \leq \frac{1}{2\pi}\max_{\zeta\in\Gamma}|\varphi(\zeta)|\frac{L}{(R_0-\rho)^2} \leq \frac{LR_0^{1-\alpha}}{2\pi(R_0-\rho)^2}\max_{\zeta\in\gamma}|\varphi(\zeta)|\delta^{\alpha-1} .$$

Altogether we have

$$|\phi'(z)| \leq C\delta^{\alpha-1} ,$$

where the constant C depends on φ, α, ρ , R_0, ω_0 and

$$C \leq M(\alpha,\rho,\Gamma)\left[\max_{\zeta\in\Gamma}|\varphi(\zeta)| + H(\varphi;\Gamma,\alpha)\right] .$$

Lemma 3. *For any non-negative x and y and $0<\alpha<1$*

$$|x^\alpha - y^\alpha| \leq |x-y|^\alpha, \quad x^\alpha + y^\alpha \leq 2^{1-\alpha}(x+y)^\alpha$$

hold.

Proof. We only prove the first inequality leaving the second as an exercise. Assuming without restrictions $x<y$ and setting $t = \frac{x}{y}$ the first inequality is seen to be equivalent to

$$1 - t^\alpha \leq (1-t)^\alpha, \quad 0 \leq t < 1 .$$

Because the derivative of

$$g(t) := \frac{1-t^\alpha}{(1-t)^\alpha}$$

is

$$g'(t) = \frac{\alpha(1-t^{\alpha-1})}{(1-t)^{\alpha-1}} < 0, \quad 0 \leq t < 1 ,$$

the function g is monotone non-increasing, i.e.

$$\frac{1-t^\alpha}{(1-t)^\alpha} = g(t) \leq g(0) = 1 .$$

Theorem 5. $\phi \in C^\alpha(\overline{D^+}), \phi \in C^\alpha(\overline{D^-})$.

Proof. We only prove the first part, the second can be shown in a similar way. This will be done in three steps. At first the HÖLDER condition is verified when one point lies on Γ, secondly when one point is in a neighborhood of Γ and lastly for both points outside this neighborhood. The case when both points are on Γ was considered in the preceding theorem already.

1. $\tau \in \Gamma, z \in D^+$.

For $0 \leq \beta < \alpha$ we consider a branch Ψ of the multi-valued function

$$\Psi(z) := \frac{\phi(z) - \phi^+(\tau)}{(z-\tau)^\beta} .$$

This branch is single-valued analytic in D^+. Its boundary values

$$\Psi^+(\zeta) = \frac{\phi^+(\zeta) - \phi^+(\tau)}{(\zeta-\tau)^\beta}$$

are continuous on Γ because $\phi^+ \in C^\alpha(\Gamma)$. Moreover,

$$|\Psi^+(\zeta)| \leq H|\zeta - \tau|^{\alpha-\beta} .$$

From the CAUCHY representation formula we have for $z \in D^+$ with $0 < \varepsilon < |z-\tau|$

$$\Psi(z) = \frac{1}{2\pi i} \int\limits_{\partial(D^+ \setminus \overline{K_\varepsilon(\tau)})} \Psi^+(\zeta) \frac{d\zeta}{\zeta - z} ,$$

where

$$K_\varepsilon(\tau) := \{z : |z-\tau| < \varepsilon\}$$

and

$$\Psi^+(z) := \begin{cases} \Psi(z), & z \in D^+ \\ \Psi^+(z), & z \in \Gamma = \partial D^+ \end{cases} .$$

In order to show that the limit of the right-hand side of this representation formula exists when ε tends to zero we observe

$$|\Psi(z)| \leq \frac{C}{|z-\tau|^\beta} \quad \text{for} \quad z \in D^+, |z-\tau| \leq 2\varepsilon$$

with a proper positive constant C. This holds because $\phi(z)$ tends to $\phi^+(\tau)$ when $z \in D^+$ tends to τ (see Theorem 3.). Hence, for $|z - \tau| > \varepsilon$

$$\left| \int\limits_{\partial(K_\varepsilon(\tau) \cap D^+)} \Psi^+(\zeta) \frac{d\zeta}{\zeta - z} \right| \leq C \int\limits_0^{2\pi} \frac{\varepsilon^{1-\beta} d\varphi}{|\varepsilon e^{i\varphi} + \tau - z|} \leq \frac{2\pi C \varepsilon^{1-\beta}}{|\tau - z| - \varepsilon} .$$

Therefore, in D^+

$$\Psi(z) = \frac{1}{2\pi i} \int\limits_{\partial D^+} \Psi^+(\zeta) \frac{d\zeta}{\zeta - z} ,$$

where the improper integral exists in the ordinary sense. Thus from Theorems 2 and 3 we see that $\Psi^+ \in C(\overline{D^+})$.

From the maximum principle for analytic functions, see [Burc79], p. 128 or [Tsuj59], p. 2 then

$$\left|\frac{\phi(z)-\phi^+(\tau)}{(z-\tau)^\beta}\right| \le \max_{\zeta\in\Gamma}|\Psi^+(\zeta)| \le H(\phi^+;\Gamma,\alpha)\max_{\zeta\in\Gamma}|\zeta-\tau|^{\alpha-\beta} \le H(\phi^+;\Gamma,\alpha)d(\Gamma),$$

where $H(\phi^+;\Gamma,\alpha)$ is the HÖLDER constant (see Definition 1) of ϕ^+ and

$$d(\Gamma) := \max\{1, \operatorname{diam}\Gamma\}\ , \quad \operatorname{diam}\Gamma := \max_{\zeta_1,\zeta_2\in\Gamma}|\zeta_1-\zeta_2|\ .$$

Setting $H_1 := H(\phi^+;\Gamma,\alpha)d(\Gamma)$ which is independent of β we get from the last estimate

$$|\phi(z)-\phi^+(\tau)| \le H_1|z-\tau|^\alpha$$

by letting β tend to α.

2. $z, z_0 \in D^+, \operatorname{dist}(z_0,\Gamma) \le \rho < R_0$.

Here $R_0 = R_0(\alpha_0)$ is the standard radius of Γ. Let $\tau \in \Gamma$ be such that

$$|z_0-\tau| = \operatorname{dist}(z_0,\Gamma) := \min_{\zeta\in\Gamma}|z_0-\zeta|\ .$$

Consider the function

$$\Psi_0(z) := \frac{\phi(z)-\phi(z_0)}{(z-z_0)^\alpha}$$

which we want to show to be bounded in D^+. In order to obtain a single-valued branch of Ψ_0 we consider a branch of Ψ_0 in $D^+\setminus\ \overrightarrow{\tau z_0}$, where $\overrightarrow{\tau z_0}$ is the straight line from τ to z_0. In order to estimate Ψ_0 on (both sides of) the line $\overrightarrow{\tau z_0}$ Lemma 2 is applied. For $z \in \overrightarrow{\tau z_0}$ we have

$$\phi(z)-\phi(z_0) = \int_{z_0}^{z}\phi'(\zeta)d\zeta\ .$$

Because the line $\overrightarrow{\tau z_0}$ is perpendicular to the tangent of Γ at τ Lemma 2 may be applied giving

$$|\phi(z)-\phi(z_0)| \le C\int_{z_0}^{z}|\zeta-\tau|^{\alpha-1}|d\zeta| = \frac{C}{k_0}\left|\int_{|z_0-\tau|}^{|z-\tau|}t^{\alpha-1}dt\right|$$

$$= \frac{C}{\alpha k_0}\big||z-\tau|^\alpha - |z_0-\tau|^\alpha\big| \le \frac{C}{\alpha k_0}|z-z_0|^\alpha.$$

Here for the last estimate Lemma 3 is used. Thus, Ψ_0^+ is bounded on the line $\overrightarrow{\tau z_0}$ as it is on Γ via part 1 of this proof.

Again applying the maximum principle to Ψ_0 in $D^+ \setminus \overrightarrow{\tau z_0}$ together with the estimate of Ψ_0 on $\overrightarrow{\tau z_0}$ we have

$$|\phi(z) - \phi(z_0)| \leq \max\left\{H_1, \frac{C}{\alpha k_0}\right\}|z - z_0|^\alpha$$

for $z, z_0 \in D^+, \operatorname{dist}(z_0, \Gamma) \leq \rho < R_0$.

3. $z, z_0 \in D^+, \rho \leq \operatorname{dist}(z, \Gamma), \rho \leq \operatorname{dist}(z_0, \Gamma), 0 < \rho < R_0$.

If $|z - z_0| < \frac{1}{2}\rho$ then $\left\{\zeta : |\zeta - z_0| < \frac{1}{2}\rho\right\} \subset D^+$. Integrating along the line $\overrightarrow{\tau z_0}$ we have

$$\phi(z) - \phi(z_0) = \int_{z_0}^{z} \phi'(\tilde{\zeta})d\tilde{\zeta}, \quad \phi'(\tilde{\zeta}) = \frac{1}{2\pi i}\int_\Gamma \phi^+(\zeta)\frac{d\zeta}{(\zeta - \tilde{\zeta})^2}\,.$$

From

$$|\phi'(\tilde{\zeta})| \leq \frac{1}{2\pi}\max_{\zeta\in\Gamma}|\phi^+(\zeta)|\frac{4L}{\rho^2}, \quad \tilde{\zeta} \in D^+, \frac{1}{2}\rho \leq \operatorname{dist}(\tilde{\zeta}, \Gamma)\,,$$

then

$$|\phi(z) - \phi(z_0)| \leq \frac{2L}{\pi\rho^2}\max_{\zeta\in\Gamma}\left|\phi^+(\zeta)\right||z - z_0|\,.$$

If $|z - z_0| \geq \frac{1}{2}\rho$ then

$$|\phi(z) - \phi(z_0)| \leq 2\max_{\zeta\in\Gamma}|\phi^+(\zeta)| \leq 4\max_{\zeta\in\Gamma}|\phi^+(\zeta)|\frac{|z - z_0|}{\rho}\,.$$

Therefore

$$|\phi(z) - \phi(z_0)| \leq \frac{2}{\rho}\max\left\{\frac{L}{\pi\rho}, 2\right\}\max_{\zeta\in\Gamma}|\phi^+(\zeta)|d(\Gamma)|z - z_0|^\alpha\,.$$

In connection with the HÖLDER continuity of CAUCHY integrals we mention a result from PRIVALOV, see [Cohi53/62], p. 380, 401–403.

Theorem 6. (PRIVALOV). *Let $w = u + iv$ be analytic in the unit disc $\mathbb{D}$, where v is continuous in the closure $\overline{\mathbb{D}}$ and* HÖLDER *continuous on the boundary $\partial\mathbb{D}$ satisfying*

$$|v(\zeta) - v(\tau)| \leq H|\zeta - \tau|^\alpha, \quad \zeta, \tau \in \partial\mathbb{D}\,.$$

Then w is HÖLDER *continuous in $\overline{\mathbb{D}}$ with the same exponent and the constant kH where k only depends on α, i.e.*

$$|w(z) - w(z_0)| \le kH|z - z_0|^\alpha, \quad z, z_0 \in \overline{\mathbb{D}} .$$

Proof. For $|\tau| = 1$ the function $(1 - z\overline{\tau})^\alpha$ is multi-valued analytic in $z \in \mathbb{D}$. Choosing the branch which at the origin $z = 0$ is equal to 1 leads to a single-valued analytic function in $\mathbb{D}$ the real part of which

$$f(z) := \mathrm{Re}(1 - z\overline{\tau})^\alpha, \quad f(0) = 1 ,$$

is harmonic in $\mathbb{D}$. Obviously,

$$f(z) = |\tau - z|^\alpha \cos(\alpha|\arg(\tau - z) - \arg\tau|)$$

and

$$|\arg(\tau - z) - \arg\tau| \le \frac{\pi}{2}$$

because τ is perpendicular to $\partial\mathbb{D}$. Therefore

$$|\tau - z|^\alpha \le \frac{f(z)}{\cos\frac{\alpha\pi}{2}} ,$$

so that for $|z| = 1$

$$|v(z) - v(\tau)| \le H|z - \tau|^\alpha \le H\frac{f(z)}{\cos\frac{\alpha\pi}{2}} .$$

Applying the maximum principle for harmonic functions, see [Burc79], p. 128, [Tsuj59], p. 2, to

$$\pm(v(z) - v(\tau)) - \frac{H}{\cos\frac{\alpha\pi}{2}}f(z)$$

leads to

$$|v(z) - v(\tau)| \le H\frac{f(z)}{\cos\frac{\alpha\pi}{2}}, \quad |z| \le 1, |\tau| = 1 .$$

In order to estimate the first order derivatives of v we observe that with $0 \le r < 1$

$$|v(z) - v(\tau)| \le \frac{H}{\cos\frac{\alpha\pi}{2}}|\tau - z|^\alpha \le \frac{H}{\cos\frac{\alpha\pi}{2}}(|\tau - r\tau| + |r\tau - z|)^\alpha \le \frac{2^\alpha H}{\cos\frac{\alpha\pi}{2}}|1 - r|^\alpha$$

for $|r\tau - z| \le 1 - r$. The POISSON formula, see p. 31, or e.g. [Burc79], p. 134, applied to the disc $|z - r\tau| < 1 - r$ for the function $v(z) - v(\tau)$ gives for $|z - r\tau| < 1 - r$

$$v(z) - v(\tau) = \frac{1}{2\pi}\int\limits_{|\zeta - r\tau| = 1 - r}(v(\zeta) - v(\tau))\mathrm{Re}\frac{\zeta + z - 2r\tau}{\zeta - z}d\arg(\zeta - r\tau) .$$

Differentiating with respect to x, interchanging the order of differentiation and integration gives

$$\begin{aligned} v_x(z) &= \frac{1}{2\pi} \int\limits_{|\zeta - r\tau| = 1-r} (v(\zeta) - v(\tau)) \mathrm{Re} \frac{\partial}{\partial x} \left(\frac{2(\zeta - r\tau)}{\zeta - z} - 1 \right) d \arg(\zeta - r\tau) \\ &= \frac{1}{2\pi} \int\limits_{|\zeta - r\tau| = 1-r} (v(\zeta) - v(\tau)) \mathrm{Re} \frac{2(\zeta - r\tau)}{(\zeta - z)^2} d \arg(\zeta - r\tau) \,, \end{aligned}$$

especially for $z = r\tau$

$$v_x(r\tau) = \frac{1}{2\pi} \int\limits_{|\zeta - r\tau| = 1-r} (v(\zeta)) - v(\tau)) \mathrm{Re} \frac{2}{\zeta - r\tau} d \arg(\zeta - r\tau) \,.$$

Thus,

$$|v_x(r\tau)| \leq \frac{2^{1+\alpha} H}{\cos \frac{\alpha\pi}{2}} |1 - r|^{\alpha - 1} \,.$$

Similarly,

$$|v_y(r\tau)| \leq \frac{2^{1+\alpha} H}{\cos \frac{\alpha\pi}{2}} |1 - r|^{\alpha - 1} \,.$$

From

$$w'(z) = (u_x + iv_x)(z) = (v_y + iv_x)(z)$$

we find

$$|w'(z)| \leq \frac{2^{\frac{3}{2}+\alpha} H}{\cos \frac{\alpha\pi}{2}} (1 - |z|)^{\alpha - 1}, \quad |z| < 1 \,.$$

Let now $z, z_0, z \neq z_0$, be two points on $\overline{\mathbb{D}}$. Integrating w' along the straight line form z_0 to z gives

$$w(z) - w(z_0) = \int\limits_{z_0}^{z} w'(\zeta) d\zeta = \int\limits_0^1 w'(z_0 + t(z - z_0))(z - z_0) dt \,,$$

$$|w(z) - w(z_0)| \leq \frac{2^{\frac{3}{2}+\alpha} H}{\cos \frac{\alpha\pi}{2}} \int\limits_0^1 \frac{|z - z_0| dt}{(1 - |z_0 + t(z - z_0)|)^{1-\alpha}} \,.$$

For $z_0 = 0$ the integral becomes

$$\int\limits_0^1 \frac{|z| dt}{(1 - |z| t)^{1-\alpha}} = \frac{1}{\alpha}(1 - (1 - |z|)^\alpha) \leq \frac{1}{\alpha} |z|^\alpha \,.$$

The last inequality is an application of Lemma 3.
Let therefore $z_0 \neq 0$. Then putting $\zeta := \dfrac{z}{z_0}$ and without loss of generality assuming $|\zeta| \leq 1$

$$\begin{aligned} z_0 + t(z - z_0) &= z_0(1 + t(\zeta - 1)) , \\ |1 + t(\zeta - 1)|^2 &= 1 + 2t\mathrm{Re}(\zeta - 1) + t^2|\zeta - 1|^2 \\ &= 1 - 2t|\zeta - 1|\cos\varphi + t^2|\zeta - 1|^2 , \end{aligned}$$

where $\varphi = \pi - \arg(\zeta - 1)$. As $|\zeta| \leq 1$ we know for $\zeta \neq 1$

$$\frac{\pi}{2} < \arg(\zeta - 1) < \frac{3}{2}\pi ,$$

i.e. $|\varphi| < \frac{\pi}{2}$. Moreover, for $|\zeta| \leq 1, \zeta \neq 1$, we have $|\varphi| \leq \varphi_0$, where $\cos\varphi_0 = \dfrac{|\zeta - 1|}{2}$.

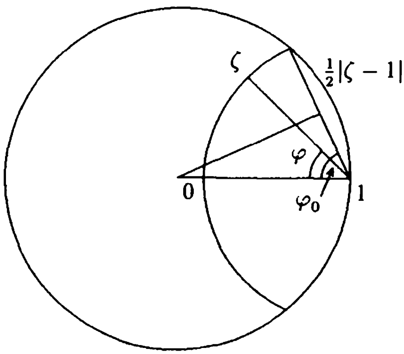

Figure 3.

So

$$|1 + t(\zeta - 1)|^2 \leq 1 - t|\zeta - 1|^2 + t^2|\zeta - 1|^2 = 1 - t(1 - t)|\zeta - 1|^2$$

$$= 1 - \frac{1}{4}|\zeta - 1|^2 + |\zeta - 1|^2\left(\frac{1}{2} - t\right)^2 ,$$

$$|1 + t(\zeta - 1)| \leq \sqrt{1 - \frac{1}{4}|\zeta - 1|^2} + |\zeta - 1|\left|\frac{1}{2} - t\right| .$$

Therefore

$$\int_0^1 \frac{dt}{(1-|z_0||1+t(\zeta-1)|)^{1-\alpha}} \leq 2\int_0^{\frac{1}{2}} \frac{dt}{\left(1-|z_0|\sqrt{1-t(1-t)|\zeta-1|^2}\right)^{1-\alpha}}$$

$$\leq 2\int_0^{\frac{1}{2}} \frac{dt}{\left(1-|z_0|\left(\sqrt{1-\frac{1}{4}|\zeta-1|^2}+|\zeta-1|\left(\frac{1}{2}-t\right)\right)\right)^{1-\alpha}}$$

$$= \frac{2}{\alpha}\frac{1}{|z_0||\zeta-1|}\left[1-|z_0|\left(\sqrt{1-\frac{1}{4}|\zeta-1|^2}+\left(\frac{1}{2}-t\right)|\zeta-1|\right)\right]^{\alpha}\Bigg|_0^{\frac{1}{2}}$$

$$= \frac{2}{\alpha}\frac{1}{|z-z_0|}\left[\left(1-|z_0|\sqrt{1-\frac{1}{4}|\zeta-1|^2}\right)^{\alpha}\right.$$

$$\left.-\left(1-|z_0|\left(\sqrt{1-\frac{1}{4}|\zeta-1|^2}+\frac{1}{2}|\zeta-1|\right)\right)^{\alpha}\right]$$

$$\leq \frac{2}{\alpha}\frac{1}{|z-z_0|}\frac{|z_0|^{\alpha}|\zeta-1|^{\alpha}}{2^{\alpha}} = \frac{2^{1-\alpha}}{\alpha}|z-z_0|^{\alpha-1},$$

where again Lemma 3 was used. Thus

$$\int_0^1 \frac{|z-z_0|dt}{(1-|z_0+t(z-z_0)|)^{1-\alpha}} \leq \frac{2^{1-\alpha}}{\alpha}|z-z_0|^{\alpha}$$

and

$$|w(z)-w(z_0)| \leq \frac{2^{\frac{5}{2}}H}{\alpha\cos\frac{\alpha\pi}{2}}|z-z_0|^{\alpha}$$

for any $z, z_0 \in \overline{\mathbb{D}}$.

1.2 Green functions and Schwarz operators

Theorem 7. (Riemann mapping theorem). *Any simply connected domain D of the complex* Riemann *sphere $\hat{\mathbb{C}}$ having at least two boundary points is conform equivalent to the unit disc $\mathbb{D}$.*

Remark. Two domains are called conform equivalent if there is a bijective mapping from one onto the other domain being analytic. Functions of this kind are called schlicht or univalent. The mapping function in the Riemann mapping theorem is uniquely given if for some $z_0 \in \mathbb{D}\, f(z_0) = 0$ and $f'(z_0) > 0$ is prescribed.

Proof. Let a, b be two points of ∂D distinct from one another and

$$S := \{w : w : D \longrightarrow \mathbb{D} \quad \text{schlicht}\}$$

be the class of schlicht mappings from D into $\mathbb{D}$.

1. $S \neq 0$: Consider the multi-valued function $w_1(\zeta) := \sqrt{\dfrac{1-a\zeta}{1-b\zeta}}$ in the neighborhood $U(0)$ of the origin and choose that branch satisfying

$$\lim_{\zeta \to 0} \sqrt{\frac{1-a\zeta}{1-b\zeta}} = 1$$

which is single-valued and analytic in $U(0)$. We will show

$$w(z) := w_1\left(\frac{1}{z}\right) = \sqrt{\frac{z-a}{z-b}}$$

being single-valued and analytic in the neighborhood of infinity to be single-valued and schlicht in D. That w is single-valued follows from $a, b \in \partial D$, that w is analytic is obvious . In order to prove schlichtness let for $z_1, z_2 \in D$

$$w(z_1) = w(z_2), \quad \text{i.e.} \quad \frac{z_1 - a}{z_1 - b} = \frac{z_2 - a}{z_2 - b} \,.$$

Because $a \neq b$ from here $z_1 = z_2$ follows.
Let now $w(z_0) = w_0$ for some $z_0 \in D$. Then

$$w(z) \neq -w_0 = |w_0| \exp(i(\arg w_0 + \pi)) \,.$$

This holds because w is one of the branches of the square root. Hence, there exists a $c \in \mathbb{C}$ satisfying $w(z) \neq c$ in D. Moreover, because we may assume $c \in -w[D]$ and because $-w[D]$ is an open set there exists an $\eta > 0$ such that

$$0 < \eta \leq |w(z) - c| \quad \text{in} \quad D \,.$$

Then we may choose A, B such that

$$w_1(z) := \frac{A}{w(z) - c} + B \,,$$

$w_1(z_1) = 0, |w_1(z)| < 1$ for some fixed $z_1 \in D$ and all $z \in D$. It is easy to see that $w_1 \in S$.

2. In order to show the existence of a schlicht function mapping D onto $\mathbb{D}$ choose $z_0 \in D$ arbitrarily and consider the class

$$S_0 := \left\{ f : f(z) = \frac{w(z) - w(z_0)}{1 - \overline{w(z_0)}w(z)}, w \in S \right\} .$$

Any $f \in S_0$ satisfies $f(z_0) = 0$ and $|f(z)| < 1$. The function $f \in S_0$ is holomorphic in

$$K_0 := \{z : |z - z_0| < r_0(z_0)\} \subset D ,$$

where $r_0(z_0)$ is the holomorphy radius of z_0, i.e. maximal such that $K_0 \subset D$. Because in K_0 we have $|f(z)| < 1$ and $f(z_0) = 0$ the SCHWARZ Lemma (see [Burc79],p. 191) implies

$$|f(z)| \leq \frac{|z - z_0|}{r_0(z_0)}, \quad |z - z_0| < r_0(z_0) ,$$

and

$$|f'(z_0)| \leq \frac{1}{r_0(z_0)} .$$

Hence,

$$C := \sup_{f \in S_0} |f'(z_0)| \leq \frac{1}{r_0(z_0)} < +\infty .$$

Let (f_n) be a sequence in S_0 satisfying

$$\lim_{n \to +\infty} |f_0'(z_0)| = C ,$$

where because of the schlichtness of f and the fact that $S_0 \neq \emptyset$ we have $0 < C$. Because $|f_n(z)| \leq 1$ in D by the Theorem of ARZELÀ-ASCOLI-MONTEL (see e.g. [Burc79], p. 254) there exists a subsequence (f_{n_k}) of (f_n) converging uniformly on any compact subset of D. The limit f either being constant or a schlicht function (see [Ding61], p. 256) satisfies $|f'(z_0)| = C > 0$. Hence, f is schlicht in D.

It remains to show $f[D] = \mathbb{D}$. Assume this is not true. Then there exists a $d \in \mathbb{D} \backslash f[D] \neq \emptyset$ satisfying $d \neq 0$ because $f(z_0) = 0$. Then any branch of the function

$$f_1(z) := \sqrt{\frac{f(z) - d}{1 - \bar{d}f(z)}}, \quad f_1(z_0) = \sqrt{-d}$$

is single-valued and schlicht in D different from 0 and ∞. Choosing one of the branches we consider

$$f_0(z) := \frac{f_1(z) - f_1(z_0)}{1 - \overline{f_1(z_0)}f_1(z)}, \quad z \in D ,$$

which is in S_0 as $f_1 \in S$. Now

$$f_0'(z_0) = \frac{f_1'(z_0)}{1 - |f_1(z_0)|^2} = \frac{f_1'(z_0)}{1 - |d|}, \qquad f_1'(z_0) = \frac{1 - |d|^2}{2\sqrt{-d}} f'(z_0),$$

so that

$$f_0'(z_0) = \frac{1 + |d|}{2\sqrt{-d}} f'(z_0), \quad |f_0'(z_0)| = \frac{1 + |d|}{2\sqrt{|d|}} C > C.$$

But $|f_0'(z_0)| > C$ is a contradiction to $f_0 \in S_0$ and the definition of C. Hence, $f[D] = \mathbb{D}$.

3. To show the mapping function f being uniquely given by $f(z_0) = 0$, $f'(z_0) > 0$ let f and φ be two schlicht mappings from D onto $\mathbb{D}$ with $f(z_0) = \varphi(z_0) = 0, f'(z_0) = \varphi'(z_0) > 0$. Let $z(\varphi)$ be the inverse mapping of $\varphi = \varphi(z)$ mapping $\mathbb{D}$ schlicht onto D. Then $w = f \circ z$ is a schlicht mapping from $\mathbb{D}$ onto D satisfying $w(0) = 0$ and

$$w'(0) = \frac{d}{d\varphi} f(z(\varphi)) \bigg|_{\varphi=0} = f'(z_0) z'(0) = \frac{f'(z_0)}{\varphi'(z_0)} = 1.$$

Applying the SCHWARZ Lemma gives

$$w(\varphi) = \varphi \quad (\varphi \in \mathbb{D}), \quad \text{i.e.}\, f(z) = \varphi(z) \quad (z \in D).$$

Theorem 8. *Let $w(z, z_0)$ be the* RIEMANN *mapping function from the domain D onto the unit disc $\mathbb{D}$ for fixed $z_0 \in D$, satisfying*

$$w(z_0, z_0) = 0, \quad w'(z_0, z_0) > 0.$$

Then

$$g(z, z_0) := -\log |w(z, z_0)|, \quad z \in D,$$

has the following properties.

i. $g(z, z_0)$ *is harmonic in* $z \in D \backslash \{z_0\}$,

ii. $\log |z - z_0| + g(z, z_0)$ *is harmonic in the neighborhood of* z_0,

iii. $\lim_{z \to \partial D} g(z, z_0) = 0.$

Remark. This result can be reverted. If $g(z, z_0)$ is the GREEN function of a simply connected domain D, see Definition 3, and $h(z, z_0)$ its harmonic conjugate, see p. 32, then $w(z, z_0) := \exp[-(g + ih)(z, z_0)]$ is the RIEMANN mapping function for D, satisfying $\lim_{z \to \zeta} |w(z, z_0)| = 1$ for $\zeta \in \partial D$ and $w(z_0, z_0) = 0$, $w'(z_0, z_0) \neq 0$.

Proof.

i. Applying the LAPLACE operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \overline{z}} ,$$

where

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

gives in $D\backslash\{z_0\}$

$$\begin{aligned} \Delta g(z, z_0) &= -4\frac{\partial^2}{\partial z \partial \overline{z}} \frac{1}{2} \log |w(z, z_0)|^2 \\ &= -2\frac{\partial^2}{\partial z \partial \overline{z}} \left(\log w(z, z_0) + \log \overline{w(z, z_0)}\right) = 0 . \end{aligned}$$

ii.

$$\log |z - z_0| + g(z, z_0) = \log \left|\frac{z - z_0}{w(z, z_0)}\right| = \text{Re } \log \frac{z - z_0}{w(z, z_0)} .$$

Because $\dfrac{w(z, z_0)}{z - z_0}$ is analytic and non-vanishing in the neighborhood of z_0 the function $\log \left|\dfrac{z - z_0}{w(z, z_0)}\right|$ is harmonic there.

iii. Let (z_n) be a sequence in D with all its accumulation points on the boundary ∂D. Then all accumulation points of the image sequence $(w_n), w_n = w(z_n, z_0) \in \mathbb{D}$, are lying on $\partial \mathbb{D}$. For if otherwise $\widehat{w}, |\widehat{w}| < 1$ would be an accumulation point of (w_n) there would exist a subsequence (w_{n_k}) of (w_n) with limit $\widehat{w} \in \mathbb{D}$. Let $\widehat{z} \in D$ be the preimage of $\widehat{w}, \widehat{w} = w(\widehat{z}, z_0)$ and z_{n_k} those of $w_{n_k}, w_{n_k} = w(z_{n_k}, z_0)$. The inverse mapping of $w = w(z, z_0)$ which will shortly be denoted by $z = z(w)$ is analytic in the neighborhood of $\widehat{z}$, too. By continuity we have

$$\widehat{z} = z(\widehat{w}) = \lim_{k \to +\infty} z(w_{n_k}) = \lim_{k \to +\infty} z_{n_k} \in \Gamma .$$

This contradicts $\widehat{z} \in D$. Therefore all accumulation points of (w_n) are on $\partial \mathbb{D}$. This proves property iii.

Definition 3. *A real-valued function in a domain D of $\mathbb{C}$ having properties i. to iii. of Theorem 8 is called the* GREEN *function of D, more exactly the* GREEN *function of D for the* LAPLACE *operator.*

Theorem 9. *The* GREEN *function of D has the additional properties*

1. $0 < g(z, z_0)$,
2. $g(z, z_0) = g(z_0, z)$,
3. *It is uniquely given by properties i. to iii.* ,
4. *If ϕ is any schlicht mapping from D onto $\mathbb{D}$, then the* GREEN *function is*

$$g(z, z_0) = -\log \left| \frac{\phi(z) - \phi(z_0)}{1 - \overline{\phi(z_0)}\phi(z)} \right| .$$

Proof.

1. This property of g follows from the maximum principle for harmonic functions applied to $g(\cdot, z_0)$ in the domain $D \backslash \{z : |z - z_0| < \varepsilon\}$ for small enough positive ε.

2. We prove the symmetry only in the case when $g(z, z_0) = -\log|w(z, z_0)|$ as in Theorem 8. The function

$$f(\zeta) := \frac{w(\zeta, z_0) - w(z, z_0)}{1 - \overline{w(z, z_0)}w(\zeta, z_0)}$$

maps D onto $\mathbb{D}$ for any fixed $z, z_0 \in D$ with $z \neq z_0$. Moreover, $f(z) = 0$ and the function $\dfrac{w(\zeta, z)}{f(\zeta)}$ has a removable singularity at $\zeta = z$ with

$$\lim_{\zeta \to z} \frac{w(\zeta, z)}{f(\zeta)} = \frac{w'(z, z)}{f'(z)} = 1 - |w(z, z_0)|^2 \leq 1 ,$$

where for $z \neq z_0$ we have a strong inequality. Applying the maximum principle by observing

$$\lim_{\zeta \to \partial D} \left| \frac{w(\zeta, z)}{f(\zeta)} \right| = 1$$

we find

$$|w(\zeta, z)| \leq |f(\zeta)|, \quad \zeta \in D ,$$

especially for $\zeta = z_0$

$$|w(z_0, z)| \leq |w(z, z_0)| .$$

By interchanging the roles of z and z_0 the inverse inequality can be proved in the same way. Hence, equality holds in the last relation giving

$$g(z, z_0) = g(z_0, z) .$$

For a proof in the general case see e.g. [Tsuj59], p. 17, [Ding61], p. 267, [Cour50], p. 250.

3. Let g_1, g_2 be two functions with properties i. to iii., then $g_1 - g_2$ is a harmonic function in the entire domain D with vanishing boundary values of ∂D. Thus $g_1 - g_2 = 0$ in D by the maximum principle.

4. Together with ϕ the linear transformation $\dfrac{\phi - \phi(z_0)}{1 - \overline{\phi(z_0)}\phi}$ maps D onto $\mathbb{D}$. Obviously,

$$-\log \frac{|\phi - \phi(z_0)|}{|1 - \overline{\phi(z_0)}\phi|}$$

has the properties i. to iii., so it is the GREEN function of D.

Remark. The existence of the GREEN function for a given domain D can be proved if the DIRICHLET problem for harmonic functions can be solved for D. The DIRICHLET problem demands us to find a harmonic function in D attaining prescribed boundary values on the boundary ∂D. In case of continuous boundary values this problem can be shown to be (uniquely) solvable for a wide class of domains by the method of PERRON–RADO–RIESZ. If then $u(z, z_0)$ is the harmonic function in D satisfying

$$\lim_{z \to \zeta} u(z, z_0) = \log |\zeta - z_0|, \quad \zeta \in \partial D ,$$

for $z_0 \in D$ fixed then

$$g(z, z_0) := \log \frac{1}{|z - z_0|} + u(z, z_0)$$

is the GREEN function for D. See [Tsuj59], p. 4, [Ding61], p. 263.
There are domains without a GREEN function. If e.g. D has an isolated boundary point a GREEN function should vanish at this point but would be positive in a neighborhood which is impossible because of the minimum principle, see [Tsuj59], p. 2. Another class of domains having no GREEN function are domains whose boundary has vanishing capacity, see [Ding61], p. 281, [Tsuj59], p. 54.

Definition 4. *A real-valued function N in D is called* NEUMANN *function (for the* LAPLACE *operator) if it satisfies*

i. *$N(z, z_0)$ is harmonic in $z \in D \setminus \{z_0\}, z_0 \in D$,*

ii. *$N(z, z_0) + \log |z - z_0|$ is harmonic in the neighborhood of z_0,*

iii. $\dfrac{\partial}{\partial \boldsymbol{n}} N(z, z_0) = -\dfrac{2\pi}{L}$ *on ∂D, where L is the total length of ∂D and $\boldsymbol{n}$ is the outer normal direction.*

Remark. From iii.

$$-\frac{1}{2\pi}\int\limits_{\partial D}\frac{\partial}{\partial \boldsymbol{n}_z}N(z,z_0)ds_z = 1$$

follows. N is not uniquely defined by i. to iii., it is only given up to an arbitrary additive constant. This constant can be fixed by asking

iv. $\int\limits_{\partial D} N(z,z_0)ds_z = 0$ for all $z_0 \in D$,

so that with iv. N is uniquely given, see [Cour50], p. 261.
If ϕ is a conformal mapping from D onto $\mathbb{D}$ then

$$N(z,z_0) = -\log|(\phi(z)-\phi(z_0))(1-\overline{\phi(z_0)}\phi(z))|\,.$$

We only verify i. to iii. in the special case where $D = \mathbb{D}$, i.e.

$$N(z,z_0) = -\log|(z-z_0)(1-\overline{z_0}z)|\,.$$

Because i. and ii. are obvious we only consider iii. On the boundary $\partial\mathbb{D}$ the outward normal direction coincides with the radial direction, so that

$$\frac{\partial}{\partial \boldsymbol{n}}N(z,z_0) = -\frac{\partial}{\partial r}\log|(z-z_0)(1-\overline{z_0}z)| \quad \text{for} \quad |z|=1\,.$$

In general

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x}+\frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \frac{x}{r}\frac{\partial}{\partial x}+\frac{y}{r}\frac{\partial}{\partial y} = \frac{z}{r}\frac{\partial}{\partial z}+\frac{\overline{z}}{r}\frac{\partial}{\partial \overline{z}}\,,$$

i.e.

$$r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y} = z\frac{\partial}{\partial z}+\overline{z}\frac{\partial}{\partial \overline{z}} = 2\mathrm{Re}\, z\frac{\partial}{\partial z}\,,$$

where the last equality holds for real-valued functions only. Thus, on $|z|=1$

$$\begin{aligned}\frac{\partial}{\partial \boldsymbol{n}}N(z,z_0) &= -\mathrm{Re}\, z\frac{\partial}{\partial z}\log(|z-z_0|^2|1-\overline{z_0}z|^2)\\ &= -\mathrm{Re}\left(\frac{z}{z-z_0}-\frac{\overline{z_0}z}{1-\overline{z_0}z}\right)\\ &= -\mathrm{Re}\left(\frac{z}{z-z_0}-\frac{\overline{z_0}}{\overline{z}-\overline{z_0}}\right)\\ &= -1 = -\frac{2\pi}{2\pi}\,.\end{aligned}$$

We also can verify iv.,

$$\int\limits_{\partial \mathbb{D}} N(z,z_0)ds = -\int\limits_{|z|=1} \log|1-\overline{z_0}z|^2 d\arg z = -2\int\limits_{|\zeta|=|z_0|} \log|1-\zeta| d\arg\zeta = 0\,.$$

The last equality holds because of the mean value property, see [Ding61], p. 202, [Tsuj59], p. 1 ,

$$u(z_0) = \frac{1}{2\pi}\int\limits_0^{2\pi} u(z_0 + re^{i\varphi})d\varphi$$

for harmonic functions. Obviously $\log|1-\zeta|$ is harmonic for $|\zeta| < |z_0| < 1$. Condition iii. of Definition 4 may be replaced by

iii′. $\dfrac{\partial}{\partial n}N(z,z_0) = -\dfrac{\sigma(s)}{\Sigma}, \quad z = z(s) \in \partial D,\ z_0 \in D,$

where s is the arc length parameter of $z \in \partial D, 0 \le s \le L$, and σ is a positive normalizing function with

$$\Sigma := \int\limits_0^L \sigma(s)ds > 0\,.$$

In order to get N uniquely defined instead of iv. now

iv′. $\displaystyle\int\limits_0^L N(z(s),z_0)\sigma(s)ds = 0, \quad z_0 \in D,$

is demanded, see [Hawe72], p. 113, [Wend79], p. 5.
The NEUMANN function is sometimes called the second GREEN function, $g(z,z_0)$ the first GREEN function and $G^I(z,z_0) := g(z,z_0), G^{II}(z,z_0) := N(z,z_0)$ used.

Remark. A regular curve is an open or closed continuously differentiable curve

$$\Gamma := \{z : z = z(t),\ 0 \le t \le 1\}$$

given by a continuously differentiable parameter representation $z = z(t)$ with $z'(t) \neq 0$. Let

$$s = s(t) = \int\limits_0^t |z'(\tau)|d\tau, \quad 0 \le s \le L := \int\limits_0^1 |z'(\tau)|d\tau$$

be the arc length parameter, where L is the total length of Γ. Then the tangent of Γ at the point $z(s)$ is represented by

$$\frac{dz}{ds} = \frac{dx}{ds} + i\frac{dy}{ds}$$

which is a unimodular number for any s, i.e.

$$\left|\frac{dx}{ds} + i\frac{dy}{ds}\right| + \left|\frac{x'(t)}{|z'(t)|} + i\frac{y'(t)}{|z'(t)|}\right| = 1 \,.$$

This complex number represents a unit vector pointing into the direction of Γ at the point $z = z(s)$. We here consider z as a function of the arc length parameter rather than of the originally given parameter t.

The normal vector to the curve Γ at $z(s)$ is a vector perpendicular to the tangent vector. There are two possibilities. The inner normal $\boldsymbol{\nu}$ is a unit vector originating from the tangent unit vector by a rotation about $90°$ or $\frac{\pi}{2}$ counterclockwise while the opposite direction is called the outer normal direction $\boldsymbol{n}$. Hence, if

$$\frac{dz}{ds} = \frac{dx}{ds} + i\frac{dy}{ds} = \cos\alpha + i\sin\alpha$$

then the inner normal vector is

$$\cos\left(\alpha + \frac{\pi}{2}\right) + i\sin\left(\alpha + \frac{\pi}{2}\right) = -\sin\alpha + i\cos\alpha = i\frac{dz}{ds}$$

which means

$$\frac{\partial z}{\partial \nu} = \frac{\partial x}{\partial \nu} + i\frac{\partial y}{\partial \nu} = i\frac{\partial z}{\partial s} = -\frac{\partial y}{\partial s} + i\frac{\partial x}{\partial s}$$

or

$$\frac{\partial x}{\partial \nu} = -\frac{\partial y}{\partial s}, \quad \frac{\partial x}{\partial s} = \frac{\partial y}{\partial \nu} \,.$$

We here prefer to use partial derivative symbols because now directional derivatives are involved with respect to normal and tangential directions, respectively. This system reminds us of the CAUCHY–RIEMANN system. In fact this system not only holds for the x and y axes directions but also for any other two directions originated by a rotation of this axes system.

Let $w = u + iv$ be an analytic function, so that the CAUCHY–RIEMANN system

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

holds. Then

$$\begin{aligned}
\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial v}{\partial y}\frac{\partial y}{\partial \nu} + \frac{\partial v}{\partial x}\frac{\partial x}{\partial \nu} = \frac{\partial v}{\partial \nu}\,, \\
\frac{\partial u}{\partial \nu} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \nu} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \nu} = -\frac{\partial v}{\partial y}\frac{\partial y}{\partial s} - \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} = -\frac{\partial v}{\partial s}\,.
\end{aligned}$$

Both the GREEN and NEUMANN functions are so-called fundamental solutions (to the LAPLACEian) used for solving boundary value problems via integral representation formulas for solutions. These integral formulas originate from GREEN integral formulas.

Theorem 10. (GAUSS) *Let $f \in C^1(\overline{D};\mathbb{R}^2)$ be a continuously differentiable vector-valued function from $\overline{D}$ into $\mathbb{R}^2$, where $D \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, then*

$$\int_D \nabla \cdot f dxdy = \int_{\partial D} f \cdot \boldsymbol{n} ds .$$

This theorem holds in $\mathbb{R}^m$ too with $m \geq 2$ for proper domains. Here $dxdy$ is to be replaced by $dx = dx_1 dx_2 \dots dx_m$. The nabla operator ∇ (sometimes called gradient operator) is

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_m}\right) ,$$

$\boldsymbol{n}$ is the outward normal unit vector given by

$$\boldsymbol{n} = (X_1, \dots, X_n) ,$$

$$X_\mu := \frac{(-1)^{\mu-1}}{J} \frac{\partial(x_1, \dots, x_{\mu-1}, x_{\mu+1}, \dots, x_m)}{\partial(t_1, \dots, t_{m-1})}, \quad 1 \leq \mu \leq m ,$$

$$ds = J dt_1 dt_2 \dots dt_{m-1} ,$$

$$J := \left[\sum_{\mu=1}^m \left(\frac{\partial(x_1, \dots, x_{\mu-1}, x_{\mu+1}, \dots, x_m)}{\partial(t_1, \dots, t_{m-1})}\right)^2\right]^{\frac{1}{2}} .$$

$$\nabla \cdot f = div f = \sum_{\mu=1}^m \frac{\partial f_\mu}{\partial x_\mu}, \quad f \cdot \boldsymbol{n} = \sum_{\mu=1}^m f_\mu X_\mu .$$

The variable $t = (t_1, \dots, t_{m-1})$ is a parametrization of the $(m-1)$–dimensional manifold ∂D and

$$\frac{\partial(y_1, \dots, y_m)}{\partial(x_1, \dots, x_m)} = \left|\left(\frac{\partial y_\nu}{\partial x_\mu}\right)_{1 \leq \nu,\mu \leq m}\right|$$

is the determinant of the functional matrix called JACOBIan functional determinant. In order to handle these integrals it is convenient to use the calculus of alternating differential forms given by E. CARTAN, see e.g. [Spiv65], p. 89.
The proof of this so–called divergence theorem is based on the exhaustion of D by sets of axes parallel net cubes

$$Q := \{\mathbf{x} = (x_1, \dots, x_m) : a_\mu \leq x_\mu \leq b_\mu, 1 \leq \mu \leq m\} \subset D .$$

The projection of Q in the x_μ–direction $(1 \leq \mu \leq m)$ is

$$Q(\mu) := \{(x_1, \dots, x_{\mu-1}, x_{\mu+1}, \dots, x_m) : a_\nu \leq x_\nu \leq b_\nu, 1 \leq \nu \leq m, \nu \neq \mu\} \subset \mathbb{R}^{m-1} .$$

If $g \in C^1(Q; I\!R)$ is a real continuously differentiable function in Q then

$$\int\limits_Q \frac{\partial g(\boldsymbol{x})}{\partial x_\mu} dx_1 \wedge \ldots \wedge dx_m$$

$$= \int\limits_{Q(\mu)} (-1)^{\mu-1} \int\limits_{a_\mu}^{b_\mu} \frac{\partial g(\boldsymbol{x})}{\partial x_\mu} dx_\mu \wedge dx_1 \wedge \ldots \wedge dx_{\mu-1} \wedge dx_{\mu+1} \wedge \ldots \wedge dx_m$$

$$= \int\limits_{Q(\mu)} (-1)^{\mu-1} \Big(g(\boldsymbol{x})|_{x_\mu=b_\mu} - g(\boldsymbol{x})|_{x_\mu=a_\mu} \Big) dx_1 \wedge \ldots \wedge dx_{\mu-1} \wedge dx_{\mu+1} \wedge \ldots \wedge dx_m$$

$$= \int\limits_{\partial Q} g(\boldsymbol{x}) X_\mu ds \; .$$

Replacing g by $f_\mu, 1 \leq \mu \leq m$, and adding we get

$$\int\limits_Q \nabla \cdot f dx_1 \wedge \ldots \wedge dx_m = \int\limits_{\partial Q} f \cdot \boldsymbol{n} ds \; ,$$

i.e. the GAUSS theorem for cubes Q.

Theorem 11. (GREEN) *For* $u \in C^1(\overline{D}; I\!R), v \in C^2(\overline{D}; I\!R)$

$$\int\limits_D (u \Delta v + \nabla u \cdot \nabla v) d\boldsymbol{x} = \int\limits_{\partial D} u \frac{\partial v}{\partial \boldsymbol{n}} ds \; .$$

This formula is called the first GREEN *formula.*

Proof. Apply the GAUSS formula to $f = u \nabla v$ and observe

$$\nabla \cdot f = \nabla u \cdot \nabla v + v \Delta u$$

and

$$u \nabla v \cdot \boldsymbol{n} = u \frac{\partial v}{\partial \boldsymbol{n}} \; .$$

Theorem 12. (GREEN) *For* $u, v \in C^2(\overline{D}; I\!R)$

$$\int\limits_D (u \Delta v - v \Delta u) d\boldsymbol{x} = \int\limits_{\partial D} \left(u \frac{\partial v}{\partial \boldsymbol{n}} - v \frac{\partial u}{\partial \boldsymbol{n}} \right) ds \; .$$

This is the second GREEN formula.

Proof. Substract the first GREEN formula for v and u from that for u and v.

Remark. It is enough to assume $u, v \in C^2(D;\mathbb{R}) \cap C^1(\overline{D};\mathbb{R})$ for the second Green formula.

In the sequal we return to the two-dimensional case although the next theorem holds in higher dimensions, too and again use complex variables.

Theorem 13. (GREEN). *Let $D \subset \mathbb{C}$ be a bounded domain with (piecewise) smooth boundary and having a* GREEN *function $g(z, z_0)$. Then for any harmonic function u which on ∂D is still continuous*

$$u(z) = -\frac{1}{2\pi} \int_{\partial D} u(\zeta) \frac{\partial g(\zeta, z)}{\partial \boldsymbol{n}_\zeta} ds_\zeta \quad (z \in D) .$$

This is the GREEN *representation formula for harmonic functions.*

Proof. Applying the second GREEN formula to $u(\zeta)$ and $g(\zeta, z)$ for $z \in D$ fixed in the domain $D \backslash \overline{K_\varepsilon(z)}$ where again $K_\varepsilon(z)$ is the open disc with small enough radius ε and center z gives

$$- \int_{D \backslash \overline{K_\varepsilon(z)}} g(\zeta, z) \Delta u(\zeta) d\xi d\eta = \int_{\partial D} u(\zeta) \frac{\partial g(\zeta, z)}{\partial \boldsymbol{n}_\zeta} ds_\zeta$$
$$- \int_{|\zeta - z| = \varepsilon} \left(u(\zeta) \frac{\partial g(\zeta, z)}{\partial \boldsymbol{n}_\zeta} - g(\zeta, z) \frac{\partial u(\zeta)}{\partial \boldsymbol{n}_\zeta} \right) ds_\zeta, \quad \zeta = \xi + i\eta .$$

Observing that $\boldsymbol{n}$ is the outer normal and that

$$\omega(\zeta, z) = g(\zeta, z) + \log |\zeta - z|$$

is harmonic in D we get

$$\int_{|\zeta - z| = \varepsilon} u(\zeta) \frac{\partial g(\zeta, z)}{\partial \boldsymbol{n}_\zeta} ds_\zeta = \int_0^{2\pi} u(z + \varepsilon e^{i\varphi}) \frac{\partial g(z + \varepsilon e^{i\varphi}, z)}{\partial \varepsilon} \varepsilon d\varphi$$

$$= - \int_0^{2\pi} u(z + \varepsilon e^{i\varphi}) d\varphi + \varepsilon \int_0^{2\pi} u(z + \varepsilon e^{i\varphi}) \frac{\partial \omega(z + \varepsilon e^{i\varphi}, z)}{\partial \varepsilon} d\varphi .$$

Letting ε tend to zero the first integral tends to $2\pi u(z)$ while the second term tends to zero by a continuity argument. Moreover, for a harmonic function u the first integral equals $2\pi u(z)$ for small enough ε by the mean value property of harmonic functions. Because

$$\int_{|\zeta - z| = \varepsilon} g(\zeta, z) \frac{\partial u(\zeta)}{\partial \boldsymbol{n}_\zeta} ds_\zeta = \varepsilon \int_0^{2\pi} (\omega(z + \varepsilon e^{i\varphi}, z) - \log \varepsilon) \frac{\partial u(z + \varepsilon e^{i\varphi}, z)}{\partial \varepsilon} d\varphi$$

also tend to zero together with ε and because

$$\lim_{\varepsilon\to 0} \int\limits_{D\setminus\overline{K_\varepsilon(z)}} g(\zeta,z)\Delta u(\zeta)d\xi d\eta = \int\limits_D g(\zeta,z)\Delta u(\zeta)d\xi d\eta$$

for any $u \in C^2(D)$ we arrive at

$$u(z) = -\frac{1}{2\pi}\int\limits_{\partial D} u(\zeta)\frac{\partial g(\zeta,z)}{\partial \boldsymbol{n}_\zeta}ds_\zeta - \frac{1}{2\pi}\int\limits_D g(\zeta,z)\Delta u(\zeta)d\xi d\eta, \quad z \in D\,.$$

If in particular u is harmonic the last term vanishes. Introducing polar coordinates about the point z it is not difficult to show that the singularity of $g(\zeta,z)$ is weak with respect to area integrals. Therefore

$$\int\limits_D g(\zeta,z)\Delta u(\zeta)d\xi d\eta$$

exists as a proper integral.

This representation formula for harmonic functions gives the unique solution to the DIRICHLET problem for harmonic functions. Let $\sigma \in C(\partial D;\mathbb{R})$ be given. Then

$$u(z) := -\frac{1}{2\pi}\int\limits_{\partial D}\sigma(\zeta)\frac{\partial g(\zeta,z)}{\partial \boldsymbol{n}_\zeta}ds_\zeta, \quad z \in D\,,$$

is a harmonic function with boundary values σ on ∂D, see [Tsuj59], p. 22.
Let us consider the special case $D = \mathbb{D}$. Then

$$g(\zeta,z) = -\log\left|\frac{\zeta - z}{1-\overline{z}\zeta}\right|\,.$$

Applying on $\{|\zeta| = \rho = 1\}$

$$\frac{\partial}{\partial \boldsymbol{n}_\zeta} = \frac{\partial}{\partial\rho} = \rho\frac{\partial}{\partial\rho} = \zeta\frac{\partial}{\partial\zeta} + \overline{\zeta}\frac{\partial}{\partial\overline{\zeta}} = 2\mathrm{Re}\zeta\frac{\partial}{\partial\zeta}$$

we get

$$\frac{\partial}{\partial \boldsymbol{n}_\zeta}g(\zeta,z) = 2\mathrm{Re}\,\zeta\frac{\partial}{\partial\zeta}(\log|1-\overline{z}\zeta| - \log|\zeta - z|)$$

$$= -\mathrm{Re}\left(\frac{\overline{z}\zeta}{1-\overline{z}\zeta} + \frac{\zeta}{\zeta - z}\right) = -\mathrm{Re}\left(\frac{\overline{z}}{\overline{\zeta}-\overline{z}} + \frac{\zeta}{\zeta - z}\right) = -\frac{1-|z|^2}{|\zeta - z|^2}\,.$$

Therefore in $\mathbb{D}$ any harmonic function being continuous in $\overline{\mathbb{D}}$ may be represented by

$$u(z) = \frac{1}{2\pi}\int\limits_{|\zeta|=1} u(\zeta)\frac{1-|z|^2}{|\zeta - z|^2}d\arg\zeta\,.$$

This is the POISSON formula with the POISSON kernel

$$\mathrm{Re}\frac{\zeta+z}{\zeta-z}=\frac{1-|z|^2}{|\zeta-z|^2}=\frac{1-r^2}{1-2r\cos(\vartheta-\varphi)+r^2},\quad z=re^{i\varphi},\varphi=e^{i\vartheta},$$

which for an arbitrary circle $|\zeta-a|<\rho$ is

$$\mathrm{Re}\frac{\zeta+z-2a}{\zeta-z}=\frac{\rho^2-r^2}{\rho^2-2r\rho\cos(\vartheta-\varphi)+r^2},\quad z=a+re^{i\varphi},\zeta=a+\rho e^{i\vartheta}.$$

Solving the CAUCHY–RIEMANN equations with a given harmonic function the conjugate harmonic function can locally be calculated. This conjugate harmonic function in general is not a single–valued function. But it is when the domain D under consideration is simply connected.

The conjugate harmonic function to the GREEN function locally, say in the neighborhood of a point $a\in D$ is given by

$$\begin{aligned} h(z,z_0) &= \int_a^z\left(-\frac{\partial g(\zeta,z_0)}{\partial\eta}d\xi+\frac{\partial g(\zeta,z_0)}{\partial\xi}d\eta\right)+\text{const.}\\ &= \int_a^z\frac{\partial g(\zeta,z_0)}{\partial \boldsymbol{n}_\zeta}ds_\zeta+\text{const.}\end{aligned}$$

because for $g=g(z,z_0)$ on some smooth curve γ in D we have

$$-\frac{\partial g}{\partial y}dx+\frac{\partial g}{\partial x}dy=\left(-\frac{\partial g}{\partial y}\frac{\partial x}{\partial s}+\frac{\partial g}{\partial x}\frac{\partial y}{\partial s}\right)ds=\left(\frac{\partial g}{\partial y}\frac{\partial y}{\partial \boldsymbol{n}}+\frac{\partial g}{\partial x}\frac{\partial x}{\partial \boldsymbol{n}}\right)ds=\frac{\partial g}{\partial \boldsymbol{n}}ds.$$

Observe that $\boldsymbol{n}$ is the outer normal while above, p. 27, the CAUCHY–RIEMANN system for the function (x,y) was written with respect to s and the inner normal $\boldsymbol{\nu}$. The CAUCHY–RIEMANN system for the function (g,h) leads to $\frac{\partial g}{\partial \boldsymbol{n}}=\frac{\partial h}{\partial s},\frac{\partial g}{\partial s}=-\frac{\partial h}{\partial \boldsymbol{n}}$.
The integral in the above representation of h can be taken along the straight line from a to z. h is determined up to an arbitrary additive constant. It is a multi–valued function. The so–called complex GREEN function [Mikh35], see [Gakh66], p. 209,

$$M(z,z_0)=g(z,z_0)+ih(z,z_0)$$

in the neighborhood of z_0 behaves like $-\log(z-z_0)+$ analytic function.

Remark. For simply connected domains the function $f(z,z_0):=\exp(-M(z,z_0))$ is a single–valued analytic function mapping D conformally onto $\mathbb{D}$ with $f(z_0,z_0)=0$. For more details see [Gakh66], pp. 209, 332.

Lemma 4. *The conjugate harmonic function to a function u which is harmonic in D and continuous in $\overline{D}$ is*

$$v(z)=-\frac{1}{2\pi}\int_{\partial D}u(\zeta)\frac{\partial h(\zeta,z)}{\partial \boldsymbol{n}_\zeta}ds_\zeta,\quad z\in D,$$

and

$$w(z) := (u + iv)(z) = -\frac{1}{2\pi} \int\limits_{\partial D} u(\zeta) \frac{\partial M(\zeta, z)}{\partial \boldsymbol{n}_\zeta} ds_\zeta, \quad z \in D ,$$

is analytic in D.

Proof. As above for $h(z, z_0)$ we have for v the representation

$$v(z) = \int\limits_a^z (-u_\eta(\zeta) d\xi + u_\xi(\zeta) d\eta) + c = \int\limits_a^z \frac{\partial u(\zeta)}{\partial \boldsymbol{n}_\zeta} ds_\zeta + c \, , \; c \in I\!R \, .$$

Using Theorem 13 and interchanging the orders of integrations and differentiations gives

$$v(z) = -\frac{1}{2\pi} \int\limits_{\partial D} u(t) \frac{\partial}{\partial \boldsymbol{n}_t} \int\limits_a^z \frac{\partial g(t, \zeta)}{\partial \boldsymbol{n}_\zeta} ds_\zeta ds_t + c = -\frac{1}{2\pi} \int\limits_{\partial D} u(\zeta) \frac{\partial h(\zeta, z)}{\partial \boldsymbol{n}_\zeta} ds_\zeta + c \, ,$$

where the symmetry $g(z, z_0) = g(z_0, z)$ of the GREEN function D was applied.

Remark. If in these integrals $u(\zeta)$ is replaced by a continuous function $\sigma(\zeta)$ on ∂D then w is an analytic function in D satisfying

$$\operatorname{Re} w(\zeta) = \sigma(\zeta), \quad \zeta \in \partial D \, .$$

See also Lemma 8, p. 51, where the DIRICHLET problem is handled as a special RIEMANN–HILBERT problem for simply connected domains. If D is multiply connected w in general is multi-valued and the DIRICHLET problem for single-valued functions is not always solvable, see 1.4.

Defintion 5. *An operator*

$$S : C(\partial D; I\!R) \longrightarrow \mathcal{A}(D) \cap C(\overline{D}; \mathbb{C})$$

from the space of real-valued continuous functions on ∂D into the space of analytic functions in D being continuous on the closure $\overline{D}$ of D satisfying

$$\operatorname{Re} S\sigma = \sigma \quad on \quad \partial D$$

is called SCHWARZ *operator.*

Remark. If D has a GREEN function then S is given by

$$(S\sigma)(z) := -\frac{1}{2\pi} \int\limits_{\partial D} \sigma(\zeta) \frac{\partial M(\zeta, z)}{\partial \boldsymbol{n}_\zeta} ds_\zeta \, . \tag{1.2.1}$$

Moreover, it is clear that S is given only up to an imaginary additive constant. This constant can be fixed by demanding

$$\operatorname{Im}(S\sigma)(a) = 0$$

for some fixed $a \in D$.
In case of $D = \mathbb{D}$ the SCHWARZ operator with the SCHWARZ-POISSON kernel is

$$(S\sigma)(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \sigma(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} + ic, \quad z \in \mathbb{D}, \tag{1.2.2}$$

where $c \in \mathbb{R}$ is arbitrary and

$$\operatorname{Im}(S\sigma)(0) = c .$$

The SCHWARZ operator can be given another form, see [Dzhu92a]. If $\zeta(s) = \xi(s) + i\eta(s)$ is the parameter representation of ∂D then the outer normal derivative is

$$\frac{\partial}{\partial \boldsymbol{n}_\zeta} = \eta'(s) \frac{\partial}{\partial \xi} - \xi'(s) \frac{\partial}{\partial \eta} = -i \left(\zeta'(s) \frac{\partial}{\partial \zeta} - \overline{\zeta'(s)} \frac{\partial}{\partial \overline{\zeta}} \right) .$$

Thus the GREEN representation formula (Theorem 13) becomes

$$u(z) = -\frac{1}{2\pi i} \int_{\partial D} u(\zeta) \left[\frac{\partial g(\zeta, z)}{\partial \zeta} d\zeta - \frac{\partial g(\zeta, z)}{\partial \overline{\zeta}} d\overline{\zeta} \right], \quad z \in D .$$

Differentiating with respect to z gives by the symmetry of $g(z, \zeta)$

$$u_z(z) = \frac{1}{4i} \int_{\partial D} u(z) \left[L(z, \zeta) d\zeta - K(z, \overline{\zeta}) d\overline{\zeta} \right]$$

where

$$L(z, \zeta) := -\frac{2}{\pi} \frac{\partial^2 g(z, \zeta)}{\partial z \partial \zeta}, \quad K(z, \overline{\zeta}) := -\frac{2}{\pi} \frac{\partial^2 g(z, \zeta)}{\partial z \partial \overline{\zeta}} .$$

Definition 6. *For a domain $D \subset \mathbb{C}$ with* GREEN *function $g(z, \zeta)$ the function*

$$K(z, \overline{\zeta}) := -\frac{2}{\pi} \frac{\partial^2 g(z, \zeta)}{\partial z \partial \overline{\zeta}}$$

is called the BERGMAN *kernel function of D.*

It was introduced by St. BERGMAN, see [Berg50,Besc53,Cour50].
Because $g(z, \zeta)$ vanishes for $\zeta \in \partial D$ identically in $z \in D$

$$\frac{\partial g(z, \zeta)}{\partial z} = 0, \; \frac{\partial^2 g(z, \zeta)}{\partial z \partial \zeta} \zeta'(s) + \frac{\partial^2 g(z, \zeta)}{\partial t \partial \overline{\zeta}} \overline{\zeta'(s)} = 0, \quad \zeta = \zeta(s) \in \partial D ,$$

so that for $z \in D, \zeta \in \partial D$

$$L(z,\zeta)\zeta'(s) + K(z,\overline{\zeta})\overline{\zeta'(s)} = 0 .$$

Hence,

$$u_z(z) = \frac{1}{2i} \int\limits_{\partial D} u(\zeta) L(z,\zeta) d\zeta .$$

As u is harmonic u_z is analytic. Realizing that u is real so that $u_{\overline{z}} = \overline{u_z}$ this leads to

$$\begin{aligned} u(z) - u(a) &= \int\limits_a^z \{u_\zeta((\zeta)d\zeta + u_{\overline{\zeta}}(\zeta)d\overline{\zeta}\} = 2\int\limits_a^z \operatorname{Re} u_\zeta((\zeta)d\zeta \\ &= \operatorname{Re}\left\{\frac{1}{i}\int\limits_{\partial D} u(\zeta)\int\limits_a^z L(t,\zeta)dt d\zeta\right\} . \end{aligned}$$

Introducing

$$J_L(z,\zeta) := \int\limits_a^z L(t,\zeta)dt \ , \ J_K(z,\overline{\zeta}) := \int\limits_a^z K(t,\overline{\zeta})dt$$

this becomes

$$u(z) = \operatorname{Re}\left\{\frac{1}{i}\int\limits_{\partial D} J_L(z,\zeta)u(\zeta)d\zeta\right\} + u(a) .$$

Here a is an arbitrarily chosen fixed point in D. It also can be chosen on the boundary ∂D if u is continuous in $\overline{D}$. Integration is taken along an arc connecting a with z. Thus the functions $J_L(\cdot,\zeta), J_K(\cdot,\overline{\zeta})$ being analytic in $D\backslash\{\zeta\}$ are multi-valued.
Let v be the harmonic conjugate to u. Then by the CAUCHY–RIEMANN system

$$u_x = v_y \ , \ u_y = -v_x$$

we have

$$\begin{aligned} v(z) &= \int\limits_a^z \{-u_\eta(\zeta)d\xi + u_\xi(\zeta)d\eta\} + v(a) \\ &= 2\operatorname{Im}\left\{\int\limits_a^z u_\zeta((\zeta)d\zeta\right\} + v(a) \\ &= \operatorname{Im}\left\{\frac{1}{i}\int\limits_{\partial D} u(\zeta)J_L(z,\zeta)d\zeta\right\} + v(a) . \end{aligned}$$

Hence, the analytic function $w = u + iv$ is representable as

$$w(z) = \frac{1}{i} \int\limits_{\partial D} u(\zeta) J_L(z,\zeta) d\zeta + w(a) . \tag{1.2.3}$$

In general v and w, too are multi-valued functions.

Remarks.

1. If the solution to the DIRICHLET problem

$$\operatorname{Re} w(z) = \sigma(z), \quad z \in \partial D ,$$

exists it is given by

$$w(z) = \frac{1}{i} \int\limits_{\partial D} J_L(z,\zeta)\sigma(\zeta) d\zeta - \frac{1}{2\pi i} \int\limits_{\partial D} \sigma(\zeta) \left[\frac{\partial g(\zeta,a)}{\partial \zeta} d\zeta - \frac{\partial g(\zeta,a)}{\partial \overline{\zeta}} d\overline{\zeta} \right] + ic$$

with an arbitrary real c and $a \in D$ fixed.
For $a \in \partial D$

$$w(z) = \frac{1}{i} \int\limits_{\partial D} J_L(z,\zeta)\sigma(\zeta) d\zeta + \sigma(a) + ic .$$

2. For the unit disc $D = \mathbb{D}$

$$g(z,\zeta) = \log \left| \frac{1 - z\overline{\zeta}}{z - \zeta} \right| , L(z,\zeta) = \frac{1}{\pi} \frac{1}{(\zeta - z)^2} ,$$

$$K(z,\overline{\zeta}) = \frac{1}{\pi} \frac{1}{(1 - z\overline{\zeta})^2} , \ J_L(z,\zeta) = \frac{1}{\pi} \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - a} \right) .$$

To verify the two preceding representation formulas observe for $a \in \mathbb{D}$

$$\frac{1}{i} \int\limits_{\partial \mathbb{D}} J_L(z,\zeta)\sigma(\zeta) d\zeta = \frac{1}{2\pi i} \int\limits_{\partial \mathbb{D}} \left(\frac{\zeta + z}{\zeta - z} - \frac{\zeta + a}{\zeta - a} \right) \sigma(\zeta) \frac{d\zeta}{\zeta} ,$$

$$\frac{1}{2\pi i} \int\limits_{\partial \mathbb{D}} \sigma(\zeta) \left[\frac{\partial g(\zeta,a)}{\partial \zeta} d\zeta - \frac{\partial g(\zeta,a)}{\partial \overline{\zeta}} d\overline{\zeta} \right] = -\operatorname{Im} \frac{1}{2\pi} \int\limits_{\partial \mathbb{D}} \sigma(\zeta) \frac{\zeta + a}{\zeta - a} \frac{d\zeta}{\zeta}$$

so that

$$w(z) = \frac{1}{2\pi i} \int\limits_{\partial D} \sigma(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} + i\widehat{c}$$

with

$$\widehat{c} = c - \operatorname{Im} \frac{1}{2\pi i} \int\limits_{\partial D} \sigma(\zeta) \frac{\zeta + a}{\zeta - a} \frac{d\zeta}{\zeta} .$$

If $a \in \partial \mathbb{D}$ then the same formula holds, since

$$\frac{1}{i} \int_{\partial D} J_L(z,\zeta)\sigma(\zeta)d\zeta = \frac{1}{2\pi i} \int_{\partial D} \sigma(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta} - \sigma(a) - \frac{1}{2\pi} \int_{\partial \mathbb{D}} \sigma(\zeta)\mathrm{Im}\frac{\zeta + a}{\zeta - a}\frac{d\zeta}{\zeta} .$$

3. For a general domain D

$$L(z,\zeta) = \frac{1}{\pi}\frac{1}{(\zeta - z)^2} - \ell(z,\zeta),\ \ell(z,\zeta) := \frac{2}{\pi}\frac{\partial^2 \omega(z,\zeta)}{\partial z \partial \zeta} .$$

Obviously ℓ is an analytic function of both variables z and ζ in D. Integrating gives

$$J_L(z,\zeta) = \frac{1}{\pi}\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - a}\right) - J_\ell(z,\zeta)$$

where

$$J_\ell(z,\zeta) := \int_a^z \ell(t,\zeta)dt$$

is multi-valued in general. While $L(z,\zeta)$ has a second order pole in $z = \zeta$ the BERGMAN kernel $K(z,\overline{\zeta})$ is analytic in z and $\overline{\zeta}$ for z and ζ in D,

$$K(z,\overline{\zeta}) := -\frac{2}{\pi}\frac{\partial^2 g(z,\zeta)}{\partial z \partial \overline{\zeta}} = -\frac{2}{\pi}\frac{\partial^2 \omega(z,\zeta)}{dz \partial \overline{\zeta}} ,$$

having a singularity for $z = \zeta \in \partial D$. Here, as before

$$\omega(z,\zeta) := g(z,\zeta) + \log|z - \zeta|$$

is harmonic for z and for ζ in D.

Lemma 5. *As a function of ζ the kernel $L(z,\zeta)$ is orthogonal to all bounded analytic functions f in D for any fixed $z \in D$, i.e.*

$$(f, L(z,\cdot)) := \int_D f(\zeta)\overline{L(z,\zeta)}d\xi d\eta = 0 .$$

Proof. Let f be bounded and analytic in D. Then

$$(f, L(z,\cdot)) = \frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{(\overline{\zeta - z})^2} - \int_D f(\zeta)\overline{\ell(z,\zeta)}d\xi d\eta .$$

Applying the GREEN formula for the domain

$$D_\varepsilon := D \backslash \{\zeta : |\zeta - z| \leq \varepsilon\}$$

to the integral $f(\zeta)\overline{L(z,\zeta)}$ gives

$$\begin{aligned}\int\limits_{D_\epsilon} f(\zeta)\overline{L(z,\zeta)}d\xi d\eta &= -\frac{2}{\pi}\int\limits_{D_\epsilon} f(\zeta)\frac{\partial^2 g(z,\zeta)}{\partial\overline{z}\partial\overline{\zeta}}d\xi d\eta \\ &= \frac{1}{2\pi i}\int\limits_{\partial D_\epsilon} 2f(\zeta)\frac{\partial g(z,\zeta)}{\partial\overline{z}}d\zeta = -\frac{1}{\pi i}\int\limits_{|\zeta-z|=\epsilon} f(\zeta)\frac{\partial g(z,\zeta)}{\partial\overline{z}}d\zeta \\ &= -\frac{1}{2\pi i}\int\limits_{|\zeta-z|=\epsilon} f(\zeta)\left[2\frac{\partial\omega(z,\zeta)}{\partial\overline{z}} - \frac{1}{\overline{\zeta-z}}\right]d\zeta\,.\end{aligned}$$

Letting ε tend to zero this last integral tends to

$$\frac{f(z)}{2\pi}\int\limits_0^{2\pi} e^{2i\varphi}d\varphi = 0\,.$$

Thus $(f, L(z,\cdot)) = 0$. Observing $\ell(z,\zeta) = \ell(\zeta,z)$ which follows from the symmetry of the GREEN function then

$$\frac{1}{\pi}\int\limits_D f(\zeta)\frac{d\xi d\eta}{\overline{(\zeta-z)}^2} = \int\limits_D f(\zeta)\overline{\ell(\zeta,z)}d\xi d\eta, \quad z\in D\,,$$

follows.
Repeating the preceding proof with $K(z,\overline{\zeta})$ instead of $\overline{L(z,\zeta)}$ the respective limit by letting ε tend to zero instead of becoming zero is $f(z)$. This result is called the reproducing property of the BERGMAN kernel in the space of bounded analytic functions in D.

Lemma 6. *For any bounded analytic function f in D*

$$f(z) = \int\limits_D f(\zeta)K(z,\overline{\zeta})d\xi d\eta, \quad z\in D\,.$$

From the symmetry of the GREEN function

$$K(z,\overline{\zeta}) = \overline{K(\zeta,\overline{z})}$$

follows immediately.

1.3 RIEMANN boundary value problem

Let Γ be a smooth simple closed curve (or a set of finitely many such curves) in the complex plane $\mathbb{C}$. In the following for simplicity we mainly will consider simply closed curves. In that case D^+ denotes the bounded domain with boundary Γ and $D^- := \widehat{\mathbb{C}}\setminus\overline{D^+}$.

Definition 7. *Let $G \in C(\Gamma;\mathbb{C})$ and $G(\zeta) \neq 0$ on Γ. Then the index κ of G with respect to Γ is the mean variation of* $\arg G(\zeta)$ *while ζ varies on Γ in the positive direction passing any point once,*

$$\kappa := \mathrm{ind}G = \frac{1}{2\pi}\int_\Gamma d\arg G(\zeta) = \frac{1}{2\pi i}\int_\Gamma d\log G(\zeta) .$$

Because Γ is closed and G is continuous κ is an entire number. The index has the following properties.

1. $\mathrm{ind}\,(G_1G_2) = \mathrm{ind}\,G_1 + \mathrm{ind}\,G_2$, $\mathrm{ind}(1/G) = -\mathrm{ind}\,G = \mathrm{ind}\,\overline{G}$.
2. If D is a domain with smooth boundary and G is an analytic function in D up to isolated poles with continuous non-vanishing boundary values on ∂D then

   $$\mathrm{ind}G = n(0) - n(\infty) ,$$

   where $n(0)$ is the number of zeroes and $n(\infty)$ the number of poles of G each counted with respect to its multiplicity. This follows from the argument principle, in the case of an unbounded domain D applied to $D \cap \{|z| < R\}$ for sufficiently large R and then passing with R to infinity.

Riemann problem. *Let Γ be a smooth simply closed curve and $G, g \in C^\alpha(\Gamma;\mathbb{C}), 0 < \alpha < 1$, with $G(\zeta) \neq 0$ on Γ. Find analytic functions ϕ^+ in D^+ and ϕ^- in D^- such that*

$$\phi^+(\zeta) = G(\zeta)\phi^-(\zeta) + g(\zeta), \quad \zeta \in \Gamma .$$

Remark. This problem is sometimes called the problem of linear conjugacy.

Theorem 14. *For $0 \leq \kappa$ the homogeneous* RIEMANN *boundary value problem $(g = 0)$ has $\kappa + 1$ linearly independent solutions*

$$\begin{aligned} \phi_\kappa^+(z) &= z^k e^{\gamma^+(z)}, \quad z \in D^+ , \\ & \qquad\qquad\qquad\qquad 0 \leq k \leq \kappa , \\ \phi_\kappa^-(z) &= z^{k-\kappa} e^{\gamma^-(z)}, \quad z \in D^-, \\ \gamma(z) &:= \frac{1}{2\pi i}\int_\Gamma \log\{\zeta^{-\kappa}G(\zeta)\}\frac{d\zeta}{\zeta - z}, \quad z \notin \Gamma . \end{aligned}$$

The general solution contains $\kappa+1$ arbitrary complex constants. The space of solutions vanishing at infinity contains κ linearly independent solutions, namely the preceding *ones for* $0 \leq k \leq \kappa - 1$. *To fix a solution* $\kappa + 1$ *side conditions are necessary. For* $\kappa < 0$ *the homogeneous problem* $(g = 0)$ *is unsolvable.*

Proof.

i. $0 \leq \kappa$.
Let ϕ^+, ϕ^- be a solution and $N^\pm$ the number of zeroes of $\phi^\pm$ in $D^\pm$. Then

$$N^+ = \mathrm{ind}\phi^+ = \mathrm{ind}(G\phi^-) = \mathrm{ind}\, G + \mathrm{ind}\phi^- = \kappa - N^- ,$$

$$0 \leq \kappa = N^+ + N^- .$$

If $\kappa = 0$ then $N^+ = N^- = 0$ and $\log \phi^\pm$ is a single-valued analytic function in $D^\pm$. The function $\log G(\zeta)$ is single-valued too because

$$\int_\Gamma d \log G = 0 .$$

From the PLEMELJ–SOKHOTZKI formulae (1.1.7) we see that the solution to the problem

$$\log \phi^+ = \log G + \log \phi^- \quad \text{on} \quad \Gamma$$

is given by the Cauchy integral

$$\log \phi(z) = \frac{1}{2\pi i} \int_\Gamma \log G(\zeta) \frac{d\zeta}{\zeta - z} =: \gamma(z), \quad z \notin \Gamma . \tag{1.3.1}$$

Hence $\phi(z) = e^{\gamma(z)}$, i.e. $\phi^\pm(z) = e^{\gamma^\pm(z)}, z \in D^\pm$, satisfies

$$\phi^+ = G\phi^- \quad \text{on} \quad \Gamma, \quad \phi^-(\infty) = 1 ,$$

and $\widetilde{\phi}^\pm(z) = ae^{\gamma^\pm(z)}$ satisfies

$$\widetilde{\phi}^+ = G\widetilde{\phi}^- \quad \text{on} \quad \Gamma, \quad \widetilde{\phi}^-(\infty) = a .$$

For $a = 0$ there is only the trivial solution.
If $0 < \kappa$ then $\zeta^{-\kappa}G(\zeta)$ is a function continuous on ∂D, because $0 \in D^+$ i.e. $0 \notin \Gamma$, with vanishing index

$$\mathrm{ind}(\zeta^{-\kappa}G(\zeta)) = \mathrm{ind}\zeta^{-\kappa} + \mathrm{ind}G(\zeta) = -\kappa + ind G(\zeta) = 0 .$$

Rewriting the boundary condition as

$$\phi^+(\zeta) = \zeta^\kappa(\zeta^{-\kappa}G(\zeta))\phi^-(\zeta) \quad \text{on} \quad \Gamma$$

and using

$$\gamma(\zeta) := \frac{1}{2\pi i} \int_{\Gamma} \log\{\zeta^{-\kappa} G(\zeta)\} \frac{d\zeta}{\zeta - z} ,$$

satisfying

$$e^{\gamma^+(\zeta)} = \zeta^{-\kappa} G(\zeta) e^{\gamma^-(\zeta)} \quad \text{on} \quad \Gamma$$

with

$$X^+(z) := e^{\gamma^+(z)} \ (z \in D^+), \quad X^-(z) := z^{-\kappa} e^{\gamma^-(z)} \quad (z \in D^-)$$

we find

$$\frac{\phi^+(\zeta)}{X^+(\zeta)} = \frac{\phi^-(\zeta)}{X^+(\zeta)} \quad \text{on} \quad \Gamma .$$

While $\dfrac{\phi^+(z)}{X^+(z)}$ is analytic in D^+, $\dfrac{\phi^-(z)}{X^-(z)}$ is analytic in D^- up to a pole at infinity. Because both functions coincide on Γ they are analytic continuations from one another forming together an entire analytic function with pole at infinity of order at most κ. From the general LIOUVILLE theorem this function is seen to be a polynomical P_κ of degree at most κ. Thus,

$$\begin{aligned} \phi^+(z) &= P_\kappa(z) e^{\gamma^+(z)}, & z \in D^+ , \\ \phi^-(z) &= z^{-\kappa} P_\kappa(z) e^{\gamma^-(z)}, & z \in D^- . \end{aligned}$$

ii. $\kappa < 0$.
There is no solution in this case because of the relation $\kappa = N^+ + N^-$. But the functions $X^\pm$ in $D^\pm$ satisfy on Γ the RIEMANN condition

$$X^+(\zeta) = G(\zeta) X^-(\zeta)$$

in this case too. It does not form a solution because X^- fails to be analytic at infinity where it rather has a pole of order $-\kappa$.

Defintion 8. *The function*

$$X(z) = \begin{cases} X^+(z), & z \in D^+ \\ X^-(z), & z \in D^- \end{cases}$$

is called canonical function of the RIEMANN *problem.*

The canonical solution will be important for solving the inhomogeneous problem. Dividing ϕ by X gives

$$\frac{\phi^+}{X^+} = \frac{\phi^-}{X^-} + \frac{g}{X^+} \quad \text{on} \quad \Gamma .$$

As $g \in C^\alpha(\Gamma)$ and $X^+ \in C^\alpha(\Gamma)$ by Theorem 4, we have $\dfrac{g}{X^+} \in C^\alpha(\Gamma)$. The solution to this jump problem is

$$\psi(z) := \frac{1}{2\pi i} \int_\Gamma \frac{g(\zeta)}{X^+(\zeta)} \frac{d\zeta}{\zeta - z}, \quad z \notin \Gamma ,$$

namely ψ satisfies

$$\psi^+(\zeta) = \psi^-(\zeta) + \frac{g(\zeta)}{X^+(\zeta)} \quad \text{on} \quad \Gamma .$$

Hence $\dfrac{\phi}{X} - \psi$ is continuous on Γ

$$\frac{\phi^+}{X^+} - \psi^+ = \frac{\phi^-}{X^-} - \psi^- \quad \text{on} \quad \Gamma .$$

Again $\dfrac{\phi}{X} - \psi$ is an analytic function having a pole of order $\leq \kappa$ if $0 < \kappa$ but a zero if $\kappa < 0$ at infinity.
In order that $\phi = X\psi$ for $\kappa < 0$ is a solution i.e. behaves regular at infinity it is necessary and sufficient that ψ^- has a zero of order κ. From its series representation near ∞

$$\psi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{g(\zeta)}{X^+(\zeta)} \frac{d\zeta}{\zeta - z} = \sum_{k=1}^{\infty} c_k z^{-k} , \; c_k := -\frac{1}{2\pi i} \int_\Gamma \frac{g(\zeta)}{X^+(\zeta)} \zeta^{k-1} d\zeta \quad (1 \leq k)$$

these solvability conditions are seen to be

$$\int_\Gamma \frac{g(\zeta)}{X^+(\zeta)} \zeta^{k-1} d\zeta = 0, \quad 1 \leq k \leq -\kappa - 1 .$$

In case when solutions vanishing at ∞ are looked for moreover $c_{-\kappa}$ has to vanish. Thus we have the following result.

Theorem 15. *For $0 \leq \kappa$ the general solution to the inhomogeneous* RIEMANN *boundary value problem is*

$$\phi(z) = X(z)[\psi(z) + P_\kappa(z)] ,$$

where P_κ is a polynomial of order at most κ with arbitrary coefficients.
For $\kappa < 0$ it is solvable if and only if the above solvability conditions are satisfied. Then

$$\phi(z) = X(z)\psi(z)$$

is the solution.

Remark. HASEMAN, a Ph.D. student from HILBERT in Göttingen, solved the following problem.

Haseman problem. *Let Γ be a smooth simply closed curve, $G, g, \in C^\beta(\Gamma), 0 < \beta < 1, g(\zeta) \neq 0$ on $\Gamma, \alpha \in C^1(\Gamma;\Gamma)$ a bijective mapping from Γ onto itself, preserving the orientation and satisfying $\alpha'(\zeta) \neq 0$ on Γ. Find an analytic function in $\hat{\mathbb{C}} \setminus \Gamma$ satisfying*

$$\phi^+(\alpha(\zeta)) = G(\zeta)\phi^-(\zeta) + g(\zeta) \quad \text{on} \quad \Gamma\,.$$

This problem is called RIEMANN problem with shift or delay, too. For a solution see [Gakh66], p. 121. It is solved by reducing the problem to the RIEMANN problem. There is still a lot of new research done on RIEMANN and related boundary value problems.

Next we will consider the RIEMANN problem for a multiply connected domain. Let $\Gamma_\mu, 0 \leq \mu \leq m$, be $m+1$ mutually disjoint smooth simply closed curves such that Γ_0 positively oriented surrounds the other $\Gamma_\mu (1 \leq \mu \leq m)$ being negatively oriented. Denote by D^+ the bounded domain with Γ as its boundary and assume without loss of generality $0 \in D^+$,

$$D^- := \hat{\mathbb{C}} \setminus \overline{D^+}\,.$$

The solution of the simple jump condition, $G = 1$,

$$\phi^+(\zeta) = \phi^-(\zeta) + g(\zeta) \quad \text{on} \quad \Gamma\,,$$

is again given by the CAUCHY integral

$$\phi(z) = \frac{1}{2\pi i}\int_\Gamma g(\zeta)\frac{d\zeta}{\zeta - z}, \quad z \notin \Gamma\,,$$

which is true just because of the local behaviour. For arbitrary G the index κ with respect to Γ, obviously, is the sum of the indices κ_μ of G with respect to Γ_μ,

$$\kappa = \sum_{\mu=0}^{m} \kappa_\mu\,.$$

1. $\kappa_\mu = 0$ for all $0 \leq \mu \leq m$.
   Then for an $a \in \mathbb{C}$

   $$\phi(z) = a e^{\gamma(z)}, \quad \gamma(z) := \frac{1}{2\pi i}\int_\Gamma \log G(\zeta)\frac{d\zeta}{\zeta - z}$$

   is a solution to the homogeneous problem.

2. κ_μ arbitrary.
This case again is reduced to case 1. Denote by D_μ^- the bounded domain with boundary $\partial D_\mu^- = -\Gamma_\mu$ and let $z_\mu \in D_\mu^-$ be an arbitrary point, $1 \le \mu \le m$. Then

$$\frac{1}{2\pi i}\int_{\Gamma_\nu} d\log(\zeta - z_\mu) = -\delta_{\mu\nu} := \begin{cases} -1, & \text{for} \quad \mu = \nu \\ 0, & \text{for} \quad \mu \ne \nu \end{cases},$$

$$\frac{1}{2\pi i}\int_{\Gamma_\nu} d\log\left\{G(\zeta)\prod_{\mu=1}^{m}(\zeta - z_\mu)^{\kappa_\mu}\right\} = 0, \quad 1 \le \nu \le m\,,$$

$$\frac{1}{2\pi i}\int_{\Gamma_0} d\log\left\{G(\zeta)\prod_{\mu=1}^{m}(\zeta - z_\mu)^{\kappa_\mu}\right\} = \kappa_0 + \sum_{\mu=1}^{m}\kappa_\mu = \kappa\,,$$

$$\frac{1}{2\pi i}\int_{\Gamma_\mu} d\log\zeta = \delta_{0\mu} = \begin{cases} 1, & \text{for} \quad \mu = 0 \\ 0, & \text{for} \quad \mu \ne 0 \end{cases},$$

$$\frac{1}{2\pi i}\int_{\Gamma} d\log\left\{\zeta^{-\kappa}\prod_{\mu=1}^{m}(\zeta - z_\mu)^{\kappa_\mu}G(\zeta)\right\} = 0\,.$$

Proceeding as in the case of a simply connected domain the homogeneous jump condition is written as

$$\phi^+(\zeta) = \zeta^\kappa\prod_{\mu=1}^{m}(\zeta - z_\mu)^{-\kappa_\mu}\left\{\zeta^{-\kappa}\prod_{\mu=1}^{m}(\zeta - z_\mu)^{\kappa_\mu}G(\zeta)\right\}\phi^-(\zeta) \quad \text{on} \quad \Gamma\,.$$

Using

$$\gamma(z) := \frac{1}{2\pi i}\int_{\Gamma}\log\left\{\zeta^{-\kappa}\prod_{\mu=1}^{m}(\zeta - z_\mu)^{\kappa_\mu}G(\zeta)\right\}\frac{d\zeta}{\zeta - z}, \quad z \notin \Gamma\,,$$

and

$$\begin{aligned} X^+(z) &= \prod_{\mu=1}^{m}(z - z_\mu)^{-\kappa_\mu}e^{\gamma(z)}, \quad z \in D^+\,, \\ X^-(z) &= z^{-\kappa}e^{\gamma(z)}, \quad z \in D^-\,, \end{aligned}$$

we see $X^+ = GX^-$ on Γ. X again is called canonical function. In the case $\kappa < 0$ it has a pole of order $-\kappa$ at ∞. From the condition

$$\frac{\phi^+}{X^+} = \frac{\phi^-}{X^-} \quad \text{on} \quad \Gamma$$

the analytic function $\dfrac{\phi}{X}$ is seen to be a polynomial P_κ as before if $0 \le \kappa$. The general solution is given by $\phi = XP_\kappa$. For $\kappa < 0$ the homogeneous *problem* is unsolvable corresponding to the fact that then $\frac{\phi^-}{X^-}$ would vanish at ∞ so that ϕ only could be identically zero, i.e. being the trivial solution to the homogeneous problem. By the canonical function the inhomogeneous condition is reduced to

$$\frac{\phi^+(\zeta)}{X^+(\zeta)} = \frac{\phi^-(\zeta)}{X^-(\zeta)} + \frac{g(\zeta)}{X^+(\zeta)} \quad \text{on} \quad \Gamma\,.$$

Hence Theorem 15 holds for multiply connected domains of the above type too if the canonical function is changed as above.

Remark. The same is true if D^+ is an unbounded domain. Let in the above notation $\Gamma_0 = \emptyset$ so that $\infty \in \mathbb{D}^+$. Then the index of

$$\prod_{\mu=1}^{m} (\zeta - z_\mu)^{\kappa_\mu} G(\zeta), \quad \zeta \in \Gamma\,,$$

with respect to any $\Gamma_\mu (1 \le \mu \le m)$ is zero. Therefore if in the above considerations the factor $\zeta^{-\kappa}$ is cancelled everything holds again.

1.4 Riemann–Hilbert boundary value problem

Let D be a simply connected bounded domain with (piecewise) smooth boundary. Contrary to the Riemann problem here the simple connectivity of D is important. Although the Riemann–Hilbert problem may be solved even for multiply connected domains as for simply connected ones, the solution in general is a multi-valued function and further considerations are necessary to find single-valued solutions, see [Gakh66], p. 326. Because this problem does not occur for simply connected domains, here we stay with these.

Riemann–Hilbert problem. *For given $\lambda, \varphi \in C^\alpha(\partial D; \mathbb{C}), 0 < \alpha < 1$, with $\lambda(\zeta) \neq 0$ on ∂D find an analytic function w in D such that*

$$\mathrm{Re}\{\overline{\lambda(\zeta)} w(\zeta)\} = \varphi(\zeta), \quad \zeta \in \partial D\,.$$

Remark. With $\lambda = \mu + i\nu, w = u + iv$ we have

$$\mathrm{Re}\{\overline{\lambda} w\} = \mu u + \nu v = \varphi \quad \text{on} \quad \partial D\,.$$

While the Dirichlet problem, coinciding with the Riemann–Hilbert problem if $\lambda = 1$ just prescribes the real part of the analytic function on the boundary in the Riemann–Hilbert boundary condition a linear combination of the real and imaginary parts of the function looked for is given. Although this problem is more general

the solution will be found by reducing it to the DIRICHLET problem. This latter problem is solvable via the SCHWARZ operator, see Definition 5 in 1.2. Principally, the domain D can be mapped conformally onto the unit disc mapping ∂D onto $\partial \mathbb{D}$, see [Golu69], p. 44, so that it is enough to study the case $D = \mathbb{D}$. The solution then is found by combining the solution of the transformed problem with the inverse conformal mapping.
In the sequel we always will assume $|\lambda(\zeta)| = 1$ on $\partial \mathbb{D}$ which is no loss of generality as can be seen by dividing the boundary condition by $|\lambda(\zeta)|$. At first we prove a connection of the RIEMANN problem with the RIEMANN–HILBERT problem, see [Gakh66], p. 228.

Theorem 16. *The solution of the* RIEMANN *problem*

$$\phi^+(\zeta) = G(\zeta)\phi^-(\zeta) + g(\zeta), \quad |\zeta| = 1\,, \tag{1.4.1}$$

with

$$G(\zeta) := \frac{\lambda(\zeta)}{\overline{\lambda(\zeta)}}, \quad g(\zeta) = \frac{2\varphi(\zeta)}{\overline{\lambda(\zeta)}}$$

is a solution of the RIEMANN–HILBERT *problem*

$$\mathrm{Re}\{\overline{\lambda(\zeta)}\phi^+(\zeta)\} = \varphi(\zeta), \quad |\zeta| = 1\,, \tag{1.4.2}$$

if the free complex parameters of this solution are chosen properly.

Proof. Let $\kappa := \operatorname{ind}\lambda$, then $\operatorname{ind} G = 2\kappa$.

1. $\kappa = 0$.
   The solution to (1.4.1) is

$$\phi(z) = X(z)\left\{\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{g(\zeta)}{X^+(\zeta)}\frac{d\zeta}{\zeta - z} + c\right\}, \quad c \in \mathbb{C}\,,$$

   where

$$\begin{aligned} X^+(z) &= \exp\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log G(\zeta)\frac{d\zeta}{\zeta - z} = \exp\frac{1}{\pi}\int\limits_{|\zeta|=1}\arg\lambda(\zeta)\frac{d\zeta}{\zeta - z} \\ &= \exp\left\{\frac{1}{2\pi}\int\limits_{|\zeta|=1}\arg\lambda(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta} + \frac{1}{2\pi}\int\limits_{|\zeta|=1}\arg\lambda(\zeta)\frac{d\zeta}{\zeta}\right\} \end{aligned}$$

$$= \quad e^{iC}e^{i\gamma(z)} \;, \quad \gamma(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \arg \lambda(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} \;,$$

$$C := \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \arg \lambda(\zeta) \frac{d\zeta}{\zeta} \in \mathbb{R} \;.$$

We remark that $\gamma = S \arg \lambda$, where S is the SCHWARZ operator (1.2.2) for $\mathbb{D}$ satisfying $\mathrm{Re}\, S \arg \lambda = \arg \lambda$ on $\partial \mathbb{D}$ and $\mathrm{Im}\,(S \arg \lambda)(0) = 0$. Thus

$$\phi^+(z) = e^{i\gamma(z)} \left\{ \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{g(\zeta)}{e^{i\gamma(\zeta)}} \frac{d\zeta}{\zeta - z} + c \right\} , \quad |z| < 1 \;,$$

where $c \in \mathbb{C}$ is arbitrary

$$\int\limits_{|\zeta|=1} \frac{g(\zeta)}{e^{i\gamma(\zeta)}} \frac{d\zeta}{\zeta - z} = \int\limits_{|\zeta|=1} \frac{2\varphi(\zeta)}{\overline{\lambda(\zeta)} e^{i\gamma(\zeta)}} \frac{d\zeta}{\zeta - z} = \int\limits_{|\zeta|=1} \frac{\varphi(\zeta)}{\overline{\lambda(\zeta)} e^{i\gamma(\zeta)}} \left(\frac{\zeta + z}{\zeta - z} + 1 \right) \frac{d\zeta}{\zeta} \;.$$

$$\overline{\lambda(\zeta)} e^{i\gamma(\zeta)} = \overline{\lambda(\zeta)} e^{i\mathrm{Re}\gamma(\zeta) - \mathrm{Im}\gamma(\zeta)} = e^{-\mathrm{Im}\gamma(\zeta)} \overline{\lambda(\zeta)} e^{i \arg \lambda(\zeta)} = e^{-\mathrm{Im}\gamma(\zeta)}, \quad |\zeta| = 1 \;.$$

$$\int\limits_{|\zeta|=1} \frac{g(\zeta)}{e^{i\gamma(\zeta)}} \frac{d\zeta}{\zeta - z} = \int\limits_{|\zeta|=1} \varphi(\zeta) e^{\mathrm{Im}\gamma(\zeta)} \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} + \int\limits_{|\zeta|=1} \varphi(\zeta) e^{\mathrm{Im}\gamma(\zeta)} \frac{d\zeta}{\zeta} \;.$$

Thus

$$\phi^+(z) = e^{i\gamma(z)} \left\{ \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \varphi(\zeta) e^{\mathrm{Im}\gamma(\zeta)} \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} + \tilde{c} \right\} \;,$$

where $\tilde{c} \in \mathbb{C}$ is imaginary if and only if c is chosen such that

$$\mathrm{Re}\, c + \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \varphi(\zeta) e^{\mathrm{Im}\gamma(\zeta)} \frac{d\zeta}{\zeta} = 0 \;.$$

That this $\phi^+(z)$ is a solution to (1.4.2) follows by direct calculation. We have for $|\zeta| = 1$

$$\overline{\lambda(\zeta)} e^{i\gamma(\zeta)} = e^{-\mathrm{Im}\gamma(\zeta)} \;,$$

so that

$$\mathrm{Re} \left\{ \overline{\lambda(\zeta)} \phi^+(\zeta) \right\} = e^{-\mathrm{Im}\gamma(\zeta)} \mathrm{Re} \left\{ (S\varphi e^{\mathrm{Im}\gamma})(\zeta) + \tilde{c} \right\} = \varphi(\zeta) \;.$$

2. $0 < \kappa$.

We proceed as in 1. The solution to (1.4.1) is

$$\phi(z) = X(z)\left\{\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{g(\zeta)}{X^+(\zeta)}\frac{d\zeta}{\zeta - z} + P_{2\kappa}(z)\right\},$$

where for $|z| < 1$

$$\begin{aligned}
X^+(z) &= \exp\frac{1}{2\pi i}\int_{|\zeta|=1}\log\left\{\zeta^{-2\kappa}\frac{\lambda(\zeta)}{\overline{\lambda(\zeta)}}\right\}\frac{d\zeta}{\zeta - z} \\
&= \exp\frac{1}{2\pi}\int_{|\zeta|=1}\{\arg\lambda(\zeta) - \kappa\arg\zeta\}\left\{\frac{\zeta + z}{\zeta - z} + 1\right\}\frac{d\zeta}{\zeta} = e^{iC}e^{i\gamma(z)}
\end{aligned}$$

with

$$\gamma(z) := \frac{1}{2\pi i}\int_{|\zeta|=1}\{\arg\lambda(\zeta) - \kappa\arg\zeta\}\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta} = S(\arg\lambda(\zeta) - \kappa\arg\zeta)(z)\,;$$

$$C := \frac{1}{2\pi i}\int_{|\zeta|=1}\{\arg\lambda(\zeta) - \kappa\arg\zeta\}\frac{d\zeta}{\zeta} \in I\!R\,.$$

On $|\zeta| = 1$

$$\overline{\lambda(\zeta)}e^{i\gamma(\zeta)} = \overline{\lambda(\zeta)}e^{i\mathrm{Re}\gamma(\zeta) - \mathrm{Im}\gamma(\zeta)} = \overline{\lambda(\zeta)}e^{i(\arg\lambda(\zeta) - \kappa\arg\zeta) - \mathrm{Im}\gamma(\zeta)} = \zeta^{-\kappa}e^{-\mathrm{Im}\gamma(\zeta)}\,.$$

Thus for $|z| < 1$

$$\begin{aligned}
\phi^+(z) &= e^{i\gamma(z)}\left\{\frac{1}{2\pi i}\int_{|\zeta|=1}\overline{\lambda(\zeta)}g(\zeta)e^{\mathrm{Im}\gamma(\zeta)}\zeta^{\kappa}\frac{d\zeta}{\zeta - z} + P_{2\kappa}(z)\right\} \\
&= e^{i\gamma(z)}\left\{\frac{1}{2\pi i}\int_{|\zeta|=1}\varphi(\zeta)e^{\mathrm{Im}\gamma(\zeta)}\zeta^{\kappa}\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta}\right. \\
&\qquad\left. + \frac{1}{2\pi i}\int_{|\zeta|=1}\varphi(\zeta)e^{\mathrm{Im}\gamma(\zeta)}\zeta^{\kappa}\frac{d\zeta}{\zeta} + P_{2\kappa}(z)\right\}
\end{aligned}$$

with an arbitrary polynomial $P_{2\kappa}$ of degree at most 2κ. Because

$$\begin{aligned}\frac{\zeta+z}{\zeta-z}\frac{\zeta^\kappa-z^\kappa}{\zeta} &= \left(1+\frac{z}{\zeta}\right)\sum_{k=0}^{\kappa-1}\zeta^{\kappa-1-k}z^k \\ &= \sum_{k=0}^{\kappa-1}\zeta^{\kappa-1-k}z^k+\sum_{k=0}^{\kappa-1}\zeta^{\kappa-2-k}z^{k+1} = p_\kappa(z,\zeta)\end{aligned}$$

is a polynomial in z of degree κ we can write

$$\phi^+(z) = e^{i\gamma(z)}\left\{\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\varphi(\zeta)e^{\mathrm{Im}\gamma(\zeta)}z^\kappa\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}+z^\kappa Q_\kappa(z)\right\}.$$

Here Q_κ has the form

$$Q_\kappa(z) = \sum_{k=-\kappa}^{\kappa} c_k z^k$$

with coefficients depending among others on those of $P_{2\kappa}$. If

$$\mathrm{Re}\,Q_\kappa(\zeta) = 0 \quad \text{on} \quad |\zeta| = 1$$

then ϕ^+ is a solution to (1.4.2) as is seen immediately. For $|\zeta| = 1$ we have

$$\mathrm{Re}\{\overline{\lambda(\zeta)}\zeta^\kappa e^{i\gamma(\zeta)}\} = e^{-\mathrm{Im}\gamma(\zeta)}$$

and hence

$$\begin{aligned}\mathrm{Re}\{\overline{\lambda(\zeta)}\phi^+(\zeta)\} &= e^{-\mathrm{Im}\gamma(\zeta)}\mathrm{Re}\{\varphi(\zeta)e^{\mathrm{Im}\gamma(\zeta)}+Q_\kappa(\zeta)\} \\ &= \varphi(\zeta)+e^{-\mathrm{Im}\gamma(\zeta)}\mathrm{Re}\,Q_\kappa(\zeta)\,.\end{aligned}$$

The condition $\mathrm{Re}\,Q_\kappa(\zeta) = 0$ on $|\zeta| = 1$ imposes conditions on $P_{2\kappa}(\zeta)$.

3. $\kappa < 0$.

Problem (1.4.1) is solvable if

$$\int\limits_{|\zeta|=1}\frac{g(\zeta)}{X^+(\zeta)}\zeta^{k-1}d\zeta = 0, \quad 1 \le k \le -2\kappa-1\,,$$

i.e.

$$\int\limits_{|\zeta|=1}\varphi(\zeta)e^{\mathrm{Im}\gamma(\zeta)}\zeta^{\kappa+k-1}d\zeta = 0, \quad 1 \le k \le -2\kappa-1\,,$$

or equivalently

$$\int\limits_{|\zeta|=1} \varphi(\zeta)e^{\operatorname{Im}\gamma(\zeta)}\zeta^{-k-1}d\zeta = 0, \quad \kappa+1 \le k \le -\kappa-1\,.$$

The solution then especially for $|z|<1$, similarly as before is

$$\phi^+(z) = \frac{X^+(z)}{2\pi i}\int\limits_{|\zeta=1} \frac{g(\zeta)}{X^+(\zeta)}\frac{d\zeta}{\zeta-z} = \frac{e^{i\gamma(z)}}{2\pi i}\int\limits_{|\zeta|=1}\varphi(\zeta)e^{\operatorname{Im}\gamma(\zeta)}\zeta^{\kappa}\left(\frac{\zeta+z}{\zeta-z}+1\right)\frac{d\zeta}{\zeta}\,.$$

As

$$\begin{aligned}
\frac{\zeta+z}{\zeta-z}\frac{\zeta^\kappa-z^\kappa}{\zeta} &= \frac{\zeta+z}{\zeta}\frac{\frac{1}{z}-\frac{1}{\zeta}}{\zeta-z}\frac{(\frac{1}{\zeta})^{-\kappa}-(\frac{1}{z})^{-\kappa}}{\frac{1}{\zeta}-\frac{1}{z}} \\
&= -\frac{\zeta+z}{\zeta^2 z}\sum_{k=0}^{-\kappa-1}\left(\frac{1}{z}\right)^k\left(\frac{1}{z}\right)^{-\kappa-1-k} \\
&= -\frac{1}{\zeta}\sum_{k=0}^{-\kappa-1}(\zeta^{-k}z^{\kappa+k}+\zeta^{-k-1}z^{\kappa+k+1})
\end{aligned}$$

and

$$\zeta^\kappa - z^\kappa = \left(\frac{1}{\zeta}-\frac{1}{z}\right)\frac{(\frac{1}{\zeta})^{-\kappa}-(\frac{1}{z})^{-\kappa}}{\frac{1}{\zeta}-\frac{1}{z}} = \sum_{k=0}^{-\kappa-1}(\zeta^{-k-1}z^{\kappa+k+1}-\zeta^{-k}z^{\kappa+k})$$

we find because of the solvability conditions

$$\int\limits_{|\zeta|=1}\varphi(\zeta)e^{\operatorname{Im}\gamma(\zeta)}\zeta^\kappa\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}$$

$$= z^\kappa\int\limits_{|\zeta|=1}\varphi(\zeta)e^{\operatorname{Im}\gamma(\zeta)}\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} - z^{\kappa+1}\int\limits_{|\zeta|=1}\varphi(\zeta)e^{\operatorname{Im}\gamma(\zeta)}\zeta^{-1}\frac{d\zeta}{\zeta}\,,$$

$$\int\limits_{|\zeta|=1}\varphi(\zeta)e^{\operatorname{Im}\gamma(\zeta)}\zeta^\kappa\frac{d\zeta}{\zeta}$$

$$= z^\kappa\int\limits_{|\zeta|=1}\varphi(\zeta)e^{\operatorname{Im}\gamma(\zeta)}\frac{d\zeta}{\zeta} + z^{\kappa+1}\int\limits_{|\zeta|=1}\varphi(\zeta)e^{\operatorname{Im}\gamma(\zeta)}\zeta^{-1}\frac{d\zeta}{\zeta}\,.$$

Here in both formulas the last integral vanishes if $\kappa<-1$, for $\kappa=-1$ it might not. But adding both formulas we get

$$\phi^+(z) = \frac{e^{i\gamma(z)}z^\kappa}{2\pi i}\int\limits_{|\zeta|=1}\varphi(\zeta)e^{\operatorname{Im}\gamma(\zeta)}\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}$$

which as before can be shown to solve (1.4.2).

Because the SCHWARZ integral is an analytic function especially in $z = 0$, from the above solvability conditions only those for $0 \le k \le -\kappa - 1$ are important for ϕ^+ to be regular in $z = 0$. But because φ is a real function taking the complex conjugate of

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \varphi(\zeta) e^{\operatorname{Im}\gamma(\zeta)} \zeta^{-k} \frac{d\zeta}{\zeta} = 0, \quad 0 \le k \le -\kappa - 1 ,$$

this leads to the same conditions for $\kappa + 1 \le k \le 0$.

The RIEMANN–HILBERT problem will now be studied using the SCHWARZ operator directly. At first we modify the DIRICHLET problem by allowing the solution to have an isolated pole of order not greater than a given natural number $n \ge 0$ at the origin.

Lemma 7. *The general solution to the* DIRICHLET *problem*

$$\operatorname{Re} w(\zeta) = \varphi(\zeta) \quad \text{on} \quad |\zeta| = 1$$

in the space of functions analytic in $\mathbb{D}\backslash\{0\}$ having a pole at most of order $n \in \mathbb{N}$ at $z = 0$ is

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \varphi(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} + ic_0 + \sum_{k=1}^{n} (c_k z^k - \overline{c_k} z^{-k})$$

with arbitrary coefficients $c_0 \in \mathbb{R}, c_k \in \mathbb{C}$ $(1 \le k \le n)$.

Proof. We only need to consider the homogeneous problem. If it is solvable the solution has the form

$$P(z) = \sum_{k=-n}^{+\infty} c_k z^k \; (|z| < 1), \quad c_k = a_k + ib_k \in \mathbb{C} \;\; (-n \le k) ,$$

and satisfying at $z = e^{i\vartheta}$

$$\operatorname{Re} P(e^{i\vartheta}) = \operatorname{Re} \sum_{k=-n}^{+\infty} c_k e^{ik\vartheta} = \sum_{k=-n}^{+\infty} \{a_k \cos k\vartheta - b_k \sin k\vartheta\}$$

$$= a_0 + \sum_{k=1}^{n} \{(a_k + a_{-k}) \cos k\vartheta + (b_{-k} - b_k) \sin k\vartheta\} + \sum_{k=n+1}^{+\infty} \{a_k \cos k\vartheta - b_k \sin k\vartheta\} = 0 .$$

Thus

$$a_0 = 0 \quad a_k + a_{-k} = 0, \quad b_{-k} - b_k = 0 \quad (1 \le k \le n) , \; a_k = b_k = 0 \quad (n + 1 \le k) ,$$

i.e.

$$c_{-k} = -\overline{c_k} \quad (0 \le k \le n), \quad c_k = 0 \quad (n+1 \le k)$$

and

$$P(z) = ib_0 + \sum_{k=1}^{n} \{c_k z^k - \overline{c_k} z^{-k}\} .$$

This function, obviously, has a pole at most of order n at $z = 0$ and its real part vanishes at $|z| = 1$. A special solution to the inhomogeneous problem is given by (1.2.2) with $c = 0$.

Remark. $P(z)$ is a solution to the homogeneous DIRICHLET problem for $\mathbb{C} \setminus \mathbb{D}$, too. In that case ∞ is a pole at most of order n. The homogeneous DIRICHLET problem in the class of analytic functions which vanish at the origin (infinity) only is trivially solvable. Moreover, this result is true for any point of $\mathbb{D}$ (of $\mathbb{C} \setminus \mathbb{D}$) replacing $z = 0$. Let $\omega(z)$ be a conformal mapping from $\mathbb{D}$ onto itself mapping z_0 onto 0. Then

$$\widetilde{P}(z) := ic_0 + \sum_{k=1}^{n} \{c_k \omega^k(z) - \overline{c_k} \omega^{-k}(z)\}, \quad c_0 \in \mathbb{R}, c_k \in \mathbb{C}\ , \ 1 \le k \le n\ ,$$

is a solution of the homogeneous DIRICHLET problem with a pole at z_0, and z_0 cannot be a zero of $\widetilde{P}$ if $\widetilde{P}$ is not identically zero. This even holds when ω is the conformal mapping from some domain onto $\mathbb{D}$.

Corollary 1. *The general solution to the* DIRICHLET *problem for analytic functions in the unit disc* $\mathbb{D}$ *is*

$$w(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \varphi(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} + ic_0, \quad c_0 \in \mathbb{R}\ .$$

Lemma 8. *Let D be a simply connected domain in $\mathbb{C}$ with* GREEN *function $g(z, z_0)$. Then the solution of the* DIRICHLET *problem for analytic functions is*

$$w(z) = -\frac{1}{2\pi} \int_{\partial D} \varphi(\zeta) \frac{\partial M(\zeta, z)}{\partial \boldsymbol{n}_\zeta} ds_\zeta + ic_0, \quad c_0 \in \mathbb{R}\ ,$$

where $M(z, z_0) = g(z, z_0) + ih(z, z_0)$ is the complex GREEN *function.*

Proof. From the DIRICHLET condition

$$u = \operatorname{Re} w = \varphi \quad \text{on} \quad \partial D$$

the function u is known as

$$u(z) = -\frac{1}{2\pi} \int_{\partial D} \varphi(\zeta) \frac{\partial g(\zeta, z)}{\partial \boldsymbol{n}_\zeta} ds_\zeta, \quad z \in D\ .$$

The conjugate harmonic function v related to u by the CAUCHY–RIEMANN system

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{in} \quad D\,,$$

is locally given by

$$v(z) = \int_a^z \left\{ \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right\} + c_0 = \int_a^z \left\{ -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right\} + c_0, \quad c_0 \in \mathbb{R}\,.$$

If D is simply connected then the integral on the right–hand side is path–independent because the integrability condition

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is satisfied. Introducing the integral representation for u into this integral and interchanging differentiation and integration with one another leads to

$$v(z) = -\frac{1}{2\pi} \int_{\partial D} \varphi(\zeta) \frac{\partial h(\zeta, z)}{\partial \boldsymbol{n}_\zeta} ds_\zeta + c_0,$$

with, see p. 32,

$$h(\zeta, z) = \int_a^\zeta \frac{\partial g(t, z)}{\partial \boldsymbol{n}_t} ds_t + c, \quad c \in \mathbb{R}\,.$$

Thus $w = u + iv$ is representable as claimed in the lemma.

Theorem 17. *The general* RIEMANN–HILBERT *problem for a simply connected domain $D, 0 \in D$ with* SCHWARZ *operator S is solvable for non–negative index $\kappa \geq 0$. The general solution then is*

$$w(z) = z^\kappa e^{i\gamma(\zeta)} \left[(S\widetilde{\varphi})(z) + ic_0 + \sum_{k=1}^{\kappa} (c_k \omega^k(z) - \overline{c_k} \omega^{-k}(z)) \right], \quad z \in D\,,$$

where

$$\gamma(z) \;:=\; (S \arg(\zeta^{-\kappa} \lambda(\zeta)))(z), \quad z \in D\,,$$

$$\widetilde{\varphi}(\zeta) \;:=\; e^{\operatorname{Im}\gamma(\zeta)} |\zeta|^{-\kappa} \varphi(\zeta), \quad \zeta \in \partial D\,,$$

and ω is a conformal mapping from D onto $\mathbb{D}, \omega(0) = 0, c_0 \in \mathbb{R}, c_k \in \mathbb{C}, 1 \leq k \leq \kappa$, are arbitrary constants.
For negative index $\kappa < 0$ the problem is solvable if and only if the analytic function $S\widetilde{\varphi}$ has a zero of order $-\kappa$ at $z = 0$. If these $-\kappa$ conditions are satisfied then

$$w(z) = z^\kappa e^{i\gamma(\zeta)} (S\widetilde{\varphi})(z), \quad \zeta \in D\,,$$

is the solution.

Remark. When the SCHWARZ operator is explicitly known then the conditions for the problem to be solvable in the negative index case can be written down explicitly. For the unit disc for example, they are

$$\int\limits_{|\zeta|=1} \tilde{\varphi}(\zeta)\frac{d\zeta}{\zeta^{k+1}} = 0, \quad 0 \le k \le -\kappa - 1 .$$

Proof. Because $0 \notin \partial D$ and $\int\limits_{\partial D} d\log\{\zeta^{-\kappa}\lambda(\zeta)\} = 0$ then $\log\{\zeta^{-\kappa}\lambda(\zeta)\}$ is a single-valued continuous function on ∂D and

$$\gamma(z) := (S\arg(\zeta^{-\kappa}\lambda(\zeta)))(z)$$

is an analytic function in D satisfying

$$\operatorname{Re}\gamma(\zeta) = \arg\{\zeta^{-\kappa}\lambda(\zeta)\} = \arg\lambda(\zeta) - \kappa\arg\zeta \quad \text{on} \quad \partial D .$$

Thus on ∂D

$$e^{-i\gamma(\zeta)} = e^{Im\gamma(\zeta) - i\arg\lambda(\zeta) + i\kappa\arg\zeta} = \overline{\lambda(\zeta)}\zeta^{\kappa}|\zeta|^{-\kappa}e^{\operatorname{Im}\gamma(\zeta)}$$

and for a solution w

$$\begin{aligned}\operatorname{Re}\{\zeta^{-\kappa}e^{-i\gamma(\zeta)}w(\zeta)\} &= e^{\operatorname{Im}\gamma(\zeta)}|\zeta|^{-\kappa}\operatorname{Re}\{\overline{\lambda(\zeta)}w(\zeta)\}\\ &= e^{\operatorname{Im}\gamma(\zeta)}|\zeta|^{-\kappa}\varphi(\zeta) =: \tilde{\varphi}(\zeta) .\end{aligned}$$

As $z^{-\kappa}e^{-i\gamma(z)}w(z)$ will be an analytic function in D with the possible exception of $z = 0$, where eventually in the case $\kappa > 0$ there will be a pole of order not greater than κ. Using Lemma 7 for $\kappa \ge 0$ we get the above form of the solution. If $\kappa < 0$ we have to find the solution within the class of analytic functions having a zero of order $-\kappa$ in $z = 0$. But the solution of the DIRICHLET problem

$$\operatorname{Re}\{\zeta^{-\kappa}e^{-i\gamma(\zeta)}w(\zeta)\} = \tilde{\varphi}(\zeta), \quad \zeta \in \partial D ,$$

then is uniquely given by $S\tilde{\varphi}$. This leads to the solvability conditions. If they are satisfied then, obviously,

$$w(z) = z^{\kappa}e^{i\gamma(z)}(S\tilde{\varphi})(z)$$

is an analytic function in all of D satisfying

$$\operatorname{Re}\{\overline{\lambda(\zeta)}w(\zeta)\} = \varphi(\zeta), \quad \zeta \in \partial D .$$

In order to get a uniquely given solution in case $\kappa \ge 0$ we impose some side conditions on the solution.

Corollary 2. *For nonnegative index the* RIEMANN–HILBERT *problem together with the side conditions*

$$\operatorname{Im}\{\overline{\lambda(a_k)}w(a_k)\} = b_k, \quad 0 \le k \le 2\kappa \ ,$$

is uniquely solvable. Here $a_k \in \partial D, a_k \neq a_l$ $(k \neq l)$, are given points and $b_k \in I\!R$ are prescribed, $0 \le k \le 2\kappa$.

Proof. It has to be shown that the free coefficients $c_0 \in I\!R, c_k \in \mathbb{C}\ , 1 \le k \le \kappa$, are uniquely given by the side conditions. We have

$$\begin{aligned} b_k &= \operatorname{Im}\{\overline{\lambda(a_k)}w(a_k)\} = \operatorname{Im}\{a_k^{\kappa}e^{i\gamma(a_k)}\overline{\lambda(a_k)}[(S\widetilde{\varphi})(a_k) + Q_\kappa(a_k)]\} \\ &= e^{-\operatorname{Im}\gamma(a_k)}|a_k|^{\kappa}\left[\operatorname{Im} S\widetilde{\varphi}(a_k) + c_0 + \sum_{\mu=1}^{\kappa}\operatorname{Im}\{c_\mu\omega^\mu(a_k) - \overline{c_\mu}\omega^{-\mu}(a_k)\}\right] , \end{aligned}$$

where

$$Q_\kappa(z) := ic_0 + \sum_{k=1}^{\kappa}[c_k\omega^k(z) - \overline{c_k}\omega^{-k}(z)] \ .$$

Thus

$$c_0 - i\sum_{\mu=1}^{\kappa}\{c_\mu\omega^\mu(a_k) - \overline{c_\mu}\omega^{-\mu}(a_k)\} = b_k e^{\operatorname{Im}\gamma(a_k)}|a_k|^{-\kappa} - \operatorname{Im} S\widetilde{\varphi}(a_k) =: \widetilde{b_k}$$

or with $z_k := \omega(a_k) \in \partial I\!D, z_k \neq z_l, k \neq l, d_\mu := -ic_\mu, 1 \le \mu \le \kappa$,

$$c_0 + \sum_{\mu=1}^{\kappa}\{d_\mu z_k^\mu + \overline{d_\mu}z_k^{-\mu}\} = \widetilde{b_k}, \quad 0 \le k \le 2\kappa \ .$$

The determinant of this linear system is

$$\begin{bmatrix} 1 & z_0 & \dots & z_0^\kappa & z_0^{-1} & \dots & z_0^{-\kappa} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & z_{2\kappa} & \dots & z_{2\kappa}^\kappa & z_{2\kappa}^{-1} & \dots & z_{2\kappa}^{-\kappa} \end{bmatrix} = (-1)^{\sum_{\nu=\kappa+1}^{2\kappa}\nu}\begin{bmatrix} 1 & z_0 & \dots & z_0^{2\kappa} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & z_{2\kappa} & \dots & z_{2\kappa}^{2\kappa} \end{bmatrix}\prod_{\nu=0}^{2\kappa} z_\nu^{-\kappa}$$

$$= (-1)^{\frac{3\kappa^2+\kappa}{2}}\prod_{0\le i\le k\le 2\kappa}(z_i - z_k)\prod_{\nu=0}^{2\kappa} z_\nu^{-\kappa} \neq 0$$

and hence the system is uniquely solvable. From the fact $|z_k| = 1$ and $b_k \in I\!R$ by using the CRAMER rule the solution $\overline{c_\mu}$ can be shown to be the complex conjugate of c_μ.

In order to handle the solvability conditions for $\kappa < 0$ the boundary condition is modified.

Definition 9. *The* RIEMANN–HILBERT *problem*

$$\mathrm{Re}\,\{\overline{\lambda(\zeta)}w(\zeta)\} = \varphi(\zeta) + h(\zeta), \quad \zeta \in \partial D\,,$$

where h is identically zero if $\kappa \geq 0$ and for $\kappa < 0$

$$h(z) = \sum_{k=\kappa+1}^{-\kappa} h_k \omega^k(z), \quad z \in \overline{D}\,,$$

is called the modified RIEMANN–HILBERT *boundary value problem. The coefficients h_k are restricted to*

$$h_{-k} = \overline{h_k}, \quad |k| \leq -\kappa - 1\,, \tag{1.4.3}$$

and have to be determined properly so that the solvability conditions are satisfied.

Theorem 18. *The modified* RIEMANN–HILBERT *problem is uniquely solvable for $\kappa < 0$.*

Remark. Together with w the h_k have to be determined. Because for $z \in \partial D$ we have $\omega(z) \in \partial \mathbb{D}$, so that for $z \in \partial D$

$$h(z) = h_0 + \sum_{k=1}^{-\kappa-1} \{h_k \omega^k(z) + h_{-k}\omega^{-k}(z)\} = \mathrm{Re}\, h_0 + 2\mathrm{Re} \sum_{k=1}^{-\kappa-1} h_k \omega^k(z)$$

is real. Observe $h_0 = \overline{h_0} = \mathrm{Re}\, h_0$.
If the h_k are found then the solution w to the RIEMANN–HILBERT problem

$$\mathrm{Re}\,\{\overline{\lambda(\zeta)}w(\zeta)\} = \varphi(\zeta) + h(\zeta) \quad \text{on} \quad \partial D\,,$$

is uniquely given by

$$w(z) = z^\kappa e^{i\gamma(z)} S(e^{\mathrm{Im}\gamma(\zeta)}|\zeta|^{-\kappa}(\varphi(\zeta) + h(\zeta)))(z), \quad z \in D\,.$$

Proof. The solvability conditions for the modified problem in the case $D = \mathbb{D}$ are

$$\int\limits_{|\zeta|=1} e^{\mathrm{Im}\gamma(\zeta)}|\zeta|^{-\kappa}(\varphi(\zeta) + h(\zeta))\zeta^{-l-1}d\zeta = 0, \quad 0 \leq l \leq -\kappa - 1\,.$$

We will only consider this case. It has to be shown that the h_k are uniquely determined by these equations. Taking the complex conjugate of

$$\sum_{k=\kappa+1}^{-\kappa-1} h_k \int\limits_{|\zeta|=1} \zeta^{k-l-1} e^{\mathrm{Im}\gamma(\zeta)} d\zeta = -\int\limits_{|\zeta|=1} \varphi(\zeta) e^{\mathrm{Im}\gamma(\zeta)} \zeta^{-l-1} d\zeta, \quad 0 \leq l \leq -\kappa - 1\,,$$

and replacing k by $-k$ gives

$$\sum_{k=\kappa+1}^{-\kappa-1} h_k \int\limits_{|\zeta|=1} \zeta^{k+l-1} e^{\mathrm{Im}\gamma(\zeta)} d\zeta = - \int\limits_{|\zeta|=1} \varphi(\zeta) e^{\mathrm{Im}\gamma(\zeta)} \zeta^{l-1} d\zeta, \quad 0 \le l \le -\kappa - 1 .$$

Hence, we have to solve the linear system

$$\sum_{k=\kappa+1}^{-\kappa-1} h_k \int\limits_{|\zeta|=1} \zeta^{k-l-1} e^{\mathrm{Im}\gamma(\zeta)} d\zeta = - \int\limits_{|\zeta|=1} \varphi(\zeta) e^{\mathrm{Im}\gamma(\zeta)} \zeta^{-l-1} d\zeta, \quad |l| \le -\kappa - 1 .$$

The determinant of this system is the GRAM determinant of the system

$$\{\zeta^{\nu} e^{\mathrm{Im}\gamma(\zeta)} : |\nu| \le -2\kappa - 2\} .$$

Because this system is linearly independent its GRAM determinant does not vanish, see [Cohi53], p. 62. We will give another function theoretic proof for this system to be uniquely solvable. Consider the homogeneous system

$$\sum_{k=\kappa+1}^{-\kappa-1} h_k \int\limits_{|\zeta|=1} e^{\mathrm{Im}\gamma(\zeta)} \zeta^{k-l-1} d\zeta = 0, \quad 0 \le l \le -\kappa - 1 ,$$

and assume it is not trivially solvable. Then

$$h(z) := \sum_{k=\kappa+1}^{-\kappa-1} h_k z^k \not\equiv 0 \quad \text{in} \quad \mathbb{D}, \quad \mathrm{Im}\, h(\zeta) = 0 \quad \text{on} \quad \partial\mathbb{D} ,$$

$$\int\limits_{|\zeta|=1} e^{\mathrm{Im}\gamma(\zeta)} h(\zeta) \zeta^{-l-1} d\zeta = 0, \quad 0 \le l \le -\kappa - 1 ,$$

such that

$$H(z) := \frac{1}{2\pi i} \int\limits_{|\zeta|=1} e^{\mathrm{Im}\gamma(\zeta)} h(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} = \frac{z^{-\kappa}}{\pi i} \int\limits_{|\zeta|=1} e^{\mathrm{Im}\gamma(\zeta)} h(\zeta) \zeta^{\kappa} \frac{d\zeta}{\zeta - z} ,$$

is an analytic function in $\mathbb{D}$ having a zero at the origin at least of order $-\kappa$. Hence, $\mathrm{Re}\, H(z) = 0$ describes near the origin $-\kappa$ level lines. To see this rewrite

$$H(z) = z^{-\kappa} \widehat{H}(z), \quad \widehat{H}(z) \quad \text{analytic}, \quad z = re^{i\varphi} ,$$

and note

$$\mathrm{Re}\, H(z) = r^{-\kappa} \cos(-\kappa\varphi) \mathrm{Re}\, \widehat{H}(re^{i\varphi}) - r^{-\kappa} \sin(-\kappa\varphi) \mathrm{Im}\, \widehat{H}(re^{i\varphi}) = 0 ,$$

from which

$$\cot(-\kappa\varphi) = \frac{\operatorname{Im} \widehat{H}(z)}{\operatorname{Re} \widehat{H}(z)} .$$

Letting r tend to 0 and assuming that the right-hand side tends to some finite limit we get

$$\varphi = \frac{\varphi_0 + k\pi}{-\kappa}, \quad 1 \leq k \leq -2\kappa .$$

The assumption on the existence of the limit is no restriction of generality because $\widehat{H}(0) \neq 0$ may be assumed or we may factor out another power of z from $\widehat{H}(z)$. If then $\operatorname{Re} \widehat{H}(0) = 0$ we know $\operatorname{Im} \widehat{H}(0) \neq 0$ and instead of $\cot(-\kappa\varphi)$ we pass to $\tan(-\kappa\varphi)$ leading to the same result.

Consider now these at least -2κ level lines. They cannot intersect themselves in $\overline{\mathbb{D}}$, because otherwise $H(z) \equiv 0$ would follow by the maximum principle and the CAUCHY–RIEMANN equations. Thus the level lines meet $\partial\mathbb{D}$ in points different from one another. As on $\partial\mathbb{D}$

$$\operatorname{Re} H(\zeta) = e^{\operatorname{Im} \gamma(\zeta)} h(\zeta), \quad \operatorname{Im} h(\zeta) = 0$$

in these at last -2κ points we have $h(\zeta) = 0$. But $z^{-\kappa-1}h(z)$ is a polynomial of degree at most $-2\kappa - 2$. Having at least -2κ zeroes this forces h to vanish identically. Thus the homogeneous linear system is only trivially solvable, $h_k = 0, |k| \leq -\kappa - 1$. This also shows the inhomogeneous system to be uniquely solvable.

Remark. If ∂D is HÖLDER continuous, i.e. consists of finitely many HÖLDER continuous simple closed JORDAN curves, and if λ and φ are HÖLDER continuous functions then the solution to the RIEMANN–HILBERT problem is HÖLDER continuous too when it exists. This can immediately be seen for the unit disc, see Theorem 5. If D is simply connected and ω is the conformal map from D into $\mathbb{D}$ and w the solution to the RIEMANN–HILBERT problem then $w \circ \omega^{-1}$ satisfies

$$\operatorname{Re} \{\overline{\lambda(\omega^{-1}(z))} w(\omega^{-1}(z))\} = \varphi(\omega^{-1}(z)), \quad z \in \partial\mathbb{D} .$$

From Kellogg's result, see [Golu69], chap. X, §1, which asserts the continuity of $d/dz\, \omega^{-1}(z)$ in $\overline{D}$ and of $d/d\zeta\, \omega(\zeta)$ in $\overline{\mathbb{D}}$, the coefficients of this boundary condition are seen to be HÖLDER continuous. Therefore $w \circ \omega^{-1}$ is HÖLDER continuous and thus w is too by the boundedness of $d/d\zeta\, \omega(\zeta)$ in $\mathbb{D}$.

If D is multiply connected w can be represented by the CAUCHY integral

$$w(z) = \frac{1}{2\pi i} \int\limits_{\partial D} w(\zeta) \frac{d\zeta}{\zeta - z} = \sum_{\mu=0}^{m} w_\mu(z), w_\mu(z) := \frac{1}{2\pi i} \int\limits_{\Gamma_\mu} w(\zeta) \frac{d\zeta}{\zeta - z}, \quad 0 \leq \mu \leq m .$$

While w_0 is analytic in the bounded domain D_0 with $\partial D_0 = \Gamma_0, w_\mu, 1 \leq \mu \leq m$, is analytic in the unbounded domain D_μ containing z_∞ with $\partial D_\mu = \Gamma_\mu$. Because these domains are simply connected and contain the curves Γ_ν for $0 \le \nu \le m, \nu \ne \mu, w_\mu$ is HÖLDER continuous not only on Γ_μ but also in the other Γ_ν, i.e. on all of Γ. Thus w is HÖLDER continuous on Γ and hence on $\overline{D}$.

For multiply connected bounded domains only two simple RIEMANN–HILBERT problems will be considered, the DIRICHLET problem and a problem which turns out to be "adjoint" to the DIRICHLET problem, see [Veku62], p. 228.

Definition 10. *The adjoint* RIEMANN–HILBERT *boundary condition to*

$$\mathrm{Re}\,\{\overline{\lambda(\zeta)}w(\zeta)\} = \varphi(\zeta), \quad \zeta \in \partial D\,,$$

is the homogeneous RIEMANN–HILBERT *boundary condition*

$$\mathrm{Re}\,\{\lambda(\zeta)\zeta'(s)\omega(\zeta)\} = 0, \quad \zeta = \zeta(s) \in \partial D\,,$$

where s is the arc length parameter of ∂D.

In order to motivate this definition let w and ω be solutions to the respective RIEMANN–HILBERT problems, i.e. w and ω are analytic functions satisfying the above conditions, respectively. From the CAUCHY theorem then especially

$$\mathrm{Re}\,\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\omega(\zeta)d\zeta = 0\,.$$

Observing

$$|\lambda(\zeta)| = 1, \quad \overline{\lambda(\zeta)}w(\zeta) = \varphi(\zeta) + i\mu(\zeta), \quad \zeta \in \partial D\,,$$

with a proper real function μ we get

$$\begin{aligned}
&\mathrm{Re}\int_{\partial D}(\mu(\zeta) - i\varphi(\zeta))\lambda(\zeta)\omega(\zeta)\zeta'(s)ds \\
&= \int_{\partial D}\mu(\zeta)\mathrm{Re}\,\{\lambda(\zeta)w(\zeta)\zeta'(s)\}ds + \int_{\partial D}\varphi(\zeta)\mathrm{Im}\,\{\lambda(\zeta)\omega(\zeta)\zeta'(s)\}ds \\
&= -i\int_{\partial D}\varphi(\zeta)\lambda(\zeta)\omega(\zeta)\zeta'(s)ds\,.
\end{aligned}$$

This partly gives the following result.

Theorem 19. *A necessary and sufficient condition for the inhomogeneous* RIEMANN–HILBERT *problem for analytic functions to be solvable is that*

$$\int_{\partial D}\varphi(\zeta)\lambda(\zeta)\omega(\zeta)d\zeta = 0 \tag{1.4.4}$$

for any solution ω to the adjoint problem.

Proof. The consideration from before shows that (1.4.4) is a necessary condition. In order to show this condition to be sufficient the theory of singular integral equations is needed, see [Veku62], pp. 230–236, where generalized analytic rather than just analytic functions are treated. Here these considerations are exploited for the case of analytic functions.

From $\operatorname{Re}\{\overline{\lambda}w\} = \varphi, \operatorname{Re}\{\lambda\zeta'\omega\} = 0$ on ∂D it follows $\overline{\lambda}w = \varphi + i\mu$ and $\lambda\zeta'\omega = i\chi$, i.e. $w = \lambda\varphi + i\lambda\mu$ and $\omega = i\overline{\lambda\zeta'}\chi$ on ∂D with real functions μ and χ. From the CAUCHY formula

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int_{\partial D}\lambda(\zeta)\varphi(\zeta)\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi}\int_{\partial D}\lambda(\zeta)\mu(\zeta)\frac{d\zeta}{\zeta - z} =: w_1(z) + w_2(z), z \in D\,, \\ \omega(z) &= \frac{1}{2\pi}\int_{\partial D}\overline{\lambda(\zeta)\zeta'(s)}\chi(\zeta)\frac{d\zeta}{\zeta - z}, z \in D\,, \end{aligned}$$

and by the PLEMELJ–SOKHOTZKI formula

$$\begin{aligned} w^+(\zeta) &= \frac{1}{2}\lambda(\zeta)(\varphi(\zeta) + i\mu(\zeta)) + w_1(\zeta) + w_2(\zeta)\,,\ z \in \partial D\,, \\ \omega^+(\zeta) &= \frac{i}{2}\overline{\lambda(\zeta)\zeta'(s)}\chi(\zeta) + \frac{1}{2\pi}\int_{\partial D}\overline{\lambda(t)t'(\sigma)}\chi(t)\frac{dt}{t-\zeta} \\ &= \frac{i}{2}\overline{\lambda(\zeta)\zeta'(s)}\chi(\zeta) + \frac{1}{2\pi}\int_{\partial D}\overline{\lambda(t)}\chi(t)\frac{d\sigma}{t-\zeta}\,,\ \zeta = \zeta(s) \in \partial D, t = t(\sigma)\,. \end{aligned}$$

Here the respective integrals are CAUCHY principal value integrals. From the boundary conditions then

$$\begin{aligned} \varphi &= \operatorname{Re}\{\overline{\lambda}w^+\} = \operatorname{Re}\frac{1}{2}(\varphi + i\mu) + \operatorname{Re}\{\overline{\lambda}(w_1 + w_2)\} \\ &= \frac{1}{2}\varphi + \operatorname{Re}\{\overline{\lambda}w_1\} + \operatorname{Re}\{\overline{\lambda}w_2\}\,, \\ 0 &= \operatorname{Re}\{\lambda\zeta'\omega^+\} = \operatorname{Re}\frac{i}{2}\chi + \int_{\partial D}\operatorname{Re}\left\{\frac{1}{2\pi}\,\frac{\lambda(\zeta)\zeta'(s)\overline{\lambda(t)}}{t-\zeta}\right\}\chi(t)d\sigma\,. \end{aligned}$$

Setting

$$K_1(\zeta, t) := \operatorname{Re}\frac{1}{2\pi}\,\frac{\overline{\lambda(\zeta)t'(\sigma)}\lambda(t)}{\zeta - t}$$

then

$$\begin{aligned}
\operatorname{Re}\{\overline{\lambda(\zeta)}w_2(\zeta)\} &= -\int_{\partial D} K_1(\zeta,t)\mu(t)d\sigma = \frac{1}{2}\varphi(\zeta) - \operatorname{Re}\{\overline{\lambda(\zeta)}w_1(\zeta)\} \\
&= \frac{1}{2}\varphi(\zeta) - \operatorname{Re}\left\{\overline{\lambda(\zeta)}\left[w_1^+(\zeta) - \frac{1}{2}\lambda(\zeta)\varphi(\zeta)\right]\right\} \\
&= \varphi(\zeta) - \operatorname{Re}\{\overline{\lambda(\zeta)}w_1^+(\zeta)\} \\
&= -\operatorname{Re}\{\overline{\lambda(\zeta)}w_1^-(\zeta)\} =: \varphi_0 .
\end{aligned}$$

This is a singular integral equation for μ

$$-\int_{\partial D} K_1(\zeta,t)\mu(t)d\sigma = \varphi_0 . \tag{1.4.5}$$

The preceding equation for χ is of the same kind, namely

$$\int_{\partial D} K_1(t,\zeta)\chi(t)d\sigma = 0 . \tag{1.4.6}$$

For treating these equations the following excerpt from the theory of singular integral equations is given, see [Veku62], p. 230-236, [Mipr86], chapter 3, [Musk53], chapter 2. For given $a, f \in C^\alpha(\partial D;\mathbb{C}), K \in C^\alpha(\partial D \times \partial D;\mathbb{C})$, $0 < \alpha \leq 1$,

$$\mathbf{K}\varphi := a(\zeta)\varphi(\zeta) + \frac{1}{\pi i}\int_{\partial D} K(\zeta,t)\varphi(t)\frac{dt}{t-\zeta} = f(\zeta),\ \zeta \in \partial D ,$$

is called a singular integral equation.

$$\mathbf{K}'\psi := a(\zeta)\psi(\zeta) - \frac{1}{\pi i}\int_{\partial D} K(t,\zeta)\psi(t)\frac{dt}{t-\zeta} = 0,\ \zeta \in \partial D ,$$

is called adjoint to $\mathbf{K}\varphi = f$. If

$$a(\zeta) + K(\zeta,\zeta) \neq 0, a(\zeta) - K(\zeta,\zeta) \neq 0 \text{ on } \partial D$$

then

$$\kappa := \frac{1}{2\pi}\int_{\partial D} d\arg\frac{a(\zeta) - K(\zeta,\zeta)}{a(\zeta) + K(\zeta,\zeta)}$$

is an entire number which is called the index of $\mathbf{K}$. If $\mathbf{K}\varphi = 0$ has k and $\mathbf{K}'\psi = 0$ has k' linearly independent solutions then both k and k' are finite numbers related by

$k - k' = \kappa$ with one another. The inhomogeneous equation $\mathbf{K}\varphi = f$ is solvable if and only if f satisfies

$$\int\limits_{\partial D} f(\zeta)\psi_j(\zeta)d\zeta = 0\,,\ 1 \le j \le k'\,,$$

for any basis $\{\psi_j := 1 \le j \le k'\}$ for the solution space to the adjoint equation $\mathbf{K}'\psi = 0$.
Applying this result for the above equations (1.4.5), (1.4.6), observing $|\lambda(\zeta)| = |\zeta'(s)| = 1$, and $\kappa = 0$ we see that (1.4.5) is solvable if and only if

$$\int\limits_{\partial D} \varphi_0(\zeta)\chi_j(\zeta)ds = 0\ ,\ 1 \le j \le k\,, \tag{1.4.7}$$

for a basis $\{\chi_j : 1 \le j \le k\}$ for the solution space to (1.4.6). If it can be shown that these conditions are equivalent to the conditions (1.4.4) then it is evident that (1.4.5) has a solution μ determining the solution w via $w = \lambda\varphi + i\lambda\mu$ on ∂D. From here w is uniquely given in D by the CAUCHY formula, see the remark below.
We consider some χ_j, and observe

$$\omega_j(z) = \frac{1}{2\pi}\int\limits_{\partial D}\overline{\lambda(\zeta)\zeta'(s)\chi_j(\zeta)}\frac{d\zeta}{\zeta - z} = \frac{1}{2\pi}\int\limits_{\partial D}\overline{\lambda(\zeta)\chi_j(\zeta)}\frac{ds}{\zeta - z}, 1 \le j \le k\,.$$

Let us assume there is a function ϕ_μ analytic in the bounded domain D_μ with $\partial D_\mu = \Gamma_\mu, 1 \le \mu \le m$, and ϕ_0 analytic in the unbounded domain D_0 with $\partial D_0 = \Gamma_0$ vanishing at infinity, $\phi_0(\infty) = 0$, such that $\phi_\mu(\zeta) = \overline{\lambda(\zeta)\zeta'(s)\chi_j(\zeta)}$ on Γ_μ for $0 \le \mu \le m$. Then by the CAUCHY formula

$$\omega_j(z) = \frac{1}{2\pi}\int\limits_{\Gamma_0}\frac{\phi_0(\zeta)}{\zeta - z}d\zeta - \sum_{\mu=1}^{m}\frac{1}{2\pi}\int\limits_{\Gamma_\mu}\frac{\phi_\mu(\zeta)}{\zeta - z}d\zeta = 0$$

in D, so that ω_j is the trivial solution to the homogeneous adjoint boundary value problem. Let $\chi_1, \dots, \chi_\ell$, $0 \le \ell \le k$, be those elements of the above basis corresponding to linearly independent solutions $\omega_1, \dots, \omega_\ell$. For the remaining χ_j the related ω_j is identically zero and hence by the PLEMELJ–SOKHOTZKI formula we have an analytic function ϕ_j in $\widehat{\mathbb{C}}\setminus\overline{D}$ satisfying $\phi_j(\infty) = 0$ and $\phi_j^-(\zeta) = -i\overline{\lambda(\zeta)\zeta'(s)\chi_j(\zeta)}$ on ∂D, i.e. $\chi_j(\zeta) = i\lambda(\zeta)\zeta'(s)\phi_j^-(\zeta)$ on ∂D. Thus ϕ_j is a solution to the so–called concomitant problem

$$\operatorname{Re}\{\lambda(\zeta)\zeta'(s)\phi_j^-(\zeta)\} = 0 \quad \text{on } \partial D$$

in the class of analytic functions in $\widehat{\mathbb{C}}\setminus\overline{D}$ vanishing at infinity. It has $k - \ell$ linearly independent solutions because the χ_j are linearly independent.
Let us now consider the conditions (1.4.7). Inserting

$$\chi_j(\zeta) = \begin{cases} -i\lambda(\zeta)\zeta'(s)\omega_j(\zeta) &,\ 1 \le j \le \ell\,, \\ i\lambda(\zeta)\zeta'(s)\phi_j^-(\zeta) &,\ \ell + 1 \le j \le k\,, \end{cases} \qquad \text{on } \partial D$$

leads for $1 \le j \le \ell$ to

$$\begin{aligned}\int_{\partial D} \varphi_0(\zeta)\chi_j(\zeta)ds &= -i\int_{\partial D} \varphi(\zeta)\lambda(\zeta)\zeta'(s)\omega_j(\zeta)ds\\ &\quad i\int_{\partial D} \mathrm{Re}\,\{\overline{\lambda(\zeta)}w_1^+(\zeta)\}\lambda(\zeta)\zeta'(s)\omega_j(\zeta)ds\\ &= -i\int_{\partial D} \varphi(\zeta)\lambda(\zeta)\omega_j(\zeta)d\zeta + \mathrm{Re}\left\{i\int_{\partial D} w_1^+(\zeta)\omega_j(\zeta)d\zeta\right\}.\end{aligned}$$

Because $w_1^+\omega_j$ is analytic in D the last integral vanishes so that (1.4.4) for $\omega = \omega_j$ implies

$$\int_{\partial D} \varphi(\zeta)\lambda(\zeta)\omega_j(\zeta)d\zeta = 0\,,\ 1 \le j \le \ell\,.$$

Similarly, for $\ell + 1 \le j \le k$ we get

$$\begin{aligned}\int_{\partial D} \varphi_0(\zeta)\chi_j(\zeta)ds &= -i\int_{\partial D} \mathrm{Re}\,\{\overline{\lambda(\zeta)}w_1^-(\zeta)\}\lambda(\zeta)\zeta'(s)\phi_j^-(\zeta)ds\\ &= -\mathrm{Re}\left\{i\int_{\partial D} w_1^-(\zeta)\phi_j^-(\zeta)d\zeta\right\} = 0\end{aligned}$$

again by the CAUCHY theorem, this time applied to the function $w_1^-\phi_j^-$ analytic in $\widehat{\mathbb{C}}\setminus\overline{D}$ and vanishing at infinity.
Thus (1.4.4) together with these last $k-\ell$ conditions imply (1.4.7) and hence (1.4.5) is solvable for μ.

Remark. Having determined μ as a solution to (1.4.5) it has to be verified that

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} \lambda(\zeta)(\varphi(\zeta) + i\mu(\zeta))\frac{d\zeta}{\zeta - z} = w_1(z) + w_2(z)\,,\ z \in D\,,$$

indeed solves the RIEMANN–HILBERT problem. (1.4.5) can be written as

$$\mathrm{Re}\,\overline{\lambda(\zeta)}(w_1^+(\zeta) + w_2(\zeta)) = \varphi(\zeta)\,.$$

This and the formula

$$w_2^+(\zeta) = \frac{i}{2}\lambda(\zeta)\mu(\zeta) + w_2(\zeta)$$

show

$$\mathrm{Re}\,\{\overline{\lambda(\zeta)}w^+(\zeta)\} = \varphi(\zeta)\,.$$

The index of the adjoint problem to the RIEMANN–HILBERT problem is

$$\frac{1}{2\pi i}\int_{\partial D} d\log\overline{\lambda(\zeta)\zeta'(s)} = -\frac{1}{2\pi i}\int_{\partial D} d\log\lambda(\zeta) - \frac{1}{2\pi i}\int_{\partial D} d\log\zeta'(s)\,.$$

Here the first term on the right-hand side is the negative index of the original problem, $-\kappa$, while the second term is $m-1$ where $m+1$ is the connectivity of the multiply connected bounded domain D. $\zeta'(s)$ is the tangent unit vector on ∂D. When ζ varies along Γ_0 in the positive direction the argument of $\zeta'(s)$ increases from a starting angle φ_0 to $\varphi_0+2\pi$. Because of the negative orientation of the $\Gamma_\mu, 1\le\mu\le m$, similarly on Γ_μ the tangent vector starting with some angle φ_μ ends up with $\varphi_\mu-2\pi$ after a revolution of ζ around Γ_μ.
Later on we will be interested in the adjoint problem to the DIRICHLET problem i.e. where $\lambda(\zeta)\equiv 1$. Then the index of the adjoint problem is $m-1$ and the solution space to this special RIEMANN–HILBERT problem has dimension m, see [Veku62], p. 259. To show this m over $\mathbb{R}$ linearly independent solutions are constructed which span the solution space.
Let $u_\mu, 1\le\mu\le m$, be the so-called harmonic measure of the boundary curve Γ_μ with respect to the domain D, i.e. the uniquely given harmonic function satisfying the boundary condition

$$u_\mu=\delta_{\mu\nu}:=\begin{cases}0, & \nu\neq\mu\\ 1, & \nu=\mu\end{cases}\qquad \text{on}\quad \Gamma_\mu, 0\le\nu\le m\,.$$

Obviously,

$$u_\mu(z)=\frac{1}{2\pi}\int_{\Gamma_\mu}\frac{\partial g(\zeta,z)}{\partial \boldsymbol{n}_\zeta}ds_\zeta\,.$$

On any level set $\{z: u_\mu(z)=const.\}$, especially on ∂D we have

$$\frac{du_\mu(z)}{ds}=u_{\mu z}(z)z'(s)+u_{\mu\bar z}(z)\overline{z'(s)}=2\mathrm{Re}\,\{z'(s)u_{\mu z}(z)\}=0$$

when s denotes the arc length parameter. Thus the function (see section 1.2)

$$\phi_\mu(z):=2u_{\mu z}(z)=\frac{1}{i}\int_{\Gamma_\mu}L(z,\zeta)d\zeta$$

which because of u_μ being harmonic is analytic, is a solution for the problem adjoint to the DIRICHLET problem.
Let $\alpha_\mu\in\mathbb{R}$ be such that $\sum_{\mu=1}^m\alpha_\mu\phi_\mu(z)=0$ in D. From $\partial/\partial z\sum_{\mu=1}^m\alpha_\mu u_\mu(z)=0$ it follows that the real function $\sum_{\mu=1}^m\alpha_\mu u_\mu(z)$ must be antiholomorphic i.e. a holomorphic function of $\bar z$ and hence constant. But this constant has to be equal to

any of the α_μ and at the same time equal to zero as can be seen by restricting z to $\Gamma_\nu, 0 \leq \nu \leq m$. It can be shown that the number of linearly independent solutions is at most m, [Veku62], p. 259. Thus any solution to the adjoint problem is a real linear combination of the $\phi_\mu, 1 \leq \mu \leq m$. This can be shown directly. Let w be a solution to the adjoint problem $\text{Re}\{z'(s)w(z)\} = 0$ to the DIRICHLET problem. Let u be a harmonic function in D such that $2u_z = w$. For some fixed point $a \in D$ we have

$$\begin{aligned} u(z) &= \int_a^z \{u_\zeta(\zeta)d\zeta + u_{\overline{\zeta}}(\zeta)d\overline{\zeta}\} + u(a) \\ &= \frac{1}{2}\int_u^z \{w(\zeta)d\zeta + \overline{w(\zeta)}d\overline{\zeta}\} + u(a) = \text{Re}\int_a^z w(\zeta)d\zeta + u(a)\,. \end{aligned}$$

This functions single-valued. To see this let $\widetilde{\Gamma}_\mu$ be some simply closed curve in D homological to Γ_μ. From the CAUCHY theorem

$$\int_{\widetilde{\Gamma}_\mu} w(\zeta)d\zeta = \int_{\Gamma_\mu} w(\zeta)d\zeta$$

we have

$$\int_{\widetilde{\Gamma}_\mu} \{u_\zeta(\zeta)d\zeta + u_{\overline{\zeta}}(\zeta)d\overline{\zeta}\} = \int_{\Gamma_\mu} \{u_\zeta(\zeta)d\zeta + u_{\overline{\zeta}}(\zeta)d\overline{\zeta}\} = \int_{\Gamma_\mu} \frac{du(\zeta(s))}{ds}ds = 0$$

because

$$2\frac{du(\zeta(s))}{ds} = w(\zeta)\zeta'(s) + \overline{w(\zeta)\zeta'(s)} = 2\text{Re}\{\zeta'(s)w(\zeta)\} = 0 \quad \text{on } \Gamma_\mu\,.$$

This last equation also shows $u(\zeta)$ to be constant on any Γ_μ. Let $u(\zeta) = \alpha_\mu \in \mathbb{R}$ for $0 \leq \mu \leq m$. Then $u_0 := u - \alpha_0$ is a harmonic function in D satisfying $u_0 = (\alpha_\mu - \alpha_0)u_\mu$ on Γ_μ for $0 \leq \mu \leq m$ and hence, obviously, $u_0 = \sum_{\mu=1}^m (\alpha_\mu - \alpha_0)u_\mu$ in D as a consequence of the maximum–minimum principle. Thus $w = 2u_z = 2u_{0_z} = \sum_{\mu=1}^m (\alpha_\mu - \alpha_0)\phi_\mu$. The harmonic functions u_μ give some insight into the nature of the multi-valuedness of the harmonic conjugate $h(z,\zeta)$ to the GREEN function $g(z,\zeta)$ of D. This multi-valued function defined as

$$h(z, z_0) = \int_a^z \frac{\partial g(\zeta, z_0)}{\partial n_\zeta} ds_\zeta + const.$$

acquires an increment equal to $2\pi u_\mu(z_0)$ as z describes a closed curve $\widetilde{\Gamma}_\mu$ homological to Γ_μ which does not contain the point z_0. To see this apply the first GREEN formula, Theorem 11, to $u = 1$ and the harmonic function $v = g(\cdot, z_0)$ in the ring domain $\widetilde{D}_\mu \subset D$ with $\partial\widetilde{D}_\mu = \widetilde{\Gamma}_\mu \cup \Gamma_\mu$ giving

$$0 = \int_{\widetilde{D}_\mu} \Delta_\zeta g(\zeta, z_0) d\xi d\eta = \int_{\Gamma_\mu} \frac{\partial g(\zeta, z_0)}{\partial n_\zeta} ds_\zeta - \int_{\Gamma_\mu} \frac{\partial g(\zeta, z_0)}{\partial n_\zeta} ds_\zeta .$$

If on the other hand z describes a simply closed curve $\widetilde{\Gamma}_\mu$ bounding a subset of D to which z_0 belongs then $h(z, z_0)$ acquires an increment equal to 2π. This can be seen by observing that $g(\zeta, z_0) + \log|\zeta - z_0|$ is harmonic in the subset of D bounded by $\widetilde{\Gamma}_\mu$, see the proof of Theorem 13.

Corollary 3. *1. The* DIRICHLET *problem for analytic functions w in D*

$$\operatorname{Re} w(z) = h_0(z) \quad \text{on} \quad \partial D$$

is solvable if and only if with the basis elements $\phi_\mu, 1 \le \mu \le m$, of the solution space to the adjoint problem

$$\operatorname{Re}\{z'(s)\phi(z)\} = 0$$

the conditions

$$\int_{\partial D} h_0(z)\phi_\mu(z) dz = 0 , \quad 1 \le \mu \le m ,$$

are satisfied.

2. The RIEMANN–HILBERT *problem for analytic functions ϕ in D*

$$\operatorname{Re}\{z'(s)\phi(z)\} = h^0(z) \quad \text{on} \quad \partial D$$

is solvable if and only if

$$\int_{\partial D} h^0(z) ds_z = 0 .$$

Remark. The solution to the DIRICHLET problem is given in connection with (1.2.3). The homogeneous DIRICHLET problem has the solution $ic, c \in I\!R$. If the RIEMANN–HILBERT problem

$$\operatorname{Re}\{z'(s)\phi(z)\} = h^0(z) \quad \text{on} \quad \partial D$$

is solvable the general solution has the form

$$\phi(z) = \phi_0(z) + \sum_{\mu=1}^{m} \gamma_\mu \phi_\mu, \quad \gamma_\mu \in I\!R, \quad 1 \le \mu \le m ,$$

with the particular solution

$$\phi_0(z) := \frac{1}{\pi i} \int_{\partial D} \left(\underset{\sim}{\ell}\, (z,\zeta) + \frac{1}{\zeta - z} \right) h^0(\zeta) ds_\zeta, \quad \underset{\sim}{\ell}(z,\zeta) := \pi \int_a^\zeta \ell(z,t) dt .$$

Compare p. 37 for the definition of $\ell(z,\zeta)$. To verify this representation formula the form of the particular solution has to be deduced. Let $a \in \Gamma_0$ be that boundary point corresponding to $s = 0, a = z(0)$. Define the multi-valued analytic function

$$\Phi_0(z) := \int_a^z \phi_0(\zeta)d\zeta$$

for a particular solution ϕ_0. On ∂D it satisfies

$$\frac{d}{ds}\Phi_0(z) = z'(s)\phi_0(z) .$$

Defining $H(z(s))$ such that $H(a_0) = 0$ and

$$\frac{d}{ds}H(z(s)) = h^0(z(s))$$

then Φ_0 satisfies the DIRICHLET problem

$$\operatorname{Re}\Phi_0(z) = H(z), z \in \partial D .$$

Applying the representation formula for the solution to this DIRICHLET problem gives

$$\Phi_0(z) = \frac{1}{i}\int_{\partial D} J_L(z,\zeta)H(\zeta)d\zeta + ic .$$

Differentiating this and using

$$L(z,\zeta) = \frac{1}{\pi(z-\zeta)^2} - \ell(z,\zeta) ,$$

shows

$$\begin{aligned}
\phi_0(z) &= \frac{1}{i}\int_{\partial D} L(z,\zeta)H(\zeta)d\zeta = \frac{1}{\pi i}\int_{\partial D} \frac{H(\zeta)}{(\zeta-z)^2}d\zeta - \frac{1}{i}\int_{\partial D} \ell(z,\zeta)H(\zeta)d\zeta \\
&= -\frac{1}{\pi i}\int_{\partial D} \frac{\partial}{\partial\zeta}\left(\underset{\sim}{\ell}(z,\zeta) - \frac{1}{\zeta - z}\right) H(\zeta)d\zeta \\
&= \frac{1}{\pi i}\int_{\partial D} \left(\underset{\sim}{\ell}(z,\zeta) + \frac{1}{\zeta - z}\right) \frac{\partial H(\zeta)}{\partial\zeta}d\zeta ,
\end{aligned}$$

which is the above formula.

2. Inhomogeneous CAUCHY–RIEMANN systems

2.1 Integral representations

If $w = u + iv$ is analytic then u, v satisfy the CAUCHY–RIEMANN system. If $u, v \in C^1(D)$ satisfy this system, then w is analytic in D. Let

$$u_x = v_y, \quad u_y = -v_x \,.$$

Introducing the partial complex derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

then

$$2w_z = u_x + iv_x - iu_y + v_y = 2(u_x + iv_x) = w', 2w_{\overline{z}} = u_x + iv_x + iu_y - v_y = 0 \,.$$

Thus the complex form of the CAUCHY–RIEMANN system is

$$w_{\overline{z}} = 0 \,.$$

Lemma 9. *Let $u, v \in C^1(\overline{D}; I\!R), w = u + iv$. Then*

$$\begin{aligned}\int_{\partial D} w(z)dz + \int_D w_{\overline{z}} dz d\overline{z} &= 0, \\ \int_{\partial D} w(z)d\overline{z} - \int_D w_z(z) dz d\overline{z} &= 0 \,.\end{aligned}$$

Proof. From the GAUSS Theorem (Theorem 10) we have

$$\int_D (u_x + v_y) dx dy = \int_{\partial D} (u\,dy - v\,dx) \,.$$

From

$$\begin{aligned} w\,dz &= (u + iv)(dx + i\,dy) = u\,dx - v\,dy + i(u\,dy + v\,dx) \,, \\ w\,d\overline{z} &= (u + iv)(dx - i\,dy) = u\,dx + v\,dy - i(u\,dy - v\,dx) \end{aligned}$$

and

$$dz d\overline{z} = \frac{d(z, \overline{z})}{d(x, y)} dx dy = \begin{vmatrix} 1 & 1 \\ i & -i \end{vmatrix} dx dy = -2i dx dy$$

both formulas follow.

Remark. If w is analytic then from the first formula of Lemma 7 the CAUCHY integral theorem

$$\int_{\partial D} w(z)dz = 0$$

follows.

Theorem 20. $w \in C^1(\overline{D}; \mathbb{C})$. *Then*

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_{D} w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}, \quad \zeta = \xi + i\eta, z \in D\,. \tag{2.1.1}$$

Proof. For $z_0 \in D$ let

$$K_\varepsilon(z_0) := \{z : |z - z_0| < \varepsilon\} \subset D, \quad D_\varepsilon := D\backslash\overline{K_\varepsilon(z_0)}\,.$$

Applying Lemma 7,

$$\int_{\partial D_\varepsilon} w(\zeta)\frac{d\zeta}{\zeta - z_0} - 2i\int_{D_\varepsilon} w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z_0} = 0$$

follows. From

$$\int_{\partial D_\varepsilon} w(\zeta)\frac{d\zeta}{\zeta - z_0} = \int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z_0} - i\int_0^{2\pi} w(z_0 + \varepsilon e^{i\varphi})d\varphi$$

letting ε tend to zero (2.1.1) is obtained because the area integral exists as can be seen by introducing polar coordinates about z_0,

$$\int_{K_\varepsilon(z_0)} w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z_0} = \int_0^{\varepsilon}\int_0^{2\pi} w_{\overline{\zeta}}(z_0 + te^{i\varphi})e^{-i\varphi}d\varphi dt\,.$$

Remark. (2.1.1) is called CAUCHY–POMPEIU formula, see [Pomp13]. Similarly, from the second formula of Lemma 7

$$w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}} - \frac{1}{\pi}\int_{D} w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta} - \overline{z}}, \quad z \in D\,, \tag{2.1.1'}$$

follows. (2.1.1′) can be deduced from (2.1.1) by applying (2.1.1) to $\overline{w}$ and taking complex conjugation. In a similarly way the SCHWARZ–POISSON formula can be extended.

Theorem 21. $w \in C^1(\overline{\mathbb{D}};\mathbb{C})$. *Then*

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \operatorname{Re} w(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}$$

$$+\frac{1}{2\pi}\int\limits_{|\zeta|=1} \operatorname{Im} w(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{|\zeta|<1}\left\{\frac{w_{\bar\zeta}(\zeta)}{\zeta-z} + \frac{z\overline{w_{\bar\zeta}(\zeta)}}{1-z\bar\zeta}\right\}d\xi d\eta, \quad |z|<1\,. \tag{2.1.2}$$

Proof. Adding formula (2.1.1) for $D = \mathbb{D}$ and the complex conjugate of

$$0 = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} w(\zeta)\frac{\bar z d\zeta}{1-\bar z\zeta} - \frac{1}{\pi}\int\limits_{|\zeta|<1} w_{\bar\zeta}(\zeta)\frac{\bar z}{1-\bar z\zeta}d\xi d\eta, \quad |z|<1\,,$$

which follows directly from Lemma 7 for $D = \mathbb{D}$ we get

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\left\{\frac{w(\zeta)}{\zeta-z} + \frac{z\overline{w(\zeta)}}{\zeta((\zeta-z)}\right\}d\zeta$$

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1}\left\{\frac{w_{\bar\zeta}(\zeta)}{\zeta-z} + \frac{z\overline{w_{\bar\zeta}(\zeta)}}{1-z\bar\zeta}\right\}d\xi d\eta,\ |z|<1\,,$$

when using $\bar\zeta\, d\zeta = -\zeta\, d\bar\zeta$ on $|\zeta| = 1$. From

$$\frac{1}{\zeta-z} - \frac{1}{\zeta} = \frac{z}{\zeta(\zeta-z)} = \frac{1}{2\zeta}\left(\frac{\zeta+z}{\zeta-z} - 1\right)\,,$$

then

$$\int\limits_{|\zeta|=1}\left\{\frac{z(\zeta)}{\zeta-z} + \frac{w\overline{w(\zeta)}}{\zeta((\zeta-z)}\right\}d\zeta = \int\limits_{|\zeta|=1}\left\{\operatorname{Re} w(\zeta)\frac{\zeta+z}{\zeta-z} + i\operatorname{Im} w(\zeta)\right\}\frac{d\zeta}{\zeta}$$

follows.

Remark.

$$\frac{1}{2\pi}\int\limits_{|\zeta|=1} \operatorname{Im} w(\zeta)\frac{d\zeta}{\zeta} = ic_0, \quad c_0 \in \mathbb{R}\,.$$

If w is analytic in $\mathbb{D}$ then (2.1.2) is the SCHWARZ–POISSON formula. (2.1.2) may be called SCHWARZ–POISSON–POMPEIU formula. It may be given a more symmetric form by subtracting $i \operatorname{Im} w(0)$ from both sides giving, see [Behi93],

$$w(z) - i \operatorname{Im} w(0)$$

$$= \frac{1}{2\pi i} \int_{\partial D} \operatorname{Re} w(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} - \frac{1}{2\pi} \int_{\mathbb{D}} \left\{ \frac{w_{\bar\zeta}(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} + \frac{\overline{w_{\bar\zeta}(\zeta)}}{\bar\zeta} \frac{1 + z\bar\zeta}{1 - z\bar\zeta} \right\} d\xi d\eta \qquad (2.1.2')$$

In order to show that this formula will give a solution to the DIRICHLET problem for the equation $w_{\bar\zeta} = f$ for some given f in $\mathbb{D}$ we have to consider the area integral in (2.1.2). Generally we will study

$$Tf(z) := -\frac{1}{2\pi} \int_D f(\zeta) \frac{d\xi d\eta}{\zeta - z} .$$

Before doing this we will generalize the representation formula (2.1.2) to any simply connected smoothly bounded finite domain $D, 0 \in D$. Let ω be the conformal mapping from D onto $\mathbb{D}$ with $\omega(0) = 0, \omega'(0) > 0$. We will transform formula (2.1.2) for functions in D. In order to do so we use the first and second GREEN function of D, see section 1.2,

$$\begin{aligned} G^I(z,\zeta) &= -\frac{1}{2\pi} \log \left| \frac{\omega(\zeta) - \omega(z)}{1 - \omega(\zeta)\overline{\omega(z)}} \right| , \\ G^{II}(z,\zeta) &= -\frac{1}{2\pi} \log \left| (\omega(\zeta) - \omega(z))(1 - \omega(\zeta)\overline{\omega(z)}) \right| \end{aligned}$$

where here we add the factor $\dfrac{1}{2\pi}$. We have

$$\begin{aligned} G^I(z,\zeta) + G^{II}(z,\zeta) &= -\frac{1}{2\pi} \log |\omega(\zeta) - \omega(z)|^2 , \\ G^I(z,\zeta) - G^{II}(z,\zeta) &= \frac{1}{2\pi} \log |1 - \omega(\zeta)\overline{\omega(z)}|^2 , \\ \frac{\partial}{\partial\zeta}(G^I(z,\zeta) + G^{II}(z,\zeta)) &= -\frac{1}{2\pi} \frac{\omega'(\zeta)}{\omega(\zeta) - \omega(z)} , \\ \frac{\partial}{\partial\zeta}(G^I(z,\zeta) - G^{II}(z,\zeta)) &= -\frac{1}{2\pi} \frac{\omega'(\zeta)\overline{\omega(z)}}{1 - \omega(\zeta)\overline{\omega(z)}} , \end{aligned}$$

$$\begin{aligned}
dG^{II}(z,\zeta) &= G^{II}_\zeta(z,\zeta)d\zeta + G^{II}_{\overline{\zeta}}(z,\zeta)d\overline{\zeta} = 2\mathrm{Re}\,[G^{II}_\zeta(z,\zeta)d\zeta] \\
&= -\frac{1}{2\pi}\mathrm{Re}\left[\frac{\omega'(\zeta)d\zeta}{\omega(\zeta)-\omega(z)} - \frac{\overline{\omega(z)}\omega'(\zeta)d\zeta}{1-\omega(\zeta)\overline{\omega(z)}}\right], \\
d_nG^{II}(z,\zeta) &= -i\left[G^{II}_\zeta(z,\zeta)d\zeta - G^{II}_{\overline{\zeta}}(z,\zeta)d\overline{\zeta}\right] = 2\,\mathrm{Im}\,\left[G^{II}_\zeta(z,\zeta)d\zeta\right] \\
&= -\frac{1}{2\pi}\mathrm{Im}\left[\frac{\omega'(\zeta)d\zeta}{\omega(\zeta)-\omega(z)} - \frac{\overline{\omega(z)}\omega'(\zeta)d\zeta}{1-\omega(\zeta)\overline{\omega(z)}}\right], \\
d_nG^{I}(z,\zeta) &= -i\left[G^{I}_\zeta(z,\zeta)d\zeta - G^{I}_{\overline{\zeta}}(z,\zeta)d\overline{\zeta}\right] = 2\,\mathrm{Im}\,[G^{I}_\zeta(z,\zeta)d\zeta] \\
&= -\frac{1}{2\pi}\,\mathrm{Im}\left[\frac{\omega'(\zeta)d\zeta}{\omega(\zeta)-\omega(z)} + \frac{\overline{\omega(z)}\omega'(\zeta)d\zeta}{1-\omega(\zeta)\overline{\omega(z)}}\right], \\
d_nG^{I}(z,\zeta) - i\,dG^{II}(z,\zeta) &= \frac{i}{2\pi}\left[\frac{\omega'(\zeta)d\zeta}{\omega(\zeta)-\omega(z)} - \frac{\omega(z)\overline{\omega'(\zeta)}\,\overline{d\zeta}}{1-\overline{\omega(\zeta)}\omega(z)}\right]; \\
d_nG^{I}(z,\zeta) - i\,dG^{II}(z,\zeta) &= -\frac{1}{2\pi i}\left[\frac{1}{\omega(\zeta)-\omega(z)} + \frac{\omega(z)}{(\omega(\zeta)-\omega(z))\omega(\zeta)}\right]\omega'(\zeta)d\zeta \\
&= -\frac{1}{2\pi i}\,\frac{\omega(\zeta)+\omega(z)}{\omega(\zeta)-\omega(z)}\,\frac{d\omega(\zeta)}{\omega(\zeta)}, \quad \zeta \in \partial D, \\
d_nG^{II}(z,\zeta) &= -\frac{1}{2\pi}\,\mathrm{Im}\left[\frac{\omega'(\zeta)d\zeta}{\omega(\zeta)-\omega(z)} + \frac{\omega(z)\overline{\omega'(\zeta)}\,\overline{d\zeta}}{1-\overline{\omega(\zeta)}\omega(z)}\right] = -\frac{1}{2\pi}\,\mathrm{Im}\,\frac{\omega'(\zeta)d\zeta}{\omega(\zeta)} \\
&= -\frac{1}{2\pi i}\,\frac{d\omega(\zeta)}{\omega(\zeta)}, \quad \zeta \in \partial D\,.
\end{aligned}$$

Theorem 22. *Let D be a simply connected bounded domain with smooth boundary, $0 \in D$, and $w \in C^1(\overline{D};\mathbb{C})$. Then for $z \in D$*

$$w(z) = -\int_{\partial D} \mathrm{Re}\,w(\zeta)\{d_nG^{I}(z,\zeta) - i\,dG^{II}(z,\zeta)\} - i\int_{\partial D}\mathrm{Im}\,w(\zeta)d_nG^{II}(z,\zeta)$$

$$+2\int_D \left\{w_{\overline{\zeta}}(\zeta)\left[G^{I}_\zeta(z,\zeta) + G^{II}_\zeta(z,\zeta)\right] + \overline{w_{\overline{\zeta}}(\zeta)}\left[G^{I}_{\overline{\zeta}}(z,\zeta) - G^{II}_{\overline{\zeta}}(z,\zeta)\right]\right\}d\xi d\eta\,, \tag{2.1.3}$$

Proof. Applying (2.1.2) to $w(\omega^{-1}(\hat{z}))$ in $\mathbb{D}$ and transforming the integrals by $\hat{\zeta} = \omega(\zeta)$ we arrive at (2.1.3).

Remark. Let $\varphi \in C(\partial D; I\!R), c_0 \in I\!R$, and f be integrable over D then

$$w(z) = -\int\limits_{\partial D} \varphi(\zeta)\,(d_n G^I - i\,dG^{II})\,(z,\zeta) - ic_0$$

$$+2\int\limits_{\partial D} \left\{ f(\zeta)\,(G^I_\zeta + G^{II}_\zeta)\,(z,\zeta) + \overline{f(\zeta)}\left(G^I_{\bar\zeta} - G^{II}_{\bar\zeta}\right)(\zeta,z)\right\} d\xi d\eta, \quad z \in D\,,$$

on ∂D satisfies $\operatorname{Re} w = \varphi$ because on ∂D

$$\int\limits_D \left\{ f(\zeta)\,(G^I_\zeta + G^{II}_\zeta)\,(z,\zeta) + \overline{f(\zeta)}\left(G^I_{\bar\zeta} - G^{II}_{\bar\zeta}\right)(z,\zeta)\right\} d\xi d\eta$$

$$= -\frac{1}{2\pi}\int\limits_{\partial D}\left\{ f(\zeta)\frac{\omega'(\zeta)}{\omega(\zeta)-\omega(z)} + \overline{f(\zeta)}\frac{\overline{\omega'(\zeta)}}{\overline{\omega(z)}-\overline{\omega(\zeta)}}\right\} d\xi d\eta$$

$$= \frac{1}{\pi i}\operatorname{Im}\int\limits_{\partial D}\frac{f(\zeta)\omega'(\zeta)}{\omega(\zeta)-\omega(z)}d\xi d\eta\,.$$

Later we will show $w_{\bar z} = f$ so that the above formula gives a solution to the DIRICHLET problem for the inhomogeneous CAUCHY–RIEMANN system.

2.2 Properties of integral operators

Definition 11. *For $f \in L_1(\overline{D})$ Tf is given by*

$$(Tf)(z) = (T_D f)(z) := -\frac{1}{\pi}\int\limits_D f(\zeta)\frac{d\xi d\eta}{\zeta - z}, \quad z \in \mathbb{C}\,.$$

Theorem 23. (I.N. VEKUA). *Let D be a bounded domain, $f \in L_p(\overline{D}), 2 < p$. Then*

$$|Tf(z)| \le M(p,D)\|f\|_p,\ z \in \mathbb{C}\,; |Tf(z)| \le M(p,D)\|f\|_p|z|^{-1}, 0 < 2R \le |z|\,;$$

$$|Tf(z_1) - Tf(z_2)| \le M(p)\|f\|_p|z_1 - z_2|^{\alpha_0}, \quad \alpha_0 = \frac{p-2}{p}, z_1, z_2 \in \mathbb{C}\,.$$

Here $\|f\|_p$ is the L_p-norm

$$\|f\|_p := \left(\int\limits_D |f(z)|^p dxdy\right)^{\frac{1}{p}}, \quad 1 \le p\,.$$

$M(\ldots)$ always denotes a nonnegative constant, depending on the quantities in the parentheses.

Proof. Applying the HÖLDER inequality

$$\|f_1 f_2\|_p \le \|f_1\|_{p_1} \|f_2\|_{p_2} ,$$

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \le 1, \quad f_\nu \in L_{p_\nu}(\overline{D}), \quad \nu = 1,2,$$

gives

$$|Tf(z)| \le \frac{1}{\pi}\|f_p\| \left(\int_D \frac{d\xi d\eta}{|\zeta - z|^q} \right)^{\frac{1}{q}} , \quad \frac{1}{p} + \frac{1}{q} = 1 .$$

Let $d = \operatorname{diam} D := \sup\limits_{\zeta, z \in D} |\zeta - z|$ denote the diameter of D, then from

$$q = \frac{p}{p-1} < 2, \quad 2 - q = \frac{p-2}{p-1} = \alpha_0 q$$

we see

$$\int_D |\zeta - z|^{-q} d\xi d\eta \le \int_{|\zeta - z| \le d} |\zeta - z|^{-q} d\xi d\eta = 2\pi \int_0^d t^{1-q} dt = \frac{2\pi}{2-q} d^{2-q} .$$

Thus

$$M(p, D) := \frac{1}{\pi} \left(\frac{2\pi}{\alpha_0 q} \right)^{\frac{1}{q}} d^{\alpha_0} .$$

$$|Tf(z_1) - Tf(z_2)| = \left| \frac{z_1 - z_2}{\pi} \int_D \frac{d\xi d\eta}{(\zeta - z_1)(\zeta - z_2)} \right|$$

$$\le \frac{|z_1 - z_2|}{\pi} \|f\|_p \left(\int_D (|\zeta - z_1||\zeta - z_2|)^{-q} d\xi d\eta \right)^{\frac{1}{q}} .$$

Consider more generally for $\alpha < 2, \beta < 2, z_1, z_2 \in \mathbb{C}, z_1 \ne z_2$

$$J(\alpha, \beta) = \int_D |\zeta - z_1|^{-\alpha} |\zeta - z_2|^{-\beta} d\xi d\eta .$$

Let $\rho_0 > 0$ be so large that

$$\overline{D} \subset \{z : |z - z_1| < 2\rho_0\} .$$

Then

$$J_1 := \int_{2|z_1 - z_2| \le |\zeta - z_1| < 2\rho_0} |\zeta - z_1|^{-\alpha} |\zeta - z_2|^{-\beta} d\xi d\eta \le 2^{1+\beta}\pi \int_{2|z_1 - z_2|}^{2\rho_0} t^{1-\alpha-\beta} dt$$

because for these ζ in question

$$2|\zeta - z_2| \geq 2|\zeta - z_1| - 2|z_1 - z_2| \geq 2|\zeta - z_1| - |\zeta - z_1| = |\zeta - z_1| .$$

Thus the HADAMARD estimate is shown, see [Veku62], p. 39:

$$J_1 \leq 2^{1+\beta}\pi \begin{cases} \dfrac{2^{2-\alpha-\beta}}{\alpha+\beta-2}|z_1 - z_2|^{2-\alpha-\beta}, & \text{if} \quad 2 < \alpha + \beta , \\ \log \dfrac{\rho_0}{|z_1 - z_2|}, & \text{if} \quad \alpha + \beta = 2 , \\ \dfrac{2^{2-\alpha-\beta}}{2-\alpha-\beta}\rho_0^{2-\alpha-\beta}, & \text{if} \quad \alpha + \beta < 2 . \end{cases}$$

$$J_2 := \int\limits_{|\zeta - z_1| \leq 2|z_1 - z_2|} |\zeta - z_1|^{-\alpha}|\zeta - z_2|^{-\beta} d\xi d\eta = |z_1 - z_2|^{2-\alpha-\beta} \int\limits_{|\zeta| \leq 2} |\zeta|^{-\alpha}|\zeta + 1|^{-\beta} d\xi d\eta$$

because for those ζ in question

$$\frac{|\zeta - z_2|}{|z_1 - z_2|} = \left| \frac{\zeta - z_1}{z_1 - z_2} + \frac{z_1 - z_2}{z_1 - z_2} \right| = \left| \frac{\zeta - z_1}{z_1 - z_2} + 1 \right| .$$

Hence, for $2 < \alpha + \beta$

$$J(\alpha, \beta) \leq J_1 + J_2 \leq M(\alpha, \beta)|z_1 - z_2|^{2-\alpha-\beta} ,$$

where

$$M(\alpha, \beta) := \frac{2^{2-\alpha-\beta}}{\alpha + \beta - 2} + \int\limits_{|\zeta| \leq 2} |\zeta|^{-\alpha}|\zeta + 1|^{-\beta} d\xi d\eta .$$

From this estimate applied for $\alpha = \beta = q, 1 < q < 2\,(2 < 2q)$

$$|Tf(z_1) - Tf(z_2)| \leq \frac{|z_1 - z_2|}{\pi} \|f\|_p (M(q, q)|z_1 - z_2|^{2-2q})^{\frac{1}{q}}$$

$$= M(p)\|f\|_p |z_1 - z_2|^{\alpha_0} ,$$

where

$$\alpha_0 := \frac{p-2}{p} = \frac{2p-2}{p} - 1 = \frac{2}{q} - 1 .$$

For $0 < 2R \leq |z|$ the estimates is obvious.
Theorem 23 shows that T is a completely continuous linear operator from $L_p(\overline{D})$ into $C^{\alpha_0}(\mathbb{C})$ if $2 < p$ and $\alpha_0 = \dfrac{p-2}{p}$. Moreover, the HÖLDER norm of Tf

$$C_{\alpha_0}(Tf; \mathbb{C}) := \sup_{z \in \mathbb{C}} |Tf(z)| + \sup_{z_1, z_2 \in \mathbb{C}} \frac{|Tf(z_1) - Tf(z_2)|}{|z_1 - z_2|^{\alpha_0}} \leq M(p, D)\|f\|_p . \quad (2.2.1)$$

Definition 12. *A complex function f is said to belong to $L_{p,\nu}(\mathbb{C})$ if $f \in L_p(\overline{\mathbb{D}})$ and $f_\nu \in L_p(\overline{\mathbb{D}})$, where $f_\nu(z) := |z|^{-\nu} f\left(\frac{1}{z}\right)$. Here $1 < p$ and $\nu \in \mathbb{R}$. The norm for $f \in L_{p,\nu}(\mathbb{C})$ is defined as*

$$\|f\|_{p,\nu} := \|f\|_{p,\mathbb{D}} + \|f_\nu\|_{p,\mathbb{D}} .$$

Remark. Integrability of $|f_\nu|^p$ means that $\left|z^{-\nu} f\left(\frac{1}{z}\right)\right|^p$ near 0 does not increase too much, namely weaker than $|z|^{-2}$, i.e. $\left|f\left(\frac{1}{z}\right)\right|$ near 0 behaves as $|z|^{-\mu}$ where $\mu < \dfrac{(2-\nu p)}{p}$. This means $|f(z)|$ near infinity increases not more than $|z|^{\frac{1}{p}-\nu}$.
From

$$\int\limits_{1\le|\zeta|} |f(\zeta)|^p d\xi d\eta = \int\limits_{|\zeta|\le 1} |\zeta|^{-4} \left|f\left(\frac{1}{\zeta}\right)\right|^p d\xi d\eta = \int\limits_{|\zeta|\le 1} \left(|\zeta|^{-\frac{4}{p}} \left|f\left(\frac{1}{\zeta}\right)\right|\right)^p d\xi d\eta$$

we see $L_p(\mathbb{C}) = L_{p,\frac{4}{p}}(\mathbb{C})$.

Theorem 24. (I.N. VEKUA). $f \in L_{p,2}(\mathbb{C}), 2 < p, \alpha_0 := \dfrac{p-2}{p}$. *Then*

$$(Tf)(z) = (T_{\mathbb{C}} f)(z) := -\frac{1}{\pi} \int\limits_{\mathbb{C}} f(\zeta) \frac{d\xi d\eta}{\zeta - z}, \quad z \in \mathbb{C} ,$$

satisfies

$$\begin{aligned} |Tf(z)| &\le M(p)\|f\|_{p,2}, & z \in \mathbb{C} ,\\ |Tf(z_1) - T(z_2)| &\le M(p)\|f\|_{p,2}|z_1 - z_2|^{\alpha_0}, & z_1, z_2 \in \mathbb{C} ,\\ |Tf(z)| &\le M(p,R)\|f\|_{p,2}|z|^{-\alpha_0}, & 1 < R \le |z| . \end{aligned}$$

Proof.

$$Tf(z) = -\frac{1}{\pi} \int\limits_{|\zeta|\le 1} f(\zeta) \frac{d\xi d\eta}{\zeta - z} - \frac{1}{\pi} \int\limits_{|\zeta|\le 1} |\zeta|^{-4} f\left(\frac{1}{\zeta}\right) \frac{d\xi d\eta}{\frac{1}{\zeta} - z} .$$

Rewriting

$$\frac{1}{(1 - z\zeta)\zeta} = \frac{z}{1 - z\zeta} + \frac{1}{\zeta} = \frac{1}{\frac{1}{z} - \zeta} + \frac{1}{\zeta}$$

the second integral can be written as

$$g_2(0) - g_2\left(\frac{1}{z}\right), \quad g_2(z) := -\frac{1}{\pi} \int\limits_{|\zeta|\le 1} \overline{\zeta}^{-2} f\left(\frac{1}{\zeta}\right) \frac{d\xi d\eta}{\zeta - z}\,.$$

Setting

$$g_1(z) := -\frac{1}{\pi} \int\limits_{|\zeta|\le 1} f(\zeta) \frac{d\xi d\eta}{\zeta - z} = T_{\mathbb{D}} f(z)$$

we have

$$Tf(z) = g_1(z) + g_2(0) - g_2\left(\frac{1}{z}\right)\,.$$

Therefore from Theorem 23 we have for $z \in \mathbb{C}$

$$|Tf(z)| \le M(p)\left\{\|f\|_{p,\mathbb{D}} + \left\|\overline{\zeta}^{-2} f\left(\frac{1}{\zeta}\right)\right\|_{p,\mathbb{D}}\right\} \le M(p)\|f\|_{p,2}\,,$$

$$|g_1(z_1) - g_1(z_2) \le M(p)\|f\|_{p,\mathbb{D}}|z_1 - z_2|^{\alpha_0} \le M(p)\|f\|_{p,2}|z_1 - z_2|^{\alpha_0}\,,$$

$$\left|g_2\left(\frac{1}{z_1}\right) - g_2\left(\frac{1}{z_2}\right)\right| \le \frac{|z_1 - z_2|}{\pi} \int\limits_{|\zeta|\le 1} \left|\overline{\zeta}^{-2} f\left(\frac{1}{\zeta}\right)\right| \frac{d\xi d\eta}{|1 - z_1\zeta||1 - z_2\zeta|}\,.$$

To estimate the last integral we distinguish three cases.

i. $|z_1|, |z_2| \le \frac{1}{2}$. For $|\zeta| \le 1$ then

$$\frac{1}{2} \le 1 - |\zeta||z_k| \le |1 - \zeta z_k|\,,$$

so that because of $|z_1 - z_2| \le 1$

$$\left|g_2\left(\frac{1}{z_1}\right) - g_2\left(\frac{1}{z_2}\right)\right| \le M(p)\left\|\overline{\zeta}^{-2} f\left(\frac{1}{\zeta}\right)\right\|_{p,\mathbb{D}} |z_1 - z_2| \le M(p)\|f\|_{p,2}|z_1 - z_2|^{\alpha_0}\,.$$

ii. $|z_1| \le \dfrac{1}{2} \le |z_2|$. Then

$$\frac{|z_1 - z_2|^{\frac{2}{p}}}{|z_2|} \le 2^{\alpha_0}\left|\frac{z_1 - z_2}{z_2}\right|^{\frac{2}{p}} \le 2^{\alpha_0}\left(\left|\frac{z_1}{z_2}\right| + 1\right)^{\frac{2}{p}} \le 2\,,$$

so that

$$\left|g_2\left(\frac{1}{z_1}\right) - g_2\left(\frac{1}{z_2}\right)\right| \le \frac{2}{\pi}\frac{|z_1 - z_2|}{|z_2|} \int\limits_{|\zeta|\le 1} \left|\overline{\zeta}^{-2} f\left(\frac{1}{\zeta}\right)\right| \frac{d\xi d\eta}{|\frac{1}{z_2} - \zeta|}$$

$$\le M(p)\|f\|_{p,2}|z_1 - z_2|^{\alpha}.$$

iii. $\frac{1}{2} \le |z_1|, |z_2|$. Then from step i.

$$\left|g_2\left(\frac{1}{z_1}\right) - g_2\left(\frac{1}{z_2}\right)\right| = \frac{|\frac{1}{z_1} - \frac{1}{z_2}|}{\pi}\left|\int_{|\zeta|\le 1} \overline{\zeta}^{-2} f\left(\frac{1}{\zeta}\right)\frac{d\xi d\eta}{(\zeta - \frac{1}{z_1})(\zeta - \frac{1}{z_2})}\right|$$

$$\le M(p)\|f\|_{p,2}\left|\frac{1}{z_1} - \frac{1}{z_2}\right|^{\alpha_0} = M(p)\|f\|_{p,2}|z_1 - z_2|^{\alpha_0} .$$

In order to prove the last estimation of the theorem consider $1 < |z|$. Then

$$|g_1(z)| \le \frac{1}{\pi(|z| - 1)}\int_{|\zeta|<1} |f(\zeta)| d\xi d\eta \le M(p)\|f\|_p \frac{1}{|z| - 1} .$$

Applying the second estimate of Theorem 23 gives

$$\begin{aligned}\left|g_2(0) - g_2\left(\frac{1}{z}\right)\right| &\le M(p)\left\|\overline{\zeta}^{-2} f\left(\frac{1}{\zeta}\right)\right\|_{p,\mathbb{D}}\left|\frac{1}{z}\right|^{\alpha_0} \\ &\le M(p)\|f\|_{p,2}|z|^{-\alpha_0} .\end{aligned}$$

Thus

$$|Tf(z)| \le M(p)\|f\|_{p,2}\left(\frac{1}{|z| - 1} + |z|^{-\alpha_0}\right) \le M(p, R)\|f\|_{p,2}|z|^{-\alpha_0} ,$$

because for $1 < R \le |z|$

$$\frac{|z|^{\alpha_0}}{|z| - 1} \le \frac{|z|}{|z| - 1} \le \frac{R}{R - 1} .$$

Remark. For $f \in L_{p,2}(\mathbb{C})$ the function $Tf \in C^{\alpha_0}(\mathbb{C}), Tf(z)$ vanishes at infinity as $|z|^{-\alpha_0}$. T again is a completely continuous operator on $L_{p,2}(\mathbb{C})$ into $C^{\alpha_0}(\mathbb{C})$. Moreover,

$$C_{\alpha_0}(Tf, \mathbb{C}) \le M(p)\|f\|_{p,2} .$$

We are now interested in differentiability properties of Tf. For that reason we need the concept of weak derivatives, that are in other word generalized derivatives in the distributional or SOBOLEV sense.

Lemma 10. *$f \in L_p(\overline{D}), 2 \le p, D$ a bounded domain, $f = 0$ in $\hat{\mathbb{C}} \setminus D$. Then for $\lambda < 2 < p(2 - \lambda)$*

$$g(z) := \int_D f(\zeta)\frac{d\xi d\eta}{|\zeta - z|^\lambda} = \int_{\mathbb{C}} f(\zeta + z)\frac{d\xi d\eta}{|\zeta|^\lambda}$$

is continuous in $\mathbb{C}$.

Proof. Let $z_1 \in \mathbb{C}$ be arbitrarily fixed and $|z_2 - z_1| \leq 1$. Let $R > 0$ be such that the supports of $f(\zeta + z_1)$ and of $f(\zeta + z_2)$ are in $\{z : |z| < R\}$. Then with $\frac{1}{p} + \frac{1}{q} = 1$

$$|g(z_1) - g(z_2)| = \left| \int\limits_{|\zeta|<R} [f(\zeta + z_1) - f(\zeta + z_2)] \frac{d\xi d\eta}{|\zeta|^\lambda} \right|$$

$$\leq \left(\int\limits_{|\zeta|<R} |f(\zeta + z_1) - f(\zeta + z_2)|^p d\xi d\eta \right)^{\frac{1}{p}} \left(\int\limits_{|\zeta|<R} |\zeta|^{-\lambda q} d\xi d\eta \right)^{\frac{1}{q}} ,$$

$$\int\limits_{|\zeta|<R} |\zeta|^{-\lambda q} d\xi d\eta = 2\pi \int\limits_0^R t^{1-\lambda q} dt = \frac{2\pi}{2 - \lambda q} R^{2-\lambda q} ,$$

$$\lambda q = \frac{\lambda p}{p - 1} < \lambda \frac{2}{2 - (2 - \lambda)} = 2 .$$

We have to show

$$\int\limits_{|\zeta|<R} |f(\zeta + z_1) - f(\zeta + z_2)|^p d\xi d\eta = \int\limits_{|\zeta|<R+|z_1|} |f(\zeta) - f(\zeta + z_2 - z_1)|^p d\xi d\eta$$

becomes small if $z_1 - z_2$ does, i.e. that

$$\int\limits_{|\zeta|<R_1} |f(\zeta) - f(\zeta + z)|^p d\xi d\eta, \quad R_1 := R + |z_1| ,$$

becomes small if z does, see [Sobo63], p. 12.
If f would be uniformly continuous the assertion would hold. Because f is integrable in the LEBESGUE sense there exists for any $\hat{\delta} > 0$ a closed set F in the open disc $K := \{z : |z| < R_1\}$ such that f is continuous on F and hence uniformly continuous there and $m(K \backslash F) < \frac{1}{2}\hat{\delta}$ where m is the LEBESGUE measure. Because $F \subset K$ is compact $0 < \operatorname{dist}(F, \partial K) =: d$. For $|z| < d$ the set

$$F - z := \{\zeta - z : \zeta \in F\} \subset K .$$

Let

$$F^* := F \cap \{F - z\} ,$$

then $\zeta \in F^*$ means $\zeta \in F$ and $\zeta + z \in F$. Let now $|z| < \delta(\varepsilon)$ such that for all $\zeta \in F^*$.

$$|f(\zeta) - f(\zeta + z)| < \varepsilon(2\pi R_1^2)^{-\frac{1}{p}} .$$

Moreover, from

$$K\backslash F^* = \{K\backslash F\} \cup \{K\backslash\{F - z\}\}$$

we see

$$m(K\backslash F^*) \le m(K\backslash F) + m(K\backslash\{F - z\}) = 2(mK - mF) = 2m(K\backslash F) < \hat{\delta}\,.$$

Hence, from the property of the LEBESGUE integral

$$\int\limits_{K\backslash F^*} |f(\zeta)|^p d\xi d\eta < \frac{1}{2}\left(\frac{\varepsilon}{2}\right)^p$$

follows if $\hat{\delta}$ is small enough, $\hat{\delta} = \hat{\delta}(\varepsilon)$. Thus,

$$\int\limits_K |f(\zeta) - f(\zeta + z)|^p d\xi d\eta \le \int\limits_{F^*} |f(\zeta) - f(\zeta + z)|^p d\xi d\eta$$

$$+ \left[\left(\int\limits_{K\backslash F^*} |f(\zeta)|^p d\xi d\eta\right)^{\frac{1}{p}} + \left(\int\limits_{K\backslash F^*} |f(\zeta + z)|^p d\xi d\eta\right)^{\frac{1}{p}}\right]^p \le \frac{\varepsilon^p}{2\pi R_1^2} mF^* + \frac{\varepsilon^p}{2} < \varepsilon^p\,.$$

Theorem 25. *Let D be a bounded domain, $f \in L_1(\overline{D})$. Then $Tf \in L_p(\overline{D_0})$ for any bounded domain D_0 and $1 \le p \le 2$.*

Proof. The function

$$g_1(z) = \int\limits_{D_0} |g(\zeta)| \frac{d\xi d\eta}{|\zeta - z|}$$

is continuous according to the preceding lemma if $g \in L_p(\overline{D_0})$ for some $p \ge 2$. Therefore $fg_1 \in L_1(\overline{D})$ for $f \in L_1(\overline{D})$. Applying the FUBINI theorem, see [Rogo52], p. 121

$$\int\limits_D |f(z)| g_1(z) dxdy = \int\limits_D |f(z)| \int\limits_{D_0} |g(\zeta)| \frac{d\xi d\eta}{|\zeta - z|} dxdy$$

$$= \int\limits_{D_0} |g(\zeta)| \int\limits_D |f(z)| \frac{dxdy}{|\zeta - z|} d\xi d\eta = \int\limits_{D_0} |g(\zeta)| f_1(\zeta) dxdy\,,$$

where

$$f_1(z) := \int\limits_D |f(\zeta)| \frac{d\xi d\eta}{|\zeta - z|}\,.$$

From $|g| f_1 \in L_1(\overline{D_0})$ and $g \in L_p(\overline{D_0})$ it follows $f_1 \in L_q(\overline{D_0})$ for $\dfrac{1}{p} + \dfrac{1}{q} = 1$ i.e. $1 \le q \le 2$. From $|Tf| \le f_1$ then $Tf \in L_q(\overline{D_0})$.

Definition 13. *A function φ is said to have a compact support if there exists a compact set such that φ vanishes outside this set. The closure of $\{z : \varphi(z) \neq 0\}$ then is called the support of φ denoted by supp φ. The set of functions in $C^n(D)$ with compact support in D are denoted by $C_0^n(D)$.*

Theorem 26. *If $f \in L_1(\overline{D})$ then for all $\varphi \in C_0^1(D)$*

$$\int_D Tf(z)\varphi_{\bar{z}}(z)dxdy + \int_D f(z)\varphi(z)dxdy = 0 . \tag{2.2.2}$$

Proof. From the CAUCHY-POMPEIU formula (Theorem 20)

$$\varphi(z) = \frac{1}{2\pi i}\int_{\partial D} \varphi(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D \varphi_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z} = (T\varphi_{\bar{\zeta}})(z)$$

follows, so that

$$\int_D Tf(z)\varphi_{\bar{z}}(z)dxdy = -\frac{1}{\pi}\int_D f(\zeta)\int_D \varphi_{\bar{z}}(z)\frac{dxdy}{\zeta - z}d\xi d\eta = -\int_D f(\zeta)\varphi(\zeta)d\xi d\eta . \tag{2.2.3}$$

Definition 14. (SOBOLEV). *$f, g \in L_1(\overline{D})$. Then f is called generalized derivative of g with respect to $\bar{z}\,(z)$ if for all $\varphi \in C_0^1(D)$*

$$\int_D g(z)\varphi_{\bar{z}}(z)dxdy \;+\; \int_D f(z)\varphi(z)dxdy \;=\; 0$$

$$\left(\int_D g(z)\varphi_z(z)dxdy \;+\; \int_D f(z)\varphi(z)dxdy \;=\; 0\right).$$

This derivative is denoted by $f = g_{\bar{z}} = \dfrac{\partial g}{\partial \bar{z}} \left(f = g_z = \dfrac{\partial g}{\partial z}\right)$.

Remark. For $g \in C^1(D)$ and $f = g_{\bar{z}}$ we have for $\varphi \in C_0^1(D)$ by the GAUSS Theorem in complex form (see Lemma 7)

$$\int_D (g(z)\varphi_{\bar{z}}(z) + f(z)\varphi(z))dxdy = \int_D (g(z)\varphi(z))_{\bar{z}}dxdy = -\frac{1}{2i}\int_{\partial D} g(z)\varphi(z)dz = 0 .$$

This means that if g is differentiable in the classical sense it is differentiable in the SOBOLEV sense and its generalized derivative is the classical one.

Definition 15. *$\mathbf{D}_{\bar{z}}(D)$ is the set of functions having generalized derivatives with respect to $\bar{z}$ in D. Similarly, $\mathbf{D}_z(D)$ is defined. Moreover, $\mathbf{D}_1(D) := \mathbf{D}_{\bar{z}}(D) \cap \mathbf{D}_z(D)$.*

Obviously, $g \in \mathbf{D}_{\bar{z}}(D)$ implies $\overline{g} \in \mathbf{D}_z(D)$ and vice versa. Theorem 26 shows $Tf \in \mathbf{D}_{\bar{z}}(D)$ for $f \in L_1(\overline{D})$ and

$$\frac{\partial Tf}{\partial \overline{z}} = f .$$

Theorem 27. *If $g \in \mathbf{D}_{\bar{z}}(D)$ and $g_{\bar{z}} = 0$ in D then g is analytic in D.*

Proof. It is enough to prove g to be analytic in the neighborhood of some arbitrary $z_0 \in D$. Without loss of generality we assume $z_0 = 0$ and for proper R

$$K := \{z : |z| < R\} \subset D .$$

Let for $z, \zeta \in K$

$$Z(z,\zeta) := 2|z-\zeta|^2 \log \frac{|R^2 - z\overline{\zeta}|}{R|z-\zeta|} - (R^2 - |z|^2)\left(1 - \frac{|\zeta|^2}{R^2}\right) ,$$

which is the GREEN function for K with respect to Δ^2, i.e. as a function of z is a biharmonic function,

$$\Delta_z^2 Z(z,\zeta) = 0 \quad \text{in} \quad K \backslash \{\zeta\} .$$

$Z \in C^1(\overline{K})$ as can be seen by direct calculations, and

$$Z(z,\zeta) = Z_x(z,\zeta) = Z_y(z,\zeta) = 0, \quad |z| = R, |\zeta| < R .$$

Hence,

$$\varphi(z) := \begin{cases} Z(z,\zeta), & z \in \overline{K} \\ 0, & z \notin K \end{cases} , \zeta \in K \quad \text{fixed},$$

belongs to $C_0^1(D)$. From $f := g_{\bar{z}} = 0$ we have

$$\int_D g(z)\varphi_{\bar{z}}(z)dxdy = \int_K g(z)\frac{\partial Z(z,\zeta)}{\partial \overline{z}}dxdy = -\int_K f(z)Z(z,\zeta)dxdy = 0 .$$

This result holds for any $\zeta \in K$.

$$\frac{\partial^2}{\partial\zeta\partial\overline{\zeta}} \int_K g(z)\frac{\partial Z(z,\zeta)}{\partial \overline{z}}dxdy = \int_K g(z)\frac{\partial^3 Z(z,\zeta)}{\partial\zeta\partial\overline{\zeta}\partial\overline{z}}dxdy = 0 ,$$

when we used that $Z \in C^\infty(D)$ is at least three times continuously differentiable. Because

$$\frac{\partial^3 Z(z,\zeta)}{\partial\zeta\partial\overline{\zeta}\partial\overline{z}} = \frac{1}{\overline{\zeta} - \overline{z}} + \frac{R^2 z - 2R^2\zeta + \overline{z}\zeta^2}{(R^2 - \overline{z}\zeta)^2} + \frac{z^2\overline{\zeta}}{R^2(R^2 - z\overline{\zeta})}$$

we find

$$\overline{(T\overline{g})(\zeta)} := -\frac{1}{\pi}\int_K g(z)\frac{dxdy}{\overline{z}-\overline{\zeta}} = \phi(\zeta) + \overline{\phi_1(\zeta)},$$

with

$$\begin{aligned}\phi(\zeta) &:= -\frac{1}{\pi}\int_K g(z)\frac{R^2 z - 2R^2\zeta + \overline{z}\zeta^2}{(R^2 - \overline{z}\zeta)^2}dxdy, \\ \phi_1(\zeta) &:= -\frac{1}{R^2\pi}\int_K \overline{g(\zeta)}\frac{\overline{z}^2\zeta}{R^2 - \overline{z}\zeta}dxdy.\end{aligned}$$

Obviously, ϕ and ϕ_1 are analytic functions in K. Thus, differentiating the last equation in the weak sense with respect to ζ gives

$$\phi'(\zeta) = \frac{\partial}{\partial\zeta}\overline{(T\overline{g})(\zeta)} = \overline{\frac{\partial}{\partial\overline{\zeta}}(T\overline{g})(\zeta)} = \overline{\overline{g(\zeta)}} = g(\zeta), \quad \zeta \in K.$$

Hence, g is analytic in K.

Theorem 28. *If $g \in \mathbf{D}_{\overline{z}}(D)$ and $f = g_{\overline{z}} \in L_1(\overline{D})$ then*

$$g(z) = \phi(z) + (T_D f)(z) = \phi(z) - \frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{\zeta - z}, \quad z \in D,$$

where ϕ is an analytic function in D. The reverse of this statement is true, too: If ϕ is analytic and $f \in L_1(\overline{D})$ then $g = \phi + T_D f \in \mathbf{D}_{\overline{z}}(D)$ and $g_{\overline{z}} = f$.

Proof.

1. $$\frac{\partial}{\partial\overline{z}}(g - T_D f) = g_{\overline{z}} - f = 0,$$
   so that by Theorem 27 the function $g - T_D f$ is analytic in D.

2. $$g_{\overline{z}} = \frac{\partial}{\partial\overline{z}}(\phi + T_D f) = \phi_{\overline{z}} + f = f$$
   because ϕ is analytic.

Theorem 29. *Let D be a $C^{1+\alpha}$ domain, $0 < \alpha < 1$, and $f \in C^\alpha(\overline{D})$. Then $Tf \in C^{1+\alpha}(\overline{D})$, T is a completely continuous operator from $C^\alpha(\overline{D})$ into $C^{1+\alpha}(\overline{D})$. Moreover,*

$$\frac{\partial}{\partial x}Tf = f + \Pi f, \quad \frac{\partial}{\partial y}Tf = -if + i\Pi f$$

with

$$(\Pi f)(z) := -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}\,.$$

Πf is singular integral to be taken in the CAUCHY *principal sense, $\Pi f \in C^\alpha(\overline{D})$ for $f \in C^\alpha(\overline{D})$, Π is a bounded linear operator from $C^\alpha(\overline{D})$ into itself.*

Corollary 4. *Under the conditions of Theorem 29*

$$\frac{\partial Tf(z)}{\partial \overline{z}} = f(z), \quad \frac{\partial Tf(z)}{\partial z} = \Pi f(z)\,.$$

These equations hold for $z \in \mathbb{C}$ if for $z \notin \overline{D}$ the function value $f(z)$ is replaced by 0.

Corollary 5. *If $f(z_0) = 0, f \in C^\alpha(\overline{D})$ then Tf is complex differentiable and*

$$\frac{dTf}{dz}(z_0) = (\Pi f)(z_0)\,,$$

where the integral Πf exists in the ordinary sense as an improper integral.

Remark. $D \in C^{1+\alpha}$ or D a $C^{1+\alpha}$ domain means D is bounded and the boundary is a finite set of smooth curves with HÖLDER continuous tangent with exponent α.

Proof.

1. At first the Π-operator is studied,

$$\Pi f(z) = -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2} = -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{D_\varepsilon} f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{D_\varepsilon}\frac{f(z) - f(\zeta)}{(\zeta - z)^2}d\xi d\eta - \lim_{\varepsilon\to 0}\frac{f(z)}{\pi}\int_{D_\varepsilon}\frac{d\xi d\eta}{(\zeta - z)^2}\,,$$

$$D_\varepsilon = D\backslash K_\varepsilon(z), \quad K_\varepsilon(z) := \{\zeta : |\zeta - z| < \varepsilon\}\,.$$

Since $f \in C^\alpha(\overline{D})$

$$\int_D \frac{f(\zeta) - f(z)}{(\zeta - z)^2}d\xi d\eta = \lim_{\varepsilon\to 0}\int_{D_\varepsilon}\frac{f(\zeta) - f(z)}{(\zeta - z)^2}d\xi d\eta$$

exists.

$$\frac{1}{\pi}\int_{D_\varepsilon}\frac{d\xi d\eta}{(\zeta - z)^2} = -\frac{1}{\pi}\int_{D_\varepsilon}\frac{d}{d\zeta}\frac{1}{\zeta - z}d\xi d\eta$$

$$= \frac{1}{2\pi i}\int_{\partial D_\varepsilon}\frac{d\overline{\zeta}}{\zeta - z} = \frac{1}{2\pi i}\int_{\partial D}\frac{d\overline{\zeta}}{\zeta - z} - \frac{1}{2\pi i}\int_{|\zeta - z| = \varepsilon}\frac{d\overline{\zeta}}{\zeta - z}\,.$$

By integration by parts

$$\frac{1}{2\pi i}\int_{\partial D}\frac{d\overline{\zeta}}{\zeta - z} = \frac{1}{2\pi i}\int_{\partial D}\frac{\overline{\zeta}d\zeta}{(\zeta - z)^2} = \phi'(z)\,,$$

where

$$\phi(z) := \frac{1}{2\pi i}\int_{\partial D}\frac{\overline{\zeta}d\zeta}{\zeta - z}$$

is seen to be an analytic function in D and in $\hat{\mathbb{C}}\setminus D$.

Especially, when $D = K_{z_0}(R)$ for $|z - z_0| < R$

$$\phi(z) = \frac{1}{2\pi i}\int_{|\zeta - z_0|=R}\frac{\overline{\zeta - z_0}}{\zeta - z}d\zeta + \frac{\overline{z_0}}{2\pi i}\int_{|\zeta - z_0|=R}\frac{d\zeta}{\zeta - z}$$

$$= \frac{R^2}{2\pi i}\int_{|\zeta - z_0|=R}\frac{d\zeta}{(\zeta - z)(\zeta - z_0)} + \overline{z_0}$$

$$= \frac{R^2}{2\pi i}\int_{|\zeta - z_0|=R}\frac{1}{z - z_0}\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0}\right)d\zeta + \overline{z_0} = \overline{z_0}\,,$$

so that $\phi'(z) \equiv 0$ in $|z - z_0| < R$ in that special case. Moreover,

$$\frac{1}{2\pi i}\int_{|\zeta - z|=\varepsilon}\frac{d\overline{\zeta}}{\zeta - z} = -\frac{1}{2\pi i}\int_{|\zeta - z|=\varepsilon}\frac{d\zeta}{(\zeta - z)^3} = 0\,.$$

Thus, in general for any $0 < \varepsilon$

$$\frac{1}{\pi}\int_{D_\varepsilon}\frac{d\xi d\eta}{(\zeta - z)^2} = \phi'(z)\,,$$

so that

$$\Pi f(z) = -\frac{1}{\pi}\int_{D}\frac{f(\zeta) - f(z)}{(\zeta - z)^2}d\xi d\eta - f(z)\phi'(z)\,.$$

This proves that $\Pi f(z)$ exists as a CAUCHY principal value integral in $\overline{D}$. For $z \notin \overline{D}$ the integral, obviously, exists as an ordinary integral. In fact as was mentioned earlier it is analytic as Tf.

2. Next we will prove $\Pi f \in C^\alpha(\overline{D})$ if $f \in C^\alpha(\overline{D})$. Let $z_1, z_2 \in \overline{D}, z_1 \neq z_2$. Then

$$\Pi f(z_2) - \Pi f(z_1)$$

$$= -\frac{1}{\pi}\int_D \left\{ \frac{f(\zeta)}{(\zeta - z_2)^2} - \frac{f(\zeta)}{(\zeta - z_1)(\zeta - z_2)} + \frac{f(\zeta)}{(\zeta - z_1)(\zeta - z_2)} - \frac{f(\zeta)}{(\zeta - z_1)^2} \right\} d\xi d\eta$$

$$= -\frac{z_2 - z_1}{\pi}\int_D \frac{f(\zeta) - f(z_2)}{(\zeta - z_2)^2(\zeta - z_1)} d\xi d\eta - \frac{z_2 - z_1}{\pi}\int_D \frac{f(\zeta) - f(z_1)}{(\zeta - z_1)^2(\zeta - z_2)} d\xi d\eta$$

$$-\frac{z_2 - z_1}{\pi}\int_D \frac{f(z_2)}{(\zeta - z_2)^2(\zeta - z_1)} d\xi d\eta - \frac{z_2 - z_1}{\pi}\int_D \frac{f(z_1)}{(\zeta - z_1)^2(\zeta - z_2)} d\xi d\eta \, .$$

$$\int_D \frac{(z_1 - z_2) d\xi d\eta}{(\zeta - z_1)^2(\zeta - z_2)} = \int_D \left\{ \frac{1}{(\zeta - z_1)^2} - \frac{1}{z_1 - z_2}\left(\frac{1}{\zeta - z_1} - \frac{1}{\zeta - z_2}\right)\right\} d\xi d\eta.$$

Applying the CAUCHY–POMPEIU formula (2.1.1) (Theorem 20) to $\overline{z}$ gives

$$\overline{z} = \frac{1}{2\pi i}\int_{\partial D} \frac{\overline{\zeta} d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D \frac{d\xi d\eta}{\zeta - z} = \phi(z) - \frac{1}{\pi}\int_D \frac{d\xi d\eta}{\zeta - z} \, .$$

Moreover, observing

$$\phi'(z) = \frac{1}{\pi}\int_D \frac{d\xi d\eta}{(\zeta - z)^2}$$

we see

$$\frac{(z_1 - z_2)}{\pi}\int_D \frac{d\xi d\eta}{(\zeta - z_1)^2(\zeta - z_2)} = \phi'(z_1) + \frac{1}{z_1 - z_2}[\overline{z_1} - \phi(z_1) - \overline{z_2} + \phi(z_2)] \, .$$

In order to see $\phi' \in C^\alpha(\overline{D})$ rewrite

$$\phi'(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{d\overline{\zeta}}{\zeta - z} = \frac{1}{2\pi i}\int_{\partial D} \frac{\overline{\zeta'(s)}}{\zeta'(s)} \frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{\partial D} \overline{\zeta'(s)}^2 \frac{d\zeta}{\zeta - z} \, .$$

Here s is the arc length parameter on ∂D. As $\zeta'(s) \in C^\alpha(\partial D)$ we have, see proof of Lemma 1, with $\zeta_k = \zeta(s_k), k = 1, 2$,

$$|\zeta'(s_1) - \zeta'(s_2)| \le H|s_1 - s_2|^\alpha \le \frac{1}{k_0^\alpha} H |\zeta_1 - \zeta_2|^\alpha \, .$$

Hence, $\phi \in C^{1+\alpha}(\overline{D})$ (and $\phi \in C^{1+\alpha}(\widehat{\mathbb{C}} \setminus D)$) follows from Theorem 5.

$$\left| \frac{1}{\pi}\int_D \frac{f(\zeta) - f(z)}{(\zeta - z)^2(\zeta - z_0)} d\xi d\eta \right| \le \frac{H_\alpha(f)}{\pi}\int_D \frac{d\xi d\eta}{|\zeta - z|^{2-\alpha}|\zeta - z_0|}$$

$$\le H_\alpha(f) M(\alpha) |z - z_0|^{\alpha - 1}$$

(see Proof of Theorem 23). Hence,

$$\begin{aligned}
|\Pi f(z_2) - \Pi f(z_1)| &\le M(\alpha)H_\alpha(f)|z_2 - z_1|^\alpha \\
&\quad + \Big| f(z_2)\Big[\phi'(z_2) + \frac{1}{z_2 - z_1}(\overline{z_2} - \phi(z_2) - \overline{z_1} + \phi(z_1))\Big] \\
&\quad - f(z_1)\Big[\phi'(z_1) + \frac{1}{z_1 - z_2}(\overline{z_1} - \phi(z_1) - \overline{z_2} + \phi(z_2))\Big]\Big| \\
&\le M(\alpha)H_\alpha(f)|z_2 - z_1|^\alpha + |f(z_2)||\phi'(z_2) - \phi'(z_1)| \\
&\quad + |f(z_2) - f(z_1)|\left|\frac{\overline{z_2 - z_1}}{z_2 - z_1} - \frac{\phi(z_2) - \phi(z_1)}{z_2 - z_1} + \phi'(z_1)\right| \\
&\le [(M(\alpha) + C_1(\phi; D))H_\alpha(f) + C_0(f; D)H_\alpha(\phi')]|z_2 - z_1|^\alpha \\
&= M(\alpha, D)C_\alpha(f; D)|z_2 - z_1|^\alpha .
\end{aligned}$$

In other words $\Pi f \in C^\alpha(\overline{D})$ and

$$C_\alpha(\Pi f; \overline{D}) \le M(\alpha, D)C_\alpha(f; \overline{D}) . \tag{2.2.4}$$

3. For differentiating Tf we consider

$$\begin{aligned}
&\frac{Tf(z_0) - Tf(z)}{z_0 - z} - \Pi f(z) \\
&= -\frac{1}{\pi(z_0 - z)}\int_D \left\{\frac{1}{\zeta - z_0} - \frac{1}{\zeta - z} - \frac{z_0 - z}{(\zeta - z)^2}\right\} f(\zeta)d\xi d\eta \\
&= -\frac{z_0 - z}{\pi}\int_D \frac{f(\zeta)}{(\zeta - z_0)(\zeta - z)^2} d\xi d\eta \\
&= -\frac{z_0 - z}{\pi}\int_D \frac{f(\zeta) - f(z)}{(\zeta - z_0)(\zeta - z)^2} d\xi d\eta \\
&\quad + f(z)\left[\phi'(z) + \frac{\overline{z - z_0}}{z - z_0} - \frac{\phi(z) - \phi(z_0)}{z - z_0}\right] ,
\end{aligned}$$

so that

$$\begin{aligned}
&\left|\frac{Tf(z_0) - Tf(z)}{z_0 - z} - \Pi f(z) - \frac{\overline{z_0 - z}}{z_0 - z} f(z)\right| \\
&\le M(\alpha)H_\alpha(f)|z_0 - z|^\alpha + |f(z)|\left|\frac{\phi(z_0) - \phi(z)}{z_0 - z} - \phi'(z)\right| .
\end{aligned}$$

Taking the directional limit from this equation

$$\lim_{\substack{|z_0-z|\to 0\\ z_0-z=|z_0-z|e^{i\varphi}}} \left\{\frac{Tf(z_0)-Tf(z)}{z_0-z} - \Pi f(z) - \frac{\overline{z_0-z}}{z_0-z} f(z)\right\} = 0$$

follows for any direction φ, i.e.

$$\lim_{\substack{|z_0-z|\to 0\\ z_0-z=|z_0-z|e^{i\varphi}}} \frac{Tf(z_0)-Tf(z)}{z_0-z} = \Pi f(z) + f(z)e^{-2i\varphi} .$$

For $\varphi = 0$ and $\varphi = \frac{\pi}{2}$ we get

$$\frac{\partial Tf(z)}{\partial x} = \Pi f(z) + f(z), \quad \frac{\partial Tf(z)}{\partial y} = i\Pi f(z) - if(z) ,$$

respectively, or

$$\begin{aligned} \frac{\partial Tf(z)}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) Tf(z) = f(z) , \\ \frac{\partial Tf(z)}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) Tf(z) = \Pi f(z) . \end{aligned}$$

Remark. From Theorems 23 and 29 we have

$$\begin{aligned} C_{1+\alpha}(Tf;\overline{D}) &:= C_0(Tf;\overline{D}) + C_\alpha((Tf)_z;\overline{D}) + C_\alpha((Tf)_{\overline{z}};\overline{D}) \\ &\leq M(\alpha, D) C_\alpha(f;\overline{D}) . \end{aligned}$$

Hence T is a completely continuous operator from $C^\alpha(\overline{D})$ into $C^{1+\alpha}(\overline{D})$.

Theorem 30. *Let D be a $C^{1+m+\alpha}$ domain, $f \in C^{m+\alpha}(\overline{D}), m \in \mathbb{N}_0, 0 < \alpha < 1$. Then $Tf \in C^{1+m+\alpha}(\overline{D})$ and*

$$C_{m+\alpha}(\Pi f;\overline{D}) \leq C_{m+1+\alpha}(Tf;\overline{D}) \leq M(m,\alpha,D) C_{m+\alpha}(f;\overline{D}) .$$

Remark. The norm of $f \in C^{m+\alpha}(\overline{D})$ is defined by

$$C_{m+\alpha}(f;D) := \sum_{\mu=0}^{m} \sum_{\nu=0}^{\mu} \max_{z\in\overline{D}} \left|\frac{\partial^\mu f(z)}{\partial x^{\mu-\nu}\partial y^\nu}\right| + \sum_{\nu=0}^{m} H_\alpha\left(\frac{\partial^m f(z)}{\partial x^{m-\nu}\partial y^\nu}\right) .$$

Here, obviously, the derivatives with respect to x and y may be replaced by those with respect to z and $\overline{z}$.

Proof. We only consider $m = 1$. Similarly, by induction we can proceed to the general case. As

$$C_{1+\alpha}(Tf;\overline{D}) \leq M(\alpha, D)C_\alpha(f;\overline{D})$$

just was shown and $\dfrac{\partial}{\partial \overline{z}}Tf = f$ it is enough to estimate $\dfrac{\partial}{\partial z}Tf = \Pi f$. For this aim we rewrite

$$\begin{aligned}
\Pi f(z) &= -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{D_\varepsilon} f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2} = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{D_\varepsilon} f(\zeta)\frac{\partial}{\partial\zeta}\frac{1}{\zeta - z}d\xi d\eta \\
&= -\frac{1}{\pi}\lim_{\varepsilon\to 0}\left[\int_{D_\varepsilon}\frac{\partial f(\zeta)}{\partial\zeta}\frac{d\xi d\eta}{\zeta - z} + \frac{1}{2i}\lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon} f(\zeta)\frac{d\overline{\zeta}}{\zeta - z}\right].
\end{aligned}$$

Now

$$\frac{1}{2\pi i}\int_{|\zeta - z| = \varepsilon} f(\zeta)\frac{d\overline{\zeta}}{\zeta - z} = -\frac{1}{2\pi}\int_0^{2\pi}(f(z + \varepsilon e^{i\varphi}) - f(z))e^{-2i\varphi}d\varphi - \frac{f(z)}{2\pi}\int_0^{2\pi} e^{-2i\varphi}d\varphi$$

tends to zero when ε does. Thus,

$$\Pi f(z) = \left(T\frac{\partial f}{\partial\zeta}\right)(z) - \frac{1}{2\pi i}\int_{\partial D} f(\zeta)\frac{d\overline{\zeta}}{\zeta - z}.$$

Because of the line integral being an analytic function

$$\frac{\partial}{\partial\overline{z}}\Pi f = \frac{\partial}{\partial\overline{z}}\left(T\frac{\partial f}{\partial\zeta}\right) = \frac{\partial f}{\partial z}, \quad \frac{\partial}{\partial z}\Pi f = \Pi\frac{\partial f}{\partial\zeta} - \frac{1}{2\pi i}\int_{\partial D} f(\zeta)\frac{d\overline{\zeta}}{(\zeta - z)^2}.$$

From here because of $f \in C^{1+\alpha}(\overline{D})$ and thus

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial D} f(\zeta)\frac{d\overline{\zeta}}{(\zeta - z)^2} &= -\frac{1}{2\pi i}\int_{\partial D} f(\zeta)\frac{\partial}{\partial\zeta}\frac{1}{\zeta - z}d\overline{\zeta} \\
&= \frac{1}{2\pi i}\int_{\partial D}\frac{\partial[f(\zeta)\overline{\zeta'(s)}^2]}{\partial\zeta}\frac{d\zeta}{\zeta - z},
\end{aligned}$$

which is HÖLDER continuous in $\overline{D}$ with exponent α because $f \in C^{1+\alpha}(\overline{D})$ and $\partial D \in C^{1+\alpha}$, we see $\Pi f \in C^{1+\alpha}(\overline{D})$.

Moreover,

$$
\begin{aligned}
C_\alpha\left(\frac{\partial}{\partial z}\Pi f;\overline{D}\right) &\le C_\alpha\left(\Pi\frac{\partial f}{\partial z};\overline{D}\right) + M(\alpha,D)H_\alpha\left(\frac{\partial f}{\partial z}\right)\\
&\le M(\alpha,D)\left[C_\alpha\left(\frac{\partial f}{\partial z};\overline{D}\right) + H_\alpha\left(\frac{\partial f}{\partial z}\right)\right]\\
&\le M(\alpha,D)C_{1+\alpha}(f;\overline{D})\,,\\
C_\alpha\left(\frac{\partial}{\partial \overline{z}}\Pi f;\overline{D}\right) &= C_\alpha\left(\frac{\partial f}{\partial z};\overline{D}\right) \le C_{1+\alpha}(f;\overline{D})\,,\\
C_0(\Pi f;\overline{D}) &\le C_\alpha(\Pi f;\overline{D}) \le M(\alpha,D)C_\alpha(f;\overline{D})\\
&\le M(\alpha,D)C_{1+\alpha}(f;\overline{D})\,.
\end{aligned}
$$

This, obviously, proves the assertion of Theorem 30 in the case $m = 1$.

Remark. While Tf is continuous in the whole plane $\mathbb{C}$ (Theorem 23) even if $f \in L_p(\overline{D}), 2 < p$, this is not true for Πf in general. As was just shown Πf is continuous in $\overline{D}$ under proper assumptions on f and D. It is analytic and hence continuous in $\widehat{\mathbb{C}}\setminus\overline{D}$. But it is discontinuous in general by passing through points of ∂D. Let $f \in C^{1+\alpha}(\overline{D})$ and let ∂D be $C^{2+\alpha}$. For $z \in D$

$$\Pi f(z) = \left(T\frac{\partial f}{\partial \zeta}\right)(z) - \frac{1}{2\pi i}\int_{\partial D} f(\zeta)\frac{d\overline{\zeta}}{\zeta - z}\,.$$

If $z \notin \overline{D}$ then the same relation holds where Πf is a proper integral. This can be shown as for the domain D_ε. Hence, for $\zeta \in \partial D, \zeta = \zeta(s), s$ arc length parameter,

$$(\Pi f)^+(\zeta) - (\Pi f)^-(\zeta) = -f(\zeta)\overline{\zeta'(s)}^2\,. \tag{2.2.5}$$

Next we study differentiability properties of the T operator when $D = \mathbb{C}$.

Theorem 31. *If $f \in L_p(\mathbb{C}) \cap C^\alpha(\mathbb{C})$ then*

$$Tf \in L_p(\mathbb{C}) \cap C^{1+\alpha}(\mathbb{C}), \quad \Pi f \in C^\alpha(\mathbb{C})\,.$$

Proof.

$$\phi(z) := \frac{1}{2\pi i}\int_{|\zeta|=R}\frac{\overline{\zeta}d\zeta}{\zeta - z} = \frac{R^2}{2\pi i z}\int_{|\zeta|=R}\left(\frac{1}{\zeta - z} - \frac{1}{\zeta}\right)d\zeta = 0, \quad 0 < |z| < R\,,$$

$$\phi(0) = \frac{R^2}{2\pi i} \int\limits_{|\zeta|=R} \frac{d\zeta}{\zeta^2} = 0 .$$

$$(\Pi_R f)(z) := -\frac{1}{\pi} \int\limits_{|\zeta|<R} f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}, \quad \Pi f = \lim_{R\to+\infty} \Pi_R f .$$

As in the proof of Theorem 29, step 2, $|\Pi_R f(z_2) - \Pi_R f(z_1)|$ can be estimated. Because $\phi = 0$ in $|z| < R$ the essential difference here is that the constant M is independent of the domain i.e. of R. We have

$$|\Pi_R f(z_2) - \Pi_R f(z_1)| \le M(\alpha)C_\alpha(f;\mathbb{C})|z_2 - z_1|^\alpha, \quad |z_1|, |z_2| < R ,$$

Letting R tend to infinity yields

$$|\Pi f(z_2) - \Pi f(z_1)| \le M(\alpha)C_\alpha(f;\mathbb{C})|z_2 - z_1|^\alpha, \quad z_1, z_2 \in \mathbb{C} .$$

From

$$\Pi_R f(z) = -\frac{1}{\pi} \int\limits_{|\zeta|<R} \frac{f(\zeta) - f(z)}{(\zeta - z)^2} d\xi d\eta - f(z)\phi'(z), \quad |z| < R ,$$

where $\phi'(z) \equiv 0$ it follows

$$\Pi f(z) = -\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{f(\zeta + z) - f(z)}{\zeta^2} d\xi d\eta - \frac{1}{\pi} \int\limits_{1\le|\zeta|} \frac{f(\zeta + z)}{\zeta^2} d\xi d\eta$$

because

$$\int\limits_{1\le|\zeta|} \frac{d\xi d\zeta}{\zeta^2} = \int_1^\infty \int_0^{2\pi} e^{-2i\varphi} d\varphi \frac{dr}{r} = 0 .$$

Hence,

$$|\Pi f(z)| \le \frac{2}{\alpha} H_\alpha(f) + \frac{1}{\pi} \|f\|_p \left(\int\limits_{1\le|\zeta|} |\zeta|^{-2q} d\xi d\eta \right)^{\frac{1}{q}}$$

$$= \frac{2}{\alpha} H_\alpha(f) + \frac{1}{\pi} \left(\frac{\pi}{q-1}\right)^{\frac{1}{q}} \|f\|_p \le M(\alpha, p)(\|f\|_p + H_\alpha(f)) .$$

Together with the above estimate of $H_\alpha(\Pi f)$ this gives

$$C_\alpha(\Pi f;\mathbb{C}) \le M(\alpha,p)(\|f\|_p + H_\alpha(f)) \le M(\alpha,p)(\|f\|_p + C_\alpha(f;\mathbb{C})) .$$

From (see proof of Theorem 29, step 3)

$$\left| \frac{T_R f(z_0) - T_R f(z)}{z_0 - z} - \Pi_R f(z) - \frac{\overline{z_0 - z}}{z_0 - z} f(z) \right| \le M(\alpha) H_\alpha(f) |z_0 - z|^\alpha ,$$

where $\phi = 0$ is again observed, letting R tend to infinity we see for $z \in \mathbb{C}$

$$\frac{\partial}{\partial \bar{z}} Tf(z) = f(z), \quad \frac{\partial}{\partial z} Tf(z) = \Pi f(z) .$$

At last we will extend the operator Π to a linear mapping from $L_p(\overline{D})$ into itself.

Lemma 11. *For $f, g \in C_0^\infty(\mathbb{C})$ we have*

$$(\Pi f, g) = (f, \overline{\Pi} g) .$$

Remark. The $\overline{\Pi}$–operator is defined by

$$(\overline{\Pi} f)(z) := -\frac{1}{\pi} \int_{\mathbb{C}} f(\zeta) \frac{d\xi d\eta}{\overline{(\zeta - z)^2}}$$

and (f, g) denotes the inner product

$$(f, g) := \int_{\mathbb{C}} f(z) \overline{g(z)} dx dy .$$

Π and $\overline{\Pi}$ are adjoint operators in $C_0^\infty(\mathbb{C})$. This is what Lemma 11 says.

Proof. For $f \in C_0^\infty(\mathbb{C})$ and $0 < R$ big enough

$$\Pi f = \frac{\partial T f}{\partial z} = T\left(\frac{\partial f}{\partial \zeta}\right) , \quad \frac{\partial \Pi f}{\partial \bar{z}} = \frac{\partial f}{\partial z} ,$$

$$\Pi_R f = \frac{\partial T_R f}{\partial z} = T_R\left(\frac{\partial f}{\partial \zeta}\right) , T_R f(z) := -\frac{1}{\pi} \int_{|\zeta|<R} f(\zeta) \frac{d\xi d\eta}{\zeta - z} .$$

The first of these formulae follows from (see Proof of Theorem 30)

$$\Pi_R f(z) = T_R\left(\frac{\partial f}{\partial \zeta}\right)(z) - \frac{1}{2\pi i} \int_{|\zeta|=R} f(\zeta) \frac{d\bar{\zeta}}{\zeta - z}$$

because f vanishes on $|\zeta| = R$ for big enough R. For the same reason for those R by interchanging the order of integrations

$$\begin{aligned} \int_{|z|<R} \bar{g} \Pi f dx dy &= \int_{|z|<R} \frac{\partial T f}{\partial z} \bar{g} dx dy &= - \int_{|z|<R} T f \frac{\partial \bar{g}}{\partial z} dx dy \\ &= \int_{\mathbb{C}} f T_R \left(\frac{\partial \bar{g}}{\partial \zeta}\right) dx dy &= \int_{\mathbb{C}} f \Pi_R \bar{g} dx dy = (f, \overline{\Pi_R} g) . \end{aligned}$$

Letting R tend to infinity then $(\Pi f, g) = (f, \overline{\Pi} g)$ follows.

Lemma 12. *For $f \in C_0^\infty(\mathbb{C})$ we have $\overline{\Pi}\,\Pi f = f$.*

Proof. Again we choose R so large that

$$\operatorname{supp} f \subset \{z : |z| < R\}$$

and observe

$$\overline{\Pi}\,\Pi f = \frac{\partial}{\partial \bar{z}} \overline{T}\left(\frac{\partial}{\partial z} T f\right) . \tag{2.2.6}$$

From the CAUCHY–POMPEIU formula (2.1.1′) applied to Tf,

$$Tf(z) = -\frac{1}{2\pi i} \int\limits_{|\zeta|=R} Tf(\zeta) \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{z}} - \frac{1}{\pi} \int\limits_{|\zeta|<R} \frac{\partial Tf(\zeta)}{\partial \zeta} \frac{d\xi d\eta}{\bar{\zeta} - \bar{z}} ,$$

by letting R tend to infinity we get

$$Tf(z) = -\frac{1}{\pi} \int\limits_{\mathbb{C}} \frac{\partial Tf(\zeta)}{\partial \zeta} \frac{d\xi d\eta}{\overline{\zeta - z}} = \overline{T}\left(\frac{\partial Tf}{\partial z}\right) = \overline{T}(\Pi f) .$$

This holds because for $R < |z|$

$$|Tf(z)| = \left| \frac{1}{\pi} \int\limits_{|\zeta|<R} f(\zeta) \frac{d\xi d\eta}{\zeta - z} \right| \leq \frac{R^2}{|z| - R} \max_{\zeta \in \mathbb{C}} |f(\zeta)| ,$$

so that $zTf(z)$ is bounded at infinity and hence

$$\lim_{R \to +\infty} \int\limits_{|\zeta|=R} Tf(\zeta) \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{z}} = 0 .$$

From

$$Tf = \overline{T}\left(\frac{\partial Tf}{\partial z}\right)$$

by differentiating we get

$$f = \frac{\partial}{\partial \bar{z}} Tf = \frac{\partial}{\partial \bar{z}} \overline{T}\left(\frac{\partial Tf}{\partial \zeta}\right) = \overline{\Pi}\,\Pi f .$$

Remark. $\overline{\Pi}\,\Pi$ is the identity operator on $C_0^\infty(\mathbb{C})$. In other words the adjoint operator $\overline{\Pi}$ is the inverse operator of Π and hence Π is a unitary operator in this space endowed with the L_2–norm.

Lemma 13. *For $f, h \in C_0^\infty(\mathbb{C})$ we have $(\Pi f, \Pi h) = (f, h)$.*

Proof. If we could apply Lemma 11 to f and Πh then we would have

$$(\Pi f, \Pi h) = (f, \overline{\Pi}\,\Pi h)\ .$$

But in general $\Pi h \notin C_0^\infty(\mathbb{C})$. On the other hand for $h \in C_0^\infty(\mathbb{C})$ we have (observe $\operatorname{supp} h \subset K_{R_0}(0)$)

$$|\Pi h(z)| \ \le\ \frac{1}{\pi} \int\limits_{|\zeta|<R_0} |h(\zeta)| \frac{d\xi d\eta}{|\zeta - z|^2} \ \le\ \frac{M R_0^2}{(|z| - R_0)^2},\quad R_0 < |z|\,,$$

$$|Th(z)| \ \le\ \frac{1}{\pi} \int\limits_{|\zeta|<R_0} |h(\zeta)| \frac{d\xi d\eta}{|\zeta - z|} \ \le\ \frac{M R_0^2}{|z| - R_0},\quad R_0 < |z|\,,$$

with $|h(z)| \le M$. This is enough to show

$$\lim_{R \to +\infty} \frac{1}{2\pi i} \int\limits_{|z|=R} \Pi h(\zeta) \frac{d\bar\zeta}{\zeta - z} = 0 \quad \text{for} \quad |z| < R$$

and

$$\lim_{R\to+\infty} \int\limits_{|z|<R} \frac{\partial}{\partial z}(Tf\overline{\Pi h})dxdy = -\lim_{R\to+\infty} \frac{1}{2i} \int\limits_{|z|=R} Tf\overline{\Pi h}d\bar z = 0\ .$$

Hence, similarly as in the proof of Lemma 11 we have $\Pi(\overline{\Pi h}) = T(\frac{\partial}{\partial z}\overline{\Pi h})$ as well as

$$(\Pi f, \Pi h) = -\int\limits_{\mathbb{C}} Tf\frac{\partial}{\partial z}\overline{\Pi h}dxdy = \int\limits_{\mathbb{C}} fT\left(\frac{\partial}{\partial z}\overline{\Pi h}\right)d\xi d\eta = \int\limits_{\mathbb{C}} f\Pi(\overline{\Pi h})d\xi d\eta = (f, \overline{\Pi}\Pi h).$$

Applying then Lemma 12 to h proves Lemma 13.

Corollary 6. *For $f \in C_0^\infty(\mathbb{C})$ we have $\|\Pi f\|_2 = \|f\|_2$.*

Proof. For $f = h$ Lemma 13 reads

$$(\Pi f, \Pi f) = (f, f,)$$

which is the assertion.

Remark. The L_2–norm of the linear operator Π is defined by

$$\|\Pi\|_2 = \sup_{f\neq0} \frac{\|\Pi f\|_2}{\|f\|_2} = 1\ .$$

Because $C_0^\infty(\mathbb{C})$ is a subset of $L_2(\mathbb{C})$ which is dense there, Π as well as $\overline{\Pi}$ are uniquely continuable onto all of $L_2(\mathbb{C})$ (see any book from functional analysis e.g. [Tayl58] or [Dusc66]).

Theorem 32. *For $f \in L_p(\mathbb{C})$ we have $\Pi f \in L_p(\mathbb{C})$ and*

$$\|\Pi f\|_p \leq \Lambda_p \|f\|_p \quad (1 < p) .$$

λ_p being the smallest such constant has the following property. Λ_p^p is a logarithmically convex function of p for $1 < p$ satisfying $\Lambda_2 = 1$.

For the proof we refer to [Veku62], p. 66–72.

Corollary 7. *For any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that*

$$|\Lambda_p - 1| < \varepsilon \quad \text{for} \quad |p - 2| < \delta(\varepsilon) .$$

Remark. $\Lambda_p = \|\Pi\|_p$ is the L_p–norm of the Π–operator. $f(x)$ is called logarithmically convex if $\log f(x)$ is convex, i.e.

$$\log f(x) \leq \frac{x - x_1}{x_2 - x_1} \log f(x_2) + \frac{x_2 - x}{x_2 - x_1} \log f(x_1)$$

$$= \log f(x_1) + \frac{\log f(x_2) - \log f(x_1)}{x_2 - x_1}(x - x_1), \quad x_1 < x < x_2 .$$

In Corollary 3 we have seen $\frac{\partial}{\partial z} Tf = \Pi f$ holds for $f \in C^\alpha(\overline{D})$. Now this result is extended to $L_p(\overline{D})$.

Theorem 33. *For $f \in L_p(\mathbb{C}), 1 < p$, then Tf has a generalized derivative with respect to z being equal to Πf.*

Remark. $\frac{\partial}{\partial \overline{z}} Tf = f$ for $f \in L_1(\overline{D})$ is shown in Theorem 26.

Proof. It has to be proved that for all $\varphi \in C_0^\infty(\mathbb{C})$

$$\int\limits_{\mathbb{C}} \left((Tf) \frac{\partial}{\partial z} \varphi + \varphi \Pi f \right) dx dy = 0 .$$

Let $f_n \in C_0^\infty(\mathbb{C})(n \in \mathbb{N})$ such that $\lim\limits_{n \to +\infty} \|f - f_n\|_p = 0$. Because with $D := \operatorname{supp} \varphi$

$$\int\limits_{\mathbb{C}} \left(Tf_n \frac{\partial}{\partial z} \varphi + \varphi \Pi f_n \right) dx dy = 0$$

and with $\frac{1}{p} + \frac{1}{q} = 1$

$$\left| \int\limits_{\mathbb{C}} \left[T(f - f_n) \frac{\partial}{\partial z} \varphi + \varphi \Pi (f - f_n) \right] dx dy \right|$$

$$\begin{aligned} &\leq \|T(f - f_n)\|_{p,D} \left\| \frac{\partial}{\partial z} \varphi \right\|_p + \|\Pi(f - f_n)\|_{p,D} \|\varphi\|_q \\ &\leq M(p, D) \|f - f_n\|_p \left\| \frac{\partial}{\partial z} \varphi \right\|_q + \Lambda_p \|f - f_n\|_p \|\varphi\|_q \end{aligned}$$

which becomes small for fixed φ when n is large, the assertion holds.

Theorem 34. *For $f \in L_p(\mathbb{C}), 1 < p$,*

$$\int\limits_{\mathbb{C}} f(\zeta) \frac{d\xi d\eta}{(\zeta - z)^2}$$

exists in the CAUCHY *principal value sense almost everywhere in $\mathbb{C}$ and*

$$\Pi f(z) = -\frac{1}{\pi} \int\limits_{\mathbb{C}} f(\zeta) \frac{d\xi d\eta}{(\zeta - z)^2} . \tag{2.2.7}$$

The proof is based on involved results of CALDERON–ZYGMUND [Cazy52,56], see [Veku62], p. 72.

3. Boundary value problems for generalized BELTRAMI equations

3.1 Generalized BELTRAMI equation

The CAUCHY-RIEMANN system, in complex form written as $w_{\bar{z}} = 0$ is a special form of an elliptic system of two real first order partial differential equations. The BELTRAMI system is a more general system of the same type and has in complex notation the form

$$w_{\bar{z}} = \mu w_z ,$$

where μ is a measurable function satisfying

$$|\mu(z)| \leq q_0 < 1 .$$

This condition guaranteeing strong ellipticity of the system is called ellipticity condition. Solutions to the BELTRAMI equation are quasiconformal mappings, a central subject in geometrical function theory. The main part of a general first order elliptic system is in complex form

$$w_{\bar{z}} + \mu_1 w_z + \mu_2 \overline{w_z}$$

with

$$|\mu_1(z)| + |\mu_2(z)| \leq q_0 < 1 .$$

Admitting lower order terms we get the equation

$$w_{\bar{z}} + \mu_1 w_z + \mu_2 \overline{w_z} + aw + b\overline{w} + c = 0 \tag{3.1.1}$$

which is the general form of the generalized BELTRAMI equation. For equations

$$w_{\bar{z}} + aw + b\overline{w} = 0$$

the theory of pseudoanalytic and of generalized analytic functions is developed. If $a, b \in C^{\alpha}(\overline{D})$ then w is pseudo-analytic, see [Bers53]. For $a, b \in L_p(\overline{D})$ the solutions are generalized analytic, see [Veku62]. Basic research of this equation is done by HAACK, too, see [Hawe72].

In this chapter results on RIEMANN, on RIEMANN–HILBERT and on related boundary value problems will be discussed. Before doing this some results on generalized BELTRAMI equations are reported on but not all of them will be proved here.

Theorem 35. *Let μ be a measurable function on $\mathbb{C}$, $\mu \in L_p(\mathbb{C}), 2 \leq p$, satisfying $|\mu(z)| \leq q_0 < 1, q_0\Lambda_p < 1$. Then there exists a homeomorphism of $\zeta(z)$ of $\mathbb{C}$ onto itself being a solution of the* BELTRAMI *equation*

$$\zeta_{\bar{z}} + \mu\zeta_z = 0$$

and being HÖLDER *continuous in $\mathbb{C}$.*

Remark. A homeomorphism of $\mathbb{C}$ onto itself is called a complete homeomorphism.

The proof of this theorem can be found in [Veku67], Chapter II, 5.

Proof. The existence of a solution $w(z)$ for $\mu \in L_p(\mathbb{C})$ for p close enough to 2 can easily be shown. One is looking for a solution ζ in the form

$$\zeta(z) = z + T\varphi(z)$$

with an unknown function $\varphi \in L_p(\mathbb{C})$ to be determined. Using the differentiability properties of T gives the integral equation for φ

$$\varphi(z) + \mu(z)\Pi\varphi(z) = -\mu(z) .$$

If p is so close to 2 that

$$q\Lambda_p < 1, \quad \Lambda_p := \|\Pi\|_p ,$$

then $\mu\Pi$ is a contractive mapping on $L_p(\mathbb{C})$. Hence, this equation is uniquely solvable. The solution φ may be found as a NEUMANN series

$$\varphi = -\sum_{n=0}^{+\infty} \mu(-\Pi\mu)^n$$

and satisfies

$$\|\varphi\|_p \leq q_0\Lambda_p\|\varphi\|_p + \|\mu\|_p$$

i.e.

$$\|\varphi\|_p \leq \frac{\|\mu\|_p}{1 - q_0\Lambda_p} .$$

Obviously, $\zeta(\infty) = \infty$ and $\lim_{z\to\infty} z^{-1}\zeta(z) = 1$, which follows from (see Theorem 23)

$$|T\varphi(z) - T\varphi(0)| \leq M(p)\|\varphi\|_p|z|^{\frac{p-2}{p}} .$$

These conditions determine the homeomorphism up to an arbitrary additive constant uniquely. That ζ takes every value of the complex plane exactly once follows from the argument principle. In order to prove this the following result is needed.

Lemma 14. *If w is a solution to the* BELTRAMI *equation*

$$w_{\bar{z}} + \mu w_z = 0$$

for $\mu \in L_p(\overline{D})$ in a domain D and ζ a homeomorphism of it then $w(z) = W(\zeta(z))$ with an analytic function W.

Proof.

$$\begin{aligned} w_{\bar{z}} + \mu w_z &= W_\zeta\zeta_{\bar{z}} + W_{\bar{\zeta}}\overline{\zeta_z} + \mu(W_\zeta\zeta_z + W_{\bar{\zeta}}\overline{\zeta_{\bar{z}}}) \\ &= W_\zeta(\zeta_{\bar{z}} + \mu\zeta_z) + W_{\bar{\zeta}}(\overline{\zeta_z} + \mu\overline{\zeta_{\bar{z}}}) = W_{\bar{\zeta}}(1 - |\mu|^2)\overline{\zeta_z} = 0 , \end{aligned}$$

compare [Mona83], p. 268. Because $\zeta_z \neq 0$ (the JACOBIan of ζ is $|\zeta_z|^2(1-|\mu|^2) \neq 0$) and $|\mu| < 1$ we see $W_{\overline{\zeta}} = 0$.

Remark. What is said about the solvability of the BELTRAMI equation and the representation of the solutions holds for the generalized equation

$$w_{\overline{z}} + \mu_1 w_z + \mu_2 \overline{w_z} = 0 \,,$$

too, if $\mu_1, \mu_2 \in L_p(\mathbb{C})$ and

$$|\mu_1(z)| + |\mu_2(z)| \leq q_0 < 1 \,.$$

We immediately see that

$$\widehat{\Pi}\varphi := \mu_1 \Pi\varphi + \mu_2 \overline{\Pi\varphi}$$

is a contraction on $L_p(\mathbb{C})$ for p close to 2. If w is a solution, then it satisfies

$$w_{\overline{z}} + \mu w_z = 0 \,,$$

$$\mu(z) := \begin{cases} \mu_1(z) + \mu_2(z)\dfrac{\overline{w_z(z)}}{w_z(z)} \,, & w_z(z) \neq 0 \,, \\ \mu_1(z) + \mu_2(z) \,, & w_z(z) = 0 \,, \end{cases}$$

where again μ is measurable and $|\mu(z)| \leq q_0 < 1$ but now depends on the solution w. But still $w(z) = W(\zeta(z))$ where ζ is a complete homeomorphism of $\zeta_{\overline{z}} + \mu\zeta_z = 0$ and W is analytic. The homeomorphism now depends on the solution w.

Continuation of Proof of Theorem 35. The proof is completed in the following three steps.

i. At first the argument principle is shown to hold for solutions to the BELTRAMI equation when $\mu \in L_p(\mathbb{C})$ is HÖLDER–continuous. Applying this result ζ can be shown to take every value of $\mathbb{C}$ just once.

ii. Assuming $\mu \in L_p(\mathbb{C})$ having compact support and approximating μ by HÖLDER–continuous functions the existence of a complete homeomorphism being HÖLDER–continuous itself is shown.

iii. The general case $\mu \in L_p(\mathbb{C})$ finally is treated by reduction to case ii.

Step i. In order to prove the argument principle the zeroes of solutions to the BELTRAMI equation have to be shown to be isolated. For this purpose we first prove the existence of a local homeomorphism $\zeta \in C^{1+\alpha}(\overline{U})$ for the BELTRAMI equation $w_{\overline{z}} = \mu w_z$ where $\mu \in C^\alpha(U_0), 0 < \alpha < 1$. Here U_0 is a given neighborhood of some point $z_0 \in \mathbb{C}$ and $U \subset U_0$ is a proper neighborhood of z_0. This is done as follows. Denote $\mu_0 = \mu(z_0)$ and

$$t := z - z_0 + \mu_0(\overline{z - z_0}) .$$

Then $W(t) := w(z(t)), z(t) := z_0 + (t - \mu_0\bar{t})/(1 - |\mu_0|^2)$, satisfies

$$w_{\bar{z}} - \mu w_z = W_t(t_{\bar{z}} - \mu t_z) + W_{\bar{t}}(\overline{t_z} - \mu\overline{t_{\bar{z}}}) = (1 - \overline{\mu}_0\mu)\left(W_{\bar{t}} - \frac{\mu - \mu_0}{1 - \overline{\mu_0}\mu}W_t\right) = 0 ,$$

i.e.

$$W_{\bar{t}} - qW_t = 0, \quad q(t) = \frac{\mu(z(t)) - \mu_0}{1 - \overline{\mu_0}\mu(z(t))} .$$

From $|\mu(z)| \le q_0 < 1$ it follows

$$\left|\frac{\mu - \mu_0}{1 - \overline{\mu_0}\mu}\right| = \left|\frac{\mu - \mu_0}{q_0^2 - \overline{\mu_0}\mu}\right| \left|\frac{q_0^2 - \overline{\mu_0}\mu}{1 - \overline{\mu_0}\mu}\right| \le \frac{1}{q_0}\left|\frac{q_0^2 - \overline{\mu_0}\mu}{1 - \overline{\mu_0}\mu}\right| .$$

Let $\alpha := \overline{\mu_0}\mu$ and observe $|\alpha| \le q_0^2$, then

$$|q_0^2 - \alpha|^2 - |q_0(1 - \alpha)|^2 = (1 - q_0^2)(|\alpha|^2 - q_0^2) \le -q_0^2(1 - q_0^2)^2 .$$

Hence,

$$\frac{|q_0^2 - \alpha|^2}{q_0^2|1 - \alpha|^2} \le 1 - \frac{q_0^2(1 - q_0^2)^2}{q_0^2|1 - \alpha|^2} \le 1 - \frac{(1 - q_0^2)^2}{(1 + q_0^2)^2} = \frac{4q_0^2}{(1 + q_0^2)^2} ,$$

so that

$$|q(t)| \le \frac{2q_0}{1 + q_0^2} =: \widetilde{q}_0 < 1 .$$

Moreover, from

$$|q(t_1) - q(t_2)| = \left|\frac{(\mu_1 - \mu_2)(1 - |\mu_0|^2)}{(1 - \overline{\mu_0}\mu_1)(1 - \overline{\mu_0}\mu_2)}\right| \le \frac{|\mu_1 - \mu_2|(1 - |\mu_0|^2)}{(1 - |\mu_0|q_0)^2} \le \frac{|\mu_1 - \mu_2|}{1 - q_0^2}$$

where $\mu_k := \mu(z(t_k)), k = 1, 2$, it follows $q \in C^\alpha(\widetilde{U}_0), \widetilde{U}_0 := t[U_0]$. Because $q(0) = 0$ thus $|q(t)| \le M|t|^\alpha$ as well as $|q(t_1) - q(t_2)| \le M|t_1 - t_2|^\alpha$ for $|t|, |t_1|, |t_2| \le \delta \le 1$ and $0 < M$ with δ and M properly chosen. Then defining $\widetilde{q}_\delta(t)$ by $q(t)$ for $2|t| < \delta$, by 0 for $\delta < |t|$ and by $2q(t)(1 - |t|/\delta)$ for $\delta \le 2|t| \le 2\delta$ this function belongs to $C^\alpha(\mathbb{C})$ satisfying

$$|\widetilde{q}_\delta(t)| \le M|t|^\alpha, |\widetilde{q}_\delta(t_1) - \widetilde{q}_\delta(t_2)| \le 3M|t_1 - t_2|^\alpha .$$

$$|\widetilde{q}_\delta(t_1) - \widetilde{q}_\delta(t_2)| \le |q(t_1) - q(t_2)| + \frac{1}{\delta}|q(t_2)||t_2 - t_1| \le 3M|t_2 - t_1|^\alpha$$
$$\text{for } \delta/2 \le |t_1|, |t_2| \le \delta ,$$

$$|\widetilde{q}_\delta(t_1) - \widetilde{q}_\delta(t_2)| \le |q(t_1) - q(t_2)| + |q(t_2)|(2|t_2|/\delta - 1) \le 3M|t_2 - t_1|^\alpha$$
$$\text{for } |t_1| < \delta/2 \le |t_2| \le \delta ,$$

$$|\widetilde{q}_\delta(t_1) - \widetilde{q}_\delta(t_2)| = |q(t_1)| \le M|t_1|^\alpha \le M(\delta/2)^\alpha \le M|t_2 - t_1|^\alpha \text{ for } 2|t_1| < \delta < |t_2| ,$$

$$|\widetilde{q}_\delta(t_1) - \widetilde{q}_\delta(t_2)| = 2|q(t_1)|(1 - \frac{|t_1|}{\delta}) \le 2M|t_1|^\alpha(|t_2| - |t_1|)^\alpha/\delta^\alpha \le 2M||t_2 - t_1|^\alpha$$
$$\text{for } \delta/2 \le |t_1| \le \delta < |t_2| .$$

Denoting the set of HÖLDER–continuous functions in $\mathbb{C}$ with compact support in $K_\delta := \{|t| < \delta\}$ by $C_0^\alpha(K_\delta)$ the operator Π_δ defined by

$$\Pi_\delta f(t) := \widetilde{q}_\delta(t)\Pi f(t)\,,\ f \in C_0^\alpha(K_\delta)\,,$$

maps $C_0^\alpha(K_\delta)$ linearly into itself. Moreover, Π_δ is a bounded operator there because for $f \in C_0^\alpha(K_\delta)$

$$\Pi f(t) = -\frac{1}{\pi}\int\limits_{\mathbb{C}} \frac{f(\zeta)}{(\zeta - t)^2} d\xi d\eta = -\frac{1}{\pi}\int\limits_{K_\delta} \frac{f(\zeta) - f(z)}{(\zeta - t)^2} d\xi d\eta$$

implies

$$|\Pi f(t)| \le \frac{1}{\pi} H_\alpha(f) \int\limits_{K_\delta} \frac{d\xi d\eta}{|\zeta - t|^{2-\alpha}} \le 2\frac{(2\delta)^\alpha}{\alpha} H_\alpha(f) \le \frac{4\delta^\alpha}{\alpha} H_\alpha(f)\,,$$

$$|\Pi f(t_1) - \Pi f(t_2)| \le M(\alpha) H_\alpha(f) |t_1 - t_2|^\alpha\,,$$

see p. 88. Let now $\delta > 0$ be chosen so small that

$$C_\alpha(\widetilde{q}_\delta \Pi f; K_\delta) \le \widetilde{M}(\delta, \alpha) C_\alpha(f; K_\delta)$$

with $\widetilde{M}(\delta, \alpha) < 1$.
We are now looking for a solution $W = W(t)$ to the BELTRAMI equation $W_{\bar t} - \widetilde{q}_\delta W_t = 0$ in the form

$$W(t) = t - \frac{1}{\pi}\int\limits_{K_\delta} \frac{f(\zeta)}{\zeta - t} d\xi d\eta = t + Tf\,,\ f \in C_0^\alpha(K_\delta)\,.$$

Then f satisfies the singular integral equation

$$f - \widetilde{q}_\delta \Pi f = \widetilde{q}_\delta\,.$$

As Π_δ is a contractive mapping on $C_0^\alpha(K_\delta)$ this equation is uniquely solvable. As $\widetilde{q}_\delta(0) = q(0) = 0$ also $f(0) = 0$. Moreover, the solution satisfies the estimate

$$C_\alpha(f; K_\delta) \le \frac{C_\alpha(\widetilde{q}_\delta; K_\delta)}{1 - \widetilde{M}(\delta, \alpha)} < \frac{1}{4}\frac{\alpha}{(2\delta)^\alpha}$$

where the last inequality can be achieved by choosing $\delta > 0$ small enough. Therefore $W(t) = t + Tf$ belongs to $C^{1+\alpha}(K_\delta)$. For the JACOBIan

$$\begin{aligned} |W_t|^2 - |W_{\bar t}|^2 &= (1 - |\widetilde{q}_\delta|^2)|1 + \Pi f|^2 \\ &\ge (1 - \widetilde{q}_0^2)(1 - 2\frac{(2\delta)^\alpha}{\alpha} C_\alpha(f; K_\delta))^2 > 0 \end{aligned}$$

the function $W(t)$ is one-to-one in some neighborhood K_{δ_0} of the origin, $0 < \delta_0 \le \delta$. Hence, because $t = t(z)$ is an affine transformation

$$\zeta(z) := W(z - z_0 + \mu(z_0)(\overline{z - z_0}))$$

is a bijective mapping in some neighborhood $U \subset U_0$ of z_0, mapping U onto $\zeta[U]$ and satisfying $\zeta_{\bar z} = \mu\zeta_z$.

On the basis of this result the zeroes of a non-constant solution to the BELTRAMI equation $w_{\bar z} - \mu w_z = 0$ in a domain D can be shown to be isolated when $\mu \in C^\alpha(D)$. Moreover, if $w(z_0) = 0$ for some $z_0 \in D$ then locally - in the neighborhood of z_0 - w can be represented as

$$w(z) = [z - z_0 + \mu(z_0)(\overline{z - z_0})]^n \tilde{w}(z)\,,\ \tilde{w}(z_0) \ne 0\,,$$

where $n \in I\!N$ is uniquely given and $\tilde{w}$ is some proper HÖLDER-continuous function. To show this we utilize a local homeomorphism ζ in the neighborhood of z_0 and apply Lemma 14. This guarantees the representation $w(z) = \phi(\zeta(z))$ in the neighborhood of z_0 where ϕ is some analytic function. Hence, w has isolated zeroes. Denoting the order of the zero of ϕ in $\zeta(z_0)$ by n we have

$$w(z) = (\zeta(z) - \zeta(z_0))^n \phi_0(\zeta(z)), \phi_0(\zeta(z_0)) \ne 0\,,$$

with some proper analytic function ϕ_0 in the neighborhood of $\zeta(z_0)$. Using the above notations

$$\begin{aligned}
\zeta(z) - \zeta(z_0) &= W(z - z_0 + \mu(z_0)(\overline{z - z_0})) - W(0) = W(t(z)) - W(0) \\
&= t(z) + Tf(t(z)) - Tf(0) = t(z)\left(1 - \frac{1}{\pi}\int_{K_\delta} f(\tilde\zeta)\frac{d\tilde\xi d\tilde\eta}{\tilde\zeta(\tilde\zeta - t(z))}\right) \\
&= t(z)\widetilde{W}(t(z))\,.
\end{aligned}$$

From the HÖLDER condition $|f(t)| \le H_\alpha(f)|t|^\alpha$ - observe $f(0) = 0$ - we have $t^{-1}f(t) \in L_p(I\!\!C)$ for $1 \le p < 2(1-\alpha)^{-1}$, so that $\widetilde{W}(t) \in C^{\alpha_0}(I\!\!C)$ with $\alpha_0 := \dfrac{p-2}{p}$ for $2 < p$. Because

$$\begin{aligned}
|\widetilde{W}(0)| &\ge 1 - \frac{1}{\pi}\int_{K_\delta} |f(\tilde\zeta)|\frac{d\tilde\xi d\tilde\eta}{|\tilde\zeta|^2} \ge 1 - \frac{C_\alpha(f;K_\delta)}{\pi}\int_{K_\delta}\frac{d\tilde\xi d\tilde\eta}{|\tilde\zeta|^{2-\alpha}} \\
&= 1 - \frac{(2\delta)^\alpha}{\alpha}C_\alpha(f;K_\delta) \ge 1/2
\end{aligned}$$

for small enough $\delta > 0$ it follows $\widetilde{W}(0) \ne 0$. On the basis of this local behaviour the argument principle can be proved.

Argument Principle. *Any non-constant solution w to $w_{\bar z} + \mu w_z = 0$ in $D, \mu \in C^\alpha(D)$, being continuous on $\overline{D}$ and non-vanishing on ∂D has only finitely many zeroes* *in D. Counting with respect to multiplicities their total number in D is*

$$N = \frac{1}{2\pi} \int_{\partial D} d \arg w(z) .$$

Because the zeroes are isolated there are only finitely many. Let $w(z_\nu) = 0, 1 \le \nu \le N$, where any zero is listed with respect to its multiplicity. The local behaviour of w near z_ν shows

$$w(z) = w_0(z) \prod_{\nu=1}^{N} [z - z_\nu + \mu(z_\nu)(\overline{z - z_\nu})]$$

with some continuous function w_0 not vanishing in $\overline{D}$. Therefore

$$\frac{1}{2\pi} \int_{\partial D} d \arg w(z) = \frac{1}{2\pi} \int_{\partial D} d \arg w_0(z) + \sum_{\nu=1}^{N} \frac{1}{2\pi} \int_{\partial D} d \arg[z - z_\nu + \mu(z_\nu)(\overline{z - z_\nu})] = N .$$

We are now in the position where we are able to prove that the non-constant solution $\zeta(z) = z + T\varphi(z)$ to the BELTRAMI equation with HÖLDER-continuous coefficient μ takes any complex value once. Obviously, from the properties of the T-operator, see Theorem 23, $\zeta(z) = z(1 + O(|z|^{-2/p}))$ as z tends to infinity. This estimation holds too for $\zeta(z) - \zeta_0$ for any finite complex ζ_0, this function being a solution to $w_{\bar{z}} - \mu w_z = 0$ as is $\zeta(z)$. For sufficiently large $R > 0$ we have

$$\frac{1}{2\pi} \int_{|z|=R} d \arg(\zeta(z) - \zeta_0) = \frac{1}{2\pi} \int_{|z|=R} d \arg z = 1 .$$

By the argument principle thus ζ is seen to have just one ζ_0-point in $|z| < R$ for sufficiently large R i.e. for any $R > R_0$ with a proper $R_0 \ge 0$. From the asymptotic behaviour at infinity we also see $\zeta(\infty) = \infty$ and $\lim_{z \to \infty} z^{-1}\zeta(z) = 1$.

Step ii. Let us assume $\mu \in L_p(\mathbb{C})$ satisfies $\mu(z) = 0$ in $|z| \ge R$ but is not necessarily HÖLDER-continuous in $K_R := \{|z| < R\}$. From

$$\left(\int_{|z|<R} |\mu(z)|^{p'} dx dy \right)^{1/p'} \le (\pi R^2)^{(p-p')/pp'} \left(\int_{|z|<R} |\mu(z)|^{p} dx dy \right)^{1/p} , p' < p ,$$

it follows $\mu \in L_{p'}(\mathbb{C})$ for $1 \le p' \le p$, which only holds because of μ having compact support in $\mathbb{C}$. As $C_0^\alpha(K_R)$ is a dense subset in $L_{p'}(K_R)$, see [Adam75], p. 31, there is a sequence $(\mu_n), \mu_n \in C_0^\alpha(K_R)$ such that $\lim_{n \to \infty} \mu_n = \mu$ in $L_{p'}(K_R)$ for any $1 \le p'$. Although (μ_n) may depend on the space $L_{p'}(K_R)$ considered, from

$$\|\mu_n - \mu\|_{p'} \le M(p', p, R) \|\mu_n - \mu\|_p, M(p', p, R) := (\pi R^2)^{(p-p')/pp'} ,$$

it follows that it converges to μ in any $L_{p'}(K_R)$ if it does in $L_p(K_R), 1 \le p' \le p$. Let us now assume (μ_n) converges to μ in $L_{\widetilde{p}}(K_R)$ where $2 \le p < \widetilde{p}, \widetilde{q}$ and $1/p = 1/\widetilde{p} + 1/\widetilde{q}, q_0\Lambda_{\widetilde{q}} < 1$. Denote $\zeta_n(z) := z + T\varphi_n(z)$ where φ_n is the solution to $\varphi_n - \mu_n \Pi\varphi_n = \mu_n$ i.e. ζ_n is a homeomorphism to the BELTRAMI equation $\zeta_{n\bar{z}} = \mu_n \zeta_{nz}$, see step i. Then

$$\begin{aligned}
\|\varphi_n\|_p &= \frac{\|\mu_n\|_p}{1 - q_0\Lambda_p} \le \frac{q_0(\pi R^2)^{1/p}}{1 - q_0\Lambda_p}, \; q_0\Lambda_p < 1 , \\
\varphi_n - \varphi_m &= \mu_n \Pi(\varphi_n - \varphi_m) + (\mu_n - \mu_m)\Pi\varphi_m + \mu_n - \mu_m , \\
\|\varphi_n - \varphi_m\|_p &\le \frac{1}{1 - q_0\Lambda_p}[\|(\mu_n - \mu_m)\Pi\varphi_m\|_p + \|\mu_n - \mu_m\|_p] \\
&\le \frac{1}{1 - q_0\Lambda_p}[\|\mu_n - \mu_m\|_{\widetilde{p}}\|\Pi\varphi_m\|_{\widetilde{q}} + M(p, \widetilde{p}, R)\|\mu_n - \mu_m\|_{\widetilde{p}}] \\
&\le \frac{1}{1 - q_0\Lambda_p}\left[\frac{q_0\Lambda_{\widetilde{q}}(\pi R^2)^{1/\widetilde{q}}}{1 - q_0\Lambda_{\widetilde{q}}} + M(p, \widetilde{p}, R)\right]\|\mu_n - \mu_m\|_{\widetilde{p}} .
\end{aligned}$$

Hence, (φ_n) is a CAUCHY-sequence in $L_p(K_R)$ which because of the completeness of this space converges to $\varphi \in L_p(K_R)$, say. This limit function turns out to satisfy $\varphi - \mu\Pi\varphi = \mu$. Setting $\zeta(z) := z + T\varphi(z)$, then $\zeta - \zeta_n = T(\varphi - \varphi_n)$ and

$$C_{\alpha_0}(\zeta - \zeta_n; \mathbb{C}) \le M(p, q_0)\|\varphi - \varphi_n\|_p, \alpha_0 := \frac{p-2}{p} .$$

Therefore (ζ_n) converges uniformly on K_R to $\zeta \in C^{\alpha_0}(\mathbb{C})$. It remains to show that ζ is a homeomorphism from $\mathbb{C}$ onto $\mathbb{C}$. We know already that ζ_n is such a homeomorphism. Let $z_n(\zeta)$ denote its inverse mapping. From $z_n(\zeta_n(z)) \equiv z$ we see

$$z_{n\zeta}\zeta_{n\bar{z}} + z_{n\bar{\zeta}}\overline{\zeta_{nz}} = z_{n\bar{z}} = 0 , \; z_{n\zeta}\zeta_{nz} + z_{n\bar{\zeta}}\overline{\zeta_{n\bar{z}}} = z_{nz} = 1 ,$$

so that

$$z_{n\zeta} = \frac{\overline{\zeta_{nz}}}{|\zeta_{nz}|^2 - |\zeta_{n\bar{z}}|^2} , \; z_{n\bar{\zeta}} = -\frac{\zeta_{n\bar{z}}}{|\zeta_{nz}|^2 - |\zeta_{n\bar{z}}|^2} , \; J_n := |\zeta_{nz}|^2 - |\zeta_{n\bar{z}}|^2$$

i.e.

$$z_{n\bar{\zeta}} + \mu_n(z_n(\zeta))\overline{z_{n\zeta}} = 0 .$$

This is a quasilinear special generalized BELTRAMI equation for $z_n(\zeta)$. But when $z_n(\zeta)$ is known it satisfies a linear BELTRAMI equation

$$z_{n\bar{\zeta}} + \widetilde{\mu}_n z_{n\zeta} = 0 , \; \widetilde{\mu}_n(\zeta) := \mu_n(z_n(\zeta))\frac{\overline{z_{n\zeta}(\zeta)}}{z_{n\zeta}(\zeta)} , \; |\widetilde{\mu}_n(\zeta)| \le q_0 < 1 .$$

Next (z_n) will be shown to be compact. From

$$|\zeta_n(z) - z| = |T\varphi_n(z)| \leq M(p,R)\|\varphi_n\|_p \leq M(p,q_0,R) \quad \text{for} \quad |z| < R$$

we have with $\zeta = \zeta_n(z), z = z_n(\zeta)$ for those ζ for which $|z_n(\zeta)| < R$ the estimate

$$|\zeta - z_n(\zeta)| \leq M(p,q_0,R)\ .$$

For these ζ then

$$|\zeta| \leq |z_n(\zeta)| + |\zeta - z_n(\zeta)| \leq R + M(p,q_0,R) =: R_1\ .$$

If now $R_1 < |\zeta|$ then $R < |z_n(\zeta)|$. But there $z_{n\bar{\zeta}} = 0$ because μ_n vanishes outside K_R, so that z_n is analytic outside K_R. Denote $z_{n\bar{\zeta}}$ by $\widetilde{\varphi}_n$. Then $\phi_n(\zeta) := z_n(\zeta) - \zeta - T\widetilde{\varphi}_n$ is analytic in the complex plane. Moreover, from the boundedness of $z_n(\zeta) - \zeta$ in $|\zeta| \leq R_1$ and that of $T\widetilde{\varphi}_n$ in $\mathbb{C}$ and

$$\lim_{\zeta\to\infty}[z_n(\zeta) - \zeta] = \lim_{z\to\infty}[z - \zeta_n(z)] = 0\ ,\ \lim_{\zeta\to\infty} T\widetilde{\varphi}_n(\zeta) = 0$$

it follows $\phi_n = 0$. Hence $z_n(\zeta) = \zeta + T\widetilde{\varphi}_n(\zeta)$ where $\widetilde{\varphi}_n(\zeta) = 0$ outside K_{R_1}. Inserting $z_{n\bar{\zeta}} = \widetilde{\varphi}_n, z_{n\zeta} = 1 + \Pi\widetilde{\varphi}_n$ in the differential equation for z_n we get the singular integral equation

$$\widetilde{\varphi}_n = -\mu_n(z_n(\zeta))\overline{\Pi\widetilde{\varphi}_n} - \mu_n(z_n(\zeta))\ .$$

With $|\mu_n(z_n(\zeta))| \leq q_0 < 1$ and $q_0\Lambda_p < 1$ this equation shows

$$\|\widetilde{\varphi}_n\|_p \leq \frac{q_0(\pi R_1^2)^{1/p}}{1 - q_0\Lambda_p}\ .$$

Because T is a compact operator on $L_p(K_{R_1}), 2 < p, z_n(\zeta) = \zeta + T\widetilde{\varphi}_n(\zeta)$ is compact. Let (z_{n_k}) be a convergent subsequence with limit $z(\zeta) \in C^{\alpha_0}(\mathbb{C}), \alpha_0 := 1 - 2/p$. From $\zeta_{n_k}(z_{n_k}(\zeta)) \equiv \zeta, z_{n_k}(\zeta_{n_k}(z)) \equiv z, \lim\limits_{k\to\infty}\zeta_{n_k}(z) = \zeta(z), \lim\limits_{k\to\infty} z_{n_k}(\zeta) = z(\zeta)$ it follows $\zeta(z(\zeta)) \equiv \zeta, z(\zeta(z)) \equiv z$. Hence, ζ is invertible with inverse $z(\zeta)$ and thus a homeomorphism from $\mathbb{C}$ onto $\mathbb{C}$ with infinity as fixed point as is its inverse $z(\zeta)$, too. As $\zeta(z) = \lim\limits_{n\to\infty}\zeta_n(z)$ the entire sequence $(z_n(\zeta))$ converges to $z(\zeta)$.

Step iii. Let now $\mu \in L_p(\mathbb{C})$ be arbitrary, $|\mu(z)| \leq q_0 < 1$. Define

$$\mu_R(z) := \begin{cases} \mu(z) & ,\quad |z| \leq R \\ 0 & ,\quad R < |z| \end{cases}$$

and let w_R be the complete homeomorphism of the BELTRAMI equation $w_{R\bar{z}} + \mu_R w_{Rz} = 0, w_R \in C^{\alpha_0}(\mathbb{C})$, see step ii. Then $\zeta(z) := [w_R(z) - w_R(0)]^{-1}, \zeta(0) = \infty, \zeta(\infty) = 0$ is a homeomorphism of the same BELTRAMI equation too, mapping the complex sphere onto itself. Let $z(\zeta)$ be its inverse. Then the transformation $\widetilde{w}(\zeta) := w(z(\zeta))$

converts a solution w to the BELTRAMI equation $w_{\bar z} + \mu w_z = 0$ into a solution $\widetilde{w}$ to $\widetilde{w}_{\bar\zeta} - \mu_1 \widetilde{w}_\zeta = 0, \mu_1 := \dfrac{\mu - \mu_R}{1 - \overline{\mu_R}\mu}\dfrac{\zeta_z}{\overline{\zeta_z}}$ where $|\mu_1| \le q_0' < 1$ with $q_0' := \dfrac{2q_0}{1 + q_0^2} < 1$, see step i. This can be seen from

$$w_{\bar z} = \widetilde{w}_\zeta \zeta_{\bar z} + \widetilde{w}_{\bar\zeta}\overline{\zeta_z}\,,\; w_z = \widetilde{w}_\zeta \zeta_z + \widetilde{w}_{\bar\zeta}\overline{\zeta_{\bar z}}$$

which implies

$$\begin{aligned}\widetilde{w}_\zeta &= -\frac{w_{\bar z}\overline{\zeta_{\bar z}} - w_z\overline{\zeta_z}}{|\zeta_z|^2 - |\zeta_{\bar z}|^2} = \frac{1 - \mu\overline{\mu_R}}{|\zeta_z|^2 - |\zeta_{\bar z}|^2}\overline{\zeta_z} w_z\,,\\ \widetilde{w}_{\bar\zeta} &= -\frac{w_z\zeta_{\bar z} - w_{\bar z}\zeta_z}{|\zeta_z|^2 - |\zeta_{\bar z}|^2} = \frac{\mu - \mu_R}{|\zeta_z|^2 - |\zeta_{\bar z}|^2}\zeta_z w_z\,.\end{aligned}$$

As

$$\mu_1(z) = \begin{cases} 0 & , \ |z| \le R \\ \mu(z) & , \ R < |z| \end{cases}$$

the function $\mu_1(z(\zeta))$ vanishes in the neighborhood $R_1 < |\zeta|$ of infinity because the inverse mapping $z(\zeta)$ to $\zeta(z)$ maps neighborhoods of infinity onto vicinities of zero and vice versa as $\zeta(z)$ does, too. Again the result from step ii guarantees the existence of a complete homeomorphism $\widetilde{w}_1 = \widetilde{w}_1(\zeta)$ of the BELTRAMI equation $\widetilde{w}_{1\bar\zeta} = \mu_1\widetilde{w}_{1\zeta}$. Then $\widehat{w}(z) := \widetilde{w}_1(\zeta(z))$ is a homeomorphism *too*, satisfying $\widehat{w}_{\bar z} = \mu\widehat{w}_z$ as follows from

$$\widehat{w}_z = \widetilde{w}_{1\zeta}\zeta_z + \widetilde{w}_{1\bar\zeta}\overline{\zeta_{\bar z}}\,,\; \widehat{w}_{\bar z} = \widetilde{w}_{1\zeta}\zeta_{\bar z} + \widetilde{w}_{1\bar\zeta}\overline{\zeta_z}\,.$$

Namely, for $R < |z|$ where $\mu_{R_1} = 0$ we have $\zeta_{\bar z} = 0$ so that

$$\widehat{w}_z = \widetilde{w}_{1\zeta}\zeta_z\,,\; \widehat{w}_{\bar z} = \widetilde{w}_{1\bar\zeta}\overline{\zeta_z} : \widehat{w}_{\bar z} = \mu_1\frac{\overline{\zeta_z}}{\zeta_z}\widehat{w}_z = \mu\widehat{w}_z\,,$$

and for $|z| \le R$ where $\mu_1 = 0$ we have $\widetilde{w}_{1\bar\zeta} = 0$ so that

$$\widehat{w}_z = \widetilde{w}_{1\zeta}\zeta_z\,,\; \widehat{w}_{\bar z} = \widetilde{w}_{1\zeta}\zeta_{\bar z} : \widehat{w}_{\bar z} = \mu_R\widehat{w}_z = \mu\widehat{w}_z\,.$$

But as $\widehat{w}(\infty) = \widetilde{w}_1(0), \widehat{w}(0) = \widetilde{w}_1(\infty) = \infty$ we again take the reciprocal

$$w(z) := [\widetilde{w}_1(\zeta(z)) - \widetilde{w}_1(0)]^{-1}\,.$$

This function is a complete homeomorphism of the BELTRAMI equation $w_{\bar z} = \mu w_z$, leaving infinity fixed and being HÖLDER-continuous.

Remark. Let $\zeta(z)$ be a particular homeomorphism of the BELTRAMI equation $w_{\bar z} = \mu w_z$ mapping $\mathbb{C}$ onto itself then any homeomorphism $\widetilde{\zeta}$ of the same kind is a linear transformation of $\zeta(z)$

$$\widetilde{\zeta}(z) = \frac{\alpha\zeta(z) + \beta}{\gamma\zeta(z) + \delta}\,,\; \alpha\delta - \gamma\beta \ne 0\,.$$

This follows from the representation formula $\tilde{\zeta} = \phi \circ \zeta$ with an analytic function ϕ which has to be a schlicht mapping of the RIEMANN sphere onto itself i.e. a linear transformation. If both ζ and $\tilde{\zeta}$ leaving infinity fixed then $\tilde{\zeta}(z) = \alpha\zeta(z) + \beta$. The condition $\lim\limits_{z\to\infty} z^{-1}\tilde{\zeta}(z) = \lim\limits_{z\to\infty} z^{-1}\zeta(z) = 1$ forces α to be 1. Finally if, moreover, $\tilde{\zeta}$ coincides with ζ in some finite point e.g. $\tilde{\zeta}(0) = \zeta(0)$ then $\beta = 0$. These two conditions $\lim\limits_{z\to\infty} z^{-1}\zeta(z) = 1, \zeta(0) = 0$ prescribe the homeomorphism of the BELTRAMI equation uniquely.

Lemma 15. *For $f \in L_{(p,p')}(\overline{D}) := L_p(\overline{D}) \cap L_{p'}(\overline{D})$ where D is an unbounded domain and $1 < p' < 2 < p$ we have*

$$\begin{aligned} |Tf(z)| &\le M(p,p')\,(\|f\|_p + \|f\|_{p'})\ , \quad z \in \mathbb{C}\ , \\ |Tf(z_1) - Tf(z_2)| &\le M(p,p')\,(\|f\|_p + \|f\|_{p'})\,|z_1 - z_2|^{\alpha_0}\ , \\ \alpha_0 &= \frac{p-2}{p}\ , \quad z_1, z_2 \in \mathbb{C}\ . \end{aligned}$$

Definition 16. *For $f \in L_{(p,p')}(\overline{D})$ we understand*

$$\|f\|_{(p,p')} := \|f\|_p + \|f\|_{p'}\ .$$

Proof. Let $f = 0$ outside $\overline{D}$. Then with $\frac{1}{p} + \frac{1}{q} = \frac{1}{p'} + \frac{1}{q'} = 1, q < 2 < q'$,

$$\begin{aligned} Tf(z) &= -\frac{1}{\pi}\int\limits_{|\zeta|\le 1} \frac{f(\zeta+z)}{\zeta}d\xi d\eta - \frac{1}{\pi}\int\limits_{1<|\zeta|} \frac{f(\zeta+z)}{\zeta}d\xi d\eta\ , \\ |Tf(z)| &\le \frac{1}{\pi}\left(\int\limits_{|\zeta|\le 1} |f(\zeta+z)|^p d\xi d\eta\right)^{1/p}\left(\int\limits_{|\zeta|\le 1} |\zeta|^{-q} d\xi d\eta\right)^{1/q} \\ &\quad + \frac{1}{\pi}\left(\int\limits_{1<|\zeta|} |f(\zeta+z)|^{p'} d\xi d\eta\right)^{1/p'}\left(\int\limits_{1<|\zeta|} |\zeta|^{-q'} d\xi d\eta\right)^{1/q'} \\ &\le \frac{1}{\pi}\left(\frac{2\pi}{2-q}\right)^{1/q}\|f\|_p + \frac{1}{\pi}\left(\frac{2\pi}{q'-2}\right)^{1/q'}\|f\|_{p'} \le M(p,p')\|f\|_{p,p'}\ . \end{aligned}$$

The second inequality of the lemma follows from that one in Theorem 23.

Theorem 36. *Suppose μ_1, μ_2 are satisfying*

$$|\mu_1(z)| + |\mu_2(z)| \le q_0 < 1, \quad z \in \mathbb{C}\ ,$$

$1 < p' < 2 < p$ such that

$$q_0 \max\{\Lambda_p, \Lambda_{p'}\} < 1 \,,$$

and $a, b \in L_{(p,p')}(\mathbb{C})$. Then any solution w to the homogeneous generalized BELTRAMI *equation (3.1.1) with $c = 0$ is representable in the form*

$$w(z) = W(\zeta(z))e^{\varphi(z)}, \quad z \in \mathbb{C} \,.$$

Here ζ is a complete homeomorphism of a BELTRAMI *equation, W is analytic and φ is a* HÖLDER*-continuous function.*

Proof. Let w be a solution to (3.1.1) with $c = 0$.
Define

$$\mu(z) := \begin{cases} \mu_1(z) + \mu_2(z)\dfrac{\overline{w_z(z)}}{w_z(z)}\,, & \text{if} \quad w_z(z) \neq 0\,, \\ \mu_1(z) + \mu_2(z)\,, & \text{if} \quad w_z(z) = 0\,, \end{cases}$$

and

$$h(z) := \begin{cases} a(z) + b(z)\dfrac{\overline{w(z)}}{w(z)}\,, & \text{if} \quad w(z) \neq 0\,, \\ a(z) + b(z)\,, & \text{if} \quad w(z) = 0\,. \end{cases}$$

Then obviously $|\mu(z)| \leq q < 1$ and $h \in L_{(p,p')}(\mathbb{C})$ and

$$w_{\bar{z}} + \mu w_z + hw = 0 \,.$$

Consider the integral equation

$$\omega + \mu\Pi\omega + h = 0 \,.$$

It is uniquely solvable in $L_{(p,p')}(\mathbb{C})$ because $\mu\Pi$ is a contraction in this space. The function

$$W := w \exp(-T\omega)$$

then is a solution to a BELTRAMI equation,

$$\begin{aligned} W_{\bar{z}} + \mu W_z &= \exp(-T\omega)[w_{\bar{z}} + \mu w_z - (\omega + \mu\Pi\omega)w] \\ &= \exp(-T\omega)[w_{\bar{z}} + \mu w_z + aw + b\overline{w}] = 0 \,. \end{aligned}$$

Using Lemma 14 this proves the assertion.

Lemma 16. 1. *If f is bounded and measurable on $\mathbb{C}$ and*

$$f(z) = O(|z|^{-1-\varepsilon}) \text{ as } z \to \infty$$

for some $\varepsilon, 0 < \varepsilon < 1$, then *(i)* $f \in L_{p,2}(\mathbb{C})$ *for* $1 \leq p \leq \dfrac{2}{1-\varepsilon}$,

(ii) $f \in L_p(\mathbb{C})$ *for* $\dfrac{2}{1+\varepsilon} < p$.

2. *If* μ *is measurable on* $\mathbb{C}$ *satisfying*

$$|\mu(z)| \leq q_0 < 1 \text{ on } \mathbb{C}\,, \mu(z) = O(|z|^{-\varepsilon}) \text{ as } z \to \infty$$

and $g \in L_p(\mathbb{C})$ *for some* $p, 1 \leq p < \dfrac{4}{2-\varepsilon}$ *then* $\mu g \in L_p(\mathbb{C}) \cap L_{p,2}(\mathbb{C})$.

Proof. 1. There exist constants K and $R, 0 \leq K, 1 \leq R$ such that

$$|z|^{1+\varepsilon}|f(z)| \leq K \text{ for } R \leq |z| \text{ or } |z|^{-1-\varepsilon}|f(\frac{1}{z})| \leq K \text{ for } |z| \leq \frac{1}{R} \leq 1 \,.$$

Hence,

$$(i) \quad \int\limits_{|z|\leq 1} \left[|z|^{-2}\left|f\left(\frac{1}{z}\right)\right|\right]^p dxdy \leq K^p \int\limits_{|z|\leq \frac{1}{R}} |z|^{(\varepsilon-1)p}dxdy + R^{2p}\int\limits_{\frac{1}{R}}^{1}\left|f\left(\frac{1}{z}\right)\right|^p dxdy\,,$$

where the last integral is bounded, because f is bounded, while the first is less than or equal to

$$2\pi K^p \int\limits_0^1 t^{1-(1-\varepsilon)p}dt = \frac{2\pi K^p}{2-(1-\varepsilon)p}\,,$$

$$(ii) \quad \int\limits_{\mathbb{C}} |f(z)|^p dxdy = \int\limits_{|z|\leq R} |f(z)|^p dxdy + K^p \int\limits_{R<|z|} |z|^{-(1+\varepsilon)p}dxdy\,,$$

where the first integral is bounded, because f is bounded, while the last is less than or equal to

$$2\pi K^p \int\limits_1^{+\infty} t^{1-(1+\varepsilon)p}dt = \frac{2\pi K^p}{(1+\varepsilon)p-2}\,.$$

2. There exist constants K and $R, 0 \leq K, 1 \leq R$, such that

$$|z|^{\varepsilon}|\mu(z)| \leq K \text{ for } 1 \leq R \leq |z|\,.$$

Hence,

$$\begin{aligned}\int\limits_{|z|\leq 1}\left|z^{-2}\mu\left(\frac{1}{z}\right)f(\frac{1}{z})\right|^p dxdy &= \int\limits_{1\leq|z|} |z^2\mu(z)f(z)|^p|z|^{-4}dxdy \\ &\leq K^p\int\limits_{1\leq|z|}|z|^{(2-\varepsilon)p-4}|f(z)|^p dxdy \leq K^p\int\limits_{1\leq|z|}|f(z)|^p dxdy\,.\end{aligned}$$

Remark. The integrals

$$\int_{|z|\le 1} |f(z)|^p dxdy, \int_{|z|\le 1} |\mu(z)g(z)|^p dxdy$$

not considered under 1 (i) and 2, respectively are obviously finite under the assumptions.

Theorem 37. *Suppose μ_1, μ_2 satisfy*

$$\begin{aligned} |\mu_1(z)| + |\mu_2(z)| &\le q_0 < 1 \quad (z \in \mathbb{C}), \\ |\mu_1(z)| + |\mu_2(z)| &= 0(|z|^{-\varepsilon}) \quad (z \to \infty) \end{aligned}$$

for some $\varepsilon, 0 < \varepsilon < 1$ and $a, b, c \in L_{(p,p')}(\mathbb{C}) \cap L_{p,2}(\mathbb{C})$ with

$$\frac{2}{1+\varepsilon} < p' < 2 < p < \frac{4}{2-\varepsilon}, \quad q_0 \max\{\Lambda_p, \Lambda_{p'}\} < 1 .$$

Then there exists a unique solution of the generalized BELTRAMI *equation (3.1.1) vanishing at infinity. Moreover, for this solution $w_z, w_{\bar z} \in L_{(p,p')}(\mathbb{C}), w_{\bar z} \in L_{p,2}(\mathbb{C})$ and*

$$\begin{aligned} &C_0(w; \mathbb{C}) + \|w_z\|_{(p,p')} + \|w_{\bar z}\|_{(p,p')} \\ &\le M(p,p',q_0) \exp\left\{M(p,p',q_0)\left(\|a\|_{(p,p')} + \|b\|_{(p,p')}\right)\right\} \|c\|_{(p,p')} . \end{aligned} \tag{3.1.2}$$

Proof.

1. At first the case $a = b = 0$ is considered.

   Uniqueness of the solution in this case is clear, since the difference of two solutions would satisfy the homogeneous equation

   $$w_{\bar z} + \mu_1 w_z + \mu_2 \overline{w_z} = 0$$

   and vanish at infinity and hence would vanish identically in $\mathbb{C}$ by the remark behind Lemma 14.
   In order to prove existence we look for a solution of the form

   $$w = T\omega, \quad \omega \in L_{(p,p')}(\mathbb{C}) .$$

   The differential equation then leads to the integral equation

   $$\omega + \widehat{\Pi}\omega + c = 0 , \ \widehat{\Pi}\omega := \mu_1 \Pi\omega + \mu_2 \overline{\Pi\omega} .$$

Because $\widehat{\Pi}$ is a contraction in $L_{(p,p')}(\mathbb{C})$ under the above assumptions this integral equation is uniquely solvable in this space. The solution satisfies the estimate

$$\|\omega\|_{(p,p')} \leq (1 - q_0\Lambda_{(p,p')})^{-1}\|c\|_{(p,p')} ,$$

from which by estimating $T\omega$ and $\Pi\omega$

$$C_0(w;\mathbb{C}) + \|w_z\|_{(p,p')} + \|w_{\bar{z}}\|_{(p,p')} \leq M(p,p',q_0)\|c\|_{(p,p')} .$$

Since also $c \in L_{p,2}(\mathbb{C})$ and because of the growth condition on μ_1, μ_2 and $\Pi\omega \in L_{(p,p')}(\mathbb{C})$ we have $\mu_k\Pi\omega \in L_{p,2}(\mathbb{C})$, $k = 1,2$, see Lemma 16. Thus $\omega \in L_{p,2}(\mathbb{C})$ follows from the integral equation. Therefore $w = T\omega$ vanishes at infinity.

2. Uniqueness of the solution for the general equation follows similarly as in step 1, this time from Theorem 36 rather than from Lemma 14.
In order to prove existence we again look for it in the form $w = T\omega$ leading to the integral equation

$$\omega + \mu_1\Pi\omega + \mu_2\overline{\Pi\omega} + aT\omega + b\overline{T\omega} + c = 0 . \tag{3.1.3}$$

To this equation the FREDHOLM alternative holds, see [Cohi53], p. 116. In fact denoting the inverse operator to $I + \widehat{\Pi}$ by R we get

$$\omega + R(aT\omega) + R(b\overline{T\omega}) + Rc = 0$$

which we rewrite as

$$\omega + R_1\omega + Rc = 0 .$$

Here I is the identity operator and

$$R_1\omega := R(aT\omega) + R(b\overline{T\omega}) .$$

Because T is completely continuous R_1 is too and hence the FREDHOLM theory applies to our equation, see [Dusc67], p. 609.
We therefore have to show that the homogeneous equation, $c = 0$, only has the trivial solution. Then the inhomogeneous equation is uniquely solvable. But a solution $\omega \in L_{(p,p')}(\mathbb{C})$ of the homogeneous problem turns out to belong to $L_{p,2}(\mathbb{C})$ too, because $\mu_k\Pi\omega \in L_{p,2}(\mathbb{C})(k = 1,2)$ by the conditions on the μ_k and p and p' and $aT\omega, b\overline{T\omega} \in L_{p,2}(\mathbb{C})$ because a and b are, and $T\omega$ is bounded for $\omega \in L_{(p,p')}(\mathbb{C})$. Therefore $w = T\omega$ being a solution of the homogeneous equation (3.1.3) vanishes at infinity. By the representation of w given in Theorem 36 then it follows that w vanishes indentically.

3. What remains to be proved is the a priori estimates (3.1.2) for the solution. Let w be the solution. Then we rewrite the equation as before as

$$w_{\bar{z}} + \mu w_z + hw + c = 0 .$$

The case $h = 0$ was considered in step 1 already. We now apply this result to the equation

$$v_{\bar z} + \mu v_z + h = 0, \quad v(\infty) = 0$$

giving the estimate

$$C_0(v; \mathbb{C}) + \|v_z\|_{(p,p')} + \|v_{\bar z}\|_{(p,p')} \le M(p, p', q_0)\|h\|_{(p,p')} .$$

The function $f := w \exp(-v)$ turns out to be a solution to

$$f_{\bar z} + \mu f_z = [w_{\bar z} + \mu w_z - w(v_{\bar z} + \mu v_z)] \exp(-v) = -c \exp(-v)$$

satisfying $f(\infty) = 0$.
Again applying the result of step 1 we receive

$$\begin{aligned} C_0(f; \mathbb{C}) + \|f_z\|_{(p,p')} + \|f_{\bar z}\|_{(p,p')} &\le M(p, p', q_0)\|c \exp(-v)\|_{(p,p')} \\ &\le M(p, p', q_0) e^{C_0(v;\mathbb{C})} \|c\|_{(p,p')} . \end{aligned}$$

Then (3.1.2) follows from $w = f \exp v$,

$$w_{\bar z} = (f_{\bar z} + f v_{\bar z}) \exp v, \quad w_z = (f_z + f v_z) \exp v .$$

Remark. The uniqueness of the solution follows at once from the a priori estimate (3.1.2). The difference of two solutions would solve the equation with $c = 0$ and hence would be 0 by (3.1.2).
A more detailed consideration shows that (3.1.2) holds even if $C_0(w; \mathbb{C})$ is replaced by the HÖLDER norm $C_{\alpha_0}(w; \mathbb{C}), \alpha_0 := \dfrac{p-2}{p}$.

3.2 RIEMANN boundary value problem

We are interested in finding a solution to

$$w_{\bar z} + \mu_1 w_z + \mu_2 \overline{w_z} + aw + b\overline{w} + c = 0 \quad \text{in} \quad \mathbb{C} \setminus \Gamma ,$$

$$w^+ = Gw^- + g \quad \text{on} \quad \Gamma, \quad w(\infty) = 0 ,$$

where Γ is a smooth curve or a system of smooth curves in the complex plane without multiple points. The coefficients are assumed to satisfy with $0 < \varepsilon < 1$

$$\begin{aligned} |\mu_1(z)| + |\mu_2(z)| &\le q_0 < 1 , z \in \mathbb{C} , \\ |\mu_1(z)| + |\mu_2(z)| &= O(|z|^{-\varepsilon}) \text{ as } z \to \infty , \end{aligned}$$

$$a, b, c \in L_{(p,p')}(\mathbb{C}) \cap L_{p,2}(\mathbb{C}), \frac{2}{2+\varepsilon} < p' < 2 < p < \frac{4}{2-\varepsilon} ,$$

$$G, g \in C^\alpha(\Gamma) , \ \frac{1}{2} + \frac{\varepsilon}{4} < \alpha < 1, \quad G(\zeta) \ne 0 , \ \zeta \in \Gamma .$$

Lemma 17. *For* $\varphi \in C^\alpha(\Gamma), \frac{1}{2} < \alpha < 1$ *the* CAUCHY *integral*

$$\phi(z) := \frac{1}{2\pi i} \int_\Gamma \varphi(\zeta) \frac{d\zeta}{\zeta - z}, \quad z \notin \Gamma ,$$

with the derivative

$$\phi'(z) = \frac{1}{2\pi i} \int_\Gamma \varphi(\zeta) \frac{d\zeta}{(\zeta - z)^2}, \quad z \notin \Gamma ,$$

satisfies $\phi' \in L_r(\mathbb{C}), 1 < r < \frac{1}{(1-\alpha)}$. *Moreover,*

$$\|\phi'\|_r \leq M(\alpha, r, \Gamma) C_\alpha(\varphi; \Gamma) .$$

Remark. Obviously,

$$|\phi(z)| = O(|z|^{-1}), \quad |\phi'(z)| = O(|z|^{-2}) \quad \text{as} \quad z \to \infty .$$

In more detail we have for $R \leq |z|$

$$|\phi'(z)| \leq \frac{C_0(\varphi;\Gamma)L}{2\pi(|z| - R + \rho)^2} \leq \frac{R^2}{\rho^2} \frac{C_0(\varphi;\Gamma)L}{2\pi|z|^2}$$

where $\Gamma \subset \{|z| < R - \rho\}, 0 < \rho < R$, and $L = L(\Gamma)$ is the total length of Γ.

Proof. In Lemma 2 it is shown that

$$|\phi'(z)| \leq M(\alpha, p, \Gamma) C_\alpha(\varphi; \Gamma) \delta^{\alpha - 1}$$

where

$$\delta = \delta(z) = \operatorname{dist}(z, \Gamma) = \min_{\zeta \in \Gamma} |z - \zeta| \leq \rho .$$

Here R_0 is the standard radius of Γ and $0 < \rho < R_0$. The set

$$S := \bigcup_{\zeta \in \Gamma} \{z : |z - \zeta| \leq \rho\}$$

is a strip around Γ. Assuming Γ to be closed we denote by S^+ the part of S inside Γ and by S^- that outside Γ. In S^+ as well as in S^- we introduce a rectangular coordinate system (s, δ) where $\delta = \delta(z)$ and $s := \min\{\sigma : |\zeta(\sigma) - z| = \delta, 0 \leq \sigma \leq L\}$. Then there is a one-to-one relation between (s, δ) and $z = x + iy$ in $S^\pm \cup \Gamma$. Moreover, with the inner normal direction $\boldsymbol{\nu}$ on Γ

$$dxdy = \begin{vmatrix} x_s & y_s \\ x_\nu & y_\nu \end{vmatrix} dsd\delta = \begin{vmatrix} x_s & y_s \\ -y_s & x_s \end{vmatrix} dsd\delta = dsd\delta .$$

Thus we have with $M = M(\alpha, r, \Gamma)$

$$\int_S |\phi'(z)|^r dxdy \leq 2(MC_\alpha(\varphi;\Gamma))^r \int_0^L \int_0^\rho \delta^{r(\alpha-1)} d\delta ds = \frac{2(MC_\alpha(\varphi;\Gamma))^r L\rho^{1-r(1-\alpha)}}{1-r(1-\alpha)} .$$

Let now $S \subset K_0(R) := \{z : |z| < R\}$. Because for $z \notin S$ we have $\rho \leq \delta(z)$ then

$$\int_{K_0(R)\setminus S} |\phi'(z)|^r dxdy \leq \left(\frac{C_0(\varphi;\Gamma)L}{2\pi\rho^2}\right)^r \pi R^2 ,$$

$$\int_{R\leq|z|} |\phi'(z)|^r dxdy \leq 2\pi\left(\frac{C_0(\varphi;\Gamma)L}{2\pi}\right)^r \left(\tfrac{R}{\rho}\right)^{2r} \int_R^\infty t^{1-2r}dt$$

$$= \frac{\pi}{r-1}\left(\frac{C_0(\varphi;\Gamma)L}{2\pi}\right)^r \frac{R^2}{\rho^{2r}} .$$

Altogether this proves Lemma 17.

Let X be the analytic factorization of G i.e. X is analytic in $\mathbb{C}\setminus\Gamma$ satisfying $X^+ = GX^-$ on Γ and eventually having a pole at infinity. Then $\widehat{w} := \dfrac{w}{X}$ satisfies a generalized BELTRAMI equation and a simple jump condition. We have

$$0 = w_{\bar z} + \mu_1 w_z + \mu_2 \overline{w_z} + aw + b\overline{w} + c$$

$$= X\left[\widehat{w}_{\bar z} + \mu_1\widehat{w}_z + \mu_2\overline{\widehat{w}_z}\frac{\overline{X}}{X} + a\widehat{w} + b\overline{\widehat{w}}\frac{\overline{X}}{X}\right] + c + \mu_1 X'\widehat{w} + \mu_2\overline{X'\widehat{w}}$$

i.e.

$$\widehat{w}_{\bar z} + \mu_1\widehat{w}_z + \mu_2\frac{\overline{X}}{X}\overline{\widehat{w}_z} + \left(a + \mu_1\frac{X'}{X}\right)\widehat{w} + \left(b\frac{\overline{X}}{X} + \mu_2\frac{\overline{X'}}{X}\right)\overline{\widehat{w}} + \frac{c}{X} = 0 \quad \text{in} \quad \mathbb{C}\setminus\Gamma$$

and

$$\widehat{w}^+ = \widehat{w}^- + \frac{g}{X^+} \quad \text{on} \quad \Gamma .$$

But $\widehat{w}$ does not necessarily vanish at infinity. It does for sure if the index $\kappa \leq 0$ while for $\kappa > 0$ $\widehat{w}$ vanishes only if w has a zero at infinity at least of order $\kappa + 1$.
The coefficients of the equation for $\widehat{w}$ satisfy the same conditions as those of the equation for w eventually up to the inhomogeneous term. This follows from

$$\frac{X'}{X} = \begin{cases} \gamma'(z), & \text{in} \quad D^+ , \\ \gamma'(z) - \kappa z^{-1}, & \text{in} \quad D^- , \end{cases}$$

where

$$\gamma(z) := \frac{1}{2\pi i}\int_\Gamma \log[\zeta^{-\kappa}G(\zeta)]\frac{d\zeta}{\zeta - z} ,$$

and from the properties of μ_1 and μ_2. More precisely, we have

$$\left|\mu_1\frac{X'}{X}\right|,\quad \left|\mu_2\frac{\overline{X'}}{X}\right| = O(|z|^{-1-\varepsilon}),\quad z\to\infty\,,$$

and, moreover, $\gamma'\in L_r, 1<r<\frac{1}{1-\alpha}$, by Lemma 17, where $2<\frac{1}{1-\alpha}$, so that these functions are in $L_p(\mathbb{C})\cap L_{p,2}(\mathbb{C})$ for $1<p<\frac{4}{2-\varepsilon}$.

The problem

$$w_{\bar z}+\mu_1 w_z+\mu_2\overline{w_z}+aw+b\overline{w}+c=0 \quad\text{in}\quad \mathbb{C}\setminus\Gamma\,,$$
$$w^+=w^-+g\quad\text{on}\quad\Gamma\,,\quad w(\infty)=0$$

can easily be reduced by the transformation

$$\widetilde{w}=w-\phi,\quad \phi(z):=\frac{1}{2\pi i}\int\limits_{\Gamma} g(\zeta)\frac{d\zeta}{\zeta-z}$$

to

$$\widetilde{w}_{\bar z}+\mu_1\widetilde{w}_z+\mu_2\overline{\widetilde{w}_z}+a\widetilde{w}+b\overline{\widetilde{w}}+c+\mu_1\phi'+\mu_2\overline{\phi'}+a\phi+b\overline{\phi}=0\quad\text{in}\quad\mathbb{C}\setminus\Gamma\,,$$
$$\widetilde{w}^+=\widetilde{w}^-\quad\text{on}\quad\Gamma\,,\quad\widetilde{w}(\infty)=0\,.$$

Because of Lemma 16 the inhomogeneous coefficient again is in $L_{(p,p')}(\mathbb{C})\cap L_{p,2}(\mathbb{C})$. The solution to this problem is continuous in Γ. Thus we have reduced the RIEMANN problem for the generalized BELTRAMI equation to the case of finding the entire solution to an equation of the same kind vanishing at infinity. In the case when the coefficients vanish quickly enough when z tends to infinity then one can find solutions behaving asymptotically as a given complex polynomial, see section 4.3 and [Behi83].

Applying the a priori estimate (3.1.2.) to $\widetilde{w}$ leads to an estimation of w. Analysing the proofs of Theorems 4 and 5 we see

$$C_\alpha(\phi;\mathbb{C}):=C_\alpha(\phi;\overline{D^+})+C_\alpha(\phi;\overline{D^-})\le M(\Gamma,\alpha)C_\alpha(g;\Gamma).$$

From Lemma 17 and $\frac{1}{2}+\frac{\varepsilon}{4}<\alpha<1<p'<p<\frac{4}{2-\varepsilon}$ we find $\phi'\in L_{(p,p')}(\mathbb{C})$ and

$$\|\phi'\|_{(p,p')}\le M(p,p',\varepsilon,\Gamma,\alpha)C_\alpha(g;\Gamma)\,.$$

Hence, with $\|a\|_{(p,p')}+\|b\|_{(p,p')}\le K$

$$C_\alpha(w;\mathbb{C})+\|w_z\|_{(p,p')}+\|w_{\bar z}\|_{(p,p')}\le M(p,p',q_0,\varepsilon,\Gamma,K)\{C_\alpha(g;\Gamma)+\|c\|_{(p,p')}\},\quad(3.2.1)$$

see section 4.3 and [Behi83].

3.3 Riemann–Hilbert boundary value problem

In 2.1 for $w \in C^1(\overline{D})$ the complex form of the Gauss theorem

$$\frac{1}{2i}\int_{\partial D} w(z)dz = \int_D w_{\bar z}(z)dxdy$$

was proved. In the same way as the Cauchy formula for analytic functions from here

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\bar\zeta}(z)dxdy \tag{3.3.1}$$

can be deduced. In what follows (3.3.1) will be adjusted to the Riemann–Hilbert boundary conditions. At first we are doing this with regard to the Dirichlet condition in the same way as the Poisson formula was deduced. Let us therefore consider the case where the domain D is the unit disc $\mathbb{D}$. Applying for $z \in \mathbb{D}$ fixed the Gauss theorem to $\dfrac{\bar z w(\zeta)}{1 - \bar z\zeta}$ we get

$$0 = \frac{1}{2i}\int_{\partial\mathbb{D}} w(\zeta)\frac{\bar z d\zeta}{1 - \bar z\zeta} - \int_{\mathbb{D}} \frac{\bar z w_{\bar\zeta}(\zeta)}{1 - \bar z\zeta}d\xi d\eta \,.$$

Taking the complex conjugate and adding it to (3.3.1) gives for $z \in \mathbb{D}$

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\left(\frac{w(\zeta)}{\zeta - z} + \frac{\overline{w(\zeta)}z\bar\zeta}{\zeta(1 - z\bar\zeta)}\right)d\zeta - \frac{1}{\pi}\int_{\mathbb{D}}\left\{\frac{w_{\bar\zeta}(\zeta)}{\zeta - z} + \frac{z\overline{w_{\bar\zeta}(\zeta)}}{1 - z\bar\zeta}\right\}d\xi d\eta \,,$$

i.e.

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \operatorname{Re} w(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta}$$

$$+\frac{1}{2\pi}\int_{\partial\mathbb{D}} \operatorname{Im} w(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int_{\mathbb{D}}\left\{\frac{w_{\bar\zeta}(\zeta)}{\zeta - z} + \frac{z\overline{w_{\bar\zeta}(\zeta)}}{1 - z\bar\zeta}\right\}d\xi d\eta \,. \tag{3.3.2}$$

The operator

$$\tilde T g(z) := -\frac{1}{\pi}\int_{\mathbb{D}}\left\{\frac{g(\zeta)}{\zeta - z} + \frac{z\overline{g(\zeta)}}{1 - z\bar\zeta}\right\}d\xi d\eta$$

has similar properties as T, see [Veku62], p. 210. Especially

$$\frac{\partial\tilde T g}{\partial\bar z} = g \,, \quad \frac{\partial\tilde T g}{\partial z}(z) = \tilde\Pi g(z) := -\frac{1}{\pi}\int_{\mathbb{D}}\left\{\frac{g(\zeta)}{(\zeta - z)^2} + \frac{\overline{g(\zeta)}}{(1 - z\bar\zeta)^2}\right\}d\xi d\eta$$

where

$$\widetilde{\Lambda}_p = \|\widetilde{\Pi}\|_p$$

has the same properties as $\Lambda_p = \|\Pi\|_p$. This last fact follows directly from $\widetilde{\Pi}g = \Pi\widetilde{g}$ with

$$\widetilde{g}(z) := \begin{cases} g(z), & |z| < 1\,, \\ \overline{\dfrac{1}{z^2} g\left(\dfrac{1}{\overline{z}}\right)}, & 1 \leq |z|\,. \end{cases}$$

To show $\widetilde{\Pi}$ is a unitary operator in $L_2(\overline{\mathbb{D}})$ observe

$$\int\limits_{\mathbb{D}} \frac{z\overline{g(\zeta)}}{1 - z\overline{\zeta}} d\xi d\eta = \int\limits_{\mathbb{C}\setminus\mathbb{D}} \overline{\frac{1}{\overline{\zeta}^2} g\left(\frac{1}{\overline{\zeta}}\right)} \frac{d\xi d\eta}{\zeta - z} - \int_{\mathbb{C}\setminus\mathbb{D}} \overline{g\left(\frac{1}{\overline{\zeta}}\right)} \frac{d\xi d\eta}{\zeta\overline{\zeta}^2}$$

so that

$$\widetilde{T}g(z) = T\widetilde{g}(z) + \frac{1}{\pi} \int\limits_{\mathbb{C}\setminus\mathbb{D}} \widetilde{g}(\zeta)\frac{d\xi d\eta}{\zeta}, T\widetilde{g}(z) := -\frac{1}{\pi}\int\limits_{\mathbb{C}} \widetilde{g}(\zeta)\frac{d\xi d\eta}{\zeta - z}\,.$$

Differentiating with respect to z gives $\widetilde{\Pi}g = \Pi\widetilde{g}$ and thus

$$\|\widetilde{\Pi}g\|_{2,\mathbb{C}} = \|\Pi\widetilde{g}\|_{2,\mathbb{C}} = \|\widetilde{g}\|_{2,\mathbb{C}}\,.$$

Because

$$\|\widetilde{\Pi}g\|^2_{2,\mathbb{C}} = 2\|\widetilde{\Pi}g\|^2_{2,\mathbb{D}}\,,\ \|\widetilde{g}\|^2_{2,\mathbb{C}} = 2\|g\|^2_{2,\mathbb{D}}\,,$$

which follow from

$$\begin{aligned} \int\limits_{\mathbb{C}\setminus\mathbb{D}} |\widetilde{\Pi}g(z)|^2 dxdy &= \int\limits_{\mathbb{D}} \left|\widetilde{\Pi}g\left(\frac{1}{\overline{z}}\right)\right|^2 \frac{dxdy}{|z|^4} \\ &= \int\limits_{\mathbb{D}} \left|\frac{1}{\pi}\int\limits_{\mathbb{D}} \left[\frac{g(\zeta)}{(1-\overline{z}\zeta)^2} + \frac{\overline{g(\zeta)}}{(\overline{\zeta - z})^2}\right] d\xi d\eta\right|^2 dxdy \\ &= \int\limits_{\mathbb{D}} |\widetilde{\Pi}g(z)|^2 dxdy\,, \\ \int\limits_{\mathbb{C}\setminus\mathbb{D}} \left|\overline{\frac{1}{\overline{\zeta}^2} g\left(\frac{1}{\overline{\zeta}}\right)}\right|^2 d\xi d\eta &= \int\limits_{\mathbb{D}} |g(\zeta)|^2 d\xi d\eta\,, \end{aligned}$$

we have

$$\|\widetilde{\Pi}g\|_{2,\mathbb{D}} = \|g\|_{2,\mathbb{D}}\,.$$

Formula (3.3.2) can be directly transformed to more general domains by utilizing the GREEN functions G^I, G^{II}, see 1.2. In the sequel a more general form is needed. To get this formula let $\sigma \in C(\partial \mathbb{D}; \mathbb{R})$ be a positive function with

$$\Sigma := \int\limits_{\partial \mathbb{D}} \sigma(\zeta)|d\zeta| \neq 0$$

and

$$\widetilde{\sigma}(z) := \frac{1}{2\pi i} \int\limits_{\partial \mathbb{D}} \sigma(\zeta) \frac{d\zeta}{\zeta(\zeta - z)} ,$$

which is analytic in $\widehat{\mathbb{C}} \setminus \partial \mathbb{D}$. By

$$\frac{2z}{\zeta(\zeta - z)} = \frac{2}{\zeta - z} - \frac{2}{\zeta} = \left(\frac{\zeta + z}{\zeta - z} - 1 \right) \frac{1}{\zeta}$$

then

$$2z\widetilde{\sigma}(z) = \frac{1}{2\pi i} \int\limits_{\partial \mathbb{D}} \sigma(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} - \frac{\Sigma}{2\pi} ,$$

and

$$2 \operatorname{Re} \{ \zeta \widetilde{\sigma}(\zeta) \} = \sigma(\zeta) - \frac{\Sigma}{2\pi}, \quad |\zeta| = 1 ,$$

where on $\partial \mathbb{D}$ the function $\widetilde{\sigma}$ is understood as $\widetilde{\sigma}^+$.
Applying the GAUSS theorem to $\widetilde{\sigma} w$ we find

$$\frac{1}{2i} \int\limits_{\partial \mathbb{D}} w(\zeta) \widetilde{\sigma}(\zeta) d\zeta = \int\limits_{\mathbb{D}} w_{\bar{\zeta}}(\zeta) \widetilde{\sigma}(\zeta) d\xi d\eta .$$

Hence

$$\begin{aligned}
&\int\limits_{\mathbb{D}} \left\{ w_{\bar{\zeta}}(\zeta) \widetilde{\sigma}(\zeta) - \overline{w_{\bar{\zeta}}(\zeta)}\, \overline{\widetilde{\sigma}(\zeta)} \right\} d\xi d\eta \\
&= 2i \operatorname{Im} \frac{1}{2i} \int\limits_{\partial \mathbb{D}} w(\zeta) \widetilde{\sigma}(\zeta) d\zeta = i \operatorname{Im} \int\limits_{\partial \mathbb{D}} w(\zeta) \zeta \widetilde{\sigma}(\zeta) |d\zeta| \\
&= \frac{i}{2} \int\limits_{\partial \mathbb{D}} \operatorname{Im} w(\zeta) \left(\sigma(\zeta) - \frac{\Sigma}{2\pi} \right) |d\zeta| + \frac{i}{2} \int\limits_{\partial \mathbb{D}} \operatorname{Re} w(\zeta) \frac{1}{2\pi i} \int\limits_{\partial \mathbb{D}} \sigma(t) \operatorname{Im} \frac{t + \zeta}{t - \zeta} \frac{dt}{t} |d\zeta| \\
&= \frac{1}{2} \int\limits_{\partial \mathbb{D}} \operatorname{Re} w(\zeta) \frac{1}{2\pi i} \int\limits_{\partial \mathbb{D}} \sigma(t) \operatorname{Im} \frac{t + \zeta}{t - \zeta} \frac{dt}{t} \frac{d\zeta}{\zeta} + \frac{1}{2} \int\limits_{\partial \mathbb{D}} \operatorname{Im} w(\zeta) \sigma(\zeta) \frac{d\zeta}{\zeta} - \frac{\Sigma}{4\pi} \int\limits_{\partial \mathbb{D}} \operatorname{Im} w(\zeta) \frac{d\zeta}{\zeta}
\end{aligned}$$

or

$$\frac{2}{\Sigma} \int\limits_{\mathbb{D}} \left\{ w_{\bar{\zeta}}(\zeta) \widetilde{\sigma}(\zeta) - \overline{w_{\bar{\zeta}}(\zeta)}\, \overline{\widetilde{\sigma}(\zeta)} \right\} d\xi d\eta$$

$$= \frac{2}{\Sigma}\int\limits_{\partial \mathbb{D}} \operatorname{Re} w(\zeta)\operatorname{Im}[\zeta\tilde{\sigma}(\zeta)]\frac{d\zeta}{\zeta} + \frac{1}{\Sigma}\int\limits_{\partial \mathbb{D}} \operatorname{Im} w(\zeta)\sigma(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{2\pi}\int\limits_{\partial \mathbb{D}} \operatorname{Im} w(\zeta)\frac{d\zeta}{\zeta}\,.$$

Remark. For $\sigma(\zeta) \equiv 1$ we have $\tilde{\sigma}(z) \equiv 0$ in $\mathbb{D}$.

If we add this formula for general σ to (3.3.2) then

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial \mathbb{D}} \operatorname{Re} w(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta}$$

$$+\frac{2}{\Sigma}\int\limits_{\partial \mathbb{D}} \operatorname{Re} w(\zeta)\operatorname{Im}[\zeta\tilde{\sigma}(\zeta)]\frac{d\zeta}{\zeta} + \frac{1}{\Sigma}\int\limits_{\partial \mathbb{D}} \operatorname{Im} w(\zeta)\sigma(\zeta)\frac{d\zeta}{\zeta}$$

$$-\frac{1}{\pi}\int\limits_{\mathbb{D}} \left\{ w_{\bar{\zeta}}(\zeta)\left(\frac{1}{\zeta - z} + \frac{2\pi}{\Sigma}\tilde{\sigma}(\zeta)\right) + \overline{w_{\bar{\zeta}}(\zeta)}\left(\frac{z}{1 - z\bar{\zeta}} - \frac{2\pi}{\Sigma}\overline{\tilde{\sigma}(\zeta)}\right)\right\} d\xi d\eta\,. \qquad (3.3.3)$$

This formula can be given a simplified form by introducing the first and the second GREEN function. We saw in 1.2

$$G^{I}(z,\zeta) = -\frac{1}{2\pi}\log\left|\frac{\zeta - z}{1 - \bar{z}\zeta}\right|\,, \quad G^{II}(z,\zeta) = -\frac{1}{2\pi}\log|(\zeta - z)(1 - \bar{z}\zeta)|$$

are these functions for $\mathbb{D}$. But here we want to apply G^{II} related to the condition

$$\frac{\partial}{\partial \boldsymbol{n}_z} G^{II}(z,\zeta) = -\frac{\sigma(z)}{\Sigma}$$

rather than to

$$\frac{\partial}{\partial \boldsymbol{n}_z} G^{II}(z,\zeta) = -\frac{1}{2\pi}\,.$$

Remark. For convenience we here incorporate the factor $\dfrac{1}{2\pi}$ into G^{I} and G^{II}.

From the properties of G^{II} one can find

$$G^{II}(z,\zeta) = -\frac{1}{2\pi}\log|(\zeta - z)(1 - \bar{z}\zeta)| + V(z,\zeta)\,,$$

$$V(z,\zeta) := \frac{1}{\pi\Sigma}\int\limits_{\partial \mathbb{D}} \sigma(t)\log|(t - z)(t - \zeta)||dt|$$

$$-\frac{1}{\pi\Sigma^2}\int\limits_{\partial \mathbb{D}}\int\limits_{\partial \mathbb{D}} \sigma(t)\sigma(\tau)\log|t - \tau||dt||d\tau|\,,$$

see [Hawe72], p. 119. Instead of deducing this form for V we just verify the appropriately modified properties i, ii, iii′, iv′ for G^{II}, see Definition 4. Because V is harmonic everywhere in z and ζ, we only have to look at the last two conditions. The outward normal derivative on $\partial\mathbb{D}$ applied to real functions is just

$$\frac{\partial}{\partial r} = \zeta\frac{\partial}{\partial\zeta} + \overline{\zeta}\frac{\partial}{\partial\overline{\zeta}} = 2\mathrm{Re}\,\zeta\frac{\partial}{\partial\zeta}\,.$$

We here prefer to consider ζ as variable and z as fixed. Obviously G^{II} is symmetric in z and ζ as defined above. Now

$$\zeta\frac{\partial}{\partial\zeta}G^{II}(z,\zeta) = -\frac{1}{4\pi}\left[\frac{\zeta}{\zeta - z} - \frac{\overline{z}\zeta}{1-\overline{z}\zeta}\right] + \frac{1}{2\pi\Sigma}\int\limits_{\partial\mathbb{D}}\sigma(t)\frac{\zeta|dt|}{\zeta - t}\,.$$

Again rewriting

$$\frac{2\zeta}{t(\zeta - t)} = \left(\frac{\zeta+t}{\zeta-t}+1\right)\frac{1}{t}$$

we have

$$2\int\limits_{\partial\mathbb{D}}\sigma(t)\frac{\zeta|dt|}{\zeta - t} = -\int\limits_{\partial\mathbb{D}}\sigma(t)\frac{t+\zeta}{t-\zeta}\frac{dt}{t} + i\Sigma\,.$$

Hence for $|\zeta| = 1$

$$\begin{aligned} 2\mathrm{Re}\,\zeta\frac{\partial}{\partial\zeta}G^{II}(z,\zeta) &= -\mathrm{Re}\left[\frac{1}{2\pi}\left[\frac{\zeta}{\zeta - z} - \frac{\overline{z}}{\overline{\zeta}-\overline{z}}\right] + \frac{1}{2\pi i\Sigma}\int\limits_{\partial\mathbb{D}}\sigma(t)\frac{t+\zeta}{t-\zeta}\frac{dt}{t} - \frac{1}{2\pi}\right] \\ &= -\left[\frac{1}{2\pi} + \frac{\sigma(\zeta)}{\Sigma} - \frac{1}{2\pi}\right] = -\frac{\sigma(\zeta)}{\Sigma}\,. \end{aligned}$$

Property iv′, i.e.

$$\int\limits_{\partial\mathbb{D}} G^{II}(z,\zeta)\sigma(\zeta)|d\zeta| = 0\,,$$

follows immediately by direct integration. In order to express the representation formula (3.3.3) by the GREEN functions we calculate

$$\begin{aligned} \frac{\partial}{\partial\zeta}\left[G^{I}(z,\zeta) + G^{II}(z,\zeta)\right] &= -\frac{1}{2\pi}\frac{\partial}{\partial\zeta}\log|\zeta - z|^2 + \frac{\partial}{\partial\zeta}V(z,\zeta) \\ = -\frac{1}{2\pi}\frac{1}{\zeta - z} + \frac{1}{2\pi\Sigma}\int\limits_{\partial\mathbb{D}}\sigma(t)\frac{|dt|}{\zeta - t} &= -\frac{1}{2\pi}\frac{1}{\zeta - z} - \frac{\tilde{\sigma}(\zeta)}{\Sigma}\,, \\ \frac{\partial}{\partial\overline{\zeta}}\left[G^{I}(z,\zeta) - G^{II}(z,\zeta)\right] &= \frac{1}{2\pi}\frac{\partial}{\partial\overline{\zeta}}\log|1-\overline{z}\zeta|^2 - \frac{\partial}{\partial\overline{\zeta}}V(z,\zeta) \end{aligned}$$

$$= -\frac{1}{2\pi}\frac{z}{1-z\overline{\zeta}} - \frac{1}{2\pi\Sigma}\int_{\partial\mathbb{D}} \sigma(t)\frac{|dt|}{\overline{\zeta}-\overline{t}} = -\frac{1}{2\pi}\frac{z}{1-z\overline{\zeta}} + \frac{\overline{\tilde{\sigma}(\zeta)}}{\Sigma},$$

and for $\zeta \in \partial\mathbb{D}$ with $\overline{\zeta}d\zeta = -\zeta d\overline{\zeta}$ and

$$d = \frac{\partial}{\partial\zeta}d\zeta + \frac{\partial}{\partial\overline{\zeta}}d\overline{\zeta}\,,\; d_n = -i\left(\frac{\partial}{\partial\zeta}d\zeta - \frac{\partial}{\partial\overline{\zeta}}d\overline{\zeta}\right),$$

$$d_nG^I(z,\zeta) - i\,dG^{II}(z,\zeta)$$

$$= -i\left[\frac{\partial G^I(z,\zeta)}{\partial\zeta}d\zeta - \frac{\partial G^I(z,\zeta)}{\partial\overline{\zeta}}d\overline{\zeta} + \frac{\partial G^{II}(z,\zeta)}{\partial\zeta}d\zeta + \frac{\partial G^{II}(z,\zeta)}{\partial\overline{\zeta}}d\overline{\zeta}\right]$$

$$= -i\left[\frac{\partial}{\partial\zeta}(G^I(z,\zeta)+G^{II}(z,\zeta))d\zeta - \frac{\partial}{\partial\overline{\zeta}}(G^I(z,\zeta)-G^{II}(z,\zeta))d\overline{\zeta}\right]$$

$$= \frac{i}{2\pi}\left[\frac{d\zeta}{\zeta-z} - \frac{zd\overline{\zeta}}{1-z\overline{\zeta}}\right] + \frac{i}{\Sigma}\left[\tilde{\sigma}(\zeta)d\zeta + \overline{\tilde{\sigma}(\zeta)}d\overline{\zeta}\right]$$

$$= -\frac{1}{2\pi i}\left[\frac{\zeta}{\zeta-z} + \frac{z}{\zeta-z}\right]\frac{d\zeta}{\zeta} + \frac{i}{\Sigma}\left[\zeta\tilde{\sigma}(\zeta) - \overline{\zeta}\,\overline{\tilde{\sigma}(\zeta)}\right]\frac{d\zeta}{\zeta}$$

$$= -\frac{1}{2\pi i}\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} - \frac{2i}{\Sigma}\mathrm{Im}\,[\zeta\tilde{\sigma}(\zeta)]\frac{d\zeta}{i\zeta}\,,$$

$$d_nG^{II}(z,\zeta) = -i\left[\frac{\partial G^{II}(z,\zeta)}{\partial\zeta}d\zeta - \frac{\partial G^{II}(z,\zeta)}{\partial\overline{\zeta}}d\overline{\zeta}\right] = 2\mathrm{Im}\,\frac{\partial G^{II}(z,\zeta)}{\partial\zeta}d\zeta$$

$$= -\frac{1}{2\pi}\mathrm{Im}\left\{\left[\frac{1}{\zeta-z} - \frac{\overline{z}}{1-\overline{z}\zeta}\right]d\zeta\right\} - 2\mathrm{Im}\left\{\frac{\tilde{\sigma}(\zeta)}{\Sigma}d\zeta\right\}$$

$$= -\frac{1}{2\pi}\mathrm{Im}\left\{\left[\frac{\zeta}{\zeta-z} - \frac{\overline{z}}{\overline{\zeta}-\overline{z}}\right]\frac{d\zeta}{\zeta}\right\} - \frac{2}{\Sigma}\mathrm{Im}\left\{\zeta\tilde{\sigma}(\zeta)\frac{d\zeta}{\zeta}\right\}$$

$$= -\frac{1}{2\pi}\mathrm{Im}\left\{i\left[1 + \frac{z}{\zeta-z} - \frac{\overline{z}}{\overline{\zeta}-\overline{z}}\right]\right\}\frac{d\zeta}{i\zeta} - \frac{2}{\Sigma}\mathrm{Re}\,\{\zeta\tilde{\sigma}(\zeta)\}\frac{d\zeta}{i\zeta}$$

$$= -\frac{1}{2\pi i}\frac{d\zeta}{\zeta} - \left[\frac{\sigma(\zeta)}{\Sigma i} - \frac{1}{2\pi i}\right]\frac{d\zeta}{\zeta} = -\frac{\sigma(\zeta)}{\Sigma}\frac{d\zeta}{i\zeta} = -\frac{\sigma(\zeta)}{\Sigma}|d\zeta|\,.$$

Thus (3.3.3) can be rewritten as

$$w(z) = -\int_{\partial\mathbb{D}} \mathrm{Re}\,w(\zeta)\left[d_nG^I(z,\zeta) - i\,dG^{II}(z,\zeta)\right] - i\int_{\partial\mathbb{D}} \mathrm{Im}\,w(\zeta)d_nG^{II}(z,\zeta)$$

$$+2\int_{\mathbb{D}}\left\{w_{\overline{\zeta}}(\zeta)\left[G^I_\zeta(z,\zeta)+G^{II}_\zeta(z,\zeta)\right] + \overline{w_{\overline{\zeta}}(\zeta)}\left[G^I_{\overline{\zeta}}(z,\zeta) - G^{II}_{\overline{\zeta}}(z,\zeta)\right]\right\}d\xi d\eta\,. \quad (3.3.4)$$

The function

$$\Theta(z) := -\int_{\partial \mathbb{D}} \operatorname{Re} w(\zeta)\left[d_n G^I(z,\zeta) - i\, dG^{II}(z,\zeta)\right] - i\int_{\partial \mathbb{D}} \operatorname{Im} w(\zeta) d_n G^{II}(z,\zeta)$$

is an analytic function in $\mathbb{D}$ satisfying

$$\operatorname{Re}\Theta(\zeta) = \operatorname{Re} w(\zeta),\ \zeta \in \partial\mathbb{D}\,,\ \int_{\partial\mathbb{D}} \operatorname{Im}\Theta(\zeta)\sigma(\zeta)|d\zeta| = \int_{\partial\mathbb{D}} \operatorname{Im} w(\zeta)\sigma(\zeta)|d\zeta|\,.$$

This is immediately seen from (3.3.3) rather than from (3.3.4).
Moreover, for $z \in \partial\mathbb{D}$

$$\frac{\partial}{\partial\zeta}\left(G^I(z,\zeta) + G^{II}(z,\zeta)\right) = -\overline{\frac{\partial}{\partial\overline{\zeta}}(G^I(z,\zeta) - G^{II}(z,\zeta))}$$

so that the area integral in (3.3.4) on $\partial\mathbb{D}$ equals

$$4i\int_{\mathbb{D}} \operatorname{Im}\left\{w_{\overline{\zeta}}(\zeta)\left[G^I_\zeta(z,\zeta) + G^{II}_\zeta(z,\zeta)\right]\right\} d\xi d\eta\,.$$

Hence

$$w(z) := \Theta(z)$$

$$+2\int_{\mathbb{D}}\left\{f(\zeta)\left[G^I_\zeta(z,\zeta) + G^{II}_\zeta(z,\zeta)\right] + \overline{f(\zeta)}\left[G^I_{\overline{\zeta}}(z,\zeta) - G^{II}_{\overline{\zeta}}(z,\zeta)\right]\right\} d\xi d\eta \qquad (3.3.5)$$

where

$$\Theta(z) := -\int_{\partial\mathbb{D}} \psi(\zeta)\left[d_n G^I(z,\zeta) - i\, dG^{II}(z,\zeta)\right] - ic_0$$

is a solution to the problem

$$w_{\overline{z}} = f \quad \text{in } \mathbb{D}\,,\ \operatorname{Re} w = \psi \quad \text{on } \partial\mathbb{D}\,,\ \frac{1}{\Sigma}\int_{\partial\mathbb{D}} \operatorname{Im} w(\zeta)\sigma(\zeta)|d\zeta| = c_0\,.$$

Here ψ is a real continuous function on $\partial\mathbb{D}$ and f is a complex integrable function in $\mathbb{D}$. That $w_{\overline{z}} = f$ follows again from (3.3.3) from where we see that w is equal to Tf up to an additive analytic function in $\mathbb{D}$.

Remark. Formula (3.3.4) holds for any domain D which is conformally equivalent to $\mathbb{D}$. If ω is a conformal map from D onto $\mathbb{D}$ then the GREEN functions of $D, G^j(z,\zeta)$ are given by those for $\mathbb{D}, \widetilde{G}^j(\tilde{z},\widetilde{\zeta})$, via

$$G^j(z,\zeta) = \widetilde{G}^j(\omega(z),\omega(\zeta)),\quad j = I, II\,.$$

See in this regard [Hawe72], section 10.4 and [Wend79], section 1.1.

In order to get a representation formula of the above kind related to the RIEMANN–HILBERT rather than to the DIRICHLET boundary condition we consider that condition in the case of non-negative index $\kappa \geq 0$ together with some side conditions:

$$\operatorname{Re}\left\{\overline{\lambda(\zeta)}w(\zeta)\right\} = \varphi(\zeta) \quad \text{on} \quad \partial\mathbb{D}\,, \tag{3.3.6a}$$

$$\frac{1}{\Sigma}\int\limits_{\partial\mathbb{D}} \operatorname{Im}\left\{\overline{\lambda(\zeta)}w(\zeta)\right\}\sigma(\zeta)|d\zeta| = c_0\,, \tag{3.3.6b}$$

$$w(z_k) = b_k, \quad 1 \leq k \leq \kappa\,, \tag{3.3.6c}$$

where $|\lambda(\zeta)| = 1, \lambda \in C^\alpha(\partial\mathbb{D};\mathbb{C}), \varphi \in C^\alpha(\partial\mathbb{D};\mathbb{R}), 0 < \alpha < 1, \sigma \in C(\partial\mathbb{D};\mathbb{R}^+)$ with $\Sigma := \int\limits_{\partial\mathbb{D}} \sigma(\zeta)|d\zeta| > 0, c_0 \in \mathbb{R}, z_k \in \mathbb{D}, z_k \neq z_l$, for $k \neq l, b_k \in \mathbb{C}, 1 \leq k, l \leq \kappa$ where $\kappa := \operatorname{ind}\lambda \geq 0, w \in \mathbf{D}_{\bar{z}}(\mathbb{D}) \cap C^\alpha(\overline{\mathbb{D}};\mathbb{C})$, to be determined in connection with a differential equation.

At first we study homogeneous point conditions $b_k = 0, 1 \leq k \leq \kappa$. These conditions can be taken care of by the transformation

$$w(z) = \prod_{k=1}^{\kappa}(z - z_k)\check{w}(z)\,.$$

But this would mean that also $w_{\bar{z}}$ vanishes at the z_k which by no means is justified by (3.3.5). To correct this we define $\check{w}$ by

$$w(z) = \prod_{k=1}^{\kappa}(z - z_k)\check{w}(z) + \sum_{k=1}^{\kappa} w_{\bar{z}}(z_k)\overline{P_k(z)} \tag{3.3.7}$$

where P_k are polynomials of degree 2κ uniquely defined by

$$P_k(z_l) = 0, \quad P_k'(z_l) = \delta_{kl}, \quad 1 \leq k, l \leq \kappa\,.$$

For $\check{w}$ we find

$$\operatorname{Re}\left\{\overline{\check{\lambda}(\zeta)}\check{w}(\zeta)\right\} = \check{\varphi} \quad \text{on } \partial\mathbb{D}\,, \; \frac{1}{\check{\Sigma}}\int\limits_{\partial\mathbb{D}} \operatorname{Im}\left\{\overline{\check{\lambda}(\zeta)}\check{w}(\zeta)\right\}\check{\sigma}(\zeta)|d\zeta| = \check{c}_0\,,$$

where

$$\check{\lambda}(\zeta) := \lambda(\zeta)\prod_{k=1}^{\kappa}\frac{\overline{\zeta - z_k}}{|\zeta - z_k|}\,,$$

$$\check{\kappa} = \operatorname{ind}\check{\lambda} = \operatorname{ind}\lambda - \sum_{k=1}^{\kappa}\operatorname{ind}\frac{\zeta - z_k}{|\zeta - z_k|} = 0\,,$$

$$\check{\sigma}(\zeta) := \prod_{k=1}^{\kappa} |\zeta - z_k| \sigma(\zeta), \quad \check{\Sigma} := \int_{\partial \mathbb{D}} \check{\sigma}(\zeta)|d\zeta| > 0 \,,$$

$$\check{\varphi}(\zeta) := \varphi(\zeta) \prod_{k=1}^{\kappa} |\zeta - z_k|^{-1} - \operatorname{Re}\left\{\overline{\lambda(\zeta)} \prod_{k=1}^{\kappa} |\zeta - z_k|^{-1} \sum_{\nu=1}^{\kappa} w_{\bar z}(z_\nu)\overline{P_\nu(\zeta)}\right\} \,,$$

$$\check{c}_0 := \frac{\Sigma}{\check{\Sigma}} c_0 - \frac{1}{\check{\Sigma}} \int_{\partial \mathbb{D}} \operatorname{Im}\left\{\overline{\lambda(\zeta)} \sum_{k=1}^{\kappa} w_{\bar z}(z_k)\overline{P_k(\zeta)}\right\} \sigma(\zeta)|d\zeta| \,.$$

We thus arrive at a condition of vanishing index. In order to get rid of the factor $\check{\lambda}$ we consider

$$\widehat{w}(z) := e^{-i\gamma(z)}\check{w}(z) \,, \; \gamma(z) := \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \arg \check{\lambda}(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta}$$

which satisfies

$$\operatorname{Re} \widehat{w}(\zeta) = e^{\operatorname{Im}\gamma(\zeta)} \operatorname{Re}\left\{\overline{\check{\lambda}(\zeta)}\check{w}(\zeta)\right\} = e^{\operatorname{Im}\gamma(\zeta)}\check{\varphi}(\zeta) \quad \text{on} \quad \partial \mathbb{D} \,,$$

$$\frac{1}{\widehat{\Sigma}} \int_{\partial \mathbb{D}} \operatorname{Im} \widehat{w}(\zeta)\widehat{\sigma}(\zeta)|d\zeta| = \frac{\check{\Sigma}}{\widehat{\Sigma}} \check{c}_0 \,,$$

where on $\partial \mathbb{D}$

$$\widehat{\sigma}(\zeta) = e^{-\operatorname{Im}\gamma(\zeta)}\check{\sigma}(\zeta), \quad \widehat{\Sigma} := \int_{\partial \mathbb{D}} \widehat{\sigma}(\zeta)|d\zeta| > 0 \,.$$

We now may apply the representation formula (3.3.4) where we have to adjust the second GREEN function to the weight function $\widehat{\sigma}$. This GREEN function will be denoted by $\widehat{G}^{II}(z,\zeta)$. Applying (3.3.4) to

$$\widehat{w}(z) = e^{-i\gamma(z)} \left[w(z) - \sum_{\nu=1}^{\kappa} w_{\bar z}(z_\nu)\overline{P_\nu(z)}\right] \prod_{k=1}^{\kappa} (z - z_k)^{-1}$$

and multiplying this formula by $e^{i\gamma(z)}\prod_{k=1}^{\kappa}(z - z_k)$ yields

$$w(z) = \sum_{k=1}^{\kappa} w_{\bar z}(z_k)\overline{P_k(z)} + ie^{i\gamma(z)} \prod_{k=1}^{\kappa} (z - z_k) \frac{\check{\Sigma}}{\widehat{\Sigma}} \check{c}_0$$

$$-e^{i\gamma(z)} \prod_{k=1}^{\kappa} (z - z_k) \int_{\partial \mathbb{D}} e^{\operatorname{Im}\gamma(\zeta)}\check{\varphi}(\zeta) \left[d_n G^I(z,\zeta) - i\, d\widehat{G}^{II}(z,\zeta)\right]$$

$$+2 \int_{\mathbb{D}} \left\{ e^{i(\gamma(z)-\gamma(\zeta))} \left[w_{\bar\zeta}(\zeta) - \sum_{k=1}^{\kappa} w_{\bar z}(z_k)\overline{P_k'(\zeta)}\right] \prod_{k=1}^{\kappa} \frac{z - z_k}{\zeta - z_k} \left[G^I_\zeta(z,\zeta) + \widehat{G}^{II}_\zeta(z,\zeta)\right]\right.$$

$$+e^{i(\gamma(z)+\overline{\gamma(\zeta)})}\left[\overline{w_{\overline{\zeta}}(\zeta)}-\sum_{k=1}^{\kappa}\overline{w_{\overline{z}}(z_k)}P_k'(\zeta)\right]\prod_{k=1}^{\kappa}\frac{z-z_k}{\overline{\zeta}-\overline{z_k}}\left[G_{\overline{\zeta}}^{I}(z,\zeta)-\widehat{G}_{\overline{\zeta}}^{II}(z,\zeta)\right]\Bigg\}\,d\xi d\eta\,.$$

The analytic function on the right-hand side of this formula is

$$e^{i\gamma(z)}\prod_{k=1}^{\kappa}(z-z_k)\Theta_\kappa(z)$$

with

$$\Theta_\kappa(z):=\Theta(z)$$

$$+\int_{\partial\mathbb{D}}e^{\operatorname{Im}\gamma(\zeta)}\operatorname{Re}\left\{\overline{\lambda(\zeta)}\prod_{k=1}^{\kappa}|\zeta-z_k|^{-1}\sum_{\nu=1}^{\kappa}w_{\overline{z}}(z_\nu)\overline{P_\nu(\zeta)}\right\}\left[d_nG^I(z,\zeta)-i\,d\widehat{G}^{II}(z,\zeta)\right]$$

$$-\frac{i}{\widehat{\Sigma}}\int_{\partial\mathbb{D}}\operatorname{Im}\left\{\overline{\lambda(\zeta)}\sum_{\kappa=1}^{\kappa}w_{\overline{z}}(z_k)\overline{P_k(\zeta)}\right\}\widehat{\sigma}(\zeta)|d\zeta|\,,$$

$$\Theta(z):=-\int_{\partial\mathbb{D}}e^{\operatorname{Im}\gamma(\zeta)}\varphi(\zeta)\prod_{k=1}^{\kappa}|\zeta-z_k|^{-1}\left[d_nG^I(z,\zeta)-i\,d\widehat{G}^{II}(z,\zeta)\right]+i\frac{\Sigma}{\widehat{\Sigma}}c_0\,. \tag{3.3.8}$$

Hence, we have the representation formula

$$w(z)=\sum_{k=1}^{\kappa}w_{\overline{z}}(z_k)\overline{P_k(z)}+\Theta_\kappa(z)e^{i\gamma(\zeta)}\prod_{k=1}^{\kappa}(z-z_k)$$

$$+2\int_{\mathbb{D}}\Bigg\{e^{i[\gamma(z)-\gamma(\zeta)]}\left[w_{\overline{\zeta}}(\zeta)-\sum_{k=1}^{\kappa}w_{\overline{z}}(z_k)\overline{P_k'(\zeta)}\right]\prod_{k=1}^{\kappa}\frac{z-z_k}{\zeta-z_k}\left[G_\zeta^I(z,\zeta)+\widehat{G}_\zeta^{II}(z,\zeta)\right]$$

$$+e^{i[\gamma(z)+\overline{\gamma(\zeta)}]}\left[\overline{w_{\overline{\zeta}}(\zeta)}-\sum_{k=1}\overline{w_{\overline{z}}(z_k)}P_k'(\zeta)\right]\prod_{k=1}^{\kappa}\frac{z-z_k}{\overline{\zeta}-\overline{z_k}}\left[G_{\overline{\zeta}}^I(z,\zeta)-\widehat{G}_{\overline{\zeta}}^{II}(z,\zeta)\right]\Bigg\}\,d\xi d\eta \tag{3.3.9}$$

for $z\in\mathbb{D}$. At last we have to get rid of the restriction $w(z_k)=0, 1\le k\le\kappa$. For $w(z_k)=b_k, 1\le k\le\kappa$, the function

$$w(z)-\sum_{k=1}^{\kappa}b_k\prod_{l\neq k}\frac{z-z_l}{z_k-z_l}$$

vanishes at the z_k's. Applying (3.3.9) to this function leads to a representation formula for $w(z)$.

We are interested in the properties of the area integral operator in (3.3.9). Using the above expressions for the GREEN functions it can be written as

$$T_\kappa g(z):=-\frac{1}{\pi}\int_{\mathbb{D}}\left\{g(\zeta)\prod_{k=1}^{\kappa}\frac{z-z_k}{\zeta-z_k}\left[\frac{1}{\zeta-z}+2\pi\frac{\tilde{\widehat{\sigma}}(\zeta)}{\widehat{\Sigma}}\right]\right.$$

$$+\overline{g(\zeta)}\prod_{k=1}^{\kappa}\frac{\overline{z-z_k}}{\overline{\zeta-z_k}}\left[\frac{z}{1-z\overline{\zeta}}-2\pi\frac{\overline{\widehat{\sigma}(\zeta)}}{\widehat{\Sigma}}\right]\Bigg\}\,d\xi d\eta\,.$$

It, obviously, satisfies the boundary and side conditions

$$\mathrm{Re}\left\{\prod_{k=1}^{\kappa}\frac{\overline{z-z_k}}{|z-z_k|}T_\kappa g(z)\right\}=0 \quad \text{on} \quad \partial\mathbb{D}\,,$$

$$\int_{\partial\mathbb{D}}\mathrm{Im}\left\{\prod_{k=1}^{\kappa}\frac{\overline{z-z_k}}{|z-z_k|}T_\kappa g(z)\right\}\widehat{\sigma}(z)|dz|=0$$

as well as $(T_\kappa g)_{\overline{z}}=g$ and $(T_\kappa g)_z=\Pi_\kappa g$, where

$$\Pi_\kappa g(z):=\sum_{k=1}^{\kappa}\frac{T_\kappa g(z)}{z-z_k}-\frac{1}{\pi}\int_{\mathbb{D}}\left\{\frac{g(z)}{(\zeta-z)^2}\prod_{k=1}^{\kappa}\frac{z-z_k}{\zeta-z_k}+\frac{\overline{g(\zeta)}}{(1-z\overline{\zeta})^2}\prod_{k=1}^{\kappa}\frac{\overline{z-z_k}}{\overline{\zeta-z_k}}\right\}d\xi d\eta\,.$$

This operator Π_κ is a linear operator from $L_p(\mathbb{D})$ into itself but unfortunately the L_2–norm $\Lambda_{\kappa,2}$ of Π_κ is greater than 1 for $1\le\kappa$. Only for $\kappa=0$, where $\Pi_0=\widetilde{\Pi}$, we have $\|\Pi_0\|_2=1$, see [Veku62], p. 210. Again the RIESZ convexity theorem, see [Dusc66], p. 525, assures the continuity of $\Lambda_{\kappa,p}$ by the logarithmic convexity of $\Lambda^p_{\kappa,p}$ for $1<p$.

Lemma 18. *For $2<p$*

$$|T_\kappa g(z)|\le M(p,\sigma,z_k,\mathbb{D})\|g\|_p\,,\ \|\Pi_\kappa g\|_p\le M(p,\sigma,z_k,\mathbb{D})\|g\|_p\,.$$

Proof. The first estimation follows from

$$\prod_{k=1}^{\kappa}\frac{z-z_k}{\zeta-z_k}\frac{1}{\zeta-z}=\frac{1}{\zeta-z}-\sum_{k=1}^{\kappa}\prod_{\nu\neq k}\frac{z-z_\nu}{z_k-z_\nu}\frac{1}{\zeta-z_k}$$

and

$$\prod_{k=1}^{\kappa}\frac{\overline{z-z_k}}{\overline{\zeta}-\overline{z_k}}\frac{z}{1-z\overline{\zeta}}=\prod_{k=1}^{\kappa}\frac{\overline{z-z_k}}{\overline{z}-\overline{z_k}}\frac{z}{1-z\overline{\zeta}}-\sum_{k=1}^{\kappa}\prod_{\nu\neq k}\frac{\overline{z-z_\nu}}{\overline{z_k}-\overline{z_\nu}}\frac{\overline{z-z_k}}{\overline{z}-\overline{z_k}}\frac{\overline{\zeta}-\overline{z}}{1-z\overline{\zeta}}\frac{z}{\overline{\zeta}-\overline{z_k}}$$

because of the boundedness of the coefficients of $\dfrac{1}{\zeta-z_k}$ and $\dfrac{1}{\overline{\zeta-z_k}}$ on the right–hand sides and the boundedness of $\dfrac{z-\zeta}{1-z\overline{\zeta}}$ in $\mathbb{D}\times\mathbb{D}$ and because of the related estimation

of the T–operator, see Theorem 23.
For the second inequality we observe

$$\left[\sum_{k=1}^{\kappa}\frac{1}{z-z_k}-\frac{1}{\zeta-z}\right]\frac{1}{\zeta-z}\prod_{\nu=1}^{\kappa}\frac{z-z_\nu}{\zeta-z_\nu}$$

$$=\frac{1}{(\zeta-z)^2}-\sum_{k=1}^{\kappa}\sum_{\nu\neq k}\prod_{\mu\neq\nu,k}\frac{z-z_\mu}{z_\nu-z_\mu}\frac{1}{z_k-z_\nu}\left[\frac{1}{\zeta-z}-\frac{1}{\zeta-z_\nu}\right],$$

$$\sum_{k=1}^{\kappa}\frac{1}{z-z_k}\prod_{\nu=1}^{\kappa}\frac{z-z_\nu}{\overline{\zeta}-\overline{z_\nu}}\frac{z}{1-z\overline{\zeta}}$$

$$=\sum_{k=1}^{\kappa}\prod_{\nu\neq k}\frac{z-z_\nu}{\overline{z}-\overline{z_\nu}}\left[\frac{1}{\overline{\zeta}-\overline{z}}-\sum_{\nu\neq k}\prod_{\mu\neq\nu,k}\frac{\overline{z}-\overline{z_\mu}}{\overline{z_k}-\overline{z_\mu}}\frac{1}{\overline{\zeta}-\overline{z_\nu}}\right]\frac{z}{\overline{\zeta}-\overline{z_k}}\frac{\overline{\zeta}-\overline{z}}{1-z\overline{\zeta}},$$

and

$$\prod_{k=1}^{\kappa}\frac{z-z_k}{\overline{\zeta}-\overline{z_k}}\frac{1}{(1-z\overline{\zeta})^2}$$

$$=\prod_{k=1}^{\kappa}\frac{z-z_k}{\overline{z}-\overline{z_k}}\frac{1}{(1-z\overline{\zeta})^2}-\sum_{k=1}^{\kappa}\prod_{\nu\neq k}\frac{z-z_\nu}{\overline{z_k}-\overline{z_\nu}}\frac{z-z_k}{\overline{z}-\overline{z_k}}\left(\frac{\overline{\zeta}-\overline{z}}{1-z\overline{\zeta}}\right)^2\frac{1}{\overline{\zeta}-\overline{z_k}}\frac{1}{\overline{\zeta}-\overline{z}}$$

together with the appropriate property of the Π–operator, see Theorem 32, and the boundedness of the coefficients in $\mathbb{D}$.

Lemma 19. $1\leq\Lambda_{\kappa,2}=\|\Pi_\kappa\|_2$. *For* $0<\kappa$, *moreover,* $1<\Lambda_{\kappa,2}$.

Proof. The proof only needs the following properties of Π_κ.

1. $\Pi_\kappa g=\dfrac{\partial}{\partial z}T_\kappa g,\quad \dfrac{\partial T_\kappa g}{\partial\overline{z}}=g\,.$

2. $\operatorname{Re}\{\overline{\lambda}T_\kappa g\}=0$ on $\partial\mathbb{D}, 0\leq\operatorname{ind}\lambda=\kappa, \lambda(z):=\Pi_{k=1}^{\kappa}\dfrac{z-z_k}{|z-z_k|}\,.$

From the GAUSS formula, Theorem 10, for $g \in C_0^\infty(\mathbb{D})$

$$
\begin{aligned}
(\Pi_\kappa g, \Pi_\kappa g) &:= \int_{\mathbb{D}} \frac{\partial T_\kappa g}{\partial z} \overline{\frac{\partial T_\kappa g}{\partial z}} dxdy = \int_{\mathbb{D}} \left[\frac{\partial}{\partial \overline{z}} \left(\overline{T_\kappa g} \frac{\partial T_\kappa g}{\partial z} \right) - \overline{T_\kappa g} \frac{\partial^2 T_\kappa g}{\partial z \partial \overline{z}} \right] dxdy \\
&= \frac{1}{2i} \int_{\partial \mathbb{D}} \overline{T_\kappa g} \frac{\partial T_\kappa g}{\partial z} dz - \int_{\mathbb{D}} \overline{T_\kappa g} \frac{\partial g}{\partial z} dxdy \\
&= \frac{1}{2i} \int_{\partial \mathbb{D}} \overline{T_\kappa g} \frac{\partial T_\kappa g}{\partial z} dz + \frac{1}{2i} \int_{\partial \mathbb{D}} g \overline{T_\kappa g} d\overline{z} + \int_{\mathbb{D}} g \overline{g} dxdy \\
&= \frac{1}{2i} \int_{\partial \mathbb{D}} \overline{T_\kappa g} dT_\kappa g + (g,g) = I + (g,g) .
\end{aligned}
$$

Because on $\partial \mathbb{D}$

$$\overline{\lambda} T_\kappa g + \lambda \overline{T_\kappa g} = 0$$

we have

$$
\begin{aligned}
I &:= \frac{1}{2i} \int_{\partial \mathbb{D}} \overline{T_\kappa g} dT_\kappa g = \frac{1}{2i} \int_{\partial \mathbb{D}} \overline{\lambda} \lambda \overline{T_\kappa g} dT_\kappa g = -\frac{1}{2i} \int_{\partial \mathbb{D}} \overline{\lambda}^2 T_\kappa g dT_\kappa g = -\frac{1}{4i} \int_{\partial \mathbb{D}} \overline{\lambda}^2 d(T_\kappa g)^2 \\
&= \frac{1}{4i} \int_{\partial \mathbb{D}} (T_\kappa g)^2 d\overline{\lambda}^2 = -\frac{1}{2i} \int_{\partial \mathbb{D}} \lambda |T_\kappa g|^2 d\overline{\lambda} = \frac{1}{2i} \int_{\partial \mathbb{D}} \overline{\lambda} |T_\kappa g|^2 d\lambda = \overline{I} .
\end{aligned}
$$

Hence, I is real. Moreover, because

$$\overline{\lambda} d\lambda = i\, d \arg \lambda := i \sum_{k=1}^{\kappa} d \arg(z - z_k)$$

we see that I is non–negative. In order to show $\Lambda_{\kappa,2} = 1$ we ought to have I vanishing for arbitrary $g \in C_0^\infty(\mathbb{D})$. But because $|T_\kappa g| \geq 0$ in general not identically vanishing I will not be zero for all $g \in C_0^\infty(\mathbb{D})$. Therefore $1 \leq \Lambda_{\kappa,2}$ and $1 < \Lambda_{\kappa,2}$ for $0 < \kappa$. In case $\lambda = 1$ i.e. $\kappa = 0$ we have $I = 0$ and then $\Lambda_2 = \Lambda_{0,2} = 1$.
That the relation

$$(g,g) \leq (\Pi_\kappa g, \Pi_\kappa g)$$

proved for $g \in C_0^\infty(\mathbb{D})$ holds in $L_2(\overline{D})$, too follows from the fact that $C_0^\infty(\mathbb{D})$ is dense in $L_2(\overline{\mathbb{D}})$.
As $1 < \Lambda_{\kappa,2}$ for $0 < \kappa$ we cannot treat the boundary value problem in the same way as for $\kappa = 0$. Demanding

$$q \Lambda_{\kappa,p} < 1$$

would put restrictions on the coefficients μ_1, μ_2 of the differential equation. But then the proof from the case $\kappa = 0$ could be repeated here, see [Behs83].

Theorem 38. *Let* $0 \le q_0 < 1 < 2\alpha < 2 < p < 1/(1-\alpha), \alpha_1 := \min\{\alpha, 1-2/p\}, 0 \le K, \mu_1, \mu_2$ *be measurable functions in* $\overline{\mathbb{D}}, a, b, c \in L_p(\mathbb{D})$,

$$|\mu_1(z)| + |\mu_2(z)| \le q_0 < 1, \quad \|a\|_p + \|b\|_p \le K, \quad q_0\Lambda_p < 1 ,$$

$\varphi, \lambda \in C^\alpha(\partial\mathbb{D}), \sigma \in C(\partial\mathbb{D}), c_0 \in \mathbb{R}$,

$$\kappa := \frac{1}{2\pi}\int\limits_{\partial\mathbb{D}} d\arg\lambda \ge 0\,,\ \Sigma := \int\limits_{\partial\mathbb{D}} \sigma\, ds > 0\,,\ |\lambda(\zeta)| = 1, \quad \zeta \in \partial\mathbb{D}\,,$$

$z_k \in \mathbb{D}, a_k \in \mathbb{C}\,(1 \le k \le \kappa)$. *Then there exist constants* $\beta, \gamma_1, \gamma_2, \delta$ *depending on* $\alpha, p, q_0, \lambda, \sigma, z_k, K$ *but not on* $\mu_1, \mu_2, a, b, c, \varphi, c_0, a_k, w$ *such that any solution of*

$$w_{\bar z} + \mu_1 w_z + \mu_2\overline{w_z} + aw + b\overline{w} + c = 0 \quad \text{in } \mathbb{D}\,,$$

$$\operatorname{Re}\{\overline{\lambda}w\} = \varphi \quad \text{on } \partial\mathbb{D}\,,\ \frac{1}{\Sigma}\int\limits_{\partial\mathbb{D}} \operatorname{Im}\{\overline{\lambda}w\}\sigma\, ds = c_0\,,\ w(z_k) = a_k,\ 1 \le k \le \kappa\,,$$

satisfies the a priori estimate

$$C_{\alpha_1}(w;\mathbb{D}) + \|w_z\|_p + \|w_{\bar z}\|_p \le \beta C_\alpha(\varphi;\partial\mathbb{D}) + \gamma_1|c_0| + \gamma_2\sum_{k=1}^{\kappa}|a_k| + \delta\|c\|_p\,. \quad (3.3.10)$$

Proof. Before proving this estimate we reduce the problem to a special one.

i. Taking

$$\check\lambda(\zeta) := \lambda(\zeta)\prod_{k=1}^{\kappa}\frac{\overline{\zeta - z_k}}{|\zeta - z_k|}\,,\ \gamma(z) := \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \arg\check\lambda(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta}\,,$$

$$\check\sigma(\zeta) := \sigma(\zeta)e^{-\operatorname{Im}\gamma(\zeta)}\,,\ \check\Sigma := \int\limits_{\partial\mathbb{D}} \check\sigma(\zeta)ds$$

as before we see that $\check\omega(z) := w(z)e^{i\gamma(z)}$ satisfies

$$\check\omega_{\bar z} + \mu_1\check\omega_z + \mu_2 e^{-2i\operatorname{Re}\gamma}\overline{\check\omega_z} + (a + i\mu_1\gamma')\check\omega$$

$$+(b - i\mu_2\overline{\gamma'})e^{-2i\operatorname{Re}\gamma}\overline{\check\omega} + ce^{-i\gamma} = 0 \quad \text{in } \mathbb{D}\,,$$

$$\operatorname{Re}\left\{\prod_{k=1}^{\kappa}\frac{\overline{\zeta - z_k}}{|\zeta - z_k|}\check\omega(\zeta)\right\} = e^{\operatorname{Im}\gamma(\zeta)}\varphi(\zeta) \quad \text{on} \quad \partial\mathbb{D}\,,$$

$$\frac{1}{\check\Sigma}\int\limits_{\partial\mathbb{D}} \operatorname{Im}\left\{\prod_{k=1}^{\kappa}\frac{\overline{\zeta - z_k}}{|\zeta - z_k|}\check\omega(\zeta)\right\}\check\sigma(\zeta)ds = \frac{\Sigma}{\check\Sigma}c_0\,,$$

$$\check{\omega}(z_k) = a_k e^{-i\gamma(z_k)},\ 1 \le k \le \kappa\,.$$

The coefficients of this problem have the same properties as those of the original problem. Obviously,

$$C_{\alpha_1}(w; \mathbb{D}) \le M(\lambda, z_k) C_{\alpha_1}(\check{\omega}; \mathbb{D})\,,$$

$$\|w_z\|_p \le M(\lambda, z_k)(\|\check{\omega}_z\|_p + C_0(\check{\omega}; \mathbb{D}))\,,\ \|w_{\bar z}\|_p \le M(\lambda, z_k)\|\check{\omega}_{\bar z}\|_p\,.$$

ii. Subtracting from w the analytic function $\widehat{\Theta}$ satisfying the same boundary and side conditions we arrive at the homogeneous boundary problem. $\widehat{\Theta}$ is explicitly given as

$$\widehat{\Theta}(z) := \Theta(z) e^{i\gamma(z)} \prod_{k=1}^{\kappa} (z - z_k) + \sum_{k=1}^{\kappa} b_k \Pi_{l \ne k} \frac{z - z_l}{z_k - z_l}\,,$$

with Θ as given in (3.3.8). For $\widehat{\omega} := w - \widehat{\Theta}$ we get

$$\widehat{\omega}_{\bar z} + \mu_1 \widehat{\omega}_z + \mu_2 \overline{\widehat{\omega}_z} + a\widehat{\omega} + b\overline{\widehat{\omega}} + \mu_1 \widehat{\Theta}' + \mu_2 \overline{\widehat{\Theta}'} + a\widehat{\Theta} + b\overline{\widehat{\Theta}} + c = 0 \quad \text{in} \quad \mathbb{D}\,,$$

$$\mathrm{Re}\,\{\bar{\lambda}\widehat{\omega}\} = \varphi - \mathrm{Re}\,\{\bar{\lambda}\widehat{\Theta}\} = 0 \quad \text{on } \partial\mathbb{D}\,,$$

$$\frac{1}{\Sigma} \int_{\partial\mathbb{D}} \mathrm{Im}\,\{\bar{\lambda}\widehat{\omega}\}\sigma\, ds = c_0 - \frac{1}{\Sigma} \int_{\partial\mathbb{D}} \mathrm{Im}\,\{\bar{\lambda}\widehat{\Theta}\}\sigma\, ds = 0\,,\ \omega(z_k) = 0\,,\ 1 \le k \le \kappa\,.$$

Using PRIVALOV's theorem (Theorem 6) and Lemma 17 we see

$$C_\alpha(\widehat{\Theta}; \mathbb{D}) \le M(\alpha, \lambda, z_k) C_\alpha(\Theta; \mathbb{D}) + M(\alpha, z_k) \sum_{k=1}^{\kappa} |b_k|\,,$$

$$C_\alpha(\Theta; \mathbb{D}) \le M(\alpha, \lambda, z_k) C_\alpha(\varphi; \partial\mathbb{D}) + M(\lambda, \sigma, z_k)|c_0|\,,$$

$$\|\widehat{\Theta}'\|_p \le M(\lambda, z_k)(\|\Theta'\|_p + C_0(\Theta; \mathbb{D})) + M(z_k) \sum_{k=1}^{\kappa} |b_k|\,,$$

$$\|\Theta'\|_p \le M(\alpha, p, \lambda, z_k) C_\alpha(\varphi; \partial\mathbb{D})\,.$$

iii. Let now w be a solution to the homogeneous boundary problem for the inhomogeneous equation where especially

$$\lambda(\zeta) := \prod_{k=1}^{\kappa} \frac{\zeta - z_k}{|\zeta - z_k|}\,.$$

Then as before introducing

$$\mu(z) := \begin{cases} \mu_1(z) + \mu_2(z) \dfrac{\overline{w_z(z)}}{w_z(z)}, & \text{if} \quad w_z(z) \ne 0\,, \\ \mu_1(z) + \mu_2(z), & \text{if} \quad w_z(z) = 0\,, \end{cases}$$

$$h(z) := \begin{cases} a(z) + b(z)\dfrac{\overline{w(z)}}{w(z)}, & \text{if} \quad w(z) \neq 0 \,, \\ a(z) + b(z), & \text{if} \quad w(z) = 0 \,, \end{cases}$$

the differential equation reads

$$w_{\bar z} + \mu w_z + hw + c = 0 \,.$$

α. $h = 0$

Let w be a solution of the equation

$$w_{\bar z} + \mu w_z + c = 0 \quad \text{in} \quad \mathbb{D}$$

with $w_z, w_{\bar z} \in L_p(\overline{D})$ where $|\mu(z)| \leq q_0 \leq 1, c \in L_p(\overline{D})$ and let w satisfy the above homogeneous boundary and side conditions. Then

$$w - T_\kappa w_{\bar z}$$

is an analytic function satisfying homogeneous boundary and side conditions. Hence it vanishes identically, i.e.

$$w = T_\kappa w_{\bar z} \;,\; w_z = \Pi_\kappa w_{\bar z} \,.$$

From the properties of T_κ we get

$$C_{\alpha_1}(w; \mathbb{D}) \leq M(\alpha, p) C_{\alpha_0}(w; \mathbb{D}) \leq M(\alpha, p, \sigma, z_k) \|w_{\bar z}\|_p, \quad \alpha_0 := \frac{p-2}{p} \,,$$

$$\|w_z\|_p \leq \Lambda_{\kappa,p} \|w_{\bar z}\|_p \,.$$

If we have shown

$$\|w_{\bar z}\|_p \leq M \|c\|_p$$

for some proper constant M then in this special case we are done. The existence of M is shown by reductio ad absurdum. Assuming such a constant M does not exist. Then there exist sequences $(\mu_n), (c_n), (w_n)$ with the same properties as μ, c, w, especially

$$|\mu_n(z)| \leq q_0 < 1 \,,$$

satisfying

$$\begin{aligned} w_{n\bar z} + \mu_n w_{nz} + c_n &= 0 \quad \text{in} \quad \mathbb{D} \,, \\ \operatorname{Re}\left\{ \prod_{k=1}^{\kappa} \frac{\overline{\zeta - z_k}}{|\zeta - z_k|} w_n \right\} &= 0 \quad \text{on} \quad \partial\mathbb{D} \,, \\ \int_{\partial\mathbb{D}} \operatorname{Im}\left\{ \prod_{k=1}^{\kappa} \frac{\zeta - z_k}{|\zeta - z_k|} w_n(\zeta) \right\} \sigma(\zeta) ds &= 0 \,, \\ w_n(z_k) &= 0, \quad 1 \leq k \leq \kappa \,, \end{aligned}$$

and

$$n\|c_n\|_p \leq \|w_{n\bar{z}}\|_p .$$

Assuming $0 < \|w_{n\bar{z}}\|_p$ and setting

$$\omega_n := \frac{w_n}{\|w_{n\bar{z}}\|_p} , \quad \varepsilon_n := \frac{c_n}{\|w_{n\bar{z}}\|_p}$$

such that

$$\|\omega_{n\bar{z}}\|_p = 1, \quad \|\varepsilon_n\|_p < \frac{1}{n} \quad (n \in I\!N)$$

one has

$$\begin{aligned}
\omega_{n\bar{z}} + \mu_n \omega_{nz} + \varepsilon_n &= 0 \quad \text{in} \quad I\!D , \\
\operatorname{Re}\left\{\prod_{k=1}^{\kappa} \frac{\overline{\zeta - z_k}}{|\zeta - z_k|}\omega_n(\zeta)\right\} &= 0 \quad \text{on} \quad \partial I\!D , \\
\int_{\partial I\!D} \operatorname{Im}\left\{\prod_{k=1}^{\kappa} \frac{\overline{\zeta - z_k}}{|\zeta - z_k|}\omega_n(\zeta)\right\}\sigma(\zeta)ds &= 0 , \\
\omega(z_k) &= 0, \quad 1 \leq k \leq \kappa .
\end{aligned}$$

Let $\psi_n = T\chi_n$ be a special solution of this inhomogeneous problem. Using the representation given in Theorem 36 then

$$\omega_n(z) = \Omega_n(\zeta_n(z)) + \psi_n(z)$$

where ζ_n is a complete homeomorphism of

$$\zeta_{n\bar{z}} + \mu_n \zeta_{nz} = 0$$

of the form

$$\zeta_n(z) = z + T\theta_n(z) , \quad \theta_n + \mu_n \Pi\theta_n + \mu_n = 0$$

so that for any compact set $K \subset \mathbb{C}$

$$C_{\alpha_0}(\zeta_n; K) \leq M(K), \quad \alpha_0 := \frac{p-2}{p} .$$

The function $\chi_n \in L_p(\overline{I\!D})$ has to satisfy

$$\chi_n + \mu_n \Pi\chi_n + \varepsilon_n = 0 \quad \text{in} \quad I\!D .$$

Hence χ_n is the unique solution to this equation satisfying

$$\|\chi_n\|_p \leq \frac{\|\varepsilon_n\|_p}{1 - q_0\Lambda_p} .$$

From this estimate

$$\|\psi_{n\bar{z}}\|_p \le \frac{1}{1-q_0\Lambda_p}\,\frac{1}{n}\,, \quad \|\psi_{nz}\|_p \le \frac{\Lambda_p}{1-q_0\Lambda_p}\,\frac{1}{n}\,,$$

$$C_{\alpha_0}(\psi_n;\boldsymbol{D}) \le \frac{M(p)}{1-q_0\Lambda_p}\,\frac{1}{n}\,.$$

Furthermore, Ω_n is analytic in $\zeta_n[\boldsymbol{D}]$ satisfying

$$\begin{aligned} &\operatorname{Re}\left\{\prod_{k=1}^{\kappa}\frac{t-z_k}{|t-z_k|}\Omega_n(\zeta_n(t))\right\} \\ = &\; -\operatorname{Re}\left\{\prod_{k=1}^{\kappa}\frac{t-z_k}{|t-z_k|}\psi_n(t)\right\} =: \varphi_n(t) \quad \text{on} \quad \partial\boldsymbol{D}\,, \\ &\int\limits_{\partial\boldsymbol{D}}\operatorname{Im}\left\{\prod_{k=1}^{\kappa}\frac{t-z_k}{|t-z_k|}\Omega_n(\zeta_n(t))\right\}\sigma(t)ds \\ = &\; -\int\limits_{\partial\boldsymbol{D}}\operatorname{Im}\left\{\prod_{k=1}^{\kappa}\frac{t-z_k}{|t-z_k|}\psi_n(t)\right\}\sigma(t)ds =: \rho_n\,, \\ &\Omega_n(\zeta_n(z_k)) = -\psi_n(z_k), \quad 1\le k\le\kappa\,. \end{aligned}$$

Using the inverse homeomorphism $z_n(\zeta)$ of $\zeta_n(z)$, see [Veku62], p. 95, we find

$$\operatorname{Re}\left\{\prod_{k=1}^{\kappa}\frac{z_n(\tau)-z_k}{|z_n(\tau)-z_k|}\Omega_n(\tau)\right\} = \varphi_n(z_n(\tau)) \quad \text{on} \quad \zeta_n[\partial\boldsymbol{D}]\,,$$

$$\int\limits_{\zeta_n[\partial\boldsymbol{D}]}\operatorname{Im}\left\{\prod_{k=1}^{\kappa}\frac{z_n(\tau)-z_k}{|z_n(\tau)-z_k|}\Omega_n(\tau)\right\}\sigma(z_n(\tau))\left|z_{n\tau}(\tau)\frac{d\tau}{ds}+z_{n\bar{\tau}}(\tau)\frac{\overline{d\tau}}{ds}\right|ds = \rho_n,$$

$$\Omega_n(\zeta_{nk}) = -\psi_n(z_k), \quad \zeta_{nk} := \zeta_n(z_k), \quad 1\le k\le\kappa\,.$$

We have for the right-hand sides

$$C_{\alpha_0^2}(\varphi_n(z_n(\cdot));\zeta_n[\partial\boldsymbol{D}]) \le M(\alpha,p,z_k)C_{\alpha_0}(\psi_n;\boldsymbol{D})\,,$$

$$|\rho_n| \le M(\sigma,z_k)C_0(\psi_n;\partial\boldsymbol{D}), \quad |\psi_n(z_k)| \le C_0(\psi_n;\boldsymbol{D})\,.$$

Hence, for the analytic function Ω_n we have

$$C_0(\Omega_n;\zeta_n[\overline{\boldsymbol{D}}]) \le M(\alpha,p,q_0,\sigma,z_k)C_{\alpha_0}(\psi_n;\partial\boldsymbol{D}) \le M\frac{1}{n}\,.$$

The sequence (ζ_n) satisfying

$$C_{\alpha_0}(\zeta_n;K) \le M(K)$$

for any compact $K \subset \mathbb{C}$ may, by the ARZELÀ–ASCOLI–MONTEL theorem, be assumed to converge uniformly on compact subsets of $\mathbb{C}$ to, say ζ_0. By the argument principle, ζ_0 is a homeomorphism of $\mathbb{C}$, too. Because (Ω_n) converges uniformly on compact subsets of $\zeta_0[\mathbb{D}]$ to zero as for any compact $\widetilde{K} \subset \zeta_0[\mathbb{D}]$ there exists an $n \in \mathbb{N}$ such that $\widetilde{K} \subset \zeta_n[\mathbb{D}]$, so does the sequence (Ω_n') of the derivatives. Therefore, from

$$\omega_{n\bar{z}}(z) = \Omega_n'(\zeta_n(z))\zeta_{n\bar{z}}(z) + \varphi_{n\bar{z}}(z)$$

we obtain for any compact set $K \subset \mathbb{D}$

$$\begin{aligned}\|\omega_{n\bar{z}}\|_{p,K} &\le C_0(\Omega_n'; \zeta_n[K])\|\theta_n\|_{p,K} + \|\chi_n\|_{p,K}\\ &\le M(p,q)C_0(\Omega_n'; \zeta_n[K]) + M(p,q)\frac{1}{n}\,.\end{aligned}$$

As $\zeta_n[K]$ converges to $\zeta_0[K]$ and for compact $K \subset \mathbb{D}$ the set $\zeta_0[K]$ is compact, $\zeta_n[K] \subset K_0 \subset \zeta_0[\mathbb{D}]$ with compact K_0 for large enough n. Therefore

$$\lim_{n\to+\infty} \|\omega_{n\bar{z}}\|_{p,K} = 0$$

for any compact $K \subset \mathbb{D}$. This contradicts

$$\|\omega_{n\bar{z}}\|_{p,\mathbb{D}} = 1$$

for all n. Hence, there exists a constant M such that for any solution of the special inhomogeneous BELTRAMI equation with homogeneous boundary and side conditions satisfies

$$\|w_{\bar{z}}\|_p \le M\|c\|_p\,.$$

β. $h \neq 0$

This last estimate shall now be generalized to the equation

$$w_{\bar{z}} + \mu w_z + hw + c = 0 \quad \text{in} \quad \mathbb{D}$$

with again homogeneous conditions.
Let v be a solution of

$$v_{\bar{z}} + \mu v_z + h = 0 \quad \text{in } \mathbb{D},\ \operatorname{Im} v = 0 \quad \text{on } \partial\mathbb{D},\ \int_{\partial\mathbb{D}} \operatorname{Re} v\sigma ds = 0\,.$$

Then from the above considerations

$$C_{\alpha_0}(v; \mathbb{D}) + \|v_z\|_p + \|v_{\bar{z}}\|_p \le M\|h\|_p\,.$$

Let f_0 be a solution of

$$f_{0\bar{z}} + \mu f_{0z} = -\frac{\mu}{a_0}\varphi_0' \quad \text{in} \quad \mathbb{D}\,,$$

$$\operatorname{Re}\left\{\prod_{k=1}^{\kappa} \frac{\overline{\zeta - z_k}}{|\zeta - z_k|} f_0(\zeta)\right\} = 0 \quad \text{on} \quad \partial\mathbb{D}\,,$$

$$\int_{\partial\mathbb{D}} \operatorname{Im}\left\{\prod_{k=1}^{\kappa} \frac{\overline{\zeta - z_k}}{|\zeta - z_k|} f_0(\zeta)\right\} \sigma(\zeta)ds = 0\,,$$

$$f_0(z_k) = 0, \quad 1 \le k \le \kappa\,,$$

with

$$\varphi_0(z) := \frac{i}{\check{\Sigma}} \prod_{k=1}^{\kappa}(z - z_k),\ \check{\sigma}(\zeta) := \prod_{k=1}^{\kappa} |\zeta - z_k|\sigma(\zeta),\ \check{\Sigma} := \int_{\partial\mathbb{D}} \check{\sigma} ds\,,$$

$$a_0 := \int_{\partial\mathbb{D}} e^{\operatorname{Re} v(\zeta)}\check{\sigma}(\zeta)ds \left[2\check{\Sigma}M \int_{\partial\mathbb{D}} e^{\operatorname{Re} v(\zeta)}\sigma(\zeta)ds\right]^{-1}.$$

Again applying the above estimate we get

$$C_{\alpha_0}(f_0; \mathbb{D}) + \|f_{0z}\|_p + \|f_{0\bar{z}}\|_p \le M\,.$$

Consider now

$$f := we^{-v} - A(a_0 f_0 + \varphi_0)$$

with

$$A := \int_{\partial\mathbb{D}} \operatorname{Im}\left\{\prod_{k=1}^{\kappa} \frac{\overline{\zeta - z_k}}{|\zeta - z_k|} w(\zeta)\right\} e^{-\operatorname{Re} v(\zeta)}\sigma(\zeta)ds\,,$$

satisfying

$$f_{\bar{z}} + \mu f_z + ce^{-v} = 0 \quad \text{in} \quad \mathbb{D}\,,$$

$$\operatorname{Re}\left\{\prod_{k=1}^{\kappa} \frac{\overline{\zeta - z_k}}{|\zeta - z_k|} f(\zeta)\right\} = 0 \quad \text{on} \quad \partial\mathbb{D}\,,$$

$$\int_{\partial\mathbb{D}} \operatorname{Im}\left\{\prod_{k=1}^{\kappa} \frac{\overline{\zeta - z_k}}{|\zeta - z_k|} f(\zeta)\right\} \sigma(\zeta)ds = 0\,,$$

$$f(z_k) = 0, \quad 1 \le k \le \kappa\,.$$

Once more applying the estimate from before

$$C_{\alpha_0}(f; \mathbb{D}) + \|f_z\|_p + \|f_{\bar{z}}\|_p \le M\|ce^{-v}\|_p \le Me^{M\|h\|_p}\|c\|_p\,.$$

From the definition of f we find

$$A = -B \int_{\partial \mathbb{D}} \operatorname{Im} \left\{ \prod_{k=1}^{\kappa} \frac{\overline{\zeta - z_k}}{|\zeta - z_k|} f(\zeta) \right\} e^{\operatorname{Re} v(\zeta)} \sigma(\zeta) ds \,,$$

$$B := \left[a_0 \int_{\partial \mathbb{D}} \operatorname{Im} \left\{ \prod_{k=1}^{\kappa} \frac{\overline{\zeta - z_k}}{|\zeta - z_k|} f_0(\zeta) \right\} e^{\operatorname{Re} v(\zeta)} \sigma(\zeta) ds + \frac{1}{\check{\Sigma}} \int_{\partial \mathbb{D}} e^{\operatorname{Re} v(\zeta)} \check{\sigma}(\zeta) ds \right]^{-1} ,$$

and

$$\begin{aligned} |A| &\leq 2e^{C_0(v;\mathbb{D})} \left| \int_{\partial \mathbb{D}} \operatorname{Im} \left\{ \prod_{k=1}^{\kappa} \frac{\overline{\zeta - z_k}}{|\zeta - z_k|} f(\zeta) \right\} e^{\operatorname{Re} v(\zeta)} \sigma(\zeta) ds \right| \\ &\leq 2\Sigma e^{2C_0(v;\mathbb{D})} C_0(f;\mathbb{D}) \,. \end{aligned}$$

Combining these estimates gives

$$C_{\alpha_0}(w;\mathbb{D}) + \|w_z\|_p + \|w_{\bar{z}}\|_p \leq M e^{M\|h\|_p} \|c\|_p \leq M e^{M(\|a\|_p + \|b\|_p)} \|c\|_p \,.$$

Theorem 39. *The* RIEMANN–HILBERT *problem formulated in Theorem 38 is uniquely solvable.*

Proof. From the a priori estimate (3.3.10) at once the RIEMANN–HILBERT problem of non-negative index for the generalized BELTRAMI equation is seen to be uniquely solvable if any solution exists at all.
Finally we will prove the existence of a solution. As was mentioned before it is enough to consider homogeneous boundary and side conditions. With these conditions in mind we are looking for a solution to the equation

$$w_{\bar{z}} + \mu_1 w_z + \mu_2 \overline{w_z} + aw + b\overline{w} + c = 0 \,.$$

Introducing a real parameter $t, 0 \leq t \leq 1$, by

$$w_{\bar{z}} + t\{\mu_1 w_z + \mu_2 \overline{w_z} + aw + b\overline{w}\} + c = 0 \tag{3.3.11}$$

the solution then depends on this parameter too, $w = w(z,t)$. For $t = 1$ the original equation is attained. For $t = 0$ the equation is the inhomogeneous CAUCHY–RIEMANN system, the solution of which is given by $w(z,0) = T_\kappa c(z)$ in the case of λ being specified as above. We may assume that for some $t_0, 0 \leq t_0 < 1$ there exists a solution $w(z,t_0)$ to (3.3.11). Taking $w(z,t_0)$ as a first approximation for a solution to (3.3.11) for some $t > t_0$ we can inductively construct a sequence of approximate solutions $w_n(z,t)$ by asking $w_{n+1}(z,t)$ for $w_n(z,t)$ given to be a solution to

$$w_{n+1\bar{z}} + t_0[\mu_1 w_{n+1z} + \mu_2 \overline{w_{n+1z}} + aw_{n+1} + b\overline{w_{n+1}}]$$

$$+(t-t_0)[\mu_1 w_{nz} + \mu_2 \overline{w_{nz}} + a w_n + b\overline{w_n}] + \mathrm{c} = 0\,.$$

As $w_n \in W_p^1(\overline{D})$ where especially $w_n \in C^{\alpha_0}(\overline{D}), w_{nz} \in L_p(\overline{D}), \alpha_0 = 1 - 2/p, 2 < p$, the factor of $(t - t_0)$ as well as c belong to $L_p(\overline{D})$. By assumption this equation has a solution $w_{n+1}(z,t)$. In order to prove convergence consider $\omega_{n+1} := w_{n+1} - w_n$ satisfying for $n \in \mathbb{N}$

$$\omega_{n+1\bar{z}} + t_0[\mu_1 \omega_{n+1z} + \mu_2 \overline{\omega_{n+1z}} + a\omega_{n+1} + b\overline{\omega_{n+1}}]$$

$$+(t-t_0)[\mu_1 \omega_{nz} + \mu_2 \overline{\omega_{nz}} + a\omega_n + b\overline{\omega_n}] = 0\,.$$

Applying the a priori estimate (3.3.10) to this equation with homogeneous boundary and side conditions gives

$$\|\omega_{n+1}\| \le (t-t_0)\delta\,[\|\mu_1 \omega_{nz} + \mu_2 \overline{\omega_{nz}} + a\omega_n + b\overline{\omega_n}\|_p]$$

$$\le (t-t_0)\delta\,[q + \|a\|_p + \|b\|_p]\,\|\omega_n\|\,,$$

where

$$\|\omega\| := C_{\alpha_1}(\omega; D) + \|w_z\|_p + \|w_{\bar{z}}\|_p\,.$$

Choosing now $t - t_0 > 0$ so small that

$$(t-t_0)\delta\,[q + \|a\|_p + \|b\|_p] < 1 \tag{3.3.12}$$

this inequality guarantees convergence of (w_n) in the norm $\|\cdot\|$. Denoting the limit by $w(z,t)$ it is seen to be a solution to (3.3.11) for $t > t_0$ satisfying (3.3.10). Because the bound in (3.3.12) to the step–width $t - t_0$ is independent of t_0 repeating this procedure finitely many times serves to find a solution to (3.3.11) for $t = 1$ giving a solution to the original equation.

Remark. The restriction to homogeneous boundary and side conditions are unnecessary for the above reasoning because in the convergence proof we would in any case be involved with homogeneous data. Inhomogeneous data only change the upper bound of the solutions.

One easily figures out

$$\|\omega_1\| \le (t-t_0)\widehat{\delta}\|w_0\|,\ \|w_0\| \le \delta\|\mathrm{c}\|_p,\ \widehat{\delta} := \delta\,[q + \|a\|_p + \|b\|_p]$$

where in the case of nonhomogeneous data the bound for $\|w_0\|$ has to be altered. From

$$w_n = w_0 + \sum_{k=1}^{n} \omega_k\,, \quad \|\omega_k\| \le (t-t_0)^{k-1}\widehat{\delta}^{k-1}\|\omega_1\| \le (t-t_0)^k\widehat{\delta}^k\|w_0\|$$

for $(t - t_0)\widehat{\delta} < 1$ convergence follows.

Theorem 40. *The modified* RIEMANN–HILBERT *problem*

$$\operatorname{Re}\{\overline{\lambda}w\} = \varphi + h \quad \text{on} \quad \partial\mathbb{D}$$

where

$$\varphi, \lambda \in C^\alpha(\partial\mathbb{D}), |\lambda(\zeta)| = 1 \quad \text{on } \partial\mathbb{D}, \kappa := \frac{1}{2\pi}\int_{\partial\mathbb{D}} d\arg\lambda < 0\,,$$

$$h(z) := \sum_{k=\kappa+1}^{-\kappa-1} h_k z^k, h_{-k} = \overline{h_k}, \quad |k| \le -\kappa - 1\,,$$

with undetermined coefficients h_k for the generalized BELTRAMI *equation*

$$w_{\bar z} + \mu_1 w_z + \mu_2 \overline{w_z} + aw + b\overline{w} + c = 0 \quad \text{in} \quad \mathbb{D}$$

satisfying

$$|\mu_1(z)| + |\mu_2(z)| \le q_0 < 1, \quad \|a\|_{p_0} + \|b\|_{p_0} \le K\,,$$

where

$$0 \le q_0 < 1 < \frac{1}{\alpha_0} := \frac{p_0}{p_0 - 2} < 2\alpha < 2, 4 < p_0\,,\ 0 \le K\,,$$

is uniquely solvable. The solution satisfies the a priori estimate

$$C_{\alpha\alpha_0^2}(w;\mathbb{D}) + \|w_z\|_{\tilde p} + \|w_{\bar z}\|_{\tilde p} \le \beta C_\alpha(\varphi;\mathbb{D}) + \delta\|c\|_{p_0} \tag{3.3.13}$$

when β and δ are nonnegative constants depending on $\alpha, p, p_0, q_0, K, \lambda, 2 < p \le p_0$ such that $q_0\Lambda_p < 1$, and $\tilde p := \dfrac{p^2}{2(p-1)}, 2 < \tilde p < p$.

Proof. Let ψ be the solution to

$$\psi_{\bar z} + \mu_1\psi_z + \mu_2\overline{\psi_z} + a\psi + b\overline{\psi} + c = 0 \quad \text{in } \mathbb{D}\,,$$

$$\operatorname{Im}\psi = 0 \quad \text{on } \partial\mathbb{D}, \int_{\partial\mathbb{D}} \operatorname{Re}\psi(\zeta)ds = 0\,.$$

Applying Theorems 38 and 39 to $\widehat{\psi} := i\psi$ shows that ψ is uniquely given and satisfies

$$C_{\alpha_0}(\psi,\mathbb{D}) + \|\psi_z\|_p + \|\psi_{\bar z}\|_p \le \delta\|c\|_{p_0}, q_0\Lambda_p < 1, 2p \le p_0\,.$$

Let w be a solution to the problem formulated in the theorem for a proper function h. Then $v := w - \psi$ satisfies

$$v_{\bar z} + \mu_1 v_z + \mu_2\overline{v_z} + av + b\overline{v} = 0 \quad \text{in } \mathbb{D}\,,$$

$$\operatorname{Re}\{\overline{\lambda}v\} = \varphi + h - \operatorname{Re}\lambda\psi \quad \text{on } \partial\mathbb{D}\,.$$

Rewriting this differential equation as

$$v_{\bar z} + \mu v_z + \alpha v = 0 \quad \text{in} \quad \mathbb{D}$$

where

$$\mu := \begin{cases} \mu_1 + \mu_2 \dfrac{\overline{v_z}}{v_z} \quad , & \text{if} \quad v_z \neq 0 \\ \mu_1 \quad , & \text{if} \quad v_z = 0 \end{cases} ,$$

$$a_1 := \begin{cases} a + b\dfrac{\overline{v}}{v} \quad , & \text{if} \quad v \neq 0 \\ a \quad , & \text{if} \quad v = 0 \end{cases}$$

and similarly as before utilising the solution χ to

$$\chi_{\bar z} + \mu \chi_z + a_1 = 0 \quad \text{in } \mathbb{D} ,$$

$$\operatorname{Im} \chi = 0 \quad \text{on } \partial\mathbb{D}, \qquad \int\limits_{\partial \mathbb{D}} \operatorname{Re} \chi(\zeta) ds = 0$$

satisfying

$$C_{\alpha_0}(\chi; \mathbb{D}) + \|\chi_z\|_p + \|\chi_{\bar z}\|_p \leq \delta \|a_1\|_{p_0} \leq \delta K ,$$

then $f := ve^{-\chi}$ is a solution to

$$f_{\bar z} + \mu f_z = 0 \quad \text{in } \mathbb{D} ,$$

$$\operatorname{Re}\{\overline{\lambda} f\} = \{\varphi + h - \operatorname{Re} \lambda \psi\} e^{-\operatorname{Re} \chi} = \widetilde{\varphi} + \widetilde{h} \quad \text{on } \partial \mathbb{D}$$

with

$$\widetilde{\varphi} := \{\varphi - \operatorname{Re} \lambda \psi\} e^{-\operatorname{Re} \chi}, \ \widetilde{h}(z) = \sum_{k=\kappa+1}^{-\kappa-1} \widetilde{h}_k z^k, \ \widetilde{h}_k := h_k e^{-\operatorname{Re} \chi(z)} .$$

By Lemma 14 f can be represented as $f = \phi \circ \zeta$ where ζ is a homeomorphism of the BELTRAMI equation $\zeta_{\bar z} + \mu \zeta_z = 0$ and ϕ is analytic in $\zeta[\mathbb{D}]$.
We may assume $\zeta[\mathbb{D}] = \mathbb{D}$. An arbitrary complete homeomorphism maps $\mathbb{D}$ onto a simply connected domain $\zeta[\mathbb{D}]$. Let ω be a conformal mapping from $\zeta[\mathbb{D}]$ onto $\mathbb{D}$, see Theorem 7, then $\zeta_1 := \omega \circ \zeta$ is a homeomorphism from $\mathbb{D}$ onto $\mathbb{D}$ satisfying

$$\zeta_{1\bar z} + \mu \zeta_{1z} = 0 .$$

The inverse mapping $z = z(\zeta)$ of $\zeta = \zeta(z)$ is a solution to the BELTRAMI equation

$$z_{\bar\zeta} - \mu \frac{\zeta_z}{\overline{\zeta_z}} \bigg|_{z=z(\zeta)} z_\zeta = 0 .$$

From the estimates in the proof of Theorem 35 and the respective property of the T-operator, see Theorem 23,

$$C_{\alpha_0}(\zeta; \mathbb{D}) + \|\zeta_z\|_p + \|\zeta_{\bar{z}}\|_p \leq M = M(p, p_0, q_0), \alpha_0 := \frac{p_0 - 2}{p_0}, 2 < p \leq p_0, q_0\Lambda_p < 1 .$$

Similarly,

$$C_{\alpha_0}(z; \mathbb{D}) + \|z_\zeta\|_p + \|z_{\bar{\zeta}}\|_p \leq M .$$

Thus, ϕ is an analytic function of ζ in $\mathbb{D}$ satisfying

$$\mathrm{Re}\,\{\overline{\lambda(z(\zeta))}\phi(\zeta)\} = \widetilde{\varphi}(z(\zeta)) + \widetilde{h}(z(\zeta)), \zeta \in \partial\mathbb{D} .$$

By Theorem 18 this problem is uniquely solvable when fixing the yet undetermined coefficients $h_k, |k| \leq -\kappa - 1$, by the system

$$\int_{\partial\mathbb{D}} [\widetilde{\varphi}(z(t)) + \widetilde{h}(z(t))] e^{\mathrm{Im}\,\gamma(t)} t^{-\ell-1} dt = 0, \quad 0 \leq \ell \leq -\kappa - 1 ,$$

where

$$\gamma(\zeta) := \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \arg\left\{t^{-\kappa}\lambda(z(t))\right\} \frac{t + \zeta}{t - \zeta} \frac{dt}{t}, \quad \zeta \in \mathbb{D} .$$

Then ϕ is given by the SCHWARZ integral

$$\begin{aligned} \phi(\zeta) &= \frac{\zeta^\kappa e^{i\gamma(\zeta)}}{2\pi i} \int_{\partial\mathbb{D}} [\widetilde{\varphi}(z(t)) + \widetilde{h}(z(t))] e^{\mathrm{Im}\,\gamma(t)} \frac{t + \zeta}{t - \zeta} \frac{dt}{t} \\ &= \frac{e^{i\gamma(\zeta)}}{\pi i} \int_{\partial\mathbb{D}} [\widetilde{\varphi}(z(t)) + \widetilde{h}(z(t))] e^{\mathrm{Im}\,\gamma(t)} \frac{t^\kappa}{t - \zeta} dt, \quad \zeta \in \mathbb{D} . \end{aligned}$$

For the last equality

$$\zeta^\kappa \frac{t + \zeta}{t - \zeta} \frac{1}{t} = \frac{\zeta^\kappa}{t} + 2 \sum_{\nu=1}^{-\kappa-1} \frac{\zeta^{\nu+\kappa}}{t^{\nu+1}} + \frac{t^\kappa}{t - \zeta}$$

is used. From the estimates in the proof of Theorem 5

$$C_{\alpha\alpha_0}(\phi; \mathbb{D}) \leq M(\alpha, \alpha_0) C_{\alpha\alpha_0}(e^{i\gamma}; \mathbb{D}) C_{\alpha\alpha_0}([\widetilde{\varphi}(z(t)) + \widetilde{h}(z(t))] e^{\mathrm{Im}\,\gamma(t)} t^\kappa; \partial\mathbb{D})$$

follows so that

$$C_{\alpha\alpha_0}(\phi; \mathbb{D}) \leq M(\alpha, p_0, K, \lambda) C_\alpha(\varphi; \partial\mathbb{D}) .$$

Therefore $f = \phi \circ \zeta \in C^{\alpha\alpha_0^2}(\overline{D})$ satisfies

$$C_{\alpha\alpha_0^2}(f; \mathbb{D}) \leq M(\alpha, p_0, K, \lambda) C_\alpha(\varphi; \partial\mathbb{D}) .$$

Moreover, by Lemma 17

$$\begin{aligned}\|\phi'\|_p &\le M(\lambda)C_{\alpha\alpha_0}(\widetilde{\varphi}(z(\cdot))+\widetilde{h}(z(\cdot));\mathbb{D})\\ &\le M(\alpha,p_0,K,\lambda)\,[C_{\alpha\alpha_0}(\varphi(z(\cdot));\mathbb{D})+\|c\|_p]\\ &\le M(\alpha,p_0,K,\lambda)\,[C_\alpha(\varphi;\mathbb{D})+\|c\|_{p_0}]\ ,\end{aligned}$$

where $2<p<\dfrac{1}{1-\alpha\alpha_0}, q_0\Lambda_p<1$. Hence, for $\widetilde{p}:=\dfrac{p^2}{2(p-1)}, 2<\widetilde{p}<p$, we have with $p':=\dfrac{2(p-1)}{p}, q'=\dfrac{2(p-1)}{p-2}$

$$\begin{aligned}\|f_z\|_{\widetilde{p}}^{\widetilde{p}} &= \|\phi'(\zeta(z))\zeta_z\|_{\widetilde{p}}^{\widetilde{p}}=\int_{\mathbb{D}}|\phi'(\zeta)|^{\widetilde{p}}|\zeta_z|^{\widetilde{p}}\left(|z_\zeta|^2-|z_{\overline{\zeta}}|^2\right)d\xi d\eta\\ &= \frac{1}{1-q^2}\int_{\mathbb{D}}|\phi'(\zeta)|^{\widetilde{p}}|\zeta_z|^{\widetilde{p}-2}d\xi d\eta\\ &\le \frac{1}{1-q^2}\left(\int_{\mathbb{D}}|\phi'(\zeta)|^{\widetilde{p}p'}d\xi d\eta\right)^{1/p'}\left(\int_{\mathbb{D}}|\zeta_z|^{(\widetilde{p}-2)q'}d\xi d\eta\right)^{1/q'}\ ,\end{aligned}$$

where

$$z_\zeta(|\zeta_z|^2-|\zeta_{\overline{z}}|^2)=\overline{\zeta_z},\quad z_{\overline{\zeta}}(|\zeta_z|^2-|\zeta_{\overline{z}}|^2)=-\zeta_{\overline{z}}$$

is used. Because $\widetilde{p}p'=p$ and

$$\int_{\mathbb{D}}|\zeta_z|^{(\widetilde{p}-2)q'}d\xi d\eta=\int_{\mathbb{D}}|\zeta_z|^{(\widetilde{p}-2)q'}(|\zeta_z|^2-|\zeta_{\overline{z}}|^2)dxdy$$

$$=(1-q^2)\int_{\mathbb{D}}|\zeta_z|^{(\widetilde{p}-2)q'+2}dxdy=(1-q^2)\int_{\mathbb{D}}|\zeta_z|^p dxdy$$

we have

$$\|f_z\|_{\widetilde{p}}\le\|\phi'\|_p\|\zeta_z\|_p^{\frac{p-2}{p}}\ .$$

By a similar procedure or from the differential equation for f we can estimate $\|f_{\overline{z}}\|_{\widetilde{p}}$. Thus

$$C_{\alpha\alpha_0^2}(f;\mathbb{D})+\|f_z\|_{\widetilde{p}}+\|f_{\overline{z}}\|_{\widetilde{p}}\le M(\alpha,p,p_0,q_0,K,\lambda)[C_\alpha(\varphi;\mathbb{D})+\|c\|_{p_0}]\ .$$

From $w=fe^\chi+\psi$ we find by the preceding estimates and from $\|\cdot\|_{\widetilde{p}}\le\|\cdot\|_p$ for $1\le\widetilde{p}\le p$

$$C_0(w;\mathbb{D})+\|w_z\|_{\widetilde{p}}+\|w_{\overline{z}}\|_{\widetilde{p}}\le M(\alpha,p,p_0,q_0,K,\lambda)[C_\alpha(\varphi;\partial\mathbb{D})+\|c\|_{p_0}]\ ,$$

where $C_0(w;\mathbb{D})$ can be replaced by $C_{\alpha\alpha_0^2}(w;\mathbb{D})$, too.

3.4 POINCARÉ boundary value problem

If in the RIEMANN–HILBERT boundary condition

$$\mathrm{Re}\,\{\overline{\lambda(\zeta)}w(\zeta)\} = \varphi(\zeta), \zeta \in \partial D\ ,$$

the function w is replaced by its z-derivative w_z this problem is called the oblique derivative or POINCARÉ problem. We will treat this problem for the linear generalized BELTRAMI equation, i.e. the problem

$$w_{\bar z} + \mu_1 w_z + \mu_2 \overline{w_z} + aw + b\overline{w} + c = 0 \quad \text{in } D\ ,$$

$$\mathrm{Re}\,\{\overline{\lambda} w_z\} = \varphi + h \quad \text{on } \partial D\ ,$$

$$w(1) = 0\ ,$$

where we assume $1 \in \overline{D}$. For simplicity again D is assumed to be the unit disc. This can always be achieved by a conformal mapping if D is a bounded simply connected domain.

Introducing $u := w_z$ and observing

$$w_{\bar z} = -[\mu_1 u + \mu_2 \overline{u} + aw + b\overline{w} + c]$$

we see

$$w(z) = \int_1^z \left\{u(t)dt - [\mu_1 u(t) + \mu_2 \overline{u(t)} + aw(t) + b\overline{w(t)} + c]d\bar t\right\}\ . \tag{3.4.1}$$

This integral is path-independent because

$$u_{\bar z} = w_{\bar z z} = -[\mu_1 u + \mu_2 \overline{u} + aw + b\overline{w} + c]_z\ .$$

Obviously, we have to assume that the coefficients of the differential equation are (weakly) differentiable. Differentiating the generalized BELTRAMI equation leads to

$$u_{\bar z} + \mu_1 u_z + \mu_2 \overline{u_{\bar z}} + au + b\overline{w_{\bar z}} + \mu_{1z} u + \mu_{2z}\overline{u} + a_z w + b_z \overline{w} + c_z = 0$$

and its complex conjugate

$$\overline{u_{\bar z}} + \overline{\mu_1}\,\overline{u_z} + \overline{\mu_2} u_{\bar z} + \overline{a}\,\overline{u} + \overline{b} w_{\bar z} + \overline{\mu_{1z}}\,\overline{u} + \overline{\mu_{2z}} u + \overline{a_z}\,\overline{w} + \overline{b_z} w + \overline{c_z} = 0\ .$$

Solving this system with the two unknowns $u_{\bar z}$ and $\overline{u_{\bar z}}$ for $u_{\bar z}$ gives

$$u_{\bar z} + q_1 u_z + q_2 \overline{u_z} + Au + B\overline{u} + H(w) = 0\ , \tag{3.4.2}$$

where

$$q_1 := \frac{\mu_1}{1-|\mu_2|^2}\,,\; q_2 := -\frac{\mu_2\overline{\mu_1}}{1-|\mu_2|^2}\,,\; |q_1|+|q_2| \le q_0 < 1\,,$$

$$A := \frac{a+\mu_{1z}-\mu_2\overline{\mu_{2z}}}{1-|\mu_2|^2},\; B := \frac{\mu_{2z}-\mu_2\overline{a}-\mu_2\overline{\mu_{1z}}}{1-|\mu_2|^2}\,,$$

$$H(w) := \frac{1}{1-|\mu_2|^2}\left[-\mu_2\overline{b}w_{\overline{z}} + b\overline{w_{\overline{z}}} + (a_z-\mu_2\overline{b_z})w + (b_z-\mu_2\overline{a_z})\overline{w} + c_z - \mu_2\overline{c_z}\right]\,.$$

Together with the boundary condition

$$\operatorname{Re}\{\overline{\lambda}u\} = \varphi + h \tag{3.4.3}$$

this is just a RIEMANN–HILBERT problem for u where in H the unknown function w and its derivative $w_{\overline{z}}$ is incorporated.
Introducing the side conditions

$$\frac{1}{\Sigma}\int_{\partial\mathbb{D}}\{\operatorname{Im}\overline{\lambda(\zeta)}u(\zeta)\}\sigma(\zeta)ds = c_0, \quad u(z_k) = a_k, 1\le k\le\kappa\,, \tag{3.4.4}$$

or alternatively, see Corollary 2,

$$\operatorname{Im}\{\overline{\lambda(a_k)}u(a_k)\} = b_k, \quad 0\le k\le 2\kappa\,,$$

in the case where the index is nonnegative ($0\le\kappa$) and assuming there is a solution (w,u) for problem (3.4.1) – (3.4.4) the a priori estimate (3.3.10) from Theorem 38 for the first set of side conditions in case $0\le\kappa$ and (3.3.13) from Theorem 40 for $\kappa<0$ gives

$$C_{\alpha\alpha_0^2}(u;\mathbb{D}) + \|u_z\|_{\tilde{p}} + \|u_{\overline{z}}\|_{\tilde{p}} \le \beta\, C_\alpha(\varphi;\partial\mathbb{D}) + \gamma_1|c_0| + \gamma_2\sum_{k=1}^{\kappa}|a_k| + \delta\|H(w)\|_{p_0}\,,$$

where for $0\le\kappa$ $\tilde{p}=p=p_0$ and for $\kappa<0$ the constants γ_1 and γ_2 formally are replaced by 0. Here the constants $\beta,\gamma_1,\gamma_2,\delta$ depend among others especially on p,p_0 and the constant K_1 satisfies

$$\|A\|_{p_0} + \|B\|_{p_0} \le \frac{1}{1-q_0}\left[\|\mu_{1z}\|_{p_0} + \|\mu_{2z}\|_{p_0} + \|a\|_{p_0}\right] \le K_1\,.$$

As

$$\|H(w)\|_{p_0} \le \frac{1}{1-q_0}\left[C_0(b;\mathbb{D})\|w_{\overline{z}}\|_{p_0} + (\|a_z\|_{p_0} + \|b_z\|_{p_0})C_0(w;\mathbb{D}) + \|c_z\|_{p_0}\right]$$

and from the integral equation for w

$$C_0(w;\mathbb{D}) \le 2(1+q_0)C_0(u;\mathbb{D}) + 2(C_0(a;\mathbb{D}) + C_0(b;\mathbb{D}))C_0(w;\mathbb{D}) + 2C_0(c;\mathbb{D})$$

we have under the assumption

$$2\,(C_0(a;\boldsymbol{D})+C_0(b;\boldsymbol{D}))\le\eta,\quad 2(\|a_z\|_{p_0}+\|b_z\|_{p_0})\le\eta,\quad \eta<1\,,$$

the estimates

$$C_0(w;\boldsymbol{D})\le\frac{2(1+q_0)}{1-\eta}C_0(u;\boldsymbol{D})+\frac{2}{1-\eta}C_0(c;\boldsymbol{D})$$

and

$$\|H\|_{p_0}\le\frac{(1+q_0)\eta}{(1-q_0)(1-\eta)}C_0(u;\boldsymbol{D})+\frac{\eta}{2(1-q_0)}\|w_{\bar z}\|_{p_0}+\frac{\eta}{(1-q_0)(1-\eta)}C_0(c;\boldsymbol{D})\,.$$

From the generalized BELTRAMI equation

$$\begin{aligned}C_0(w_{\bar z};\boldsymbol{D})&\le q_0C_0(u;\boldsymbol{D})+\frac{1}{2}\eta C_0(w;\boldsymbol{D})+C_0(c;\boldsymbol{D})\\&\le\left(q_0+\frac{\eta(1+q_0)}{1-\eta}\right)C_0(u;\boldsymbol{D})+\left(1+\frac{\eta}{1-\eta}\right)C_0(c;\boldsymbol{D})\,,\end{aligned}$$

so that because of $\|w_{\bar z}\|_p\le 2C_0(w_{\bar z};\boldsymbol{D})$ for $2<p$

$$\begin{aligned}\|H\|_{p_0}&\le\frac{\eta}{1-q_0}\left(q_0+\frac{(1+q_0)(1+\eta)}{1-\eta}\right)C_0(u;\boldsymbol{D})\\&\quad+\frac{\eta}{1-q_0}\left(1+\frac{1+\eta}{1-\eta}\right)C_0(c;\boldsymbol{D})+\frac{1}{1-q_0}\|c_z\|_{p_0}\\&=\frac{\eta}{1-q_0}\,\frac{2q_0+1+\eta}{1-\eta}C_0(u;\boldsymbol{D})+\frac{2\eta}{(1-q_0)(1-\eta)}C_0(c;\boldsymbol{D})+\frac{1}{1-q_0}\|c_z\|_{p_0}\end{aligned}$$

Inserting this into the a priori estimate and neglecting $\|u_z\|_{\tilde p}+\|u_{\bar z}\|_{\tilde p}$ gives

$$\begin{aligned}\left[1-\frac{\eta\delta(2q_0+1+\eta)}{(1-q_0)(1-\eta)}\right]C_0(u;\boldsymbol{D})&\le\beta\,C_\alpha(\varphi;\partial\boldsymbol{D})+\gamma_1|c_0|+\gamma_2\sum_{k=1}^{\kappa}|a_k|\\&\quad+\frac{2\eta\delta}{(1-q_0)(1-\eta)}C_0(c;\boldsymbol{D})+\frac{\delta}{1-q_0}\|c_z\|_{p_0}\,.\end{aligned}$$

Assuming

$$\eta\delta(2q_0+1+\eta)<(1-q_0)(1-\eta)$$

which is satisfied for example for

$$\eta<\frac{1-q_0}{2\delta(1+q_0)+1-q_0}$$

then

$$C_0(u;\mathbb{D}) \le \widetilde{\beta} C_\alpha(\varphi;\partial\mathbb{D}) + \widetilde{\gamma}_1|c_0| + \widetilde{\gamma}_2\sum_{k=1}^{\kappa}|a_k| + \widetilde{\delta}[C_0(c;\mathbb{D}) + \|c_z\|_{p_0}] .$$

Collecting the respective estimates leads to the a priori estimate

$$C_0(w;\mathbb{D}) + C_0(w_z;\mathbb{D}) + C_0(w_{\bar z};\mathbb{D}) \le \widehat{\beta} C_\alpha(\varphi;\partial\mathbb{D}) + \widehat{\gamma}_1|c_0| + \widehat{\gamma}_2\sum_{k=1}^{\kappa}|a_k|$$

$$+\widehat{\delta}[C_0(c;\mathbb{D}) + \|c_z\|_{p_0}] . \tag{3.4.5}$$

Of course the maximum norms for the first order derivatives for w can be replaced by the $L_p(\overline{\mathbb{D}})$-norms. Moreover, on the left-hand side $\|w_{zz}\|_{\tilde p} + \|w_{z\bar z}\|_{\tilde p}$ may be added. We even may add $\|w_{\bar z\bar z}\|_{\tilde p}$, too if

$$\|\mu_{1\bar z}\|_{p_0} + \|\mu_{2\bar z}\|_{p_0} \le K_1 , \|a_{\bar z}\|_{p_0} + \|b_{\bar z}\|_{p_0} \le K_1$$

is assumed and on the right-hand side $\widehat{\delta}\|c_{\bar z}\|_{p_0}$ is added. This follows just from the differential equation and the preceding estimates. Thus

$$C_0(w;\mathbb{D}) + C_0(w_z;\mathbb{D}) + C_0(w_{\bar z};\mathbb{D}) + \|w_{\bar z\bar z}\|_{\tilde p} + \|w_{\bar z z}\|_{\tilde p} + \|w_{zz}\|_{\tilde p}$$

$$\le \widehat{\beta} C_\alpha(\varphi;\partial\mathbb{D}) + \widehat{\gamma}_1|c_0| + \widehat{\gamma}_2\sum_{k=1}^{\kappa}|a_k| + \widehat{\delta}[C_0(c;\mathbb{D}) + \|c_z\|_{p_0} + \|c_{\bar z}\|_{p_0}] . \tag{3.4.6}$$

It remains to show that the POINCARÉ problem is solvable at all. From the a priori estimates it then follows that the solution is unique.
Let $w_0 \in C^0(\overline{\mathbb{D}})$ satisfy $C_0(w_0;\mathbb{D}) \le K$ where K is a constant not less then

$$\frac{2(1+q_0)}{1-(1+(1+q_0)\delta)\eta}\left[\beta C_\alpha(\varphi;\partial\mathbb{D}) + \gamma_1|c_0| + \gamma_2\sum_{k=1}^{\kappa}|a_k| + \delta(\|c_z\|_{p_0} + C_0(c;\mathbb{D}))\right]$$

and $[1+(1+q_0)\delta]\eta < 1$ is assumed. Let u be a solution to

$$u_{\bar z} + q_1 u_z + q_2\overline{u_z} + Au + B\overline{u} + H(w_0) = 0 \quad \text{in } \mathbb{D} \tag{3.4.7}$$

satisfying the boundary condition

$$\operatorname{Re}\{\overline{\lambda} u\} = \varphi + h \quad \text{on } \partial\mathbb{D}$$

and in case of nonnegative index κ the side conditions (3.4.4). Moreover, let w be defined by

$$w(z) := \int_1^z \left\{u(t)dt - [\mu_1(t)u(t) + \mu_2(t)\overline{u(t)} + a(t)w_0(t) + b(t)\overline{w_0(t)} + c(t)]d\bar t\right\}$$

for $z \in \mathbb{D}$. Then from the above estimates

$$
\begin{aligned}
C_0(u;\mathbb{D}) &\le \beta C_\alpha(\varphi;\mathbb{D}) + \gamma_1|c_0| + \gamma_2\sum_{k=1}^{\kappa}|a_k| + \delta\|c_z\|_{p_0} \\
&\quad + \frac{1}{2}\delta\eta\left[C_0(w_0;\mathbb{D}) + \|w_{0\bar z}\|_{p_0}\right] , \\
C_0(w;\mathbb{D}) &\le 2(1+q_0)\left[\beta C_\alpha(\varphi;\partial\mathbb{D}) + \gamma_1|c_0| + \gamma_2\sum_{k=1}^{\kappa}|a_k| + \delta\|c_z\|_{p_0} + C_0(c;\mathbb{D})\right] \\
&\quad + (1+(1+q_0)\delta)\eta C_0(w_0;\mathbb{D}) \le K .
\end{aligned}
$$

In this way a mapping L is defined from $C_0(\overline{\mathbb{D}})$ into itself which, obviously, is linear and will be shown to be a contraction. For this purpose let $w_0, w_0' \in C_0(\overline{\mathbb{D}})$ be given and $w = Lw_0, w' = Lw_0'$. Then the related functions u and u' as solutions to (3.4.7) and the boundary (and side –) conditions lead to a solution $u - u'$ of

$$
(u-u')_{\bar z} + q_1(u-u')_z + q_2\overline{(u-u')_z} + A(u-u') + B(\overline{u-u'}) + H(w) - H(w') = 0 \quad \text{in } \mathbb{D} ,
$$

$$
\operatorname{Re}\{\overline{\lambda}(u-u')\} = h \quad \text{on } \partial\mathbb{D}
$$

and eventuelly homogeneous side conditions. Hence,

$$
C_0(u-u';\mathbb{D}) \le \frac{1}{2}\delta\eta\left[C_0(w_0 - w_0';\mathbb{D}) + \|(w_0 - w_0')_{\bar z}\|_{p_0}\right]
$$

and

$$
C_0(w-w';\mathbb{D}) \le (1+(1+q_0)\delta)\eta\, C_0(w_0 - w_0';\mathbb{D}) .
$$

Because the coefficient on the right-hand of the last inequality is less than 1, this estimate shows L being a contraction. Hence, the BANACH fixed point theorem proves the existence of a unique solution $w \in C^0(\overline{\mathbb{D}})$ to the equation $w = Lw$. From (3.4.1) then $w(1) = 0$ and by the continuity of the integrand

$$
w_z = u, w_{\bar z} + \mu_1 u + \mu_2\overline{u} + aw + b\overline{w} + c ,
$$

follow in $\mathbb{D}$. Because u satisfies the modified boundary and the side-conditions w is a solution to the POINCARÉ problem.
Thus the following result is proved.

Theorem 41. *The modified* POINCARÉ *boundary value problem for the generalized* BELTRAMI *equation is uniquely solvable under the following assumptions.*

$$
\mu_1, \mu_2, a, b, c \in C^0(\overline{\mathbb{D}}) \cap \mathbf{D}_z(\mathbb{D}),\ \mu_{1z}, \mu_{2z} \in L_{p_0}(\overline{\mathbb{D}}) ,
$$

$$
\|\mu_{1z}\|_{p_0} + \|\mu_{2z}\|_{p_0} + \|a\|_{p_0} \le K_1 ,\ |\mu_1(z)| + |\mu_2(z)| \le q_0 < 1, \quad q_0\Lambda_{p_0} < 1 ,
$$

where

$$2 < p_0 = \tilde{p} = p \quad \text{for} \quad 0 \le \kappa\,,$$

$$2 < \tilde{p} := \frac{p^2}{2(p-1)} < p \le p_0, 4 < p_0\,, \quad \text{for} \quad \kappa < 0\,,$$

$$2\left(\|a_z\|_{p_0} + \|b_z\|_{p_0}\right) \le \eta\,,\ 2[C_0(a;\mathbb{D}) + C_0(b;\mathbb{D})] \le \eta\,.$$

Here q_0, η, K_1 *are nonnegative constants and* η *is so small that*

$$\eta \le \frac{1-q_0}{1-q_0+2(1+q_0)\delta}$$

where δ *is the constant appearing in the a priori estimate (3.3.10) and (3.3.13), respectively, depending among others especially on* p_0 *and* K_1 *rather than on* p *and* K. *The solution satisfies the a priori estimate (3.4.5) where* $\widehat{\beta}, \widehat{\gamma}_1, \widehat{\gamma}_2, \widehat{\delta}$ *are nonnegative constants depending on* $\alpha, p, p_0, q_0, \lambda, \eta, K_1$, *and also on* $\sigma, z_k (1 \le k \le \kappa)$ *for nonnegative index.*

If, moreover, $\mu_1, \mu_2, a, b, c \in \mathbf{D}_{\bar z}(\mathbb{D})$, *and*

$$\|\mu_{1\bar z}\|_{p_0} + \|\mu_{2\bar z}\|_{p_0} \le K_1, \quad \|a_{\bar z}\|_{p_0} + \|b_{\bar z}\|_{p_0} \le K_1$$

the solution then satisfies (3.4.6). In case of negative index again formally $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$.

A basic paper for the POINCARÉ problem for generalized BELTRAMI systems especially for generalized analytic functions is [Dani 62].

3.5 Discontinuous boundary value problems

The conditions on the coefficients of the RIEMANN–HILBERT boundary conditions can be weakened. They might have discontinuities of first kind (λ) or even of second kind (φ). For analytic functions this discontinuous RIEMANN–HILBERT problem was investigated by MUSKHELISHVILI, see [Musk53]. The discontinuous POINCARÉ problem for generalized BELTRAMI equations was studied in [Bewe88,89]. In this section a priori estimates for solutions to these discontinuous problems for generalized BELTRAMI equations are developed. Again we will deal with the unit disc $\mathbb{D}$. The coefficients of the boundary condition

$$\operatorname{Re}\{\overline{\lambda} w\} = \varphi \quad \text{on } \partial\mathbb{D}$$

are assumed to satisfy the following conditions. There are finitely many consecutively ordered points $\{c_\mu : 1 \le \mu \le m\}$ on $\partial\mathbb{D}$ subdividing $\partial\mathbb{D}$ into m disjoint open arcs $\{\Gamma_\mu : 1 \le \mu \le m\}$ such that

$$\lambda \in C^\beta(\overline{\Gamma_\mu}),\ 1 \le \mu \le m, 0 < \beta < 1\,,$$

$$\lambda(c_\mu - 0) = e^{i\vartheta_\mu}\lambda(c_\mu + 0),\ \varphi_\mu := \frac{1}{\pi}\vartheta_\mu - k_\mu, k_\mu := \left[\frac{1}{\pi}\vartheta_\mu\right] + I_\mu\,,$$

where

$$\lambda(c_\mu \pm 0) := \lim_{t\to\pm 0} \lambda(c_\mu e^{it}),\ I_\mu \in \{0,1\} \quad \text{such that} \quad -1 < \varphi_\mu < 1\,,$$

$$\varphi(z) := \varphi_0(z) \prod_{\mu=1}^{m} |z-c_\mu|^{-\beta_\mu},\ 0 \le \beta_\mu, \varphi_\mu + \beta_\mu < 1,\ 1 \le \mu \le m\,,$$

with $\varphi_0 \in C^\beta(\overline{\Gamma_\mu})$, $1 \le \mu \le m$.
The index κ of this discontinuous problem is defined by the equation

$$2\kappa := \sum_{\mu=1}^{m} k_\mu\,.$$

Later on the entire part $[\kappa]$ of κ, i.e. the largest entire number less than or equal to κ will be used. In case when no discontinuities at the points $c_\mu (1 \le \mu \le m)$ occur we have $\vartheta_\mu = 2\kappa_\mu \pi, I_\mu = 0, \varphi_\mu = 0$ with $\kappa_\mu \in \mathbb{Z}$. Hence, $\kappa = \sum_{\mu=1}^{m} \kappa_\mu \in \mathbb{Z}$ coincides with

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}} d \log \lambda(\zeta)\,.$$

If λ is discontinuous in c_μ then I_μ can be chosen in such a way that k_μ is an even number. But this will influence the behaviour of the solution near c_μ. We thus have to consider the two cases when 2κ is an even or an odd number. If k_μ is even then $\widetilde{\lambda}_\mu(z) := \lambda(z)(\overline{z-c_\mu})^{\varphi_\mu}|z-c_\mu|^{-\varphi_\mu}$ is continuous at $z = c_\mu$ as a function of $z \in \partial\mathbb{D}$. Here $(z-c_\mu)^{\varphi_\mu}$ on $\partial\mathbb{D}$ is considered as the boundary values of an analytic branch of this function for $z \in \mathbb{D}$. If k_μ is odd then the function $\widetilde{\lambda}_\mu$ has different signs on the two sides of c_μ on $\partial\mathbb{D}$. This can be seen from

$$\begin{aligned}
\lim_{z\to c_\mu \pm 0} \lambda(z) \frac{(\overline{z-c_\mu})^{\varphi_\mu}}{|z-c_\mu|^{\varphi_\mu}} &= \lim_{\substack{t\to 0\\ t>0}} \lambda\left(c_\mu e^{\pm it}\right) \frac{\overline{c_\mu}^{\varphi_\mu} |e^{\mp it}-1|^{\varphi_\mu}}{|e^{\mp it}-1|^{\varphi_\mu}} \\
&= \lambda(c_\mu \pm 0)\overline{c_\mu}^{\varphi_\mu} \left(\mp e^{i\frac{\pi}{2}}\right)^{\varphi_\mu}
\end{aligned}$$

from which

$$\begin{aligned}
\lim_{z\to c_\mu+0} \widetilde{\lambda}_\mu(z) &= \lambda(c_\mu+0)\overline{c_\mu}^{\varphi_\mu} e^{i\frac{3\pi}{2}\varphi_\mu} = \lambda(c_\mu-0)\overline{c_\mu}^{\varphi_\mu} e^{i\frac{1}{2}\pi\varphi_\mu} e^{i(\pi\varphi_\mu-\vartheta_\mu)} \\
&= (-1)^{k_\mu} \lim_{z\to c_\mu-0} \widetilde{\lambda}_\mu(z)
\end{aligned}$$

follows. We therefore distinguish two kinds of points of discontinuity,

$$C_1 := \{c_\mu : k_\mu + 1 \in 2\mathbb{Z}\}\,,\ C_2 := \{c_\mu : k_\mu \in 2\mathbb{Z}\}\,.$$

In $c_\mu \in C_2$ the function $\widetilde{\lambda}(z) := \lambda(z)\Pi_{\mu=1}^m \frac{(\overline{z - c_\mu})^{\varphi_\mu}}{|z - c_\mu|^{\varphi_\mu}}$ is continuous while in $c_\mu \in C_1$ it undergoes a sign jump. In order to transform $\widetilde{\lambda}$ into a continuous function on ∂D consider

$$C_1 = \{c_{\mu_\nu} : 1 \leq \nu \leq n\} , \quad \mu_\nu < \mu_{\nu+1}(1 \leq \nu < n \leq m) .$$

It now is appropriate to distinguish two cases

i. $n \in 2\mathbb{N}$, so that $\kappa \in \mathbb{Z}$,

ii. $n + 1 \in 2\mathbb{N}$, so that $2\kappa + 1 \in 2\mathbb{Z}$.

In case i. denote

$$\widetilde{\Gamma}_{\mu_\nu} := \{z : z \in \partial\mathbb{D}, \arg c_{\mu_\nu} \leq \arg z \leq \arg c_{\mu_\nu+1}\} , \quad 1 \leq \nu < n ,$$

$$\widetilde{\Gamma}_{\mu_n} := \{z : z \in \partial\mathbb{D}, \arg c_{\mu_n} \leq \arg z \leq \arg c_{\mu_1} + 2\pi\}$$

and define

$$\iota(z) := (-1)^{\nu-1}, z \in \Gamma_{\mu_\nu}, \quad 1 \leq \nu \leq n .$$

In case ii. choose $c_0 \in \partial\mathbb{D}\setminus\{c_1, \ldots, c_m\}$ and proceed as in case i. with $K_1 \cup \{c_0\}$. Then

$$\lambda_0(z) := \begin{cases} \iota(z)\widetilde{\lambda}(z) , & \text{in case i.} \\ \iota(z)\dfrac{(\overline{z - c_0})}{|z - c_0|}\widetilde{\lambda}(z) , & \text{in case ii.} \end{cases}$$

is continuous on $\partial\mathbb{D}$ where ι is defined with respect to C_1 and to $C_1 \cup \{c_0\}$, respectively. The discontinuous boundary condition

$$\operatorname{Re}\{\overline{\lambda(z)}w(z)\} = \varphi(z) \quad \text{on } \partial\mathbb{D}$$

therefore can be transformed into the continuous one

$$\operatorname{Re}\{\overline{\lambda_0(z)}w_0(z)\} = \varphi_0(z) \quad \text{on } \partial\mathbb{D}$$

for

$$w_0 := X_1^{-1}w , \quad X_1 := \begin{cases} \prod_1^m (z - c_\mu)^{\varphi_\mu} , & \text{in case i.} \\ (z - c_0)\prod_1^m (z - c_\mu)^{\varphi_\mu} , & \text{in case ii.} \end{cases}$$

with

$$\lambda_0 := \iota\lambda\frac{\overline{X_1}}{|X_1|} , \quad \varphi_0 := \frac{\iota}{|X_1|}\varphi$$

the index of which in both cases is $\kappa - \frac{1}{2}n \in \mathbb{Z}$.
In the following ι is assumed to be identically 1 and therefore $n = 0$ or $n = 1$ which can be achieved just by relabeling λ and φ. Thus the two cases are

$$\text{i.} \quad \kappa \in \mathbb{Z}\,, \quad \text{ii.} \quad \kappa - \frac{1}{2} \in \mathbb{Z}\,.$$

The solution w to the discontinuous RIEMANN–HILBERT problem for generalized BELTRAMI equations in both cases is looked for in the space $C(\overline{\mathbb{D}}) \cap \mathbf{D}_1(\mathbb{D})$.

Lemma 20. *For $\lambda \in C^\beta(\overline{\Gamma_\mu})$ the function $\lambda_0 \in C^\beta(\overline{\Gamma_\mu})$.*

Proof. Any factor of $X_1|X_1|^{-1}$ turns out to be LIPSCHITZ-continuous. This shows $\lambda_0 \in C^\beta(\overline{\Gamma_\mu})$. To prove $(z-c)^\varphi|z-c|^{-\varphi}$ for fixed c, $|c| = 1$, and φ real to be LIPSCHITZ-continuous on $\partial\mathbb{D}\backslash\{c\}$ let $z_1, z_2 \in \partial\mathbb{D}$, $z_1 \neq c$, $z_2 \neq c$. Then

$$(z_k - c)^\varphi|z_k - c|^{-\varphi} = e^{i\varphi \arg(z_k - c)}, k = 1, 2\,,$$

$$\begin{aligned} |(z_2 - c)^\varphi|z_2 - c|^{-\varphi} - (z_1 - c)^\varphi|z_1 - c|^{-\varphi}| &= \left|e^{i\varphi[\arg(z_2 - c) - \arg(z_1 - c)]} - 1\right| \\ &\leq |\varphi||\arg(z_2 - c) - \arg(z_1 - c)|e^{|\varphi||\arg(z_2 - c) - \arg(z_1 - c)|} \\ &\leq |\varphi|e^{2\pi|\varphi|}|\arg(z_2 - c) - \arg(z_1 - c)|\,. \end{aligned}$$

From the triangle $0, 1, e^{i\alpha}$ we see $\arg(e^{i\alpha} - 1) = \frac{1}{2}(\alpha + \pi)$. Hence

$$\arg(e^{i\alpha} - e^{i\alpha_0}) = \alpha_0 + \frac{1}{2}(\alpha - \alpha_0 + \pi), \ \arg(z_2 - c) - \arg(z_1 - c) = \frac{1}{2}(\arg\, z_2 - \arg\, z_1)\,.$$

Applying the cosine theorem for the triangle with corners $0, z_1, z_2$, we see since $|\arg z_2 - \arg z_1| \leq \pi$

$$\frac{2}{\pi}\,|\arg z_2 - \arg z_1| \leq 2\sin\frac{1}{2}\,|\arg z_2 - \arg z_1| \leq \sqrt{2(1 - \cos(\arg z_2 - \arg z_1))} = |z_1 - z_2|.$$

Thus

$$|\arg(z_2 - c) - \arg(z_1 - c)| \leq \frac{\pi}{4}|z_1 - z_2|\,.$$

This estimate holds even if say z_1 tends to c when $\arg(z_1 - c)$ is replaced by its limit $\arg c + \frac{\pi}{2}$. Therefore z_1 and z_2 may take the value c when $\arg(z_2 - c)$ both for $k = 1, 2$ is replaced by the tangent direction or both by the opposite directions. Hence, λ_0 satisfies for any $z_1, z_2 \in \Gamma_\mu$ a HÖLDER condition

$$|\lambda_0(z_1) - \lambda_0(z_2)| \leq H|z_1 - z_2|^\beta$$

with a HÖLDER constant H independent of the location of z_1 and z_2 with respect to the end points of Γ_μ. As $\lambda_0(z)$ is continuous, we may pass with z_1 to any endpoint $c \in \{c_\mu, c_{\mu+1}\}$, $c_{m+1} = c_1$, getting

$$|\lambda_0(c) - \lambda_0(z_2)| \leq H|c - z_2|^\beta \, .$$

Remark. Later we will see by Lemma 24 that $\lambda_0 \in C^\beta(\partial \mathbb{D})$ because of its continuity at the c_μ's .

Theorem 42. *Let λ and φ satisfy the above conditions. Then the modified discontinuous* RIEMANN–HILBERT *problem for analytic functions ϕ,*

$$\operatorname{Re}\{\overline{\lambda}\phi\} = \varphi + h \quad \text{on } \partial\mathbb{D} \, ,$$

$$\operatorname{Im}\{\overline{\lambda(z_k)}\phi(z_k)\} = a_k \, , \; 1 \leq k \leq 2\kappa + 1, \quad \text{if } 0 \leq \kappa \, ,$$

where $z_k \in \partial\mathbb{D}\backslash\{c_\mu : 1 \leq \mu \leq m\}$, $a_k \in \mathbb{C}$, $1 \leq k \leq 2\kappa + 1$, are given points and $h = 0$ if $0 \leq \kappa$ and

$$h(z) := \begin{cases} \displaystyle\sum_{\kappa+1}^{-\kappa-1} h_k z^k, & h_k = \overline{h_{-k}}, \quad |k| \leq -\kappa - 1, \quad if \quad -\kappa \in \mathbb{N} \, , \\ \displaystyle\sum_{[x]+2}^{-[x]} h_k z^{k-\frac{1}{2}}, & h_k = \overline{h_{1-k}}, \quad |k| \leq [-\kappa], \quad\quad if \quad \tfrac{1}{2} - \kappa \in \mathbb{N} \, , \end{cases}$$

if $\kappa < 0$ with undetermined coefficients h_k, is uniquely solvable. The solution has the form

$$\phi(z) = \frac{iX(z)}{2\pi i} \int_{\partial\mathbb{D}} \frac{\lambda(t)(\varphi(t) + h(t))}{iX(t)} \frac{t+z}{t-z} \frac{dt}{t} + iX(z)Q(z) \, , \tag{3.5.1}$$

where

$$X(z) := iz^{[\kappa]} e^{i\tau(z)} X_1(z) \, ,$$

$$\tau(z) := \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \{\arg \lambda_0(t) - [\kappa] \arg t\} \frac{t+z}{t-z} \frac{dt}{t} \, ,$$

$$\lambda_0(z) := \lambda(z) \frac{\overline{X_1(z)}}{|X_1(z)|} \, ,$$

and

$$Q(z) = \sum_{-[\kappa]}^{[\kappa]} d_k z^k + \begin{cases} 0, & \text{if } \kappa \in \mathbb{N}_0 \, , \\ d^* \dfrac{c_0 + z}{c_0 - z}, & \text{if } \kappa - \tfrac{1}{2} \in \mathbb{N}_0 \, , \end{cases}$$

$$Q(z) = \begin{cases} 0, & \text{if } -\kappa \in I\!N\,, \\ d^* \dfrac{c_0 + z}{c_0 - z}, & \text{if } \frac{1}{2} - \kappa \in I\!N\,, \end{cases}$$

with coefficients satisfying

$$d_{-k} = -\overline{d_k}, \quad |k| \le [\kappa], d^* = \overline{d^*}\,,$$

being uniquely defined by the side conditions for $0 \le \kappa$ *and* d^* *being undetermined for* $\kappa < 0$ *an odd multiple of* $(-1/2)$. *(3.5.1) is the solution for* $-\kappa \in I\!N$ *if the* h_k *are determined by*

$$\sum_{\mu=\kappa+1}^{-\kappa-1} A_{k\mu}h_\mu = B_k, A_{k\mu} := \frac{1}{2\pi i}\int\limits_{\partial D} \frac{e^{\operatorname{Im}\tau(t)}}{|X_1(t)|} t^{\mu-k-1}dt,$$

$$B_k := -\frac{1}{2\pi i}\int\limits_{\partial D} \frac{e^{\operatorname{Im}\tau(t)}}{|X_1(t)|}\varphi(t)\frac{dt}{t^{k+1}}\,, |\mu| \le -\kappa - 1, \quad 0 \le k \le -\kappa - 1\,,$$

and for $1/2 - \kappa \in I\!N$ *if the* h_k *and* d^* *are determined by*

$$\sum_{\mu=[\kappa]+2}^{[-\kappa]} A_{k\mu}h_\mu + \frac{d^*}{c_0^k} = B_k, A_{k\mu} := \frac{1}{2\pi i}\int\limits_{\partial D} \frac{e^{\operatorname{Im}\tau(t)}}{|X_1(t)|} t^{\mu-k-3/2}dt\,,$$

$$B_k := -\frac{1}{2\pi i}\int\limits_{\partial D} \frac{e^{\operatorname{Im}\tau(t)}}{|x_1(t)|}\varphi(t)\frac{dt}{t^{k+1}}\,, \; |\mu| \le [-\kappa], \quad 0 \le k \le [-\kappa]\,.$$

Proof. Because $\varphi_\mu + \beta_\mu < 1$ the integral in (3.5.1) exists and ϕ will be seen to satisfy the boundary condition in $\partial D \backslash \{c_\mu : 1 \le \mu \le m\}$. Near the c_μ's ϕ in general will turn out to be unbounded. This last assertion will be proved in the following lemmas. We first concentrate on proving formula (3.5.1).
For $z \in \partial D$ we have

$$\operatorname{Re}\{\overline{\lambda(z)}X(z)\} = \operatorname{Re}\{i\overline{\lambda(z)}\lambda_0(z)e^{-\operatorname{Im}\tau(z)}X_1(z)\} = 0\,,$$

so that X is a canonical solution to the homogeneous boundary problem. A solution ϕ to the inhomogeneous boundary problem then would satisfy

$$\operatorname{Re}\frac{\phi}{iX} = \operatorname{Re}\frac{\overline{\lambda}\phi}{-e^{-\operatorname{Im}\tau}|X_1|} = \frac{\lambda\varphi}{-\lambda e^{-\operatorname{Im}\tau}|X_1|} = \frac{\lambda\varphi}{iX} \quad \text{on } \partial D$$

because on ∂D

$$X = i\lambda e^{-\operatorname{Im}\tau}|X_1|\,.$$

But $iX^{-1}\phi$ is analytic in $\mathbb{D}$ up to a pole of order at most $[\kappa]$ at the origin if $0 < \kappa$, and a pole of order one in $z = c_0$ if 2κ is odd, and with a zero at $z = 0$ of order at least $|[\kappa]|$ if $\kappa < 0$. Thus ϕ has the form (3.5.1) at least if φ is continuous. This is clear from section 1.4, proof of Lemma 7, for $\kappa \in \mathbb{N}_0$. If $\kappa - 1/2 \in \mathbb{N}_0$ then Q having the same singularities as $-iX^{-1}\phi$ has a pole of order at most $[\kappa]$ at the origin and a pole of order one at c_0 on $\partial\mathbb{D}$. Moreover, $\operatorname{Re} Q$ vanishes on $\partial\mathbb{D}$. From

$$Q(z) := \frac{d}{z - c_0} + \sum_{k=-[\kappa]}^{\infty} d_k z^k$$

we find on $\partial\mathbb{D}$

$$Q(z) + \overline{Q(z)} = \frac{d - \overline{d}c_0 z}{z - c_0} + \sum_{k=-[\kappa]}^{\infty} [d_k z^k + \overline{d_k} z^{-k}]$$

$$= \frac{d - \overline{d}c_0 z}{z - c_0} + d_0 + \overline{d_0} + \sum_{k=1}^{[\kappa]} (d_k + \overline{d_{-k}}) z^k + \sum_{k=[\kappa]+1}^{\infty} d_k z^k + \sum_{k=-\infty}^{-[\kappa]-1} \overline{d_{-k}} z^k = 0\,.$$

Hence, $d_k = 0$ for $[\kappa] < k, d_k + \overline{d_{-k}} = 0$ for $1 \le k \le [\kappa], d = (d_0 + \overline{d_0})c_0$ so that

$$Q(z) = \frac{d_0 + \overline{d_0}}{2} \frac{z + c_0}{z - c_0} + \frac{d_0 - \overline{d_0}}{2} + \sum_{k=1}^{[\kappa]} \left[d_k z^k + d_k z^{-k}\right]$$

which is of the above form. The coefficients are determined by the side conditions which give

$$i\frac{z_k + c_0}{z_k - c_0} d^* + \sum_{\nu=-[\kappa]}^{[\kappa]} i z_k^\nu d_\nu = \frac{a_k e^{\operatorname{Im}\tau(z_k)}}{|X_1(z_k|} - \frac{1}{2\pi i} \int\limits_{\partial\mathbb{D}} \frac{\varphi(t) e^{\operatorname{Im}\tau(t)}}{|X_1(t)|} \operatorname{Im} \frac{t + z_k}{t - z_k} \frac{dt}{t}\,,\ 1 \le k \le 2\kappa+1\,,$$

where formally $d^* = 0$ if $\kappa \in N_0$. The determinant of this system of linear equations is

$$2c_0^{[\kappa]+1} \prod_{\nu=1}^{2\kappa+1} (z_\nu - c_0)^{-1} z_\nu^{-\kappa} \Pi_{1\le\mu<\nu\le 2\kappa+1} (z_\mu - z_\nu)\,.$$

If $\kappa \in \mathbb{N}_0$ the determinant of the system is

$$\prod_{\nu=1}^{2\kappa+1} z_\nu^{-\kappa} \prod_{1\le\mu<\nu\le 2\kappa+1} (z_\mu - z_\nu)\,.$$

Thus the coefficients of Q and d^* are uniquely given if $\kappa \ge 0$ and being the solutions to the above system of linear equations can be estimated by

$$|d_\nu|, |d^*| \le M(z_k, c_\mu, \lambda) \sum_{\mu=1}^{2\kappa+1} \left[|a_\mu| + \left| \frac{1}{2\pi i} \int\limits_{\partial\mathbb{D}} \frac{\varphi(t) e^{\operatorname{Im}\tau(t)}}{|X_1(t)|} \operatorname{Im} \frac{t + z_\mu}{t - z_\mu} \frac{dt}{t} \right| \right]\,. \tag{3.5.2}$$

For $\kappa < 0$ coefficients of h and d^* too, if 2κ is odd, are determined by means of the solvability conditions. They arise because the solution ϕ is demanded to behave regular at the origin where X has a pole of order $|[\kappa]|$. In order to show that the linear systems

$$\sum_{\mu=\kappa+1}^{-\kappa-1} A_{k\mu}h_\mu = B_k,\ 0 \le k \le -\kappa-1;\ \sum_{\mu=[\kappa]+2}^{[-\kappa]} A_{k\mu}h_\mu + \frac{d^*}{c_0^k} = B_k,\ 0 \le k \le [-\kappa]\,,$$

are uniquely solvable the related homogeneous systems are shown to be only trivially solvable.
Let $\widehat{\delta}_\mu$ satisfy $\widehat{\delta}_{-\mu} = \overline{\widehat{\delta}_\mu}, |\mu| \le -\kappa-1$, and δ_μ and δ satisfy $\delta_{-\mu} = \overline{\delta_{\mu+1}}, \delta = \overline{\delta}, |\mu| \le [-\kappa]$, and

$$\sum_{\mu=\kappa+1}^{-\kappa-1} A_{k\mu}\widehat{\delta}_\mu = 0,\ 0 \le k \le -\kappa-1;\ \sum_{\mu=[\kappa]+2}^{[-\kappa]} \Delta_{k\mu}\delta_\mu + \frac{\delta}{c_0^k} = 0, 0 \le k \le [-\kappa]\,.$$

Then

$$\widehat{g}(z) := \sum_{\mu=\kappa+1}^{-\kappa-1} \widehat{\delta}_\mu z^\mu\,;\quad g(z) := \sum_{\mu=[\kappa]+2}^{[-\kappa]} \delta_\mu z^{\mu-\frac{1}{2}}$$

are satisfying

$$\mathrm{Im}\,\widehat{g}(z) = 0;\ \mathrm{Im}\,g(z) = 0 \quad \text{on } \partial\mathbb{D}$$

and

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{e^{\mathrm{Im}\,\tau(t)}}{|X_1(t)|}\widehat{g}(t)\frac{dt}{t^{k+1}} &= 0,\ 0 \le k \le -\kappa-1\,;\\ \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{e^{\mathrm{Im}\,\tau(t)}}{|X_1(t)|}g(t)\frac{dt}{t^{k+1}} &= -\frac{\delta}{c_0^k},\ 0 \le k \le [-\kappa]\,.\end{aligned}$$

Using in the last case the first equality $k = 0$ to express δ through g the other equations of this system become

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{e^{\mathrm{Im}\,\tau(t)}}{|X_1(t)}g(t)\left(\frac{1}{t^k} - \frac{1}{c_0^k}\right)\frac{dt}{t} = 0, \quad 1 \le k \le [-\kappa]\,.$$

From the first system we see that

$$\widehat{G}(z) := \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{e^{\mathrm{Im}\,\tau(t)}}{|X_1(t)|}\widehat{g}(t)\frac{t+z}{t-z}\frac{dt}{t} = \frac{z^{-\kappa}}{\pi i}\int_{\partial\mathbb{D}} \frac{e^{\mathrm{Im}\,\tau(t)}}{|X_1(t)|t^{-\kappa}}\widehat{g}(t)\frac{dt}{t-z}$$

is an analytic function having a zero of order at least $-\kappa$ at $z = 0$. Hence, at least -2κ level curves $\mathrm{Re}\,G(z) = 0$ of its real part which coincides with

$$\mathrm{Re}\,\widehat{G}(z) = \frac{e^{\mathrm{Im}\,\tau(z)}}{|X_1(z)|}\widehat{g}(z)$$

on $\partial\mathbb{D}$ pass through $z = 0$, see the proof of Theorem 18. These level lines cannot intersect one another within $\overline{\mathbb{D}}\setminus\{0\}$ if G is not vanishing identically. They therefore intersect $\partial\mathbb{D}$ in different points. Thus $\widehat{G}(z)$ and hence $\widehat{g}(z)$ have at least -2κ zeroes on $\partial\mathbb{D}$. But $z^{-\kappa-1}\widehat{g}(z)$ is a polynomial of degree at most $-2\kappa - 2$. Hence $\widehat{g}(z) \equiv 0$. From the second of the above two systems we see that

$$\begin{aligned} G(z) &:= \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{e^{\operatorname{Im}\tau(t)}}{|X_1(t)|}g(t)\left[\frac{t+z}{t-z} - \frac{c_0+z}{c_0-z}\right]\frac{dt}{t} \\ &= \frac{z^{[-\kappa]+1}}{\pi i}\int_{\partial\mathbb{D}} \frac{e^{\operatorname{Im}\tau(t)}}{|X_1(t)|}g(z)\left[\frac{t^{-[-\kappa]}}{t-z} - \frac{c_0^{-[-\kappa]}}{c_0-z}\right]\frac{dt}{t} \end{aligned}$$

is an analytic function in $\mathbb{D}$ with a zero of order at least $[-\kappa]+1$ at $z = 0$. Therefore at least $2[-\kappa]+2$ level lines $\operatorname{Re}G(z) = 0$ of its real part passing through the origin intersect $\partial\mathbb{D}$ in $2[-\kappa]+2$ points if G does not vanish identically. But on $\partial\mathbb{D}$

$$\operatorname{Re}G(z) = \frac{e^{\operatorname{Im}\tau(z)}}{|X_1(z)|}g(z) + \delta\operatorname{Re}\frac{c_0+z}{c_0-z} = \frac{e^{\operatorname{Im}\tau(t)}}{|X_1(z)|}g(z)\,,$$

so that g has $2[-\kappa]+2$ zeroes on $\partial\mathbb{D}$. This is only possible if $g(z) \equiv 0$ because $z^{-\kappa-1}g(z)$ is a polynomial of degree at most $-2\kappa-2 = 2[-\kappa]-1$. This proves that the homogeneous systems are only trivially solvable. The unique solution to the inhomogeneous systems can be estimated by

$$|h_\nu|, |d^*| \le M(c_\mu, \lambda)\sum_{k=0}^{[-\kappa-\frac{1}{2}]}\left|\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{\varphi(t)e^{\operatorname{Im}\tau(t)}}{|X_1(t)|}\frac{dt}{t^{k+1}}\right|\,. \tag{3.5.3}$$

We are now looking for an a priori estimate for the solution ϕ in (3.5.1) which will charaterize the behaviour near the c_μ's , too. In order to prove this a priori estimate the behaviour of CAUCHY integrals with discontinuous density has to be studied. The results are due to N.I. MUSKHELISHVILI, see [Musk53], p. 83 and [Mona83], p. 21. In the following Γ_{ab} denotes a smooth bounded curve in the complex plane $\mathbb{C}$ with end points a and b. The proofs of the next lemmas follow the argumentation in [Musk65], §23 - §25.

Lemma 21. $\varphi^* \in C^\alpha(\Gamma_{ab}), 0 < \alpha \le \mu \le \delta \le 1-\alpha, \alpha+\mu \le \delta$ *or* $\alpha < \mu = \delta$,

$$\varphi(t_0) := \frac{1}{2\pi i}\int_{\Gamma_{ab}}\frac{\varphi^*(t)dt}{(t-a)^\mu(t-t_0)}\,,$$

$N(a)$ a neighborhood of $a, \Gamma := \Gamma_{ab}\cap N(a)$. Then $(t_0-a)^\delta\varphi(t_0) \in C_\alpha(\Gamma)$ and

$$C_\alpha((t-a)^\delta\varphi(t);\Gamma) \le M(\alpha,\mu,\delta,\Gamma_{ab},N(a))C_\alpha(\varphi^*;\Gamma_{ab}).$$

Proof. Decompose φ into the form

$$\varphi(t_0) = (\varphi^*(a) + \psi(t_0))\Omega(t_0) + \frac{1}{2\pi i}\int\limits_{\Gamma_{ab}} \frac{\psi(t) - \psi(t_0)}{t - t_0} \frac{dt}{(t-a)^\mu}$$

where $\psi(t_0) = \varphi^*(t_0) - \varphi^*(a)$ and

$$\frac{(t_0 - a)^\delta}{2\pi i}\int\limits_{\Gamma_{ab}} \frac{\psi(t) - \psi(t_0)}{t - t_0} \frac{dt}{(t-a)^\mu} = \Psi(t_0) + \frac{1}{2\pi i}\int\limits_{\Gamma_{ab}} \frac{\psi(t) - \psi(t_0)}{t - t_0}(t-a)^{\delta-\mu} dt \, .$$

The function

$$\Omega(z) := \frac{1}{2\pi i}\int\limits_{\Gamma_{ab}} \frac{(t-a)^{-\mu}}{t-z} dt$$

satisfies by the PLEMELJ–SOKHOTZKI formula

$$\Omega^+(t_0) - \Omega^-(t_0) = (t_0 - a)^{-\mu}, \; t_0 \in \Gamma_{ab} \setminus \{a\} \, .$$

Choose the branch of $(z-a)^{-\mu}$ being single valued analytic in $\mathcal{N}\setminus\Gamma$ satisfying

$$((z-a)^{-\mu})^+ = (t_0 - a)^{-\mu}, \; t_0 \in \Gamma\setminus\{a\} \, .$$

Then

$$((z-a)^{-\mu})^- = e^{-2\pi i \mu}(t_0 - a)^{-\mu}, \; t_0 \in \Gamma\setminus\{a\} \, ,$$

so that

$$\omega := \frac{(z-a)^{-\mu}}{1 - e^{-2\pi i \mu}} = \frac{e^{i\pi\mu}}{2i \sin \pi\mu}(z-a)^{-\mu}$$

satisfies

$$\omega^+(t_0) - \omega^-(t_0) = (t_0 - a)^\mu, \; t_0 =\in \Gamma\setminus\{a\} \, .$$

Hence, $\Omega - \omega$ is continuous on $\Gamma\setminus\{a\}$ and thus analytic there. In order to show a to be a removable singularity let $\mu < \nu < 1$. Then

$$|z-a|^\nu \Omega(z) = \frac{1}{2\pi i}\int\limits_{\Gamma_{ab}} \frac{|z-a|^\nu - |t-a|^\nu}{(t-a)^\mu (t-z)} dt + \frac{1}{2\pi i}\int\limits_{\Gamma_{ab}} \frac{|t-a|^{\nu-\mu} e^{-i\mu \arg(t-a)}}{t-z} dt \, .$$

Because the numerator of the integrand in the second integral is HÖLDER–continuous, see [Musk53], p. 18, this integral is bounded, see Theorem 5. The first integral is bounded by

$$\frac{1}{2\pi}\int\limits_{\Gamma_{ab}} \frac{|dt|}{|t-a|^\mu |t-z|^{1-\nu}} \le \frac{1}{2\pi k_0^2}\int\limits_0^R \frac{dr}{r^\mu |r - r_0|^{1-\nu}}$$

where k_0 is the constant from the proof of Lemma 1, $r := |t-a|, r_0 := |z-a|$ and $|r - r_0| \leq |t-z|$. Assuming $r_0 < R$ one sees

$$\begin{aligned}
\int_0^{r_0/2} \frac{dr}{r^\mu |r-r_0|^{1-\nu}} &\leq \frac{1}{1-\mu}\left(\frac{r_0}{2}\right)^{\nu-\mu}, \\
\int_{r_0/2}^{r_0} \frac{dr}{r^\mu |r-r_0|^{1-\nu}} &\leq \frac{1}{\nu}\left(\frac{r_0}{2}\right)^{\nu-\mu}, \\
\int_{r_0}^{R} \frac{dr}{r^\mu |r-r_0|^{1-\nu}} &\leq \frac{1}{\nu}\frac{(R-r_0)^\nu}{r_0^\mu},
\end{aligned}$$

so that the first integral is bounded, too. Using a weakened form of the RIEMANN theorem for removable singularities, see [Ding61], p. 45, [Ash71], p. 78, $h := \Omega - \omega$ is analytic in N. Therefore

$$\Omega(z) = \frac{e^{i\mu z}}{2i \sin \mu\pi}(z-a)^{-\mu} + h(z), \quad z \in N,$$

where $h(z)$ is some analytic function in N. Moreover, from

$$2\Omega(t_0) = \Omega^+(t_0) + \Omega^-(t_0), \ t_0 \in \Gamma,$$

it follows

$$\Omega(t_0) = \frac{1}{2i} \cot \pi\mu (t_0 - a)^{-\mu} + h(t_0), \ t_0 \in \Gamma \backslash \{a\}.$$

Therefore $(t_0 - a)^\delta \Omega(t_0)$ is HÖLDER-continuous on Γ with exponent α where $\alpha < \mu$ if $\delta = \mu$ or $\alpha \leq \delta - \mu$ if $\mu < \delta$, and in the latter case

$$\begin{aligned}
C_\alpha((t-a)^\delta \Omega(t); \Gamma) &\leq M(\alpha, \mu, \delta, \Gamma_{ab}) C_{\delta-\mu}((t-a)^\delta \Omega(t); \Gamma) \\
&\leq M(\alpha, \mu, \delta, \Gamma_{ab}, N(a)),
\end{aligned}$$

while in the first case the $C_\mu(\cdot; \Gamma)$ norm has to be used leading to the same estimate. The proof of the estimate

$$C_\alpha(\Psi; \Gamma_{ab}) \leq M(\alpha, \mu, \delta, \Gamma_{ab}) H_\alpha(\varphi^*)$$

for

$$\Psi(t_0) := \frac{1}{2\pi i} \int_{\Gamma_{ab}} \frac{\psi(t) - \psi(t_0)}{t - t_0} \frac{(t_0 - a)^\delta - (t-a)^\delta}{(t-a)^\mu} dt$$

technically involved and lengthy is given in [Musk65], p. 84–88. Finally

$$C_\alpha\left(\frac{1}{2\pi i}\int\limits_{\Gamma_{ab}}\frac{\psi(t)-\psi(t_0)}{t-t_0}\,\frac{dt}{(t-a)^{\delta-\mu}};\Gamma\right)\le M(\alpha,\mu,\delta,\Gamma_{ab})C_\alpha(\varphi^*;\Gamma_{ab})$$

follows from the proof of Theorem 4. In order to apply this result we supplement Γ_{ab} with a curve Γ_{ba} such that $\Gamma_{ab}\cup\Gamma_{ba}$ is a smooth simply closed curve and extend ψ on this curve such that it becomes a HÖLDER–continuous function coinciding with ψ on Γ_{ab} and vanishing identically on Γ_{ba}. This is possible because $\psi(a)=0$ and N is assumed to be a small neighborhood of a. These estimates give the estimate in Lemma 21.

Lemma 22. $\varphi\in C^\alpha(\Gamma_{ab}), 0<\alpha\le\mu\le\delta\le 1-\alpha, \alpha+\mu\le\delta,$

$$\Phi(t_0):=\frac{1}{2\pi i}\int\limits_{\Gamma_{ab}}\frac{\varphi(t)dt}{|t-a|^\mu(t-t_0)},$$

$\Gamma:=\Gamma_{ab}\cap\overline{N(a)}, \overline{N(a)}$ *the closure of* $N(a)$. *Then* $(t_0-a)^\delta\Phi(t_0)\in C^\alpha(\Gamma)$ *and*

$$C_\alpha((t-a)^\delta\Phi(t);\Gamma)\le M(\alpha,\mu,\delta,\Gamma_{ab},N(a))C_\alpha(\varphi;\Gamma_{ab})\,.$$

Proof. Decompose Φ into

$$\Phi(t_0)=\frac{1}{2\pi i}\int\limits_{\Gamma_{ab}}\frac{\varphi^*(t)dt}{(t-a)^\mu(t-t_0)}+\frac{\varphi(a)}{2\pi i}\int\limits_{\Gamma_{ab}}\frac{dt}{|t-a|^\mu(t-t_0)}$$

where

$$\varphi^*(t)=[\varphi(t)-\varphi(a)]e^{i\mu\arg(t-a)}$$

is a $C^\alpha(\Gamma_{ab})$–function, see [Musk65],§6. Moreover,

$$\frac{1}{2\pi i}\int\limits_{\Gamma_{ab}}\frac{dt}{(t-a)^\mu(t-t_0)}=\frac{1}{2\pi i}\int\limits_{\Gamma_{ab}}\frac{\varphi_1^*(t)-\varphi_1^*(a)}{(t-a)^{\mu+\frac{1}{2}\alpha}(t-t_0)}dt$$

with

$$\varphi_1^*(t):=e^{i\mu\arg(t-a)}(t-a)^{\alpha/2}\,,\quad \varphi_1^*(a)=0\,,$$

in $C^{\frac{1}{2}\alpha}(\Gamma_{ab})$. As in the preceding proof

$$C_\alpha\left(\frac{(t_0-a)^\delta}{2\pi i}\int\limits_{\Gamma_{ab}}\frac{\varphi^*(t)dt}{(t-a)^\mu(t-t_0)};\Gamma\right)\le M(\alpha,\mu,\delta,\Gamma_{ab},N(a))H_\alpha(\varphi^*)$$

where $H_\alpha(\varphi^*) \le C_\alpha(\varphi; \Gamma_{ab})$ and

$$C_\alpha \left(\frac{(t_0 - a)^\delta}{2\pi i} \int_{\Gamma_{ab}} \frac{\varphi_1^*(t) - \varphi_1^*(a)}{(t-a)^{\mu+\frac{1}{2}\alpha}(t-t_0)} dt; \Gamma \right) \le M(\alpha, \mu, \delta, \Gamma_{ab}, N(a)) .$$

Remark. Both these results hold with respect to the end point b , too. The only difference is that in this case Ω has to be replaced by

$$-\frac{e^{i\mu\pi}}{2i \sin \mu\pi}(z-b)^{-\mu} + h(z)$$

and the boundary values $\Omega(t_0)$ by

$$-\frac{1}{2i} \cot \mu\pi (t_0 - b)^{-\mu} + h(t_0) .$$

Lemma 23. $\varphi \in C^\alpha(\Gamma_{ab}), 0 < \alpha < 1, 0 < \mu < 1, \nu := \min\{\alpha, \mu\}$. *Then the boundary values of*

$$\Phi(z) := \frac{1}{2\pi i} \int_{\Gamma_{ab}} \frac{\varphi(t)}{|t-c|^\mu (t-z)} dt, \quad c \in \{a, b\} ,$$

on

$$\Gamma_{\delta_0} := \{\zeta : \zeta \in \Gamma_{ab}, \delta_0 \le |\zeta - a|, \delta_0 \le |\zeta - b|\}, \quad 0 < 4\delta_0 < |b-a| ,$$

belong to $C^\nu(\Gamma_{\delta_0})$ *and*

$$C_\nu(\Phi^\pm; \Gamma_{2\delta_0}) \le M(\alpha, \mu, \delta_0, \Gamma_{ab}) C_\alpha(\varphi; \Gamma_{ab}) .$$

Proof. Decompose Φ by splitting Γ_{ab} into Γ_{δ_0} and $\Gamma_{ab} \backslash \Gamma_{\delta_0}$. Denote the integral over Γ_{δ_0} by Φ_1 and $\Phi_2 := \Phi - \Phi_1$. Let a', b' be the end points of Γ_{δ_0} and

$$\varphi^*(t) := |t-c|^{-\mu} \varphi(t) .$$

Then

$$\Phi_1(z) := \frac{1}{2\pi i} \int_{\Gamma_{\delta_0}} \frac{\varphi^*(t) - \varphi^*(t_0)}{t-z} dt + \frac{\varphi^*(t_0)}{2\pi i} \log \frac{z-b'}{z-a'}, \quad t_0 \in \Gamma_{\delta_0} ,$$

where we choose the branch of $\log \dfrac{z-b'}{z-a'}$ in $\mathbb{C} \backslash \Gamma_{\delta_0}$ which vanishes at infinity. Letting z tend to t_0 from each side of Γ_δ we see from Theorem 2 and

$$\left(\log \frac{z-b'}{z-a'} \right)^+ = \log \left| \frac{t_0 - b'}{t_0 - a'} \right| + i[\arg(b' - t_0) - \arg(a' - t_0)] ,$$

$$\left(\log \frac{z-b'}{z-a'} \right)^- = \log \left| \frac{t_0 - b'}{t_0 - a'} \right| + i[\arg(b' - t_0) - 2\pi - \arg(a' - t_0)]$$

that on Γ_{δ_0}

$$\begin{aligned}
\Phi_1^+(t_0) &= \frac{1}{2\pi i}\int\limits_{\Gamma_{\delta_0}} \frac{\varphi^*(t)-\varphi^*(t_0)}{t-t_0}dt + \frac{\varphi^*(t_0)}{2\pi i}\log\frac{t_0-b'}{t_0-a'} = \frac{1}{2}\varphi^*(t_0)+\Phi_1(t_0)\,, \\
\Phi_1^-(t_0) &= \frac{1}{2\pi i}\int\limits_{\Gamma_{\delta_0}} \frac{\varphi^*(t)-\varphi^*(t_0)}{t-t_0}dt + \frac{\varphi^*(t_0)}{2\pi i}\log\frac{t_0-b'}{t_0-a'} - \varphi^*(t_0) \\
&= -\frac{1}{2}\varphi^*(t_0)+\Phi_1(t_0)\,.
\end{aligned}$$

From $\varphi^* \in C^\nu(\Gamma_{\delta_0})$ and the proof of Theorem 4

$$C_\nu(\Phi_1^\pm;\Gamma_{\delta_0}) \le M(\alpha,\mu,\delta_0,\Gamma_{ab})C_\alpha(\varphi;\Gamma_{\delta_0})$$

follows. But Theorem 4 applies to closed curves. We therefore again as in the preceding proof supplement Γ_{ab} by some Γ_{ba}, define φ^* by zero on Γ_{ba} and as a linear function on $\Gamma_{ab}\backslash\Gamma_{\delta_0}$ so that φ^* extended in this way becomes HÖLDER–continuous with exponent ν on the closed curve $\Gamma_{ab}\cup\Gamma_{ba}$.
For $z, z_0 \in \Gamma_{2\delta_0}$

$$\begin{aligned}
|\Phi_2(z)| &\le \frac{1}{\pi\delta_0}\int\limits_{\Gamma_{ab}\backslash\Gamma_{\delta_0}} \frac{|\varphi(t)|}{|t-c|^\mu}dt \le M(\mu,\delta_0,\Gamma_{ab})C_0(\varphi;\Gamma_{ab})\,, \\
|\Phi_2(z)-\Phi_2(z_0)| &= \left|\frac{1}{2\pi i}\right|\int\limits_{\Gamma_{ab}\backslash\Gamma_{\delta_0}} \left|\frac{\varphi(t)(z-z_0)dt}{|t-c|^\mu(t-z)(t-z_0)}\right| \\
&\le M(\mu,\delta_0,\Gamma_{ab})C_0(\varphi;\Gamma_{ab})|z-z_0|\,,
\end{aligned}$$

i.e.

$$C_1(\Phi_2;\Gamma_{2\delta_0}) \le M(\mu,\delta_0,\Gamma_{ab})C_0(\varphi;\Gamma_{ab})$$

such that

$$C_\alpha(\Phi_2;\Gamma_{2\delta_0}) \le M(\alpha,\mu,\delta_0,\Gamma_{ab})C_0(\varphi;\Gamma_{ab})\,.$$

Lemma 24. *Let Γ_{ac} and Γ_{cb} be two smooth curves with endpoints a and c and c and b, respectively. Let $\Gamma_{ac}\cap\Gamma_{cb}=\{c\}, \Gamma_{ab}:=\Gamma_{ac}\cup\Gamma_{cb}$ and the nonobtuse angle α_0 between Γ_{ac} and Γ_{cb} at the point c satisfy $0<\alpha_0\le\pi$. If $\varphi\in C^\alpha(\Gamma_{ac})$ and $\varphi\in C^\alpha(\Gamma_{ab})$ then $\varphi\in C^\alpha(\Gamma_{cb})$.*

Proof. It is enough to estimate the HÖLDER coefficient of φ in the neighborhood of c on Γ_{ab}. Let R_0 be the common standard radius for Γ_{ac} and Γ_{cb} related to $\alpha_0/4$, $R_0 = R_0(\alpha_0/4)$, see section 1.1. Let $t\in\Gamma_{ac}, t'\in\Gamma_{cb}$ with $|t-c|, |t'-c| < R_0$. Then the angle between the tangent of Γ_{ac} in c and the straight line through c and t *is less* than $\alpha_0/4$. The same statement holds for t' and Γ_{cb}. Hence, the angle at the corner c of the triangle t, c, t' is at least $\alpha_0/2$. The cosine theorem then gives

$$|t-t'|^2 \geq |t-c|^2 + |t'-c|^2 - 2|t-c||t'-c|\cos\alpha_0/2 \geq 2[|t-c|^2 + |t'-c|^2]\sin^2\alpha_0/4 \; .$$

Thus

$$\begin{aligned}\frac{|\varphi(t)-\varphi(t')|}{|t-t'|^\alpha} &\leq \frac{|\varphi(t)-\varphi(c)|}{|t-c|^\alpha}\frac{|t-c|^\alpha}{|t-t'|^\alpha} + \frac{|\varphi(c)-\varphi(t')|}{|c-t'|^\alpha}\frac{|c-t'|^\alpha}{|t-t'|^\alpha} \\ &\leq \frac{|t-c|^\alpha + |c-t'|^\alpha}{|t-t'|^\alpha}\max\{H(\varphi;\Gamma_{ac},\alpha), H(\varphi;\Gamma_{cb},\alpha)\} \; .\end{aligned}$$

Applying Lemma 3 and the above estimate to get

$$|t-c|^\alpha + |c-t'|^\alpha \leq 2^{1-\alpha/2}[|t-c|^2 + |c-t'|^2]^{\alpha/2} \leq 2^{1-\alpha}|t-t'|^\alpha \sin^{-\alpha}\alpha_0/4$$

shows that φ is HÖLDER–continuous on Γ_{ab} near c. Therefore φ is HÖLDER–continuous on Γ_{ab}.

Lemma 25. $\varphi \in C^\alpha(\Gamma_{ab}), 0 < \alpha \leq \mu_k \leq \delta_k \leq 1-\alpha, \alpha + \mu_k \leq \delta_k, k = 1, 2$. *Then the boundary values of*

$$\Phi(z) := \frac{1}{2\pi i}\int\limits_{\Gamma_{ab}} \frac{\varphi(t)dt}{|t-a|^{\mu_1}|t-b|^{\mu_2}(t-z)}$$

on Γ_{ab} *are* HÖLDER*–continuous with possible exceptions of the end points* a *and* b *and*

$$C_\alpha((t-a)^{\delta_1}(t-b)^{\delta_2}\Phi^\pm(t);\Gamma_{ab}) \leq M(\alpha,\delta_1,\delta_2,\Gamma_{ab})C_\alpha(\varphi;\Gamma_{ab}) \; .$$

Proof. Applying Lemma 23 for a fixed $\delta_0 > 0$ small enough, e.g. $4\delta_0 = |b-a|$, one finds

$$C_\alpha(\Phi^\pm;\widetilde{\Gamma}_{\delta_0}) \leq M(\alpha,\delta_1,\delta_2,\delta_0,\Gamma_{ab})\; C_\alpha(\varphi;\Gamma_{ab})$$

and from Lemma 21 applied to $\varphi(t)|t-b|^{-\mu_2}$ in $N(a)$ and to $\varphi(t)|t-a|^{-\mu_1}$ in $N(b)$

$$C_\alpha((t-a)^{\delta_1}(t-b)^{\delta_2}\Phi(t);\Gamma_{ab}\backslash\widetilde{\Gamma}_{\delta_0}) \leq M(\alpha,\delta_1,\delta_2,\delta_0,\Gamma_{ab})\; C_\alpha(\varphi;\Gamma_{ab}) \; .$$

Using Lemma 23 then

$$C_\alpha((t-a)^{\delta_1}(t-b)^{\delta_2}\Phi(t);\Gamma_{ab}) \leq M(\alpha,\delta_1,\delta_2,\delta_0,\Gamma_{ab})\; C_\alpha(\varphi;\Gamma_{ab}) \; .$$

Lemma 26. *Let* $c_\mu \in \partial\mathbb{D}(1 \leq \mu \leq m), \arg c_\mu < \arg c_{\mu+1}(1 \leq \mu < m)$, $\arg c_m < \arg c_1 + 2\pi$,

$$\Gamma_\mu := \{|\zeta| = 1, \arg c_\mu \leq \arg\zeta \leq \arg c_{\mu+1}\}, 1 \leq \mu < m \; ,$$

$$\Gamma_m := \{|\zeta| = 1, \arg c_m \leq \arg\zeta \leq \arg c_1 + 2\pi\} \; ,$$

$\rho \in C^{\alpha}(\Gamma_\mu), 0 < \alpha \le \varphi_\mu \le \delta_\mu \le 1-\alpha, \alpha + \varphi_\mu \le \delta_\mu (1 \le \mu \le m)$. *Then*

$$\varphi(z) := \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \rho(t) \prod_{1}^{m} |t - c_\mu|^{-\varphi_\mu} \frac{dt}{t-z}$$

satisfies

$$C_\alpha \left(\prod_{1}^{m} (t - c_\mu)^{\delta_\mu} \varphi^+(t); \partial \mathbb{D} \right) \le M(\alpha, \varphi_\mu, c_\mu, \delta_\mu)\, C_\alpha(\rho)$$

where

$$C_\alpha(\rho) := \max_{1 \le \mu \le m} C_\alpha(\rho; \Gamma_\mu) .$$

Proof. From Lemma 25 for ($c_{m+1} := c_1, \varphi_{m+1} := \varphi_1$ etc.!)

$$\varphi_\mu(z) := \frac{1}{2\pi i} \int_{\Gamma_\mu} \frac{\rho_\mu(t)}{|t-c_\mu|^{\varphi_\mu} |t - c_{\mu+1}|^{\varphi_{\mu+1}}} \frac{dt}{t-z}, \rho_\mu(t) := \rho(t) \prod_{k \ne \mu, \mu+1} |t - c_k|^{-\varphi_k}$$

one gets

$$C_\alpha((t-c_\mu)^{\delta_\mu} (t - c_{\mu+1})^{\delta_{\mu+1}} \varphi_\mu^+(t); \Gamma_\mu) \le M(\alpha, \varphi_\lambda, c_\lambda, \Gamma_\mu)\, C_\alpha(\rho; \Gamma_\mu)$$

so that

$$C_\alpha \left(\prod_{1}^{m} (t_0 - c_k)^{\delta_k} \varphi_\mu^+(t_0); \Gamma_\mu \right) \le M(\alpha, \varphi_\lambda, c_\lambda, \Gamma_\mu)\, C_\alpha(\rho; \Gamma_\mu) .$$

In order to estimate φ_μ on $\partial \mathbb{D} \setminus \Gamma_\mu$ we proceed as in the proof of Lemma 23 considering

$$z, z_0 \in \Gamma_{\mu\delta_0} := \{\zeta \in \partial \mathbb{D} \setminus \Gamma_\mu^0, \delta_0 \le |\zeta - c_\mu|, \delta_0 \le |\zeta - c_{\mu+1}|\}$$

where Γ_μ^0 denotes Γ_μ without the end points and $\delta_0 > 0$ is small enough. We get

$$\begin{aligned} C_1(\varphi_\mu; \Gamma_{\mu\delta_0}) &\le M(\varphi_\lambda, c_\lambda, \delta_0)\, C_0(\rho; \Gamma_\mu) , \\ C_\alpha(\varphi_\mu; \Gamma_{\mu\delta_0}) &\le M(\alpha, \varphi_\lambda, c_\lambda, \delta_0)\, C_0(\rho; \Gamma_\mu) . \end{aligned}$$

For $z \in \partial \mathbb{D} \setminus \Gamma_\mu^0$ with $|z - c_\mu| \le \delta_0$ decompose Γ_μ at its middle point into the two parts $\Gamma_{\mu 1}$ and $\Gamma_{\mu 2}$, let the middle point belong to both, and $\varphi_\mu = \varphi_{\mu 1} + \varphi_{\mu 2}$ with

$$\begin{aligned} \varphi_{\mu 1}(z) &:= \frac{1}{2\pi i} \int_{\Gamma_{\mu 1}} \frac{\rho_{\mu 1}(t) dt}{|t - c_\mu|^{\varphi_\mu}(t-z)}, \qquad \rho_{\mu 1}(t) := \rho(t) \textstyle\prod_{k \ne \mu} |t - c_k|^{-\varphi_k} , \\ \varphi_{\mu 2}(z) &:= \frac{1}{2\pi i} \int_{\Gamma_{\mu 2}} \frac{\rho_{\mu 2}(t) dt}{|t - c_{\mu+1}|^{\varphi_{\mu+1}}(t-z)}, \quad \rho_{\mu 2}(t) := \rho(t) \textstyle\prod_{k \ne \mu+1} |t - c_k|^{-\varphi_k} . \end{aligned}$$

Obviously,

$$C_1(\varphi_{\mu 2}; (\partial \mathbb{D}\backslash\Gamma_\mu^0) \cap \{|z - c_\mu| \le \delta_0\}) \;\le\; M(\varphi_\lambda, c_\lambda, \delta_0, \Gamma_\mu)\, C_\alpha(\rho; \Gamma_\mu)\,,$$

$$C_\alpha(\varphi_{\mu 2}; (\partial \mathbb{D}\backslash\Gamma_\mu^0) \cap \{|z - c_\mu| \le \delta_0\}) \;\le\; M(\alpha, \varphi_\lambda, c_\lambda, \delta_0, \Gamma_\mu)\, C_\alpha(\rho; \Gamma_\mu)\,.$$

Arguing as in the proofs of Lemma 21 and Lemma 22 using

$$\frac{1}{2\pi i}\int\limits_{\Gamma_{\mu 1}} \frac{dt}{(t - c_\mu)^{\varphi_\mu}(t - z)} = \frac{e^{i\mu\pi}}{2i\sin\mu\pi}(z - c_\mu)^{\varphi_\mu} + h_\mu(z)$$

with h_μ analytic in $N(c_\mu)\backslash\Gamma_\mu$ one can find

$$C_\alpha((t - c_\mu)^{\delta_\mu}\varphi_{\mu 1}(t); (\partial \mathbb{D}\backslash\Gamma_\mu) \cap \{|z - c_\mu| \le \delta_0\}) \le M(\alpha, \delta_\mu, \delta_0, \varphi_\lambda, c_\lambda, \Gamma_\mu)\, C_\alpha(\rho; \Gamma_\mu)\,.$$

The same argumentation for the other part in $|z - c_{\mu+1}| \le \delta_0$ finishes the estimation of φ_μ. From

$$C_\alpha\left(\prod_1^m (t - c_k)^{\delta_k}\varphi_\mu(t); \partial \mathbb{D}\right) \le M(\alpha, \varphi_\lambda, c_\lambda, \delta_0, \delta_\lambda, \Gamma_\mu)\, C_\alpha(\rho; \Gamma_\mu)$$

because $\varphi = \sum\limits_1^m \varphi_\mu$ by summation

$$C_\alpha\left(\prod_1^m (t_0 - c_\mu)^{\delta_\mu}\varphi(t_0); \partial \mathbb{D}\right) \le M(\alpha, \varphi_\mu, c_\mu, \delta_\mu, \delta_0)\, C_\alpha(\rho)\,.$$

Corollary 8. *Under the assumptions of the preceding lemma*

$$C_\alpha\left(\prod_1^m (z - c_\mu)^{\delta_\mu}\varphi(z); \overline{\mathbb{D}}\right) \le M(\alpha, \varphi_\mu, c_\mu, \delta_\mu)\, C_\alpha(\rho; \partial \mathbb{D})\,.$$

This follows from the PRIVALOV theorem for analytic functions, see Theorem 5.

Lemma 27. *Let c_μ, Γ_μ and ρ be as in Lemma 25 and $|\varphi_\mu| < 1$. If then α and δ_μ are chosen according to*

$$0 < 2\alpha \le \min\{\min_{\varphi_\mu \ne 0} |\varphi_\mu|, \min_{0<\varphi_\mu}(1 - \varphi_\mu)\}\,,$$

$$\alpha + \varphi_\mu \le \delta_\mu \le 1 - \alpha\,, \quad if \quad 0 < \varphi_\mu\,,$$

$$\alpha < \delta_\mu \le 1 - \alpha\,, \qquad if \quad \varphi_\mu = 0\,,$$

$$\delta_\mu = 0 \quad \text{or} \quad \alpha \le \delta_\mu\,, \quad if \quad \varphi_\mu < 0\,,$$

then

$$C_\alpha\left(\prod_1^m (z-c_\mu)^{\delta_\mu}\varphi(z);\overline{\mathbb{D}}\right) \leq M(\alpha,\varphi_\mu,c_\mu,\delta_\mu)\, C_\alpha(\rho)\,.$$

Proof.

1. Let some μ be zero. In the situations of Lemma 21 and 22 then instead of Ω

$$\frac{1}{2\pi i}\int\limits_{\Gamma_{ab}}\frac{dt}{t-z} = \frac{1}{2\pi i}\log\frac{b-z}{a-z}\ (\log 1 = 0!)$$

occurs. This function is single-valued in $\mathbb{C}\setminus\Gamma_{ab}$ with boundary values

$$\left(\frac{1}{2\pi i}\int\limits_{\Gamma_{ab}}\frac{dt}{t-t_0}\right)^{\pm} = \frac{1}{2\pi i}\log\frac{b-t_0}{a-t_0} - \frac{1\mp 1}{2}\,,\ t_0\in\Gamma_{ab}\setminus\{a,b\}\,,$$

holds. Moreover,

$$\frac{(t_0-a)^\delta}{2\pi i}\log\frac{b-t_0}{a-t_0}\in C^\alpha(\Gamma),\ a<\delta\,,$$

and

$$C_\alpha\left(\frac{(t_0-a)^\delta}{2\pi i}\int\limits_{\Gamma_{ab}}\frac{dt}{t-t_0};\Gamma\right)\leq M(\alpha,\delta,\Gamma_{ab},N(a))\,.$$

Again by the PLEMELJ–PRIVALOV theorem, Theorem 4, observing that the numerator vanishes at $t=a$ it follows

$$C_\alpha\left(\frac{1}{2\pi i}\int\limits_{\Gamma_{ab}}\frac{\varphi^*(t)-\varphi^*(a)}{t-t_0}dt;\Gamma\right)\leq M(\alpha,\Gamma_{ab})H_\alpha(\varphi^*)\,.$$

Lemma 23 holds for $\mu=0$ too if ν is replaced by α. Namely, for $\mu=0$ one has $\varphi^*\in C^\alpha(\Gamma_{\delta_0})$ so that

$$C_\alpha(\Phi_1^\pm;\Gamma_{\delta_0})\leq M(\alpha,\delta_0,\Gamma_{ab})\,C_\alpha(\varphi;\Gamma_{\delta_0})\,.$$

Moreover, for $z,z_0\in\Gamma_{2\delta_0}$

$$\begin{aligned}|\Phi_2(z)| &\leq \frac{1}{\pi\delta_0}\int\limits_{\Gamma_{ab}\setminus\Gamma_{\delta_0}}|\varphi(t)|\,|dt|\leq M(\delta_0,\Gamma_{ab})C_\alpha(\varphi;\Gamma_{ab})\,,\\ |\Phi_2(z)-\Phi_2(z_0)| &\leq \frac{|z-z_0|}{2\pi}\int\limits_{\Gamma_{ab}\setminus\Gamma_{\delta_0}}\frac{|\varphi(t)|\,|dt|}{|t-z|\,|t-z_0|}\\ &\leq M(\delta_0,\Gamma_{ab})C_0(\varphi;\Gamma_{ab})|z-z_0|\,.\end{aligned}$$

Obviously, Lemma 25 may be reformulated where in the case $\mu_1 = \mu_2 = 0$ the number $\alpha > 0$ is less than $1/2$ and δ_k is chosen according to $\alpha < \delta_k \leq 1 - \alpha$ for $k = 1, 2$. If $\mu_1 = 0, \mu_2 \neq 0$ then α has to be restricted by $0 < \alpha \leq \mu_2$ and δ_1 and δ_2 are chosen so that $\alpha < \delta_1 \leq 1 - \alpha$ and $\alpha + \mu_2 \leq \delta_2 \leq 1 - \alpha$. For $\mu_1 \neq 0, \mu_2 = 0$ corresponding conditions hold.
These arguments allow the φ_μ in Lemma 26 to vanish. For a vanishing φ_μ the conditions from Lemma 26 have to be changed to $\alpha < \delta_\mu \leq 1 - \alpha$. Moreover, α is restricted by

$$0 < \alpha \leq \min_{\substack{1 \leq \mu \leq m \\ 0 < \varphi_\mu}} \varphi_\mu, \quad \alpha < 1/2 .$$

This condition is no restriction on the function ρ because if $\rho \in C^{\alpha'}(\Gamma_\mu)$ then $\rho \in C^\alpha(\Gamma_\mu)$ for any $\alpha, 0 < \alpha \leq \alpha'$, and

$$C_\alpha(\rho; \Gamma_\mu) \leq M(\alpha, \alpha', \Gamma_\mu)\, C_{\alpha'}(\rho; \Gamma_\mu) .$$

2. Let a particular μ be negative. For $\varphi \in C^\alpha(\Gamma_{ab})$ and $\alpha \leq -\mu$ from the PLEMELJ–PRIVALOV theorem

$$\Phi(t_0) = \frac{1}{2\pi i} \int_{\Gamma_{ab}} \frac{\varphi(t)|t-a|^{-\mu}}{t - t_0} dt$$

is seen to be HÖLDER-continuous on $\Gamma = \Gamma_{ab} \cap N(a)$ and

$$C_\alpha(\Phi; \Gamma) \leq M(\alpha, \mu, \Gamma_{ab}, N(a))\, C_\alpha(\varphi; \Gamma_{ab}) .$$

Moreover,

$$C_\alpha(\Phi; \Gamma_{\delta_0}) \leq M(\alpha, \mu, \delta_0, \Gamma_{ab})\, C_\alpha(\varphi; \Gamma_{ab}) .$$

This estimation can also be achieved from

$$\Phi(t_0) - \Phi(a) = \frac{t_0 - a}{2\pi i} \int_{\Gamma_{ab}} \frac{\varphi^*(t) dt}{(t-a)^{\mu+1}(t - t_0)}, \quad \varphi^*(t) := \varphi(t) e^{i\mu \arg(t-a)} ,$$

by applying Lemma 20 to Lemma 21.
If in the situation of Lemma 24 one exponent, say μ, is negative then with $0 < \alpha \leq -\mu, 2\alpha \leq \nu + \alpha \leq \delta \leq 1 - \alpha$

$$\Phi(t) := \frac{1}{2\pi i} \int_{\Gamma_{ab}} \frac{\varphi(t) dt}{|t-a|^\mu |t-b|^\nu (t - z)}$$

satisfies

$$C_\alpha((t-b)^\delta \Phi(t); \Gamma_{ab}) \leq M(\alpha, \mu, \nu, \delta, \Gamma_{ab})\, C_\alpha(\varphi; \Gamma_{ab}) .$$

If both μ and ν are negative then with $0 < \alpha \leq \min\{-\mu, -\nu\}$

$$C_\alpha(\Phi; \Gamma_{ab}) \leq M(\alpha, \mu, \nu, \Gamma_{ab})\, C_\alpha(\varphi; \Gamma_{ab}) .$$

But then for any $\delta_k \geq \alpha (k = 1, 2)$

$$C_\alpha((t-a)^{\delta_1}(t-b)^{\delta_2}\Phi(t); \Gamma_{ab}) \leq M(\alpha, \mu, \nu, \delta_1, \delta_2, \Gamma_{ab})\, C_\alpha(\varphi; \Gamma_{ab}) .$$

Remark. If in the case $0 < \varphi_\mu (1 \leq \mu \leq m)$ instead of $\delta_\mu > \varphi_\mu$ we would take $\delta_\mu = \varphi_\mu$ and would not restrict α then we only would get an estimation of the form

$$C_{\nu-\varepsilon}\left(\prod_1^m (t_0 - c_\mu)^{\varphi_\mu}\varphi(t_0); \partial \mathbb{D}\right) \leq M(\alpha, \varepsilon, \varphi_\mu, c_\mu)\, C_\alpha(\rho) ,$$

where $\nu := \min\{\alpha, \varphi_1, \dots, \varphi_m\}$ and $0 < \varepsilon$ is arbitrarily small as follows from the consideration in [Musk65], §25. But this estimation is not good enough for the kind of problems we are involved with.

On the basis of Lemma 26 the solution (3.5.1) to the modified discontinuous RIEMANN–HILBERT problem for analytic functions can be estimated.

Theorem 43. *There exist constants δ_1 and δ_2 independent of $\varphi, a_k\ (1 \leq k \leq 2\kappa+1)$ if $0 \leq \kappa, h_k(|k| \leq -\kappa - 1)$ if $\kappa < 0$, and of ϕ itself such that with $\beta + \beta_\mu \leq \gamma_\mu$ and $0 < \beta \leq \beta_\mu + \varphi_\mu \leq \gamma_\mu + \varphi_\mu \leq 1 - \beta (1 \leq \mu \leq m)$ the analytic solution ϕ to the modified discontinuous* RIEMANN–HILBERT *problem satisfies the a priori estimate*

$$C_\beta\left(\prod_{\mu=1}^m (z - c_\mu)^{\gamma_\mu}\phi(z); \overline{\mathbb{D}}\right) \leq \delta_1 C_\beta(\varphi_0; \partial\mathbb{D}) + \delta_2 \sum_{k=1}^{2\kappa+1} |a_k| ,$$

where the second term does not occur for $\kappa < 0$.

Proof. 1. Applying Lemma 26 to the representation (3.5.1) of ϕ, i.e. to

$$\phi(z) = z^{[\kappa]} e^{i\tau(z)} X_1(z) \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{\varphi(t) + h(t)}{|X_1(t)|} e^{\operatorname{Im}\tau(t)} \frac{t+z}{t-z}\frac{dt}{t} - z^{[\kappa]} e^{i\tau(z)} X_1(z) Q(z)$$

with

$$X_1(z) = \prod_{\mu=1}^m (z - c_\mu)\varphi_\mu \text{ if } 2\kappa \text{ is even, } X_1(z) = (z - c_0)\prod_{\mu=1}^m (z - c_\mu)^{\varphi_\mu} \text{ if } 2\kappa \text{ is odd}$$

gives for 2κ even and nonnegative

$$C_\beta\left(\prod_{\mu=1}^m (z-c_\mu)^{\gamma_\mu}\phi;\overline{\mathbb{D}}\right) \le C_\beta\left(z^{[\kappa]}e^{i\tau(z)};\overline{\mathbb{D}}\right)$$

$$\times C_\beta\left(\prod_{\mu=1}^m (z-c_\mu)^{\gamma_\mu+\varphi_\mu}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{\varphi_0(t)}{\prod_{\mu=1}^m |t-c_\mu|^{\beta_\mu+\varphi_\mu}}\left(\frac{2}{t-z}-\frac{1}{t}\right)dt;\overline{\mathbb{D}}\right)$$

$$+C_\beta(z^{[\kappa]}e^{i\tau(z)};\overline{\mathbb{D}})C_\beta\left(\prod_{\mu=1}^m (z-c_\mu)^{\gamma_\mu+\varphi_\mu}Q(z);\overline{\mathbb{D}}\right)$$

$$\le \delta_1 C_\beta(\varphi_0;\partial\mathbb{D}) + \delta_2\sum_{k=1}^{2\kappa+1}|a_k|\,.$$

Here for the last estimate (3.5.2) is used and

$$\left|\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{\varphi_0(t)e^{\operatorname{Im}\tau(t)}}{\prod_{\mu=1}^m |t-c_\mu|^{\beta_\mu+\varphi_\mu}}\operatorname{Im}\frac{t+z_k}{t-z_k}\frac{dt}{t}\right| \le MC_\beta(\varphi_0;\partial\mathbb{D})$$

which follows from Lemma 22 because $z_k \ne c_\mu, 1 \le k \le 2\kappa+1, 1 \le \mu \le m$. The same estimate holds for 2κ odd and nonnegative. The only difference is the additional factor $z-c_0$ in X_1. In this case it is more convenient to rewrite the second term in the sum on the right-hand side of (3.5.2) as

$$\left|\operatorname{Im}\left[\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{\lambda(t)\varphi(t)}{t^{[\kappa]}e^{i\tau(t)}X_1(t)}\frac{t+z_\mu}{t-z_\mu}\frac{dt}{t}\right]\right|$$

$$\le\left|\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{\lambda(t)t^{-[\kappa]}e^{-i\tau(t)}\varphi_0(t)}{\prod_{\mu=1}^m |t-c_\mu|^{\beta_\mu+\varphi_\mu}}\left[\frac{2}{z_k-c_0}\left(\frac{1}{t-z_k}-\frac{1}{t-c_0}\right)+\frac{1}{c_0}\left(\frac{1}{t-c_0}-\frac{1}{t}\right)\right]dt\right|.$$

Integrals appearing here are of the same kind as before and we just have to observe $z_k \ne c_\mu$ for $1 \le k \le 2\kappa+1, 0 \le \mu \le m$.

2. If $\kappa < 0$ the estimates (3.5.3) and the same argumentation as before show that

$$C_\beta\left(\prod_{\mu=1}^m (z-c_\mu)^{\gamma_\mu+\varphi_\mu}Q(z);\overline{\mathbb{D}}\right) \le M(c_\mu,\lambda)C_\beta(\varphi_0;\partial\mathbb{D})$$

and h being a trigonometric polynomial satisfies especially

$$C_\beta(h;\partial\mathbb{D}) \le M(\beta,c_\mu,\lambda)C_\beta(\varphi_0;\partial\mathbb{D}).$$

Therefore in this case

$$C_\beta\left(\prod_{\mu=1}^{m}(z-c_\mu)^{\gamma_\mu}\phi(z);\overline{\mathbb{D}}\right)\leq\delta_1 C_\beta(\varphi_0;\partial\mathbb{D})\,.$$

Using the representation of solutions to generalized BELTRAMI equations by analytic functions as given in the proof of Theorem 38 on the basis of Theorems 42 and 43 the discontinuous RIEMANN–HILBERT problem can be solved for the generalized BELTRAMI equation. For the proof the following lemma is needed.

Lemma 28. *Let D be a bounded domain and $f\in C^{\alpha_0}(\overline{D})$ and $0<\alpha<\alpha_0\leq 2\alpha$. Then for any fixed $t_0\in D$ the function $\varphi(t):=\dfrac{f(t)-f(t_0)}{|t-t_0|^\alpha}$ belongs to $C^{\alpha_0-\alpha}(\overline{D})$ satisfying $C_{\alpha_0-\alpha}(\varphi;\overline{\mathbb{D}})\leq(1+2^{\alpha_0-\alpha}+(diam D)^{\alpha_0-\alpha})H_{\alpha_0}(f)$.*

Proof. From

$$\varphi(t)=\frac{f(t)-f(t_0)}{|t-t_0|^{\alpha_0}}|t-t_0|^{\alpha_0-\alpha}$$

it can be seen that $\varphi(t_0)=0$, $\|\varphi\|_{0,D}\leq H_\alpha(f)(diam\,D)^{\alpha_0-\alpha}$, and

$$\frac{|\varphi(t)-\varphi(t_0)|}{|t-t_0|^{\alpha_0-\alpha}}=\frac{|f(t)-f(t_0)|}{|t-t_0|^{\alpha_0}}\leq H_\alpha(f)\,.$$

Let now t_1 and t_2 be two different points from D. Then

$$\begin{aligned}\frac{|\varphi(t_1)-\varphi(t_2)|}{|t_1-t_2|^{\alpha_0-\alpha}} &= \left|\frac{f(t_1)-f(t_2)}{|t_1-t_2|^{\alpha_0-\alpha}|t_1-t_0|^\alpha}+\frac{f(t_2)-f(t_0)}{|t_1-t_2|^{\alpha_0-\alpha}}\frac{|t_2-t_0|^\alpha-|t_1-t_0|^\alpha}{|t_1-t_0|^\alpha|t_2-t_0|^\alpha}\right|\\ &\leq H_{\alpha_0}(f)\left[\frac{|t_1-t_2|^\alpha}{|t_1-t_0|^\alpha}+\frac{|t_2-t_0|^{\alpha_0-\alpha}}{|t_1-t_2|^{\alpha_0-\alpha}}\frac{|t_1-t_2|^\alpha}{|t_1-t_0|^\alpha}\right]\,.\end{aligned}$$

We distinguish three cases.

(i) If $|t_2-t_0|\leq|t_1-t_2|\leq|t_1-t_0|$, then

$$\frac{|\varphi(t_1)-\varphi(t_2)|}{|t_1-t_2|^{\alpha_0-\alpha}}\leq 2H_{\alpha_0}(f)\,.$$

(ii) If $|t_1-t_2|\leq|t_1-t_0|,|t_2-t_0|$, then

$$|t_2-t_0|\leq|t_2-t_1|+|t_1-t_0|\leq 2|t_1-t_0|$$

so that

$$\frac{|\varphi(t_1)-\varphi(t_2)|}{|t_1-t_2|^{\alpha_0-\alpha}}\leq H_{\alpha_0}(f)\left[1+2^{\alpha_0-\alpha}\frac{|t_1-t_0|^{\alpha_0-2\alpha}}{|t_1-t_2|^{\alpha_0-2\alpha}}\right]\leq(1+2^{\alpha_0-\alpha})H_{\alpha_0}(f)\,.$$

(iii) If $|t_1 - t_0|, |t_2 - t_0| \le |t_1 - t_2|$ then

$$\begin{aligned}\frac{|\varphi(t_1) - \varphi(t_2)|}{|t_1 - t_2|^{\alpha_0 - \alpha}} &\le \frac{|f(t_1) - f(t_0)|}{|t_1 - t_2|^{\alpha_0-\alpha}|t_1 - t_0|^{\alpha}} + \frac{|f(t_2) - f(t_0)|}{|t_1 - t_2|^{\alpha_0-\alpha}|t_2 - t_0|^{\alpha}} \\ &\le \frac{|f(t_1) - f(t_0)|}{|t_1 - t_0|^{\alpha_0}} + \frac{|f(t_2) - f(t_0)|}{|t_1 - t_0|^{\alpha_0}} \le 2H_{\alpha_0}(f) .\end{aligned}$$

Theorem 44. *The discontinuous* RIEMANN–HILBERT *problem*

$\mathrm{Re}\,\{\overline{\lambda(z)}w(z)\} = \varphi(z) + h(z)$ on $\partial \boldsymbol{D}$, $\mathrm{Im}\,\{\overline{\lambda(z_k)}w(z_k)\} = a_k, 1 \le k \le 2\kappa + 1$ if $0 < \kappa$,

for the generalized BELTRAMI *equation*

$$w_{\bar z} + \mu_1 w_z + \mu_2 \overline{w_z} + aw + b\overline{w} + c = 0 \quad \text{in } \boldsymbol{D}$$

is uniquely solvable. The solution satisfies the a priori estimate

$$C_{\alpha_0^2\beta_0}\left(\prod_{\mu=1}^{m}(z - c_\mu)^{\gamma_\mu} w(z); \overline{\boldsymbol{D}}\right) \le \delta_1 C_\beta(\varphi_0; \partial\boldsymbol{D}) + \delta_2 \sum_{k=1}^{2\kappa+1} |a_k| + \delta_3 \|c\|_{p_0} \tag{3.5.4}$$

where $\delta_1, \delta_2, \delta_3$ are nonnegative constants not depending on $w, \mu_1, \mu_2, a, b, c, \varphi_0, a_k, \delta_2 = 0$ formally if $\kappa < 0, \beta_0 := \min\{\alpha_0, \beta\}$ and

$$2\alpha_0^2\beta_0 < 4\beta_\mu + 2\alpha_0\varphi_\mu < \alpha_0^2\gamma_\mu + 2\alpha_0\varphi_\mu < 2\alpha_0(1 - \alpha_0\beta_0), 2\alpha_0^2\beta_0 + 4\beta_\mu \le \alpha_0^2\gamma_\mu .$$

Proof. Proceeding as in the proof for Theorem 38 w is represented as

$$w(z) = \phi(\zeta(z))e^{\chi(z)} + \psi(z)$$

where

$$\begin{aligned} C_{\alpha_0}(\psi, \boldsymbol{D}) + \|\psi_z\|_p + \|\psi_{\bar z}\|_p &\le \delta\|c\|_{p_0} , \ \alpha_0 := \frac{p_0 - 2}{p_0}, 2 < p \le p_0, q_0\Lambda_p < 1 , \\ C_{\alpha_0}(\chi; \boldsymbol{D}) + \|\chi_z\|_p + \|\chi_{\bar z}\|_p &\le \delta(\|a\|_{p_0} + \|b\|_{p_0}) , \end{aligned}$$

ζ is a homeomorphism of a BELTRAMI equation mapping $\boldsymbol{D}$ onto itself satisfying together with its inverse mapping z

$$C_{\alpha_0}(\zeta; \boldsymbol{D}) + \|\zeta_z\|_p + \|\zeta_{\bar z}\|_p \le M(p, p_0, q_0), C_{\alpha_0}(z; \boldsymbol{D}) + \|z_\zeta\|_p + \|z_{\bar\zeta}\|_p \le M(p, p_0, q_0)$$

and ϕ is analytic in $\boldsymbol{D}$ satisfying

$$\mathrm{Re}\,\{\overline{\lambda(z(\zeta))}\phi(\zeta)\} = \widetilde{\varphi}(\zeta) + \widetilde{h}(\zeta) \quad \text{on } \partial\boldsymbol{D}$$

with
$$\widetilde{\varphi}(\zeta) := \{\varphi(z(\zeta)) - \operatorname{Re}\lambda(z(\zeta))\psi(z(\zeta))\}e^{-\operatorname{Re}\chi(z(\zeta))},$$
$$\widetilde{h}(\zeta) := h(z(\zeta))e^{-\operatorname{Re}\chi(z(\zeta))},$$
$\operatorname{Im}\{\overline{\lambda(z(\zeta_k))}\phi(\zeta_k)\} = \{a_k - \operatorname{Im}\overline{\lambda(z_k)}\psi(z_k)\}e^{\operatorname{Re}\chi(z_k)}, \zeta_k = \zeta(z_k), 1 \le k \le 2\kappa+1$, if $0 < \kappa$. Let $c^*_\mu := \zeta(c_\mu)$ be the points corresponding to the points of discontinuity for $\lambda(z(\cdot))$ and $\varphi(z(\cdot))$. Obviously, $\lambda \circ z, \varphi_0 \circ z, \tau \circ z \in C^{\alpha_0\beta}(\overline{\Gamma^*_\mu})$ where $\Gamma^*_\mu := \zeta[\Gamma_\mu], 1 \le \mu \le m$, and $\psi \circ z, \chi \circ z \in C^{\alpha_0^2}(\overline{\boldsymbol{D}})$. $\lambda \circ z$ has the same discontinuity at c^*_μ as λ has at c_μ characterized by φ_μ. As $\beta_0 := \min\{\alpha_0, \beta\}$ then instead of $C^{\alpha_0\beta}(\overline{\Gamma^*_\mu}), C^{\alpha_0^2}(\overline{\boldsymbol{D}})$ we may work with $C^{\alpha_0\beta_0}(\overline{\Gamma^*_\mu})$ and $C^{\alpha_0\beta_0}(\overline{\boldsymbol{D}})$. Then $\widetilde{\varphi} \in C^{\alpha_0\beta_0}(\overline{\Gamma^*_\mu})$ and e.g.

$$C_{\alpha_0\beta_0}(\varphi_0 \circ z; \partial\boldsymbol{D}) \le M(\alpha_0, \beta_0, \beta)C_{\alpha_0\beta}(\varphi_0 \circ z; \partial\boldsymbol{D}) \le M(\alpha_0, \beta_0, \beta, p, p_0, q_0)C_\beta(\varphi_0; \partial\boldsymbol{D}).$$

By Lemma 27, $\zeta \in C^{\alpha_0}(\overline{\Gamma_\mu})$ implies $\dfrac{\zeta(z) - \zeta(c_\mu)}{|z - c_\mu|^\alpha} \in C^{\alpha_0 - \alpha}(\overline{\Gamma^*_\mu})$ and

$$\left.\frac{\zeta(z) - \zeta(c_\mu)}{|z - c_\mu|^\alpha}\right|_{z=z(\zeta)} = \frac{\zeta - c^*_\mu}{|z(\zeta) - z(c^*_\mu)|^\alpha} \in C^{(\alpha_0-\alpha)\alpha_0}(\overline{\Gamma^*_\mu})$$

and
$$\left|\frac{\zeta - c^*_\mu}{|z(\zeta) - z(c^*_\mu)|^\alpha}\right|^{\beta_\mu/\alpha} \in C^{(\alpha_0-\alpha)\beta_\mu\alpha_0/\alpha}(\overline{\Gamma^*_\mu})$$
where $0 < \alpha < \alpha_0 \le 2\alpha$. Choosing $2\alpha = \alpha_0$ then $(\alpha_0 - \alpha)\beta_\mu\alpha_0/\alpha = \alpha_0\beta_\mu$. Then

$$\varphi(z(\zeta)) = \widehat{\varphi}_0(\zeta)\prod_{\mu=1}^m |\zeta - c^*_\mu|^{-\beta_\mu/\alpha}, \quad \widehat{\varphi}_0(\zeta) := \varphi_0(z(\zeta))\prod_{\mu=1}^m \left|\frac{\zeta - c^*_\mu}{|z(\zeta) - c_\mu|^\alpha}\right|^{2\beta_\mu/\alpha_0}.$$

Assuming $\beta_0 \le \min\limits_{1\le\mu\le m}\{\beta_\mu : 0 < \beta_\mu\}$ then $\widehat{\varphi}_0 \in C^{\alpha_0\beta}(\overline{\Gamma^*_\mu})$. Observing $\widehat{\varphi}(c^*_\mu) = 0$ even $\widehat{\varphi}_0 \in C^{\alpha_0\beta_0}(\partial\boldsymbol{D})$ is seen. Applying Theorem 43 then the a priori estimate

$$\begin{aligned}
&C_{\alpha_0\beta_0}\left(\prod_{\mu=1}^m(\zeta - c^*_\mu)^{\widetilde{\gamma}_\mu}\phi(\zeta); \overline{\boldsymbol{D}}\right) \\
&\le \widetilde{\delta}_1 C_{\alpha_0\beta_0}\left(\left\{\widehat{\varphi}_0(\zeta) - \prod_{\mu=1}^m |\zeta - c^*_\mu|^{2\beta_\mu/\alpha_0}\operatorname{Re}\lambda(z(\zeta))\psi(z(\zeta))\right\}e^{-\operatorname{Re}\chi(z(\zeta))}; \partial\boldsymbol{D}\right) \\
&\quad + \widetilde{\delta}_2\sum_{k=1}^{2\kappa+1}\left|\left\{a_k - \operatorname{Im}\overline{\lambda(z_k)}\psi(z_k)\right\}e^{\operatorname{Re}\chi(z_k)}\right| \\
&\le \delta_1 C_\beta(\varphi_0; \partial\boldsymbol{D}) + \delta_2\sum_{k=1}^{2\kappa+1}|a_k| + \delta_3\|c\|_{p_0}
\end{aligned}$$

where $\delta_1, \delta_2, \delta_3$ are nonnegative constants not depending on $\phi, \varphi, \mu_1, \mu_2, a, b, c, a_k$ and the $\widetilde{\gamma}_\mu$ satisfy

$$0 < \alpha_0^2\beta_0 < 2\beta_\mu + \alpha_0\varphi_\mu \le \alpha_0\widetilde{\gamma}_\mu + \alpha_0\varphi_\mu < \alpha_0(1-\alpha_0\beta_0), \alpha_0^2\beta_0 + 2\beta_\mu \le \alpha_0\widetilde{\gamma}_\mu \,.$$

Thus setting $\alpha_0\gamma_\mu = 2\widetilde{\gamma}_\mu$

$$\begin{aligned} &C_{\alpha_0^2\beta_0}\left(\prod_{\mu=1}^m (z-c_\mu)^{\gamma_\mu}\phi(\zeta(z)); \boldsymbol{D}\right) \\ &= C_{\alpha_0^2\beta_0}\left(\prod_{\mu=1}^m \left(\frac{z-c_\mu}{|\zeta(z)-c_\mu^*|^{\alpha_0/2}}\right)^{2\widetilde{\gamma}_\mu/\alpha_0} \prod_{\mu=1}^m |\zeta(z)-c_\mu^*|^{\widetilde{\gamma}_\mu}\phi(\zeta(z)); \boldsymbol{D}\right) \\ &\le C_{\alpha_0^2\beta_0}\left(\prod_{\mu=1}^m \left(\frac{z-c_\mu}{|\zeta(z)-c_\mu^*|^{\alpha_0/2}}\right)^{2\widetilde{\gamma}_\mu/\alpha_0}; \boldsymbol{D}\right) C_{\alpha_0^3\beta_0}\left(\prod_{\mu=1}^m |\zeta-c_\mu^*|^{\widetilde{\gamma}_\mu}\phi(\zeta); \boldsymbol{D}\right) . \end{aligned}$$

Because

$$C_{\alpha_0^3\beta_0}\left(\prod_{\mu=1}^m (\zeta-c_\mu^*)^{\widetilde{\gamma}_\mu}\phi(\zeta); \boldsymbol{D}\right) \le M(\alpha_0,\beta_0) C_{\alpha_0\beta_0}\left(\prod_{\mu=1}^m (\zeta-c_\mu^*)^{\widetilde{\gamma}_\mu}\phi(\zeta); \boldsymbol{D}\right)$$

these estimations and the representation

$$w(z) = \phi(\zeta(z))e^{\chi(z)} + \psi(z)$$

lead to the a priori estimate stated in the theorem.

Remark. These estimates are only of qualitative value because the bounds for the γ_μ seem not to be sharp. The function $\widehat{\varphi}_0$ is constructed in such a way that it vanishes in the critical points c_μ^*. This is much more than is needed to apply the estimate from Theorem 43. In (3.5.4) just a weighted subnorm for the solution is estimated. This estimate can be completed by adding weighted L_p-norms of the first derivatives of w. Because this will again be involved we stay with (3.5.4).
Combining these results with the POINCARÉ problem the discontinuous POINCARÉ problem can be studied. This is done in [Bewe89], see also [Bewe88] for the related nonlinear problem.

4. Other equations and systems related to the BELTRAMI equation

4.1 Initial boundary value problem for pseudoparabolic equations

If L is an elliptic differential operator with respect to the space variable x, M a differential operator with respect to x of lower order than L, and t is a time variable then $L\dfrac{\partial}{\partial t} + M$ is called a pseudoparabolic operator. Besides pseudoparabolic also metaparabolic operators are investigated. A metaparabolic operator has the form $L + M\dfrac{\partial}{\partial t}$. In this section a pseudoparabolic equation will be studied which is related to the generalized BELTRAMI equation,

$$w_{t\bar{z}}+\mu_1 w_{tz}+\mu_2\overline{w_{tz}}+a_1 w_t+a_2\overline{w}_t+b_1 w_{\bar{z}}+b_2\overline{w_{\bar{z}}}+b_3 w_z+b_4\overline{w}_z+c_1 w+c_2\overline{w}+d = 0.\tag{4.1.1}$$

A related metaparabolic equation would be

$$w_{\bar{z}} + \mu_1 w_z + \mu_2\overline{w_z} + a_1 w_t + a_2\overline{w_t} + c_1 w + c_2\overline{w} + d = 0 \,.$$

There is not too much known about metaparabolic equations, see [Gisc78], [Obol85]. The coefficients in (4.1.1) as well as the unknown w are functions of z in some domain D of the complex plane $\mathbb{C}$ and of $t \in I$ with a finite closed real intervall $I = [0, T]$. For simplicity we restrict ourselves to the special case of the unit disc $D = \mathbb{D}$. The BANACH space of the continuous and of the continously differentiable mappings of I into a BANACH space V is denoted by $C(I; V)$ and $C^1(I; V)$ respectively, their norms are

$$\|f\|_{0;V} = \sup_{t\in I}\|f(t)\|_V, \|f\|_{1;V} := \|f\|_{0;V} + \|f'\|_{0;V} \,,$$

respectively. Here $\|f(t)\|_V$ denotes the norm of $f(t)$ in V. As vector spaces V there will appear $C^\alpha(\partial\mathbb{D}), C^1(I), L_p(\overline{D}), W^1_p(\mathbb{D})$ of complex valued functions. $W^1_p(\mathbb{D})$ is the space of functions in $\mathbb{D}$ with weak first order derivatives in $L_p(\overline{\mathbb{D}})$,

$$\|f\|_{W^1_p(\mathbb{D})} := \|f\|_{C(\overline{\mathbb{D}})} + \|f_z\|_{L_p(\overline{\mathbb{D}})} + \|f_{\bar{z}}\|_{L_p(\overline{\mathbb{D}})} \,.$$

These notations differ slightly from those used before.

Lemma 29. *Let the real numbers α and p satisfy $1 < 2\alpha < 2 < p \le \dfrac{1}{1-\alpha}$, let $\lambda \in C^\alpha(\partial\mathbb{D}), \lambda(z) \ne 0$ on $\partial\mathbb{D}, \kappa := \dfrac{1}{2\pi i}\displaystyle\int_{\partial\mathbb{D}} d\log\lambda(z) \in \mathbb{Z}$, $\mu_1, \mu_2 \in C(I; L_\infty(\overline{\mathbb{D}})), a_1, a_2, c \in C(I; L_p(\overline{\mathbb{D}}))$,*

$$\|\mu_1(t,\cdot)\|_{L_\infty(\overline{\mathbb{D}})} + \|\mu_2(t,\cdot)\|_{L_\infty(\overline{\mathbb{D}})} \le q < 1 \,,$$

$$\|a_1(t,\cdot)\|_{L_p(\overline{\mathbb{D}})} + \|a_2(t,\cdot)\|_{L_p(\overline{\mathbb{D}})} \le K \,,$$

$$\varphi \in C(I, C^\alpha(\partial \mathbb{D})), z_k \in \partial \mathbb{D}, \alpha_k \in C(I) \textit{ for } 1 \le k \le 2\kappa + 1 \textit{ if } 0 \le \kappa .$$

Then the problem

$$w_{\bar z}(t,z) + \mu_1(t,z) w_z(t,z) + \mu_2(t,z)\overline{w_z(t,z)} + a_1(t,z)w(t,z)$$

$$+a_2(t,z)\overline{w(t,z)} + c(t,z) = 0 \quad \textit{in } I \times \mathbb{D} ,$$

$$\mathrm{Re}\,\{\overline{\lambda(z)} w(t,z)\} = \varphi(t,z) + h(t,z) \quad \textit{in} \quad I \times \partial\mathbb{D} \tag{4.1.2}$$

$$\mathrm{Im}\,\{\overline{\lambda(z_k)} w(t,z_k)\} = \alpha_k(t) \ , \ 1 \le k \le 2\kappa + 1 \ , \quad \textit{if} \quad 0 \le \kappa \ , \ \text{in } I$$

is uniquely solvable. Here $h(t,z) = 0$ *if* $0 \le \kappa$ *and for* $\kappa < 0$

$$h(t,z) := \sum_{\nu=\kappa+1}^{-\kappa-1} h_\nu(t) z^\nu \ , \ h_{-\nu}(t) = \overline{h_\nu(t)} \ , \ |\nu| \le -\kappa - 1 \ ,$$

with coefficients $h_\nu \in C(I)$ *to be determined properly together with the unknown* w. *The solution* $w \in C(I; W^1_p(\mathbb{D}))$ *satisfies the a priori estimate*

$$\|w(t,\cdot)\|_{W^1_p(\mathbb{D})} \le \beta \|\varphi(t,\cdot)\|_{C^\alpha(\partial\mathbb{D})} + \gamma \sum_{k=1}^{2\kappa+1} |\alpha_k(t)| + \delta \|c(t,\cdot)\|_{L_p(\overline{\mathbb{D}})}$$

with nonnegative constants depending only on $t, \alpha, p, q, z_k, \lambda, K$. *The second term on the right-hand side does not appear if* $\kappa < 0$.

Proof. The existence of the solution together with the a priori estimate follows from Theorem 37 applied for fixed $t \in I$. It remains to prove $w \in C(I; W^1_p(\mathbb{D}))$. Let $\omega(z) := w(t,z) - w(t_0,z)$ for $t, t_0 \in I$. Then

$$\omega_{\bar z} + \mu_1(t,z)\omega_z + \mu_2(t,z)\overline{\omega_z} + a_1(t,z)\omega + a_2(t,z)\overline{\omega}$$

$$+(\mu_1(t,z) - \mu_1(t_0,z)) w_z(t_0,z) + (\mu_2(t,z) - \mu_2(t_0,z))\overline{w_z(t_0,z)}$$

$$+(a_1(t,z) - a_1(t_0,z)) w(t_0,z) + (a_2(t,z) - a_2(t_0,))\overline{w(t_0,z)}$$

$$+c(t,z) - c(t_0,z) = 0 \quad \text{in} \quad I \times \mathbb{D} \ ,$$

$$\mathrm{Re}\,\{\overline{\lambda(z)}\omega(z)\} = \varphi(t,z) - \varphi(t_0,z) + h(t,z) \quad \text{in } I \times \partial\mathbb{D}$$

$$\mathrm{Im}\,\{\overline{\lambda(z_k)}\omega(z_k)\} = \alpha_k(t) - \alpha_k(t_0), \ 1 \le k \le 2\kappa + 1, \quad \text{if} \quad 0 \le \kappa, \quad \text{in} \quad I \ .$$

Applying the a priori estimate from Lemma 29 to this problem gives

$$\begin{aligned}
\|\omega\|_{W_p^1(\mathbb{D})} \;\le\; & \beta\|\varphi(t,\cdot)-\varphi_0(t,\cdot)\|_{C^\alpha(\mathbb{D})}+\gamma\sum_{k=1}^{2\kappa+1}|\alpha_k(t)-\alpha_k(t_0)| \\
& +\delta\Bigg\{\Big(\|\mu_1(t,\cdot)-\mu_1(t_0,\cdot)\|_{L_\infty(\mathbb{D})} \\
& \qquad +\|\mu_2(t,\cdot)-\mu_2(t_0,\cdot)\|_{L_\infty(\mathbb{D})}\Big)\|w_z(t_0,\cdot)\|_{L_p(\overline{\mathbb{D}})} \\
& \qquad +\Big(\|a_1(t,\cdot)-a_1(t_0,\cdot)\|_{L_p(\overline{\mathbb{D}})} \\
& \qquad\qquad +\|a_2(t,\cdot)-a_2(t_0,\cdot)\|_{L_p(\overline{\mathbb{D}})}\Big)\|w(t_0,\cdot)\|_{C(\overline{\mathbb{D}})} \\
& \qquad +\|c(t,\cdot)-c(t_0,\cdot)\|_{L_p(\overline{\mathbb{D}})}\Bigg\}.
\end{aligned}$$

From the continuity of the involved coefficients as functions of t in the respective spaces and the boundedness of $\|w(t_0,\cdot)\|_{W_p^1(\mathbb{D})}$ the continuity of the mapping $w: I\to W_p^1(\mathbb{D})$ follows.

Lemma 30. *There exists a constant $M=M(p,T)$ such that*

$$\left\|\int_0^t w(\tau,\cdot)d\tau\right\|_{W_p^1(\mathbb{D})}\le M\left(\int_0^t \|w(\tau,\cdot)\|^p_{W_p^1(\mathbb{D})}d\tau\right)^{1/p}$$

for any $w\in C(I;W_p^1(D))$.

Proof.

$$\begin{aligned}
\left\|\int_0^t w(\tau,\cdot)d\tau\right\|_{C(\overline{\mathbb{D}})} &\le \int_0^t \|w(\tau,\cdot)\|_{C(\overline{D})}d\tau\le T^{1-1/p}\left(\int_0^t \|w(\tau,\cdot)\|^p_{C(\overline{\mathbb{D}})}d\tau\right)^{1/p} \\
&\le T^{1-1/p}\left(\int_0^t \|w(\tau,\cdot)\|^p_{W_p^1(\mathbb{D})}d\tau\right)^{1/p},
\end{aligned}$$

$$\left\|\int_0^t w_z(\tau,\cdot)d\tau\right\|_{L_p(\overline{\mathbb{D}})} = \left(\int_{\mathbb{D}}\left|\int w_z(\tau,z)d\tau\right|^p dxdy\right)^{1/p}$$

$$\leq \left(\int_{\mathbb{D}} T^{p-1} \int_0^t |w_z(\tau, z)|^p d\tau \, dx dy \right)^{1/p}$$

$$\leq T^{1-1/p} \left(\int_0^t \|w_z(\tau, \cdot)\|^p_{L_p(\overline{\mathbb{D}})} d\tau \right)^{1/p}$$

$$\leq T^{1-1/p} \left(\int_0^t \|w(\tau, \cdot)\|^p_{W_p^1(\overline{\mathbb{D}})} d\tau \right)^{1/p} .$$

The last inequality holds for $w_{\overline{z}}$, too. Hence

$$\left\| \int_0^t w(\tau, \cdot) d\tau \right\|_{W_p^1(\mathbb{D})} \leq 3T^{1-1/p} \left(\int_0^t \|w(\tau, \cdot)\|^p_{W_p^1(\mathbb{D})} d\tau \right)^{1/p} .$$

Lemma 31. (GRONWALL Lemma). *Let f be a continuous nonnegative function on $I = [0, T]$ satisfying the integral inequality*

$$f(t) \leq c_1 + c_2 \int_0^t f(\tau) d\tau, \quad t \in I ,$$

with nonnegative constants $c_k, k = 1, 2$. Then

$$f(t) \leq c_1 e^{c_2 t}, \quad t \in I .$$

Proof. Introducing

$$F(t) := c_1 + c_2 \int_0^t f(\tau) d\tau$$

one sees

$$F'(t) \leq c_2 F(t)$$

so that by integration

$$f(t) \leq F(t) = F(0) e^{c_2 t} = c_1 e^{c_2 t} .$$

Theorem 45. *If the coefficients of equation (4.1.1) satisfy*

$$\|\mu_1\|_{0,L_\infty(\mathbb{D})} + \|\mu_2\|_{0,L_\infty(\overline{\mathbb{D}})} \leq q < 1 , \ \|a_1\|_{0,L_p(\overline{\mathbb{D}})} + \|a_2\|_{0,L_p(\overline{\mathbb{D}})} \leq K ,$$

$$\sum_{k=1}^{4} \|b_k\|_{=,L_\infty(\overline{\mathbb{D}})} \leq K , \ \|c_1\|_{0,L_p(\overline{\mathbb{D}})} + \|c_2\|_{0,L_p(\overline{\mathbb{D}})} \leq K$$

and those from the boundary condition (4.1.2)

$$\varphi \in C^1(I; C^\alpha(\partial \mathbb{D})), \alpha_k \in C^1(I), 1 \leq k \leq 2\kappa + 1 \ , \quad \text{if } 0 \leq \kappa \ ,$$

if, moreover, with $d_0 \in W_p^1(\mathbb{D})$ w *satisfies the initial condition*

$$w(0, z) = d_0(z) \quad \text{in} \quad \mathbb{D} \tag{4.1.3}$$

and the compatibility conditions

$$\begin{aligned} \operatorname{Re}\{\overline{\lambda(z)} d_0(z)\} \quad &= \quad \varphi(0, z) + h(0, z) \quad \text{on } \partial \mathbb{D} \ , \\ \operatorname{Im}\{\overline{\lambda(z_k)} d_0(z_k)\} \quad &= \quad \alpha_k(0) \ , \ 1 \leq k \leq 2\kappa + 1 \ , \quad \text{if } 0 \leq \kappa \ , \end{aligned}$$

are fulfilled then problem (4.1.1) - (4.1.3) is uniquely solvable. The solution belongs to $c^1(I; W_p^1(\mathbb{D}))$ *and satisfies the a priori estimate*

$$\|w\|_{1,W_p^1(\mathbb{D})} \leq \beta \|\varphi\|_{1,C^\alpha(\partial \mathbb{D})} + \gamma \sum_{k=1}^{2\kappa+1} \|\alpha_k\|_{C^1(I)} + \delta_1 \|d\|_{0,L_p(\overline{\mathbb{D}})} + \delta_2 \|d_0\|_{W_p^1(\overline{\mathbb{D}})} \tag{4.1.4}$$

with coefficients independent of the solution w *as well as of the coefficients of the equation and the right-hand sides of the boundary, the side, and the initial condition.*

Proof.

(1) A priori estimate. Let $w \in C^1(I; W_p^1(D))$ be a solution to the initial boundary problem (4.1.2) then applying Lemma 28 to w_z we have

$$\begin{aligned} \|w_t(t, \cdot)\|_{W_p^1(\mathbb{D})} \quad &\leq \quad \beta \|\varphi_t(t, \cdot)\|_{C^\alpha(\partial \mathbb{D})} + \gamma \sum_{k=1}^{2\kappa+1} |\alpha_k'(t)| \\ &\quad + \delta \|b_1 w_{\overline{z}} + b_2 \overline{w_{\overline{z}}} + b_3 w_z + b_4 \overline{w_z} + c_1 w + c_2 \overline{w} + d\|_{L_p(\overline{\mathbb{D}})} \\ &\leq \quad \beta \|\varphi_t(t, \cdot)\|_{C^\alpha(\partial \mathbb{D})} + \gamma \sum_{k=1}^{2\kappa+1} |\alpha_k'(t)| \\ &\quad + \delta K \|w(t, \cdot)\|_{W_p^1(\mathbb{D})} + \delta \|d(t, \cdot)\|_{L_p(\overline{\mathbb{D}})} \ . \end{aligned}$$

Taking (4.1.3) into account w can be represented as

$$w(t, z) = d_0(z) + \int_0^t w_\tau(\tau, z) d\tau$$

so that by Lemma 30

$$\|w(t,\cdot)\|_{W_p^1(\mathbb{D})} \leq \|d_0\|_{W_p^1(\mathbb{D})} + M \left(\int_0^t \|w_\tau(\tau,\cdot)\|^p_{W_p^1(\mathbb{D})} d\tau \right)^{1/p} . \tag{4.1.5}$$

Inserting this on the right–hand side of the preceding estimate taking the pth power and observing, see Lemma 3,

$$(\sigma_1 + \sigma_2)^p \leq 2^{p-1}(\sigma_1^p + \sigma_2^p)$$

for $0 \leq \sigma_1, \sigma_2, 1 \leq p$, then

$$\begin{aligned} \|w_t(t,\cdot)\|^p_{w_p^1(\mathbb{D})} \quad \leq \quad & 2^{p-1}\Bigg\{ \beta\|\varphi_t(t,\cdot)\|_{C^\alpha(\partial\mathbb{D})} + \gamma \sum_{k=1}^{2\kappa+1} |\alpha_k'(t)| \\ & + \delta\|d(t,\cdot)\|_{L_p(\overline{\mathbb{D}})} + \delta K \|d_0\|_{W_p^1(\mathbb{D})} \Bigg\}^p \\ & + 2^{p-1}(\delta K M)^p \int_0^t \|w_\tau(\tau,\cdot)\|^p_{W_p^1(\mathbb{D})} d\tau . \end{aligned}$$

Applying now Lemma 31 thus

$$\begin{aligned} \|w_t(t,\cdot)\|_{W_p^1(\mathbb{D})} \quad \leq \quad & 2^{1-1/p}\Bigg\{ \beta\|\varphi_t(t,\cdot)\|_{C^\alpha(\partial\mathbb{D})} + \gamma \sum_{k=1}^{2\kappa+1} |\alpha_k'(t)| \\ & + \delta\|d(t,\cdot)\|_{L_p(\overline{\mathbb{D}})} + \delta K \|d_0\|_{W_p^1(\mathbb{D})} \Bigg\} e^{\frac{1}{p}2^{p-1}(\delta K M)^p t} \end{aligned}$$

so that

$$\begin{aligned} \|w_t(t,\cdot)\|_{0,W_p^1(\mathbb{D})} \quad \leq \quad & 2^{1-1/p} e^{2^{p-1}(\delta K M)^p T} \Bigg\{ \beta\|\varphi\|_{1,C^\alpha(\partial\mathbb{D})} + \gamma \sum_{k=1}^{2\kappa+1} \|\alpha_k\|_{C^1(I)} \\ & \delta\|d\|_{0,L_p(\overline{\mathbb{D}})} + \delta K \|d_0\|_{W_p^1(\mathbb{D})} \Bigg\} . \end{aligned}$$

Using again (4.1.5) the a priori estimate (4.1.4) follows.

(2) Existence. In order to prove existence of a solution successive approximation is used. Let $w_0(t,z) = d_0(z)$ and $w_{n+1}(t,z)$ for $n \in N_0$ be a solution to

$$\begin{aligned} w_{n+1t\bar z} &+ \mu_1 w_{n+1tz} + \mu_2 \overline{w_{n+1tz}} \\ &+ a_1 w_{n+1t} + a_2 \overline{w_{n+1t}} + b_1 w_{n\bar z} + b_2 \overline{w_{n\bar z}} + b_3 w_{nz} + b_4 \overline{w_{nz}} \\ &+ c_1 w_n + c_2 \overline{w_n} + d = 0 \quad \text{in } I \times \boldsymbol{D}\,, \\ \operatorname{Re}\{\overline{\lambda(z)} w_{n+1}(t,z)\} &= \varphi(t,z) + h(t,z) \quad \text{in } I \times \partial \boldsymbol{D}\,, \\ \operatorname{Im}\{\overline{\lambda(z)} w_{n+1}(t,z)\} &= \alpha_k(t)\,,\ 1 \le k \le 2\kappa+1, \quad \text{if } 0 \le \kappa, \text{ in } I\,, \\ w_{n+1}(0,z) = d_0(z) &\quad \text{in } \boldsymbol{D}\,. \end{aligned}$$

Differentiating the boundary and side conditions with respect to t Lemma 29 shows that $w_{n+1t} \in W_p^1(\boldsymbol{D})$ exists and is uniquely given. Thus by the initial condition $w_{n+1} \in C^1(I; W_p^1(\boldsymbol{D}))$ is unique. By the a priori estimate in Lemma 29

$$\begin{aligned} \|w_{n+1t}(t,\cdot)\|_{W_p^1(\boldsymbol{D})} &\le \beta \|\varphi_t(t,\cdot)\|_{C^\alpha(\partial \boldsymbol{D})} + \gamma \sum_{k=1}^{2\kappa+1} |\alpha_k'(t)| \\ &\quad + \delta \|d(t,\cdot)\|_{L_p(\overline{\boldsymbol{D}})} + \delta K \|w_n(t,\cdot)\|_{W_p^1}(\boldsymbol{D})\,, \end{aligned}$$

where the constants are independent of n. As in step (1) using Lemma 3 form here

$$\begin{aligned} \|w_{n+1t}(t,\cdot)\|_{W_p^1(\boldsymbol{D})}^p &\le 2^{p-1} \Big\{ \beta \|\varphi\|_{1,C^\alpha(\partial \boldsymbol{D})} + \gamma \sum_{k=1}^{2\kappa+1} \|\alpha_k\|_{C^1(I)} \\ &\quad + \delta \|d\|_{0,L_p(\overline{\boldsymbol{D}})} + \delta K \|d_0\|_{W_p^1(\boldsymbol{D})} \Big\}^p \\ &\quad + 2^{p-1} (\delta K M)^p \int_0^t \|w_{n\tau}(\tau,\cdot)\|_{W_p^1(\boldsymbol{D})}^p d\tau\,, \end{aligned}$$

where within the brackets the supremum-norms have been introduced. By induction

$$\begin{aligned} \|w_{n+1t}(t,\cdot)\|_{W_p^1(\boldsymbol{D})}^p &\le 2^{p-1} \Big\{ \beta \|\varphi\|_{1,C^\alpha(\partial \boldsymbol{D})} + \gamma \sum_{k=1}^{2\kappa+1} \|\alpha_k\|_{C^1(I)} + \delta \|d\|_{0,L_p(\overline{\boldsymbol{D}})} \\ &\quad + \delta K \|d_0\|_{W_p^1(\boldsymbol{D})} \Big\}^p \sum_{\nu=0}^{n} \frac{2^{\nu(p-1)} (\delta K M)^{\nu p} t^\nu}{\nu!} \\ &\quad + \frac{1}{(n+1)!} 2^{(n+1)(p-1)} (\delta K M)^{(n+1)p} t^{n+1} \|d_0\|_{W_p^1(\boldsymbol{D})}^p \end{aligned}$$

or

$$\|w_{n+1}(t,\cdot)\|_{0,W_p^1(\mathbb{D})} \leq 2e^{2^{p-1}\frac{1}{p}(\delta KM)^pT}\left\{\beta\|\varphi\|_{1,C^\alpha(\partial\mathbb{D})} + \gamma\sum_{k=1}^{2\kappa+1}\|\alpha_k\|_{C^1(I)}\right.$$
$$\left. +\delta\|d\|_{0,L_p(\overline{\mathbb{D}})} + \delta K\|d_0\|_{W_1^p(\mathbb{D})}\right\}.$$

By (4.1.5) we again deduce

$$\|w_{n+1}\|_{1,W_p^1(\mathbb{D})} \leq \beta\|\varphi\|_{1,C^\alpha(\partial\mathbb{D})} + \gamma\sum_{k=1}^{2\kappa+1}\|\alpha_k\|_{C^1(I)}$$
$$+\delta_1\|d\|_{0,L_p(\overline{\mathbb{D}})} + \delta_2\|d_0\|_{W_1^p(\mathbb{D})}\,. \tag{4.1.6}$$

Hence, (w_n) is a bounded sequence in $C^1(I;W_p^1(\mathbb{D}))$. In order to prove its convergence, $\omega_n := w_n - w_{n-1}, n \in \mathbb{N}$, is considered. Besides the boundary, side, and initial conditions

$$\operatorname{Re}\{\overline{\lambda(z)}\omega_1(t,z)\} = \varphi(t,z) - \varphi(0,z) + h(t,z) \quad \text{in } I\times\partial\mathbb{D}\,,$$

$$\operatorname{Im}\{\overline{\lambda(z_k)}\omega_1(t_1,z_k)\} = \alpha_k(t) - \alpha_k(0)\,,\ 1\leq k\leq 2\kappa+1, \quad \text{if } 0\leq\kappa, \quad \text{in } I\,,$$

$$\omega_1(0,z) = 0 \quad \text{in } \mathbb{D}\,,$$

and for $n > 1$

$$\operatorname{Re}\{\overline{\lambda(z)}\omega_n(t,z)\} = 0 \quad \text{in } I\times\partial\mathbb{D}\,,$$

$$\operatorname{Im}\{\overline{\lambda(z_k)}\} = 0\,,\ 1\leq k\leq 2\kappa+1, \quad \text{if } 0\leq\kappa, \quad \text{in } I\,,$$

$$\omega_n(0,z) = 0 \quad \text{in } \mathbb{D}\,,$$

the function ω_n for $n\in\mathbb{N}$ in $I\times\mathbb{D}$ satisfies the linear equation

$$\omega_{1t\bar z} + \mu_1\omega_{1tz} + \mu_2\overline{\omega_{1tz}} + a_1\omega_{1t} + a_2\overline{\omega_{1t}}$$
$$+b_1w_{0\bar z} + b_2\overline{w_{0\bar z}} + b_3w_{0z} + b_4\overline{w_{0z}} + c_1w_0 + c_2\overline{w_0} + d = 0$$

and for $n > 1$

$$\omega_{nt\bar z} + \mu_1\omega_{ntz} + \mu_2\overline{\omega_{ntz}} + a_1\omega_{nt} + a_2\overline{\omega_{nt}}$$
$$+b_1\omega_{n-1\bar z} + b_2\overline{\omega_{n-1\bar z}} + b_3\omega_{n-1z} + b_4\overline{\omega_{n-1z}} + c_1\omega_{n-1} + c_2\overline{\omega_{n-1}} = 0\,.$$

Again by the a priori estimate of Lemma 29

$$\begin{aligned}\|\omega_{1t}(t,\cdot)\|_{W_p^1(\mathbb{D})} &\leq \beta\|\varphi_t(t,\cdot)\|_{C^\alpha(\partial\mathbb{D})} + \gamma\sum_{k=1}^{2\kappa+1}|\alpha_k'(t)| + \delta\|d(t,\cdot)\|_{L_p(\overline{\mathbb{D}})} \\ &\quad +\delta K\|d_0\|_{W_p^1(\mathbb{D})} \\ &\leq \beta\|\varphi\|_{1,C^\alpha(\partial\mathbb{D})} + \gamma\sum_{k=1}^{2\kappa+1}\|\alpha_k\|_{C^1(I)} + \delta_1\|d\|_{0,L_p(\overline{\mathbb{D}})} \\ &\quad +\delta_2\|d_0\|_{W_p^1(\mathbb{D})}\end{aligned}$$

and for $n > 1$

$$\|\omega_{nt}(t,\cdot)\|_{W_p^1(\mathbb{D})} \leq \delta K\|\omega_{n-1}\|_{W_p^1(\mathbb{D})} ,$$

or observing $\omega_n(0,z) = 0$ and (4.1.5)

$$\|\omega_{nt}(t,\cdot)\|^p_{W_p^1(\mathbb{D})} \leq (\delta KM)^p\int_0^t\|\omega_{n-1\tau}(\tau,\cdot)\|_{W_p^1(\mathbb{D})}d\tau .$$

By iteration thus

$$\begin{aligned}\|\omega_{nt}(t,\cdot)\|^p_{W_p^1(\mathbb{D})} &\leq \Bigg\{\beta\|\varphi\|_{1,C^\alpha(\partial\mathbb{D})} + \gamma\sum_{k=1}^{2\kappa+1}\|\alpha_k\|_{C^1(I)} \\ &\quad +\delta_1\|d\|_{0,L_p(\overline{\mathbb{D}})} + \delta_2\|d_0\|_{W_p^1(\mathbb{D})}\Bigg\}^p\frac{(\delta KM)^{(n-1)p}t^{n-1}}{(n-1)!} .\end{aligned}$$

Integrating and applying Lemma 30 gives

$$\begin{aligned}\|\omega_n(t,\cdot)\|_{W_p^1(\mathbb{D})} &\leq M\Bigg\{\beta\|\varphi\|_{1,C^\alpha(\partial\mathbb{D})} + \gamma\sum_{k=1}^{2\kappa+1}\|\alpha_k\|_{C^1(I)} \\ &\quad +\delta_1\|d\|_{0,L_p(\overline{\mathbb{D}})} + \delta_2\|d_0\|_{W_p^1(\mathbb{D})}\Bigg\}\left(\frac{(\delta KM)^{np}t^n}{n!}\right)^{1/p} .\end{aligned}$$

These last two estimates give

$$\|\omega_n\|_{1,W_p^1(\mathbb{D})} \leq C\left(\frac{(\delta KM)^{(n-1)p}T^{n-1}}{(n-1)!}\right)^{1/p}$$

with

$$C := M(1+(\delta KM)^p T)^{1/p}\left\{\beta\|\varphi\|_{1,C^\alpha(\partial\mathbb{D})}+\gamma\sum_{k=1}^{2\kappa+1}\|\alpha_k\|_{C^1(I)}\right.$$
$$\left.+\delta_1\|d_0\|_{0,L_p(\overline{\mathbb{D}})}+\delta_2\|d_0\|_{W_p^1(\mathbb{D})}\right\}.$$

From here convergence of (w_n) in $C^1(I;W_p^1(\mathbb{D}))$ follows. Passing to the limit in the equation and the conditions for w_{n+1} shows that $w=\lim\limits_{n\to+\infty} w_n$ is a solution to problem (4.1.1) - (4.1.3).

(3) Uniqueness. Let $w_1, w_2 \in C^1(I;W_p^1(\mathbb{D}))$ be two solutions to (4.1.1) - (4.1.3) then $\omega := w_1 - w_2$ satisfies the homogeneous problem

$$\omega_{t\bar z}+\mu_1\omega_{tz}+\mu_2\overline{\omega_{tz}}+a_1\omega_t+a_2\overline{\omega_t}+b_1\omega_{\bar z}+b_2\overline{\omega_{\bar z}}+b_3\omega_z+b_4\overline{\omega_z}+c_1\omega+C_2\overline{\omega}=0 \text{ in } I\times\mathbb{D}\,,$$

$$\begin{aligned}
\operatorname{Re}\{\overline{\lambda(z)}\omega(t,z)\} &= 0 \quad \text{in } I\times\partial\mathbb{D}\,,\\
\operatorname{Im}\{\overline{\lambda(z_k)}\omega(t,z_k)\} &= 0, \quad 1\le k\le 2\kappa+1, \text{ if } 0\le\kappa, \text{ in } I\,,\\
\omega(0,z) &= 0 \quad \text{in } \mathbb{D}\,,
\end{aligned}$$

and hence by the a priori estimate from step (1) $\|\omega\|_{1,W_p^1(\mathbb{D})}=0$. Hence, w_1 and w_2 are identical.

Remark The assumption $p(1-\alpha)\le 1$ is unnecessary . In case when $1<2\alpha<2<p$ are arbitrary the estimation (4.1.4) holds if the norm on the left-hand side is replaced by $\|w\|_{1,W^1_{p_0}(\mathbb{D})}$ with any $p_0, 2<p_0\le\min\{p,1/(1-\alpha)\}$. The considerations hold too for any simply connected bounded smooth domain D. Quasilinear as well as nonlinear pseudoparabolic equations were investigated in [Plus87], [Dai90] and [Beda92], see also [Bege85a,b] and [Begi78] where the RIEMANN problem is solved, or [Bege93], chap. VIII, §4.

4.2 Initial boundary value problem for a composite type system

Systems of first order partial differential equations in two real variables of composite type were independently from one another at first introduced by A. DZHURAEV and Ch. VIDIC, see [Dzhu72], [Vidi69]. A system

$$\sum_{\nu=1}^{n} a_{\mu\nu}u_{\nu x}+b_{\mu\nu}u_{\nu y}=f_\mu(x,y,u_1,\cdots,u_n)\,,\ 1\le\mu\le n$$

of n first order partial differential equations with real coefficients, real variables and real unknowns is called elliptic if

$$A(\kappa) := \det[a_{\mu\nu}\kappa + b_{\mu\nu}] \ , \ \det[a_{\mu\nu}] \neq 0 \ ,$$

does not vanish for real κ. It is called hyperbolic if $A(\kappa)$ has n real zeroes different from each other and it is called of composite type if $A(\kappa)$ has real as well as non real roots. Obviously, the system can be elliptic only if n is even. The simplest cases of composite type systems occur if $n = 3$ and $n = 4$. They are extensively studied in [Dzhu72]. [Vidi69], [Bege77,79] studied the case $n = 3$, [Gisc79] the case $2n + 1$ where $A(\kappa)$ has one real and n complex roots, see also [Begi93] chap. VI, §3. For further references see [Dzhu72].

A motivation for considering composite type systems is the fact that any elliptic second order equation can at least be reduced to a first order system of composite type of three equations. The second order equation

$$a\phi_{xx} + 2b\phi_{xy} + c\phi_{yy} + f(x, y, \phi, \phi_x, \phi_y) = 0$$

is called elliptic if $0 < ac - b^2$ where without loss of generality $0 < a$. Introducing

$$u := \phi_y \ , \ v := \phi_x$$

the equation can be written as the first order system

$$\begin{aligned} &\phi_x - v = 0 \ , \\ &u_x - v_y = 0 \ , \\ &av_x + 2bv_y + cu_y + f(x, y, \phi, v, u) = 0 \ . \end{aligned}$$

Here

$$A(\kappa) := \begin{vmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 1 \\ 0 & -c & a\kappa - 2b \end{vmatrix} = a\kappa \left[\left(\kappa - \frac{b}{a} \right)^2 + \frac{ac - b^2}{a^2} \right]$$

which obviously only has the real zero $\kappa = 0$. In case f is independent of ϕ only the last two equations of the first order system have to be solved. They form an elliptic first order system. Having solved them for u and v then ϕ is determined up to an additive constant locally or in simply connected domains by

$$\int_{(x_0,y_0)}^{(x,y)} \{v dx + u dy\} \ .$$

This curve integral is path-independent because the integrability condition is just $u_x - v_y = 0$.

In this section only the case $n = 3$ will be investigated. The system

$$\sum_{\nu=1}^{3} a_{\mu\nu}u_{\nu x} + b_{\mu\nu}u_{\nu y} = f_\mu(x, y, u_1, u_2, u_3) , \tag{4.2.1}$$

in matrix form written as $Au_x + Bu_y = f$ where

$$D(\kappa) := \det[a_{\mu\nu}\kappa + b_{\mu\nu}]_{0\leq\mu,\nu\leq 3} = 0$$

has the only real root κ_3 and the pair of complex conjugate roots $\kappa_0 := \kappa_1 + i\kappa_2, \overline{\kappa_0} = \kappa_1 - i\kappa_2$ with real κ_1, κ_2 and $\kappa_2 \neq 0$. The eigenvectors associated to $\kappa_0, \overline{\kappa_0}, \kappa_3$ say $c_0, \overline{c_0}, c_3$ are linearly independent over $I\!R$. Writing them as row vectors the respective column vectors form a regular matrix $C^T = [c_0^T, \overline{c}_0^T, c_3^T]$ and satisfy

$$(A^T\kappa_\mu + B^T)c_\mu^T = 0 \ , \ \mu = 0, 3 \ .$$

Here the upper index T denotes transposition of matrices. The equation for $\overline{c_0}^T$ is the complex conjugate of that for c_0^T. Hence,

$$A^T C^T \begin{bmatrix} \kappa_0 & 0 & 0 \\ 0 & \overline{\kappa_0} & 0 \\ 0 & 0 & \kappa_3 \end{bmatrix} + B^T C^T = 0$$

or

$$\begin{bmatrix} \kappa_0 & 0 & 0 \\ 0 & \overline{\kappa_0} & 0 \\ 0 & 0 & \kappa_3 \end{bmatrix} CA + CB = 0 \ .$$

We introduce the new unknown $U := CAu$ the components of which are

$$U_1 = c_0 Au \ , \ U_2 = \overline{c_0}Au \ , \ U_3 = c_3 Au \ .$$

Obviously, $U_2 = \overline{U_1}$ and $U_3 = \overline{U_3}$. Moreover, $u = A^{-1}C^{-1}U$. Multiplying the system for u in matrix form from the left by C gives

$$CAu_x + CBu_y = Cf$$

or in the new unknown U

$$U_x + CBA^{-1}C^{-1}U_y = Cf + [(CA)_x + CBA^{-1}C^{-1}(CA)_y]A^{-1}C^{-1}U$$

that is

$$U_x - \begin{bmatrix} \kappa_0 & 0 & 0 \\ 0 & \overline{\kappa_0} & 0 \\ 0 & 0 & \kappa_3 \end{bmatrix} U_y = Cf + \left[(CA)_x - \begin{bmatrix} \kappa_0 & 0 & 0 \\ 0 & \overline{\kappa_0} & 0 \\ 0 & 0 & \kappa_3 \end{bmatrix} (CA)_y\right] A^{-1}C^{-1}U =: \tilde{f} \ .$$

Because the second component of this system is the complex conjugate of the first the system is reduced to a complex and a real scalar equation,

$$U_{1x} - \kappa_0 U_{1y} = \widetilde{f}_1 \,, \; U_{3x} - \kappa_3 U_{3y} = \widetilde{f}_3 \,. \tag{4.2.2}$$

Introducing complex variables $z := x + iy, \overline{z} = x - iy$ the first equation can be written as

$$U_{1\overline{z}} + \frac{1 - i\kappa_0}{1 + i\kappa_0} U_{1z} = \frac{1}{1 + i\kappa_0} \widetilde{f}_1$$

where $\mu := \dfrac{1 - i\kappa_0}{1 + i\kappa_0} \neq 1$. We have $|\mu| < 1$ if $\kappa_2 < 0$ and $1 < |\mu|$ if $0 < \kappa_2$. This equation therefore is elliptic and in fact for $\kappa_2 < 0$ a BELTRAMI equation. For $0 < \kappa_2$ it can be written as a BELTRAMI equation, too by introducing the complex variables $z := x - iy, \overline{z} = x + iy$, in other words just by reversing the roles of z and $\overline{z}$. In order to rewrite the second equation in (4.2.2) as an equation for a directional derivative of U_3 the ordinary differential equation

$$\frac{dy}{dx} + \kappa_3(x, y) = 0$$

is considered. Its solutions are the real characteristics of system (4.2.1). Because the reduction of (4.2.1) to the normal form (4.2.2) is only possible if the coefficients are differentiable their differentiability is assumed. Thus κ_3 is differentiable too and even continuous differentiability is assumed. Then κ_3 is LIPSCHITZ continuous at least on compact subsets so that initial value problems for this ordinary differential equation are uniquely solvable. Consider a characteristic curve γ. If t is some parameter for this curve the direction on it is given by $\left(\dfrac{dx}{dt}, \dfrac{dy}{dt}\right)$. For $t = x$ this vector is $(1, -\kappa_3(x, y))$. Rewriting this vector in complex form and normalizing it gives $\dfrac{1 - i\kappa_3}{\sqrt{1 + \kappa_3^2}}$ as the direction on γ. If this direction is denoted by σ then $\cos(\sigma, x) = \dfrac{1}{\sqrt{1 + \kappa_3^2}}, \cos(\sigma, y) = \dfrac{-\kappa_3}{\sqrt{1 + \kappa_3^2}}$ are the direction cosines and the directional derivative is

$$\frac{\partial}{\partial \sigma} = \cos(\sigma, x) \frac{\partial}{\partial x} + \cos(\sigma, y) \frac{\partial}{\partial y} \,.$$

Therefore the second equation in (4.2.2) can be written as

$$\frac{\partial U_3}{\partial \sigma} = \frac{1}{\sqrt{1 + \kappa_3^2}} \widetilde{f}_3 \,.$$

Under some geometrical assumptions on the domain in this normal form systems of composite type of three equations are handled in [Vidi68],[Bege79]. There is another transformation of (4.2.2) by a transformation of the independent variables which maps the real characteristics onto lines parallel to the imaginary axis. This can be achieved at least locally i.e. in the neighborhood of any point in the domain under consideration. The first equation remains of the same kind with the new variables and thus again may be written as a BELTRAMI equation. To do this let $\xi(x,y) = const.$ be a parameter-free representation of a real characteristic of (4.2.1), that is the solution $y = y(x)$ to this implicit equation satisfying $\xi_y \neq 0$ is a solution to our ordinary differential equation and hence

$$\xi_x - \kappa_3 \xi_y = 0 .$$

Choose an arbitrary continuously differentiable function $\eta(x,y)$ such that the JACOBIan

$$J := \left| \frac{d(\xi,\eta)}{d(x,y)} \right| = \xi_x \eta_y - \xi_y \eta_x > 0 .$$

Then $(x,y) \to (\xi,\eta)$ is a diffeomorphism such that

$$\begin{aligned} \frac{\partial}{\partial x} &= \xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta} , \; \frac{\partial}{\partial y} = \xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta} , \\ J\frac{\partial}{\partial \xi} &= \eta_y \frac{\partial}{\partial x} - \eta_x \frac{\partial}{\partial y} , \; J\frac{\partial}{\partial \eta} = \xi_x \frac{\partial}{\partial y} - \xi_y \frac{\partial}{\partial x} = -\xi_y \left(\frac{\partial}{\partial x} - \kappa_3 \frac{\partial}{\partial y} \right) . \end{aligned}$$

With this change of variables (4.2.2) becomes

$$(\xi_x - \kappa_0 \xi_y) U_{1\xi} + (\eta_x - \kappa_0 \eta_y) U_{1\eta} = \widetilde{f}_1 \, , \; (\xi_x - \kappa_3 \xi_y) U_{3\xi} + (\eta_x - \kappa_3 \eta_y) U_{3\eta} = \widetilde{f}_3$$

or

$$U_{1\xi} + \frac{\eta_x - \kappa_0 \eta_y}{(\kappa_3 - \kappa_0)\xi_y} U_{1\eta} = \frac{\widetilde{f}_1}{(\kappa_3 - \kappa_0)\xi_y} \, , \; U_{3\eta} = \frac{\widetilde{f}_3}{\eta_x - \kappa_3 \eta_y} . \tag{4.2.3}$$

Note that

$$J = \xi_y (\kappa_3 \eta_y - \eta_x) > 0 .$$

Treating the first equation as that from (4.2.2) by introducing the complex variables $\zeta := \xi + i\eta, \overline{\zeta} = \xi - i\eta$, again a BELTRAMI equation is attained. This in fact is the case because the coefficient is not real. We have

$$|\kappa_3 - \kappa_0|^2 \xi_y \frac{\eta_x - \kappa_0 \eta_y}{(\kappa_3 - \kappa_0)\xi_y} = \kappa_3 \eta_x + |\kappa_0|^2 \eta_y - \kappa_0 \kappa_3 \eta_y - \overline{\kappa_0} \eta_x$$

and therefore with $\kappa_0 = \kappa_1 + i\kappa_2, \kappa_2 \neq 0$,

$$|\kappa_3 - \kappa_0|^2 \mathrm{Im} \frac{\eta_x - \kappa_0 \eta_y}{\kappa_3 - \kappa_0} = \kappa_2 (\eta_x - \kappa_3 \eta_y) \neq 0 .$$

The reason that this transformation in general is possible only locally is that $\xi_y \neq 0$ may hold only in some neighborhood of a point (x_0, y_0) of the domain. The normal form (4.2.3) is due to DZHURAEV, see [Dzhu72], [Bege93]. Methodically there is no difference in treating initial boundary value problems for systems in form (4.2.2) or (4.2.3). We will for simplicity deal with (4.2.3) in complex form and again will consider the unit disc $\mathbb{D}$, see [Bewz91a,b].

Lemma 32. *Let* $c_2, d_2 \in C^\alpha(\overline{\mathbb{D}}; \mathbb{R})$, $\Gamma := \{z = x + iy : |z| = 1, y \leq 0\}$ *and* $\psi \in C^\alpha(\Gamma; \mathbb{R}), 0 < \alpha < 1$. *Then the solution to the initial value problem*

$$\omega_y + c_2\omega + d_2 = 0 \quad \text{in } \mathbb{D} ,$$

$$\omega = \psi \quad \text{on } \Gamma$$

is uniquely given in $C^{\alpha_2}(\overline{\mathbb{D}}; \mathbb{R}) \cap C^1_y(\overline{\mathbb{D}}; \mathbb{R})$ *satisfying*

$$C_*(\omega; \overline{\mathbb{D}}) := C_{\alpha_2}(\omega; \overline{\mathbb{D}}) + C_0(\omega_y; \overline{\mathbb{D}}) \leq \beta_2 C_\alpha(\psi; \Gamma) + \delta_2 C_\alpha(d_2; \overline{\mathbb{D}}) , \tag{4.2.4}$$

where $\alpha_2 := \min\left\{\frac{1}{2}, \alpha\right\}$ *and* β_2 *and* δ_2 *are nonnegative constants depending on* α *and some upper bound* K *for* $C_\alpha(c_2; \overline{\mathbb{D}})$.

Proof. As is known from the theory of ordinary differential equations, see any text-book, the solution to the initial value problem is

$$\begin{aligned} \omega(z) \;=\; & \Bigg[\psi(x - i\sqrt{1-x^2}) \\ & - \int\limits_{-\sqrt{1-x^2}}^{y} d_2(x+it) \exp\Bigg\{ \int\limits_{-\sqrt{1-x^2}}^{t} c_2(x+i\tau)d\tau \Bigg\} dt \Bigg] \exp\Bigg\{ - \int\limits_{-\sqrt{1-x^2}}^{y} c_2(x+it)dt \Bigg\} . \end{aligned}$$

This can be easily verified by direct calculations. From here we see

$$\begin{aligned} C_0(\omega; \overline{\mathbb{D}}) &\leq \left\{C_0(\psi; \Gamma) + 2C_0(d_2; \overline{\mathbb{D}})\right\} e^{2C_0(c_2; \overline{\mathbb{D}})} , \\ C_0(\omega_y; \overline{\mathbb{D}}) &\leq C_0(c_2; \overline{\mathbb{D}}) C_0(\omega; \overline{\mathbb{D}}) + C_0(d_2; \overline{\mathbb{D}}) \\ &\leq C_0(c_2; \overline{\mathbb{D}}) e^{2C_0(c_2; \overline{\mathbb{D}})} C_0(\psi; \Gamma) + \left(1 + 2C_0(c_2; \overline{\mathbb{D}}) e^{2C_0(c_2; \overline{\mathbb{D}})}\right) C_0(d_2; \overline{\mathbb{D}}). \end{aligned}$$

To show ω to be HÖLDER-continuous we observe that with $f \in C^\alpha(\overline{\mathbb{D}})$ the integral

$$F(z) := \int\limits_{-\sqrt{1-x^2}}^{y} f(x+it)dt \; , \; z = x + iy \in \overline{\mathbb{D}} ,$$

is HÖLDER-continuous with exponent $\alpha_2 := \min\{1/2, \alpha\}$. We verify this by estimating

$$F(z) - F(z_0) = \int\limits_{-\sqrt{1-x^2}}^{-\sqrt{1-x_0^2}} f(x+it)dt + \int\limits_{-\sqrt{1-x_0^2}}^{y} [f(x+it) - f(x_0+it)]dt + \int\limits_{y_0}^{y} f(x_0+it)dt\,.$$

Assuming f is vanishing identically outside $\overline{\mathbb{D}}$ we do not have to distinguish different cases assuring $x_0 + iy, x - i\sqrt{1-x_0^2} \in \mathbb{D}$.

$$\begin{aligned}|F(z) - F(z_0)| &\leq C_0(f;\mathbb{D})\left[|\sqrt{1-x^2} - \sqrt{1-x_0^2}| + |y - y_0|\right] \\ &\quad + H_\alpha(f)|x - x_0|^\alpha|y + \sqrt{1-x_0^2}| \\ &\leq 2\sqrt{2}C_0(f;\mathbb{D})|z - z_0|^{1/2} + 2H_\alpha(f)|z - z_0|^\alpha \\ &\leq 6C_\alpha(f;\mathbb{D})|z - z_0|^{\min\{\frac{1}{2},\alpha\}}\,.\end{aligned}$$

Applying this estimate to each of the factors in the integral representation for ω shows $\omega \in C^{\alpha_2}(\overline{\mathbb{D}};\mathbb{R})$ and

$$C_*(\omega;\overline{\mathbb{D}}) := C_{\alpha_2}(\omega;\overline{\mathbb{D}}) + C_0(\omega_y;\mathbb{D}) \leq \beta_2 C_\alpha(\psi;\Gamma) + \delta_2 C_\alpha(d_2;\overline{\mathbb{D}})$$

where β_2 and δ_2 are constants depending on the upper bound K for $C_\alpha(c_2;\mathbb{D})$ and on α.

Lemma 33. *Let the coefficients of the generalized* BELTRAMI *equation*

$$w_{\bar z} + \mu_1 w_z + \mu_2 \overline{w_z} + a_1 w + b_1 \overline{w} + d_1 = 0 \quad \text{in } \mathbb{D}$$

and the boundary and side conditions

$$\operatorname{Re}\{\overline{\lambda} w\} = \varphi + h \quad \text{on } \partial\mathbb{D}\,,$$

$$\operatorname{Im}\{\overline{\lambda(z_k)} w(z_k)\} = a_k\,,\ 1 \leq k \leq 2\kappa + 1\,,\ \text{if } 0 \leq \kappa\,,$$

satisfy μ_1, μ_2 *measurable in* $\overline{\mathbb{D}}, |\mu_1(z)| + |\mu_2(z)| \leq q_0 < 1$ *in* $\mathbb{D}, a_1, b_1, d_1 \in L_{p_0}(\overline{\mathbb{D}};\mathbb{C})$, $\|a_1\|_{p_0} + \|b_1\|_{p_0} \leq K, \lambda \in C^{\alpha_1}(\partial\mathbb{D};\mathbb{C}), |\lambda(z)| = 1$ *on* $\partial\mathbb{D}, \varphi \in C^{\alpha_1}(\partial\mathbb{D};\mathbb{R}), z_k \in \partial\mathbb{D}$, $z_k \neq z_\ell$ *for* $k \neq \ell, a_k \in \mathbb{R}, 1 \leq k, \ell \leq 2\kappa + 1, \kappa := \operatorname{ind}\lambda, h(z) := \sum\limits_{k=\kappa+1}^{-\kappa-1} h_k z^k, h_{-k} = \overline{h_k}, |k| \leq -\kappa - 1$, *if* $\kappa < 0, h = 0$, *if* $0 \leq \kappa$. *Then the solution* w *is uniquely given in* $W^1_{\tilde p}(\mathbb{D})$ *satisfying*

$$C_\alpha(w;\mathbb{D}) + \|w_z\|_{\tilde p} + \|w_{\bar z}\|_{\tilde p} \leq \beta_1 C_{\alpha_1}(\varphi;\partial\mathbb{D}) + \gamma_1 \sum_{k=1}^{2\kappa+1} |a_k| + \delta_1 \|d_1\|_{p_0}\,. \quad (4.2.5)$$

Here $\tilde{p} := p = p_0$ *if* $0 \leq \kappa$ *and* $2 < \tilde{p} < p \leq p_0, \tilde{p} := \dfrac{p^2}{2(p-1)}, 4 < p_0, q_0\Lambda_p < 1, 1 < 2\alpha_1\alpha_0$ *if* $\kappa < 0$, *where* $\alpha_0 := \dfrac{p_0 - 2}{p_0}$, *and* $0 < \alpha_1 := \alpha \leq \alpha_0$ *if* $0 \leq \kappa, 0 < \alpha_1 := \alpha\alpha_0^{-2} < 1$ *if* $\kappa < 0$. *The nonnegatvie constants* $\beta_1, \gamma_1, \delta_1$ *depend on all the constants especially on an upper bound for* $\|a_1\|_{p_0} + \|b_1\|_{p_0}$ *but not on the particular coefficients up to* λ *and not on the solution* $w; \gamma_1 = 0$ *formally if* $\kappa < 0$.
This result is just a conclusion of Theorems 38 and 39, see also section 3.5.

Theorem 46. *Let the coefficients of the composite system*

$$\begin{aligned} w_{\bar{z}} + \mu_1 w_z + \mu_2 \overline{w_z} + a_1 w + b_1 \overline{w} + c_1\omega + d_1 = 0\,, \\ \omega_y + a_2 w + \overline{a_2 w} + c_2\omega + d_2 = 0 \end{aligned} \quad \text{in } \mathbb{D} \tag{4.2.6}$$

and of the boundary, side, and initial conditions

$$\begin{aligned} &\operatorname{Re}\{\overline{\lambda} w\} = \varphi + h \quad \text{on } \partial\mathbb{D}\,, \\ &\operatorname{Im}\{\overline{\lambda(z_k)} w(z_k)\} = a_k\,,\ 1 \leq k \leq 2\kappa + 1, \quad \text{if } 0 \leq \kappa\,, \\ &\omega = \psi \quad \text{on } \Gamma \end{aligned} \tag{4.2.7}$$

besides the conditions in Lemma 32 and 33 satisfy

$$c_1 \in L_{p_0}(\overline{\mathbb{D}}; \mathbb{R}), a_2 \in C^\alpha(\overline{\mathbb{D}}; \mathbb{C}), \|c_1\|_{p_0} \leq \eta_1, C_\alpha(a_2; \mathbb{D}) \leq \eta_2$$

where η_1, η_2 *are nonnegative constants satisfying with* δ_1, δ_2 *appearing in the a priori estimates (4.2.4) and (4.2.5)*

$$2\delta_1\delta_2\eta_1\eta_2 < 1\,.$$

Then this problem is uniquely solvable in $W_p^1(\mathbb{D}; \mathbb{C}) \times C^*(\overline{\mathbb{D}}; \mathbb{R})$ *satisfying the a priori estimate*

$$C_\alpha(w; \overline{\mathbb{D}}) + C_{\alpha_2}(\omega; \overline{\mathbb{D}}) + \|w_z\|_{\tilde{p}} + \|w_{\bar{z}}\|_{\tilde{p}} + C_0(\omega_y; \overline{\mathbb{D}}) \tag{4.2.8}$$

$$\leq \beta_1 C_{\alpha_1}(\varphi; \partial\mathbb{D}) + \beta_2 C_\alpha(\psi; \Gamma) + \gamma \sum_{k=1}^{2\kappa+1} |a_k| + \delta_1\|d_1\|_{p_0} + \delta_2 C_\alpha(d_2; \overline{\mathbb{D}})\,.$$

Here instead of $W_p^1(\mathbb{D})$ *the notation* $W_p^1(\mathbb{D}; \mathbb{C})$ *is used and* $C^*(\overline{\mathbb{D}}; \mathbb{R})$ *is the set of real functions in* $\overline{\mathbb{D}}$ *being continuously differentiable with respect to* y *and in* $C^{\alpha_2}(\mathbb{D}; \mathbb{R})$.

Proof.

i. From (4.2.4) and (4.2.5) $C_*(\omega; \mathbb{D})$ and

$$\|w\|_* := C_\alpha(w; \mathbb{D}) + \|w_z\|_{\tilde{p}} + \|w_{\bar{z}}\|_{\tilde{p}}$$

are seen to satisfy

$$\|w\|_* \le \beta_1 C_{\alpha_1}(\varphi;\partial\mathbb{D}) + \gamma_1 \sum_{k=1}^{2\kappa+1} |a_k| + \delta_1 \|d_1\|_{p_0} + \delta_1 \|c_1\|_{p_0} C_0(\omega;\overline{\mathbb{D}}) \, ,$$

$$C_0(\omega;\overline{\mathbb{D}}) \le C_*(\omega;\overline{\mathbb{D}}) \le \beta_2 C_\alpha(\psi;\Gamma) + \delta_2 C_\alpha(d_2;\overline{\mathbb{D}}) + 2\delta_2 C_\alpha(a_2;\overline{\mathbb{D}}) C_\alpha(w;\overline{\mathbb{D}}) \, .$$

Inserting the last into the first inequality shows

$$\begin{aligned} \|w\|_* \ \le\ & \beta_1 C_{\alpha_1}(\varphi;\partial\mathbb{D}) + \gamma_1 \sum_{k=1}^{2\kappa+2} |a_k| + \delta_1 \|d_1\|_{p_0} + \delta_1 \eta_1 \beta_2 C_\alpha(\psi;\Gamma) \\ & + \delta_1 \delta_2 \eta_1 C_\alpha(d_2;\overline{\mathbb{D}}) + \delta_1 \delta_2 \eta_1 \eta_2 \|w\|_* \end{aligned}$$

and the first into the last

$$\begin{aligned} C_*(\omega;\overline{\mathbb{D}}) \ \le\ & \beta_2 C_\alpha(\psi;\Gamma) + \delta_2 C_\alpha(d_2;\overline{\mathbb{D}}) + \delta_2 \eta_2 \beta_1 C_{\alpha_1}(\varphi;\partial\mathbb{D}) \\ & + \delta_2 \eta_2 \gamma_1 \sum_{k=1}^{2\kappa+1} |a_k| + \delta_1 \delta_2 \eta_2 \|d_1\|_{p_0} + \delta_1 \delta_2 \eta_1 \eta_2 C_*(\omega;\overline{\mathbb{D}}) \, . \end{aligned}$$

Because of $\delta_1 \delta_2 \eta_1 \eta_2 < 1$ this gives

$$\|w\|_* \le \beta_I C_{\alpha_1}(\varphi;\partial\mathbb{D}) + \beta_I' C_\alpha(\psi;\Gamma) + \gamma_I \sum_{k=1}^{2\kappa+1} |a_k| + \delta_I \|d_1\|_{p_0} + \delta_I' C_\alpha(d_2;\overline{\mathbb{D}})$$

and

$$\begin{aligned} C_*(\omega;\overline{\mathbb{D}}) \ \le\ & \beta_{II} C_{\alpha_1}(\varphi;\partial\mathbb{D}) + \beta_{II}' C_\alpha(\psi;\Gamma) \\ & + \gamma_{II} \sum_{k=1}^{2\kappa+1} |a_k| + \delta_{II} \|d_1\|_{p_0} + \delta_{II}' C_\alpha(d_2;\overline{\mathbb{D}}) \end{aligned}$$

and hence (4.2.8).
From this estimate again the uniqueness of the solution follows at once.

ii. The existence of a solution is shown with the method used in the proof of Theorem 39 already. Instead of (4.2.6) we consider the composite system

$$\begin{aligned} & w_{\bar z} + \mu_1 w_z + \mu_2 \overline{w_z} + a_1 w + b_1 \overline{w} + t c_1 \omega + d_1 = 0 \, , \\ & \omega_y + t a_2 w + t \overline{a_2}\, \overline{w} + c_2 \omega + d_2 = 0 \end{aligned} \tag{4.2.9}$$

in $\mathbb{D}$ with a real parameter $t, 0 \le t \le 1$, together with the conditions (4.2.7). For $t = 0$ the system is uncoupled and hence solvable in $W_p^1(\mathbb{D};\mathbb{C}) \times C^*(\overline{\mathbb{D}};\mathbb{R})$ as follows by Lemmas 32 and 33.

Let the problem be solvable for some $t_0, 0 \le t_0 < 1$, and $(w_0, \omega_0) \in W_p^1(\mathbb{D}; \mathbb{C}) \times C^*(\overline{\mathbb{D}}; \mathbb{R})$ be a solution. With this initial element a sequence of approximative solutions $(w_n, \omega_n) \in W_p^1(\mathbb{D}, \mathbb{C}) \times C^*(\overline{D}; \mathbb{R}), n \in \mathbb{N}_0$, to the problem for some $t, t_0 < t \le 1$ is constructed as solutions to the systems

$$w_{n+1\bar{z}} + \mu_1 w_{n+1z} + \mu_2 \overline{w_{n+1z}} + a_1 w_{n+1} + b_1 \overline{w_{n+1}} + t_0 c_1 \omega_{n+1} + (t-t_0) c_1 \omega_n + d_1 = 0$$

$$\omega_{n+1y} + t_0 a_2 w_{n+1} + t_0 \overline{a_2}\, \overline{w_{n+1}} + c_2 \omega_{n+1} + (t-t_0) a_2 w_n + (t-t_0) \overline{a_2}\, \overline{w_n} + d_2 = 0 \,,$$

in $\mathbb{D}$ satisfying (4.2.7). For $(w_n, \omega_n) \in W_p^1(\mathbb{D}; \mathbb{C}) \times C^*(\overline{D}; \mathbb{R})$ given this problem is solvable for (w_{n+1}, ω_{n+1}) by the Lemmas 32 and 33. Using the notation

$$\|(w, \omega)\|_* := \|w\|_* + C_*(\omega; \mathbb{D}) \,, \ (w, \omega) \in W_p^1(\mathbb{D}; \mathbb{C}) \times C^*(\overline{\mathbb{D}}; \mathbb{R})$$

(w_{n+1}, ω_{n+1}) can be shown to be bounded by some constant independent of n. From (4.2.8) it follows

$$\begin{aligned} \|(w_{n+1}, \omega_{n+1})\|_* \ \le \ & \beta_1 C_{\alpha_1}(\varphi; \partial\mathbb{D}) + \beta_2 C_\alpha(\psi; \Gamma) + \gamma \sum_{k=1}^{2\kappa+1} |a_k| + \delta_1 \|d_1\|_{p_0} \\ & + \delta_2 C_\alpha(d_2; \overline{\mathbb{D}}) + (t-t_0)[\delta_1 \eta_1 + 2\delta_2 \eta_2] \|(w_n, \omega_n)\|_* \,. \end{aligned}$$

Rewriting this inequality as

$$\|(w_{n+1}, \omega_{n+1})\|_* \le M + \delta(t-t_0) \|(w_n, \omega_n)\|_*$$

one can show inductively

$$\begin{aligned} \|(w_{n+1}, \omega_{n+1})\|_* \ \le \ & M \sum_{k=0}^{n} \delta^k (t-t_0)^k + \delta^{k+1}(t-t_0)^{n+1} \|(w_0, \omega_0)\|_* \\ = \ & M \frac{1 - \delta^{n+1}(t-t_0)^{n+1}}{1 - \delta(t-t_0)} + \delta^{n+1}(t-t_0)^{n+1} \|(w_0, \omega_0)\|_* \,. \end{aligned}$$

Let $(t-t_0)$ be so small that $\delta(t-t_0) < 1$, then

$$\|(w_{n+1}, \omega_{n+1})\|_* \le \frac{M}{1 - \delta(t-t_0)} + \|(w_0, \omega_0)\|_* \,.$$

In order to show (w_n, w_n) converges, we apply (4.2.8) to $(u_{n+1}, v_{n+1}) := (w_{n+1} - w_n, \omega_{n+1} - \omega_n), n \in \mathbb{N}_0$, getting

$$\begin{aligned} \|(u_{n+1}, v_{n+1})\|_* \ \le \ & \delta_1 (t-t_0) \eta_1 C_0(v_n; \overline{\mathbb{D}}) + 2\delta_2 (t-t_0) \eta_2 C_\alpha(u_n; \mathbb{D}) \\ \le \ & (\delta_1 \eta_1 + 2\delta_2 \eta_2)(t-t_0) \|(u_n, v_n)\| \,. \end{aligned}$$

Again choosing $0 < t - t_0 < \delta^{-1} = (\delta_1 \eta_1 + 2\delta_2 \eta_2)^{-1}$ the sequence (u_n, v_n) can be seen to converge in $W_p^1(\mathbb{D}; \mathbb{C}) \times C^*(\overline{\mathbb{D}}; \mathbb{R})$, the limit being a solution to system (4.2.9) for $t_0 < t \le 1$. Hence, by the same reasoning as in the proof of Theorem 39 problem (4.2.6), (4.2.7) turns out to be solvable in $W_p^1(\mathbb{D}; \mathbb{C}) \times C^*(\overline{\mathbb{D}}, \mathbb{R})$.

4.3 Entire solution to nonlinear generalized BELTRAMI equations

In the preceding sections and chapters repeatedly a priori estimates were developed for different kinds of equations and boundary conditions. They all serve to solve related nonlinear equations under the same linear or even related nonlinear boundary conditions. Side and initial conditions if appearing may be nonlinear, too, see e.g. [Behi82,83], [Behs80,81,82,83,87], [Bewz91a,b], [Bewe88,89], [Beda92], [Dai90], [Plus87], [Wend78], [Wen80a,b,85a], [Vino58a,b], [Webe90], [Tuts76,78], [Wols72], [Waro70]. But the results concerning nonlinear boundary conditions are minor satisfactory. The nonlinearity is always assumed to be LIPSCHITZ-continuous and - and this is the point - its LIPSCHITZ constant has to be small enough.
The aim of this section is just to introduce the method, a combination of SCHAUDER's imbedding method and the NEWTON approximation procedure providing a constructive method for solving nonlinear problems (see [Wack70]). It can be applied to nonlinear problems related to any of the problems studied in this book. In connection with RIEMANN-HILBERT problems it was at first used in [Wend74], see also [Wend78,79]. In order to concentrate on the equation, we neglect any boundary conditions and just study nonlinear equations in the entire plane. Solutions which exist in the entire complex plane are called entire solutions which does not mean that they are entire functions in the sense of complex function theory. In principal we are thus treating the RIEMANN problem for the nonlinear equation because we have seen that the RIEMANN condition can be always transformed using proper analytic functions to entire solutions to a differential equation of the same kind of course under proper assumptions. And even for entire solutions we have to observe a kind of boundary behaviour namely the asymtotic behaviour at infinity. We thus have to prescribe some growth condition the simplest of which would be to ask the solution to have a certain limit or just be bounded at infinity. We also will allow polynomial growth asking the solution to grow not faster than some fixed power of z. Other growth restrictions as for example functions of finite order or of finite lower order etc. have not yet been studied. The following results are given in [Behi83].

Theorem 47. *Let H be a measurable complex valued function of three complex variables $(z,w,v) \in \mathbb{C}^3$ satisfying*

$$|H(z,w_1,v) - H(z,w_2,v)| \le K(z)|w_1 - w_2|\,,$$

$$|H(z,w,v_1| - H(z,w,v_2)| \le q(z)|v_1 - v_2|\,,$$

where K and q are nonnegative functions. Suppose

$$H(\cdot,0,0), K \in L_{(p,p')}(\mathbb{C}) \cap L_{p,2}(\mathbb{C})\,,\ \frac{2}{1+\varepsilon} < p' < 2 < p < \frac{4}{2-\varepsilon}\,,$$

$$0 \le q(z) \le q_0 < 1\,,\ q(z) = 0(|z|^{-\varepsilon}) \quad \text{as } z \to \infty\,,$$

with some $0 < \varepsilon < 1$ and $q_0 \max\{\Lambda_p, \Lambda_{p'}\} < 1$. Then there is one and only one solution to

$$w_{\bar{z}} = H(z_1 w, w_z) \tag{4.3.1}$$

vanishing at infinity. It satisfies the a priori estimate

$$C_0(w, \mathbb{C}) + \|w_z\|_{(p,p')} + \|w_{\bar{z}}\|_{(p,p')} \leq K_0 ,$$

where

$$K_0 := M(p, p', q_0)\|H(z, 0, 0)\|_{(p,p')} \exp[M(p, p', q_0)\|K\|_{(p,p')}] .$$

Proof.

i. In order to prove existence of a solution a real parameter $t, 0 \leq t \leq 1$, is introduced. With

$$R(z, w, v) := H(z, w, v) - H(z, 0, 0)$$

the equation

$$w_{\bar{z}} = tR(z, w, w_z) + C \tag{4.3.2}$$

is studied for any $C \in L((_{p,p'})(\mathbb{C}) \cap L_{p,2}(\mathbb{C})$. If $C = H(z, 0, 0)$ and $t = 1$ *this* is just the equation to be solved. We are looking for solutions vanishing at infinity. Hence, we can do this in the form

$$w = T\rho \ , \ \rho \in L_{(p,p')}(\mathbb{C}) \cap L_{p,2}(\mathbb{C}) \ ,$$

observing that by Theorem 23 this function vanishes at infinity. For $t = 0$ this is easily done by taking $\rho = C$.
We assume that (4.3.2) is solvable for any $t, 0 \leq t \leq t_0 < 1$, and any function $C \in L_{(p,p')}(\mathbb{C}) \cap L_{p,2}(\mathbb{C})$. Let ρ_0 be a solution of (4.3.2) corresponding to $t = t_0$. We construct a sequence of successive approximations according to the scheme

$$\rho_{n+1} = t_0 R(z, T\rho_{n+1}, \Pi\rho_{n+1}) + (t - t_0)R(z, T\rho_n, \Pi\rho_n) + C \ , \ n \in \mathbb{N}_0 \, .$$

As

$$|R(z, T\rho_n, \Pi\rho_n)| \leq K(z)|T\rho_n| + q(z)|\Pi\rho_n|$$

and since $\rho_n, \Pi\rho_n \in L_{(p,p')}(\mathbb{C})$ and $T\rho_n$ is bounded by the assumptions on K and q it follows via Lemma 16 that $R(z, T\rho_n, \Pi\rho_n) \in L_{(p,p')}(\mathbb{C}) \cap L_{p,2}(\mathbb{C})$. Thus, by the assumption that (4.3.2) is solvable for $t = t_0$ and any $C \in L_{(p,p')}(\mathbb{C}) \cap L_{p,2}(\mathbb{C})$ a solution ρ_{n+1} exists. The difference $r_{n+1} := \rho_{n+1} - \rho_n, n \in \mathbb{N}_0$, satisfies

$$\begin{aligned} r_{n+1} &= t_0\widehat{\mu}(z, T\rho_n, \Pi\rho_{n+1}, \Pi\rho_n)\Pi\, r_{n+1} + t_0\widehat{A}(z, T\rho_{n+1}, T\rho_n, \Pi\rho_{n+1})Tr_{n+1} \\ &\quad +(t - t_0)[R(z, T\rho_n, \Pi\rho_n) - R(z, T\rho_{n-1}, \Pi\rho_{n-1})] \ , \end{aligned}$$

$$\widehat{\mu}(z,w,v_1,v_2) := \begin{cases} \dfrac{R(z,w,v_1)-R(z,w,v_2)}{v_1-v_2} & , \text{ if } v_1 \neq v_2 , \\ 0 & , \text{ if } v_1 = v_2 , \end{cases}$$

$$\widehat{A}(z,w_1,w_2,v) := \begin{cases} \dfrac{R(z,w_1,v)-R(z,w_2,v)}{w_1-w_2} & , \text{ if } w_1 \neq w_2 , \\ 0 & , \text{ if } w_1 = w_2 . \end{cases}$$

Applying the a priori estimate (3.1.2) to Tr_{n+1} observing for $w_1 := T\rho_{n+1}$, $w_2 := T\rho_n$,

$$\begin{aligned} |\widehat{\mu}(z,w_2,w_{1z},w_{2z})| &\leq q(z) \leq q_0 < 1 , \\ |\widehat{A}(z,w_1,w_2,w_{1z})| &\leq K(z) , \end{aligned}$$

we get

$$\|r_{n+1}\|_{(p,p')} \leq M(p,p',q_0)\exp[M(p,p',q_0)\|K\|_{(p,p')}](t-t_0)\|g_n\|_{(p,p')}$$

with

$$g_n := R(z,T\rho_n,\Pi\rho_n) - R(z,T\rho_{n-1},\Pi\rho_{n-1}) .$$

From

$$|g_n| \leq q(z)|\Pi r_n| + K|Tr_n| , \; C_0(Tr_n;\mathbb{C}) \leq M(p,p')\|r_n\|_{(p,p')} ,$$

we see

$$\|g_n\|_{(p,p')} \leq q_0\Lambda_{(p,p')}\|r_n\|_{(p,p')} + \|K\|_{(p,p')}M(p,p')\|r_n\|_{(p,p')} .$$

Inserting this into the preceding inequality leads to

$$\|r_{n+1}\|_{(p,p')} \leq \delta(t-t_0)\|r_n\|_{(p,p')}$$

with some positive constant δ depending on p, p', q_0, and $\|K\|_{(p,p')}$. If $\delta(t-t_0) < 1$ the sequence (ρ_n) converges in $L_{(p,p')}(\mathbb{C})$ to a function ρ. In order to show ρ to satisfy

$$\rho = t\,R(z,T\rho,\Pi\rho) + C$$

we observe

$$\begin{aligned} |\rho - t\,R(z,T\rho,\Pi\rho) - C| &\leq |\rho-\rho_{n+1}| + t|R(z,T\rho_n,\Pi\rho_n) - R(z,T\rho,\Pi\rho)| \\ &\quad + t_0|R(z,T\rho_{n+1},\Pi\rho_{n+1}) - R(z,T\rho_n,\Pi\rho_n)| \\ &\leq |\rho-\rho_{n+1}| + q|\Pi(\rho_n-\rho)| + K|T(\rho_n-\rho)| \\ &\quad + q|\Pi(\rho_{n+1}-\rho_n)| + K(T(\rho_{n+1}-\rho_n)| , \end{aligned}$$

so that

$$
\begin{aligned}
&\|\rho - t\,R(z,T\rho,\Pi\rho) - C\|_{(p,p')} \\
&\qquad \le \|\rho - \rho_{n+1}\|_{(p,p')} + q_0\Lambda_{(p,p')}[\|\rho_{n+1} - \rho_n\|_{(p,p')} + \|\rho - \rho_n\|_{(p,p')}] \\
&\qquad\quad + \|K\|_{(p,p')}M(p,p')[\|\rho_{n+1} - \rho_n\|_{(p,p')} + \|\rho - \rho_n\|_{(p,p')}] \\
&\qquad \le \|\rho - \rho_{n+1}\|_{(p,p')} \\
&\qquad\quad + [q_0\Lambda_{(p,p')} + \|K\|_{(p,p')}M(p,p')][\|\rho_{n+1} - \rho_n\|_{(p,p')} + \|\rho - \rho_n\|_{(p,p')}] \,.
\end{aligned}
$$

Since c $\in L_{p,2}(\mathbb{C})$ and from

$$|R(z,T\rho,\Pi\rho)| \le q|\Pi\rho| + K|T\rho|$$

in view of Lemma 16 we have $R(z,T\rho,\Pi\rho) \in L_{p,2}(\mathbb{C})$, the function ρ belongs to $L_{p,2}(\mathbb{C})$, too. $w = T\rho$ satisfies (4.3.2) and vanishes at infinity. This holds for any t such that $0 \le t - t_0 < \delta^{-1}$. Thus after finitely many steps from $t = 0$ one will end up at a solution to (4.3.2) for $t = 1$ vanishing at infinity.

ii. For a uniqueness proof assume there are two solutions w_1, w_2 to (4.3.1) vanishing at infinity. Then $\omega := w_1 - w_2$ would vanish at infinity and satisfy

$$\omega_{\bar z} = \widehat{\mu}(z, w_2, w_{1z}, w_{2z})\omega_z + \widehat{A}(z, w_1, w_2, w_{1z})\omega \,.$$

This is a linear equation to which Theorem 37 and hence (3.1.2) may be applied showing ω vanishing identically.

iii. The a priori estimate follows at once from (3.1.2) because the solution to (4.3.1) satisfies the quasilinear equation

$$w_{\bar z} = \widehat{\mu}(z, 0, w_z, 0)w_z + \widehat{A}(z, w, 0, w_z)w + H(z, 0, 0) \,.$$

Corollary 9. *The difference between the exact solution w to (4.3.1) and the approximative solution $w_n = T\rho_n, n \in \mathbb{N}_0$, for some $t, 0 \le t_0 \le t \le 1$ can be estimated by*

$$
\begin{aligned}
&C_0(w - w_n; \mathbb{C}) + \|(w - w_n)_z\|_{(p,p')} + \|(w - w_n)_{\bar z}\|_{(p,p')} \\
&\le \gamma K_0 \left[\frac{1 - \gamma^n(t - t_0)^n}{1 - \gamma(t - t_0)}(1 - t) + (1 - t_0)\gamma^n(t - t_0)^n\right] \,,
\end{aligned}
$$

with

$$\gamma := M(p,p',q_0)[q_0 + \|K\|_{(p,p')}]\exp[M(p,p',q_0)\|K\|_{(p,p')}]$$

for $\gamma(t - t_0) < 1$.

Proof. Choosing $C = H(\cdot,0,0)$ in (4.3.2) we abreviate $\omega_n := w - w_n, n \in I\!N_0$. It satisfies

$$\begin{aligned}
\omega_{n+1\bar z} &= (1-t)R(z,w,w_z) + t_0[R(z,w,w_z) - R(z,w_{n+1},w_{n+1z})] \\
&\quad +(t-t_0)[R(z,w,w_z) - R(z,w_n,w_{nz})] \\
&= t_0\widehat{\mu}(z,w_{n+1},w_z,w_{n+1z})\omega_{n+1z} + t_0\widehat{A}(z,w,w_{n+1},w_z)\omega_{n+1} \\
&\quad +(1-t)R(z,w,w_z) + (t-t_0)[R(z,w,w_z) - R(z,w_n,w_{nz})]
\end{aligned}$$

and vanishes at infinity. Again using (3.1.2) we find

$$\|\omega_{n+1}\|_* \le \gamma(1-t)\|w\|_* + (t-t_0)\|\omega_n\|_* ,$$

where for $f \in W^1_{(p,p')}(\mathbb{C}) = W^1_p(\mathbb{C}) \cap W^1_{p'}(\mathbb{C}), f(\infty) = 0,$

$$\|f\|_* := C_0(f;\mathbb{C}) + \|w_z\|_{(p,p')} + \|w_{\bar z}\|_{p,p'} .$$

By iteration one has

$$\|\omega_{n+1}\|_* \le (1-t)\gamma\|w\|_* \frac{1-(\gamma(t-t_0))^{n+1}}{1-\gamma(t-t_0)} + \|\omega_0\|_*(\gamma(t-t_0))^{n+1} .$$

For estimating ω_0 we observe

$$\begin{aligned}
\omega_{0\bar z} &= (1-t_0)R(z,w,w_z) + t_0[R(z,w,w_z) - R(z,w_0,w_{0z})] \\
&= t_0\widehat{\mu}(z,w_0,w_z,w_{0z})\omega_{0z} + t_0\widehat{A}(z,w,w_0,w_z)\omega_0 + (1-t_0)R(z,w,w_z) ,
\end{aligned}$$

so that by 3.1.2 and the a priori estimate of Theorem 47

$$\|\omega_0\|_* \le (1-t_0)\gamma\|w\|_* \le (1-t_0)\gamma K_0 .$$

Combining these last two estimates proves the inequality from the corollary. Since $t - t_0$ can be chosen so small that $\gamma(t-t_0) < 1$ the error ω_n becomes small for big enough n and t close enough to 1.

On the basis of the result of Theorem 47 the general RIEMANN problem can be treated for nonlinear BELTRAMI systems. The solutions are admitted to have polynomial growth at infinity.

Theorem 48. *Suppose H satisfies both* HÖLDER*-conditions from Theorem 47 and*

$$H(\cdot,0,0) \in L_{\infty,loc}(\mathbb{C}),\ \ |H(z,0,0)| = O(|z|^{-1-\varepsilon-\kappa}) \quad \text{as } z \to \infty ,$$

$$K(\cdot) \in L_{\infty,loc}(\mathbb{C}),\ \ K(z) = O(|z|^{-1-\varepsilon-m}) \quad \text{as } z \to \infty ,$$

$$0 \le q(z) \le q_0 < 1,\ \ q(z) = O(|z|^{-\varepsilon-m}) \quad \text{as } z \to \infty ,$$

for some $m \in I\!N_0$. Suppose further $G, g \in C_\alpha(\Gamma), 1 < 2\alpha < 2, G(z) \neq 0$ on Γ, where Γ is a finite set of simply closed smooth bounded curves dividing the complex plane $\mathbb{C}$ in a bounded domain $D^+, 0 \in D^+$ and $\partial D^+ = \Gamma$, and $D^- := \widehat{\mathbb{C}} \setminus \overline{D^+}, \kappa := indG$. Let P be a complex polynomial of degree $\leq m$. Then there exists a unique solution w of the problem

$$w_{\bar z} = H(z, w, w_z) \quad \text{in } \mathbb{C} \setminus \Gamma ,$$

$$w^+ = Gw^- + g \quad \text{on } \Gamma ,$$

$$\lim_{z\to\infty} \{X^{-1}(z)w(z) - P(z)\} = 0 .$$

Here X is the canonical function of the RIEMANN *problem defined by G, see Definition 8.*

Proof. If w is a solution then

$$\omega := X^{-1}w - P - \psi \, , \; \psi(z) := \frac{1}{2\pi i} \int\limits_\Gamma \frac{g(\zeta)}{X^+(\zeta)} \frac{d\zeta}{\zeta - z} \, , \; z \notin \Gamma ,$$

has the following properties:

$$\begin{aligned} \omega_{\bar z} &= \widehat{H}(z, \omega, \omega_z) \quad \text{in } \mathbb{C} \setminus \Gamma , \\ \omega^+ &= \omega^- \quad \text{on } \Gamma , \\ \omega(\infty) &= 0 , \end{aligned}$$

where

$$\widehat{H}(z, \omega, u) := X^{-1}(z) H(z, \widehat{\omega}, \widehat{u}) ,$$

$$\widehat{\omega} := X(z)[\omega + P(z) + \psi(z)] ,$$

$$\widehat{u} := X'(z)[\omega + P(z) + \psi(z)] + X(z)[u + P'(z) + \psi'(z)] .$$

It has to be shown that the nonlinear differential equation for ω holds throughout $\mathbb{C}$ instead of just in $\mathbb{C} \setminus \Gamma$. Since $\omega_{\bar z} = X^{-1} w_{\bar z}$ and $w_{\bar z} \in L_{(p,p')}(\mathbb{C}) \subset L_p(\mathbb{C})$ – for this argumentation $w_{\bar z} \in L_{r,loc}(\mathbb{C})$ for some $2 < r$ would be sufficient, see [Behi83] – we also have $\omega_{\bar z} \in L_p(\{|z| \leq R\})$ for any $0 < R$. Choose R so large that $\Gamma \subset \{|z| < R\}$ and define

$$F(z) = \begin{cases} \widehat{H}(z, \omega, w_z) & , \text{ for } |z| \leq R , \\ 0 & , \; for R < |z| . \end{cases}$$

Then $F \in L_p(\mathbb{C})$ and $\varphi := \omega - TF$ satisfies

$$\varphi_{\bar z} = \omega_{\bar z} - F = 0 \quad \text{in } \{|z| \leq R\} \setminus \Gamma .$$

But ω as well as TF are continuous functions in $|z| \leq R$ hence, continuous especially across Γ. Therefore φ is analytic in $|z| < R$ and then we have

$$\omega_{\bar{z}} = \varphi_{\bar{z}} + (TF)_{\bar{z}} = F \, .$$

This proves that the differential equation for ω holds in $\mathbb{C}$, ω being continuous across Γ and vanishing at infinity.
Conversely, from a function ω having just these properties it can be shown that

$$w := X(\omega + P + \psi)$$

is a solution to the above RIEMANN problem with polynomial growth at infinity. In order to apply Theorem 47 to the problem for ω the function $\hat{H}$ has to be shown to satisfy the corresponding properties for H. We find

$$|\hat{H}(z,\omega_1,u) - \hat{H}(z,\omega_2,u)| \leq \hat{K}(z)|\omega_1 - \omega_2| \, ,$$

$$\hat{K}(z) := K(z) + q(z)\frac{|X'(z)|}{|X(z)|} \, .$$

Lemma 16 shows $K \in L_{(p,p')}(\mathbb{C}) \cap L_{p,2}(\mathbb{C})$ provided p and p' satisfy the appropriate restrictions from Theorem 47.

$$\frac{|X'(z)|}{|X(z)|} = \begin{cases} |\gamma'(z)| & , \text{ in } D^+ \, , \\ |\gamma'(z) - \kappa z^{-1}| & , \text{ in } D^- \, . \end{cases}$$

By Lemma 17 $\gamma' \in L_r(\mathbb{C})$ for $1 < r < (1-\alpha)^{-1}$. Because $1 < 2\alpha < 2$ we have $2 < (1-\alpha)^{-1}$. Again Lemma 16 serves to see $\hat{K} \in L_{(p,p')}(\mathbb{C}) \cap L_{p,2}(\mathbb{C})$ for $p' < 2 < p$ both, p and p' close enough to 2.
We also have

$$|\hat{H}(z,\omega,u_1) - \hat{H}(z,\omega,u_2)| \leq q(z)|u_1 - u_2| \, .$$

Finally,

$$|\hat{H}(z,0,0)| \leq F_1(z) + F_2(z) + F_3(z) + F_4(z) \, ,$$

with

$$\begin{aligned} F_1(z) &= |X(z)|^{-1}|H(z,0,0)| \, , \\ F_2(z) &= K(z)[|P(z)| + |\psi(z)|] \, , \\ F_3(z) &= q(z)\frac{|X'(z)|}{|X(z)|}[|P(z)| + |\psi(z)|] \, , \\ F_4(z) &= q(z)[|P'(z)| + |\psi'(z)|] \, . \end{aligned}$$

All these functions can be seen to belong to $L_{(p,p')}(\mathbb{C}) \cap L_{p,2}(\mathbb{C})$ for $p' < 2 < p$, and p, p' close enough to 2. This follows again via Lemma 16 and the observations

$$\begin{aligned} X(z) &= O(|z|^{-\kappa}), |\psi(z)| = O(|z|^{-1}), |P(z)| = O(|z|^{m}) \, , \\ \psi'(z) &= O(|z|^{-2}), P'(z) = O(|z|^{m-1}) \quad \text{as } z \to \infty \end{aligned}$$

and $\psi' \in L_r(\mathbb{C})$ for $1 < r < (1-\alpha)^{-1}$.

Remark. From the representation

$$w = X(\omega + P + \psi)$$

and $\omega(\infty) = \psi(\infty) = 0, X(z) = O(|z|^{-\kappa})$ as z tends to ∞, one can see when bounded solutions may occur. For negative index $\kappa < 0$ one cannot conclude that bounded solutions exist and we know from Theorems 14, 15 that they need not exist. For nonnegative index $0 \leq \kappa$ one obtains a bounded solution for each polynomial the degree of which is less than or equal to κ.

5. Higher order complex differential equations and equations in several complex variables

5.1 Elliptic second order equations

The theory of complex elliptic second order equations is not yet far developed. Of course the LAPLACE equation

$$\Delta w = 4\,\frac{\partial^2}{\partial\bar{z}\partial z}w = 0$$

is extensively studied in close connection to the theory of analytic functions but more real methods are applied because they are available for higher dimensions than 2, too. Harmonic functions, i.e. solutions to the LAPLACE equation, are known to be uniquely defined by their boundary values. In other words the homogeneous DIRICHLET problem (with vanishing boundary data) for the LAPLACE equation is only trivially solvable. BITSADZE [Bica48] pointed out that this is a particular property of the LAPLACE equation and will not hold in general. His counterexample is the equation

$$\frac{\partial^2}{\partial\bar{z}^2}w = 0$$

in the unit disc $\mathbb{D}$. There are countably many over $\mathbb{C}$ linearly independent solutions vanishing at $\partial\mathbb{D}$. They are

$$w_k := (1 - z\bar{z})z^k\;,\;k \in \mathbb{N}_0\;,$$

as one can easily verify.
Some early papers where complex methods were applied to second order equations are [Hou58], [Boja60]. Several members of a research group at the Fudan University e.g. [Li78], [Xu81] draw their attention to this subject and in [Wen85], [Weta83] involved boundary conditions in multiply connected domains are handled. In [Dzhu87] the theory of the BERGMAN kernel function was applied to treat some natural boundary value problems for complex second order equations in multiply connected domains. This seems to be the adequate method for these kinds of problems and will be introduced here.
An arbitrary complex second order equation is of the form

$$aw_{\bar{z}\bar{z}} + bw_{\bar{z}z} + cw_{zz} + \alpha\overline{w_{\bar{z}\bar{z}}} + \beta\overline{w_{\bar{z}z}} + \gamma\overline{w_{zz}} + f(z,w,w_z,w_{\bar{z}}) = 0\;.$$

Since here only some special second order equations will be studied, we restrict our attention at first to equations where $\alpha = \beta = \gamma = 0$.

Definition 17. *A second order differential operator with main part of the form*

$$a\frac{\partial^2}{\partial\bar{z}^2} + b\frac{\partial^2}{\partial\bar{z}\partial z} + c\frac{\partial^2}{\partial z^2}$$

is called elliptic if the symbol

$$a\zeta^2 + b\zeta\overline{\zeta} + c\overline{\zeta}^2 \neq 0 \; for \; \zeta \neq 0 \, .$$

Lemma 34. *The main part of any second order elliptic differential operator L can be transformed to a multiple of the* LAPLACE *operator* $\frac{\partial^2}{\partial\overline{\zeta}\partial\zeta}$ *or to a multiple of*

$$\frac{\partial^2}{\partial\overline{\zeta}^2} + \mu\frac{\partial^2}{\partial\overline{\zeta}\partial\zeta} \quad \text{or} \quad \frac{\partial^2}{\partial\zeta^2} + \mu\frac{\partial^2}{\partial\overline{\zeta}\partial\zeta}$$

where $|\mu| \neq 1$.

Proof 1. $a = c = 0, b \neq 0 : L = b\frac{\partial^2}{\partial\overline{z}\partial z}$.

2. $a \neq 0$. Consider the symbol of L

$$a\left(\frac{\zeta}{\overline{\zeta}}\right)^2 + b\frac{\zeta}{\overline{\zeta}} + c = a(\rho - \rho_1)(\rho - \rho_2), \rho := \frac{\zeta}{\overline{\zeta}} \, .$$

The ellipticity condition shows $|\rho_1| \neq 1, |\rho_2| \neq 1$. Moreover,

$$L = a\left\{\left(\frac{\partial}{\partial\overline{z}} - \rho_1\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial\overline{z}} - \rho_2\frac{\partial}{\partial z}\right) + \left(\frac{\partial\rho_2}{\partial\overline{z}} - \rho_1\frac{\partial\rho}{\partial z}\right)\frac{\partial}{\partial z}\right\} \, .$$

Let $\zeta = \zeta(z)$ be a complete homeomorphism of

$$\frac{\partial\zeta}{\partial\overline{z}} - \rho_2\frac{\partial\zeta}{\partial z} = 0 \, ,$$

then if $1 - \rho_1\overline{\rho_2} \neq 0$

$$\begin{aligned} L \;&=\; a(1 - |\rho_2|^2)(1 - \rho_1\overline{\rho_2})\left(\frac{\overline{\partial\zeta}}{\partial z}\right)^2\left[\frac{\partial^2}{\partial\overline{\zeta}^2} + q\frac{\zeta_z}{\overline{\zeta_z}}\frac{\partial^2}{\partial\overline{\zeta}d\zeta}\right] \\ &\quad + a\left(\frac{\partial\rho_2}{\partial\overline{z}} - \rho_1\frac{\partial\rho_2}{\partial z}\right)\left(\frac{\partial\zeta}{\partial z}\frac{\partial}{\partial\zeta} + \overline{\rho_2}\frac{\overline{\partial\zeta}}{\partial z}\frac{\partial}{\partial\overline{\zeta}}\right) \end{aligned}$$

where

$$q := \frac{\rho_2 - \rho_1}{1 - \rho_1\overline{\rho_2}}\frac{\partial_z}{\overline{\zeta_z}}, |q| \neq 1 \, .$$

If $1 - \rho_1\overline{\rho_2} = 0$ then

$$L = a(\rho_2 - \rho_1)|\zeta_z|^2(1 - |\rho_2|^2)\frac{\partial^2}{\partial\overline{\zeta}\partial\zeta} + a\left(\frac{\partial\rho_2}{\partial\overline{z}} - \rho_1\frac{\partial\rho_2}{\partial z}\right)\left(\frac{\partial\zeta}{\partial z}\frac{\partial}{\partial\zeta} + \overline{\rho_2}\frac{\overline{\partial\zeta}}{\partial z}\frac{\partial}{\partial\overline{\zeta}}\right) \, .$$

3. If $c \neq 0$ the operator

$$\overline{L} := \overline{c}\frac{\partial^2}{\partial \overline{z}^2} + \overline{b}\frac{\partial^2}{\partial \overline{z}\partial z} + \overline{a}\frac{\partial^2}{\partial z^2}$$

can be transformed as L was before.
Thus there are two main cases:

i. $L = \dfrac{\partial^2}{\partial \overline{\zeta}^2} + \mu \dfrac{\partial^2}{\partial \overline{\zeta}\partial \zeta}, |\mu| < 1$,

ii. $L = \varepsilon \dfrac{\partial^2}{\partial \overline{\zeta}^2} + \mu \dfrac{\partial^2}{\partial \overline{\zeta}\partial \zeta}, 0 \leq \varepsilon \leq 1 < |\mu|$.

As in the case of the generalized BELTRAMI equation where the solutions were expressed through $Tw_{\overline{z}}$ we are looking for an integral operator playing the same role for second order equations as T does for first order. At once we realize that there will be more than just one such operator. Already for first order equations there are two operators; for the BELTRAMI operator $w_{\overline{z}} + \mu w_z, |\mu| < 1$, the T–operator is appropriate while if $1 < |\mu|$ the $\overline{T}$–operator has to be used, see (2.1.1) and (2.1.1′). This symmetry does occur for second order operators, too, as is seen from the two operators

$$\frac{\partial^2}{\partial \overline{z}^2} + \mu \frac{\partial^2}{\partial \overline{z}\partial z}\, , \ \frac{\partial^2}{\partial z^2} + \mu \frac{\partial^2}{\partial \overline{z}\partial z} \quad , \ |\mu| < 1 \, .$$

But now there is a third case, namely of $1 < |\mu|$ where the leading term is $\dfrac{\partial^2}{\partial \overline{z}\partial z}$.
The basic idea in finding these operators is iterating the T–operator. We start from the CAUCHY–POMPEIU formula (2.1.1) in Theorem 20 applied to w as well as to $w_{\overline{z}}$ under proper assumption on w,

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} + Tw_{\overline{\zeta}}\, , \\ w_{\overline{z}}(z) &= \frac{1}{2\pi i}\int_{\partial D} w_{\overline{\zeta}}(\zeta)\frac{d\zeta}{\zeta - z} + Tw_{\overline{\zeta}\overline{\zeta}}\, . \end{aligned}$$

Inserting the last in the former gives

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi^2 i}\int_{\partial D} w_{\overline{t}}(t)\int_{D} \frac{d\xi d\eta}{(t-\zeta)(\zeta - z)}dt + TTw_{\overline{\zeta}\overline{\zeta}}\, .$$

Since

$$\frac{1}{\pi}\int_{D} \frac{d\xi d\eta}{(t-\zeta)(\zeta - z)} = \frac{1}{\pi(t-z)}\int_{D}\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - t}\right)d\xi d\eta = \frac{\overline{t} - \overline{z}}{t - z}$$

we get

$$w(z) = \frac{1}{2\pi i} \int\limits_{\partial D} \left[\frac{w(\zeta)}{\zeta - z} + \frac{\overline{\zeta - z}}{\zeta - z} w_{\overline{\zeta}}(\zeta) \right] d\zeta + \frac{1}{\pi} \int\limits_{D} \frac{\overline{\zeta - z}}{\zeta - z} w_{\overline{\zeta}\overline{\zeta}}(\zeta) d\xi d\eta \, .$$

Here the boundary integral represents a polyanalytic function of the form $\psi(z) + \overline{z}\phi(z)$ with analytic functions ψ and ϕ while the area integral

$$S\rho(z) := T_{0,2}\rho(z) := \frac{1}{\pi} \int\limits_{D} \frac{\overline{\zeta - z}}{\zeta - z} \rho(\zeta) d\xi d\eta$$

represents the integral operator, we were looking for. Obviously, $\psi + \overline{z}\phi$ is the general solution to the homogeneous equation $w_{\overline{z}\overline{z}} = 0, T_{0,2}\rho$ being a particular solution to the inhomogeneous equation $w_{\overline{z}\overline{z}} = \rho$ in D.

In order to find the second integral operator we are iterating the $T-$ with the $\overline{T}-$operator, i.e.

$$\begin{aligned} w(z) &= \frac{1}{2\pi i} \int\limits_{\partial D} w(\zeta) \frac{d\zeta}{\zeta - z} + T w_{\overline{\zeta}} \, , \\ w_{\overline{z}}(z) &= -\frac{1}{2\pi i} \int\limits_{\partial D} w_{\overline{z}}(\zeta) \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}} + \overline{T} w_{\overline{\zeta}\zeta} \, , \end{aligned}$$

leading to

$$w(z) = \frac{1}{2\pi i} \int\limits_{\partial D} w(\zeta) \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi^2 i} \int\limits_{\partial D} w_{\overline{t}}(t) \int\limits_{D} \frac{d\xi d\eta}{(\overline{t} - \overline{\zeta})(\zeta - z)} d\overline{t} + T\overline{T} w_{\overline{\zeta}\zeta} \, .$$

Applying (2.1.1) to $\log|z-t|^2$ in $D \backslash \{|z-t| \le \varepsilon\}$ for $0 < \varepsilon$ small enough, we see

$$-\frac{1}{\pi} \int\limits_{D} \frac{1}{\overline{\zeta - t}} \frac{d\xi d\eta}{\zeta - z} = \log|z-t|^2 - \frac{1}{2\pi i} \int\limits_{\partial D} \log|\zeta - t|^2 \frac{d\zeta}{\zeta - z} + \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int\limits_{|\zeta - t| = \varepsilon} \log|\zeta - t|^2 \frac{d\zeta}{\zeta - z} .$$

As

$$\left| \frac{1}{2\pi i} \int\limits_{|\zeta - t| = \varepsilon} \log|\zeta - t|^2 \frac{d\zeta}{\zeta - z} \right| \le 2 \log \varepsilon \frac{\varepsilon}{|z-t| - \varepsilon} \quad \text{for } \varepsilon < |z-t|$$

the limit vanishes in the last equation and

$$w(z) = \frac{1}{2\pi i} \int\limits_{\partial D} w(\zeta) \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int\limits_{\partial D} w_{\overline{\zeta}}(\zeta) \log|\zeta - z|^2 d\overline{\zeta} - \frac{1}{\pi} \int\limits_{D} \log|\zeta - z|^2 w_{\overline{\zeta}\zeta}(\zeta) d\xi d\eta \, .$$

Rewriting the second boundary integral as the sum

$$\frac{1}{2\pi i} \int\limits_{\partial D} w_{\overline{\zeta}}(\zeta) \log(\zeta - z) d\zeta - \overline{\frac{1}{2\pi i} \int\limits_{\partial D} \overline{w_{\overline{\zeta}}(\zeta)} \log(\zeta - z) d\zeta}$$

the function w is represented as the sum

$$w = \psi + \overline{\phi} + T_{1,1} w_{\overline{\zeta}\zeta} \,,$$

where ϕ and ψ are analytic functions in D, again forming the general solution to the homogeneous - LAPLACE - equation, and

$$T_{1,1,}\rho(z) := \frac{2}{\pi} \int_D \log|\zeta - z| \rho(\zeta) d\xi d\eta$$

is a particular solution to the inhomogeneous - POISSON - equation $w_{\bar{z}z} = \rho$ in D. One can proceed in this manner to higher order equations, see the next section and [Behi93], compare [Dzhu87], too. We get an equivalent operator to $T_{1,1}$ by introducing the GREEN function

$$g(z, \zeta) = -\log|\zeta - z| + \omega(z, \zeta)$$

where ω is harmonic in z and ζ,

$$\begin{aligned} T_{1,1}\rho(z) &= -\frac{2}{\pi} \int_D g(z, \zeta) \rho(\zeta) d\xi d\eta + \frac{2}{\pi} \int_D \omega(z, \zeta) \rho(\zeta) d\xi d\eta \\ S_1 \rho(z) &:= -\frac{2}{\pi} \int_D g(z, \zeta) \rho(\zeta) d\xi d\eta \end{aligned}$$

represents a particular solution to $w_{\bar{z}z} = \rho$, the second integral is a harmonic function in D and as such can be represented by an analytic function φ in the form $\varphi + \overline{\varphi}$. We collect these results in the next theorem.

Theorem 49. *Any $w \in C^2(\overline{D}; \mathbb{C})$ can be represented as*

$$w(z) = \frac{1}{2\pi i} \int_{\partial D} \left[\frac{w(\zeta)}{\zeta - z} + \frac{\overline{\zeta - z}}{\zeta - z} w_{\overline{\zeta}}(\zeta) \right] d\zeta + \frac{1}{\pi} \int_D \frac{\overline{\zeta - z}}{\zeta - z} w_{\overline{\zeta}\overline{\zeta}}(\zeta) d\xi d\eta \;,\; z \in D \,,$$

$$w(z) = \frac{1}{2\pi i} \int_{\partial D} \left[\frac{w(\zeta)}{\zeta - z} d\zeta + w_{\overline{\zeta}}(\zeta) \log|\zeta - z|^2 d\overline{\zeta} \right] - \frac{2}{\pi} \int_D \log|\zeta - z| w_{\overline{\zeta}\zeta} d\xi d\eta, z \in D. \quad (5.1.1)$$

Corollary 10. *Any $w \in C^2(\overline{D}; \mathbb{C})$ can be represented as*

$$\begin{aligned} w(z) &= \psi(z) + \overline{z}\phi(z) + (S w_{\overline{\zeta}\overline{\zeta}})(z) \quad \text{in } D \,, \\ w(z) &= \psi(z) + \overline{\phi(z)} + (S_1 w_{\overline{\zeta}\zeta})(z) \quad \text{in } D \,, \end{aligned}$$

where ϕ and ψ are analytic functions in D.

We are going to determine these arbitrary analytic functions by some boundary conditions on w. Only one particular set of two natural boundary conditions will be investigated which were in this combination introduced in [Dzhu87], see also [Bege93]. For more general conditions see [Dzhu92].

Boundary value problem. Let D be a smooth multiply connected bounded domain, $\partial D = \cup_{\mu=0}^{m}\Gamma_\mu, m \in \mathbb{N}_0$, where the Γ_μ are mutually disjoint smooth curves and Γ_0 is surrounding to other Γ_μ. Find w as a solution to a second order elliptic equation in D satisfying

$$\operatorname{Re} w_{\bar z}(z) = 0\ ,\ \operatorname{Re}\{z'(s)w(z)\} = 0 \quad \text{on } \partial D\ .$$

Before approaching this boundary value problem some lemmas preparing for later calculations will be proved.

Lemma 35. *Let D be a C^4 domain and $\omega(z,\zeta)$ the regular part of the* GREEN *function for D. Then for $\varphi \in C^\alpha(\overline{D})$*

$$\widehat{\varphi}(z) := -\frac{2}{\pi}\int\limits_D \omega_{zz}(z,\zeta)\varphi(\zeta)d\xi d\eta$$

belongs to $C^\alpha(\overline{D})$ and satisfies

$$C_\alpha(\widehat{\varphi};\overline{D}) \le M(\alpha,D)C_\alpha(\varphi;\overline{D})\ .$$

Proof.

i. $D = \mathbb{D}$. Then

$$\widehat{\varphi}(z) = \frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{\overline{\zeta}^2}{(1-z\overline{\zeta})^2}\varphi(\zeta)d\xi d\eta\ .$$

Because

$$\int\limits_{\mathbb{D}} \frac{\overline{\zeta}}{1-z\overline{\zeta}}d\xi d\eta = 0 \quad \text{and} \quad \int\limits_{\mathbb{D}} \frac{\overline{\zeta}^2}{(1-z\overline{\zeta})^2}d\xi d\eta = 0$$

we have

$$\widehat{\varphi}(z) = \frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{\overline{\zeta}^2}{(1-z\overline{\zeta})^2}(\varphi(\zeta)-\varphi(z))d\xi d\eta$$

and

$$\begin{aligned}\widehat{\varphi}(z_1)-\widehat{\varphi}(z) &= \frac{1}{\pi}\int\limits_{\mathbb{D}}\left\{\frac{\overline{\zeta}^2(\varphi(\zeta)-\varphi(z_1))}{(1-z_1\overline{\zeta})^2} - \frac{\overline{\zeta}^2(\varphi(\zeta)-\varphi(z_1))}{(1-z_1\overline{\zeta})(1-z\overline{\zeta})}\right.\\ &\qquad\left.+\frac{\overline{\zeta}^2(\varphi(\zeta)-\varphi(z))}{(1-z_1\overline{\zeta})(1-z\overline{\zeta})} - \frac{\overline{\zeta}^2(\varphi(\zeta)-\varphi(z))}{(1-z\overline{\zeta})^2}\right\}d\xi d\eta.\end{aligned}$$

Using

$$\left|\frac{\zeta - z}{1-z\overline{\zeta}}\right| \le 1 \quad \text{for} \quad |z|,|\zeta| \le 1$$

and the estimate for $J(\alpha,\beta)$ from the proof to Theorem 23 thus

$$|\widehat{\varphi}(z_1)-\widehat{\varphi}(z)|\le M(\alpha)H_\alpha(\varphi)|z_1-z|^\alpha\,.$$

Therefore

$$H_\alpha(\widehat{\varphi})\le M(\alpha)H_\alpha(\varphi)\,.$$

Similarly,

$$C_0(\widehat{\varphi};\mathbb{D}):=\max_{z\in\mathbb{D}}|\widehat{\varphi}(z)|\le M(\alpha)H_\alpha(\varphi)$$

follows. In the same way

$$C_\alpha\left(\frac{1}{\pi}\int\limits_{\mathbb{D}}\frac{\overline{\zeta}\varphi(\zeta)}{1-z\overline{\zeta}}d\xi d\eta;\mathbb{D}\right)\le M(\alpha)H_\alpha(\varphi)$$

can be proved.

ii. D simply connected. Let σ be a conformal mapping from D onto $\mathbb{D}$. Assuming the domain being C^4, this means that the boundary ∂D is four times continuously differentiable, then σ together with its derivatives up to the fourth order can be continuously extended on ∂D mapping ∂D onto $\partial\mathbb{D}$, see [Golu69], p. 417. The GREEN function of D is

$$g(z,\zeta):=\log\left|\frac{1-\sigma(z)\overline{\sigma(\zeta)}}{\sigma(z)-\sigma(\zeta)}\right|=-\log|z-\zeta|+\omega(z,\zeta)$$

with

$$\omega(z,\zeta):=\log\left|\frac{z-\zeta}{\sigma(z)-\sigma(\zeta)}[1-\sigma(z)\overline{\sigma(\zeta)}]\right|\,.$$

We have

$$\begin{aligned}
2\omega_z(z,\zeta) &= \frac{1}{z-\zeta}-\frac{\sigma'(z)}{\sigma(z)-\sigma(\zeta)}-\frac{\sigma'(z)\overline{\sigma(\zeta)}}{1-\sigma(z)\overline{\sigma(\zeta)}}\,,\\
2\omega_{zz}(z,\zeta) &= -\frac{1}{(z-\zeta)^2}+\frac{\sigma'(z)^2-\sigma''(z)[\sigma(z)-\sigma(\zeta)]}{[\sigma(z)-\sigma(\zeta)]^2}\\
&\quad -\frac{[\sigma'(z)\overline{\sigma(\zeta)}]^2+\sigma''(z)\overline{\sigma(\zeta)}[1-\sigma(z)\overline{\sigma(\zeta)}]}{[1-\sigma(z)\overline{\sigma(\zeta)}]^2}\\
&= h_2(z,\zeta)-\frac{[\sigma'(z)\overline{\sigma(\zeta)}]^2}{[1-\sigma(z)\overline{\sigma(\zeta)}]^2}-\frac{\sigma''(z)\overline{\sigma(\zeta)}}{1-\sigma(z)\overline{\sigma(\zeta)}}
\end{aligned}$$

with a function

$$
\begin{aligned}
h_2(z,\zeta) &:= -\left[\frac{z-\zeta}{\sigma(z)-\sigma(\zeta)}\right]^2 \frac{\left[\frac{\sigma(z)-\sigma(\zeta)}{z-\zeta}\right]^2 - \sigma'(z)^2 + \sigma''(z)[\sigma(z)-\sigma(\zeta)]}{(\zeta-z)^2}, \\
h_2(z,z) &:= \frac{1}{4}\left[\frac{\sigma''(z)}{\sigma'(z)}\right]^2 - \frac{1}{6}\frac{\sigma'''(z)}{\sigma'(z)}
\end{aligned}
$$

analytic in z and ζ from D, bounded in $\overline{D}\times\overline{D}$ together with its first derivatives. Hence,

$$\widehat{\varphi}_1(z) := -\frac{2}{\pi}\int\limits_D h_2(z,\zeta)\varphi(\zeta)d\xi d\eta$$

satisfies

$$C_\alpha(\widehat{\varphi}_1;\overline{D}) \le M(\alpha;D)C_\alpha(\varphi;D)\ .$$

Since σ' and σ'' both belong to $C_\alpha(\overline{D};\mathbb{C}\,)$ what follows from the boundedness of σ''' in $\overline{D}$, it remains to estimate

$$\widehat{\varphi}_2(z) := \frac{1}{\pi}\int\limits_D \frac{\overline{\sigma(\zeta)}^2\varphi(\zeta)}{[1-\sigma(z)\overline{\sigma(\zeta)}]^2}d\xi d\eta$$

and

$$\widehat{\varphi}_3(z) := \frac{1}{\pi}\int\limits_D \frac{\overline{\sigma(\zeta)}\varphi(\zeta)}{1-\sigma(z)\overline{\sigma(\zeta)}}d\xi d\eta\ .$$

Let $\zeta=\zeta(\sigma)$ be the inverse mapping from $\sigma=\sigma(\zeta)$. Then

$$\widehat{\varphi}_2(z) = \frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{\overline{\sigma}^2\varphi(\zeta(\sigma))}{[1-\sigma(z)\overline{\sigma}]^2}|\zeta'(\sigma)^2 d\sigma_1 d\sigma_2\ ,\ \sigma := \sigma_1 + i\sigma_2\ .$$

As in step i. writing

$$\widehat{\varphi}_2(z) = \frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{\overline{\sigma}^2[\varphi(\zeta(\sigma))|\zeta'(\sigma)|^2 - \varphi(z)|\zeta'(\sigma(z))|^2]}{[1-\sigma(z)\overline{\sigma}]^2}d\sigma_1 d\sigma_2$$

shows

$$C_\alpha(\widehat{\varphi}_2;\overline{D}) \le M(\alpha)H_\alpha((\varphi\circ\zeta)|\zeta'|^2)H_1(\sigma)^\alpha = M(\alpha;D)H_\alpha(\varphi)\ .$$

Function $\widehat{\varphi}_3$ is treated in the same way.

iii. D multiply connected. Let $N_\mu \subset D$ be a neighborhood of Γ_μ in D for any $0 \le \mu \le m, \overline{N_\mu} \cap \overline{N_\nu} = \emptyset$ for $\mu \neq \nu$, and $\widehat{N}_\mu := D \backslash \cup_{\nu \neq \mu} N_\nu$. Let D_μ denote the simply connected domain with boundary Γ_μ and containing D and hence $\widehat{N}_\mu$. The GREEN function of D_μ is denoted by

$$g_\mu(z,\zeta) = -\log|z-\zeta| + \omega_\mu(z,\zeta) .$$

Then

$$g_\mu(z,\zeta) - g(z,\zeta) = \omega_\mu(z,\zeta) - \omega(z,\zeta)$$

is harmonic in D and vanishing at Γ_μ. From

$$\omega_\mu(z,\zeta) - \omega(z,\zeta) = -\frac{1}{2\pi} \int\limits_{\partial D \backslash \Gamma_\mu} [g_\mu(t,\zeta) - g(t,\zeta)] \frac{\partial g(t,z)}{\partial \boldsymbol{n}_t} ds_t ,$$

see Theorem 13, we see

$$\frac{\partial^2}{\partial z^2}[\omega_\mu(z,\zeta) - \omega(z,\zeta)] = -\frac{1}{2\pi} \int\limits_{\partial D \backslash \Gamma_\mu} \frac{\partial g_{zz}(t,z)}{\partial \boldsymbol{n}_t} [g_\mu(t,\zeta) - g(t,\zeta)] ds_t ,$$

i.e.

$$\omega_{zz}(z,\zeta) = \omega_{\mu zz}(z,\zeta) + \frac{1}{2\pi} \int\limits_{\partial D \backslash \Gamma_\mu} \frac{\partial g_{zz}(t,z)}{\partial \boldsymbol{n}_t} [g_\mu(t,\zeta) - g(t,\zeta)] ds_t .$$

Thus

$$\begin{aligned} \widehat{\varphi}(z) &= -\frac{2}{\pi} \int\limits_D \omega_{\mu zz}(z,\zeta)\varphi(\zeta) d\xi d\eta \\ &\quad -\frac{1}{\pi^2} \int\limits_D \int\limits_{\partial D \backslash \Gamma_\mu} \frac{\partial g_{zz}(t,z)}{\partial \boldsymbol{n}_t} [g_\mu(t,\zeta) - g(t,\zeta)] \varphi(\zeta) ds_t d\xi d\eta . \end{aligned}$$

The first form satisfies an estimate as is claimed in the lemma. This follows by the estimations from step ii. applied to $\omega_{\mu zz}$. We just have to realize $D \subset D_\mu$. The last term, say $J(z)$, being analytic in the closure of $\widehat{N}_\mu$ since $t \in \partial D \backslash \Gamma_\mu$ and $(\partial D \backslash \Gamma_\mu) \cap \overline{\widehat{N}_\mu} = \emptyset$ and $g_\mu(t,\zeta) - g(t,\zeta)$ is bounded on $(\partial D \backslash \Gamma_\mu) \times D$, is LIPSCHITZ continuous there. Hence, this integral is LIPSCHITZ continuous in $\overline{\widehat{N}_\mu}$ satisfying

$$C_\alpha(J; \overline{\widehat{N}_\mu}) \le M(\alpha, D, \widehat{N}_\mu) C_\alpha(\varphi; D) .$$

Since $\overline{D} = \cup_{\mu=0}^m \overline{\widehat{N}_\mu}$ we have

$$C_\alpha(\widehat{\varphi}; \overline{D}) \le \max_{0 \le \mu \le m} C_\alpha(\widehat{\varphi}; \overline{\widehat{N}_\mu}) \le M(\alpha, D) C_\alpha(\varphi; D) .$$

To verify this let $z, z' \in \overline{D}$ with $2|z - z'| < d_0 := \min_{\left(\substack{0 \le \mu,\nu \le m \\ \mu \ne \nu}\right)} dist(\overline{N_\mu}, \overline{N_\nu})$. If $z \in \overline{N_\mu}$ then $z, z' \in \overline{\widehat{N}_\mu}$ and hence

$$\frac{|\widehat{\varphi}(z) - \widehat{\varphi}(z')|}{|z - z'|^\alpha} \le M(\alpha, \widehat{N}_\mu) C_\alpha(\varphi; D) \,.$$

If $z \notin \cup_{\mu=0}^{m} \overline{N_\mu}$ then $z, z' \in \overline{\widehat{N}_\mu}$ for at least one $\mu, 0 \le \mu \le m$ and the same estimate hold.

Corollary 11. *The statement of the lemma holds if in the definition of* $\widehat{\varphi}$ *the function* ω_{zz} *is replaced by either* $\underset{\sim}{\ell}_z$ *or by* $2\omega_{zz} - \underset{\sim}{\ell}_z$.

Proof.

i. $D = D$. Then $\underset{\sim}{\ell}(z, \zeta) = 0$.

ii. D simply connected. Then

$$\begin{aligned}
\underset{\sim}{\ell}(z,\zeta) &= 2\int_{z_0}^{\zeta} \omega_{zt}(z,t)dt \\
&= 2\left(\frac{1}{z-\zeta} - \frac{\sigma'(z)}{\sigma(z)-\sigma(\zeta)}\right) - 2\left(\frac{1}{z-z_0} - \frac{\sigma'(z)}{\sigma(z)-\sigma(z_0)}\right) , \\
\frac{1}{2}\,\underset{\sim}{\ell}_z(z,\zeta) &= -\frac{1}{(z-\zeta)^2} + \frac{\sigma'(z)^2}{(\sigma(z)-\sigma(\zeta))^2} - \frac{\sigma''(z)}{\sigma(z)-\sigma(\zeta)} \\
&\quad + \frac{1}{(z-z_0)^2} - \frac{\sigma'(z)}{(\sigma(z)-\sigma(z_0))^2} + \frac{\sigma''(z)}{\sigma(z)-\sigma(z_0)} \\
&= h_2(z,\zeta) - h_2(z,z_0) \,, \quad 2\omega_{zz}(z,\zeta) - \underset{\sim}{\ell}_z(z,\zeta) \\
&= -h_2(z,\zeta) + 2h_2(z,z_0) - \left(\frac{\sigma'(z)\overline{\sigma(\zeta)}}{1-\sigma(z)\overline{\sigma(\zeta)}}\right)^2 - \frac{\sigma''(z)\overline{\sigma(\zeta)}}{1-\sigma(z)\overline{\sigma(\zeta)}} \,.
\end{aligned}$$

Following the proof for the lemma and observing the HÖLDER continuity of $h_2(z, z_0)$ for $z \in \overline{D}$ the integral $\widehat{\varphi}$ is seen to satisfy the desired estimation.

iii. D multiply connected. The proof based on the argumentation under step ii. can be given as in the proof for the lemma.

In the same manner the next lemma is proved.

Lemma 36. *Let D be a bounded domain with C^4 boundary. Then there exist positive constants C and C_1 such that for $(z,\zeta) \in \overline{D} \times \overline{D}$*

$$|\ell(z,\zeta)| \leq C, |\zeta - z||\omega_z(z,\zeta)| \leq C, \ |\zeta - z|^2 |\omega_{zz}(z,\zeta)| \leq C$$

and if the path of integration has a winding number between -1 and 1 with respect to any point in $\mathbb{C} \setminus D$

$$|\underset{\sim}{\ell}(z,\zeta)| \leq C_1, |\underset{\sim}{\ell}_z(z,\zeta)| \leq C_1 .$$

Proof.

i. $D = \mathbb{D}$. Then

$$\begin{aligned} \omega(z,\zeta) &= \log|1 - z\overline{\zeta}| \quad , \omega_z(z,\zeta) = -\frac{1}{2}\frac{\overline{\zeta}}{1 - z\overline{\zeta}} , \\ \omega_{zz}(z,\zeta) &= -\frac{1}{2}\frac{\overline{\zeta}^2}{(1 - z\overline{\zeta})^2} \quad , \omega_{z\zeta}(z,\zeta) = \ell(z,\zeta) = \underset{\sim}{\ell}(z,\zeta) = 0 . \end{aligned}$$

The functions ω_z and ω_{zz} become singular for $z = \zeta \in \partial\mathbb{D}$ and in this case $2C = 1$ and $C_1 = 0$.

ii. D simply connected with C^4 boundary. Let σ be a conformal mapping from D onto $\mathbb{D}$ as in the preceding proof.
From there we know

$$\begin{aligned} 2\omega_z(z,\zeta) &= h_1(z,\zeta) - \frac{\sigma'(z)\overline{\sigma(\zeta)}}{1 - \sigma(z)\overline{\sigma(\zeta)}} \\ h_1(z,\zeta) : &= -\frac{\zeta - z}{\sigma(\zeta) - \sigma(z)} \frac{\sigma(\zeta) - \sigma(z) - \sigma'(z)(\zeta - z)}{(\zeta - z)^2} \\ 2\omega_z(z,z) &= -\frac{1}{2}\frac{\sigma''(z)}{\sigma'(z)} - \frac{\sigma'(z)\overline{\sigma(z)}}{1 - |\sigma(z)|^2} ; \\ 2\omega_{z\zeta}(z,\zeta) &= \frac{1}{(z - \zeta)^2} - \frac{\sigma'(z)\sigma'(\zeta)}{(\sigma(z) - \sigma(\zeta))^2} \\ &= \left(\frac{\zeta - z}{\sigma(\zeta) - \sigma(z)}\right)^2 \frac{\left(\frac{\sigma(\zeta) - \sigma(z)}{\zeta - z}\right)^2 - \sigma'(z)\sigma'(\zeta)}{(\zeta - z)^2} , \\ 2\omega_{z\zeta}(z,z) &= -\frac{1}{4}\left[\frac{\sigma''(z)}{\sigma'(z)}\right]^2 + \frac{1}{3}\frac{\sigma'''(z)}{\sigma'(z)} ; \end{aligned}$$

and an expression for w_{zz} given in the last proof. Since ∂D is C^3 the functions h_1, h_2 and $\omega_{z\zeta}$ are continuous on $\overline{D} \times \overline{D}$ while the remaining terms in $2\omega_z$ and $2\omega_{zz}$ are singular. For $\zeta \in \partial D$

$$\begin{aligned} I_1 &:= \frac{(\zeta - z)\sigma'(z)\overline{\sigma(\zeta)}}{1 - \sigma(z)\overline{\sigma(\zeta)}} = \frac{\sigma'(z)(\zeta - z)}{\sigma(\zeta) - \sigma(z)}, \\ I_2 &:= \frac{(\zeta - z)^2[\sigma'(z)\overline{\sigma(\zeta)})^2 - \sigma''(z)\overline{\sigma(\zeta)}(1 - \sigma(z)\overline{\sigma(\zeta)})]}{[1 - \sigma(z)\overline{\sigma(\zeta)}]^2} \\ &= \frac{(\zeta - z)^2[(\sigma'(z))^2 - \sigma''(z)(\sigma(\zeta) - \sigma(z))]}{[\sigma(\zeta) - \sigma(z)^2]}. \end{aligned}$$

For $\zeta \in \overline{D}$, moreover,

$$\begin{aligned} I_1 &= \frac{\sigma(\zeta) - \sigma(z)}{1 - \sigma(z)\overline{\sigma(\zeta)}} \frac{(\zeta - z)\sigma'(z)\overline{\sigma(\zeta)}}{\sigma(\zeta) - \sigma(z)}, \\ I_2 &= \left[\frac{\sigma(\zeta) - \sigma(z)}{1 - \sigma(z)\overline{\sigma(\zeta)}}\right]^2 \left[\frac{\sigma'(z)\overline{\sigma(\zeta)}(\zeta - z)}{\sigma(\zeta) - \sigma(z)}\right]^2 - \frac{\sigma(\zeta) - \sigma(z)}{1 - \sigma(z)\overline{\sigma(\zeta)}} \frac{\sigma''(z)\overline{\sigma(\zeta)}(\zeta - z)^2}{\sigma(\zeta) - \sigma(z)}. \end{aligned}$$

These expressions are bounded on $\overline{D} \times \overline{D}$. From the boundedness of $\ell(z,\zeta)$ that of $\underset{\sim}{\ell}(z,\zeta)$ follows with the restriction made on the path of integration. In order to estimate $\underset{\sim}{\ell}_z(z,\zeta)$ consider

$$\omega_{zz\zeta}(z,\zeta) = \frac{1}{(\zeta - z)^3} - \frac{1}{2} \frac{\sigma''(z)\sigma'(\zeta)}{(\sigma(\zeta) - \sigma(z))^2} - \frac{(\sigma'(z))^2\sigma'(\zeta)}{(\sigma(\zeta) - \sigma(z))^3}$$

$$= \left[\frac{\zeta - z}{\sigma(\zeta) - \sigma(z)}\right]^3 \frac{\left[\frac{\sigma(\zeta) - \sigma(z)}{\zeta - z}\right]^3 - \frac{1}{2}\sigma''(z)\sigma(\zeta)[\sigma(\zeta) - \sigma(z)] - (\sigma'(z))^2\sigma'(\zeta)}{(\zeta - z)^3}$$

$$\omega_{zz\zeta}(z,z) = \frac{(\sigma'(z))^2\sigma^{(4)}(z) + 3(\sigma''(z))^3}{24(\sigma'(z))^3}.$$

This is an analytic function in $D \times D$, continuous on $\overline{D} \times \overline{D}$.

iii. D multiply connected with C^4 boundary. With the notations from the last proof besides the formulas from there we have

$$\frac{\partial}{\partial z}[\omega_\mu(z,\zeta)-\omega(z,\zeta)] = -\frac{1}{2\pi}\int\limits_{\partial D\setminus\Gamma_\mu}\frac{\partial g_z(z,t)}{\partial \boldsymbol{n}_t}[g_\mu(t,\zeta)-g(t,\zeta)]ds_t\,,$$

$$\frac{\partial^2}{\partial z\partial\zeta}[\omega_\mu(z,\zeta)-\omega(z,\zeta)] = -\frac{1}{2\pi}\int\limits_{\partial D\setminus\Gamma_\mu}\frac{\partial g_z(z,t)}{\partial \boldsymbol{n}_t}\frac{\partial}{\partial\zeta}[g_\mu(t,\zeta)-g(t,\zeta)]ds_t\,,$$

$$\frac{\partial^3}{\partial z^2\partial\zeta}[\omega_\mu(z,\zeta)-\omega(z,\zeta)] = -\frac{1}{2\pi}\int\limits_{\partial D\setminus\Gamma_\mu}\frac{\partial g_{zz}(z,t)}{\partial \boldsymbol{n}_t}\frac{\partial}{\partial\zeta}[g_\mu(t,\zeta)-g(t,\zeta)]ds_t\,.$$

For z and ζ near Γ_μ or on Γ_μ itself the integrands are continuous, even continuously differentiable since $t\in\partial D\setminus\Gamma_\mu$. Thus the functions on the left-hand sides are bounded for z and ζ on or near Γ_μ. Because ω_μ and its respective derivatives satisfy the above inequalities in $\overline{D_\mu}\times\overline{D_\mu}$ as follows from the considerations under ii. they hold for ω and its derivatives, too. Since $\mu, 0\le\mu\le m$, is arbitrary the estimates hold on the entire boundary ∂D and hence on $\overline{D}$.

Theorem 50. *Let D be a bounded smooth multiply connected domain and $\rho\in L_p(\overline{D}), 2<p$, satisfying*

$$\int\limits_D\left(\rho(\zeta)+\overline{\Pi\rho(\zeta)}\right)\psi_\mu(\zeta)d\xi d\eta = 0\,,\ 1\le\mu\le m\,,$$

where $\{\psi_\mu : 1\le\mu\le m\}$ is a basis for the solution space of the boundary value problem

$$\operatorname{Re}\{z'(s)\psi\}=0\quad\text{on }\partial D$$

for analytic functions, and

$$S\rho(z) := \frac{1}{\pi}\int\limits_D\frac{\overline{\zeta-z}}{\zeta-z}\rho(\zeta)d\xi d\eta\,.$$

Then there exists a unique solution w to the boundary value problem

$$\operatorname{Re} w_{\overline{z}} = \operatorname{Re}\{z'(s)w\}=0\quad\text{on }\partial D$$

within the set of functions representable in the form

$$w=\psi+\overline{z}\phi+S\rho$$

with in D analytic functions ϕ and ψ being continuous in $\overline{D}$. The solution w is given by

$$w(z)=\sum_{\mu=1}^m\gamma_\mu\psi_\mu(z)+T(\phi+T\rho)(z)$$

$$-\frac{1}{\pi}\int\limits_D\{\ell(z,\zeta)(\phi(\zeta)+T\rho(\zeta))+(2\omega_z(z,\zeta)-\underset{\sim}{\ell}(z,\zeta))\overline{(\phi(\zeta)+T\rho(\zeta))}\}d\xi d\eta\,,$$

where $\gamma_\mu, 1 \le \mu \le m$, are arbitrary real constants and

$$\begin{aligned}\phi(z) &= -T\rho(z) - \int_D \left\{ J_L(z,\zeta)\rho(\zeta) + J_K(z,\overline{\zeta})\overline{\rho(\zeta)} \right\} d\xi d\eta \\ &+ i\,\mathrm{Im}\frac{1}{|D|}\int_D\int_D \left\{ J_L(z,\zeta)\rho(\zeta) + J_K(z,\overline{\zeta})\overline{\rho(\zeta)} \right\} d\xi d\eta dx dy \,.\end{aligned}$$

Proof. Differentiating $w = \psi + \overline{z}\phi + S\rho$ leads to

$$\begin{aligned}w_{\overline{z}} &= \phi + T\rho, T\rho := -\frac{1}{\pi}\int_D \rho(\zeta)\frac{d\xi d\eta}{\zeta - z}\,, \\ w_z &= \psi' + \overline{z}\phi' + \frac{\partial}{\partial z}S\rho, \frac{\partial}{\partial z}S\rho = \frac{1}{\pi}\int_D \rho(\zeta)\frac{\overline{\zeta - z}}{(\zeta - z)^2}d\xi d\eta\,, \\ w_{\overline{z}z} &= \phi' + \Pi\rho, \Pi\rho := -\frac{1}{\pi}\int_D \rho(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}\,, \\ w_{zz} &= \psi'' + \overline{z}\phi'' + \frac{\partial^2}{\partial z^2}S\rho, \frac{\partial^2}{\partial z^2}S\rho = \frac{2}{\pi}\int_D \rho(\zeta)\frac{\overline{\zeta - z}}{(\zeta - z)^3}d\xi d\eta\,, \\ w_{\overline{z}\overline{z}} &= \rho\,.\end{aligned}$$

Inserting $w_{\overline{z}}$ into the boundary condition $\mathrm{Re}\, w_{\overline{z}} = 0$ gives

$$\mathrm{Re}\,\phi = -\mathrm{Re}\,T\rho =: h_0 \quad \text{on } \partial D\,. \tag{5.1.2}$$

This problem for analytic ϕ is solvable if and only if

$$\int_{\partial D} h_0(\zeta)\psi_\mu(\zeta)d\zeta = 0\,,\ 1 \le \mu \le m\,,$$

see Corollary 3.1. Applying GREEN'S theorem these conditions become

$$\begin{aligned}\int_{\partial D}(T\rho + \overline{T\rho})\psi_\mu d\zeta &= 2i\int_D \frac{\partial}{\partial\overline{\zeta}}(T\rho + \overline{T\rho})\psi_\mu d\xi d\eta \\ &= 2i\int_D(\rho + \overline{\Pi\rho})\psi_\mu d\xi d\eta = 0\,,\ 1 \le \mu \le m\,.\end{aligned}$$

If they are satisfied the solution is by Remark 1 on p. 36 with some fixed $z_0 \in \partial D$

$$\phi = \phi_0 + ic_0, \phi_0(z) := -\frac{1}{i}\int_{\partial D} J_L(z,\zeta)\mathrm{Re}\,T\rho(\zeta)d\zeta - \mathrm{Re}\,T\rho(z_0). \tag{5.1.3}$$

We have

$$\phi_0(z) = \frac{1}{2\pi i}\int\limits_D\int\limits_{\partial D}\left[\frac{J_L(z,\zeta)}{\tilde{\zeta}-\zeta}\rho(\tilde{\zeta}) - \frac{J_K(z,\overline{\zeta})}{\overline{\tilde{\zeta}-\zeta}}\,\frac{\overline{\zeta'(s)}}{\zeta(s)}\overline{\rho(\tilde{\zeta})}\right] d\zeta\, d\tilde{\xi}\, d\tilde{\eta} - \operatorname{Re} T\rho(z_0)\,,$$

where the integral version

$$J_L(z,\zeta)\zeta'(s) + J_K(z,\overline{\zeta})\overline{\zeta'(s)} = 0$$

of

$$L(z,\zeta)\zeta'(s) + K(z,\overline{\zeta})\overline{\zeta'(s)} = 0$$

is used. With

$$J_L(z,\zeta) = \frac{1}{\pi}\frac{1}{\zeta - z} - \frac{1}{\pi}\frac{1}{\zeta - z_0} - J_\ell(z,\zeta)\,,\ J_\ell(z,\zeta) := \int\limits_{z_0}^{z} \ell(t,\zeta)dt$$

and the CAUCHY formula then

$$\phi_0(z) = \int\limits_D \left\{J_\ell(z,\zeta)\rho(\zeta) - J_K(z,\overline{\zeta})\overline{\rho(\zeta)}\right\} d\xi d\eta - \operatorname{Re} T\rho(z_0)\,,$$

$$\phi_0(z) + T\rho(z) = -\int\limits_D \left\{J_L(z,\zeta)\rho(\zeta) + J_K(z,\overline{\zeta})\overline{\rho(\zeta)}\right\} d\xi d\eta + i\operatorname{Im} T\rho(z_0)\,. \quad (5.1.4)$$

This is just $w_{\overline{z}}$ up to the additive constant ic_0.
Inserting the above representation for w into the second boundary condition $\operatorname{Re}\{z'(s)w\} = 0$ leads to

$$\operatorname{Re}\{z'(s)\psi\} = -\operatorname{Re}\{z'(s)[\overline{z}\phi + S\rho]\} =: h^0 \quad \text{on } \partial D.$$

The solvability condition

$$\int\limits_{\partial D} h^0(\zeta)ds = 0\,,$$

see Corollary 3.2, serves to determine c_0 and hence to satisfy this condition just by the proper choice for c_0. We have with $|D| := measD$ the area of D

$$\begin{aligned} 0 &= \operatorname{Re}\int\limits_{\partial D}\{\overline{z}\phi(z) + S\rho(z)\}dz = \operatorname{Re} 2i\int\limits_D \frac{\partial}{\partial \overline{z}}\{\phi(z) + S\rho(z)\}dxdy \\ &= \operatorname{Re} 2i\int\limits_D\{\phi(z) + T\rho(z)\}dxdy \\ &= \operatorname{Re}\, 2i\int\limits_D\int\limits_D\left(J_\ell(z,\zeta)\rho(z) - J_K(z,\overline{\zeta})\overline{\rho(\zeta)}) - \frac{\rho(\zeta)}{\pi(\zeta - z)}\right) d\xi d\eta dxdy - 2|D|c_0\,, \end{aligned}$$

so that

$$c_0 = -\mathrm{Im}\frac{1}{|D|}\int\limits_D\int\limits_D\left\{\left(J_\ell(z,\zeta)-\frac{1}{\pi(\zeta-z)}+\frac{1}{\pi(\zeta-z_0)}\right)\rho(\zeta)-J_K(z,\overline{\zeta})\,\overline{\rho(\zeta)}\right\}d\xi d\eta dx dy$$

$$= \mathrm{Im}\frac{1}{|D|}\int\limits_D\int\limits_D\{J_L(z,\zeta)\rho(\zeta)+J_K(z,\overline{\zeta})\overline{\rho(\zeta)}\}d\xi d\eta dx dy - \mathrm{Im}\,T\rho(z_0)\,. \qquad (5.1.5)$$

The solution to the boundary value problem then is

$$\psi = \psi_0 + \sum_{\mu=1}^{m}\gamma_\mu\psi_\mu,\, \psi_0(z) = \frac{1}{\pi i}\int\limits_{\partial D}\left(\underset{\sim}{\ell}(z,\zeta)+\frac{1}{\zeta-z}\right)h^0(\zeta)ds_\zeta, \gamma_\mu \in \mathbb{R},\ 1\le\mu\le m\,,$$

see Remark on p. 66 . Although

$$\underset{\sim}{\ell}(z,\zeta) := \pi\int\limits_{z_0}^{\zeta}\ell(z,t)dt$$

is a multi-valued function for multiply connected domain D condition

$$\int\limits_{\partial D} h^0(\zeta)ds = 0$$

guarantees the single–valuedness of ψ_0. Because if $\underset{\sim}{\ell}(z,\zeta)$ for fixed $z\in D$ is replaced by $\underset{\sim}{\ell}(z,\zeta)$ plus a linear combination of the modules of periodicity this will result in the same value for $\psi_0(z)$.
Inserting h^0 in the integral representation for ψ_0 gives

$$\psi_0(z) = -\frac{1}{2\pi i}\int\limits_{\partial D}\left(\frac{1}{\zeta-z}+\underset{\sim}{\ell}(z,\zeta)\right)\left[(\overline{\zeta}\phi(\zeta)+S\rho(\zeta))d\zeta+(\zeta\overline{\phi(\zeta)}+\overline{S\rho(\zeta)})d\overline{\zeta}\right], \quad (5.1.6)$$

where ϕ is given in (5.1.3). Applying the GREEN formula with respect to the domain $D_\varepsilon := D\backslash\{\zeta : |\zeta-z|\le\varepsilon\}$ this function can be written as

$$\psi_0(z) = I + \lim_{\varepsilon\to 0}J_1(\varepsilon)\,,$$

$$I := -\frac{1}{\pi}\int\limits_D\left\{\frac{\partial}{\partial\overline{\zeta}}\left[\left(\frac{1}{\zeta-z}+\underset{\sim}{\ell}(z,\zeta)\right)(\overline{\zeta}\phi(\zeta)+S\rho(\zeta))\right]\right.$$
$$\left.-\frac{\partial}{\partial\zeta}\left[\left(\frac{1}{\zeta-z}+\underset{\sim}{\ell}(z,\zeta)\right)(\zeta\overline{\phi(\zeta)}+\overline{S\rho(\zeta)})\right]\right\}d\xi d\eta$$

$$
\begin{aligned}
&= -\frac{1}{\pi}\int_D \left\{\left(\frac{1}{\zeta - z} + \underset{\sim}{\ell}(z,\zeta)\right)(\phi(\zeta) + T\rho(\zeta) - \overline{\phi(\zeta)} - \overline{T\rho(\zeta)}) \right. \\
&\qquad \left. + \left(\frac{1}{(\zeta - z)^2} - \pi\ell(z,\zeta)\right)(\zeta\overline{\phi(\zeta)} + \overline{S\rho(\zeta)})\right\} d\xi d\eta \\
&= -\frac{2i}{\pi}\int_D \left(\frac{1}{\zeta - z} + \underset{\sim}{\ell}(z,\zeta)\right)\mathrm{Im}\,(\phi(\zeta) + T\rho(\zeta)) d\xi d\eta \\
&\quad - \int_D L(z,\zeta)(\zeta\overline{\phi(\zeta)} + \overline{S\rho(\zeta)}) d\xi d\eta
\end{aligned}
$$

because

$$\ell(z,\zeta) := \frac{1}{\pi(z-\zeta)^2} - L(z,\zeta)\,.$$

From the definition of $L(z,\zeta)$ the last term on the right-hand side turns out as

$$
\begin{aligned}
&+\frac{2}{\pi}\int_D \frac{\partial^2 g(z,\zeta)}{\partial z \partial \zeta}(\zeta\overline{\phi(\zeta)} + \overline{S\rho(\zeta)}) d\xi d\eta \\
&= -\frac{1}{\pi i}\int_{\partial D} g_z(z,\zeta)(\zeta\overline{\phi(\zeta)} + \overline{S\rho(\zeta)}) d\overline{\zeta} \\
&\qquad - \frac{2}{\pi}\int_D g_z(z,\zeta)(\overline{\phi(\zeta)} + \overline{T\rho(\zeta)}) d\xi d\eta + \lim_{\varepsilon\to 0} J_2(\varepsilon) \\
&= -\frac{1}{\pi}\int_D \left(2\omega_z(z,\zeta) - \frac{1}{z-\zeta}\right)(\overline{\phi(\zeta)}) + \overline{T\rho(\zeta)}) d\xi d\eta\,,
\end{aligned}
$$

since

$$
\begin{aligned}
J_2(\varepsilon) &= \frac{1}{2\pi i}\int_{|\zeta - z| = \varepsilon} \left(2\omega_z(z,\zeta) + \frac{1}{\zeta - z}\right)(\overline{\zeta\phi(\zeta)} + \overline{S\rho(\zeta)}) d\overline{\zeta} \\
&= -\frac{1}{2\pi}\int_0^{2\pi} (2\omega_z(z,\zeta)\varepsilon e^{i\varphi} + 1)(\zeta\overline{\phi(\zeta)} + S\rho(\zeta)) e^{2i\varphi} d\varphi\,,\ \zeta := z + \varepsilon e^{i\varphi}\,,
\end{aligned}
$$

tends to zero with ε.

$$J_1(\varepsilon) := -\frac{1}{2\pi i}\int_{|\zeta - z| = \varepsilon} \left(\frac{1}{\zeta - z} + \underset{\sim}{\ell}(z,\zeta)\right)[(\overline{\zeta}\phi(\zeta) + S\rho(\zeta)) d\zeta + (\zeta\overline{\phi(\zeta)} + \overline{S\rho(\zeta)}) d\overline{\zeta}]\,.$$

Similarly as $J_2(\varepsilon)$ the second term here tends to zero with ε, so that only $\widetilde{J}_1(\varepsilon)$ has to be considered where

$$\begin{aligned}
\widetilde{J}_1(\varepsilon) &:= -\frac{1}{2\pi i}\int\limits_{|\zeta-z|=\varepsilon}\left(\frac{1}{\zeta-z}+\underset{\sim}{\ell}(z,\zeta)\right)(\overline{\zeta}\phi(\zeta)+S\rho(\zeta))d\zeta \\
&= -\frac{1}{2\pi}\int\limits_0^{2\pi}(1+\varepsilon e^{i\varphi}\,\underset{\sim}{\ell}(z,\zeta))(\overline{\zeta}\phi(\zeta)+S\rho(\zeta))d\varphi\ ,\ \zeta := z+\varepsilon e^{i\varphi}\ , \\
\lim_{\varepsilon\to 0}\widetilde{J}_1(\varepsilon) &= -[\overline{z}\phi(z)+S\rho(z)]\ .
\end{aligned}$$

Hence,

$$\begin{aligned}
\psi_0(z) = -\frac{1}{\pi}\int\limits_D\Bigg\{&\left(\frac{1}{\zeta-z}+\underset{\sim}{\ell}(z,\zeta)\right)(\phi(\zeta)+T\rho(\zeta)) \\
&+(2\omega_z(z,\zeta)-\underset{\sim}{\ell}(z,\zeta))(\overline{\phi(\zeta)}+\overline{T\rho(\zeta)})\Bigg\}d\xi d\eta-\overline{z}\phi(z)-S\rho(z)\ ,
\end{aligned}$$

$$\psi_0(z)+\overline{z}\phi(z)+S\rho(z) = T\phi(z)+TT\rho(z)$$

$$-\frac{1}{\pi}\int\limits_D\left\{\underset{\sim}{\ell}(z,\zeta)(\phi(\zeta)+T\rho(\zeta))+(2\omega_z(z,\zeta)-\underset{\sim}{\ell}(z,\zeta))(\overline{\phi(\zeta)}+\overline{T\rho(\zeta)})\right\}d\xi d\eta\ . \quad (5.1.7)$$

This formula represents w up to the additive term $\sum_{\mu=1}^m \gamma_\mu\psi_\mu$. For the derivatives of w the derivatives of ϕ and ψ are needed.

$$\begin{aligned}
\phi'(z) &= \int\limits_D\{\ell(z,\zeta)\rho(\zeta)-K(z,\overline{\zeta})\overline{\rho(\zeta)}\}d\xi d\eta\ , \\
\psi_0'(z)+\overline{z}\phi'(z)+\frac{\partial}{\partial z}S\rho(z) &= \Pi(\phi+T\rho)(z)
\end{aligned}$$

$$-\frac{1}{\pi}\int\limits_D\left\{\underset{\sim}{\ell}_z(z,\zeta)(\phi(\zeta)+T\rho(\zeta))+(2\omega_{zz}(z,\zeta))-\underset{\sim}{\ell}_z(z,\zeta))(\overline{\phi(\zeta)}+\overline{T\rho(\zeta)})\right\}d\xi d\eta\ .(5.1.8)$$

This last expression is w_z up to the term $\sum_{\mu=1}^m \gamma_\mu\psi_\mu'$. For $w_{\overline{z}z}$ we see it coincides with

$$\begin{aligned}
S^0\rho := \phi'+\Pi\rho &= \int\limits_D\left\{\left[\ell(z,\zeta)-\frac{1}{\pi}\frac{1}{(\zeta-z)^2}\right]\rho(\zeta)-K(z,\overline{\zeta})\overline{\rho(\zeta)}\right\}d\xi d\eta \\
&= -\int\limits_D\{L(z,\zeta)\rho(\zeta)+K(z,\overline{\zeta})\overline{\rho(\zeta)}\}d\xi d\eta\ . \quad (5.1.9)
\end{aligned}$$

Because of Lemma 34 there is no need in finding w_{zz}. In fact up to the additive term $\sum_{\mu=1}^{m}\gamma_\mu\psi_\mu''$ the function w_{zz} is equal to

$$\psi_0''(z)+\bar{z}\phi''(z)+\frac{\partial^2}{\partial z^2}S\rho(z)=\Pi(\phi'+T\rho)(z)-\frac{1}{2\pi i}\int_{\partial D}[\phi(\zeta)+T\rho(\zeta)]\frac{d\bar{\zeta}}{(\zeta-z)^2}$$
$$-\frac{1}{\pi}\int_D\left\{\underset{\sim}{\ell}_{zz}(z,\zeta)[\phi(\zeta)+T\rho(\zeta)]+2\omega_{zzz}(z,\zeta)-\underset{\sim}{\ell}_{zz}(z,\zeta)(\overline{\phi(\zeta)}+\overline{T\rho(\zeta)})\right\}d\xi d\eta\ .$$

The operator on the right-hand side acting on ρ is involved. Its L_2-norm is not likely to be 1. For this reason the term w_{zz} is excluded from the differential operator to be considered later on. But S^0 can be shown to have L_2-norm 1, see [Dzhu92]. We reproduce the proof from [Bege93].

Lemma 37.

$$\|S^0\|_{L_2(\overline{D})}=1\ .$$

Proof. Let $\rho\in C_0^\infty(D)$ and denote

$$\begin{aligned}T^0\rho:=\phi_0+T\rho &= \int_D\left\{\left(J_\ell(z,\zeta)-\frac{1}{\pi}\frac{1}{\overline{\zeta-z}}\right)\rho(\zeta)-J_K(z,\overline{\zeta})\overline{\rho(\zeta)}\right\}d\xi d\eta\\ &= -\int_D\left\{J_L(z,\zeta)\rho(\zeta)+J_K(z,\overline{\zeta})\overline{\rho(\zeta)}\right\}d\xi d\eta+T\rho(z_0)\ .\end{aligned}$$

Then

$$\begin{aligned}\|S^0\rho\|^2_{L_2(\overline{D})} &= \int_D S^0\rho\overline{S^0}dxdy=\int_D\frac{\partial}{\partial z}T^0\rho\frac{\overline{\partial}}{\partial z}\overline{T^0\rho}dxdy\\ &= \int_D\left\{\frac{\partial}{\partial\bar{z}}\left(\frac{\partial T^0\rho}{\partial z}\overline{T^0\rho}\right)-\frac{\partial\rho}{\partial z}\overline{T^0\rho}\right\}dxdy\end{aligned}$$

$$=\frac{1}{2i}\int_{\partial D}\frac{\partial T^0\rho}{\partial z}\overline{T^0\rho}dz-\int_D\left\{\frac{\partial}{\partial z}(\rho\overline{T^0\rho})-\rho\bar{\rho}\right\}dxdy$$
$$=\frac{1}{2i}\int_{\partial D}\frac{\partial T^0\rho}{\partial z}\overline{^0\rho}dz+\frac{1}{2i}\int_{\partial D}\rho\overline{^0\rho}d\bar{z}+\int_D\rho\bar{\rho}dxdy=\|\rho\|^2_{L_2}\ ,$$

since

$$\frac{\partial}{\partial z}T^0\rho=S^0,\frac{\partial}{\partial\bar{z}}T^0\rho=\rho\ .$$

In order to verify the last equality we observe $\rho=0$ on ∂D and show

$$\frac{1}{2i}\int_{\partial D}\frac{\partial T^0\rho}{\partial z}\overline{T^0\rho}dz=0\ .$$

$$\frac{1}{2i}\int_{\partial D}\frac{\partial T^0\rho}{\partial z}\overline{T^0\rho}dz$$
$$=\frac{1}{2i}\int_{\partial D}\int_D\int_D\left\{\left(\overline{J_\ell(z,\zeta)}-\frac{1}{\pi}\frac{1}{\overline{\zeta-z}}\right)\overline{\rho(\zeta)}-\overline{J_K(z,\overline{\zeta})\rho(\zeta)}\right\}$$
$$\times\left\{\left(\ell(z,\widetilde{\zeta})-\frac{1}{\pi}\frac{1}{(\widetilde{\zeta}-z)^2}\right)\rho(\widetilde{\zeta})-K(z,\overline{\widetilde{\zeta}})\overline{\rho(\widetilde{\zeta})}\right\}d\xi d\eta d\widetilde{\xi}d\widetilde{\eta}dz\,.$$

(i) $$\frac{1}{2i}\int_{\partial D}\ell(z,\widetilde{\zeta})\overline{J_\ell(z,\zeta)}dz=\int_D\frac{\partial}{\partial\overline{z}}(\ell(z,\widetilde{\zeta})\overline{J_\ell(z,\zeta)})dxdy=\int_D\ell(z,\widetilde{\zeta})\overline{\ell(z,\zeta)}dxdy$$
$$=\frac{1}{\pi}\int_D\frac{\overline{\ell(z,\zeta)}}{(z-\widetilde{\zeta})^2}dxdy=\frac{\partial}{\partial\widetilde{\zeta}}\frac{1}{\pi}\int_D\frac{\partial}{\partial\overline{z}}\left(\frac{1}{z-\widetilde{\zeta}}\overline{J_\ell(z,\zeta)}\right)dxdy$$
$$=\frac{\partial}{\partial\widetilde{\zeta}}\left\{\frac{1}{2\pi i}\int_{\partial D}\frac{1}{z-\widetilde{\zeta}}\overline{J_\ell(z,\zeta)}dz-\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{|z-\widetilde{\zeta}|=\epsilon}\frac{1}{z-\widetilde{\zeta}}\overline{J_\ell(z,\zeta)}dz\right\}$$
$$=\frac{1}{2\pi i}\int_{\partial D}\frac{1}{(z-\widetilde{\zeta})^2}\overline{J_\ell(z,\zeta)}dz-\frac{\partial}{\partial\widetilde{\zeta}}\overline{J_\ell(\widetilde{\zeta},\zeta)}=\frac{1}{2\pi i}\int_{\partial D}\frac{1}{(z-\widetilde{\zeta})^2}\overline{J_\ell(z,\zeta)}dz\,.$$

To obtain here the third equality

$$\int_D\ell(z,\widetilde{\zeta})\overline{f(z)}dxdy=\frac{1}{\pi}\int_D\frac{\overline{f(z)}}{(\widetilde{\zeta}-z)^2}dxdy\ ,\ z\in D\ ,$$

see proof of Lemma 5, is applied for $f(z)=\ell(z,\zeta)$. Hence,

$$\int_{\partial D}\left(\ell(z,\widetilde{\zeta})-\frac{1}{\pi}\frac{1}{(\widetilde{\zeta}-z)^2}\right)\overline{J_\ell(z,\zeta)}dz=0\,.$$

(ii) $$\frac{1}{2\pi i}\int_{\partial D}\frac{\ell(z,\widetilde{\zeta})}{\overline{\zeta-z}}dz=\frac{1}{\pi}\int_D\frac{\partial}{\partial\overline{z}}\frac{\ell(z,\widetilde{\zeta})}{\overline{\zeta-z}}dxdy=\frac{1}{\pi}\int_D\frac{\ell(z,\widetilde{\zeta})}{(\overline{\zeta}-z)^2}dxdy$$

which follows from applying the GREEN formula to D_ϵ. On the other hand from the definition of $\ell(z,\widetilde{\zeta})$

$$\frac{1}{2\pi i}\int_{\partial D}\frac{\ell(z,\widetilde{\zeta})}{\overline{\zeta-z}}dz=\frac{1}{2\pi^2 i}\int_{\partial D}\frac{dz}{(z-\widetilde{\zeta})^2(\overline{\zeta-z})}-\frac{1}{2\pi i}\int_{\partial D}\frac{L(z,\widetilde{\zeta})}{\overline{\zeta-z}}dz$$

follows and by $L(z,\zeta)d\zeta+K(z,\overline{\zeta})d\overline{\zeta}=0$ for $\zeta\in\partial D$, $z\in D$, $K(z,\overline{\zeta})=\overline{K(\zeta,\overline{z})}$ and the analyticity of $K(\cdot,\overline{\widetilde{\zeta}})$

$$\frac{1}{2\pi i}\int_{\partial D}\frac{L(z,\widetilde{\zeta})}{\overline{\zeta-z}}dz=\overline{\frac{1}{2\pi i}\int_{\partial D}\frac{K(z,\overline{\widetilde{\zeta}})}{\zeta-z}dz}=-\overline{K(\zeta,\overline{\widetilde{\zeta}})}=-K(\widetilde{\zeta},\overline{\zeta}).$$

Hence,

$$\frac{1}{2\pi i}\int\limits_{\partial D}\left\{\ell(z,\tilde{\zeta}) - \frac{1}{\pi}\frac{1}{(\tilde{\zeta}-z)^2}\right\}\frac{dz}{\overline{\zeta - z}} = K(\tilde{\zeta},\bar{\zeta})\,.$$

(*iii*)
$$\frac{1}{2i}\int\limits_{\partial D}\left(\overline{J_\ell(z,\zeta)} + \frac{1}{\pi}\frac{1}{\overline{z-\zeta}}\right)K(z,\bar{\tilde{\zeta}})dz$$
$$= \int\limits_{D}\frac{\partial}{\partial\bar{z}}\left(\overline{J_\ell(z,\zeta)} + \frac{1}{\pi}\frac{1}{\overline{z-\zeta}}\right)K(z,\bar{\tilde{\zeta}})dxdy$$
$$= \int\limits_{D}\left(\overline{\ell(z,\zeta)} - \frac{1}{\pi}\frac{1}{\overline{(z-\zeta)^2}}\right)K(z,\bar{\tilde{\zeta}})dxdy = 0\,,$$

where again Lemma 5 is applied.

(*iv*)
$$\frac{1}{2i}\int\limits_{\partial D}\left(\ell(z,\tilde{\zeta}) - \frac{1}{\pi}\left(\frac{1}{\tilde{\zeta}-z}\right)^2\right)\overline{J_K(z,\bar{\zeta})}dz$$
$$= \frac{\partial}{\partial\tilde{\zeta}}\frac{1}{2i}\int\limits_{\partial D}\left(\frac{1}{\pi}\,\underset{\sim}{\ell}(z,\tilde{\zeta}) + \frac{1}{\pi}\frac{1}{\tilde{\zeta}-z}\right)\overline{J_K(z,\bar{\zeta})}dz$$
$$= \frac{\partial}{\partial\tilde{\zeta}}\int\limits_{D}\frac{\partial}{\partial\bar{z}}\left(\frac{1}{\pi}\,\underset{\sim}{\ell}(z,\tilde{\zeta}) + \frac{1}{\pi}\frac{1}{\tilde{\zeta}-z}\right)\overline{J_K(z,\bar{\zeta})}dxdy$$
$$+\lim_{\epsilon\to 0}\frac{\partial}{\partial\tilde{\zeta}}\frac{1}{2i}\int\limits_{|z-\tilde{\zeta}|=\epsilon}\left(\frac{1}{\pi}\,\underset{\sim}{\ell}(z,\tilde{\zeta}) + \frac{1}{\pi}\frac{1}{\tilde{\zeta}-z}\right)\overline{J_K(z,\bar{\zeta})}dz$$
$$= \int\limits_{D}\left(\ell(z,\tilde{\zeta}) - \frac{1}{\pi}\frac{1}{(\tilde{\zeta}-z)^2}\right)\overline{K(z,\bar{\zeta})}dxdy - \frac{\partial}{\partial\tilde{\zeta}}\overline{J_K(\tilde{\zeta},\bar{\zeta})} = 0$$

once more by applying Lemma 5 in complex conjugate form and because of the analyticity of $J_K(z,\bar{\zeta})$.

(*v*)
$$\frac{1}{2i}\int\limits_{\partial D}K(z,\bar{\tilde{\zeta}})J_K(z,\bar{\zeta})dz = \int\limits_{D}\frac{\partial}{\partial\bar{z}}(K(z,\bar{\tilde{\zeta}})J_K(z,\bar{\zeta}))dxdy$$
$$= \int\limits_{D}(K(z,\bar{\tilde{\zeta}})\,\overline{K(z,\bar{\zeta})})dxdy$$
$$= \int\limits_{D}K(z,\bar{\tilde{\zeta}})K(\zeta,\bar{z})dxdy = K(\zeta,\bar{\tilde{\zeta}})$$

in view of the reproducing property of the BERGMAN kernel K.
These estimates (i) to (v) show

$$\frac{1}{2i}\int_{\partial D}\frac{\partial}{\partial z}T^0\rho\overline{T^0\rho}dz = -\int_D\int_D\left[K(\tilde{\zeta},\overline{\zeta})\overline{\rho(\zeta)}\rho(\tilde{\zeta}) - K(\zeta,\overline{\tilde{\zeta}})\rho(\zeta)\overline{\rho(\tilde{\zeta})}\right]d\xi d\eta d\tilde{\xi}d\tilde{\eta} = 0 .$$

After these preparations we are leading our attention towards the differential equation

$$w_{\overline{z}\overline{z}} + \mu_1 w_{\overline{z}z} + \mu_2\overline{w_{\overline{z}z}} + a_1 w_{\overline{z}} + a_2\overline{w_{\overline{z}}} + b_1 w_z + b_2\overline{w_z} + c_1 w + c_2\overline{w} + d = 0 \text{ in } D \quad (5.1.10)$$

and combine this with the boundary value problem

$$\operatorname{Re} w_{\overline{z}} = \operatorname{Re}\{z'(s)w\} = 0 \quad \text{on } \partial D .$$

Assuming this problem to have a solution w we set $\rho := w_{\overline{z}\overline{z}}$ and notice by the preceding considerations that thus w is representable by analytic functions ϕ and ψ in the form $w = \psi + \overline{z}\phi + S\rho$, where ϕ and ψ are given as stated in Theorem 50. Inserting this representation into the differential equation gives us an integral equation for the density ρ, namely

$$\rho + \mu_1 S^0\rho + \mu_2\overline{S^0\rho} + K^0\rho + d_0 = 0 . \quad (5.1.11)$$

Here S^0 is the singular integral operator from (5.1.9) and

$$\begin{aligned} K^0\rho &:= a_1(\phi + T\rho) + a_2(\overline{\phi} + \overline{T\rho}) + b_1\left(\psi_0' + \overline{z}\phi' + \frac{\partial}{\partial z}S\rho\right) + b_2\left(\overline{\psi_0'} + z\overline{\phi'} + \overline{\frac{\partial}{\partial z}S\rho}\right) \\ &\quad + c_1(\psi_0 + \overline{z}\phi + S\rho) + c_2(\overline{\psi_0} + z\overline{\phi} + \overline{S\rho}) , \\ d_0 &:= d + \sum_{\mu=1}^{m}\gamma_\mu[b_1\psi_\mu' + b_2\overline{\psi_\mu'} + c_1\psi_\mu + c_2\overline{\psi_\mu}] . \end{aligned}$$

Here we have to apply some results from the theory of integral equations the proof of which cannot be reproduced here. A classical reference for the FREDHOLM theory is [Cohi53]. Singular – one–dimensional – integral equations are studied in [Musk53], [Mipr80], [Proe78]. For a reduction of a singular integral equation to some FREDHOLM equation see [Musk53], p. 149. The operator in (5.1.11) turns out to be quasi-FREDHOLM. The inverse to $(I + \widehat{S})\rho := I\rho + \mu_1 S^0\rho + \mu_2\overline{S^0\rho}$, where I is the identity operator, exists in $L_p(\overline{D})$ for $0 < p-2$ small enough so that $q_0\|S^0\|_{L_p(\overline{D})} < 1$. Applying this inverse to (3.5.4) reduces this singular integral equation to an equivalent integral equation

$$\rho + K^1\rho + (I + \widehat{S})^{-1}d_0 = 0 \quad (5.1.12)$$

with a linear operator

$$K^1 := (I + \widehat{S})^{-1} K^0 = \sum_{k=0}^{+\infty} (-1)^k \widehat{S}^k K^0$$

which maps $L_p(\overline{D})$ into $L_p(\overline{D})$ where $2 < p$ satisfies $q_0 \|S^0\|_{L_p(\overline{D})} < 1$.

Definition 18. *A linear operator K from a* BANACH *space A into a* BANACH *space B is called compact or completely continuous if the image KA_0 of any bounded subset A_0 of A is a compact set in B.*

Lemma 38. *Let $a_1, a_2, b_1, b_2, c_1, c_2 \in C^\alpha(\overline{D}; \mathbb{C})$. Then if $0 < \alpha \leq \alpha_0 := \dfrac{p-2}{p}, 2 < p, K^0$ is a compact operator from $L_p(\overline{D})$ into $C^\alpha(D)$.*

Proof. The compactness of the T-operator on $L_p(\overline{D})$ follows from Theorem 23 by the ARZELÀ–ASCOLI theorem, see [Tayl58]. From $S\rho = \overline{z} T\rho - T(\overline{\zeta}\rho)$ the S-operator is compact, too. For showing ϕ to represent a compact operator let $z_1, z_2 \in \partial D$ both lie on one continuum of ∂D. Then choosing the path of integration along ∂D from z_1 to z_2

$$\begin{aligned}
J_\ell(z_2,\zeta) - J_\ell(z_1,\zeta) &= \int_{z_1}^{z_2} \ell(t,\zeta)dt\,, \\
J_K(z_2,\zeta) - J_K(z_1,\zeta) &= \int_{z_1}^{z_2} K(t,\overline{\zeta})dt = -\frac{2}{\pi}\int_{z_1}^{z_2} \frac{\partial}{\partial t} g_{\overline{\zeta}}(t,\zeta)dt = \frac{2}{\pi}\int_{z_1}^{z_2} \frac{\partial}{\partial \overline{t}} g_{\overline{\zeta}}(t,\zeta)\overline{dt} \\
&= \frac{2}{\pi}\int_{z_1}^{z_2} \overline{\frac{\partial}{\partial t} g_\zeta(t,\zeta)dt} = -\int_{z_1}^{z_2} \overline{L(t,\zeta)dt} = \overline{J_L(z_1,\zeta)} - \overline{J_L(z_2,\zeta)} \\
&= \overline{J_\ell(z_2,\zeta)} - \overline{J_\ell(z_1,\zeta)} + \frac{1}{\pi}\left(\frac{1}{\overline{\zeta - z_1}} - \frac{1}{\overline{\zeta - z_2}}\right).
\end{aligned}$$

Thus for $z_1, z_2 \in \partial D$

$$\begin{aligned}
\phi(z_2) - \phi(z_1) &= 2i\mathrm{Im}\int_D (J_\ell(z_2,\zeta) - J_\ell(z_1,\zeta))\rho(\zeta)d\xi d\eta \\
&\quad + \frac{1}{\pi}\int_D \left(\frac{1}{\overline{\zeta - z_2}} - \frac{1}{\overline{\zeta - z_1}}\right) d\xi d\eta \\
&= 2i\mathrm{Im}\int_D \int_{z_1}^{z_2} \ell(t,\zeta)dt\rho(\zeta)d\xi d\eta + \overline{T\rho(z_2)} - \overline{T\rho(z_1)}\,.
\end{aligned}$$

From the boundedness of $\ell(z,\zeta)$ and well known properties of the T–operator, see Theorem 23,

$$|\phi(z_2) - \phi(z_1)| \leq M(p, D)\|\rho\|_{L_p(\overline{D})}|z_2 - z_1|^{\alpha_0}, \alpha_0 = \frac{p-2}{p} ,$$

follows. The PRIVALOV theorem, Theorem 6, then guarantees that ϕ satisfies the last inequality on $\overline{D}$ rather than just on ∂D. It also ensures that for smooth boundary ∂D the boundary integral (5.1.6) defining ψ_0 satisfies an estimate of this kind, too. Since $\phi + T\rho$ is HÖLDER continuous and

$$C_\alpha(\Pi f; \overline{D}) \leq M(\alpha, D)C_\alpha(f; \overline{D}) \quad \text{for} \quad f \in C^\alpha(\overline{D}) ,$$

see proof of Theorem 29, the operator $\Pi(\phi + T\rho)$ is compact for $\rho \in L_p(\overline{D}), 2 < p$. For the other term in (5.1.8) this follows from Corollary 11. That the operator on the right–hand side of (5.1.7) is compact in $L_p(\overline{D})$ follows besides from properties of the T–operator from analyticity. Thus

$$\begin{aligned} C_\alpha(K^0\rho; \overline{D}) \;&\leq\; [C_\alpha(a_1; \overline{D}) + C_\alpha(a_2; \overline{D})]C_\alpha(\phi + T\rho; \overline{D}) \\ &\quad +[C_\alpha(b_1; \overline{D}) + C_\alpha(b_2; \overline{D})]C_\alpha(\psi_0' + \overline{z}\phi' + \frac{\partial}{\partial z}S\rho; \overline{D}) \\ &\quad +[C_\alpha(c_1; \overline{D}) + C_\alpha(c_2; \overline{D})]C_\alpha(\psi_0 + \overline{z}\phi + S\rho; \overline{D}) \\ &\leq\; M(\alpha, p, K_1, K_2, D)\|\rho\|_{L_p(\overline{D})} , \end{aligned}$$

where K_1, K_2 are nonnegative constants such that

$$C_\alpha(a_1; \overline{D}) + C_\alpha(a_2; \overline{D}) \leq K_1 ,$$

$$C_\alpha(b_1; \overline{D}) + C_\alpha(b_2; \overline{D}) + C_\alpha(c_1; \overline{D}) + C_\alpha(c_2; \overline{D}) \leq K_2$$

and $\alpha \leq \alpha_0 := \dfrac{p-2}{p}$.

Lemma 39. *Let $a_1, a_2, b_1, b_2, c_1, c_2 \in L_p(\overline{D}), 2 < p$. Then K^0 is a compact operator from $L_p(\overline{D})$ into $L_p(\overline{D})$.*

Proof. For $\rho \in L_p(\overline{D})$ we have

$$\begin{aligned} \|K^0\rho\|_{L_p(\overline{D})} \;&\leq\; \left[\|a_1\|_{L_p(\overline{D})} + \|a_2\|_{L_p(\overline{D})}\right] C_\alpha(\phi + T\rho; \overline{D}) \\ &\quad + \left[\|b_1\|_{L_p(\overline{D})} + \|b_2\|_{L_p(\overline{D})}\right] C_\alpha(\psi' + \overline{z}\phi' + \frac{\partial}{\partial z}S\rho; \overline{D}) \\ &\quad + \left[\|c_1\|_{L_p(\overline{D})} + \|c_2\|_{L_p(\overline{D})}\right] C_\alpha(\psi_0 + \overline{z}\phi + S\rho; \overline{D}) \\ &\leq\; M\|\rho\|_{L_p(\overline{D})} . \end{aligned}$$

Let (ρ_n) be a bounded sequence in $L_p(\overline{D})$ then $(\phi_n + T\rho_n)$ where $\phi_n = \phi(\rho_n)$, is bounded in $C_\alpha(D)$ and hence by the ARZELÀ- ASCOLI theorem there is a convergent subsequence $(\phi_{n_k} + T\rho_{n_k})$. Then $(a_1(\phi_{n_k} + T\rho_{n_k}) + a_2(\overline{\phi_{n_k} + T\rho_{n_k}}))$ is a convergent sequence in $L_p(\overline{D})$. The same argument holds for the other terms.

Lemma 40. *The operator $K^1 := (I + \widehat{S})^{-1}K^0$ is a compact operator from $L_p(\overline{D})$ into itself if K^0 is.*

Proof. Let $A \subset L_p(\overline{D})$ be a bounded subset i.e. $\|f\|_{L_p(\overline{D})} \leq M$ for all $f \in A$. Then $K^0A \subset C^\alpha(D)$ is compact, since K^0 is compact, see Lemma 38. Hence there exists a sequence $(K^0 f_k)$ in $K^0A, f_k \in A, k \in I\!N$, which is convergent on $C^\alpha(D)$. Since $(I + \widehat{S})^{-1}$ is a bounded linear operator mapping $C^\alpha(D)$ continuously into $L_p(\overline{D})$ the sequence $((I + \widehat{S})^{-1}K^0 f_k)$ is convergent in $L_p(\overline{D})$.

Hence equation (5.1.12) is a FREDHOLM equation, see [Cohi53], [Musk53], [Veku62], [Mipr80] and the FREDHOLM alternative applies.

Fredholm alternative. Let $K\rho = f$ be a FREDHOLM equation and $\widetilde{K}\sigma = 0$ its adjoint. Then

i. The homogeneous equation $K\rho = 0$ has only finitely many linearly independent solutions. The adjoint equation $\widetilde{K}\sigma = 0$ has exactly the same number of linearly independent solutions.

ii. The inhomogeneous equation $K\rho = f$ is solvable for any right-hand side f, if and only if the adjoint equation $\widetilde{K}\sigma = 0$ is only trivially solvable. Otherwise the inhomogeneous equation is solvable if and only if f is orthogonal to the solution space to the adjoint problem $\widetilde{K}\sigma = 0$. There are as many solvability conditions as the homogeneous problem has linearly independent solutions.

Theorem 51. *Let the coefficients of the second order equation (5.1.10) satisfy*

$$|\mu_1(z)| + |\mu_2(z)| \leq q_0 < 1, \|a_1\|_{L_p(\overline{D})} + \|a_2\|_{L_p(\overline{D})} \leq K_1 \ ,$$

$$\|b_1\|_{L_p(\overline{D})} + \|b_2\|_{L_p(\overline{D})} + \|c_1\|_{L_p(\overline{D})} + \|c_2\|_{L_p(\overline{D})} \leq K_2, 2 < p, q_0\Lambda_p < 1 \ .$$

Then the homogeneous boundary value probelm

$$\operatorname{Re} w_{\overline{z}} = \operatorname{Re}\{z'(s)w\} = 0 \quad \text{on } \partial D$$

for equation (5.1.10) has the FREDHOLM *property.*

Proof. It was just explained that for the integral equation (5.1.12) the FREDHOLM alternative does apply. Let $\rho^0 \in L_p(\overline{D})$ be a solution to (5.1.12) in the homogeneous case $d_0 = 0$ and denote by ψ_0^0, ϕ^0 the related functions from (5.1.7) and (5.1.4), respectively. Then

$$w^0 := \psi_0^0 + \overline{z}\phi^0 + S\rho^0$$

is a solution to the homogeneous problem $d = 0$. If the coefficients b_1, b_2, c_1, c_2, too vanish identically (5.1.10) is just the generalized BELTRAMI equation for $w_{\bar z}$. Because of the boundary condition $\operatorname{Re} w_{\bar z} = 0$ by Theorem 38 it follows that w is the trivial solution $w = 0$. In case when b_1, b_2, c_1, c_2 are small enough w can again be shown to vanish identically. This can be done on the basis of the a priori estimate (3.3.10). The same is true for the singular integral equation (5.1.12).
But in general the homogeneous problem (5.1.12), $d_0 = 0$, has $N \geq 0$ nontrivial over $\mathbb{R}$ linearly independent solutions $\rho_\nu, 1 \leq \nu \leq N$, while the inhomogeneous problem is solvable if and only if the right–hand side d_0 satisfies N orthogonality conditions

$$\operatorname{Re} \int_D d_0(z)\sigma_\nu(z)dxdy = 0,\ 1 \leq \nu \leq N.$$

Here $\sigma_1, \cdots, \sigma_N$ are linearly independent solutions to the adjoint equation to 5.1.11. These conditions lead to a linear algebraic system of N equations for the real coefficients $\gamma_\mu, 1 \leq \mu \leq m$, in the general solution ψ.
Let the rank of the coefficients matrix of this linear system be $r \geq 0$. Then r of the $\gamma_\mu' s$ can be expressed by the remaining $m - r$ ones leaving $N - r$ equations as solvability conditions on d and the remaining $\gamma_\mu' s$. Together with the m conditions for (5.1.2) to be solvable where ρ has to be replaced by

$$\rho_p + \sum_{\nu=1}^{N} c_\nu \rho_\nu$$

there are $N - r + m$ solvability conditions. Here ρ_p is a particular solution to the inhomogeneous equation (5.1.11) expressed through d_0.
The number of linearly independent solutions to the homogeneous problem is $N+m-r$, too. This can be seen by considering (5.1.11) for $d = 0$ and $\gamma_\mu = \delta_{\mu\nu}$ where ν varies over the indices of the remaining $m - r$ coefficients γ_μ.

We are now turning to the second representation formula in Theorem 49 and Corollary 10.

Theorem 52. *Under the same assumption as in Theorem 50 there exists a unique solution to the boundary value problem*

$$\operatorname{Re} w_{\bar z} = \operatorname{Re}\{z'(s)w\} = 0 \text{ on } \partial D$$

within the set of functions representable in the form

$$w = \psi + \overline{\phi} + S_1\overline{\rho},\ S_1\rho(z) := -\frac{2}{\pi}\int_D g(z,\zeta)\rho(\zeta)d\xi d\eta\ ,$$

with functions ϕ and ψ analytic in D and continuous in $\overline{D}$. The solution is given by

$$w(z) = \sum_{\mu=1}^{m} \gamma_\mu \psi_\mu(z) + T(\overline{\widetilde{\phi} + T\rho})(z)$$
$$-\frac{1}{\pi}\int_D \left\{ \underset{\sim}{\ell}(z,\zeta)(\overline{\widetilde{\phi}(\zeta) + T\rho(\zeta)}) + (2\omega_z(z,\zeta) - \underset{\sim}{\ell}(z,\zeta))(\widetilde{\phi}(\zeta) + T\rho(\zeta))\right\} d\xi d\eta ,$$

where $\gamma_\mu \in \mathbb{R}, 1 \le \mu \le m, \widetilde{\phi}$ is identical with ϕ from Theorem 50 and

$$\phi'(z) = \widetilde{\phi}(z) + \frac{2}{\pi}\int_D \omega_z(z,\zeta)\rho(\zeta) d\xi d\eta .$$

Proof. Differentiating the representation formula for w gives

$$\begin{aligned}
w_z &= \psi' - \frac{2}{\pi}\int_D \omega_z(z,\zeta)\overline{\rho(\zeta)} d\xi d\eta - \frac{1}{\pi}\int_D \overline{\rho(\zeta)}\frac{d\xi d\eta}{\zeta - z} = \widetilde{\psi} + T\overline{\rho} ,\\
w_{\overline{z}} &= \overline{\phi'} - \frac{2}{\pi}\int_D \omega_{\overline{z}}(z,\zeta)\overline{\rho(\zeta)} d\xi d\eta - \frac{1}{\pi}\int_D \overline{\rho(\zeta)\frac{d\xi d\eta}{\zeta - z}} = \overline{\widetilde{\phi}} + \overline{T\rho} ,
\end{aligned}$$

with

$$\begin{aligned}
\widetilde{\psi} &:= \psi' - \frac{2}{\pi}\int_D \omega_z(z,\zeta)\overline{\rho(\zeta)} d\xi d\eta ,\\
\widetilde{\phi} &:= \phi' - \frac{2}{\pi}\int_D \omega_z(z,\zeta)\rho(\zeta) d\xi d\eta ,
\end{aligned}$$

and

$$\begin{aligned}
w_{\overline{z}\overline{z}} &= \overline{\phi''} - \frac{2}{\pi}\int_D \omega_{\overline{z}\overline{z}}(z,\zeta)\overline{\rho(\zeta)} d\xi d\eta - \frac{1}{\pi}\int_D \overline{\rho(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}} = \overline{\widetilde{\phi}'} + \overline{\Pi\rho} ,\\
w_{zz} &= \psi'' - \frac{2}{\pi}\int_D \omega_{zz}(z,\zeta)\overline{\rho(\zeta)} d\xi d\eta - \frac{1}{\pi}\int_D \overline{\rho(\zeta)}\frac{d\xi d\eta}{(\zeta - z)^2} = \widetilde{\psi}' + \Pi\overline{\rho} ,\\
w_{\overline{z}z} &= \overline{\rho} .
\end{aligned}$$

The first boundary condition implies

$$\mathrm{Re}\,\widetilde{\phi} = -\mathrm{Re}\,T\rho =: h_0 \text{ on } \partial D .$$

This is just (5.1.2) now for $\widetilde{\phi}$ rather than ϕ. Under the same conditions this problem is solvable and $\widetilde{\phi}$ is given by (5.1.3) again replacing ϕ there. The constant c_0 still has to be determined.
The second boundary condition means

$$\mathrm{Re}\,\{z'(s)\psi\} = -\mathrm{Re}\,\{\overline{z'(s)}(\phi + S_1\rho)\} = h^0 \text{ on } \partial D .$$

This problem is solvable if and only if

$$\int_{\partial D} h^0(\zeta)\overline{\zeta'(s)}d\zeta = \int_{\partial D} h^0(\zeta)ds = 0 \,,$$

i.e.

$$-\mathrm{Re}\int_{\partial D}(\phi(\zeta)+S_1\rho(\zeta))d\overline{\zeta} = \mathrm{Re}\, 2i\int_D \frac{\partial}{\partial\zeta}(\phi(\zeta)+S_1\rho(\zeta))d\xi d\eta = 0$$

or

$$\mathrm{Im}\int_D \{\widetilde{\phi}(\zeta)+T\rho(\zeta)\}d\xi d\eta = 0 \,.$$

Because $S_1\rho$ is vanishing on ∂D the term $S_1\rho$ can be neglected in the boundary condition. The advantage for keeping it is that then the solvability condition is expressed through $\widetilde{\phi}$ rather than through ϕ'. Plugging in $\widetilde{\phi}$ given as ϕ in (5.1.3) and (5.1.4) leads to the same equation for c_0 which therefore turns out to be (5.1.5). As before then

$$\psi = \widetilde{\psi}_0 + \sum_{\mu=1}^{m}\gamma_\mu\psi_\mu \,,$$

$$\widetilde{\psi}_0 = -\frac{1}{2\pi i}\int_{\partial D}\left[\frac{1}{\zeta - z} + \underset{\sim}{\ell}(z,\zeta)\right]\left[\left(\phi(\zeta)+S_1\rho(\zeta)\right)d\overline{\zeta} + \left(\overline{\phi(\zeta)} + \overline{S_1\rho(\zeta)}\right)d\zeta\right] \,. \quad (5.1.13)$$

Although ϕ only given through its derivative via $\widetilde{\phi}$ in the multliply connected domain D in general is multi-valued the sum $\overline{\phi}+\widetilde{\psi}_0$ is uniquely defined single-valued function. This follows from

$$\frac{1}{2\pi i}\int_{\partial D}\underset{\sim}{\ell}(z,\zeta)d\overline{\zeta} = -\frac{2}{\pi}\int_D \frac{\partial^2 h(z,\zeta)}{\partial z\partial\zeta}d\xi d\eta = \frac{1}{\pi i}\int_{\partial D} h_z(z,\zeta)d\overline{\zeta} = -\frac{1}{2\pi i}\int_{\partial D}\frac{d\overline{\zeta}}{\zeta - z}$$

and

$$\frac{1}{2\pi i}\int_{\partial}\underset{\sim}{\ell}(z,\zeta)d\zeta = 0 \,.$$

Therefore for $c \in \mathbb{C}$

$$\frac{1}{2\pi i}\int_{\partial D}\left[\frac{1}{\zeta - z} + \underset{\sim}{\ell}(z,\zeta)\right][c d\overline{\zeta} + \overline{c}d\zeta] = \overline{c} \,.$$

Thus, if ϕ is replaced by $\phi + c$ for constant c then $\widetilde{\psi}_0$ switches to $\widetilde{\psi}_0 - \overline{c}$. Again as before by the GREEN formula applied for the domain $D_\varepsilon = D\backslash\{\zeta : |\zeta - z| \le \varepsilon\}$ the function $\widetilde{\psi}_0$ is rewritten as

$$\widetilde{\psi}_0 = \widetilde{J} + \lim_{\varepsilon\to 0}\widetilde{J}_1(\varepsilon) \,.$$

$$\begin{aligned}
\tilde{J} &= \frac{1}{\pi}\int_D \left\{ \frac{\partial}{\partial\zeta}\left[\left(\frac{1}{\zeta - z}\ \underset{\sim}{\ell}(z,\zeta)\right)(\phi(\zeta) + S_1\rho(\zeta))\right]\right. \\
&\qquad \left. -\frac{\partial}{\partial\overline{\zeta}}\left[\left(\frac{1}{\zeta - z} + \underset{\sim}{\ell}(z,\zeta)\right)(\overline{\phi(\zeta)} + \overline{S_1\rho(\zeta)})\right]\right\} d\xi d\eta \\
&= \int_D \left\{ \left(\ell(z,\zeta) - \frac{1}{\pi(\zeta - z)^2}\right)(\phi(\zeta) + S_1\rho(\zeta)) \right. \\
&\qquad \left. +\frac{1}{\pi}\left(\frac{1}{\zeta - z} + \underset{\sim}{\ell}(z,\zeta)\right)(\tilde{\phi}(\zeta) + T\rho(\zeta) - \overline{\tilde{\phi}(\zeta)} - \overline{T\rho(\zeta)})\right\} d\xi d\eta \\
&= -\int_D L(z,\zeta)(\phi(\zeta) + S_1\rho(\zeta))d\xi d\eta \\
&\qquad +\frac{2i}{\pi}\int_D \left(\frac{1}{\zeta - z} + \ \underset{\sim}{\ell}(z,\zeta)\right)\operatorname{Im}(\tilde{\phi}(\zeta) + T\rho(\zeta))d\xi d\eta\ ,
\end{aligned}$$

$$\begin{aligned}
-\int_D L(z,\zeta)(\phi(\zeta) + S_1\rho(\zeta))d\xi d\eta &= \frac{2}{\pi}\int_D \frac{\partial^2 g(z,\zeta)}{\partial z\partial\zeta}(\phi(\zeta) + S_1\rho(\zeta))d\xi d\eta \\
&= -\frac{1}{\pi i}\int_{\partial D} g_z(z,\zeta)(\phi(\zeta) + S_1\rho(\zeta))d\overline{\zeta} \\
&\qquad -\frac{2}{\pi}\int_D g_z(z,\zeta)(\tilde{\phi}(\zeta) + T\rho(\zeta))d\xi d\eta + \lim_{\epsilon\to 0}\tilde{J}_2(\epsilon) \\
&= -\frac{1}{\pi}\int_D \left(2\omega_z(z,\zeta) - \frac{1}{z - \zeta}\right)(\tilde{\phi}(\zeta) + T\rho(\zeta))d\xi d\eta\ ,
\end{aligned}$$

since

$$\tilde{J}_2(\varepsilon) := \frac{1}{2\pi i}\int_{|\zeta - z| = \epsilon}\left(2\omega_z(z,\zeta) + \frac{1}{\zeta - z}\right)(\phi(\zeta) + S_1\rho(\zeta))d\overline{\zeta}$$

tends to zero with ε.

$$\begin{aligned}
\tilde{J}_1(\epsilon) &:= -\frac{1}{2\pi i}\int_{|\zeta - z| = \epsilon}\left[\frac{1}{\zeta - z} + \underset{\sim}{\ell}(z,\zeta)\right][(\phi(\zeta) + S_1\rho(\zeta))d\overline{\zeta} \\
&\qquad\qquad +(\overline{\phi(\zeta)} + \overline{S_1\rho(\zeta)})d\zeta],
\end{aligned}$$

$$\lim_{\epsilon\to 0}\tilde{J}_1(\epsilon) = -[\overline{\phi(z)} + S_1\overline{\rho}(z)]\ .$$

Hence

$$\tilde{\psi}_0(z) + \overline{\phi(z)} + S_1\overline{\rho}(z) = \frac{1}{\pi}\int_D \left\{ (\underset{\sim}{\ell}(z,\zeta) - 2\omega_z(z,\zeta))(\tilde{\phi}(\zeta) + T\rho(\zeta)) \right.$$

$$-\left(\frac{1}{\zeta - z} + \underset{\sim}{\ell}(z,\zeta)\right)(\overline{\tilde{\phi}(\zeta) + T\rho(\zeta)})\Big\}\, d\xi d\eta$$

$$= T\overline{\tilde{\phi}}(z) + T\overline{T\rho}(z) - \frac{1}{\pi}\int_D \Big\{ \underset{\sim}{\ell}(z,\zeta)(\overline{\tilde{\phi}(\zeta)} + \overline{T\rho(\zeta)})$$

$$+(2\omega_z(z,\zeta) - \underset{\sim}{\ell}(z,\zeta))(\tilde{\phi}(\zeta) + T\rho(\zeta))\Big\}\, d\xi d\eta\,. \quad (5.1.14)$$

This formula gives a particular solution to the boundary value problem in the desired form. It is quite similar to (5.1.7). Differentiation and applying the formula, see the proof of Theorem 30,

$$\Pi f = T\left(\frac{\partial f}{\partial \zeta}\right) - \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\overline{\zeta}$$

gives

$$\tilde{\psi}_0'(z) = \frac{2}{\pi}\int_D \omega_z(z,\zeta)\overline{\rho(\zeta)}d\xi d\eta - T\overline{\rho}(z) + T\left(\frac{\partial}{\partial\zeta}\overline{T\rho}\right) - \frac{1}{2\pi i}\int_{\partial D}(\overline{\tilde{\phi}(\zeta)} + \overline{T\rho(\zeta)})\frac{d\overline{\zeta}}{\zeta - z}$$

$$-\frac{1}{\pi}\int_D \Big\{ \underset{\sim}{\ell}_z(z,\zeta)(\overline{\tilde{\phi}(\zeta)} + \overline{T\rho(\zeta)}) + (2\omega_{zz}(z,\zeta) - \underset{\sim}{\ell}_z(z,\zeta))(\tilde{\phi}(\zeta) + T\rho(\zeta))\Big\}\, d\xi d\eta$$

$$= \frac{2}{\pi}\int_D \omega_z(z,\zeta)\overline{\rho(\zeta)}d\xi d\eta - \frac{1}{2\pi i}\int_{\partial D}(\overline{\tilde{\phi}(\zeta)} + \overline{T\rho(\zeta)})\frac{d\overline{\zeta}}{\zeta - z}$$

$$-\frac{1}{\pi}\int_D \Big\{ \underset{\sim}{\ell}_z(z,\zeta)(\overline{\tilde{\phi}(\zeta)} + \overline{T\rho(\zeta)}) + (2\omega_{zz}(z,\zeta) - \underset{\sim}{\ell}_z(z,\zeta))(\tilde{\phi}(\zeta) + T\rho(\zeta))\Big\}\, d\xi d\eta\,.$$

As is shown in the Lemmas 35, 36 the integral operators in the representations of $\tilde{\phi}, \tilde{\psi}_0, \tilde{\psi}_0'$ are compact operators in $L_p(\overline{D})$ for $2 < p$.
If we are involved with a differential equation analogous to (5.1.10) namely

$$w_{\overline{z}z} + \mu_1 w_{\overline{z}\overline{z}} + \mu_2\overline{w_{\overline{z}\overline{z}}} + a_1 w_{\overline{z}} + a_2\overline{w_{\overline{z}}} + b_1 w_z + b_2\overline{w_z} + c_1 w + c_2\overline{w} + d = 0 \text{ in } D (5.1.15)$$

then we get a singular integral equation of the kind as (5.1.11) with the same singular integral operator S^0 but for $\overline{\rho}$ instead of ρ. The compact operator K^0 is given by

$$\begin{aligned} K^0\rho &= a_1(\overline{\tilde{\phi} + T\rho}) + a_2(\tilde{\phi} + T\rho) + b_1\left(\tilde{\psi}_0' - \frac{2}{\pi}\int_D \omega_z(z,\zeta)\overline{\rho(\zeta)}d\xi d\eta + T\overline{\rho}\right) \\ &+ b_2\left(\overline{\tilde{\psi}_0' - \frac{2}{\pi}\int_D \omega_z(z,\zeta)\overline{\rho(\zeta)}d\xi d\eta + T\overline{\rho}}\right) + c_1(\tilde{\psi}_0 + \overline{\tilde{\phi}} + S_1\overline{\rho}) \\ &+ c_2(\overline{\tilde{\psi}_0} + \phi + S_1\rho)\end{aligned}$$

and

$$d_0 = d + \sum_{\mu=1}^{m} \gamma_\mu [b_1 \psi'_\mu + b_2 \overline{\psi'_\mu} + c_1 \psi_\mu + c_2 \overline{\psi}_\mu] \,.$$

Hence, the boundary value problem for (5.1.15) has the FREDHOLM property, too.

In [Bege93] the case where D is the unit disc $\mathbb{D}$ is explicitly worked out.

5.2 Higher order equations

As in the preceding section where iteration of the T–operator with itself and the $\overline{T}$–operator, respectively led to the operators $T_{0,2}$ and $T_{1,1}$ further iterations will produce a hierarchy of integral operators, see [Behi93]. They are related to the differential operators $\partial^{m+n}/\partial\overline{z}^m\partial z^n$ in the same way as the T–operator is to $\partial/\partial\overline{z}$. While these results for general domains just will be listed we will again concentrate on the unit disc, see [Behi93]. For the disc $\mathbb{D}$ in (2.1.2′) instead of the T–operator

$$S_1\rho(z) := -\frac{1}{2\pi} \int_{\mathbb{D}} \left\{ \frac{\zeta + z}{\zeta - z} \frac{\rho(\zeta)}{\zeta} + \frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}} \frac{\overline{\rho(\zeta)}}{\overline{\zeta}} \right\} d\xi d\eta \ , \ z \in \mathbb{D} \ ,$$

appears which is the T–operator adjusted to the DIRICHLET boundary condition and some normalization (side condition). For $\rho \in L_p(\overline{\mathbb{D}}), 1 \leq p$, we have

$$\frac{\partial S_1 \rho}{\partial \overline{z}} = \rho \ , \ \operatorname{Re} S_1\rho = 0 \text{ on } \partial\mathbb{D}, \operatorname{Im} S_1\rho(0) = 0 \ ,$$

and

$$\frac{\partial S_1\rho(z)}{\partial z} = -\frac{1}{\pi} \int_{\mathbb{D}} \left\{ \frac{\rho(\zeta)}{(\zeta - z)^2} + \frac{\overline{\rho(\zeta)}}{(1 - z\overline{\zeta})^2} \right\} d\xi d\eta \ .$$

Definition 19. *Let $\rho \in L_p(\overline{\mathbb{D}}), 1 \leq p$, and $k \in \mathbb{N}$. Then for $z \in D$*

$$S_k\rho(z) := \frac{(-1)^k}{2\pi(k-1)!} \int_{\mathbb{D}} [2\operatorname{Re}(\zeta - z)]^{k-1} \left[\frac{\zeta + z}{\zeta - z} \frac{\rho(\zeta)}{\zeta} + \frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}} \frac{\overline{\rho(\zeta)}}{\overline{\zeta}} \right] d\xi d\eta \ . \qquad (5.2.1)$$

Theorem 53. $S_k\rho = S_1^k\rho \ , \ k \in \mathbb{N}.$

Proof. For $k = 1$ there is nothing to prove. Let $2 \leq k$ then

$$\begin{aligned} \frac{\partial}{\partial \overline{z}} S_k\rho(z) &= \frac{(-1)^{k-1}}{2\pi(k-2)!} \int_{\mathbb{D}} [2\operatorname{Re}(\zeta - z)]^{k-2} \left[\frac{\zeta + z}{\zeta - z} \frac{\rho(\zeta)}{\zeta} + \frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}} \frac{\overline{\rho(\zeta)}}{\overline{\zeta}} \right] d\xi d\eta \\ &= S_{k-1}\rho(z) \ . \end{aligned}$$

Moreover, since

$$S_k\rho(0) = \frac{(-1)^k}{2\pi(k-1)!}\int\limits_{\mathbb{D}} (2\mathrm{Re}\,\zeta)^{k-1}\left[\frac{\rho(\zeta)}{\zeta} + \frac{\overline{\rho(\zeta)}}{\overline{\zeta}}\right] d\xi d\eta$$

we have $\mathrm{Im}\, S_k\rho(0) = 0$, and since on $\partial\mathbb{D}$

$$\begin{aligned} S_k\rho(z) &= \frac{(-1)^k}{2\pi(k-1)!}\int\limits_{\mathbb{D}} [2\mathrm{Re}\,(\zeta - z)]^{k-1}\left[\frac{\zeta+z}{\zeta-z}\frac{\rho(\zeta)}{\zeta} + \frac{\overline{z+\zeta}}{\overline{z-\zeta}}\frac{\overline{\rho(\zeta)}}{\overline{\zeta}}\right] d\xi d\eta \\ &= \frac{(-1)^k i}{\pi(k-1)!}\int\limits_{\mathbb{D}} [2\mathrm{Re}\,(\zeta - z)]^{k-1}\mathrm{Im}\left[\frac{\zeta+z}{\zeta-z}\frac{\rho(\zeta)}{\zeta}\right] d\xi d\eta \end{aligned}$$

thus $\mathrm{Re}\, S_k\rho(z) = 0$ for $z \in \partial\mathbb{D}$. Applying the modified CAUCHY–SCHWARZ–POMPEIU formula to $S_k\rho$ gives

$$\begin{aligned} S_k\rho(z) &= -\frac{1}{2\pi}\int\limits_{\mathbb{D}}\left\{\frac{\zeta+z}{\zeta-z}\frac{S_{k-1}\rho(\zeta)}{\zeta} + \frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\frac{\overline{S_{k-1}\rho(\zeta)}}{\overline{\zeta}}\right\} d\xi d\eta \\ &= S_1(S_{k-1}\rho)(z)\,,\ z \in \mathbb{D}\,. \end{aligned}$$

Hence, for $2 \le k$

$$S_k = S_1 S_{k-1}\,.$$

Theorem 54. *For $\rho \in L_p(\overline{\mathbb{D}}, 1 \le p, k \in \mathbb{N}, S_k\rho$ has the following properties:*

$$\frac{\partial^\ell}{\partial\overline{z}^\ell}S_k\rho = S_{k-\ell}\rho, 1 \le \ell \le k, \text{ if } S_0\rho := \rho\,,$$

$$\mathrm{Re}\,\frac{\partial^\ell}{\partial\overline{z}^\ell}S_k\rho = 0 \text{ on } \partial\mathbb{D}\,,\ 0 \le \ell \le k-1\,,$$

$$\mathrm{Im}\,\frac{\partial^\ell}{\partial\overline{z}^\ell}S_k\rho(0) = 0\,,\ 0 \le \ell \le k-1\,.$$

Moreover, $\partial^\ell/\partial z^\ell S_k\rho$ is a weakly singular integral if $0 \le \ell \le k-1$, while for $\ell = k$ it is a singular integral,

$$\begin{aligned} \frac{\partial^k}{\partial z^k}S_k\rho(z) = \frac{(-1)^k k}{\pi}\int\limits_{\mathbb{D}}\Bigg\{&\left(\frac{\overline{\zeta-z}}{\zeta-z}\right)^{k-1}\frac{\rho(\zeta)}{(\zeta-z)^2} \\ &+\left(\frac{\zeta-z+\overline{\zeta-z}}{1-z\overline{\zeta}}\overline{\zeta}-1\right)^{k-1}\frac{\overline{\rho(\zeta)}}{(1-z\overline{\zeta})^2}\Bigg\} d\xi d\eta,\ z \in \mathbb{D}. \quad (5.2.2) \end{aligned}$$

Proof. One can inductively show by the LEIBNIZ rule

$$\frac{\partial^\ell}{\partial z^\ell} S_k\rho(z)$$

$$= \frac{1}{\pi}\int_{\mathbb{D}} \sum_{\lambda=0}^{\ell} \binom{\ell}{\lambda} a_{\lambda\ell k} [2\mathrm{Re}\,(\zeta - z)]^{k-\lambda-1} \left[\frac{\rho(\zeta)}{(\zeta - z)^{\ell-\lambda+1}} + \frac{\overline{\zeta}^{\ell-\lambda-1}\overline{\rho(\zeta)}}{(1 - z\overline{\zeta})^{\ell-\lambda+1}}\right] d\xi d\eta$$

$$-\frac{1}{2\pi}\int_{\mathbb{D}} \frac{(-1)^{k-\ell}}{(k-\ell-1)!} [2\mathrm{Re}\,(\zeta - z)]^{k-\ell-1} \left[\frac{\rho(\zeta)}{\zeta} + \frac{\overline{\rho(\zeta)}}{\overline{\zeta}}\right] d\xi d\eta \ , \ 0 \le \ell \le k-1, z \in \mathbb{D} \ ,$$

where $a_{\lambda\ell k} := \dfrac{(-1)^{k-\lambda}(\ell - \lambda)!}{(k - \lambda - 1)!}$. Differentiating this formula for $\ell = k - 1$ once more gives $\partial^k/\partial z^k S_k\rho$ in the above form. Obviously, the integral for the derivative $\partial^\ell/\partial z^\ell S_k\rho$ is weakly singular since $k - \lambda - 1 - (\ell - \lambda + 1) = k - \ell - 2 \ge -1$ for $0 \le \ell \le k-1$, while for $\ell = k$ this expression becomes -2 so that $\partial^k/\partial z^k S\rho$ is a singular integral.

In order to avoid running again into involved technical estimations, here only a simple *kth* order equation will be considered, see [Behi93].

Theorem 55. *The* DIRICHLET *problem*

$$\mathrm{Re}\,\frac{\partial^\ell w}{\partial \overline{z}^\ell} = 0 \text{ on } \partial\mathbb{D}, 0 \le \ell \le k-1, \ \mathrm{Im}\,\frac{\partial^\ell w}{\partial \overline{z}^\ell}(0) = 0 \ , \ 0 \le \ell \le k-1 \ ,$$

for the differential equation

$$\frac{\partial^k w}{\partial \overline{z}^k} = \rho$$

has the only solution $w = S_k\rho$.

Proof. Similar to the first representation formula in Corollary 10 the general solution to the differential equation is

$$w(z) = \sum_{\ell=0}^{k-1} \phi_\ell(z)\overline{z}^\ell + S_k\rho(z) \ .$$

Here the ϕ_ℓ are analytic functions. While $S_k\rho$ is a particular solution the other part is the general solution to the homogeneous differential equation $\partial^k/\partial\overline{z}^k w = 0$. Such a function is called polyanalytic see [Balk91]. From the boundary behaviour of $S_k\rho$ we find

$$\mathrm{Re}\,\frac{\partial^{k-1} w}{\partial \overline{z}^{k-1}} = \mathrm{Re}\,\{(k-1)!\phi_{k-1} + S_1\rho\} = (k-1)!\mathrm{Re}\,\phi_{k-1} = 0 \text{ on } \partial\mathbb{D} \ .$$

Thus $\phi_{k-1}(z) = ic_{k-1}$ with a real constant c_{k-1}. This constant is zero since from the side condition $\mathrm{Im}\,\phi_{k-1}(0) = 0$ follows. Proceeding in this manner $\phi_\ell(z) = 0$ is shown for any $\ell, 0 \le \ell \le k-1$.

The higher order operators for general domains are defined by means of some kernel functions.

Definition 20. *For $m,n \in \mathbb{Z}$ with $0 \le m+n$ but $(m,n) \neq (0,0)$ the kernels $K_{m,n}(z)$ are*

$$K_{m,n}(z) := \begin{cases} \dfrac{(-1)^m(-m)!}{\pi(n-1)!} z^{m-1}\overline{z}^{n-1} \text{ if } m \le 0\,, \\ \dfrac{(-1)^n(-n)!}{\pi(m-1)!} z^{m-1}\overline{z}^{n-1} \text{ if } n \le 0\,, \\ \dfrac{z^{m-1}\overline{z}^{n-1}}{\pi(m-1)!(n-1)!}\left[\log|z|^2 - \sum\limits_{\mu=1}^{m-1}\dfrac{1}{\mu} - \sum\limits_{\nu=1}^{n-1}\dfrac{1}{\nu}\right] \text{ if } 1 \le m,n\,. \end{cases} \tag{5.2.3}$$

Definition 21. *Let $D \subset \mathbb{C}$ be a domain and $w \in L_1(\overline{D})$. Then for $m,n + \mathbb{Z}$ with $0 \le m+n$*

$$T_{0,0}w(z) := w(z) \text{ if } (m,n) = (0,0)\,,$$

$$T_{m,n}w(z) := \int\limits_D K_{m,n}(z-\zeta)w(\zeta)d\xi d\eta \text{ if } (m,n) \neq (0,0)\,. \tag{5.2.4}$$

Sometimes $T_{m,n,D}$ is used instead of just $T_{m,n}$.
Obviously, $T_{0,1} = T, T_{1,0} = \overline{T}, T_{-1,1} = \Pi, T_{1,-1} = \overline{\Pi}$ in the notation of chapter 2. Analogously to the POMPEIU formulas (2.1.1) and (2.1.1′) higher order representation formulas are available [Behi93].

Lemma 41. *Let D be a bounded domain with smooth boundary and $w \in C^1(\overline{D};\mathbb{C})$. Then for $0 < m+n$ and $z \in \mathbb{C} \setminus \partial D$*

$$T_{m,n}w(z) = T_{m,n+1}w_{\overline{z}}(z) - \frac{1}{2i}\int\limits_{\partial D} K_{m,n+1}(z-\zeta)w(\zeta)d\zeta\,, \tag{5.2.5}$$

$$T_{m,n}w(z) = T_{m+1,n}w_z(z) + \frac{1}{2i}\int\limits_{\partial D} K_{m+1,n}(z-\zeta)w(\zeta)d\overline{\zeta}\,. \tag{5.2.5'}$$

Proof. As in the proof of Theorem 20 for $z_0 \in D$ consider $D_\epsilon := D \setminus \overline{K_\epsilon(z_0)}$ for small enough $\varepsilon > 0$. Applying Lemma 9 gives

$$\begin{aligned}
&\int_{D_\varepsilon} K_{m,n}(z-\zeta)w(\zeta)d\xi d\eta \\
&= -\int_{D_\varepsilon} \frac{\partial}{\partial\overline{\zeta}}[K_{m,n+1}(z-\zeta)w(\zeta)]d\xi d\eta \quad + \int_{D_\varepsilon} K_{m,n+1}(z-\zeta)\frac{\partial}{\partial\overline{\zeta}}w(\zeta)d\xi d\eta \\
&= -\frac{1}{2i}\int_{\partial D_\varepsilon} K_{m,n+1}(z-\zeta)w(\zeta)d\zeta \quad + \int_{D_\varepsilon} K_{m,n+1}(z-\zeta)\frac{\partial}{\partial\overline{\zeta}}w(\zeta)d\xi d\eta \,.
\end{aligned}$$

Passing with ε to zero and observing $0 < m+n$ formula (5.2.5) is obtained. If $z_0 \neq \overline{D}$ then instead of D_ε we can just work with D in this argumentation.

Theorem 56. *Let D be a bounded domain with smooth boundary and $w \in C^m(\overline{D};\mathbb{C})$ for $1 \leq m$. Moreover, let $(\mu_k,\nu_k), 0 \leq k \leq m$, be a set of double indices, satisfying*

$$\mu_0 = \nu_0 = 0, \mu_{k-1} \leq \mu_k, \nu_{k-1} \leq \nu_k, \mu_{k-1} + \nu_{k-1} + 1 = \mu_k + \nu_k, 1 \leq k \leq m \,.$$

Then

$$w(z) = T_{\mu_m,\nu_m}\frac{\partial^{\mu_m+\nu_m}w(z)}{\partial z^{\mu_m}\partial\overline{z}^{\nu_m}}$$

$$+\sum_{k=0}^{m-1}\frac{1}{2}\int_{\partial D} K_{\mu_{k+1},\nu_{k+1}}(z-\zeta)\frac{\partial^{\mu_k+\nu_k}w(\zeta)}{\partial\zeta^{\mu_k}\partial\overline{\zeta}^{\nu_k}}d\left[(i\zeta)^{\nu_{k+1}-\nu_k}(-i\overline{\zeta})^{\mu_{k+1}-\mu_k}\right] \,. \tag{5.2.6}$$

Proof. For $m = 1$ there are two possibilities $(\mu_1,\nu_1) = (0,1)$ or $(\mu_1,\nu_1) = (1,0)$. In these cases (5.2.6) coincides with (2.1.1) and (2.1.1′), respectively.

Assuming (5.2.6) holds for m, we will show it holds for $m+1$, too. Let $w \in C^{m+1}(\overline{D};\mathbb{C})$ and choose $\mu_{m+1} = \mu_m$ and $\nu_{m+1} = \nu_m + 1$. From (5.2.5) applied to $\partial^{\mu_m+\nu_m}w/\partial^{\mu_m}z\partial^{\nu_m}\overline{z}$ we see

$$T_{\mu_m,\nu_m}\frac{\partial^{\mu_m+\nu_m}}{\partial z^{\mu_m}\partial\overline{z}^{\nu_m}} = T_{\mu_{m+1},\nu_{m+1}}\frac{\partial^{\mu_{m+1}+\nu_{m+1}}}{\partial z^{\mu_{m+1}}\partial\overline{z}^{\nu_{m+1}}}$$

$$+\frac{1}{2}\int_{\partial D} K_{\mu_{m+1},\nu_{m+1}}(z-\zeta)\frac{\partial^{\mu_m+\nu_m}w(\zeta)}{\partial\zeta^{\mu_m}\partial\overline{\zeta}^{\nu_m}}d(i\zeta) \,.$$

Inserting this formula into (5.2.6) shows that it holds for $m+1$, too in this particular case. When $\mu_{m+1} = \mu_m + 1$ and $\nu_{m+1} = \nu_m$ an analogue argument holds.

Corollary 12. In the particular case $\mu_k = 0, \nu_k = k, 0 \leq k \leq m$, any $w \in$

$C^m(\overline{D}; \mathbb{C})$ is representable as

$$w(z) = \sum_{k=0}^{m-1} \phi_k(z)\overline{z}^k + \frac{1}{(m-1)!\pi} \int_D \frac{(\overline{z-\zeta})^{m-1}}{z-\zeta} \frac{\partial^m w(\zeta)}{\partial \overline{\zeta}^m} d\xi d\eta ,$$

with analytic functions ϕ_k in D.

Proof. From (5.2.6) it is seen

$$\begin{aligned} w(z) &= T_{0,m}\frac{\partial^m w(z)}{\partial \overline{z}^m} - \sum_{k=0}^{m-1} \frac{1}{2i} \int_{\partial D} K_{0,k+1}(z-\zeta) \frac{\partial^k w(\zeta)}{\partial \overline{\zeta}^k} d\zeta \\ &= \frac{1}{(m-1)!\pi} \int_D \frac{(\overline{z-\zeta})^{m-1}}{z-\zeta} \frac{\partial^m w(\zeta)}{\partial \overline{\zeta}^m} d\xi d\eta \\ &\quad - \sum_{k=0}^{m} \frac{1}{k!} \frac{1}{2\pi i} \int_{\partial D} \frac{(\overline{z-\zeta})^k}{z-\zeta} \frac{\partial^k w(\zeta)}{\partial \overline{\zeta}^k} d\zeta . \end{aligned}$$

The boundary integrals are obviously of the form as stated in the corollary.

In connection with the RIEMANN jump problem we only mention that for a system of smooth curves Γ and $f_k \in C^\alpha(\Gamma), 0 \le k \le m-1, 0 < \alpha < 1$ the function

$$\phi(z) := \sum_{k=0}^{m-1} \frac{1}{2\pi i k!} \int_\Gamma \frac{(\overline{z-\zeta})^k}{\zeta - z} f_k(\zeta) d\zeta$$

is a polyanalytic function, $\partial^m \phi / \partial \overline{z}^m = 0$, in $\mathbb{C} \setminus \Gamma$ satisfying

$$\left(\frac{\partial^k w}{\partial \overline{z}^k}\right)^+ - \left(\frac{\partial^k w}{\partial \overline{z}^k}\right)^- = f_k, 0 \le k \le m-1 .$$

This follows from the fact that, see [Behi93],

$$\int_\Gamma \frac{(\overline{z-\zeta})}{\zeta - z} f_k(\zeta) d\zeta$$

is continuous on Γ for $1 \le k$ while for $k = 0$ it is just a CAUCHY–type integral. Moreover,

$$\frac{\partial^\ell \phi(z)}{\partial \overline{z}^\ell} = \sum_{k=\ell}^{m-1} \frac{1}{2\pi i (k-\ell)!} \int_\Gamma \frac{(\overline{z-\zeta})^{k-\ell}}{\zeta - z} f_k(\zeta) d\zeta , \ 0 \le \ell \le m-1 .$$

Properties of the integral operators $T_{m,n}$ similarly as for $T = T_{0,1}$ studied in section 2.2 are investigated in [Behi93] in detail. Besides continuity, differentiability, and behaviour at inifinity the norm of the singular operators $T_{m,-m}, m \in \mathbb{Z}$, are considered. They all turn out to be unitary operators in $L_2(\mathbb{C})$ as is the Π–operator, $\Pi = T_{-1,1}$.

5.3 First order systems in two complex variables with analytic coefficients

In this section simple well-posed boundary value problems for systems of first order partial differential equations in two complex variables are considered. The result then is applied to solve a higher order equation again of simple type. In several complex variables there is almost no theory for boundary value problems. Here the solutions are determined by RIEMANN–HILBERT conditions on some complex one-dimensional subset of the boundary. The basic idea is due to A. DZHURAEV, see [Bedz93b]. The method is working only for equations and systems with analytic coefficients.
Let A be an $N \times N$ matrix function in a domain D_0 of $\mathbb{C}^2$ with analytic entries. The multiplicities n_κ of the eigenvalues $\lambda_\kappa, 1 \le \kappa \le k, \sum_{\kappa=1}^k n_\kappa = N$, are assumed to be constant on D_0. Then there is a nonsingular matrix B with analytic entries such that $\Lambda := B^{-1}AB$ has JORDAN normal form,

$$\Lambda = \begin{bmatrix} \Lambda_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \cdot & \\ & & & & \Lambda_k \end{bmatrix}, \; \Lambda_\kappa := \begin{bmatrix} \lambda_\kappa & & & & 0 \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & 0 & & \cdot & \cdot \\ & & & 1 & \lambda_\kappa \end{bmatrix}, 1 \le \kappa \le k \,.$$

The Λ_κ are $n_\kappa \times n_\kappa$ matrices with analytic entries. Let $\boldsymbol{f}$ be a given vector-function in $D_0 \times \mathbb{C}^N$. Then $\omega = B\boldsymbol{w}$ transforms the first order system

$$\boldsymbol{w}_{z_2} + A\boldsymbol{w}_{z_1} = \boldsymbol{f}(z_1, z_2, \boldsymbol{w}) \tag{5.3.1}$$

into the system

$$\omega_{z_2} + \Lambda\omega_{z_1} = B\boldsymbol{f} + (B_{z_2} + \Lambda B_{z_1})B\boldsymbol{w} =: \tilde{\boldsymbol{f}}(z_1, z_2, \omega) \,.$$

Let us at first assume that this system decomposes into the k systems

$$\omega_{\kappa z_2} + \Lambda_\kappa \omega_{\kappa z_1} = \boldsymbol{f}_\kappa(z_1, z_2, \omega_\kappa) \,, \; 1 \le \kappa \le k \,.$$

Any system of the form

$$\boldsymbol{w}_{z_2} + \Lambda \boldsymbol{w}_{z_1} = \boldsymbol{f}(z_1, z_2, \boldsymbol{w}) \,,$$

where

$$\Lambda = \begin{bmatrix} \lambda & & & & 0 \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ 0 & & & 1 & \lambda \end{bmatrix} , \tag{5.3.2}$$

has the component form

$$w_{0z_2} + \lambda w_{0z_1} = f_0 \ , \ w_{\nu z_2} + \lambda w_{\nu z_1} + w_{\nu-1 z_1} = f_\nu \ , \ 1 \le \nu \le n-1 \ .$$

Here $\boldsymbol{f}$ and $\boldsymbol{w}$ are vectors in $\mathbb{C}^n$ and λ is analytic in D_0 . In the following the differential form $dz_1 - \lambda dz_2$ is assumed to have an integrating factor $\mu(z_1, z_2)$ different from zero and analytic in D_0 such that

$$d\zeta_1 := \mu dz_1 - \mu\lambda dz_2$$

is a total differential i.e. an exact differential form. Such factor can be found by finding a particular solution to the partial differential equation

$$(\mu\lambda)_{z_1} + \mu_{z_2} = 0 \ .$$

Let ζ_1 be a function with this total differential $d\zeta_1$ such that $\zeta_{1z_1} = \mu, \zeta_{1z_2} = -\mu\lambda$. Choosing $\zeta_2(z_1, z_2) \equiv z_2$ then the mapping $\boldsymbol{\zeta} = (\zeta_1, \zeta_2)$ maps D_0 one-to-one onto a domain $G \subset \mathbb{C}^2$. The JACOBIan of this mapping is

$$\begin{vmatrix} \zeta_{1z_1} & \zeta_{1z_2} \\ \zeta_{2z_1} & \zeta_{2z_2} \end{vmatrix} = \begin{vmatrix} \mu & -\mu\lambda \\ 0 & 1 \end{vmatrix} = \mu \ne 0$$

in D_0 while the JACOBIan of its inverse is

$$\begin{vmatrix} z_{1\zeta_1} & z_{1\zeta_2} \\ z_{2\zeta_1} & z_{2\zeta_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{\mu} & \lambda \\ 0 & 1 \end{vmatrix} = \frac{1}{\mu} \ .$$

Transforming an analytic vector $\boldsymbol{w}$ by $\boldsymbol{z} = \boldsymbol{z}(\boldsymbol{\zeta})$ gives an analytic vector function $\boldsymbol{\omega}$ of $\boldsymbol{\zeta}, \boldsymbol{\omega}(\boldsymbol{\zeta}) = \boldsymbol{w}(\boldsymbol{z}(\boldsymbol{\zeta}))$, satisfying

$$\begin{aligned} \boldsymbol{\omega}_{\zeta_1} &= \boldsymbol{w}_{z_1} z_{1\zeta_1} + \boldsymbol{w}_{z_2} z_{\zeta_1} &= \frac{1}{\mu}\boldsymbol{w}_{z_1} \ , \\ \boldsymbol{\omega}_{\zeta_2} &= \boldsymbol{w}_{z_1} z_{1\zeta_2} + \boldsymbol{w}_{z_2} z_{2\zeta_2} &= \lambda \boldsymbol{w}_{z_1} + \boldsymbol{w}_{z_2} \ . \end{aligned}$$

Hence the above system is equivalent to

$$\omega_{0\zeta_2} = \widetilde{f}_0, \ \omega_{\nu\zeta_2} + \mu\omega_{\nu-1\zeta_1} = \widetilde{f}_\nu, \ 1 \le \nu \le n-1 \ ,$$

with $\tilde{f}_\nu(\zeta, \omega) := f_\nu(z(\zeta), \omega)$.
Let us consider a system in this reduced form

$$w_{0z_2} = f_0,\ w_{\nu z_2} + \mu w_{\nu-1 z_1} = f_\nu,\ 1 \le \nu \le n-1\ ,$$

in a domain D_0 of $\mathbb{C}^2$. Assuming the hyperplane $z_2 = 0$ intersects D_0 in the plane domain G_0 of $\mathbb{C}$, $G_0 = D_0 \cap \{(z_1, z_2) \in \mathbb{C}^2 : z_2 = 0\}$, integration leads to

$$\begin{aligned} w_0(z_1, z_2) &= \varphi_0(z_1) + \int_0^{z_2} f_0(z_1, t, \boldsymbol{w})dt\ , \\ w_\nu(z_1, z_2) &= \varphi_\nu(z_1) + \int_0^{z_2} \{f_\nu(z_1, t, \boldsymbol{w}) - \mu(z_1, t) w_{\nu-1 z_1}(z_1, t)\}dt\ ,\ 1 \le \nu \le n-1\ , \end{aligned}$$

where the $\varphi_\nu, 0 \le \nu \le n-1$, are analytic functions in the domain G_0 of $\mathbb{C}$. In vector form the system becomes

$$\boldsymbol{w}_{z_2} + \begin{bmatrix} 0 & & & & 0 \\ \mu & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ 0 & & & \mu & 0 \end{bmatrix} \boldsymbol{w}_{z_1} = f\ . \tag{5.3.3}$$

Introducing the nilpotent element, see [Doug53],

$$e = \begin{bmatrix} 0 & & & & 0 \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ 0 & & & 1 & 0 \end{bmatrix}\ ,\ e^n = 0\ ,$$

together with the hypercomplex quantities

$$\boldsymbol{w} = \sum_{\nu=0}^{n-1} w_\nu e^\nu,\ \boldsymbol{f} = \sum_{\nu=0}^{n-1} f_\nu e^\nu$$

the system is

$$\sum_{\nu=0}^{n-1} w_{\nu z_2} e^\nu + \mu e \sum_{\nu=0}^{n-1} w_{\nu z_1} e^\nu = \sum_{\nu=0}^{n-1} f_\nu e^\nu$$

i.e.

$$\sum_{\nu=0}^{n-1} w_{\nu z_2} e^\nu + \sum_{\nu=1}^{n-1} \mu w_{\nu-1 z_1} e^\nu = \sum_{\nu=0}^{n-1} f_\nu e^\nu .$$

Defining the hypercomplex first order differential operator

$$D := \frac{\partial}{\partial z_2} + \mu e \frac{\partial}{\partial z_1}$$

the system can be written as

$$\boldsymbol{D w} = \boldsymbol{f} . \tag{5.3.4}$$

Let the hypercomplex function $\boldsymbol{t} = \sum_{\nu=0}^{n-1} t_\nu e^\nu$ be a solution of the homogeneous equation, $\boldsymbol{f} = 0$, satisfying $t_0(z_1, z_2) = z_1$. Then

$$t_\nu = t_\nu(z_1, z_2) = (-1)^\nu \mu_\nu(z_1, z_2) \ , \ 1 \le \nu \le n-1 \ ,$$

where

$$\mu_0(z_1, z_2) := z_1, \mu_\nu(z_1, z_2) = \int_0^{z_2} \mu(z_1, t) \mu_{\nu-1 z_1}(z_1, t) dt \ , \ 1 \le \nu \le n-1 \ .$$

Let us denote this function in the sequel as $\boldsymbol{t}_1$ rather than just $\boldsymbol{t}$ and denote by $\boldsymbol{t}_2$ the formally hypercomplex function $\boldsymbol{t}_2(z_1, z_2) \equiv z_2$. Introducing the hypercomplex variables $(\boldsymbol{t}_1, \boldsymbol{t}_2)$ instead of the complex variable (z_1, z_2) equation (5.3.4) can formally be simplified. The JACOBIan of this transformation is

$$J := \begin{vmatrix} \boldsymbol{t}_{1z_1} & \boldsymbol{t}_{1z_2} \\ \boldsymbol{t}_{2z_1} & \boldsymbol{t}_{2z_2} \end{vmatrix} = \boldsymbol{t}_{z_1} = 1 + \sum_{\nu=1}^{n-1} t_{\nu z_1} e^\nu \neq 0 \ .$$

Moreover from,

$$\begin{aligned} \frac{\partial}{\partial z_1} &= \boldsymbol{t}_{1z_1} \frac{\partial}{\partial \boldsymbol{t}_1} + \boldsymbol{t}_{2z_1} \frac{\partial}{\partial \boldsymbol{t}_2} = J \frac{\partial}{\partial \boldsymbol{t}_1} \ , \\ \frac{\partial}{\partial z_2} &= \boldsymbol{t}_{1z_2} \frac{\partial}{\partial \boldsymbol{t}_1} + \boldsymbol{t}_{2z_2} \frac{\partial}{\partial \boldsymbol{t}_2} = -\mu e J \frac{\partial}{\partial \boldsymbol{t}_1} + \frac{\partial}{\partial \boldsymbol{t}_2} \ , \end{aligned}$$

we find

$$\frac{\partial}{\partial \boldsymbol{t}_2} = \frac{\partial}{\partial z_2} + \mu e \frac{\partial}{\partial z_1} = D \ , \ \frac{\partial}{\partial \boldsymbol{t}_1} = \frac{1}{J} \frac{\partial}{\partial z_1} \ .$$

Obviously, $D\boldsymbol{t}_2 = 1$, $D\boldsymbol{t}_1 = 0$ which just is $\partial \boldsymbol{t}_2 / \partial \boldsymbol{t}_2 = 1$, $d\boldsymbol{t}_1 / \partial \boldsymbol{t}_2 = 0$. We also see

$$\frac{\partial \boldsymbol{t}_2}{\partial \boldsymbol{t}_1} = \frac{1}{J} \frac{\partial \boldsymbol{t}_2}{\partial z_1} = \frac{1}{J} \frac{\partial z_2}{\partial z_1} = 0 \ .$$

Thus, t_1, t_2 are independent hypercomplex variables. With these hypercomplex variables (5.3.4) becomes

$$\frac{\partial}{\partial t_2} w(z_1(t_1,t_2),t_2) = f(z_1(t_1,t_2),t_2) \ ,$$

where f is assumed independent of the unknown w. A particular solution to this equation is

$$w_0(t_1,t_2) := \int_0^{t_2} f(z_1(t_1,t),t)dt \ .$$

The general solution to the homogeneous equation, $f = 0$, is given by an arbitrary analytic hypercomplex function φ as $\varphi(t_1(z_1,z_2))$. Hence, the general solution is

$$w(z_1,z_2) = w_0(t_1(z_1,z_2),z_2) + \varphi(t_1(z_1,z_2)) \ .$$

Thus the following result holds.

Theorem 57. *Let μ and f be analytic in D_0. Then*

$$w(z_1,z_2) = \varphi(t_1(z_1,z_2)) + \int_0^{z_2} f(z_1(t_1(z_1,z_2),t)dt \qquad (5.3.5)$$

is the general solution to (5.3.3), where φ is an arbitrary analytic hypercomplex function in G_0.

Proof. In order to show (5.3.5) to be a solution we differentiate w with respect to z_1 and z_2, getting

$$\begin{aligned} w_{z_1}(z_1,z_2) &= \left[\varphi'(t_1(z_1,z_2)) + \frac{1}{J}\int_0^{z_2} f_{z_1}(z_1(t_1(z_1,z_2),t),t)dt\right] J \ , \\ w_{z_2}(z_1,z_2) &= \left[\varphi'(t_1(z_1,z_2)) + \frac{1}{J}\int_0^{z_2} f_{z_1}(z_1(t_1(z_1,z_2),t),t)dt\right] [-\mu(z_1,z_2)eJ] \\ &\quad + f(z_1,z_2) \ , \end{aligned}$$

and hence (5.3.3). That (5.3.5) is the general solution follows from the above considerations. Any solution to the homogeneous problem $Dw = 0$ is an analytic function in t_1 in particular independent of t_2.

Remark. In order to fix φ the RIEMANN–HILBERT boundary condition

$$\operatorname{Re}\{\overline{\tau(z_1)}w(z_1,0)\} = g(z_1) \text{ on } \partial G_0$$

can be imposed where $\tau, g \in C^\alpha(\partial G_0)$ and $|\tau(z_1)| = 1$ on ∂G_0. For $z_1 \in G_0, z_2 = 0$ then from (5.3.5)

$$w(z_1, 0) = \varphi(t_1(z_1, 0)) = \varphi(z_1)$$

follows, so that

$$\mathrm{Re}\,\{\overline{\tau(z_1)}\varphi(z_1)\} = g(z_1) \text{ on } \partial G_0$$

fixes φ eventually not uniquely or only under some solvability conditions, see section 1.4.

We are returning to our original problem which now will not be assumed to decompose into subsystems independent from one another. The systems are now weakly coupled and semilinear. According to the JORDAN normal form we get the system

$$w_{\kappa z_2} + \Lambda_\kappa w_{\kappa z_1} = f_\kappa,\ 1 \le \kappa \le k\ ,$$

where

$$w^T = (w_1^T, w_2^T, \ldots, w_k^T) \in \mathbb{C}^N, w_\kappa^T = (w_{\kappa 0}, w_{\kappa 1}, \ldots, w_{\kappa n_\kappa - 1}) \in \mathbb{C}^{n_\kappa},\ 1 \le \kappa \le k\ ,$$

and similarly for f. Here the upper index T denotes transposition of matrices. The right-hand side f is assumed to be analytic in $(z_1, z_2) \in D_0 \subset \mathbb{C}^2$ for any $w \in \mathbb{C}^N$ and analytic in w, too. If f would be independent of w using the transformations

$$(z_1, z_2) \to (\zeta_{\kappa 1}, \zeta_{\kappa 2}) \to (t_{\kappa 1}, t_{\kappa 2})$$

as above for any of the subsystems and the respective inverse transformations the general solution would have the form

$$w_\kappa(z_1, z_2) = \varphi_\kappa(t_{\kappa 1}(\zeta_{\kappa 1}(z_1, z_2), z_2))$$

$$+ \int_0^{z_2} f_\kappa(z_1(\zeta_{\kappa 1}(t_{\kappa 1}(\zeta_{\kappa 1}(z_1, z_2), z_2), t), t), t)dt, 1 \le \kappa \le k\ , \tag{5.3.6}$$

with arbitrary analytic hypercomplex or vectors φ_κ. The changes of variables obey the rules

$$t_{\kappa 2} = \zeta_{\kappa 2} = z_2$$

$$\frac{\partial \zeta_{\kappa 1}}{\partial z_1} = \mu_\kappa, \frac{\partial z_1}{\partial \zeta_{\kappa 1}} = \frac{1}{\mu_\kappa}, \frac{\partial t_{\kappa 1}}{\partial \zeta_{\kappa 1}} = J_\kappa\, \frac{\partial \zeta_{\kappa 1}}{\partial t_{\kappa 1}} = \frac{1}{J_\kappa}, \frac{\partial t_{\kappa 1}}{\partial z_2} = -\mu_\kappa e, \frac{\partial \zeta_{\kappa 1}}{\partial z_2} = -\mu_\kappa \lambda_\kappa$$

with

$$d\zeta_{\kappa 1} = \mu_\kappa dz_1 - \mu_\kappa \lambda_\kappa dz_2\ ,$$

$$t_{\kappa 1} = \zeta_{\kappa 1} + \sum_{\nu=1}^{n_\kappa - 1} t_{k\nu} e^\nu, t_{\kappa\nu}(\zeta_{\kappa 1}, \zeta_{\kappa 2}) = (-1)^k \mu_{\kappa\nu}(\zeta_{\kappa 1}, z_2)\ , \mu_{\kappa 0}(\zeta_{\nu 1}, z_2) = \zeta_{\kappa_1}\ ,$$

$$\mu_{\kappa\nu}(\zeta_{\kappa 1}, z_2) := \int_0^{z_2} \mu_\kappa(z_1(\zeta_{\kappa 1}, t), t)\mu_{\kappa\nu-1\zeta_{\kappa 1}}(z_1(\zeta_{\kappa 1}, t), t)dt,\ 1 \le \nu \le n_\kappa - 1\ .$$

For checking that w_κ is a solution again by differentiation we see

$$\begin{aligned} w_{\kappa z_1} &= \int_0^{z_2} \frac{\partial f_\kappa}{\partial z_1} dt + \mu_\kappa J_\kappa \varphi'_\kappa \,, \\ w_{\kappa z_2} &= f_\kappa - (\lambda_\kappa + e) \int_0^{z_2} \frac{\partial f_\kappa}{\partial z_1} dt - (\lambda_\kappa + e)\mu_\kappa J_\kappa \varphi'_\kappa \,, \end{aligned}$$

which gives

$$w_{\kappa z_2} + (\lambda_\kappa + e) w_{\kappa z_1} = f_\kappa \,.$$

For the coupled system, $f = f(z_1, z_2, w)$ equations (5.3.6) give a system of VOLTERRA type integral equations

$$w_\kappa(z_1, z_2) = \varphi_\kappa(t_{\kappa 1}(\zeta_{\kappa 1}(z_1, z_2), z_2))$$

$$+ \int_0^{z_2} f_\kappa(z_1(\zeta_{\kappa 1}(t_{\kappa 1}(\zeta_{k1}(z_1, z_2), z_2), t), t), t, w(z_1(\zeta_{\kappa 1}(t_{\kappa 1}(\zeta_{\kappa 1}(z_1, z_2), z_2), t), t), t))\, dt \,,$$

$$1 \le \kappa \le k \,, \quad (5.3.7)$$

which we shortly write in vector form as

$$w(z_1, z_2) = \varphi(z_1, z_2) + \int_0^{z_2} \widehat{f}(z_1, z_2, t, \widehat{w}(z_1, z_2, t))dt$$

where $\widehat{f}^T = (\widehat{f}_1^T, \dots, \widehat{f}_k^T), \varphi^T = (\varphi_1^T, \dots, \varphi_k^T), \widehat{w}^T = (\widehat{w}_1^T, \dots, \widehat{w}_k^T)$ and the $\widehat{f}_\kappa, \varphi_\kappa, \widehat{w}$ depend on $(z_1, z_2, t, \widehat{w})(z_1, z_2)$ and (z_1, z_2, t) respectively as indicated in (5.3.7). In order to solve this nonlinear integral equation some condition on the domain D_0 is imposed.

Definition 22. *A plane domain $G \subset \mathbb{C}$ is called k_0-quasi-starlike, $0 < k_0$, with respect to $z_0 \in G$ if for any $z \in G$ there exists a rectifiable arc γ in G connecting z_1 with z_0 such that on γ*

$$|dt| \le k_0 d|t| \,.$$

Obviously, any starlike domain is quasi-starlike with $k = 1$.

Because f is analytic in all of its variables it satisfies a LIPSCHITZ condition especially with respect to the variable w. In general the LIPSCHITZ constant depends on the distance of w from the boundary of the domain of definition. It increases when w gets closer to the boundary. If this domain is the entire space $\mathbb{C}^N$ then f is a linear function of w, see [Scsc73].

Theorem 58. *Let $D_0 \subset \mathbb{C}^2$ be a domain such that for any $z_{01} \in proj_{z_2} D_0$ the intersection of D_0 with the hyperplane $z_1 = z_{01}$ is k_0-quasi-starlike with respect to the point $(z_{01}, 0)$. Let $\boldsymbol{w}_0$ and φ be analytic functions in D_0 and $\boldsymbol{f}(z_1, z_2, \boldsymbol{w})$ be analytic in*

$$\Omega := D_0 \times \{\boldsymbol{w} : |\boldsymbol{w} - \boldsymbol{w}_0(z_1, z_2)| < Ce^{2k_1|z_2|}\}$$

with some positive constants $C \leq +\infty, k_1$, satisfying the LIPSCHITZ *condition*

$$|\boldsymbol{f}(z_1, z_2, \boldsymbol{w}) - \boldsymbol{f}(z_1, z_2, \omega)| \leq L(z_1, z_2)|\boldsymbol{w} - \omega|$$

for any $(z_1, z_2, \boldsymbol{w}), (z_1, z_2, \omega) \in \Omega$ and

$$\sup_{(z_1,z_2)\in D_0} |\varphi(z_1, z_2) + \int_0^{z_2} \widehat{\boldsymbol{f}}(z_1, z_2, t, \widehat{\boldsymbol{w}}_0(z_1, z_2, t))dt - \boldsymbol{w}_0(z_1, z_2)|e^{-2k_1|z_2|} \leq C\,. \quad (5.3.8)$$

Here the integral is taken along an arc from $(z_1, 0)$ to (z_1, z_2) in the hyperplane $z_1 =$ const. satisfying $|dt| \leq k_0 d|t|$. Moreover, if $\widehat{L}$ is connected with L in the same manner as $\widehat{\boldsymbol{f}}$ is with $\boldsymbol{f}$,

$$\sup_{z_1 \in proj_{z_2} D_0} 2k_0 \int_0^{z_2} \widehat{L}(z_1, z_2, t)e^{2k_1|t|}d|t| \leq e^{2k_1|z_2|}$$

is assumed. Then the operator

$$(T\boldsymbol{w})(z_1, z_2) := \varphi(z_1, z_2) + \int_0^{z_2} \widehat{\boldsymbol{f}}(z_1, z_2, t, \widehat{\boldsymbol{w}}(z_1, z_2, t))dt$$

is bounded on

$$\mathcal{H}_C(D_0) := \{\boldsymbol{w} : \boldsymbol{w} \text{ analytic in } D_0, \|\boldsymbol{w} - \boldsymbol{w}_0\| \leq C\}\,,$$

mapping this set into itself and providing a contraction, where

$$\|\omega\| := \sup_{(z_1,z_2)\in D_0} |\omega(z_1, z_2)|e^{-2k_1|z_2|}\,.$$

Proof. Direct computation shows

$$\|(T\boldsymbol{w})(z_1, z_2) - \boldsymbol{w}_0(z_1, z_2)\| \leq C\,,$$

and

$$\begin{aligned} |(T\boldsymbol{w} - T\omega)(z_1, z_2)| &\leq k_0 \int_0^{z_2} \widehat{L}(z_1, z_2, t)|\widehat{\boldsymbol{w}}(z_2, z_2, t) - \widehat{\omega}(z_1, z_2, t)|d|t| \\ &\leq \|\boldsymbol{w} - \omega\| k_0 \int_0^{z_2} \widehat{L}(z_1, z_2, t)e^{2k_1|t|}d|t| \leq \frac{1}{2}\|\boldsymbol{w} - \omega\|e^{2k_1|z_2|} \end{aligned}$$

so that

$$\|Tw - T\omega\| \le \frac{1}{2}\|w - \omega\| .$$

Remark. If L is constant then (5.3.8) is satisfied and one may choose

$$k_1 = k_0 \sup_{(z_1,z_2)\in D_0} L(z_1, z_2) .$$

On the basis of this result the integral equation can be solved by successive approximation. The sequence of approximative solutions converges in the norm stated in the theorem, leading to an analytic vector function satisfying the integral equation. This limit vector function is a solution to the semilinear coupled system of partial differential equations satisfying

$$w(z_1, 0) = \varphi(z_1),\ z_1 \in G_0 .$$

In order to explain the usefulness of this somewhat involved procedure a simple higher order equation will be solved. The solution of course can be obtained in a different way, too. Any higher order equation is reducible to some first order system. The solution to this system leads to the solution of the original equation. We consider the equation

$$\frac{\partial^m u}{\partial z_2^m} - \frac{\partial u}{\partial z_1} = 0 \text{ in } |z_1|^2 + |z_2|^2 < 1 \tag{5.3.9}$$

for $2 \le m$. Introducing

$$w_\mu := \frac{\partial^{m-\mu-1} u}{\partial z_2^{m-\mu-1}}, 0 \le \mu \le m-2, w_{m-1} := \frac{\partial u}{\partial z_1} ,$$

and $w^T := (w_0, \dots, w_{m-1})$ equation (5.3.9) becomes

$$w_{z_2} - Aw_{z_1} - Bw = 0$$

with the $m \times m$ matrices

$$A = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & \cdots & 0 & 0 & 1 \\ 1 & & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} .$$

A has JORDAN form with the $(m-2) \times (m-2)$ and 2×2 block matrices, respectively,

$$\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} .$$

Decomposing this equation into the two systems

$$\boldsymbol{w}^1_{z_2} - e_{m-2}\boldsymbol{w}^1 = \begin{bmatrix} w_{m-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \boldsymbol{w}^2_{z_2} - e_2\boldsymbol{w}^2 = \begin{bmatrix} w_{m-3} \\ 0 \end{bmatrix},$$

where $\boldsymbol{w}^{1T} := (w_0, \dots, w_{m-3}), \boldsymbol{w}^{2T} := (w_{m-2}, w_{m-1})$, and

$$e_{m-2} := \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 1 & & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}, \quad e_2 := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

are nilpotent $(m-2) \times (m-2)$ and 2×2 matrices, respectively. For these systems the systems of integral equations (5.3.7) become

$$\boldsymbol{w}^1(z_1, z_2) = \int_0^{z_2} \left\{ e_{m-2}\boldsymbol{w}^1(z_1, t) + \begin{bmatrix} w_{m-1}(z_1, t) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\} dt + \boldsymbol{\phi}(z_1),$$

$$\boldsymbol{w}^2(z_1, z_2) = \int_0^{z_2} \begin{bmatrix} w_{m-3} \\ 0 \end{bmatrix} (z_1 + e_2 z_2 - e_2 t, t) dt + \boldsymbol{\varphi}(z_1 + e_2 z_2). \qquad (5.3.10)$$

Here $\boldsymbol{\phi}^T = (\varphi_0, \dots, \varphi_{m-3}), \boldsymbol{\varphi}^T = (\varphi_{m-2}, \varphi_{m-1})$ are analytic vectors.

Replacing the complex variables z_1 in the power series expansion of $\boldsymbol{\varphi}$ and $\begin{bmatrix} w_{m-3} \\ 0 \end{bmatrix}$ by the matrix variable $z_1 + e_2 z_2 = \begin{bmatrix} z_1 & 0 \\ z_2 & z_1 \end{bmatrix}$ we see

$$\begin{aligned} \boldsymbol{\varphi}(z_1 + e z_2) &= \begin{bmatrix} \varphi_{m-2}(z_1) \\ \varphi'_{m-2}(z_1) z_2 + \varphi_{m-1}(z_1) \end{bmatrix}, \\ \begin{bmatrix} w_{m-3} \\ 0 \end{bmatrix} (z_1 + e z_2, t) &= \begin{bmatrix} w_{m-3}(z_1, t) \\ w_{m-3 z_1}(z_1, t) z_2 \end{bmatrix}. \end{aligned}$$

System (5.3.10) is solved by iteration starting with

$$\boldsymbol{w}^1_0(z_1, z_2) = \boldsymbol{\phi}(z_1), \ \boldsymbol{w}^2_0(z_1, z_2) = \boldsymbol{\varphi}(z_1 + e_2 z_2).$$

After μ steps one gets

$$w_\mu^1(z_1,z_2) = \begin{bmatrix} \varphi_0(z_1)+\varphi_{m-1}(z_1)z_2+\varphi'_{m-2}(z_1)\frac{z_2^2}{2}+\cdots+\varphi'_{m-\mu-1}(z_1)\frac{z_2^{\mu+1}}{(\mu+1)!} \\ \varphi_1(z_1)+\varphi_0(z_1)z_2+\varphi_{m-1}(z_1)\frac{z_2^2}{2}+\cdots+\varphi'_{m-\mu}(z_1)\frac{z_2^{\mu+1}}{(\mu+1)!} \\ \cdots \\ \varphi_{\mu-1}(z_1)+\varphi_{\mu-2}(z_1)z_2 \\ \qquad +\cdots+\varphi_0(z_1)\frac{z_2^{\mu-1}}{(\mu-1)!}+\varphi_{m-1}(z_1)\frac{z_2^\mu}{\mu!}+\varphi'_{m-2}(z_1)\frac{z_2^{\mu+1}}{(\mu+1)!} \\ \varphi_\mu(z_1)+\varphi_{\mu-1}(z_1)z_2+\cdots+\varphi_1(z_1)\frac{z_2^{\mu-1}}{(\mu-1)!}+\varphi_0(z_1)\frac{z_2^\mu}{\mu!} \\ \cdots \\ \varphi_{m-3}(z_1)+\varphi_{m-4}(z_1)z_2+\cdots+\varphi_{m-\mu-3}\frac{z_2^\mu}{\mu!} \end{bmatrix},$$

$$w_\mu^2(z_1,z_2) = \begin{bmatrix} \varphi_{m-2}(z_1)+\varphi_{m-3}(z_1)z_2+\cdots+\varphi_{m-\mu-2}\frac{z_2^\mu}{\mu!} \\ \varphi_{m-1}(z_1)+\varphi'_{m-2}(z_1)z_2+\cdots+\varphi'_{m-\mu-2}\frac{z_2^{\mu+1}}{(\mu+1)!} \end{bmatrix}.$$

This can be shown inductively for $0 \le \mu \le m-3$. Iterating two or three more times suggests the vector $w, w^T = (w^{1T}, w^{2T})$ to be of the form

$$w(z_1,z_2) \tag{5.3.11}$$

$$
= \sum_{\nu=0}^{+\infty}
\begin{bmatrix}
\varphi^{(\nu)}_{m-1}(z_1)\frac{z_2^{\nu m+1}}{(\nu m+1)!} + \varphi^{(\nu+1)}_{m-2}(z_1)\frac{z_2^{\nu m+2}}{(\nu m+2)!} + \cdots + \varphi^{(\nu+1)}_{2}(z_1)\frac{z_2^{\nu m+m-2}}{((\nu+1)m-2)!} \\
\qquad + \varphi^{(\nu+1)}_{1}(z_1)\frac{z_2^{\nu m+m-1}}{((\nu+1)m-1)!} + \varphi^{(\nu)}_{0}(z_1)\frac{z_2^{\nu m}}{(\nu m)!} \\
\varphi^{(\nu)}_{m-1}(z_1)\frac{z_2^{\nu m+2}}{(\nu m+2)!} + \varphi^{(\nu+1)}_{m-2}(z_1)\frac{z_2^{\nu m+3}}{(\nu m+3)!} + \cdots + \varphi^{(\nu+1)}_{2}(z_1)\frac{z_2^{\nu m+m-1}}{((\nu+1)m-1)!} \\
\qquad + \varphi^{(\nu)}_{1}(z_1)\frac{z_2^{\nu m}}{(\nu m)!} + \varphi^{(\nu)}_{0}(z_1)\frac{z_2^{\nu m+1}}{(\nu m+1)!} \\
\cdots \\
\varphi^{(\nu)}_{m-1}(z_1)\frac{z_2^{\nu m+m-2}}{((\nu+1)m-2)!} + \varphi^{(\nu+1}_{m-2}(z_1)\frac{z_2^{\nu m+m-1}}{((\nu+1)m-1)!} + \varphi^{(\nu)}_{m-3}(z_1)\frac{z_2^{\nu m}}{(\nu m)!} \\
\qquad + \cdots + \varphi^{(\nu)}_{m-\mu}(z_1)\frac{z_2^{\nu m+\mu-3}}{(\nu m+\mu-3)!} + \cdots + \varphi^{(\nu)}_{0}(z_1)\frac{z_2^{\nu m+m-3}}{((\nu+1)m-3)!} \\
\varphi^{(\nu)}_{m-1}(z_1)\frac{z_2^{\nu m+m-1}}{((\nu+1)m-1)!} + \varphi^{(\nu)}_{m-2}(z_1)\frac{z_2^{\nu m}}{(\nu m)!} + \varphi^{(\nu)}_{m-3}(z_1)\frac{z_2^{\nu m+1}}{(\nu m+1)!} \\
\qquad + \cdots + \varphi^{(\nu)}_{m-\mu}(z_1)\frac{z_2^{\nu m+\mu-2}}{(\nu m+\mu-2)!} + \cdots + \varphi^{(\nu)}_{0}(z_1)\frac{z_2^{\nu m+m-2}}{((\nu+1)m-2)!} \\
\varphi^{(\nu)}_{m-1}(z_1)\frac{z_2^{\nu m}}{(\nu m)!} + \varphi^{(\nu+1)}_{m-2}(z_1)\frac{z_2^{\nu m+1}}{(\nu m+1)!} + \varphi^{(\nu+1)}_{m-3}(z_1)\frac{z_2^{\nu m+2}}{(\nu m+2)!} \\
\qquad + \cdots + \varphi^{(\nu+1)}_{m-\mu}(z_1)\frac{z_2^{\nu m+\mu-1}}{(\nu m+\mu-1)!} + \cdots + \varphi^{(\nu+1)}_{0}(z_1)\frac{z_2^{\nu m+m-1}}{((\nu+1)m-1)!}
\end{bmatrix}
$$

Applying the integral operators from (5.3.9) to this series this vector is seen to solve this system. If the $\varphi_\mu(z_1), 0 \leq \mu \leq m-1$, are analytic functions in the unit disc $|z_1| < 1$ by the CAUCHY coefficient estimation the series is seen to converge for any $z_2 \in \mathbb{C}$. The equations

$$u_{z_2} = w_{m-2},\ u_{z_1} = w_{m-1}\ ,$$

then give the solution to (5.3.9) in the form

$$
\begin{aligned}
u(z_1, z_2) &= \sum_{\nu=0}^{\infty}\sum_{\mu=2}^{m} \varphi^{(\nu)}_{m-\mu}(z_1)\frac{z_2^{\nu m+\mu-1}}{(\nu m+\mu-1)!} \\
&\quad + \sum_{\nu=0}^{\infty} \varphi^{(\nu)}_{m-1}(z_1)\frac{z_2^{(\nu+1)m}}{((\nu+1)m!} + \int_0^{z_1} \varphi_{m-1}(t)dt + \widetilde{c}_{m-1}
\end{aligned}
$$

with an arbitrary $\tilde{c}_{m-1} \in \mathbb{C}$. Obviously, from here

$$u(z_1, 0) = \int_0^{z_1} \varphi_{m-1}(t)dt + c_{m-1}, \frac{\partial^\mu u}{\partial z_2^\mu}(z_1, 0) = \varphi_{m-\mu-1}(z_1), 1 \le \mu \le m-1, \tag{5.3.12}$$

follow. In order to determine the analytic functions $\varphi_\mu, 0 \le \mu \le m-1$, the following boundary conditions are prescribed.

Theorem 59. *The solution to (5.3.9) satisfying*

$$\operatorname{Re} \frac{\partial^\mu u}{\partial z_2^\mu}(z_1, 0) = h_{m-\mu-1}(z_1) \text{ on } |z_1| = 1,\ 0 \le \mu \le m-1 ,$$

is given by

$$u(z_1, z_2) = \sum_{\mu=1}^{m} \left\{ \sum_{\nu=0}^{\infty} \frac{\nu!}{\pi i} \int_{|\zeta|=1} h_{m-\mu}(\zeta) \frac{d\zeta}{(\zeta - z_1)^{\nu+1}} \frac{z_2^{\nu m+\mu-1}}{(\nu m + \mu - 1)!} \right.$$

$$\left. -\frac{1}{2\pi i} \int_{|\zeta|=1} h_{m-\mu}(\zeta) \frac{d\zeta}{\zeta} \frac{z_2^{\mu-1}}{(\mu - 1!)} + i c_{m-\mu} \frac{z_2^{\mu-1}}{(\mu-1)!} \right\} \tag{5.3.13}$$

with arbitrary constants $c_\mu \in \mathbb{R}, 0 \le \mu \le m-2, c_{m-1} \in \mathbb{C}$.

Proof. The boundary conditions and the relations (5.3.12) imply DIRICHLET boundary conditions on the analytic functions φ_μ,

$$\operatorname{Re} \varphi_\mu(z_1) = h_\mu(z_1) \text{ on } |z_1| = 1,\ 0 \le \mu \le m-2 ,$$

so that these functions are given up to some purely imaginary constants by the SCHWARZ–POISSON integral

$$\begin{aligned} \varphi_\mu(z_1) &= \frac{1}{2\pi i} \int_{|\zeta|=1} h_\mu(\zeta) \frac{\zeta + z_1}{\zeta - z_1} \frac{d\zeta}{\zeta} + i c_\mu \\ &= \frac{1}{\pi i} \int_{|\zeta|=1} h_\mu(\zeta) \frac{d\zeta}{\zeta - z_1} - \frac{1}{2\pi i} \int_{|\zeta|=1} h_\mu(\zeta) \frac{d\zeta}{\zeta} + i c_\mu,\ c_\mu \in \mathbb{R}, |z_1| < 1 , \end{aligned}$$

from which for $0 < \nu$

$$\varphi_\mu^{(\nu)}(z_1) = \frac{\nu!}{\pi i} \int_{|\zeta|=1} h_\mu(\zeta) \frac{d\zeta}{(\zeta - z_1)^{\nu+1}} , |z_1| < 1 ,$$

follows. For φ_{m-1} one observes that with $z_1 = e^{it}$

$$\operatorname{Re}\frac{\partial}{\partial t}u(z_1,0) = \operatorname{Re}\{iz_1\frac{\partial u}{\partial z_1}(z_1,0)\} = \operatorname{Re}\{iz_1\varphi_{m-1}(z_1)\} = \frac{d}{dt}h_{m-1}(z_1)\,.$$

Because

$$\int_0^{2\pi}\frac{d}{dt}h_{m-1}(e^{it})dt = 0$$

this problem is solvable; $iz_1\varphi_{m-1}$ again is given by the SCHWARZ–POISSON formula giving ($\zeta = e^{it}$)

$$\varphi_{m-1}(z_1) = -\frac{1}{\pi}\int_{|\zeta|=1}\frac{d}{dt}h_{m-1}(\zeta)\frac{d\zeta}{(\zeta - z_1)\zeta} = \frac{1}{\pi i}\int_{|\zeta|=1}h_{m-1}(\zeta)\frac{d\zeta}{(\zeta - z_1)^2},\ |z_1| < 1\,,$$

and for $0 \le \nu$

$$\varphi_{m-1}^{(\nu)}(z_1) = \frac{(\nu+1)!}{\pi i}\int_{|\zeta|=1}h_{m-1}(\zeta)\frac{d\zeta}{(\zeta - z_1)^{\nu+2}},\ |z_1| < 1\,.$$

Inserting these representations in the formula for u gives (5.3.13), where

$$ic_{m-1} := \tilde{c}_{m-1} + \frac{1}{2\pi i}\int_{|\zeta|=1}h_{m-1}(\zeta)\frac{d\zeta}{\zeta}\,.$$

As was mentioned before there is another way to treat (5.3.8) which is more common for this type of equation. It is reduced to the integro–differential equation

$$u(z_1,z_2) = \int_0^{z_2}\frac{(z_2 - t)^{m-1}}{(m-1)!}u_{z_1}(z_1,t)dt + \sum_{\mu=0}^{m-1}\varphi_\mu(z_1)z_2^\mu$$

where φ_μ are arbitrary analytic functions. Solving this equation iteratively gives the solution from Theorem 58 more easily.

5.4 The SCHWARZ–POISSON formula for polydiscs

The SCHWARZ–POISSON formula for analytic functions in the unit disc of the complex plane was found in Chapter 1 – addressed as the SCHWARZ operator (1.2.2) for the unit disc – by means of the complex GREEN function. A simple purely complex analytic deduction only using the CAUCHY formula for analytic functions is contained in the proof of formula (3.3.2) in the case where w is analytic.

From this formula and the CAUCHY formula we can inductively get a SCHWARZ–POISSON formula for analytic functions in the unit polydisc of $\mathbb{C}^n$ for any n, see [Beku93]. For $w \in C(\overline{\mathbb{D}};\mathbb{C})$ analytic in $\mathbb{D}$ we have the CAUCHY formula

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w(\zeta)\frac{d\zeta}{\zeta - z}$$

which we rewrite as

$$2w(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w(\zeta)\left(\frac{\zeta+z}{\zeta-z}+1\right)\frac{d\zeta}{\zeta}$$

and the SCHWARZ–POISSON formula

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \mathrm{Re}\, w(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} + i\mathrm{Im}\, w(0)\,.$$

We also have

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \overline{w(\zeta)}\frac{d\zeta}{\zeta - z} = -\overline{\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta - z}}} = \overline{\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w(\zeta)\frac{d\zeta}{\zeta(1-\overline{z}\zeta)}} = \overline{w(0)}\,.$$

Theorem 60. *Let $w \in C(\overline{\mathbb{D}}^n;\mathbb{C})$ be analytic in the polydisc $\mathbb{D}^n = \{(z_1,\dots,z_n) : |z_\nu| < 1, 1 \le \nu \le n\}$ of $\mathbb{C}^n$. Then*

$$2^{n-1}w(z_1,\dots,z_n) = \left(\frac{1}{2\pi i}\right)^n \int\limits_{|\zeta_1|=1}\cdots\int\limits_{|\zeta_n|=1} \mathrm{Re}\, w(\zeta_1,\dots,\zeta_n)\prod_{\nu=1}^{n}\frac{\zeta_\nu+z_\nu}{\zeta_\nu-z_\nu}\frac{d\zeta_\nu}{\zeta_\nu} \tag{5.4.1}$$

$$+\sum_{k=2}^{n}(-1)^k 2^{n-k}\sum_{\substack{\nu_1,\dots,\nu_{k-1}=1\\ \nu_\kappa\neq\nu_\lambda,\kappa\neq\lambda}}^{n} w(z_1,\dots,z_n)\Big|_{z_{\nu_1}=\dots=z_{\nu_{k-1}}=0} - \frac{1}{2}[\overline{w(0,\dots,0)}+(-1)^n w(0,\dots,0)].$$

Proof. For $n = 1$ the formula (5.4.1) is the known SCHWARZ–POISSON formula in $\mathbb{C}$ since the sum on the right-hand side does not occur. Assume (5.4.1) is valid for some $n \ge 1$. Let $w \in C(\overline{\mathbb{D}}^{n+1};\mathbb{C})$ be analytic in $\mathbb{D}^{n+1}$ then by the above CAUCHY formula and by (5.4.1)

$$2^n w(z_1,\ldots,z_n,z_{n+1}) = \frac{1}{2\pi i}\int\limits_{|\zeta_{n+1}|=1} 2^{n-1}w(z_1,\ldots,z_n,\zeta_{n+1})\left(\frac{\zeta_{n+1}+z_{n+1}}{z_{n-1}-z_{n+1}}+1\right)\frac{d\zeta_{n+1}}{\zeta_{n+1}}$$

$$= \left(\frac{1}{2\pi i}\right)^{n+1}\int\limits_{|\zeta_1|=1}\cdots\int\limits_{|\zeta_{n+1}|=1} \operatorname{Re} w(\zeta_1,\ldots,\zeta_{n+1})\prod_{\nu=1}^{n+1}\frac{\zeta_\nu+z_\nu}{\zeta_\nu-z_\nu}\frac{d\zeta_\nu}{\zeta_\nu}$$

$$+\left(\frac{1}{2\pi i}\right)^{n+1}\int\limits_{|\zeta_1|=1}\cdots\int\limits_{|\zeta_{n+1}|=1} \operatorname{Re} w(\zeta_1,\ldots,\zeta_{n+1})\prod_{\nu=1}^{n}\frac{\zeta_\nu+z_\nu}{\zeta_\nu-z_\nu}\frac{d\zeta_\nu}{\zeta_\nu}\frac{d\zeta_{n+1}}{\zeta_{n+1}}$$

$$+\sum_{k=2}^{n}(-1)^k 2^{n-k}\sum_{\nu_1,\ldots,\nu_{k-1}=1}^{n}\int\limits_{|\zeta_{n+1}|=1} w(z_1,\ldots,z_n,\zeta_{n+1})\Big|_{z_{\nu_1}=\cdots=z_{\nu_{k-1}}=0}\frac{2d\zeta_{n+1}}{\zeta_{n+1}-z_{n+1}}$$

$$-\frac{1}{2\pi i}\int\limits_{|\zeta_{n+1}=1}\left[\overline{w(0,\ldots,0,\zeta_{n+1})}\right]+(-1)^n w(0,\ldots,0,\zeta_{n+1})\frac{d\zeta_{n+1}}{\zeta_{n+1}-z_{n+1}}\,.$$

Here the second term on the right-hand side is

$$\left(\frac{1}{2\pi i}\right)^{n}\int\limits_{|\zeta_1|=1}\cdots\int\limits_{|\zeta_n|=1} \operatorname{Re} w(\zeta_1,\ldots,\zeta_n,0)\prod_{\nu=1}^{n}\frac{\zeta_\nu+z_\nu}{\zeta_\nu-z_\nu}\frac{d\zeta_\nu}{\zeta_\nu}$$

which by (5.4.1) can be replaced by

$$2^{n-1}w(z_1,\ldots,z_n,0)+\sum_{k=2}^{n}(-1)^{k+1}2^{n-k}\sum_{\nu_1,\ldots,\nu_{k-1}=1}^{n} w(z_1,\ldots,z_n,0)\Big|_{z_{\nu_1}=\cdots=z_{\nu_{k-1}}=0}$$

$$+\frac{1}{2}\left[\overline{w(0,\ldots,0)}+(-1)^n w(0,\ldots,0)\right]$$

$$= 2^{n-1}w(z_1,\ldots,z_n,0)+\sum_{k=3}^{n+1}(-1)^k 2^{n+1-k}\sum_{\nu_1,\ldots,\nu_{k-2}=1}^{n} w(z_1,\ldots,z_{n+1})\Big|_{z_{\nu_1}=\ldots=z_{\nu_{k-2}}=z_{n+1}=0}$$

$$+\frac{1}{2}\left[\overline{w(0,\ldots,0)}+(-1)^n w(0,\ldots,0)\right]\,.$$

The last term in the above formula for $w(z_1,\ldots,z_{n+1})$ is

$$-\overline{w(0,\ldots,0)}+(-1)^{n-+1}w(0,\ldots,0,z_{n+1})\,.$$

Hence,

$$2^n w(z_1,\ldots,z_{n+1}) = \left(\frac{1}{2\pi i}\right)^{n+1} \int\limits_{|\zeta_1|=1} \cdots \int\limits_{|\zeta_{n+1}|=1} w(\zeta_1,\ldots,\zeta_{n+1}) \prod_{\nu=1}^{n+1} \frac{\zeta_\nu + z_\nu}{\zeta_\nu - z_\nu} \frac{d\zeta_\nu}{\zeta_\nu}$$

$$+2^{n-1} w(z_1,\ldots,z_n,0) + \sum_{k=3}^{n+1} (-1)^k 2^{n+1-k} \sum_{\nu_1,\ldots,\nu_{k-2}=1}^{n} w(z_1,\ldots,z_n,0)\Big|_{z_{\nu_1}=\ldots=z_{\nu_{k-2}}=0}$$

$$+\sum_{k=2}^{n+1} (-1)^k 2^{n+1-k} \sum_{\nu_1,\ldots,\nu_{k-1}=1}^{n} w(z_1,\ldots,z_{n+1})\Big|_{z_{\nu_1}=\ldots=z_{\nu_{k-1}}=0}$$

$$-\frac{1}{2}\left[\overline{w(0,\ldots,0)} + (-1)^{n+1} w(0,\ldots,0)\right] .$$

Up to the first and last term this right-hand side is identical to

$$2^{n-1} \sum_{\nu=1}^{n+1} w(z_1,\ldots,z_{n+1})\Big|_{z_{\nu_1}=0} + \sum_{k=3}^{n+1} (-1)^k 2^{n+1-k} \sum_{\nu_1,\ldots,\nu_{k-1}=1}^{n} w(z_1,\ldots,z_{n+1})\Big|_{z_{\nu_1}=\ldots=z_{\nu_{k-1}}=0}$$

$$+\sum_{k=3}^{n+1} (-1)^k 2^{n+1-k} \sum_{\nu_1,\ldots,\nu_{k-2}=1}^{n} w(z_1,\ldots,z_n,z_{n+1})\Big|_{z_{\nu_1}=\ldots=z_{\nu_{k-2}}=z_{n+1}=0}$$

$$= 2^{n-1} \sum_{\nu=1}^{n+1} w(z_1,\ldots,z_{n+1})\Big|_{z_{\nu_1}=0}$$

$$+\sum_{k=3}^{n+1} (-1)^k 2^{n+1-k} \sum_{\nu_1,\ldots,\nu_{k-1}=1}^{n+1} w(z_1,\ldots,z_{n+1})\Big|_{z_{\nu_1}=\ldots=z_{\nu_{k-1}}=0}$$

$$= \sum_{k=2}^{n+1} (-1)^k 2^{n+1-k} \sum_{\nu_1,\ldots,\nu_{k-1}=1}^{n+1} w(z_1,\ldots,z_{n+1})\Big|_{z_{\nu_1}=\ldots=z_{\nu_{k-1}}=0} .$$

This gives (5.4.1) for $n+1$ rather than for n.

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