LERAY–SCHAUDER TYPE ALTERNATIVES, COMPLEMENTARITY PROBLEMS AND VARIATIONAL INEQUALITIES

Nonconvex Optimization and Its Applications

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LERAY–SCHAUDER TYPE ALTERNATIVES, COMPLEMENTARITY PROBLEMS AND VARIATIONAL INEQUALITIES

By

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To my family

The motion of Truth is cyclical,
The way of Truth is pliant

The Works of Lao Zi
Truth and Nature

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PREFACE

This book deals with the Leray–Schauder Principle, the study of complementarity problems and the study of variational inequalities. The first is given by the following classical result.

Theorem 1 [Leray–Schauder Principle]. *Let $(E,\|\cdot\|)$ be a Banach space, $\Omega \subset E$ an open bounded set such that $0 \in \Omega$ and $f:\overline{\Omega} \to E$ a continuous compact mapping. If $f(x) \neq \lambda x$ for all $x \in \partial\Omega$ and $\lambda > 1$, then f has a fixed point.*

From *Theorem 1* we deduce the following result.

Theorem 2 [Leray–Schauder alternative]. *Let $(E,\|\cdot\|)$ be a Banach space, $\Omega \subset E$ an open bounded subset such that $0 \in \Omega$ and $f:\overline{\Omega} \to E$ a continuous compact mapping. Then:*

(1) *either f has a fixed point in $\overline{\Omega}$ or*
(2) *there exists an element $x_* \in \partial\Omega$ and a real number $\lambda_* \in]0, 1[$ such that $x_* = \lambda_* f(x_*)$.*

Theorems 1 and *2* are considered to be the most important results in nonlinear analysis and lead to applications in the study of nonlinear functional equations.

Complementarity theory is a relatively new domain in applied mathematics with deep connections with several aspects of fundamental mathematics. The main goal of complementarity theory is the study of *complementarity problems* from several points of view. Complementarity problems represent a wide class of mathematical models related to optimization, economics, mechanics and engineering. In many mathematical models the complementarity condition is used to determine the equilibrium as used in physics or in economics. There exist few books dedicated to the study of complementarity problems: Some of these are (Cottle, R. W., Pang, J. S. and Stone, R. E. [1]), (Isac, G. [12] and [26]), (Hyers, D. H., Isac, G. and Rassias, Th. M. [1]) and (Isac, G., Bulavski, W. A. and Kalashnikov, V. V. [2])

The study of *variational inequalities* is another domain of applied mathematics. Variational inequalities have many applications to the study of certain problems with unilateral conditions, and there are many papers and books dedicated to this subject. A complementarity problem is associated with a mapping and a closed convex cone, whereas a variational inequality is associated with a mapping and a closed convex set. It is known that a variational inequality associated with a mapping and a closed convex cone is equivalent to a complementarity problem. Until now all applications of the Leray–Schauder Principle [Theorem 1] have been exclusively dedicated to the study of existence of fixed points or of existence of solutions of nonlinear equations. See, for example, the books (O'Regan, D. and Precup, R. [1] and (Precup, R. [1]).

Considering these applications from the point of view of the Leray–Schauder alternative [Theorem 2], we observe that the authors considered only the conclusion (1) of *Theorem 2.* In this book we show that *conclusion* (2) of *Theorem 2* has also interesting applications. By using this conclusion we introduce the notion of *an exceptional family of elements* for a mapping. This notion is related to a complementarity problem or to a variational inequality. The property of being *without an exceptional family of elements* is a kind of *coercivity property,* which is more general than the classical notion of coercivity.

The notion of an *exceptional family of elements* introduced in this book by the Leray–Schauder alternative is the same notion that was introduced in 1997 in our paper, (Isac, G., Bulavski, V. A. and Kalashnikov, V. V. [1]), by using the family's topological degree. In this book we replace the topological degree by Leray–Schauder alternatives, because in this way we can define the notion of *exceptional family of elements* for classes of mappings for which the topological degree is not defined. The investigation method based on this notion is simpler and elegant.

Our notion of *exceptional family of elements* contains as a particular case the notion of "exceptional sequence of elements" which was introduced with respect to $\mathbb{R}^n_+$ in (Smith, T. E. [1]) and has no relation with the main result proved in (Eaves, B. C. [2]). Moreover, the main result proved by Eaves is strongly related to the fact that the convex cone $\mathbb{R}^n_+$ has a bounded base; his result can not be extended to an arbitrary cone in a Hilbert space or in a Banach space.

The notion of *exceptional family of elements* presented in this book has deep relations with fundamental notions of nonlinear analysis and shows promise of other new developments. In particular, research shows that the investigation method based on this notion is a remarkable method for complementarity theory and for the theory of variational inequalities with respect to unbounded closed convex sets. The study of existence of solutions for complementarity problems and for variational inequalities is unified by this method.

Now, let us briefly describe the content of this book.

Chapter 1 is dedicated to the preliminary notions that are used systematically in this book.

Chapter 2 defines the complementarity problems and the variational inequalities used in this book and their equivalences.

Chapter 3 presents the Leray–Schauder type alternatives. The alternatives are given by their proofs.

Chapter 4 contains several results and facts considered as the origin of the notions of *exceptional family of elements* presented in **Chapters 5–8**.

Chapter 5 is dedicated to the results obtained for complementarity problems by the topological method based on the notion of *exceptional family of elements.*

Chapter 6 introduces the notion of *infinitesimal exceptional family of elements*. Here we apply scalar derivatives to the study of complementarity problems.

Chapter 7 presents several special notions and results related to the notion of *exceptional family of elements*. In this chapter we show that the notion of *exceptional family of elements* can be defined for more general classes of mappings and for this definition the Leray–Schauder alternatives are not necessary. In this chapter we give also a necessary and sufficient condition for the non-existence of an exceptional family of elements. This result is the starting point for new and interesting results.

Finally, **Chapter 8** is dedicated to the study of variational inequalities by the method presented in this book. The last subject of this chapter is the study of variational inequalities with integral operators.

We note that the Bibliography contains not only the cited papers but other papers related to this subject.

The goal of many books is to present a collection of the most significant results on some subjects, obtained in a period of time, but the main goal of our book is to show a new method, applicable to the study of complementarity problems and variational inequalities. We would like the reader to consider this book as a starting point of a new topological method applicable to the study of complementarity problems and of variational inequalities. Certainly, this method can be improved, and many new developments based on ideas presented in this book are possible. In particular, the study of *order complementarity problems* by the method presented in this book is a completely open subject. Considering the fact that mathematics is a collective work, perhaps other authors will improve and develop our method.

It is impossible to finish this preface without to say many, many thanks to my wife Viorica, for her excellent work. She has carefully prepared the manuscript of this book with unlimited and constant enthusiasm. I will keep in my heart her real support.

To conclude, I would like to say that I appreciated very much the excellent assistance offered me by the staff of Springer Publishers.

June 1, 2005

Prof. G. Isac

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PRELIMINARY NOTIONS

The reader of this book must have a minimum background of a course in general topology and a course in functional analysis. However, to facilitate the lecture we recall in this chapter several preliminary notions. Certainly, other special notions related to the results presented in this book will be introduced in each chapter.

1.1 Topological spaces. Some fundamental notions

Let X, Y be arbitrary sets. We use the standard notations $x \in X$ for "x is an element of X", $X \subset Y$ for "X is a subset of Y" and $X = Y$ for " $X \subset Y$ and $Y \subset X$". The complement of X relative to Y is the set $C_Y X = \{x \in Y : x \notin X\}$. The set of all subsets of X is denoted by $\mathcal{P}(X)$. Let $\{X_i\}_{i \in I}$ be a family of sets. For the union of this family we use the notation $\bigcup_{i \in I} X_i$ and for intersection the notation $\bigcap_{i \in I} X_i$. If $I = \mathbb{N}$ we have a sequence of sets and we use respectively the notations $\bigcup_{n=1}^{\infty} X_n$ and $\bigcap_{n=1}^{\infty} X_n$. A mapping f of X into Y is denoted by $f : X \to Y$. The domain of f is X and the image of X under f is called the range of f. For any $A \subset X$, we write $f(A)$ to denote the set $\{f(x) : x \in A\} \subset Y$. For any $B \subset Y$, $f^{-1}(B) = \{x \in X : f(x) \in B\}$. If

$f : X \to Y$ and $g : Y \to Z$ are mappings, the composition mapping $x \to g(f(x))$ is denoted by $g \circ f$. We denote the empty set by ϕ.

DEFINITION 1.1.1. *Let X be any non-empty set. A subset τ of $\mathcal{P}(X)$ is said to be a topology on X if the following axioms are satisfied:*

(i) *X and ϕ are members of τ,*

(ii) *the intersection of any two members of τ is a member of τ,*

(iii) *the union of any family of members of τ is again in τ.*

We say that the couple (X, τ) is *a topological space*. If τ is a topology on X the members of τ are then said to be *τ-open subsets* of X, or merely *open subsets* of X if no confusion may result. The subset $\tau_t = \{\phi, X\}$ of $\mathcal{P}(X)$ is a topology on X called the *trivial topology*. It is easy to show that $\tau_d = \mathcal{P}(X)$ is a topology on X called the *discrete topology*. The topologies τ_t and τ_d are not interesting. An interesting topology τ on X must be such that $\tau_t \subset \tau \subset \tau_d$.

DEFINITION 1.1.2. *In a topological space (X, τ), we say that a subset F of X is τ-closed (or merely closed) if $F = C_X U$, where U is a τ-open set.*

The *closed subsets* of X have the following properties:

(1) *X and ϕ are closed subsets of X,*

(2) *the union of any two closed subsets of X is again a closed subset of X,*

(3) *the intersection of any family of closed subsets of X is again a closed subset of X.*

Remark. There exist subsets that are not open and not closed.

Given a non-empty subset $A \subset X$, the open set int$A$ which is the union of all open subsets of A, is called the *interior* of A. The interior of a set may be empty. The closed set $\overline{A}$, the intersection of all closed sets containing A, is called *the closure* of A. An element $x \in$ int$A$ is called an *interior point* of A. An element $x \in \overline{A}$ is called *an adherent point* of A. We say that a subset V of X is a *τ-neighborhood (or merely neighborhood)* of a point $x \in X$ if there exists an open set U such that $x \in U \subseteq V$. Let $(I, \leq)$ be any partially ordered set. It is said to be a *directed set* if given any i and any j in I there is $k \in I$ such that $i \leq k$ and $j \leq k$. Note that any totally ordered set is directed. In particular the set N of natural real numbers is a directed set.

Let (X, τ) be a topological space and I be a directed set. A function x from I into X is said to be a *net* in X. The expression $x(i)$ is usually denoted by x_i, and the net itself is denoted by $\{x_i\}_{i \in I}$.

DEFINITION 1.1.3. *A net $\{x_i\}_{i \in I}$ is said to be convergent to a point $x_* \in X$ if for any neighborhood V of x_*, there exists an index $i_V \in I$ such that for any $i \in I$ satisfying $i_V \leq i$, we have that $x_i \in V$.*

If a net $\{x_i\}_{i \in I}$ is convergent to x_*, we write $\lim_{i \in I} x_i = x_*$. It is known that a subset A of X is closed, if and only if for any net $\{x_i\}_{i \in I}$ in A the condition $\lim_{i \in I} x_i = x_0$ implies $x_0 \in A$.

DEFINITION 1.1.4. *We say that a topological space (X, τ) is a Hausdorff space, if and only if given any two distinct points x and y of X, there are open sets U and V such that $x \in U$, $y \in V$* and *$U \cap V = \phi$.*

It is known that a topological space (X, τ) is Hausdorff, if and only if given any convergent net $\{x_i\}_{i \in I}$ in X the limit of $\{x_i\}_{i \in I}$ is unique. In this book we will consider only Hausdorff topological spaces.

Let $(X, \tau_1), (Y, \tau_2)$ be topological spaces and let $f : X \to Y$ be a mapping.

DEFINITION 1.1.5. *We say that f is continuous at a point $x \in X$, if for each τ_2-neighborhood V of $y = f(x)$, $f^{-1}(V)$ is a τ_1-neighborhood of x.*

If f is continuous at any $x \in X$, then in this case we say that f is *continuous* on X.

The following statements are equivalent:

(1) *f is continuous on X,*

(2) *for any subset A of X, we have* $f(\overline{A}) \subseteq \overline{f(A)}$,

(3) *if* $F \subset Y$ *is* τ_2*-closed, then* $f^{-1}(F)$ *is* τ_1*-closed in X,*

(4) *if* $U \subset Y$ *is* τ_2*-open, then* $f^{-1}(U)$ *is* τ_1*-open in X.*

Convergent nets can characterize the continuity of a mapping. In this sense we have the following classical result.

A mapping f: X → Y is continuous on X if and only if for every net $\{x_i\}_{i\in I}$ *in X such that* $\{x_i\}_{i\in I}$ *is convergent to x, the net* $\{f(x_i)\}_{i\in I}$ *in Y converges to f(x).*

1.2 Metric spaces

First, we note that a metric space is a set in which we have a measure of the closedness or proximity of two arbitrary elements in the set. This measure is obtained by a "distance".

DEFINITION 1.2.1. *Let X be an arbitrary non-empty set. We say that a function* $d : X \times X \to \boldsymbol{R}$ *is said to be a metric (distance) on X if:*

(1) $d(x,y) \geq 0$, *for all* $x, y \in X$,

(2) $d(x,y) = d(y,x)$, *for all* $x, y \in X$,

(3) $d(x,y) = 0$ *if and only if* $x = y$,

(4) $d(x,z) \leq d(x,y) + d(y,z)$, *for all* $x, y, z \in X$.

The couple (X, d) is said to be a *metric space*. If (X, d) is a metric space, we can define on X a topology by the following method. For any $x \in X$ and any positive real number ε, the $d - \varepsilon$-ball is the set:

$$B(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}.$$

Consider the following collection of subsets of X,

$$\tau_d = \{U \subset X : \textit{for any } x \in U \textit{ there exists } \varepsilon > 0 \textit{ such that } B(x,\varepsilon) \subset U\}.$$

Obviously X, $\phi \in \tau_d$. Indeed, if $x \in X$ and $\varepsilon > 0$, then $B(x, \varepsilon) \subset X$. Since ϕ contains no points, it is true that for each $x \in \phi$ (there is no such x) and any $\varepsilon > 0$, $B(x,\varepsilon) \subset \phi$. We can prove that τ_d is a topology on X, named the topology defined by the metric d. Therefore, any metric space is a topological space, but the converse is not true. Moreover, any metric space is a Hausdorff topological space.

Let (X, d) be a metric space and $\{x_n\}_{n\in N}$, be a sequence in X.

DEFINITION 1.2.2. *The sequence* $\{x_n\}_{n\in N}$ *is said to be convergent to a point* x_* *of* X *if given any positive real number* ε, *there is a natural number* n_ε *such that if* $n > n_\varepsilon$, *then* $d(x_*, x_n) < \varepsilon$.

If $\{x_n\}_{n\in N}$ converges to x_*, then we write $\{x_n\}_{n\in N} \to x_*$, or $x_* = \lim_{n\to\infty} x_n$. The element x_* is said to be the limit of $\{x_n\}_{n\in N}$. In a metric space the limit of a sequence is unique.

Let (X_1, d_1) and (X_2, d_2) be metric spaces.

DEFINITION 1.2.3. *A mapping* $f : X_1 \to X_2$ *is said to be continuous at a point* $x_0 \in X_1$ *if, given any positive real number* $\varepsilon > 0$, *there is a positive real number* δ_ε *such that if* $d_1(x_0, x) < \delta_\varepsilon$, *then* $d_2(f(x_0), f(x)) < \varepsilon$.

Let (X, d) be a metric space and $\{x_n\}_{n\in N}$ a sequence in X.

DEFINITION 1.2.4. *The sequence* $\{x_n\}_{n\in N}$ *is said to be a Cauchy sequence if given any positive real number* ε *there is a natural number* n_ε *such that if* m, n *are natural numbers and* $m, n > n_\varepsilon$, *then* $d(x_n, x_m) < \varepsilon$.

DEFINITION 1.2.5. *A metric space* (X, d) *is said to be complete if every Cauchy sequence in* X *converges to a point of* X.

It is known that any incomplete metric space can be densely immersed in a complete metric space.

1.3 Some classes of topological vector spaces

In this book we will use only real vector spaces. Given a real vector space E and a topology τ on E, the pair (E, τ) (or often denoted by $E(\tau)$) is called a *topological vector space if the following axioms are satisfied:*

(1) $(x,y) \to x+y$ *is continuous on* $E \times E$ *into* E,

(2) $(\lambda, x) \to \lambda x$ *is continuous on* $\mathbb{R} \times E$ *into* E.

An important class of topological vector spaces is the class of normed vector spaces.

Normed vector spaces

A real vector space E is said to be a *normed space* if to every $x \in E$ there is associated a non-negative real number $\|x\|$, called the *norm* of x such that the following axioms are satisfied:

(n_1) $\|x+y\| \le \|x\| + \|y\|$, for all x and y in E,

(n_2) $\|\lambda x\| = |\lambda| \|x\|$, for all $x \in E$ and $\lambda \in \boldsymbol{R}$,

(n_3) $\|x\| = 0$ if and only if $x = 0$.

Remark. From axiom (n_3) we have that $\|x\| > 0$ if $x \neq 0$.

A normed vector space will be denoted by $(E, \|\cdot\|)$. Every normed vector space $(E, \|\cdot\|)$ may be regarded as a metric space, in which the distance between x and y is $d(x,y) = \|x-y\|$. The topology defined on E by this distance is called the topology of the normed vector space $(E, \|\cdot\|)$.

The sets $B(0,1) = \{x \in E : \|x\| < 1\}$ *and* $\bar{B}(0,1) = \{x \in E : \|x\| \le 1\}$ are the *open unit ball* and the *closed unit ball* of E, respectively.

Banach space

A *Banach space* is a normed vector space, which is *complete* in the metric defined by its norm, that is, every Cauchy sequence is convergent.

Many of the best-known function spaces, used in practical problems are Banach spaces. We mention just a few types: spaces of continuous functions on compact spaces, the well-known L_p-spaces, certain spaces of differentiable functions, spaces of continuous linear mappings from one Banach space into another etc.

Let $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ be Banach spaces. A mapping $L: E_1 \to E_2$ is called a *linear mapping* if $L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$ for all $x, y \in E_1$ and all real numbers α, β. The linear mapping L is called *continuous* at $x_0 \in E_1$ if for any sequence $\{x_n\}_{n\in N}$ of elements of E_1 such that $\|x_n - x_0\|_1 \to 0$ we have that $\|L(x_n) - L(x_0)\| \to 0$. If L is continuous at every $x \in E_1$, then we say that L is continuous on E_1. A mapping $L: E_1 \to E_2$ is called *bounded* if there exists a number ρ such that $\|L(x)\|_2 \le \rho \|x\|_1$, for all $x \in E_1$. We denote by $\mathcal{L}(E_1, E_2)$ the set of all continuous mappings from E_1 into E_2. It is known that a linear mapping from E_1 into E_2 is continuous, if and only if it is bounded.

If we take $(E_2, \|\cdot\|_2) = (\mathbb{R}, |\cdot|)$, where $|x|$ is the absolute value of $x \in \mathbb{R}$, then we denote by $E_1^* = \mathcal{L}(E_1, \mathbb{R})$ and we say that E_1^* is the *topological dual* of E_1. If for any $L \in \mathcal{L}(E_1, E_2)$ we define

$$\|L\| = \sup_{\|x\|=1} \|L(x)\|,$$

then we have that $L \to \|L\|$ is a norm on $\mathcal{L}(E_1, E_2)$ and it is known that $(\mathcal{L}(E_1, E_2), \|\cdot\|)$ is a Banach space. Consequently for any Banach space $(E, \|\cdot\|)$, its topological dual E^* is also a Banach space.

Hilbert space

The class of Hilbert spaces is an important subclass of Banach spaces. In this book, we will consider only real Hilbert spaces. D. Hilbert (in his paper: Grundzuge einer allgemeinen Theory der linearen Integralgleichungen, Leipzig, 1912) initiated the theory of Hilbert spaces. After many years, John von Neumann (1903-1957) became the first to formulate an axiomatic theory of Hilbert spaces.

Let E be a real vector space.

DEFINITION 1.3.1. *We say that a mapping* $\langle\cdot,\cdot\rangle : E \times E \to \mathbb{R}$ *is an inner-product in E if for any x, y, z* $\in$ *E and* α, $\beta \in \mathbb{R}$ *the following axioms are satisfied:*

(1) $\langle x, y\rangle = \langle y, x\rangle$,

(2) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$,

(3) $\langle x, x\rangle \geq 0$ *and* $\langle x, x\rangle = 0$ *if and only if* $x = 0$.

A real vector space with an inner-product is called an *inner-product space,* or a pre-Hilbert space.

Examples

I. The real field $\mathbb{R}$ is an inner-product space. The inner-product is defined by $\langle x, y\rangle = x \cdot y$.

II. The n-dimensional real vector space $\mathbb{R}^n$, with the *inner-product* defined by $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$, where $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$, is an inner-product space.

III. The space ℓ^2 of all sequences $(x_1, x_2, ..., x_n, ...)$ of real numbers such that $\sum_{k=1}^{\infty} |x_k|^2 < +\infty$, with the inner-product defined by $\langle x, y\rangle = \sum_{k=1}^{\infty} x_k y_k$, is an infinite dimensional inner-product space. This space is between the most important examples of inner-product spaces.

IV. Let E be the real vector space of sequences $(x_1, x_2, ..., x_n, ...)$ of real numbers such that only a finite number of terms is non-zero. This is an inner-product space with the inner-product defined by $\langle x, y\rangle = \sum_{k=1}^{\infty} x_k y_k$, where $x = (x_1, x_2, ..., x_k, ...)$ and y = $(y_1, y_2, ..., y_k, ...)$.

V. The real vector space $C([a,b],\mathbb{R})$ of all continuous real-valued functions on the interval $[a, b] \subset \mathbb{R}$, with the inner-product $\langle f,g\rangle = \int_a^b f(x)g(x)dx$ is an inner-product space.

VI. The real vector space $L^2(\Omega)$ with the inner-product defined by $\langle f,g\rangle = \int_\Omega f(x)g(x)dx$ is a very important inner-product space.

VII. Let E be the Cartesian product of Hilbert spaces $(E_1,\langle\cdot,\cdot\rangle_1),\ldots,(E_n,\langle\cdot,\cdot\rangle_n)$, i.e.,
$E = E_1 \times E_2 \times \cdots \times E_n = \{(x_1,x_2,\ldots,x_n) : x_1 \in E_1,\ldots,x_n \in E_n\}$
is a Hilbert space with the inner-product defined by: $\langle(x_1,\ldots,x_n),(y_1,\ldots,y_n)\rangle = \langle x_1,y_1\rangle_1 + \langle x_2,y_2\rangle_2 + \cdots + \langle x_n,y_n\rangle$.

Let $(E,\langle\cdot,\cdot\rangle)$ be an inner-product space. Two vectors x and y in E are called *orthogonal*, denoted by $x \perp y$, if $\langle x, y\rangle = 0$. Any inner-product space $(E,\langle\cdot,\cdot\rangle)$ is a normed vector space with the norm defined by

$$\|x\| = \sqrt{\langle x,x\rangle},\ \textit{for any } x \in E.$$

The norm of any inner-product space $(E,\langle\cdot,\cdot\rangle)$ satisfies the following important properties:

Schwartz's inequality. *For any two elements x and y in E we have $|\langle x,y\rangle| \le \|x\|\|y\|$. The equality $|\langle x,y\rangle| = \|x\|\|y\|$ holds, if and only if x and y are linearly dependent.*

Parallelogram law. *For any two elements x and y in E we have* $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

A consequence of the parallelogram law is the Pythagorean Formula: if $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

DEFINITION 1.3.2. *A complete inner-product space is called a Hilbert space. (By the completeness of an inner-product space $(E,\langle\cdot,\cdot\rangle)$, we mean the completeness of E as a normed space).*

The examples I-III and VI-VII are Hilbert spaces. The examples described in IV and V are not Hilbert spaces, since these spaces as normed vector spaces are not complete. We will denote a Hilbert space by $(H, \langle\cdot,\cdot\rangle)$.

The *topological dual* of a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ can be identified (by an isomorphism) with *H*. We recall also that a Hilbert space is called *separable* if it contains a complete orthonormal sequence. (An orthonormal sequence $\{x_n\}_{n\in N}$ in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ is said to be *complete* if for every $x \in H$ we have $x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$.)

Let $E(\tau)$ be a topological vector space.

DEFINITION 1.3.3. *A subset D of E is called bounded if for each 0-neighborhood U in E, there exists* $\lambda \in \mathbb{R}$ *such that* $D \subseteq \lambda U$.

For example, in a normed vector space $(E, \langle\cdot,\cdot\rangle)$, the sets $B(0, 1)$ and $\overline{B}(0,1)$ are bounded sets. The following notions are also useful. We say that a subset $D_1 \subset E$ *absorbs* a subset $D_2 \subset E$ if there exists $\lambda_0 \in \mathbb{R}$ such that $D_2 \subseteq \lambda D_1$ whenever $|\lambda| \geq |\lambda_0|$. A subset $D \subset E$ is called *radial* (*absorbing*), if *D* absorbs every finite subset of *E*. A subset $D \subset E$ is *circled* if $\lambda D \subseteq D$, whenever $|\lambda| \leq 1$. If $A \subset E$ the *circled hull* of *A* is the intersection of all circled subsets of *E* containing *A*.

DEFINITION 1.3.4. *We say that a subset D of E is convex if* $x \in D$ *and* $y \in D$ *imply that* $\lambda x + (1-\lambda) y \in D$ *for all scalars satisfying* $0 < \lambda < 1$.

It is known that the sets

$$\{\lambda x + (1-\lambda) y : 0 \leq \lambda \leq 1\} \text{ and } \{\lambda x + (1-\lambda) y : 0 < \lambda < 1\}$$

are called the *closed* and *open line segments*. It is easy to show that convexity of a subset $D \subset E$ is preserved under translation, i.e., *D* is convex if and only if $x_0 + D$ is convex for every $x_0 \in E$. If *A, B* are convex subsets of the space *E*, then $\operatorname{int}(A)$, $\overline{A}$, $A + B$ and $\lambda A (\lambda \in \boldsymbol{R})$ are convex.

The union of two convex sets generally is not a convex set, but the intersection of any family of convex sets is a convex set.

Let A be a subset of the space E. The *convex hull* of A, denoted by *conv*(A), is the intersection of all convex sets containing the set A. It is known that

$$conv(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : \lambda_i > 0, \sum_{i=1}^{n} \lambda_i = 1 \text{ and } n \in N \right\}.$$

DEFINITION 1.3.5. *If $D \subset E$ is any radial subset, the non-negative real function on E*

$$x \to p_D(x) = \inf\{\lambda > 0 : x \in \lambda D\}$$

is called the gauge, or Minkowski functional of D.

A *semi-norm* on E is the gauge of a *radial, circled* and *convex* subset of E. The analytical description of semi-norms is given by the following definition.

DEFINITION 1.3.6. *A real-valued function p on E is a semi-norm if and only if*

(1) $p(x+y) \le p(x) + p(y)$ *for any* $x, y \in E$,

(2) $p(\lambda x) = |\lambda| p(x)$ *for any* $\lambda \in \mathbb{R}$ *and* $x \in E$.

Obviously if p is a semi-norm on E then $p(0) = 0$ and $p(x) \ge 0$ for any $x \in E$. If $D \subset E$ is a radial, convex, circled set, then the semi-norm p on E is the gauge of D if and only if $D_0 \subseteq D \subseteq D_1$ where $D_0 = \{x \in E : p(x) < 1\}$, $D_1 = \{x \in E : p(x) \le 1\}$. It is known also that if p is a semi-norm on E, then p is continuous at $0 \in E$ if and only if $D_0 = \{x \in E : p(x) < 1\}$ is open in E, and also, if and only if p is uniformly continuous on E. The following two notions are also useful in this book.

Let $E(\tau)$ be a topological vector space. A subset D of E is said to be *star-shaped* if there is at least one $x_0 \in D$ such that $(1-\lambda)x_0 + \lambda x \in D$ for all $x \in D$ and $0 < \lambda < 1$.The point $x_0 \in D$ is said to be the *star centre* of D. Every convex set is star shaped but not conversely.

A subset D of E is called *contractible* if there is a continuous mapping $h : D \times [0,1] \to D$ such that $h(x,0) = x$ for all $x \in D$ and $h(x,1) = x_0$ for some $x_0 \in D$. Every star shaped set $D \subset E$ is contractible since the mapping $h(x,t) = tx_0 + (1-t)x,\ (x,t) \in D \times [0,1]$ where x_0 is the star centre of D.

Locally convex spaces

A topological vector space E over $\mathbb{R}$ will be called *locally convex* if it is a Hausdorff space such that every neighborhood of any $x \in E$ contains a convex neighborhood of x. We can show that E is a *locally convex topological vector space* if the convex neighborhoods of 0 form a base at 0 with intersection $\{0\}$.

Analytically, a locally convex topology on E is determined by an arbitrary family $\{p_\alpha\}_{\alpha\in\mathcal{A}}$ of semi-norms as follows: for each $\alpha \in \mathcal{A}$, let $U_\alpha = \{x \in E : p_\alpha(x) \le 1\}$ and consider the family $\left\{\frac{1}{n}U\right\}$, where $n \in N$ and U ranges over all finite intersections of sets $U_\alpha(\alpha \in \mathcal{A})$. This family $\mathcal{V}$ satisfies the conditions indicated above and hence is a base at 0 for a locally convex topology τ on E, called *the topology generated by the family* $\{p_\alpha\}_{\alpha\in\mathcal{A}}$; equivalently, $\{p_\alpha\}_{\alpha\in\mathcal{A}}$ is said to be a *generating family* of semi-norms for τ. We denote a locally convex space by $E(\tau)$ or $\left(E(\tau), \{p_\alpha\}_{\alpha\in\mathcal{A}}\right)$.

Conversely every locally convex topology on E is generated by a suitable family of semi-norms; it suffices to take the gauge functions of a family of convex, circled 0-neighborhoods whose positive multiples form a subbase at 0. Obviously, every member of a generating family of semi-norms is continuous for τ.

We note that, we can prove that τ is Hausdorff if and only if for each $x \in E$, $x \neq 0$ and each family $\mathcal{P}$ of semi-norms generating τ, there exists $p \in \mathcal{P}$ such that $p(x) > 0$. Any Banach space is a locally convex vector space, but the converse is not true. There exist topological vector spaces that are not locally convex spaces. The general topological vector spaces are not very much used in mathematical modeling of practical problems, but the notion of topological vector space is a fundamental notion in mathematics.

1.4 Compactness and compact operators

Let (X,τ) be a topological space. We say that a family $\{U_i\}_{i\in I}$ of open subsets of X is an *open cover* of X if $X=\bigcup_{i\in I} U_i$. Let $\{U_i\}_{i\in I}$ be an open cover of the space X. A collection $\{V_j\}_{j\in J}$, is said to be an *open subcover* of $\{U_i\}_{i\in I}$ if $\{V_j : j\in J\}\subset\{U_i : i\in I\}$, (that is, each V_j is a U_i and $\{V_j\}_{j\in J}$, is itself an open cover of X.

DEFINITION 1.4.1. *A topological space (X,τ) is said to be compact if given any open cover $\{U_i\}_{i\in I}$, of X, there is a finite subcover of $\{U_i\}_{i\in I}$.*

Let (X,τ) be a topological space and $A\subset X$ a non-empty subset. An *open cover* of A is a collection $\{U_i\}_{i\in I}$ of open subsets of X such that $A\subset\bigcup_{i\in I} U_i$. Equivalently, $\{U_i\}_{i\in I}$ is an open cover of A if $\{U_i\bigcap A\}_{i\in I}$ is an open cover of the subspace A. We say that A is *a compact* subset of X if every open cover of A has a finite subcover. Equivalently, A is a compact subset if the topological subspace (A,τ_A) is compact, where τ_A is the topology on A induced by the topology τ. The following theorem is a classical result.

THEOREM 1.4.1. *Let (X,τ) be a topological space. The following statements are equivalent:*

(1) *(X,τ) is a compact topological space,*

(2) *for any family $\{F_i\}_{i\in I}$ of closed subsets of* X *such that the intersection of any finite number of the F_i is non-empty we have that $\bigcap_{i\in I} F_i \neq \phi$,*

(3) *every net in X has a subnet convergent to an element of X.*

If A is a subset of topological space (X,τ), we say that A is *relatively compact* in X, if $\overline{A}$ is compact. Suppose we are given two topological spaces (X_1,τ_1) and (X_2,τ_2).

DEFINITION 1.4.2. *We say that a mapping* $f: X_1 \to X_2$ *is compact with respect to a non-empty subset A of* X_1 *if* $\overline{f(A)}$ *is compact in* X_2 *(i.e., if f(A) is relatively compact in* X_2*).*

Remark. For nonlinear mappings (when X_1, X_2 are Banach spaces) the continuity is not a consequence of compactness. This is true only for linear mappings.

Let $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ be two Banach spaces. Let $T: E_1 \to E_2$ be a mapping (linear or nonlinear).

DEFINITION 1.4.3. *We say that T is completely continuous if and only if the following two properties are satisfied:*

(1) *T is a continuous mapping,*

(2) *for any bounded subset A in* E_1*, the set T(A) is relatively compact in* E_2.

Remark. If T is linear then, in this case property (2) implies property (1), but for nonlinear operators this implication is not true.

Compactness and *complete continuity* are two fundamental notions in topology and in functional analysis (linear and nonlinear). Complete continuity will be very much used in this book.

DEFINITION 1.4.4. *We say that a mapping* $f: E_1 \to E_1$ *is a completely continuous field if and only if, there is a completely continuous operator* $T: E_1 \to E_1$ *such that f has the representation* $f(x) = x - T(x)$, *for any* $x \in E_1$ *(or shortly,* $f = I - T$, *where* $I: E_1 \to E_1$ *is the identity mapping).*

The notion of a *completely continuous field* is related to the notion of Leray–Schauder degree.

1.5 Measures of noncompactness and condensing operators

In this section we consider the basic notions connected with measures of noncompactness and condensing mappings. We give on this subject only the elementary properties necessary in this book.

The first notion of "*measure of noncompactness*" was introduced by K. Kuratowski in 1930, (Kuratowski, K. [1]). The theory of measures of noncompactness and of condensing operators received a new impetus after the work of G. Darbo (Darbo, G. [1]). Now, there exist some expository articles and books on this subject [(Sadovskii, B. N., [1]), (Daneš, J., [1]), (Banaś, J. and Goebel, K., [1]), (Akhmerov, R. R., Kamenskii, M. I., Potapov, A. S., Rodkina, A. E. and Sadovskii, B. N., [1]).

We give the notions of noncompactness in a general Banach space. Let $(E,\|\cdot\|)$ be a Banach space and let Ω be a subset of E. Let A be a non-empty subset of E. We recall that by *diameter* of A, (denoted by diam(A)) one means the number $\sup\{\|x-y\|: x, y \in A\}$. We use $B = B(0, 1)$ to denote the open unity ball in E.

DEFINITION 1.5.1. *The Kuratowski measure of noncompactness* $\alpha(\Omega)$ *of the set* Ω *is the number* $\inf\{d > 0 : \Omega$ *admits a finite covering of sets of diameter smaller than* $d\}$.

We say that a set $D \subset E$ is an *ε-net* of Ω if

$$\Omega \subset D + \varepsilon \bar{B} = \{x + \varepsilon b : x \in D, b \in \bar{B}\}.$$

DEFINITION 1.5.2. *The Hausdorff measure of noncompactness* $\chi(\Omega)$ *of the set* Ω *is the number* $\inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon-\text{net in } E\}$.

Now, we indicate some of the properties of the Kuratowski and Hausdorff measures of noncompactness (denoted below by ψ).

Property 1 (Regularity). $\psi(\Omega) = 0$ *if and only if* $\bar{\Omega}$ *is compact.*

Property 2 (Nonsingularity). ψ *is equal to zero on every one-element set.*

Property 3 (Monotonicity). $\Omega_1 \subset \Omega_2$ *implies* $\psi(\Omega_1) \leq \psi(\Omega_2)$.

Property 4 (Semi-additivity). $\psi(\Omega_1 \cup \Omega_2) = \max\{\psi(\Omega_1), \psi(\Omega_2)\}$.

Property 5 (Lipschitzianity). $|\psi(\Omega_1)-\psi(\Omega_2)|\le L_\psi \rho(\Omega_1,\Omega_2)$, *where* $L_\chi=1, L_\alpha=2$ *and* ρ *denotes the Hausdorff semi-metric, i.e.,* $\rho(\Omega_1,\Omega_2)=\inf\{\varepsilon>0:\Omega_1+\varepsilon\bar{B}\supset\Omega_2 \text{ and } \Omega_2+\varepsilon\bar{B}\supset\Omega_1\}$.

Property 6 (Continuity). *For any* $\Omega\subset E$ *and any* $\varepsilon>0$ *there is a* $\delta>0$ *such that* $|\psi(\Omega)-\psi(\Omega_1)|<\varepsilon$ *for all* Ω_1 *satisfying* $\rho(\Omega,\Omega_1)<\varepsilon$.

Property 7 (Semi-homogeneity). $\psi(\lambda\Omega)=|\lambda|\psi(\Omega)$ *for any real number* λ.

Property 8 (Algebraic Semi-additivity). $\psi(\Omega_1+\Omega_2)\le\psi(\Omega_1)+\psi(\Omega_2)$.

Property 9 (Invariance under Translations). $\psi(\Omega+x_0)=\psi(\Omega)$ *for any* $x_0\in E$.

The following properties are important but the proof of each requires some technicalities.

THEOREM 1.5.1. *The Kuratowski and Hausdorff measures of noncompactness are invariant under passage to the closure and to the convex hull, i.e.,* $\psi(\Omega)=\psi(\bar{\Omega})=\psi(conv(\Omega))$.

COROLLARY 1.5.2. *We have the following useful formula:*

$$\alpha\left(\bigcup_{0\le\lambda\le\lambda_0}\lambda\Omega\right)=\lambda_0\alpha(\Omega).$$

Proof. This formula is a consequence of properties (1), (3), (4) of Theorem 1.5.1 and of the fact that $\bigcup_{0\le\lambda\le\lambda_0}\lambda\Omega\subset conv(\lambda_0\Omega\cup\{0\})$. □

THEOREM 1.5.3. *Let* $B = B(0, 1)$ *be the unit ball in E. Then* $\alpha(B)=\chi(B)=0$ *if* $\dim(E)<\infty$ *and* $\alpha(B)=2, \chi(B)=1$ *if E is an infinite dimensional Banach space.*

THEOREM 1.5.4. *The Kuratowski and Hausdorff measures of noncompactness are related by the inequalities* $\chi(\Omega) \leq \alpha(\Omega) \leq 2\chi(\Omega)$.

Now, we give the definition and some properties of condensing operators. A condensing operator is a mapping under which the image of any set is in a certain sense "more compact" than the set itself. The degree of noncompactness of a set is estimated by a measure of noncompactness. Contractive maps and the completely continuous maps are condensing.

Let $(E_1, \|\cdot\|_1), (E_2, \|\cdot\|_2)$ and $(E_3, \|\cdot\|_3)$ be Banach spaces. Suppose we are given on each space a measure of noncompactness denoted respectively by μ_1, μ_2, μ_3. We denote by $\mathcal{B}_{E_i}$ the bounded sets in E_i ($i = 1, 2, 3$).

DEFINITION 1.5.3. *We say that a mapping* $f: E_1 \to E_2$ *satisfies the Darbo condition with a constant* $k \geq 0$, *with respect to the measures of noncompactness* μ_1, μ_2 *if for any* $D \in \mathcal{B}_{E_1}$ *we have* $f(D) \in \mathcal{B}_{E_2}$ *and* $\mu_2(f(D)) \leq k\mu_1(D)$

Remark. We note that if f satisfies the Darbo condition with a constant k, we say also that f is a *k-set Lipschitz mapping.*

If $0 \leq k < 1$ and f satisfies the Darbo condition with the constant k, then in this case we say that f is a *k-set contraction.* If f satisfies the Darbo condition, the smallest constant k such that $\mu_2(f(D)) \leq k\mu_1(D)$ will be denoted by $k(\mu_1, \mu_2, f)$. In the case $E_1 = E_2$ and $\mu_1 = \mu_2 = \mu$, we shall write $k(\mu, f)$, instead of $k(\mu_1, \mu_2, f)$.

The following propositions describe the basic properties of mapping satisfying the Darbo property.

PROPOSITION 1.5.5. *If* $f: E_1 \to E_2$ *and* $g: E_2 \to E_3$ *satisfy the Darbo condition with respect to* (μ_1, μ_2) *and* (μ_2, μ_3) *respectively, then we have* $k(\mu_1, \mu_3, g \circ f) \leq k(\mu_1, \mu_2, f) \cdot k(\mu_2, \mu_3, g)$.

PROPOSITION 1.5.6. *If $f_1, f_2 : E_1 \to E_2$ satisfy the Darbo condition, then $f = \lambda f_1 + (1-\lambda) f_2$, with $0 \le \lambda \le 1$ also satisfy the Darbo condition and we have $k(\mu_1, \mu_2, f) \le \lambda k(\mu_1, \mu_2, f_1) + (1-\lambda) k(\mu_1, \mu_2, f_2)$.*

PROPOSITION 1.5.7. *If $f, g : E_1 \to E_2$ and μ_2 is semi-homogeneous and semi-additive, then we have*

$$k(\mu_1, \mu_2, f+g) \le k(\mu_1, \mu_2, f) + k(\mu_1, \mu_2, g)$$

and for any $\lambda \in \mathbb{R}$, also we have the formula,

$$k(\mu_1, \mu_2, \lambda f) = |\lambda| k(\mu_1, \mu_2, f).$$

Let $(E_1, \|\cdot\|_1), (E_2, \|\cdot\|)$ be Banach spaces. Suppose that on E_1 (resp. on E_2) is defined a measure of noncompactness μ_1 (resp. μ_2), with values in some partially ordered set $(Q, \le)$.

DEFINITION 1.5.4. *A continuous mapping $f : E_1 \to E_2$ is said to be (μ_1, μ_2)-condensing if $\Omega \subset E_1$ and $\mu_2[f(\Omega)] \ge \mu_1(\Omega)$ imply that Ω is relatively compact.*

Remarks.

1. In Definition 1.5.4 we can have $f : D(f) \to E_2$, where $D(f) \subset E_1$ is the domain of definition of f and $D(f)$ is such that $D(f) \ne E_1$.In this case we must take $\Omega \subset D(f)$.
2. The mapping f is said also to be (μ_1, μ_2)-*condensing in the proper sense* if $\mu_2[f(\Omega)] < \mu_1(\Omega)$ for any $\Omega \subset D(f)$ with the property that $\overline{\Omega}$ is not compact. We note that in a partially ordered set $(Q, \le)$ the strict inequality $\alpha < \beta$ means $\alpha \le \beta$ and $\alpha \ne \beta$. If the set Q is totally ordered by the ordering "$\le$", then in this case the two notions of condensing mapping coincide. Obviously, a completely continuous mapping and a contractive mapping, both are condensing with respect to the Kuratowski measure of noncompactness and any continuous and compact mapping is also condensing with respect to the Hausdorff measure of noncompactness.

Suppose that $(Q, \le)$ is an ordered convex cone in a Banach space.

DEFINITION 1.5.5. *A continuous mapping $f: D(f) \subset E_1 \to E_2$ is said to be (k, μ_1, μ_2)-bounded if $\mu_2[f(\Omega)] \leq k\mu_1(\Omega)$ for any set $\Omega \subset D(f)$*, (We suppose $0 < k$.)

When we have $E_1 = E_2 = E$ and $\mu_1 = \mu_2 = \mu$ we say that f is (k, μ)-bounded. In the case $0 < k < 1$, (k, μ)-bounded means, *μ-condensing* with constant k. We indicate the following elementary properties of condensing mappings. We suppose again that $(Q, \leq)$ is an ordered convex cone in a Banach space.

PROPOSITION 1.5.8. *If the measure of noncompactness μ_1 is regular, then any (k, μ_1, μ_2)-bounded operator with $0 < k < 1$ is (μ_1, μ_2)-condensing in the proper sense.*

PROPOSITION 1.5.9. *The composition $f_1 \circ f_2$ of a (k_1, μ_1, μ_2)-bounded mapping f_1 and a (k_2, μ_2, μ_3)-bounded mapping is a $(k_1 k_2, \mu_1, \mu_3)$-bounded mapping.*

PROPOSITION 1.5.10. *If the measure of noncompactness μ_2 is monotone and algebraically semi-additive, then the sum $f_1 + f_2$ of a (k_1, μ_1, μ_2)-bounded f_1* and a *(k_2, μ_1, μ_2)-bounded f_2 (where $f_1, f_2 : E_1 \to E_2$) is a $(k_1 + k_2,$, $\mu_1, \mu_2)$-bounded operator.*

PROPOSITION 1.5.11. *If f_1 is a (μ_1, μ_2)–condensing mapping and f_2 is a (μ_2, μ_3)–condensing mapping that maps relatively compact sets into relatively compact sets, μ_1 and μ_2 are regular measures of noncompactness and $Q = \mathbb{R}_+$, then the composition $f_2 \circ f_1$ is a (μ_1, μ_3)–condensing mapping.*

1.6 Topological degrees

A fundamental mathematical tool in nonlinear analysis is the notion of *topological degree,* because one of the most important tasks in mathematical analysis is to compute the number of solutions $x_* \in \Omega$ of an equation $f(x) = y_0$, where Ω is a subset of some vector space E, y_0 is an element of a range space F and $f : E \to F$ is a mapping. We denote this number of solutions by $N(f,\Omega,y_0)$. We consider the following elementary example.

Let E be the real field $\mathbb{R}$ and $E = F$. Let $\Omega =]a, b[$ where a, $b \in \mathbb{R}$ and $a < b$ and $f(x) = a_0x^n + a_n,x^{n-1} + \cdots + a_{n-1}x + a_n$, a polynomial function with real coefficients defined on the real field $\mathbb{R}$. In this case by the classical theorem of Sturm, we have a procedure to calculate the number $N(f,\Omega,y_0)$, for any $y_0 \in \mathbb{R}$. For a general situation, the estimation of $N(f,\Omega,y_0)$ is a hard problem. We remark that the number $N(f,\Omega,y_0)$ may not be continuous in dependence on y_0 or f. The number $N(f,\Omega,y_0)$ suggested the idea to introduce a numerical indicator of the existence of solutions of an equation $f(x) = y_0$ in a given set Ω.

L. E. J. Brouwer introduced this indicator, named "*topological degree*" in 1912 and M. Nagumo gave the analytic viewpoint on this notion in 1951. We note that topological degrees have developed as a means of examining the solution set of the equation $f(x) = y_0$ in the sense of obtaining information on the existence of solutions, their number and their nature, when f is a member of some special classes of mappings (continuous fields or functions in $\mathbb{R}^n$, completely continuous fields or functions satisfying condition $(S)_+$ in Banach spaces). Now the theory of topological degree is used in the study of both ordinary and partial differential equations and in that of more general functional equations. For this book we need to recall Brouwer's topological degree, Leray–Schauder's degree and the topological degree defined by I. V. Skrypnik.

I. Degree theory in finite dimensional space (Brouwer's topological degree)

Let $\mathbb{R}^n = \{x = (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R},\ i =, 2, \ldots, n\}$ be the n-dimensional Euclidean space. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open non-empty subset and y_0 a point of $\mathbb{R}^n$. We denote by $\overline{\Omega}$ the closure of Ω and by $\partial\Omega$ its boundary. Let $C(\overline{\Omega})$ be the linear space of continuous functions from $\overline{\Omega}$ into $\mathbb{R}^n$ with the norm

$$\|f\| = \sup_{x \in \Omega} |f(x)|,$$

where $|\cdot|$ is the norm $|x| = \max\{|x_i|: i = 1, 2, ..., n\}$ which is equivalent to the Euclidean norm.

If $f'(x)$ is the derivative of the function f at the point x, we denote by $J_f(x)$ the Jacobian determinant of f at x, i.e., $J_f(x) = \det f'(x)$. The vector space $C^1(\overline{\Omega})$ is the space defined as follows: $f \in C^1(\overline{\Omega})$ if $f \in C(\overline{\Omega})$ and there is an extension $\widehat{f}$ of f defined on an open set $U(f)$ containing $\overline{\Omega}$ such that $\widehat{f}$ has continuous first order partial derivatives in $U(f)$. The norm on $C^1(\overline{\Omega})$ is

$$\|f\|_1 = \sup_{\substack{x \in \Omega \\ 1 \le i \le n}} |f_i(x)| + \sup_{\substack{x \in \Omega \\ 1 \le i, j \le n}} \left| \frac{\partial f_i}{\partial x_j}(x) \right|.$$

Let $f \in C^1(\overline{\Omega})$ be an arbitrary mapping. We say that $x_0 \in \overline{\Omega}$ is a *critical point* of f if $J_f(x_0) = 0$. In this case, $f(x_0)$ is *a critical value* of f. We define $S_f = S_f(\Omega) = \{x_0 \in \Omega: J_f(x_0) = 0\}$. It is known that $\mu_n(f(S_f)) = 0$, where μ_n denotes the n-dimensional Lebesgue measure and if $y_0 \notin f(S_f)$, then $f^{-1}(y_0)$ is a finite set.

DEFINITION 1.6.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $f \in C^1(\overline{\Omega})$ and $y_0 \in \mathbb{R}^n \setminus f(\partial\Omega \cup S_f)$. Then we define the (Brouwer's) degree of f at y_0 relative to Ω to be $d(f, \Omega, y_0)$, where*

$$d(f, \Omega, y_0) = \sum_{x \in f^{-1}(y_0)} sign J_f(x).$$

Remarks.

(1) $d(f, \Omega, y_0)$ is an integer number, i.e., $d(f, \Omega, y_0) \in \mathbb{Z}$.

(2) The condition $y_0 \notin f(\partial\Omega)$ is essential; it cannot be removed.

(3) Often the degree defined by Definition 1.6.1 is also called the *topological degree*.

(4) From Definition 1.6.1 we have that $d(I, \Omega, y_0) = 1$, if $y_0 \in \Omega$ and $d(I, \Omega, y_0) = 0$, if $y_0 \notin \overline{\Omega}$, where I denotes the identity mapping.

If $\mathcal{D}_* = \left\{ \begin{array}{l} (f,\Omega,y) : \Omega \subset \mathbb{R}^n \textit{ open and bounded}, \\ f : \overline{\Omega} \to \mathbb{R}^n \textit{ continuous and } y \in \mathbb{R}^n \setminus f(\partial\Omega) \end{array} \right\}$, then it is known that there exists only one function $d : \mathcal{D}_* \to \mathbb{Z}$ (where $\mathbb{Z}$ is the set of integer numbers), satisfying the following properties:

i) $d(I,\Omega,y_0) = 1$, for any $y_0 \in \Omega$,

ii) $d(f,\Omega,y_0) = d(f,\Omega_1,y_0) + d(f,\Omega_2,y_0)$ where Ω_1 ,Ω_2 are disjoint open subsets of Ω such that $y \notin f\left(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)\right)$,

iii) $d(h(t,\cdot),\Omega,y(t))$ is independent of $t \in [0, 1]$ whenever $h : [0,1] \times \overline{\Omega} \to \mathbb{R}^n$ is continuous, $y : [0,1] \to \mathbb{R}^n$ is continuous and $y(t) \notin h(t,\partial\Omega)$ for all $t \in [0, 1]$.

It is known that this unique function *d*, satisfying conditions (i) – (iii) is exactly the topological degree if we extend Definition 1.6.1 from functions of class C^1 to continuous functions. Denote by ρ the induced distance by the norm $|\cdot|$ considered on $\mathbb{R}^n$. If $A \subset \mathbb{R}^n$ is a non-empty subset, we denote by $\rho(y, A)$ the distance from a point $y \in \mathbb{R}^n \setminus A$ to the set A.

DEFINITION 1.6.2. *Let* $f \in C(\overline{\Omega})$ *be an arbitrary function and* $y_0 \notin f(\partial\Omega)$. *Define the degree* $d(f,\Omega,y_0)$ *to be the degree* $d(g,\Omega,y_0)$, *where g is any function* $C^1(\overline{\Omega})$ *satisfying the inequality*

$$|f(x) - g(x)| < \rho(y_0, f(\partial\Omega)). \tag{1.6.1}$$

Remark. In Definition 1.6.2 the integer number $d(g,\Omega,y_0)$ is the same for all $g \in C^1(\overline{\Omega})$ satisfying inequality (1.6.1). We note also that in the definition of the degree $d(g,\Omega,y_0)$, for $g \in C^1(\overline{\Omega})$ satisfying inequality (1.6.1) it is sufficient to have that $y_0 \notin g(\partial\Omega)$ and not that $y_0 \in \mathbb{R}^n \setminus g(\partial\Omega \cup S_g)$.

Properties of topological degree in $\mathbb{R}^n$

We recall only some fundamental properties necessary in this book.

Property 1 [Existence]. *Let* $f \in C(\overline{\Omega})$ *be an arbitrary function. If* $d(f,\Omega,y_0)$ *is defined and non-zero, then the equation* $f(x) = y_0$ *has a solution in* Ω.

Property 2 [Rouché's Theorem]. *Suppose that* $f, g \in C(\overline{\Omega})$ *and* $y_0 \notin f(\partial\Omega)$. *If* $\|f-g\| < \rho(y_0, f(\partial\Omega))$, *then* $d(g,\Omega,y_0)$ *is defined and* $d(g,\Omega,y_0) = d(f,\Omega,y_0)$.

Property 3 [Homotopy Invariance]. *If* $H(t,x) \equiv h_t(x)$ *is a homotopy and* $y_0 \notin h_t(\partial\Omega)$ *for any* $t \in [0, 1]$, *then* $d(h_t,\Omega,y_0)$ *is independent of* $t \in [0, 1]$.

Property 4 [Poincaré–Bohl]. *If* $f, g \in C(\overline{\Omega})$ *and for all* $x \in \partial\Omega$ *the line segment* $[f(x), g(x)]$ *does not contain* y_0, *then* $d(f,\Omega,y_0) = d(g,\Omega,y_0)$.

Property 5 [Domain Decomposition]. *Suppose that* $f \in C(\overline{\Omega})$ *and* $y_0 \notin f(\partial\Omega)$. *If* Ω *is the disjoint union of open sets* Ω_i $(i = 1, 2, \ldots.)$, *then* $d(f,\Omega,y_0) = \sum_i d(f,\Omega_i,y_0)$. *(Note that for any* $y_0 \notin f(\partial\Omega)$, *the summation is finite).*

Property 6 [Excision]. *If* $f \in C(\overline{\Omega})$, $y_0 \notin f(\partial\Omega)$ *and* $y_0 \notin f(\Omega_0)$, *where* $\Omega_0 \subset \overline{\Omega}$ *is closed, then* $d(f,\Omega,y_0) = d(f,\Omega \setminus \Omega_0,y_0)$.

Property 7 [Boundary Value Dependence]. *If* $f, g \in C(\overline{\Omega})$, $y_0 \notin f(\partial\Omega)$ *and* $f = g$ *on* $\partial\Omega$, *then* $d(f,\Omega,y_0) = d(g,\Omega,y_0)$.

Property 8. $d(f,\Omega,\cdot)$ *is constant on connected components of* $\mathbb{R}^n \setminus f(\partial\Omega)$.

II. Leray–Schauder degree

By the Leray–Schauder degree we extend to the infinite dimensional case the topological degree presented in the previous section in $\mathbb{R}^n$.

Let $(E, \|\cdot\|)$ be a Banach space, Ω an open bounded subset of E and y_0 an arbitrary element in E. Our aim, in this section is to define for a suitable class of mappings $f : \overline{\Omega} \to E$, an integer $d(f, \Omega, y_0)$ which satisfies the most important properties of the topological degree defined in $\mathbb{R}^n$.
It is known that, it is impossible to define a topological degree $d(f, \Omega, y_0)$ for any continuous mapping in an arbitrary Banach space. Therefore, in an infinite dimensional Banach space it is necessary to impose some restrictions to the mapping $f : \overline{\Omega} \to E$, before defining a topological degree for f with respect to the set Ω and the element $y_0 \in E$.

Let D be a bounded subset in E and $T : E \to E$ a completely continuous mapping. In this case $\overline{T(D)}$ is a compact set.

A classical result says that, for any $\varepsilon > 0$, there is a continuous mapping $T_\varepsilon : D \to E$ whose range $T_\varepsilon(D)$ is finite dimensional such that $\|T(x) - T_\varepsilon(x)\| < \varepsilon$, for any $x \in D$.

Let $f : E \to E$ be a *completely continuous field* of the form $f = I - T$, where T is a completely continuous mapping. In this case $T_{|\overline{\Omega}} : \overline{\Omega} \to E$ is a compact mapping. Having found a mapping T_ε (for some $\varepsilon > 0$) which approximates the mapping T and which has finite dimensional range, we will use the mapping T_ε to define $d(f, \Omega, y_0)$ using the degree of $I - T_\varepsilon$ relative to an appropriate finite dimensional subset of Ω. We have the following definition.

DEFINITION 1.6.3. *Let Ω be an open bounded subset of E and $f = I - T$ a mapping such that $T : \overline{\Omega} \to E$ is continuous and $\overline{T(\overline{\Omega})}$ is compact. Suppose that $y_0 \in E \setminus f(\partial\Omega)$. Consider the mapping $f_* = I - T_*$, where T_* is a continuous mapping defined on $\overline{\Omega}$ with finite dimensional range such that $\|T(x) - T_*(x)\| < \rho(y_0, f(\partial\Omega)), (x \in \overline{\Omega})$. Consider a finite dimensional* *vector subspace E_f in E containing $T_*(\overline{\Omega})$ and y_0. Let $\Omega_f = \Omega \cap E_f$. Then define*

$$d(f, \Omega, y_0) := d(f_*, \Omega_f, y_0).$$

Remark. In the theory of topological degree it is proved that Definition 1.6.3 is correct and all are well defined. The integer number $d(f, \Omega, y_0)$ defined by Definition 1.6.3 is known by the name of "*Leray–Schauder degree*". Therefore the *Leray–Schauder degree* is defined for any completely continuous field $f = I - T : E \to E$ with respect to any open bounded subset $\Omega \subset E$ and any $y_0 \in E \setminus f(\partial\Omega)$.

Properties of the Leray–Schauder degree

Let $(E, \|\cdot\|)$ be a Banach space and $\Omega \subset E$ an open bounded subset. Denote by $\mathbb{K}(\overline{\Omega})$ the set of compact mappings from $\overline{\Omega}$ into E and define $\mathbb{K}_1(\overline{\Omega}) = \{f : f = I - T, T \in \mathbb{K}(\overline{\Omega})\}$. First, we remark that from the definition of Leray–Schauder degree we have the following elementary result.

If $y_0 \in \Omega$, then $d(f, \Omega, y_0) = 1$ and if $y_0 \notin \Omega$, then $d(f, \Omega, y_0) = 0$.

Property 1 [Existence]. *If $f \in \mathbb{K}_1(\overline{\Omega})$ and $d(f, \Omega, y_0) \neq 0$, then there is $x_0 \in \Omega$ such that $f(x_0) = y_0$.*

Property 2 [Rouché's Theorem]. *Suppose $f, g \in \mathbb{K}_1(\overline{\Omega})$ and $y_0 \notin f(\partial\Omega)$. If $\|f(x) - g(x)\| < \rho(y_0, f(\partial\Omega))$, for all $x \in \overline{\Omega}$, then $y_0 \notin g(\partial\Omega)$ and $d(f, \Omega, y_0) = d(g, \Omega, y_0)$.*

Suppose that h maps the interval [0, 1] into $\mathbb{K}(\overline{\Omega})$. We say that h is a *homotopy* of compact transformation on Ω if, given $\varepsilon > 0$ and a (bounded) subset A of Ω, there is $\delta = \delta(\varepsilon, A) > 0$ such that

$$\|(h(t))(x) - (h(s))(x)\| < \varepsilon, \text{ for any } x \in A \text{ and } t, s \text{ with } |t-s| < \delta.$$

Property 3 [Homotopy Invariance]. *Let* Ω *be a bounded open subset of E and let h(t) be a homotopy of compact transformations on* $\overline{\Omega}$ *such that if* $f_t = I - h(t)$, *then* $y_0 \notin f_t(\partial\Omega)$, $(0 \le t \le 1)$. *Then* $d(f_t, \Omega, y_0)$ *is independent of* $t \in [0, 1]$.

Property 4 [Poincaré–Bohl]. *Let* Ω *be a bounded open subset of E. Suppose given* f_1, f_2 *in* $\mathbb{K}_1(\overline{\Omega})$ *and consider* $f_t = (1-t)f_1 + tf_2$ *for any* $t \in [0, 1]$. If $y_0 \notin f_t(\partial\Omega)$ *for any* $t \in [0, 1]$, *then* $d(f_t, \Omega, y_0)$ *is independent of t.*

Property 5 [Domain Decomposition]. *Suppose* $f \in \mathbb{K}_1(\overline{\Omega})$ *and* $y_0 \notin f(\partial\Omega)$. *If* Ω *is the disjoint union of open sets* Ω_i $(i = 1, 2, \ldots)$ *then* $d(f, \Omega, y_0) = \sum_i d(f, \Omega_i, y_0)$.

Property 6 [Excision]. *Suppose* $f \in \mathbb{K}_1(\overline{\Omega})$ *and* $y_0 \notin f(\partial\Omega)$. *If* $\Omega_0 \subset \overline{\Omega}$ *is closed and* $y_0 \notin f(\Omega_0)$, *then* $d(f, \Omega, y_0) = d(f, \Omega \setminus \Omega_0, y_0)$.

Property 7 [Boundary Value Dependence]. *Suppose that f, g* $\in \mathbb{K}_1(\overline{\Omega})$ *and f = g on* $\partial\Omega$. *Then* $d(f, \Omega, y_0) = d(f, \Omega, y_0)$.

Property 8. *Suppose* $f \in \mathbb{K}_1(\overline{\Omega})$. *Then* $d(f, \Omega, y_0)$ *is the same for all* y_0 *in the same connected component of* $E \setminus f(\partial\Omega)$.

Remark. We conclude that the properties (1)–(8) of Brouwer's degree are valid also for the Leray–Schauder degree, but replacing the continuous mapping by completely continuous fields.

III Skrypnik degree

Let $(E, \|\cdot\|)$ be a real reflexive Banach space and $\Omega \subset E$ a bounded open set. *We say that a mapping* $f : \overline{\Omega} \to E^*$ *is demicontinuous if for any sequence* $\{x_n\}_{n\in N} \subset \overline{\Omega}$, *strongly convergent to* $x_0 \in \overline{\Omega}$, *we have that*

$$\lim_{n\to\infty}\langle f(x_n),v\rangle=\langle f(x_0),v\rangle, \textit{ for any } v\in E.$$

(We denote by $\langle\cdot,\cdot\rangle$ the natural duality between E and its topological dual E^*).

DEFINITION 1.6.4. *If $D\subset\Omega$ is a subset, then we say that the mapping f is of class $(S)_+$ with respect to D if for any sequence $\{x_n\}_{n\in N}\subset D$ weakly convergent to x_0, ($x_0\in D$) and $\limsup_{n\to\infty}\langle f(x_n),x_n-x_0\rangle\leq 0$ we have that $\{x_n\}_{n\in N}$ is norm convergent to x_0.*

For more information about mappings satisfying condition $(S)_+$ the reader is referred to (F. E. Browder [1]), (I. V. Skrypnik, [1], [2]) and (G. Isac and M. S. Gawda [1]).

We denote by $\mathcal{F}(E)$ the set of all *finite-dimensional* subspaces F of E such that $\Omega\cap F\neq\phi$. Let $F\in\mathcal{F}(E)$ and let $u_1,u_2,\ldots,u_m$ be a basis in F. We define the finite-dimensional mapping

$$f_F(x)=\sum_{i=1}^{m}\langle f(x),u_i\rangle u_i, \textit{ for } x\in\overline{\Omega_F}, \text{ where } \Omega_F=\Omega\cap F.$$

The topological degree defined by I. V. Skrypnik is a topological degree for mappings satisfying condition $(S)_+$. The definition of this topological degree is based on the following result proved in (I. V. Skrypnik [1], [2]).

THEOREM 1.6.1. *Let $f:\overline{\Omega}\to E^*$ be a demicontinuous mapping satisfying condition $(S)_+$ with respect to $\partial\Omega$ and $f(x)\neq 0$ for $x\in\partial\Omega$. Then there exists a subspace $F_0\in\mathcal{F}(E)$ such that any subspace $F\in\mathcal{F}(E)$ with $F_0\subseteq F$ satisfies the following properties:*

(i) *the equation $f_F(x)=0$ has no solution belonging to $\partial\Omega_F$,*

(ii) $\deg(f_F,\Omega_F,0)=\deg(f_{F_0},\Omega_{F_0},0)$, *where deg is the Brouwer degree of the finite-dimensional mapping.*

Now we can give the following definition.

DEFINITION 1.6.5. *Under the condition of Theorem 1.6.1, the number* $\deg(f,\Omega,0) := \deg(f_{F_0},\Omega_{F_0},0)$ *is called the degree (Skrypnik degree) of the mapping f on the set* Ω *with respect to the point* $0 \in E^*$.

Remark. In Definition 1.6.5, f_{F_0}, Ω_F are defined as above and F_0 is the finite-dimensional subspace of E determined by Theorem 1.6.1.

The degree defined by Definition 1.6.5 can be extended also to *pseudomonotone* (in Brezis's sense) mappings (I. V. Skrypnik [2]). Therefore on a Hilbert space $(H,\langle\cdot,\cdot\rangle)$, for each mapping f of class $(S)_+$ defined on $\overline{\Omega}$ (where $\Omega \subset H$ is a bounded open set, without zero on its boundary $\partial\Omega$) there is defined an integer deg(f, Ω, 0), named the *Skrypnik degree*, which has the usual properties of the Brouwer and Leray–Schauder degree. More precisely, deg(f, Ω, 0) has the following properties:

Property 1 [Kronecker]. $\deg(f,\Omega,0)=1$, *if* $0\in\Omega$.

Property 2. *If* $\Omega=\Omega_1\cup\Omega_2$ *and* f *has no zero on the set* $\partial\Omega_1\cup\partial\Omega_2\cup(\Omega_1\cap\Omega_2)$, *then* $\deg(f,\Omega,0) = \deg(f,\Omega_1,0) + \deg(f,\Omega_2,0)$.

Property 3. *If* f_0 *and* f_1 *are homotopic on* Ω, *then* $\deg(f_0,\Omega,0) = \deg(f_1,\Omega,0)$. *In this property we say that* f_0 *and* f_1 *are homotopic on* Ω *if there exists a family of mappings* $f(\lambda,\cdot)\,(0\le\lambda\le1)$ *of class* $(S)_+$, *defined on* $\overline{\Omega}$ *and demicontinuous with respect to both variables such that* $f(0,\cdot)=f_0$, $f(1,\cdot)=f_1$ *and* $f(\lambda,x)\neq 0\ (0\le\lambda\le1, x\in\partial\Omega)$.

Remark. In Property 3 the condition $(S)_+$ can be replaced by "to be a *zero-closed mapping*" or to be a "*quasi-monotone mapping*".

We recall (see Carbone, A. and Zabreiko, P. P. [1], [2]) that a mapping $h : H \to H$ is *zero-closed* if the convergence in norm of $\{h(x_n)\}_{n\in N}$ to zero implies that there exists a point $x_* \in \overline{conv}\{x_n\}$ such that $h(x_*)=0$ holds. Also, we say that h: $H \to H$ is *quasi-monotone* if each sequence $\{x_n\}_{n\in N}$ from *H*, which weakly converges to x_*, satisfies the condition

$$\liminf_{n\to\infty}\langle h(x_n), x_n - x_*\rangle \geq 0.$$

If *f* satisfies condition $(S)_+$ then *f* is *zero-closed* and *quasi-monotone*, the converse is not true (see I. V. Skrypnik [2]; see also M. A. Krasnoselskii and P. P. Zabreiko [1])

Property 4. *If f has no zero on the boundary* $\partial\Omega$ *of* Ω *and the degree* $d(f,\Omega,0)$ *is non-zero, then there exists at least one zero* x_* *of f in* Ω.

For the proof of Properties 1–4 see I. V. Skrypnik [2].

1.7 Zero-epi mappings

We present in this section the concept of *zero-epi mapping.* This notion was defined in 1980 in (Furi, M., Martelli, M. and Vignoli, A. [1]), and developed by the *Italian School* (Furi, M. and Pera, M. P. [1]–[3]), (Furi, M. and Vignoli, A. [1]), (Furi, M., Pera, M. P. and Vignoli, A. [1]), (Ize, J., Massabo, I., Pejsachowicz J., and Vignoli, A, [1]), Massabo, I., Nistri, P and Pera, M. P., [1]), (Pera, M. P. [1]–[3]). Applications to optimization and to complementarity theory are given in (Isac, G. [1], [19], [20]) and a generalization of the notion of zero-epi mapping to *k*-set-contractions was presented in (Tarafdar, E. U. and Thompson, H. B. [1]).

We find important, recent contributions to the development of the theory of zero-epi mappings in (Väth, M., [1] and in (Giorgieri, E. and Väth, M. [1]). We note that the notion of zero-epi mapping has some relations with the notion of *essential compact vector field* introduced in 1962 in (Granas, A. [1]). The notion of zero-epi mapping has been applied to the study of some problems related to differential equations and to some problems considered in nonlinear analysis.

Some deep relations between the notion of zero-epi mapping and the notion of topological degree exist and are interesting. About this fact the reader is referred to (Väth, M. [1]) and (Giorgieri, E. and Väth, M. [1]). The notion of zero-epi mapping is based on homotopy theory, on Urysohn's

Lemma and on the Schauder Fixed Point Theorem, while the theory of topological degree is more complicated and is based on advanced calculus and on some special results of nonlinear analysis. For more results about the notion of zero-epi mapping the reader is also referred to (Hyers, D. H., Isac, G. and Rassias, Th. M. [1]).

Let $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ be Banach spaces. First, we recall the following classical theorems.

THEOREM 1.7.1. [Schauder]. *If Ω is a convex (not necessarily closed) subset of a Banach space $(E, \|\cdot\|)$, then each continuous compact mapping $f : \Omega \to \Omega$ has at least one fixed point, i.e., there exists at least an element $x_* \in \Omega$ such that $f(x_*) = x_*$.*

Proof. For a proof of this result, the reader is referred to (Dugundji, J. and Granas, A. [1]). (See also Chapter 3 of this book). □

We recall that a topological space (X, τ) is *normal* if it is Hausdorff, and for all closed subsets A, $B \subset X$ such that $A \cap B = \phi$, there exist two open subsets U and V such that $A \subset U$, $B \subset V$ and $U \cap V = \phi$. It is known that every normed vector space is normal.

THEOREM 1.7.2. [Urysohn]. *A Hausdorff topological space (X, τ) is normal if and only if, for every two closed subsets A and B such that $A \cap B = \phi$, there exists a continuous function $h : X \to [0, 1]$ such that $h(x) = 0$ for every $x \in A$ and $h(x) = 1$, for every $x \in B$.*

Proof. A proof of this fundamental result of the general topology is given in (Bourbaki, N. [1]). □

DEFINITION 1.7.1. *Let $\Omega \subset E$ be a bounded subset and $f : \overline{\Omega} \to F$ a continuous mapping. We say that f is zero-epi (shortly 0-epi) if and only if the following properties are satisfied:*

1. $0 \notin f(\partial\Omega)$ *(i.e., f is 0-admissible),*
2. *for any continuous compact mapping $h : \overline{\Omega} \to F$, such that $h(x) = 0$ for every $x \in \partial\Omega$, the equation $f(x) = h(x)$ has a solution in Ω.*

The notion of zero-epi mapping was defined for the first time in (Furi, M., Martelli, M and Vignoli, A. [1]) and studied by many authors. The fundamental properties of zero-epi mappings are similar to the properties of Brouwer's topological degree.

Remark. If for an arbitrary element $p \in F$ we have that $p \notin f(\partial\Omega)$ and the mapping $f - p$ defined by $(f - p)(x) = f(x) - p$ is 0-epi, then in this case we say that f is *p-epi*, with respect to Ω.

Property 1 [Existence]. *If $f : \overline{\Omega} \to F$ is p-epi, then the equation $f(x) = p$ has a solution in Ω.*

Proof. The property is a consequence of the definition. □

Property 2 [Normalization]. *The inclusion $i : \overline{\Omega} \to E$ (i.e., $i(x) = x$ for any $x \in \overline{\Omega}$) is p-epi if and only if $p \in \Omega$.*

Proof. If the inclusion $i : \overline{\Omega} \to E$ is p-epi, then by the existence property (Property 1) we have that $p \in \Omega$. Conversely, we suppose that $p \in \Omega$. It is sufficient to suppose that $0 \in \Omega$ and to show that the inclusion $i : \overline{\Omega} \to E$ is 0-epi. Indeed, let $h : E \to E$ be a continuous and compact mapping such that $h(x) = 0$ for any $x \notin \Omega$. Since $0 \in \Omega$, the equation $i(x) = h(x)$ has a solution in Ω if and only if the mapping $h : E \to E$ has a fixed point. But, since $\overline{h(E)}$ is compact, applying Schauder's Fixed Point Theorem, (Theorem 1.7.1), we deduce that h has a fixed point and the proof is complete. □

Property 3 [Localization]. *If $f : \overline{\Omega} \to F$ is 0-epi and $\Omega_1 \subset \Omega$ is an open set such that $f^{-1}(0) \subset \Omega_1$, then the restriction of f to $\overline{\Omega_1}$, i.e., $f_{|\overline{\Omega_1}} : \overline{\Omega_1} \to F$ is 0-epi.*

Proof. Because $f^{-1}(0) \subset \Omega_1$ and $\Omega_1 \cap \partial\Omega_1 = \phi$, we have that $0 \notin f(\partial\Omega_1)$. Let $h : \overline{\Omega_1} \to F$ be a continuous compact mapping such that $h(x) = 0$ for every $x \in \partial\Omega_1$. Let h_* be the extension of h to $\overline{\Omega}$ given by

$$h_*(x) = \begin{cases} 0, \text{ if } x \in \overline{\Omega} \setminus \Omega_1, \\ h(x), \text{ if } x \in \Omega_1. \end{cases}$$

We have that h_* is a continuous and compact mapping. By assumption, the equation $f(x) = h_*(x)$ has a solution $x_* \in \Omega$. Since $f^{-1}(0) \subset \Omega_1$, we must have that $x_* \in \Omega_1$, and the property is proved. □

Property 4 [Homotopy]. *Let* $f : \overline{\Omega} \to F$ *be a 0-epi mapping and let* $h : \overline{\Omega} \times [0,1] \to F$ *be a continuous and compact mapping such that* $h(x, 0) = 0$ *for any* $x \in \overline{\Omega}$. *If* $f(x) + h(x,t) \neq 0$ *for all* $x \in \partial\Omega$ *and for any* $t \in [0, 1]$, *then the mapping* $f(\cdot) + h(\cdot,1) : \overline{\Omega} \to F$ *is 0-epi.*

Proof. Consider a continuous compact mapping $g : \overline{\Omega} \to F$ such that $g(x) = 0$ for all $x \in \partial\Omega$. The set $D = \{x \in \overline{\Omega} : f(x) + h(x,t) = g(x) \textit{ for some } t \in [0,1]\}$ is a closed set since $[0, 1]$ is compact. By Urysohn's Theorem, there exists a continuous function $\psi : \overline{\Omega} \to [0,1]$ such that $\psi(x) = 1$ for every $x \in D$ and $\psi(x) = 0$ for all $x \in \partial\Omega$. Considering the equation

$$f(x) = g(x) - h(x, \psi(x)), \tag{1.7.1}$$

we have that the mapping $h_* : \overline{\Omega} \to F$ defined by

$$h_*(x) = g(x) - h(x, \psi(x))$$

is continuous, compact and vanishes on $\partial\Omega$, then, since f is 0-epi, there exists a solution x_* of equation (1.7.1). We observe that $x_* \in D$ and hence $\psi(x_*) = 1$. Obviously $f(x_*) + h(x_*, 1) = g(x_*)$ and the proof is complete.

□

Property 5 [Boundary Dependence]. *If* $f : \overline{\Omega} \to F$ *is 0-epi and* $g : \overline{\Omega} \to F$ *is a continuous compact mapping such that* $g(x) = 0$ *for all* $x \in \partial\Omega$, *then* $f + g : \overline{\Omega} \to F$ *is 0-epi.*

Proof. This property is a consequence of Definition 1.7.1. □

The notion of 0-epi mapping is obviously simpler than the notion of topological degree. We must put in evidence the fact that the notion of 0-epi mapping is more refined than the notion of topological degree, in the sense that we may have a mapping that has the topological degree zero but it is 0-epi. (See (Furi, M., Martelli, M. and Vignoli, A. [1]).

There exist several results about the relation between topological degree and the property to be 0-epi. In this sense, we cite the following results.

THEOREM 1.7.3. *Let* $(E, \|\cdot\|)$ *be a Banach space and* $\Omega \subset E$ *an open bounded set. Let* $f : \overline{\Omega} \to E$ *be a continuous compact vector field (i.e.,* $f = I - T$*, where* $T : \Omega \to E$ *is compact) such that* $p \notin f(\partial\Omega)$*. If the Leray–Schauder degree,* $\deg(f, \Omega, p) \neq 0$*, then f is p-epi.*

Proof. For a proof of this result the reader is referred to (Furi, M., Martelli, M and Vignoli, A. [1]) or to (Hyers, D. H., Isac, G. and Rassias, Th. M.[1]). □

Let $(E, \|\cdot\|)$ be a Banach space. We denote by γ the Hausdorff or the Kuratowski measure of noncompactness. Recall that a non-empty bounded open set $\Omega \subset E$ is called a *Jordan domain* if $E \setminus \overline{\Omega}$ is connected. We say that $f : \overline{\Omega} \to E$ is *countably k-condensing* (on $\overline{\Omega}$ with respect to γ) if all countable sets $D \subset \overline{\Omega}$ with $\gamma(f(D)) \geq k\gamma(D)$ are precompact.

THEOREM 1.7.4. *If* $\Omega \subset E$ *is a Jordan domain and* $f : \overline{\Omega} \to E$ *is continuously countably* $\frac{1}{2}$*- condensing without fixed points on* $\partial\Omega$*, then* $h = I - f$ *is 0-epi if and only if* $\deg(h, \Omega) \neq 0$.

Proof. For a proof of this theorem and the definition of $\deg(h, \Omega)$ the reader is referred to (Väth, M. [1]). Other similar results are presented in (Väth, M. [1]) and in (Giorgieri, E. and Väth, M. [1]). □

The notion of 0-epi mapping was extended to functions defined on unbounded sets and in particular defined on a closed convex cone. (See the references cited in the introduction of this section or see (Hyers, D. H., Isac, G. and Rassias, Th. M.[1]). For this book, we need to present also the extension of the notion of 0-epi mapping to *k-set contraction*. The extension to *k*-set contractions is due to (Tarafdar, E. U. and Thompson, H. B. [1]).

If $(E,\|\cdot\|)$ is a Banach space and $D \subset E$ is a bounded subset, then we recall that the measure of noncompactness in Kuratowski's sense of the set D is

$$\alpha(D) = \inf\left\{ \varepsilon > 0 \,\middle|\, \begin{array}{l} D \textit{ can be covered by a finite number of sets} \\ \textit{of diameter less than } \varepsilon \end{array} \right\}.$$

We presented in section 1.5 of this chapter the properties of the measure of noncompactness α.

Let $(E,\|\cdot\|)$ and $(F,\|\cdot\|)$ be Banach spaces and $f: E \to F$ a continuous mapping. Denoting by α on both spaces the measure of noncompactness we recall that f is called a *k-set contraction*, if for each bounded subset $D \subset E$, we have that $\alpha(f(D)) \le k\alpha(D)$, where $k \ge 0$. We know that the concept of *zero-epi* mapping is strongly based on Schauder's Fixed Point Theorem and on Urysohn's Theorem. We note that the concept of (p, k)-epi mapping is based on Darbo's Fixed Point Theorem and also on Urysohn's Theorem.

THEOREM 1.7.5 [Darbo]. *If $(E,\|\cdot\|)$ is a Banach space and $\Omega \subset E$ is a closed bounded convex set, then any k-set contraction $f: \Omega \to \Omega$ with $k \in [0, 1[$, has a fixed point in Ω.*

Let $(E,\|\cdot\|)$ and $(F,\|\cdot\|)$ be Banach spaces and $\Omega \subset E$ an open bounded subset of E. Let $p \in F$ be an element and $k \ge 0$ a real number.

DEFINITION 1.7.2. *We say that a continuous mapping $f: \overline{\Omega} \to F$ is (p,k)-epi if:*

1. $p \notin f(\partial\Omega)$,
2. *for each k-set contraction $h: \overline{\Omega} \to F$ with $h(x) = 0$ on $\partial\Omega$ we have that the equation $f(x) - p = h(x)$ has a solution in Ω.*

When $p = 0$, we say that f is $(0, k)$-epi.

The (p, k)-epi mappings have the following fundamental properties:

Property 1 [Existence]. *If $f: \overline{\Omega} \to F$ is a (p, k)-epi mapping, then the equation $f(x) = p$ has a solution in Ω.*

Property 2 [Normalization]. *The inclusion mapping* $i : \overline{\Omega} \to E$ *(i.e.,* $i(x) = x$, *for any* $x \in \Omega$*) is (p,k)-epi with* $k \in [0, 1[$ *if and only if* $p \in \Omega$.

Property 3 [Localization]. *If* $f : \overline{\Omega} \to F$ *is a (0, k) –epi mapping and* $f^{-1}(0)$ *is contained in an open set* $\Omega_1 \subset \Omega$, *then f restricted to* Ω_1 *is also a (0, k)-epi mapping.*

Property 4 [Homotopy]. *Let* $f : \overline{\Omega} \to F$ *be a (0, k)-epi mapping and* $h:[0,1]\times\overline{\Omega}\to F$ *a β-set contraction with* $0\le\beta\le k<1$, *such that* $h(0,x) = 0$ *for all* $x \in \overline{\Omega}$. *If* $f(x)+h(t,x)\neq 0$ *for all* $x\in\partial\Omega$ *and for all* $t\in[0,1]$, *then* $f(\cdot)+h(1,\cdot):\overline{\Omega}\to F$ *is a (0, k–β) –epi mapping.*

Property 5 [Boundary Dependence]. *Let* $f : \overline{\Omega} \to F$ *be a (0, k)-epi mapping and* $g : \overline{\Omega} \to F$ *a β-set contraction with* $0\le\beta\le k<1$ *and such that* $g(x) = 0$ *for all* $x \in \partial\Omega$. *Then* $f + g : \overline{\Omega} \to F$ *is (0, k – β)-epi.*

The proofs of properties (1)–(5) of (*p*, *k*)-epi mappings are similar to the proofs of *p*-epi mappings, but with several technical details, specific to *k*-set contractions. For the proofs of these properties, the reader is referred to (Tarafdar, E. U and Thompson, H. B. [1]) or (Hyers, D. H., Isac, G. and Rassias, Th. M. [1]).

1.8 Convex cones

We recall in this section several notions and results related to convex cones in topological vector spaces. Let $E(\tau)$ be a real topological vector space. We suppose that E is endowed with an *order structure* defined by a reflexive, transitive and anti-symmetric binary relation, denoted by "≤" and such that the following axioms are satisfied:

O_1) *$x \le y$ implies $x + z \le y + z$ for all $x, y, z \in E$,*

O_2) *$x \le y$ implies $\lambda x \le \lambda y$, for all $x, y \in E$ and $\lambda \in \mathbb{R}_+ \setminus \{0\}$.*

Obviously, the set $E_+ = \{x\in E \mid x\ge 0\}$ satisfies the following properties:

c$_1$) $E_+ + E_+ \subseteq E_+$,

c$_2$) $\lambda E_+ \subseteq E_+$, *for all* $\lambda\in\mathbb{R}_+$, c₃) $E_+ \cap (-E_+) = \{0\}$.

We say in this case that E_+ *is a pointed convex cone* in *E*. Generally we suppose also that E_+ is a *closed* set with respect to the topology τ given on *E*.

Now, we introduce the following notion.

DEFINITION 1.8.1. *We say that a non-empty subset* $\mathbb{K} \subset E$ *is a convex cone if the following assumptions are satisfied:*

k_1) $\mathbb{K} + \mathbb{K} \subseteq \mathbb{K}$,

k_2) $\lambda \mathbb{K} \subseteq \mathbb{K}$, *for any* $\lambda \in \mathbb{R}_+$.

We say that the convex cone $\mathbb{K} \subset E$ *is pointed if* $\mathbb{K}$ *satisfies also the following assumption:*

k_3) $\mathbb{K} \cap (-\mathbb{K}) = \{0\}$.

Given a pointed convex cone $\mathbb{K} \subset E$, we can define an order structure on *E* by, $x \leq y \Leftrightarrow y - x \in \mathbb{K}$. This ordering is compatible with the vectorial structure of *E*. In a topological vector space, we will consider only closed, pointed convex cones. An ordered vector space will be denoted by $(E, \mathbb{K})$ and an ordered topological vector space by $(E(\tau), \mathbb{K})$. In this book we will consider only closed pointed convex cones in a Hilbert space $(H, \langle \cdot, \cdot \rangle)$ or in a Banach space $(E, \|\cdot\|)$. We recall also that an ordered vector space $(E, \mathbb{K})$ is a vector lattice if and only if, for every pair $(x, y) \in E \times E$, the *supremum* (denoted by $x \vee y$ and the *infimum* (denoted by $x \wedge y$), with respect to the ordering "≤" defined by $\mathbb{K}$, exist in *E*.

Let $(E, \|\cdot\|)$ be a Banach space and let E^* be the topological dual of *E*. If $\langle \cdot, \cdot \rangle : E \times E^* \to \mathbb{R}$ is a bilinear form satisfying the *separation axioms, that is:*

s_1) $\langle x_0, y \rangle = 0$ *for all* $y \in E^*$ *implies* $x_0 = 0$,

s_2) $\langle x, y_0 \rangle = 0$ *for all* $x \in E$ *implies* $y_0 = 0$,

then in this case we say that $(E, E^*, \langle \cdot, \cdot \rangle)$ is a *dual system*, or a *duality* between *E* and E^*. A dual system of Banach spaces will be denoted by $\langle E, E^* \rangle$.

Let $\langle E, E^* \rangle$ be a dual system of Banach spaces. If $\mathbb{K} \subset E$ is a pointed convex cone, we define the dual of $\mathbb{K}$ by:

$$\mathbb{K}^* = \{ y \in E^* : \langle x, y \rangle \geq 0 \; for\ any\ x \in \mathbb{K} \}.$$

The set $\mathbb{K}^*$ is a closed convex cone. The polar of $\mathbb{K}$ is defined by

$$\mathbb{K}^0 = \{ y \in E^* : \langle x, y \rangle \leq 0 \; for\ any\ x \in \mathbb{K} \}.$$

We have that $\mathbb{K}^0 = -\mathbb{K}^*$ and using the Bipolar Theorem (Peressini, A. L. [1]), we can show that $\mathbb{K}^{**} = (\mathbb{K}^*)^* = \mathbb{K}$ (because we supposed that $\mathbb{K}$ is closed). The duality of cones is more interesting in Hilbert spaces than in Banach spaces, since the dual is in the same space. Indeed, let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\mathbb{K} \subset H$ a closed pointed convex cone. In this case we have

$$\mathbb{K}^* = \{ y \in H : \langle x, y \rangle \geq 0 \; for\ any\ x \in \mathbb{K} \}.$$

The following result has some consequences for relations between $\mathbb{K}$ and $\mathbb{K}^*$.

THEOREM 1.8.1. *If $(H, \langle \cdot, \cdot \rangle)$ is an arbitrary Hilbert space and $\mathbb{K} \subset H$ is a closed pointed convex cone, such that $\mathbb{K} \neq \{0\}$, then $\mathbb{K} \cap \mathbb{K}^* \neq \{0\}$.*

Proof. Consider an arbitrary element $u \in \mathbb{K} \setminus \{0\}$. Using a classical separation theorem (see Schaefer, H. H. [2]) for $\{-u\}$ and $\mathbb{K}$, (since $-u \notin \mathbb{K}$) we obtain a continuous linear functional φ such that $\varphi(-u) < -1$ and $\varphi(x) \geq -1$ for all $x \in \mathbb{K}$. The functional $\Psi : \mathbb{K} \to \mathbb{R}$ defined by:

$$\Psi(x) = \frac{\|x\|^2}{2} - \varphi(x), \; for\ all\ x \in \mathbb{K}$$

is strictly convex weakly lower semicontinuous and coercive (i.e., $\lim_{\|x\| \to \infty} \Psi(x) = +\infty$). By a classical variational result, we obtain an element $x_* \in \mathbb{K}$ such that $\Psi(x_*) = \inf_{x \in \mathbb{K}} \Psi(x)$, which implies

$$\frac{d}{dt} \Psi(x_* + tx)_{|t=0} \geq 0, \; for\ all\ x \in \mathbb{K},$$

that is ,

$$\langle x_*, x \rangle \geq \varphi(x) \geq 0 \; for\ all\ x \in \mathbb{K} \text{ and } \langle x_*, u \rangle \geq \varphi(u) > 0.$$

Therefore we have $x_* \neq 0$ and $x_* \in \mathbb{K} \cap \mathbb{K}^*$. □

An immediate consequence of Theorem 1.8.1 is the fact that in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$, given a closed pointed convex cone $\mathbb{K} \subset H$, we can have one of the following interesting situations:

(i) $\mathbb{K} \subset \mathbb{K}^*$ ($\mathbb{K}$ is sub-adjoint),

(ii) $\mathbb{K} \supset \mathbb{K}^*$ ($\mathbb{K}$ is super-adjoint),

(iii) $\mathbb{K} = \mathbb{K}^*$ ($\mathbb{K}$ is self-adjoint).

Remark. The general situation, i.e., $\mathbb{K} \cap \mathbb{K}^* \neq \mathbb{K}$ and $\mathbb{K} \cap \mathbb{K}^* \neq \mathbb{K}^*$, is also possible.

A particular class of cones, in an arbitrary Banach space $(E, \|\cdot\|)$, with many applications is the class of *well-based cones.* Suppose that $\mathbb{K} \subset E$ is a closed convex cone. Let $B \subset \mathbb{K}$ be a non-empty convex subset. We say that $\mathbb{K}$ is generated by B if

$$\mathbb{K} = \bigcup_{\lambda \geq 0} \lambda B = \{x = \lambda b : \lambda \in \mathbb{R}_+ \text{ and } b \in B\}.$$

DEFINITION 1.8.2. *We say that a non-empty convex subset B of* $\mathbb{K}$ *is a base for* $\mathbb{K}$ *if each element* $x \in \mathbb{K} \setminus \{0\}$ *has a unique representation of the form* $x = \lambda b$, *with* $\lambda > 0$ *and* $b \in B$.

The following results are known.

THEOREM 1.8.2. *Let* $E(\tau)$ *be a locally convex space and* $\mathbb{K} \subset E$ *a convex cone. A subset* $B \subset \mathbb{K}$ *is a base for* $\mathbb{K}$ *if and only if there is a strictly positive linear functional f on E (i.e.* $f(x) > 0$ *for any* $x \in \mathbb{K} \setminus \{0\}$*) such that* $f^{-1}(1) \cap \mathbb{K} = B$.

THEOREM 1.8.3 [Krein–Rutman]. *In a separable Banach space every closed pointed convex cone has a base.*

Proof. A proof of this result is in (Krein, M. G. and Rutman, M. A. [1]). □

Remark. *Any closed convex cone, which has a base, is pointed.*

Let $E(\tau)$ be a locally convex space.

DEFINITION 1.8.3. *We say that a convex cone* $\mathbb{K} \subset E$ *is well based, if it has a bounded base B such that* $0 \notin \overline{B}$.

It is known that, if a pointed convex cone has a closed base *B*, then $\mathbb{K}$ *is closed.*

The next theorem is a characterization of a well-based cone $\mathbb{K}$ by a topological property of its dual $\mathbb{K}^*$.

THEOREM 1.8.4. *Let* $E(\tau)$ *be a locally convex space and* $\mathbb{K} \subset E$ *a pointed convex cone. The cone* $\mathbb{K}$ *is well based if and only if its dual* $\mathbb{K}^*$ *has an interior point with respect to the strong topology* $\beta(E^*, E)$.

Proof. The reader can see a proof of this result in [(Isac, G. [20]) or in (Jameson, G. [1])]. □

The locally compact cones form a particular sub-class of the class of well-based cones. It is known that if $E(\tau)$ is a locally convex space, then a pointed convex cone $\mathbb{K} \subset E$ is locally compact if and only if there exists a τ-neighborhood U of zero such that $U \cap \mathbb{K}$ is a compact set.

THEOREM 1.8.5 [Klee]. *Let* $E(\tau)$ *be a locally convex space and* $\mathbb{K} \subset E$ *a pointed convex cone. The cone* $\mathbb{K}$ *is locally compact, if and only if it has a compact base.*

Proof. For a proof of this result the reader is referred to (G. Isac [20]). □

Another particular class of convex cones in Banach spaces is the class of *Bishop–Phelps cones.*

Let $(E, \|\cdot\|)$ be a Banach space and let E^* be its topological dual. Given $0 < k < 1$ and $f \in E^*$ with $\|f\| = 1$, we consider the set

$$\mathbb{K}(f,k) = \{x \in E : k\|x\| \leq f(x)\}.$$

We can show that $\mathbb{K}(f,k)$ is a pointed convex cone with non-empty interior. Moreover, $\mathbb{K}(f,k)$ has a bounded base. The cone $\mathbb{K}(f,k)$ is called the Bishop-Phelps cone. It is known that if $1 < k$, then

$\mathbb{K}(f,k) = \{0\}$. Finally, if $(H,\langle\cdot,\cdot\rangle)$ is a Hilbert space and $\{x_n\}_{n\in N}$ is a complete orthogonal system, then in this case we know that any element $x \in H$ has a representation of the form $x=\sum_{n=1}^{\infty}\langle x,x_n\rangle x_n$. We can prove that the set $\mathbb{K}=\{x\in H:\langle x,x_n\rangle\geq 0 \; for\; any\; n\in N\}$ is a closed convex cone in $\mathbb{H}$. The reader can find other examples of convex cones in the books (Isac, G. [20]), (Schaefer, H. H. [2]) and (Peressini, A. L. [1]) among others.

1.9 Projection operators

The *projection operators* play an important role in this book. The main results presented in Chapters 3–8 are based, in particulars, on projection operators. We will give some results on projections operators onto closed convex sets in Hilbert spaces, onto closed convex cones, onto arbitrary closed sets in Hilbert spaces and we will introduce the notion of *generalized projection* in Alber's sense. First, we recall the following result.

Proposition 1.9.1. *Let $(E,\|\cdot\|)$ be a Banach space and let E^* be the dual of E. If $\{x_n\}_{n\in N}$ is a sequence in E weakly convergent to an element $x_0 \in E$, then we have $\|x_0\|\leq \liminf_{n\to\infty}\|x_n\|$.*

Proof. The sequence $\{x_n\}_{n\in N}$ is bounded (because it is weakly convergent). (see (Brezis, H. [1], Proposition III.5). For any $f \in E^*$, the sequence $\{\langle f,x_n\rangle\}_{n\in N}$ is convergent to $\langle f,x_0\rangle$ (and in particular it is bounded). If f is an arbitrary element in E^*, then we have:

$$|\langle f,x_n\rangle|\leq\|f\|\|x_n\|, \text{ for any } n\in N,$$

and computing the lim inf we obtain

$$|\langle f,x_0\rangle|\leq\|f\|\liminf_{n\to\infty}\|x_n\|.$$

Considering (Brezis, H. [1], Corollary 1.4) we obtain

$$\|x_0\|=\sup_{\|f\|\leq 1}|\langle f,x_0\rangle|\leq\|f\|\liminf_{n\to\infty}\|x_n\|\leq\liminf_{n\to\infty}\|x_n\|.$$

□

THEOREM 1.9.2. *Let $(E,\|\cdot\|)$ be a reflexive Banach space and $D \subset E$ a closed convex set. For every $x \in E$, there exists an element $x_0 \in D$ such that $\|x - x_0\| \le \|x - y\|$, for any $y \in D$. Moreover, if E is a Hilbert space, then the element x_0 is unique.*

Proof. If $x \in D$, then the element x_0 is x itself. Suppose that $x \in E \setminus D$. Consider the continuous function $\Phi : D \to \mathbb{R}$ defined by

$$\Phi(y) = \frac{1}{2}\|x - y\|^2, \textit{ for every } y \in D.$$

We have $-\infty < \alpha = \inf_{y \in D} \Phi(y)$. It is sufficient to show that there exists an element $x_0 \in D$ such that $\Phi(x_0) = \alpha$. Indeed, the definition of the greatest lower bound implies that for every $n \in \mathbb{N}$ there exists $x_n \in D$ such that

$$\Phi(x_n) \le \alpha + \frac{1}{n}.$$

The sequence $\{x_n\}_{n \in N}$ is bounded, and because E is reflexive there exists a subsequence $\{x_{n_k}\}_{k \in N}$ of $\{x_n\}_{n \in N}$ weakly convergent to an element x_0. Because D is closed and convex it is weakly closed, and hence we have that $x_0 \in D$. Applying Proposition 1.9.1 to the sequence $\{x - x_{n_k}\}_{k \in N}$, considering the fact that $\alpha + \frac{1}{n_K} \ge \Phi(x_{n_k})$ and applying to the last inequality the operator lim inf, we obtain $\alpha \ge \Phi(x_0) \ge \alpha$. Therefore we have that $\|x - x_0\| \le \|x - y\|$ for any $y \in D$. When E is a Hilbert space, in this case the function $\Phi(y)$ is *strictly convex*, which implies that x_0 is unique. □

The element x_0 defined in Theorem 1.9.2 is called *a projection* of x onto D and it is denoted by $x_0 \in P_D(x)$. In the case when E is a Hilbert space, we denote $x_0 = P_D(x)$, and we have

$$\|x - P_D(x)\| \le \|x - y\|, \textit{ for any } y \in D.$$

Remark. It is known that the projection of any element $x \in E$ onto a closed convex set $D \in E$ is also unique if $(E,\|\cdot\|)$ is a uniformly convex Banach space. In this case, the existence and the uniqueness are obtained by another proof, not similar to the proof given above.

Now, we consider the case of Hilbert spaces.

THEOREM 1.9.3. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $D \subset H$ a closed convex set and $x \in H$ an arbitrary element. The following statements are equivalent:*

i) $\langle x - P_D(x), P_D(x) - y\rangle \geq 0$, *for all* $y \in D$,

ii) $\|x - P_D(x)\| \leq \|x - y\|$ *for all* $y \in D$.

Proof. Indeed, if (i) is satisfied, then we have

$$\|x - P_D(x)\|^2 - \|x - y\|^2 = \|x - P_D(x)\|^2 - \|(x - P_D(x)) + (P_D(x) - y)\|^2$$

$$= -2\langle x - P_D(x), P_D(x) - y\rangle - \|P_D(x) - y\|^2 \leq 0,$$

which implies that (ii) is satisfied. Conversely, suppose that (ii) is satisfied. In this case, for an arbitrary $y \in D$ and all $t \in]0, 1]$ we have

$$\|x - P_D(x)\|^2 - \|x - [ty + (1-t)P_D(x)]\|^2$$

$$= -2t\langle x - P_D(x), P_D(x) - y\rangle - t^2\|P_D(x) - y\|^2.$$

Dividing by t and computing $\lim_{t\to 0}$ we obtain formula (i). □

Remark. The projection operator P_D satisfies also the following properties:

iii) $\|P_D(x_1) - P_D(x_2)\| \leq \|x_1 - x_2\|$, for any $x_1, x_2 \in H$,

iv) $\langle P_D(x_1) - P_D(x_2), x_1 - x_2\rangle \geq \|P_D(x_1) - P_D(x_2)\|^2$, for any $x_1, x_2 \in H$.

For a proof of these properties, the reader is referred to (Baiocchi, C. and Capelo, A. [1]). See also (Zarantonello, E. H. [1]). From property (iii) we deduce that P_D is a *non-expansive* operator and property (iv) means that P_D is a *monotone operator.*

Now, we give an important characterization of the projection operator P_D using the notion of *normal cone*. First, we remark that Theorem 1.9.3 can be put in the following equivalent form:

THEOREM 1.9.3 b. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $D \subset H$ a closed convex set and $x \in H$ an arbitrary element. Then the projection of x onto D*

(denoted by $P_D(x)$) is the unique element $z \in D$ such that $\langle x-z, x-u\rangle \geq 0$, for any $u \in D$.

If $D \subset H$ is a closed convex set and $x_* \in D$, then the *normal cone* of the set D at the point x_* is by definition

$$N_D(x_*) = \{\xi \in H : \langle \xi, u - x_*\rangle \leq 0, \textit{ for all } u \in D\}.$$

THEOREM 1.9.4. *If $D \subset H$ is a closed convex set and $x \in H$ is an arbitrary element, then we have that $z = P_D(x)$ if and only if $x \in z + N_D(z)$.*

Proof. The theorem is a consequence of the definition of the normal cone and of Theorem 1.9.3.b. Now, we consider the particular case, when the set D is a closed convex cone $\mathbb{K}$ in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$. In this case the projection operator $P_{\mathbb{K}}(\cdot)$ has some particular properties. □

THEOREM 1.9.5 [Moreau's Decomposition Theorem]. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K}_1$, $\mathbb{K}_2$ two closed convex cones in H. If $\mathbb{K}_1$ and $\mathbb{K}_2$ are mutually polar, i.e., $\mathbb{K}_1 = \mathbb{K}_2^0$ and $\mathbb{K}_1^0 = \mathbb{K}_2$, then for any x, y, $z \in H$ the following properties are equivalent:*

1) $z = x + y$, $x \in \mathbb{K}_1$, $y \in \mathbb{K}_2$ *and* $\langle x, y\rangle = 0$,
2) $x = P_{\mathbb{K}_1}(z)$ *and* $y = P_{\mathbb{K}_2}(z)$.

Proof. Let x, y, $z \in H$ be arbitrary elements satisfying property (1). In this case we have $\langle z-x, u-x\rangle = \langle y, u-x\rangle = \langle y, u\rangle \leq 0$, for all $u \in \mathbb{K}_1$, and by Theorem 1.9.3 a, (i) we have that $x = P_{\mathbb{K}_1}(z)$. Similarly we can show that $y = P_{\mathbb{K}_2}(z)$. Hence, property (1) implies property (2). Conversely, we have that (2) implies (1). Indeed, if $z \in H$ is an arbitrary element, we put $x = P_{\mathbb{K}_1}(z)$ and $y' = z - x$. For every $u \in \mathbb{K}_1$, by Theorem 1.9.3 a, (i) we have

$$\langle z-x, u-x\rangle \leq 0. \tag{1.9.1}$$

If $u = \lambda x$, with $\lambda \geq 0$, then from (1.9.1) we deduce

$$(\lambda - 1)\langle y', x\rangle \leq 0. \tag{1.9.2}$$

Since $\lambda - 1$ can be positive or negative, we obtain (using (1.9.2)) that $\langle y', x\rangle = 0$, which implies (using (1.9.1)) that $\langle y', u\rangle \leq 0$ for all $u \in \mathbb{K}_1$, that is, $y' \in \mathbb{K}_2$. Hence, x, y', z satisfy property (1) and by a similar calculus, as in the proof of implication (1) $\Rightarrow$ (2) we obtain that $y' = P_{\mathbb{K}_2}(z)$, and the proof is complete. □

A consequence of Theorem 1.9.5 is the following result.

COROLLARY 1.9.6. *If $\mathbb{K} \subset H$ is a closed convex cone, then $P_{\mathbb{K}}(\cdot)$ is a positive homogeneous operator, i.e., $P_{\mathbb{K}}(\alpha x) = \alpha P_{\mathbb{K}}(x)$ for all $\alpha \in \mathbb{R}_+$ and all $x \in H$.*

Proof. Indeed if we take in Theorem 1.9.5 $\mathbb{K}_1 = \mathbb{K}$ and $\mathbb{K}_2 = \mathbb{K}^0$, then we obtain $x = P_{\mathbb{K}}(x) + P_{\mathbb{K}^0}(x)$, with $\langle P_{\mathbb{K}}(x), P_{\mathbb{K}^0}(x)\rangle = 0$. For an arbitrary $\alpha \in \mathbb{R}_+$ we have $\alpha x = \alpha P_{\mathbb{K}}(x) + \alpha P_{\mathbb{K}^0}(x)$, where $\alpha P_{\mathbb{K}}(x) \in \mathbb{K}$, $\alpha P_{\mathbb{K}^0}(x) \in \mathbb{K}^0$ and $\langle \alpha P_{\mathbb{K}}(x), \alpha P_{\mathbb{K}^0}(x)\rangle = 0$. Since the decomposition given by Theorem 1.9.5 is unique, we obtain in particular that $\alpha P_{\mathbb{K}}(x) = P_{\mathbb{K}}(\alpha x)$. □

THEOREM 1.9.7. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\mathbb{K} \subset H$ a closed convex cone. For $x \in H$ and $x_* \in \mathbb{K}$ we have that $x_* = P_{\mathbb{K}}(x)$, if and only if the following properties are satisfied:*

1) $x_* - x \in \mathbb{K}^*$,
2) $\langle x_*, x_* - x\rangle = 0$.

Proof. First, we suppose that x^* satisfies (1) and (2). In this case, for every $y \in \mathbb{K}$ we have

$$\|y - x\|^2 = \|y - x_*\|^2 + 2\langle y, x_* - x\rangle + \|x_* - x\|^2 \geq \|x_* - x\|^2.$$

The uniqueness of $P_{\mathbb{K}}(x)$ implies that $x_* = P_{\mathbb{K}}(x)$.

Conversely, suppose that $x_* = P_{\mathbb{K}}(x)$. If (1) is not satisfied, then there exists an element $u \in \mathbb{K}$ such that $\langle P_{\mathbb{K}}(x) - x, u\rangle < 0$ and for some $t > 0$ we have

$$2t\langle P_{\mathbb{K}}(x) - x, u\rangle + t^2\|u\|^2 < 0$$

which implies

$$\left\|P_{\mathbb{K}}(x)+tu-x\right\|^2<\left\|P_{\mathbb{K}}(x)-x\right\|^2. \qquad (1.9.3)$$

Because $P_{\mathbb{K}}(x)+tu\in\mathbb{K}$, and considering the definition of $P_{\mathbb{K}}(x)$, we observe that (1.9.3) is impossible. Therefore, relation (1) is true. Also, if (2) is not satisfied, then we have $\langle P_{\mathbb{K}}(x),P_{\mathbb{K}}(x)-x\rangle>0$ (since (1) is satisfied). Then there exists $\rho>0$ such that

$$-t\langle P_{\mathbb{K}}(x),P_{\mathbb{K}}(x)-x\rangle+t^2\left\|P_{\mathbb{K}}(x)\right\|^2<0, \text{ for all } t\in\left]0,\rho\right[,$$

which implies

$$\left\|(1-t)P_{\mathbb{K}}(x)-x\right\|^2<\left\|P_{\mathbb{K}}(x)-x\right\|^2, \text{ for some } t\in\left]0,1\right[.$$

But the last inequality is in contradiction with the definition $P_{\mathbb{K}}(x)$ [since $(1-t)P_{\mathbb{K}}(x)\in\mathbb{K}$]. □

Finally, for closed convex cones we remark also the following property

(v) $\left\|P_{\mathbb{K}}(x)\right\|\leq\|x\|$ for any $x\in H$.

Property (v) is a consequence of property (iii). Indeed, we have that $x_1=x$ and $x_2=0\in\mathbb{K}$.

Now, we consider the last situation, the case of a generalized *projection operator*, useful in the transformation of a *variational inequality* in a *fixed point problem,* when the mapping is from a Banach space to its topological dual. First, we need to recall some well-known notions in the theory of the *geometry of Banach spaces.*

Let $(E,\|\cdot\|)$ be a Banach space. We say that E is *strictly convex*, if for two elements x, $y\in E$ which are linearly independent, we have $\|x+y\|<\|x\|+\|y\|$. The strict convexity is equivalent to the following condition:

$$\|x\|=\|y\|=1, \quad x\neq y\Rightarrow\left\|\frac{x+y}{2}\right\|<1.$$

The Banach space E is said to be *uniformly convex*, if for any two sequences $\{x_n\}_{n\in N}$, $\{y_n\}_{n\in N}$ in E such that $\|x_n\|=\|y_n\|=1$ and $\lim_{n\to\infty}\|x_n+y_n\|=2$, $\lim_{n\to\infty}\|x_n-y_n\|=0$ holds. The *uniform convexity* is equivalent to the following property: *for any* $\varepsilon>0$ *with* $0<\varepsilon\leq 2$, *there exists* $\delta>0$ *depending only on* $\varepsilon>0$ *such that*

$$\left\| \frac{x+y}{2} \right\| \leq 1-\delta$$

for any $x, y \in E$ *with* $\|x\| = \|y\| = 1$ *and* $\|x - y\| \geq \varepsilon$.

It is known (Takahashi, W. [1]) that *any uniformly convex Banach space is strictly convex.* Consider the set $S_1 = \{x \in E : \|x\| = 1\}$. We say that the norm of E is *uniformly Fréchet differentiable* (and we say in this case that E is *uniformly smooth)* if the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$$

is attained uniformly for $(x, y) \in S_1 \times S_1$.

Let E^* be the topological dual of E. To each $x \in E$ we associate the set

$$J(x) = \left\{ f \in E^* : f(x) = \|x\|^2 = \|f\|^2 \right\}.$$

The multivalued mapping $J : E \to E^*$ is called the *duality mapping of E*. For each $x \in E$, $J(x)$ is a non-empty bounded, closed and convex set, $J(0) = \{0\}$ and for any $x \in E$ and any real number α we have $J(\alpha x) = \alpha J(x)$. In the definition and the applications of the generalized projection we need to have *E a uniformly convex and a uniformly smooth Banach space.* In this case the duality mapping J is *a single-valued mapping norm-to-norm continuous.* We cite as uniformly convex and uniformly smooth Banach spaces, the spaces l^p, L^p and W_m^p, $p \in]1, \infty[$. Also, it is known (Takahashi, W. [1]) that the duality mapping J is a *monotone operator* and it is *strictly monotone,* if the space E is *strictly convex.* About the proofs of the results presented above (related to the geometry of Banach spaces) the reader is referred to (Takahashi, W. [1]) and (Ciorănescu, I. [1]). Now, we can define the *generalized projection,* using the construction given by Y. Alber (Alber, Y. I. [1]).

Let $(E, \|\cdot\|)$ be a uniformly convex and uniformly smooth Banach space. Denote by E^* the topological dual of E. We introduce the functional:

$$V(\varphi, x) = \|\varphi\|_{E^*}^2 - 2\langle \varphi, x \rangle + \|x\|_E^2, \textit{ for any } (\varphi, x) \in E^* \times E.$$

We have that $V : E^* \times E \to \mathbb{R}$. The functional $V(\cdot,\cdot)$ has several nice properties, but we will put in evidence only the properties necessary to define the generalized projection:

1. $V(\varphi, x)$ *is continuous,*

2. *$V(\varphi, x)$ is differentiable with respect to φ and x,*
3. *$V(\varphi, x) \geq 0$, for all $(\varphi, x) \in E^* \times E$,*
4. *for any φ (fixed) we have that $V(\varphi, x) \to \infty$ if $\|x\| \to \infty$,*
5. *$V(\varphi, x) = 0$ if and only if $\varphi = J(x)$.*

Let $\Omega \subset E$ be a closed convex set. Using property (4) of the functional $V(\varphi, x)$, Theorem 1.2 from (Vladimirov, A. A., Nesterov, Yu. E. and Chekanov, Yu. N. [1]), property (5) of the functional $V(\varphi, x)$ and the strict monotonicity of the mapping J, we have that the minimization problem

$$\begin{cases} \text{given } \varphi \in E^*, \text{ find } x_\varphi \in \Omega \subset E, \\ \text{such that } V(\varphi, x_\varphi) = \inf_{x \in \Omega} V(\varphi, x_\varphi) \end{cases}$$

has a solution and the solution is unique. The operator $\Pi_\Omega : E^* \to \Omega \subset E$ defined by $\Pi_\Omega(\varphi) = x_\varphi$ is called the *generalized projection operator.* The generalized projection operator has several interesting properties, but we need to put in evidence only the following properties.

THEOREM 1.9.8. *Let $(E, \|\cdot\|)$ be a uniformly convex and uniformly smooth Banach space. Let $\Omega \subset E$ be a closed convex set. Then the following properties hold:*

1) *The operator Π_Ω is J-fixed in each point $x \in \Omega$, i.e., $\Pi_\Omega(J(x)) = x$.*
2) *Π_Ω is monotone in E^*, i.e., for all $\varphi_1, \varphi_2 \in E^*$ we have $\langle \varphi_1 - \varphi_2, \Pi_\Omega(\varphi_1) - \Pi_\Omega(\varphi_2) \rangle \geq 0$.*
3) *For any $\varphi \in E^*$ we have $\langle \varphi - J(\Pi_\Omega(\varphi)), \Pi_\Omega(\varphi) - x \rangle \geq 0$ for all $x \in \Omega$.*
4) *Π_Ω is uniformly continuous on each bounded subset of E.*

Proof. For the proof of the properties 1–4 the reader is referred to (Alber, Y. I. [1]) (See also Chapter 2, section 2.3). □

2

COMPLEMENTARITY PROBLEMS AND VARIATIONAL INEQUALITIES

We present in this chapter two classes of mathematical models, used in applied mathematics. The first class comprises *complementarity problems* and the second class *variational inequalities*. We present the necessary definitions and some important relations between complementarity problems, variational inequalities and the fixed-point problem.

2.1. Complementarity problems

The study of *complementarity problems* has developed sufficiently to call it *Complementarity Theory*. Now we consider it as a new domain of Applied Mathematics, having deep relations with several domains of fundamental mathematics and with numerical analysis. Complementarity problems represent a wide class of mathematical models related to optimization, economics, engineering, mechanics, elasticity, fluid mechanics and game theory.

It is important to note that the *complementarity condition* is a kind of *general equilibrium* concept that includes the equilibria of physics and economics. Equilibrium in physics has long been well known. Equilibrium in economics has become central to the understanding of competitive systems. One example is the general economic equilibrium problem in which all commodity prices are to be determined. A second example is the general financial equilibrium of markets in which firms compete to determine their profit-maximizing production outputs.

Many authors have studied equilibria of economic systems by several mathematical methods and from several points of view, but the recent development of Complementarity Theory helps us to understand better a number of more complex aspects of economic equilibrium. In this sense we cite the books (Isac, G. [20] and (Isac, G., Bulavsky, V. A. and Kalashnikov, V. V. [2]). A deep study of equilibrium in Economics help us to understand better the non-equilibrium state of particular economical systems.

There exist several kinds of complementarity problems [see books (Isac, G. [12], [20]), (Isac, G., Bulavski, V. A. and Kalashnikov, V. V. [2]), (Hyers, D. H., Isac, G. and Rassias, Th. [1])]. In this book we present only the most important kinds of complementarity problems, from the point of view of applications and related to the Leray–Schauder type alternatives. We must keep in mind the fact that Complementarity Theory stands at a point on the crossroads of *applied mathematics, fundamental mathematics and experimental mathematics* related to numerical solvability. The connection of Complementarity Theory with Variational Inequalities Theory, with Fixed Point Theory and with Nonlinear Analysis is an important factor in its development as a theory. The literature on complementarity problems is now huge [See the references cited in [(Cottle, R. W., Pang, J. S. and Stone, R. E. [1]), (Isac, G. [12], 20]), (Isac, G., Bulavski, V. A. and Kalashnikov, V. V. [2]), (Hyers, D. H., Isac, G. and Rassias, Th. [1]), (Murty, K. G. [1])].

A. The classical complementarity problem

First, we note that many problems arising in fields such as economics, game theory, mathematical programming, mechanics, elasticity theory and engineering, several equilibrium problems can be stated in the following unified form.

Consider the vector space $\mathbb{R}^n$ and the classical inner-product $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$, $x = (x_i)$, $y = (y_i) \in \mathbb{R}^n$. Suppose that $\mathbb{R}^n$ is ordered by the closed pointed convex cone $\mathbb{R}^n_+$ and suppose given a function $f : \mathbb{R}^n_+ \to \mathbb{R}$. The classical complementarity problem defined by the function f and the convex cone $\mathbb{R}^n_+$ is

$$CP(f, \mathbb{R}^n_+): \begin{cases} \textit{find } x_0 \in \mathbb{R}^n_+ \textit{such that} \\ f(x_0) \in \mathbb{R}^n_+ \textit{ and } \langle x_0, f(x_0)\rangle = 0. \end{cases}$$

The origin of this problem is perhaps in the Kuhn–Tucker Theorem, known in nonlinear programming (which gives the necessary optimality conditions, under some differentiability assumptions), or perhaps in the old and neglected paper by Du Val published in 1940 (Du Val, P. [1]). We note also that the origin of the term "complementarity" is in the paper by Cottle (Cottle, R. W. [1]) published in 1964. Initially, this problem was called, in the linear case (i.e., when $f(x) = Ax + b$, where A is a matrix and b is a vector), the "copositive problem", the "fundamental problem of mathematical programming" and the "complementarity problem". It seems that the term "complementarity problem" was proposed by R. W. Cottle in 1965 and used in the papers of R. W. Cottle, G. J. Habetler and C. E. Lemke. From the mathematical point of view, the origin of the term "complementarity" is the following fact.

Let $x_* = \{x_{*_i}\}_{i=1}^n$ be a solution of $CP(f, \mathbb{R}_+^n)$. We say that x_* is *nondegenerate* if at most n components of a $2n$-components vector $(x_*, f(x_*))$ are equal to zero. Otherwise, it is a *degenerate solution.* Denote it by $N_n = \{1, 2, \ldots, n\}$. If x_* is a nondegenerate solution and $y_* = \{y_{*_i}\}_{i=1}^n$, where $y_* = f(x_*)$, then the sets $A = \{i : x_{*_i} > 0\}$ and $B = \{i : y_{*_i} > 0\}$ are complementary subsets of N_n, that is $A = C_{N_n} B$.

If the function f has the form $f(x) = Ax + b$, where A is an $n \times n$-*matrix* and $b \in \mathbb{R}^n$, then in this case $CP(f, \mathbb{R}_+^n)$ is called the *linear complementarity problem* defined by A, b and $\mathbb{R}_+^n$, and it is denoted by

$$LCP(A, b, \mathbb{R}_+^n): \begin{cases} \textit{find } x_* \in \mathbb{R}_+^n \textit{ such that} \\ Ax_* + b \in \mathbb{R}_+^n \textit{ and} \\ \langle x_*, Ax_* + b \rangle = 0. \end{cases}$$

We note that the linear complementarity problem was initially defined as a basic mathematical model that unified linear and quadratic programs, as well as the bimatrix game problem. Specifically, W. S. Dorn in 1961 proved that if A is a *positive-definite* (but not necessarily symmetric) *matrix* then the *minimum value* of the quadratic programming problem

$$(P):\begin{cases} minimize\ \langle x, Ax+b\rangle, \\ x\in \mathcal{F}, \\ where\ \mathcal{F}=\{x\in \mathbb{R}^n_+ : Ax+b\in \mathbb{R}^n_+\}\ and\ b\in \mathbb{R}^n \end{cases}$$

is zero. [See (Dorn, W. S., [1])]. We note that Dorn's paper was the first step in treating the linear complementarity problem as an independent problem.

In 1963 G. B. Dantzig and R. W. Cottle generalized Dorn's result to the case when all the principal minors of the matrix A are positive (Dantzig, G. B. and Cottle, R. W. [1]). R. W. Cottle studied problem (P) in 1964, under the assumption that A is a *positive semi-definite* matrix and he remarked that, in this case it is not true that (P) must have an optimal solution. [See Cottle, R. W. [2]). Cottle proved that, if the matrix A is *positive semi-definite* and the set $\mathcal{F}=\{x\in \mathbb{R}^n_+ : Ax+b\in \mathbb{R}^n_+\}$ *where* $b\in \mathbb{R}^n$ (called the *feasible set*) is non-empty, then an optimal solution for (P) exists and again $\min_{x\in\mathcal{F}}\langle x, Ax+b\rangle = 0$.

After some time, G. B. Dantzig and R. W. Cottle constructively showed that if A is a square (not necessarily symmetric) matrix with all *the principal minors* positive, then problem (P) has an optimal solution x_* such that $\langle x_*, Ax_*+b\rangle = 0$. This result is in (Dantzig, G. B. and Cottle, R. W. [1]). In 1966 R. W. Cottle generalized this result. His generalization is the following:

Let $f:\mathbb{R}^n\to\mathbb{R}^n$ *be a continuously differentiable mapping. We say that* f *has a positively bounded Jacobian matrix* $J_f(x)$, *if there exists a real number* $0<\delta<1$ *such that for every* $x\in\mathbb{R}^n$ *each principal minor of* $J_f(x)$ *is an element of the interval* $[\delta, \delta^{-1}]$.

We recall that a solution (y, x) of the equation $y-f(x)=0$ is said to be *nondegenerate* if at most n of the $2n$ components are zero.

THEOREM [Cottle]. *If* $f:\mathbb{R}^n\to\mathbb{R}^n$ *is a continuous differentiable mapping such that the solutions of equation* $y-f(x)=0$ *are nondegenerate, and if* f *has a positively bounded Jacobian matrix* $J_f(x)$, *then the problem*

$$NCP(f,\mathbb{R}_+):\quad \begin{cases} find\ x_0\in\mathbb{R}^n_+\ such\ that \\ f(x_0)\in\mathbb{R}^n_+\ and\ \langle x_0, f(x_0)\rangle = 0, \end{cases}$$

has a solution.

A proof of this theorem is in (Cottle, R. W., [3]) where he defined the *nonlinear complementarity problem* by f and the convex cone $\mathbb{R}^n_+$, and denoted it by $NCP\left(f,\mathbb{R}^n_+\right)$.

In studying the origin of the *Complementarity Theory* we must consider the papers (Lemke, C. E. [1]–[6]) and (Ingleton, A. W. [1]). Lemke proposed, in 1965, the complementarity problem as a method for solving matrix games (Lemke, C. E. [1]). His contribution to the development of complementarity theory was remarkable, because his algorithm for solving complementarity problems, known as *Lemke's algorithm*, has been widely used in many practical applications, (Lemke, C. E. [1]–[6]), (Lemke, C. E. and Howson, J, T. [1]).

In 1966, A. Ingleton showed the importance of complementarity problems in engineering (Ingleton, A. W. [1], [2]). Certainly, a strong influence on the development of complementarity theory is also found in (Eaves, B. C, [1]-[7]), (Eaves, B. C. and Lemke, C. E. [1], [2]), (Karamardian, S. [1]-[5]), (Kaneko, I, [1]-[13]) and (Kojima, M. [1]-[4]).

After 1970 the complementarity theory enjoyed a strong and ascending development from theoretical, numerical solvability and applicability points of view. Now, the literature on this subject is vast. To see this, the reader is referred to the books (Cottle, R. W., Pang, J. S. and Stone, R. E, [1]), (Isac, G. [12], [20]), (Isac, G., Bulavski, V. A. and Kalashnikov, V. V. [2]), (Murty, K. G. [1]) among others. Now, it is unanimously accepted that the study of complementarity problems is a necessary domain in applied mathematics and a stimulant for fundamental mathematics.

B. The general nonlinear complementarity problem

Let $\left\langle E,E^*\right\rangle$ be a dual system of locally convex spaces and let $\mathbb{K}\subset E$ be a closed pointed convex cone. If $f:\mathbb{K}\to E^*$ is a given mapping, the (general) nonlinear complementarity problem defined by f and $\mathbb{K}$ is:

$$NCP\left(f,\mathbb{K}\right):\ \begin{cases} \textit{find } x_* \in \mathbb{K} \textit{ such that} \\ f\left(x_*\right)\in \mathbb{K}^* \textit{ and } \left\langle x_*, f\left(x_*\right)\right\rangle = 0. \end{cases}$$

$NCP(f,\mathbb{K})$ contains as a particular case the classical complementarity problem $NCP(f,\mathbb{R}^n_+)$, where $f:\mathbb{R}^n_+ \to \mathbb{R}^n$. Also the (general) linear complementarity problem $LCP(T, b, \mathbb{K})$, where $T: E \to E^*$ is a linear operator and $b \in E^*$ can be considered as a particular case of the problem $NCP(f,\mathbb{K})$.

The problem $NCP(f,\mathbb{K})$ has many applications in optimization, game theory, economics, engineering, mechanics, etc. We will see in this chapter that the problem $NCP(f,\mathbb{K})$ is related to variational inequalities and in Hilbert spaces it is related to the *Fixed Point Problem.* The fixed-point problem represents an important chapter in nonlinear analysis. (Isac, G. [20]).

C. The multivalued complementarity problem

First, we note that the *multivalued complementarity problem* is necessary in the study of some problems in economics in the sensitivity analysis of classical complementarity problems and in numerical computation of solutions of practical complementarity problems, because of the accidental corruption of the problem data. Also, the *multivalued* complementarity problem is related with the theory of quasi-variational inequalities defined by set-valued mappings. Variational inequalities with set-valued mappings are used in the study of equilibrium in economics.

Let $\langle E,E^*\rangle$ be a dual system of locally convex spaces, $\mathbb{K} \subset E$ a pointed closed convex cone and $f:\mathbb{K} \to 2^{E^*}$ a set-valued mapping. The multivalued complementarity problem defined by *f* and $\mathbb{K}$ is:

$$MCP(f,\mathbb{K}): \begin{cases} \textit{find } x_0 \in \mathbb{K} \textit{ and } y_0 \in E^* \textit{ such that} \\ y_0 \in f(x_0) \cap \mathbb{K}^* \textit{ and } \langle x_0, y_0\rangle = 0. \end{cases}$$

This complementarity problem has been the subject of several papers as for example: (Chang, S. S. and Huang, N. J., [1]–[4]), (Gowda, M. S. and Pang, J. S., [1]), (Huang, N. J., [1]), (Isac, G. [12], [20]), (Isac, G. and Kostreva, M. M., [2]), (Isac, G. and Kalashnikov, V. V. [1]), (Luna, G. [1]), (Parida, J. and Sen, A., [1]), (Saigal, R., [1]).

D. Implicit complementarity problem

Another class of complementarity problems is *the class of implicit* *complementarity problems.* It seems that the origin of *implicit complementarity problems* is the dynamic programming approach of stochastic impulse and of continuous optimal control (Bensoussan, A., [1]), (Bensoussan, A. and Lions, J. L., [1]–[3]), (Bensoussan, A., Gourset, M. and Lions, J. L. [1]), (Capuzzo–Dolcetta. I. and Mosco, U., [1]), (Mosco, U. [1]), (Mosco, U. and Scarpini, F., [1].

The study of *implicit complementarity problems* has been stimulated by the applications of this class of mathematical models to the study of various free boundary problems associated to some particular differential operators. This class of complementarity problems has been studied by many authors as for example: (Pang, J. S. [1]–[2]), (Chan, D. and Pang J. S., [1]), (Noor, M. A., [1]), (Capuzzo–Dolcetta, I., Lorenzani, M. and Spizziachino, F. [1]), (Isac, G. and Nemeth, S. Z. [1]), (Kalashnikov, V. V. and Isac. G. [1]). We note that there exist deep and interesting relations between the *implicit complementarity problems and the quasivariational inequalities theory.*

Now, we give the most important kind of implicit complementarity problems. Let $E(\tau)$ be a locally convex space and let $\mathbb{K} \subset E$ be a closed convex cone. Suppose given an element $b \in E$ and two mappings A, $M: E \to E$. If $\langle \cdot, \cdot \rangle$ is a bilinear functional defined on $E \times E$ then the *implicit complementarity problem* is:

$$ICP(A,M,b,\mathbb{K}): \quad \begin{cases} \text{find } x_0 \in E \text{ such that} \\ M(x_0) - x_0 \in \mathbb{K}, b - A(x_0) \in \mathbb{K} \\ \text{and } \langle A(x_0) - b, x_0 - M(x_0) \rangle = 0. \end{cases} \tag{2.1.1}$$

The *implicit complementarity problem* (2.1.1) has the following variant for a dual system. Let $\langle E, E^* \rangle$ be a dual system of locally convex spaces, $\mathbb{K} \subset E$ a closed pointed convex cone, $M : E \to E$ and $A : E \to E^*$ arbitrary mappings and $b \in E^*$ an arbitrary element. In this case the problem (2.1.1) has the following form:

$$ICP(A,M,b,\mathbb{K}): \quad \begin{cases} \text{find } x_0 \in E \text{ such that} \\ M(x_0) - x_0 \in \mathbb{K}, b - A(x_0) \in \mathbb{K}^* \\ \text{and } \langle A(x_0) - b, x_0 - M(x_0) \rangle = 0. \end{cases} \tag{2.1.2}$$

Obviously, if $E = H$, where $(H,\langle\cdot,\cdot\rangle)$ is a Hilbert space with respect to an inner-product $\langle\cdot,\cdot\rangle$, then the problem (2.1.2) is exactly the problem (2.1.1).

The most general form of the *implicit complementarity* problem is the following. Let $\langle E,E^*\rangle$ be a dual system of locally convex spaces, $\mathbb{K}\subset E$ a pointed closed convex cone and $D\subset E$ a non-empty subset. If $f:D\to E^*$ and $g:D\to E^*$ are arbitrary mappings, then the *generalized implicit complementarity problem* defined by *f, g, D* and $\mathbb{K}$ is:

$$GICP(f,g,D,\mathbb{K}):\quad \begin{cases} \text{find } x_0\in D \text{ such that} \\ g(x_0)\in\mathbb{K},\ f(x_0)\in\mathbb{K}^* \text{ and} \\ \langle g(x_0),f(x_0)\rangle=0. \end{cases}$$

Finally, *the generalized implicit complementarity problem* has the following *multivalued* variant. Let $D\subset E$ be a non-empty subset, $\mathbb{K}\subset E$ a closed pointed convex cone and $f:D\to 2^{E^*}$, $g:D\to 2^{E}$ set-valued mappings. The *multivalued generalized implicit complementarity problem* is:

$$MGICP(f,g,D,\mathbb{K}):\quad \begin{cases} \text{find } x_0\in D \text{ such that} \\ \text{there exist } x_*\in g(x_0)\cap\mathbb{K} \text{ and} \\ y_*\in f(x_0)\cap\mathbb{K}^*, \text{satisfying}^* \\ \langle x_*,y_*\rangle=0. \end{cases}$$

E. Order complementarity problem

A new chapter in complementarity theory is the study of complementarity problems with respect to an ordering. The introduction of *order complementarity problems in complementarity theory* is justified by two reasons.

(i) In the study of some particular classical complementarity problems the essential fact is not the orthogonality in the sense of an inner-product, but the lattice orthogonality. Therefore, in some circumstances it is useful to represent the classical complementarity problem as an order complementarity problem.

(ii) In some practical problems, we must use the complementarity condition simultaneously with respect to several operators.

Denote by $E(\tau)$ [respectively by $(E,\|\cdot\|)$ or by $(E,\langle\cdot,\cdot\rangle)$] a locally convex space (respectively, a Banach space or a Hilbert space). Suppose that, E is ordered by a closed, pointed convex cone $\mathbb{K}$. Denote by "$\leq$" the ordering defined by $\mathbb{K}$, that is $x \leq y$ if and only if $y - x \in \mathbb{K}$. Assume that the ordered vector space $(E,\mathbb{K})$ is a *vector lattice*, i.e., for every pair $(x,y)\in E\times E$, *the supremum* $\vee(x,y)$ and the *infimum* $\wedge(x,y)$ with respect to the ordering $\leq$ exist in E. In this case, for every $x_1,x_2,x_3\in E$ we have the following formulas:

(1) $\vee(x_1,x_2)+x_3=\vee(x_1+x_3,x_2+x_3)$,

(2) $\wedge(x_1,x_2)+x_3=\wedge(x_1+x_3,x_2+x_3)$,

(3) $\vee(x_1,x_2,x_3)=\vee(\vee(x_1,x_2),x_3)=\vee(\vee(x_1,x_2),\vee(x_2,x_3))$.

If $x_1,x_2,\ldots,x_n\in E$, then by induction $\vee(x_1,x_2,\ldots,x_n)$ and $\wedge(x_1,x_2,\ldots,x_n)$ are well defined for any $n \in N$, considering also the formula $\wedge(x,y)=-\vee(-x,-y)$.

Let $\langle\cdot,\cdot\rangle:E\times E\to\mathbb{R}$ be a bilinear form. We say that the bilinear form $\langle\cdot,\cdot\rangle$ is $\mathbb{K}$*-local*, if and only if $\langle x,y\rangle=0$, whenever $x, y \in \mathbb{K}$ and $\wedge(x,y)=0$. (The term $\mathbb{K}$*-local* is used in the axiomatic potential theory). Let D be a non-empty subset of E. In particular the set D can be the cone $\mathbb{K}$. Given m, linear or nonlinear mappings $f_1,f_2,\ldots,f_m:E\to E$, the *order complementarity problem* defined by the family of mappings $\{f_i\}_{i=1}^m$ and the set D is:

$$OCP\left(\{f_i\}_{i=1}^m,D\right)=\begin{cases} \textit{find } x_0\in D \textit{ such that} \\ \wedge\left(f_1(x_0),f_2(x_0),\ldots,f_m(x_0)\right)=0. \end{cases}$$

In (Isac, G. and Goeleven, D. [1]) this problem is called the *implicit general order complementarity problem.* We have several interesting particular cases:

(1) If $m = 2$, $D = E$, $f_1 = I$ (the identity mapping) and $f_2(x) = T(x)+q$, where $T : E \to E$ is a linear mapping and q is an element in E, we have *the linear order complementarity problem* denoted by

LOCP(*T, q*). This problem was studied systematically for the first time in 1989 in (Borwein, J. M. and Dempster, M. A. H., [1]), where several interesting new classes of linear operators were introduced. We find for example the operators of class (H^{1}), (S), (Z), ($\mathbb{K}$), (P) and (A).

(2) If m is an arbitrary natural number and $f_i, (i = 1, 2, \ldots, m)$ are affine mappings we have *the generalized linear order complementarity problem.* Several results about this problem are in (Gowda, M. S. and Sznajder, R., [1]), (Isac, G. and Goeleven, D., [1]), and (Sznajder, R. [1]).

(3) If $m = 2$, $D = \mathbb{K}$ and f_1, f_2 are nonlinear mappings, then in this case we have *the nonlinear order complementarity problem,* studied for the first time in 1986 (Isac, G. [5]).

(4) If $m = 3$, $D = E$, $f_1 = I$ (the identity mapping) and f_2, f_3 are nonlinear we have *an order complementarity problem.* In 1986 Oh, K. P introduced this notion in lubrication theory. (Oh, K. P., [1]). This interesting order complementary problem is the following. Consider the mixed lubrication in the context of a journal bearing with elastic support. The problem is to study the contact pressure X. In this case $E = H^1(\Omega)$ (defined over $L^2(\Omega)$) and the cone is $\mathbb{K} = \{u \in H^1(\Omega) | u \geq 0 \text{ } a.e., on \text{ } \Omega\}$. We have two operators, $T_1(X)$ and $T_2(X)$, where T_1 is the Reynolds partial differential operator. For the definition of these operators, the reader is referred to (Oh, K. P., [1]), (Isac, G. and Kostreva, M., [1], (Isac, G. and Goeleven, D., [1]). In this case, there are three distinct functions, which cause the decomposition of the spatial area into three disjoint regios: *the innermost region* (solid-to-solid contact), *the elasto-hydrodynamic lubrication region* (solid-to-fluid contact) and *the cavity region* (in which the pressure returns to the ambient value). The complementarity formulation is based on the observation that the contact pressure X satisfies the following equation specified for every region:

(i) $X \geq 0$, $T_1(X) = 0$, $T_2(X) \geq 0$ (solid-to-solid contact),
(ii) $X = 0$, $T_1(X) \geq 0$, $T_2(X) \geq 0$ (cavity point),
(iii) $X \geq 0$, $T_1(X) \geq 0$, $T_2(X) = 0$ (lubrication point).

The problem of finding the contact pressure X is equivalent to solvability of *OCP*(I, T_1, T_2; $\mathbb{K}$). This problem, defined in 1986 in (Oh, K. P., [1]) is theoretical not yet solved, but it has many interesting applications. In practical problems this mathematical

model is implemented by simulation and by numerical approximations. Finally, we note that the *order complementarity problems* are used also in the study of the *global reproduction of economic systems working with several technologies,* in the study of *discrete dynamic complementarity problems.* (Isac, G. [20]), and in the study of the fold *complementarity problems* (Isac, G. [15]) and (Isac, G. and Kostreva, M. [3]).
If m is an arbitrary natural number, $D = \mathbb{K}$, $f_1 = I$ (the identity mapping) and $f_2, f_3, \dots, f_m$ are nonlinear but having the form $f_i(x) = x - T_i(x), (i = 1, 2, 3, \dots, m)$, with T_i nonlinear mappings, then we have the *generalized order complementarity problem* studied systematically in (Isac, G. and Kostreva, M. [1]) and for set valued mappings in (Isac, G. and Kostreva, M. [2]) and (Huang, N. J. and Fang, Y. P. [1]). Some numerical methods for the order complementarity problem can be found in (Isac, G. [11]) and in (Isac, G. and Goeleven, D. [2]).

2.2. Variational inequalities

Another important domain of applied mathematics is the study of variational inequalities, which is deeply related to complementarity theory. It seems that the notion of *variational inequality* was introduced in the papers of G. Stampacchia and G. Stampacchia and P. Hartman. For references the reader is referred to the books (Stampacchia, G. [1]), (Kinderlehrer, D, [1]), (Baiocchi, C. and Capelo, A, [1]), (Duvaut, G. and Lions, J. L., [1]) and (Lions, J. L. and Magenes, E., [1]).

The theory of variational inequalities had from the beginning a rapid development and a prolific growth of its applications. Initially, one of the attractions of the theory of variational inequalities was its applications to many questions of physical interest, as for example: the lubrication theory, the steady filtration of a liquid through a porous membrane, the motion of a fluid past a given profile and the small deflections of an elastic beam etc. Many remarkable mathematicians added their contributions to the development of the variational inequalities theory as for example: H. Brezis, C. Baiocchi, L. Caffarelly, D. Kinderlehrer, H. Lewy, J. L. Lions and E. Magenes, among others. Now, the literature on variational inequalities is huge and contains several variations. We consider in this book only the classical variational inequalities.

Let $\langle E, E^* \rangle$ be a duality of locally convex spaces, i.e., E is a locally convex space, E^* is the topological dual of E and $\langle \cdot, \cdot \rangle$ is a bilinear form on $E \times E^*$ satisfying the following separation axioms:

(s_1) $\langle x_0, y \rangle = 0$ *for all* $y \in E^*$ *implies* $x_0 = 0$,

(s_2) $\langle x, y_0 \rangle = 0$ *for all* $x \in E$ *implies* $y_0 = 0$.

Let $f: E \to E^*$ be a mapping. We recall the following classical notions.

(a) We say that f is *monotone*, if for any x, $y \in E$ we have $\langle x - y, f(x) - f(y) \rangle \geq 0$.

(b) We say that f is *pseudomonotone* (in Karamardian's sense) if for any x, $y \in E$ we have that $\langle x - y, f(y) \rangle \geq 0$ implies $\langle x - y, f(x) \rangle \geq 0$.

We have similar definitions if $f : \Omega \to E^*$, where Ω is an arbitrary non-empty subset of E. The *Hartman–Stampacchia variational inequality* defined by f and Ω is:

$$HSVI(f, \Omega): \begin{cases} \text{find } x_* \in \Omega \text{ such that} \\ \langle x - x_*, f(x_*) \rangle \geq 0 \text{ for all } x \in \Omega, \end{cases}$$

and the *Minty variational inequality* defined by f and Ω is:

$$MVI(f, \Omega): \begin{cases} \text{find } x_* \in \Omega \text{ such that} \\ \langle x - x_*, f(x) \rangle \geq 0 \text{ for all } x \in \Omega. \end{cases}$$

For more information about Minty's variational inequality the reader is referred to (Minty, G. J., [1]). The Hartman–Stampacchia variational inequality has many applications in physics, engineering and in economics, while the Minty variational inequality is important in the study of solvability of *HSVI(f, Ω)*.

About the solvability of problem *HSVI(f, Ω)*, first we note the following classical result, which is a generalization to locally convex spaces of Hartman–Stampacchia's theorem (proved initially in Euclidean space).

THEOREM 2.2.1 [Hartman–Stampacchia]. *Let* Ω *be a compact convex subset of a locally convex space* E *and let* $f : \Omega \to E^*$ *be a continuous mapping, (with respect to the strong topology). Then, there exists an element* $x_* \in \Omega$ *such that* $\langle x - x_*, f(x_*) \rangle \geq 0$ *for all* $x \in \Omega$.

Proof. A proof of this result is in (Holmes, R. B., [1]). The proof is based on the Fan–Kakutani Fixed Point Theorem. □

Remark. The study of solvability of problem *HSVI*(f, Ω) in the case when Ω is unbounded, generally is based on special mathematical tools. In this book we develop a new method to study variational inequalities with respect to unbounded closed convex sets.

The following result establishes a relation between problems *HSVI*(f, Ω) and *MVI*(f, Ω). If $\Omega \subset E$ is a convex set and $f : \Omega \to E^*$ is a mapping, we say that f is *hemicontinuous* if it is continuous from the line segments of Ω to the *weak topology* of E^*.

THEOREM 2.2.2. *Let* $E(\tau)$ *be a locally convex space,* $\Omega \subset E$ *a closed convex set and* $f : \Omega \to E^*$ *a pseudomonotone, hemicontinuous mapping. Then, an element* $u_0 \in \Omega$ *is a solution to the problem HSVI*(f, Ω), *if and only if* u_0 *is a solution to the problem MVI*(f, Ω).

Proof. Suppose that $u_0 \in \Omega$ is a solution to the problem *HSVI*(f, Ω). Then, in this case we have,

$$\langle x - u_0, f(u_0) \rangle \geq 0 \text{, for all } x \in \Omega$$

and the pseudomonotonicity implies that

$$\langle x - u_0, f(x) \rangle \geq 0 \text{, for all } x \in \Omega,$$

i.e., u_0 is a solution to the problem *MVI*(f, Ω).

Conversely, suppose that an element $u_0 \in \Omega$ is a solution to the problem *MVI*(f, Ω). In this case, if $x \in \Omega$ is an arbitrary element, we denote it by

$$x_t = (1-t)u_0 + tx, \; t \in]0,1[.$$

If we put x_t in the definition of the problem *MVI*(f, Ω), then we have

$$\langle x_t - u_0, f(x_t) \rangle \geq 0,$$

which implies

$$\langle t(x_t - u_0), f(x_t) \rangle \geq 0,$$

and finally,

$$\langle x - u_0, f(x_t) \rangle \geq 0.$$

Supposing that $t \to 0$ and using the hemicontinuity of f we obtain that $f(x_t)$ is weakly convergent to $f(u_0)$, which implies that

$$\langle x - u_0, f(u_0) \rangle \geq 0 \text{, for any } x \in \Omega,$$

i.e., u_0 is a solution to the problem $HSVI(f, \Omega)$ and the proof is complete. □

Obviously, the variational inequalities $HSVI(f, \Omega)$ and $MVI(f, \Omega)$ can be defined for set-valued mappings. Indeed, let f be a set-valued mapping from Ω into E^*, i.e., $f : \Omega \to 2^{E^*}$. The *multivalued Hartman–Stampacchia variational inequality* defined by f and Ω is:

$$MHSVI(f, \Omega): \begin{cases} \text{find } x_* \in \Omega \text{ and } y_* \in E^* \\ \text{such that } y_* \in f(x_*) \text{ and} \\ \langle x - x_*, y_* \rangle \geq 0 \text{ for all } x \in \Omega \end{cases}$$

and the *multivalued Minty variational inequality* defined by f and Ω is:

$$MMVI(f, \Omega): \begin{cases} \text{find } x_* \in \Omega \text{ such that} \\ \text{for any } x \in \Omega \text{ there exists} \\ y_x \in f(x) \text{ satisfying} \\ \langle x - x_*, y_x \rangle \geq 0. \end{cases}$$

Finally, we consider in this book a special *implicit* variational inequality.

Consider again a dual system $\langle E, E^* \rangle$ of locally convex spaces. $\Omega \subset E$ a closed convex cone and two mappings $S : \Omega \to \Omega$ and $f : \Omega \to E^*$. The *implicit variational inequality* defined by S, f and Ω is:

$$IVI(f, S, \Omega): \begin{cases} \text{find } x_0 \in \Omega \text{ such that} \\ \langle x - S(x_0), f(x_0) \rangle \geq 0, \text{ for all } x \in \Omega. \end{cases}$$

The problem $IVI(f, S, \Omega)$ is a special variational inequality. It is implicit in the sense of implicit variational inequalities presented in (Mosco, U., [1]). Obviously, if $S(x) = x$ for every $x \in \Omega$, the problem $IVI(f, S, \Omega)$ is exactly the problem $HIVI(f, \Omega)$. We note that the problem $IVI(f, S, \Omega)$ is related to the problem $GICP(f, g, D, \mathbb{K})$ when $g = S$ and $D = \mathbb{K}$.

2.3 Complementarity problems, variational inequalities, equivalences and equations

We present in this section some equivalences between complementarity problems and variational inequalities. We show also, how a complementarity problem or a variational inequality can be transformed in an equation. These equations are essential for the next chapters of this book.

Let $\langle E, E^* \rangle$ be a dual system of locally convex spaces. Let $\mathbb{K} \subset E$ be a closed convex cone and $f: E \to E^*$ a mapping.

THEOREM 2.3.1. *The problems $NCP(f, \mathbb{K})$ and $HSVI(f, \mathbb{K})$ are equivalent.*

Proof. Indeed, if x_* is a solution to the problem $HSVI(f, \mathbb{K})$, then we have

$$\langle x - x_*, f(x_*) \rangle \geq 0, \textit{ for all } x \in \mathbb{K}. \tag{2.3.1}$$

Let $y \in \mathbb{K}$ be an arbitrary element. If we put $x = y + x_*$ in (2.3.1), then we obtain

$$\langle y, f(x_*) \rangle \geq 0, \textit{ for all } y \in \mathbb{K},$$

which implies that $f(x_*) \in \mathbb{K}^*$.

If we consider $x = 2x_*$ in (2.3.1) then we deduce that $\langle x_*, f(x_*) \rangle = 0$, i.e., x_* is a solution to the $NCP(f, \mathbb{K})$.

Conversely, if we suppose that $x_* \in \mathbb{K}$ is a solution to the problem $NCP(f, \mathbb{K})$, then we have $\langle x_*, f(x_*) \rangle = 0$ and $\langle x, f(x_*) \rangle \geq 0$ for every $x \in \mathbb{K}$, which obviously imply $\langle x - x_*, f(x_*) \rangle \geq 0, \textit{ for all } x \in \mathbb{K}$, that is, x_* is a solution to the problem $HSVI(f, \mathbb{K})$. □

Now, we consider the following problems. Let $\langle E, E^* \rangle$ be a duality of locally convex spaces and $\mathbb{K} \subset E$ a pointed closed convex cone. Suppose given two mappings, $f: E \to E^*$ and $g: E \to E$. The next theorem is related to the following two problems:

$$IVI(f, g, \mathbb{K}): \begin{cases} \textit{find } x_* \in E \textit{ such that} \\ g(x_*) \in \mathbb{K} \textit{ and} \\ \langle x - g(x_*), f(x_*) \rangle \geq 0 \textit{ for all } x \in \mathbb{K}, \end{cases}$$

$$ICP(f,g,\mathbb{K}): \begin{cases} \text{find } x_* \in E \text{ such that} \\ g(x_*) \in \mathbb{K}, f(x_*) \in \mathbb{K}^* \text{ and} \\ \langle g(x_*), f(x_*) \rangle = 0. \end{cases}$$

THEOREM 2.3.2. *The problems $IVI(f,g,\mathbb{K})$ and $ICP(f,g,\mathbb{K})$ are equivalent.*

Proof. Indeed if $x_* \in E$ is a solution to the problem $ICP(f,g,\mathbb{K})$, then we have $g(x_*) \in \mathbb{K}$, $f(x_*) \in \mathbb{K}^*$ and $\langle g(x_*), f(x_*) \rangle = 0$ which imply

$$\langle x, f(x_*) \rangle \geq 0 \text{ for all } x \in \mathbb{K} \tag{2.3.2}$$

and

$$\langle g(x_*), f(x_*) \rangle = 0. \tag{2.3.3}$$

By using (2.3.2) and (2.3.3) we obtain $g(x_*) \in \mathbb{K}$, and $\langle x - g(x_*), f(x_*) \rangle \geq 0$ for any $x \in \mathbb{K}$, that is, x_* is a solution to the problem $IVI(f,g,\mathbb{K})$.

Conversely, we suppose that $x_* \in E$ is a solution to the problem $IVI(f,g,\mathbb{K})$. Then, we have $g(x_*) \in \mathbb{K}$, and $\langle x - g(x_*), f(x_*) \rangle \geq 0$ for all $x \in \mathbb{K}$. If we take $x = y + g(x_*)$, then we obtain that $\langle y, f(x_*) \rangle \geq 0$, which implies that $f(x_*) \in \mathbb{K}^*$. If we consider $x = 2g(x_*)$ in $ICP(f,g,\mathbb{K})$, then we obtain $\langle g(x_*), f(x_*) \rangle \geq 0$ and considering $x = 0$, we obtain $\langle g(x_*), f(x_*) \rangle \leq 0$. Therefore $\langle g(x_*), f(x_*) \rangle = 0$ and we have that x_* is a solution to the problem $ICP(f,g,\mathbb{K})$. □

For the method developed in this book, it is important to transform a complementarity problem or variational inequality in a fixed-point problem or in an equation. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\Omega \subset H$ a closed convex cone. Given f, g: $H \to H$ two arbitrary mappings, we consider the following implicit variational inequality:

$$IVI(f,g,\Omega): \begin{cases} \text{find } x_* \in H \text{ such that} \\ g(x_*) \in \Omega \text{ and} \\ \langle y - g(x_*), f(x_*) \rangle \geq 0 \text{ for all } y \in \Omega. \end{cases}$$

We have the following result.

THEOREM 2.3.3. *An element $x_* \in H$ is a solution to the problem IVI(f, g, Ω) if and only if, x_* is a solution to the coincidence problem*

$$CP(f,g,\Omega): \begin{cases} \text{find } x_* \in H \text{ such that} \\ g(x_*) = P_\Omega\left(g(x_*) - f(x_*)\right). \end{cases}$$

Proof. Indeed, if $x_* \in H$ and $g(x_*) = P_\Omega\left(g(x_*) - f(x_*)\right)$, then we have that $g(x_*) \in \Omega$ and $g(x_*) - f(x_*) \in g(x_*) + N_\Omega\left(g(x_*)\right)$. [We used *Theorem 1.9.4*]. Therefore $\langle -f(x_*), y - g(x_*)\rangle \leq 0$ for all $y \in \Omega$ and $g(x_*) \in \Omega$, that is x_* is a solution to the problem *IVI(f, g, Ω)*. Conversely, if $x_* \in H$, $g(x_*) \in \Omega$ and $\langle f(x_*), y - g(x_*)\rangle \geq 0$ for all $y \in \Omega$, then we have

$$\langle -f(x_*), y - g(x_*)\rangle \leq 0 \text{ for all } y \in \Omega,$$

or

$$g(x_*) - f(x_*) \in g(x_*) + N_\Omega\left(g(x_*)\right),$$

which implies that

$$g(x_*) = P_\Omega\left(g(x_*) - f(x_*)\right),$$

[using again Theorem 1.9.4]. □

COROLLARY 2.3.4. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed convex cone and f: $H \to H$ a mapping. The problem $NCP(f, \mathbb{K})$ has a solution if and only if the mapping Ψ : $H \to H$ defined by $\Psi_{\mathbb{K}}(x) = P_{\mathbb{K}}\left(x - f(x)\right)$ has a fixed point, i.e., there exists an element $x_* \in H$ such that $x_* = P_{\mathbb{K}}\left(x_* - f(x_*)\right)$.*

Proof. We take in Theorem 2.3.2 and Theorem 2.3.3, $g(x) = x$, for any $x \in H$ and $\Omega = \mathbb{K}$. □

Also for problem *HSVI(f, Ω)* we have the following result.

COROLLARY 2.3.5. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\Omega \subset H$ a closed convex set and f: $H \to H$ a mapping. The problem HSVI(f, Ω) has a solution if and only if the mapping $\Psi_\Omega : H \to H$ defined by $\Psi_\Omega(x) = P_\Omega\left(x - f(x)\right)$*

has a fixed point, i.e., there exists an element $x_* \in H$ *such that* $x_* = P_\Omega\left(x_* - f(x_*)\right)$.

Proof. We take in Theorem 2.3.3 $g(x) = x$, for any $x \in H$. □

Remark. We can prove Corollary 2.3.4 using Theorem 1.9.7.

The reader can extend Corollary 2.3.4 (resp. Corollary 2.3.5) to the case when f is a set-valued mapping, that is when $f: H \to 2^H$, but in this case the mapping $\Psi_{\mathbb{K}}$ (resp. Ψ_Ω) will be a set-valued mapping. Therefore, we have the following result, related to the problems:

$$MCP(f,\mathbb{K}):\begin{cases} \textit{find } x_0 \in \mathbb{K} \textit{ and} \\ y_0 \in f(x_0) \cap \mathbb{K}^* \textit{ such that} \\ \langle x_0, y_0 \rangle = 0, \end{cases}$$

and

$$MHSVI(f,\Omega):\begin{cases} \textit{find } x_0 \in \Omega \textit{ and} \\ y_0 \in f(x_0) \textit{ such that} \\ \langle x - x_0, y_0 \rangle \geq 0 \textit{ for all } x \in \Omega. \end{cases}$$

COROLLARY 2.3.6. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed convex set and* $f: H \to H$ *a set-valued mapping. The problem* $MCP(f,\mathbb{K})$ *(resp. the problem* $MHSVI(f,\Omega)$*) has a solution if and only if the set-valued mapping* $\Psi_{\mathbb{K}}(x) = P_{\mathbb{K}}(x - f(x))$ *(resp.* $\Psi_\Omega(x) = P_\Omega(x - f(x))$*) has a fixed point, i.e., there exists an element* $x_0 \in H$ *such that* $x_0 \in \Psi_{\mathbb{K}}(x_0) = P_{\mathbb{K}}(x_0 - f(x_0))$ *(resp.* $x_0 \in \Psi_\Omega(x_0) = P_\Omega(x_0 - f(x_0))$*).* □

Now, we introduce the normal operator and we will show that the solvability of a complementarity problem or a variational inequality is equivalent to the solvability of an equation. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\Omega \subset H$ a closed convex set. Let $f: H \to H$ be an arbitrary mapping. Consider again the problem:

$$HSVI(f,\Omega):\ \begin{cases} \textit{find } x_0 \in \Omega \textit{ such that} \\ \langle x - x_0, f(x_0) \rangle \geq 0, \textit{ for all } x \in \Omega. \end{cases}$$

The operator $\mathcal{N}_f : H \to H$ defined by

$$\mathcal{N}_f(z) = f(P_\Omega(z)) + z - P_\Omega(z) \text{ for all } z \in H$$

is called the *normal operator defined by f and* Ω.

Remark. In 1992, S. M. Robinson introduced the name *normal operator* [see (Robinson S. M., [1]–[4]), and this operator was used in several papers to transform a variational inequality in an equation of the form $\mathcal{N}_f(z) = 0$. In 1988, G. Isac used the same operator in complementarity theory, but in the form $z = P_{\mathbb{K}}(z) - f(P_{\mathbb{K}}(z))$. He used this operator to transform the solvability of the complementarity problem in a fixed-point problem. (Isac, G. [7]).

THEOREM 2.3.7. *An element* $z_* \in H$ *is a solution to the equation*

$$\mathcal{N}_f(z) = 0$$

if and only if, $x_* = P_\Omega(z_*)$ *is a solution to the problem HSVI(f,* Ω*).*

Proof. First, by Theorem 1.9.3 a, we have that $x_* = P_\Omega(z_*)$ if and only if

$$\langle z_* - x_*, x_* - x \rangle \geq 0, \text{ for all } x \in \Omega. \tag{2.3.4}$$

If $\mathcal{N}_f(z_*) = 0$, then we have

$$f(P_\Omega(z_*)) + z_* - P_\Omega(z_*) = 0$$

or

$$-f(x_*) = z_* - x_*$$

which implies [using (2.3.4)]

$$\langle -f(x_*), x_* - x \rangle \geq 0, \text{ for all } x \in \Omega,$$

and finally

$$\langle f(x_*), x_* - x \rangle \geq 0, \text{ for all } x \in \Omega,$$

that is (using the commutativity of the inner-product, we have that x_* is a solution to the problem *HSVI(f,* Ω*)*.

Conversely, suppose that $z_* = x_* - f(x_*)$ and $\langle -f(x_*), x - x_* \rangle \geq 0$, for any $x \in \Omega$. We have $z_* - x_* = -f(x_*)$ or $f(x_*) = x_* - z_*$, which implies

$$\langle x_* - z_*, x - x_* \rangle \geq 0, \text{ for all } x \in \Omega,$$

or

$$\langle z_* - x_*, x_* - x \rangle \geq 0, \text{ for all } x \in \Omega,$$

which implies that $x_* = P_\Omega(z_*)$. Therefore, we have

$$f\left(P_\Omega(z_*)\right) + z_* - P_\Omega(z_*) = 0,$$

that is, $\mathcal{N}_f(z_*) = 0$, and the proof is complete. □

Now, we consider the case when the Hilbert space is replaced by a Banach space. In this case we must replace the projection operator defined in a Hilbert space by the projection operator in Alber's sense (See Chapter 1 of this book).

Let $(E, \|\cdot\|)$ be a *uniformly convex and uniformly smooth* Banach space. Let E^* be the topological dual of *E*. Denote by $\langle \cdot, \cdot \rangle$ the natural duality between E^* and *E*, that is, $\langle y, x \rangle = y(x)$, for all $y \in E^*$ and all $x \in E$. Let $\Omega \subset E$ be a closed convex set. Denote by $\|\cdot\|_*$ the norm on E^*. Let $J : E \to E^*$ be the duality mapping (See Chapter 1). We consider the mapping $V : E^* \times E \to \mathbb{R}$, defined by:

$$V(x, y) = \|y\|_*^2 - 2\langle y, x \rangle + \|x\|^2, \textit{ for any } (y, x) \in E^* \times E.$$

We know that the minimization problem:

$$\begin{cases} \text{given } y \in E^*, \text{ find } x_y \in \Omega \subset E \text{ such that} \\ V(y, x_y) = \inf_{x \in \Omega} V(y, x) \end{cases}$$

has a unique solution. (see Chapter 1). The mapping $\Pi_\Omega : E^* \to \Omega \subset E$, defined by $\Pi_\Omega(y) = x_y$, is called the *generalized projection operator* (or the *Alber projection*). We need to use the following properties of mapping *V*.

(i) *V(y, x) is convex with respect to y, when x is fixed and with respect to x, when y is fixed.*

(ii) $grad_x V(y, x) = 2(J(x) - y)$, *(because E is a smooth Banach space).*

For the proof of properties (i) and (ii) the reader is referred to (Alber, Y. I. [1]). We recall also the following property, well known in convex analysis.

(iii) *A differentiable mapping $\varphi : E \to \mathbb{R}$, is convex if and only if, for any x and x_0 in E we have*

$$\varphi(x) - \varphi(x_0) \geq \langle grad\varphi(x_0), x - x_0 \rangle.$$

THEOREM 2.3.8. *An element $y_* \in \Omega$ is the generalized projection of an element $y \in E^*$ (i.e., $y_* = \Pi_\Omega(y)$, if and only if,*

$$\langle y - J(y_*), y_* - u\rangle \geq 0, \text{ for all } u \in \Omega. \tag{2.3.5}$$

Proof. Considering the definition of $y_* = \Pi_\Omega(y)$ we have

$$V(y, y_*) \leq V(y, y_* + t(u - y_*)),$$

where $t \in]0, 1]$ and $y_* + t(u - y_*) \in \Omega$, because of the convexity of Ω. Using the properties (i), (ii) and (iii) we have

$$0 \geq V(y, y_*) - V(y, y_* + t(u - y_*))$$
$$\geq 2\langle J(y_* + t(u - y_*)) - y, y_* - y_* - t(u - y_*)\rangle,$$

which implies

$$\langle J(y_* + t(u - y_*)) - y, u - y_*\rangle \geq 0.$$

Letting $t \to 0$, we have

$$\langle J(y_*) - y, u - y_*\rangle \geq 0, \text{ for all } u \in \Omega,$$

or

$$\langle y - J(y_*), y_* - u\rangle \geq 0, \text{ for all } u \in \Omega,$$

that is condition (2.3.5) is satisfied.

Conversely, if condition (2.3.5) is satisfied, then we have (using properties (i), (ii) and (iii)),

$$V(y, u) - V(y, y_*) \geq 2\langle J(y_*) - y, u - y_*\rangle \geq 0, \text{ for any } x \in \Omega,$$

which implies

$$V(y, u) \geq V(y, y_*), \text{ for any } u \in \Omega.$$

Therefore $y_* = \Pi_\Omega(y)$ and the proof is complete. □

Let $f: E \to E^*$ be an arbitrary mapping. Consider again the problem

$$HSVI(f, \Omega): \begin{cases} \text{find } x_* \in \Omega \text{ such that} \\ \langle f(x_*), u - x_*\rangle \geq 0, \text{ for all } u \in \Omega. \end{cases}$$

THEOREM 2.3.9. *Let f be a mapping from E to E^*, $\Omega \subset E$ a closed convex set and α an arbitrary fixed positive real number. Then an element $x_* \in \Omega$ is a solution to the problem $HSVI(f, \Omega)$ if and only if x_* is a fixed point of the mapping $\Psi_\Omega(x) = \Pi_\Omega[J(x) - \alpha f(x)]$, i.e., $x_* = \Pi_\Omega[J(x_*) - \alpha f(x_*)]$.*

Proof. Indeed, we observe that the problem *HSVI(f, Ω)* has the following representation:

$$\langle J(x_*) - \alpha f(x_*) - J(x_*), x_* - u \rangle \geq 0, \textit{ for all } u \in \Omega .$$

Considering this representation, and taking into account Theorem 2.3.8, $y = J(x_*) - \alpha f(x_*) \in E^*$ and $y_* = x_* \in \Omega \subset E$, we obtain the conclusion of the theorem. □

3

LERAY–SCHAUDER ALTERNATIVES

We present in this book a topological method applicable to the study of solvability of complementarity problems and of variational inequalities. This special method is based on several Leray-Schauder type alternatives. The classical *Leray–Schauder Alternative* is based on the *Leray–Schauder Continuation Theorem*, which is a remarkable result in nonlinear analysis. We note that there exist several *continuation* theorems, which have many applications to the study of nonlinear functional equations (O'Regan, D. and Precup, R. [1]), (Precup, R. [1]).

The *Continuation Theorem* is based on the idea of obtaining a solution of a given equation, starting from one of the solutions of a simpler equation. The essential part of this theorem is the *"Leray–Schauder boundary condition"*. It seems the *"continuation method"*, was initiated by H. Poincaré and S. Bernstein (Poincaré, H. [1], [2]), (Bernstein, S. [1]). Certainly, J. Leray and J. Schauder in 1934 gave the first abstract formulation of *"continuation principle"* using the topological degree. Now we recall this result.

Let $(E,\|\cdot\|)$ be a Banach space, $U \subset E$ a bounded open set and $\mathcal{H}: \overline{U} \times [0,1] \to E$ a *compact* mapping, i.e., $\mathcal{H}$ is continuous and $\mathcal{H}(\overline{U} \times [0,1])$ is relatively compact. Denote by I the identity mapping of E and by $\deg(I - \mathcal{H}(\cdot,0), U, 0)$ the Leray–Schauder degree of $\mathcal{H}(\cdot,0)$ with respect to U and the origin of E.

THEOREM A [Leray–Schauder]. *If the following conditions are satisfied:*

(i) $\mathcal{H}(x,t) \neq x$, *for all* $x \in \partial U$ *and* $t \in [0,1]$,

(ii) $\deg\left(I - \mathcal{H}(\cdot,0), U, 0\right) \neq 0$,

then there exists at least one $x_0 \in U$ *such that* $\mathcal{H}(x_0, 1) = x_0$.

Proof. To prove this result it is sufficient to show that

$$\deg\left(I - \mathcal{H}(\cdot,0), U, 0\right) = \deg\left(I - \mathcal{H}(\cdot,1), U, 0\right). \qquad \square$$

The *continuation theorem* is an expression of the homotopy invariance of the degree. There exist many books and papers presenting Theorem A, using generalizations of Leray-Schauder degree, as for example (Deimling, K. [1]), (Gaines, R. E. and Mawhin, J. [1]), (Krasnoselskii, M. A. [1]), (Krasnoselskii, M. A. and Zabreiko, P. P. [1]), (Lloyd, N. G. [1]), (Nussbaum, R. D. [1]), (Petryshyn, V. [1]) and (Rothe, E. H. [1]) among others. In 1955, H. H. Schaefer proved a variant of Theorem A in a Banach space, using the Schauder fixed-point theorem (Schaefer, H. H. [1]). A version of Theorem A without degree for the general case is due to A. Granas. The result proved by Granas in 1959 is based on the notion of *essential map* (Granas, A. [1]).

In this chapter we will present several Leray–Schauder type alternatives. The alternatives will be with respect to an open set in a Hilbert or in a Banach space. We will suppose that the open set contains the origin of the space. This case is related to applications to the study of complementarity problems and to the study of variational inequalities.

3.1 The Leray–Schauder alternative by topological degree

We give in this section the classical Leray–Schauder alternative, for completely continuous mappings in Banach spaces.

THEOREM 3.1.1 [Leray–Schauder]. *Let* $(E, \|\cdot\|)$ *be a Banach space,* $\Omega \subset E$ *an open bounded subset such that* $0 \in \Omega$ *and* $f : \overline{\Omega} \to E$ *a compact mapping. If the following assumption is satisfied:*

$f(x) \neq \lambda x$, *for all* $x \in \partial\Omega$ *and all* $\lambda > 1$, *then f has a fixed point.*

Proof. We consider the homotopy $h(\cdot,\cdot):\overline{\Omega}\times[0,1]\to E$ defined by

$$h(x,\lambda)=x-\lambda f(x),\ \textit{for all}\ x\in\overline{\Omega}\ \textit{and all}\ \lambda\in[0,1].$$

We have two possibilities:

(a) f has a fixed point in $\partial\,\Omega$,

(b) f has no fixed point in $\partial\,\Omega$.

If (a) is satisfied, in this case there is nothing to prove. Now, we suppose that (b) is satisfied. In this case we can suppose that $0\notin h(\partial\Omega,\lambda)$ for $0\le\lambda\le 1$. Indeed, if $\lambda = 0$ and $0 \in h(\partial\Omega, 0)$, then $x = 0$ for some $x \in \partial\,\Omega$, which is impossible. If $\lambda = 1$ and $0 \in h(\partial\,\Omega, 1)$, then $0 = x - f(x)$, for some $x \in \partial\Omega$, which is impossible, since we suppose that f has no fixed point in $\partial\Omega$.

Finally, we suppose that $h(x, \lambda) = 0$ for some $x \in \partial\,\Omega$ and some $0 < \lambda < 1$. Then in this case we have $\frac{1}{\lambda}x=f(x)$, where $0 < \lambda < 1$ and $x \in \partial\,\Omega$, which is in contradiction with our assumption. Therefore, $0\notin h(\partial\Omega,\lambda)$ for $0\le\lambda\le 1$. Applying Property 3 (Homotopy Invariance) of the Leray–Schauder degree, we have

$$d(I-f,\Omega,0)=d(I,\Omega,0)=1\ (\text{since } 0\in\Omega),$$

which implies that there exists an element $x \in \Omega$ such that $f(x) = x$, and the proof is complete. □

From Theorem 3.1.1 we deduce the following result.

THEOREM 3.1.2 [Leray–Schauder alternative]. *Let* $(E,\|\cdot\|)$ *be a Banach space,* $\Omega\subset E$ *an open bounded subset such that* $0\in\Omega$ *and* $f:\overline{\Omega}\to E$ *a compact mapping. Then:*

(1) *either* f *has a fixed point in* $\overline{\Omega}$, *or*

(2) *there exist an element* $x_*\in\partial\Omega$ *and a real number* $\lambda_*\in\,]0,1[$ *such that* $x_*=\lambda_* f(x_*)$. □

3.2 The Leray–Schauder alternative by the fixed point theory

The main result of this section is based on the following classical results.

THEOREM 3.2.1 [Schauder]. *Let $(E,\|\cdot\|)$ be a Banach space, $D \subset E$ a non-empty convex compact subset and f: $D \to D$ a continuous mapping. Then f has at least one fixed point.*

Proof. A proof of this theorem can be found in (Schauder, J., [1]) or in (Dugundji, J and Granas, A., [1]). □

LEMMA 3.2.2 [Mazur]. *Let $(E,\|\cdot\|)$ be a Banach space. If $D \subset E$ is a relatively compact subset, then $conv(D)$ is also a relatively compact subset in E.*

Proof. This result is also true in a locally convex space. See (Schaefer, H. H., [2], Theorem 4.3, page 50). □

The next result is a more general variant of Theorem 3.1.1. If Ω and U are subsets of E and $U \subset \Omega$, then in this case we denote by $\partial_\Omega U$ the boundary of U with respect to the topology of Ω.

THEOREM 3.2.3 [Leray–Schauder]. *Let $(E,\|\cdot\|)$ be a Banach space, $\Omega \subset E$ a closed convex subset, $U \subset \Omega$ a bounded set, open (with respect to the topology Ω) and such that $0 \in U$. Let $f : \overline{U} \to \Omega$ be a completely continuous mapping. If the following assumption is satisfied:*

$$\lambda f(x) \neq x, \text{ for all } x \in \partial_\Omega U \text{ and all } \lambda \in \,]0,1[,$$

then f has a fixed point in $\overline{U}$.

Proof. First we observe that the assumption is satisfied also for $\lambda = 0$ (since $0 \in U$). If the assumption is satisfied for $\lambda = 1$, then in this case we have a fixed-point in $\partial_\Omega U$ and there is nothing to prove.

In conclusion, we can suppose that the assumption is satisfied for any $x \in \partial_\Omega U$ and any $\lambda \in [0, 1]$. Let D be the set defined by

$$D = \{x \in \overline{U} : x = \lambda f(x), \text{ for some } \lambda \in [0,1]\}.$$

The set D is non-empty, because $0 \in U$ and the continuity of f implies that D is closed. We have that $D \cap \partial_\Omega U = \varnothing$.

By Theorem 1.7.2 (Urysohn's Lemma), there exists a function $g \in C(\overline{U},[0,1])$ such that

$$g(x) = \begin{cases} 0 & if \quad x \in \partial_\Omega U, \\ 1 & if \quad x \in D. \end{cases}$$

The mapping $f^* : \Omega \to \Omega$ defined by

$$f^*(x) = \begin{cases} g(x) f(x) & if \quad x \in U, \\ 0 & if \quad x \in \Omega \setminus U \end{cases}$$

is continuous and $f^*(\Omega) \subset conv\left(\{0\} \cup f(\overline{U})\right)$. The complete continuity of f implies that $f(\overline{U})$ is relatively compact. Applying Lemma 3.2.2 we have that the set $D_* = \overline{conv}\left(\{0\} \cup f(\overline{U})\right)$ is convex and compact. Moreover $f^*(D_*) \subseteq D_*$. By Theorem 3.2.1 we obtain the existence of an element $x_0 \in D_*$ such that $f^*(x_0) = x_0$. By the definition of f^*, x_0 must be an element of the set U. Then, $x_0 = g(x_0) f(x_0)$, which implies that $x_0 \in D$ and so, $g(x_0) = 1$. Therefore $f(x_0) = x_0$ and the proof is complete. □

Remark. It seems that the idea to prove Theorem 3.2.3 by using the fixed-point theory is due to A. Granas.

From Theorem 3.2.3 we deduce the following alternative.

THEOREM 3.2.4 [Leray–Schauder alternative]. *Let $(E, \|\cdot\|)$ be a Banach space, $\Omega \subset E$ a closed convex subset, $U \subset \Omega$ a bounded set, open (with respect to the topology of Ω) and such that $0 \in U$. If $f : \overline{U} \to \Omega$ is a completely continuous mapping then:*

(1) *either f has a fixed point in $\overline{U}$, or*
(2) *there exist an element $x_* \in \partial_\Omega U$ and a real number $\lambda_* \in]0,1[$ such that $x_* = \lambda_* f(x_*)$.* □

Remark. The condition used in Theorem 3.1.1, and in Theorem 3.2.3 is known in nonlinear analysis under the name of *the Leray–Schauder boundary condition.*

3.3 The Leray–Schauder alternative by the topological transversality theory

The general proof of the Leray–Schauder alternative can be given by the *Topological Transversality Theory.* A. Granas introduced the notion of *topological transversality* [See (Granas, A. [2, 3, 4])]. We follow his ideas. The main idea of *topological transversality* is the following.

Let $(E,\|\cdot\|)$ be a normed vector space, $X \subset E$ a non-empty subset and $f\colon X \to E$ a compact mapping satisfying a "boundary condition" on a closed subset $D \subset X$. A method for determining whether or not the equation $f(x) = x$ has a solution is to deform f and possibly also the boundary value $f_{|D}$ to a simpler mapping g and to reduce the problem to that for the equation $g(x) = x$. Geometrically, one deforms the graph of f to that of g and seeks to conclude, from the nature of the deformation, that if the graph of g meets the *diagonal* $\Delta \subset X \times E \subset E \times E$, then the graph of f must also do so. Now, we give *the topological transversality theorem.* This theorem gives conditions under which such a conclusion is valid.

Let C be a non-empty convex subset of E. We denote by (X, D) a pair of subsets, such that X is a subset of C and $D \subset X$ is a closed subset of X, i.e., $D \subset X \subset C$ and D is closed in X. We say in this case that (X, D) is a pair in the convex set $C \subset E$. Let Y be another subset of E and $I = [0, 1]$. We say that a homotopy $H : X \times I \to Y$ is a compact homotopy if it is a compact mapping. If $X \subset Y$, the homotopy H is called *fixed-point free* on $D \subset X$ if for each $t \in I$, the map $H_{|D\times\{t\}} : D \to Y$ has no fixed point. Also, we denote by $\mathcal{H}_D(X,C)$ the set of all compact maps $f : X \to C$ such that the restriction $f_{|D} : D \to C$ is fixed-point free.

DEFINITION 3.3.1. *We say that two mappings* $f,g \in \mathcal{H}_D(X,C)$ *are homotopic (and we denote* $f \simeq g$*) in* $\mathcal{H}_D(X,C)$ *if there is a compact homotopy* $H : X \times I \to C$ *which is fixed-point free on* D *for each* $t \in [0, 1]$ *and such that* $H(\cdot,0) = f$ *and* $H(\cdot,1) = g$.

The following result is from (Dugundji, J. and Granas, A., [1]).

PROPOSITION 3.3.1. *Let* $f,g \in \mathcal{H}_D(X,C)$ *be two mappings. If one of the following conditions holds:*

(1) $tg(x)+(1-t)f(x) \neq x$ *for each* $(x,t) \in D \times [0,1]$,

(2) $\sup_{x\in D}\|f(x)-g(x)\| \leq \inf_{x\in D}\|x-f(x)\|$,

then $f \simeq g$ *in* $\mathcal{H}_D(X,C)$.

Proof. First, we observe that assumption (2) implies that given $x \in D$, the segment $[f(x),g(x)]$ does not contain x, which is exactly assumption (1). Thus it is sufficient to show that assumption (1) implies $f \simeq g$ in $\mathcal{H}_D(X,C)$. Indeed,

$$H(x,t) = tg(x)+(1-t)f(x), \textit{ for } (x,t) \in X \times [0,1]$$

is a compact homotopy which is fixed-point free on D and such that $H(\cdot,0) = f$ *and* $H(\cdot,1) = g$. □

DEFINITION 3.3.2. *Let* (X,D) *be a pair in a convex set* $C \subset E$*. We say that a mapping* $f \in \mathcal{H}_D(X,C)$ *is transverse or essential, provided every* $g \in \mathcal{H}_D(X,C)$ *such that* $f_{|D} = g_{|D}$ *has a fixed-point. A mapping, which is not transverse, is called inessential.*

Remark. In geometric terms, a compact mapping f: $X \to C$ is *transverse*, if the graph of $f_{|D}$ does not meet the diagonal $\Delta \subset X \times C$ but the graph of every compact mapping g: $X \to C$ that coincides with f on D must cross (i.e. traverse) the diagonal Δ.

The following result [See (Dugundji, J. and Granas, A., [1])] is a characterization of inessential mappings in terms of homotopy, and it implies the topological transversality principle.

THEOREM 3.3.2. *Let (X,D) be a pair in a convex set $C \subset E$. The following conditions on $f \in \mathcal{H}_D(X,C)$ are equivalent:*

(1) *f is inessential,*

(2) *there is a fixed-point free mapping $g \in \mathcal{H}_D(X,C)$ such that $f \simeq g$ in $\mathcal{H}_D(X,C)$,*

(3) *f is homotopic in $\mathcal{H}_D(X,C)$ to a fixed-point free $f_* \in \mathcal{H}_D(X,C)$ by a homotopy, keeping $f_{|D}$ pointwise fixed.*

Proof.

(1) ⇒ (2) Let $g \in \mathcal{H}_D(X,C)$ be a fixed-point free mapping such that $f_{|D} = g_{|D}$. The compact homotopy

$$H(x,t) = tg(x) + (1-t)f(x).$$

joins f to g and is fixed-point free on D.

(2) ⇒ (3) Let $H : X \times [0, 1] \to C$ be a compact homotopy from g to f (i.e., $H(\cdot,0) = g$ and $H(\cdot,1) = f$), such that $H_{|D\times\{t\}}$ is fixed-point free for each $t \in [0, 1]$. We consider the set $D_0 = \{x : x = H(x,t) \text{ for some } t \in [0,1]\}$. There is no loss of generality in supposing that D_0 is non-empty. Then D_0 is a closed subset of the compact set $\overline{H(X \times [0,1])}$, so is compact and therefore closed in X. Since $D \cap D_0 = \phi$, because H is fixed-point free on D, then by Theorem 1.7.2 (Urysohn's Lemma), there is a continuous function $\Psi : X \to [0,1]$, with $\Psi(D) = 1$ and $\Psi(D_0) = 0$. We define a mapping f_* by $f_*(x) = H(x, \Psi(x))$. Obviously, f_* is compact, it is also fixed-point free. Because, if $f_*(x) = H(x,\Psi(x)) = x$ we have $x \in D_0$, we deduce that $\Psi(x) = 0$ and $x = H(x,0) = g(x)$, which contradicts the assumption that g is fixed-point free.

We consider the compact homotopy $H_*(x,t) = H(x, 1-t+t\Psi(x))$ to show that, f_* is homotopic to f keeping $f_{|D}$ pointwise fixed. Then we have

$$H_*(x,0) = H(x,1) = f(x) \text{ and } H_*(x,1) = H(x,\Psi(x)) = f_*(x).$$

Moreover, $\Psi(x) = 1$ for all $x \in D$, therefore $H_*(x,t) = H(x,1) = f(x)$, for all $t \in [0,1]$. For each $t \in [0, 1]$, $H_*(x, t)$ is obviously fixed-point free on D.

(3) ⇒ (1) The proof of this implication is elementary. □

An immediate consequence is the following important result, due to A. Granas.

THEOREM 3.3.3 [Topological transversality]. *Let (X, D) be a pair in a convex set $C \subset E$ and f, g two mappings in $\mathcal{H}_D(X,C)$ such that $f \simeq g$ in $\mathcal{H}_D(X,C)$. Then, f is essential if and only if g is essential.* □

We note that the concept of *topological transversality,* which is invariant under fixed-point free deformations on D is also invariant under small modifications of f on D. This fact is presented in the following result.

THEOREM 3.3.4. *Let (X, D) be a pair in a convex set $C \subset E$. If $f \in \mathcal{H}_D(X,C)$ is an essential mapping, then there exists an $\varepsilon > 0$ such that:*

(1) *any compact mapping $g : X \to C$ satisfying $\|g(x) - f(x)\| < \varepsilon$ for all $x \in D$, is in $\mathcal{H}_D(X,C)$ and*

(2) *g is essential.*

Proof. Because f is compact and fixed-point free on D, there is an $\varepsilon > 0$ such that $\|x - f(x)\| \geq \varepsilon$ for all $x \in D$. If $g : X \to C$ satisfies the inequality

$$\|g(x) - f(x)\| < \varepsilon \textit{ for all } x \in D,$$

then g is fixed-point free. Indeed, if for some $x_0, g(x_0) = x_0$ we have $\|x_0 - f(x_0)\| < \varepsilon$, which is impossible. By Proposition 3.3.1 we have $f \simeq g$ in $\mathcal{H}_D(X,C)$. Now, the theorem follows from Theorem 3.3.2. □

THEOREM 3.3.5. *Let $C \subset E$ be a convex set and U an open subset of C. Let $(\overline{U}, \partial U)$ be the pair consisting of the closure of U in C and the boundary of U in C. Then, for any $x_0 \in C$, the constant mapping $f(x) = x_0$, for any $x \in \overline{U}$ is essential in $\mathcal{H}_{\partial U}(\overline{U}, C)$.*

Proof. Indeed, the theorem is proved, if we show that any compact mapping $g : \overline{U} \to C$ with $g(\partial U) = x_0$ has a fixed-point. Let g_* be the extension of g to C defined by

$$g_*(x) = \begin{cases} g(x) & \text{if } x \in \overline{U}, \\ x_0 & \text{if } x \in C \setminus \overline{U}. \end{cases}$$

The mapping $g_* : C \to C$ is (continuous) compact and by *Schauder's fixed point theorem* [the general version given in (Dugundji, J, and Granas, A. [1] Theorem 3.2 pg. 57)] it has a fixed point x_*, which must be in U. Therefore, we have $g(x_*) = x_*$. □

Now, we apply the transversality theory to the study of equation $f(x) = x$, where f is compact and we obtain the Leray-Schauder nonlinear alternative.

THEOREM 3.3.6. *Let $C \subset E$ be a convex set, $U \subset C$ an open subset (in U) such that $0 \in U$. Then, each compact mapping $f : \overline{U} \to C$ has at least one of the following properties:*

(1) *f has a fixed point,*

(2) *there exist an element $x_* \in \partial U$ and a real number $\lambda_* \in]0, 1[$ such that $x_* = \lambda_* f(x_*)$.*

Proof. If property (1) is satisfied, there is nothing to prove. Therefore, we can assume that $f_{|\partial U}$ is fixed-point free. Let $g : \overline{U} \to C$ be the constant mapping $g(x) = 0$, for any $x \in \overline{U}$. We consider the compact homotopy $H : \overline{U} \times [0, 1] \to C$ defined by $H(x, t) = tf(x)$. The homotopy $H(\cdot,\cdot)$ joins g with f. Either this homotopy is fixed-point free on ∂U or it is not. If it is fixed-point free, then by Theorem 3.3.3 and Theorem 3.3.5 we find that f must have a fixed point. If the homotopy is not fixed-point free on ∂U, then there is an $x_* \in \partial U$ with $x_* = \lambda_* f(x_*)$ and $\lambda_* \in [0, 1]$. We observe that $\lambda_* \neq 0$, because $0 \notin \partial U$ and $\lambda_* \neq 1$, because $f_{|\partial U}$ has been assumed to be fixed-point free and the proof is complete. □

Remark. In Theorem 3.3.6, the closure and the boundary of U are with respect to C.

3.4. Some classes of mappings and Leray–Schauder type alternatives

We present in this section some *Leray–Schauder type alternatives* for mappings, which are not completely continuous fields.

Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space and $\mathbb{K} \subset H$ a closed pointed convex cone. For any real number $r > 0$ we denote $\mathbb{K}_r = \{x \in \mathbb{K} : \|x\| \le r\}$. We denote by α the Kuratowski measure of noncompactness, i.e., for any bounded set $B \subset H$,

$$\alpha(B) = \inf\left\{\begin{array}{l}\varepsilon > 0 : B = \\ \bigcup_{i=1}^{m} B_i \ \textit{for some } m \in \mathbb{N} \textit{ and some } B_i \textit{ with } diam(B_i) \le \varepsilon\end{array}\right\}.$$

(See Chapter 1). We recall that a mapping f: $\mathbb{K}_r \to H$ is *α-condensing* (see also Definition 1.5.4) if f is continuous, bounded and $\alpha(f(B)) < \alpha(B)$ for any $B \subset \mathbb{K}_r$, bounded and such that $\alpha(B) > 0$. It is known that any completely continuous mapping and any contraction are α-condensing. For any $r > 0$, we denote by P_r the *radial projection* of $\mathbb{K}$ onto $\mathbb{K}_r$, i.e., $P_r : \mathbb{K} \to \mathbb{K}_r$ and

$$P_r(x) = \begin{cases} x, & if \quad \|x\| \le r, \\ \dfrac{rx}{\|x\|}, & if \ \|x\| > r. \end{cases}$$

Since $\alpha(P_r(B)) \le \alpha(B)$ for any bounded set $B \subset \mathbb{K}$, we have that $f \circ P_r$ is α-condensing if f: $\mathbb{K}_r \to \mathbb{K}$ is α-condensing. Indeed, in this case we have

$$\alpha(f \circ P_r(B)) < \alpha(P_r(B)) \le \alpha(B), \text{ for any bounded set } B \subset \mathbb{K}.$$

Let $D \subset H$ be a closed convex set and f: $D \to H$ a mapping. The set

$I_D(x) = x + \{\lambda(y - x) : \lambda \ge 0, y \in D\} = \{(1-\lambda)x + \lambda y : \lambda \ge 0, y \in D\}$, is the *inward set* of $x \in D$ with respect to D. We say that the mapping f is *weakly inward* if $f(x) \in \overline{I_D(x)}$ for every $x \in D$. It is known that f is *weakly inward* if and only if

$$\lim_{t \to 0_+} \frac{\rho\left(x + t\left(f(x) - x\right), D\right)}{t} = 0, \textit{ for all } x \in D,$$

where ρ denotes the distance to D.

The following results are well known.

THEOREM 3.4.1. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $D \subset H$ a closed bounded convex set and $f : D \to H$ a mapping. If f is α-condensing and weakly inward, then f has a fixed point.*

Proof. This result is valid in an arbitrary Banach space and a proof is given in (Deimling, K., [1], Theorem 18.3). □

THEOREM 3.4.2. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $D \subset H$ a closed bounded convex set and $f : D \to H$ a continuous mapping. If the following assumptions are satisfied:*

(1) $\|f(x)\| \le c$ *for any* $x \in D$, *where c is a positive real number,*

(2) $\alpha(f(B)) \le k\alpha(B)$ *for some* $k > 0$ *and all subsets* $B \subset D$,

(3) $\lim_{t \to 0_+} \dfrac{\rho\left(x + tf(x), D\right)}{t} = 0$ *for all* $x \in \partial D$,

then the initial value problem $\begin{cases} u' = f(u), \\ u(0) = x \in D \end{cases}$ *has a solution on* $J = [0, a]$ *for each* $a > 0$.

Proof. For the proof of this result, the reader is referred to (Deimling, K., [1], Lemma 18.3). We note that this result is valid in any Banach space. □

The following fixed-point theorem is due to K. Deimling.

THEOREM 3.4.3. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $f : \mathbb{K}_r \to H$ an α-condensing mapping. If the following assumptions are satisfied:*

(1) *if* $x \in \partial\mathbb{K}$, $\|x\| \le r$, $x^* \in \mathbb{K}^*$ and $x^*(x) = 0$, *then* $x^*\left(f(x)\right) \ge 0$,

(2) $f(x) \ne \lambda x$ *for all* $\lambda > 1$ *and all* x *with* $\|x\| = r$,

then f has a fixed point (*in* $\mathbb{K}_r$).

Proof. We follow the ideas of the proof given in (Deimling, K., [2]). It is known [see (Deimling, K., [2])] that assumption (1) implies that $f \circ P_r$ is weakly inward on $\mathbb{K}$. We can show that there exists $\delta > 1$ such that $\|f \circ P_r(x)\| \le \delta - 1$ on $\mathbb{K}$. Hence it is enough to show that $f \circ P_r$ is weakly inward on $\mathbb{K}_\delta$, because applying Theorem 3.3.1 we obtain a fixed point in $\mathbb{K}_\delta$ for $f \circ P_r$, which must be in $\mathbb{K}_r$, a fact implied by assumption (2). Indeed, since $f \circ P_r$ is weakly inward on $\mathbb{K}$, we have

$$\rho\left(x + t\left(f \circ P_r(x) - x\right), \mathbb{K}\right) = o(t) \text{ as } t \to 0_+ \text{ for every } x \in \mathbb{K}_r.$$

(We recall that $\rho(y, C) = \inf\{\|y - z\| : z \in C\}$). Because $\alpha\left(\left(f \circ P_r - I\right)(B)\right) \le 2\alpha(B)$ for all bounded sets $B \subset \mathbb{K}$, the initial value problem

$$\begin{cases} u' = f \circ P_r(u) - u, \\ u(0) = x \in \mathbb{K} \end{cases}$$

has a solution u in $\mathbb{K}$ on $J = [0, 1]$. This fact is obtained by applying Theorem 3.4.2. This solution cannot leave $\mathbb{K}_\delta$, since we have

$$\langle f \circ P_r(x) - x, x\rangle = \langle f \circ P_r(x), x\rangle - \|x\|^2 \le (\delta - 1)\delta - \delta^2 \text{ for } \|x\| = \delta.$$

Hence $u(t) \in \mathbb{K}_\delta$ and $u(t) = x + t\left(f \circ P_r(x) - x\right) + o(t)$; *as* $t \to 0_+$ imply

$$\rho\left(x + t\left(f \circ P_r(x) - x\right), \mathbb{K}_\delta\right) = o(t); \text{ as } t \to 0_+$$

and therefore $f \circ P_r(x) \in \overline{I_{\mathbb{K}_\delta}(x)}$, that is $f \circ P_r$ is weakly inward on $\mathbb{K}_\delta$. □

Remark. Theorem 3.4.3 is valid in an arbitrary Banach space (Deimling, K. [2]).

From Theorem 3.4.3 we obtain the following alternative.

THEOREM 3.4.4 [Leray–Schauder type alternative]. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed pointed convex cone and* $h : H \to H$ *a mapping such that* $h(x) = x - T(x)$, *for all* $x \in H$, *where* $T : H \to H$ *is an* α-*condensing mapping. Then, for any* $r > 0$, *for the mapping* $f(x) = P_{\mathbb{K}}[x - h(x)]$ *at least one of the following two situations is satisfied:*

(1) *h has a fixed-point in* $\mathbb{K}_r$,

(2) *there exist* x_* *with* $\|x_*\| = r$ *and* $\lambda_* \in]0, 1[$ *such that* $x_* = \lambda_* f(x_*)$.

Proof. Since f is continuous, bounded and

$$\alpha\left(P_{\mathbb{K}}\left[T(B)\right]\right) \le \alpha\left(T(B)\right) < \alpha(B) \text{ for all } B \subset \mathbb{K}_r \text{ with } \alpha(B) > 0,$$

we deduce that f is α-condensing. The theorem is now a consequence of Theorem 3.4.3. □

Now, we consider mappings of the form $f(x) = x - T(x)$, where $T : H \to H$ is a nonexpansive mapping, i.e., for any $x_1, x_2 \in H$ we have

$$\|T(x_1) - T(x_2)\| \le \|x_1 - x_2\|.$$

We say that a mapping $g: H \to H$ is *semi-closed* if its graph is sequentially closed in the product of the weak topology on H with the norm topology on H. Because a Hilbert space is a uniformly convex Banach space, it is known that any mapping of the form $f(x) = x - T(x)$, with T nonexpansive is demi-closed (Penot, J. P., [1]). This means that if $\{x_n\}_{n\in n} \subset H$ is weakly convergent to an element $x_0 \in H$ and $\{x_n - T(x_n)\}_{n\in n}$ is convergent in norm to an element $y_0 \in H$ then $x_0 - T(x_0) = y_0$. We show that Theorem 3.4.3 is valid also for nonexpansive mappings.

THEOREM 3.4.5. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed pointed convex cone and* $f : \mathbb{K}_r \to H$ *a nonexpansive mapping. If the following assumptions are satisfied:*

(1) *if* $x \in \partial\mathbb{K}, \|x\| \le r, x^* \in \mathbb{K}^*$ *and* $x^*(x) = 0$, *then* $x^*(f(x)) \ge 0$,

(2) $f(x) \ne \lambda(x)$ *for all* $\lambda > 1$ *and all* x *with* $\|x\| = r$,

then f has a fixed point (in $\mathbb{K}_r$*).*

Proof. We consider a sequence of real numbers $\{\alpha_n\}_{n\in N} \subset]0,1[$ such that $\lim_{n\to\infty} \alpha_n = 1$ and for any $n \in N$ the mapping $f_n(x) = \alpha_n f(x)$. The mapping f_n is a contraction and consequently it is α-condensing. We can show that, for any $n \in N$, the mapping f_n satisfies the assumptions of *Theorem 3.4.3*. Therefore, for any $n \in N$, there exists an element $x_n \in \mathbb{K}_r$ such that $x_n = \alpha_n f(x_n)$. Because $\mathbb{K}_r$ is a bounded set and H is a reflexive space, we have that the sequence $\{x_n\}_{n\in N}$ has a subsequence $\{x_{n_k}\}_{k\in N}$, weakly convergent to an element $x_* \in \mathbb{K}_r$. We have also,

$$\left\|x_{n_k} - f(x_{n_k})\right\| = \left\|\alpha_{n_k} f(x_{n_k}) - f(x_{n_k})\right\| = \left|\alpha_{n_k} - 1\right| \left\|f(x_{n_k})\right\|, \text{ for any } k \in N.$$

Because f is nonexpansive it is bounded and consequently there exists a number $M > 0$ such that

$$\left\|x_{n_k} - f(x_{n_k})\right\| = \left|\alpha_{n_k} - 1\right| \left\|f(x_{n_k})\right\| \leq \left|\alpha_{n_k} - 1\right| M,$$

which implies that $\lim_{k\to\infty} \left\|x_{n_k} - f(x_{n_k})\right\| = 0$. Now, using the fact that the mapping $g(x) = x - f(x)$ is demi-closed we deduce that $f(x_*) = x_*$. □

Remark. From Theorem 3.4.5 we obtain some similar results proved in (Frigon, M., Granas, A and Guennoun, Z. E. A., [1]).

From Theorem 3.4.5 we obtain the following alternative.

THEOREM 3.4.6 [Leray–Schauder type alternative]. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $h : H \to H$ a mapping such that $h(x) = x - T(x)$, for any $x \in H$, where $T : H \to H$ is a nonexpansive mapping. Then for any $r > 0$ the mapping $f(x) = P_{\mathbb{K}}[x - h(x)]$ has at least one of the following two properties:*

(1) *f has a fixed point in $\mathbb{K}_r$,*

(2) *there exist x_* with $\|x_*\| = r$ and $\lambda_* \in]0, 1[$ such that $x_* = \lambda_* f(x_*)$.*

Proof. The mapping $f(x) = P_{\mathbb{K}}[x - h(x)] = P_{\mathbb{K}}[T(x)]$ is nonexpansive. If f has a fixed point in $\mathbb{K}_r$ the proof is complete. Suppose the f has no fixed point in $\mathbb{K}_r$. If property (2) is not satisfied, we have that assumptions (1) and (2) of Theorem 3.4.5 are satisfied which implies that f has a fixed-point in $\mathbb{K}_r$ and a contradiction follows. □

Remark. A variant of Theorem 3.4.5 was proved in 1974 in (Gatica, J. A. and Kirk, W. A. [1]).

Now, we consider mappings that are demi-continuous and pseudo-contractant. Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space. We recall that a mapping $f : H \to H$ is *monotone* if for any $x, y \in H$ we have $\langle f(x)-f(y),x-y\rangle \geq 0$. We say that a mapping $T : H \to H$ is *pseudo-contractant* if the mapping $f(x)=x-T(x)$ is monotone. We need to introduce some notations. If L is a linear operator defined on a vector subspace of H, we denote by $D(L)$ the maximal domain of definition of L. We have:

$L:D(L)\subset H \to H$.

$\mathcal{N}(L)=\{x\in H:L(x)=0\}=Ker(L)$ *the null space.*

$\mathcal{R}(L)=\{y\in H:L(x)=y \text{ for some } x\in H\}$. The set $\mathcal{R}(L)$ is *the range* of L.

We denote by $\mathcal{L}$ the class of linear operators $L : D(L) \subset H \to H$ such that $\mathcal{R}(L)=[\mathcal{N}(L)]^{\perp}$ (the orthogonal complement of $\mathcal{N}(L)$.). We suppose that, for each $L \in \mathcal{L}$, $D(L)$ is dense in H. Obviously we have that $I \in \mathcal{L}$ and $O \in \mathcal{L}$ (where I is the identity operator and O is the null operator). If $L \in \mathcal{L}$ we denote by P the orthogonal projection onto $\mathcal{N}(L)$, and by $\mathcal{K}$ the continuous inverse of the restriction of L with respect to $D(L)\cap\mathcal{R}(L)$. We denote by $\mathcal{F}$ the class of finite dimensional subspaces of $\mathcal{N}(L)$. For any $F\in \mathcal{F}$ we denote by P_F the orthogonal projection onto F and we put $H_F=\mathcal{R}(I-P+P_F)$. The following notation was defined in (Willem, M., [1]). If $\{x_n\}_{n\in N}$ is a sequence in H, we denote by $\{x_n\}_{n\in N}\xrightarrow{\|\cdot\|}x_*$ its convergence with respect to the norm given on H and by $\{x_n\}_{n\in N}\xrightarrow{(w)}x_*$ its convergence with respect to the weak topology, if this sequence is respectively convergent to an element $x_* \in H$.

DEFINITION 3.4.1. *We say that a mapping T: H → H is pseudomonotone with respect to a linear operator L ∈ $\mathcal{L}$ if the following conditions are satisfied:*

(i) *if*

$$P(x_n) \xrightarrow{(w)} x_*, (I-P)(x_n) \xrightarrow{\|\cdot\|} y_* \text{ and}$$
$$\limsup_{n\to\infty} \langle T(x_n), x_n - (x_* + y_*) \rangle \leq 0,$$
$$\text{then } \langle T(x_n), x_n - (x_* + y_*) \rangle \to 0 \text{ and } T(x_n) \xrightarrow{(w)} T(x_* + y_*).$$

(ii) $\mathcal{K}(I-P)T$ *is a completely continuous mapping,*

(iii) $(I-P)T$ *is a bounded mapping, i.e., if* $M \subset H$ *is a bounded set, then* $[(I-P)](M)$ *is bounded,*

(iv) T *is demicontinuous, i.e.,* $\{x_n\}_{n\in N} \xrightarrow{\|\cdot\|} x_*$ *implies* $\{T(x_n)\}_{n\in N} \xrightarrow{(w)} T(x_*)$,

(v) *for any* $F \in \mathcal{F}$ *the restriction of* $P_F T$ *with respect to* H_F *is a bounded mapping.*

Examples

(1) Any completely continuous mapping is pseudomonotone with respect to the identity operator *I*.

(2) *T*: *H*→ *H* is pseudomonotone with respect to the null operator *O*, if and only if *T* is pseudomonotone in Brezis's sense [See (Brezis, H., [2]) and demicontinuous.

(3) Any monotone and demicontinuous mapping is pseudomonotone with respect to the null operator *O*

(4) Other examples of pseudomonotone mappings with respect to a linear operator $I \in L$ are considered in (Willem, M., [1])

Remark. The pseudomonotonicity defined by Definition 3.4.1 is a generalization of the classical pseudomonotonicity defined in (Brezis, H., [2]).

The following result is due to M. Willem.

THEOREM 3.4.7. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space $T_0 : H \to H$ a mapping and $\Omega \subset H$ an open bounded set such that $0 \in \Omega$. If the following assumptions are satisfied:*

(1) *T_0 is pseudomonotone with respect to a linear operator $L \in \mathcal{L}$,*

(2) *for any $\mu \in]0, 1[$ and any $x \in D(L) \cap \partial\Omega$ we have* $L(x)+(1-\mu)P(x)+\mu T_0(x)\neq 0$,

then for any $\lambda \in]0, 1[$ there exists $x \in \Omega$ such that

$$L(x)+(1-\lambda)P(x)+\lambda T_0(x)=0.$$

Proof. A proof of this result is in (Willem, M., [1]). We note that the proof is long and it is based on several intermediate results. □

The following result is also due to M. Willem.

THEOREM 3.4.8. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space and $T : H \to H$ a demicontinuous pseudo-contractant mapping. Let $\Omega \subset H$ be an open bounded set such that $0 \in \Omega$. If for any $\lambda \in]0, 1[$ and any $x \in \partial\Omega$ we have $x \neq \lambda T(x)$, then T has at least a fixed point in $\overline{\Omega}$.*

Proof. We denote by $T_0 = I - T$. The mapping T_0 is monotone and demicontinuous. Because for any $\lambda \in]0, 1[$ and any $x \in \delta\Omega$ we have that $x \neq \lambda T(x)$ we deduce that, for any $\lambda \in]0, 1[$ and any $x \in \delta\Omega$ we have

$$(1-\lambda)x+\lambda T_0(x)\neq 0.$$

We have also that T_0 is pseudomonotone with respect to the null operator O. Let $\{\lambda_n\}_{n\in N}$ be a strictly increasing sequence in $]0, 1[$, convergent to 1. Applying Theorem 3.4.7 for any $n \in N$ we obtain for any $n \in N$ an element $x_n \in \Omega$ such that

$$(1-\lambda_n)x_n+\lambda_n T_0(x_n)=0.$$

If we denote $\alpha_n=\dfrac{1-\lambda_n}{\lambda_n}$, for any $n \in N$, then from the last relation we obtain

$$\alpha_n x_n+T_0(x_n)=0.$$

Using the fact that T_0 is monotone we deduce that for any $n, m \in N$ we have

$$\langle\alpha_m x_m-\alpha_n x_n, x_m-x_n\rangle=-\langle T_0(x_m)-T_0(x_n), x_m-x_n\rangle\leq 0. \qquad (3.4.1)$$

We observe that the sequence $\{\alpha_n\}_{n\in N}$ is decreasing. By an algebraic computation we can show that for any $m<n$ we have

$$(\alpha_m + \alpha_n)\|x_m - x_n\|^2 + (\alpha_m - \alpha_n)\left(\|x_m\|^2 - \|x_n\|^2\right) = 2\langle \alpha_m x_m - \alpha_n x_n, x_m - x_n \rangle. \tag{3.4.2}$$

From (3.4.2) we obtain that the sequence $\{\|x_n\|\}_{n\in N}$ is increasing. Because this sequence is bounded it is convergent. From (3.4.1) and (3.4.2) we deduce that for any *m, n* satisfying $m > n$ we have

$$\|x_m - x_n\|^2 \leq \frac{\alpha_m - \alpha_n}{\alpha_m + \alpha_n}\left(\|x_n\|^2 - \|x_m\|^2\right) \leq \|x_n\|^2 - \|x_m\|^2,$$

which implies that $\{x_n\}_{n\in N}$ is a Cauchy sequence. Therefore $\{x_n\}_{n\in N}$ is norm convergent to an element $x_* \in \overline{\Omega}$. Then we have $\{T_0(x_n)\}_{n\in N} \xrightarrow{(w)} T_0(x_*)$ and because $\{T_0(x_n)\}_{n\in N} \xrightarrow{\|\cdot\|} 0$, we have that $x_* = T(x_*)$ and the proof is complete. □

For applications to complementarity problems and to variational inequalities defined on unbounded closed convex sets, it is interesting to know under what conditions the mapping $I(x) = P_{\mathbb{K}}\left[x - f(x)\right]$ is pseudo-contractant where $\mathbb{K} \subset H$ is a closed convex cone or an unbounded closed convex set and $f : H \to H$ is a given mapping. In this sense we have the following result.

PROPOSITION 3.4.9. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed convex cone or a closed convex set and* $f : H \to H$, *a mapping. If* $f(x) = x - \varphi(x,)$, *where* $\varphi : H \to H$ *is non-expansive, then the mapping* $\Phi(x) = P_{\mathbb{K}}\left[x - f(x)\right]$ *is pseudo-contractant.*

Proof. Indeed, the mapping $\Psi(x) = x - \Phi(x,)$ is monotone, if for any $x_1, x_2 \in H$ we have

$$\begin{aligned}&\langle x_1 - x_2, \Psi(x_1) - \Psi(x_2)\rangle \\ &= \langle x_1 - x_2, x_1 - x_2\rangle - \langle x_1 - x_2, P_{\mathbb{K}}[x_1 - f(x_1)] - P_{\mathbb{K}}[x_2 - f(x_2)]\rangle \geq 0,\end{aligned}$$

which is equivalent to

$$\langle x_1 - x_2, P_{\mathbb{K}}[x_1 - f(x_1)] - P_{\mathbb{K}}[x_2 - f(x_2)]\rangle \leq \|x_1 - x_2\|^2. \tag{3.4.3}$$

From our assumptions we have,

$$\langle x_1 - x_2, P_{\mathbb{K}}[x_1 - f(x_1)] - P_{\mathbb{K}}[x_2 - f(x_2)]\rangle$$
$$\leq \|x_1 - x_2\| \|\varphi(x_1) - \varphi(x_2)\| \leq \|x_1 - x_2\|^2,$$

which implies that (3.4.3) is true and the proof is complete. □

From Theorem 3.4.8 we deduce the following result.

THEOREM 3.4.10 [Leray–Schauder type alternative]. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\Omega \subset H$ a bounded open set such that $0 \in \Omega$. If $T : H \to H$ is a demicontinuous pseudo-contractant mapping, then at least one of the following situations is true:*

(1) *T has a fixed point in $\overline{\Omega}$,*

(2) *There exist $\lambda_* \in]0,1[$ and $x_* \in \partial\Omega$ such that $x_* = \lambda_* T(x_*)$.*

Proof. The theorem is a direct consequence of Theorem 3.4.8. □

COROLLARY 3.4.11. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed convex cone or a closed convex set and $f : H \to H$ a continuous mapping. Let $\Omega \subset H$ be a bounded open set such that $0 \in \Omega$. If $f(x) = x - \varphi(x)$, where $\varphi : H \to H$ is non-expansive then for the mapping $\Phi(x) = P_{\mathbb{K}}[x - f(x)]$ at least one of the following situations is true:*

(1) *Φ has a fixed point in $\overline{\Omega}$,*

(2) *there exist $\lambda_* \in]0,1[$ and $x_* \in \partial\Omega$ such that $x_* = \lambda_* \Phi(x_*)$.*

Proof. The corollary is a consequence of Proposition 3.4.9 and of Theorem 3.4.10. □

Remark. We note that Theorems 3.5, 3.6, 3.8 and 3.9 proved in (O'Regan, D., [1]) are Leray–Schauder type alternatives and all are particular cases of Theorem 3.4.10.

3.5 An implicit Leray–Schauder alternative

We indicated that in this book we will develop a topological method based on Leray–Schauder type alternatives applicable to the study of solvability of complementarity problems. This method can be extended to the study of solvability of variational inequalities. The unification of both theories is realized by an implicit Leray–Schauder alternative. For this aim we present in this section an implicit Leray–Schauder alternative. The following result, which is valid in locally convex spaces, is the most general form of Schauder's fixed-point theorem, proposed by R. Cauty in 2001, as a solution to Schauder's conjecture. [See (Mauldin, R. D. [1])]

THEOREM 3.5.1. *Let $E(\tau)$ be a topological Hausdorff vector space, $C \subset E$ a convex subset and $f : C \to C$ a continuous mapping. If $f(C)$ is contained in a compact subset of C, then f has a fixed point.*

Proof. It seems that the proof of this result proposed in (Cauty, R. [1]) has a gap. T. Dobrowolski remarked this fact in the international conference "Fixed Point Theory and its Applications", August 01-05, 2005, Stefan Banach International Mathematical Center, Poland. □

Waiting a new proof, if such proof exists for Theorem 3.5.1, we present Theorems 3.5.2, 3.5.3 and 3.5.4 in a general Hausdorff topological vector space. We note that Theorems 3.5.2, 3.5.3 and 3.5.4 are valid in any locally convex topological vector spaces.

Let $E(\tau)$ be a Hausdorff topological vector space. We recall the following notions. If A and B are subsets of E, we say that *A absorbs B* if there exists $\lambda_0 \in \mathbb{R}$ such that $B \subset \lambda A$ whenever $|\lambda| \geq |\lambda_0|$. We say that a subset $U \subset E$ is *radial* if U absorbs every finite subset of E. We say that U is *circled* if $\lambda U \subset U$ whenever $|\lambda| \leq 1$. We denote by ∂U (respectively by $\text{int}U$) the boundary (respectively the interior) of U.

THEOREM 3.5.2 [Rothe type]. *Let $E(\tau)$ be a Hausdorff topological vector space, in particular a locally convex space and $B \subset E$ a closed convex subset such that the zero of E is contained in the interior of B. Let $h : B \to E$ be a continuous mapping with $h(B)$ relatively compact in E. If $h(\partial B) \subset B$, then there is a point $x_* \in B$ such that $h(x_*) = x_*$.*

Proof. First we recall that, because $E(\tau)$ is a Hausdorff topological vector space, we have that the topology τ possess a zero-neighborhood base $\mathcal{U}$ such that any $V \subset \mathcal{U}$ is *radial* and *circled* (Schaefer, H. H., [2], Theorem 1.2). Then, because $\text{int}\, B \subset B$, we have that B is *radial*. Let p_B the Minkowski functional of B, i. e., for any $x \in E$ $p_B(x) = \inf\{\lambda > 0 : x \in \lambda B\}$. The functional p_B is positive homogeneous. Indeed, first $p_B(0) = 0$. Let $x \in E$ be arbitrary and $\lambda > 0$. We have,

$$p_B(\lambda x) = \inf\{\mu > 0 : \lambda x \in \mu B\} = \inf\{\mu > 0 : x \in \lambda^{-1}\mu B\}$$
$$= \inf\{\lambda\mu_1 : x \in \mu_1 B\} = \lambda p_B(x).$$

Now, we show that p_B is continuous. The continuity of p_B is a consequence of the following facts. Let $\varepsilon > 0$ be an arbitrary real number. From (Schaefer, H. H., [2], Theorem 1.2) there exists a radial and circled zero-neighbourhood U such that $0 \in U \subset \operatorname{int} B \subset B$. Let P_U be the Minkowski functional of U. We have $p_B \leq p_U$. Because B is convex, p_B is subadditive and we can show that for any $x, y \in E$ we have,

$$p_B(x) - p_B(y) \leq p_B(x-y) \leq p_U(x-y)$$

and

$$p_B(y) - p_B(x) \leq p_B(y-x) \leq p_U(y-x).$$

Because U is circled $p_U(x-y) = p_U(y-x)$. If x, y are such that $x - y \in \varepsilon U$, then we have,

$$|p_B(x) - p_B(y)| \leq \varepsilon,$$

which implies the continuity of p_B.

Now, we consider the mapping $g: E \to E$ defined by

$$g(x) = \left[\max\{1, p_B(x)\}\right]^{-1} \cdot x, \textit{ for any } x \in E.$$

The mapping g is continuous and $g(E) \subset B$. We define the mapping $f: B \to B$ by $f = g \circ h$. The mapping f is continuous and $f(B)$ is relatively compact in E. By Theorem 3.5.1 there exists an element $x_* \in B$ such that $f(x_*) = x_*$. We have two possibilities:

(i) $x_* \in \operatorname{int} B$ or

(ii) $x_* \in \partial B$.

If (i) holds, then we have

$$1 > p_B(x_*) = p_B(f(x_*)) = \left[\max\{1, p_B(h(x_*))\}\right]^{-1} p_B(h(x_*)),$$

which implies that we must have $p_B(h(x_*)) < 1$ and consequently

$$f(x_*) = g(h(x_*)) = h(x_*).$$

Therefore $h(x_*) = x_*$. If (ii) holds, then we have

$$x_* = f(x_*) = \left[\max\left\{1, p_B\left(h(x_*)\right)\right\}\right]^{-1} \cdot h(x_*)$$

and

$$\left[\max\left\{1, p_B\left(h(x_*)\right)\right\}\right]^{-1} \cdot p_B\left(h(x_*)\right) = 1.$$

If $p_B\left(h(x_*)\right) < 1$ then $1 = p_B\left(h(x_*)\right) < 1$ which is a contradiction. Thus we must have $p_B\left(h(x_*)\right) = 1$ (since $h(\partial B) \subset B$). But $p_B\left(h(x_*)\right) = 1$ implies $f(x_*) = h(x_*)$, that is we have again $h(x_*) = x_*$ and the proof is complete.

□

Remark. Our Theorem 3.5.2 is a generalization to an arbitrary Hausdorff topological vector space of a similar result proved in 1972 in (Potter, A. J. B., [1]) in a locally convex topological vector space. The next result is an implicit Leray–Schauder type theorem in an arbitrary topological vector space with respect to a closed convex set B with $0 \in \operatorname{int} B \subset B$.

THEOREM 3.5.3 [Leray–Schauder (implicit form)]. *Let $E(\tau)$ be a Hausdorff topological vector space, in particular a locally convex space, $B \subset E$ a closed convex set such that $0 \in \operatorname{int}B$. Let $f : [0, 1] \times B \to E$ be a continuous mapping ($[0, 1] \times B$ is endowed with the product topology). The set $f\left([0,1] \times B\right)$ is supposed to be relatively compact in E. If the following assumptions are satisfied:*

(1) $f(t,x) \neq x$ *for all* $x \in \partial B$ *and* $t \in [0,1]$,

(2) $f\left(\{0\} \times \partial B\right) \subset B$,

then there is an element $x_* \in B$ such that $f(1, x_*) = x_*$.

Proof. For any $n \in N$ we consider the mapping $f_n : B \to E$ defined by

$$f_n(x) = \begin{cases} f\left(\dfrac{1 - p_B(x)}{\varepsilon_n}, \dfrac{x}{p_B(x)}\right), & \text{if } 1 - \varepsilon_n \le p_B(x) \le 1, \\ f\left(1, \dfrac{x}{1 - \varepsilon_n}\right), & \text{if } p_B(x) < 1 - \varepsilon_n, \end{cases}$$

where p_B is the Minkowski functional of B and $\{\varepsilon_n\}_{n \in N}$ is a sequence of real numbers such that $\lim_{n \to \infty} \varepsilon_n = 0$ and $0 < \varepsilon_n < \frac{1}{2}$ for any $n \in N$. We observe that for each $n \in N$ the mapping f_n is continuous on B and $f_n(B)$ is relatively compact in E. From assumption (2) we have that $f_n(\partial B) \subset B$, and the assumptions of Theorem 3.5.2 are satisfied. Then for each $n \in N$ there exists an element $u_n \in B$ such $f_n(u_n) = u_n$.

We suppose that an infinite number of the elements u_n satisfy

$$1 - \varepsilon_n \le p_B(u_n) \le 1 . \qquad (3.5.1)$$

Because $f_n(B)$ is relatively compact and considering the definition of mappings f_n, we have that $\{u_n\}_{n \in N}$ is contained in a compact set in E. It is known that any compact set is countably compact [(Gemignani, M. C. [1], page 179)] and every countable infinite subset of a countable compact set has at least an accumulation point [(Gemignani, M. C. [1], page 179)]. We consider the sequence $\{t_n\}_{n \in N}$ defined by

$$t_n = \frac{1 - p_B(u_n)}{\varepsilon_n}, \text{ for any } n \in N.$$

We have that $\{t_n\}_{n \in N} \subset [0, 1]$. Considering eventually a subsequence, we suppose that $\lim_{n\to\infty} t_n = t \in [0,1]$. The corresponding subsequence of $\{u_n\}_{n \in N}$ is denoted again by $\{u_n\}$ and it satisfies the inequalities (3.5.1). From (3.5.1) we have that $\lim_{n\to\infty} p_B(u_n) = 1$.

Let u_* be an accumulation point of $\{u_n\}_{n \in N}$. We know that $\{u_n\}_{n \in N}$ has a net converging to u_*. Using this fact we can show that (t, u_*, u_*) is an accumulation point of the sequence $\left\{\left(t_n, \frac{u_n}{p_B(u_n)}, u_n\right)\right\}_{n \in N}$ in $[0, 1] \times E \times E$. Considering the net convergent to u_*, the continuity of f and the equations $f\left(t_n, \frac{u_n}{p_B(u_n)}\right) = u_n$ for any $n \in N$, we obtain $f(t, u_*) = u_*$. This fact is a contradiction of assumption (1). Indeed, $p_B(u_*) = 1$ (since $\lim_{i \in I} p_B(u_i) = 1, \{u_i\}_{i \in I}$ being a net of $\{u_n\}_{n \in N}$ convergent to u_*), and $u_* \in \partial B$. Therefore, it is impossible to have satisfied (3.5.1). Then (3.5.1) can be satisfied only for a finite number of elements of the sequence $\{u_n\}_{n \in N}$. Hence, we can suppose that

$$p_B(u_n) < 1 - \varepsilon_n, \text{ for all } n \in N. \qquad (3.5.2)$$

Since $\lim_{n\to\infty}(1-\varepsilon_n)=1$, selecting an accumulation point u_* for $\{u_n\}_{n\in N}$ and using a net of $\{u_n\}_{n\in N}$ convergent to u_*, we obtain by continuity and considering the equations $f\left(1,\frac{u_n}{1-\varepsilon_n}\right)=u_n, n\in N$ that $f(1,u_*)=u_*$. By this conclusion the proof is complete. □

Remark. Theorem 3.5.3 is a generalization to an arbitrary Hausdorff topological vector space of a similar result proved in (Potter, A. J. B. [1]). We note that the proof given in (Potter, A. J. B. [1]) has some inaccuracies. (The notion of accumulation point is badly used.)

THEOREM 3.5.4 [Implicit Leray–Schauder type alternative]. *Let $E(\tau)$ be a Hausdorff topological vector space, in particular a locally convex space, $B\subset E$ a closed convex set such that $0\in \text{int}B$. Let $f\colon [0,1]\times B\to E$ be a continuous mapping such that $f([0,1]\times B)$ is relatively compact in E. We consider an $[0,1]\times B$ the product topology. If the following assumptions are satisfied:*

(1) $f(\{0\}\times\partial B)\subset B$,

(2) $f(0,x)\neq x$, *for any* $x\in\partial B$,

then at least one of the following properties is satisfied:

(i) *there exists $x_*\in B$ such that $f(1,x_*)=x_*$,*

(ii) *there exists $(\lambda_*,x_*)\in\,]0,1[\,\times\partial B$ such that $f(\lambda_*,x_*)=x_*$.*

Proof. The theorem is an immediate consequence of Theorem 3.5.3. □

3.6 Leray–Schauder type alternatives for set-valued mappings

In many applications of complementarity problems or of variational inequalities, we need to replace a single-valued mapping by a set-valued mapping. The reason is the fact that, in many practical problems, the mappings used in mathematical modelling are not single-valued and the interest is to study, for example complementarity problems or variational inequalities with set-valued mappings. It is clear that the set-valued mappings are also related to the presence of perturbations, to approximate definitions of values of functions or to uncertainty. While many results have been obtained for complementarity problems defined by single-valued mappings, few results have been published for complementarity problems defined by set-valued mappings. In this book we will show that the Leray–Schauder type alternatives can be applied to the study of complementarity problems defined by set-valued mappings. To do this, we need to give a Leray–Schauder type alternative for set-valued mappings. This is the aim of this section in which all topological vector spaces are assumed to be real Hausdorff spaces.

Given a set X, we denote by $\mathcal{P}(X)$ the family of all non-empty subsets of X. Let E and F be topological vector spaces, and $X \subset E$, $Y \subset F$ non-empty subsets. We recall that *the boundary, the interior and the convex hull* of the subset X are denoted by ∂X, int(X) and $conv(X)$. Let $f : X \to Y$ be a set-valued mapping (i.e., $f : X \to \mathcal{P}(Y)$).

DEFINITION 3.6.1. *We say that the set-valued mapping f is upper semicontinuous (u.s.c.) on X if the set $\{x \in X : f(x) \subset V\}$ is open in X, whenever V is an open subset in Y.*

The following result is well known: *if $f : X \to Y$ is (u.s.c.) and $f(x)$ is compact for every $x \in X$, then, if $D \subset X$ is compact we have that $f(D) = \bigcup_{x \in D} f(x)$ is compact* (Berge, C., [1]). From this result we deduce that if f is a set-valued mapping with $f(x)$ compact for any $x \in X$ and there exists a compact set D such that $f(D)$ is not compact, then f is not (u.s.c.). Obviously, if f is single-valued then the upper semicontinuity is the classical continuity. As for single-valued mappings, we say that a set-valued mapping $f : X \to Y$, is *compact* if $f(X)$ is relatively compact in Y. We recall that a single-valued mapping $\varphi : X \to Y$, is a *selection* of a set-valued mapping $f : X \to Y$, if for any $x \in X$, $\varphi(x) \in f(x)$.

We recall that a Hausdorff topological space Ω is a *completely regular* space, if for each closed subset $A \subset \Omega$ and each $x_0 \in \Omega \setminus A$ there exists a continuous function $\psi : \Omega \to [0,1]$ such that $\psi(x_0) = 1$ and $\psi(x) = 0$ for any $x \in A$. It is known [see (Schaefer, H. H. [2]), page 16] that *any Hausdorff topological vector space is completely regular.* A set D in a topological space Ω is called a *neighbourhood retract* if and only if, D is a *retract* of some of its neighbourhood U, ($D \subset U \subset \Omega$). This is equivalent to being able *to extend the identity mapping* $i : D \to D$ (i.e., $i(x) = x$ for any $x \in D$) to a continuous mapping r. The mapping r is called a *retract.* It is known that *every closed convex set in a Banach space E is a neighbourhood retract.* By an *ANR* we mean a compact metric space Ω with the universal property that every homeomorphic image of Ω in a separable metric space is a neighbourhood retract. The prototype for *ANR* spaces, are compact convex sets in Banach spaces.

Let $(\Omega, \leq)$ be an ordered set. We suppose that Ω is a lattice with a minimal element denoted by 0. We recall that a function $\Phi : \mathcal{P}(E) \to \Omega$ is a *measure of noncompactness* (see Chapter 1) if the following conditions hold for any $X_1, X_2 \in \mathcal{P}(E)$:

(1) $\Phi\left(\overline{conv}(X_1)\right) = \Phi(X_1)$,

(2) $\Phi(X_1) = 0$ *if and only if* X_1 *is precompact,*

(3) $\Phi(X_1 \cup X_2) = \max\{\Phi(X_1), \Phi(X_2)\}$.

We recall that a subset D of a Hausdorff topological vector space is *precompact* if and only if the closure of D in the completion $\widetilde{E}$ of E is compact. *A set-valued mapping* $f : X \to Y$ *is said to be* Φ*-condensing if for any subset D of X with* $\Phi(f(D)) \geq \Phi(D)$ *we have that D is relatively compact.*

There exist Φ-condensing mappings $f : X \to E$ only if for the subsets of X precompactness coincides with relative compactness. A compact set-valued mapping $f : X \to E$ is Φ-condensing if either the domain X is complete, or if E is quasicomplete. (We recall that a topological vector space E is *quasicomplete* if every bounded closed subset of E is complete.) Obviously, every mapping defined on a compact set is necessarily Φ-condensing.

Let $E(\tau)$ be a locally convex topological vector space. We suppose that E has a fundamental system $\mathcal{U}(0)$ of convex symmetric neighborhoods of the origin. The following notions are fundamental for the main result of this section.

Let X and Y be non-empty subsets of E.

DEFINITION 3.6.2. *If $f : X \to Y$ is a set-valued mapping and if $U, V \in \mathcal{U}(0)$, then in this case we say that a function $\varphi : X \to Y$ is a (U, V)-approximative selection of f if for any $x \in X$,*

$$\varphi(x) \in \left(f\left[(x+U)\cap X\right]+V\right)\cap Y.$$

This notion was introduced in (Ben-El-Mechaiekh, H. Chebbi, S and Florenzano, M., [1]), but the reader is also referred to (Ben-El-Mechaiekh, H. and Deguire, P., [1]), (Ben-El-Mechaiekh, H. and Idzik, A., [1]) and for the metric space to (Gorniewicz, L., Granas, A. and Kryszewski, W., [1]).

DEFINITION 3.6.3. *We say that a set-valued mapping $f : X \to Y$ is approachable if it has a continuous (U, V)-approximative selection for any $(U, V) \in \mathcal{U}(0) \times \mathcal{U}(0)$.*

We denote by $\mathcal{A}(X, Y)$ the class of approachable set-valued mappings from X into Y. When $X = Y$, we write $\mathcal{A}(X)$ for $\mathcal{A}(X, X)$.

DEFINITION 3.6.4. *We say that a set-valued mapping $f: X \to Y$ is approximable if its restriction $f_{|\mathbb{K}}$ to any compact subset $\mathbb{K}$ of X is approachable.*

It is known (Ben-El-Mechaiekh, H., [1]) that an approachable set-valued mapping is approximable. For examples of approachable and approximable mappings the reader is referred to (Ben-El-Mechaiekh, H. and Deguire, P., [1]), (Ben-El-Mechaiekh, H. and Idzik, A., [1]), (Ben-El-Mechaiekh, H. Chebbi, S and Florenzano, M., [1]), (Ben-El-Mechaiekh, H. and Isac, G., [1]), (Cellina, A., [1]) and (Gorniewicz, L., Granas, A. and Kryszewski, W., [1])

For the main result of this section we need to indicate only the following examples.

We suppose that $f: X \to \mathcal{P}(Y)$ is an (u.s.c.) mapping

(i) **Convex case**. If X is a topological space, Y is a convex subset in a locally convex space F and if the values of f are convex, then f is approximable. (Cellina, A., [1]), (Ben-El-Mechaiekh, H. and Idzik, A., [1]).

(ii) **Nonconvex case.** If X is contained in a topological vector space E and Y is contained in a locally convex space F, then f admits a (U, V)-approximative continuous selection for any open neighborhoods of the origin U and V in E and F respectively if the following condition is satisfied: *X is a compact ANR, Y is an ANR and the values of f are compact and contractible.* (Ben-El-Mechaiekh, H. and Deguire, P., [1]). Obviously, in this case f is approximable.

Now, to give the Leray–Schauder type alternative for set-valued mappings we follow the steps and the ideas used in (Ben-El-Mechaiekh, H. Chebbi, S and Florenzano, M., [1]). First, we will prove some useful results.

PROPOSITION 3.6.1. *Let $f: X \to E$ be a compact approximable mapping. For any $Y \in \mathcal{U}(0)$ there exists a finite subset D_V of $\overline{f(X)}$ and $f_V : X \to conv(D_V)$ such that $f_V(x) \subset f(x) + V$, for any $x \in X$. Moreover, f_V is u.s.c. with non-empty closed values whenever f has the same properties. If $f \in \mathcal{A}(X, E)$ takes its values in a convex compact subset $\mathbb{K}$ of E, then $f \in \mathcal{A}(X, \mathbb{K})$.*

Proof. Let V be an arbitrary neighborhood in $\mathcal{U}(0)$ and let $D_V = \{u_1, u_2, \ldots, u_n\}$ be a finite subset in $\overline{f(X)}$ with the property that the sets $\left\{u_i + \frac{1}{6}V : i = 1, 2, ..n\right\}$ form an open cover of the compact set $\overline{f(X)}$.

For any $i = 1, 2, \ldots, n$ and any $u \in \bigcup_{i=1}^{n}\left(u_i + \frac{1}{3}V\right)$ we define

$$\mu_i(u) = \max\left\{0, 1 - p_{\frac{1}{3}V}(u - u_i)\right\},$$

where $p_{\frac{1}{3}V}$ is the Minkowski functional of $\frac{1}{3}V$. We define the Schauder projection.

$$\Pi_V(u) = \frac{1}{\sum_{i=1}^{n}\mu_i(u)}\sum_{i=1}^{n}\mu_i(u)u_i, \text{ for all } u \in \bigcup_{i=1}^{n}\left(u_i + \frac{1}{3}V\right).$$

We have that $\Pi_V : \bigcup_{i=1}^{n}\left(u_i + \frac{1}{3}V\right) \to conv(D_V)$. We can show that

$$\Pi_V(u) - u \in \frac{1}{3}V, \textit{ for all } u \in \bigcup_{i=1}^{n}\left(u_i + \frac{1}{3}V\right).$$

Now, we define the mapping $f_V : X \to conv(D_V)$, as the composition product $f_V = \Pi_V \circ f$. If f is approximable, then f_V is approximable because its restriction to any compact subset of X is approachable as the composition product of two approachable mappings (Ben-El-Mechaiekh, H., [1], Proposition 2.5). We have also that, $f_V(x) \subset f(x) + V$ for all $x \in X$, f_V is u.s.c. and for any $x \in X$, $f_V(x)$ is compact, whenever f is u.s.c. and, for any $x \in X$, $f(x)$ is compact.

Now, we suppose that $f \in \mathcal{A}(X, E)$ and for any $x \in X$, $f(x) \subset \mathbb{K}$, where $\mathbb{K} \subset E$ is a compact convex set. In this case, for a given $U \in \mathcal{U}(0)$ let $s : X \to E$ be a continuous $\left(U, \frac{1}{6}V\right)$-approximative selection of f. Then for all $x \in X$ we have

$$s(x) \in f((x+U) \cap X) + \frac{1}{6}V \subset \bigcup_{i=1}^{n}\left(u_i + \frac{1}{6}V\right) + \frac{1}{6}V = \bigcup_{i=1}^{n}\left(u_i + \frac{1}{3}V\right),$$

$$\Pi_V(s(x)) \in s(x) + \frac{1}{3}V \subset f((x+U) \cap X) + \frac{1}{6}V + \frac{1}{3}V$$
$$\subset f((x+U) \cap X) + V.$$

Therefore, $\Pi_V \circ s$ is a continuous (U, V)-approximative selection of f with values in $conv(D_V) \subset \mathbb{K}$. □

PROPOSITION 3.6.2. *Let $X \subset E$ be a convex compact subset and $f \in \mathcal{A}(X)$. If f is u.s.c. with non-empty closed values, then f has a fixed-point, i.e., there exists a point $x_0 \in X$ such that, $x_0 \in f(x_0)$.*

Proof. This proposition is a generalization of the classical Fan–Kakutani fixed-point theorem, and a proof is in (Ben-El-Mechaiekh, H. and Deguire, P., [1]). □

The following useful result is related to a similar result used in (Petryshyn, W. V. and Fitzpatrick, P. M. [1]). We follow the proof given in (Ben-El-Mechaiekh, H. Chebbi, S and Florenzano, M., [1]).

PROPOSITION 3.6.3. *If X is a non-empty subset of E and $f : X \to E$ is a Φ-condensing mapping, then there exists a non-empty compact and convex subset $\mathbb{K} \subset E$ such that $f(\mathbb{K} \cap X) \subset \mathbb{K}$.*

Proof. Indeed, let $x_0 \in X$ be a fixed element. We consider the family $\mathcal{F}$ of all closed convex subsets D of E such that $x_0 \in D$ and $f(D \cap X) \subset D$. Obviously $\mathcal{F}$ is non-empty, since $\overline{conv}(f(X) \cup \{x_0\}) \in \mathcal{F}$. We denote $\mathbb{K} = \bigcap_{D \in \mathcal{F}} D$. We have that $\mathbb{K}$ is closed convex and $x_0 \in \mathbb{K}$. If $x \in \mathbb{K} \cap X$, then $f(x) \subset D$ for all $D \in \mathcal{F}$ and hence $f(\mathbb{K} \cap X) \subset \mathbb{K}$. Therefore we have that $\mathbb{K} \in \mathcal{F}$. The proof will be complete if we prove that $\mathbb{K}$ is compact. If $\mathbb{K}$ is not compact, then $\Phi(\mathbb{K}) \not\leq \Phi(f(\mathbb{K}))$, since f is Φ-condensing. Denoting by $\mathbb{K}_* = \overline{conv}(\{x_0\} \cup f(\mathbb{K} \cap X))$, we have that $\mathbb{K}_* \subset \mathbb{K}$, which implies that $f(\mathbb{K}_* \cap X) \subset f(\mathbb{K} \cap X) \subset \mathbb{K}_*$. Therefore $\mathbb{K}_* \in \mathcal{F}$ and $\mathbb{K} \subset \mathbb{K}_*$. Because $\mathbb{K} = \mathbb{K}_*$,

$$\Phi(\mathbb{K}) = \Phi(\mathbb{K}_*) = \Phi(f(\mathbb{K} \cap X)) \leq \Phi(f(\mathbb{K}))$$

and we have a contradiction. □

A main result is the following theorem.

Theorem 3.6.4 [Leray–Schauder set-valued alternative]. *Let X be a closed subset of E such that* $0 \in \text{int}(X)$.Let f: *$X \to E$ be a Φ-condensing or a compact u.s.c. set-valued mapping. If f is approximable with non-empty closed values, then one of the following properties holds:*

(1) *there exists an element $x_0 \in X$ such that $x_0 \in f(x_0)$,*
(2) *there exist $x_* \in \partial X$ and $\lambda_* \in]0, 1[$ such that $x_* \in \lambda_* f(x_*)$.*

Proof. First, we suppose that f is Φ-condensing. In this case, we suppose that for each $x \in X$, with $x \notin f(x)$ and for each $(\lambda, x) \in]0,1[\times \partial X$, $x \notin \lambda f(x)$. Applying, Proposition 3.6.3 we deduce the existence of a non-empty convex and compact subset $\mathbb{K}$ of E such that $f(\mathbb{K} \cap X) \subset \mathbb{K}$. We can suppose that $0 \in \mathbb{K}$. Since $\mathbb{K} \cap X$ is compact, $f_{|K \cap X} \in \mathcal{A}(\mathbb{K} \cap X, E)$ and by Proposition 3.6.1, $f_{|K \cap X} \in \mathcal{A}(\mathbb{K} \cap X, \mathbb{K})$.

We consider the set-valued mapping $f_1 : \mathbb{K} \to \mathbb{K}$ defined by

$$f_1(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{K} \text{ and } x \in \operatorname{int}(X), \\ \mathbb{K} & \text{if } x \in \mathbb{K} \text{ and } x \notin \operatorname{int}(X) \end{cases}$$

and we can show that f_1 is *u.s.c.*, with non-empty closed values. We have that $f_1 \in \mathcal{A}(\mathbb{K})$. Indeed, let $(U, V) \in \mathcal{U}(0) \times \mathcal{U}(0)$ be arbitrary and $s : \mathbb{K} \cap X \to \mathbb{K}$ be a continuous $\left(U, \frac{1}{2}V\right)$-approximative selection of $f_{|\mathbb{K} \cap X}$. By [(Ben-El-Mechaiekh, H. and Deguire, P., [1]), Proposition 1.6] there exists a continuous function $s_1 : \mathbb{K} \to \mathbb{K}$ such that s and $s_{1|\mathbb{K} \cap X}$ are $\frac{1}{2}V$-near (i.e., for any $x \in \mathbb{K} \cap X, s_1(x) - s(x) \in \frac{1}{2}V$). Therefore s_1 is a (U, V)-approximative selection of f_1.

Now, we consider the set

$$D = \left\{x \in X \cap \mathbb{K} \,\middle|\, x \in \lambda f(x) \text{ for some } \lambda \in [0,1]\right\}.$$

Because $0 \in D$, we have that D is non-empty and D is closed since f is *u.s.c.* and $f(X \cap \mathbb{K}) \subset \mathbb{K}$, hence compact. Because E is a Hausdorff locally convex space, we have that E is completely regular [(Schaefer, H.H., [2]), page 16]. Since $D \cap (E \setminus \operatorname{int}(X)) = \phi$, there is a continuous function $\varphi : E \to [0,1]$, such that $\varphi(x) = 1$ for $x \in D$ and $\varphi(x) = 0$ for $x \in E \setminus \operatorname{int}(X)$.

Let $g : \mathbb{K} \to \mathbb{K}$ be the mapping defined by:

$$g(x) = \varphi(x) f_1(x).$$

The mapping g is *u.s.c.* with non-empty closed values and by [(Ben-El-Mechaiekh, H., [1]), Proposition 2.4 and Proposition 2.5] we have that $g \in \mathcal{A}(\mathbb{K})$. By Proposition 3.6.2, g has a fixed-point $x_0 \in \mathbb{K}$, i.e., $x_0 \in \varphi(x_0) f_1(x_0)$. If $x_0 \notin \operatorname{int}(X)$, $\varphi(x_0) = 0$ and $x_0 = 0$, which contradicts the hypothesis $0 \in \operatorname{int}(X)$. If $x_0 \in \operatorname{int}(X)$, $x_0 \in \varphi(x_0) f(x_0)$, hence $x_0 \in D$, $\varphi(x_0) = 1$ and x_0 is a fixed-point of f, another contradiction.

Now, we consider the case when f is compact. In this case, let $V \in \mathcal{U}(0)$ be arbitrary but fixed. Let D_V be the finite subset of $\overline{f(X)}$ and the approximable mapping $f_V : X \to conv(D_V)$, verifying $f_V(x) \subset f(x) + V$ for all $x \in X$, both provided by Proposition 3.6.1. Without loss of generality, we can assume that $0 \in conv(D_V)$ (otherwise, we replace $conv(D_V)$ by $conv(\{0\} \cup D_V)$. We note that

$$f_{V|X \cap conv(D_V)} \in \mathcal{A}(X \cap conv(D_V), conv(D_V))$$

The same proof as in the first case, but replacing $f_{|X \cap \mathbb{K}}$ by $f_{V|X \cap conv(D_V)}$, gives us the following alternative:

(i) *there exists* $x_V \in X$, *with* $x_V \in f_V(x_V)$; *or*

(ii) *there exists* $(x_V, \lambda_V) \in \partial X \times]0,1[$, *with* $x_V \in \lambda_V f_V(x_V)$.

Using the compactness of f, its upper semicontinuity and the closedness of its values we conclude that the proof is complete. [For more details the reader is referred to (Ben-El-Mechaiekh, H. and Idzik, A. [1]). □

Remark. In the proof of Theorem 3.6.4 we followed the ideas used in (Ben-El-Mechaiekh, H., Chebbi, S. and Florenzano, M., [1])

A variant of Theorem 3.6.4 is the following result due to S. Park. We recall that a set-valued mapping $f : X \to Y$ is said to be *closed* if it has a closed graph $Gr(f) \subset X \times Y$, where

$$Gr(f) = \{(x, y) \in X \times Y : x \in X, y \in Y \text{ and } y \in f(x)\}$$

It is known that if f is *u.s.c.*, then it is closed (Berge, C. [1]).

THEOREM 3.6.5 [Leray–Schauder set-valued alternative]. *Let X be a closed subset of E such that $0 \in$ int(X). If $f : E \to E$ is a closed compact approximable set-valued mapping, then either:*

(1) *h has a fixed-point, or*

(2) *there exist $x_* \in \partial X$ and $\lambda_* > 1$ such that $\lambda_* x_* \in f(x_*)$.*

Proof. A proof of this theorem is given in (Park, S. [1]) and it is similar to the second part of the proof of Theorem 3.6.4. □

For the results presented in this book the following variant of Theorem 3.6.4. is useful.

THEOREM 3.6.6. *Let X be a closed subset of a Banach space $(E,\|\cdot\|)$, such that $0 \in \text{int}(X)$. Let $f : X \to E$ be a compact u.s.c. set-valued mapping with non-empty compact contractible values. If f is fixed-point free, then there exists $(\lambda_*, x_*) \in]0,1[\times \partial X$ such that $x_* \in \lambda_* f(x_*)$.*

Proof. This result is a corollary of Theorem 3.6.4. □

Remark. We note that Theorem 3.6.6 follows immediately from Corollary 3.3 proved in (Gorniewicz, L. and Slosarski, M., [1]), using the notion of *essential set-valued mapping.*

Now, we present another notion. First, we need to introduce some notation and definitions. Let X be a topological space. We denote by H the Čech homology functor with compact carriers and rational coefficients Q (See (Gorniewicz, L. and Slosarski, M., [1]). We say that X is *acyclic* if

$$H_n(X) = \begin{cases} 0 \text{ if } n > 0, \\ Q \text{ if } n = 0. \end{cases}$$

It is known that a *contractible space is acyclic.* Let Z be another topological space. We say that a continuous mapping $p : Z \to X$ is a *Vietoris mapping if the following conditions are satisfied*:

(1) *the set $p^{-1}(x)$ is acyclic, for each $x \in X$,*

(2) *p is proper, i.e., $p^{-1}(\mathbb{K})$ is compact for any compact set $\mathbb{K} \subset X$.*

Let X and Y be subsets of a Banach space $(E,\|\cdot\|)$. A set-valued mapping $\psi : X \to Y$ *is called admissible if there exists a topological space Z and two continuous mappings $p : Z \to X$ and $q: Z \to Y$ such that the following conditions are satisfied:*

(1) *p is a Vietoris mapping,*

(2) *$\psi(x) = q(p^{-1}(x))$ for any $x \in X$.*

It is known that all *u.s.c.* set-valued mappings with acyclic compact values and all compositions of such set-valued mappings are admissible.

Now, we suppose that $(E,\|\cdot\|)$ and $(F,\|\cdot\|)$ are Banach spaces. Let $U \subset E$ be an open bounded subset. We introduce the following notation:

$\mathcal{A}_{\partial U}(U,F) = \{\psi : \overline{U} \to F; \psi \text{ is admissible and } 0 \notin \psi(\partial U)\}$,

$\mathcal{A}_c(U,F) = \{\psi : \overline{U} \to F; \psi \text{ is admissible and compact}\}$,

$\mathcal{A}_0(U,F) = \{\psi : \overline{U} \to F;\ \psi \in \mathcal{A}_c(U,F) \text{ and } \psi(x) = \{0\} \text{ for all } x \in \partial U\}$.

DEFINITION 3.6.5. *We say that a set-valued mapping* $f \in \mathcal{A}_{\partial U}(U,F)$ *is essential if for every* $\psi \in \mathcal{A}_0(U,F)$ *there exists a point* $x \in U$ *such that* $f(x) \cap \psi(x) \neq \phi$.

We enumerate several properties of the notion of essential set-valued mappings.

Property 3.6.5 [Existence]. *If* $f \in \mathcal{A}_{\partial U}(U,F)$ *is an essential mapping, then there exists a point* $x \in U$ *such that* $0 \in f(x)$.

Property 3.6.6 [Compact Perturbation]. *If* $f \in \mathcal{A}_{\partial U}(U,F)$ *is essential and* $g \in \mathcal{A}_0(U,F)$, *then* $(f+g) \in \mathcal{A}_{\partial U}(U,F)$ *is an essential set-valued mapping.*

Property 3.6.7 [Coincidence]. *If* $f \in \mathcal{A}_{\partial U}(U,F)$ *is essential,* $g \in \mathcal{A}_c(U,F)$ *and* $A \subset U$, *where* $A = \{x \in \overline{U} : f(x) \cap (tg)(x) \neq \phi, \text{ for some } t \in [0,1]\}$, *then* f *and* g *have a coincidence point, i.e., there exists* $x_0 \in A$ *such that* $f(x_0) \cap g(x_0) \neq \phi$.

Property 3.6.8 [Normalization]. *If* $0 \notin \partial U$ *and* $\overline{U}$ *is an absolute retract space, then the inclusion mapping* $i : \overline{U} \to E$ *defined by* $i(x) = x$, *for any* $x \in \overline{U}$ *is essential if and only if* $0 \in U$.

Property 3.6.9 [Localization]. *Let* $f \in \mathcal{A}_{\partial U}(U,F)$ *be an essential set-valued mapping. If* V *is an open subset of* U *satisfying the following conditions:*

(i) $f^{-1}(\{0\}) \subset U$,

(ii) $\overline{V}$ *is an absolute retract space*,

then the restriction $f_{|V} : V \to E$ *of* f *is an essential set-valued mapping.*

We recall that a topological space X is an *absolute retract*, if for each space Y and each homeomorphism $h: X \to Y$ such that $h(X)$ is a closed subset of Y, the set $h(X)$ is a retract of Y. We note that a convex subset of a space is an *absolute retract.*

Property 3.6.10 [Homotopy]. *Let* $f \in \mathcal{A}_{\partial U}(U,F)$ *be an essential set-valued mapping. If* $h:\overline{U}\times[0,1]\to F$ *is a compact admissible set-valued mapping such that:*

(i) $h(x,0) = \{0\}$ *for every* $x \in \partial U$,

(ii) $\{x\in\overline{U}: f(x)\cap h(x,t)\neq\phi, \text{ for some } t\in[0,1]\}\subset U$,

then $(f-h(\cdot,1)):\overline{U}\to F$ *is an essential set-valued mapping.*

Property 3.6.11 [Continuation]. *Let* $f \in \mathcal{A}_{\partial U}(U,F)$ *be an essential set-valued mapping. Assume that* f *is proper, i.e.,* $f^{-1}(\mathbb{K})$ *is compact for any compact set* $\mathbb{K}\subset F$. *If* $h:\overline{U}\times[0,1]\to F$ *is a compact admissible set-valued mapping such that* $h(x, 0) = \{0\}$, *for every* $x \in \partial U$, *then there exists a positive real number* ε_0 *such that the mapping* $(f-h(\cdot,\lambda)):\overline{U}\to F$ *is essential for each* $\lambda\in\,]-\varepsilon_0,\varepsilon_0[$.

For Definition 3.6.5 and the proofs of Properties 3.6.5-3.6.11, the reader is referred to (Gorniewicz, L. and Slosarski, M. [1]).

Remark. The notion of *essential set-valued mapping* is similar to the notion of *zero-epi mapping* defined for single-valued mappings (see Chapter 1).

Now, using the essentiality we can give a Leray–Schauder alternative for coincidence.

THEOREM 3.6.7 [Leray–Schauder] [Alternative for coincidence]. *Let* $f \in \mathcal{A}_{\partial U}(U,F)$, $g\in\mathcal{A}_c(U,F)$ *be arbitrary set-valued mappings. If* f *is essential and* $f(x)\cap g(x)=\phi$ *for any* $x \in \partial U$, *then at least one of the following conditions is satisfied:*

(1) *there exists a coincidence point of* f *and* g, *i.e., there exists a point* $x_0 \in U$, *such that* $f(x_0)\cap g(x_0)\neq\phi$, *or*

(2) *there exists* $\lambda_0 \in\,]0, 1[$ *and* $x_0 \in \partial U$ *such that*

$$f(x_0)\cap[\lambda_0 g(x_0)]\neq\phi.$$

Proof. We obtain the conclusion of the theorem if we apply Property 3.6.10 considering the set-valued mappings f and h, where $h(x,t) = tg(x)$, for $x \in \overline{U}$ and $t \in [0, 1]$. □

Comment. There exist in the literature other kinds of Leray–Schauder alternatives, but we selected in this chapter only the Leray–Schauder type alternatives, which are useful for complementarity problems and variational inequalities.

4

THE ORIGIN OF THE NOTION OF EXCEPTIONAL FAMILY OF ELEMENTS

We informed the reader that we present in this book a topological method applicable to the study of solvability of complementarity problems and of variational inequalities. This method is based on the notion of "*exceptional family of elements*" associated to a mapping and to a closed convex cone or more generally to an unbounded closed convex set. A mapping can have, or have not an exceptional family of elements. When a mapping is without an exceptional family of elements, we have for this mapping a kind of general coercivity condition. This general coercivity condition implies the solvability of complementarity problems and of variational inequalities. We explain in this chapter how this notion was introduced in $\mathbb{R}^n$ using the topological degree. In the next chapters we will extend this method to more general situations using Leray-Schauder type alternatives. By Leray-Schauder type alternatives, this method becomes simpler and more elegant.

4.1 Exceptional family of elements, topological degree and nonlinear complementarity problems in $\mathbb{R}^n$.

Let $\left(\mathbb{R}^n, \langle\cdot,\cdot\rangle\right)$ be the n-dimensional Euclidean space. We denote by $\mathbb{K}$ a closed pointed convex cone in $\mathbb{R}^n$, and by $\mathbb{K}^*$ its dual. The cone $\mathbb{K}$ defines an ordering on $\mathbb{R}^n$ by $x \le y$ if and only if $y - x \in \mathbb{K}$. We say that the ordered vector space $(\mathbb{R}^n, \mathbb{K})$ is a vector lattice if and only if, for every pair (x, y) of elements of $\mathbb{R}^n$ the supremum $x \vee y$ and the infimum $x \wedge y$ exist in $\mathbb{R}^n$. If $(\mathbb{R}^n, \mathbb{K})$ is a vector lattice, we define for every $x \in \mathbb{R}^n$, $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x^+ + x^-$. Other properties of x^+, x^- and $|x|$ are presented and proved in (Peressini, A. L., [1]). We say that the n-dimensional Euclidean space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle, \mathbb{K})$ is a *Hilbert lattice* if and only if:

(h$_1$) $\mathbb{R}^n$ *is a vector lattice,*

(h$_2$) $\| |x| \| = \|x\|$ *for every* $x \in \mathbb{R}^n$,

(h$_3$) $0 \leq x \leq y$ *implies* $\|x\| \leq \|y\|$ *for every* $x, y \in \mathbb{K}$.

We denote by $P_{\mathbb{K}}$ the projection onto $\mathbb{K}$ (see Chapter 1). We say that $\mathbb{K}$ is an *isotone projection cone* if and only if, for every $x, y \in \mathbb{R}^n$, $x \leq y$ implies $P_{\mathbb{K}}(x) \leq P_{\mathbb{K}}(y)$. The following result is known.

THEOREM 4.1.1. *If $(\mathbb{R}^n, \langle \cdot, \cdot \rangle, \mathbb{K})$ is a Hilbert lattice, then $\mathbb{K}$ is an isotone projection cone and moreover, $P_{\mathbb{K}}(x) = x^+$ for every $x \in \mathbb{R}^n$.*

Proof. The theorem is a particular case of Theorem 1.50 proved in (Isac, G., [20]). □

This result justifies some of our notation in this chapter and in particular we have that $P_{\mathbb{R}^n_+}(x) = x^+$ for every $x \in \mathbb{R}^n$. Let $\mathbb{K} \subset \mathbb{R}^n$ be an arbitrary closed pointed convex cone and $f : \mathbb{K} \to \mathbb{R}^n$ a continuous mapping. We consider the *nonlinear complementarity problem* defined by f and $\mathbb{K}$, i.e.,

$$NCP(f, \mathbb{K}) : \begin{cases} \text{find } x_* \in \mathbb{K} \text{ such that} \\ f(x_*) \in \mathbb{K}^* \text{ and } \langle x_*, f(x_*) \rangle = 0. \end{cases}$$

We recall that the polar cone $\mathbb{K}^0$ of $\mathbb{K}$ is defined by

$$\mathbb{K}^0 = \left\{ x \in \mathbb{R}^n \,\middle|\, \langle x, y \rangle \leq 0 \text{ for all } y \in \mathbb{K} \right\}.$$

If $Q = \mathbb{K}^0$, then by the Bipolarity Theorem (Schaefer, H., [2]) it follows that $\mathbb{K} = \overline{\mathbb{K}} = Q^0$ and hence $\mathbb{K}$ and Q are mutually polar. By Moreau's Decomposition Theorem (Theorem 1.9.5), each vector z $\in \mathbb{R}^n$ has a unique decomposition of the form $z = z^+ - z^-$, where $z^+ = P_{\mathbb{K}}(z)$ and $z^- = -P_{\mathbb{K}^0}(z)$. (Note that $-z^-$ is the orthogonal complement to z^+.) Obviously, $z^- = z^+ - z$. The following result is useful in this chapter.

PROPOSITION 4.1.2. *Given a mapping $f : \mathbb{K} \to \mathbb{R}^n$, the complementarity problem NCP(f, $\mathbb{K}$) has a solution if and only if, the mapping*

$$\Psi(x) = P_{\mathbb{K}}(x) - f\left(P_{\mathbb{K}}(x)\right), \text{ for all } x \in \mathbb{R}^n$$

has a fixed point in $\mathbb{R}^n$. If x_0 is a fixed point of Ψ, $x_* = P_{\mathbb{K}}(x_0)$ is a solution to the problem NCP(f, $\mathbb{K}$).

Proof. This proposition is Theorem 2.3.7 considered for the case $(H, \langle \cdot, \cdot \rangle) = (\mathbb{R}^n, \langle \cdot, \cdot \rangle)$. □

Let $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ be a continuous mapping.

DEFINITION 4.1.1. *We say that a family of elements $\{x^r\}_{r>0} \subset \mathbb{R}^n_+$ is an exceptional family of elements for f, if $\|x^r\| \to +\infty$ as $r \to +\infty$ and for each real number r> 0 there exists a real number $\mu_r > 0$ such that:*

(i) $f_i(x^r) = -\mu_r x_i^r$ *if* $x_i^r > 0$,

(ii) $f_i(x^r) \geq 0$ *if* $x_i^r = 0$.

DEFINITION 4.1.2. *We say that an exceptional family of elements $\{x^r\}_{r>0}$ for f is regular if $\|x^r\| = r$ for every $r > 0$.*

The importance of the notion introduced in Definition 4.1.1 is given by the following result. First, we consider the problem *NCP(f, $\mathbb{K}$)* with $\mathbb{K} = \mathbb{R}^n_+$.

THEOREM 4.1.3. *For a continuous mapping* $f:\mathbb{R}^n_+ \to \mathbb{R}^n$, *there exists either a solution to the problem* $NCP\left(f,\mathbb{R}^n_+\right)$ *or an exceptional family of elements for f.*

Proof. Applying Proposition 4.1.2 we have that the solvability of the problem $NCP\left(f,\mathbb{R}^n_+\right)$ is equivalent to the problem of finding a fixed point for the mapping $\Psi(x)=P_{\mathbb{R}^n_+}(x)-f\left(P_{\mathbb{R}^n_+}(x)\right)$, $\left(x\in\mathbb{R}^n\right)$. Consequently, we consider the equation $\Psi(x)=x$, or

$$f\left(P_{\mathbb{R}^n_+}(x)\right)+x-P_{\mathbb{R}^n_+}(x)=0. \tag{4.1.1}$$

Since $\left(\mathbb{R}^n,\langle\cdot,\cdot\rangle,\mathbb{R}^n_+\right)$ is a Hilbert lattice by Theorem 4.1.1 we have that $P_{\mathbb{R}^n_+}(x)=x^+$, and because $x-x^+=x^-$, equation (4.1.1) becomes

$$f\left(x^+\right)-x^-=0. \tag{4.1.2}$$

If we denote $F(x)=f\left(x^+\right)-x^-$, now the problem is to solve the equation

$$F(x)=f\left(x^+\right)-x^-=0. \tag{4.1.3}$$

Obviously, the mapping $F:\mathbb{R}^n\to\mathbb{R}^n$ is continuous. For any real number $r>0$, we consider the spheres of radius r:

$$S_r=\left\{x\in\mathbb{R}^r:\ \|x\|=r\right\}$$

and the open ball of radius r:

$$B_r=\left\{x\in\mathbb{R}^r:\ \|x\|<r\right\}.$$

Obviously $\partial B_r = S_r$. We consider the homotopy between the identity mapping I and F defined by:

$$\begin{cases} H(x,t)=tx+(1-t)F(x), \\ \textit{for any } (x,t)\in\partial B_r\times[0,1]. \end{cases} \tag{4.1.4}$$

We apply the Poincaré–Bohl Theorem (this is Property 4 of topological degree, in part I of Chapter 1), with $y_0=0$ and $\Omega=B_r$. We have

$$\begin{aligned} H(x,t)&=tx+(1-t)f\left(x^+\right)-(1-t)x^- \\ &=t\left(x+x^-\right)+(1-t)f\left(x^+\right)-x^- \\ &=tx^+ +(1-t)f\left(x^+\right)-x^-, \end{aligned}$$

and for this homotopy the following cases are possible:

(i) There exists an $r>0$ such that $0\notin H(x,t), x\in S_r, t\in[0,1]$. Then the Poincaré–Bohl Theorem implies that

$$\deg(F, B_r, 0) = \deg(I, B_r, 0).$$

Because deg(I, B_r, 0) = 1, we have that deg(F, B_r, 0) = 1. This means that the ball $\overline{B_r}$ contains at least one solution to the equation $F(x) = 0$ [cf. Kronecker's Theorem, Property 1 of topological degree, Chapter 1)]. Therefore the problem $NCP(f, \mathbb{R}^n_+)$ has a solution.

(ii) For each $r > 0$, there exist a point $u_r \in S_r$ and a scalar $t_r \in [0, 1]$ such that

$$H(u_r, t_r) = 0. \tag{4.1.5}$$

We remark that $\|u_r\|^2 = \langle u_r^+ - u_r^-, u_r^+ - u_r^- \rangle = \|u_r^+\|^2 + \|u_r^-\|^2 = r^2$. If $t_r = 0$, then u_r solves equation (4.1.3), which implies again that the problem $NCP(f, \mathbb{R}^n_+)$ has a solution. Otherwise, if $t_r > 0$, then the definition of $H(x, t)$ and (4.1.5) yield

$$t_r u_r^+ + (1 - t_r) f(u_r^+) = u_r^-. \tag{4.1.6}$$

From (4.1.6) we have

$$(1 - t_r) f_i(u_r^+) = -t_r (u_r^+)_i, \ if\ (u_r)_i > 0 \tag{4.1.7}$$

and

$$(1 - t_r) f_i(u_r^+) = (u_r^-)_i, \ if\ (u_r)_i \le 0. \tag{4.1.8}$$

Now, we put $x^r = u_r^+$ and we rearrange (4.1.7) and (4.1.8) as follows:

$$f_i(x^r) = -\frac{t_r}{1 - t_r} x_i^r, \ if\ x_i^r > 0; \tag{4.1.9}$$

and

$$f_i(x^r) = -\frac{1}{1 - t_r}(u_r^-)_i \ge 0, \ if\ x_i^r = 0. \tag{4.1.10}$$

If we put $\mu_r = \frac{t_r}{1 - t_r}$ we have that (4.1.9) and (4.1.10) represent relation (i) and (ii) from Definition 4.1.1. To have that $\{x^r\}_{r>0}$ is an exceptional family of elements, we must show that $\|x^r\| \to +\infty$ when $r \to +\infty$. Indeed, if we suppose that the set $\{u_r^+\}_{r>0}$ is bounded, then in this case it follows that $\|u_r^-\| = \sqrt{r^2 - \|u_r^+\|^2} \to +\infty$ which means that the right-hand side of (4.1.6) is unbounded, On the other hand, the left-hand side of (4.1.6) is bounded since the set $\{u_r^+\}_{r>0}$ is supposed to be bounded and f is a continuous mapping. This contradiction completes the proof. □

We have a similar result for regular exceptional families.

THEOREM 4.1.4. *For any continuous mapping $f:\mathbb{R}^n_+ \to \mathbb{R}^n$ there exists either a solution to the problem $NCP\left(f,\mathbb{R}^n_+\right)$ or a regular exceptional family of elements for f.*

Proof. As in the proof of Theorem 4.1.3, we consider the equation

$$F(x) = f\left(x^+\right) - x^- = 0. \qquad (4.1.11)$$

For each $r > 0$ we define the set

$$D_r = W_r \cap \overline{B_r}, \text{ where } W_r = \left\{x \in \mathbb{R}^n : \left\|x^+\right\| \le r\right\}$$

and the number

$$\delta = \sqrt{\left(\max\left\{r, M_r\right\}\right)^2 + r^2 + 1}, \text{ with } M_r = \max_{x \in W_r}\left\|f\left(x^+\right)\right\|.$$

The number δ is well defined since

$$M_r \le \max_{x \in B_r \cap \mathbb{R}^n_+}\left\|f(x)\right\| < +\infty.$$

As in the proof of Theorem 4.1.3 we apply the Poincaré–Bohl Theorem to the mappings I, F and to the set D_r. It is sufficient to consider two cases:

(i) There exists an $r > 0$ such that $0 \notin H(x,t), x \in \partial D_r, t \in [0,1]$. By the same arguments used in the proof of Theorem 4.1.3 we obtain a solution to the problem $NCP\left(f,\mathbb{R}^n_+\right)$

(ii) For each $r > 0$ there exist a point $u_r \in \partial D_r$ and a real number $t_r \in [0, 1]$ such that $H\left(u_r, t_r\right) = 0$. If $t_r = 0$, then u_r is a solution of equation (4.1.11), which implies that the problem $NCP\left(f,\mathbb{R}^n_+\right)$ has a solution. Otherwise, if $t_r > 0$ we obtain as in the proof of Theorem 4.1.3 that $x^r = u_r^+$ satisfies conditions (i) and (ii) of Definition 4.1.1. In order to show that $\left\|x^r\right\| = r$, we examine the structure of the frontier ∂ D_r. We can show that $\partial D_r = V_r \cup U_\delta$, where

$$V_r = \left\{x \in \mathbb{R}^n : r = \left\|x^+\right\| \le \delta\right\} = \partial W_r \cap \overline{B_r} \text{ and } U_\delta = W_r \cap S_\delta.$$

We have that $u_r \notin U_\delta$. Indeed from (4.1.6) it follows that

$$\|u_r^-\| \le \max\left\{\|u_r^+\|, \|f(u_r^+)\|\right\} \le \max\{r, M_r\}.$$

Hence,

$$\|u_r\|^2 = \|u_r^+\|^2 + \|u_r^-\|^2 \le r^2 + \left(\max\{r, M_r\}\right)^2 = (\delta - 1)^2$$

which implies that $\|u_r\| < \delta$. Thus, $u_r \in V_r$ and consequently $\|x^r\| = \|u_r^+\| = r$. This means that $\{x_r\}_{r>0}$ is a regular exceptional family of elements and the theorem is completely proved. □

Remarks.

1. G. Isac in 1991–1992 defined the notion of *exceptional family of elements* in some unpublished notes, under the name of *radial (or asymptotic) family of elements.* T. E. Smith considered a similar notion. The notion considered by Smith is not a family of elements of the form $\{x^r\}_{r>0}$ but is a sequence $\{x_n\}_{n\in N}$ (Smith, T. E., [1]). This sequence was defined using some special properties of the polyhedral cone $\mathbb{R}_+^n$. Smith's notion cannot be related to the topological degree and cannot be extended to any closed convex cone. Because the sequence he defined was named, *exceptional sequence of elements*, we named our notion "*exceptional family of elements*", and we developed this notion in several papers, published after the paper (Isac, G., Bulavski, V. and Kalashnikov, V. [1]).
2. We remark that Theorem 4.1.4 can be derived by using the Hartman–Stampacchia Theorem and the Karush–Kuhn–Tucker conditions, used in optimization, but this method cannot be extended to general situations and in particular to infinite dimensional Hilbert spaces. Our method based on the notion of exceptional family of elements can be extended to general situations because it is based on the topological degree. In Chapter 7 we will give a more general construction without topological degree.

From Theorem 4.1.3 and 4.1.4 we deduce immediately the following result.

THEOREM 4.1.5. *If* $f : \mathbb{R}_+^n \to \mathbb{R}$ *is an arbitrary continuous mapping without an exceptional family of elements, then the problem* $NCP\left(f, \mathbb{R}_+^n\right)$ *has at least a solution.* □

From Theorem 4.1.5 we deduce the following natural question: *Is the class of functions f such that f is without an exceptional family of elements empty, or non-empty?*
In the theory of variational inequalities we find the notion of "coercive mapping". We recall that a mapping $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ is said to be *coercive* on $\mathbb{R}^n_+$ if and only if

$$\frac{\langle f(x) - f(x_0), x - x_0 \rangle}{\|x - x_0\|} \to +\infty \ \textit{as} \ \|x\| \to +\infty, \ x \in \mathbb{R}^n_+, \ \textit{for some} \ x_0 \in \mathbb{R}^n_+ .$$

We proved in (Isac, G., Bulavski, V. and Kalashnikov, V. [1]) that coercive continuous mappings do not have regular exceptional families of elements with respect to $\mathbb{R}^n_+$ and also, there exist noncoercive mappings without exceptional families of elements.
In a later chapter, we will consider again the relation between *coercivity* and the property of being *without an exceptional family of elements.*

Now, we consider the case of a general closed convex cone in $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping and $\mathbb{K} \subset \mathbb{R}^n$ a closed convex cone.

DEFINITION 4.1.3. *We say that a set of elements* $\{x^r\}_{r>0} \subset \mathbb{R}^n$ *is an exceptional family of elements for f (with respect to* $\mathbb{K}$*) if the following conditions are satisfied:*

(1) $\left\|(x^r)^+\right\| \to +\infty \ \textit{as} \ r \to +\infty$,

(2) *for each* $r > 0$, $f\left((x^r)^+\right)$ *belongs to the open ray*

$$\mathcal{O}\left((x^r)^- ; s_r\right) = \left\{ y = (x^r)^- + \mu s_r \,|\, \mu > 0 \right\} \ \textit{where} \ s_r = (x^r)^- - (x^r)^+ .$$

Remarks.

(a) If in particular, $x^r \in \mathbb{K}$, then from condition (2) of Definition 4.1.3, we have the equality

$$f(x^r) = -\mu_r (x^r), \ \textit{for some} \ \mu_r > 0 .$$

(b) If $\mathbb{K} = \mathbb{R}^n_+$, then from Definition 4.1.3 we do not obtain exactly Definition 4.1.1. In Definition 4.1.1 there is more information about $f_i(x^r)$, because of the particularities of the cone $\mathbb{R}^n_+$.

(c) For a general cone $\mathbb{K} \subset \mathbb{R}^n_+$ there is also the concept of *regular exceptional family of elements. We say that an exceptional family of elements* $\{x^r\}_{r>0}$ *for f (with respect to* $\mathbb{K}$*) is regular if* $\left\|\left(x^r\right)^+\right\| = r$, *for every* $r > 0$.

THEOREM 4.1.6. *For any continuous mapping* $f : \mathbb{R}^n \to \mathbb{R}^n$ *and any closed pointed convex cone* $\mathbb{K} \subset \mathbb{R}^n$, *there exists either a solution to the problem* $NCP(f, \mathbb{K})$ *or an exceptional family of elements (in the sense of Definition 4.1.3) for f.*

Proof. Using Proposition 4.1.2 and Theorem 1.9.5 (Moreau's Decomposition Theorem), we have that the solvability of the problem $NCP(f, \mathbb{K})$, is equivalent to the solvability of the equation

$$F(x) = f\left(x^+\right) - x^- = 0, \text{ where } x^+ = P_{\mathbb{K}}(x) \text{ and } x^- = -P_{\mathbb{K}^0}(x).$$

Now, repeating exactly the proof of Theorem 4.1.3 we obtain either that there exists a solution to the problem $NCP(f, \mathbb{K})$, or for each $r > 0$ there exist a point $u_r \in S_r$ and real number $t_r \in]0,1[$ such that the equality $t_r u_r^+ + (1 - t_r) f(u_r^+) = u_r^-$ is true. Dividing both sides of that equality by $(1 - t_r)$ and rearranging, one obtains the relation

$$f\left(u_r^+\right) = \frac{1}{1 - t_r} u_r^- - \frac{t_r}{1 - t_r} u_r^+ = u_r^- + \frac{t_r}{1 - t_r}\left(u_r^- - u_r^+\right),$$

which means that $f\left(u_r^+\right) \in \mathcal{O}\left(u_r^-; s_r\right)$. The fact that, $\left\|u_r^+\right\| \to +\infty$ *as* $r \to +\infty$ is established in the same way as in the proof of Theorem 4.1.3. Thus $\{x^r\}_{r>0}$, where $x^r = u_r^+$ is an exceptional family of elements for *f* (with respect to $\mathbb{K}$) and this complete the proof. □

Remark. We have a variant of Theorem 4.1.6 for regular exceptional families of elements with respect to a closed convex cone $\mathbb{K} \subset \mathbb{R}^n$.

4.2. Exceptional family of elements, topological degree and implicit complementarity problems in $\mathbb{R}^n$

We present in this section a notion of exceptional family of elements for a couple of continuous mappings in $\mathbb{R}^n$. This notion is applicable to the study of *implicit complementarity problems.*

Let $\mathbb{K} \subset \mathbb{R}^n$ be a closed convex cone and $f, g : \mathbb{R}^n \to \mathbb{R}^n$ continuous mappings. In particular the convex cone $\mathbb{R}^n$ may be $\mathbb{R}^n_+$. If $D \subset \mathbb{R}^n$ is a non-empty subset, the *implicit complementarity problem* defined by f, g, D and $\mathbb{K}$ is:

$$ICP(f,g,\mathbb{K}): \begin{cases} \text{find } x_0 \in D \text{ such that} \\ g(x_0) \in \mathbb{K}, f(x_0) \in \mathbb{K}^* \text{ and} \\ \langle g(x_0), f(x_0) \rangle = 0. \end{cases}$$

When $D = \mathbb{K}$, we denote this problem by $ICP(f, g, \mathbb{K})$.

In complementarity theory, the study of implicit complementarity problems is a big chapter (Isac, G. [12], [20]), (Hyers, D. H., Isac, G. and Rassias, Th. M. [1]). Several authors have studied implicit complementarity problems from several points of view. [See the references of the book (Isac, G, [20]).

DEFINITION 4.2.1. *We say that a family of elements* $\{x^r\}_{r>0} \subset \mathbb{R}^n$ *is an exceptional family of elements for the couple (f, g) with respect to* $\mathbb{R}^n_+$ *if the following conditions are satisfied:*

(1) $\|x^r\| \to +\infty$ *as* $r \to +\infty$,

(2) $g(x^r) \geq 0$ *for each* $r > 0$,

(3) *for each* $r > 0$, *there exists* $\mu_r > 0$ *such that for* $i = 1, 2, \ldots, n$ *we have*

(i) $f_i(x^r) = -\mu_r g_i(x^r)$, *if* $g_i(x^r) > 0$,

(ii) $f_i(x^r) \geq 0$ *if* $g_i(x^r) = 0$.

We have the following result.

THEOREM 4.2.1. *Let f, g :* $\mathbb{R}^n \to \mathbb{R}^n$ *be continuous mappings. If the following assumptions are satisfied:*

(1) *there exists an element* $b \in \mathbb{R}^n$ *such that* $g(x) = 0$ *if and only if* $x = b$,

(2) *g maps a neighborhood of the point b homeomorphically onto a neighborhood of the origin,*

then, there exists either a solution to the problem $ICP(f, g, \mathbb{R}^n_+)$, *or an exceptional family of elements (in the sense of Definition 4.2.1) for the couple (f, g).*

Proof. We consider the mapping $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ defined by

$$F(z,x) = \begin{pmatrix} f(x) - z^- \\ g(x) - z^+ \end{pmatrix}; \textit{for any } (z,x) \in \mathbb{R}^n \times \mathbb{R}^n$$

and the equation

$$F(z, x) = 0. \tag{4.2.1}$$

The problem $ICP(f, g, \mathbb{R}^n_+)$ is equivalent to the solvability of equation (4.2.1). Indeed, if (z, x) solves (4.2.1), then x is a solution to the problem $ICP(f, g, \mathbb{R}^n_+)$. Conversely, if x is a solution to the problem $ICP(f, g, \mathbb{R}^n_+)$, then (z, x) is a solution of (4.2.1) where

$$z_i = \begin{cases} g_i(x), \textit{if } g_i(x) > 0, \\ -f_i(x), \textit{if } g_i(x) = 0, \end{cases} \quad i = 1, 2.3, \ldots, n.$$

Obviously the mapping $F(z, x)$ is continuous over $\mathbb{R}^{2n}$. For any $r > 0$, let S_r be the $(2n$-$1)$-dimensional sphere

$$S_r = \{(z,x) \in \mathbb{R}^{2n} : \|(z, x - b)\| = r\},$$

and B_r the open ball of radius r, i.e.,

$$B_r = \{(z,x) \in \mathbb{R}^{2n} : \|(z, x - b)\| < r\}.$$

Now, we consider the homotopy $H(z, x, t)$ of the mappings $F(z, x)$ and $G(z,x) = \begin{pmatrix} z \\ g(x) \end{pmatrix}$, defined by

$$H(z,x,t) = tG(z,x) + (1-t)F(z,x)$$
$$= \begin{pmatrix} tz + (1-t)f(x) - (1-t)z^- \\ tg(x) + (-x)g(x) - (-t)z^+ \end{pmatrix} = \begin{pmatrix} tz^+ + (1-t)f(x) - z^- \\ g(x) - (1-t)z^+ \end{pmatrix}.$$

Hence we have

$$H(z,x,t)=\begin{pmatrix} tz^{+}+(1-t)f(x)-z^{-} \\ g(x)-(1-t)z^{+} \end{pmatrix}. \tag{4.2.2}$$

Two cases are possible:

(A) There exists an $r > 0$ such that

$$H(z,x,t)\neq 0, \; \textit{for all} \; (z,x)\in S_r \; \textit{and} \; t\in[0,1].$$

In this case, Property 4 [Poincaré-Bohl] of topological degree implies the equality

$$\deg(F,B_r,0)=\deg(G,B_r,0).$$

Since $|\deg(G,B_r,0)|=1$, we have that $\deg(F,B_r,0)=\pm 1$. Because of this fact, we conclude that the ball $\overline{B_r}$ contains at least one solution of the equation (4.2.1) and the solvability of the problem $ICP(f,g,\mathbb{R}^n_+)$ is proved.

(B) For $r > 0$ there exist a point $(z_r, x^r)\in S_r$ and a scalar $t_r \in [0, 1]$ such that

$$H(z_r,x^r,t_r)=0. \tag{4.2.3}$$

We have

$$\left\|(z_r,x^r-b)\right\|_{2n}^2=\left\|z_r^{+}\right\|_n^2+\left\|z_r^{-}\right\|_n^2+\left\|x^r-b\right\|_n^2=r^2. \tag{4.2.4}$$

If $t_r = 0$, then x_r solves the problem $ICP(f,g,\mathbb{R}^n_+)$. If $t_r > 0$, from (4.2.2) and (4,2,3) we obtain

$$t_r z_r^{+}+(1-t_r)f(x^r)=z_r^{-}, \tag{4.2.5}$$

and

$$z_r^{+}=\frac{g(x^r)}{1-t_r}. \tag{4.2.6}$$

Substituting expression (4.2.6) for z_r^{+} into (4.2.5) we obtain

$$\frac{t_r}{1-t_r}g(x^r)+(1-t_r)f(x^r)=z_r^{-},$$

which implies for $i = 1, 2, \ldots, n$,

$$f_i(x^r) = \begin{cases} -\dfrac{t_r g(x^r)}{(1-t_r)^2}, & if \ (z_r)_i > 0, \\ \dfrac{(z_r^-)_i}{1-t_r}, & if \ (z_r)_i \leq 0. \end{cases} \quad (4.2.7)$$

Taking $\mu_r = \dfrac{t_r}{(1-t_r)^2}$, we obtain from (4.2.7) that the family of elements $\{x^r\}_{r>0}$ is an exceptional family of elements for the couple (f, g), if we prove that $\|x^r\| \to +\infty$ when $r \to +\infty$. To prove this fact we suppose on the contrary, that the family $\{x^r\}_{r>0}$ is bounded, hence it has a finite accumulation point x_*. Note that the respective scalar limit t_* cannot be equal to 1 (otherwise (4.2.4) contradicts (4.2.5)). But if $t_* < 1$, then the continuity of f and g combined with (4.2.5) and (4.2.6) imply the boundedness of the family $\{z_r\}_{r>0}$, which again contradicts (4.2.4) as $r \to +\infty$. Thus we must have that $\|x^r\| \to +\infty$ as $r \to +\infty$, and the proof is complete. □

Remarks.

1. We remark that, because of assumption (2) in Theorem 4.2.1, the notion of exceptional family of elements for a pair of continuous mappings is not so natural. Later, we will replace this notion by another, more natural and more flexible.
2. In the next section we will extend Definition 4.2.1 to an arbitrary closed convex cone $\mathbb{K} \subset \mathbb{R}^n$.

4.3 A general notion of an exceptional family of elements for continuous mappings

In this section we introduce another notion of *exceptional family of elements*, for a continuous mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, with respect to an arbitrary closed convex cone $\mathbb{K} \subset \mathbb{R}^n$. This notion is more natural and more flexible than the similar notion introduced by Definition 4.1.3.

Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping and $\mathbb{K} \subset \mathbb{R}^n$ a closed pointed convex cone. Let $\mathbb{K}^*$ be the dual of the cone $\mathbb{K}$ (see Chapter 1). If x is an arbitrary element in $\mathbb{R}^n$, then $x^+ = P_{\mathbb{K}}(x)$ is well defined. We denote $x^- = x^+ - x$. By Theorem 1.9.7 we have that $x^- \in \mathbb{K}^*$ and $\langle x^+, x^- \rangle = 0$, that is, x^- is a *normal vector* to a supporting hyperplane of the cone $\mathbb{K}$ at the point x^+.

DEFINITION 4.3.1. *We say that a family of elements* $\{x_r\}_{r>0} \subset \mathbb{K}$, *is an exceptional family of elements for f with respect to the cone* $\mathbb{K}$, *if the following conditions are satisfied*:

(1) $\|x_r\| \to \infty$ *as* $r \to +\infty$,

(2) *for each* $r > 0$ *there exists a scalar* $\mu_r > 0$ *such that* $s_r = f(x_r) + \mu_r x_r \in \mathbb{K}^*$ *and* $\langle x_r, s_r \rangle = 0$.

Remark. If $\{x_r\}_{r>0}$ is an exceptional family of elements for f with respect to $\mathbb{K}$, then from condition (1) and (2) of Definition 4.3.1 we deduce that for any $r > 0$, the vector $s_r = f(x_r) + \mu_r x_r$ is the normal one to a supporting hyperplane of the cone $\mathbb{K}$ at the point x_r.

We have the following result, which justifies the importance of the notion of *exceptional family* of elements (in the sense of Definition 4.3.1).

THEOREM 4.3.1. *If* $\mathbb{K} \subset \mathbb{R}^n$ *is an arbitrary closed pointed convex cone and* f: $\mathbb{K} \to \mathbb{R}^n$ *is a continuous mapping, then either the problem NCP(f,* $\mathbb{K}$*) has a solution or f has an exceptional family of elements with respect to* $\mathbb{K}$ *(in the sense of Definition 4.3.1).*

Proof. We consider the mapping $F(x) = f(x^+) - x^-$ where $x^+ = P_{\mathbb{K}}(x)$ and $x^- = x^+ - x$, and we remark that the solvability of equation $F(x) = 0$ and the solvability of the *NCP(f,* $\mathbb{K}$*)* are equivalent in the following sense: if x_* is a solution of equation $F(x) = 0$, then $x_0 = P_{\mathbb{K}}(x_*)$ solves the problem $NCP(f, \mathbb{K})$, and conversely, if x_0 is a solution to the problem $NCP(f, \mathbb{K})$, then $x_* = x_0 - f(x_0)$ is a solution of equation $F(x) = 0$. In order to investigate equation

$$F(x) = f(x^+) - x^- = 0. \tag{4.3.1}$$

we consider, for any $r > 0$ the spheres S_r and the open balls B_r:

$$S_r = \{x \in \mathbb{R}^n : \|x\| = r\},$$

$$B_r = \{x \in \mathbb{R}^n : \|x\| < r\}.$$

Let $G(x) = x$ be the identity mapping and $H(x, t)$ the following homotopy defined by G and F:

$$H(x,t) = tx + (1-t)F(x),\ t \in [0,1].$$

In order to apply the Poincaré–Bohl Theorem (Property 4 of topological degree) for $y_0 = 0$ and $\Omega = B_r$, we consider the expression $H(x, t)$ for arbitrary $x \in \partial B_r = S_r$ and $t \in [0,1]$:

$$H(x,t) = tx^+ + (1-t)f(x^+) - x^-. \tag{4.3.2}$$

We have two possibilities:

(A) There is a scalar $t > 0$ such that

$$H(x, t) \neq 0, \text{ for all } x \in S_r \text{ and all } t \in [0, 1].$$

Then by the Poincaré–Bohl Theorem we obtain that

$$\deg(F, B_r, 0) = \deg(G, B_r, 0) = 1.$$

Therefore, because $\deg(F, B_r, 0) = 1$, by Property 1 of topological degree (Kronecker's Theorem), we have that equation (4.3.1) has a solution in B_r and consequently the $NCP(f, \mathbb{K})$ has a solution.

(B) For every $r > 0$ there exist a point $x_r = S_r$ and a scalar $t_r \in [0, 1[$ such that

$$H(x_r, t_r) = 0. \tag{4.3.3}$$

If $t_r = 0$, then x_r solves (4.3.1) which again implies the solvability of the $NCP(f, \mathbb{K})$. Otherwise, if $t_r > 0$, then it follows from (4.3.2) and (4.3.3) that

$$t_r x_r^+ + (1-t_r) f(x_r^+) = x_r^-. \tag{4.3.4}$$

Dividing both parts of equation (4.3.4) by $(1 - t_r)$ we obtain

$$f(x_r^+) + \frac{t_r}{1-t_r} x_r^+ = \frac{1}{1-t_r} x_r^-. \tag{4.3.5}$$

If we denote $\mu_r = \frac{t_r}{1-t_r}$, we obtain $\{x_r^+\}_{r>0}$ as an exceptional family of elements for f with respect to $\mathbb{K}$. Indeed, from (4.3.5) it follows that the vector $s_r = f(x_r^+) + \mu_r x_r^+$ is in $\mathbb{K}^*$ and $\langle x_r^+, s_r\rangle = 0$. Moreover, $\|x_r^+\| \to +\infty$ as $r \to +\infty$, because if the contrary is true, the family $\{x_r^+\}_{r>0}$ must have a finite accumulation point. On the other hand, the equality $\|x_r^-\| = \sqrt{r^2 - \|x_r^+\|^2}$ implies that the right-hand side of (4.3.4) comprises an unbounded sequence of elements. On the other hand, the respective vectors in the left-hand side of (4.3.4) compose a bounded family due to the continuity of the mapping f. This contradiction completes the proof. □

The notion of *exceptional family of elements* introduced by Definition 4.3.1 can be extended to a pair of mappings. Indeed, let $\mathbb{K} \subset \mathbb{R}^n$ be a closed pointed convex cone and $f, g : \mathbb{R}^n \to \mathbb{R}^n$ continuous mappings.

DEFINITION 4.3.2. *We say that a family of elements $\{x_r\}_{r>0} \subset \mathbb{R}^n$ is an exceptional family of elements for the pair (f, g) of continuous mappings with respect to the cone $\mathbb{K}$, if the following conditions are satisfied:*

(1) $\|x_r\| \to +\infty$ *as* $r \to +\infty$,

(2) $g(x_r) \in \mathbb{K}$ *for any* $r > 0$,

(3) *for every* $r > 0$, *there exists* $\mu_r > 0$ *such that*
$$s_r = f(x_r) + \mu_r g(x_r) \in \mathbb{K}^* \text{ and } \langle g(x_r), s_r\rangle = 0.$$

Remark. The notion of an exceptional family of elements for a couple (f, g) of continuous mappings was introduced as a mathematical tool for the study of the problem $ICP(f, g, \mathbb{K})$. If the cone $\mathbb{K}$ is self-adjoint, i.e., $\mathbb{K} = \mathbb{K}^*$, then in this case the notion of exceptional family of elements can be formulated for the couple (g, f).

Definition 4.3.2 allows us to state the following result.

THEOREM 4.3.2. *Let f, g : $\mathbb{R}^n \to \mathbb{R}^{n\ n}$ be continuous mappings, $\mathbb{K} \subset \mathbb{R}^n$ a closed convex cone. Let $b \in \mathbb{R}^n$ be a unique solution to equation f(x) = 0. Moreover, let g map homeomorphically some neighborhood of the element b onto a particular neighborhood of the origin. Then either the problem ICP(f, g, $\mathbb{K}$) has a solution, or the couple (f, g) has an exceptional family of elements (in the sense of Definition 4.3.1).*

Proof. We consider the mapping $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ defined by

$$F(z,x) = \begin{pmatrix} f(x) - z^- \\ g(x) - z^+ \end{pmatrix}, \text{ where } z^+ = P_{\mathbb{K}}(z)\ \ z^- = z - z^+.$$

We observe that equation

$$F(z, x) = 0, \tag{4.3.6}$$

is equivalent to the solvability of the problem *ICP(f, g, $\mathbb{K}$)* in the following sense. If (*z*, *x*) solves (4.3.6), then *x* is a solution to the problem *ICP(f, g, $\mathbb{K}$)*. Conversely, given a solution *x* to the problem *ICP(f, g, $\mathbb{K}$)*, then the pair (*z*, *x*) with $z = g(x) - f(x)$ is a solution to equation (4.3.6). Obviously, the mapping *F* is continuous over $\mathbb{R}^{2n}$. Let S_r be a (2*n* – 1)-dimensional sphere of radius *r* with its centre at the point (0, *b*):

$$S_r = \{(z,x) \in \mathbb{R}^{2n} : \|(z, x - b)\| = r\},$$

and B_r an open ball with the same radius and centre, i.e.,

$$B_r = \{(z,x) \in \mathbb{R}^{2n} : \|(z, x - b)\| < r\}.$$

We consider the sets S_r and B_r for any $r > 0$. Considering the mapping $G(z,x) = \begin{pmatrix} z \\ g(x) \end{pmatrix}$ we define the following standard homotopy defined with *F* and *G* by:

$$\begin{cases} H(z,x,t) = tG(z,x) + (1-t)F(z,x) \\ with\ \ t \in [0,1]. \end{cases} \tag{4.3.7}$$

We have

$$H(z,x,t)=\begin{pmatrix} tz+(1-t)f(x)-(1-t)z^- \\ tg(x)+(1-t)g(x)-(1-t)z^+ \end{pmatrix}$$
$$=\begin{pmatrix} tz^+ +(1-t)f(x)-z^- \\ g(x)-(1-t)z^+ \end{pmatrix}.$$

We have two possible cases:

(A) There exists $r>0$ such that

$$H(z,x,t)\neq 0, \textit{ for any } (z,x)\in S_r, \textit{ and any } t\in[0,1].$$

In this case, Property 4 of topological degree (Poincaré–Bohl Theorem) implies

$$\deg(F,B_r,0)=\deg(G,B_r,0). \qquad (4.3.8)$$

Using the assumptions of our theorem we can verify that $\deg(G,B_r,0)=\pm 1$. By taking (4.3.8) into account we also obtain that $\deg(F,B_r,0)=\pm 1$. Now by Property 1 of topological degree (Kronecker's Theorem), we conclude that $\overline{B_r}$ contains at least one solution of (4.3.8). Therefore, the problem *ICP(f, g, $\mathbb{K}$)* has a solution.

(B) For every $r>0$, there exist a pair $(z_r,x_r)\in S_r$ and a scalar $t_r\in[0,1[$ such that

$$H(z_r,x_r,t_r)=0. \qquad (4.3.9)$$

Note that

$$\|(z_r,x_r-b)\|^2_{\mathbb{R}^{2n}}=\|z_r^+\|^2_{\mathbb{R}^n}+\|z_r^-\|_{\mathbb{R}^n}+\|x_r-b\|^2_{\mathbb{R}^n}=r^2. \quad (4.3.10)$$

If $t_r=0$, then (z_r,x_r) solves equation (4.3.6) and consequently, x_r is a solution to the problem *ICP(f, g, $\mathbb{K}$)*. Otherwise, if $t_r>0$, then (4.3.7) and (4.3.9) imply the following equalities:

$$t_r z_r^+ +(1-t_r)f(x_r)=z_r^-, \qquad (4.3.11)$$

$$z_r^+=\frac{g(x_r)}{(1-t_r)}. \qquad (4.3.12)$$

If we put z_r^+ given by (4.3.12) in (4.3.11), we obtain

$$\frac{t_r}{1-t_r}g(x_r)+(1-t_r)f(x_r)=z_r^-.$$

Dividing both sides by $(1 - t_r)$, and denoting by $\mu_r = \dfrac{t_r}{(1-t_r)^2} > 0$ we have

$$f(x_r) + \mu_r g(x_r) = \frac{1}{1-t_r} z_r^- .$$

From the last equality we obtain that the family of elements $\{x_r\}_{r>0}$ is an exceptional family of elements for the couple (f, g) if we show that $\|x_r\|_{\mathbb{K}^n} \to +\infty$ as $r \to +\infty$. In order, to prove this, we suppose on the contrary, that the family $\{x_r\}_{r>0}$ has a finite accumulation point x_*. Note that the respective scalar limit t_* cannot be equal to 1, otherwise (4.3.10) contradicts (4.3.11). But if $t_* < 1$, then the continuity of mappings f and g combined with (4.3.11) and (4.3.12) imply the boundedness of the family of elements $\{z_r\}_{r>0}$, which again contradicts (4.3.10) as $r \to +\infty$. Thus, it is shown that $\|x_r\| \to +\infty$ and the proof is complete. □

4.4 Exceptional family of elements, zero-epi mappings and nonlinear complementarity problems in Hilbert spaces

In this section we extend the concept of exceptional family of elements to infinite dimensional Hilbert spaces, for k-set fields. We realize this extension using the concept of *k-set contraction* and (0, *k*)-*epi mapping,* presented in sections 1.5 and 1.7 of Chapter 1. The properties of these concepts will be used in this section.

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\mathbb{K} \subset H$ a closed pointed convex cone. We recall that if A is a subset of H, the *Kuratowski measure* of noncompactness of A is defined by:

$$\alpha(A) = \inf \left\{ \begin{array}{l} \varepsilon > 0 : A \textit{ can be convered by} \\ \textit{a finite number of sets of diameter less than } \varepsilon \end{array} \right\}.$$

It is known (see Chapter 1) that $\alpha(A) = 0$ if and only if A is relatively compact. Let D be a subset of H and $f\colon D \to H$ a continuous mapping. We recall that f *is said to be a k-set contraction if for each bounded subset A of D* we have $\alpha(f(A)) \leq k\alpha(A)$, where $k \geq 0$. For more information about the measure of noncompactness α and about k-set contractions, the reader is referred to Chapter 1. Let $f: H \to H$ be an arbitrary mapping. We repeat Definition 4.3.1 but in a general Hilbert space

DEFINITION 4.4.1. *We say that a family of elements* $\{x_r\}_{r>0} \subset \mathbb{K}$ *is an exceptional family of elements for f, with respect to* $\mathbb{K}$, *if and only if, for every real number* $r > 0$ *there exists a real number* $\mu_r > 0$ *such that the vector* $u_r = f(x_r) + \mu_r x_r$ *satisfies the following conditions*:

(1) $u_r \in \mathbb{K}^*$,

(2) $\langle u_r, x_r \rangle = 0$,

(3) $\|x_r\| \to +\infty$ *as* $r \to +\infty$.

Remark. *We say that an exceptional family of elements* $\{x_r\}_{r>0}$ *is regular if for any* $r > 0$ *we have* $\|x_r\| = r$.

DEFINITION 4.4.2. *We say that a mapping* $f: H \to H$ *is a k-set field if* $f(x) = x - T(x)$, *where* $T: H \to H$ *is a k-set contraction with* $0 \leq k < 1$.

Remark. If in Definition 4.4.2 the mapping T is completely continuous, we have that f is a *completely continuous field.*

If $f: H \to H$ is a mapping and $\mathbb{K} \subset H$ is a closed pointed convex cone, we consider again the problem:

$$NCP(f, \mathbb{K}) = \begin{cases} \text{find } x_* \in \mathbb{K} \text{ such that} \\ f(x_*) \in \mathbb{K}^* \text{ and } \langle x_*, f(x_*) \rangle = 0. \end{cases}$$

THEOREM 4.4.1. *If* $f: H \to H$ *is an arbitrary k-set field, then there exists either a solution to the problem NCP(f,* $\mathbb{K}$*) or an exceptional family of elements for f with respect to* $\mathbb{K}$ *(in the sense of Definition 4.4.1).*

Proof. We consider the mapping $\Phi: H \to H$ defined by

$$\Phi(x) = x - P_{\mathbb{K}}[x - f(x)], \text{ for any } x \in H.$$

From Chapter 2 (Corollary 2.3.4) we know that the problem *NCP(f, $\mathbb{K}$)* has a solution if and only if, the equation $\Phi(x) = 0$ has a solution. For any $r > 0$ we consider the sets:

$$S_r = \{x \in H : \|x\| = r\} \text{ and } B_r = \{x \in H : \|x\| < r\}.$$

We remark that the identity mapping on H, denoted by I, is a (0, k)-epi mapping on any B_r with $k \in [0, 1[$. Let $h : [0,1] \times \overline{B_r} \to H$ be the mapping defined by

$$h(t,x) = t\left(x - P_{\mathbb{K}}\left[x - f(x)\right] - x\right) = t\left(-P_{\mathbb{K}}\left[x - f(x)\right]\right).$$

The mapping h is a k-set contraction such that $h(0, x) = 0$, for all $x \in \overline{B_r}$.
We have only the following two situations.

(A) There exists $r > 0$ such that $x + t\left(-P_{\mathbb{K}}\left[x - f(x)\right]\right) \neq 0$, for all $x \in S_r$ and all $t \in [0, 1]$. In this case, applying Property 4 [Homotopy] of (0, k)-epi mappings we have that $x + \left(-P_{\mathbb{K}}\left[x - f(x)\right]\right) = 0$ has a solution in B_r, that is there exists $x_* \in B_r$ such that $x_* = P_{\mathbb{K}}\left[x_* - f(x_*)\right]$, which implies that x_* is a solution to the problem *NCP(f, $\mathbb{K}$)*.

(B) For every $r > 0$ there exists $x_r \in S_r$ and $t_r \in [0, 1]$ such that

$$x_r + t_r\left(-P_{\mathbb{K}}\left[x_r - f(x_r)\right]\right) = 0.$$

If $t_r = 0$, we have that $x_r = 0$, which is impossible since $x_r \in S_r$. If $t_r = 1$, then $x_r - P_{\mathbb{K}}\left[x_r - f(x_r)\right] = 0$, which is equivalent to saying that the problem *NCP(f, $\mathbb{K}$)* has a solution. Hence, we can say that either the problem *NCP(f, $\mathbb{K}$)* has a solution or for any $r > 0$, there exist $x_r \in S_r$ and $t_r \in]0, 1[$ such that

$$x_r - t_r P_{\mathbb{K}}\left[x_r - f(x_r)\right] = 0. \tag{4.4.1}$$

From (4.4.1) we have

$$\frac{1}{t_r} x_r = P_{\mathbb{K}}\left[x_r - f(x_r)\right]. \tag{4.4.2}$$

From (4.4.2) and using the properties (1) and (2) of projection operator $P_{\mathbb{K}}$, given in Theorem 1.9.7, we deduce:

$$\begin{cases} \left\langle \frac{1}{t_r} x_r - (x_r - f(x_r)), y \right\rangle \geq 0 \ \ for\ \ all\ \ y \in \mathbb{K} \\ and \\ \left\langle \frac{1}{t_r} x_r - (x_r - f(x_r)), \frac{1}{t_r} x_r \right\rangle = 0, \end{cases} \tag{4.4.3}$$

which implies

$$\begin{cases} \left\langle \left(\frac{1}{t_r} - 1 \right) x_r + f(x_r), y \right\rangle \geq 0 \ \ for\ \ all\ \ y \in \mathbb{K} \\ and \\ \left\langle \left(\frac{1}{t_r} - 1 \right) x_r + f(x_r), \frac{1}{t_r} x_r \right\rangle = 0. \end{cases} \tag{4.4.4}$$

If in (4.4.3) and (4.4.4) we put $\mu_r = \frac{1}{t_r} - 1$, it follows that

$$\mu_r x_r + f(x_r) \in \mathbb{K}^*, \ \langle \mu_r x_r + f(x_r), x_r \rangle = 0$$

and since for every $r > 0$, $\|x_r\| = r$, we have that $\|x_r\| \to +\infty$ as $r \to +\infty$. Thus, the family of elements $\{x_r\}_{r>0}$ is an exceptional family of elements for f with respect to $\mathbb{K}$ and the proof is complete. □

Remark. Looking at the proof of Theorem 4.4.1 we remark that the exceptional family of elements $\{x_r\}_{r>0}$ is a *regular exceptional family of elements.*

COROLLARY 4.4.2. *If $f : H \to H$ is a k-set field without an exceptional family of elements, with respect to $\mathbb{K}$, then the problem NCP(f, $\mathbb{K}$) has a solution.*

COROLLARY 4.4.3. *If $f : H \to H$ is a completely continuous field without an exceptional family of elements, with respect to $\mathbb{K}$, then the problem NCP(f, $\mathbb{K}$) has a solution.*

COROLLARY 4.4.4. *If $\mathbb{K} \subset \mathbb{R}^n$ is an arbitrary closed pointed convex cone, then for any continuous mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, there exists either a solution to the problem NCP(f, $\mathbb{K}$), or an exceptional family of elements for f, with respect to $\mathbb{K}$.*

Remark. Theorem 4.4.1 is valid even if the k-set field f is defined only on the cone $\mathbb{K}$. Indeed, in this case we consider the k-set field $h(x) = x - T(P_{\mathbb{K}}(x))$, *for all* $x \in H$.

4.5 Two applications

We will close this chapter with two applications of the notion of *exceptional family of elements* to the study of two particular complementarity problems. We consider the problem $NCP(f, \mathbb{R}^n_+)$, where $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ is a P_0-function. The notions of P_0-function (resp. P-function) were introduced by J. J. Moré and W. Rheinboldt as a natural extension of the notions of P_0-matirx (resp. P-matrix). For more details about P_0 and P-functions the reader is referred to (Moré, J. J. and Rheinboldt, W. [1]).

We recall that a matrix is a P_0-matrix (resp. P-matrix) if all its principal minors are nonnegative (resp. positive). Let D be a subset of $\mathbb{R}^n$ and $f : D \to \mathbb{R}^n$ a function. We say that f is a P_0-function (resp. P-function) on D if for all $x, y \in D$, $x \neq y$, there exists an index $i = i(x, y)$, such that $x_i \neq y_i$ and $(x_i - y_i)(f_i(x) - f_i(y)) \geq 0$ (resp.

$$(x_i - y_i)(f_i(x) - f_i(y)) > 0).$$

Considering the problem $NCP(f, \mathbb{R}_+^n)$, we denote by $\mathcal{F}$ the set of all feasible solutions, i.e.,

$$\mathcal{F} = \{x \in \mathbb{R}_+^n : f_i(x) \geq 0, \textit{for all } i = 1, 2, ... n\}.$$

We say that $u \in \mathcal{F}$ is *strictly feasible* if $f_i(u) > 0$ for all $i = 1, 2, ..., n$. It is known that if f is a monotone mapping, i.e., $(x-y)(f(x)-f(y)) \geq 0$ for all $x, y \in \mathbb{R}_+^n$ and $\mathcal{F}$ contains at least one strictly feasible point, then the $NCP(f, \mathbb{R}_+^n)$ has a solution (Moré, J. J. [1]). This result cannot be extended to the class of P_0-functions (even P-function), as the following simple example shows.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by $f_1(x) = \psi(x_1) + x_2$ and $f_2(x) = x_2$, where $\psi : \mathbb{R} \to \mathbb{R}$ is $\psi(t) = -\frac{1}{2}e^{-t}$. The function f is a continuous P-function and $\mathcal{F} = \left\{(x_1, x_2) \in \mathbb{R}_+^2 : x_1 \geq 0, x_2 \geq \frac{1}{2}e^{-x_1}\right\}$. The point $u = (0, 1)$ is a strictly feasible point, but the only point which satisfies the complementarity condition associated with f is $(0, 0)$ which is not a solution to the problem $NCP(f, \mathbb{R}_+^n)$ since $(0, 0) \notin \mathcal{F}$.

The next result shows that the problem $NCP(f, \mathbb{R}_+^n)$ associated to a P_0-function is solvable if the set $\mathcal{F}$ contains n points of a particular form.

THEOREM 4.5.1. *Let $f : \mathbb{R}_+^n \to \mathbb{R}^n$ be a P_0-function. If the feasible set $\mathcal{F}$ contains n points $e^{(j)}, j = 1, 2, ..., n$ such that $e_j^{(j)} > 0$ and $e_i^{(j)} = 0$ for all $i \neq j$, then the problem $NCP(f, \mathbb{R}_+^n)$ has a solution.*

Proof. If, for a particular j, $f_j(e^{(j)}) = 0$, we have that $e^{(j)}$ is a solution to the problem $NCP(f, \mathbb{R}_+^n)$. Hence, we can suppose that for every $j \in \{1, 2, ..., n\}$, $f_j(e^{(j)}) > 0$. The theorem will be proved, if we show that f is without *exceptional families of elements* in the sense of Definition 4.1.1

with respect to $\mathbb{R}^n_+$. Indeed, we suppose that f has an exceptional family of elements $\{x^r\}_{r>0} \subset \mathbb{R}^n_+$. We have the following facts:

(i_1) $\|x^r\| \to +\infty$ *as* $r \to +\infty$,

(i_2) *for every* $r > 0$ *there exists* $\mu_r > 0$, such that

(a) $f_i(x^r) = -\mu_r x_i^r$, *if* $x_i^r > 0$,

(b) $f_i(x^r) \geq 0$ *if* $x_i^r = 0$.

By property (i_1) there exists an index $r > 0$ such that

$$\|x^r\| > \sqrt{\sum_{j=1}^{n} \left(e_j^{(j)}\right)^2}. \tag{4.5.1}$$

From (4.5.1) we have that there exists $j_0 \in \{1, 2, \ldots, n\}$ such that $x_{j_0}^r > e_{j_0}^{(j_0)}$. We observe that

$$x^r \neq e^{(j_0)} = \left(0, 0, .., e_{j_0}^{(j_0)}, 0, 0, ..., 0\right). \tag{4.5.2}$$

Since f is a P_0-function, there exists $i = i\left(x^r, e^{(j_0)}\right)$ such that $x_i^r \neq e_i^{(j_0)}$ and

$$\left(x_i^r - e_i^{(j_0)}\right)\left(f_i(x^r) - f_i\left(e^{(j_0)}\right)\right) \geq 0. \tag{4.5.3}$$

If $i = j_0$, then we have

$$\left(x_{j_0}^r - e_{j_0}^{(j_0)}\right)\left(f_{j_0}(x^r) - f_{j_0}\left(e^{(j_0)}\right)\right) < 0,$$

which is a contradiction of (4.5.3). If $i \neq j_0$, then we have $e_i^{(j_0)} = 0$ and $x_i^r > 0$, which imply again

$$\left(x_i^r - e_i^{(j_0)}\right)\left(f_i(x^r) - f_i\left(e^{(j_0)}\right)\right) < 0.$$

The last inequality is also a contradiction of (4.5.3). We conclude that f is without an exceptional family of elements with respect to $\mathbb{R}^n_+$, in the sense of Definition 4.1.1, and by Theorem 4.1.3, the problem $NCP\left(f, \mathbb{R}^n_+\right)$ has a solution. □

Theorem 4.5.1 can be used to obtain an existence theorem for the *Generalized Linear Complementarity Problem*, associated to a matrix M and a vector q. We recall the definition of this problem. By a vertical bloc, matrix M of type $(m_1, m_2, ..., m_n)$ we mean a matrix

$$M = \begin{pmatrix} M^1 \\ M^2 \\ \vdots \\ M^j \\ \vdots \\ M^n \end{pmatrix}$$

where the j^{th} block M^j has order $m_j \times n$. Thus for $m = \sum_{j=1}^{n} m_j$, the matrix M is of order $m \times n$. Let q be a vector in $\mathbb{R}^m$ partitioned conformably with M, i.,e.,

$$q = \begin{pmatrix} q^1 \\ q^2 \\ \vdots \\ q^j \\ \vdots \\ q^n \end{pmatrix}$$

with $q^j \in \mathbb{R}^{m_j}$.

The Generalized Linear Complementarity Problem (associated with M and q), denoted by $GLCP(M, q)$, is

$$GLCP(M,q): \begin{cases} \text{find } z \in \mathbb{R}^n \text{ such that}: \\ z \geq 0, M^j z + q^j \geq 0_{m_j} \text{ and} \\ z_j \prod_{i=1}^{mj} \left(M^j z + q^j\right)_i = 0, \ (j = 1, 2, \ldots, n), \end{cases}$$

where 0_{m_j} is the null vector in $\mathbb{R}^{m_j}$. This clearly agrees with the *Linear Complementarity Problem* when $m_j = 1$ and M^j is the j^{th} row of M (j = 1, 2, ...,n). The problem $GLCP(M, q)$ was defined in (Cottle, R. W. and Dantzig, G. B., [1] and it was studied in (Isac, G. and Carbone, A. [1]) (Carbone, A. and Isac, G. [1]), (Ebiefung, A. A [1]), (Ebiefung, A. A and Kostreva, M. M. [1]), (Mohan, S. R., Neogy, S. K. and Sridhar, R., [1]), (Szank, B. P., [1]), (Sznajder, R. and Gowda, M. S., [1]) and (Vandeberge, L., De Moor, B. L. and Vanderwalle, J. [1]) among others.

Now, we recall some notions on rectangular matrices. Let M be a vertical block matrix of type $(m_1, m_2, ..., m_n)$. An $n \times n$ submatrix N of M is called a *representative submatrix* if its j^{th} row is drawn from the j^{th} block, M^j of M. The properties of M are based on properties of its representative submatrices. Having this concept, we can talk about principal submatrices of the rectangular matrix M. Obviously a vertical block matrix M of type $(m_1, m_2, ..., m_n)$ has $\prod_{j=1}^{n} m_j$ representative submatrices.

Let M be a vertical block matrix of type $(m_1, m_2, ..., m_n)$. Consider a principal submatrix of M. The determinant of such a matrix is a *principal minor* of M. A vertical block matrix M of type $(m_1, m_2, ..., m_n)$ is called a P_0-matrix (resp. P-matrix) if and only if all its principal minors are nonnegative (resp. strictly positive).

The next result is an existence theorem for the problem $GLCP(M, q)$ when M is a P_0-vertical block matrix. This existence theorem is more general than some existence results for this problem obtained by other authors.

THEOREM 4.5.2. *Let M be a P_0-vertical block matrix of type $(m_1, m_2, ..., m_n)$ and $q \in \mathbb{R}^m$ a vector partitioned conformably with M, $m = \sum_{j=1}^{n} m_j$. Assume that there exists n vectors $x^{(l)} = (x_k^l), l = 1, 2, ..., n$, $k = 1, 2, \ldots, n$ such that*

$$\begin{cases} \text{for each } l = 1, 2, ..., n \\ x_k^l = 0 \ \text{ for } k \neq l, \ x_l^l > 0 \text{ and} \\ \min\limits_{1 \le i \le m_j} \left\{ \left(M^j x^{(l)} \right)_i + q_i^j \right\} \ge 0, \text{ for } j = 1, ..., n. \end{cases} \quad (4.5.4)$$

Then the problem $GLCP(M, q)$ has a solution.

Proof. We consider the piecewise linear function $f : \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$f_j(x) = \min_{1 \le i \le m_j} \left\{ \left(M^j x \right)_i + q_i^j \right\}, \ j = 1, 2, ..., n.$$

Clearly, the solvability of the *GLCP*(*M*, *q*) is equivalent to the solvability of the problem $NCP\left(f, \mathbb{R}^n_+\right)$. As already observed by A. A. Ebiefung, the assumptions on *M*, imply that f is a P_0-function (Ebiefung, A. A., [1]). Condition (4.5.4) implies that the assumptions of Theorem 4.5.1 hold for f defined above and $e^{(1)} = x^{(1)}, e^{(2)} = x^{(2)}, \ldots., e^{(n)} = x^{(n)}$. Hence, the result follows from Theorem 4.5.1. □

5

LERAY–SCHAUDER TYPE ALTERNATIVES. EXISTENCE THEOREMS

Considering the results presented in Chapter 4, we conclude that the notion of *exceptional family of elements* can be used to study the solvability of *complementarity problems*. This concept is supported by the notion of *topological degree* and by the notion of *zero-epi mapping*, which is a kind of topological degree, but more refined than the classical notion of topological degree.

It is useful, from the point of view of applications to extend the investigation methods based on the notion of exceptional family of elements to other classes of mappings, different than the mappings used in Chapter 4, and to set-valued mappings. To do this, the topological degree can be an obstacle. Because of this fact, in this chapter we will establish some relations between the notion of exceptional family of elements and *Leray–Schauder type alternatives*. The Leray–Schauder type alternatives are not based on topological degree (see Chapter 3). Moreover, there exist Leray–Schauder type alternatives for set-valued mappings, (see again Chapter 3).

By establishing relations between the notion of *exceptional family of elements* and Leray–Schauder type alternatives we will attain also two major goals. The first goal is the fact that the method based on the notion of exceptional family of elements will be founded on Leray–Schauder type alternatives. In this way, we obtain a simpler method, which is open to new developments, related to new classes of mappings and also related to other kinds of applications. The second goal is to give a new direction of applications of Leray–Schauder type alternatives. By this method we introduce a *very general notion of coercivity*. The *coercivity conditions* are used in nonlinear analysis and in optimization theory.

In the Leray–Schauder Alternative Theorem (see Chapter 3) the essential idea is to join the operator f to the constant operator $\mathcal{O}(x) = 0$ by means of the *homotopy* $H : \overline{\Omega} \times [0,1] \to E$ defined by $H(x,\lambda) = \lambda f(x)$, in a such way that the unique fixed-point of $H(\cdot,0)$ can be "continued" in a fixed-point of $H(\cdot,\lambda)$, for each $\lambda \in [0, 1]$ and in particular in a fixed-point of $H(\cdot,1) = f$. This continuation process is possible, if all operators $H(\cdot,\lambda)$, for $\lambda \in [0, 1]$ are fixed-point free on the boundary of Ω. In applications, the Leray–Schauder Principle is usually used together with the so-called *a priori bounds techniques*. In many problems related to *complementarity problems* or to *variational inequalities,* it is hard to locate the solution and hence to use the *a priori bounds principle*. Establishing a relation between the notion of *exceptional family of elements* and the *Leray–Schauder Principle*, we give a new kind of application of this powerful classical principle, well known in nonlinear analysis. This chapter is dedicated to this development.

5.1 Nonlinear complementarity problems in arbitrary Hilbert spaces

Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $f : H \to H$ a mapping. We consider in this section the problem

$$NCP(f,\mathbb{K}) : \begin{cases} \text{find } x_* \in \mathbb{K} \text{ such that} \\ f(x_*) \in \mathbb{K}^* \text{ and } \langle x_*, f(x_*)\rangle = 0. \end{cases}$$

For any real number $r > 0$ we consider the sets:

$$B_r = \{x \in H : \|x\| < r\},$$

$$S_r = \{x \in H : \|x\| = r\}.$$

Obviously, we have that $\partial B_r = S_r$. Let $P_{\mathbb{K}}$ be the projection operator onto $\mathbb{K}$. We know that $P_{\mathbb{K}}(x)$ is well defined for any $x \in H$. We recall that a mapping $f : H \to H$ is a *completely continuous field* if f has a representation of the form $f(x) = x - T(x)$, where $T : H \to H$ is a *completely continuous mapping*. Similarly, we say that a mapping $f : H \to H$ is an *α-condensing* *field* if f has a representation of the form $f(x) = x - T(x)$, where $T : H \to H$ *is an* α*-condensing mapping*. Also, we say that a mapping $f : H \to H$ is a *nonexpansive field* if f has a representation of the form $f(x) = x - T(x)$ where $T : H \to H$ is a *nonexpansive mapping*.

We recall that a mapping $f : H \to H$ is said to be *monotone* if for any $x, y \in H$, we have $\langle x - y, f(x) - f(y) \rangle \geq 0$ and a mapping $T : H \to H$ is *pseudo-contractant* if the mapping $f(x) = x - T(x)$ is monotone. Also, a mapping $f : H \to H$ is said to be *demicontinuous* if for any sequence $\{x_n\}_{n \in N} \subset H$, convergent in norm to an element $x_* \in H$, we have that the sequence $\{f(x_n)\}_{n \in N}$ is weakly convergent to $f(x_*)$. Obviously any continuous mapping is *demicontinuous*.

Let $\mathbb{K} \subset H$ be a closed pointed convex cone and $f : H \to H$ a mapping. Let $\Phi : H \to \mathbb{K}$ be the mapping defined by $\Phi_{\mathbb{K}}(x) = P_{\mathbb{K}}[x - f(x)]$.

DEFINITION 5.1.1. *We say that a continuous mapping $f : H \to H$ is a projectionally Leray–Schauder mapping with respect to $\mathbb{K}$, if for any $r > 0$, the condition*

$$(LS):\ x \neq \lambda \Phi_{\mathbb{K}}(x) \text{ for any } (x, \lambda) \in \partial B_r \times]0,1[,$$

implies that, $\Phi_{\mathbb{K}}$ has a fixed-point in $\overline{B_r}$. (Obviously the fixed-point is in $\overline{B_r} \cap \mathbb{K}$.)

Examples

(1) If $\mathbb{K}$ has a compact base (i.e., $\mathbb{K}$ is a locally compact cone), then in this case, any bounded mapping $f : H \to H$ is a *projectionally Leray–Schauder mapping*, with respect to $\mathbb{K}$. (In this example, f is a bounded mapping means that for any bounded subset $D \subset H$, we have that $f(D)$ is a bounded set).

(2) If $f : H \to H$ is a completely continuous field, then f is a projectionally Leray–Schauder mapping with respect to any closed convex cone $\mathbb{K} \subset H$. (In particular, in $\mathbb{R}^n$, any continuous mapping is

a projectionally Leray–Schauder mapping with respect to any closed convex cone.) This example is a consequence of Theorem 3.2.4.

(3) If $f : H \to H$ is an α-condensing field, then f is a projectionally Leray–Schauder mapping with respect to any closed convex cone $\mathbb{K} \subset H$. This result is a consequence of Theorem 3.4.4.

(4) If $f : H \to H$ is a nonexpansive field, then f is a projectionally Leray–Schauder mapping, with respect to any closed convex cone $\mathbb{K} \subset H$. This example is a consequence of *Theorem 3.4.6*.

(5) We say that a mapping $f : H \to H$ is a *projectionally pseudo-contractant field with respect to* $\mathbb{K}$ if the mapping $\Phi_{\mathbb{K}}(x) = P_{\mathbb{K}}[x - f(x)]$ is *pseudo-contractant*. If f is a continuous and projectionally pseudo-contractant field with respect to $\mathbb{K}$, then f is a projectionally Leray–Schauder mapping with respect to $\mathbb{K}$. This result is a consequence of Theorem 3.4.10.

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $f : H \to H$ a mapping.

DEFINITION 5.1.2. *We say that a family of elements* $\{x_r\}_{r>0} \subset \mathbb{K}$ *is an exceptional family of elements (denoted shortly by EFE) for the mapping f, with respect to* $\mathbb{K}$, *if for every real number* $r > 0$, *there exists a real number* $\mu_r > 0$ *such that the vector* $u_r = \mu_r x_r + f(x_r)$ *satisfies the following conditions:*

(1) $u_r \in \mathbb{K}^*$,

(2) $\langle u_r, x_r \rangle = 0$,

(3) $\|x_r\| \to +\infty$ *as* $r \to +\infty$.

Related to the notion of *EFE* we have the following alternative theorem for nonlinear complementarity problems.

THEOREM 5.1.1. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed pointed convex cone and* $f : H \to H$ *a continuous mapping. If* f *is a* *projectionally Leray–Schauder mapping with respect to $\mathbb{K}$, then there exists either a solution to the problem NCP(f, $\mathbb{K}$), or f has an EFE with respect to $\mathbb{K}$.*

Proof. We consider the mapping $\Phi_{\mathbb{K}}(x) = P_{\mathbb{K}}\left[x - f(x)\right]$, for any $x \in H$. From complementarity theory (see also Chapter 2), we know that the problem *NCP(f, $\mathbb{K}$)* has a solution if and only if the mapping $\Phi_{\mathbb{K}}$ has a fixed-point. Therefore, if the mapping $\Phi_{\mathbb{K}}$ has a fixed-point, this fixed-point must be in $\mathbb{K}$ and the problem *NCP(f, $\mathbb{K}$)* has a solution. The converse is also true. If the problem *NCP(f, $\mathbb{K}$)* has a solution, then the proof is complete.

Suppose that the problem *NCP(f, $\mathbb{K}$)* is without solution. Obviously, in this case the mapping $\Phi_{\mathbb{K}}$ is fixed-point free. Because f is a projectionally Leray–Schauder mapping we have that for any $r > 0$ there exist x_r with $\|x_r\| = r$ and $\lambda_r \in \,]0, 1[$ such that $x_r = \lambda_r P_{\mathbb{K}}\left[x_r - f(x_r)\right]$. Because $\mathbb{K}$ is a cone, we have that $x_r \in \mathbb{K}$. We have

$$\frac{1}{\lambda_r} x_r = P_{\mathbb{K}}\left[x_r - f(x_r)\right].$$

Applying the properties of operator $P_{\mathbb{K}}$ we obtain

$$\begin{cases} \left\langle x_r/\lambda_r - \left(x_r - f(x_r)\right), y\right\rangle \geq 0, \text{ for all } y \in \mathbb{K}, \\ \left\langle x_r/\lambda_r - \left(x_r - f(x_r)\right), x_r/\lambda_r\right\rangle = 0, \end{cases}$$

which implies

$$\begin{cases} \left\langle (1/\lambda_r - 1)x_r + f(x_r), y\right\rangle \geq 0 \text{ for all } y \in \mathbb{K}, \\ \left\langle (1/\lambda_r - 1)x_r + f(x_r), x_r\right\rangle = 0. \end{cases}$$

If we put $\mu_r = \left(\frac{1}{\lambda_r} - 1\right)$ it follows that $\mu_r x_r + f(x_r) \in \mathbb{K}^*$, $\left\langle \mu_r x_r + f(x_r), x_r\right\rangle = 0$, and since $\|x_r\| = r$, for any $r > 0$, we have that

$\{x_r\}_{r>0}$ is an *EFE* for the mapping *f*, with respect to $\mathbb{K}$, and the proof is complete. □

A consequence of Theorem 5.1.1 is the following result.

THEOREM 5.1.2 [Existence theorem]. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K}\subset H$ a closed pointed convex cone and $f: H\to H$ a continuous mapping. If f is a projectionally Leray–Schauder mapping without an EFE with respect to $\mathbb{K}$, then the problem NCP(f, $\mathbb{K}$) has a solution.*

COROLLARY 5.1.3. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K}\subset H$ a closed pointed convex cone and $f: H\to H$ a continuous mapping. If f is a completely continuous field without an EFE, with respect to $\mathbb{K}$, then the problem NCP(f, $\mathbb{K}$) has a solution.*

COROLLARY 5.1.4. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K}\subset H$ a closed pointed convex cone and $f: H\to H$ a mapping. If f is an α-condensing field without an EFE, with respect to $\mathbb{K}$, then the problem NCP(f, $\mathbb{K}$) has a solution.*

COROLLARY 5.1.5. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K}\subset H$ a closed pointed convex cone and $f: H\to H$ a mapping. If f is a nonexpansive field without an EFE, with respect to $\mathbb{K}$, then the problem NCP(f, $\mathbb{K}$) has a solution.*

COROLLARY 5.1.6. *Let $(\mathbb{R}^n,\langle\cdot,\cdot\rangle)$ be the n-dimensional Euclidean space, $\mathbb{K}\subset\mathbb{R}^n$ a closed pointed convex cone. If $f:\mathbb{R}^n$ is a continuous mapping without EFE, with respect to $\mathbb{K}$, then the problem NCP(f, $\mathbb{K}$) has a solution.*

A consequence of the results presented above, is the fact that for applications, it is useful to have tests that can be used to decide if a given mapping *does not have exceptional families of elements.* A test is given by the following definition.

DEFINITION 5.1.3. *We say that a mapping $f: H \to H$ satisfies condition (θ) with respect to a closed convex cone $\mathbb{K} \subset H$ if there exists $\rho > 0$ such that for each $x \in \mathbb{K}$ with $\|x\| > \rho$ there exists $y \in \mathbb{K}$ with $\|y\| < \|x\|$ such that* $\langle x - y, f(x) \rangle \geq 0$.

Remark. Condition (θ) is due to G. Isac. For more details the reader is referred to (Isac, G. [16], [17], [26]), (Isac, G. and Carbone, A. [1]).

The importance of condition (θ) is given by the following result.

THEOREM 5.1.7. *If $f: H \to H$ satisfies condition (θ) with respect to a closed pointed convex cone $\mathbb{K} \subset H$, then f is without an EFE (with respect to $\mathbb{K}$).*

Proof. Indeed, we suppose that f has an *EFE* $\{x_r\}_{r>0} \subset \mathbb{K}$. Then for all $r > 0$ we have $u_r = \mu_r x_r + f(x_r) \in \mathbb{K}^*, \langle x_r, u_r \rangle = 0$ and $\|x_r\| \to +\infty$ as $r \to +\infty$. We take $r > 0$ such that $\|x_r\| > \rho$, where ρ is the positive real number defined in condition (θ). Since f satisfies condition (θ), there exists $y_r \in \mathbb{K}$ such that $\|y_r\| < \|x_r\|$ and $\langle x_r - y_r, f(x_r) \rangle \geq 0$.
We have

$$0 \leq \langle x_r - y_r, f(x_r) \rangle = \langle x_r - y_r, u_r - \mu_r x_r \rangle$$
$$= \langle x_r - y_r, u_r \rangle - \mu_r \|x_r\|^2 + \mu_r \langle y_r, x_r \rangle \leq -\mu_r \|x_r\| \left[\|x_r\| - \|y_r\| \right] < 0,$$

which is a contradiction. Hence f is without an *EFE* and the proof is complete. □

We recall the classical notion of *coercivity.*

DEFINITION 5.1.4. *We say that a mapping $f: H \to H$ is coercive with respect to a closed pointed convex cone $\mathbb{K} \subset H$, if there exists an element $x_0 \in \mathbb{K}$ such that*

$$\lim_{x\in\mathbb{K},\|x\|\to\infty} \frac{\langle x - x_0, f(x) - f(x_0)\rangle}{\|x - x_0\|} = +\infty.$$

THEOREM 5.1.8. *If $f: H \to H$ is a coercive mapping with respect to a closed pointed convex cone $\mathbb{K}$, then f satisfies condition (θ) and consequently f is without an EFE.*

Proof. Let k_0 be a real number such that $\|f(x_0)\| \leq k_0$. The coercivity of f implies that there exists $\rho > 0$ such that for any $x \in \mathbb{K}$ with $\|x\| > \rho$ we have

$$\frac{\langle x - x_0, f(x) - f(x_0)\rangle}{\|x - x_0\|} \geq k_0$$

or

$$\langle x - x_0, f(x) - f(x_0)\rangle \geq k_0 \|x - x_0\|.$$

We can take ρ such that $\rho > \|x_0\|$. We have

$$\langle x - x_0, f(x)\rangle \geq \langle x - x_0, f(x_0)\rangle + k_0 \|x - x_0\|$$
$$\geq -\|x - x_0\|\|f(x_0)\| + k_0 \|x - x_0\| = \|x - x_0\|\left[k_0 - \|f(x_0)\|\right] \geq 0.$$

If for any $x \in \mathbb{K}$ with $\|x\| > \rho$ we take $y = x_0$, we obtain that f satisfies condition (θ). Applying Theorem 5.1.7 we obtain that f is without an *EFE*, with respect to $\mathbb{K}$. □

Remark. There exist mappings that are not coercive with respect to a closed pointed convex cone but without an *EFE*. In this sense we have the following example. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\mathbb{K} \subset H$ a closed pointed convex cone. We consider the mapping $f: H \to H$ defined by $f(x) = \frac{x}{\|x\| + 1}$. The mapping f cannot be coercive with respect to $\mathbb{K}$, because for any $x_0 \in \mathbb{K}$, we have

$$\frac{\langle x-x_0, f(x)-f(x_0)\rangle}{\|x-x_0\|} \le \|f(x)-f(x_0)\| \le \left\| \frac{x}{\|x\|+1} - \frac{x_0}{\|x_0\|+1} \right\| \le 2,$$

for any $x \in \mathbb{K}$. The mapping f is without an *EFE*. Indeed if $\{x_r\}_{r>0} \subset \mathbb{K}$ is an *EFE*, then we have that for any $r > 0$ there exists $\mu_r > 0$ such that $u_r = \mu_r x_r + f(x_r) \in \mathbb{K}^*$ and $\langle x_r, u_r \rangle = 0$. In this case we have

$$0 = \langle x_r, u_r \rangle = \left\langle x_r, \mu_r x_r + \frac{x_r}{\|x_r\|+1} \right\rangle = \|x_r\|^2 \left[\mu_r + \frac{1}{\|x_r\|+1} \right],$$

which implies that $\|x_r\| = 0$ for any $r > 0$, and it is impossible to have $\|x_r\| \to +\infty$ as $r \to \infty$.

Therefore, the property for a mapping of being "without an EFE" is a kind of coercivity property but strictly more general than the classical coercivity property. Obviously, condition *(θ) implies this generalized coercivity condition. Considering the results presented above we deduce that if* f *is a projectionally Leray–Schauder mapping and satisfies condition (θ), then the problem NCP(f, $\mathbb{K}$) has a solution.*

Now, we show that if $\mathbb{K}$ is a locally compact convex cone (i.e., if $\mathbb{K}$ has a compact base), then condition (θ) gives also information about the norm of solution. This information about the solution is important in the study of equilibrium problems.

THEOREM 5.1.9. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone. The cone $\mathbb{K}$ is supposed to be locally compact. Let $f : H \to H$ be a continuous bounded mapping such that $f(0) \notin \mathbb{K}^*$. If f satisfies condition (θ) with respect to $\mathbb{K}$, then the problem NCP(f, $\mathbb{K}$) has a solution x_* such that $\|x_*\| \le \rho$ (where ρ is defined by condition (θ)).*

Proof. For every $\varepsilon > 0$, consider the Tihonov regularization $f_\varepsilon(x) = f(x) + \varepsilon x$, for all $x \in H$. The mapping f_ε satisfies condition (θ), but with a strict inequality, i.e., there exists $\rho > 0$ such that for all $x \in \mathbb{K}$ with $||x|| > \rho$, there exists $y \in \mathbb{K}$ with $||y|| < ||x||$ such that

$$\langle x - y, f_\varepsilon(x)\rangle > 0. \tag{5.1.1}$$

Indeed, let $\rho > 0$ be the real number given by condition (θ), for f. Let $x \in \mathbb{K}$ be an element such that $||x|| > \rho$. By condition (θ) there exists $y \in \mathbb{K}$ with $||y|| < ||x||$ such that,

$$\langle x - y, f_\varepsilon(x)\rangle = \langle x - y, f(x) + \varepsilon x\rangle \geq \langle x - y, \varepsilon x\rangle$$
$$\geq \varepsilon\left[\|x\|^2 - \|x\|\|y\|\right] = \varepsilon\|x\|\left[\|x\| - \|y\|\right] > 0.$$

We note that the mapping f_ε is bounded and because $\mathbb{K}$ is locally compact, f_ε is a projectionally Leray–Schauder mapping. We keep this ρ and we take $\varepsilon_* > 0$. By Theorem 5.1.2 and 5.1.7, for every $\varepsilon \in\,]0, \varepsilon_*[$ there exists a solution $x_*(\varepsilon)$ to the problem $NCP(f_\varepsilon, \mathbb{K})$. Condition (5.1.1), implies that for $x_*(\varepsilon)$ we must have $||x_*(\varepsilon)|| \leq \rho$. If $0 < \varepsilon_1 < \varepsilon_2 \leq \varepsilon_*$, then we have $x_*(\varepsilon_1) \neq x_*(\varepsilon_2)$. Indeed, $\varepsilon_2 = \varepsilon_1 + r$ with $r > 0$. If $x_*(\varepsilon_2) = x_*(\varepsilon_1)$, then we have

$$0 = \langle f(x_*(\varepsilon_2)) + \varepsilon_2 x_*(\varepsilon_2), x_*(\varepsilon_2)\rangle$$
$$= \langle f(x_*(\varepsilon_1)) + (\varepsilon_1 + r)x_*(\varepsilon_1), x_*(\varepsilon_1)\rangle = r\|x_*(\varepsilon_1)\|^2,$$

which implies that $x_*(\varepsilon_1) = 0$, and finally $f(0) \in \mathbb{K}^*$, which is impossible.

We have that $\{x_*(\varepsilon)\}_{0<\varepsilon<\varepsilon_*}$ is a branch of solutions, where $x_*(\varepsilon)$ is a solution to the problem $NCP(f_\varepsilon, \mathbb{K})$ and it is a subset of $\overline{B_\rho} \cap \mathbb{K}$. We take $\varepsilon_n = \frac{1}{n}, n \in N$ and $\varepsilon_* = 1$. Because $\mathbb{K}$ is a locally compact cone, we have that the sequence $\left\{x_*\left(\frac{1}{n}\right)\right\}_{n\in N}$ has a convergent subsequence $\left\{x_*\left(\frac{1}{n_k}\right)\right\}_{k\in N}$. Let x_* be its limit. By continuity we obtain that x_* is a solution of the problem $NCP(f, \mathbb{K})$ such that $\|x_*\| \leq \rho$ and the proof is complete. □

The following result is an application of Theorem 5.1.2 and Theorem 5.1.7 to the fixed-point theory.

THEOREM 5.1.10. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K}\subset H$ a closed pointed convex cone and $h:\mathbb{K}\to\mathbb{K}$ a mapping. If the mapping $f(x)=x-h(x)$ is a projectionally Leray–Schauder mapping which satisfies condition (θ) with respect to $\mathbb{K}$, then h has a fixed point in $\mathbb{K}$.*

Proof. From the complementarity theory it is known that the mapping $h:\mathbb{K}\to\mathbb{K}$ has a fixed point in $\mathbb{K}$ if and only if the problem $NCP(I-h,\mathbb{K})$ has a solution. Since by Theorem 5.1.2 and 5.1.7 the problem $NCP(I-h,\mathbb{K})$ has a solution, the conclusion of the theorem follows. □

By the following theorems, we put in evidence some classes of mappings that satisfy condition (θ) that is mappings without an *EFE*.

THEOREM 5.1.11. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K}\subset H$ a closed pointed convex cone and $f:H\to H$ a mapping. Let $\varphi:[0,+\infty[\to[0,+\infty[$ be a mapping such that $\lim_{t\to+\infty}\varphi(t)=+\infty$ and $\varphi(t)>0$ for any $t>0$. If $\langle x-y,f(x)-f(y)\rangle\geq\|x-y\|\varphi(\|x-y\|)$, for any x, $y\in\mathbb{K}$, then the mapping f satisfies condition (θ).*

Proof. Obviously, we suppose that f is not a trivial mapping. Let $y_0\in\mathbb{K}$ be an arbitrary element such that $\|f(y_0)\|>0$. We denote $\rho_0=\|f(y_0)\|$. By assumption, there exists $\rho>0$ such that $\varphi(\|x-y_0\|)\geq\rho_0$ for any $x\in\mathbb{K}$ with $\|x-y_0\|>\rho$. If $x\in\mathbb{K}$ and $\|x\|>\rho+\|y_0\|>\|y_0\|$, then we have $\|x\|>\|y_0\|$ and $\langle x-y_0,f(x)-f(y_0)\rangle\geq\|x-y_0\|\varphi(\|x-y_0\|)$ which implies

$$\begin{aligned}\langle x-y_0,f(x)\rangle&\geq\langle x-y_0,f(y_0)\rangle+\|x-y_0\|\varphi(\|x-y_0\|)\\&\geq\|x-y_0\|\varphi(\|x-y_0\|)-\|x-y_0\|\|f(y_0)\|\\&=\|x-y_0\|\left[\varphi(\|x-y_0\|)-\|f(y_0)\|\right]\\&=\|x-y_0\|\left[\varphi(\|x-y_0\|)-\rho_0\right]\geq 0.\end{aligned}$$

If for any $x \in \mathbb{K}$, satisfying $\|x\| > \rho + \|y_0\|$, we take $y = y_0$, we obtain that f satisfies condition (θ). □

DEFINITION 5.1.5. *We say that a mapping $f: H \to H$ satisfies the weak Karamardian's condition with respect to a closed pointed convex cone $\mathbb{K} \subset H$, if there exists a bounded set $D \subset \mathbb{K}$, such that for all $x \in \mathbb{K} \setminus D$ there exists $y \in D$ such that $\langle x - y, f(x) \rangle \geq 0$.*

Remark. The classical Karamardian's condition supposes in Definition 5.1.5 that D is a compact convex set. (Karamardian, S. [1]].

THEOREM 5.1.12. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $f: H \to H$ a mapping. If f satisfies the weak Karamardian's condition with respect to $\mathbb{K}$, then f satisfies condition (θ).*

Proof. Let $D \subset \mathbb{K}$ be the set defined by the weak Karamardian's condition. Since D is bounded, there exists $\rho > 0$ such that $D \subset \overline{B_\rho} \cap \mathbb{K}$. For any $x \in \mathbb{K}$ such that $\|x\| > \rho$, there exists $y \in D$ (that is such that $\|y\| \leq \rho < \|x\|$) verifying $\langle x - y, f(x) \rangle \geq 0$. Hence, condition ($\theta$) is satisfied. □

Remark. Condition (θ) is a strict generalization of *Karamardian's condition*. Indeed, consider the Euclidean space $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$, the cone $\mathbb{K} = \mathbb{R}^2_+$ and the function $f(x_1, x_2) = (x_1, -x_1^2)$ for all $(x_1, x_2) \in \mathbb{R}^2$. We can show that f satisfies condition (θ), but not Karamardian's condition.

DEFINITION 5.1.6. *We say that $f: H \to H$ is a ρ-copositive mapping with respect to a closed pointed convex cone $\mathbb{K} \subset H$, if there exists $\rho > 0$ such that for all $x \in \mathbb{K}$ with $\|x\| > \rho$ we have $\langle x, f(x) \rangle \geq 0$.*

THEOREM 5.1.13. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K}\subset H$ a closed pointed convex cone and $f: H\to H$ a mapping. If f is ρ-copositive with respect to $\mathbb{K}$, then f satisfies condition (θ).*

Proof. Indeed, we consider the set $D=\overline{B_\rho}\cap\mathbb{K}$. Because D is bounded and $\langle x-0,f(x)\rangle=\langle x,f(x)\rangle\geq 0$ for any $x\in\mathbb{K}$ such that $\|x\|>\rho$, we have that f satisfies a weak Karamardian's condition and we apply Theorem 5.1.12.

□

Remark. The mapping $f:\mathbb{R}^2\to\mathbb{R}^2$ defined by $f(x_1,x_2)=(x_1,-x_1^2)$ is not ρ-copositive with respect to the convex cone $\mathbb{R}^2_+$, with some $\rho>0$. Indeed, if we suppose that f is ρ-copositive with respect to $\mathbb{R}^2_+$, we take $x=(x_1,x_2)$ with $x_1>0$ and $x_2>\max\{\rho,1\}$ and we have

$$\langle x,f(x)\rangle=\langle(x_1,x_2),(x_1,-x_1^2)\rangle=x_1^2-x_2x_1^2=x_1^2(1-x_2)<0,$$

which is impossible. From Theorem 5.1.13 we obtain the following result.

COROLLARY 5.1.14. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K}\subset H$ a closed pointed convex cone and $f: H\to H$ a mapping. If there exists a bounded subset C of $\mathbb{K}$ such that $\langle x,f(x)\rangle\geq 0$ for any $x\in\mathbb{K}\setminus C$, then the mapping f satisfies condition (θ).*

Proof. Indeed, because the set C is bounded, we can show that f is ρ-copositive with respect to $\mathbb{K}$ and Theorem 5.1.13 is applicable. □

The condition used in the definition of ρ-copositivity can be replaced in some particular cases by a weak condition using a radial retract. Let $(E,\|\cdot\|)$ be a Banach space and $r>0$ a real number. By definition the *radial retract* associated to the number r is:

$$\mathcal{R}_r(x)=\begin{cases} x, & \text{if } \|x\|\leq r,\\ r\dfrac{x}{\|x\|}, & \text{if } \|x\|>r.\end{cases}$$

THEOREM 5.1.15. *For any $r > 0$, the radial projection $\mathcal{R}_r$ is a continuous mapping.*

Proof. We prove that $\mathcal{R}_r$ is a *2*-Lipschitzian mapping, considering three possible situations:

(I) $\|x\| \le r$ *and* $\|y\| \le r$. In this case we have

$$\|\mathcal{R}_r(x) - \mathcal{R}_r(y)\| = \|x - y\| \le 2\|x - y\|.$$

(II) $\|x\| > r$ *and* $\|y\| \le r$. Then we have

$$\|\mathcal{R}_r(x) - \mathcal{R}_r(y)\| = \left\|\frac{rx}{\|x\|} - y\right\| \le \frac{r}{\|x\|}\|x - y\| + \left\|\frac{ry}{\|x\|} - y\right\|$$

$$\le \|x - y\| + \frac{\|y\|}{\|x\|}(\|x\| - r) \le \|x - y\| + \|x\| - \|y\| \le 2\|x - y\|.$$

(III). $\|x\| > r$ *and* $\|y\| > r$. In this last case, we have

$$\|\mathcal{R}_r(x) - \mathcal{R}_r(y)\| = \left\|\frac{rx}{\|x\|} - \frac{ry}{\|y\|}\right\| \le \frac{r}{\|x\|}\|x - y\| + r\|y\|\left|\frac{1}{\|x\|} - \frac{1}{\|y\|}\right|$$

$$\le \|x - y\| + \frac{r}{\|x\|}\left|\|y\| - \|x\|\right| \le 2\|x - y\|.$$

Therefore, the mapping $\mathcal{R}$ is a continuous mapping. □

DEFINITION 5.1.7. *We say that $f : H \to H$ is a strictly ρ-copositive mapping with respect to a closed pointed convex cone , $\mathbb{K} \subset H$, if there exists $\rho > 0$ such that for all $x \in \mathbb{K}$ with $\|x\| > \rho$ we have $\langle x, f(x)\rangle > 0$.*

THEOREM 5.1.16. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $f : H \to H$ a continuous mapping. If there exists $\rho > 0$ such that $\langle x, f(x)\rangle \ge 0$ for any $x \in \mathbb{K}$ with $\|x\| = \rho$, then the mapping $h : H \to H$ defined by $h(x) = f(\mathcal{R}_\rho(x)) + \|x - \mathcal{R}_\rho(x)\| x$, is continuous and strictly ρ-copositive with respect to K. Moreover, if $\mathbb{K}$ has a compact base and f is a bounded mapping, then the problem NCP(f, $\mathbb{K}$) has a solution x_* such that $\|x_*\| \le \rho$.*

Proof. Obviously, the mapping h is continuous. We remark also that $\mathcal{R}_\rho(\mathbb{K}) \subseteq \mathbb{K}$. Let $x \in \mathbb{K}$ be an arbitrary element such that $\|x\| > \rho$. Then $\|\mathcal{R}_\rho(x)\| = \rho > 0$ and $x = \frac{\|x\|}{\rho}\mathcal{R}_\rho(x)$. We have

$$\langle x, h(x)\rangle = \langle x, f(\mathcal{R}_\rho(x))\rangle + \|x - \mathcal{R}_\rho(x)\|\|x\|^2$$
$$= \frac{\|x\|}{\rho}\langle \mathcal{R}_\rho(x), f(\mathcal{R}_\rho(x))\rangle + \|x - \mathcal{R}_\rho(x)\|\|x\|^2 > 0.$$

Therefore $h(x)$ is strictly ρ-copositive with respect to $\mathbb{K}$ and consequently without an *EFE* with respect to $\mathbb{K}$. The mapping h is bounded if f is bounded. If the cone $\mathbb{K}$ has a compact base and f is bounded, we have that h is a projectionally Leray-Schauder mapping and consequently the problem *NCP*(h, $\mathbb{K}$) has a solution x_*. Because h is strictly ρ-copositive, the solution x_* must satisfy the condition $\|x_*\| \leq \rho$. But in this case $h(x_*) = f(x_*)$ and we have that x_* is a solution to the problem *NCP*(f, $\mathbb{K}$). □

COROLLARY 5.1.17. *Let $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ be n-dimensional Euclidean space, $\mathbb{K} \subset \mathbb{R}^n$ a closed pointed convex cone and $f : \mathbb{R}^n \to \mathbb{R}^n$ a continuous mapping. If there exists $\rho > 0$ such that $\langle x, f(x)\rangle \geq 0$ for any $x \in \mathbb{K}$ with $\|x\| = \rho$, then the problem NCP(f, $\mathbb{K}$) has a solution x_* such that $\|x_*\| \leq \rho$.*

THEOREM 5.1.18. *Let $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ be n-dimensional Euclidean space, $\mathbb{K} \subset \mathbb{R}^n$ a closed convex cone and $f : \mathbb{K} \to \mathbb{R}^n$ a continuous function. If there exists $\rho > 0$ such that for all $x \in \mathbb{K}$ with $\|x\| = \rho$, there exists u with $\|u\| < \rho$, such that $\langle x - u, f(x)\rangle \geq 0$, then the problem NCP($f$, $\mathbb{K}$) has a solution.*

Proof. For any $x \in K$ with $\|x\| > \rho$, we denote by $T_\rho(x)$ the radial projection of x onto $S_\rho^+ = \{x \in \mathbb{K} : \|x\| = \rho\}$, *i.e.*, $T_\rho(x) = \frac{x\rho}{\|x\|}$. We consider the function $g : \mathbb{K} \to \mathbb{R}^n$, defined by

$$g(x) = \begin{cases} f(x), & \text{if } \|x\| \le \rho, \\ f(T_\rho(x)) + \|x - T_\rho(x)\|, & \text{if } \|x\| > \rho. \end{cases}$$

For any $x \in \mathbb{K}$ with $\|x\| > \rho$ there exists $\lambda_x > 0$ such that $x = \lambda_x T_\rho(x)$. By assumption, for $T_\rho(x)$ there exists u_ρ^x with $\|u_\rho^x\| < \rho$ such that $\langle T_\rho(x) - u_\rho^x, f(T_\rho(x))\rangle \ge 0$. We have the following relations:

$$\begin{aligned} &\langle x - \lambda_x u_\rho^x, g(x)\rangle = \langle \lambda_x T_\rho(x) - \lambda_x u_\rho^x, g(x)\rangle \\ &= \langle \lambda_x T_\rho(x) - \lambda_x u_\rho^x, f(T_\rho(x)) + \|x - T_\rho(x)\| x\rangle \\ &= \lambda_x \langle T_\rho(x) - u_\rho^x, f(T_\rho(x))\rangle + \|x - T_\rho(x)\|\|x\|^2 - \|x - T_\rho(x)\|\langle \lambda_x u_\rho^x, x\rangle \\ &\ge \|x - T_\rho(x)\|\left[\|x\|^2 - \langle \lambda_x u_\rho^x, x\rangle\right] \\ &\ge \|x - T_\rho(x)\|\left[\lambda_x^2 \|T_\rho(x)\|^2 - \lambda_x^2 \|u_\rho^x\|\|T_\rho(x)\|\right] \\ &= \|x - T_\rho(x)\|\lambda_x^2 \|T_\rho(x)\|\left[\|T_\rho(x)\| - \|u_\rho^x\|\right] > 0. \end{aligned}$$

If for a given x we take $y = \lambda_x u_\rho^x$, we have that g satisfies condition (θ) with respect to $\mathbb{K}$.. Because we can show that g is continuous, we have that the problem *NCP*(g, $\mathbb{K}$) has a solution, $x_* \in \mathbb{K}$. The solution x_* is such that $\|x_*\| \le \rho$. Indeed, if $\|x_*\| > \rho$ we must have $\langle x_* - \lambda_{x_*} u_\rho^{x_*}, g(x_*)\rangle > 0$ or $\langle \lambda_{x_*} u_\rho^{x_*} - x_*, g(x_*)\rangle < 0$, which is impossible, because the problem *NCP*(g, $\mathbb{K}$) is equivalent to the variational inequality $\langle \lambda_{x_*} u_\rho^{x_*} - x_*, g(x_*)\rangle \ge 0$. Hence, $\|x_*\| \le \rho$ and in this case $g(x_*) = f(x_*)$, that is x_* is a solution to the problem *NCP*(f, $\mathbb{K}$). □

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $D \subset H$ a closed convex set. We say that D has a *retraction* if there exists a continuous mapping $\mathcal{R}_D : H \to H$

such that $\mathcal{R}_D(x) \in D$ for any $x \in H$ and $\mathcal{R}_D(x) = x$ for any $x \in D$. If D has a retraction we say that D is a retract of the space H. It is known (Zeidler, E. [1]) that every closed convex subset D of a Banach space $(E, \|\cdot\|)$ is a retract of E. We have the following result, which is a generalization of Theorem 5.1.16.

THEOREM 5.1.19. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed pointed convex cone and* $f: H \to H$ *a continuous mapping. Let* $D \subset K$ *be a closed bounded convex set such that* $0 \in D$. *If for any* $x \in K \setminus D$ *and for any* $y \in D$ *we have* $\langle x, f(y) \rangle \geq 0$, *then for any retraction* $\mathcal{R}_D : H \to H$ *we have that the mapping* $h: H \to H$ *defined by*

$$h(x) = f(\mathcal{R}_D(x)) + \|x - \mathcal{R}_D(x)\| x$$

satisfies the condition $\langle x, h(x) \rangle > 0$ *for any* $x \in \mathbb{K} \setminus D$. *Moreover, the mapping* h *satisfies condition* (θ) *with respect to* $\mathbb{K}$. *If* $\mathbb{K}$ *has a compact base and* f *is bounded, then the problem* $NCP(f, \mathbb{K})$ *has a solution* x_* *such that* $x_* \in D$.

Proof. We observe that the mapping h is a continuous mapping. Let x be an arbitrary element in $\mathbb{K} \setminus D$. We have

$$\langle x, h(x) \rangle = \langle x, f(\mathcal{R}_D(x)) + \|x - \mathcal{R}_D(x)\| x \rangle$$
$$= \langle x, f(\mathcal{R}_D(x)) \rangle + \|x - \mathcal{R}_D(x)\| \cdot \|x\|^2 > 0.$$

Because D is bounded, there exists $\rho > 0$ such that for any $x \in D$ we have $\|x\| \leq \rho$. Therefore for any $x \in \mathbb{K}$ with $\|x\| > \rho$ we have $\langle x, h(x) \rangle > 0$, which implies that h is strictly ρ-copositive with respect to $\mathbb{K}$ and it satisfied condition (θ). Consequently h is without an *EFE* with respect to $\mathbb{K}$. *If* $\mathbb{K}$ has a compact base and f is bounded, we have that h is bounded, and because $\mathbb{K}$ is locally compact, the mapping h is a projectionally Leray–Schauder mapping and the problem $NCP(f, \mathbb{K})$ has a solution x_*. Because $\langle x, h(x) \rangle > 0$ for any $x \in K \setminus D$, we must have that $x_* \in D$ and the proof is complete. □

COROLLARY 5.1.20. *Let* $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ *be the n-dimensional Euclidean space,* $\mathbb{K} \subset \mathbb{R}^n$ *a closed pointed convex cone and* $f : \mathbb{R}^n \to \mathbb{R}^n$ *a continuous mapping. Let* $D \subset \mathbb{K}$ *be a closed bounded convex set such that* $0 \in D$. *If for any* $x \in \mathbb{K} \setminus D$ *and any* $y \in D$ *we have* $\langle x, f(y)\rangle \geq 0$, *then the problem NCP(f,* $\mathbb{K}$*) has a solution* x_* *such that* $x_* \in D$. □

In (Ding, X, P. and Tan, K. K. [1]) was introduced the following condition which is more general than Karamardian's condition.

DEFINITION 5.1.8. *We say that* $f : H \to H$ *satisfies condition (DT) with respect to* $\mathbb{K}$*, if there exist a non-empty compact convex subset* $D_0 \subset \mathbb{K}$ *and a non-empty compact subset* $D_* \subset \mathbb{K}$ *such that for each* $x \in K \setminus D_*$, *there is a* $y \in conv(D_0 \cup \{x\})$ *such that* $\langle x - y, f(x)\rangle > 0$.

We have the following result.

THEOREM 5.1.21. *If* $f : H \to H$ *satisfies condition (DT) with respect to* $\mathbb{K}$*, then f satisfies condition* (θ) *and consequently f is without an EFE with respect to* $\mathbb{K}$.

Proof. Since D_0 and D_* are bounded sets, there exists $\rho > 0$ such that $D_0, D_* \subset \overline{B_\rho} \cap \mathbb{K}$. If $x \in \mathbb{K}$ is such that $\|x\| > \rho$, then by condition (*DT*) there exists $y \in conv(D_0 \cup \{x\})$ such that $\langle x - y, f(x)\rangle > 0$. We have $y = \lambda d_0 + (1-\lambda)x$, with $\lambda \in [0, 1]$ and $d_0 \in D_0$, which implies

$$\|y\| \leq \lambda\|d_0\| + (1-\lambda)\|x\| < \lambda\|x\| + (1-\lambda)\|x\| = \|x\|,$$

(since $\|d_0\| \leq \rho < \|x\|$). Therefore, *f* satisfies condition (θ). □

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\mathbb{K} \subset H$ a closed pointed convex cone. Let $\varphi : [0,+\infty[\to [0,+\infty[$ be a function such that $\lim_{t\to+\infty} \varphi(t) = +\infty$ and $u \in \mathbb{K}$ an arbitrary element.

DEFINITION 5.1.9. *We say that a mapping $f : H \to H$ is asymptotically (u, g, φ)-monotone on $\mathbb{K}$, if there exists $\rho > 0$ and a mapping $g : \mathbb{K} \to H$ such that $\langle x-u, f(x)-g(u)\rangle \geq \|x-u\|\varphi(\|x-u\|)$, for all $x \in \mathbb{K}$ with $\|x\| > \rho$.*

For this kind of mapping we have the following result.

THEOREM 5.1.22. *If $f : H \to H$ is an asymptotically (u, g, φ)-monotone mapping with respect to $\mathbb{K}$, then f satisfies condition (θ).*

Proof. For every $x \in \mathbb{K}$ with $\|x\| > \max(\rho, \|u\|)$, we have

$$\langle x-u, f(x)-g(u)\rangle \geq \|x-u\|\varphi(\|x-u\|),$$

which implies

$$\langle x-u, f(x)\rangle \geq \langle x-u, g(u)\rangle + \|x-u\|\varphi(\|x-u\|).$$

Since $\|x\| > \|u\|$ we have $\|x - u\| > 0$ and we deduce

$$\langle x-u, f(x)\rangle \geq \|x-u\|\left[\left\langle \frac{x-u}{\|x-u\|}, g(u)\right\rangle + \varphi(\|x-u\|)\right].$$

Since $S_1 = \{x \in H \mid \|x\| = 1\}$ is a bounded set and for u fixed, considering $g(u)$ as a continuous linear functional on H, we have that there exists $\gamma \in \mathbb{R}$ such that $\left\langle \frac{x-u}{\|x-u\|}, g(u)\right\rangle \geq \gamma$ for any $x \in \mathbb{K}$ with $\|x\| > \max(\rho, \|u\|)$.

Because $\lim_{t\to+\infty} \varphi(t) = +\infty$, we have that there exists $\rho_* > 0$ such that $\|x - u\| > \rho_*$ implies $\varphi(\|x-u\|) \geq -\gamma$, that is $\langle x-u, f(x)\rangle \geq 0$. If for any $x \in \mathbb{K}$ satisfying $\|x\| > \max(\rho_* + \|u\|, \rho)$ we take $y = u$, we have immediately that f satisfies *condition* (θ), with respect to $\mathbb{K}$ and the proof is complete. □

COROLLARY 5.1.23. *Let $f : H \to H$ be an asymptotically (u, g, φ)-monotone mapping with respect to $\mathbb{K}$. If f is projectionally Leray–Schauder with respect to $\mathbb{K}$, then the problem $NCP(f, \mathbb{K})$ has a solution.*

Remark. The class of asymptotically (u, g, φ)-monotone operators contains as a particular case the strongly monotone operators.

We consider again an arbitrary Hilbert space $(H, \langle \cdot, \cdot \rangle)$, a closed pointed convex cone $\mathbb{K} \subset H$ and an arbitrary mapping, $f: H \to H$. We know that the problem $NCP(f, \mathbb{K})$ has a solution if and only if the equation

$$x = P_{\mathbb{K}}\left[x - f(x)\right] \tag{5.1.2}$$

has a solution in H (which is necessarily an element of $\mathbb{K}$). Therefore, equation (5.1.2) has a solution if and only if, the optimization problem

$$\|x - (x - f(x))\| = \min\left\{\|y - (x - f(x))\| : \ y \in \mathbb{K}\right\}$$

has a solution in $\mathbb{K}$. The above equation can be rewritten as the following variational inequality:

$$\|f(x)\| \leq \|y - x + f(x)\|, \textit{for all } y \in \mathbb{K}.$$

Considering this variational inequality we define the following condition.

DEFINITION 5.1.10. *We say that a mapping* $f: H \to H$ *satisfies condition* $M(D)$ *with respect to* $\mathbb{K}$ *if there exists a non-empty bounded subset* $D \subset \mathbb{K}$ *such that the set* $M(D)$ *defined by*

$$M(D) = \bigcap_{y \in D}\left\{x \in K : \ \|f(x)\| \leq \|y - (x - f(x))\|\right\}$$

is a bounded set.

A natural question arises: *Is there is a relation between condition* $M(D)$ *and condition* (θ)? An answer to this question is given by the following result.

THEOREM 5.1.24. *If a mapping* $f: H \to H$ *satisfies condition* $M(D)$ *with respect to* $\mathbb{K}$, *then* f *satisfies condition* (θ).

Proof. We consider the bounded set $A = D \cup M(D)$. If $x_0 \in \mathbb{K} \setminus A$, then $x_0 \notin \bigcap_{y \in D}\left\{x \in \mathbb{K} : \|f(x)\| \leq \|y - (x - f(x))\|\right\}$ which implies that there exist $y_0 \in D$ with the property that

$$\|f(x_0)\| > \|y_0 - (x_0 - f(x_0))\|. \tag{5.1.3}$$

From (5.1.3) we have

$$\|f(x_0)\|^2 > \|y_0 - (x_0 - f(x_0))\|^2$$

or

$$\langle f(x_0), f(x_0)\rangle > \langle y_0 - (x_0 - f(x_0)), y_0 - (x_0 - f(x_0))\rangle$$
$$= \langle y_0 - x_0, y_0 - x_0\rangle - 2\langle y_0 - x_0, f(x_0)\rangle + \langle f(x_0), f(x_0)\rangle,$$

which implies

$$\langle x_0 - y_0, f(x_0)\rangle > \frac{1}{2}\|y_0 - x_0\|^2 > 0.$$

Therefore, the weak Karamardian's condition is satisfied and by Theorem 5.1.24, we have that the mapping f satisfies *condition* (θ).

COROLLARY 5.1.25. *If $f: H \to H$ is a projectionally Leray–Schauder mapping with respect to a closed pointed convex cone $\mathbb{K} \subset H$ and satisfies condition M(D), then the problem NCP(f, $\mathbb{K}$) has a solution.*

COROLLARY 5.1.26. *Let $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ be n-dimensional Euclidean space, $\mathbb{K} \subset \mathbb{R}^n$ a closed pointed convex cone. If $f: \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping and it satisfies condition M(D), then the problem NCP(f, $\mathbb{K}$) has a solution.*

Remark. If in *condition M(D)* we have that D and $M(D)$ are compact sets, and f is a continuous mapping, then in this case we take, in the proof of Theorem 5.1.24, $A = \overline{conv}\left(D \cup \overline{M(D)}\right)$ and by the classical Karamardian's Theorem we have that the problem *NCP(f, $\mathbb{K}$)* has a solution. About this result the reader is referred to (Isac, G. and Li, J. [1]).

Now, we give a variant of *condition* (θ) in an arbitrary Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Let $\mathbb{K} \subset H$ be a pointed closed convex cone and $f: H \to H$ a mapping.

DEFINITION 5.1.11. *We say that f satisfies condition* (θ - *S*) *with respect to* $\mathbb{K}$, *if for any family of elements* $\{x_r\}_{r>0} \subset \mathbb{K}$, *such that* $\|x_r\| \to +\infty$ *as* $r \to +\infty$, *there exists* $y_* \in \mathbb{K}$ *such that* $\langle x_r - y_*, f(x_r)\rangle \geq 0$ *for some* $r > 0$ *such that* $\|x_r\| > \|y_*\|$.

Remark. We observe that *condition* (θ) implies condition (θ – *S*). If *f* is positive homogeneous, then condition (θ – *S*) implies condition (θ). (In this case we take for any $r > 0$, $x_r = rx$ *if* $x \in \mathbb{K}$). As for condition (θ – *S*) we have the following result.

THEOREM 5.1.27. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed pointed convex cone and* $f : H \to H$ *a mapping. If f satisfies condition* (θ – *S*) *with respect to* $\mathbb{K}$, *then f is without an EFE with respect to* $\mathbb{K}$.

Proof. Indeed, we suppose that *f* has an *EFE* with respect to $\mathbb{K}$, namely $\{x_r\}_{r>0} \subset \mathbb{K}$. Since *f* satisfies condition (θ – *S*) there exists $y_* \in \mathbb{K}$ such that $\langle x_r - y_*, f(x_r)\rangle \geq 0$ for some $r > 0$ for which we have $\|y_*\| < \|x_r\|$. In this case we have

$$\begin{aligned} 0 &\leq \langle x_r - y_*, f(x_r)\rangle = \langle x_r - y_*, u_r - \mu_r x_r\rangle \\ &= \langle x_r, u_r\rangle - \langle y_*, u_r\rangle - \mu_r \langle x_r, x_r\rangle + \mu_r \langle y_*, x_r\rangle \\ &\leq \mu_r \left[\langle y_*, x_r\rangle - \|x_r\|^2\right] \leq \mu_r \left[\|x_r\|\|y_*\| - \|x_r\|^2\right] \\ &= \mu_r \|x_r\|\left[\|y_*\| - \|x_r\|\right] < 0, \end{aligned}$$

which is a contradiction. Therefore *f* is without an *EFE* with respect to $\mathbb{K}$. □

Remark. Our condition (θ – *S*) is more general than the condition used in (Zhao, Y. B. and Han, J. Y. [1], Theorem 3.1), since in condition (θ – *S*), the element y_* is dependent on the family $\{x_r\}_{r>0}$, while in (Zhao, Y. B. and Han, J. Y. [1]) the element $\hat{x}$ is independent on the family $\{x_r\}_{r>0}$. In 1990 P. T. Harker and J. S. Pang studied the solvability of variational inequalities in $\mathbb{R}^n$ and by using an interesting condition, they obtained some existence theorems for variational inequalities (Harker, P. T. and Pang, J. S. [1]).

Now, we consider this condition in an arbitrary Hilbert space, but for complementarity problems. We will denote this condition by (*HP*).

DEFINITION 5.1.12. *We say that a mapping $f: H \to H$ satisfies condition* (*HP*) *with respect to a closed pointed convex cone $\mathbb{K} \subset H$, if there exists an element $x_* \in \mathbb{K}$ such that the set $\mathbb{K}(x_*) = \{x \in \mathbb{K} : \langle f(x), x - x_* \rangle < 0\}$ is bounded (or empty).*

Considering this condition we have the following result.

THEOREM 5.1.28. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $f: H \to H$ a mapping. If f satisfies condition* (*HP*) *with respect to $\mathbb{K}$, then f satisfies condition* ($\theta - S$) *and consequently f is without an EFE with respect to $\mathbb{K}$.*

Proof. Let $\{x_r\}_{r>0} \subset \mathbb{K}$ be a family of elements such that $\|x_r\| \to +\infty$ *as* $r \to +\infty$. If there exists an element $x_* \in \mathbb{K}$ such that the set $\mathbb{K}(x_*)$ is bounded (or empty), then for $r > 0$ sufficiently large, we have that $x_r \notin \mathbb{K}(x_*)$ implies that $\langle x_r - x_*, f(x_r) \rangle \geq 0$. We can take r sufficiently large satisfying also the condition $\|x_r\| > \|x_*\|$. Therefore f satisfies condition ($\theta - S$). By Theorem 5.1.27, f is a mapping without an *EFE* with respect to $\mathbb{K}$. □

PROPOSITION 5.1.29. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $f: H \to H$ a mapping. If f has an EFE with respect to $\mathbb{K}$, then for any point $x_* \in \mathbb{K}$, the set*

$$\mathbb{K}(x_*) = \{x \in \mathbb{K} : \langle f(x), x - x_* \rangle < 0\}$$

is non-empty and unbounded.

Proof. This result is a consequence of Theorem 5.1.27 and 5.1.28. □

The following notion was considered in (Zhao, Y. B. and Han, J. Y. [1]).

DEFINITION 5.1.13. *We say that a mapping $f: H \to H$ is (x_*, p)-coercive with respect to a closed pointed convex cone $\mathbb{K} \subset H$ if there exists some $p \in]-\infty, 1[$ and an element $x_* \in \mathbb{K}$ such that*

$$\lim_{x \in \mathbb{K}, \|x\| \to \infty} \frac{\langle f(x), x - x_* \rangle}{\|x\|^p} = +\infty .$$

Remark. The case $p = 1$ is covered by the classical notion of coercivity. Any coercive mapping is p-coercive, but the converse is not true. For example if we take $H = \mathbb{R}$, $\mathbb{K} = \mathbb{R}_+$, the mapping $f(x) = \frac{x^\alpha}{1 + x^\alpha}$, with $\alpha > 0$ and x_* any element such that $x_* \geq 1$, is p-coercive for any $p \in]-\infty, 1[$, but f is not coercive since

$$\lim_{x \geq 1, x \to +\infty} \frac{f(x)(x - x_*)}{x} = 1.$$

THEOREM 5.1.30. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $f: H \to H$ a mapping. If f is (x_*, p)-coercive with $\infty < p < 1$, then f satisfies condition $(\theta - S)$ and consequently f is without an EFE.*

Proof. Indeed, if $0 \leq p < 1$ then we have

$$\lim_{x \in \mathbb{K}, \|x\| \to \infty} \frac{\langle f(x), x - x_* \rangle}{\|x\|^p} = +\infty , \tag{5.1.4}$$

with $x_* \in K$ defined by the (x_*, p)-coercivity. Relation (5.1.4) implies

$$\lim_{x \in \mathbb{K}, \|x\| \to \infty} \langle f(x), x - x_* \rangle = +\infty .$$

Which has as a consequence the fact that condition $(\theta - S)$ is satisfied with respect to $\mathbb{K}$. If $-\infty < p < 0$, then for every family of elements $\{x_r\}_{r>0} \subset \mathbb{K}$, with $\|x_r\| \to +\infty$ as $r \to +\infty$, we have (using Definition 5.1.3) that $\langle f(x_r), x_r - x_* \rangle > 0$ for $r > 0$ sufficiently large. Therefore, again condition $(\theta - S)$ is satisfied and the proof is complete. □

DEFINITION 5.1.14. *Let $f: H \to H$ be a mapping and $K \subset H$ a closed pointed convex cone. We say that a mapping $T: H \to H$ is an (x_*, p)-scalar asymptotic derivative of f with respect to K, if there exists an element $x_* \in K$ and a real number $p \in]-\infty, 1[$ such that*

$$\lim_{x \in \mathbb{K}, \|x\| \to +\infty} \frac{\langle f(x) - T(x), x - x_* \rangle}{\|x\|^p} = 0.$$

The importance of Definition 5.1.14 is given by the following result.

THEOREM 5.1.31. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $f: H \to H$ a mapping and $\mathbb{K} \subset H$ a closed pointed convex cone. If f has an (x_*, p)-scalar asymptotic derivative T with respect to $\mathbb{K}$, and T is (x_*, p)-coercive, then f is without an EFE with respect to $\mathbb{K}$.*

Proof. The theorem is a consequence of Theorem 5.1.30 and of the relation

$$\lim_{x \in \mathbb{K}, \|x\| \to +\infty} \frac{\langle f(x), x - x_* \rangle}{\|x\|^p}$$

$$= \lim_{x \in \mathbb{K}, \|x\| \to +\infty} \frac{\langle f(x) - T(x), x - x_* \rangle}{\|x\|^p} + \lim_{x \in \mathbb{K}, \|x\| \to +\infty} \frac{\langle T(x), x - x_* \rangle}{\|x\|^p} = +\infty.$$

□

For the next result, we need to recall the following notion. We say that a mapping $f: H \to H$ is *pseudo-monotone* on $\mathbb{K}$ if for any $x, y \in \mathbb{K}$, $x \neq y$ we have that $\langle y - x, f(x) \rangle \geq 0$ implies $\langle y - x, f(y) \rangle \geq 0$.

DEFINITION 5.1.15. *We say that a mapping $f: H \to H$ is weakly proper on $\mathbb{K}$ if for any family of elements $\{x_r\}_{r>0} \subset \mathbb{K}$, with $\|x_r\| \to +\infty$ as $r \to +\infty$, there exists an element $x_* \in \mathbb{K}$ such that for some $r > 0$, with $\|x_*\| < \|x_r\|$ we have $\langle f(x_*), x_r - x_* \rangle \geq 0$.*

We have the following interesting result.

THEOREM 5.1.32. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed pointed convex cone and* $f : H \to H$ *a pseudo-monotone mapping. If f is a projectionally Leray–Schauder mapping, then the problem NCP(f,* $\mathbb{K}$*) has a solution if and only if f is weakly proper with respect to* $\mathbb{K}$*.*

Proof. We suppose that the problem *NCP(f,* $\mathbb{K}$*)* has a solution x_0. Because the fact that the solvability of the problem *NCP(f,* $\mathbb{K}$*)* is equivalent to the solvability of the variational inequality

$$VI(f, \mathbb{K}) : \begin{cases} \textit{find } x_0 \in \mathbb{K} \textit{ such that} \\ \langle f(x_0), x - x_0 \rangle \geq 0, \textit{ for all } x \in \mathbb{K}, \end{cases}$$

we have that $\langle f(x_0), x - x_0 \rangle \geq 0$, *for all* $x \in \mathbb{K}$. Obviously, if we take in Definition 5.1.15, $x_* = x_0$, we deduce that f is weakly proper on $\mathbb{K}$.

Conversely, assume that f is weakly proper on $\mathbb{K}$. In this case Definition 5.1.15 implies that for each family of elements $\{x_r\}_{r>0} \subset \mathbb{K}$, with $\|x_r\| \to +\infty$ as $r \to +\infty$, there exists an element $x_* \in \mathbb{K}$ such that $\langle f(x_*), x_r - x_* \rangle \geq 0$ for some $r > 0$ such that $\|x_*\| < \|x_r\|$. Since f is pseudo-monotone we have that $\langle f(x_r), x_r - x_* \rangle \geq 0$, which implies that f satisfies condition (θ – *S*), with respect to $\mathbb{K}$. By Theorem 5.1.27, we have that f is without an *EFE* with respect to $\mathbb{K}$. Applying Theorem 5.1.2, we obtain that the problem *NCP(f,* $\mathbb{K}$*)* has a solution and the proof is complete. □

Now, we introduce a generalization of the Harker–Pang condition. We will denote this condition by (*HPT*), (Harker–Pang Type).

DEFINITION 5.1.16. *We say that a mapping* $f : H \to H$ *satisfies condition (HPT) with respect to* $\mathbb{K}$ *if there exists a bounded set* $D \subset \mathbb{K}$ *such that the set* $\mathbb{K}(D) = \{x \in \mathbb{K} : \langle y - x, f(x) \rangle \geq 0, \textit{ for all } y \in D\}$ *is bounded.*

By using this notion we have the following result.

THEOREM 5.1.33. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed pointed convex cone and* $f : H \to H$ *a mapping. If f satisfies condition (HPT) with respect to* $\mathbb{K}$*, then f is without an EFE. Moreover, if f is a continuous projectionally Leray–Schauder mapping, then the problem NCP(f,* $\mathbb{K}$*) has a solution.*

Proof. Indeed because f satisfies *condition* (*HPT*) with respect to $\mathbb{K}$, then considering the sets D and $\mathbb{K}(D)$ defined by *condition* (*HPT*), we have that the set $M = D \cup \mathbb{K}(D)$ is a bounded subset of $\mathbb{K}$. If x is an arbitrary element in $\mathbb{K} \setminus M$, then $x \notin \mathbb{K}(D)$, which implies that there exists an element $y \in D$ such that $\langle y - x, f(x) \rangle < 0$, or $\langle x - y, f(x) \rangle > 0$. Because $y \in M$, we have that the weak Karamardian's condition is satisfied and by Theorem 5.1.12 we have that f satisfies *condition* (θ). Applying Theorem 5.1.8 we obtain that f is without an *EFE* with respect to $\mathbb{K}$. If f is a continuous projectionally Leray–Schauder mapping with respect to $\mathbb{K}$, by Theorem 5.1.2 we have that the problem *NCP*(f, $\mathbb{K}$) has a solution. □

Remarks.

1. If the set D has only one element x_*, that is $D = \{x_*\}$, we obtain from Definition 5.1.16 the condition (*HP*).
2. If D and $\mathbb{K}(D)$ are compact and f is continuous, we obtain that f satisfies the classical Karamardian condition which implies that the problem *NCP*(f, $\mathbb{K}$) has a solution.
3. We remark that $\mathbb{K}(D) = \bigcap_{y \in D} \{x \in \mathbb{K} : \langle y - x, f(x) \rangle \geq 0\}$ which implies that $\mathbb{K}(D)$ can be bounded without each set $\{x \in \mathbb{K} : \langle y - x, f(x) \rangle \geq 0\}$ being bounded.

We consider again a general Hilbert space $(H,\langle\cdot,\cdot\rangle)$ and $\mathbb{K}\subset H$ a closed pointed convex cone.

DEFINITION 5.1.17. *We say that a mapping* $f : H \to H$ *is scalarly increasing to infinity with respect to* $\mathbb{K}$*, if for each* $y \in \mathbb{K}$ *there exists a real number* $\rho(y) > 0$ *such that for all* $x \in K$ *with* $\|x\|\geq\rho(y)$ *we have* $\langle x-y, f(x)\rangle\geq 0$.

THEOREM 5.1.34. *If* $f: H\to H$ *is a mapping that is scalarly increasing to infinity with respect to* $\mathbb{K}$*, then* f *is without an exceptional family of elements with respect to* $\mathbb{K}$*.*

Proof. Let $\{x_r\}_{r>0}\subset\mathbb{K}$ be an exceptional family of elements for f with respect to $\mathbb{K}$. We have $u_r = f(x_r)+\mu_r x_r \in\mathbb{K}^*, \langle x_r, u_r\rangle = 0$ for any $r > 0$ and $\|x_r\| \to +\infty$ as $r \to +\infty$. For any $r > 0$ we have $\mu_r > 0$. We show that in this case the converse of Definition 5.1.17 is satisfied, i.e., there exists $y \in \mathbb{K}$ such that for each $\rho > 0$ there exists $x \in \mathbb{K}$ with $\|x\| > \rho$ and $\langle x-y, f(x)\rangle<0$. Indeed, if $y \in \mathbb{K}\setminus \{0\}$ is an arbitrary element, then we have

$$\begin{aligned}\langle x_r - y, f(x_r)\rangle &= \langle x_r - y, u_r - \mu_r x_r\rangle\\ &= \langle x_r, u_r\rangle - \langle y, u_r\rangle - \mu_r\langle x_r, x_r\rangle + \langle y, \mu_r x_r\rangle\\ &\leq -\mu_r\|x_r\|^2 + \mu_r\langle y, x_r\rangle,\end{aligned}$$

which implies

$$\langle x_r - y, f(x_r)\rangle \leq \mu_r\|x_r\|\big[\|y\| - \|x_r\|\big]. \tag{5.1.5}$$

Let $y_0 \in \mathbb{K}\setminus \{0\}$ be an arbitrary element, $\rho > 0$ an arbitrary real number. Let $r > 0$ be a real number such that $\|x_r\| > \rho$ and $\|x_r\| > \|y_0\|$. Using (5.1.5) we obtain

$$\langle x_r - y_0, f(x_r)\rangle \leq \mu_r\|x_r\|\big[\|y_0\| - \|x_r\|\big] < 0.$$

Therefore, this fact is in contradiction with Definition 5.1.17, which implies that f is without an *EFE*. □

DEFINITION 5.1.18. *We say that a mapping* $h : H \to H$ *is monotonically decreasing on rays with respect to* $\mathbb{K}$ *if there exists* $t_0 > 0$ *such that, for every* $x \in \mathbb{K}$ *and every* s, t *with the property* $s \geq t \geq t_0$ *we have* $\langle x, h(tx)\rangle \geq \langle x, h(sx)\rangle$.

PROPOSITION 5.1.35. *A mapping* $h : H \to H$ *is monotonically decreasing on rays with respect to* $\mathbb{K}$, *if and only if, for every* $\alpha \geq 1$ *and every* $x \in \mathbb{K}$ *we have:*

$$\langle x, h(x)\rangle \geq \langle x, h(\alpha x)\rangle . \tag{5.1.6}$$

Proof. We suppose that h is monotonically decreasing on rays with respect to $\mathbb{K}$. For $x = 0$, inequality (5.1.6) is satisfied. We consider $\alpha \geq 1$ and $x \in \mathbb{K} \setminus \{0\}$. We can put $\alpha = \frac{s}{t}$ (where $s \geq t \geq t_0$). For $t \geq t_0$, $s = \alpha t$. Let $x_* = \frac{1}{t}x$. We have $x = tx_*$. Since h is monotonically decreasing on rays we have

$$\langle x_*, h(tx_*)\rangle \geq \langle x_*, h(sx_*)\rangle$$

which implies

$$\langle tx_*, h(tx_*)\rangle \geq \langle tx_*, h(\alpha tx_*)\rangle ,$$

and finally we have

$$\langle x, h(x)\rangle \geq \langle x, h(\alpha x)\rangle .$$

Conversely, we suppose that (5.1.6) is satisfied for every $\alpha \geq 1$ and every $x \in \mathbb{K}$. Let $x_* \in \mathbb{K} \setminus \{0\}$ be an arbitrary element. Take $t_0 = 1$ and $s \geq t \geq 1$. Using (5.1.6) with $\alpha = \frac{s}{t}$ and $x = tx_*$, we obtain

$$\langle tx_*, h(tx_*)\rangle \geq \left\langle tx_*, h\left(\frac{s}{t}(tx_*)\right)\right\rangle ,$$

which implies

$$\langle x_*, h(tx_*)\rangle \geq \langle x_*, h(sx_*)\rangle ,$$

that is h is monotonically decreasing on rays with respect to $\mathbb{K}$. □

THEOREM 5.1.36. *If the mapping $h : H \to H$ is bounded and monotonically decreasing on rays, with respect to $\mathbb{K}$, then $f(x) = x - h(x)$ is without exceptional family of elements, with respect to $\mathbb{K}$.*

Proof. We suppose that f has an exceptional family of elements, with respect to $\mathbb{K}$, namely $\{x_r\}_{r>0} \subset \mathbb{K}$. Then, for any $r > 0$ there exists a real number $\mu_r > 0$ such that $u_r = f(x_r) + \mu_r x_r \in \mathbb{K}^*, \langle x_r, u_r \rangle = 0$ and $\|x_r\| \to +\infty$ as $r \to +\infty$. From Proposition 5.1.35 we have

$$\begin{cases} \langle x, h(x) \rangle \geq \langle x, h(\alpha x) \rangle, \, \textit{for all } x \in \mathbb{K} \\ \textit{and all } \alpha \geq 1. \end{cases} \tag{5.1.7}$$

For every x_r with $\|x_r\| \geq 1$ we consider in (5.1.7) $\alpha = \|x_r\|$ and $x = \frac{x_r}{\|x_r\|}$ and we obtain

$$\langle \alpha x, h(x) - h(x_r) \rangle \geq 0,$$

which implies

$$\begin{cases} \langle x_r, h(x) - h(x_r) \rangle \geq 0 \textit{ for all } r > 0 \\ \textit{such that } \|x_r\| \geq 1. \end{cases} \tag{5.1.8}$$

The expression (5.1.8) is equivalent with the inequality

$$\begin{cases} \langle x_r, h(x) - h(x_r) + x_r - x_r \rangle \geq 0 \textit{ for all } r > 0 \\ \textit{such that } \|x_r\| \geq 1. \end{cases} \tag{5.1.9}$$

Because h is supposed to be bounded and $x = \frac{x_r}{\|x_r\|}$, there exists $M > 0$ such that $\|h(x)\| \leq M$. (Since $\|x\| = 1$, M is independent of r.) Using the fact that $\{x_r\}_{r>0}$ is an exceptional family of elements for $f(x) = x - h(x)$, we have from (5.1.9),

$$\begin{aligned} 0 &\leq \langle x_r, h(x) - x_r + u_r - \mu_r x_r \rangle \\ &= \langle x_r, h(x) \rangle - \|x_r\|^2 + \langle x_r, u_r \rangle - \mu_r \|x_r\|^2 \\ &= -(1 + \mu_r)\|x_r\|^2 + \langle x_r, h(x) \rangle \\ &\leq -(1 + \mu_r)\|x_r\|^2 + \|x_r\| M = \|x_r\| \left[M - (1 + \mu_r)\|x_r\| \right], \end{aligned}$$

which implies $\|x_r\| \le \dfrac{M}{1+\mu_r} \le M$, for all $r > 0$ such that $\|x_r\| \ge 1$. Obviously, the last inequality is impossible, since $\|x_r\| \to +\infty$ as $r \to +\infty$. We conclude from this contradiction that f is without an *EFE* with respect to $\mathbb{K}$ and the proof is complete. □

The following condition was defined in (Zhao, Y. B. [3]) under the name of *Isac–Gowda condition.* We note that this condition was initially used in (Isac, G. and Gowda, M. S. [1]).

DEFINITION 5.1.19. *We say that a mapping $f: H \to H$ satisfies condition (IG) with respect to $\mathbb{K}$, if there exists a real number $p > 0$ such that the mapping $T(x) = \|x\|^{p-1} \cdot x - f(x)$ is monotone decreasing on rays with respect to $\mathbb{K}$.*

Remark. Y. B. Zhao used condition (*IG*) in $\mathbb{R}^n$, *with respect to the cone* $\mathbb{R}^n_+$ in relation with the notion of a *d-oriented family of elements.* [See (Zhao, Y. B. [3])].

THEOREM 5.1.37. *Let $(H, \langle \cdot,\cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $f: H \to H$ a mapping. If the mapping f satisfies condition (IG), then f is without an EFE with respect to $\mathbb{K}$.*

Proof. We suppose that f has an exceptional family of elements $\{x_r\}_{r>0} \subset \mathbb{K}$ and we consider the mapping $T(x) = \|x\|^{p-1} \cdot x - f(x)$, defined for any $x \in H$. By Proposition 5.1.35 we have

$$\langle x, T(x) - T(\alpha x) \rangle \ge 0 \textit{ for all } x \in \mathbb{K} \textit{ and } \alpha \ge 1. \qquad (5.1.10)$$

Setting $\alpha = \|x_r\|$ and $x = \dfrac{x_r}{\|x_r\|}$ in (5.1.10) we have $\left\langle \dfrac{x_r}{\|x_r\|}, T\left(\dfrac{x_r}{\|x_r\|}\right) - T(x_r) \right\rangle \ge 0$, for all $r > 0$ such that $\|x_r\| \ge 1$, which is equivalent with $\left\langle x_r, T\left(\frac{x_r}{\|x_r\|}\right) - \|x_r\|^{p-1} \cdot x_r + f(x_r)\right\rangle \geq 0$, for all $r > 0$ such that $\|x_r\| \geq 1$. Using the definition of an *EFE* and the last inequality, we obtain $\left\langle x_r, T\left(\frac{x_r}{\|x_r\|}\right) - \|x_r\|^{p-1} \cdot x_r + u_r - \mu_r x_r\right\rangle \geq 0$ for all $r > 0$ such that $\|x_r\| \geq 1$, which implies $\left\langle x_r, T\left(\frac{x_r}{\|x_r\|}\right)\right\rangle - \|x_r\|^{p+1} - \mu_r \|x_r\|^2 \geq 0$, for all $r > 0$, such that $\|x_r\| \geq 1$. Finally we have $\left\langle x_r, T\left(\frac{x_r}{\|x_r\|}\right)\right\rangle - \|x_r\|^{p+1} \geq \mu_r \|x_r\|^2 \geq 0$, for all $r > 0$, such that $\|x_r\| \geq 1$, or $\|x_r\|^{p+1} \leq \left\langle x_r, T\left(\frac{x_r}{\|x_r\|}\right)\right\rangle$, for all $r > 0$, such that $\|x_r\| \geq 1$. Since T is bounded, there exists a real number $M > 0$ such that $T\left(\frac{x_r}{\|x_r\|}\right) \leq M$, for all $r > 0$ such that $\|x_r\| \geq 1$. Therefore we have $\|x_r\|^p \leq M$, for all $r > 0$ such that $\|x_r\| \geq 1$, which is impossible, since $\|x_r\| \to +\infty$ as $r \to +\infty$. This contradiction implies that f is without an *EFE* with respect to $\mathbb{K}$. □

DEFINITION 5.1.20. *We say that a mapping $f : H \to H$ is p-order generalized coercive with respect to $\mathbb{K}$ if there exists an element $x_* \in \mathbb{K}$ and a real number $p \in\ ]-\infty, 1]$ such that*

$$\limsup_{\|x\| \to +\infty, x \in \mathbb{K}} \frac{\langle f(x), x - x_* \rangle}{\|x\|^p} > 0. \tag{5.1.11}$$

THEOREM 5.1.38. *If $f : H \to H$ is a p-order generalized coercive mapping with respect to $\mathbb{K}$, then f is without an EFE.*

Proof. Indeed, we suppose that f has an *EFE* with respect to $\mathbb{K}$. Let $\{x_r\}_{r>0}$ be this family. In this case we have

$$\frac{\langle f(x_r), x_r - x_* \rangle}{\|x_r\|^p} = \frac{\langle u_r - \mu_r x_r, x_r - x_* \rangle}{\|x_r\|^p} \le \frac{-\mu_r \|x_r\|^2 + \mu_r \|x_r\| \|x_*\|}{\|x_r\|^p}$$
$$= \frac{\mu_r \left[\|x_*\| - \|x_r\|\right]}{\|x_r\|^{p-1}}$$

which implies that for r sufficiently large we have

$$\frac{\langle f(x_r), x_r - x_* \rangle}{\|x_r\|^p} \le \frac{\mu_r \left[\|x_*\| - \|x_r\|\right]}{\|x_r\|^{p-1}} < 0,$$

which is a contradiction of (5.1.11). Therefore, f is without an *EFE* with respect to $\mathbb{K}$. □

COROLLARY 5.1.39. *If f is coercive with respect to $\mathbb{K}$, that is, there exists $x_* \in \mathbb{K}$ such that*

$$\lim_{x \in \mathbb{K} \|x\| \to \infty,} \frac{\langle f(x), x - x_* \rangle}{\|x\|} = +\infty,$$

then f is without an EFE with respect to $\mathbb{K}$.

COROLLARY 5.1.40. *If f satisfies the condition*

$$\liminf_{\|x\| \to \infty, x \in \mathbb{K}} \langle x, f(x) \rangle > 0,$$

then f is without an EFE with respect to $\mathbb{K}$.

Remark. In (Zhao, Y. B. [3]) was proved a result similar to Theorem 5.1.38 but for a *d*-oriented family with respect to the cone $\mathbb{R}^n_+$.

We close this section with another variant of *condition* (θ).

DEFINITION 5.1.21. *We say that a mapping $f: H \to H$ satisfies condition $(\tilde{\theta})$ with respect to $\mathbb{K}$, if there exists $\rho > 0$ such that for any $x \in \mathbb{K}$ with $\|x\| > \rho$, there exists $y \in \mathbb{K}$ such that $\langle y, x \rangle < \|x\|^2$ and $\langle x - y, f(x) \rangle \ge 0$.*

Remark. If f satisfies *condition* (θ), then it satisfies *condition* $(\tilde{\theta})$. Indeed, if there is $\rho > 0$ such that for any $x \in \mathbb{K}$ with $\|x\| > \rho$, there exists $y \in \mathbb{K}$ with $\|y\| > \|x\|$ and , $\langle x - y, f(x)\rangle \geq 0$, then we have

$$\langle y, x\rangle \leq \|y\| \cdot \|x\| < \|x\|^2$$

which implies $\langle x - y, f(x)\rangle \geq 0$. About *condition* $(\tilde{\theta})$ we have the following result.

THEOREM 5.1.41. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed pointed convex cone and* $f : H \to H$ *a mapping. If* f *satisfies condition* $(\tilde{\theta})$, *then* f *is without an EFE with respect to* $\mathbb{K}$.

Proof. Suppose that f has an *EFE* with respect to $\mathbb{K}$, namely $\{x_r\}_{r>0} \subset \mathbb{K}$. We have that $\|x_r\| \to +\infty$ as $r \to +\infty$, $u_r = \mu_r x_r + f(x_r) \in \mathbb{K}^*$ and $\langle x_r, u_r\rangle = 0$, for any $r > 0$, (with $\mu_r > 0$). We take $r > 0$ such that $\|x_r\| > \rho$. Because f satisfies *condition* $(\tilde{\theta})$, for such r, there exists $y_r \in \mathbb{K}$, such that $\langle y_r, x_r\rangle < \|x_r\|^2$ and $0 \leq \langle x_r - y_r, f(x_r)\rangle$. In this case we have

$$0 \leq \langle x_r - y_r, f(x_r)\rangle = \langle x_r - y_r, u_r - \mu_r x_r\rangle$$

$$= \langle x_r, u_r\rangle - \langle y_r, u_r\rangle - \mu_r \|x_r\|^2 + \mu_r \langle y_r, x_r\rangle \leq \mu_r \left[\langle y_r, x_r\rangle - \|x_r\|^2\right] < 0,$$

which is impossible. Therefore f is without an *EFE* with respect to $\mathbb{K}$. □

Remark. We can show that if $g : H \to H$ satisfies *condition* $(\tilde{\theta})$ with respect to a closed convex cone $\mathbb{K} \subset H$, then for any $\alpha > 0$, $\beta > 0$, the mapping $f(x) = \alpha x + \beta g(x)$ satisfies also *condition* $(\tilde{\theta})$. We have the same property for *condition* (θ).

5.2 Implicit complementarity problems

In Chapter 4 we presented the extension of the notion of *EFE* from a mapping to a pair of mappings and we applied this notion to the study of implicit complementarity problems. In this section we present a variant of this notion. This variant will be introduced by a little modification of the notion of *EFE* defined in Chapter 4 and by *replacing the topological degree* by the Leray–Schauder alternative. In this way we obtain a general alternative, which implies an existence theorem for implicit complementarity problems.

Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K}\subset H$ a closed pointed convex cone and $f, g : H \to H$ two arbitrary mappings. We consider the following *implicit complementarity problem* defined by the (ordered) pair of mappings (f, g) and the cone $\mathbb{K}$.

$$ICP(f,g,\mathbb{K}):\begin{cases} \textit{find } x_* \in H \textit{ such that} \\ f(x_*)\in\mathbb{K}^*, g(x_*)\in\mathbb{K} \textit{ and} \\ \langle g(x_*), f(x_*)\rangle = 0. \end{cases}$$

In Chapter 4 , we considered this problem supposing that the Hilbert space $(H,\langle\cdot,\cdot\rangle)$ is the Euclidean space $(\mathbb{R}^n,\langle\cdot,\cdot\rangle)$, $\mathbb{K}$ is a closed convex cone in $\mathbb{R}^n$ and $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are continuous mappings. We recall the definition of *EFE* for the pair (f, g) of mappings (Definition 4.3.2). *We say that a family of elements* $\{x_r\}_{r>0}\subset\mathbb{R}^n$ *is an EFE for the pair* (f, g), *with respect to the cone* $\mathbb{K}$, *if the following conditions are satisfied:*

(i) $\|x_r\|\to+\infty$ *as* $r\to+\infty$,

(ii) $g(x_r)\in\mathbb{K}$ *for any* $r>0$,

(iii) *for every* $r>0$, *there exists* $\mu_r>0$ *such that* $s_r = f(x_r)+\mu_r g(x_r)\in\mathbb{K}^*$ *and* $\langle g(x_r), s_r\rangle = 0$.

Using the topological degree we established the following result (Theorem 4.3.2). *If f, g : $\mathbb{R}^n \to \mathbb{R}^n$ are continuous mappings and the following assumptions are satisfied:*

(1) *the equation g(x) = 0 has a unique solution, namely $b \in \mathbb{R}^n$,*

(2) *the mapping g maps homeomorphically a neighborhood of the element b onto a neighborhood of the origin,*

then, there exists either a solution to the problem ICP(f, g, $\mathbb{K}$) or an EFE for the couple (f,g) (in the sense of Definition 4.3.2) with respect to $\mathbb{K}$.

In this result, conditions (1) and (2) are strong conditions from the practical point of view. Because of this fact, our goal in this section is to introduce a new definition for the notion of an *EFE* associated to a pair of mappings (*f, g*). We will realize this by a little modification of the notion introduced by Definition 4.3.2 and replacing the topological degree by the Leray–Schauder alternative.

Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $f, g : H \to H$ continuous mappings.

DEFINITION 5.2.1. *We say that a family of elements $\{x_r\}_{r>0} \subset H$ is an exceptional family of elements (an EFE) for the pair (f, g) with respect to $\mathbb{K}$, if the following conditions are satisfied:*

(1) $\|x_r\| \to +\infty$ *as* $r \to +\infty$,

(2) *for any r > 0, there exists $\mu_r \geq 0$ such that*

$$s_r = \mu_r x_r + f(x_r) \in \mathbb{K}^*,\ v_r = \mu_r x_r + g(x_r) \in \mathbb{K} \text{ and } \langle v_r, s_r \rangle = 0.$$

This notion will be used in this section. We have the following result.

THEOREM 5.2.1. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and f, g : $H \to H$ completely continuous fields, such that $f(x) = x - T(x)$ and $g(x) = x - S(x)$, where T, S : $H \to H$ are completely continuous mappings. Then, there exists either a solution to the problem ICP(f, g, $\mathbb{K}$), or an exceptional family of elements $\{x_r\}_{r>0}$ for the* *pair (f, g). Moreover, if $S(\mathbb{K}) \subseteq \mathbb{K}$, then we have, either an exceptional family of element $\{x_r\}_{r>0} \subset \mathbb{K}$ for the pair (f, g) or the problem ICP(f, g, $\mathbb{K}$) has a solution in $\mathbb{K}$.*

Proof. We know (See Chapter 2) that the problem *ICP(f, g, $\mathbb{K}$)* has a solution, if and only if the equation

$$g(x) = P_{\mathbb{K}}\left[g(x) - f(x)\right] \tag{5.2.1}$$

has a solution in H. Considering the mapping

$$\Phi(x) = x - g(x) + P_{\mathbb{K}}\left[g(x) - f(x)\right]$$

defined for any $x \in H$, we observe that equation (5.2.1) has a solution if and only if, the mapping Φ has a fixed point in H. From assumptions we have

$$\Phi(x) = x - g(x) + P_{\mathbb{K}}\left[g(x) - f(x)\right] = S(x) + P_{\mathbb{K}}\left[-S(x) + T(x)\right].$$

We note that Φ is a completely continuous mapping. If the mapping Φ has a fixed point, then the problem *ICP(f, g, $\mathbb{K}$)* has a solution and the proof is complete. We suppose that Φ has no fixed point in the space H. For any $r > 0$, we consider the set $U_r = B_r = \{x \in H : \|x\| < r\}$ and we observe that the restriction of Φ onto the set U_r is a continuous compact mapping without fixed points. Applying Theorem 3.2.4 (the classical Leray–Schauder alternative) to the mapping Φ and the set $\Omega = H$ and $U = U_r$, we obtain that for all $r > 0$,there exist $x_r \in \partial U_r = \{x \in H : \|x\| = r\}$ and $\lambda_r \in]0,1[$ such that

$$x_r = \lambda_r\left[S(x_r) + P_{\mathbb{K}}\left[T(x_r) - S(x_r)\right]\right]. \tag{5.2.2}$$

From (5.2.2) we deduce

$$\frac{1}{\lambda_r} x_r - S(x_r) = P_{\mathbb{K}}\left[T(x_r) - S(x_r)\right],$$

which implies (using the properties of $P_{\mathbb{K}}$),

$$\begin{cases} \langle \lambda_r^{-1} x_r - T(x_r), y \rangle \geq 0, \textit{ for all } y \in \mathbb{K} \textit{ and} \\ \langle \lambda_r^{-1} x_r - T(x_r), \lambda_r^{-1} x_r - S(x_r) \rangle = 0. \end{cases}$$

Therefore we have that

$$\begin{cases} \lambda_r^{-1} x_r - T(x_r) \in \mathbb{K}^* \textit{ and} \\ \lambda_r^{-1} x_r - S(x_r) \in \mathbb{K}. \end{cases}$$

If we denote by $\mu_r = \lambda_r^{-1} - 1 > 0$, for any $r > 0$, we obtain

$$s_r = \mu_r x_r + f(x_r) = \left(\frac{1}{\lambda_r} - 1\right) x_r + x_r - T(x_r) \in \mathbb{K}^*,$$

and

$$v_r = \mu_r x_r + g(x_r) = \left(\frac{1}{\lambda_r} - 1\right) x_r + x_r - S(x_r) \in \mathbb{K}.$$

Obviously we have also the orthogonality condition $\langle v_r, s_r \rangle = 0$ for any $r > 0$. According to Definition 5.2.1, the family of elements $\{x_r\}_{r>0}$ is an *EFE* and the first conclusion of the theorem is proved. If $S(\mathbb{K}) \subseteq \mathbb{K}$, then $\Phi(\mathbb{K}) \subseteq \mathbb{K}$. In this case we apply Theorem 3.2.4 to the set $\Omega = \mathbb{K}$ and for any $r > 0$ we consider $U = U_r(\mathbb{K}) = \{x \in \mathbb{K} : \|x\| < r\}$. If the problem *ICP*($f$, g, $\mathbb{K}$) has a solution in $\mathbb{K}$, then the proof is complete. Otherwise, as in the first part of the proof, we construct the family $\{x_r\}_{r>0}$, where $x_r \in \mathbb{K}$ for each $r > 0$, and the proof of the second conclusion of the theorem is also complete. Therefore, the theorem is proved. □

COROLLARY 5.2.2. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed convex cone and $f, g : H \to H$ completely continuous fields. If the pair (f, g) is without an EFE, then the problem ICP(f, g, $\mathbb{K}$) has a solution.*

COROLLARY 5.2.3. *Let $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ be n-dimensional Euclidean space, $\mathbb{K} \subset \mathbb{R}^n$ a closed convex cone and $f, g : \mathbb{R}^n \to \mathbb{R}^n$ continuous mappings. If the pair (f, g) is without an EFE, then the problem ICP(f, g, $\mathbb{K}$) has a solution.*

In view of Corollary 5.2.2, now, we give some examples of pairs of mappings without an *EFE*. We suppose again that $(H, \langle \cdot, \cdot \rangle)$ is an arbitrary Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and f, $g : H \to H$ continuous mappings.

DEFINITION 5.2.2. *We say that the pair (f, g) of mappings satisfies condition* (θ_g) *with respect to* $\mathbb{K}$ *if there exists* $\rho > 0$ *such that for any* $x \in \mathbb{K}$ *with* $\|x\| > \rho$, *there exists* $y \in \mathbb{K}$ *such that:*

(1) $\langle g(x) - y, f(x) \rangle \geq 0$ *and*

(2) $\langle g(x) - y, x \rangle > 0$.

This notion implies the following result.

THEOREM 5.2.4. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed pointed convex cone and f, g : H → H two mappings. If the pair (f, g) satisfies condition* (θ_g), *then the pair (f, g) is without an EFE with respect to* $\mathbb{K}$.

Proof. Indeed, we suppose that the pair (*f*, *g*) has an *EFE*, $\{x_r\}_{r>0} \subset \mathbb{K}$. For any $r > 0$ such that $\|x_r\| > \rho$ we have an element $y_r \in \mathbb{K}$ such that condition (θ_g) is satisfied, i.e.,

$$\begin{cases} \langle g(x_r) - y_r, f(x_r) \rangle \geq 0 \\ \langle g(x_r) - y_r, x_r \rangle > 0. \end{cases}$$

From the definition of an *EFE* for a pair of mappings we have

$$s_r = \mu_r x_r + f(x_r) \in \mathbb{K}^*,$$

$$v_r = \mu_r x_r + g(x_r) \in \mathbb{K} \text{ and } \langle v_r, s_r \rangle = 0.$$

We deduce

$$\begin{aligned} 0 &\leq \langle g(x_r) - y_r, f(x_r) \rangle = \langle v_r - \mu_r x_r - y_r, s_r - \mu_r x_r \rangle \\ &= \langle v_r, s_r \rangle - \langle \mu_r x_r, s_r \rangle - \langle y_r, s_r \rangle - \langle v_r, \mu_r x_r \rangle + \mu_r^2 \|x_r\|^2 + \langle y_r, \mu_r x_r \rangle \\ &\leq -\langle v_r, \mu_r x_r \rangle + \mu_r^2 \|x_r\|^2 + \langle y_r, \mu_r x_r \rangle \\ &= -\langle \mu_r x_r + g(x_r), \mu_r x \rangle + \mu_r^2 \|x_r\|^2 + \langle y_r, \mu_r x_r \rangle \\ &= -\mu_r^2 \|x_r\|^2 - \langle g(x_r), \mu_r x_r \rangle + \mu_r^2 \|x_r\|^2 + \langle y_r, \mu_r x_r \rangle \\ &= -\langle g(x_r), \mu_r x_r \rangle + \langle y_r, \mu_r x_r \rangle = -\mu_r \left[\langle g(x_r) - y_r, x_r \rangle \right] < 0, \end{aligned}$$

which is a contradiction. Therefore, the pair (f, g) is without an *EFE*, with respect to $\mathbb{K}$. □

COROLLARY 5.2.5. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $f, g : H \to H$ completely continuous fields. If the pair (f, g) satisfies condition (θ_g), then the problem ICP$(f, g, \mathbb{K})$ has a solution.*

COROLLARY 5.2.6. *Let $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ be n-dimensional Euclidean space, $\mathbb{K} \subset \mathbb{R}^n$ a closed pointed convex cone and $f, g : \mathbb{R}^n \to \mathbb{R}^n$ continuous mappings. If the pair (f, g) satisfies condition (θ_g), then the problem ICP$(f, g, \mathbb{K})$ has a solution.*

Remark. For a pair of mappings (f, g), condition (θ_g) is an extension of condition (θ) from a mapping to pair of mappings. We consider f as the pair (f, I), where I is the identity mapping. Indeed, if $g(x) = x$ for any $x \in H$, then we have,

$$\langle g(x) - y, x \rangle = \langle x - y, x \rangle = \langle x, x \rangle - \langle y, x \rangle \geq \|x\|^2 - \|y\|\|x\| = \|x\|(\|x\| - \|y\|).$$

Hence, if $\|y\| < \|x\|$, then we have,

$$\langle g(x) - y, x \rangle = \langle x - y, x \rangle > 0.$$

In Section 5.1 we presented several classes of mappings satisfying *condition* (θ). As in the case of a single function, now, we give some examples of pairs of functions satisfying *condition* (θ_g), and consequently, pairs of functions without an *EFE*.

PROPOSITION 5.2.7. *Let $f, g : H \to H$ be two mappings. If there exists $\rho > 0$ such that for any $x \in \mathbb{K}$ with $\|x\| > \rho$, there exists $y \in \mathbb{K}$ such that*

(1) $\langle g(x) - y, f(x) \rangle \geq 0$,

(2) $\langle g(x), x \rangle \geq \alpha \|x\|^2, \alpha \in \mathbb{R}_+ \setminus \{0\}$ *and*

(3) $\|y\| < \alpha\|x\|$,

then the pair (f, g) satisfies condition (θ_g).

Proof. The assertion of the proposition is implied by the definition of *condition* (θ_g) and the following inequalities:

$$\langle g(x)-y,x\rangle=\langle g(x),x\rangle-\langle y,x\rangle\geq\alpha\|x\|^2-\|y\|\|x\|=\|x\|(\alpha\|x\|-\|y\|)>0. \quad \square$$

DEFINITION 5.2.3. *We say that a pair (f, g) of mappings from H into H, satisfies the Karamardian type condition, if there exists a bounded subset D of* $\mathbb{K}$ *such that for all* $x\in\mathbb{K}\setminus D$, *the exists a* $y\in D$ *such that*

$$\langle g(x)-y,f(x)\rangle\geq 0 \text{ and } \langle g(x),x\rangle\geq\|x\|^2.$$

PROPOSITION 5.2.8. *If the pair (f, g) satisfies the Karamardian type condition, then it also satisfies condition* (θ_g).

Proof. Because D is bounded, there exists $\rho>0$ such that $D\subseteq\{x\in\mathbb{K}:\|x\|\leq\rho\}$. In this case, for every $x\in\mathbb{K}$ with $\|x\|>\rho$, there exists $y\in D$ such that $\langle g(x)-y,f(x)\rangle\geq 0$, $\langle g(x),x\rangle\geq\|x\|^2$ and $\|y\|<\|x\|$. Hence, all the assumptions of Proposition 5.2.7 are satisfied with $\alpha=1$ and the conclusion of the proposition is a consequence of Proposition 5.2.7. □

DEFINITION 5.2.4. *We say that a pair (f, g) of mappings satisfies condition* $(HPT)_g$ *if there exists a bounded set* $D\subset\mathbb{K}$ *such that for any* $x\in\mathbb{K}\setminus D$ *we have* $\langle g(x),x\rangle\geq\|x\|^2$ *and the set*

$$\mathbb{K}(D,g)=\{x\in\mathbb{K}:\langle y-g(x),f(x)\rangle\geq 0 \text{ for any } y\in D\}$$

is bounded.

Remark. If in Definition 5.2.4, $g(x)=x$ for any $x\in H$, we obtain condition (*HPT*) defined in Section 5.1.

PROPOSITION 5.2.9. *If a pair (f, g) of mappings satisfies condition* $(HPT)_g$ *with respect to* $\mathbb{K}$, *then (f, g) satisfies condition* (θ_g).

Proof. Indeed, we consider the set $M=D\cup\mathbb{K}(D,g)$. Obviously, M is a bounded subset of $\mathbb{K}$. If $x\in\mathbb{K}\setminus M$, then $x\notin\mathbb{K}(D,g)$ and in this case there exists $y\in D$ such that $\langle y-g(x),f(x)\rangle<0$ or $\langle g(x)-y,f(x)\rangle>0$.

Because $y \in M$ and $\langle g(x), x \rangle \geq \|x\|^2$ for any $x \in \mathbb{K} \setminus M$, we have that (f, g) satisfies the Karamardian type condition. Therefore, applying Proposition 5.2.8, we have that (f, g) satisfies condition $(\theta_{g.})$. □

DEFINITION 5.2.5. *We say that a pair (f, g) of mappings from H into H is (ρ, g)-copositive, with respect to a closed convex cone $\mathbb{K} \subset H$, if there exist $\rho > 0$ and $\alpha > 0$ such that for any $x \in \mathbb{K}$, with $\|x\| > \rho$ we have*

(1) $\langle g(x), f(x) \rangle \geq 0$,

(2) $\langle g(x), x \rangle \geq \alpha \|x\|^2$.

PROPOSITION 5.2.10. *If the pair (f, g) of mappings from H into H is (ρ, g)-copositive, with respect to a closed convex cone $\mathbb{K} \subset H$, then the pair (f, g) satisfies condition $(\theta_{g.})$.*

Proof. The conclusion of this proposition follows from Proposition 5.2.7 if we take in this proposition $y = 0$. □

Let $f, g : H \to H$ be two mappings and $\mathbb{K} \subset H$ a closed convex cone. We suppose that there exist $\rho > 0$ and $\alpha > 0$, real numbers such that for any $x \in \mathbb{K}$ with $\|x\| = \rho$ we have satisfied the following relations:

(i) $\langle g(x), f(x) \rangle \geq 0$,

(ii) $\langle g(x), x \rangle \geq \alpha \|x\|^2$.

Let $\mathcal{R}_\rho$ be the radial retraction, i.e.,

$$\mathcal{R}_\rho(x) = \begin{cases} x, \text{ if } \|x\| \leq \rho, \\ \dfrac{\rho x}{\|x\|}, \text{ if } \|x\| > \rho. \end{cases}$$

The mapping $\mathcal{R}_\rho$ is continuous. We consider the following mappings:

$G(x) = g(\mathcal{R}_\rho(x))$, for any $x \in H$,

$F(x) = f(\mathcal{R}_\rho(x)) + \|x - \mathcal{R}_\rho(x)\| \mathcal{R}_\rho(x)$, for any $x \in H$.

For every $x \in \mathbb{K}$ with $\|x\| > \rho$ we have

$$\begin{aligned}\langle G(x),F(x)\rangle &= \langle g(\mathcal{R}_\rho(x)), f(\mathcal{R}_\rho(x)) + \|x-\mathcal{R}_\rho(x)\|\mathcal{R}_\rho(x)\rangle \\ &= \langle g(\mathcal{R}_\rho(x)), f(\mathcal{R}_\rho(x))\rangle + \|x-\mathcal{R}_\rho(x)\|\langle g(\mathcal{R}_\rho(x)),\mathcal{R}_\rho(x)\rangle \\ &\geq \|x-\mathcal{R}_\rho(x)\|\langle g(\mathcal{R}_\rho(x)),\mathcal{R}_\rho(x)\rangle \geq \alpha\rho^2 > 0.\end{aligned}$$

We have also,

$$\begin{aligned}\langle G(x),x\rangle = \langle g(\mathcal{R}_\rho(x)),x\rangle &= \left\langle g(\mathcal{R}_\rho(x)), \frac{\|x\|}{\rho}\mathcal{R}_\rho(x)\right\rangle \\ &\geq \alpha\frac{\|x\|}{\rho}\|\mathcal{R}_\rho(x)\|^2 = \alpha\frac{\|x\|}{\rho}\rho^2 = \alpha\rho\|x\| > 0.\end{aligned}$$

If we take in the definition of *condition* (θ_g), $y = 0$ for any $x \in \mathbb{K}$ with $\|x\| > \rho$, we obtain that the pair (G,F) satisfies condition (θ_g). By Theorem 5.2.4 the pair (G, F) is without an *EFE*. If the problem *ICP*(F, G, $\mathbb{K}$) has a solution x_*, we must have that $\|x_*\| \leq \rho$, which implies that $G(x_*) = g(x_*)$ and $F(x_*) = f(x_*)$. Therefore x_* is a solution to the problem *ICP*(f, g, $\mathbb{K}$).

Finally, we have also the following test for *condition* (θ_g).

THEOREM 5.2.11. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K}\subset H$ *a closed convex cone and* $f, g : H \to H$ *two mappings. If there exist* $\rho_* > 0$ *and* $\alpha > 0$ *such that:*

(1) $\langle g(x),x\rangle \geq \alpha\|x\|^2$ *for all* $x\in\mathbb{K}$ *with* $\|x\|\geq\rho_*$,

(2) *there exists a non-empty bounded set* $D\subset\mathbb{K}$ *such that the set*

$M = \bigcap_{y\in D}\{x\in\mathbb{K} : \|f(x)\| \leq \|y-g(x)+f(x)\|\}$ *is bounded or empty,*

then the pair (f, g) *satisfies condition* (θ_g).

Proof. Since D and M are bounded sets, there exists $\rho_0 > 0$ such that $M\cup D\subset\{x\in\mathbb{K} : \|x\|\leq\rho_0\}$. We can suppose that $0 < \alpha < 1$. Moreover, we can select α and ρ such that $0 < \alpha < 1$ and $\frac{\rho_0}{\alpha} < \rho$, i.e., $\rho_0 < \alpha\rho$ Let $\rho_1 \geq \max\{\rho,\rho_*\}$ be arbitrary. If $x \in \mathbb{K}$ is such that $\rho_1 < \|x\|$, then we have that $x \notin \bigcap_{y \in D} \{x \in \mathbb{K} : \|f(x)\| \leq \|y - g(x) + f(x)\|\}$, which implies that for some $y_0 \in D$ we have

$$\|f(x)\| > \|y_0 - g(x) + f(x)\| \text{ or } \|f(x)\|^2 > \|y_0 - g(x) + f(x)\|^2,$$

which implies

$$\langle f(x), f(x)\rangle > \langle y_0 - g(x), y_0 - g(x)\rangle + 2\langle y_0 - g(x), f(x)\rangle + \langle f(x), f(x)\rangle.$$

Therefore we have

$$\langle g(x) - y_0, f(x)\rangle > 0. \qquad (5.2.3)$$

Because $\|x\| > \rho_*$, we have $\langle g(x), x\rangle \geq \alpha \|x\|^2$ and since for any $y \in D$ we have $\|y\| \leq \rho_0 \leq \alpha\rho < \alpha\|x\|$, we deduce $\langle y, x\rangle \leq \|y\|\|x\| < \alpha\|x\|^2$ which implies that $\langle g(x) - y, x\rangle > 0$, for all $y \in D$ and in particular

$$\langle g(x) - y_0, x\rangle > 0. \qquad (5.2.4)$$

We used the fact that

$$\langle g(x) - y, x\rangle = \langle g(x), x\rangle - \langle y, x\rangle \geq \alpha\|x\|^2 - \langle y, x\rangle > 0.$$

Therefore, relations (5.2.3) and (5.2.4) say that the pair (f, g) satisfies *condition* (θ_g). □

5.3 Set-valued complementarity problems

In this section we adapt the notion of *exceptional family of elements* to the study of solvability of *multivalued complementarity problems.* All the notions used in this section were defined in Chapter 3. Now, we recall the definition of the *multivalued nonlinear complementarity problem* (i.e., the nonlinear complementarity problem defined by a closed convex cone and a set-valued mapping). (This problem was considered in Chapter 2 and it was named the *multivalued complementarity problem.*)

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $f : H \to H$ a set-valued mapping, i.e., $f : H \to 2^H$. In this section we suppose that for all $x \in H$, $f(x) \neq \phi$. We recall that f is *upper semicontinuous* (*u.s.c*) if the set $\{x \in H : f(x) \subset V\}$ is open in *H*, whenever *V* is an open subset in *H*. We say that *f* is *completely upper semicontinuous* (*c.u.s.c*) if it is upper semicontinuous and for any bounded set $B \subset H$, we have that $f(B) = \bigcup_{x \in B} f(x)$ is a relatively compact set.

Let $\mathcal{P}(H)$ be the collection of all non-empty subsets of *H*. We suppose given a *measure of noncompactness* $\Phi : \mathcal{P}(H) \to \Omega$ where, Ω is a lattice with a minimal element denoted by 0. Let *X*, *Y* be subsets of *H*. We recall that a set-valued mapping $h : X \to Y$ is Φ-*condensing* if $A \subset X$ and $\Phi(h(A)) \geq \Phi(A)$ imply that *A* is relatively compact.

DEFINITION 5.3.1. *We say that f is projectionally* Φ*-condensing (resp. projectionally approximable) with respect to* $\mathbb{K}$ *if* $P_{\mathbb{K}}(f)$ *is* Φ*-condensing (resp. approximable).*

The *multivalued nonlinear complementarity problem* defined by *f* and $\mathbb{K}$ is the following problem.

$$MNCP(f, \mathbb{K}) : \begin{cases} \text{find } x_* \in \mathbb{K} \text{ and} \\ y_* \in f(x_*) \cap \mathbb{K}^* \text{ such that} \\ \langle x_*, y_* \rangle = 0. \end{cases}$$

For more details about this problem see Chapter 3.

DEFINITION 5.3.2. *We say that a family of elements* $\{x_r\}_{r>0} \subset \mathbb{K}$ *is an exceptional family of elements (denoted shortly by EFE) for a set-valued mapping* $f : H \to H$ *if and only if, for every real number* $r > 0$ *there exist a real number* $r > 0$ *and a element* $y_r \in f(x_r)$ *such that the following properties are satisfied:*

(1) $\|x_r\| \to +\infty$ *as* $r \to +\infty$,

(2) $u_r = \mu_r x_r + y_r \in \mathbb{K}^*$,

(3) $\langle x_r, u_r \rangle = 0$.

A justification of this notion is given by the following result.

THEOREM 5.3.1. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K}\subset H$ *a closed pointed convex cone and* $f: H \multimap H$ *an u.s.c set-valued mapping with non-empty values. If the following assumptions are satisfied:*

(1) $x - f(x)$ *is projectionally* Φ*-condensing, or* $f(x) = x - T(x)$*, where* T *is a c.u.s.c. set-valued mapping with non-empty values,*

(2) $x - f(x)$ *is projectionally approximable and* $P_{\mathbb{K}}\left[x-f(x)\right]$ *is with closed values,*

then there exists either a solution to the MNCP(f, $\mathbb{K}$), or an exceptional family of elements for f with respect to $\mathbb{K}$.

Proof. If the problem *MNCP(f, $\mathbb{K}$)* has a solution, we have nothing to prove. We suppose that the problem *MNCP(f, $\mathbb{K}$)* is without solution. For any positive real number $r > 0$ we consider the set $B_r = \{x\in H : \|x\|\leq r\}$. Obviously $0\in \operatorname{int}(B_r)$. The set-valued mapping $P_{\mathbb{K}}\left[x-f(x)\right]$ is fixed-point free with respect to any set B_r. Indeed, if there exists $r > 0$ such that $P_{\mathbb{K}}\left[x-f(x)\right]$ has a fixed-point x_* in B_r, then we have $x_* \in P_{\mathbb{K}}\left[x_* - f(x_*)\right]$. Obviously, $x_* \in \mathbb{K}$ and there exists $u_* \in f(x_*)$ such that

$$x_* = P_{\mathbb{K}}\left[x_* - u_*\right]. \tag{5.3.1}$$

By using (5.3.1) and applying the properties (1) and (2) of the projection operator $P_{\mathbb{K}}$ given in Theorem 1.9.7, we have

$$\langle x_* - (x_* - u_*), y\rangle \geq 0, \text{ for all } y\in\mathbb{K} \tag{5.3.2}$$

and

$$\langle x_* - (x_* - u_*), x_*\rangle = 0. \tag{5.3.3}$$

From (5.3.2) and (5.3.3) we obtain that $u_* \in f(x_*)\cap\mathbb{K}^*$ and $\langle x_*, u_*\rangle = 0$, that is, (x_*, u_*) is a solution to the problem *MNCP(f, $\mathbb{K}$)*, which is a contradiction. Therefore, $P_{\mathbb{K}}\left[x-f(x)\right]$ is fixed-point free with respect to any set B_r with $r > 0$. Now, we observe that all the assumptions of Theorem 3.6.4 are satisfied for any set B_r with $r > 0$, and the set-valued mapping $P_{\mathbb{K}}\left[x-f(x)\right]$. Hence for any $r > 0$ there exist $x_r\in\partial B_r$ and $\lambda_r \in]0,1[$ such that

$$x_r \in \lambda_r P_{\mathbb{K}}\left[x_r - f(x_r)\right]. \tag{5.3.4}$$

From (5.3.4) we have that there exists $y_r \in f(x_r)$ such that

$$x_r = \lambda_r P_{\mathbb{K}}[x_r - y_r]. \tag{5.3.5}$$

From (5.3.5) and using again the properties (1) and (2) of the projection operator $P_{\mathbb{K}}$ (see Theorem 1.9.7), we obtain

$$\frac{1}{\lambda_r} x_r - [x_r - y_r] \in \mathbb{K}^* \tag{5.3.6}$$

and

$$\left\langle \frac{1}{\lambda_r} x_r - [x_r - y_r], \frac{1}{\lambda_r} x \right\rangle = 0, \tag{5.3.7}$$

which implies that

$$u_r = \frac{1-\lambda_r}{\lambda_r} x_r + y_r \in \mathbb{K}^*, \tag{5.3.8}$$

and

$$\langle u_r, x_r \rangle = 0. \tag{5.3.9}$$

Because $\|x_r\| = r$ we have that $\|x_r\| \to +\infty$ as $r \to +\infty$. Obviously $x_r \in \mathbb{K}$. If we denote $\mu_r = \dfrac{1-\lambda_r}{\lambda_r}$ we have, (considering (5.3.8) and (5.3.9)) that $\{x_r\}_{r>0}$ is an exceptional family of elements for f with respect to $\mathbb{K}$. □

From Theorem 5.3.1 we obtain also the following result.

THEOREM 5.3.2. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $f : H \to H$ a set-valued mapping. If f has the form $f(x) = x - T(x)$, where T is a completely upper semicontinuous set-valued mapping with non-empty compact contractible values, then there exists either a solution to the problem MNCP(f, $\mathbb{K}$), or an exceptional family of elements for f with respect to $\mathbb{K}$.*

Proof. Obviously, because $P_{\mathbb{K}}$ is continuous at any $x \in H$ and because $T(x)$ is a compact set, we have that

$$P_{\mathbb{K}}[x - f(x)] = P_{\mathbb{K}}[T(x)] \tag{5.3.10}$$

is compact for any $x \in H$ and consequently the set-valued mapping $x \to P_{\mathbb{K}}[x - f(x)]$ is with closed values. Following a proof similar to the proof of Theorem 5.3.1, but by using Theorem 3.6.6, we obtain the conclusion of our theorem if we show (using (5.3.10)) that for any $x \in H$, $P_{\mathbb{K}}\left[T(x)\right]$ is a contractible set. Indeed, if $x \in H$ is an arbitrary element, we denote $D = T(x)$. By assumption D is a contractible set, i.e., there exists a continuous function $h : D \times [0,1] \to D$ with the properties, $h(u, 0) = u$ and $h(u, 1) = u_0$ for some $u_0 \in D$.

Considering the mapping $h^* : P_{\mathbb{K}}(D) \times [0,1] \to P_{\mathbb{K}}(D)$ defined by

$$h^*\left(P_{\mathbb{K}}(u), \lambda\right) := P_{\mathbb{K}}\left[h(u), \lambda\right] \textit{ for all } P_{\mathbb{K}}(u) \in P_{\mathbb{K}}(D) \textit{ and } \lambda \in [0,1],$$

we have

$$h^*\left(P_{\mathbb{K}}(u), 0\right) = P_{\mathbb{K}}\left[h(u), 0\right] = P_{\mathbb{K}}(u)$$

and

$$h^*\left(P_{\mathbb{K}}(u), 1\right) = P_{\mathbb{K}}\left[h(u), 1\right] = P_{\mathbb{K}}(u_0).$$

The mapping h^* is a continuous mapping. Indeed, let $\left\{\left(P_{\mathbb{K}}(u_n), \lambda_n\right)\right\}_{n \in N}$ be a sequence in $P_{\mathbb{K}}(D) \times [0, 1]$ convergent to an element $\left(P_{\mathbb{K}}(u_*), \lambda_*\right)$. Because $D \times [0, 1]$ is compact, there exists a subsequence $\left\{\left(u_{n_k}, \lambda_{n_k}\right)\right\}$ of $\left\{\left(u_n, \lambda_n\right)\right\}$ convergent to an element $\left(v, \lambda_*\right) \in D \times [0,1]$. By using the continuity of $P_{\mathbb{K}}$ and the uniqueness of the limit we have

$$\lim_{k \to \infty} P_{\mathbb{K}}\left(u_{n_k}\right) = P_{\mathbb{K}}(v) = P_{\mathbb{K}}(u_*).$$

We deduce that

$$\begin{aligned}\lim_{n \to \infty} h^*\left(P_{\mathbb{K}}(u_n, \lambda_n)\right) &= \lim_{k \to \infty} h^*\left(P_{\mathbb{K}}\left(u_{n_k}, \lambda_{n_k}\right)\right) = \lim_{k \to \infty} P_{\mathbb{K}}\left(h\left(u_{n_k}, \lambda_{n_k}\right)\right) \\ &= P_{\mathbb{K}}\left[h(v, \lambda_*)\right] = h^*\left(P_{\mathbb{K}}(v), \lambda_*\right) = h^*\left(P_{\mathbb{K}}(u_*), \lambda_*\right).\end{aligned}$$

□

We recall that a mapping $F : H \to H$ is bounded if for any bounded set $B \subset H$ we have that $f(B) = \bigcup_{x \in B} f(x)$ is bounded.

COROLLARY 5.3.3. *Let $\left(H, \langle \cdot, \cdot \rangle\right)$ be a Hilbert space, $\mathbb{K} \subset H$ a locally compact pointed convex cone and $f : H \to H$ a bounded, upper semi-continuous set-valued mapping, with non-empty compact contractible* *values. Then, there exists either a solution to the problem MNCP(f, $\mathbb{K}$), or an exceptional family of elements for f with respect to $\mathbb{K}$.*

Proof. Since $\mathbb{K}$ is a locally compact cone and f is bounded upper semicontinuous, we have that $P_{\mathbb{K}}\left[x-f(x)\right]$ is completely upper semicontinuous. For every $x \in H$, $f(x)$ is compact, which implies that $x - f(x)$ and consequently $P_{\mathbb{K}}\left[x-f(x)\right]$ is compact. Hence for every $x \in H$, $P_{\mathbb{K}}\left[x-f(x)\right]$ is closed. Moreover, $P_{\mathbb{K}}\left[x-f(x)\right]$ is contractible. Indeed, to show this fact we use the function $h^*(x-u,\lambda)=x-h(u,\lambda)$, for all $u \in f(x)$ and $\lambda \in [0,1]$, where h is the continuous mapping defined by the contractibility of $f(x)$. Because $x - f(x)$ is compact and contractible, we have that $P_{\mathbb{K}}\left[x-f(x)\right]$ is contractible. (See the proof of Theorem 5.3.2). Now, the proof follows the proof of Theorem 5.3.1, but using Theorem 3.6.6. □

COROLLARY 5.3.4. *Let $(\mathbb{R}^n,\langle\cdot,\cdot\rangle)$ be n-dimensional Euclidean space and $\mathbb{K}\subset\mathbb{R}^n$ a closed pointed convex cone. If $f:\mathbb{R}^n \to \mathbb{R}^n$ is an upper semicontinuous set-valued mapping with non-empty compact contractible values, then there exists either a solution to the problem MNCP(f, $\mathbb{K}$), or an exceptional family of elements for f with respect to $\mathbb{K}$..*

To recognize if a set-valued mapping is without exceptional families of elements we adapt the condition (θ) to set-valued mappings.

DEFINITION 5.3.3. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space and $\mathbb{K}\subset H$ a closed pointed convex cone. We say that a set-valued mapping $f: H \to H$, with non-empty values satisfies condition $[\theta]_m$ with respect to $\mathbb{K}$, if there exists a real number $\rho>0$ such that for all $x \in \mathbb{K}$ with $\|x\| > \rho$, there exists $y \in \mathbb{K}$ with $\|y\| < \|x\|$ such that $\langle x-y,u\rangle \geq 0$ for all $u\in f(x)$.*

The classical Karamardian's condition for single-valued mappings has the following form for set-valued mappings.

DEFINITION 5.3.4. *We say that a set-valued mapping $f: H \to H$ satisfies the weak Karamardian's condition with respect to $\mathbb{K}$, if there exists a bounded set $D \subset \mathbb{K}$ such that for all $x \in \mathbb{K} \setminus D$ there exists $y \in D$ such that $\langle x - y, u \rangle \geq 0$ for all $u \in f(x)$.*

Obviously, if f satisfies the weak Karamardian's condition with respect to $\mathbb{K}$, then f satisfies condition $[\theta]_m$, but the converse is not true.

DEFINITION 5.3.5. *We say that a set-valued mapping $f: H \to H$ is ρ-copositive with respect to $\mathbb{K}$ if there exists $\rho > 0$ such that for all $x \in \mathbb{K}$, with $\|x\| > \rho$ we have $\langle x, u \rangle \geq 0$ for all $u \in f(x)$.*

We observe that if a set-valued mapping $f: H \to H$ is ρ-copositive with respect to $\mathbb{K}$, then f satisfies condition $[\theta]_m$. To see this fact we take $y = 0$ in the definition of condition $[\theta]_m$.

DEFINITION 5.3.6. *We say that a set-valued mapping satisfies condition $M(D)$ with respect to $\mathbb{K}$, if there exists a non-empty bounded set $D \subset \mathbb{K}$, such that the set $M(D)$ defined by*

$$M(D) = \bigcap_{y \in D} \{x \in \mathbb{K} : \text{there exists } u \in f(x), \text{ with } \|u\| \leq \|y - (x - u)\|\}$$

is a bounded set.

PROPOSITION 5.3.4. *If a set-valued mapping $f: H \to H$ satisfies condition $M(D)$ with respect to $\mathbb{K}$, then f satisfies condition $[\theta]_m$.*

Proof. Indeed, we consider the bounded set $A = D \cup M(D)$. If $x_0 \in \mathbb{K} \setminus A$, then $x_0 \notin \bigcap_{y \in D} \{x \in \mathbb{K} : \text{there exists } u \in f(x), \text{ with } \|u\| \leq \|y - (x - u)\|\}$, which implies that there exists $y_0 \in D$ with the property that, for any $u \in f(x_0)$ we have $\|u\| > \|y_0 - (x_0 - u)\|$. The last inequality implies

$$\|u\|^2 > \|y_0 - x_0\|^2 + 2\langle y_0 - x_0, u\rangle + \|u\|^2,$$

which implies

$$\langle x_0 - y_0, u\rangle > \frac{1}{2}\|y_0 - x_0\|^2 > 0.$$

Because x_0, is an arbitrary element in $\mathbb{K} \setminus A$, we deduce that f satisfies the weak Karamardian's condition with respect to $\mathbb{K}$ and consequently f satisfies condition $[\theta]_m$. □

The condition (*HPT*) can be also adapted to set-valued mappings.

DEFINITION 5.3.7. *We say that a set-valued mapping $f: H \to H$ satisfies condition $[HPT]_m$ with respect to $\mathbb{K}$ if there exists a bounded set $D \subset \mathbb{K}$ such that the set $\mathbb{K}(D) = \{x \in \mathbb{K} : \text{for any } y \in D \text{ there exist } u \in f(x),$ such that $\langle y - x, u\rangle \geq 0\}$ is a bounded set.*

Related to condition $[HPT]_m$, we have the following result.

THEOREM 5.3.5. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\mathbb{K} \subset H$ a closed pointed convex cone and $f: H \to H$ a set-valued mapping. If f satisfies condition $[HPT]_m$, with respect to $\mathbb{K}$, then f satisfies condition $[\theta]_m$.*

Proof. We consider the bounded set $M = D \cup \mathbb{K}(D)$. If $x_0 \in \mathbb{K} \setminus M$, then $x_0 \notin M$ which implies $x_0 \notin \mathbb{K}(D)$ and consequently there exists $y_0 \in D \subset M$ such that for any $u \in f(x)$ we have $\langle y_0 - x_0, u\rangle < 0$ or $\langle x_0 - y_0, u\rangle > 0$. Therefore f satisfies the weak Karamardian's condition with respect to $\mathbb{K}$ and consequently condition $[\theta]_m$. □

Now, we show that condition $[\theta]_m$ implies the non-existence of exceptional families of elements for a set-valued mapping.

THEOREM 5.3.6. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space and $\mathbb{K}\subset H$ a closed pointed convex cone. Let $f: H\to H$ be a set-valued mapping with non-empty values. If f satisfies condition $[\theta]_m$, then f is without an exceptional family of elements with respect to $\mathbb{K}$.*

Proof. We suppose that f has an exceptional family of elements with respect to $\mathbb{K}$, namely, $\{x_r\}_{r>0}\subset\mathbb{K}$. Let $r>0$ be a real number such that $\rho<\|x_r\|$. (The positive real number ρ is defined by condition $[\theta]_m$.) This is possible since $\|x_r\|\to+\infty$ *as* $r\to+\infty$. Because f satisfies condition $[\theta]_m$, there exists $y_r\in\mathbb{K}$ such that $\|y_r\|<\|x_r\|$ and

$$\langle x_r-y_r,u\rangle\geq 0,\ \textit{for all}\ u\in f(x_r). \tag{5.3.11}$$

From the definition of exceptional family of elements, we have that there exist $\mu_r>0$ and $v_r\in f(x_r)$ such that

$$\begin{cases} u_r=\mu_r x_r+v_r\in\mathbb{K}^* \\ \textit{and} \\ \langle u_r,x_r\rangle=0. \end{cases} \tag{5.3.12}$$

Considering (5.3.11) and (5.3.12) we obtain

$$0\leq\langle x_r-y_r,v_r\rangle=\langle x_r-y_r,u_r-\mu_r x_r\rangle=\langle x_r-y_r,u_r\rangle-\mu_r\|x_r\|^2$$
$$+\mu_r\langle y_r,x_r\rangle\leq-\mu_r\|x_r\|^2\left[\|x_r\|-\|y_r\|\right]<0,$$

which is a contradiction and the proof is complete. □

COROLLARY 5.3.7. *Let $(\mathbb{R}^n,\langle\cdot,\cdot\rangle)$ be n-dimensional Euclidean space and $\mathbb{K}\subset\mathbb{R}^n$ a closed pointed convex cone. Let $f:\mathbb{R}^n\to\mathbb{R}^n$ be an upper semicontinuous set-valued mapping with non-empty compact contractible values. If f satisfies condition $[\theta]_m$, then the problem MNCP(f, $\mathbb{K}$) has a solution.*

DEFINITION 5.3.8. *We say that a set-valued mapping $f: H\to H$ satisfies condition $[\theta-S]_m$ with respect to $\mathbb{K}$, if for any family of elements $\{x_r\}_{r>0}\subset\mathbb{K}$, such that $\|x_r\|\to+\infty$ as $r\to+\infty$, there exists $y_*\in\mathbb{K}$ with the* *property that for some* $r > 0$ *with* $\|y_*\| < \|x_r\|$, *we have* $\langle x_r - y_*, u\rangle \geq 0$, *for any* $u \in f(x_r)$.

THEOREM 5.3.8. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed pointed convex cone and* $f : H \to H$ *a set-valued mapping. If* f *satisfies condition* $[\theta - S]_m$, *with respect to* $\mathbb{K}$, *then* f *is without an exceptional family of elements with respect to* $\mathbb{K}$.

Proof. We suppose that f has an exceptional family of elements, namely $\{x_r\}_{r>0} \subset \mathbb{K}$. Because f satisfies condition $[\theta - S]_m$, there exists $y_* \in \mathbb{K}$ such that for some $r > 0$, with $\|y_*\| < \|x_r\|$, we have $\langle x_r - y_*, u\rangle \geq 0$ for any $u \in f(x_r)$. From the definition of exceptional family of elements there exists $y_r \in f(x_r)$ such that $u_r = \mu_r x_r + y_r \in \mathbb{K}^*$ and $\langle x_r, u_r\rangle = 0$. Then we have

$$0 \leq \langle x_r - y_*, y_r\rangle = \langle x_r - y_*, u_r - \mu_r x_r\rangle$$
$$= \langle x_r, u_r\rangle - \langle y_*, u_r\rangle - \mu_r \langle x_r, x_r\rangle + \mu_r \langle y_*, x_r\rangle$$
$$\leq \mu_r \left[\langle y_*, x_r\rangle - \|x_r\|^2\right] \leq \mu_r \|x_r\| \left[\|y_*\| - \|x_r\|\right] < 0,$$

which is a contradiction and the proof is complete. □

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone, $f : H \to H$ a set-valued mapping and $g : H \to H$ a single-valued mapping. We consider the following *multivalued implicit complementarity problem*:

$$MICP(f, g, \mathbb{K}) : \begin{cases} \textit{find } x_* \in H \textit{ and } y_* \in f(x_*) \cap \mathbb{K}^* \\ \textit{such that } g(x_*) \in \mathbb{K} \\ \textit{and } \langle g(x_*), y_*\rangle = 0. \end{cases}$$

We consider the set-valued mapping

$$\Psi_{\mathbb{K}}(x) = P_{\mathbb{K}}\left[g(x) - f(x)\right]$$

and we remark that the solvability of *MICP(f,g,$\mathbb{K}$)* is equivalent to the solvability of the following coincidence equation:

$$CE(g,\Psi_{\mathbb{K}}):\begin{cases} \text{find } x_* \in H \text{ such that} \\ g(x_*) \in \Psi_{\mathbb{K}}(x_*). \end{cases}$$

Also, we remark that if $f(x)$ is a contractible set, then $g(x) - f(x)$ is contractible. Indeed if $f(x)$ is contractible, then there exists a continuous function $h: f(x)\times[0,1] \to f(x)$ such that $h(u, 0) = u$, for any $u \in f(x)$ and there exists $u_0 \in f(x)$ such that $h(\cdot, 1) = u_0$ for any $u \in f(x)$.

Considering the function

$$h^*:(g(x)-f(x))\times[0,1]\to g(x)-f(x)$$

defined by

$$h^*(g(x)-u,\lambda)=g(x)-h(u,\lambda),$$

we can show that $g(x) - f(x)$ is a contractible set. Moreover, as in the proof of Theorem 5.3.2 we can show that if $f(x)$ is contractible, then $P_{\mathbb{K}}[g(x)-f(x)]$ is contractible too.

DEFINITION 5.3.9. *We say that a family of elements* $\{x_r\}_{r>0} \subset H$ *is an exceptional family of elements for the pair* (f, g) *with respect to* $\mathbb{K}$ *if the following properties are satisfied:*

(1) $\|x_r\| \to +\infty$ *as* $r \to +\infty$,

(2) *for any* $r > 0$ *there exist* $\mu_r > 0$ *and* $y_r \in f(x_r)$ *such that*

(i) $u_r = \mu_r g(x_r) + y_r \in \mathbb{K}^*$,

(ii) $\langle u_r, g(x_r)\rangle = 0$.

The notation and the notions used in the next theorem are defined in Chapter 3 in relation to Theorem 3.6.12.

THEOREM 5.3.9. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space and* $\mathbb{K} \subset H$ *a closed pointed convex cone* g: $H \to H$ *a single-valued continuous mapping and* $f: H \to H$ *a set-valued mapping with non-empty values. If the following assumptions are satisfied:*

(1) $g(x) \neq 0$ *for any* $x \neq 0$,

(2) g *is essential with respect to any set* $U_r = \{x \in H \mid \|x\| < r\}$, *where* $r > 0$,

(3) *f is completely upper semicontinuous with compact contractible values,*
(4) *for any bounded set $D \subset H$, $(g - f)(D)$ is relatively compact, then, at least one of the following conditions holds:*
   (i) *the problem MICP(f, g, $\mathbb{K}$) has a solution,*
   (ii) *there exists an exceptional family of elements for the pair (f, g) with respect to $\mathbb{K}$.*

Proof. If the problem *MICP(f, g, $\mathbb{K}$)* has a solution we have nothing to prove. We suppose that the problem *MICP(f, g, $\mathbb{K}$)* has no solution. We show that the assumptions of Theorem 3.6.12 are satisfied. For this we use the notation and the terminology of this theorem.

For any $r > 0$ we consider the open set $U_r = \{x \in H \mid \|x\| < r\}$ and its boundary $\partial U_r = \{x \in H \mid \|x\| = r\}$. From our assumptions we have that $0 \notin g(\partial U_r)$ and $g \in \mathcal{A}_{\partial U_r}(U_r, H)$. Also, we can show that $\Psi_{\mathbb{K}} \in \mathcal{A}_c(U_r, H)$. Because the problem *MICP(f, g, $\mathbb{K}$)* is without solution, we have that $g(x) \notin \Psi_{\mathbb{K}}(x)$ for any $x \in \partial U_r$, i.e., $g(x) \cap \Psi_{\mathbb{K}}(x) = \phi$ for any $x \in \partial U_r$, and also the conclusion (1) of Theorem 3.6.12 is not satisfied.

By assumption g is essential with respect to any U_r. Therefore, we conclude that for any $r > 0$, there exist $\lambda_r \in]0, 1[$ and $x_r \in \partial U_r$ such that

$$g(x_r) \in \lambda_r \Psi_{\mathbb{K}}(x_r) = \lambda_r P_{\mathbb{K}}[g(x_r) - f(x_r)].$$

Then, there exists $y_r \in f(x_r)$ such that

$$g(x_r) = \lambda_r P_{\mathbb{K}}[g(x_r) - y_r]$$

and considering the properties of operator $P_{\mathbb{K}}$ (given in Theorem 1.9.7) we obtain

$$u_r = \mu_r g(x_r) + y_r \in \mathbb{K}^*, \textit{ where } \mu_r = \frac{1}{\lambda_r} - 1 \textit{ and } \langle u_r, g_r \rangle = 0.$$

Because $\|x_r\| = r$, we have that $\|x_r\| \to +\infty$ *as* $r \to +\infty$ and we conclude that $\{x_r\}_{r>0}$ is an exceptional family of elements for the pair (f, g) with respect to $\mathbb{K}$, and the proof is complete. □

Remark. The following mappings g are essential with respect to any set U_r, $r > 0$.

(1) g = identity mapping.

(2) $g = L$, where $L : H \to H$ is a continuous linear isomorphism.

(3) g = a Vietoris mapping $p : H \to H$ such that $p^{-1}(\{0\}) = \{0\}$.

For more details about essentiality the reader is referred to (Gorniewicz, L. and Slosarski, M. [1]).

Obviously, we can consider the *multivalued implicit complementarity problem* defined by a closed convex cone $\mathbb{K} \subset H$ and a pair (f, g) of *set-valued mappings* from H into H. This problem is the following:

$$MICP(f, g, K): \begin{cases} \textit{find } x_* \in H \textit{ such that} \\ \textit{there exist } u \in f(x_*) \cap \mathbb{K}^*, \\ v \in g(x_*) \cap \mathbb{K} \textit{ satisfying } \langle u, v \rangle = 0. \end{cases}$$

The study of this problem in an arbitrary Hilbert space is a difficult problem. An idea is to use a selection for f and a selection for g, but the selections must satisfy some topological properties, as for example complete continuity. We note that such selection theorems are unknown at this moment. However, this idea works in the Euclidean space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$. To do this, we need to recall some definitions.

We say that a set-valued mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is *lower semicontinuous at the point* $x \in \mathbb{R}^n$, if for any arbitrary $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x - y\| < \delta$ implies $f(x) \subset f(y) + \varepsilon B$ where B is the unit ball centred at the origin. The mapping f is called *lower semicontinuous,* if it is such at every point $x \in \mathbb{R}^n$. We say that a single-valued mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a *continuous selection* if φ *is continuous* and $\varphi(x) \in f(x)$, for every $x \in \mathbb{R}^n$. We recall also the following classical result.

THEOREM 5.3.10 [Michael's theorem]. *Let X be a paracompact topological space and $(E, \|\cdot\|)$ a Banach space. If $f : X \to E$ is a set-valued* *lower semicontinuous mapping such that for any $x \in X$, $f(x)$ is a non-empty closed convex subset of E, then there exists a continuous selection $\varphi : X \to E$ for f.*

We recall that a topological space X is called *paracompact* if it is Hausdorff and from every open cover of it, one can extract a locally finite subcover. By Stone's Theorem every metric space is paracompact. A particular case of Theorem 5.3.*10* is the following result.

THEOREM 5.3.11. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a lower semicontinuous set-valued mapping such that $f(x)$ is a non-empty closed convex subset of $\mathbb{R}^n$ for each $x \in \mathbb{R}^n$. Then f has a continuous selection $\varphi : \mathbb{R}^n \to \mathbb{R}^n$.*

Now, we introduce the following notion of *exceptional family of elements* for a pair of set-valued mappings.

DEFINITION 5.3.10. *We say that a family of elements $\{x_r\}_{r>0} \subset \mathbb{R}^n$ is an exceptional family of elements for the pair (f, g) of set-valued mappings $f, g : \mathbb{R}^n \to \mathbb{R}^n$, with respect to a closed convex cone $\mathbb{K} \subset \mathbb{R}^n$ if the following conditions are satisfied:*

(1) $\|x_r\| \to +\infty$ *as* $r \to +\infty$,

(2) *for every $r > 0$, there exist a real number $\mu_r > 0$ and two elements $y_r \in f(x_r), z_r \in g(x_r)$ such that $s_r = \mu_r x_r + y_r \in \mathbb{K}^*$, $v_r = \mu_r x_r + z_r \in \mathbb{K}$ and $\langle s_r, v_r \rangle = 0$.*

We note the following result.

THEOREM 5.3.12. *Let $f, g : \mathbb{R}^n \to \mathbb{R}^n$ be lower semicontinuous set-valued mappings with non-empty convex closed values, and $\mathbb{K} \subset \mathbb{R}^n$ a pointed closed convex cone. Then there exists either a solution to the problem MICP(f, g, $\mathbb{K}$), or an exceptional family of elements in the sense of Definition 5.3.10.*

Proof. By Theorem 5.3.11, we have a continuous selection φ for f and a continuous selection ψ for g. Obviously. Because the space is n-dimensional Euclidean space, the pair (φ, ψ) of continuous mappings from $\mathbb{R}^n$ into $\mathbb{R}^n$

are completely continuous and we have that the assumptions of Theorem 5.2.1 are satisfied.

Hence, there exists either a solution to the problem *ICP*(φ, ψ, $\mathbb{K}$), which evidently solves the initial problem *MICP*(*f*, *g*, $\mathbb{K}$) too, or an exceptional family of elements $\{x_r\}_{r>0}$ for the pair (φ, ψ) in the sense of Definition 5.2.1, which is clearly an exceptional family of elements for the initial pair (*f*, *g*) in the sense of Definition 5.3.10, and the proof is complete. □

The study of the *MICP*(*f*, *g*, $\mathbb{K}$) is an interesting subject. We note also that *condition* $[\theta]_m$ can be generalized in the following form.

DEFINITION 5.3.11. *We say that a set-valued mapping* $f : H \to H$ *with non-empty values satisfies condition* $\left[\tilde{\theta}\right]_m$ *with respect to* $\mathbb{K}$, *if there exists a real number* $\rho > 0$ *such that for each* $x \in \mathbb{K}$ *there exists* $p \in \mathbb{K}$ *with* $\langle p, x \rangle < \|x\|^2$ *such that* $\langle x - p, x^f \rangle \geq 0$ *for all* $x^f \in f(x)$.

Remark. Theorem 5.3.6 is also true if we replace condition $[\theta]_m$ by condition $\left[\tilde{\theta}\right]_m$.

5.4 Exceptional family of elements and monotonicity

We know that, in a Hilbert space $(H, \langle \cdot, \cdot \rangle)$, if we have a closed pointed convex cone $\mathbb{K}$ and a mapping $f : H \to H$ which is projectionally Leray–Schauder, with respect to $\mathbb{K}$, then the problem *NCP*(*f*, $\mathbb{K}$) has a solution if *f* is without an *EFE*. A natural question is: under what conditions does the solvability of the *NCP*(*f*, $\mathbb{K}$) imply that *f* is without an *EFE?* In this section we will study this problem. First, we recall the definition of pseudomonotone mappings.

Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $f: H\to H$ a mapping and $\mathbb{K}\subset H$ a closed convex cone. We say that f is *pseudomonotone on* $\mathbb{K}$ if, for any distinct points x, $y\in\mathbb{K}$, the inequality $\langle x-y,f(y)\rangle\geq 0$ implies $\langle x-y,f(x)\rangle\geq 0$. Many authors have studied this notion from some points of view (Karamardian, S. [5]), (Karamardian, S. and Schaible, S. [1]), (Hadjisavvas, N. and Schaible, S. [1], [2]) and (Schaible, S. and Yao, J. C. [1]), (Yao, J. C. [1]). In 1976 S. Karamardian introduced the notion of *pseudomonotone mapping* in relation to the study of complementarity problems (Karamardian. S. [5]). In the cited paper, we can find a number of existence theorems for complementarity problems defined by monotone or pseudomonotone operators in $\mathbb{R}^n$ or in Hilbert space.

Also, it is natural to consider complementarity problems defined by set-valued pseudomonotone mappings. Let $f: H\to H$ be a *set-valued mapping*. We say that f is *pseudomonotone on* $\mathbb{K}$, *if for any distinct points* x, $y\in\mathbb{K}$ *and arbitrary* $u\in f(x)$ *and* $w\in f(y)$, $\langle x-y,w\rangle\geq 0$ *implies* $\langle x-y,u\rangle\geq 0$.

A monotone mapping is pseudomonotone but the converse is not true. We have the following result.

THEOREM 5.4.1. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space, and* $\mathbb{K}\subset H$ *a closed, pointed convex cone. If* $f: H\to H$ *is a set-valued pseudomonotone mapping on* $\mathbb{K}$, *and the problem MNCP(f,* $\mathbb{K}$*) has a solution, then f is without an EFE with respect to* $\mathbb{K}$ *(in the sense of Definition 5.3.2).*

Proof. Indeed, we suppose that the problem *MNCP(f,* $\mathbb{K}$*)* has a solution (x_*, y_*), i.e., $x_*\in\mathbb{K}$, $y_*\in f(x_*)\cap\mathbb{K}^*$ and $\langle x_*,y_*\rangle=0$, which is equivalent to the variational inequality

$$\langle x-x_*,y_*\rangle\geq 0 \textit{ for all } x\in\mathbb{K}. \tag{5.4.1}$$

Since f is pseudomonotone, then from (5.4.1) we have

$$\langle x-x_*,u\rangle\geq 0 \textit{ for any } x\in\mathbb{K} \textit{ and any } u\in f(x). \tag{5.4.2}$$

Suppose that f has an *EFE*, namely $\{x_r\}_{r>0} \subset \mathbb{K}$. In this case for any $r > 0$, there exist $\mu_r > 0$ and $w_r \in f(x_r)$ such that

(i) (i) $u_r = \mu_r x_r + w_r \in \mathbb{K}^*$,

(ii) $\langle x_r, u_r \rangle = 0$,

(iii) $\|x_r\| \to \infty$ *as* $r \to \infty$.

We choose x_r such that $\|x_*\| < \|x_r\|$. Making use of (5.4.2) we have

$$\begin{aligned} 0 &\le \langle x_r - x_*, w_r \rangle = \langle x_r - x_*, u_r - \mu_r x_r \rangle \\ &= \langle x_r, u_r \rangle - \langle x_*, u_r \rangle - \mu_r \|x_r\|^2 + \mu_r \langle x_*, x_r \rangle \\ &\le -\mu_r \|x_r\|^2 + \mu_r \|x_*\| \|x_r\| = \mu_r \|x_r\| \left(\|x_*\| - \|x_r\| \right) < 0 \end{aligned}$$

which is a contradiction. Therefore, f is without an exceptional family of elements (an *EFE*) with respect to $\mathbb{K}$, in the sense of Definition 5.3.1 and the proof is complete. □

We recall that $f : H \to H$ has a representation of the form $f(x) = x - T(x)$, where $T : H \to H$ is a completely upper semicontinuous set-valued mapping with compact contractible values; we say that f is *a completely upper semicontinuous field with compact contractible values.*

COROLLARY 5.4.2. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed pointed convex cone and* $f : H \to H$ *a set-valued pseudomonotone mapping with respect to* $\mathbb{K}$. *If* f *is a completely upper semicontinuous field with non-empty compact contractible values, then the problem MNCP(*f*,* $\mathbb{K}$*) has a solution, if and only if* f *is without an EFE with respect to* $\mathbb{K}$ *(in the sense of Definition 5.3.2).*

Proof. The corollary is a consequence of Theorem 5.3.2 and Corollary 5.4.2. □

COROLLARY 5.4.3. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed pointed convex cone and* $f : H \to H$ *a projectionally Leray-Schauder mapping with respect to* $\mathbb{K}$. *If* f *is pseudomonotone with respect to* $\mathbb{K}$, *then* *the problem NCP(f, $\mathbb{K}$) has a solution if and only if f is without an EFE with respect to $\mathbb{K}$ (in the sense of Definition 5.1.2).*

COROLLARY 5.4.4. *Let $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ be n-dimensional Euclidean space, and let $\mathbb{K} \subset \mathbb{R}^n$ be a closed pointed convex cone. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous pseudomonotone mapping with respect to $\mathbb{K}$, then the problem NCP(f, $\mathbb{K}$) has a solution if and only if f is without an EFE with respect to $\mathbb{K}$.*

Now we consider a more general situation. Let $f, g : H \to H$ be a pair of mappings and $\mathbb{K} \subset H$ a closed pointed convex cone.

DEFINITION 5.4.1. *We say that f is asymptotically g-pseudomonotone with respect to $\mathbb{K}$, if there exists a real number $\rho > 0$ such that for all $x, y \in \mathbb{K}$ with $\max\{\rho, \|y\|\} < \|x\|$ we have that $\langle x - y, g(y)\rangle \geq 0$ implies $\langle x - y, f(x)\rangle \geq 0$.*

Remark. Any pseudomonotone (in particular monotone) mapping $f : H \to H$, with respect to $\mathbb{K}$, is asymptotically *f*-pseudomonotone.

THEOREM 5.4.5. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $f, g : H \to H$ two mappings. If f is asymptotically g-pseudomonotone with respect to $\mathbb{K}$ and the problem NCP(g, $\mathbb{K}$) has a solution, then f is without an exceptional family of elements with respect to $\mathbb{K}$.*

Proof. Let $x_* \in \mathbb{K}$ be a solution to the problem *NCP(g, $\mathbb{K}$)*. Considering the relations between complementarity problems and variational inequalities we have that

$$\langle x - x_*, g(x_*)\rangle \geq 0, \text{ for all } x \in \mathbb{K}. \tag{5.4.3}$$

Since f is asymptotically g-pseudomonotone, we deduce that

$$\langle x - x_*, f(x) \rangle \geq 0, \textit{ for all } x \in \mathbb{K} \textit{ with } \max\{\rho, \|x_*\|\} < \|x\|. \qquad (5.4.4)$$

Suppose that f has an exceptional family of elements, that is, there exists a family of elements $\{x_r\}_{r>0} \subset \mathbb{K}$, such that for any $r > 0$ there exists a real number $\mu_r > 0$ such that the following properties are satisfied:

(i) $u_r = \mu_r x_r + f(x_r) \in \mathbb{K}^*$,

(ii) $\langle x_r, u_r \rangle = 0$,

(iii) $\|x_r\| \to +\infty$ *as* $r \to \infty$.

By property (iii) we can choose x_r such that $\max\{\rho, \|x_*\|\} < \|x_r\|$. Making use of (5.4.4) we obtain

$$\begin{aligned} 0 &\leq \langle x_r - x_*, f(x_r) \rangle = \langle x_r - x_*, u_r - \mu_r x_r \rangle \\ &= \langle x_r, u_r \rangle - \langle x_*, u_r \rangle - \mu_r \|x_r\|^2 + \mu_r \langle x_*, x_r \rangle \\ &\leq -\mu_r \|x_r\|^2 + \mu_r \|x_*\| \|x_r\| = \mu_r \|x_r\| [\|x_*\| - \|x_r\|] < 0, \end{aligned}$$

which is a contradiction. Therefore, f is without an exceptional family of elements, with respect to $\mathbb{K}$ □

Remark. We note that any (u, g, φ)-monotone mapping (see Definition 5.1.9) is asymptotically g-pseudomonotone mapping.

From Theorem 5.4.5, we deduce the following interesting result.

THEOREM 5.4.6 [Transitivity principle]. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed pointed convex cone and* f, $g : H \to H$ *a pair of mappings. If the following assumptions are satisfied:*

(1) *f is a projectionally Leray–Schauder mapping with respect to $\mathbb{K}$,*

(2) *f is asymptotically g-pseudomonotone with respect to $\mathbb{K}$,*

(3) *the problem NCP(g, $\mathbb{K}$) has a solution,*

then the problem NCP(f, $\mathbb{K}$) has a solution.

Proof. This theorem is a consequence of Theorem 5.4.5 and 5.1.2. □

Remark. It is known that complementarity problems are used as mathematical models in the study of equilibrium of economical systems. Related to this fact, Theorem 5.4.6 may have interesting applications to the study of equilibrium of two economical systems depending on each other (integrated economical systems).

COROLLARY 5.4.7. *Let f, g : $\mathbb{R}^n \to \mathbb{R}^n$ be a pair of continuous mappings and $\mathbb{K} \subset \mathbb{R}^n$ an arbitrary closed pointed convex cone. If the following assumptions are satisfied:*

(1) *f is asymptotically g-pseudomonotone with respect to $\mathbb{K}$,*

(2) *the problem $NCP(g, \mathbb{K})$ has a solution,*

then the problem $NCP(f, \mathbb{K})$ has a solution.

COROLLARY 5.4.8. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and f : $H \to H$ a projectionally Leray–Schauder mapping. If f is pseudomonotone with respect to $\mathbb{K}$, then the $NCP(f, \mathbb{K})$ has a solution, if and only if f is without an EFE with respect to $\mathbb{K}$.*

The next result is a variant of Theorem 5.4.5 where the solvability of the problem $NCP(g, \mathbb{K})$ is replaced by a *strict feasibility* condition for the problem $NCP(g, \mathbb{K})$. Before giving this result, we recall some notions. The *strict dual* of $\mathbb{K}$ is by definition:

$$\widehat{\mathbb{K}^*} = \left\{ y \in H \,\middle|\, \langle y, x \rangle > 0 \text{ for any } x \in \mathbb{K} \setminus \{0\} \right\}.$$

It is known that if $\mathbb{K}$ is well-based (see Chapter 1), then there exists a continuous linear functional φ : $H \to \mathbb{R}$ and a constant $c > 0$ such that $c\|x\| \leq \varphi(x)$, for any $x \in \mathbb{K}$ (Hyers, D. H., Isac, G. and Rassias, Th. M. [1]). Obviously, in this case $\varphi \in \widehat{\mathbb{K}^*}$ and hence, we have that $\widehat{\mathbb{K}^*}$ is non-empty. When $\widehat{\mathbb{K}^*}$ is non-empty we say that the problem $NCP(g, \mathbb{K})$ is *strictly feasible* if there exists an element $x_0 \in \mathbb{K}$ such that $g(x_0) \in \widehat{\mathbb{K}^*}$.

THEOREM 5.4.9. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K}\subset H$ *a closed well-based convex cone and* $f: H\to H$ *an asymptotically g-pseudomonotone mapping with respect to* $\mathbb{K}$. *If there exists an element* $x_0\in\mathbb{K}\setminus\{0\}$ *such that* $g(x_0)\in\widehat{\mathbb{K}^*}$, *then f is without an exceptional family of elements with respect to* $\mathbb{K}$.

Proof. We assume that f has an exceptional family of elements $\{x_r\}_{r>0}$ with respect to $\mathbb{K}$.

Then, for any $r>0$ there exists $\mu_r>0$ such that

(i) $u_r=\mu_r x_r+f(x_r)\in\mathbb{K}^*$,

(ii) $\langle x_r,u_r\rangle=0$,

(iii) $\|x_r\|\to+\infty$ *as* $r\to+\infty$.

We show that in this case, there exists $r_0>0$ such that for all $r>r_0$ we have $\langle g(x_0),x_r-x_0\rangle>0$. Indeed, since $\mathbb{K}$ is well-based there exist $\varphi\in H^*$ and a constant $c>0$ such that $c\|x\|\le\langle\varphi,x\rangle$, for all $x\in\mathbb{K}$.

The set $D=\{x\in\mathbb{K}\,|\,\langle\varphi,x\rangle=1\}$ is weakly compact in H. We have

$$\langle g(x_0),x_r-x_0\rangle=\langle g(x_0),x_r\rangle-\langle g(x_0),x_0\rangle$$
$$=\left\langle g(x_0),\frac{x_r}{\langle\varphi,x_r\rangle}\right\rangle\langle\varphi,x_r\rangle-\langle g(x_0),x_0\rangle.$$

Since $\dfrac{x_r}{\langle\varphi,x_r\rangle}\in D$, we deduce that $\left\langle g(x_0),\dfrac{x_r}{\langle\varphi,x_r\rangle}\right\rangle\ge\varepsilon>0$, where $\varepsilon=\min_{x\in D}\langle g(x_0),x\rangle>0$, due to the weak compactness of D, the weak continuity of $\langle g(x_0),x\rangle$ and the fact that $g(x_0)\in\widehat{\mathbb{K}^*}$. Therefore, we have

$$\langle g(x_0),x_r-x_0\rangle\ge\varepsilon\langle\varphi,x_r\rangle-\langle g(x_0),x_0\rangle$$
$$\ge\varepsilon c\|x_r\|-\langle g(x_0),x_0\rangle>0 \textit{ for all } r>r_0,$$

where r_0 is such that $\|x_r\| > \frac{\langle g(x_0), x_0 \rangle}{\varepsilon c}$, for all $r > r_0$.

Now, making use of the fact that f is asymptotically g-pseudomonotone we have that $\langle f(x_r), x_r - x_0 \rangle \geq 0$ for all r such that $r > r_0$ and $\|x_r\| > \max\{\rho, \|x_0\|\}$. From the last inequality and the definition of $\{x_r\}_{r>0}$ we have

$$0 \leq \langle f(x_r), x_r - x_0 \rangle = \langle u_r - \mu_r x_r, x_r - x_0 \rangle = \langle u_r, x_r \rangle - \mu_r \langle x_r, x_r \rangle$$
$$-\langle u_r, x_0 \rangle + \mu_r \langle x_r, x_0 \rangle \leq -\mu_r \|x_r\|^2 + \mu_r \|x_r\| \|x_0\| = -\mu_r \|x_r\| [\|x_r\| - \|x_0\|],$$

for all r such that $r > r_0$ and $\|x_r\| > \max\{\rho, \|x_0\|\}$.

Since $\|x_r\| \to +\infty$ as $r \to +\infty$, there exists $r_* > r_0$ such that $\|x_r\| > \max\{\rho, \|x_0\|\}$ and $0 \leq -\mu_r \|x_r\| [\|x_r\| - \|x_0\|] < 0$ for all $r > r_*$, which is impossible. Therefore, f is without an exceptional family of elements and the proof is complete. □

Remark. It is interesting to know if there exist other classes of mappings, different than the mappings considered in this section, with the property that the solvability of the complementarity problem implies the non-existence of exceptional families of elements.

5.5 Semi-definite complementarity problems

All *existence theorems* for complementarity problems and the results related to the notion of *exceptional family of elements*, presented in this chapter, can be applied in particular to the study of *semi-definite complementarity problems*. The application of the notion of exceptional family of elements to the study of semi-definite complementarity problems was considered also in (Isac, G., Bulavski, V. A. and Kalashnikov, V. V. [2]) but now we have more results. Therefore, the goal of this section is to inform the reader that the majority of results presented in this chapter are applicable to the study of *semi-definite complementarity problems*.

Let $(\mathbb{R}^{n\times n}, \langle\cdot,\cdot\rangle)$ be the $n \times n$-dimensional Euclidean space of $n \times n$-matrices endowed with the inner-product

$$\langle A, B\rangle = tr(A^t B),\ for\ any\ A, B \in \mathbb{R}^{n\times n}$$

(where $tr(A^t B)$ means the *trace* of A^tB). We introduce the *norm* on the space $\mathbb{R}^{n\times n}$ in the standard manner: $\|A\| = \langle A, A\rangle^{1/2}$. Let $S^{n\times n}$ be the linear subspace of symmetric real $(n \times n)$-matrices, and $S_+^n \subset S^{n\times n}$ the *convex cone of positive semi-definite matrices.* The notation $A \geq 0$ means $A \in S_+^n$. Interior points of this cone are positive definite matrices A, i.e., $A \geq 0$ of full rank n. We denote that by $A > 0$.

Let $F : S_+^n \to S^{n\times n}$ be a continuous mapping. The *semi-definite complementarity problem* is:

$$SDCP(F, S_+^n): \begin{cases} find\ X \in S_+^n\ such\ that \\ F(X) \in S_+^n\ and\ \langle X, F(X)\rangle = 0. \end{cases}$$

It is known that the cone S_+^n is *self-dual*, i.e.,

$$(S_+^n)^* = \{Y \in S^{n\times n} : \langle Y, X\rangle \geq 0, for\ all\ X \in S_+^n\} = S_+^n.$$

(See Schatten, R. [1]). Also we note that for A, $B \in S_+^n$, the equality $tr(AB) = 0$ is equivalent to $AB = 0$. Because of this fact, the $SDCP(F, S_+^n)$ can be given also by:

$$SDCP(F, S_+^n): \begin{cases} find\ X \in S_+^n\ such\ that \\ F(X) \in S_+^n\ and\ X \cdot F(X) = 0. \end{cases}$$

For more details about the *semi-definite complementarity problem* the reader is referred to (Kojima, M, Shindoh, S. and Hara, S. [1]) and (Bulavsky, V. A., Isac, G. and Kalashnikov, V. V. [2]).

The space $(S^{n\times n}, \langle\cdot,\cdot\rangle)$ is finite dimensional Hilbert space and S_+^n is a closed convex cone in this space. Therefore, for any matrix $Z \in S^{n\times n}$ the projection operator $P_{S_+^n}(Z)$ is well defined, as in the general case we have the following definition.

DEFINITION 5.5.1. *A family of matrices* $\{Z_r\}_{r>0} \subset S_+^n$ *is called an exceptional family of matrices with respect to the cone* S_+^n *for a mapping*

$F : S^n_+ \to S^{n\times n}$ if $\|Z_r\| \to +\infty$ as $r \to +\infty$ and for each $r > 0$ there exists a scalar $\mu_r > 0$ such that the matrix $M_r = F(Z_r) + \mu_r Z_r$ has the following property:

$$M_r \in S^n_+ \text{ and } M_r Z_r = 0.$$

This definition is justified by the following result.

THEOREM 5.5.1. *If $F : S^n_+ \to S^{n\times n}$ is a continuous mapping, then there exists either a solution to the problem $SDCP(F, S^n_+)$ or an exceptional family of matrices with respect to the cone S^n_+.*

Proof. A proof of this theorem is in (Bulavsky, V. A., Isac, G. and Kalashnikov, V. V. [2]). □

Because the space $(S^{n\times n}, \langle \cdot,\cdot \rangle)$ is a finite dimensional Hilbert space, in the existence theorems for the problem *SDCP*(F, $\mathbb{K}$), based on the notion of exceptional family of matrices, we need to have only the continuity and the fact that the mapping is without exceptional family of matrices. The *condition* (θ) is also applicable. All the existence theorems presented in this chapter are applicable to semi-definite complementarity problems.

5.6 Feasibility and an exceptional family of elements

In complementarity theory it is well known that, under some supplementary conditions, the feasibility of a complementarity problem implies its solvability [see (Isac, G. [26]), Section 5.5]. Certainly feasibility plays an important role in the study of complementarity problems. In this sense, we recall that in Theorem 4.5.1 we have that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a P_0-mapping and the feasible set with respect to $\mathbb{R}^n_+$ contains n particular points, then the mapping f is *without an exceptional family of elements* with respect to $\mathbb{R}^n_+$. Also, in Theorem 5.4.9 we have that, if a *g-pseudomonotone mapping* $f : H \to H$ (where H is Hilbert space ordered by a closed pointed convex cone $\mathbb{K} \subset H$), is such that there exists an element $x_0 \in \mathbb{K} \setminus \{0\}$ with

$g(x_0) \in \widehat{\mathbb{K}^*}$ (the strict dual of $\mathbb{K}$), then f is *without an exceptional family of elements* with respect to $\mathbb{K}$.

Now, in this section we will present other results where the strict feasibility implies the non-existence of an *exceptional family of elements* and we will introduce some notions of exceptional families of elements, which can be used to obtain the strict feasibility.

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, let $\mathbb{K} \subset H$ be a closed pointed convex cone and let $f : H \to H$ be a mapping. We consider the problem:

$$NCP(f, \mathbb{K}) : \begin{cases} \text{find } x_0 \in \mathbb{K} \text{ such that} \\ f(x_0) \in \mathbb{K}^* \text{ and } \langle x_0, f(x_0) \rangle = 0. \end{cases}$$

By definition, *the feasible* set of this problem is:

$$\mathcal{F} = \{ x \in \mathbb{K} : f(x) \in \mathbb{K}^* \}, \text{ where } \mathbb{K}^* \text{ is the dual of } \mathbb{K}.$$

The set $\mathcal{F}$ can be empty; when $\mathcal{F}$ is non-empty, we say that the *NCP(f, $\mathbb{K}$) is feasible.* If the cone $\mathbb{K}^*$ has a non-empty interior and if the set $\mathcal{F}_s = \{ x \in \mathbb{K} : f(x) \in \operatorname{int}(\mathbb{K}^*) \}$ is non-empty, we say that the *NCP(f, $\mathbb{K}$) is strictly feasible.*

We recall that the strict dual of $\mathbb{K}$ is:

$$\widehat{\mathbb{K}^*} = \{ y \in H : \langle x, y \rangle > 0 \text{ for all } x \in \mathbb{K} \setminus \{0\} \}.$$

It is known that if $\mathbb{K}$ is *well based*, then $\widehat{\mathbb{K}^*}$ is non-empty. The solvability of *NCP(f, $\mathbb{K}$)* implies its feasibility, but the converse is not true. We say that a mapping $f : H \to H$ is *quasimonotone* with respect to $\mathbb{K}$ if for every $x, y \in \mathbb{K}$ the following implication holds:

$$\langle f(x), y - x \rangle > 0 \Rightarrow \langle f(y), y - x \rangle \geq 0.$$

Obviously, *any monotone mapping is pseudomonotone and any pseudomonotone mapping is quasimonotone, but the converse is not true.* We have the following result.

THEOREM 5.6.1. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, let $\mathbb{K} \subset H$ a closed well based convex cone and let $f : H \to H$ be a mapping. If f is quasimonotone with respect to $\mathbb{K}$ and there exists an element $x_0 \in \mathbb{K}$ such that $f(x_0) \in \widehat{\mathbb{K}^*}$, then f is without an exceptional family of elements with respect to $\mathbb{K}$.*

Proof. To show that f is without an exceptional family of elements with respect to $\mathbb{K}$, is sufficient to show that f satisfies *condition* $(\theta - S)$. Indeed, let $\{x_r\}_{r>0} \subset \mathbb{K}$ be a family of elements such that $\|x_r\| \to +\infty$ *as* $r \to +\infty$.

First, we show that there exists $r_0 > 0$ such that $\langle f(x_0), x_r - x_0\rangle > 0$ for any r with $r > r_0$. Since $\mathbb{K}$ is well based, there exist $\varphi \in H^*$ and a constant $c > 0$ such that $c\|x\| \le \langle \varphi, x\rangle$ for any $x \in \mathbb{K}$, (φ can be considered as an element of H). Moreover, the set $D = \{x \in \mathbb{K} : \langle \varphi, x\rangle = 1\}$ is a weakly compact set. (Since D is a base of $\mathbb{K}$ and H is reflexive we have that $\mathbb{K}$ is weakly locally compact.) We have

$$\langle f(x_0), x_r - x_0\rangle = \langle f(x_0), x_r\rangle - \langle f(x_0), x_0\rangle$$
$$= \left\langle f(x_0), \frac{x_r}{\langle \varphi, x_r\rangle}\right\rangle \langle \varphi, x_r\rangle - \langle f(x_0), x_0\rangle.$$

Since $\frac{x_r}{\langle \varphi, x_r\rangle} \in D$, then we have $\min_{x\in D} \langle f(x_0), x\rangle = \varepsilon > 0$ (because $f(x_0) \in \widehat{\mathbb{K}^*}$ and D is weakly compact), which implies $\left\langle f(x_0), \frac{x_r}{\langle \varphi, x_r\rangle}\right\rangle \ge \varepsilon > 0$. Therefore, we have $\langle f(x_0), x_r - x_0\rangle > \varepsilon\langle \varphi, x_r\rangle - \langle f(x_0), x_0\rangle \ge \varepsilon c\|x_r\| - \langle f(x_0), x_0\rangle > 0$, for all $r > r_0$, where $r_0 > 0$ is such that $\|x_r\| > \frac{\langle f(x_0), x_0\rangle}{\varepsilon c}$, for all $r > r_0$. Now, the quasimonotonicity of f with respect to $\mathbb{K}$ implies

$$\langle f(x_r), x_r - x_0 \rangle \geq 0, \text{ for all } r > r_0.$$

If we take on $r > r_0$ and $y_* = x_0$, we obtain that f satisfies *condition* $(\theta - S)$ with respect to $\mathbb{K}$ and by Theorem 5.1.27 we obtain that f is without an exceptional family of elements with respect to $\mathbb{K}$. □

COROLLARY 5.6.2. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space, let* $\mathbb{K} \subset H$ *be a closed well based convex cone and* $f : H \to H$ *a quasimonotone mapping. If* f *is a projectionally Leray–Schauder mapping, and the problem NCP(f,* $\mathbb{K}$*) is strictly feasible, then the NCP(f,* $\mathbb{K}$*) has a solution.*

Remark. The result presented in Corollary 5.6.2, was independently proved by a different proof in (Hadjsavvas, N. and Schaible, S. [2]).

Now, we consider some results in the n-dimensional Euclidean space ordered by the cone $\mathbb{R}^n_+$.

DEFINITION 5.6.1. *We say that a mapping* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is a quasi-*P_**-mapping if there exists a constant* $\tau \geq 0$ *such that the following implication holds:*

$$\langle f(y), x - y \rangle - \tau \sum_{i \in I_+(x,y)} (x_i - y_i)[f_i(x) - f_i(y)] > 0 \Rightarrow \langle f(x), x - y \rangle \geq 0$$

for all distinct points x, y in $\mathbb{R}^n$, *where*

$$I_+(x, y) = \{ i : (x_i - y_i)[f_i(x) - f_i(y)] > 0 \}.$$

We consider also the following definition:

DEFINITION 5.6.2. *We say that a mapping* f*:* $\mathbb{R}^n \to \mathbb{R}^n$ *is a* P_**-mapping, if there exists a scalar* $k \geq 0$ *such that, for any distinct points x, y in* $\mathbb{R}^n$ *we have*

$$\langle f(x) - f(y), x - y \rangle + k \sum_{i \in I_+(x,y)} (x_i - y_i)[f_i(x) - f_i(y)] \geq 0.$$

Obviously, a P_*-mapping is a quasi- P_*-mapping, but the converse is not true. Clearly, a quasimonotone mapping, which corresponds to the case $\tau = 0$, is a quasi-P_*-mapping, but the converse is not true. It is known that the class of quasi-P_*-mappings is larger than the union of P_*-mappings and quasimonotone mapping. We note also the fact that the notion of P_*-mapping is related to the notion of P_*-matrix. An affine mapping $f(x) = Mx + q$, where M is an $(n \times n)$-matrix and $q \in \mathbb{R}^n$ is a P_*-mapping if and only if M is a P_*-matrix.

In recent years, linear complementarity problem with P_*-matrices have gained more attention in the field of interior-point algorithms, (Kojima, M., Megiddo, N., Noma, T. and Yoshise, A. [1]), (Potra, F. A. and Sheng, R., [1], [2], [3]) and (Miao, J. M., [1]).

DEFINITION 5.6.3. *We say that a mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ is a $P(\tau, \alpha, \beta)$-mapping, if there exist constants $\tau > 0$, $\alpha \geq 0$ and $0 \leq \beta < 1$ such that the following inequality holds:*

$$(1+\tau)\max_{1\leq i\leq n}(x_i - y_i)\left[f_i(x) - f_i(y)\right] + \min_{1\leq i\leq n}(x_i - y_i)\left[f_i(x) - f_i(y)\right]$$

$$\geq -\alpha\|x-y\|^{\beta}, \textit{ for any distinct points } x, y \in \mathbb{R}^n.$$

Remark. The union of all $P(\tau, 0, 0)$-mappings with $\tau \geq 0$ coincides with the class of P_*-mappings. For more details and results about the classes of mappings defined by Definitions 5.6.1, 5.6.2, and 5.6.3, the reader is referred to (Zhao, Y. B. and Isac, G. [1]). We cite without proof the following result.

THEOREM 5.6.3. *Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping. If f is a quasi-P_*-mapping or a $P(\tau, \alpha, \beta)$-mapping and there exists an element $x_0 \in \mathbb{R}^n_+$ such that $f(x_0) \in \operatorname{Int}(\mathbb{R}^n_+)$, then f is without an exceptional family of elements (in the sense of Definition 4.1.1). Moreover, the problem $NCP(f, \mathbb{R}^n_+)$ has a solution.*

Proof. The proof is given by the proofs of Theorem 2.1 and 3.1 presented in (Zhao, Y. B. and Isac, G. [1]) and are based on several technical details.

□

Remark. We recall that when the cone $\mathbb{K} \subset \mathbb{R}^n$ is reduced to the cone $\mathbb{R}^n_+$, the notion of *EFE* defined by Definition 5.1.2 is reduced to the notion of *EFE* defined by Definition 4.1.1.

By the next results we will show that the notion of *EFE* can be used to study the feasibility of nonlinear complementarity problems. Certainly, we must modify the notion of *EFE* introduced by Definition 5.1.2. Let

$(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ be n-dimensional Euclidean space, $\mathbb{K} \subset \mathbb{R}^n$ a closed pointed convex cone and f: $\mathbb{R}^n \to \mathbb{R}^n$ a continuous mapping.

DEFINITION 5.6.4. *Let* (α, β) *be a pair of real numbers such that* $0 \leq \alpha < \beta$. *We say that a family of elements* $\{x_r\}_{r>0} \subset \mathbb{R}^n$ *is an* (α, β)-*exceptional family of elements for* f *with respect to* $\mathbb{K}$ *if and only if* $\lim_{r \to +\infty} \|x_r\| = +\infty$ *and for each real number* $r > 0$, *there exists a scalar* $t_r \in]0, 1[$ *such that the vector* $u_r = (1/t_r - 1)x_r + (\beta - \alpha) f(x_r)$ *satisfies the following properties*:

(i) $u_r \in \mathbb{K}^*$,

(ii) $\langle u_r, x_r - \alpha t_r f(x_r) \rangle = 0$.

The importance of this notion is given by the following result.

THEOREM 5.6.4. *Let* (α, β) *be a pair of real numbers such that* $0 \leq \alpha < \beta$ *and let* $\mathbb{K} \subset \mathbb{R}^n$ *be a closed pointed convex cone such that* $\mathbb{K}^* \subset \mathbb{K}$ *or* $\mathbb{K}^* = \mathbb{K}$. *Then, for any continuous mapping* $f : \mathbb{R}^n \to \mathbb{R}^n$, *either the problem NCP(f, $\mathbb{K}$) is feasible or there exists an* (α, β)-*exceptional family of elements for f with respect to* $\mathbb{K}$.

Proof. For any $r > 0$ we denote

$$S_r = \{x \in \mathbb{R}^n : \|x\| = r\} \text{ and } B_r = \{x \in \mathbb{R}^n : \|x\| < r\}.$$

We consider the mapping $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ defined by:

$$\Psi(x) = \alpha f(x) + P_{\mathbb{K}}[x - \beta f(x)].$$

Obviously, Ψ is a continuous mapping. If the problem *NCP(f, $\mathbb{K}$)* is feasible we have nothing to prove.

We suppose that the problem *NCP(f, $\mathbb{K}$)* is not feasible. In this case, we apply Theorem 3.2.4 [Leray–Schauder alternative] for any $r > 0$ to the set B_r and the mapping Ψ. For any $r > 0$, Ψ does not have a fixed point in $\overline{B_r}$, because if x_* is a fixed point for Ψ in $\overline{B_r}$, then, in this case we have

$$x_* = \alpha f(x_*) + P_{\mathbb{K}}[x_* - \beta f(x_*)]$$

which implies that

$$\langle x_* - \alpha f(x_*) - [x_* - \beta f(x_*)], y\rangle \geq 0 \textit{ for all } y \in \mathbb{K}.$$

We have $\langle(\beta-\alpha) f(x_*), y\rangle \geq 0$ *for all* $y \in \mathbb{K}$, that is $f(x_*) \in \mathbb{K}^*$. Since $P_{\mathbb{K}}[x_* - \beta f(x_*)] \in \mathbb{K}$ and $\mathbb{K}^* \subseteq \mathbb{K}$, we deduce that

$$x_* = \alpha f(x_*) + P_{\mathbb{K}}[x_* - \beta f(x_*)] \in \mathbb{K}.$$

Consequently the *NCP(f, $\mathbb{K}$)* is feasible which is a contradiction. Therefore, for any $r > 0$ there exist $x_r \in S_r$ and $t_r \in]0,1[$ such that

$$x_r = t_r\left[\alpha f(x_r) + P_{\mathbb{K}}[x_r - \beta f(x_r)]\right]. \tag{5.6.1}$$

From (5.6.1) we have

$$P_{\mathbb{K}}[x_r - \beta f(x_r)] = \frac{1}{t_r} x_r - \alpha f(x_r). \tag{5.6.2}$$

Using (5.6.1) and the properties of the operator $P_{\mathbb{K}}$, we deduce

$$\left\langle \frac{1}{t_r} x_r - \alpha f(x_r) - [x_r - \beta f(x_r)], y \right\rangle \geq 0 \textit{ for all } y \in \mathbb{K}$$

and

$$\left\langle \frac{1}{t_r} x_r - \alpha f(x_r) - [x_r - \beta f(x_r)], \frac{1}{t_r} x_r - \alpha f(x_r) \right\rangle = 0 .$$

If we denote

$$u_r = \left(\frac{1}{t_r} - 1\right) x_r + (\beta - \alpha) f(x_r), \textit{ for every } r > 0,$$

we deduce that $u_r \in \mathbb{K}^*$ and $\langle u_r, x_r - \alpha t_r f(x_r)\rangle = 0$, because, for every $r > 0$, $x_r \in S_r$ we have that $\|x_r\| \to +\infty$ *as* $r \to +\infty$. Therefore, $\{x_r\}_{r>0}$ is an (α, β)-exceptional family of elements for *f* with respect to $\mathbb{K}$. □

Now, we consider the case of a general Hilbert space. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $\mathbb{K} \subset H$ be a closed pointed convex cone. Let f: $H \to H$ be a completely continuous field of the form $f(x) = \frac{1}{\beta} x - T(x)$, where $\beta > 0$ and $T : H \to H$ is a completely continuous operator. We introduce the following notion.

DEFINITION 5.6.5. *Let* $f: H \to H$ *be a completely continuous field of the form* $f(x) = \frac{1}{\beta}x - T(x)$, *for all* $x \in H$. *Given a real number* α *such that* $0 \le \alpha < \beta$, *we say that the family of elements* $\{x_r\}_{r>0} \subset H$ *is an* (α, β)-*exceptional family of elements for* f *with respect to* $\mathbb{K}$ *if and only if* $\lim_{r\to+\infty} \|x_r\| = +\infty$ *and, for every real number* $r > 0$, *there exists a scalar* $t_r \in]0, 1[$ *such that the element* $u_r = \frac{1}{\beta t_r}x_r - T(x_r)$ *satisfies the following properties*:

(i) $u_r \in \mathbb{K}^*$,

(ii) $\left\langle u_r, \frac{\beta-\alpha}{t_r}x_r + \alpha\beta T(x_r)\right\rangle = 0.$

We have the following result.

THEOREM 5.6.5. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be an arbitrary Hilbert space and* $\mathbb{K} \subset H$ *a closed pointed convex cone such that* $\mathbb{K}^* \subseteq \mathbb{K}$. *Let* $f: H \to H$ *be a completely continuous field of the form* $f(x) = \frac{1}{\beta}x - T(x)$, *where* $\beta > 0$. *Then, for any real* α *such that* $0 \le \alpha < \beta$, *either the problem* $NCP(f, \mathbb{K})$ *is feasible or there exists an* (α, β)-*exceptional family of elements in the sense of Definition 5.6.5 for* f *with respect to* $\mathbb{K}$.

Proof. For any $r > 0$ we denote

$$S_r = \{x \in H : \|x\| = r\} \text{ and } B_r = \{x \in H : \|x\| < r\}$$

and we consider the mapping $\Psi : H \to H$ defined by

$$\Psi(x) = -\frac{\beta\alpha}{\beta-\alpha}T(x) + P_{\mathbb{K}}\left[\frac{\beta^2}{\beta-\alpha}T(x)\right].$$

The mapping Ψ is completely continuous. If the problem $NCP(f, \mathbb{K})$ is feasible, then in this case we have nothing to prove. We suppose that the problem $NCP(f, \mathbb{K})$ is not feasible. For any $r > 0$ we apply Theorem 3.2.4 to the set B_r and the mapping Ψ. If the mapping Ψ has a fixed point x_* in a set $\overline{B_r}$, then in this case we have

$$x_* = -\frac{\beta\alpha}{\beta-\alpha}T(x_*) + P_{\mathbb{K}}\left[\frac{\beta^2}{\beta-\alpha}T(x_*)\right],$$

which implies that

$$x_* - \alpha f(x_*) = P_{\mathbb{K}}\left[x_* - \beta f(x_*)\right],$$

and as in the proof of Theorem 5.6.4 we deduce that $NCP(f, \mathbb{K})$ is feasible which is a contradiction.

Therefore, supposing that $NCP(f, \mathbb{K})$ is not feasible, we have (by Theorem 3.2.4) that for any $r > 0$ there exist $x_r \in S_r$ and $t_r \in]0, 1[$ such that

$$x_r = t_r\left[-\frac{\beta\alpha}{\beta-\alpha}T(x_r) + P_{\mathbb{K}}\left[\frac{\beta^2}{\beta-\alpha}T(x_r)\right]\right].$$

This relation implies (as in the proof of Theorem 5.6.4) that $\{x_r\}_{r>0}$ is an (α, β)-exceptional family of elements for f with respect to $\mathbb{K}$. □

Remark. Modulo some details, it is possible to extend *Theorem 5.6.5* from completely continuous fields of the form $f(x) = \frac{1}{\beta}x - T(x)$, to k-set fields of the form $f(x) = \frac{1}{\beta}x - T(x)$, where T is a k-set contraction with an appropriate k.

Now, we introduce another notion of *exceptional family of elements,* which can be used in the study of feasibility. When we use this notion, it is not necessary to suppose that $\mathbb{K}^* \subseteq \mathbb{K}$.

DEFINITION 5.6.6. *Given a completely continuous field f, of the form $f(x) = \frac{1}{\beta}x - T(x)$, where $\beta > 0$ and $T: H \to H$ (where T is a completely continuous mapping) and $\alpha \in [0,\beta[$, we say that a family of elements $\{x_r\}_{r>0} \subset \mathbb{K}$ is an (α, β)-exceptional family of elements for f if $\|x_r\| \to +\infty$ as $r \to +\infty$ and for each $r > 0$, there exists $t_r \in]0, 1[$ such that*

(i) $t_r x_r + \beta f(x_r) \in \mathbb{K}^*$,

(ii) $\langle t_r x_r + \beta f(x_r), (1+t_r)x_r - \alpha P_{\mathbb{K}}[T(x_r)] \rangle = 0$.

With this notion we have the following result.

THEOREM 5.6.6. *Let $(H, \langle \cdot, \cdot \rangle)$ be an arbitrary Hilbert space and $\mathbb{K} \subset H$ a closed pointed convex cone and f : H→ H a completely continuous field of the form $f(x) = \frac{1}{\beta}x - T(x)$, where $\beta > 0$. Then either the NCP(f, $\mathbb{K}$) is feasible or for each $\alpha \in [0, \beta[$, there exists an (α, β)-exceptional family of elements for f with respect to $\mathbb{K}$. (in the sense of Definition 5.6.5).*

Proof. For any $r > 0$ we denote

$$S_r = \{x \in H : \|x\| = r\} \text{ and } B_r = \{x \in H : \|x\| < r\},$$

and we consider the mapping $\Psi : H \to H$ defined for any α by:

$$\Psi(x) = \alpha P_{\mathbb{K}}[T(x)] + P_{\mathbb{K}}[x - \beta f(x) - \alpha P_{\mathbb{K}}[T(x)]].$$

The mapping Ψ is completely continuous. If the problem *NCP(f, $\mathbb{K}$)* is feasible, then in this case we have nothing to prove.

We suppose that the problem *NCP(f, $\mathbb{K}$)* is not feasible. In this case, Ψ is without a fixed point, because supposing that Ψ has a fixed point x_*, we can show that the problem *NCP(f, $\mathbb{K}$)* is feasible which is a contradiction. Applying Theorem 3.2.4 to any set B_r with $r > 0$ and to the mapping Ψ, we obtain an element $x_r \in S_r$ and a real number $t_r \in]0, 1[$ such that

$$x_r = t_r\{\alpha P_{\mathbb{K}}[T(x_r)] + P_{\mathbb{K}}[x_r - \beta f(x_r) - \alpha P_{\mathbb{K}}[T(x_r)]]\}.$$

Considering the properties of the projection operator $P_{\mathbb{K}}$, we can show that the family $\{x_r\}_{r>0}$ is an (α, β)-exceptional family of elements for f with respect to $\mathbb{K}$. □

If the Hilbert space $(H,\langle\cdot,\cdot\rangle)$ is the n-dimensional Euclidean space $(\mathbb{R}^n,\langle\cdot,\cdot\rangle)$ and $\mathbb{K}\subset\mathbb{R}^n$ is a closed pointed convex cone, then for any continuous mapping $f:\mathbb{R}^n\to\mathbb{R}^n$ and any $\alpha\geq 0$ we can introduce the following notion.

DEFINITION 5.6.7. *We say that a family of elements $\{x_r\}_{r>0}\subset\mathbb{K}$ is an α-exceptional family for f, with respect to $\mathbb{K}$, if $\|x_r\|\to+\infty$ as $r\to+\infty$ and for each $r>0$, there exists $t_r\in\,]0,1[$ such that denoting by $\eta_r=\frac{1}{t_r}-1$ we have*

(i) $\eta_r x_r+f(x_r)\in\mathbb{K}^*$,

(ii) $\langle\eta_r x_r+f(x_r),(1+\eta_r)x_r-\alpha P_{\mathbb{K}}[f(x_r)]\rangle=0$.

Considering for any $\alpha>0$ the mapping

$$\Psi(x)=\alpha P_{\mathbb{K}}[f(x)]+P_{\mathbb{K}}\left[x-f(x)-\alpha P_{\mathbb{K}}[f(x)]\right],$$

and by a proof similar to the proof of *Theorem 5.6.6* we obtain the following result.

THEOREM 5.6.7. *Let $\mathbb{K}\subset\mathbb{R}^n$ be a closed pointed convex cone and $f:\mathbb{R}^n\to\mathbb{R}^n$ a continuous mapping. Then either the problem NCP(f, $\mathbb{K}$) is feasible, or for each $\alpha\geq 0$, there exists an α-exceptional family of elements.*

Remark. Definition 5.6.7 is due to N. J. Huang, C. J. Gao and X. P. Huang [1]. About the study of feasibility by the notion of *exceptional family of elements* we cite also the paper (Zhao,Y. B. and Li, D. [1]). The strict feasibility can be studied also using a special notion of *exceptional family of elements.* In this sense we have the following notion, in $\mathbb{R}^n$.

DEFINITION 5.6.8. *Let $\delta>0$ be an arbitrary real number and let $f:\mathbb{R}^n\to\mathbb{R}^n$ be a continuous mapping. We say that $\{x_r\}_{r>0}\subset\mathbb{R}^n_+$ is a δ-exceptional family of elements for f with respect to $\mathbb{R}^n_+$ if and only if, for every $r>0$, there exists $t_r\in\,]0,1[$ such that*

(i) $\|x_r\|\to+\infty$ *as* $r\to+\infty$,

(ii) $f_i(x^r) + \mu_r x_i^r = \delta$, *for* $x_i^r > 0$, *where* $\mu_r = \dfrac{1-t_r}{2t_r}$,

(iii) $f_i(x^r) \geq \delta$, *if* $x_i^r = 0$.

We have the following result.

THEOREM 5.6.8. *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous mapping. Then, either the problem NCP(f,* $\mathbb{R}^n_+$*) is strictly feasible or for any* $\delta > 0$, *the mapping f has a* δ*-exceptional family of elements with respect to* $\mathbb{R}^n_+$.

Proof. Let $\delta > 0$ be an arbitrary real number. If the problem *NCP(f,* $\mathbb{R}^n_+$*)* is strictly feasible the proof is complete. We suppose that the problem *NCP(f,* $\mathbb{R}^n_+$*)* is not strictly feasible, and we consider the mapping $\Psi_\delta : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Psi_\delta(x) = \left[(\Psi_\delta)(x)\right]_{i=1}^n, \text{ where for any } x \in \mathbb{R}^n,$$

$$\begin{cases} (\Psi_\delta)_i(x) = \sqrt{x_i^2} - f_i(x) + \delta + \sqrt{\left[f_i(x) - \delta\right]^2}, \\ i = 1,2,\ldots,n. \end{cases}$$

The mapping Ψ_δ is continuous and $\Psi_\delta(\mathbb{R}^n_+) \subseteq \mathbb{R}^n_+$. We apply the Leray–Schauder alternative (Theorem 3.3.6) to the mapping Ψ_δ considering $C = \mathbb{R}^n_+$ and $U = U_r = \{x \in \mathbb{R}^n_+ : \|x\| < r\}$ for an arbitrary $r > 0$. The mapping Ψ_δ is fixed-point free on every set $\overline{U_r} = \{x \in \mathbb{R}^n_+ : \|x\| \leq r\}$. Indeed, if for some $r > 0$ there exists $x_*^r \in \overline{U_r}$ such that $\Psi_\delta(x_*^r) = x_*^r$, then in this case we can show that the problem *NCP(f,* $\mathbb{R}^n_+$*)* is strictly feasible, which is a contradiction.

Hence Ψ_δ is fixed-point free on any $\overline{U_r}$, $r > 0$. In this case applying Theorem 3.3.6 we obtain for any $r > 0$ an element $x^r \in \mathbb{R}^n_+$ such that $\|x^r\| = r$ and a real number $t_r \in \,]0,1[$ such that $x^r = t_r \Psi_\delta(x^r)$. Now, recalling the definition of Ψ_δ, we have that, for every $i = 1, 2, \ldots, n$,

$$x_i^r = t_r\left[\sqrt{(x_i^r)^2} - f_i(x^r) + \delta + \sqrt{\left[f_i(x^r) - \delta\right]^2}\right].$$

If $x_i^r = 0$, then in this case we have

$$-f_i\left(x^r\right)+\delta+\sqrt{\left[f_i\left(x^r\right)-\delta\right]^2}=0$$

which implies $f_i\left(x^r\right)\geq\delta$. If $x_i^r \neq 0$, i.e., $x_i^r > 0$, then in this case we have

$$\frac{1}{t_r}x_i^r - x_i^r = -f_i\left(x^r\right)+\delta+\sqrt{\left[f_i\left(x^r\right)-\delta\right]^2}$$

and finally,

$$\frac{1-t_r}{t_r}x_i^r + f_i\left(x^r\right)-\delta=\sqrt{\left[f_i\left(x^r\right)-\delta\right]^2}.$$

If in the last relation we denote

$$u=f_i\left(x^r\right)-\delta,$$

then we obtain

$$\frac{1-t_r}{t_r}x_i^r + u = \sqrt{u^2}$$

and finally,

$$u=-\left[\frac{1-t_r}{2t_r}\right]x_i^r.$$

Therefore, we have the equality

$$f_i\left(x^r\right)+\mu_r x_i^r=\delta, \textit{ where } \mu_r=\frac{1-t_r}{2t_r}>0,$$

and the proof is complete. □

COROLLARY 5.6.9. *If all the assumptions of Theorem 5.6.8 are satisfied and f is without δ-exceptional families of elements with respect to* $\mathbb{R}^n_+$, *then the problem NCP(f,* $\mathbb{R}^n_+$*) is strictly feasible.*

5.7 Paths of ε-solutions and exceptional families of elements

In this section we selected some results necessary to show how the notion of *exceptional family of elements* can be adapted to the study of interior bands of ε-solutions for complementarity problems in $\mathbb{R}^n$.

Let $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ be the n-dimensional Euclidean space ordered by the closed pointed convex cone $\mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}^n$ a continuous mapping. We consider the following nonlinear complementarity problem:

$$NCP(f, \mathbb{R}^n_+): \begin{cases} \textit{find } x_* \in \mathbb{R}^n_+ \textit{ such that} \\ f(x_*) \in \mathbb{R}^n_+ \textit{ and } \langle x_*, f(x_*)\rangle = 0. \end{cases}$$

We note that there exist several equivalent formulations of the problem $NCP(f, \mathbb{R}^n_+)$. In particular, several formulations are in the form of a nonlinear equation $F(x) = 0$, where $F: \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping (Isac, G. [20]). By using such formulation, several techniques proposed by some authors are based on the idea to perturb F to a certain $F(x, \varepsilon)$, where ε is a positive parameter and to consider the equation $F(x, \varepsilon) = 0$. If $F(x, \varepsilon) = 0$ has a unique solution, denoted by $x(\varepsilon)$, and $x(\varepsilon)$ is continuous in ε, then the solutions describe (depending on the properties of F) a short path denoted by $\{x(\varepsilon) : \varepsilon \in]0, \varepsilon_0]\}$ or a long path $\{x(\varepsilon) : \varepsilon \in]0, \infty[\}$.

We note that, if a short path $\{x(\varepsilon) : \varepsilon \in]0, \infty[\}$ is bounded, then for any sequence $\{\varepsilon_k\}$ with $\{\varepsilon_k\} \to 0$, the sequence $\{x(\varepsilon_k)\}$ has at least one accumulation point, which by continuity is a solution to the problem $NCP(f, \mathbb{R}^n_+)$. Based on this fact, several numerical methods for solving the problem $NCP(f, \mathbb{R}^n_+)$ have been developed, as for example the *interior-point path-following methods, regularization methods and noninterior path-following methods, among others.*

About such methods the reader can see the paper (Burke, J. and Xu, S. [1], [2]), (Chen, B and Chen, X. [1]), (Chen, B., Chen, X. and Kanzow, C. [1]), (Facchinei, F. and Kanzow, C. [1]), (Ferris, C. and Pang, J. S. [1]), (Gowda, M. S. and Tawhid, M. A. [1]), (Guler, O. [1]), (Kanzow, C. [1]), (Kojima, M., Megiddo, N., Noma, T. and Yoshise, A. [1]), (Megiddo, N. [1]), (Monteiro, R. D. C. and Adler, I. [1]) and (Tseng, P. [1]).

The most common interior-point path-following method is based on the notion of *central path.* The curve $\{x(\varepsilon) : \varepsilon \in]0, \infty[\}$ is said to be the *central path* if for each $\varepsilon > 0$ the vector $x(\varepsilon)$ is the unique solution to the system

$$\begin{cases} x(\varepsilon) > 0, f(x(\varepsilon)) > 0, \\ and\ X(\varepsilon) \cdot f(x(\varepsilon)) = \varepsilon e, \end{cases} \tag{5.7.1}$$

where the inequality ">" means that the components of the vector are strictly positive, $e = (1,1,...,1)^T$, $X(\varepsilon) = the\ matrix\ diag(x(\varepsilon))$ and $x(\varepsilon)$ is continuous on $]0, \infty[$.

It is well known that, for a general $NCP(f, \mathbb{R}^n_+)$, the system (5.7.1) may have multiple solutions for a given $\varepsilon > 0$, and even if the solution is unique it is not necessarily continuous in ε. Related to system (5.7.1) we consider the set-valued mapping $\mathcal{U} :]0, \infty[\to S(\mathbb{R}^n_{++})$ defined by

$$\mathcal{U}(\varepsilon) = \{x \in \mathbb{R}^n_{++} : f(x) > 0, Xf(x) = \varepsilon e\},$$

where $X = the\ matrix\ diag(x), S(\mathbb{R}^n_{++})$ is the collection of all subsets of $\mathbb{R}^n_{++}$ and $\mathbb{R}^n_{++} = \{(x = x_1, \ldots, x_n) : x_1 > 0, x_2 > 0, ..., x_n > 0\} = \mathrm{int}(\mathbb{R}^n_+)$. We say that $\mathcal{U}$ is the *interior band mapping.*

The set-valued mapping $\mathcal{U}$ was studied from several points of view in (Zhao, Y. B. and Isac, G. [2]). About the set-valued mapping $\mathcal{U}$, we are interested to know, under what conditions $\mathcal{U}$ has the following desirable properties.

(a) $\mathcal{U}(\varepsilon) \neq \phi$ *for each* $\varepsilon \in]0, \infty[$.

(b) *For any fixed* $\varepsilon_0 > 0$ *the set* $\bigcup_{\varepsilon \in]0, \varepsilon_0]} \mathcal{U}(\varepsilon)$ *is a bounded set.*

(c) *If* $\mathcal{U}(\varepsilon) \neq \phi$, *then* $\mathcal{U}(\varepsilon)$ *is upper-semicontinuous at* ε.

(d) *If* $\mathcal{U}(\cdot)$ *is single-valued, then* $\mathcal{U}(\varepsilon)$ *is continuous at* ε *provided that* $\mathcal{U}(\varepsilon) \neq \phi$.

If the mapping $\mathcal{U}(\cdot)$ satisfies properties (a), (b) and (c), then the set $\bigcup_{\varepsilon \in]0, \varepsilon_0]} \mathcal{U}(\varepsilon)$ can be viewed as an *interior band* associated with the solution set of problem $NCP(f, \mathbb{R}^n_+)$. The *interior band* can be viewed as a generalization of the concept of *the central path.* We will show in this section that a notion of *exceptional family of elements* can be used to obtain several results related to the set-valued mapping $\mathcal{U}$.

DEFINITION 5.7.1. *Let* $f: \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous mapping. Given a scalar* $\varepsilon > 0$, *we say that a family* $\{x^r\}_{r>0} \subset \mathbb{R}^n_{++}$ *is an interior-point -ε-exceptional family of elements for f if* $\|x^r\| \to +\infty$ *as* $r \to +\infty$ *and for each* x^r *there exists a real number* $\lambda_r \in]0,1[$ *such that*

$$f_i(x^r) = \frac{1}{2}\left(\lambda_r - \frac{1}{\lambda_r}\right)x_i^r + \frac{\varepsilon\lambda_r}{x_i^r}, \text{ for all } i = 1, \dots, n.$$

Using this notion we can prove the following result.

THEOREM 5.7.1. *Let* $f: \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous function. Then for each* $\varepsilon > 0$ *there exists either a point* $x(\varepsilon)$ *such that*

$$\begin{cases} x(\varepsilon) > 0, f(x(\varepsilon)) > 0 \text{ and } x_i(\varepsilon) f_i(x(\varepsilon)) = \varepsilon, \\ \text{for all } i = 1, 2, \dots, n, \end{cases}$$

or an interior-point-ε-exceptional family of elements exists for f.

We will prove a more general result than Theorem 5.7.1, using the *ε-multivalued complementarity problem.* Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a set-valued mapping with non-empty values. Suppose given a real number $\varepsilon > 0$. The *ε-multivalued complementarity problem* defined by f and the cone $\mathbb{R}^n_+$ is:

$$\varepsilon - MCP(f, \mathbb{R}^n_+): \begin{cases} \text{find } x(\varepsilon) \in \operatorname{int} \mathbb{R}^n_+ \text{ and } u(\varepsilon) \in f(x(\varepsilon)) \\ \text{such that } u(\varepsilon) > 0 \text{ and} \\ [x(\varepsilon)]_i \cdot [u(\varepsilon)]_i = \varepsilon, \text{ for all } i = 1, 2, \dots, n. \end{cases}$$

DEFINITION 5.7.2. *Let* $f: \mathbb{R}^n \to \mathbb{R}^n$ *be a set-valued mapping with non-empty values. Given a real number* $\varepsilon > 0$, *we say that* $\{x^r\}_{r>0} \subset \operatorname{int}(\mathbb{R}^n_+)$ *is an interior-point-ε exceptional family of elements for f if* $\|x^r\| \to +\infty$ *as* $r \to +\infty$, *and for each* $r > 0$, *there exist* $\lambda_r \in]0,1[$ *and* $y^r \in f(x^r)$ *such that*

$$y_i^r = \frac{1}{2}\left(\lambda_r - \frac{1}{\lambda_r}\right)x_i^r + \frac{\varepsilon\lambda_r}{x_i^r}, \text{ for all } i = 1, 2, \dots, n.$$

We have the following result.

THEOREM 5.7.2. *Let $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ be the n-dimensional Euclidean space ordered by the cone $\mathbb{R}^n_+$. Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a set-valued mapping with non-empty closed convex values. If f is lower semicontinuous, then for each $\varepsilon > 0$ there exists either a solution to the problem $\varepsilon - MCP(f, \mathbb{R}^n_+)$ or an interior-point-ε-exceptional family of elements for f.*

Proof. Let $\varepsilon > 0$ be given. Because f is a lower semicontinuous set-valued mapping, with non-empty closed, convex values, we have, by Theorem 5.3.10 (Michael's theorem) that f has a continuous selection. We denote this selection by $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$. Let $\Phi^\varepsilon(x) = (\Phi_1^\varepsilon(x), \dots, \Phi_n^\varepsilon(x))$ be the ε-Fischer–Burmeister function, i.e.,

$$\begin{cases} \Phi_i^\varepsilon(x) = x_i + \varphi_i(x) - \sqrt{x_i^2 + \varphi_i^2 + 2\varepsilon}, \\ i = 1, 2, \dots, n. \end{cases}$$

It is easy to see that if $x(\varepsilon)$ solves the nonlinear equation

$$\Phi^\varepsilon(x) = 0, \tag{5.7.2}$$

then $(x(\varepsilon), \varphi(x(\varepsilon)))$ is a solution to the problem $\varepsilon - MCP(f, \mathbb{R}^n_+)$.

Consider the continuous function $T(x) = x - \Phi^\varepsilon(x)$, defined for any $x \in \mathbb{R}^n$. Obviously, $x(\varepsilon)$ is a solution of equation (5.7.2), if and only if, $x(\varepsilon)$ is a fixed point for T. For any $r > 0$ denote by $B_r = \{x \in \mathbb{R}^n : \|x\| < r\}$ and $S_r = \{x \in \mathbb{R}^n : \|x\| = r\} = \partial B_r$. If the problem $\varepsilon - MCP(f, \mathbb{R}^n_+)$ has a solution we have nothing to prove.

Suppose that this problem has no solution. In this case, T is fixed-point free with respect to any set B_r. Applying Theorem 3.2.4 [Leray–Schauder alternative] with $\Omega = \mathbb{R}^n$ and $B_r = U$, we obtain that for any $r > 0$ there exist $x^r \in \partial B_r = S_r$ and $\lambda_r \in \,]0, 1[$ such that

$$x^r = \lambda_r \left[x^r - \Phi^\varepsilon(x^r) \right],$$

which implies

$$\begin{cases} x_i^r + \lambda_r \varphi_i(x^r) = \lambda_r \sqrt{(x_i^r)^2 + \varphi_i(x^r) + 2\varepsilon}, \\ i = 1, 2, \dots, n. \end{cases} \tag{5.7.3}$$

From (5.7.3) we deduce

$$\begin{cases} x_i^r \varphi_i\left(x^r\right) = \dfrac{1}{2}\left(\lambda_r - \dfrac{1}{\lambda_r}\right)\left(x_i^r\right)^2 + \lambda_r \varepsilon, \\ i = 1,2,\ldots,n. \end{cases} \tag{5.7.4}$$

Because $\lambda_r \in\ ]0,\ 1[$, formula (5.7.4) implies that $x_i^r \neq 0$ for all $i = 1,2,\ldots,n$ and hence we have

$$\begin{cases} \varphi_i\left(x^r\right) = \dfrac{1}{2}\left(\lambda_r - \dfrac{1}{\lambda_r}\right)x_i^r + \dfrac{\lambda_r \varepsilon}{x_i^r}, \\ i = 1,2,\ldots,n. \end{cases} \tag{5.7.5}$$

Considering (5.7.3) we have

$$\begin{cases} x_i^r + \lambda_r \varphi_i\left(x^r\right) = \lambda_r \sqrt{2\varepsilon}, \\ i = 1,2,\ldots,n. \end{cases} \tag{5.7.6}$$

Multiplying (5.7.5) by λ_r and adding for every $i = 1,2,\ldots,n$, x_i^r we obtain

$$\begin{cases} x_i^r + \lambda_r \varphi_i\left(x^r\right) = \dfrac{1}{2}\left(\lambda_r^2 + 1\right)x_i^r + \dfrac{\lambda_r^2 \varepsilon}{x_i^r}, \\ i = 1,2,\ldots,n. \end{cases} \tag{5.7.7}$$

Considering (5.7.6) we have that the right-hand side of (5.7.7) is strictly positive, which implies that $x_i^r > 0$ for every $i = 1,2,\ldots,n$. If for every $r > 0$ we denote $y_i^r = \varphi_i\left(x_r\right)$, we obtain that $y^r = \left(y_i^r\right) \in f\left(x^r\right)$ and $\left\{x^r\right\}_{r>0}$ is an interior-point-ε-exceptional family of elements for f, and the proof is complete. □

COROLLARY 5.7.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a lower semicontinuous set-valued mapping with non-empty closed convex values. If f is without an interior-point-ε-exceptional family of elements, with respect to $\mathbb{R}_+^n$, then the problem $\varepsilon - MCP\left(f, \mathbb{R}_+^n\right)$ has a solution.*

DEFINITION 5.7.3. *We say that a continuous mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the Browder–Hartman–Stampacchia condition (shortly denoted by (BHS)) on a closed convex cone $\mathbb{K} \subset \mathbb{R}^n$ if there exists $\rho > 0$ such that $\left\langle x, f\left(x\right)\right\rangle > 0$ for any $x \in \mathbb{K}$ with $\|x\| = \rho$.*

PROPOSITION 5.7.4. *If $f: \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping which satisfies condition (BHS) on $\mathbb{R}^n_+$, then the problem NCP(f, $\mathbb{R}^n_+$) has a solution.*

Proof. This proposition is a consequence of Corollary 5.1.17. □

DEFINITION 5.7.4. *We say that a continuous mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ satisfies the asymptotic Browder–Hartman–Stampacchia condition (shortly denoted by (ABHS)) on a close convex $\mathbb{K} \subset \mathbb{R}^n$, if* $\liminf\limits_{\substack{\|x\|\to+\infty \\ x\in\mathbb{K}}} \langle x, f(x)\rangle = +\infty$.

We have the following result.

THEOREM 5.7.5. *Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping. If* $\liminf\limits_{\substack{\|x\|\to+\infty \\ x\in\mathbb{R}^n_{++}}} \langle x, f(x)\rangle = +\infty$, *then the problem NCP(f, $\mathbb{R}^n_+$) has a solution,*

(1) *$\mathcal{U}(\varepsilon) \neq \phi$, for any $\varepsilon > 0$,*

(2) *for any fixed $\varepsilon_0 > 0$ the set* $\bigcup\limits_{\varepsilon\in]0,\varepsilon]} \mathcal{U}(\varepsilon)$ *is bounded.*

Proof.

(1) We can show that $\liminf\limits_{\substack{\|x\|\to+\infty \\ x\in\mathbb{R}^n_{++}}} \langle x, f(x)\rangle = +\infty$ if and only if $\liminf\limits_{\substack{\|x\|\to+\infty \\ x\in\mathbb{R}^n_{+}}} \langle x, f(x)\rangle = +\infty$. The last formula implies that f satisfies condition (*BHS*) and applying Proposition 5.7.4 we obtain that the problem *NCP(f,* $\mathbb{R}^n_+$) has a solution.

(2) By using Theorem 5.7.*1* it is sufficient to show that for any $\varepsilon > 0$, f does not have an interior-ε-exceptional family $\{x^r\}_{r>0} \subset \mathbb{R}^n_{++}$. Indeed, we suppose that f has an interior-point-ε-exceptional family $\{x^r\}_{r>0} \subset \mathbb{R}^n_{++}$. Multiplying the formula given in Definition 5.7.1 by x_l^r and summing with l from 1 to n we obtain

$$\langle x^r, f(x^r)\rangle = \frac{1}{2}\left(\lambda_r - \frac{1}{\lambda_r}\right)\|x^r\| + n\varepsilon\lambda_r,$$

where $0 < \lambda_r < 1$, for any $r > 0$. From the last equality we deduce

$$\left\langle x^r, f\left(x^r\right)\right\rangle + \frac{1}{2}\left(\frac{1}{\lambda_r} - \lambda_r\right)\left\|x^r\right\| < n\varepsilon .$$

Let $r_0 > 0$ such that $\left\|x^{r_0}\right\| > 0$. Because $\left\|x^r\right\| \to +\infty$ *as* $r \to +\infty$, we can consider a subsequence $\left\{x^{r_i}\right\}$ such that $\left\|x^{r_0}\right\| < \left\|x^{r_i}\right\|$ and $\left\|x^{r_i}\right\| \to +\infty$ *as* $i \to +\infty$. For this sequence we have

$$\frac{1}{2}\left(\frac{1}{\lambda_{r_i}} - \lambda_{r_i}\right)\left\|x^{r_0}\right\|^2 + \left\langle x^{r_i}, f\left(x^{r_i}\right)\right\rangle < n\varepsilon .$$

Computing lim inf and using the assumption of our theorem, we obtain a contradiction. Therefore by Theorem 5.7.1 we have that $\mathcal{U}(\varepsilon) \neq \phi$, for any $\varepsilon > 0$.

(3) We observe that for any $x(\varepsilon) \in \mathcal{U}(\varepsilon)$ we have $\left\langle x(\varepsilon), f\left(x(\varepsilon)\right)\right\rangle = n\varepsilon$. Now, we suppose that there is an $\varepsilon > 0$ such that $\bigcup_{\varepsilon\in]0,\varepsilon_0]} \mathcal{U}(\varepsilon)$ is not bounded. Hence, by the assumption of our theorem we have

$$\liminf_{k\to\infty}\left\langle x(\varepsilon_k), f\left(x(\varepsilon_k)\right)\right\rangle = \infty .$$

On the other hand

$$\left\langle x(\varepsilon_k), f\left(x(\varepsilon_k)\right)\right\rangle = n\varepsilon_k \leq n\varepsilon_0 ,$$

which implies

$$\liminf_{k\to\infty}\left\langle x(\varepsilon_k), f\left(x(\varepsilon_k)\right)\right\rangle \leq n\varepsilon_0$$

and we have a contradiction. Therefore $\bigcup_{\varepsilon\in]0,\varepsilon_0]} \mathcal{U}(\varepsilon)$ is bounded for any $\varepsilon_0 > 0$. □

By using Theorem 5.7.5 we can prove also the following result.

THEOREM 5.7.6. *Let* $f: \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous mapping. If*

$$\liminf_{\substack{\|x\|\to+\infty \\ x\in\mathbb{R}^n_{++}}} \frac{\left\langle x, f(x)\right\rangle}{\|x\|^2} > 0 ,$$

then $\mathcal{U}$ *satisfies properties* (a) *and* (b).

Proof. We can show that

$$\liminf_{\substack{\|x\| \to +\infty \\ x \in \mathbb{R}^n_{++}}} \langle x, f(x) \rangle = +\infty$$

and apply Theorem 5.7.5. □

The reader can see other results about the set-valued mapping $\mathcal{U}$ in (Zhao, Y. B. and Isac, G. [2]) and in (Isac, G. and Nemeth, S. Z. [5]). We cite only the following interesting result, due to Y. B. Zhao and G. Isac.

THEOREM 5.7.7. *Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a $P(\tau, \alpha, \beta)$-mapping. If the problem NCP(f, $\mathbb{R}^n_+$) is strictly feasible, the $\mathcal{U}$ satisfies properties* (a) *and* (b).

6

INFINITESIMAL EXCEPTIONAL FAMILY OF ELEMENTS

In this chapter we will introduce and we will use the notion of *infinitesimal exceptional family of elements* for a mapping. This notion is due to S. Z. Németh and it has been used in some recent papers. By this notion we establish an interesting relation between the notion of *exceptional family of elements* and the notion of *scalar derivative*, due also to S. Z. Németh. We note that by this relation we give some applications of the notion of *scalar derivative* to the study of complementarity problems.

6.1 Scalar derivatives

Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $C \subseteq H$ a non-empty set which contains at least one non-isolated point and $f, g\colon C \to H$ two mappings. Let x_0 a non-isolated point of C.

DEFINITION 6.1.1. *We say that the limit*

$$\underline{f}^{\#}(x_0) = \liminf_{x \to x_0, x \in C} \frac{\langle f(x) - f(x_0), x - x_0 \rangle}{\|x - x_0\|^2}$$

is the lower scalar derivative of f at x_0. Taking "lim sup" *in place of* "lim inf", *we obtain the upper scalar derivative* $\overline{f}^{\#}(x_0)$ *of f at x_0 similarly.*

Definition 6.1.1 can be extended for the unordered pair of mappings (f, g). In this sense we have the following notion.

DEFINITION 6.1.2. *We say that the limit*

$$\underline{(f,g)}^{\#}(x_0)=\liminf_{x\to x_0,x\in C}\frac{\langle f(x)-f(x_0),g(x)-g(x_0)\rangle}{\|x-x_0\|^2}$$

is the lower scalar derivative of the unordered pair of mappings (f, g) *at* x_0. *Taking* "lim sup" *in place of* "lim inf" *we obtain the upper scalar derivative* $\overline{(f,g)}^{\#}(x_0)$ *of* (f, g) *at* x_0 *similarly.*

The notion of scalar derivative can be also extended for set-valued mappings. Indeed, we suppose that $f, g : C \to H$ are set-valued mappings and x_0 again a non-isolated point of C.

DEFINITION 6.1.3. *We say that the limit*

$$\underline{f}^{\#}(x_0)=\liminf_{\substack{x\to x_0,x\in C\\ x^f\in f(x),x_0^f\in f(x_0)}}\frac{\langle x^f-x_0^f,x-x_0\rangle}{\|x-x_0\|^2}$$

is called the lower scalar derivative of the unordered pair of set-valued mappings (f, g) *at* x_0. *Taking* "lim sup" *in place of* "lim inf", *we can define the upper scalar derivative* $\overline{(f,g)}^{\#}(x_0)$ *of* (f, g) *at* x_0 *similarly.*

The notion of *scalar derivative* was introduced and studied by S. Z. Németh, and the reader is referred to (Németh, S. Z., [1], [2]). Several methods for computation are given in (Németh, S. Z., [3]). Applications of scalar derivatives to the study of complementarity problems, to study of fixed points and to the study of eigenvalues of nonlinear mappings are given in (Isac, G. and Németh, S. Z., [1], [4])

6.2 Infinitesimal exceptional family of elements

By Definition 5.1.6, we introduced the notion of *exceptional family of element* (*EFE*) for a mapping f, by Definition 5.2.1 the notion of *EFE* for a pair of mappings (f, g) and by Definition 5.3.2 the notion of *EFE* for a set-valued mapping and by using these notions we obtained several existence theorems for nonlinear complementarity problems, for implicit complementarity problems and for multivalued complementarity problems. Now, in this section we will introduce for each of them, a kind of *infinitesimal* *exceptional family of elements* (*EFE*). By infinitesimal forms we will establish a relation between the notion of *EFE* and the scalar derivatives.

Let $(H, \langle \cdot,\cdot \rangle)$ be a Hilbert space and $\|\cdot\|$ the norm defined by the inner-product $\langle \cdot,\cdot \rangle$.

DEFINITION 6.2.1. *The operator* $i : H \setminus \{0\} \to H \setminus \{0\}$ *defined by* $i(x) = \dfrac{x}{\|x\|^2}$ *is called inversion* (*of pole* 0).

Obviously, i is one-to-one and $i^{-1} = i$. Let $\mathbb{K} \subset H$ be a closed convex cone and $f : \mathbb{K} \to H$. Since $\mathbb{K} \setminus \{0\}$ is an invariant set of i the following definition makes sense.

DEFINITION 6.2.2. *The inversion* (*of pole* 0) *of the mapping* f *is the mapping* $\mathcal{I}(f) : \mathbb{K} \to H$ *defined by*:

$$\mathcal{I}(f)(x) = \begin{cases} \|x\|^2 (f \circ i)(x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

It is easy to see that the inversion operator $\mathcal{I}$ is a one-to-one operator on the set of mappings $\{f : f : \mathbb{K} \to H; f(0) = 0\}$ and $\mathcal{I}^{-1} = \mathcal{I}$, i.e., $\mathcal{I}(\mathcal{I}(f)) = f$.

DEFINITION 6.2.3. *Let* $(H, \langle \cdot,\cdot \rangle)$ *be a Hilbert space* $\mathbb{K} \subset H$ *a closed convex cone and* $g : \mathbb{K} \to H$ *a mapping. We say that* $\{y_r\}_{r>0} \subset \mathbb{K}$ *is an infinitesimal exceptional family of elements* (*IEFE*) *for* g *with respect to* $\mathbb{K}$, *if for every real number* $r > 0$, *there exists a real number* $\mu_r > 0$ *such that the vector* $v_r = \mu_r y_r + g(y_r)$ *satisfies the following conditions*:

(1) $v_r \in \mathbb{K}$,

(2) $\langle v_r, y_r \rangle = 0$,

(3) $y_r \to 0$ *as* $r \to +\infty$.

The following condition is similar to *condition* $(\tilde{\theta})$ (see Definition 5.1.2).

DEFINITION 6.2.4. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K}\subset H$ *a closed convex cone and* $g: H\to H$ *a mapping. We say that the mapping g satisfies condition* $({}^{i}\tilde{\theta})$ *with respect to* $\mathbb{K}$ *if there exists* $\lambda > 0$ *such that for each* $y\in\mathbb{K}\setminus\{0\}$ *with* $\|y\| < \lambda$, *there exists* $q\in\mathbb{K}$ *with* $\langle q,y\rangle<\|y\|^2$ *such that* $\langle y-q,g(y)\rangle\geq 0$.

We have the following result.

THEOREM 6.2.1. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K}\subset H$ *a closed convex cone and* $g: H\to H$ *a mapping. If g satisfies condition* $({}^{i}\tilde{\theta})$ *with respect to* $\mathbb{K}$*, then g is without an IEFE with respect to* $\mathbb{K}$.

Proof. We suppose the contrary, that is, we suppose that g has an *IEFE* $\{y_r\}_{r>0}\subset\mathbb{K}$, with respect to $\mathbb{K}$. For any $r> 0$ such that $\|y\| < \rho$ there is an element $q_r\in\mathbb{K}$ with $\langle q_r,y_r\rangle<\|y_r\|$ satisfying the relation

$$\langle y_r-q_r,g(y_r)\rangle\geq 0.$$

Since, according to Definition 6.2.3, $\langle v_r,y_r\rangle=0$ and $v_r\in\mathbb{K}^*$, we have

$$0\leq\langle y_r-q_r,g(y_r)\rangle=\langle y_r-q_r,v_r-\mu_r y_r\rangle$$
$$=-\mu_r\|y_r^2\|-\langle q_r,v_r\rangle+\mu_r\langle q_r,y_r\rangle\leq-\mu_r(\|y_r^2\|-\langle q_r,y_r\rangle)<0$$

which is a contradiction. □

THEOREM 6.2.2. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K}\subset H$ *a closed convex cone and* $f:\mathbb{K}\to H$ *a mapping. A family of elements* $\{x_r\}_{r>0}\subset\mathbb{K}\setminus\{0\}$ *is an EFE for f, with respect to* $\mathbb{K}$ *if and only if* $\{y_r\}_{r>0}\subset\mathbb{K}\setminus\{0\}$ *is an IEFE for g, with respect to K, where* $y_r = i(x_r)$ *and* $g=\mathcal{I}(f)$.

Proof. Bearing in mind the notation of Definition 6.2.3 we have

$$v_r = \mu_r y_r + \|y_r\|^2 f\left(i\left(y_r\right)\right).$$

Hence, $v_r = \|y_r\|^2\left(\mu_r i\left(y_r\right)\right) + f\left(i\left(y_r\right)\right)$. Since $i^{-1} = i$, we have

$$v_r = \frac{1}{\|x_r\|^2}\left(\mu_r y_r + f\left(x_r\right)\right).$$

Hence, $v_r = \frac{1}{\|x_r\|^2} u_r$. Therefore

$$\langle v_r, y_r\rangle = \frac{1}{\|x_r\|^4}\langle u_r, x_r\rangle, \tag{6.2.1}$$

and

$$\langle v_r, z\rangle = \frac{1}{\|x_r\|^2}\langle u_r, z\rangle, \tag{6.2.2}$$

for every $z \in \mathbb{K}$. Since $\|x_r\|\cdot\|y_r\| = 1$, $\|x_r\| \to +\infty$ if and only if $y_r \to 0$. By using (6.2.1), $\langle u_r, x_r\rangle = 0$ if and only if $\langle v_r, y_r\rangle = 0$. By using (6.2.2), $u_r \in \mathbb{K}^*$ if and only if $v_r \in \mathbb{K}^*$. □

THEOREM 6.2.3. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed convex cone* $f: H \to H$ *a mapping and* $g = \mathcal{I}(f)$. *Then f satisfies condition* $(\tilde{\theta})$ *with respect to* $\mathbb{K}$, *if and only if g satisfies condition* $({}^i\tilde{\theta})$ *with respect to* $\mathbb{K}$.

Proof. We suppose that g satisfies condition $({}^i\tilde{\theta})$ with respect to $\mathbb{K}$ and we prove that f satisfies condition $(\tilde{\theta})$ with respect to $\mathbb{K}$. We consider the constant λ defined in condition $({}^i\tilde{\theta})$ and let $\rho = \frac{1}{\lambda}$. Let $x \in \mathbb{K}$ be an arbitrary element satisfying the inequality

$$\|x\| > \rho, \tag{6.2.3}$$

and $y = i(x)$. Since $\|y\| = \frac{1}{\|x\|}$, it follows that $\|y\| < \lambda$. Hence by condition $({}^i\tilde{\theta})$, there exists $\rho \in \mathbb{K}$ with $\langle q, y\rangle < \|y^2\|$ such that $\langle y - q, g(y)\rangle \geq 0$. Let

$$p = \frac{q}{\|y\|^2}. \tag{6.2.4}$$

Since $\langle q, y\rangle < \|y\|^2$ and $i^{-1} = i$, relation (6.2.4) implies that

$$\langle p, x\rangle = \frac{\langle q, y\rangle}{\|y^4\|} < \frac{1}{\|y\|^2} = \|x\|^2. \tag{6.2.5}$$

On the other hand $\mathcal{I}^1 = \mathcal{I}$ implies that

$$\begin{aligned}\langle x - p, f(x)\rangle &= \langle x - p, \mathcal{I}(g)(x)\rangle \\ &= \left\langle x - p, \|x\|^2 g(i(x))\right\rangle = \|x\|^4 \langle y - q, g(y)\rangle \geq 0.\end{aligned} \tag{6.2.6}$$

By (6.2.3), (6.2.5) and (6.2.6) f satisfies condition $(\tilde{\theta})$ with respect to $\mathbb{K}$. Now, we suppose that f satisfies condition $(\tilde{\theta})$ with respect to $\mathbb{K}$ and we prove that g satisfies condition $({}^{\prime}\tilde{\theta})$ with respect to $\mathbb{K}$. Indeed we consider the constant $\rho > 0$ defined in condition $(\tilde{\theta})$ and let $\lambda = \frac{1}{\rho}$. Let $y \in \mathbb{K} \setminus \{0\}$ with $\|y\| < \lambda$. We have to prove that there exists $q \in \mathbb{K}$ with $\langle q, y\rangle < \|y\|^2$ such that $\langle g - q, g(y)\rangle \geq 0$. Since $f = \mathcal{I}(g)$, we can proceed as above. □

DEFINITION 6.2.5. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $\tilde{f}, \tilde{g} : H \to H$ two mappings. We say that a family of elements $\{\tilde{x}_r\}_{r>0}$ is an infinitesimal exceptional family of elements (IEFE) for the ordered pair of mappings $(\tilde{f}, \tilde{g})$ with respect to $\mathbb{K}$ if the following conditions are satisfied:*

(1) $\tilde{x}_r \to 0$ *as* $r \to +\infty$,

(2) *for any $r > 0$, there exists $\mu_r > 0$ such that $\tilde{s}_r = \mu_r \tilde{x}_r + \tilde{f}(\tilde{x}_r) \in \mathbb{K}^*$, $\tilde{v}_r = \mu_r \tilde{x}_r + \tilde{g}(x_r) \in \mathbb{K}$ and $\langle \tilde{v}_r, \tilde{s}_r\rangle = 0$.*

We have the following result.

THEOREM 6.2.4. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone, $f, g : \mathbb{K} \to H$ two mappings. A family of elements*

$\{x_r\}_{r>0} \subset \mathbb{K} \setminus \{0\}$ is an EFE (in the sense of Definition 5.2.1) for the ordered pair of mappings (f, g) with respect to $\mathbb{K}$, if and only if $\{\tilde{x}_r\}_{r>0} \subset H \setminus \{0\}$ is an IEFE for the ordered pair of mappings $(\tilde{f}, \tilde{g})$ with respect to $\mathbb{K}$, where $\tilde{x}_r = i(x_r)$, $\tilde{f} = \mathcal{I}(f)$ and $\tilde{g} = \mathcal{I}(g)$.

Proof. Considering the notions of Definition 6.2.5, we have

$$\tilde{s}_r = \mu_r \tilde{x}_r + \|\tilde{x}_r\|^2 f(i(\tilde{x}_r)) \text{ and } \tilde{v}_r = \mu_r \tilde{x}_r + \|\tilde{x}_r\|^2 g(i(\tilde{x}_r)).$$

Hence,

$$\tilde{s}_r = \|\tilde{x}_r\|^2 \left[\mu_r i(\tilde{x}_r) + f(i(\tilde{x}_r))\right] \text{ and } \tilde{v}_r = \|\tilde{x}_r\|^2 \left[\mu_r i(\tilde{x}_r) + g(i(\tilde{x}_r))\right].$$

Since $\|\tilde{x}_r\| \cdot \|x_r\| = 1$ *and* $i^{-1} = i$, we have

$$\tilde{s}_r = \frac{1}{\|x_r\|^2}\left[\mu_r x_r + f(x_r)\right] \text{ and } \tilde{v}_r = \frac{1}{\|x_r\|^2}\left[\mu_r x_r + g(x_r)\right].$$

Hence

$$\tilde{s}_r = \frac{1}{\|x_r\|^2} s_r \quad \text{and} \quad \tilde{v}_r = \frac{1}{\|x_r\|^2} v_r.$$

Therefore,

$$\langle \tilde{v}_r, \tilde{s}_r \rangle = \frac{1}{\|x_r\|^4} \langle v_r, s_r \rangle, \tag{6.2.7}$$

and

$$\langle \tilde{s}_r, y \rangle = \frac{1}{\|x_r\|^4} \langle s_r, y \rangle, \text{ for every } y \in \mathbb{K}. \tag{6.2.8}$$

By using (6.2.7), $\langle v_r, s_r \rangle = 0$ if and only if $\langle \tilde{v}_r, \tilde{s}_r \rangle = 0$.

By using (6.2.8) we have that $s_r \in \mathbb{K}^*$ if and only if $\tilde{s}_r \in \mathbb{K}^*$. By the relation between v_r and $\tilde{v}_r$ given above we obtain that $v_r \in \mathbb{K}$ if and only if $\tilde{v}_r \in \mathbb{K}$. □

DEFINITION 6.2.6. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone, $\tilde{f}, \tilde{g} : H \to H$ two mappings. We say that the mapping $\tilde{f}$ satisfies condition $({}^{i}\theta_{\tilde{g}})$ with respect to $\mathbb{K}$, if there exists $\tilde{\rho} > 0$ such that for each $\tilde{x} \in \mathbb{K} \setminus \{0\}$ with $\|\tilde{x}\| < \tilde{\rho}$ there exists $\tilde{y} \in \mathbb{K}$ such that*

$$\begin{cases} \langle \tilde{g}(\tilde{x}) - \tilde{y}, \tilde{f}(\tilde{x}) \rangle \geq 0 \text{ and} \\ \langle \tilde{g}(\tilde{x}) - \tilde{y}, \tilde{x} \rangle > 0. \end{cases} \tag{6.2.9}$$

The importance of Definition 6.2.6 is supported by the following result.

THEOREM 6.2.5. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone, $\tilde{f}, \tilde{g} : H \to H$ two mappings. If $\tilde{f}$ satisfies condition $({}^{i}\theta_{\tilde{g}})$ with respect to $\mathbb{K}$, then the pair of mappings $(\tilde{f}, \tilde{g})$ is without an IEFE with respect to $\mathbb{K}$.*

Proof. We suppose to the contrary, that $(\tilde{f}, \tilde{g})$ has an infinitesimal family of elements $\{\tilde{x}_r\}_{r>0} \subset \mathbb{K}$. For any $r > 0$ such that $\|\tilde{x}_r\| < \rho$ there is an element $\tilde{y}_r \in \mathbb{K}$ which satisfies relations (6.2.9) i.e.,

$$\langle \tilde{g}(\tilde{x}_r) - \tilde{y}_r, \tilde{f}(\tilde{x}_r) \rangle \geq 0 \text{ and } \langle \tilde{g}(\tilde{x}_r) - \tilde{y}_r, \tilde{x}_r \rangle > 0.$$

Considering Definition 6.2.5 we have

$$\tilde{s}_r = \mu_r \tilde{x}_r + \tilde{f}(\tilde{x}_r) \in \mathbb{K}^*, \tilde{v}_r = \mu_r \tilde{x}_r + \tilde{g}(\tilde{x}_r) \in \mathbb{K} \text{ and } \langle \tilde{v}_r, \tilde{s}_r \rangle = 0,$$

and we deduce that

$$\begin{aligned}
0 &\leq \langle \tilde{g}(\tilde{x}_r) - \tilde{y}_r, \tilde{f}(\tilde{x}_r) \rangle = \langle \tilde{v}_r - \mu_r \tilde{x}_r - \tilde{y}_r, \tilde{s}_r - \mu_r \tilde{x}_r \rangle \\
&= \langle \tilde{v}_r, \tilde{s}_r \rangle - \langle \mu_r \tilde{x}_r, \tilde{s}_r \rangle - \langle \tilde{y}_r, \tilde{s}_r \rangle - \langle \tilde{v}_r, \mu_r \tilde{x}_r \rangle + \mu_r \|\tilde{x}_r\|^2 + \langle \tilde{y}_r, \mu_r \tilde{x}_r \rangle \\
&\leq -\langle \tilde{v}_r, \mu_r \tilde{x}_r \rangle + \mu_r^2 \|\tilde{x}_r\|^2 + \langle \tilde{y}_r, \mu_r \tilde{x}_r \rangle \\
&= -\langle \mu_r \tilde{x}_r + \tilde{g}(\tilde{x}_r), \mu_r \tilde{x}_r \rangle + \mu_r{}^2 \|\tilde{x}_r\|^2 + \langle \tilde{y}_r, \mu_r \tilde{x}_r \rangle \\
&= -\mu^2{}_r \|\tilde{x}_r\|^2 - \langle \tilde{g}_r(\tilde{x}_r), \mu_r \tilde{x}_r \rangle + \mu_r{}^2 \|\tilde{x}_r\|^2 + \langle \tilde{y}_r, \mu_r \tilde{x}_r \rangle \\
&= -\langle \tilde{g}(\tilde{x}_r), \mu_r \tilde{x}_r \rangle + \langle \tilde{y}_r, \mu_r \tilde{x}_r \rangle \\
&= -\mu_r \langle \tilde{g}(\tilde{x}_r) - \tilde{y}_r, \tilde{x}_r \rangle < 0,
\end{aligned}$$

which is a contradiction. Hence, the pair $(\tilde{f}, \tilde{g})$ is without *IEFE* with respect to $\mathbb{K}$. □

THEOREM 6.2.6. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K}\subset H$ a closed pointed convex cone, $f,g:H\to H$ two mappings, $\tilde{f}=\mathcal{I}(f)$ and $\tilde{g}=\mathcal{I}(g)$. Then f satisfies condition (θ_g) (see Definition 5.2.2) with respect to $\mathbb{K}$ if and only if $\tilde{f}$ satisfies condition $({}^{i}\theta_{\tilde{g}})$ with respect to $\mathbb{K}$.*

Proof. We suppose that $\tilde{f}$ satisfies condition $({}^{i}\theta_{\tilde{g}})$ with respect to $\mathbb{K}$.

Let $\tilde{\rho}$ be the constant defined by condition $({}^{i}\theta_{\tilde{g}})$ and let $\rho=\dfrac{1}{\tilde{\rho}}$.

Let $x\in\mathbb{K}$ be an element such that

$$\|x\|>\rho, \tag{6.2.10}$$

and let $\tilde{x}=i(x)$. Since $\|\tilde{x}\|=\dfrac{1}{\|x\|}$ it follows that $\|\tilde{x}\|<\tilde{\rho}$. Hence, by condition $({}^{i}\theta_{\tilde{g}})$ there exists $\tilde{y}\in\mathbb{K}$ such that

$$\begin{cases}\langle\tilde{g}(\tilde{x})-\tilde{y},\tilde{f}(\tilde{x})\rangle\geq 0 \textit{ and}\\ \langle\tilde{g}(\tilde{x})-\tilde{y},\tilde{x}\rangle>0.\end{cases}$$

Let

$$y=\frac{\tilde{y}}{\|\tilde{x}\|^2}. \tag{6.2.11}$$

Since $\langle\tilde{g}(\tilde{x})-\tilde{y},\tilde{x}\rangle>0, i^{-1}=i$ and considering (6.2.11) we obtain

$$\langle g(x)-y,x\rangle=\left\langle g(i(\tilde{x}))-\frac{\tilde{y}}{\|\tilde{x}\|^2},\frac{\tilde{x}}{\|\tilde{x}\|^2}\right\rangle=\frac{1}{\|\tilde{x}\|^4}\langle\tilde{g}(\tilde{x})-\tilde{y},\tilde{x}\rangle>0. \tag{6.2.12}$$

On the other hand $\langle\tilde{g}(\tilde{x})-\tilde{y},\tilde{f}(\tilde{x})\rangle\geq 0$, $i^{-1}=i$ and considering again (6.2.11) we deduce that

$$\begin{aligned}\langle g(x)-y,f(x)\rangle&=\left\langle g(i(\tilde{x}))-\frac{\tilde{y}}{\|\tilde{x}\|^2},f(i(\tilde{x}))\right\rangle\\&=\frac{1}{\|\tilde{x}\|^4}\langle\tilde{g}(\tilde{x})-\tilde{y},\tilde{f}(\tilde{x})\rangle\geq 0.\end{aligned} \tag{6.2.13}$$

By (6.2.10), (6.2.12) and (6.2.13) we have that f satisfies condition (θ_g) with respect to $\mathbb{K}$. Since $\mathcal{I}^{-1} = \mathcal{I}$, the converse can be proved similarly. □

Now, we introduce the notion of *infinitesimal exceptional family of elements* as a mathematical tool in the study of *multivalued complementarity problems*. In this way we establish also a relation between the *scalar derivative* and the solvability of multivalued complementarity problems.

DEFINITION 6.2.7. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone, $g : \mathbb{K} \to H$ a set-valued mapping with non-empty values. We say that a family of elements $\{y_r\}_{r>0} \subset \mathbb{K}$ is an infinitesimal exceptional family of elements (IEFE) for g with respect to $\mathbb{K}$, if for every real number $r > 0$, there exist a real number $\mu_r > 0$ and an element $y_r^g \in g(y_r)$ such that the following conditions are satisfied:*

(1) $v_r = \mu_r y_r + y_r^g \in \mathbb{K}^*$,

(2) $\langle v_r, y_r \rangle = 0$,

(3) $y_r \to 0$ *as* $r \to +\infty$.

DEFINITION 6.2.8. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone. We say that a set-valued mapping $g : H \to H$ with non-empty values satisfies condition $\left[{}^i\tilde{\theta}\right]_m$ with respect to $\mathbb{K}$ if there exists a real number $\lambda > 0$ such that for each $y \in \mathbb{K} \setminus \{0\}$ with $\|y\| < \lambda$ there exists $q \in \mathbb{K}$ with $\langle q, y\rangle < \|y\|^2$ such that $\langle y - q, y^g \rangle \geq 0$ for all $y^g \in g(y)$.*

The importance of condition $\left[{}^i\tilde{\theta}\right]_m$ is given by the following result.

THEOREM 6.2.7. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $g : H \to H$ a set-valued mapping with non-empty values. If g satisfies condition $\left[{}^i\tilde{\theta}\right]_m$ with respect to $\mathbb{K}$, then it is without an IEFE with respect to $\mathbb{K}$.*

Proof. Indeed, we suppose the contrary, i.e., we suppose that g has an *IEFE* $\{y_r\}_{r>0} \subset \mathbb{K}$ with respect to $\mathbb{K}$. For any $r > 0$ such that, $\|y_r\| < \lambda$, there exists an element $q_r \in \mathbb{K}$ with $\langle q_r, y_r \rangle < \|y_r\|^2$ satisfying the relation $\langle y_r - q_r, y_r^g \rangle \geq 0$ for an arbitrary $y_r^g \in g(y_r)$. Since, according to Definition 6.2.7, $\langle v_r, y_r \rangle = 0$ and $v_r \in \mathbb{K}^*$, we have

$$\begin{aligned} 0 &\leq \langle y_r - q_r, y_r^g \rangle = \langle y_r - q_r, v_r - \mu_r y_r \rangle \\ &= -\mu_r \|y_r\|^2 - \langle q_r, v_r \rangle + \mu_r \langle q_r, y_r \rangle \\ &\leq -\mu_r \left[\|y_r\|^2 - \langle q_r, y_r \rangle \right] < 0, \end{aligned}$$

which is a contradiction. □

THEOREM 6.2.8. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed pointed convex cone and* $f : H \to H$ *a set-valued mapping with non-empty values. A family of elements* $\{x_r\}_{r>0} \subset \mathbb{K} \backslash \{0\}$ *is an EFE (in the sense of Definition 5.3.2) with respect to* $\mathbb{K}$*, if and only if* $\{y_r\}_{r>0} \subset \mathbb{K} \backslash \{0\}$ *is an IEFE for g with respect to* $\mathbb{K}$*, where* $y_r = i(x_r)$ *and* $g = \mathcal{I}(f)$.

Proof. Considering Definition 6.2.7 we have

$$v_r = \mu_r y_r + y_r^g, \text{ for some } y_r^g \in g(y_r).$$

Hence,

$$v_r = \|y_r\|^2 \left[\mu_r i(y_r) + \frac{y_r^g}{\|y_r\|^2} \right].$$

Since $i^{-1} = i$, we have

$$v_r = \frac{1}{\|x_r\|}^{2} \left[\mu_r x_r + \|x_r\|^2 y_r^g \right]. \tag{6.2.14}$$

Let

$$x_r^f := \|x_r\|^2 y_r^g. \tag{6.2.15}$$

We have $x_r^f \in f(x_r)$. Indeed,

$$x_r^f \in \|x_r\|^2 g(y_r) = \|x_r\|^2 \mathcal{I}(f)(y_r) = \|x_r\|^2 \|y_r\|^2 f(i(y_r)) = f(x_r).$$

Now, we define

$$u_r = \mu_r x_r + x_r^f. \tag{6.2.16}$$

Equations (6.2.14), (6.2.15) and (6.2.16) imply that

$$v_r = \frac{1}{\|x_r\|^2} u_r .$$

Therefore,

$$\langle v_r, y_r \rangle = \frac{1}{\|x_r\|^4} \langle u_r, x_r \rangle \tag{6.2.17}$$

and

$$\langle v_r, z \rangle = \frac{1}{\|x_r\|^2} \langle u_r, z \rangle, \tag{6.2.18}$$

for every $z \in \mathbb{K}$. Since $\|x_r\| \cdot \|y_r\| = 1$, $\|x_r\| \to +\infty$ *as* $r \to +\infty$ if and only if $y_r \to 0$ *as* $r \to +\infty$. By using (6.2.17), $\langle u_r, x_r \rangle = 0$ if and only if $\langle v_r, y_r \rangle = 0$. By using (6.2.18), $u_r \in \mathbb{K}^*$ if and only if $v \in \mathbb{K}^*$. □

THEOREM 6.2.9. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed pointed convex cone and* $f : H \to H$ *a set-valued mapping with non-empty values and* $g = \mathcal{I}(f)$. *Then, f satisfies condition* $\left[\tilde{\theta}\right]_m$ *(see Definition 5.3.11) with respect to* $\mathbb{K}$ *if and only if g satisfies condition* $\left[{}^{i}\tilde{\theta}\right]_m$ *with respect to* $\mathbb{K}$.

Proof. Since $g = \mathcal{I}(f)$ and $\mathcal{I}(\mathcal{I}(f)) = f$, it follows that

$$f = \mathcal{I}(g). \tag{6.2.19}$$

We suppose that g satisfies condition $\left[{}^{i}\tilde{\theta}\right]_m$ with respect to $\mathbb{K}$ and we prove that f satisfies condition $\left[\tilde{\theta}\right]_m$ with respect to $\mathbb{K}$. Consider the constant λ defined by condition $\left[{}^{i}\tilde{\theta}\right]_m$ and let $\rho = \dfrac{1}{\lambda}$.

Let $x \in \mathbb{K}$ with

$$\|x\| > \rho, \tag{6.2.20}$$

$y = i(x)$ and $x^f \in f(x)$. Let $y^g = \dfrac{x^f}{\|x\|^2}$. We have $y^g \in g(y)$. Indeed, by (6.2.19) we have

$$y^g \in \frac{f(x)}{\|x\|^2} = \frac{\mathcal{I}(g)(x)}{\|x\|^2} = \frac{\|x\|^2 g(i(x))}{\|x\|^2} = g(y).$$

Since $\|y\| = \dfrac{1}{\|x\|}$, it follows that $\|y\| < \lambda$.

Hence, by condition $\left[{}^i\tilde{\theta}\right]_m$, there exists $q \in \mathbb{K}$ with $\langle q, y\rangle < \|y\|^2$ such that

$$\langle y - q, y^g \rangle \geq 0. \tag{6.2.21}$$

Let

$$p = \frac{q}{\|y\|^2}. \tag{6.2.22}$$

Since $\langle q, y\rangle < \|y\|^2$ and $i^{-1} = i$, relation (6.2.22) implies that

$$\langle p, x\rangle = \frac{\langle q, y\rangle}{\|y\|^4} < \frac{1}{\|y\|^2} = \|x\|^2. \tag{6.2.23}$$

By (6.2.21) we also have

$$\langle x - p, x^f \rangle = \|x\|^2 \langle x - p, y^g \rangle = \|x\|^4 \langle y - q, y^g \rangle \geq 0. \tag{6.2.24}$$

By (6.2.20), (6.2.23) and (6.2.24), f satisfies condition $\left[\tilde{\theta}\right]_m$ with respect to $\mathbb{K}$

Now, suppose that f satisfies condition $\left[\tilde{\theta}\right]_m$ with respect to $\mathbb{K}$ and prove that g satisfies condition $\left[{}^i\tilde{\theta}\right]_m$ with respect to $\mathbb{K}$. Consider the constant ρ defined by condition $\left[\tilde{\theta}\right]_m$ and let $\lambda = \dfrac{1}{\rho}$. Let $y \in \mathbb{K} \setminus \{0\}$ with $\|y\| < \lambda$. We have to prove that there exists $q \in \mathbb{K}$ with $\langle q, y\rangle < \|y\|^2$ such that $\langle y - q, y^g \rangle \geq 0$, for all $y^g \in g(y)$. Since $f = \mathcal{I}(g)$, we can proceed as above and the proof is complete. □

6.3 Applications to complementarity theory

We present in this section some applications to complementarity problems. The results are based on the notions of *infinitesimal exceptional family of elements and scalar derivative.*

THEOREM 6.3.1. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K}\subset H$ *a closed convex cone and* $f:\mathbb{K}\to H$ *a mapping. Then* $x_*\neq 0$ *is a solution to the problem NCP(f,* $\mathbb{K}$*) if and only if* y_* *is a solution to the problem NCP(g,* $\mathbb{K}$*), where* $y_*=i(x_*)$ *is the inversion of* x_* *and g =* $\mathcal{I}(f)$ *is the inversion of f.*

Proof.

$$\langle y_*,\mathcal{I}(f)(y_*)\rangle=\langle y_*,\|y_*\|^2 f(i(y_*))\rangle.$$

Hence

$$\langle y_*,\mathcal{I}(f)(y_*)\rangle=\|y_*\|^4\langle i(y_*),f(i(y_*))\rangle.$$

Since $i^{-1}=i$, we have

$$\langle y_*,g(y_*)\rangle=\frac{1}{\|x_*\|^4}\langle x_*,f(x_*)\rangle. \tag{6.3.1}$$

It can be similarly proved that

$$\langle g(y_*),z\rangle=\frac{1}{\|x_*\|^2}\langle f(x_*),z\rangle, \tag{6.3.2}$$

for every $z\in\mathbb{K}$. By using (6.3.1) we have $\langle x_*,f(x_*)\rangle=0$ if and only if $\langle y_*,g(y_*)\rangle=0$. By using (6.3.2), $f(x_*)\in\mathbb{K}^*$ if and only if $g(y_*)\in\mathbb{K}$.

□

THEOREM 6.3.2. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K}\subset H$ *a closed convex cone and* $f:\mathbb{K}\to H$ *a projectionally Leray–Schauder mapping. If g =* $\mathcal{I}(f)$ *satisfies condition* $({}^{i}\tilde{\theta})$ *with respect to* $\mathbb{K}$*, then the problem NCP(f,* $\mathbb{K}$*) has a solution.*

Proof. By Theorem 6.2.3, f satisfies condition $(\tilde{\theta})$ with respect to $\mathbb{K}$. Hence Theorem 5.1.41 implies that f is without *EFE* with respect to $\mathbb{K}$, and finally by Theorem 5.1.2 we have that the problem *NCP*(f, $\mathbb{K}$) has a solution. □

THEOREM 6.3.3. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K}\subset H$ *a closed convex cone and* $f:\mathbb{K}\to H$ *a projectionally Leray–Schauder mapping. If there exist a* $\delta>0$ *and a mapping* $h:B(0,\delta)\cap\mathbb{K}\to\mathbb{K}$, *with* $h(0)=0$ *and* $\overline{h}^{\#}(0)<1,\underline{(1-h,\mathcal{I}(f))^{\#}}(0)>0$, *where* $B(0,\delta)=\{x\in H:\|x\|<\delta\}$, *then the problem NCP*(f, $\mathbb{K}$) *has a solution.*

Proof. Let $g=\mathcal{I}(f)$. Since $\overline{h}^{\#}(0)<1$, there is a λ_1 with $0<\lambda_1<\delta$ such that for every $y\in\mathbb{K}$ with $\|y\|<\lambda_1$ we have

$$\langle h(y),y\rangle<\|y\|^2. \tag{6.3.3}$$

Since $\underline{(I-h,g)}^{\#}(0)>0$,there is a λ_2 with $0<\lambda_2<\delta$ such that for every $y\in\mathbb{K}$ with $\|y\|<\lambda_2$ we have

$$\langle y-h(y),g(y)\rangle>0. \tag{6,3.4}$$

Let $\lambda=\min\{\lambda_1,\lambda_2\}$. Obviously,

$$\lambda>0 \tag{6.3.5}$$

for

$$\|y\|<\lambda. \tag{6.3.6}$$

Let $q=h(y)$. Then , relations (6.3.3) and (6.3.4) imply

$$\langle q,y\rangle<\|y\|^2 \tag{6.3.7}$$

and

$$\langle y-q,g(y)\rangle\geq 0 \tag{6.3.8}$$

respectively. Hence, relations (6.3.5), (6.3.6), (6.3.7) and (6.3.8) imply that g satisfies condition $({}^{\prime}\tilde{\theta})$. Hence Theorem 6.3.2 implies that the problem *NCP*(f, $\mathbb{K}$) has a solution. □

THEOREM 6.3.4. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K}\subset H$ a closed pointed convex cone and $f,g:\mathbb{K}\to H$ two mappings. Then $x_*\neq 0$ is a solution to the problem ICP(f, g, $\mathbb{K}$) if and only if $\tilde{x}_*$ is a solution to the problem $ICP(\tilde{f},\tilde{g},\mathbb{K})$, where $\tilde{x}_*=i(x_*)$, $\tilde{f}=\mathcal{I}(f)$ and $\tilde{g}=\mathcal{I}(g)$.*

Proof. We have

$$\langle\tilde{g}(\tilde{x}_*),\tilde{f}(\tilde{x}_*)\rangle=\langle\|\tilde{x}_*\|^2 g(i(\tilde{x}_*)),\|\tilde{x}_*\|^2 f(i(\tilde{x}_*))\rangle.$$

Since $\|\tilde{x}_*\|\cdot\|x_*\|=1$ and $i^{-1}=i$, we have

$$\langle\tilde{g}(\tilde{x}_*),\tilde{f}(\tilde{x}_*)\rangle=\frac{1}{\|\tilde{x}_*\|^4}\langle g(x_*),f(x_*)\rangle. \tag{6.3.9}$$

We can prove similarly that

$$\langle\tilde{f}(\tilde{x}_*),z\rangle=\frac{1}{\|\tilde{x}_*\|^2}\langle f(x_*),z\rangle, \tag{6.3.10}$$

for every $z\in\mathbb{K}$. We also have

$$\tilde{g}(\tilde{x}_*)=\frac{1}{\|x_*\|^2}g(x_*). \tag{6.3.11}$$

By using (6.3.9), $\langle g(x_*),f(x_*)\rangle=0$ if and only if $\langle\tilde{g}(\tilde{x}_*),\tilde{f}(\tilde{x}_*)\rangle=0$. By using (6.3.10), $f(x_*)\in\mathbb{K}^*$ if and only if, $\tilde{f}(\tilde{x}_*)\in\mathbb{K}^*$. By using (6.3.11), $g(x_*)\in\mathbb{K}$ if and only if $\tilde{g}(\tilde{x}_*)\in\mathbb{K}$. □

THEOREM 6.3.5. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K}\subset H$ a closed pointed convex cone and $f,g:\mathbb{K}\to H$ completely continuous fields, $\tilde{f}=\mathcal{I}(f)$ and $\tilde{g}=\mathcal{I}(g)$. If $\tilde{f}$ satisfies condition $({}^{i}\theta_{\tilde{g}})$ with respect to $\mathbb{K}$, then the problem ICP(f,g,$\mathbb{K}$) has a solution.*

Proof. Indeed, by Theorem 6.2.6, f satisfies condition (θ_g) with respect to $\mathbb{K}$. Hence Theorem 5.2.1 and Theorem 5.2.4 imply that *ICP(f,g,$\mathbb{K}$)* has a solution. □

THEOREM 6.3.6. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K}\subset H$ a closed pointed convex cone and $f,g:\mathbb{K}\to H$ completely continuous fields. If there is a $\delta>0$ and a mapping $h:B(0,\delta)\cap\mathbb{K}\to\mathbb{K}$ with $h(0)=0$ and*

$$\begin{cases}\underline{\mathcal{I}(g)^{\#}}(0)>\overline{h}^{\#}(0),\\ \underline{\left(\mathcal{I}(g)-h,\tilde{f}\right)^{\#}}(0)>0,\end{cases}$$

where $B(0,\delta)=\{x\in H:\|x\|<\delta\}$, then the problem ICP(f,g,$\mathbb{K}$) has a solution.

Proof. Let $\tilde{g}=\mathcal{I}(g)$. Since $\underline{\tilde{g}^{\#}}(0)>\overline{h}^{\#}(0)$, we have $\underline{(\tilde{g}-h)^{\#}}(0)>0$. Hence, there is a real number λ_1 with $0<\lambda_1<\delta$ such that for every $\tilde{x}\in\mathbb{K}$ with $\|\tilde{x}\|<\lambda_1$, we have

$$\langle\tilde{g}(\tilde{x})-h(\tilde{x}),\tilde{x}\rangle>0. \tag{6.3.12}$$

Since $\underline{\left(\tilde{g}-h,\tilde{f}\right)^{\#}}(0)>0$, there is a real number λ_2 with $0<\lambda_2<\delta$ such that for every $\tilde{x}\in\mathbb{K}$ with $\|\tilde{x}\|<\lambda_2$ we have

$$\left\langle\tilde{g}(\tilde{x})-h(\tilde{x}),\tilde{f}(\tilde{x})\right\rangle>0. \tag{6.3.13}$$

Let $\tilde{\rho}=\min\{\lambda_1.\lambda_2\}$. Obviously, $\tilde{\rho}>0$.

For $\|\tilde{x}\|<\tilde{\rho}$, let $\tilde{y}=h(\tilde{x})$. Then relations (6.3. 12) and (6.3.13) imply

$$\langle\tilde{g}(\tilde{x})-\tilde{y},\tilde{x}\rangle>0 \tag{6.3.14}$$

and

$$\left\langle\tilde{g}(\tilde{x})-\tilde{y},\tilde{f}(\tilde{x})\right\rangle>0 \tag{6.3.15}$$

respectively. Hence, because $\tilde{\rho}>0,\|\tilde{x}\|<\tilde{\rho}$ we have that relations (6.3.14) and (6.3.15) imply that $\tilde{f}$ satisfies condition $({}'\theta)$. Therefore Theorem 6.3.5 implies that the problem *ICP(f,g,$\mathbb{K}$)* has a solution. □

Now, we give some applications to multivalued complementarity problems.

THEOREM 6.3.7. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K}\subset H$ *a closed convex cone and* $f:\mathbb{K}\to H$ *a set-valued mapping. Then* $(x_*,x_*^f)\notin\{0\}\times H$ *is a solution to the problem MNCP(f,* $\mathbb{K}$*) if and only if* (y_*,y_*^g) *is a solution to the problem MNCP(g,* $\mathbb{K}$*), where* $y_*=i(x_*)$, $y_*^g=\frac{1}{\|x_*\|^2}x_*^f$ *and* $g=\mathcal{I}(f)$.

Proof. First, we have to prove that $y_*^g\in g(y_*)$. Indeed, dividing both sides of the relation $x_*^f\in f(x_*)$ by $\|x\|^2$ we obtain

$$y_*^g\in\frac{1}{\|x_*\|^2}f(x_*),$$

which implies $y_*^g\in\|y_*\|^2 f(i(y_*))=\mathcal{I}(f)(y_*)=g(y_*)$. It is easy to see that

$$\langle y_*,y_*^g\rangle=\frac{1}{\|x_*\|^4}\langle x_*,x_*^f\rangle \tag{6.3.16}$$

and

$$\langle y_*^g,z\rangle=\frac{1}{\|x_*\|^2}\langle x_*^f,z\rangle,\ \textit{for every } z\in\mathbb{K}. \tag{6.3.17}$$

By using (6.3.16), $\langle x_*,x_*^f\rangle=0$, if and only if $\langle y_*,y_*^g\rangle=0$. By using (6.3.17), $x_*^f\in\mathbb{K}^*$ if and only if $y_*^g\in\mathbb{K}^*$. □

THEOREM 6.3.8. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K}\subset H$ *a closed pointed convex cone and* $f:H\to H$ *an u.s.c. set-valued mapping with non-empty values such that:*

(1) $x-f(x)$ *is projectionally* Φ*-condensing, or* $f(x)=x-T(x)$*, where T is a c.u.s.c. set-valued mapping with non-empty values,*

(2) $x-f(x)$ *is projectionally approximable and* $P_{\mathbb{K}}[x-f(x)]$ *is also projectionally approximable with closed values.*

If g = $\mathcal{I}(f)$ *satisfies condition* $[{}^{i}\tilde{\theta}]_m$ *with respect to* $\mathbb{K}$*, then the problem MNCP(f,* $\mathbb{K}$*) has a solution.*

Proof. By Theorem 6.2.9, f satisfies condition $\left[\tilde{\theta}\right]_m$ with respect to $\mathbb{K}$, which implies that f is without an *EFE* (see the remark after Definition 5.3.11) Applying Theorem 5.3.1 we obtain that the problem *MNCP*(f, $\mathbb{K}$) has a solution. □

THEOREM 6.3.9. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed convex cone and* $f : H \to H$ *an u.s.c. set-valued mapping with non-empty values such that:*

(1) $x - f(x)$ *is projectionally* Φ*-condensing, or* $f(x) = x - T(x)$, *where T is a c.u.s.c. set-valued mapping with non-empty values,*

(2) $x - f(x)$ *is projectionally approximable and* $P_{\mathbb{K}}\left[x - f(x)\right]$ *with closed values.*

If there is a $\delta > 0$ *and a mapping* $h : B(0, \delta) \cap \mathbb{K} \to \mathbb{K}$ *with* $h(0) = 0$ *and*

$$\overline{h}^{\#}(0) < 1,$$

$$\underline{(I - h, \mathcal{I}(f))}^{\#}(0) > 0$$

where $B(0, \delta) = \{x \in H : \|x\| < \delta\}$, *then the problem MNCP(f,* $\mathbb{K}$*) has a solution.*

Proof. Let $g = \mathcal{I}(f)$. Since $\overline{h}^{\#}(0) < 1$, there is a λ_1 with $0 < \lambda_1 < \delta$ such that for every $y \in \mathbb{K}$ with $\|y\| < \lambda_1$ we have

$$\langle h(y), y \rangle < \|y\|^2. \tag{6.3.18}$$

Since $\underline{(I - h, g)}^{\#}(0) > 0$, there is a λ_2 with $0 < \lambda_2 < \delta$ such that for every $y \in \mathbb{K}$ with $\|y\| < \lambda_2$ we have

$$\left\langle y - h(y), y^g \right\rangle > 0, \textit{ for all } y^g \in g(y). \tag{6.3.19}$$

Let $\lambda = \min\{\lambda_1, \lambda_2\}$. Obviously $\lambda > 0$. For $\|y\| < \lambda$ let $q = h(y)$ Then relations (6.3.18) and (6.3.19) imply

$$\langle q, y \rangle < \left\| y^2 \right\|, \tag{6.3.20}$$

and

$$\left\langle y - q, y^g \right\rangle > 0, \tag{6.3.21}$$

respectively for all $y^g \in g(y)$. Hence, because $\lambda > 0$, $\|y\| < \lambda$ and considering (6.3.20) and (6.3.21) we obtain that g satisfies condition $\left[{}^i\tilde{\theta}\right]_m$. Applying Theorem 6.3.8, we obtain that the problem *MNCP(f, $\mathbb{K}$)* has a solution. □

6.4 Infinitesimal interior-point-ε-exceptional family of elements

In Section 5.7 of Chapter 5 we considered the problem $\varepsilon - MCP\left(f, \mathbb{R}^n_+\right)$ and in relation with this problem we defined and we studied the *interior band set-valued mapping* $\mathcal{U}:]0,\infty[\to S\left(\mathbb{R}^n_{++}\right)$. To study this set-valued mapping, we defined in Section 5.7 the notion of *interior-point-ε-exceptional family* for a mapping *f*. Now, we will define the *infinitesimal* variant of this notion, which is due to S.Z. Németh.

DEFINITION 6.4.1. *Given a scalar ε > 0, we say that a family* $\left\{y^r\right\}_{r>0} \subset \mathbb{R}^n_{++}$ *is an infinitesimal interior-point-ε-exceptional family of a mapping* $g: \mathbb{R}^n \to \mathbb{R}^n$ *if* $\left\|y^r\right\| \to 0$ *as* $r \to +\infty$ *and for each* y^r *there exists a positive number* $0 < \mu_r < 1$ *such that*

$$g_l\left(y^r\right) = \frac{1}{2}\left(\mu_r - \frac{1}{\mu_r}\right)y_l^r + \frac{\varepsilon\mu_r}{y_l^r}\left\|y^r\right\|^4, \text{ for all } l = 1, 2, ..., n. \qquad (6.4.1)$$

THEOREM 6.4.1. *If* $f: \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous function and* $g = \mathcal{I}(f)$ *is the inversion of f, then* $\left\{x^r\right\}_{r>0} \subset \mathbb{R}^n_{++}$ *is an interior-point-ε-exceptional family of f (in the sense of Definition 5.7.2), if and only if* $\left\{y^r\right\}_{r>0} \subset \mathbb{R}^n_{++}$ *is an infinitesimal interior-point-ε-exceptional family for g, where* $y^r = i\left(x^r\right)$ *is the inversion of* x^r *for all r > 0.*

Proof. Suppose that $\left\{x^r\right\}_{r>0} \subset \mathbb{R}^n_{++}$ is an interior-point-ε-exceptional family of f and let

$$y^r = i(x^r), \tag{6.4.2}$$

for all $r > 0$. Since $i^{-1} = i$, (equation defined in *Definition 5.7.2* and (6.4.2)) imply that

$$f_l\left(i\left(y^r\right)\right) = \frac{1}{2}\left(\mu_r - \frac{1}{\mu_r}\right)i\left(y^r\right)_l + \frac{\varepsilon\mu_r}{i\left(y^r\right)_l}, \text{ for all } l = 1,2,\ldots n. \tag{6.4.3}$$

Multiplying both sides of equation (6.4.3) by $\left\|y^r\right\|^2$ we obtain equation (6.4.1). Hence $\left\{y^r\right\}_{r>0} \subset \mathbb{R}^n_{++}$ is an infinitesimal interior-point-ε-exceptional family for *g*. Similarly we can prove that if $\left\{y^r\right\}_{r>0} \subset \mathbb{R}^n_{++}$ is an infinitesimal interior-point-ε-exceptional family for *g*, and then $\left\{x^r\right\}_{r>0} \subset \mathbb{R}^n_{++}$ is an interior-point-ε-exceptional family for *f*. □

THEOREM 6.4.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping and $\varepsilon > 0$. If there is no infinitesimal interior-point-ε-exceptional family for $g = \mathcal{I}(f)$, then there exists a point $x(\varepsilon)$ such that*

$$x(\varepsilon) > 0, f\left(x(\varepsilon)\right) > 0 \text{ and } x_l(\varepsilon) f_l\left(x(\varepsilon)\right) = \varepsilon, \tag{6.4.4}$$

for all l = 1, 2, …,n (i.e., $x(\varepsilon)$ is a solution to the problem $\varepsilon - MCP\left(f, \mathbb{R}^n_+\right)$).

Proof. We suppose that there is no point $x(\varepsilon)$ which satisfies relation (6.4.4). Then by Theorem 5.7.1, the mapping f has an interior-point-ε-exceptional family $\left\{x^r\right\}_{r>0} \subset \mathbb{R}^n_{++}$. Hence Theorem 6.4.1 implies that $\left\{y^r\right\}_{r>0} \subset \mathbb{R}^n_{++}$ is an infinitesimal interior-point-ε-exceptional family for *g*, where $y^r = i\left(x^r\right)$, for all $r > 0$. But this is in contradiction with our assumption and the proof is complete. □

THEOREM 6.4.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping and $g = \mathcal{I}(f)$. If the lower scalar derivative of g in* 0 *along $\mathbb{R}^n_{++}$ is positive, then the interior band mapping $\mathcal{U}$ has properties* (a) *and* (b).

Proof. We have

$$\underline{g}^{\#}(0) = \liminf_{\substack{y \to 0 \\ y \in \mathbb{R}^n_{++}}} \frac{\langle g(y), y \rangle}{\|y\|^2} > 0. \tag{6.4.5}$$

Let $y = i(x)$. Then we have

$$\liminf_{\substack{y \to 0 \\ y \in \mathbb{R}^n_{++}}} \frac{\langle g(y), y \rangle}{\|y\|^2} = \liminf_{\substack{\|x\| \to \infty \\ x \in \mathbb{R}^n_{++}}} \frac{\langle f(x), x \rangle}{\|x\|^2}. \tag{6.4.6}$$

Equation (6.4.5) and (6.4.6) imply

$$\underline{g}^{\#}(0) = \liminf_{\substack{\|x\| \to \infty \\ x \in \mathbb{R}^n_{++}}} \frac{\langle f(x), x \rangle}{\|x\|^2} > 0$$

which imply that Theorem 5.7.5 is applicable and our theorem is proved. □

Remarks.

(1) The results presented in this section are due to G. Isac and S. Z. Németh.

(2) For more details and results related to the subject of this chapter the reader is referred to (Isac. G. and Németh, S. Z., [1]-[5].

(3) The result presented in this chapter may be a starting point for new developments.

7

MORE ABOUT THE NOTION OF EXCEPTIONAL FAMILY OF ELEMENTS

We present in this chapter several special results related to the notion of *exceptional family of elements*. In Chapter 5 we obtained the notion of *exceptional family of elements* for a mapping applying Leray–Schauder type alternatives. Now we will show that this notion can be obtained for more general classes of mappings, which are not necessarily projectionally Leray–Schauder mappings. Moreover, we will present a necessary and sufficient condition for the non-existence of exceptional family of elements. In the last section, we will extend the notion of exceptional family of elements to functions defined on a particular class of Banach spaces and we will apply this notion to the study of complementarity problems defined on not necessarily convex cones.

7. 1 *EFE*-acceptable mappings

Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed convex cone and $f: H \to H$ a mapping. For any $r > 0$ ($r \in \mathbb{R}$) we denote

$$\mathbb{K}_r = \left\{ x \in \mathbb{K} \,\middle|\, \|x\| \le r \right\}.$$

We recall the definition of the notion of *exceptional family of element* for a mapping f with respect to $\mathbb{K}$.

DEFINITION 7.1.1. *We say that a family of elements* $\{x_r\}_{r>0} \subset \mathbb{K}$ *is an exceptional family of elements (EFE) for f with respect to* $\mathbb{K}$ *if for every* $r > 0$, *there exists a real number* $\mu_r > 0$ *such that the following conditions are satisfied:*

(1) $u_r = \mu_r x_r + f(x_r) \in \mathbb{K}^*$,

(2) $\langle x_r, u_r \rangle = 0$,

(3) $\|x_r\| \to +\infty$ *as* $r \to +\infty$.

DEFINITION 7.1.2. *We say that a mapping* $f: H \to H$ *is EFE-acceptable with respect to* $\mathbb{K}$ *if for any* $r > 0$ *the mapping* $\psi_r(x) = P_{\mathbb{K}_r}[x - f(x)]$ *has a fixed point (which necessarily is an element in* $\mathbb{K}_r$*). (The mapping* ψ_r *is considered from* K_r *into* K_r*.)*

The following result is due to M. Bianchi, N. Hadjisavvas and S. Schaible [1].

PROPOSITION 7.1.1. *If there exists* $x_* \in \mathbb{K}_r$ *such that* $\langle f(x_*), x - x_* \rangle \geq 0$ *for any* $x \in \mathbb{K}_r$ *and there exists* $y \in \mathbb{K}_r$ *with* $\|y\| < r$ *such that* $\langle f(x_*), x_* - y \rangle \geq 0$, *then we have* $\langle f(x_*), x - x_* \rangle \geq 0$ *for any* $x \in \mathbb{K}$.

Proof. We consider the convex continuous mapping

$$\varphi(x) = \langle f(x_*), x - x_* \rangle \text{ defined for any } x \in \mathbb{K}.$$

We have $\varphi(x) \geq 0$ for any $x \in \mathbb{K}_r$ and $\varphi(x_*) = 0$. Then x_* is a global minimum of φ on $\mathbb{K}_r$. Because we have

$$0 \leq \varphi(y) = \langle f(x_*), y - x_* \rangle \leq 0 = \varphi(x_*)$$

we deduce that y is also a global minimum of φ on $\mathbb{K}_r$. Therefore (since $\|y\| < r$) we have that y is a local minimum of φ on $\mathbb{K}$ and hence (because φ is convex) y is a global minimum on $\mathbb{K}$. Since $\varphi(y) = \varphi(x_*)$ we obtain that

x_* is a global minimum of φ on $\mathbb{K}$, that is, we have $\langle f(x_*), x - x_* \rangle \geq 0$ for any $x \in \mathbb{K}$. □

We have the following alternative.

THEOREM 7.1.2. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed convex cone and $f : H \to H$ an EFE-acceptable mapping with respect to $\mathbb{K}$. Then either the problem $NCP(f, \mathbb{K})$ has a solution, or the mapping f has an EFE with respect to $\mathbb{K}$.*

Proof. If the problem $NCP(f, \mathbb{K})$ has a solution we have nothing to prove. We suppose that this problem has no solution. In this case we show that f has an *EFE* with respect to $\mathbb{K}$. Indeed, because f is *EFE*-acceptable with respect to $\mathbb{K}$, then for every $r > 0$ there exists $x_r \in \mathbb{K}_r$ such that $x_r = \psi_r(x_r) = P_{\mathbb{K}_r}\left[x_r - f(x_r)\right]$. We know (see Chapter 2) that in this case we have

$$\langle f(x_r), x - x_r \rangle \geq 0 \text{ for any } x \in \mathbb{K}_r. \tag{7.1.1}$$

(Because we supposed that the problem $NCP(f, \mathbb{K})$ has no solution, we have that (7.1.1) is not satisfied for all $x \in \mathbb{K}$.)

We show [following ideas of (Bianchi, M., Hadjisavvas, N. and Schaible, S. [1], Theorem 5.1)] that $\{x_r\}_{r>0}$ is an *EFE* for f with respect to $\mathbb{K}$. For every $r > 0$ we define

$$\mu_r = -\frac{\langle f(x_r), x_r \rangle}{r^2}, \tag{7.1.2}$$

and

$$u_r = \mu_r x_r + f(x_r). \tag{7.1.3}$$

If $\|x_r\| < r$, then taking $y = x_r$ in Proposition 7.1.1, we obtain that $\langle f(x_r), x - x_r \rangle \geq 0$ for any $x \in \mathbb{K}$, i.e., x_r is a solution to the problem

$NCP(f,\mathbb{K})$ which is impossible. Therefore we must have $\|x_r\|=r$, for any $r>0$, which implies that $\|x_r\|\to+\infty$ *as* $r\to+\infty$. Also, we have

$$\langle x_r,u_r\rangle=\langle x_r,\mu_r x_r+f(x_r)\rangle=\langle x_r,\mu_r x_r\rangle+\langle x_r,f(x_r)\rangle$$
$$=-\left\langle x_r,\frac{\langle f(x_r)x_r\rangle}{r^2}x_r\right\rangle+\langle f(x_r),x_r\rangle=0.$$

The number μ_r is strictly positive. Indeed, we have $\langle f(x_r),0-x_r\rangle\geq 0$ which implies $\langle f(x_r),x_r\rangle\leq 0$ and hence $\mu_r=-\dfrac{\langle f(x_r),x_r\rangle}{r^2}\geq 0$.

If $\mu_r=0$, then $\langle f(x_r),x_r\rangle=0=\langle f(x_r),x_r-0\rangle$ and taking $y=0$ in Proposition 7.1.1 we deduce $\langle f(x_r),x_r-x\rangle\geq 0$ for any $x\in\mathbb{K}$, i.e., the $NCP(f,\mathbb{K})$ has a solution which is impossible. Therefore we have $\mu_r>0$ for any $r>0$. The theorem will be proved if we show that $u_r\in\mathbb{K}^*$ for any $r>0$. To show this, it is sufficient to prove that

$$\left\langle f(x_r),x-\frac{\langle x_r,x\rangle}{r^2}x_r\right\rangle\geq 0 \textit{ for any } x\in\mathbb{K}. \tag{7.1.4}$$

Indeed if (7.1.4) is true, then we have (because $f(x_r)=u_r-\mu_r x_r$),

$$0\leq\left\langle u_r-\mu_r x_r,x-\frac{\langle x_r,x\rangle}{r^2}x_r\right\rangle=\langle u_r,x\rangle-\left\langle u_r,\frac{\langle x_r,x\rangle}{r^2}x_r\right\rangle$$
$$-\mu_r\langle x_r,x\rangle+\mu_r\frac{\langle x_r,x\rangle}{r^2}r^2=\langle u_r,x\rangle.$$

Now, we show that (7.1.4) is true. Let $r>0$ be fixed. We denote by $y=x-\dfrac{\langle x_r,x\rangle}{r^2}x_r$ and $z_\lambda=y+\lambda x_r$ with $\lambda>\dfrac{\langle x_r,x\rangle}{r^2}$. Then $z_\lambda\in\mathbb{K}$ and $\dfrac{z_\lambda}{\|z_\lambda\|}\cdot r\in\mathbb{K}_r$. Hence, we have $\left\langle f(x_r),\dfrac{z_\lambda}{\|z_\lambda\|}\cdot r-x_r\right\rangle\geq 0$, which implies (because $y=z_\lambda-\lambda x_r$)

$$\left\langle f(x_r),y+\left(\lambda-\frac{\|z_\lambda\|}{r}\right)x_r\right\rangle\geq 0. \tag{7.1.5}$$

We can show that $\langle y, x_r \rangle = 0$, which implies that $\|z_\lambda\| = \sqrt{\|y^2\| + \lambda^2 r^2}$. We also have

$$\lim_{\lambda \to +\infty} \left[\lambda - \frac{\|z_\lambda\|}{r} \right] = \lim_{\lambda \to +\infty} \frac{r\lambda - \|z_\lambda\|}{r} = \lim_{\lambda \to +\infty} \frac{r^2\lambda^2 - \left(\|y\|^2 + \lambda^2 r^2\right)}{r\left(r\lambda + \|z_\lambda\|\right)}$$

$$= \lim_{\lambda \to +\infty} \frac{-\|y\|^2}{r^2\left(r\lambda + \|z_\lambda\|\right)} = 0.$$

Therefore, computing the limit as $\lambda \to +\infty$ in (7.1.5) we deduce that $\langle f(x_r), y \rangle \geq 0$ and we have that formula (7.1.4) is true. □

COROLLARY 7.1.3. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed convex cone and $f: H \to H$ an EFE-acceptable mapping with respect to $\mathbb{K}$. If f is without EFE, then the problem $NCP(f, \mathbb{K})$ has a solution.*

Examples

We give several examples of *EFE*-acceptable mappings.

(1) In the n-dimensional Euclidean space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, any continuous mapping is *EFE*-acceptable with respect to any closed convex cone.

(2) Let $(H, \langle \cdot, \cdot \rangle)$ be an arbitrary Hilbert space and $\mathbb{K} \subset H$ a closed convex cone with a compact base. It is known that in this case K is locally compact. Consequently, for any $r > 0$, $\mathbb{K}_r$ is a compact set. In this case, any continuous mapping f is *EFE*-acceptable with respect to $\mathbb{K}$.(This result is a consequence of Schauder's Fixed Point Theorem.)

(3) Let $(H, \langle \cdot, \cdot \rangle)$ be an arbitrary Hilbert space, $\mathbb{K} \subset H$ an arbitrary closed convex cone and $f: H \to H$ a completely continuous field, i.e., f has a representation of the form $f(x) = x - T(x)$, where $T: H \to H$ is a completely continuous operator. In this case f is an *EFE*-acceptable mapping. This result is also a consequence of Schauder's Fixed Point Theorem.

(4) Let $(H, \langle \cdot, \cdot \rangle)$ be an arbitrary Hilbert space and $\mathbb{K} \subset H$ a closed convex cone and $f: H \to H$ a nonexpansive field, i.e., f has a

representation of the form $f(x)=x-T(x)$, where $T: H \to H$ is a nonexpansive mapping. In this case also f is an *EFE*-acceptable mapping. This result is a consequence of a classical fixed point theorem for nonexpansive mapping defined on a bounded closed convex subset of a uniformly convex Banach space.

(5) Let $(H,\langle\cdot,\cdot\rangle)$ be an arbitrary Hilbert space, $\mathbb{K} \subset H$ a closed convex cone and $f: H \to H$ an α-set contraction field with respect to the α-Kuratowski measure of noncompactness. We have that $f(x)=x-T(x)$, where $T: H \to H$ is an α-set contraction. The mapping f is *EFE*-acceptable with respect to $\mathbb{K}$. This result is a consequence of Darbo's Fixed Point Theorem.

(6) Let $(H,\langle\cdot,\cdot\rangle)$ be an arbitrary Hilbert space and $\mathbb{K} \subset H$ a closed convex cone. Any mapping $f: H \to H$ with the property that for any $r > 0$, $VI(f,\mathbb{K}_r)$ has a solution is *EFE*-acceptable with respect to $\mathbb{K}$ An interesting example of such mapping is a continuous quasimonotone mapping $f:\mathbb{K} \to H$. We recall that f is *quasi-monotone* on $\mathbb{K}$ if for any $x, y \in \mathbb{K}$ the inequality $\langle f(x), y-x\rangle > 0$ implies $\langle f(y), y-x\rangle \geq 0$. Any pseudomonotone mapping (in Karamardian's sense) is quasimonotone. Also, in particular any monotone mapping is quasimonotone. From Lemma 2.1 and Proposition 2.1, both proved in (Aussel, D. and Hadjisavva, N., [1]), we deduce that for any $r > 0$ the problem $VI(f,\mathbb{K}_r)$ has a solution, since $\mathbb{K}_r$ is weakly compact. Because any solution of $VI(f,\mathbb{K}_r)$ is a fixed point for the mapping ψ_r, we have that any continuous quasimonotone mapping is *EFE*-acceptable with respect to $\mathbb{K}$. About the solvability of the problem $VI(f,\mathbb{K}_r)$ when f is quasimonotone see also [(Bianchi, M., Hadjisavvas, N. and Schaible, S. [1]), Propositions 2.2 and 2.3].

By the next theorem we will obtain other examples of *EFE-acceptable* mappings. Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space. We recall the following notion defined by G. Isac in (Isac, G. and Gowda, M. S. [1]). Let D be a subset in H.

DEFINITION 7.1.3. *We say that a mapping $f: D \to H$ satisfies condition $(S)_+^1$ if any sequence $\{x_n\}_{n\in N} \subset D$ with*

$$(w)-\lim_{n\to\infty} x_n = x_* \in H,\ (w)-\lim f(x_n) = u \in H$$

and $\limsup_{n\to\infty}\langle x_n, f(x_n)\rangle \le \langle x_*, u\rangle$, *has a sub-sequence $\{x_{n_k}\}_{k\in N}$ convergent (in norm) to x_*.*

Remark. Condition $(S)_+^1$ is related to condition $(S)_+$ introduced in nonlinear analysis by F. E. Browder. It is known that condition $(S)_+$ implies condition $(S)_+^1$ (Isac, G. and Gowda, M. S. [1]), (Isac, G. [23]). Condition $(S)_+^1$ was used and considered in several papers [see the references cited in (Isac, G. [23])].

We recall the following property of the inner-product given on H:

$$\begin{cases} \text{if a sequence } \{x_n\} \text{ is weakly convergent to an element } x_* \\ \text{and a sequence } \{y_n\} \text{ is convergent in norm to an element } y_*, \\ \text{then } \lim_{n\to\infty}\langle x_n, y_n\rangle = \langle x_*, y_*\rangle. \end{cases} \quad (7.1.6)$$

DEFINITION 7.1.4. *We say that a mapping $f: H \to H$ is scalarly compact with respect to a closed convex set $D \subset H$, if for any sequence $\{x_n\}_{n\in N} \subset D$, weakly convergent to an element $x_* \in D$, there exists a subsequence $\{x_{n_k}\}_{k\in N}$ such that* $\limsup_{x\to\infty}\langle x_{n_k} - x_*, f(x_{n_k})\rangle \le 0$.

Remark. If f is completely continuous or there exists a completely continuous operator $T: H \to H$ such that $|\langle y, f(x)\rangle| \le \langle y, T(x)\rangle$, for any x, $y \in D$, then f is scalarly compact. We recall that $f: H \to H$ is demicontinuous if for any sequence $\{x_n\}_{n\in N} \subset H$ convergent in norm to an element $x_* \in H$, $\{f(x_n)\}_{n\in N}$ is weakly convergent to $f(x_*)$.

THEOREM 7.1.4. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $T_1, T_2 : H \to H$ two demicontinuous mappings. If the following assumptions are satisfied:*

(1) T_1 *is bounded and satisfies condition* $(S)_+^1$,

(2) T_2 *is scalarly compact with respect to a closed bounded convex set* $D \subset H$,

Then the problem $VI(T_1 - T_2, D)$ *has a solution.*

Proof. Let Λ be a family of finite dimensional subspaces F of H such that $F \cap D \neq \phi$. We consider the family Λ ordered by inclusion and also we consider the mapping $h(x) = T_1(x) - T_2(x)$ defined for all $x \in D$. For each $F \in \Lambda$ we denote $D(F) = D \cap F$ and we set

$$A_F = \{y \in D : \langle x - y, h(y) \rangle \geq 0 \text{ for all } x \in D(F)\}.$$

For each $F \in \Lambda$, the set A_F is non-empty. Indeed the solution set of $VI(h, D(F))$ is a subset of A_F and the solution set of $VI(h, D(F))$ is non-empty. To obtain this fact we consider the mappings $j : F \to H$, $j^* : H^* \to F^*$ and $j^* \circ h \circ j$, where j is the inclusion and j^* is the adjoint of j. The mapping $j^* \circ h \circ j$ is continuous and

$$\langle x - y, (j^* \circ h \circ j)(y) \rangle = \langle x - y, h(y) \rangle.$$

Applying the classical Hartman–Stampacchia Theorem to the set $D(F)$ and the mapping $j^* \circ h \circ j$ we obtain that $VI(h, D(F))$ has a solution.

We denote by $\overline{A_F}^{\sigma}$ the weak closure of A_F. We have that $\bigcap_{F \in \Lambda} \overline{A_F}^{\sigma}$ is non-empty. Indeed, let $\overline{A_{F_1}}^{\sigma}, \overline{A_{F_2}}^{\sigma}, \ldots, \overline{A_{F_n}}^{\sigma}$ be a finite subfamily of the family $\left\{\overline{A_F}^{\sigma}\right\}_{F \in \Lambda}$. Let F_0 be the finite dimensional subspace of H generated by $F_1, F_2, \ldots, F_n$. Because $F_k \subset F_0$ for all $k = 1, 2, \ldots, n$, we have that $D(F_k) \subseteq D(F_0)$ for all $k = 1, 2, \ldots, n$.

We have $A_{F_0} \subseteq A_{F_k}$, which implies $\overline{A_{F_0}}^{\sigma} \subseteq \overline{A_{F_k}}^{\sigma}$, for all $k = 1, 2, \ldots, n$ and finally we deduce that $\bigcap_{k=1}^{n} \overline{A_F}^{\sigma} \neq \phi$. Because D is weakly compact we conclude that $\bigcap_{F \in \Lambda} \overline{A_F}^{\sigma} \neq \phi$.

Let $y_* \in \bigcap_{F\in\Lambda} \overline{A_F}^{\sigma}$, i.e., for every $F \in \Lambda$, $y_* \in \overline{A_F}^{\sigma}$. Let $x \in D$ be an arbitrary element. There exists some $F \in \Lambda$ such that x, $y_* \in F$. Since $y_* \in \overline{A_F}^{\sigma}$, by the Smulian Theorem there exists a sequence $\{y_n\}_{n\in N} \subset A_F$, weakly convergent to y_*. We have

$$\begin{cases} \langle y_* - y_n, h(y_n) \rangle \geq 0 \\ and \\ \langle x - y_n, h(y_n) \rangle \geq 0 \end{cases}$$

or

$$\langle y_n - y_*, T_1(y_n) \rangle \leq \langle y_n - y_*, T_2(y_n) \rangle \tag{7.1.7}$$

and

$$\langle x - y_n, T_1(y_n) \rangle \geq \langle x - y_n, T_2(y_n) \rangle . \tag{7.1.8}$$

From (7.1.7) and assumption (2), (considering eventually a subsequence) we have

$$\limsup_{n\to\infty} \langle y_n - y_*, T_1(y_n) \rangle \leq 0 . \tag{7.1.9}$$

Because T_1 is bounded, we can suppose that $\{T_1(y_n)\}_{n\in N}$ is weakly convergent to an element $v_0 \in H$. Because

$$\langle y_n, T_1(y_n) \rangle = \langle y_n - y_* + y_*, T_1(y_n) \rangle$$
$$= \langle y_n - y_*, T_1(y_n) \rangle + \langle y_*, T_1(y_n) \rangle$$

and considering (7.1.9) we obtain

$$\limsup_{n\to\infty} \langle y_n, T_1(y_n) \rangle \leq \langle y_*, v_0 \rangle .$$

Hence, by condition $(S)_+^1$ we obtain that the sequence $\{y_n\}_{n\in N}$ has a subsequence, denoted again by $\{y_n\}_{n\in N}$ convergent in norm to y_*. Because T_2 is continuous, we have $\lim_{n\to\infty} T_2(y_n) = T_2(y_*)$. From inequality (7.1.8) by using property (7.1.6) of the inner-product and computing the limit we conclude that

$$\langle x - y_*, T_1(y_*) - T_2(y_*) \rangle \geq 0 \text{ for all } x \in D$$

and the proof is complete. □

COROLLARY 7.1.5. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K}\subset H$ *a closed convex cone and* $f: H\to H$ *a mapping. If f has a decomposition of the form* $f(x)=T_1(x)-T_2(x)$ *such that*

(1) T_1 *is demicontinuous, bounded and satisfies condition* $(S)^1_+$,

(2) T_2 *is demicontinuous and scalarly compact with respect to* $\mathbb{K}$,

then f is EFE-acceptable with respect to $\mathbb{K}$.

Proof. We apply Theorem 7.1.4 to *f* and to any $\mathbb{K}_r$ with $r > 0$. □

7.2 Skrypnik topological degree and exceptional families of elements

In the previous two chapters, we presented several results related to the notion of *EFE* for *projectionally Leray–Schauder* mappings. In this section we will present another approach of the notion of *EFE* due to A. Carbone and P. P. Zabreiko [1], [2]. Their approach is based on a special topological degree defined by I. V. Skrypnik. [see (Skrypnik, I.V. [1], [2])]. By this approach we can define the notion of *EFE* (the same defined by Definition 7.1.1) for mappings, which are not completely continuous fields.

Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K}\subset H$ a closed convex cone and $f:\mathbb{K}\to H$ a completely continuous mapping. Consider the operator $Ax=P_{\mathbb{K}}(x)-f(P_{\mathbb{K}}(x))$. By Theorem 2.3.7 we know that if x_* is a fixed point of *A*, then $u_*=P_{\mathbb{K}}(x_*)$ is a solution to the problem $NCP(f,\mathbb{K})$. Consider the family of vector fields

$$\begin{cases}\Phi(\lambda)x=x-\lambda\left[P_{\mathbb{K}}(x)-f(P_{\mathbb{K}}(x))\right],\\ 0\le\lambda\le 1,\ x\in H.\end{cases}\tag{7.2.1}$$

This is a linear deformation connecting the vector field $I-A$ whose zeros define solutions to the problem $NCP(f,\mathbb{K})$ and the trivial field $\Phi_0=I$. Consider the family of sets

$$\begin{cases} \Omega_{r,\rho} = \{x \in H : \|x\| \le \rho, \|P_{\mathbb{K}}(x)\| \le r\}, \\ 0 < r < \rho < \infty. \end{cases} \tag{7.2.2}$$

Obviously, $\Omega_{r,\rho}$ is a bounded domain in H and 0 is an interior point. A geometrical image of the sets $\Omega_{r,\rho}$ is given in (Carbone, A. and Zabreiko, P. P. [1]). The boundary $\partial\Omega_{r,\rho}$ of this domain is

$$\partial\Omega_{r,\rho} = \{x \in H : \|x\| < \rho, \|P_{\mathbb{K}}(x)\| = r\} \cup \{x \in H : \|x\| = \rho, \|P_{\mathbb{K}}(x)\| \le r\}. \tag{7.2.3}$$

Let $\partial\Omega^0_{r,\rho} = \{x \in H : \|x\| < \rho, \|P_{\mathbb{K}}(x)\| = r\}$. We note that we need a special simple a priori estimate for values of the vector fields $\Phi(\lambda)$ $(0 \le \lambda \le 1)$, which shows that the part $\partial\Omega^0_{r,\rho}$ of the boundary $\partial\Omega_{r,\rho}$ is fundamental for the next results. We define

$$\mu(r) = \sup_{\|x\| \le r, u \in \mathbb{K}} \|u - f(u)\|.$$

PROPOSITION 7.2.1. *Let $\rho > \mu(r)$, then*

$$\|\Phi(\lambda)x\| \ge \rho - \mu(r) \quad (x \in \partial\Omega_{r,\rho} \setminus \partial\Omega^0_{r,\rho}).$$

In particular, the zeros of the fields $\Phi(\lambda)$ $(0 \le \lambda \le 1)$, which are situated on the boundary $\partial\Omega_{r,\rho}$ lie on $\partial\Omega^0_{r,\rho}$.

Proof. This proposition is a consequence of the inequalities

$$\|\Phi(\lambda)x\| \ge \|x\| - \|P_{\mathbb{K}}(x) - f(P_{\mathbb{K}}(x))\| \ge \rho - \mu(r) > 0 \text{ for } x \in \partial\Omega_{r,\rho} \setminus \partial\Omega^0_{r,\rho}.$$

□

Remark. From Proposition 7.2.1 we deduce that if the inequality $\rho > \mu(r)$ holds, then the zeros of $\Phi(\lambda)$ $(0 \le \lambda \le 1)$ situated on the boundary $\partial\Omega_{r,\rho}$ really lie on its part $\partial\Omega^0_{r,\rho}$. Now, we consider the family of complementarity problems

$$\begin{cases} u \in \mathbb{K}, \\ (1-\lambda)u + \lambda f(u) \in \mathbb{K}^*, \\ \langle u, (1-\lambda)u + \lambda f(u) \rangle = 0, \\ 0 \le \lambda \le 1, \end{cases} \tag{7.2.4}$$

which corresponds to the family of operators $(1-\lambda)I+\lambda f,(0\le\lambda\le1)$. It is easy to show the following result.

PROPOSITION 7.2.2. *Let f be a mapping from $\mathbb{K}$ into H and $A(\lambda)=\lambda\left[P_{\mathbb{K}}-f(P_{\mathbb{K}})\right],(0\le\lambda\le1)$. Then the complementarity problem with $(1-\lambda)I+\lambda f$ is solvable if and only if the operator $A(\lambda)$ has a fixed point in H. Moreover, if x_* is a fixed point of $A(\lambda)$, then $u_*=P_{\mathbb{K}}(x_*)$ is a solution to the complementarity problem with the mapping $(1-\lambda)I+\lambda f$.*

We will use the notion of mapping of class $(S)_+$ (see Chapter 1, Definition 1.6.4) and a notion of *quasi-monotonicity*, which is different than the quasi-monotonicity in Karamardian's sense. We say that a mapping $f: H\to H$ is *quasi-monotone* if each sequence $\{x_n\}_{n\in N}$ from H, which is weakly convergent to x_*, satisfies the condition

$$\liminf_{n\to\infty}\langle f(x_n),x_n-x_*\rangle\ge0.$$

It is known that if each mapping of class $(S)_+$ is *quasi-monotone* the converse is not true. We note that the mappings of class $(S)_+$ and quasi-monotone mappings were introduced and studied in detail by F. Browder, H. Brezis, I. V. Skrypnik and others.

PROPOSITION 7.2.3. *Let $f: \mathbb{K}\to H$ be a completely continuous mapping. Then the vector field $\Phi(\lambda),(0\le\lambda\le1)$ is of class $(S)_+$.*

Proof. Let $0\le\lambda<1$ and suppose that $\{x_n\}_{n\in N}$ is a sequence weakly convergent to an element x_* and

$$\limsup_{n\to\infty}\langle\Phi(\lambda)x_n,x_n-x_*\rangle\le0.$$

Because $\|P_{\mathbb{K}}(x_n)-P_{\mathbb{K}}(x)\|\le\|x_n-x_*\|$ we have

$$(1-\lambda)\langle x_n-x_*,x_n-x_*\rangle\le\langle x_n-x_*-\lambda P_{\mathbb{K}}(x_n)+\lambda P_{\mathbb{K}}(x_*),x_n-x_*\rangle$$
$$=\langle\Phi(\lambda)x_n,x_n-x_*\rangle-\langle x_*-\lambda P_{\mathbb{K}}(x_*),x_n-x_*\rangle-\lambda\langle f(P_{\mathbb{K}}(x_n)),x_n-x_*\rangle.$$

Without loss of generality, we can assume that the first summand in the right-hand side of this chain, as $n\to\infty$, has a non-positive limit by the

properties of the sequence $\{x_n\}_{n\in N}$. The second summand tends to 0 as $n \to \infty$, by the weak convergence of $\{x_n\}_{n\in N}$ to 0. The third summand also tends to 0 as $n \to \infty$ since $\{f(P_{\mathbb{K}}(x_n))\}$ is a bounded set.
Therefore

$$\limsup_{n\to\infty}(1-\lambda)\langle x_n - x_*, x_n - x_*\rangle \leq 0$$

which implies that $\{x_n\}_{n\in N}$ tends to x_* in norm. □

PROPOSITION 7.2.4. *If f is completely continuous, then $\Phi(1)$ is quasi-monotone.*

Proof. Let $\{x_n\}_{n\in N}$ be a sequence weakly convergent to x_*. We have (using the properties of projection operator $P_{\mathbb{K}}$),

$$\begin{aligned}\langle \Phi(1)x_n, x_n - x_*\rangle &= \langle x_n - x_* - P_{\mathbb{K}}(x_n) + P_{\mathbb{K}}(x_*), x_n - x_*\rangle \\ &\quad + \langle f(P_{\mathbb{K}}(x_n)), x_n - x_*\rangle + \langle x_* - P_{\mathbb{K}}(x_*), x_n - x_*\rangle \\ &\geq \langle f(P_{\mathbb{K}}(x_n)), x_n - x_*\rangle + \langle x_* - P_{\mathbb{K}}(x_*), x_n - x_*\rangle.\end{aligned}$$

Both summands in the right-hand side of this chain of terms tend to 0 as $n \to \infty$ by the properties of the sequence $\{x_n\}_{n\in N}$ and the operator f.
Therefore we obtain

$$\liminf_{n\to\infty}\langle \Phi(1)x_n, x_n - x_*\rangle \geq 0,$$

that is $\Phi(1)$ is quasi-monotone. □

Remark. Generally, the vector field $\Phi(1)$ is not of class $(S)_+$.

We say that a general mapping $\phi: H \to H$ is zero-closed if for any bounded sequence $\{x_n\}_{n\in N}$ such that $\{\Phi(x_n)\}$ is convergent in norm to 0, there exists $x_* \in \overline{conv}(\{x_n\})$ such that $\Phi(x_*) = 0$.

We say that a mapping $f: \mathbb{K} \to X$ is *regular*, if for each sequence $\{u_n\}_{n\in N}$, ($u_n \in \mathbb{K}$ for any $n \in N$), weakly convergent to u_* and such that the sequence $\{f(u_n)\}_{n\in N}$ converges to $v_* \in \mathbb{K}^*$, in norm, the equation $f(u_*) = v_*$ holds.

PROPOSITION 7.2.5. *If $f : \mathbb{K} \to H$ is a regular completely continuous mapping, then the vector field $\Phi(1)$ is zero-closed.*

Proof. Let $\{x_n\}$ be a bounded sequence such that $\{\Phi(1)x_n\}_{n\in N}$ is convergent in norm to 0 as $n \to \infty$. Without loss of generality we can assume that the sequence $\{x_n\}_{n\in N}$ weakly converges to an element x_* and the sequence $\{f(P_{\mathbb{K}}(x_n))\}_{n\in N}$ converges in norm to v_*. In this case the sequence $\{x_n - P_{\mathbb{K}}(x_n)\}_{n\in N}$ converges in norm to $-v_*$, since $x_n - P_{\mathbb{K}}(x_n) = -f(P_{\mathbb{K}}(x_n)) + \Phi(1)(x_n) \quad (n \geq 1)$.

By the properties of $P_{\mathbb{K}}$ we have

$$\langle x_n - P_{\mathbb{K}}(x_n) - x + P_{\mathbb{K}}(x), x_n - x \rangle \geq 0 \quad (x \in H, n \geq 1).$$

Computing the limit in this inequality as $n \to \infty$ we obtain

$$\langle -v_* - x_* + P_{\mathbb{K}}(x), x_* - x \rangle \geq 0 \quad (x \in H).$$

If we consider $x = x_* + tu \quad (u \in H,) \ (0 < t < \infty)$, and dividing by t we have

$$\langle -v_* - x_* - tu + P_{\mathbb{K}}(x_* + tu), u \rangle \leq 0 \quad (u \in H, \ 0 < t < \infty).$$

Passing to the limit as $t \to 0$, we obtain

$$\langle -v_* - x_* + P_{\mathbb{K}}(x_*), u \rangle \leq 0 \quad (u \in H).$$

Because u is arbitrary in H, we deduce that $v_* = -(x_* - P_{\mathbb{K}}(x_*)) \in \mathbb{K}^*$. Moreover, the sequence $\{u_n\}_{n\in N}$, where $u_n = P_{\mathbb{K}}(x_n)$ is weakly convergent to $u_* = P_{\mathbb{K}}(x_*)$, since

$$P_{\mathbb{K}}(x_n) = x_n - (x_n - P_{\mathbb{K}}(x_n)) \to x_* + v_* = x_* - x_* + P_{\mathbb{K}}(x_*) = P_{\mathbb{K}}(x_*).$$

Thus, the sequence $\{u_n\}$ weakly converges to u_* and the sequence $\{f(u_n)\}$ converges in norm to $v_* \in \mathbb{K}^*$. By the regularity of f we have $f(u_*) = v_*$. Therefore

$$\Phi(1)x_* = x_* - P_{\mathbb{K}}(x_*) + f(P_{\mathbb{K}}(x_*)) = x_* - P_{\mathbb{K}}(x_*) + v_* = 0$$

and the proof is complete. □

Now, we conclude that the vector fields $\Phi(\lambda)$ $(0 \le \lambda < 1)$ defined by (7.2.1) are of class $(S)_+$ and the field $\Phi(1)$ is *quasi-monotone* and moreover, $\Phi(1)$ is *zero-closed* if f is *regular*. Therefore we can apply the Skrypnik topological degree theory for studying fixed points of vector fields $\Phi(\lambda)$. The Skrypnik degree theory (Skrypnik, I. V. [2]) states that for each field Φ of class $(S)_+$ (and even zero-closed and quasi-monotone field Φ) defined on a bounded domain Ω and being without zero on the boundary of $\partial\Omega$ of the domain Ω, there is defined an integer $\gamma(\Phi, \Omega)$ with the following properties: (see Chapter 1)

(i) $\gamma(I, \Omega) = 1$ *if* $0 \in \Omega$,

(ii) *If* $\Omega = \Omega_1 \cup \Omega_2$ *and* Φ *has no zero on the set* $\partial\Omega_1 \cup \partial\Omega_2 \cup (\partial\Omega_1 \cap \partial\Omega_2)$, *then* $\gamma(\Phi,\Omega) = \gamma(\Phi,\Omega_1) + \gamma(\Phi,\Omega_2)$.

(iii) *If* Φ_0 *and* Φ_1 *are homotopic on* Ω, *then* $\gamma(\Phi_0,\Omega) = \gamma(\Phi_1,\Omega)$. *We say that* Φ_0 *and* Φ_1 *are homotopic on* Ω *if there exists a family of mappings* $\Phi(\lambda,\cdot)$ $(0 \le \lambda \le 1)$ *of class* $(S)_+$ *(or zero-closed and quasi-monotone), defined on* $\overline{\Omega}$ *and demicontinuous with respect to both variables such that*
$\Phi(0,\cdot) = \Phi_0, \Phi(1,\cdot) = \Phi_1, \Phi(\lambda,x) \neq 0 \quad (0 \le \lambda \le 1, x \in \partial\Omega)$.

The following result is known (Skrypnik, I. V. [2].

If Φ *has no zero on the boundary* $\partial\Omega$ *of the domain* Ω *and the degree* $\gamma(\Phi,\Omega)$ *of this vector field on the boundary* $\partial\Omega$ *of* Ω *is non-zero, then there exists at least one zero* x_* *of* Φ *in* Ω.

Now, we consider the family of vector fields $\Phi(\lambda)$ $(0 \le \lambda < 1)$ (defined by (7.2.1)) on the domain $\Omega_{r,\rho}$. We suppose that ρ and r are fixed positive reals and $\rho > \mu(r)$. Obviously the family of $\Phi(\lambda)$, under our assumptions, is *demicontinuous (i.e., each mapping* $\Phi(\lambda)$ maps strongly convergent sequences into weakly convergent sequences) with respect to both variables and $\Phi(0) = \Phi_0$, $\Phi(1) = \Phi_1$. We have two possibilities:

First, there exists $\lambda_* \in]0,1[$ and $x_* \in \partial\Omega_{r,\rho}$ (really $x_* \in \partial\Omega^0_{r,\rho}$) such that $\Phi(\lambda_*)x_* = 0$. Certainly, in this case $u_* = P_K(x_*)$ is a solution of the problem $NCP\left(\left(1-\lambda_*\right)I+\lambda_* f,\mathbb{K}\right)$ and this solution is situated on the set $S_r\cap\mathbb{K}$ with $S_r=\left\{u\in H:\|u\|=r\right\}$.

Second, for all $\lambda\in\left]0,1\right[$ the relations $\Phi(\lambda)x\neq 0\ \ \left(x\in\partial\Omega_{r,\rho}\right)$ hold. In this case all vector fields $\Phi(\lambda)$ $(0\leq\lambda<1)$ are homotopic on $\Omega_{r,\rho}$ and therefore they have the same degree $\gamma\left(\Phi(\lambda),\Omega_{r,\rho}\right)$ on the boundary $\partial\Omega_{r,\rho}$ of domain $\Omega_{r,\rho}$. But $\gamma\left(\Phi(0),\Omega_{r,\rho}\right)=1$ since $\Phi(0)=I$ and $0\in\Omega_{r,\rho}$. Thus in the second case we have

$$\gamma\left(\Phi(\lambda),\Omega_{r,\rho}\right)=1\quad\left(0\leq\lambda<1\right). \tag{7.2.5}$$

Moreover, if the vector field $\Phi(1)$ is zero-closed (for example if f is regular and completely continuous), and has no zero on $\partial\Omega_{r,\rho}$, then we have

$$\gamma\left(\Phi(\lambda),\Omega_{r,\rho}\right)=1\quad\left(0\leq\lambda\leq1\right). \tag{7.2.6}$$

(Obviously, if $\Phi(1)$ has a zero on $\partial\Omega_{r,\rho}$ we have that the problem $NCP(f,\mathbb{K})$ has a solution.)

Hence, if the vector field $\Phi(1)$ is zero-closed and has no zero on $\partial\Omega_{r,\rho}$, then equation (7.2.5) implies the existence of a zero of $\Phi(1)$ in the domain $\Omega_{r,\rho}$ and therefore, the solvability of the problem $NCP(f,\mathbb{K})$ in the set $B_r=\left\{u\in H:\|u\|\leq r\right\}$. We conclude with the following result.

THEOREM 7.2.6. *If $f:\mathbb{K}\to H$ is a regular completely continuous mapping and $0<r<+\infty$, then:*

(1) *either for some $\lambda_r\in\left]0,1\right[$ the complementarity problem with the mapping $\left(1-\lambda_r\right)I+\lambda_r f$ has a solution in the set $S_r\cap\mathbb{K}$,*

(2) *or the complementarity problem with the mapping f has a solution in $B_r\cap\mathbb{K}$.*

Remark. Consider the conclusion (1) of Theorem 7.2.6. If $x_r\in S_r\cap\mathbb{K}$ is the solution of the problem $NCP\left(\left(1-\lambda_r\right)I+\lambda_r f,\mathbb{K}\right)$, then we have $x_r\in\mathbb{K},\|x_r\|=r$ and

$$\begin{cases}(1-\lambda_r)x_r+\lambda_r f(x_r)\in \mathbb{K}^*,\\ \langle x_r,(1-\lambda_r)x_r+\lambda_r f(x_r)\rangle=0.\end{cases} \tag{7.2.7}$$

Dividing both relations in (7.2.7) by λ_r we obtain that $\{x_r\}_{r>0}$ is an *EFE* for *f* with respect to $\mathbb{K}$. We have the following result.

THEOREM 7.2.7. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K}\subset H$ *a closed convex cone and* $f:\mathbb{K}\to H$. *If f is regular and completely continuous then*:

(1) *either the problem NCP(f,* $\mathbb{K}$*) has a solution,*

(2) *or f has an EFE with respect to* $\mathbb{K}$.

Proof. This result is a consequence of Theorem 7.2.6 and of the remark presented above. □

Now, we give an application to the study of complementarity problems with respect to some particular nonconvex sets. Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space and $D\subset H$ a *closed* non-empty set. We define the dual D^* of the set D by

$$D^*=\{y\in H:\langle x,y\rangle\geq 0, \textit{for all } x\in D\}.$$

We say that D is star-shaped with respect to a convex set $A\subset D$, if and only if, $x\in D$ whenever $\lambda x+(1-\lambda)y\in D$ for some $y\in A$ and any $\lambda\in[0,1]$.

Let $\varepsilon>0$ be a real number, eventually very small. We say that D is ε-convex, if and only if whenever $[x,y[\,\subset conv(D)\setminus D$, we have $\|y-x\|<\varepsilon$. We recall that $[x,y[=\{\lambda y+(1-\lambda)x:\lambda\in[0,1[\}$. We denote by $\mathbb{K}(D)$ the *smallest closed convex cone* such that $D\subset\mathbb{K}(D)$. We say that a non-empty subset $D\subset H$ is a *locally compact pointed conical set* if the following properties are satisfied:

(c_1) *for all* $x\in D$ *and all* $\lambda\in\mathbb{R}_+$ *we have* $\lambda x\in D$,

(c_2) $\mathbb{K}(D)\cap(-\mathbb{K}(D))=\{0\}$,

(c_3) $\mathbb{K}(D)$ *is a locally compact convex cone.*

For some practical problems the following examples are interesting, supposing that D is a locally compact pointed conical set.

(1) $D = \bigcup_{i \in I} \mathbb{K}_i$, where for every $i \in I$, $\mathbb{K}_i$ is a polyhedral cone not necessarily convex. $\mathbb{K}(D)$ must be locally compact.

(2) $D \cap B$ is a set, star-shaped with respect to a convex set $A \subset D \cap B$, where B is a base of $\mathbb{K}(D)$.

(3) $D \cap B$ is an ε-convex set, with $\varepsilon > 0$ very small, where B is again a base of $\mathbb{K}(D)$.

If $f : H \to H$ is a mapping we can consider the *complementarity problem* defined by f and D is:

$$NCP(f, D) : \begin{cases} \text{find } x_* \in D \text{ such that} \\ f(x_*) \in D^* \text{ and } \langle x_*, f(x_*) \rangle = 0. \end{cases}$$

We say that the set $R(D; \mathbb{K}) = \mathbb{K}(D) \setminus D$ is the residual set of D with respect to $\mathbb{K}(D)$. Obviously $0 \notin R(D; \mathbb{K})$ and $R(D; \mathbb{K})$ is empty if D is a closed pointed convex cone. The following result is due to G. Isac.

THEOREM 7.2.8. *Let $D \subset H$ be a non-empty, locally compact closed and pointed conical set. D is supposed to be nonconvex. Let $f : H \to H$ be a continuous bounded mapping. If there exists $\rho > 0$ such that the following assumptions are satisfied:*

(1) *for every $x \in \mathbb{K}(D)$ with $\|x\| = \rho$, there exists $y \in \mathbb{K}(D)$ such that $\|y\| < \rho$ and $\langle f(x), x - y \rangle \geq 0$,*

(2) *for every $x \in R_\rho(D; \mathbb{K}) = \{z \in R(D; \mathbb{K}) : \|z\| < \rho\}$, there exists $y \in \mathbb{K}(D)$ with $\|y\| < \|x\|$ such that $\langle f(x), x - y \rangle \geq 0$,*

then the problem NCP(f, D) has a solution x_* such that $\|x_*\| \leq \rho$.

Proof. Let $\varepsilon > 0$ be a real number. Consider the mapping $f_\varepsilon(x) := f(x) + \varepsilon x$, for any $x \in h$. The mapping f_ε satisfies the following properties:

(i) *f_ε is continuous and bounded,*

(ii) *for every $x \in \mathbb{K}(D)$, with $\|x\| = \rho$, there exists $y \in \mathbb{K}(D)$ such that $\|y\| < \rho$ and $\langle f_\varepsilon(x), x - y \rangle > 0$,*

(iii) *for every* $x \in R_\rho(D;\mathbb{K})$, *there exists* $y \in \mathbb{K}(D)$, *such that* $\|y\| < \rho$ *and* $\langle f_\varepsilon(x), x-y\rangle > 0$.

For each $x \in \mathbb{K}(D)$, with $\|x\| > \rho$ we denote $T_\rho(x) = \frac{\rho x}{\|x\|}$ (the radial projection onto $S_\rho^+ = \{x \in \mathbb{K}(D) : \|x\| = \rho\}$).

Now, we consider the mapping g_ρ: $\mathbb{K}(D) \to H$ defined by

$$g_\varepsilon(x) = \begin{cases} f_\varepsilon(x), \text{ if } \|x\| \le \rho, \\ f_\varepsilon(T_\rho(x)) + \|x - T_\rho(x)\| x, \text{ if } \|x\| > 0. \end{cases}$$

We can show that g_ε satisfies the following property:

(iv) *for every* $x \in \mathbb{K}(D)$, *with* $\|x\| > \rho$, *there exists* $y \in \mathbb{K}(D)$ *with* $\|y\| < \|x\|$ *such that* $\langle x-y, g_\varepsilon(x)\rangle > 0$.

(For the proof of this property see (Isac, G. [30]). Therefore g_ε satisfies property (θ) and hence it is without *EFE* with respect to $\mathbb{K}(D)$. The mapping g_ε is also completely continuous and regular (Isac, G. [30]).

Applying Theorem 7.2.7 we obtain that for any $\varepsilon > 0$ the classical problem $NCP(g_\varepsilon, \mathbb{K}(D))$ has a solution x_ε^*. Because of the fact that g_ε satisfies property (iv) we must have $\|x_\varepsilon^*\| \le \rho$, which implies that $g_\varepsilon(x_\varepsilon^*) = f_\varepsilon(x_\varepsilon^*)$. Therefore, for any $\varepsilon > 0$ the problem $NCP(f_\varepsilon, \mathbb{K}(D))$ has a solution x_ε^* such that $\| x_\varepsilon^* \| \le \rho$. Considering the fact that f_ε satisfies property (iii), we have that

$$x_\varepsilon^* \in \{x \in D : \|x\| \le \rho\} = D_\rho.$$

If for any $n = 1, 2, \dots$ we take $\varepsilon = \frac{1}{n}$, we obtain a sequence $\left\{x_{\frac{1}{n}}^*\right\}_{n\in N}$ such that $x_{\frac{1}{n}}^* \in D_\rho$ and for any $n \in N$, $x_{\frac{1}{n}}^*$ is a solution to the problem $NCP(f_{\varepsilon_n}, \mathbb{K}(D))$. Because D is a closed locally compact pointed conical set, D_ρ is compact and hence the set $\left\{x^*_{\frac{1}{n}}\right\}_{n\in N}$ has a convergent subsequence $\left\{x^*_{\frac{1}{n_k}}\right\}_{k\in N}$. As a consequence, $\lim_{k\to\infty} x^*_{\frac{1}{n_k}}$ is an element of D. $\left(\mathbb{K}(D)\right)^* \subset D^*$, we obtain that x_* is a solution to the problem *NCP(f, D)* and $\|x_*\| \leq \rho$. The proof is complete. □

Remark. To include in the class of *EFE*-acceptable mappings the *regular completely continuous mappings* used in Theorem 7.2.6 and 7.2.7 it is necessary to introduce the following more general definition.

DEFINITION 7.2.1. *We say that a mapping $f: H \to H$ is REFE-acceptable with respect to $\mathbb{K}$ if either the problem NCP(f, $\mathbb{K}$) has a solution, or the mapping f has an EFE $\{x_r\}_{r>0} \subset \mathbb{K}$ with $\|x_r\| = r$, for any $r > 0$ (i.e., $\{x_r\}_{r>0}$ is a regular exceptional family of elements with respect to $\mathbb{K}$).*

This class of mappings was systematically used in (Isac, G. and Nemeth, S. Z. [6]), and many interesting results where obtained for nonlinear and linear complementarity problems.

7.3 A necessary and sufficient condition for the non-existence of an exceptional family of elements for a given mapping

In Chapters 5 and 6 we presented several sufficient conditions for the non-existence of *exceptional families of elements* for a given mapping. It is interesting to know if a *necessary* and *sufficient condition* for the non-existence of an exceptional family of elements exists. In this section we will show such a condition, which is due to G. Isac and S. Z. Németh. The proof of this result follows an idea proposed by S. Z. Németh in (Isac, G. and Nemeth, S. Z. [6]).

THEOREM 7.3.1. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K}\subset H$ a closed convex cone and $f: H\to H$ a mapping. A necessary and sufficient condition for the mapping f to be without an EFE with respect to $\mathbb{K}$ is the following:*

There exists a real number $\rho > 0$ such that for any $x\in\mathbb{K}$ with $\|x\|\geq\rho$ at least one of the following conditions holds:

(1) $\langle f(x),x\rangle\geq 0$,

(2) *there exist* $y\in\mathbb{K}$ *such that* $\|x\|^2\langle f(x),y\rangle<\langle x,y\rangle\langle f(x),x\rangle$.

Proof. First, we suppose that f is without an *EFE* with respect to $\mathbb{K}$. We prove that in this case at least one of conditions (1), (2) is satisfied. Now, we suppose to the contrary, that for any $r > 0$ there is an $x_r\in\mathbb{K}$ with $\|x\|\geq r$ such that the following conditions hold:

(i) $\langle f(x_r),x_r\rangle<0$,

(ii) $\|x_r\|^2\langle f(x_r),y\rangle\geq\langle x_r,y\rangle\langle f(x_r),x_r\rangle$, *for any* $y\in\mathbb{K}$.

We consider the real number

$$\mu_r=-\frac{\langle f(x_r),x_r\rangle}{\|x_r\|^2}.$$

Then by condition (i), $\mu_r > 0$. Let $u_r=\mu_r x_r+f(x_r)$. Then we have

$$\langle u_r,x_r\rangle=0. \tag{7.3.1}$$

Dividing condition (ii) by $\|x_r\|^2$ we have

$$\langle f(x_r),y\rangle\geq-\mu_r\langle x_r,y\rangle,\text{ for any } y\in\mathbb{K}.$$

Hence $\langle u_r,y\rangle\geq 0$ for any $y\in\mathbb{K}$, i.e., $u_r\in\mathbb{K}^*$. Since $\|x_r\|\geq r$ we have that $\|x_r\|\to+\infty$ *as* $r\to+\infty$. Therefore, the family $\{x_r\}_{r>0}$ is an exceptional family of elements for f with respect to $\mathbb{K}$.

Conversely, we suppose that at least one of conditions (1), (2) given in the theorem is satisfied and we prove that f is without an *EFE* with respect to $\mathbb{K}$. Indeed, we suppose the contrary, i.e., we suppose that

$\{x_r\}_{r>0} \subset \mathbb{K}$ is an *EFE* for f with respect to $\mathbb{K}$, with corresponding μ_r and u_r (as given in Definition 7.1.1). Because $\|x_r\| \to +\infty$ *as* $r \to +\infty$, there is an $r_0 > 0$ such that $\|x_{r_0}\| \geq \rho$. Since $u_{r_0} = \mu_{r_0} x_{r_0} + f(x_{r_0})$ and $\langle u_{r_0}, x_{r_0} \rangle = 0$, we have

$$0 < \mu_{r_0} = -\frac{\langle f(x_{r_0}), x_{r_0} \rangle}{\|x_{r_0}\|^2}.$$

Hence, $\langle f(x_{r_0}), x_{r_0} \rangle < 0$. Since $\|x_{r_0}\| \geq \rho$, the previous relation implies that condition (1) of the theorem is not satisfied. Hence, condition (2) of the theorem must hold. Because $\|x_{r_0}\| \geq \rho$, we must have

$$\|x_{r_0}\|^2 \langle f(x_{r_0}), y \rangle < \langle x_{r_0}, y \rangle \langle f(x_{r_0}), x_{r_0} \rangle, \textit{ for some } y \in \mathbb{K}. \quad (7.3.2)$$

Dividing (7.3.2) by $\|x_{r_0}\|^2$ we obtain that

$$\langle f(x_{r_0}), y \rangle < -\mu_r \langle x_{r_0}, y \rangle,$$

and therefore, $\langle u_{r_0}, y \rangle < 0$. Hence $u_{r_0} \notin \mathbb{K}^*$. But this contradicts condition (1) of Definition 7.1.1. We conclude that f is without an *EFE* with respect to $\mathbb{K}$. □

By Corollary 7.1.3 and Theorem 7.3.1 we obtain the following existence result.

THEOREM 7.3.2. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed convex cone and* $f: H \to H$ *an EFE-acceptable mapping. If there is a* $\rho > 0$ *such that for any* $x \in \mathbb{K}$ *with* $\|x\| \geq \rho$ *at least one of the following conditions holds:*

(1) $\langle f(x), x \rangle \geq 0$,

(2) *there exists an element* $y \in \mathbb{K}$ *such that*

$$\|x\|^2 \langle f(x), y \rangle < \langle x, y \rangle \langle f(x), x \rangle,$$

then the problem NCP(f, $\mathbb{K}$*) has a solution.*

From Theorem 7.3.2 we deduce the following consequence.

COROLLARY 7.3.3. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed convex cone and* $f: H \to H$ *an EFE-acceptable mapping. If there is a* $\rho > 0$ *such that for any* $x \in \mathbb{K}$ *with* $\|x\| \geq \rho$, *there is a* $y \in \mathbb{K}$ *such that* $\langle x, y \rangle \leq 0$ *and* $\langle f(x), y \rangle < 0$ *(or* $\langle x, y \rangle < 0$ *and* $\langle f(x), y \rangle \leq 0$, *then the problem NCP(f,* $\mathbb{K}$*) has a solution.*

Proof. Indeed, if $x \in \mathbb{K}$ is such that $\|x\| > \rho$, and $\langle f(x), x \rangle \geq 0$, then assumption (1) of Theorem 7.3.2 is satisfied. If $\langle f(x), x \rangle < 0$, then in this case, the assumptions of our corollary imply that assumption (2) of Theorem 7.3.2 is satisfied. Therefore the conclusion of our corollary follows from Theorem 7.3.2. □

Remark. Theorem 7.3.1 has many and interesting consequences presented in (Isac, G. and Németh, S. Z. [6]).

In this sense we give without proof the following interesting result. If $f: H \to H$ is a mapping, we define

$$\mathcal{O}(f)(x) = \|x\|^2 f(x) - \langle f(x), x \rangle x, \text{ for any } x \in H.$$

We say that $\mathcal{O}$ is the *orthogonalizer* of f and we have $\langle \mathcal{O}(f)(x), x \rangle = 0$ for all $x \in H$. If $\mathbb{K} \subset H$ is a closed convex cone, we say that a subset U of $\mathbb{K}$ is a *face* if it is a closed convex cone and if from $x \in U$, $y \in \mathbb{K}$ and $x - y \in \mathbb{K}$ it follows that $y \in U$.

THEOREM 7.3.4. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed convex cone,* $f: H \to H$ *a mapping and* $F = \mathcal{O}(f)$ *the orthogonalizer of f. Then the mapping f is without an EFE with respect to* $\mathbb{K}$ *if and only if there exists a* $\rho > 0$ *such that for any* $x \in \mathbb{K}$ *with* $\|x\| \geq \rho$ *we have:*

(1) *If* $x \in \operatorname{int}(\mathbb{K})$, *then exactly one of the following conditions holds:*
  (a) *x is not an eigenvector of f,*
  (b) *x is an eigenvector of f with nonnegative eigenvalue,*

(2) *If* $x \in \partial\mathbb{K}$, *then at least one of the following conditions holds:*

(a) $\langle f(x), x\rangle \geq 0$,

(b) *if V is the minimal face of K with respect to inclusion which contains x, then* $F(x) \notin V_\perp$, *where* $V_\perp \subset \mathbb{K}^*$ *is the orthogonal complementer face of V with respect to* $\mathbb{K}$, *i.e.,*

$$V_\perp = \{z \in \mathbb{K}^* : \langle x, z\rangle = 0 \text{ for all } x \in V\}$$

(*We can show that* $V_\perp$ *is a face of* $\mathbb{K}^*$.)

Proof. A proof of this result is given in (Isac, G. and Németh, S. Z. [6]). □

This theorem has many and very interesting consequences. We cite without proof only the following results.

THEOREM 7.3.5. *Let* $f = (f_1, f_2, ..., f_n) : \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous mapping. If there is a* $\rho > 0$ *such that for any* $x \in \mathbb{R}^n_+$ *with* $\|x\| \geq \rho$ *we have:*

(1) *If* $x \in \operatorname{int}(\mathbb{R}^n_+)$ *and x is an eigenvector of f, then its corresponding eigenvalue is nonnegative,*

(2) *if* $x = (x_1, x_2, ..., x_n) \in \partial\mathbb{R}^n_+$ *and* $\langle f(x), x\rangle < 0$, *then there exists* $i_0 \in \{1, 2, .., n\}$ *such that* $x_{i_0} = 0 \wedge f_{i_0}(x) \leq 0 \vee x_{i_0} > 0 \wedge f_{i_0}(x) \geq 0$, *where* $\wedge$ *and* $\vee$ *denotes the "logical and" and the "logical or" respectively,*

then the problem $NCP(f, \mathbb{R}^n_+)$ *has a solution.*

THEOREM 7.3.6. *Let f* = *A* + *b, where* $A : \mathbb{R}^n \to \mathbb{R}^n$ *is a linear mapping with entries* $a_{ij}, i, j \in \{1, 2, ..., n\}$ *with respect to the canonical basis of* $\mathbb{R}^n$ *and* $b = (b_1, b_2, ..., b_n)$ *is a nonzero constant vector. If there is a* $\rho > 0$ *such that for any* $x \in \mathbb{R}^n_+$ *with* $\|x\| \geq \rho$ *we have:*

(1) *if* λ *is not an eigenvalue of A and* $x = (A - \lambda I)^{-1} b \in \mathbb{R}^n_{++}$, *then* λ *is non negative,*

(2) *if* $x = (x_1, ..., x_n) \in \partial\mathbb{R}^n_+$ *and* $\sum_{i,j}^{n} a_{ij} x_j x_j + \sum_{i=1}^{n} b_i x_i < 0$, *then there exists* $i_0 \in \{1, 2, ..., n\}$ *such that*

$$x_{i_0} = 0 \wedge \sum_{j=1}^{n} a_{i_0 j} + b_{i_0} \leq 0 \ \vee x_{i_0} > 0 \wedge \sum_{j=1}^{n} a_{i_0 j} x_j + b_{i_0} \geq 0,$$

then the linear complementarity problem $LCP\left(A, b, \mathbb{R}_+^n\right)$ *has a solution.*

Proof. A proof of this result is in (Isac, G. and Nemeth, S. Z. [6]). □

For other results on this subject the reader is referred to (Isac, G. and Németh, S. Z. [6]).

7.4 Exceptional family of elements. Generalization to Banach spaces

For applications of complementarity theory to practical problems it is important to know if the notion of *EFE* can be extended from Hilbert spaces to Banach spaces. In this sense we give a generalization of the notion of *EFE* to *uniformly smooth and uniformly convex Banach space*. This generalization is obtained considering the "generalized projection operator" defined by Y. Alber presented in Chapters 1 and 2 and we use the notation and the terminology introduced in the cited chapters.

Let $(E, \|\cdot\|)$ be a *uniformly convex and uniformly smooth Banach space*. Let $f: E \to E^*$ be a mapping and $\Omega \subset E$ a closed convex set.

DEFINITION 7.4.1. *If* $x \in \Omega$ *is an arbitrary element, then the generalized normal cone of* Ω *at the point* x *is*

$$N_\Omega(x) = \left\{ y_* \in E^* : \langle y_*, u - x \rangle \leq 0, \text{ for all } u \in \Omega \right\}.$$

Remark. The generalized normal cone $N_\Omega(x)$ is a subset of the dual space E^*. If E is a Hilbert space, $\Omega \subset E$ is a closed convex cone and $x \in \Omega$, then in this case $N_\Omega(x)$ is the classical normal cone $N_\Omega(x) \subset E$ of the set Ω at the point x.

PROPOSITION 7.4.1. *An element* $y_0 \in \Omega$ *has the property that* $y_0 = \Pi_\Omega(y_*)$, *where* $y_* \in E^*$ *and* $\Pi_\Omega(\cdot)$ *is the generalized projection if and only if* $y_* \in J(y_0) + N_\Omega(y_0)$. *(J is the duality mapping.)*

Proof. Indeed, by Theorem 2.3.8 we have that $y_0 = \Pi_\Omega(y_*)$, if and only if, for any $u \in \Omega$ we have

$$\langle y_* - J(y_0), y_0 - u \rangle \geq 0,$$

or

$$\langle y_* - J(y_0), u - y_0 \rangle \leq 0 \text{ for all } u \in \Omega,$$

that is, $y_* - J(y_0) \in N_\Omega(y_0)$, i.e., $y_* \in J(y_0) + N_\Omega(y_0)$. So, the proposition is proved. □

Now, we suppose that $\Omega = \mathbb{K} \subset E$, where $\mathbb{K}$ is a closed convex cone.

DEFINITION 7.4.2. *We say that a mapping* $f: E \to E^*$ *is a J-completely continuous field, if f has a representation of the form* $f(x) = J(x) - T(x)$ *for all* $x \in E$, *where* $T: E \to E^*$ *is a completely continuous mapping.*

Now, we can define a notion of *EFE* for *J*-completely continuous fields.

DEFINITION 7.4.3. *We say that a family of elements* $\{x_r\} \subset \mathbb{K}$ *is an exceptional family of elements (EFE) for a J-completely continuous field* $f(x) = J(x) - T(x)$, *with respect to a closed convex cone* $\mathbb{K} \subset E$, *if and only if, for every real number* $r > 0$, *there exists a real number* $\mu_r > 1$ *such that*

(i) $\|x_r\| \to +\infty$ *as* $r \to +\infty$,

(ii) $T(x_r) - J(\mu_r x_r) \in N_{\mathbb{K}}(\mu_r x_r)$.

Remark. In the case of Hilbert space, the notion of *EFE* defined by Definition 7.4.3 is the notion of *EFE* defined by Definition 7.1.1. The notion of *EFE* used in this section will be in the sense of Definition 7.4.3.

We have the following result.

THEOREM 7.4.2. *Let* $(E, \|\cdot\|)$ *be a uniformly convex and uniformly smooth Banach space,* $\mathbb{K} \subset E$ *a closed convex cone and* $f: E \to E^*$ *a J-completely continuous field with the representation* $f(x) = J(x) - T(x)$. *Then there exists either a solution to the problem NCP(f,* $\mathbb{K}$*) or f has an EFE with respect to* $\mathbb{K}$ *(in the sense of Definition 7.4.3).*

Proof. Because the problem *HSVI(f,* $\mathbb{K}$*)* is equivalent to the problem *NCP(f,* $\mathbb{K}$*)*, by Theorem 2.3.9, we have that the problem *NCP(f,* $\mathbb{K}$*)* has a solution, if and only if the mapping

$$\Psi_{\mathbb{K}}(x) = \Pi_{\mathbb{K}}[J(x) - f(x)] = \Pi_{\mathbb{K}}[T(x)], \textit{ for all } x \in E,$$

has a fixed-point (which is obviously in $\mathbb{K}$). If $\Psi_{\mathbb{K}}$ has a fixed-point, the proof is completed.

Assume that the problem *NCP(f,* $\mathbb{K}$*)*, has no solution. Obviously, in this case the mapping $\Psi_{\mathbb{K}}$ is fixed-point free. We observe that $\Psi_{\mathbb{K}}$ satisfies the assumptions of Theorem 3.2.3 [Leray–Schauder alternative] with respect to each set $B_r = \{x \in E \,|\, \|x\| \le r\}$ with $r > 0$ (because T is completely continuous and $\Pi_{\mathbb{K}}$ is uniformly continuous on each bounded subset of the space). Then applying Theorem 3.2.3 to each set B_r, we obtain for each $r > 0$ that there exists $x_r \in \partial B_r$ and there is a real number $\lambda_r \in \,]0,1[$ such that $x_r = \lambda_r \Pi_{\mathbb{K}}(T(x_r))$ and we have that $x_r \in \mathbb{K}$ for each $r > 0$. From Proposition 7.4.1 we obtain that $T(x_r) \in J\left(\frac{x_r}{\lambda_r}\right) + N_{\mathbb{K}}\left(\frac{x_r}{\lambda_r}\right)$. Let $\mu_r = \frac{1}{\lambda_r}$ for all $r > 0$, then we obtain:

(a) $\|x_r\| = r$ *and* $\mu_r > 1$, *for all* $r > 0$,

(b) $\|x_r\| \to +\infty$ *as* $r \to +\infty$,

(c) $T(x_r) - J(\mu_r x_r) \in N_{\mathbb{K}}(\mu_r x_r)$,

and the conclusion of the theorem is achieved. □

Remark. A consequence of Theorem 7.4.2 is the fact that if we know that the J-completely continuous field $f: E \to E^*$ is without *EFE*, the *NCP*(f, $\mathbb{K}$) has a solution. Therefore it is interesting to have some conditions that imply the non-existence of an *EFE* for a given mapping.

Now, we give some results in this sense. First we show *condition* (θ) works also on Banach spaces. We give the definition of this condition for Banach spaces.

DEFINITION 7.4.4. *We say that a mapping $f: E \to E^*$ satisfies condition (θ) with respect to a closed convex cone $\mathbb{K} \subset E$ if there exists a real number $\rho > 0$ such that for each $x \in \mathbb{K}$ with $\|x\| > \rho$, there exists $y \in \mathbb{K}$ such that $\|y\| < \|x\|$ and $\langle f(x), x - y \rangle \geq 0$.*

THEOREM 7.4.3. *Let $(E, \|\cdot\|)$ be a uniformly convex and uniformly smooth Banach space. If $f: E \to E^*$ is a J-completely continuous field satisfying condition (θ) with respect to a closed convex cone $\mathbb{K} \subset E$, then f is without an EFE, with respect to $\mathbb{K}$, and the NCP(f, $\mathbb{K}$) has a solution.*

Proof. Suppose, by contradiction, that f has an *EFE* with respect to $\mathbb{K}$, namely $\{x_r\}_{r>0}$. Then for all $r > 0$, we have $\|x_r\| = r$, $\mu_r x_r \in \mathbb{K}$ with $\mu_r > 0$ and $J(x_r) - f(x_r) - J(\mu_r x_r) \in N_{\mathbb{K}}(\mu_r x_r)$, that is

$$\langle J(x_r) - f(x_r) - J(\mu_r x_r), y - \mu_r x_r \rangle \leq 0, \quad \textit{for all } y \in \mathbb{K}. \qquad (7.4.1)$$

Because f satisfies condition (θ), with respect to $\mathbb{K}$, we have that for any r sufficiently large, there exists $y_r \in \mathbb{K}$ such that $\|x_r\| > \rho$, $\|y_r\| < \|x_r\|$ and $\langle f(x_r), x_r - y_r \rangle \geq 0$.

Considering equation (7.4.1) and using the fact that the operator J is homogeneous we have:

$$0 \le \langle f(x_r), x_r - y_r \rangle$$
$$= \langle -J(x_r) + f(x_r) + J(\mu_r x_r) + J(x_r) - J(\mu_r x_r), x_r - y_r \rangle$$
$$= \langle -J(x_r) + f(x_r) + J(\mu_r x_r), x_r - y_r \rangle + \langle J(x_r) - J(\mu_r x_r), x_r - y_r \rangle$$
$$\le \langle J(x_r) - J(\mu_r x_r), x_r - y_r \rangle.$$

We used the following relation in the last inequality:

$$\mu_r \langle -J(x_r) + f(x_r) + J(\mu_r x_r), x_r - y_r \rangle$$
$$= \langle -J(x_r) + f(x_r) + J(\mu_r x_r), \mu x_r - \mu_r y_r \rangle$$
$$= \langle J(x_r) - f(x_r) - J(\mu_r x_r), \mu_r y_r - \mu_r x_r \rangle \le 0,$$

(because $\mu_r y_r \in \mathbb{K}$), which implies

$$\langle -J(x_r) + f(x_r) + J(\mu_r x_r), x_r - y_r \rangle \le 0 .$$

Therefore we have

$$0 \le \langle J(x_r) - J(\mu_r x_r), x_r - y_r \rangle = \langle J(x_r) - \mu_r J(x_r), x_r - y_r \rangle$$
$$= (1 - \mu_r) \langle J(x_r), x_r - y_r \rangle = (1 - \mu_r) \left(\|x_r\|^2 - \langle J(x_r), y_r \rangle \right)$$
$$= (1 - \mu_r) \|x_r\|^2 + (\mu_r - 1) \langle J(x_r), y_r \rangle$$
$$\le (1 - \mu_r) \|x_r\|^2 + (\mu_r - 1) \|J(x_r)\| \|y_r\|$$
$$= (1 - \mu_r) \|x_r\|^2 + (\mu_r - 1) \|x_r\| \|y_r\|$$
$$< (1 - \mu_r) \|x_r\|^2 + (\mu_r - 1) \|x_r\|^2$$
$$= 0$$

which is a contradiction. Hence f is without *EFE* with respect to $\mathbb{K}$. The conclusion of the theorem follows from Theorem 7.4.2, and the proof is completed. □

The following condition was considered in Chapter 5 in Hilbert spaces. Now, we give this condition for Banach spaces.

DEFINITION 7.4.5. *We say that a mapping $f: E \to E^*$ satisfies condition [DT] with respect to a closed convex cone $\mathbb{K} \subset E$ if there exist two bounded subsets D_0 and D_* in $\mathbb{K}$ such that for each $x \in \mathbb{K} \setminus D_*$, there is a $y \in conv(D_0 \cup \{x\})$ such that $\langle f(x), x - y \rangle > 0$.*

We have the following result.

PROPOSITION 7.4.4. *If $f: E \to E^*$ satisfies condition [DT] with respect to a closed convex cone $\mathbb{K} \subset E$, then f satisfies condition (θ).*

Proof. Because D_0 and D_* are bounded subsets in $\mathbb{K}$, there exists a real number $\rho > 0$ such that $D_0, D_* \subset (\overline{B}_\rho) \cap \mathbb{K}$. If $x \in \mathbb{K}$ is such that $\|x\| > \rho$, then by condition [DT], there is an element $y \in conv(D_0 \cup \{x\})$ such that $\langle f(x), x - y \rangle > 0$. We have $y = \lambda x_0 + (1-\lambda)x$ with $\lambda \in [0, 1]$ and $x_0 \in D_0$, which implies

$$\|y\| \leq \lambda \|x_0\| + (1-\lambda)\|x\| < \lambda \|x\| + (1-\lambda)\|x\| = \|x\|.$$

Therefore, f satisfies condition (θ) with respect to $\mathbb{K}$. □

DEFINITION 7.4.6. *Let $f, g : E \to E^*$ be two mappings. We say that the mapping f is asymptotically g-pseudomonotone with respect to a closed convex cone $\mathbb{K} \subset E$, if there exists a real number $\rho > 0$ such that for all $x, y \in \mathbb{K}$ with $\max\{\|y\|, \rho\} < \|x\|$, we have*

$$\langle g(y), x - y \rangle \geq 0 \ \textit{implies} \ \langle f(x), x - y \rangle \geq 0.$$

This notion implies the following result.

THEOREM 7.4.5. *Let $(E, \|\cdot\|)$ be a uniformly convex and uniformly smooth Banach space, $\mathbb{K} \subset E$ an arbitrary closed convex cone and f, g : $E \to E^*$ mappings such that f is a J-completely continuous field. If f is asymptotically g-pseudomonotone with respect to $\mathbb{K}$ and the problem NCP(g, $\mathbb{K}$) has a solution, then f is without an EFE with respect to $\mathbb{K}$ and the NCP(f, $\mathbb{K}$) has a solution.*

Proof. Let x_* be a solution to the problem *NCP*(*g*, $\mathbb{K}$). Then for all $y \in \mathbb{K}$, we have $\langle g(x_*), y - x_* \rangle \geq 0$. Since f is asymptotically *g*-pseudomonotone with respect to $\mathbb{K}$, there exists a real number $\rho > 0$ such that for all $x, y \in \mathbb{K}$ with $\max\{\|y\|, \rho\} < \|x\|$ we have that $\langle g(y), x - y\rangle \geq 0$ implies $\langle f(x), x - y\rangle \geq 0$. Take $\rho_0 = \max\{\|x_*\| + 1, \rho + 1\}$. Then for any $x \in \mathbb{K}$ with $\|x\| > \rho_0$, we may take $x_* \in \mathbb{K}$. Because $\|x_*\| < \|x\|$ and $\langle g(x_*), x - x_*\rangle \geq 0$, we have that $\langle f(x), x - x_*\rangle \geq 0$, that is, f satisfies condition (θ) with respect to $\mathbb{K}$. Now applying Theorem 7.4.3 the conclusion of this theorem is achieved. □

Remark. If in Definition 7.4.6 we take $g = f$, we have that f is asymptotically pseudomonotone, with respect to $\mathbb{K}$. Obviously, if f is monotone it is asymptotically pseudomonotone, but the converse is not true.

The following result follows from Theorem 7.4.5.

COROLLARY. 7.4.6. *Let $(E, \|\cdot\|)$ be a uniformly convex and uniformly smooth Banach space, $\mathbb{K} \subset E$ a closed convex cone and $f : E \to E^*$ a J-completely continuous field. If f is asymptotically pseudomonotone with respect to $\mathbb{K}$, then the problem NCP(f, $\mathbb{K}$) has a solution, if and only if, f is without an EFE with respect to $\mathbb{K}$.*

Remarks.

(1) The results presented in this section are due to G. Isac and J. Li and can be found in (Isac, G. and Li, J. [3])

(2) The subject of this section may be a starting point for new developments related to the notion of *EFE* and its applications to the study of complementarity problems in Banach spaces.

8

EXCEPTIONAL FAMILY OF ELEMENTS AND VARIATIONAL INEQUALITIES

In the first chapters of this book we noted that there exist deep relations between complementarity problems and variational inequalities. Considering this fact it is natural to extend the notion of *EFE* and the method based on this notion, from complementarity problems to variational inequalities. In my lectures given in 1996 at the Institute of Applied Mathematics of Academia Sinica (China) I presented the problem to do this extension. The first work dedicated to this extension was the PhD thesis presented by Y. B. Zhao in 1998, [2].

The results explored in this chapter represent the development of this subject until now. See the papers: (Bianchi, M, Hadjisavvas, N. and Schaible, S.[1]), (Isac, G. and Cojocaru, M. G. [2]), (Isac, G. and Motreanu, D. [1]), (Isac, G. and Zhao, Y. B. [1]), (Zhao, Y. B. [1], [4]), (Zhao, Y. B. and Han, J. Y. [1],), (Zhao, Y. B., Han, J. Y. and Qi, H. D. [1]), (Zhao, Y. B. and Li. D. [1]), (Zhao, Y. B. and Sun, D. [1]).

In the papers cited above the reader can find other results, which are not presented in this chapter. We note that, to extend the notion of *EFE* from complementarity problems to variational inequalities, we can follow two ways: one is to use the (explicit) Leray–Schauder alternatives and another is to use the implicit Leray–Schauder alternative.

8.1. Explicit Leray–Schauder type alternatives and variational inequalities

As we noted in the introduction of this chapter, the first extension of the notion of *EFE* from complementarity problems to variational inequalities was realized in the PhD thesis (Zhao, Y. B. [2]). In this thesis Y. B. Zhao considered variational inequalities in the n-dimensional Euclidean space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, with respect to unbounded closed convex sets, defined by inequalities and equalities (considered as constraints). The constraints are defined by continuously differentiable functions. We note that the notions of *EFE* are obtained using the topological degree and the classical optimality conditions. The notions of *EFE* obtained by this method have a long expression. Consequently it is hard to obtain existence theorems for variational inequalities using these notions. Moreover, the generalization to infinite dimensional Hilbert spaces of this notion is not so easy, even impossible.

To pass over these difficulties we use the *normal cone,* which can be associated to any closed convex set, and we replace the topological degree by the *Leray–Schauder Alternative.* In this way we obtain an elegant and simple method, as it will be developed in this chapter. However, to inform the reader about the method developed in (Zhao, Y. B. [2]) we give a few notions and results, due to Y. B. Zhao.

Let $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ the n-dimensional Euclidean space and $\Omega \subset \mathbb{R}^n$ a non-empty unbounded closed convex set. We suppose that Ω is defined by:

$$\Omega = \{x \in \mathbb{R}^n : E(x) \leq 0, H(x) = 0\},$$

where $E : \mathbb{R}^n \to \mathbb{R}^m$ and $H : \mathbb{R}^n \to \mathbb{R}^l$ are continuous and differentiable functions. The components $E_i(x)\,(i = 1, 2, \ldots, m)$ are convex functions and $H_j\,(j = 1, 2, .., l)$ are linear functions. Let f, $g : \mathbb{R}^n \to \mathbb{R}^n$ be two mappings. We consider the following finite-dimensional generalized variational inequality

$$IVI(f, g, \Omega) : \begin{cases} \textit{find } x_* \in \mathbb{R}^n \textit{ such that} \\ g(x_*) \in \Omega \textit{ and} \\ \langle f(x_*), x - g(x_*) \rangle \geq 0 \textit{ for all } x \in \Omega. \end{cases}$$

We know that this variational inequality contains as particular cases the classical (Hartman–Stampacchia) variational inequality and the nonlinear complementarity problem. We know also that the solvability of this problem is equivalent with the solvability of the nonlinear equation

$$g(x) - P_\Omega(g(x) - f(x)) = 0\,.$$

DEFINITION 8.1.1. *Let f and g be continuous mappings of* $\mathbb{R}^n$ *into* $\mathbb{R}^n$. *Let* x_0 *be an arbitrary element in* $\mathbb{R}^n$. *A family* $\{x_r\}_{r>0} \subset \mathbb{R}^n$ *is said to be an EFE with respect to* x_0, Ω, *and the pair of mappings* (f, g) *if the following conditions are satisfied:*

(1) $\|x_r\| \to +\infty$ *as* $r \to +\infty$,

(2) *for each* x_r *there exist some vectors* $\lambda_r \in \mathbb{R}^m_+$, $\mu_r \in \mathbb{R}^l$ *and some scalar* $\alpha_r > 1$ *such that* $e(x_r, \alpha_r) = \alpha_r g(x_r) + (1-\alpha_r) g(x_0) \in \Omega$, *and the following two equations hold*:

$$f(x_r) = -(\alpha_r - 1)(g(x_r) - g(x_0)),$$

$$-\frac{1}{2}\left[\nabla E(e(x_r,\alpha_r))^T \lambda_r + \nabla H(e(x_r,\alpha_r))^T \mu_r\right]$$

$$\cdot(\lambda_r)_i E_i(e(x_r,\alpha_r)) = 0, \quad i = 1,2,\ldots,m.$$

THEOREM 8.1.1. *Let f, g :* $\mathbb{R}^n \to \mathbb{R}^n$ *be two mappings. The mapping g is supposed to be one-to-one (injective). Let* x_0 *be an arbitrary element in* $\mathbb{R}^n$. *Then the problem IVI(f, g,* Ω*) has either a solution or an EFE with respect to* x_0 *and* Ω *(in the sense of Definition 8.1.1).*

Proof. Let $\Phi(x) = g(x) - P_\Omega(g(x) - f(x))$, for any $x \in \mathbb{R}^n$. It is well known that x_* solves the problem *IVI(f, g,* Ω) if and only if $\Phi(x_*) = 0$ .We consider the following homotopy between the mappings $g(x) - g(x_0)$ and $\Phi(x)$:

$$\mathcal{H}(x,t) = t\left[g(x) - g(x_0)\right] + (1-t)\Phi(x), \quad t \in [0,1].$$

We denote

$$B_r = \{x \in \mathbb{R}^n : \ \|x - x_0\| < r\},$$

$$\partial B_r = \{x \in \mathbb{R}^n : \ \|x - x_0\| = r\}.$$

Obviously, ∂B_r is the boundary of B_r. Two cases are possible.

(I) There exists an $r > 0$ such that

$$0 \notin \{\mathcal{H}(x,t) : x \in \partial B_r \text{ and } t \in [0,1]\}.$$

(II) For each $r > 0$, there exist some point $x_r \in \partial B_r$ and $t_r \in [0, 1]$ such that

$$0 = \mathcal{H}(x_r, t_r) = t_r\left[g(x_r) - g(x_0)\right] + (1-t_r)\Phi(x_r). \qquad (8.1.1)$$

If the case (I) holds, then we have

$$\deg\big(g(x)-g(x_0),B_r,0\big)=\deg\big(\Phi(x),B_r,0\big). \qquad (8.1.2)$$

Since g is one-to-one in $\mathbb{R}^n$ we have $\left|\deg\big(g(x)-g(x_0),B_r,0\big)\right|=1$. [See (Lloyd, N. G. [1])]. Then, in this case the equation $\Phi(x) = 0$ has at least a solution.

Now, we suppose that the case (II) holds. When $t_r = 0$ (8.1.1) reduces to $\Phi(x_r) = 0$. Hence, x_r is a solution to the problem $IVI(f, g, \Omega)$. If $t_r = 1$, from (8.1.1) we have $g(x_r)=g(x_0)$ which implies that $x_r = x_0$ (because g is one-to-one). This fact is impossible since $x_r \in \partial B_r$. Therefore, it suffices to consider the case that $t_r \in\ ]0, 1[$ for each $r > 1$. From the definition of Φ and the relation (8.1.1) we have

$$\frac{1}{1-t_r}g(x_r)-\frac{t_r}{1-t_r}g(x_0)=P_\Omega\big(g(x_r)-f(x_r)\big)\in\Omega. \qquad (8.1.3)$$

Let $\alpha_r=\dfrac{1}{1-t_r}$. Denote

$$e(x_r,\alpha_r)=\alpha_r g(x_r)+(1-\alpha_r)g(x_0)=\frac{1}{1-t_r}g(x_r)-\frac{t_r}{1-t_r}g(x_0).$$

By (8.1.3) and the properties of the projection operator P_Ω, $e(x_r, \alpha_r)$ is the unique solution to the following optimization problem (whose solution is completely characterized by the Karus–Kuhn–Tucker optimality conditions):

$$\begin{cases}\text{minimize } Q(x)\left(=\left\|x-\left[g(x_r)-f(x_r)\right]\right\|^2\right),\\ s.t. \quad x\in\Omega.\end{cases}$$

Consequently, there exist two vectors $\lambda_r \in \mathbb{R}^m$ and $\mu_r \in \mathbb{R}^l$ such that the following equations hold:

$$\begin{cases}\nabla Q\big(e(x_r,\alpha_r)\big)+\nabla E\big(e(x_r,\alpha_r)\big)^T\lambda_r+\nabla H\big(e(x_r,\alpha_r)\big)^T\mu_r=0,\\ (\lambda_r)_i\,E_i\big(e(x_r,\alpha_r)\big)=0, \quad i=1,2,\dots,m.\end{cases}$$

We note that $\nabla Q(x)=2\left[x-\big(g(x_r)-f(x_r)\big)\right]$ is the gradient of $Q(x)$. Rearranging the terms in the last equations, we obtain:

$$\begin{cases} f(x_r) = -(\alpha_r - 1)(g(x_r) - g(x_0)) \\ -\dfrac{1}{2}\left[\nabla E(e(x_r,\alpha_r))^T \lambda_r + \nabla H(e(x_r,\alpha_r))^T \mu_r\right], \\ (\lambda_r)_i E_i(e(x_r,\alpha_r)) = 0, \quad i = 1,2,\ldots,m. \end{cases}$$

Obviously, because $\|x_r - x_0\| = r$, we have that, $\|x_r\| \to +\infty \; as \; r \to +\infty$ and $\{x_r\}_{r>0}$ is an *EFE* with respect to x_0 and Ω for the pair (f, g) in the sense of Definition 8.1.1. □

In Theorem 8.1.1 the assumption that g is one-to-one is a strong condition. To eliminate this assumption we introduce another notion of *EFE.*

DEFINITION 8.1.2. *Let f and g be continuous mappings from* $\mathbb{R}^n$ into $\mathbb{R}^n$. *Let* x_0 *be an arbitrary element in* $\mathbb{R}^n$. *We say that a family of elements* $\{x_r\}_{r>0} \subset \mathbb{R}^n$ *is said to be an EFE with respect to* x_0, Ω *and the pair* (f, g) *if the following conditions are satisfied:*

(1) $\|x_r\| \to +\infty \; as \; r \to +\infty$,

(2) *for each* x_r, *there exist some vectors* $\lambda_r \in \mathbb{R}^m_+$, $\mu_r \in \mathbb{R}^l_+$ *and a scalar* $\alpha_r > 1$ *such that* $c(x_r,\alpha_r) = (\alpha_r - 1)(x_r - x_0) + g(x_r) \in \mathbb{K}$, *and the following two equations hold*

$$\begin{cases} f(x_r) = -(\alpha_r - 1)(x_r - x_0) \\ -\dfrac{1}{2}\left[\nabla E(c(x_r,\alpha_r))^T \lambda_r + \nabla H(c(x_r,\alpha_r))^T \mu_r\right], \\ (\lambda_r)_i E_i(c(x_r,\alpha_r)) = 0, \quad i = 1,2,\ldots,m. \end{cases}$$

We have the following result.

THEOREM 8.1.2. *Let* $f, g : \mathbb{R}^n \to \mathbb{R}^n$ *be two continuous mappings and let* $x_0 \in \mathbb{R}^n$ *be an arbitrary element. Then either the problem* $IVI(f, g, \Omega)$ *has a solution or there exists an EFE (in the sense of Definition 8.1.2) with respect to* x_0 *and* Ω.

Proof. The proof is similar to the proof of Theorem 8.1.1, but using the homotopy, $\mathcal{H}(x,t) = t(x - x_0) + (1-t)\Phi(x)$, $t \in [0,1]$. For more details the reader is referred to (Zhao, Y. B. [2]). □

From Theorem 8.1.1 and 8.1.2 we deduce the following result.

THEOREM 8.1.3. *Let f, g : $\mathbb{R}^n \to \mathbb{R}^n$ be two continuous mappings.*

(i) *If there exists an element $x_0 \in \mathbb{R}^n$ such that the pair (f, g) is without an EFE (Definition 8.1.2) with respect to x_0 and Ω, then the problem IVI(f, g, Ω) has a solution.*

(ii) *If the mapping g is one-to-one and there exists an element $x_0 \in \mathbb{R}^n$ such that the pair (f, g) is without an EFE (Definition 8.1.1) with respect to x_0 and Ω, then the problem IVI(f, g, Ω) has a solution.*

Motivated by the results presented above it is of interest to know conditions that guarantee that a pair of mappings (*f*, *g*) is without *EFE* with respect to an element $x_0 \in \mathbb{R}^n$ and to a set Ω defined as above. Several conditions in this sense are given in (Zhao, Y. B. [2]).

Now, we cite without proof only the following condition.

THEOREM 8.1.4. *Let f, g : $\mathbb{R}^n \to \mathbb{R}^n$ be two continuous mappings. The mapping g is supposed to be one-to-one. If there exists a point $x_0 \in g^{-1}(\Omega)$ such that, for each family $\{x_r\}_{r>0} \subset \mathbb{R}^n$ with $\|x_r\| \to +\infty$ as $r \to +\infty$ and $\{g(x_r)\}_{r>0} \subset \Omega$, there is an element $x_r \neq x_0$ such that*

$$\langle f(x_r), g(x_r) - g(x_0) \rangle \geq 0,$$

then the couple (f, g) is without an EFE (in the sense of Definition 8.1.1) with respect to x_0 and Ω.

Proof. A proof of this result is given in (Zhao, Y. B. [2]). □

Other similar results and other kinds of *EFE* based on optimality conditions are presented in (Zhao, Y. B. [2]) and in the papers: (Zhao, Y. B. [1], [4]), (Zhao, Y. B. and Han. J. Y. [1]), (Zhao, Y. B., Han, J. Y. and Qi, H. D. [1]), (Zhao, Y. B. and Li, D. [1]) and (Zhao, Y. B. and Sun, D. [1]).

Now, we replace the method developed by Y. B. Zhao, by our method based on Leray–Schauder alternatives. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\Omega \subset H$ a non-empty unbounded closed convex set. We denote by P_Ω the projection operator onto Ω (which is well defined) and for any real number $r > 0$ we denote $\overline{B_r} = \{x \in H : \|x\| \leq r\}$. If $x \in \Omega$, we recall that the *normal cone* of Ω at the point x is

$$N_\Omega(x) = \{\xi \in H : \langle \xi, y - x \rangle \leq 0, \textit{for all } y \in \Omega\}$$

or $N_\Omega(x) = -[T_\Omega(x)]^*$, where $T_\Omega(x)$ is the tangent cone of Ω at the point x, i.e., $T_\Omega(x) = \overline{\bigcup_{\lambda>0} \lambda(\Omega - x)}$.

Given mapping $f : H \to H$, we consider the following classical *variational inequality* defined by f and Ω.

$$VI(f,\Omega): \begin{cases} \text{find } x_* \in \Omega \text{ such that} \\ \langle x - x_*, f(x_*) \rangle \geq 0, \text{ for all } x \in \Omega. \end{cases}$$

DEFINITION 8.1.3. *We say that* $\{x_r\}_{r>0} \subset H$ *is an exceptional family of elements EFE for a completely continuous field* $f(x) = x - T(x)$ *defined on* H, *with respect to the subset* Ω, *if the following conditions are satisfied:*

(1) $\|x_r\| \to +\infty$ *as* $r \to +\infty$,

(2) *for any* $r > 0$ *there exists a real number* $\mu_r > 1$ *such that* $\mu_r x_r \in \Omega$ *and* $T(x_r) - \mu_r x_r \in N_\Omega(\mu_r x_r)$.

With respect to this notion we have the following result.

THEOREM 8.1.5. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\Omega \subset H$ *an arbitrary unbounded closed convex set and* $f : H \to H$ *a completely continuous field with a representation of the form* $f(x) = x - T(x)$, *where* $T : H \to H$ *is a completely continuous mapping(linear or nonlinear). Then the problem* $VI(f, \Omega)$ *has at least one of the following two properties*:

(1) $VI(f, \Omega)$ *has a solution,*

(2) *The completely continuous field f has an EFE with respect to* Ω (*in the sense of Definition 8.1.3*)

Proof. We know that the problem $VI(f, \Omega)$ has a solution if and only if the mapping $,\Phi(x) = P_\Omega[x - f(x)] = P_\Omega(T(x))$, $x \in H$, has a fixed-point (in H). Obviously, this fixed-point must be in Ω. We observe that the mapping Φ is completely continuous. The set $\overline{B_r}$ has a non-empty interior and $0 \in \operatorname{int}(\overline{B_r})$. Only two situations are possible:

(I) The problem $VI(f, \Omega)$ has a solution. In this case, the proof is complete.

(II) The problem $VI(f, \Omega)$ has no solution. In this case the mapping Φ is fixed-point free with respect to any set $\overline{B_r}$, because if Φ has a fixed-point in $\overline{B_r}$ we have that the problem $VI(f, \Omega)$ has a solution, which is a contradiction. Now, since $\overline{B_r}$ is bounded and Φ is completely continuous, we have that Φ restricted to $\overline{B_r}$ is a compact continuous mapping. The assumptions of Theorem 3.2.4 are satisfied. Therefore, there is an element $x_r \in \partial \overline{B_r}$ such that $x_r = \lambda_r P_\Omega(T(x_r))$ for some $\lambda_r \in [0, 1[$. We know that for each $x \in H$, $y = P_\Omega(x)$ if and only if $x \in y + N_\Omega(y)$. By using this result we have that

$$T(x_r) \in \left(\frac{1}{\lambda_r}\right)x_r + N_\Omega\left(\left(\frac{1}{\lambda_r}\right)x_r\right).$$

If we denote $\mu_r = \dfrac{1}{\lambda_r}$ for any $r > 0$, then we obtain

(i) $\|x_r\| = r$ and $\mu_r > 1$ *for* any $r > 0$,

(ii) $\mu_r x_r \in \Omega$ for any $r > 0$,

(iii) $T(x_r) - \mu_r x_r \in N_\Omega(\mu_r x_r)$ for any $r > 0$.

Since $\|x_r\| \to +\infty$ *as* $r \to +\infty$, we deduce that $\{x_r\}_{r>0}$ is an exceptional family of elements for f with respect to Ω (in the sense of Definition 8.1.3).

□

COROLLARY 8.1.6. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\Omega \subset H$ *an arbitrary unbounded closed convex set and* $f(x) = x - T(x)$ *a completely continuous field on H. If f is without an EFE with respect to* Ω *(in the sense of Definition 8.1.3), then the problem* $VI(f, \Omega)$ *has a solution.*

Theorem 8.1.5 can be extended to variational inequalities for set-valued mappings in the following manner. Let $h : H \to H$ be a set-valued mapping and $\Omega \subset H$ a non-empty unbounded closed convex set. We consider the problem

$$MVI(h,\Omega): \begin{cases} \text{find } (x_*, y_*) \in \Omega \times H \text{ such that} \\ y_* \in f(x_*) \text{ and } \langle u - x_*, y_* \rangle \geq 0, \text{ for all } u \in \Omega. \end{cases}$$

We know that the problem $MVI(h, \Omega)$ has a solution if and only if the set-valued mapping $P_\Omega[x - h(x)]$ has a fixed-point in H; i.e., there exists an element $x_* \in H$ such that $x_* \in P_\Omega[x_* - h(x_*)]$. In this case there exists $y_* \in h(x_*)$ such that $x_* \in P_\Omega[x_* - y_*]$, which implies that (x_*, y_*) is a solution to the problem $MVI(h, \Omega)$. Now, we suppose that $f : H \to H$ is a set-valued mapping of the form $f(x) = x - T(x)$, where $T : H \to H$ is a set-valued mapping. We introduce the following definition.

DEFINITION 8.1.4. *We say that* $\{x_r\}_{r>0} \subset H$ *is an EFE for the set-valued mapping* $f(x) = x - T(x)$, *with respect to the subset* Ω, *if the following conditions are satisfied:*

(1) $\|x_r\| \to +\infty$ *as* $r \to +\infty$,

(2) *for any* $r > 0$ *there exists a real number* $\mu_r > 1$ *and an element* $y_r \in T(x_r)$ *such that* $\mu_r x_r \in \Omega$, *and* $y_r - \mu_r x_r \in N_\Omega(\mu_r x_r)$.

We have the following result.

THEOREM 8.1.7. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\Omega \subset H$ *an arbitrary unbounded closed convex set and* $f : H \to H$ *a completely upper semicontinuous field with a representation of the form* $f(x) = x - T(x)$, *where* $T : H \to H$ *is a completely upper semicontinuous set-valued mapping with non-empty compact contractible values. Then the problem* $MVI(f, \Omega)$ *has at least one of the following two properties:*

(1) $MVI(f, \Omega)$ *has a solution,*

(2) *the completely upper semi-continuous field* f *has an EFE with respect to* Ω *(in the sense of Definition 8.1.4).*

Proof. The proof is similar to the proof of Theorem 8.1.5. We consider the set-valued mapping $\Phi(x) = P_\Omega[x - f(x)] = P_\Omega(T(x))$. We can show that $P_\Omega(T(x))$ is a set-valued mapping with compact contractible values, all the assumptions of Theorem 3.6.6 are satisfied and the proof follows the proof of Theorem 8.1.5. □

A consequence of Theorem 8.1.7 is the following result.

COROLLARY 8.1.8. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\Omega \subset H$ *an arbitrary unbounded closed convex set. Let* $f(x) = x - T(x)$ *be a completely upper semi-continuous field, where* $T : H \to H$ *is with non-empty compact contractible values. If* f *is without an EFE with respect to* Ω *(in the sense of Definition 8.1.4), then the problem* $MVI(f, \Omega)$ *has a solution.*

For other results related to the problem $VI(f, \Omega)$ the reader is referred to (Isac, G. and Zhao, Y. B. [1]). In the cited paper are also presented other results related to the problem $VI(f, \Omega)$, where the set Ω is defined by

$$\Omega = \{x \in H : g_i(x) \leq 0, \ldots, g_m(x) \leq 0\},$$

where $g_1, g_2, \ldots, g_m : H \to \mathbb{R}$ are continuous real-valued convex functions.

Remark. Theorem 8.1.5 is valid for k-set fields, that is for mappings with a representation of the form $f(x) = x - T(x)$, where $T : H \to H$ is a *k-set contraction* with $0 \leq k < 1$ (see Chapter 1). In this case the proof is based on the notion of $(0, k)$-epi mapping. For this extension the reader is referred to (Isac, G. and Cojocaru, M. G. [2]).

Now, we present several classes of mappings without *EFE* in the sense of Definition 8.1.3.

DEFINITION 8.1.5. *We say that a mapping $f : H \to H$ satisfies condition (θ, Ω) with respect to an unbounded closed convex set $\Omega \subset H$ if there exists $\rho > 0$ such that for each couple (x, α) with $\|x\| > \rho$, $\alpha \geq 1$ and $\alpha x \in \Omega$, there exists $y \in \Omega$ such that $\|y\| < \alpha\|x\|$ and $\langle f(x), \alpha x - y \rangle \geq 0$.*

If Ω is a closed convex cone, then in this case condition (θ, Ω) is equivalent to condition (θ) used in the study of complementarity problems (Isac, G. and Cojocaru, M. G. [2]). We recall that a mapping $f : H \to H$ is ρ-*copositive* on Ω if there exists $\rho > 0$ such that for all $x \in \Omega$, with $\|x\| > \rho$ we have $\langle x, f(x) \rangle \geq 0$.

PROPOSITION 8.1.9. *If $f : H \to H$ is ρ-copositive on Ω and there exists $x_* \in \Omega$ such that $\|x_*\| < \rho$ and $\langle x_*, f(x) \rangle \leq 0$ for all $x \in H$ with $\alpha x \in \Omega$ for $\alpha > 1$ and $\|x\| > \rho$, then f satisfies condition (θ, Ω).*

Proof. Indeed, if $x \in H$ is such that $\|x\| > \rho$ and $\alpha x \in \Omega$ for $\alpha \geq 1$, then we have $\langle x, f(x) \rangle \geq 0$. Since $\|x_*\| < \rho \leq \|\alpha x\|$ and $\langle \alpha x - x_*, f(x) \rangle \geq 0$, we have that f satisfies condition (θ, Ω). □

COROLLARY 8.1.10. *If* $f: H \to H$ *is* ρ*-copositive on* Ω *and* $0 \in \Omega$, *then* f *satisfies condition* (θ, Ω).

DEFINITION 8.1.6. *We say that* $f: H \to H$ *satisfies condition* $(\mathbb{K})$ *with respect to* Ω *if there exists a bounded set* $D \subset \Omega$ *such that for all couples* (x, α) *with* $x \in H$, $\alpha \geq 1$ *and* $\alpha x \in \Omega \setminus D$ *there exists* $y \in D$ *such that* $\langle \alpha x - y, f(x) \rangle \geq 0$.

Remark. Condition $(\mathbb{K})$ *is a Karamardian type condition.*

PROPOSITION 8.1.11. *If* $f: H \to H$ *satisfies condition* $(\mathbb{K})$ *with respect to* Ω, *then* f *satisfies condition* (θ, Ω).

Proof. Let $D \subset \Omega$ be the set defined by condition $(\mathbb{K})$. Since D is bounded there exists $\rho > 0$ such that $D \subset \{x \in \Omega \mid \|x\| \leq \rho\}$. For each couple (x, α) where $x \in H$, $\alpha \geq 1$ and $\alpha x \in \Omega$, we have $\|\alpha x\| \geq \|x\| > \rho$, which implies $\alpha x \in \Omega \setminus D$ and there exists $y \in D$ such that $\langle \alpha x - y, f(x) \rangle \geq 0$. Because $\|y\| \leq \rho < \alpha \|x\|$, we have that f satisfies condition (θ, Ω) on Ω. □

DEFINITION 8.1.7. *Let* $f, g: H \to H$ *be two mappings. We say that* f *is asymptotically strongly g-demimonotone with respect to* Ω *if there exist a mapping* $\phi: \mathbb{R}_+ \to \mathbb{R}_+$, *an element* $u \in \Omega$ *and a real number* $\rho > 0$ *such that* $\lim_{t \to +\infty} \phi(t) = +\infty$ *and*

(i) *for each couple* (x, α) *with* $x \in H$, $\|x\| > \rho$, $\alpha \geq 1$ *and* $\alpha x \in \Omega$ *we have* $\langle \alpha x - u, f(x) - g(u) \rangle \geq \|\alpha x - u\| \phi(\|\alpha x - u\|)$.

PROPOSITION 8.1.12. *If* $f: H \to H$ *is asymptotically strongly g-demimonotone with respect to* Ω, *then* f *satisfies condition* (θ, Ω).

Proof. Assume that f is asymptotically strongly g-demimonotone. For each couple (x, α) with $x \in H$, $\alpha \geq 1$, $\alpha x \in \Omega$ and $\|x\| > \max\{\rho, \|u\|\}$ we have $\|u\| < \alpha \|x\|$ and $\langle \alpha x - u, f(x) - g(u) \rangle \geq \|\alpha x - u\| \phi(\|\alpha x - u\|)$, which implies $\langle \alpha x - u, f(x)\rangle \geq \|\alpha x - u\| \left[\left\langle \frac{\alpha x - u}{\|\alpha x - u\|}, g(u) \right\rangle + \phi(\|\alpha x - u\|) \right]$

(because $\alpha\|x\| > \|u\|$ implies $\|\alpha x - u\| > 0$). Since $S_1 = \{x \in H : \|x\| = 1\}$ is bounded and considering for u fixed $g(u)$ as a continuous linear functional on H, we deduce that there exists $\gamma \in \mathbb{R}$ such that $\left\langle \frac{\alpha x - u}{\|\alpha x - u\|}, g(u) \right\rangle \geq \gamma$, for each pair (x, α) with $x \in H$, $\alpha \geq 1$, $\alpha x \in \Omega$ and $\|x\| > \max\{\rho, \|u\|\}$. Since Ω is unbounded there exist pairs (x, α) such that $x \in H$, $\alpha \geq 1$, $\alpha x \in \Omega$, $\|x\| > \max\{\rho, \|u\|\}$ and $\|\alpha x - u\| \to +\infty$ as $\|x\| \to +\infty$. Because $\lim_{t\to+\infty} \phi(t) = +\infty$ we have that there exists ρ_* such that for all pairs (x, α) with $\alpha \geq 1$, $\alpha x \in \Omega$, $\|x\| > \max\{\rho, \|u\|\}$ and $\|\alpha x - u\| > \rho_*$ we have $\phi(\|\alpha x - u\|) \geq -\gamma$, that is $\langle \alpha x - u, f(x)\rangle \geq 0$. If for any pair (x, α) with $\alpha \geq 1$, $\alpha x \in \Omega$ and $\|x\| > \max\{\rho, \|u\|\}$ we take $y = u$, we have that f satisfies condition (θ, Ω). □

DEFINITION 8.1.8. *We say that $f : H \to H$ is scalarly increasing to infinity on Ω, if for each $y \in \Omega$ there exists a real number $\rho(y) > 0$ such that for all couples (x, α) with $x \in H$, $\alpha \geq 1$, $\alpha x \in \Omega$ and $\|x\| > \rho(y)$ we have $\langle \alpha x - y, f(x)\rangle \geq 0$.*

PROPOSITION 8.1.13. *If the mapping $f : H \to H$ is scalarly increasing to infinity on Ω, then f satisfies condition (θ, Ω).*

Proof. Since f is scalarly increasing to infinity, then for each $y \in \Omega$ there exists a real number $\rho(y) > 0$ such that for all couples (x, α) with $x \in H$, $\alpha \geq 1$, $\alpha x \in \Omega$ and $\|x\| > \rho(y)$ we have $\langle \alpha x - y, f(x)\rangle \geq 0$.

Fix y_0 arbitrarily in Ω with $\|y_0\| > 0$. This is possible since Ω is unbounded. Then there exists a real number $\rho_0 := \rho(y_0) > 0$ such that for all pairs (x, α) with $x \in H$, $\alpha \geq 1$, $\alpha x \in \Omega$ and $\|x\| \geq \rho_0$ we have $\langle \alpha x - y_0, f(x)\rangle \geq 0$. If we put $\rho_* = \rho_0 + \|y_0\|$, certainly we have that the last inequality is satisfied for each (x, α) with $x \in H$, $\alpha \geq 1$, $\alpha x \in \Omega$ and

$\|x\| \geq \rho_* > \rho_0$. Obviously, for such a pair we have $\alpha\|x\| \geq \|x\| > \|y_0\|$, which implies that condition (θ, Ω) is satisfied for f with respect to Ω. □

About condition (θ, Ω) we have also the following result.

THEOREM 8.1.14. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\Omega \subset H$ *an arbitrary unbounded closed convex set and* $f : H \to H$ *a completely continuous field (or a k-set field) with a representation of the form* $f(x) = x - T(x)$. *If f satisfies condition* (θ, Ω) *with respect to* Ω, *then f is without an EFE with respect to* Ω *(in the sense of Definition 8.1.3).*

Proof. Suppose that f has an *EFE* $\{x_r\}_{r>0}$ with respect to Ω. Hence $\{x_r\}_{r>0}$ satisfies Definition 8.1.3. For each $r > 0$ we have that $\mu_r x_r \in \Omega$ where $\mu_r > 1$, and applying condition (θ, Ω), there exists y_r such that $\|y_r\| < \|\mu_r x_r\|$ and $\langle f(x_r), \mu_r x_r - y_r\rangle \geq 0$ for each $r > 0$ such that $\|x_r\| \geq \rho$. Therefore, for $r > 0$ such that $\|x_r\| \geq \rho$, we have $T(x_r) - \mu_r x_r \in N_\Omega(\mu_r x_r)$, i.e., $\xi_r = T(x_r) - \mu_r x_r$ satisfies the condition $\langle \xi_r, y - \mu_r x_r\rangle \leq 0$ for all $y \in \Omega$ and

$$\begin{aligned}
0 &\leq \langle f(x_r), \mu_r x_r - y_r\rangle = \langle x_r - T(x_r), \mu_r x_r - y_r\rangle \\
&= \langle x_r - \mu_r x_r - \xi_r, \mu_r x_r - y_r\rangle = \langle (1-\mu_r)x_r - \xi_r, \mu_r x_r - y_r\rangle \\
&= (1-\mu_r)\langle x_r, \mu_r x_r - y_r\rangle + \langle \xi_r, y_r - \mu_r x_r\rangle \\
&\leq (1-\mu_r)\left[\mu_r \|x_r\|^2 - \langle x_r, y_r\rangle\right] < 0,
\end{aligned}$$

since $1 - \mu_r < 0$ and

$$\mu_r \|x_r\|^2 - \langle x_r, y_r\rangle \geq \mu_r \|x_r\|^2 - \|x_r\|\|y_r\| = \|x_r\|\left[\mu_r \|x_r\| - \|y_r\|\right] > 0.$$

We have a contradiction, which implies that f is without *EFE* with respect to Ω. □

For other examples of classes of mappings without *EFE* in the sense of Definition 8.1.3 the reader is referred to (Isac, G. and Cojocaru, M. G. [2]).

8.2. Implicit Leray–Schauder alternatives and variational inequalities

Considering the notion of *EFE* introduced in Complementarity Theory in Chapters 3–7 and the notions of *EFE* introduced in this chapter, for variational inequalities, we may conclude that we have two different investigation methods, while a variational inequality with respect to a closed convex cone is a complementarity problem. Moreover, some results obtained in Complementarity Theory by using the notion of *EFE* cannot be extended to variational inequalities because if $\{x_r\}_{r>0}$ is an *EFE* for a variational inequality, namely *VI*(f, Ω) ,we have $\mu_r x_r \in \Omega$ for any $r > 0$ and not $x_r \in \Omega$ for any $r > 0$, as in the case of complementarity problems. In this section we will show that by using the *Implicit Leray–Schauder Alternative* we unify both notions of *EFE*. By this unification, we can extend to variational inequalities several existence results obtained previously for complementarity problems.

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $f : H \to H$ a completely continuous field with a representation of the form $f(x) = x - T(x)$, for all $x \in H$. Let $\Omega \in H$ be an unbounded closed convex set. We consider again the variational inequality *VI*(f, Ω).

DEFINITION 8.2.1. *Let* $\rho = \|P_\Omega(0)\|$. *We say that a family of elements* $\{x_r\}_{r>\rho} \subset \Omega$ *is an exceptional family of elements (EFE) with respect to* Ω, *for the completely continuous field* f, *if the following properties are satisfied:*

(1) $\|x_r\| \to +\infty$ *as* $r \to +\infty$ $(r > \rho)$,

(2) *for any* $r > \rho$ *there exists a real number* $t_r \in]0, 1[$ *such that* $t_r T(x_r) - x_r \in N_\Omega(x_r)$.

We have the following result.

THEOREM 8.2.1. *Let* $\Omega \subset H$ *be a non-empty unbounded closed convex set and* $f : H \to H$ *a completely continuous field such that* $f(x) = x - T(x)$ *for any* $x \in H$. *Then there exists either a solution to the problem VI*(f, Ω), *or* *the mapping f has an EFE (in the sense of Definition 8.2.1) with respect to* Ω.

Proof. We know that the problem *VI(f, Ω)* has a solution if and only if the mapping $\Phi(x)=P_{\Omega}\left[x-f(x)\right]$ has a fixed point. If the problem *VI(f, Ω)* has a solution, then in this case the theorem is proved.

We suppose that the problem *VI(f, Ω)* has no solution. In this case, we consider for any real number r such that $r>\rho=\left\|P_{\Omega}(0)\right\|$, the closed convex set $B_r=\{x\in H:\|x\|\leq r\}$. Obviously, $\partial B_r=\{x\in H:\|x\|=r\}$. For any $r>0$, we consider the mapping $\Phi_r:[0,1]\times B_r\to H$ defined by

$$\Phi_r(t,x)=P_{\Omega}\left[t(x-f(x))\right]=P_{\Omega}\left[t(T(x))\right].$$

We have that Φ_r is continuous and $\Phi_r([0,1]\times B_r)$ is relatively compact in H. Moreover, $\Phi_r(\{0\}\times\partial B_r)\subset B_r$ and for any $x\in\partial B_r$, we have that $\Phi_r(0,x)\neq x$.

We deduce that the assumptions of Theorem 3.5.4 (Leray–Schauder implicit alternative) are satisfied and because we supposed that the problem *VI(f, Ω)* is without solution, we have that $\Phi_r(1,x)\neq x$ for any $x\in B_r$, which implies that there exist $t_r\in\left]0,1\right[$ and $x_r\in\partial B_r$ such that $\Phi_r(t_r,x_r)=x_r$, for any $r>\rho$. We have that for any $r>\rho$ there exists $(t_r,x_r)\in\left]0,1\right[\times\partial B_r$ such that $x_r=P_{\Omega}\left[t_r(T(x_r))\right]$. Therefore we have $x_r\in\Omega$ and $t_rT(x_r)\in x_r+N_{\Omega}(x_r)$. From the last relation we obtain $t_rT(x_r)=x_r+\xi$, where $\xi\in N_{\Omega}(x_r)$. Hence, we have $t_rT(x_r)-x_r\in N_{\Omega}(x_r)$ and we have that the family $\{x_r\}_{r>\rho}$ is an *EFE* for the completely continuous field *f* and the proof is complete. □

From Theorem 8.2.1 we deduce the following existence theorem.

THEOREM 8.2.2. *Let* $\Omega\subset H$ *be a non-empty unbounded closed convex set and* $f:H\to H$ *a completely continuous field. If f is without an EFE in the sense of Definition 8.2.1, with respect to* Ω*, then the problem VI(f, Ω) has a solution.*

COROLLARY 8.2.3. *Let* $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ *be n-dimensional Euclidean space,* $\Omega \subset \mathbb{R}^n$ *an unbounded closed convex set and* $f : \mathbb{R}^n \to \mathbb{R}^n$ *a continuous mapping. If f is without an EFE, in the sense of Definition 8.2.1, with respect to* Ω *(considering* $f = I - (I - f)$*, where I is the identity mapping), then the problem* $VI(f, \Omega)$ *has a solution.* □

Remark. If $\Omega = \mathbb{K}$, where $\mathbb{K}$ is a closed convex cone in H, then in this case the notion of *EFE* defined by Definition 8.2.1 is exactly the notion of *EFE* used in Complementarity Theory. Indeed, let $\{x_r\}_{r>\rho}$ be an *EFE* as defined by Definition 8.2.1. In this case we have $\rho = \|P_{\mathbb{K}}(0)\| = 0$, i.e., $r > 0$ and for any $r > 0$ we have $t_r T(x_r) - x_r \in N_\Omega(x_r)$, which implies

$$\langle t_r T(x_r) - x_r, y - x_r \rangle \le 0, \text{ for all } y \in \mathbb{K}.$$

From the last inequality we deduce

$$\left\langle T(x_r) - \frac{1}{t_r} x_r, y - x_r \right\rangle \le 0, \text{ for all } y \in \mathbb{K},$$

or

$$\left\langle x_r - T(x_r) + \left(\frac{1}{t_r} - 1\right) x_r, y - x_r \right\rangle \ge 0, \text{ for all } y \in \mathbb{K}.$$

If we denote $\mu_r = \frac{1}{t_r} - 1 > 0$, we have $\langle \mu_r x_r + f(x_r), y - x_r \rangle \ge 0$, for all $y \in \mathbb{K}$. From the last inequality we obtain

(i) $u_r = \mu_r x_r + f(x_r) \in \mathbb{K}^*$, for all $r > 0$,

(ii) $\langle u_r, x_r \rangle = 0$, for all $r > 0$,

and because $x_r \in \mathbb{K}$ and $\|x_r\| \to +\infty$ as $r \to +\infty$, we have that $\{x_r\}_{r>0}$ is an *EFE* in the sense used in Complementarity Theory.

A consequence of Corollary 8.2.3 is the fact that we must put in evidence classes of mappings without an *EFE* in the sense of Definition 8.2.1. To realize this goal, now we present some tests that can be used as sufficient tests for the non-existence of an *EFE* for a given mapping. To do this, we need to give an equivalent form of Definition 8.2.1.

Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space and $\Omega \in H$ an arbitrary unbounded closed convex set. We denote again $\rho = \|P_\Omega(0)\|$.

DEFINITION 8.2.2. *Given a mapping $f: H \to H$, we say that a family of elements $\{x_r\}_{r>\rho} \subset \Omega$ is an EFE for f with respect to Ω, if the following properties are satisfied:*

(1) $\|x_r\| \to +\infty$ as $r \to +\infty$,

(2) *for any $r > \rho$, there exists $t_r \in]0, 1[$ such that* $-f(x_r) - \left(\frac{1}{t_r} - 1\right)x_r \in N_\Omega(x_r)$.

PROPOSITION 8.2.4. *If $f: H \to H$ is a completely continuous field with a representation of the form $f(x) = x - T(x)$ for all $x \in H$, then a family of elements $\{x_r\}_{r>\rho} \subset \Omega$ is an EFE in the sense of Definition 8.2.1, if and only if it is an EFE for f in the sense of Definition 8.2.2.*

Proof. Indeed, suppose that $\{x_r\}_{r>\rho} \subset \Omega$ is an *EFE* for f in the sense of Definition 8.2.1. Then in this case we have

(1) $\|x_r\| \to +\infty$ *as* $r \to +\infty$ *and*

(2) *for any $r > \rho$, there exists $t_r \in]0, 1[$ such that* $t_r T(x_r) - x_r \in N_\Omega(x_r)$.

We have

$$T(x_r) - \frac{1}{t_r}x_r \in \frac{1}{t_r}N_\Omega(x_r) \subseteq N_\Omega(x_r),$$

which implies

$$-f(x_r) - \left(\frac{1}{t_r} - 1\right)x_r \in N_\Omega(x_r).$$

Therefore $\{x_r\}_{r>\rho}$ is an *EFE* for f in the sense of Definition 8.2.2. Conversely, let $\{x_r\}_{r>\rho}$ be an *EFE* for f in the sense of Definition 8.2.2. We have,

(1) $\|x_r\| \to +\infty$ as $r \to +\infty$ and

(2) for any $r > \rho$, there exists $t_r \in]0, 1[$ such that

$$-f(x_r)-\left(\frac{1}{t_r}-1\right)x_r \in N_\Omega(x_r).$$

We deduce that $T(x_r)-\frac{1}{t_r}x_r \in N_\Omega(x_r)$, and finally,

$$t_rT(x_r)-x_r \in t_rN_\Omega(x_r) \subseteq N_\Omega(x_r),$$

that is, $\{x_r\}_{r>\rho}$ is an *EFE* for *f* in the sense of Definition 8.2.1. □

Now, we will show that condition (θ) also works for variational inequalities. We recall this condition.

DEFINITION 8.2.3. *We say that a mapping* $f: H \to H$ *satisfies condition* (θ) *with respect to a closed unbounded convex set* $\Omega \subset H$, *if and only if there exists* $\rho_* \geq 0$ *such that for any* $x \in \Omega$ *with* $\|x\| \geq \rho_*$, *there exists* $y \in \Omega$ *with* $\|y\| < \|x\|$ *such that* $\langle x-y, f(x)\rangle \geq 0$.

We note that any mapping, which satisfies the classical Karamardian condition, satisfies also condition (θ). Condition (*HP*) (Harker–Pang) defined initially in Euclidean space, can be extended to an arbitrary Hilbert space.

DEFINITION 8.2.4. *We say that a mapping* $f: H \to H$ *satisfies condition* (*HP*) *with respect to a closed unbounded convex set* $\Omega \subset H$, *if there exists an element* $x_* \in \Omega$ *such that the set* $\Omega(x_*)=\{x\in\Omega:\langle f(x), x-x_*\rangle<0\}$ *is bounded or empty.*

Remark. In the classical Harker–Pang Condition, the set $\Omega(x_*)$ is supposed to be *compact* or *empty* which is more restrictive than in our condition (*HP*).

PROPOSITION 8.2.5. *If* $f: H \to H$ *satisfies condition* (*HP*), *then f satisfies condition*(θ).

Proof. .If *f* satisfies condition (*HP*), then there exists $x_* \in \Omega$ such that the set $\Omega(x_*)$ is bounded or empty. In this case, there exists $\rho_0 > 0$ such that $\Omega(x_*) \subset B(0,\rho_0)$ where $B(0,\rho_0)=\{x\in H:\|x\|\leq\rho_0\}$. We take $\rho_* > \max\{\rho_0, \|x_*\|\}$. If $x \in \Omega$ is an arbitrary element such that $\|x\| > \rho_*$, then we have $x \notin \Omega(x_*)$, which implies that $\langle f(x), x - x_* \rangle \geq 0$. Obviously, if for any $x \in \Omega$ such that $\|x\| > \rho_*$, we take $y = x_*$ we obtain that f satisfies condition (θ). □

THEOREM 8.2.6. *If $f : H \to H$ satisfies condition (θ) with respect to an unbounded closed convex set $\Omega \subset H$, then f is without an EFE in the sense of Definition 8.2.2.*

Proof. We suppose that f has an *EFE*, namely $\{x_r\}_{r>\rho}$, in the sense of Definition 8.2.2, with respect to Ω. Since $\|x_r\| \to +\infty$ as $r \to +\infty$, we take x_r such that $r > \rho$ and $\|x_r\| > \rho_*$, where $\rho_* > 0$ is the real number considered in condition (θ).For this x_r, there exists $y_r \in \Omega$ such that $\|y_r\| < \|x_r\|$ and $\langle x_r - y_r, f(x_r) \rangle \geq 0$. We have also, $-f(x_r) - \left(\frac{1}{t_r} - 1 \right) x_r \in N_\Omega(x_r)$. If we denote $\mu_r = \frac{1}{t_r} - 1$, we have $\mu_r > 0$ and $-f(x_r) - \mu_r x_r = \xi \in N_\Omega(x_r)$. We deduce

$$\begin{aligned} 0 &\leq \langle x_r - y_r, f(x_r) \rangle = \langle x_r - y_r, -\mu_r x_r - \xi \rangle \\ &= \langle x_r - y_r, -\xi \rangle - \mu_r \langle x_r - y_r, x_r \rangle \\ &= -\langle x_r - y_r, \xi \rangle - \mu_r \langle x_r - y_r, x_r \rangle \\ &\leq -\mu_r \left[\|x_r\|^2 - \langle y_r, -x_r \rangle \right] < 0, \end{aligned}$$

which is a contradiction and the proof is complete. □

COROLLARY 8.2.7. *If $f : H \to H$ is a completely continuous field which satisfies condition (θ) with respect to an unbounded, closed convex set $\Omega \subset H$, then the problem VI(f, Ω) has a solution.*

DEFINITION 8.2.5. *We say that a mapping $T : H \to H$ satisfies condition ($\mathcal{R}$) with respect to Ω if for any $x \in \Omega$, with $\|x\| > 1$, we have*

$$\langle x, T(x) \rangle \leq \left\langle x, T\left(\frac{x}{\|x\|} \right) \right\rangle.$$

We denote by *coneh*(Ω) the conical hull of Ω, i.e., $coneh(\Omega) = \bigcup_{\lambda \geq 0} \lambda\Omega$.

DEFINITION 8.2.6. *We say that a mapping $T : H \to H$ is monotonically decreasing on rays, with respect to Ω, if for any $\alpha \geq 1$, and any $u \in coneh(\Omega)$ we have $\langle u, T(\alpha u)\rangle \leq \langle u, T(u)\rangle$.*

PROPOSITION 8.2.8. *If $T : H \to H$ is monotonically decreasing on rays, then T satisfies condition ($\mathcal{R}$).*

Proof. Indeed, let $x \in \Omega$ be an arbitrary element such that $\|x\| > 1$. We take $\alpha = \|x\|$ and $u = \frac{x}{\|x\|}$. We have $\alpha\ u = x$ and hence $\left\langle \frac{x}{\|x\|}, T\left(\frac{x}{\|x\|}\right)\right\rangle \geq \left\langle \frac{x}{\|x\|}, T(x)\right\rangle$, which implies $\langle x, T(x)\rangle \leq \left\langle x, T\left(\frac{x}{\|x\|}\right)\right\rangle$, that is condition ($\mathcal{R}$) is satisfied. □

THEOREM 8.2.9. *If $T : H \to H$ is a bounded mapping which satisfies condition ($\mathcal{R}$) with respect to Ω and $0 \in \Omega$, then the mapping $f(x) = x - T(x)$, $x \in H$ is without an EFE with respect to Ω in the sense of Definition 8.2.1.*

Proof. Indeed, we suppose that $\{x_r\}_{r>0}$ is an *EFE* for f with respect to Ω. In this case $\rho = \|P_\Omega(0)\| = 0$. For any x_r such that $\|x\| > 1$, by condition ($\mathcal{R}$) we have

$$\langle x_r, T(x_r)\rangle \leq \left\langle x_r, T\left(\frac{x_r}{\|x_r\|}\right)\right\rangle. \tag{8.2.1}$$

Because $\{x_r\}_{r>0}$ is an *EFE*, we have that $t_r T(x_r) - x_r \in N_\Omega(x_r)$ with $t_r \in]0, 1[$ for any $r > 0$ such that $\|x_r\| > 1$. Since $N_\Omega(x_r)$ is a cone we have that $T(x_r) - \frac{1}{t_r}x_r = \xi_r \in N_\Omega(x_r)$, and if we denote $\mu_r = \frac{1}{t_r}$, we obtain that $T(x_r) = \mu_r x_r + \xi_r$. Considering (8.2.1) we get $\langle x_r, \mu_r x_r + \xi_r\rangle \leq \left\langle x_r, T\left(\frac{x_r}{\|x_r\|}\right)\right\rangle$, for any $r > 0$ such that $\|x_r\| > 1$. Since T is bounded there exists $M > 0$ such that from the last inequality we have

$$\langle x_r, \mu_r x_r\rangle + \langle x_r, \xi_r\rangle \leq M\|x_r\|, \text{ for any } r > 1 \text{ such that } \|x_r\| > 1. \tag{8.2.2}$$

Because $0 \in \Omega$ and $\xi_r \in N_\Omega(x_r)$ we have

$$\langle \xi_r, x_r \rangle \geq 0. \tag{8.2.3}$$

From (8.2.2) and (8.2.3) we deduce that $\|x_r\| \leq \frac{1}{\mu_r} M < M$, for any $r > 0$ such that $\|x_r\| > 1$. We obtain that $\{\|x_r\|\}_{r>0}$ is bounded, which is in contradiction with the definition of the notion of *EFE* and the proof is complete. □

COROLLARY 8.2.10. *If $T : H \to H$ is completely continuous, satisfies condition $(\mathcal{R})$ and $0 \in \Omega$, then the problem $VI(f, \Omega)$ has a solution, where $f(x) = x - T(x)$, for any $x \in H$.*

COROLLARY 8.2.11. *Let $T : H \to H$ be a completely continuous mapping. Consider the completely continuous field $f_\lambda(x) = x - \lambda T(x)$, $(x \in H)$, for some $\lambda \in \mathbb{R}$. If T satisfies condition $(\mathcal{R})$ with respect to Ω (with $0 \in \Omega$), then the problem $VI(f_\lambda, \Omega)$ has a solution for any $\lambda > 0$.*

Proof. The corollary is a consequence of Corollary 8.2.10, since λT satisfies condition $(\mathcal{R})$ for any $\lambda > 0$. □

DEFINITION 8.2.7. *We say that a mapping $f : H \to H$ satisfies condition (IG) with respect to Ω if there exists a real number $p > 0$ such that the mapping $\Phi(x) = \|x\|^{p-1} \cdot x - f(x)$ defined for all $x \neq 0$ satisfies condition $(\mathcal{R})$ with respect to Ω.*

We have the following result.

THEOREM 8.2.12. *Let $T : H \to H$ be a bounded mapping. If $\Omega \subset H$ is an unbounded closed convex subset such that $0 \in \Omega$ and $f(x) = x - T(x)$ satisfies condition (IG) with respect to Ω, then f is without EFE in the sense of Definition 8.2.1 with respect to Ω.*

Proof. Assume that f has an *EFE*, namely $\{x_r\}_{r>0}$ with respect to Ω. By condition (*IG*) we have $\left\langle x, \Phi\left(\frac{x}{\|x\|}\right) - \Phi(x)\right\rangle \geq 0$, for any $x \in \Omega$ with $\|x\| > 1$. Then for any $r > 0$ such that $\|x_r\| > 1$ we obtain

$$\left\langle x_r, \Phi\left(\frac{x_r}{\|x_r\|}\right) - \Phi(x_r)\right\rangle \geq 0, \text{ which implies}$$

$$\left\langle x_r, \Phi\left(\frac{x_r}{\|x_r\|}\right) - \|x_r\|^{p-1} x_r + f(x_r)\right\rangle \geq 0. \tag{8.2.4}$$

Because $f(x_r) = x_r - T(x_r)$, we have $T(x_r) = \mu_r x_r + \xi_r$, where $\mu_r > 1$ and $\xi_r \in N_\Omega(x_r)$. The assumption $0 \in \Omega$ implies $\langle x_r, \xi_r\rangle \geq 0$. From (8.2.4) then we have

$$\left\langle x_r, \Phi\left(\frac{x_r}{\|x_r\|}\right) - \|x_r\|^{p-1} x_r + x_r - (\mu_r x_r + \xi_r)\right\rangle \geq 0, \tag{8.2.5}$$

which implies

$$\left\langle x_r, \Phi\left(\frac{x_r}{\|x_r\|}\right)\right\rangle - \|x_r\|^{p+1} + \|x_r\|^2 - \mu_r\|x_r\|^2 \geq 0,$$

or

$$\left\langle x_r, \Phi\left(\frac{x_r}{\|x_r\|}\right)\right\rangle - \|x_r\|^{p+1} \geq (\mu_r - 1)\|x_r\|^2 \geq 0,$$

and finally we have

$$\left\langle x_r, \Phi\left(\frac{x_r}{\|x_r\|}\right)\right\rangle \geq \|x_r\|^{p+1}. \tag{8.2.6}$$

Because T is bounded, there exists $M > 0$ such that $\left\|\Phi\left(\frac{x_r}{\|x_r\|}\right)\right\| \leq M$, for any $r > 0$. From (8.2.6) we have $\|x_r\|^p \leq M$, for any $r > 0$ such that $\|x_r\| > 1$, which implies that the set $\{\|x_r\|\}_{r>0}$ is bounded, which is impossible. Therefore f is without *EFE* in the sense of Definition 8.2.1 and the proof is complete. □

DEFINITION 8.2.8. *We say that a mapping* $f: H \to H$ *is δ-pseudo-monotone on* Ω *if for any* $x \in \Omega$ *there exists a real number* $\delta(x) > 0$ *such* *that for any* $y \in \Omega$ *with* $\|y\| > \delta(x)$ *we have that* $\langle x-y, f(y)\rangle \geq 0$ *implies* $\langle x-y, f(x)\rangle \geq 0$.

THEOREM 8.2.13. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $f: H \to H$ *a mapping and* $\Omega \subset H$ *an unbounded closed convex set. If f is* δ*-pseudomonotone on* Ω *and the problem* $VI(f, \Omega)$ *has a solution, then f is without an EFE with respect to* Ω*, in the sense of Definition 8.2.2.*

Proof. Indeed, let $x_* \in \Omega$ be a solution to the problem $VI(f, \Omega)$. Then we have $\langle x-x_*, f(x_*)\rangle \geq 0$, for all $x \in \Omega$. In particular we have $\langle x-x_*, f(x_*)\rangle \geq 0$ for all $x \in \Omega$ with $\|x\| > \max\{\|x_*\|, \delta(x_*), \rho\}$, where $\delta = \|P_\Omega(0)\|$. We suppose that $\{x_r\}_{r>0}$ is an *EFE* for f with respect to Ω. We take x_r with $r > \delta$ and such that $\|x\| > \max\{\|x_*\|, \delta(x_*), \rho\}$. We have $-f(x_r) - \mu_r x_r = \xi \in N_\Omega(x_r)$ and we obtain (considering the δ-pseudomonotonicity)

$$
\begin{aligned}
&0 \leq \langle x_r - x_*, f(x_r)\rangle = \langle x_r - x_*, -\mu_r x_r - \xi\rangle \\
&= \langle x_r - x_*, -\xi\rangle - \mu_r \langle x_r - x_*, x_r\rangle \\
&= \langle x_* - x_r, \xi\rangle - \mu_r \langle x_r - x_*, x_r\rangle \\
&\leq -\mu_r \langle x_r - x_*, x_r\rangle = -\mu_r \left[\|x_r\|^2 - \langle x_*, x_r\rangle\right] < 0,
\end{aligned}
$$

(since $\mu_r = \frac{1}{t_r} - 1 > 0$), which is a contradiction. Therefore f is without an *EFE* with respect to Ω in the sense of Definition 8.2.2. □

COROLLARY 8.2.14. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a Hilbert space* $f: H \to H$ *a completely continuous field and* $\Omega \subset H$*, an unbounded closed convex set. If f is* δ*-pseudomonotone (in particular pseudomonotone) on* Ω*, then the problem* $VI(f, \Omega)$ *has a solution, if and only if f is without an EFE, with respect to* Ω*.*

DEFINITION 8.2.9. *We say that a mapping* $f: H \to H$ *is weakly proper on* Ω *if there is* $\rho > 0$ *such that for any* $x \in \Omega$ *with* $\|x\| > \rho$*, there exists* $x_* \in \Omega$*, with* $\|x_*\| < \|x\|$ *such that* $\langle f(x_*), x - x_*\rangle \geq 0$.

THEOREM 8.2.15. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\Omega \subset H$, *an unbounded closed convex set and* $f : H \to H$ *a pseudomonotone mapping. If* f *is a completely continuous field, then the problem* $VI(f, \Omega)$ *has a solution, if and only if the mapping* f *is weakly proper on* Ω.

Proof. We suppose that the problem $VI(f, \Omega)$ has a solution. Let $x_0 \in \Omega$ be this solution. We have $\langle f(x_0), x - x_0\rangle \geq 0$ for any $x \in \Omega$. Obviously, if we take in *Definition 8.2.9* an arbitrary real number $\rho > \|x_0\|$ and $x_* = x_0$, then we have that f is weakly proper on Ω. Conversely, assume that f is weakly proper with respect to Ω. Then there exists $\rho > 0$ such that for any $x \in \Omega$ with $\|x\| > \rho$, we can select an element $x_* \in \Omega$ with $\|x_*\| < \|x\|$ and $\langle f(x_*), x - x_*\rangle \geq 0$. Because f is pseudomonotone we have $\langle f(x), x - x_*\rangle \geq 0$. We deduce that f satisfies condition (θ) and as a consequence, we have that f is without *EFE* with respect to Ω. Therefore, f being a completely continuous field without an *EFE*, we have that the problem $VI(f, \Omega)$ has a solution. □

DEFINITION 8.2.10. *Let* $f : H \to H$ *be a mapping and* $\Omega \subset H$ *an unbounded closed convex set. We say that a mapping* $T : H \to H$ *is an* x_*-*scalar asymptotic derivative of* f, *with respect to* Ω, *if there exists an element* $x_* \in \Omega$ *such that*

$$\lim_{\|x\|\to+\infty, x\in\Omega} \frac{\langle f(x) - T(x), x - x_*\rangle}{\|x\|^2} = 0 .$$

Considering this notion we have the following result.

THEOREM 8.2.16. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $f : H \to H$ *a mapping and* $\Omega \subset H$ *an unbounded closed convex set. If* f *has an* x_*-*scalar asymptotic derivative* $T : H \to H$ *such that*

$$\lim_{\|x\|\to+\infty, x\in\Omega} \frac{\langle T(x), x - x_*\rangle}{\|x\|^2} = \delta > 0 \left(\delta \in]0, +\infty[\right),$$

then f *is without an EFE with respect to* Ω *in the sense of Definition 8.2.2.*

Proof. We have

$$\lim_{\|x\|\to+\infty, x\in\Omega} \frac{\langle f(x), x - x_* \rangle}{\|x\|^2}$$

$$= \lim_{\|x\|\to+\infty, x\in\Omega} \frac{\langle f(x) - T(x), x - x_* \rangle}{\|x\|^2} + \lim_{\|x\|\to+\infty, x\in\Omega} \frac{\langle T(x), x - x_* \rangle}{\|x\|^2} = \delta > 0.$$

Let α, $\beta \in \mathbb{R}$ such that $0 < \alpha < \delta < \beta$. Then there exists $\rho > 0$ such that for any $x \in \Omega$ with $\|x\| > \rho$ we have $\frac{\langle f(x), x - x_* \rangle}{\|x\|^2} \in]\alpha, \beta[$. Now, we observe that f satisfies condition (θ) if we take in the definition of this condition $\rho_* > \max\{\rho, \|x_*\|\}$ and $y = x_*$. Consequently, f is without an *EFE* with respect to Ω, in the sense of Definition 8.2.2. □

8.3. Asymptotic Minty's variational inequality and condition (θ)

Let $(H, \langle \cdot, \cdot \rangle)$ be an arbitrary Hilbert space, $f: H \to H$ a mapping and $\Omega \subset H$ an unbounded closed convex set (obviously non-empty). We recall that the *variational inequality* in the sense of *Hartman* and *Stampacchia* is the following problem:

$$HSVI(f,\Omega): \begin{cases} \textit{find } x_* \in \Omega \textit{ such that} \\ \langle f(x_*), x - x_* \rangle \geq 0, \textit{ for all } x \in \Omega. \end{cases}$$

This kind of variational inequality has many applications in physics, engineering and economics. The variational inequality in Minty's sense is

$$MVI(f,\Omega): \begin{cases} \textit{find } x_* \in \Omega \textit{ such that} \\ \langle f(x), x - x_* \rangle \geq 0, \textit{ for all } x \in \Omega. \end{cases}$$

We suppose that $f: H \to H$ is pseudomonotone. In Theorem 2.2.2 *we proved that an element* $x_* \in \Omega$ *is a solution of the problem* $HSVI(f, \Omega)$ *if and only if* x_* *is a solution of the problem* $MVI(f, \Omega)$. Now, in this section we will introduce the *asymptotic Minty variational inequality* and we will show that Theorem 2.2.2 is also valid if we replace the Minty variational inequality by the asymptotic Minty variational inequality.

DEFINITION 8.3.1. *The asymptotic Minty variational inequality defined by f and Ω is the following:*

$$AMVI(f,\Omega):\begin{cases} \textit{find } \delta_* > 0 \ \textit{(eventually very large)} \\ \textit{and } x_* \in \Omega \textit{ such that} \\ \langle f(x), x - x_* \rangle \geq 0 \textit{ for any } x \in \Omega \\ \textit{satisfying } \|x\| > \delta_*. \end{cases}$$

The element x_* satisfying *Definition 8.3.1* is called a solution of the problem *AMVI(f, Ω)*.

THEOREM 8.3.1. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $\Omega \subset H$ *a non-empty, unbounded and closed convex set and* $f : H \to H$ *a completely continuous field. If f is pseudomonotone then the problem HSVI(f, Ω) has a solution if and only if the problem AMVI(f, Ω) has a solution.*

Proof. Indeed, if $x_* \in \Omega$ is a solution of the problem *HSVI(f, Ω)*, then we have

$$\langle f(x_*), x - x_* \rangle \geq 0, \textit{ for any } x \in \Omega. \tag{8.3.1}$$

Because f is pseudomonotone, from (8.3.1) we obtain

$$\langle f(x), x - x_* \rangle \geq 0, \textit{ for any } x \in \Omega.$$

Now, taking an arbitrary $\delta_* > 0$, we have that

$$\langle f(x), x - x_* \rangle \geq 0, \textit{ for any } x \in \Omega \textit{ with } \|x\| > \delta_*,$$

that is, x_* is a solution of the problem *AMVI(f, Ω)*. Conversely, let $x_* \in$ o be a solution of the problem *AMVI(f, Ω)*. Considering Theorem 8.2.2 it is sufficient to show that f is without an *EFE* with respect to Ω in the sense of Definition 8.2.1, where $\rho = \|P_\Omega(0)\|$. We suppose that f has an *EFE*, namely $\{x_r\}_{r>0}$. Since $\|x_r\| \to +\infty$ *as* $r \to +\infty$, we consider x_r with $\max\{\delta_*, \|x_*\|\} < \|x_r\|$, where δ_* is the real number used in the definition of the problem *AMVI(f, Ω)*.

Since $\|\delta_*\| < \|x_r\|$, we deduce that $\langle f(x_r), x_r - x_* \rangle \geq 0$. We have also the following relation:

$$-f(x_r) - \left(\frac{1}{t_r} - 1\right)x_r \in N_\Omega(x_r),$$

where $t_r \in]0, 1[$ (considering the definition of $\{x_r\}_{r>0}$). If we let $\mu_r = \frac{1}{t_r} - 1$, we have $\mu_r > 0$ and $-f(x_r) - \mu_r x_r = \xi \in N_\Omega(x_r)$. Then, we deduce

$$\begin{aligned} 0 &\le \langle x_r - x_*, f(x_r)\rangle = \langle x_r - x_*, -\mu_r x_r - \xi\rangle \\ &= \langle x_r - x_*, -\xi\rangle - \mu_r \langle x_r - x_*, x_r\rangle = -\langle x_r - x_*, \xi\rangle - \mu_r \langle x_r - x_*, x_r\rangle \\ &= \langle x_* - x_r, \xi\rangle - \mu_r \langle x_r - x_*, x_r\rangle \le -\mu_r \left[\|x_r\|^2 - \langle x_*, x_r\rangle\right] < 0, \end{aligned}$$

which is a contradiction. Therefore, f is without an *EFE*, with respect to Ω, in the sense of Definition 8.2.1. Applying Theorem 8.2.2 we have that the problem *AMVI*(f, Ω) has a solution, and the proof is complete. □

COROLLARY 8.3.2. *Let $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ be n-dimensional Euclidean space, $\Omega \subset \mathbb{R}^n$ a non-empty unbounded and closed convex set and $f: \mathbb{R}^n \to \mathbb{R}^n$ a continuous mapping. If f is pseudomonotone, then the problem HSVI(f, Ω) has a solution if and only if the problem AMVI(f, Ω) has a solution.*

Remark. Obviously, if the problem *MVI*(f, Ω) has a solution, then the problem *AMVI*(f, Ω) also has a solution, but the converse, generally is not true.

In this chapter we introduced the condition (θ) for variational inequalities with respect to general unbounded sets, and we proved that if f satisfies this condition (θ), then f is without *EFE* in the sense of Definitions 8.2.1 or 8.2.2 and the problem *HSVI*(f, Ω) has a solution. We recall this condition (θ). *We say that $f: H \to H$ satisfies condition (θ) with respect to Ω, if there exists $\rho > 0$ such that for any $x \in \Omega$ with $\|x\| > \rho$, there exists $y \in \Omega$ with $\|y\| < \|x\|$ such that $\langle x - y, f(x)\rangle \ge 0$. Because we can show that, if the problem AMVI(f, Ω) has a solution , then f satisfies condition (θ),* we obtain the following result.

THEOREM 8.3.3. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\Omega \subset H$ a non-empty, unbounded and closed convex set and $f: H \to H$ a completely continuous field. If f is pseudomonotone, then the problem HSVI(f, Ω) has a solution if and only if f satisfies condition (θ) with respect to Ω.* □

Remarks.

(1) It would be interesting to know if Theorem 8.3.1 is valid for other classes of mapping different from the pseudomonotone operators.

(2) From the results presented in this section, we deduce that for *condition* (θ) to be satisfied for a mapping f with respect to an unbounded set Ω, it is sufficient to show that the problem $AMVI(f, \Omega)$ has a solution. *Therefore, if f is a completely continuous field and the problem $AMVI(f, \Omega)$ has a solution, then the problem $HSVI(f, \Omega)$ has a solution too.* This result seems to be a remarkable result.

8.4. Complementarity problems and variational inequalities with integral operators

The complementarity theory and the theory of variational inequalities have many applications to mechanics, elasticity, fluid mechanics, engineering and economics. Moreover, in many applications we can have complementarity problems or variational inequalities depending on parameters. The study of such variational inequalities or of complementarity problems is related to the study of bifurcation problems. About this subject the reader is referred to the book (Le, V. K. and Schmitt, K [1]), where several kinds of bifurcation problem for variational inequalities defined by integral operators are considered. Because of this reality, we present in this section some results related to variational inequalities defined by integral operators. Our results are based on the notion of *EFE* in the sense of Definition 8.1.3.

Let Ω_* be a bounded open set in $\mathbb{R}^n$. We consider the Hilbert space $L^2(\Omega_*)$ and we recall that the norm on the space $L^2(\Omega_*)$ is

$$\|u\| = \left(\int_{\Omega_*} |u(x)|^2 \, dx \right)^{1/2}, \textit{ for all } u \in L^2(\Omega_*).$$

This notion will be used throughout this section. For simplicity, we denote the Lebesgue measure $mes\Omega_*$ by $|\Omega_*|$. We suppose we are given a function $G : \overline{\Omega_*} \times \overline{\Omega_*} \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

(i) *G is a Caratheodory function, i.e., G(x, y, u) is continuous with respect to u for almost all* $(x,y)\in\Omega_*\times\Omega_*$ *and measurable with respect to the pair of variables* $(x,y)\in\Omega_*\times\Omega_*$, *for all* $u\in\mathbb{R}$,

(ii) $|G(x,y,u)|\le\mathcal{R}(x,y)(a+b|u|)$, *a.e.* x, $y\in\Omega_*$, *for every* $u\in\mathbb{R}$, *where* $a, b > 0$, *and* $\mathcal{R}\in L^2(\Omega_*\times\Omega_*)$.

The following integral operators are used in many practical problems:

(I) $A: L^2(\Omega_*)\to L^2(\Omega_*)$ defined by $A(u)=\lambda\int_{\Omega_*}G(\bullet,y,u(y))dy$, for all $u\in L^2(\Omega_*)$ where, $\lambda\in\mathbb{R}$;

(II) $f: L^2(\Omega_*)\to L^2(\Omega_*)$ defined by
$f(u)=u-A(u)=u-\lambda\int_{\Omega_*}G(\bullet,y,u(y))dy$, for all $u\in L^2(\Omega_*)$.

First, we recall the following results.

THEOREM 8.4.1. *Let* $G:\overline{\Omega_*}\times\overline{\Omega_*}\times\mathbb{R}\to\mathbb{R}$ *be a function satisfying properties (i) and (ii). If in addition, the function* G *satisfies the following assumptions:*

(1) *for any* $\alpha>0$, *the function* $\mathcal{R}_\alpha(x,y)=\max_{|u|\le\alpha}|G(x,y,u)|$ *is summable with respect to y for almost* $x\in\Omega_*$,

(2) *for any* $\alpha>0$, $\limsup_{mesD\to 0\,|u|\le\alpha}\left\|\mathcal{P}_D\int_{\Omega_*}G(x,y,u)\,dy\right\|_{L^2(\Omega_*)}=0$, *where* $\mathcal{P}_D$ *is the operator of multiplication by the characteristic function of the set* $D\subset\Omega_*$,

(3) *for some* $\beta>0$, $\limsup_{mesD\to 0\,\|u\|\le\beta}\left\|\mathcal{P}_D\int_{\Omega_*}G(x,y,u(y))\,dy\right\|_{L^2(\Omega_*)}=0$,

then for any $\lambda\in\mathbb{R}$ *the mapping* $f(u)=u-\lambda\int_{\Omega_*}G(x,y,u(y))dy$ *is a completely continuous field from* $L^2(\Omega_*)$ *to* $L^2(\Omega_*)$.

Proof. For a proof of this classical result the reader is referred to [(Zabreyko, P. P., Koshelev, A. I., Krasnoselskii, M. A., Mikhlin, S. G., Rakovshchik, L. S. and Stesenko, V. Ya [1]), Chapter 10, Theorem 1.12.] □

We cite also the following result.

THEOREM 8.4.2. *Let* A: $L^2(\Omega_*)\to L^2(\Omega_*)$ *be the mapping defined by*

$$A(u) = \lambda \int_{\Omega_*} G(\bullet, y, u(y))dy, \textit{ for all } u \in L^2(\Omega_*) \textit{ for some } \lambda \in \mathbb{R}.$$

Then the mapping $A : L^2(\Omega_*) \to L^2(\Omega_*)$ *is continuous and bounded. Moreover, if we suppose that*

(iii) $$\begin{cases} \textit{for every } \varepsilon > 0 \textit{ there is a } \delta > 0 \textit{ such that} \\ |G(x+h, y, u) - G(x, y, u)| < \varepsilon, \textit{ for a.e. } x, y \in \Omega_*, \\ \textit{each } h \in \mathbb{R}^m \textit{ with } \|h\|_{\mathbb{R}^m} < \delta, x + h \in \Omega_*, \textit{ and each } u \in \mathbb{R}, \end{cases}$$

then the operator $A : L^2(\Omega_*) \to L^2(\Omega_*)$ *is completely continuous.*

Proof. The proof is long and based on several technical details that can be found in (Isac, G. and Motreanu, D. [1].) □

We recall *condition* (*HP*) in a general Hilbert space. Let $(H, \langle \cdot, \cdot \rangle)$ be an arbitrary Hilbert space and $\Omega \subset H$ an unbounded and closed convex set. *We say that* $f : H \to H$ *satisfies condition* (*HP*), *with respect to* Ω, *if there exists an element* $x_* \in \Omega$ *such that the set* $\Omega(x_*) = \{x \in \Omega : \langle f(x), x - x_* \rangle < 0\}$ *is bounded or empty.*

We know (see Proposition 8.2.5) that, if f satisfies condition (*HP*), then it satisfies condition (θ) and consequently, f is without an *EFE* in the sense of Definition 8.2.2. Moreover, if f is a completely continuous field and satisfies condition (*HP*) then the problem *VI*(f, Ω) has a solution. About the integral operator considered above we have the following result.

THEOREM 8.4.3. *Let* $f : L^2(\Omega_*) \to L^2(\Omega_*)$ *be the mapping defined by* $f(u) = u - \lambda \int_{\Omega_*} G(\cdot, y, u(y))dy$, *for all* $u \in L^2(\Omega_*)$, *for some* $\lambda \in \mathbb{R}$. *Suppose that the mapping G satisfies assumptions* (i), (ii), (iii) *defined above and also the following assumption:*

(iv) $$|\lambda| < \frac{1}{b\|\mathcal{R}\|_{L^2(\Omega_* \times \Omega_*)}}.$$

Then the mapping f *satisfies condition* (*HP*) *with respect to any unbounded, closed convex set* $\Omega \subset L^2(\Omega_*)$. *Moreover the problem VI*(f, Ω) *has a solution.*

Proof. By Theorem 8.4.2 we have that f is a completely continuous field. Therefore, it is sufficient to show that f satisfies property (*HP*) with respect to any unbounded, closed convex set $\Omega \subset L^2(\Omega_*)$. Indeed, let $\Omega \subset L^2(\Omega_*)$ be an arbitrary, unbounded closed convex set and let $v_* \in \Omega$, be an arbitrary element. We consider the set

$$\Omega(v_*) = \{u \in \Omega : \langle f(u), u - v_* \rangle < 0\},$$

where $\langle \cdot,\cdot \rangle$ is the inner product in $L^2(\Omega_*)$. Then, using (ii) and Cauchy–Schwartz inequality we derive that

$$\begin{aligned}
&\|u\|^2 - \langle u, v_* \rangle < \left\langle \lambda \int_{\Omega_*} G(\cdot, y, u(y))\,dy, u - v_* \right\rangle \\
&= \lambda \int_{\Omega_*} \left(G(\cdot, y, u(y))\,dy\right)(u(x) - v_*(x))\,dx \\
&\le |\lambda| \int_{\Omega_*} \left(\int_{\Omega_*} \mathcal{R}(x,y)(a + b|u(y)|)\,dy \right)(|u(x)| + |v_*(x)|)\,dx \\
&\le |\lambda| \left(\int_{\Omega_*} \left(\mathcal{R}(x,y)(a + b|u(y)|)\,dy \right)^2 dx \right)^{1/2} \left(\int_{\Omega_*} (|u(x)| + |v_*(x)|)^2\,dx \right)^{1/2} \\
&\le |\lambda| \left(\int_{\Omega_*} \left(\left(\int_{\Omega_*} \mathcal{R}(x,y)^2\,dy \right) \int_{\Omega_*} (a + b|u(y)|)^2\,dy \right) dx \right)^{1/2} (\|u\| + \|v_*\|) \\
&= |\lambda| \|\mathcal{R}\|_{L^2(\Omega_* \times \Omega_*)} \|a + b|u|\| (\|u\| + \|v_*\|) \\
&\le |\lambda| \|\mathcal{R}\|_{L^2(\Omega_* \times \Omega_*)} \left(a|\Omega_*|^{1/2} + b\|u\|\right)(\|u\| + \|v_*\|).
\end{aligned}$$

If follows that

$$\|u\|^2 \le \|u\|\|v_*\| + |\lambda| \|\mathcal{R}\|_{L^2(\Omega_* \times \Omega_*)} \left(a|\Omega_*|^{1/2} + b\|u\|\right)(\|u\| + \|v_*\|),$$

thus

$$\begin{aligned}
&\left(1 - |\lambda| b \|\mathcal{R}\|_{L^2(\Omega_* \times \Omega_*)}\right)\|u\|^2 \le \left(\|v_*\| + |\lambda| \|\mathcal{R}\|_{L^2(\Omega_* \times \Omega_*)} \left(a|\Omega_*|^{1/2} + b\|v_*\|\right)\right)\|u\| \\
&+ |\lambda| \|\mathcal{R}\|_{L^2(\Omega_* \times \Omega_*)}\, a|\Omega_*|^{1/2} \|v_*\|,
\end{aligned}$$

for all $u \in \Omega(v_*)$. Now taking into account (iv), from the above inequality we derive that $\Omega(v_*)$ is bounded. Indeed, let $\{u_n\}_{n \in N}$ be a sequence in $\Omega(v_*)$ such that $\|u_n\| \to +\infty$. Then there exists a natural number n_0 such that $\|v_*\| < \|u_n\|$ for all $n > n_0$. From the last inequality we have

$$\begin{aligned}
&\left(1 - |\lambda| b \|\mathcal{R}\|_{L^2(\Omega_* \times \Omega_*)}\right)\|u_n\| \\
&\le \left(\|v_*\| + |\lambda| \|\mathcal{R}\|_{L^2(\Omega_* \times \Omega_*)} \left(a|\Omega_*|^{1/2} + b\|v_*\|\right)\right)\|u_n\| \\
&+ |\lambda| \|\mathcal{R}\|_{L^2(\Omega_* \times \Omega_*)}\, a|\Omega_*|^{1/2} \|u_n\|, \textit{ for all } n > n_0,
\end{aligned}$$

which implies

$$\begin{aligned}&\left(1-|\lambda| b\|\mathcal{R}\|_{L^2(\Omega_*\times\Omega_*)}\right)\|u_n\| \\ &\leq\left[\|v_*\|+|\lambda|\|\mathcal{R}\|_{L^2(\Omega_*\times\Omega_*)}\left(a|\Omega_*|^{1/2}+b\|v_*\|\right)\right. \\ &\left.+|\lambda|\|\mathcal{R}\|_{L^2(\Omega_*\times\Omega_*)}\, a|\Omega_*|^{1/2}\right]\|u_n\|, \textit{ for all } n>n_0 .\end{aligned}$$

Because the last inequality implies a contradiction, the proof is complete. □

COROLLARY 8.4.4. *If all the assumptions of Theorem 8.4.3 are satisfied and* $\Omega = \mathbb{K}$, *where* $\mathbb{K}$ *is a closed convex cone in* $L^2(\Omega_*)$, *then the complementarity problem NCP(f,* $\mathbb{K}$*) has a solution.*

Now, we consider the particular case with $\Omega = \mathbb{K}$, where

$$\mathbb{K}=\left\{u \in L^2\left(\Omega_*\right): u \geq 0, \textit{a.e. in } \Omega_*\right\}.$$

We note that $\mathbb{K}$ *is a closed pointed convex cone in* $L^2(\Omega_*)$ satisfying $\mathbb{K}=\mathbb{K}^*$. In this case we have the following result using directly Definition 5.1.2.

THEOREM 8.4.5. *Let* $f: L^2(\Omega_*) \rightarrow L^2(\Omega_*)$ *be the mapping defined by*

$$f(u)=u-\lambda \int_{\Omega_*} G(\bullet, y, u(y))\, dy, \textit{ for all } u \in L^2\left(\Omega_*\right),$$

for some $\lambda \in \mathbb{R}$. *Consider* $L^2(\Omega_*)$ *the cone* $\mathbb{K}$ *defined above. If the following assumptions are satisfied:*

(1) *the mapping G satisfies assumptions* (i), (ii) *and* (iii) *used in Theorem 8.4.3,*

(2) $b|\lambda|\|\mathcal{R}\|_{L^2(\Omega_*\times\Omega_*)}<1$,

then the mapping f is without an EFE in the sense of Definition 5.1.2 and the problem NCP(f, $\mathbb{K}$*) has a solution. Moreover, if* $b = 0$, *then the problem NCP(f,* $\mathbb{K}$*) has a solution for any* $\lambda \in \mathbb{R}$.

Proof. We note that f is a completely continuous field on $L^2(\Omega_*)$. Arguing by contradiction we assume that there exists an exceptional family of elements $\{u_r\}_{r>0} \subset \mathbb{K}$ for f with respect to $\mathbb{K}$, in the sense of Definition

5.1.2. This means that for any $r > 0$ there exists $\mu_r > 0$ such that $v_r = \mu_r u_r + f(u_r)$ verifies the following conditions:

(α_1) $v_r \in \mathbb{K}^* = \mathbb{K}$ (in our case),

(α_2) $\int_{\Omega_*} v_r(x) u_r(x) dx = 0$,

(α_3) $\|u_r\| \to +\infty$ as $r \to +\infty$.

Considering (α_2), the definition of v_r and the expression of f we have

$$(\mu_r + 1)\|u_r\|^2 = \lambda \int_{\Omega_* \times \Omega_*} G(x, y, u_r(y)) u_r(x) dx\, dy.$$

By the last equality and the growth assumption (ii) we have

$$\mu_r + 1 \leq \frac{|\lambda|}{\|u_r\|^2} \int_{\Omega_* \times \Omega_*} \mathcal{R}(x, y)\left(a + b|u_r(y)|\right)|u_r(x)| dx\, dy$$

which implies (considering the Cauchy–Schwartz inequality)

$$\mu_r + 1 \leq \frac{|\lambda|}{\|u_r\|^2} \left[\begin{array}{l} a|\Omega_*|^{1/2} \|\mathcal{R}\|_{L^2(\Omega_* \times \Omega_*)} \|u_r\| \\ +b\|u_r\| \left(\int_{\Omega_*} \left(\int_{\Omega_*} \mathcal{R}(x, y) u_r(y) dy \right)^2 \right)^{1/2} \end{array} \right]$$

$$\leq \frac{|\lambda|}{\|u_r\|^2} \left[a|\Omega_*|^{1/2} \|\mathcal{R}\|_{L^2(\Omega_* \times \Omega_*)} \|u_r\| + b\|u_r\|^2 \|\mathcal{R}\|_{L^2(\Omega_* \times \Omega_*)} \right]$$

$$= |\lambda| \|\mathcal{R}\|_{L^2(\Omega_* \times \Omega_*)} \left(a|\Omega_*|^{1/2} \frac{1}{\|u_r\|} + b \right).$$

Letting $r \to +\infty$ in the last inequality and making use of (α_3) we obtain that

$$1 + \limsup_{r \to +\infty} \mu_r \leq b|\lambda| \|\mathcal{R}\|_{L^2(\Omega_* \times \Omega_*)}.$$

Now, we observe that the last inequality is in contradiction with assumption (2). Therefore, the mapping f is without an *EFE* with respect to $\mathbb{K}$ in the sense of Definition 5.1.2. By Theorem 5.1.2 we have that the problem *NCP*(f, $\mathbb{K}$) has a solution. Finally, we observe that if $b = 0$, then the problem *NCP*(f, $\mathbb{K}$) has a solution for any $\lambda \in \mathbb{R}$ (following the same proof presented above). □

Remark. The study of variational inequalities with an integral operator is opened to new developments.

8.5. Comments

We have presented in this book the main ideas developed until now about the notion of *exceptional family of elements*. A given mapping can have, or cannot have, an exceptional family of elements, with respect to an unbounded closed convex set. When a mapping is without an exceptional family of elements, this property can be considered as a generalized coercivity condition and (modulo some supplementary conditions) it implies the solvability of complementarity problems or of variational inequalities. It is evident that the notion of *exceptional family of elements* is related to deep notions and results well known in nonlinear analysis. We presented also this aspect. We recommend this book to any reader as a starting point for new developments related to this subject. In this moment we do not know if the method based on the notion of exceptional family of elements can be adapted to the study of *order complementarity problem.* We suspect that many results can be obtained on variational inequalities using the method presented in this book.

BIBLIOGRAPHY

AKHMEROV, R. R., KAMENSKII, M. I., POTAPOV, A. S., RODKINA, A. E. and SADOVSKII, B. N.

[1] Measures of Noncompactness and Condensing Operator, Birkhäuser, Verlag, Basel, Boston, Berlin (1992).

ALBER, Y. I.

[1] Metric and generalized projections operators in Banach spaces: Properties and applications, A. Kartsatos, (Ed.): Theory and Applications of Nonlinear Operators of Monotonic and Accretive Type, Marcel Dekker, New York (1996), 15-50.

AUBIN, J. P and FRANKOWSKA, H.

[1] Set-Valued Analysis, Birkhäuser, Boston (1990).

AUSSEL, D. and HADJISAVVAS, N.

[1] On quasimonotone variational inequalities, J. Opt. Theory Appl., 121 Nr. 2 (2004), 445-450.

BAIOCCHI, C. and CAPELO, A.

[1] Variational and Quasivariational Inequalities, John Wiley and Sons, Chichester, New York, (1984)

BANAS, J. and GOEBEL, K.

[1] Measures of Noncompactness in Banach Spaces, Marcel Dekker, New York, Basel (1980).

BEN-EL-MECHAIEKH, H.

[1] Continuous approximations of multifunctions, fixed points and coincidence, Approximation and Optimization in the Carribean II, Proceedings of the Second International Conference on Approximation and Optimization in the Carrribean, Florenzano et al. Eds., Verlag Peter Lang, Frankfurt, (1995), 69-97.

BEN-EL-MECHAIEKH, H. and DEGUIRE, P.

[1] Approachability and fixed points for non-convex set-valued maps, J. Math Anal. Appl. 170 (1992), 477-500.

BEN-EL-MECHAIEKH, H. and IDZIK, A.

[1] A Leray-Schauder type theorem for approximable maps, Proc. Amer. Math. Soc., 122 (1997), 105-109.

BEN-EL-MECHAIEKH, H., CHEBBI, S. and FLORENZANO, M.

[1] A Leray-Schauder type theorem for approximable maps: A simple proof, Proc. Amer. Math. Soc. 126 Nr. 8 (1998), 2345-2349.

BEN-EL-MECHAIEKH, H. and ISAC, G.

[1] Generalized multivalued variational inequalities, Analysis and Topology, (Eds. C. Andreian-Cazacu, O. Lehto and Th. M. Rassias), World Scientific (1998), 115-142.

BENSOUSSAN, A.

[1] Variational inequalities and optimal stopping time problems, In: Calculus of Variations and Control Theory (Ed. D. L. Russel), Academic Press, (1976), 219-244.

BENSOUSSAN, A. and LIONS, J. L.

[1] Problèmes de temps d'arrêt optimal et inéquations variationnelles paraboliques, Applicable Anal. (1973). 267-294.

[2] Nouvelle formulation de problèms de contrôle impulsionnel et applications, C.R. Acad. Sci. Paris, 276 (1973), A 1189-1192.

[3] Nouvellle methods en control impulsionnel, Appl. Math. Optim., 1 (1974), 289-312.

BENSOUSSAN, A., GOURSET, M. and LIONS, J. L.

[1] Contrôl impulsionnel et inequations quasi-variationnelles stationnaires, C. R. Acad. Sci. Paris 276 (1973), A 1279-1284.

BERGE, C.

[1] Topological Spaces, MacMillan, New York, (1963).

BERNSTEIN, S.

[1] Sur les équations de calcul des variations, Ann. Sci. Ecole Normale Sup. 29 (1912), 431-485.

BIANCHI, M., HADJISAVVAS, N. and SCHAIBLE, S.

[1] Minimal coercivity conditions and exceptional families of elements in quasimonotone variational inequalities, J. Opt. Theory Appl., 122 Nr. 1 (2004), 1-17.

BORWEIN, J. M. and DEMPSTER, M. A. H.

[1] The linear order complementarity problem, Math. Oper. Res. 14 (1989), 534-558.

BOURBAKI, N.

[1] Topologie Générale, Chp.1-10, Hermann, Paris, (1965).

[2] Espaces Vectoriels Topologique, Chap. 1-5, Hermann, Paris, (1965).

BREZIS, H.

[1] Analyse Fonctionnelle. Theory and Applications, Masson, Paris, (1983).

[2] Equations et inequations dans les espaces vectoriels en dualité, Ann. Inst. Fourier 18 (1968), 115-175.

BROWDER, F. E.

[1] Nonlinear elliptic boundary value problems and the generalized topological degree, Bull, Amer. Math. Soc., 76 (1976), 999-1005.

BULAVSKY, V., ISAC, G. and KALASHNIKOV, V.

[1] Application of topological degree theory to complementarity problems, Multilevel Optimization: Algorithms and Applications, (A. Migdalas et al. (Eds)), Kluwer Academic Publishers (1998), 333-358.

[2] Application of topological degree theory to semi-definite complementarity problem, Optimization, (2001), 405-423.

BURKE, J. and XU, S.

[1] The global linear convergence of a no-interior point path following algorithm for linear complemetarity problems, Meth. Oper. Res., 23 (1998), 719-734.

[2] A non-interior predictor-corrector path following algorithm for monotone linear complementarity problem, Math. Program., 87 (2000), 113-130.

CAPUZZO-DOLCETTA, I., LORENZANI, M. and SPIZZICHINO, F.

[1] Implicit complementarity problems and quasi-variational inequalities, In: Variational and Complementarity Problems. Theory and Applications (Eds. R. W. Cottle, F. Giannessi and J. L. Lions), John Wiley & Sons (1980), 271-283.

CAPUZZO-DOLCETTA, I. and MOSCO, U.

[1] A degenerate complementarity system and applications to the optimal stopping Markov chain, Boll, Un. Mat. Ital. (95), 17-B (1980), 692-703.

CARBONE, A. and ISAC. G.

[1] The generalized order complementarity problem. Applications to Economics, Nonlinear Studies, 5 Nr. 2 (1998), 129-151.

CARBONE, A. and ZABREIKO, P. P.

[1] Some remarks on complementarity problems in a Hilbert space, J. Anal. Appl., 21 (2002), 1005-1014.

[2] Explicit and implicit complementarity problems in a Hilbert space, J. Anal. Appl., 21 (2003), 31-41.

CAUTY, R.

[1] Solution du problème de point fixe de Schauder, Fund, Math., 170 (2001), 231-246.

CELLINA, A.

[1] A theorem on the approximation of compact valued mappings, Atti Accad. Naz. Lincei, Rend. ClSi Fis. Mat. Natur (8), 47 (1969), 429-433.

CHAN, D. and PANG, J. S.

[1] The generalized quasivariational inequality problem, Math. Oper. Res. 7 Nr. 2 (1982), 211-222.

CHANG, S. S and HUANG, N. J.

[1] Generalized multivalued implicit complementarity problems in Hilbert space, Math. Japonica 36 (1991), 1093-1100.

[2] Generalized random multivalued quasi-complementarity problems, Indian J. Math. 35 (1993), 305-320.

[3] Random generalized set-valued quasi-complementarity problems., Acta. Math. Appl. SINICA, 16 (1993), 396-405.

[4] Generalized complementarity problems for fuzzy mappings, Fuzzy sets and Systems, 55 (1993), 227-234.

CHEN, B. and CHEN, X.

[1] A global and local superlinear continuation –smoothing method for P_0 and R_0 NCP or monotone NCP, SIAM, J. Opt. 9 (1999), 624-645.

CHEN, B,. CHEN, X. and KANZOW, C.

[1] A Penalized Fisher-Burmeister NCP-function: Theoretical investigation and numerical results. Technical Report, Zur Angewandten Mathematik, Hamburger Beitrage (1997)

CHEN, B. and HARKER, P. T.

[1] A non-interior-point continuation method for linear complementarity problems, SIAM J. Matrix Anal. Appl., 14 (1993), 1168-1190.

[2] Smooth approximations to nonlinear complementarity problems, SIAM J. Optim. 7 (1997), 403-420.

CHEN, C and MANGASARIAN, O. L.

[1] A class of smoothing functions for nonlinear and mixed complementarity problems Comput. Optim. Appl. 5(1996), 97-138.

CIORĂNESCU, I.

[1] Geometry of Banach spaces. Duality Mappings and Nonlinear Problems, Kluwer Academic Publishers (1990)

CONWAY, J. B.

[1] A Course in Functional Analysis, (Second Edition), Springer (1990).

COTTLE, R. W.

[1] Nonlinear Program with Positively Bounded Jacobians, PhD. Thesis, Department of Mathematics, University of California, Berkley (1964).

[2] Note on a fundamental theorem in quadratic programming, SIAM J. Appl. Math., 12 (1964), 663-665.

[3] Nonlinear programs with positively bounded Jacobians, SIAM J. Appl. Math. 14 Nr. 1 (1966), 147-158.

COTTLE, R. W. and DANTZIG, G. B.

[1] A generalization of the linear complementarity problem, J. Combinatorial Theory, 8 (1970), 79-90.

COTTLE, R. W., HABETLER, G. J. and LEMKE, C. E.

[1] Quadratic forms semi-definite over convex cones, In: Proceedings of the Princeton Symposium on Mathematical Programming (H. W. Kuhn (Ed.)), Princeton University Press, Princeton, New Jersey (1970), 551-565.

[2] On classes of copositive matrices, Linear Algebra Appl., 3 (1970), 295-310.

COTTLE, R. W., PANG, J. S. and STONE, R. E.

[1] The Linear Complementarity Problem, Academic Press, Boston (1992)

DANEŠ, J.

[1] Generalized concentrative mappings and their fixed points, Comment. Math. Univ. Carolinae, 11 nr. 1 (1970), 115-136.

[2] On densifying and related mappings and their application in nonlinear functional analysis, Theory of Nonlinear Operators, Akademie-Verlag, Berlin (1974), 15-56.

DANTZIG, G. B. and COTTLE, R. W.

[1] Positive semi-definite programming, In: Nonlinear Programming (J. Abadie (Ed.)) North-Holland, Amsterdam (1967), 55-73.

DARBO, G.

[1] Punti unity in transformazioni a codominio non compatto, Rend. Sem. Math. Univ. Padova, 24 (1955), 84-92.

DAY, M. M.

[1] Normed Linear Spaces, (2nd ed.), Berlin –Göttingen-Heidelberg, (1962).

DEIMLING, K.

[1] Nonlinear Functional Analysis, Springer-Verelag, Berlin, Heidelberg, New York, Tokyo, (1985).

[2] Positive fixed points of weakly inward maps, Nonlinear Anal. Theory Meth. Appl., 12 Nr. 3 (1988), 223-226.

DING, X. P. and TAN, K. K.

[1] A minimax inequality with applications to existence of inequality with applications to existence of equilibrium point and fixed-point theorems, Colloq. Math. (1992), 233-247.

DORN, W. S.

[1] Self-dual quadratic programs, SIAM J. Appl. Math. Nr. 1 (1961), 51-54.

DUGUNDJI, J and GRANAS, A.

[1] Fixed Point Theory, PWN-Polish Scientific Publishers, Warszawa (1982)

DU VAL, P.

[1] The unloading problem for plan curves, Amer. J. Math. 62 (1940), 307-311.

DUVAUT, G. and LIONS, J. L.

[1] Les Inequations en Mechanique et en Physique, DUNOD, Paris (1972).

EAVES, B. C.

[1] The Linear Complementarity Problem in Mathematical Programming, PhD. Thesis, Department of Operations Research, Stanford University, Stanford, California (1969).

[2] On the basic theorem of complementarity, Math. Programming 1(1971), 68-75.

[3] The liner complementarity problem, Management Sci. 17 Nr. 9 (1971), 612-634.

[4] Computing stationary points, Math. Programming Study 7 (1978), 1-14.

[5] Where solving for stationary points by LCP is mixing Newton iterates. In: Homotopy Methods and Global Convergence (B.C. Eaves, F. J. Gould, H. O. Peitgen and M. J. Todd (Eds.)), Plenum (1983), 63-78.

[6] More with the Lemke complementarity algorithm, Math. Programming, 15 Nr. 2 (1978), 214-219.

[7] Thoughts on computing market equilibrium with SLCP, In: The Computation and Modelling of Economic Equilibrium, (A. Talman and G. Van der Laan (Eds.)), Elsevier Publishing Co. Amsterdam, (1987), 1-18.

EAVES, B. C. and LEMKE, C. E.

[1] Equivalence of LCP and PLC, Math. Oper. Research, 6. Nr. 4 (1981), 475-484.

[2] On the equivalence of the linear complementarity problem and system of piecewise linear equations, In: Homotopy Methods and Global Convergence, (B.C. Eaves, F. J. Gould.. H. O. Peitgen and J.J. Todd (EDS.)), Plenum Press (1983), 79-90.

EBIEFUNG, A. A.

[1] Nonlinear mapping associated with the generalized linear complementarity problem, Math. Programming, 69 (1995), 255-268.

EBIEFUNG, A. A. and KOSTREVA, M.M.

[1] The generalized Leontief input-output model and its application to the choice of new technology, Annals Oper. Res. 44 (1993), 161-172.

FACCHINEI, F.

[1] Structural and stability properties of P_0 nonlinear complementarity problems, Math. Oper. Res. 23 (1998), 735-749.

FACCHINEI, F. and KANZOW, C.

[1] Beyond monotonicity in regularization methods for nonlinear complementarity problems, SIAM J. Control Opt., 37 (1999), 1150-1161.

FERRIS, M. C. and PANG, J. S.

[1] Engineering and economic applications of complementarity problems, SIAM Rev. 39 (1997), 669-713.

FONSECA, I. and GANGBO, W.

[1] Degree Theory in Analysis and Applications, Oxford Science Publications, (1995)

FRIGON, M. and GRANAS, A.

[1] Résultats du type de Leray-Schauder pour des contractions multivoques, Topol. Methods Nonlinear Anal. 4 (1994), 197-208.

FRIGON, M., GRANAS, A. and GUENNOUN, Z.

[1] Alternative non-linéaire pour les applications contractants, Ann. Sci. Math. Québec, 19 (1995), 65-68.

FURI, M., MARTELLI, M. and VIGNOLI, A.

[1] On the solvability of nonlinear operators equations in normed spaces, Annali Mat. Pura Appl., 124 (1980), 321-343.

FURI, M. and PERA, M. P.

[1] An elementary approach to boundary value problems at resonance, Nonlinear Anal. Theory, Meth. Appl., 4 Nr. 6 (1980).

[2] On unbounded branches of solutions for nonlinear operator equations in the nonbifurcation case, Boll. Un. Mat. Ital. 1-B (1982), 919-930.

[3] Co-bifurcating branches of solutions for nonlinear eigenvalue problems in Banach spaces, Annali, Mat. Pura Appl. 135 (1983), 119-132.

FURI, M. and VIGNOLI, A.
[1] Unbounded nontrivial branches of eigenfunctions for nonlinear equations, Nonlinear Anal. Theory, Meth. Appl., 6 Nr. 11, (1982), 1267-1270.

FURI, M., PERA, M. P. and VIGNOLI, A.
[1] Components of positive solutions for nonlinear equations with several parameters, Boll. Un. Math. Ital. Serie VI, Vol. 1-C, Nr. 1 (1982), 285-302.

GAINES, R. E. and MAWHIN, J.
[1] Coincidence Degree and Nonlinear Differential Equations, Lecture Notes in Math., 568 (1977), Springer Berlin.

GATICA, J. A. and KIRK, W. A.
[1] Fixed point theorems for contraction mappings with applications to nonexpansive and contractive mappings, Rocky Mountain J. Math. 4 (1974), 69-79.

GEMIGNANI, M. C.
[1] Elementary Topology, Dover Publications Inc. New York (1972)

GIORGIERI, E. and VÄTH, M.
[1] A characterization of 0-epi maps with a degree, (Preprint), Univ, of Rome "TOR Vergata" and Univ. of Würzburg, (2001).

GORNIEWICZ, L., GRANAS, A. and KRYSZEWSKI, W.
[1] Sur la méthode de l'homotopie dans la théorie des points fixes pour les applications multivoques, Partie I, : Transversalité topologique, C. R. Acad. Sci. Paris Ser. I, Math. 307 (1988), 489-492.

GORNIEWICZ, L. and SLOSARSKI, M.
[1] Topological essentiality and differential inclusion, Bull. Austral. Math. Soc., 45 (1992), 177-193.

GOWDA, M. S. and PANG, J. S.
[1] Some existence results for multivalued complementarity problems, Mathematics Oper. Research, 17 Nr. 3 (1992), 657-669.

GOWDA, M. S. and SZNAJDER, R.
[1] The generalized order linear complementarity problem, SIAM, J. Matrix Anal. Appl., 15 Nr. 3 (1994), 779-795.

GOWDA, M. S. and TAWHID, M. A.
[1] Existence and limiting behavior of trajectories associated with P_0-equations, Comput. Optim. Appl. 12 (1999), 229-251.

GRANAS, A.
[1] The theory of compact vector fields and some applications to the theory of functional spaces, Rozprawy Matematyczne, Warszawa, 30 (1962).
[2] Homotopy extension theorem in Banach spaces and some of its applications to the theory of non-linear equations, Bull. Acad. Pol. Sci. 7(1959), 387-394.
[3] Continuation method for contractive maps, Topol. Methods Nonlinear Anal. 3 (1994), 375-379.

[4] On the Leray-Schauder alternative, Topol. Methods Nonlinear Anal. 2 (1993), 225-231.

GÜLER, O.

[1] Existence of interior points and interior-point paths in nonlinear monotone complementarity problems, Math. Oper. Res. 18 (1993), 128-147.

[2] Path following and potential reduction algorithm for nonlinear monotone complementarity problems, Technical Report Department of Management Sciences, The University of Iowa, Iowa City (1990).

HADJISAVVAS, N. and SCHAIBLE, S.

[1] On strong pseudomonotonicity. (Semi) strict quasimonotonicity, J. Opt. Theory Appl., 79 (1993), 139-156.

[2] Quasimonotone variational inequalities in Banach spaces, J. Opt. Theory Appl., 90 (1996), 95-111.

HARKER, P. T. and PANG, J. S.

[1] Finite dimensional variational inequality and nonlinear complementarity problems: a survey of theory algorithms and applications, Math. Programming 48 (1990), 161-220.

HOLMES, R. B.

[1] Geometric Functional Analysis and its Applications, Springer-Verlag (1975)

HUANG, N. J.

[1] Completely generalized strongly nonlinear quasi-complementarity problems for fuzzy mappings, Indian J. Pure Appl. Math., 28 (1997).

HUANG, N. J. and FANG, Y. P.

[1] Fixed point theorems and a new system of multivalued generalized order complementarity problems, Positivity, 7 (2003), 257-265.

HUANG, N. J, GAO, C. J. and HUANG, X. P.

[1] Exceptional family of elements and feasibility for nonlinear complementarity problems, (Report) (Dept. Math., Sichuan University, China)

HYERS, D. H., ISAC, G. and RASSIAS, TH. M.

[1] Topics in Nonlinear Analysis and Applications, World Scientific, (1997).

INGLETON, A. W.

[1] A problem in linear inequalities, Proc. London Math. Soc. 16 (1966), 516-536.

[2] The linear complementarity problem, J. London Math. Soc. (2) 2 (1970), 330-336.

ISAC, G.

[1] 0-Epi family of mappings, topological degree and optimization, J. Opt. Theory Appl., 42 Nr.1 (1984), 51-75.

[2] Problèms de Complementarité (en dimension infinie) Mincours, Université de Limoge, France (1984)

[3] Nonlinear complementarity problem and Galerkin method, J. Math. Anal. Appl., 108 Nr. 2 (1985), 563-574.

[4] On the implicit complementarity problem in Hilbert spaces, Bull. Austral. Math. Soc. 32 Nr. 2 (1985), 251-260.

[5] Complementarity problem and coincidence equations on convex cones, Boll. Unione Mat. Italiana (6), 5-B (1986), 925-943
[6] Fixed-point theory and complementarity problems in Hilbert spaces, Bull. Austral. Math. Soc. 36 Nr. 2 (1987), 295-310.
[7] Fixed point theory, coincidence equations on convex cones and complementarity problem, Contemporary Mathematics 72 (1988), 139-155.
[8] On some generalization of Karamardian's theorem on the complementarity problem, Bolletino U. M. I. (7), 2-B, (1988), 323-332.
[9] The numerical range theory and boundedness of solutions of the complementarity problems, J. Math. Anal. Appl., 142 (1989), 235-251.
[10] A special variational inequality and the implicit complementarity problem, J. Fac. Sci. Univ. Tokyo, Sec 1A Vol. 37 Nr. 1 (1990), 109-127.
[11] Iterative methods for the general order complementarity problem, App. Theory, Spline Functions and Applications (Ed. S. P. Singh).Kluwer Academic Publishers. (1992), 365-380.
[12] Complementarity Problems, Lecture Notes in Mathematics, Nr. 1528, Springer-Verlag, Berlin, Heidelberg, New York (1992).
[13] Tihonov's regularization and the complementarity problem in Hilbert spaces, J. Math. Anal. Appl., 174 Nr. 1 (1993), 53-66.
[14] Fixed-point theorems on convex cones, generalized pseudocontractive mappings and the complementarity problem, Bull. Institute Math., Acad. SINICA, 23 Nr. 1 (1995), 21-35.
[15] The fold complementarity problem, Topological Methods in Nonlinear Analysis, 8 Nr. 2 (1996), 343-358.
[16] Exceptional families of elements for k-fields in Hilbert spaces and complementarity theory, Proceed. Intern Conf Opt. Techniques Appl. (ICOTA'98) July 1-3 (1998), Perth Australia, 1135-1143.
[17] A generalization of Karamardian's condition in complementarity theory, Nonlinear Anal. Forum, 4 (1999), 49-63.
[18] On the solvability of multivalued complementarity problem: A topological method, EFDAN'99, Fourth European Workshop on Fuzzy Decision Analysis and Recognition Technology, Dortmund (Germany) June 14-15 (1999), 51-66.
[19] (0-k)-Epi mappings. Applications to complementarity theory. Math. Computer. Modeling, 32 (2000), 1433-1444.
[20] Topological Methods in Complementarity Theory, Kluwer Academic Publishers, (2000).
[21] Exceptional families of elements, feasibility and complementarity, J. Opt. Theory Appl., 104 Nr. 3 (2000), 577-588
[22] Exceptional families of elements, feasibility solvability and continuous path of ε-solutions for nonlinear complementarity problems, Approximation and Complexity in Numerical Optimization: Continuous and Discrete Problems (Eds.Pardalos) Kluwer, Academic Publishers, (2000), 323-337.
[23] Condition $(S)_+^1$, Altman's condition and the scalar asymptotic derivative applications to complementarity theory, Nonlinear Anal. Forum, 5 (2000), 1-13.
[24] On the solvability of complementarity problems by Leray-Schauder type alternatives, Libertas Mathematica 20 (2000), 15-22.
[25] Equivalence between (NCP) and fixed-point problem, Encyclopedia of Optimization (Eds. C. A. Floudas and P. M. Pardalos). Kluwer Academic Publishers (2001).

[26] Topological methods in complementarity theory, Encyclopedia of Optimization (Eds. C. A. Floudas and P. M. Pardalos). Kluwer Academic Publishers (2001).
[27] Order complementarity, Encyclopedia of Optimization (Eds. C. A. Floudas and P. M. Pardalos). Kluwer Academic Publishers (2001).
[28] Leray-Schauder type alternatives and the solvability of complementarity problems, Topological Methods in Nonlinear Analysis, 18 (2001), 191-204.
[29] An asymptotic Minty's type variational inequality with pseudomonotone operators, Nonlinear Analysis Forum, 8 (1), (2003), 55-64.
[30] On the complementarity problem with respect to a non-convex cone in Hilbert space, Fixed Point Theory and Appl. (International Journal), Vol. 4 Nr. 2 (2003), 223-236.
[31] Complementarity problems and variational inequalities, A unified approach of solvability by an implicit Leray-Schauder type alternative, J. Global Opt., 31, Nr. 3, (2005), 405-420.
[32] Solvability of nonlinear equations, global optimization and complementarity theory. An application to elasticity, Nonlinear Anal.. Forum, 10 (1) (2005), 81-96.

ISAC, G, BULAVSKI, V. A. and KALASHNIKOV, V. V.
[1] Exceptional families, topological degree and complementarity problems, J. Global Optimization 10 (1997), 207-225.
[2] Complementarity, Equilibrium, Efficiency and Economics, Kluwer Academic Publishers, (2002).

ISAC, G. and CARBONE, A.
[1] Exceptional families of elements for continuous functions: some applications to complementarity theory, J. Global Opt. 15 (1999), 181-196.

ISAC, G. and COJOCARU, M. G.
[1] Variational inequalities, complementarity problems and psuedo-monotonicity. Dynamical aspects, Seminar on Fixed Point Theory, Cluj-Napoca (Romania), Vol. 3 (2002), 41-62
[2] Functions without exceptional family of elements and the solvability of variational inequalities on unbounded sets, Topological Methods Nonlin. Anal., 20 (2002), 375-391.

ISAC, G. and GOWDA, M. S
[1] Operators of class $(S)^1_+$, Altman's condition and the complementarity problem, J. Fac. Sci. Univ. Tokyo ser. IA, 40 nr. 1 (1993), 67-74.

ISAC, G. and GOELEVEN, D.
[1] Existence theorems for the implicit complementarity problem, International J. Math. and Math. Sci. 16 Nr. 1 (1993), 67-74.
[2] The implicit general order complementarity problem, models and iterative methods, Annals. Oper. Res., 44 (1993), 63-92.

ISAC, G. and JINLU LI
[1] Complementarity problems, Karamardian's condition and a generalization of Harker-Pang condition, Nonlinear Anal. Forum, 6 Nr. 2 (2001), 383-390.

[2] The convergence property of Ishikawa iteration schemes in non-compact subsets of Hilbert spaces and its applications to complementarity theory, Computers and Math. with Appl., 47 (2004), 1745-1750.

[3] Exceptional family of elements and the solvability of complementarity problems in uniformly smooth and uniformly convex Banach spaces, J. Zhejiang Univ. Sci., (2005) 6A (4), 289-295.

ISAC, G. and KALASHNIKOV, V. V.

[1] Exceptional family of elements, Leray-Schauder alternative, pseudomonotone operators and complementarity, J. Opt. Theory Appl. 109 Nr. 1 (2001), 69-83.

ISAC, G. and KOSTREVA, M.

[1] The generalized order complementarity problem J. Opt. Theory Appl., 71 Nr. 3 (1991), 517-534.

[2] Kneser's theorem and the multivalued general order complementarity problem, Applied Math. Letters, 4 Nr. 6 (1991), 81-85.

[3] The implicit general order complementarity problem and Leontief's input-output model, Applications Mathematical, 24 Nr. 2 (1996), 113-125.

ISAC, G., KOSTREVA, M. and POLYASHUK, M.

[1] Relational complementarity problem, From Local to Global Optimization, (A. Miglalas et al. (Eds)), Kluwer Academic Publishers (2000), 327-339.

ISAC, G. and MOTREANU, D.

[1] On the solvability of complementarity problems and variational inequalities with integral operators, J. Convex and Nonlinear Analysis, 4 Nr. 3 (2003), 333-351.

ISAC, G. and NEMETH, S. Z.

[1] Duality in nonlinear complementarity theory by using inversions and scalar derivatives, (Forthcoming: Math. Inequalities Appl.)

[2] Duality of implicit complementarity problems by using inversions and scalar derivatives (Forthcoming: J. Opt. Theory Appl.)

[3] Duality in multivalued complementarity theory by using inversion and scalar derivatives (Forthcoming: J. Global Opt.)

[4] Scalar derivatives and scalar asymptotic derivatives. An Altman type fixed point theorem on convex cones and some applications, J. Math. Anal. Appl. , 290 (2004), 452-468.

[5] The asymptotic Browder-Hartman Stampacchia condition and interior bands of ε-solutions for nonlinear complementarity problems, (Report)

[6] REFE-acceptable mappings and a necessary and sufficient condition for the non-existence of the exceptional family of elements (Preprint 2005)

ISAC, G. and OBUCHOWSKA, W. T.

[1] Functions without exceptional families of elements and complementarity problems, J. Opt. Theory Appl., 99 Nr. 1 (1998), 147-163.

ISAC, G. and THERA, M.

[1] A variational principle. Application to the nonlinear complementarity problem, Nonlinear and Convex Analysis, Proceedings in honor of Ky Fan, (Eds. Bor-Luh Lin and S. Simons), Marcel Dekker, (1987), 127-145.

[2] Complementarity problem and the existence of the post critical equilibrium state of a thin elastic plate, J. Optim. Theory Appl., 58 Nr. 2 (1988), 241-257.

ISAC, G. and YUAN, G. X. Z

[1] Essential components and connectedness of solution set for complementarity problems, Progress in Optimization II (Eds. X. Yang et al) Kluwer Academic Publishers (2000), 153-165.

[2] The generic stability and existence of essentially connected components of solutions for nonlinear complementarity problems., J. Global Opt., 16 (2000), 95-105.

ISAC, G. and ZHAO, Y. B.

[1] Exceptional family of elements and the solvability of variational inequalities for unbounded sets in infinite dimensional Hilbert spaces, J. Math. Anal. Appl., 246 (2000), 544-556.

IZE, J., MASSABO, I., PEJSACHOWICZ, J. and VIGNOLI, A.

[1] Structure and dimension of global branches of solutions to multiparameter nonlinear equations, Trans. Amer. Math. Soc., 201 Nr. 2 (1985), 383-436.

JAMESON, G.

[1] Ordered Linear Spaces, Lecture Notes in Mathematics, Nr. 141, Springer-Verlag (1970).

KANEKO, I.

[1] The nonlinear complementarity problem, Term Paper, OR 340 C, Department of Operations Research, Stanford University, Stanford, Ca (1973).

[2] Parametric Complementarity Problem, PhD. Thesis, Stanford University, Stanford, California (1975).

[3] Isotone solutions of parametric linear complementarity problems, Math. Programming 12 (1977), 48-59.

[4] A parametric linear complementarity problem involving derivatives, Math. Programming 15 (1978), 146-154,

[5] A linear complementarity problem with n by $2n$ P-matrices, Math. Programming Study 7 (1978), 120-141.

[6] A maximization problem related to parametric linear complementarity, SIAM Journal Control Opt., 16 Nr. 1 (1978), 41-55.

[7] A maximization problem related to linear complementarity, Math. Programming 15 Nr. 2 (1978), 146-154.

[8] Linear complementarity problems and characterizations of Minkowski matrices, Linear Algebra and its Appl., 20 (1978), 113-130.

[9] The number of solutions of a class of linear complementarity problems, Math. Programming, 17 (1979), 104-105.

[10] A reduction theorem for the linear complementarity problem with a certain patterned matrix, Linear Algebra and its Appl., 21 (1978), 13-34.

[11] On some recent engineering applications of complementarity problems, Math. Programming Study, 17 (1982), 111-125.

[12] Complete solutions for a class of elastic-plastic structures, Comput. Methods Appl. Mech. Engineering, 21 (1980), 193-209.

[13] Piecewise linear elastic-plastic analysis, International J. Num. Methods in Engineering 14 (1979), 757-767.

KALASHNIKOW, V. V. and ISAC, G.
[1] Solvability of implicit complementarity problems, Annals of Oper. Research Vol. 116 (2002), 199-221.
KANZOW, C.
[1] Some nonlinear continuation methods for linear complementarity problems, SIAM J. Matrix Anal. Appl. 17 (1996), 851-868.

KARAMARDIAN, S.
[1] The nonlinear complementarity problem with applications, Part 1, J. Opt. Theory Appl. 4 (1969), 87-97.
[2] The nonlinear complementarity problem with applications, Part 2, J. Opt. Theory Appl. 4 (1969), 167-181.
[3] Generalized complementarity problem, J. Opt. Theory Appl. 8 (1971), 161-168.
[4] The complementarity problem, Math. Programming 2 (1972), 107-129.
[5] Complementarity problems over cones with monotone and pseudomonotone maps, J. Opt. Theory Appl., 18 Nr. 4 (1976), 445-454.

KARAMARDIAN, S. and SCHAIBLE, S.
[1] Seven kinds of monotone maps, J. Opt. Theory Appl. 66 Nr. 1 (1990), 37-46.

KINDERLEHRER, D.
[1] An Introduction to Variational Inequalities and their Applications, Academic Press (1980).

KOJIMA, M.
[1] Computational methods for solving nonlinear complementarity problems, Keio Engineering Report 27 (1974), 1-41.
[2] A unification of the existence theorem of the nonlinear complementarity problem, Math. Programming, 9 (1975), 257-277.
[3] Studied on piecewise-linear approximations of piecewise-C^1 mappings in fixed points and complementarity theory, Math. Oper. Research 3 (1978), 17-36..
[4] A complementarity pivoting approach to parametric nonlinear programming, Math. Oper. Res. 4 Nr. 4 (1979), 464-477.

KOJIMA, M., SHINDOH, S. and HARA, S.
[1] Interior-point methods for the monotone semidefinite linear complementarity problem in symmetric matrices, SIAM J. Optim. 7 (1) (1997), 86-125.

KOJIMA, M., MEGIDDO, N., NOMA, T. and YOSHISE, A.
[1] A unified Approach to Interior-Point Algorithms for Linear Complementarity Problems, Lecture Notes in Computer Sciences, Springer-Verlag, Berlin, Vol 538, (1991).

KOJIMA, M., MEGIDDO, N. and NOMA, T.
[1] Homotopy continuation methods for nonlinear complementarity problems, Math. Oper. Res. 16 (1991), 754-774.

KELLEY, J. L.
[1] General Topology, New York (1955)
KRASNOSELSKII, M. A.

[1] Topological Methods in the Theory of Nonlinear Integral Equations, (Engl Transl.), Pergamon Press, Oxford (1964).

KRASNOSELSKII, M. A. and ZABREIKO, P. P.
[1] Geometrical Methods of Nonlinear Analysis, Berlin, Springer-Verlag (1984).

KREIN, M. G. and RUTMAN, M. A.
[1] Linear operators leaving invariant a cone in a Banach space, Uspehi Matematiceskih Nauk (N. S.) 3, Nr. 1 (23), (1948), 3-95.

KURATOWSKI, K.
[1] Sur les espaces complets, Fund, Math. 15 (1930), 301-309.

LEMKE, C. E.
[1] Bimatrix equilibrium points and mathematical programming, Management Science, 11 (1965), 681-689.
[2] On complementarity pivot theory, In: Mathematics of Decision Sciences, Part 1 (G.B. Dantzig, A. F. Veinott, Jr. (Eds)) A.M.S, Providence, Rhode Island (1968), 95-114.
[3] Recent results on complementarity problems, In: Nonlinear Programming (J. B. Rosen, O. L. Mangasarian and K. Ritter, (Eds.), Academic Press, New York (1970), 349-384.
[4] Some pivot schemes for the linear complementarity problem, Math. Programming Study 7 (1978), 15-35.
[5] A brief survey of complementarity theory, In: Constructive Approaches to Mathematical Models, (C.V. Coffman and G. J. Fix (Eds)), Academic Press, New York, (1979).
[6] A survey of complementarity theory, In: Variational Inequalities and Complementarity Problems (R. W. Cottle, F. Giannessi and J.L. Lions (Eds.), John Wiley & Sons, (1980), 213-239.

LEMKE, C. E. and HOWSON, J. T.
[1] Equilibrium points of bimatrix games, SIAM J. Appl. Math. 12 (1964), 413-423.

LE, V. K. and SCHMITT, K.
[1] Global Bifurcation in Variational Inequalities, (Applications to Obstacles and Unilateral Problems) Springer-Verlag (1997).

LERAY, J. and SCHAUDER, J.
[1] Topologie et équations fonctionnelles, Ann. Sci. École Normale Sup., 51 (1934), 45-78.

LIONS, J. L. and MAGENES, E.
[1] Non-Homogeneous Boundary Value Problems and Applications, Vol. I, II, III, Springer-Verlag, (1972-1973).

LLOYD, N. G
[1] Degree Theory, Cambridge University Press, (1978).

LUNA, G.

[1] A remark on the nonlinear complementarity problem, Proc. Amer. Math. Soc. 48 Nr. 1 (1975), 132-134.

MASSABO, I, NISTRI, P. and PERA, M. P.
[1] A result on the existence of infinitely many solutions of a nonlinear elliptic boundary value problem at resonance, Boll. Un. Mat. Ital. (5), 17-A (1980), 523-530.

MARTELLI, M.
[1] Continuation Principles and Boundary Value Problems, Lecture Notes in Math.1537, Springer-Verlag (1993), 32-73.

MAULDIN, R. D. (Editor)
[1] The Scottish Book, Birkhauser, Basel-Stuttgart (1981)

MEGIDDO, N.
[1] Pathways to the optimal set in linear programming, In: Progress in Mathematical Programming: Interior-Point and Related Methods, (N. Megiddo ed.) Springer-Verlag, (1989), 131-158.

MIAO, J. .M.
[1] A quadratically convergent $O\left((k+1)\sqrt{n}L\right)$-iteration algorithm for the P_*-matrix linear complementarity problem, Math. Programming, Vol. 69, (1995), 355-374.

MINTY, G. J.
[1] On variational inequalities for monotone operators, (I), Advances in Math., 30 (1978), 1-7.

MONTEIRO, R. D. C. and ADLER, I.
[1] Interior path following primal dual algorithms, Part I: Linear programming, Math. Programming 44 (1989), 27-42.

MORÉ, J. J.
[1] Classes of functions and feasibility conditions in nonlinear complementarity problems, Math. Programming, 6 (1976), 327-338.

MORÉ, J. J. and RHEINBOLDT, W.
[1] On P and S-functions and related classes of N-dimensional nonlinear mappings, Linear Algebra and its Appl., 6 (1973), 45-68.

MOHAN, S. R., NEOGY, S. K. and SRIDHAR, R
[1] The generalized linear complementarity problem revisited, Math. Programming, 74 (1996), 197-218.

MORALES, C.
[1] Pseudo-contractive mappings and the Leray-Schauder boundary condition, Comment. Math. Univ. Carolin, 20 (1979), 745-756.

MOSCO, U.

[1] Implicit variational problems and quasi-variational inequalities, Lecture Notes in Math. Vol. 543, Springer-Verlag Berlin.
[2] On some non-linear quasi-variational inequalities and implicit complementarity problems in stochastic control theory, In: Variational Inequalities and Complementarity Problems. Theory and Applications (Eds. R.W. Cottle, F. Giannessi and J. L. Lions), John Wiley & Sons (1980), 271-283.

MOSCO, U. and SCARPINI, F.
[1] Complementarity systems and approximation of variational inequalities, RAIRO, Recherche Oper. (Analyse Numérique) (1975), 83-104.

MURTY, K. G.
[1] Linear Complementarity, Linear and Nonlinear Programming, Heldermann Verlag, Berlin (1988).

NÉMETH. S. Z.
[1] Ascalar derivative for vector functions, Rivista di Matematica Pura ed Applicata, Nr. 10 (1992), 7-24.
[2] Scalar derivatives and spectral theory, Mathematica, Vol. 35 (58), Nr. 1 (1993), 49-58.
[3] Scalar derivatives in Hilbert spaces, (Forthcoming: Positivity).

NOOR, M. A.
[1] Generalized complementarity problems, J. Math. Anal. Appl. 120 Nr. 1 (1986), 321-327.

NUSSBAUM, R. D.
[1] Degree theory for local condensing maps, J. Math. Anal. Appl., 35 (1972), 741-766.

OBUCHOWSKA, W. T.
[1] Exceptional families and existence results for nonlinear complementarity problem, J. Global Opt. 19 (2001), 183-198.

OH, K. P.
[1] The formulation of the mixed lubrication problem as a generalized nonlinear complementarity problem, J. Tribology, 108 (1986), 598-604.

ORTEGA, J. M. and RHEINBOLDT, W. C.
[1] Iterative Solutions of Nonlinear Equations in Several Variables, Academic Press, New York (1970).

O'REGAN, D.
[1] Fixed point theorems for nonlinear operators, J. Math. Anal. Appl. 202 (1996), 413-432.

O'REGAN, D. and PRECUP, R.
[1] Theorems of Leray-Schauder Type and Applications, Gordon and Breach Science Publishers, (2001).
PACHPATTE, B. G.

[1] Applications of the Leray-Schauder alternative to some Volterra integral and integrodifferential equations, Indian J. Pure Appl. Math. 26 (1995), 1161-1168.

PARIDA, J. and SEN, A.

[1] A class of nonlinear complementarity problems for multifunctions, J. Optim. Theory Appl. 53 (1987), 105-113.

PANG, J. S.

[1] The implicit complementarity problem, In: Nonlinear Programming 4 (Eds. Mangassarian, O. L., Meyer, R. R. and Robinson S. M.), Academic Press, New York, (1981), 487-518.

[2] On the convergence of a basic iterative method for the implicit complementarity problem, J. Opt. Theory Appl., 37 (1982), 149-162.

PARK, S.

[1] Fixed points of approximable maps, Proc. Amer. Math. Soc. 124 (1996), 3109-3114.

PENOT, J. P.

[1] A fixed point theorem for asymptotically contractive mappings, Proc. Amer. Math. Soc., 131 Nr. 8 (2003), 2371-2377.

PERA, M. P.

[1] Sula risolubilite di equazioni non lineari in spazi di Banch ordinati, Boll. Un. Mat. Ital., 17-B (1980), 1063-1075.

[2] A topological method for solving nonlinear equations in Banach spaces and some related results on the structure of solution set, Rend. Sem. Mat. Unives. Politecn., Torino, 41 Nr. 3 (1983), 9-30.

[3] Unbounded components of solutions of nonlinear equations at resonance, with applications to elliptic boundary value problems, Boll. Un. Mat, Ital., 2-B (1983), 469-481.

PERESSINI, A. L.

[1] Ordered Topological Vector Spaces, Harper and Row, New York (1967).

PETRYSHYN, V.

[1] Generalized Topological Degree, and Semilinear Equations, Cambridge Univ. Press, Cambridge (1995).

PETRYSHYN, V. and FITZPATRICK, P. M.

[1] A degree theory, fixed point theorems and mappings theorems for multivalued noncompact mappings, Trans. Amer. Math. Soc. 194 (1974), 1-25.

POINCARÉ, H.

[1] Sur certaines solutions particulières du problèmes de trois corps, Bull. Astronom., 1 (1984).

[2] Sur un théorème de géométrie, Rend. Circ. Mat. Palermo, 33 (1912), 357-407.

PRECUP. R.

[1] Methods in Nonlinear Integral Equation, Kluwer Academic Publishers, (2003).
[2] Foundations of the continuation principles of Leray-Schauder type, In: Proc. 23rd Conf. Geometry and Topology Cluj University, (Romania) (1994), 136-140.
[3] On the continuation principle for non expansive maps, Studia Univ. Babes-Bolyai Math. 41 Nr. 3 (1996), 85-89.

POTRA, F. A. and SHENG, R.
[1] Predictor-Corrector Algorithm for solving $P_*(\tau)$-matrix LCP from arbitrary positive starting points, Math. Programming, 76 (1996), 223-244.
[2] A large-step in feasible-interior-point method for the P_*-matrix LCP, SIAM J. Opt., 7 (1997), 318-335.
[3] Superlinearly convergent infeasible-interior-point algorithm for degenerate LCP, J. Opt. Theory Appl., 97 (1998), 249-269.

POTTER, A. J. B.
[1] An elementary version of the Leray-Schauder theorem, J. London Math. Soc. (2), 5 (1972), 414-416.

REICH, S.
[1] Fixed points of condensing functions, J. Math. Anal. Appl. 41 (1973), 460-467.
[2] A remark on set-valued mappings that satisfy the Leray-Schauder condition,
(I) Atti Acad. Naz. Lincei 61 (1976), 193-194,
(II) Atti Acad. Naz. Lincei 66 (1979), 1-2.

ROBINSON, S. M.
[1] Mathematical foundations of nonsmooth embedding methods, Math. Programming 48 (1990), 221-229.
[2] Normal maps induced by linear transformations, Math. Oper. Resesrch, 17 Nr. 3 (1992), 691-714.
[3] Homeomorphism conditions for normal maps of polyhedra, Optimization and Nonlinear Analysis, (Eds. A. Ioffe, M. Marcus and S. Reich), Longman, London (1992), 240-248.
[4] Nonsingularity and symmetry for linear normal maps, Math. Programming, 62 (1993), 415-425.

ROTHE, E. H.
[1] Introduction to Various Aspects of Degree Theory in Banach Spaces, American Math. Soc., Mathematical Surveys and Monographs, Nr. 23 (1986).

RUDIN, W.
[1] Functional Analysis, (Second Edition) McGraw-Hill, Inc., New York, (1991)

SADOVSKII, B. N.
[1] Asymptotically compact and densifying operators, Uspehi Mat. Nauk, 27 Nr. 1 (1972), 81-146.

SAIGAL, R.
[1] Extension of the generalized complementarity problem, Mathematics Oper. Research 1 Nr. 3 (1976), 260-266.

SCHAIBLE, S. and YAO, J. C.
[1] On the equivalence of nonlinear complementarity problems and least-element problems, Math. Programming, 70 (1995), 191-200.

SCHATTEN, R.

[1] Norm Ideals of Completely Continuous Operators, Springer, Berlin (1960).

SCHAEFER, H. H.

[1] Über die Methode der a-priori Schranken, Math. Anal. 129 (1955), 415-416.

[2] Topological Vector Spaces, Macmillan Company, New York (1966) (First edition).

SCHAUDER, J.

[1] Der fixpunktsatz functional raumen, Studia Math., 2 (1930), 171-180.

SCHÖNEBERG, R

[1] Leray-Schauder principles for condensing multivalued mappings in topological linear spaces, Proc. Amer. Math. Soc. 72 (1978), 268-270.

SCHWARTZ, J. T.

[1] Nonlinear Functional Analysis, Gordon and Breach, New York, (1969).

SKRYPNIK, I. V.

[1] Nonlinear Higher Order Elliptic Equations, Naukova Dumka, Kiev, (1973), (Russian)

[2] Methods for Analysis of Nonlinear Elliptic Boundary Value Problems, AMS Translations of Mathematical Monographs Vol. 139, (1994).

SMITH. T.E.

[1] A solution condition for complementarity problems with an application to spatial price equilibrium, Applied Math. and Computation, 15 Nr. 1 (1984), 61-69.

STAMPACCHIA, G.

[1] Variational Inequalities. Theory and Applications of Monotone Operators, Proc. NATO Adv. Study, Inst. Oderisi-Gubbio (1969).

SZANK, B. P.

[1] The Generalized Complementarity Problem, PhD. Thesis Rensselaer Polytechnic Institute, Troy, New York (1989).

SZNAJDER, R.

[1] Degree Theoretic Analysis of the Vertical and Horizontal Linear Complementarity Problem, PhD. Thesis, University of Maryland, USA (1994)

SZNAJDER, R. and GOWDA, M. S.

[1] Generalization of P_0- and P-properties; Extended vertical and horizontal linear complementarity problems, Linear Alg. Appl., 223/224 (1995), 695-715.

TAKAHASHI, W.

[1] Nonlinear Functional Analysis, Yokohama Publishers, (2000).

TARAFDAR, E. U. and THOMPSON, H. B.

[1] On the solvability of nonlinear, noncompact operator equations, J. Austral. Math. Soc. (Serie A), 43 (1987), 103-126.

TSENG, P.

[1] An infeasible path-following method for monotone complementarity problems, SIAM J. Opt. 7 (1997), 386-402.

VANDEBERGE, L., DeMOOR, B. L. and VANDERWALLE, J.
[1] The generalized linear complementarity problem applied to the complete analysis of resistive piecewise-linear circuits, IEEE Trans. Circuits and Syst., 36 (1989), 1382-1391.

VÄTH, M.
[1] On the connection of degree theory and 0-epi maps, J. Math. Anal. Appl., 257 (2001), 223-237.

VLADIMIROV, A. A., NESTEROV, Yu. E. and CHERKANOV, Yu. N.
[1] Uniformly convex functionals, Westnik Moscow, Univ. Ser. 15, Vychisl. Mat. Kibernet, (1978), 3-12.

WILLEM, M.
[1] Perturbation Nonlinéaire d'Opérateurs Linéaires dont le Noyau est de Dimension Infinie et Applications, Thèrse de Docteur en Sciences Appliquées, Université Catholique de Louvain, Faculté de Science Appliquées, Belgique (1979).

YAO, J. C.
[1] Multi-valued variational inequalities with *k*-pseudomonotone operators, J. Opt. Theory Appl., 83 (1994), 391-403.

ZABREYKO, P. P., KOSHELEV, A. I., KRASNOSELSKII, M. A., MIKHLIN, S. G., RAKOVSHCHIK, L. S. and STESENKO, V. Ya.
[1] Integral Equations, a Reference Text, Noordhoff International Publishing, Leyden (1975).

ZARANTONELLO, E. H.
[1] Projections on Convex Sets in Hilbert spaces and Spectral Theory. II. Spectral Theory. Contributions to Nonlinear Functional Analysis, Proc. Sympos. Math. Res. Center, Univ. Wisconsin, Madison, Wis. (1971), Academic Press, New York (1971), 343-424.

ZEIDLER, E.
[1] Nonlinear Functional Analysis and its Applications (I, Fixed-Point Theorems), Springer-Verlag (1985)

ZHAO, Y. B.
[1] Exceptional family and finite-dimensional variational inequality over polyhedral convex sets, Appl. Math. Comput. 87 (1997), 111-126.
[2] Existence Theory and Algorithms for Finite-dimensional Variational Inequality and Complementarity Problems, PhD. Dissertations, Institute of Applied Mathematics, Academia Sinica, Beijing, China (1998).
[3] D-orientation sequences for continuous functions and nonlinear complementarity problems, Appl. Math. Comput. 106 (1999), 221-235.
[4] Existence of a solution to nonlinear variational inequality under generalized positive homogeneity, Oper. Res. Letters, 25 (1999), 231-239.

ZHAO, Y. B. and HAN, J. Y.

[1] Exceptional family of elements for variational inequality problem and its applications, J. Global Optim. 14 (1999), 313-330.

ZHAO, Y. B., HAN, J. Y. and QI, H. D.

[1] Exceptional families and existence theorems for variational inequality problems, J. Opt. Theory Appl., 101 Nr. 2 (1999), 475-495.

ZHAO, Y. B. and ISAC, G.

[1] Quasi-P_* and $P(\tau, \alpha, \beta)$-maps, exceptional families of elements and complementarity problems, J. Opt. Theory Appl. 105 Nr. 1 (2000), 213-231.

[2] Properties of the multivalued mapping associated with some non-monotone complementarity problems, SIAM J. Control and Optimization, 39 Nr. 2 (2000), 571-593.

ZHAO, Y. B. and LI, D.

[1] Strict feasibility conditions in nonlinear complementarity problems, J. Opt. Theory Appl., 107 Nr. 2 (2000), 641-664.

ZHAO, Y. B. and SUN, D.

[1] Alternative theorems for nonlinear projection equations and applications to generalized complementarity problems, Nonlinear Anal. 46 (2001), 853-868.

ZHAO, Y. B. and YUAN, Y. J.

[1] An alternative theorem for generalized variational inequalities and solvability of nonlinear quasi-P_*^M-complementarity problem, Appl. Math. Computation 109 (2000), 167-182.

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