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Periodicities in Nonlinear Difference Equations

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To Lina and Maureen

Contents

Preface

"**Sharkovsky's Theorem**," the "**Period 3 implies chaos**" result of Li and Yorke, and the "**(3x+1)-Conjecture**" are beautiful and deep results showing the rich periodic character of first-order, non-linear difference equations. During the last ten years, we have been fascinated discovering non-linear difference equations of order greater than one which for certain values of their parameters have one of the following characteristics:

1. **Every solution of the equation is *periodic* with the same period.**

2. **Every solution of the equation is *eventually periodic* with a prescribed period.**

3. **Every solution of the equation *converges* to a periodic solution with the same period.**

Our goal in this monograph is to bring to the attention of the mathematical community these equations, together with **some thought-provoking questions and a great number of open problems and conjectures** which we strongly believe are worthy of investigation.

We would also like to begin the investigation of the global character of solutions of these equations for other values of their parameters and to attempt to see a more complete picture of the global behavior of their solutions.

We believe that the results in this monograph place a few more stones in the foundation of the "Basic Theory of Nonlinear Difference Equations of Order Greater Than One," where at the beginning of the third millennium, we surprisingly know so little.

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Chapter 1

PRELIMINARIES

1.1 Introduction

In this chapter we present some definitions and state some known results which will be useful in the subsequent chapters. For further details and additional references, see [5], [6], [33], [36], [37], [61], [62], [72], [101], [112], and [115].

The reader may simply glance at the results in this chapter and return for the details when they are needed in the sequel. In this way, this is a self-contained monograph, and the main prerequisite that the reader needs to understand the material in this book and to be able to attack the open problems and conjectures is a solid foundation in analysis.

1.2 Definitions of Stability and Linearized Stability

A *difference equation of order* $(k+1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}) \quad , \quad n = 0, 1, \ldots \tag{1.1}$$

where f is a continuous function which maps some set J^{k+1} into J. The set J is usually an interval of real numbers, or a union of intervals, but it may even be a discrete set such as the set of *integers* $\mathbf{Z} = \{\ldots, -1, 0, 1, \ldots\}$.

A *solution* of Eq.(1.1) is a sequence $\{x_n\}_{n=-k}^{\infty}$ which satisfies Eq.(1.1) for all $n \geq 0$. If we prescribe a set of $(k+1)$ *initial conditions*

$$x_{-k}, x_{-k+1}, \ldots, x_0 \in J$$

then

$$\begin{aligned} x_1 &= f(x_0, x_{-1}, \ldots, x_{-k}) \\ x_2 &= f(x_1, x_0, \ldots, x_{-k+1}) \\ &\vdots \end{aligned}$$

and so the solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1) exists for all $n \geq -k$ and is uniquely determined by the initial conditions.

A solution of Eq.(1.1) which is constant for all $n \geq -k$ is called an *equilibrium solution* of Eq.(1.1). If

$$x_n = \bar{x} \qquad \text{for all} \qquad n \geq -k$$

is an equilibrium solution of Eq.(1.1), then $\bar{x}$ is called an *equilibrium point*, or simply an *equilibrium*, of Eq.(1.1).

DEFINITION 1.1 **(Stability)**

(i) *We say that an equilibrium point $\bar{x}$ of Eq.(1.1) is* locally stable *if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(1.1) with*

$$|\,x_{-k} - \bar{x}\,| + |\,x_{1-k} - \bar{x}\,| + \cdots + |\,x_0 - \bar{x}\,| < \delta,$$

then

$$|\,x_n - \bar{x}\,| < \varepsilon \qquad \text{for all} \qquad n \geq -k.$$

(ii) *We say that an equilibrium point $\bar{x}$ of Eq.(1.1) is* locally asymptotically stable *if $\bar{x}$ is locally stable, and if in addition there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(1.1) with*

$$|\,x_{-k} - \bar{x}\,| + |\,x_{-k+1} - \bar{x}\,| + \cdots + |\,x_0 - \bar{x}\,| < \gamma,$$

then

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iii) *We say that the the equilibrium point $\bar{x}$ of Eq.(1.1) is a* global attractor *if for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1), we have*

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iv) *We say that the the equilibrium point $\bar{x}$ of Eq.(1.1) is* globally asymptotically stable *if $\bar{x}$ is locally stable, and $\bar{x}$ is also a global attractor of Eq.(1.1).*

(v) *We say that the the equilibrium point $\bar{x}$ of Eq.(1.1) is* unstable *if $\bar{x}$ is not locally stable.*

(vi) *We say that the the equilibrium point $\bar{x}$ of Eq.(1.1) is a* source *if there exists $r > 0$ such that for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1) with*

$$0 < |\,x_{-k} - \bar{x}\,| + |\,x_{-k+1} - \bar{x}\,| + \cdots + |\,x_0 - \bar{x}\,| < r,$$

there exists $N \geq 1$ such that

$$|\,x_N - \bar{x}\,| > r.$$

Clearly a source is an unstable equilibrium point of Eq.(1.1).

Suppose f is continuously differentiable in some open neighborhood of $\bar{x}$. Let

$$p_i = \frac{\partial f}{\partial u_i}(\bar{x}, \bar{x}, \ldots, \bar{x}) \qquad \text{for} \qquad i = 0, 1, \ldots, k$$

denote the partial derivative of $f(u_0, u_1, \ldots, u_k)$ with respect to u_i evaluated at the equilibrium point $\bar{x}$ of Eq.(1.1). Then the equation

$$y_{n+1} = p_0 z_n + p_1 z_{n-1} + \cdots + p_k z_{n-k} \quad , \qquad n = 0, 1, \ldots \tag{1.2}$$

is called the *linearized equation of Eq.(1.1) about the equilibrium point* $\bar{x}$, and the equation

$$\lambda^{k+1} - p_0 \lambda^k - \cdots - p_{k-1}\lambda - p_k = 0 \tag{1.3}$$

is called the *characteristic equation of Eq.(1.2) about* $\bar{x}$.

The following well-known result, called the *Linearized Stability Theorem*, is very useful in determining the local stability character of the equilibrium point $\bar{x}$ of Eq.(1.1). See [6], [36], [62], and [72].

THEOREM 1.1
(The Linearized Stability Theorem)
Suppose f is a continuously differentiable function defined on some open neighborhood of $\bar{x}$. Then the following statements are true:

1. *If all the roots of Eq.(1.3) have absolute value less than one, then the equilibrium point $\bar{x}$ of Eq.(1.1) is locally asymptotically stable.*
2. *If at least one root of Eq.(1.3) has absolute value greater than one, then the equilibrium point $\bar{x}$ of Eq.(1.1) is unstable.*
3. *If all the roots of Eq.(1.3) have absolute value greater than one, then the equilibrium point $\bar{x}$ of Eq.(1.1) is a source.*

The equilibrium point $\bar{x}$ of Eq.(1.1) is called *hyperbolic* if no root of Eq.(1.3) has absolute value equal to one. If there exists a root of Eq.(1.3) with absolute value equal to one, then $\bar{x}$ is called *non-hyperbolic.*

The equilibrium point $\bar{x}$ of Eq.(1.1) is called a *sink* if every root of Eq.(1.3) has absolute value less than one. Thus a sink is locally asymptotically stable, but the converse need not be true.

The equilibrium point $\bar{x}$ of Eq.(1.1) is called a *saddle point equilibrium point* if it is hyperbolic, and if in addition, there exists a root of Eq.(1.3) with absolute value less than one and another root of Eq.(1.3) with absolute value greater than one. In particular, a saddle point equilibrium point is unstable.

A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1) is called *periodic with period* p (or a period-p solution) if there exists an integer $p \geq 1$ such that

$$x_{n+p} = x_n \qquad \text{for all} \qquad n \geq -k. \tag{1.4}$$

We say that the solution is periodic with *prime period* p if p is the smallest positive integer for which Eq.(1.4) holds. In this case, a *p-tuple*

$$(x_{n+1},\ x_{n+2}, \ldots,\ x_{n+p})$$

of any p consecutive values of the solution is called a *p-cycle* of Eq.(1.1).

A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1) is called *eventually periodic with period* p if there exists an integer $N \geq -k$ such that $\{x_n\}_{n=N}^{\infty}$ is periodic with period p; that is,

$$x_{n+p} = x_n \qquad \text{for all} \qquad n \geq N.$$

The following lemma describes when a solution of Eq.(1.1) converges to a periodic solution of Eq.(1.1).

LEMMA 1.1

Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1.1), and let $p \geq 1$ be a positive integer. Suppose there exist real numbers $l_0, l_1, \ldots, l_{p-1} \in J$ such that

$$\lim_{n\to\infty} x_{pn+j} = l_j \qquad \textit{for all} \qquad j = 0, 1, \ldots, p-1.$$

Finally, let $\{y_n\}_{n=-k}^{\infty}$ be the period-p sequence of real numbers in J such that for every integer j with $0 \leq j \leq p-1$, we have

$$y_{pn+j} = l_j \qquad \textit{for all} \qquad n = 0, 1, \ldots .$$

Then the following statements are true:

1. *$\{y_n\}_{n=-k}^{\infty}$ is a period-p solution of Eq.(1.1).*
2. $\lim_{n\to\infty} x_{pn+j} = y_j \qquad$ *for every* $\qquad j \geq -k.$

PROOF It suffices to show that $\{y_n\}_{n=-k}^{\infty}$ is a solution of Eq.(1.1). Note that for $j \geq 0$, we have

$$\begin{aligned} y_{j+1} &= \lim_{n\to\infty} x_{pn+j+1} = \lim_{n\to\infty} f(x_{pn+j}, x_{pn+j-1}, \ldots, x_{pn+j-k}) \\ &= f(y_j, y_{j-1}, \ldots, y_{j-k}). \end{aligned}$$

∎

We now state Theorem 1.2 which explains the significance of Eq.(1.3) having root(s) with absolute value less than one and also root(s) with absolute value greater than one. We shall use Theorem 1.2 to show that when this occurs, even though the equilibrium $\bar{x}$ is unstable, there still exist non-trivial solutions which converge to it, and so in particular, there exist non-trivial, bounded solutions of Eq.(1.1).

Our presentation is extracted from the treatment of this topic found in [112].

We first need some notation.

Let V be a non-empty open subset of $\mathbf{R}^{k+1}$, and let $T : V \to R^{k+1}$ be a (not necessarily invertible) C^m map, where $1 \leq m \leq \infty$.

Let $\vec{p} \in V$ be an equilibrium point of T. Suppose that at least one eigenvalue of $J_T(\vec{p})$ has absolute value less than one and at least one eigenvalue of $J_T(\vec{p})$ has absolute value greater than one.

Given an open subset V_1 of V with $\vec{p} \in V_1$, the *local stable manifold* for $\vec{p}$ in the neighborhood V_1 is defined as follows:

$$\mathcal{S}(\vec{p}, V_1, T) = \{\vec{q} \in V_1 : T^n(\vec{q}) \in V_1 \text{ for all } n \geq 0 \text{ and}$$

$$\lim_{n\to\infty} ||T^n(\vec{q}) - \vec{p}|| = 0\}.$$

Define a *past history* of a point $\vec{q}$ to be a sequence of points $\{\vec{q}_{-n}\}_{n=0}^{\infty}$ such that $\vec{q}_0 = \vec{q}$ and $T(\vec{q}_{-n-1}) = \vec{q}_{-n}$ for all $n \geq 0$. The *local unstable manifold* for $\vec{p}$ in the neighborhood V_1 is defined as follows:

$$\mathcal{U}(\vec{p}, V_1, T) = \{\vec{q} \in V_1 \text{ :there exists a past history } \{\vec{q}_{-n}\}_{n=0}^{\infty} \subset V_1 \text{ of } \vec{q} \text{ such that}$$

$$\lim_{n\to\infty} ||T^{-n}(\vec{q}) - \vec{p}|| = 0\}.$$

Let S be the eigen-space of $J_T(\vec{p})$ which corresponds to eigenvalues with absolute value less than one, and let U be the eigen-space of $J_T(\vec{p})$ which corresponds to eigenvalues with absolute value greater than one.

THEOREM 1.2
There exists an open subset V_1 of V with $\vec{p} \in V_1$ such that $\mathcal{S}(\vec{p}, V_1, T)$ and $\mathcal{U}(\vec{p}, V_1, T)$ are C^m manifolds. The tangent space of $\mathcal{S}(\vec{p}, V_1, T)$ at $\vec{p}$ is S, and the tangent space of $\mathcal{U}(\vec{p}, V_1, T)$ at $\vec{p}$ is U.

Theorem 1.2 can be extended in a straightforward fashion to the case where $\vec{p}$ is a periodic point of T.

We apply Theorem 1.2 to Eq.(1.1) as follows:

Suppose I is an open sub-interval of J with the equilibrium point $\bar{x} \in I$. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1.1). For $n \geq 0$, set

$$u_n^0 = x_{n-k},\ u_n^1 = x_{n-k+1},\ \ldots,\ u_n^{k-1} = x_{n-1},\ u_n^k = x_n.$$

Then for $n \geq 0$, we have

$$\begin{aligned} u_{n+1}^0 &= x_{n-k+1} = u_n^1 \\ u_{n+1}^1 &= x_{n-k+2} = u^2 \\ &\vdots \\ u_{n+1}^{k-1} &= x_n = u_n^k \\ u_{n+1}^k &= f(x_n, x_{n-1}, \ldots, x_{n-k}) = f(u_n^k, u_n^{k-1}, \ldots, u_n^0) \end{aligned}$$

and so $T : I^{k+1} \to I^{k+1}$ is given by

$$T\left(\begin{bmatrix} u^0 \\ u^1 \\ \vdots \\ u^{k-1} \\ u^k \end{bmatrix}\right) = \begin{bmatrix} u^1 \\ u^2 \\ \vdots \\ u^k \\ f(u^k, u^{k-1}, \ldots, u^{k-1}, u^k) \end{bmatrix}.$$

Thus the eigenvalues of the Jacobian $J_T\left(\begin{bmatrix} \bar{x} \\ \bar{x} \\ \vdots \\ \bar{x} \\ \bar{x} \end{bmatrix}\right)$ at the equilibrium point $\bar{x}$ of Eq.(1.1) are the roots of Eq.(1.3).

The following three theorems give readily verifiable necessary and sufficient conditions for all the roots of a real polynomial of degree two, three, or four, respectively, to have modulus less than one.

THEOREM 1.3
Consider the second-degree polynomial equation

$$\lambda^2 + a_1\lambda + a_0 = 0 \tag{1.5}$$

where a_0 and a_1 are real numbers. Then the following statements are true:

1. *A necessary and sufficient condition for both roots of Eq.(1.5) to lie inside the open disk $|\lambda| < 1$ is*

$$|a_1| < 1 + a_0 < 2.$$

2. *A necessary and sufficient condition for Eq.(1.5) to have one root with absolute value less than one and the other with absolute value greater than one is*

$$a_1^2 - 4a_0 > 0 \qquad \text{and} \qquad |a_1| > |1 + a_0|.$$

3. *A necessary and sufficient condition for both roots of Eq.(1.5) to have absolute value greater than one is*

$$|a_0| > 1 \qquad \text{and} \qquad |a_1| < |1 + a_0|.$$

THEOREM 1.4
Consider the third-degree polynomial equation

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \tag{1.6}$$

where a_0, a_1, and a_2 are real numbers. Then a necessary and sufficient condition that all roots of Eq.(1.6) lie in the open disk $|\lambda| < 1$ is

$$|a_2 + a_0| < 1 + a_1 \quad , \quad |a_2 - 3a_0| < 3 - a_1 \quad , \quad \text{and} \quad a_0^2 + a_1 - a_0 a_2 < 1.$$

THEOREM 1.5
Consider the fourth-degree polynomial equation

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \tag{1.7}$$

where a_0, a_1, a_2, and a_3 are real numbers. Then a necessary and sufficient condition that all roots of Eq.(1.7) lie in the open disk $|\lambda| < 1$ is

$$|a_1 + a_3| < 1 + a_0 + a_2 \quad , \quad |a_1 - a_3| < 2(1 - a_0) \quad , \quad a_2 - 3a_0 < 3$$

and

$$a_0 + a_2 + a_0^2 + a_1^2 + a_0^2 a_2 + a_0 a_3^2 < 1 + 2a_0 a_2 + a_1 a_3 + a_0 a_1 a_3 + a_0^3.$$

The next result, which was given by C.W. Clark in [25] in his investigation of baleen whale populations, provides a sufficient condition that the equilibrium point $\bar{x}$ of Eq.(1.1) be a sink.

THEOREM 1.6
Assume that $p_0, p_1, \ldots, p_k$ are real numbers such that

$$|\,p_0\,| + |\,p_1\,| + \cdots + |\,p_k\,| < 1.$$

Then all roots of Eq.(1.3) lie inside the open unit disk $|\lambda| < 1$.

The following *Comparison Theorem*, the proof of which follows by induction and will be omitted, will often be useful in establishing bounds for the solutions of Eq.(1.1).

THEOREM 1.7

(Comparison Theorem)

Let m be a non-negative integer, let $\alpha_0, \alpha_1, \ldots, \alpha_m$ be non-negative real numbers, and let β be a real number. Suppose that $\{x_n\}_{n=-m}^{\infty}$, $\{y_n\}_{n=-m}^{\infty}$, and $\{z_n\}_{n=-m}^{\infty}$ are sequences of real numbers such that

$$x_n \leq y_n \leq z_n \qquad \text{for all} \qquad -m \leq n \leq 0$$

and such that

$$\left.\begin{array}{l} x_{n+1} \leq \alpha_0 x_n + \alpha_1 x_{n-1} + \cdots + \alpha_m x_{n-m} + \beta \\ y_{n+1} = \alpha_0 y_n + \alpha_1 y_{n-1} + \cdots + \alpha_m y_{n-m} + \beta \\ z_{n+1} \geq \alpha_0 z_n + \alpha_1 z_{n-1} + \cdots + \alpha_m z_{n-m} + \beta \end{array}\right\} \quad \text{for all} \quad n = 0, 1, \ldots.$$

Then

$$x_n \leq y_n \leq z_n \qquad \text{for all} \qquad n \geq -m.$$

1.3 Semi-cycle Analysis

Assume that $\bar{x}$ is an equilibrium point of Eq.(1.1), and let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1.1).

A *positive semi-cycle* of $\{x_n\}_{n=-k}^{\infty}$ consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all greater than or equal to $\bar{x}$, with $l \geq -k$ and $m \leq \infty$ such that

either $l = -k$ or $l > -k$ and $x_{l-1} < \bar{x}$

and

either $m = \infty$ or $m < \infty$ and $x_{m+1} < \bar{x}$.

A *negative semi-cycle* of $\{x_n\}_{n=-k}^{\infty}$ consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all less than $\bar{x}$, with $l \geq -k$ and $m \leq \infty$ such that

$$\text{either } l = -k \quad \text{or} \quad l > -k \text{ and } x_{l-1} \geq \bar{x}$$

and

$$\text{either } m = \infty \quad \text{or} \quad m < \infty \text{ and } x_{m+1} \geq \bar{x}.$$

A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1) is called *non-oscillatory* about $\bar{x}$ if there exists $N \geq -k$ such that

$$\text{either } x_n > \bar{x} \text{ for all } n \geq N$$

or

$$x_n < \bar{x} \qquad \text{for all } n \geq N.$$

Otherwise, $\{x_n\}_{n=-k}^{\infty}$ is called *oscillatory* about $\bar{x}$.

A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1) is called *strictly oscillatory* about $\bar{x}$ if for every $N \geq -k$ there exist $m, n \geq N$ such that $x_m < \bar{x}$ and $x_n > \bar{x}$.

We say that a positive solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1) *persists* (or is *persistent*) if there exists a positive constant m such that

$$m \leq x_n \qquad \text{for all} \qquad n \geq -k.$$

Eq.(1.1) is said to be *permanent* if there exist positive real numbers m and M such that for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1) there exists an integer $N \geq -k$ (which depends upon the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$) such that

$$m \leq x_n \leq M \qquad \text{for all} \qquad n \geq N.$$

The importance of permanence for biological systems was thoroughly reviewed by Hutson and Schmitt. See [66].

1.4 Full Limiting Sequences

The following theorem treats the method of Full Limiting Sequences. See [40], [70], [71], and [111]. On several occasions we shall utilize the method of Full Limiting Sequences to establish that all solutions of a given difference equation converge to the equilibrium of that equation.

THEOREM 1.8
Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}) \tag{1.8}$$

where $f \in C[J^{k+1}, J]$ *for some interval* J *of real numbers and some non-negative integer* k. *Let* $\{x_n\}_{n=-k}^{\infty}$ *be a solution of Eq.(1.8). Set* $I = \liminf_{n\to\infty} x_n$ *and* $S = \limsup_{n\to\infty} x_n$, *and suppose that* $I, S \in J$. *Let* $\mathcal{L}_0$ *be a limit point of the sequence* $\{x_n\}_{n=-k}^{\infty}$. *Then the following statements are true:*

1. *There exists a solution* $\{L_n\}_{n=-\infty}^{\infty}$ *of Eq.(1.8), called a full limiting sequence of* $\{x_n\}_{n=-k}^{\infty}$, *such that* $L_0 = \mathcal{L}_0$, *and such that for every* $N \in \{\ldots, -1, 0, 1, \ldots\}$, L_N *is a limit point of* $\{x_n\}_{n=-k}^{\infty}$. *In particular,*

$$I \le L_N \le S \quad \text{for all} \quad N \in \{\ldots, -1, 0, 1, \ldots\}.$$

2. *For every* $i_0 \in \{\ldots, -1, 0, 1, \ldots\}$, *there exists a subsequence* $\{x_{r_i}\}_{i=0}^{\infty}$ *of* $\{x_n\}_{n=-m}^{\infty}$ *such that*

$$L_N = \lim_{i\to\infty} x_{r_i+N} \quad \text{for every} \quad N \ge i_0.$$

PROOF Since $I = \liminf_{n\to\infty} x_n \in J$ and $S = \limsup_{n\to\infty} x_n \in J$, there exist constants $A, B \in J$ such that

$$A \le x_n \le B \quad \text{for all} \quad n \ge -k.$$

We shall first show that there exists a solution $\{l_n\}_{n=-k-1}^{\infty}$ of Eq.(1.8) such that $l_0 = \mathcal{L}_0$, and such that for every $N \ge -k-1$, l_N is a limit point of $\{x_n\}_{n=-k}^{\infty}$.

To this end, observe that there exists a subsequence $\{x_{n_i}\}_{i=0}^{\infty}$ of $\{x_n\}_{n=-k}^{\infty}$ such that

$$\lim_{i\to\infty} x_{n_i} = \mathcal{L}_0.$$

Now the subsequence $\{x_{n_i-1}\}_{i=1}^{\infty}$ of $\{x_n\}_{n=-k}^{\infty}$ also lies in the interval $[A, B]$, and so it has a limit point, which we denote by $\mathcal{L}_{-1}$. It follows that there exists a further subsequence $\{x_{n_j}\}_{j=0}^{\infty}$ of $\{x_{n_i}\}_{i=0}^{\infty}$ such that $\lim_{j\to\infty} x_{n_j-1} = \mathcal{L}_{-1}$. Thus we see that

$$\lim_{j\to\infty} x_{n_{j-1}} = \mathcal{L}_{-1} \quad \text{and} \quad \lim_{j\to\infty} x_{n_j} = \mathcal{L}_0.$$

It follows similarly to the above that after re-labeling, if necessary, we may assume that

$$\lim_{j\to\infty} x_{n_j-k-1} = \mathcal{L}_{-k-1}, \ \lim_{j\to\infty} x_{n_j-k} = \mathcal{L}_{-k}, \ \ldots, \ \lim_{j\to\infty} x_{n_j} = \mathcal{L}_0.$$

Consider the solution $\{l_n\}_{n=-k-1}^{\infty}$ of Eq.(1.8) with initial conditions

$$l_{-1} = \mathcal{L}_{-1},\ l_{-2} = \mathcal{L}_{-2},\ \ldots,\ l_{-k-1} = \mathcal{L}_{-k-1}.$$

Then

$$l_0 = f(\mathcal{L}_{-1}, \mathcal{L}_{-2}, \ldots, \mathcal{L}_{-k-1}) = \lim_{j\to\infty} f(x_{n_j-1}, x_{n_j-2}, \ldots, x_{n_j-k-1})$$

$$= \lim_{j\to\infty} x_{n_j} \ = \ \mathcal{L}_0.$$

It follows by induction that the solution $\{l_n\}_{n=-k-1}^{\infty}$ of Eq.(1.8) has the desired property that $l_0 = \mathcal{L}_0$, and that l_N is a limit point of $\{x_n\}_{n=-k}^{\infty}$ for every $N \geq -k-1$.

Let $\mathcal{S}$ be the set of all solutions $\{\tilde{L}_n\}_{n=-m}^{\infty}$ of Eq.(1.8) such that the following statements are true.

(i) $-\infty \leq -m \leq -k-1$.

(ii) $\tilde{L}_n = l_n$ for all $n \geq -k-1$.

(iii) For every $j_0 \in \text{domain}\{\tilde{L}_n\}_{n=-m}^{\infty}$, there exists a subsequence $\{x_{n_l}\}_{l=0}^{\infty}$ of $\{x_n\}_{n=-k}^{\infty}$ such that

$$\tilde{L}_N = \lim_{l\to\infty} x_{n_l+N} \qquad \text{for all} \qquad N \geq j_0.$$

Clearly $\{l_n\}_{n=-k-1}^{\infty} \in \mathcal{S}$, and so $\mathcal{S} \neq \emptyset$. Given $y, z \in \mathcal{S}$, we say that $y \preceq z$ if $y \subset z$. It follows that $(\mathcal{S}, \preceq)$ is a partially ordered set which satisfies the hypotheses of Zorn's Lemma, and so we see that $\mathcal{S}$ has a maximal element, which clearly is the desired solution $\{L_n\}_{n=-\infty}^{\infty}$. ∎

1.5 Convergence Theorems

The following lemma provides sufficient conditions for establishing the boundedness of solutions of certain equations.

LEMMA 1.2

Let J be an interval of real numbers, and assume that $f \in C[J^{k+1}, J]$ is non-decreasing in each of its arguments. Suppose also that every point c in J is an equilibrium point of Eq.(1.1); that is,

$$f(c, c, \ldots, c) = c \qquad \textit{for every} \qquad c \in I.$$

Let $\{x_n\}_{n=-k}^{\infty}$ *be a solution of Eq.(1.1). Set*

$$m = \min\{x_{-k}, x_{-k+1}, \ldots, x_0\} \qquad \textit{and} \qquad M = \max\{x_{-k}, x_{-k+1}, \ldots, x_0\}.$$

Then

$$m \le x_n \le M \qquad \textit{for all} \qquad n \ge -k.$$

PROOF Clearly

$$m \le x_n \le M \qquad \text{for all} \qquad -k \le n \le 0.$$

Now $x_1 = f(x_0, x_{-1}, \ldots, x_{-k})$, and

$$m = f(m, m, \ldots, m) \le f(x_0, x_{-1}, \ldots, x_{-k}) \le f(M, M, \ldots, M) = M.$$

It follows by induction that $m \le x_n \le M$ for all $n \ge -k$. ▯

The following convergence result will provide some powerful applications in certain rational equations. See [40] and [41].

THEOREM 1.9
Let J *be an interval of real numbers. Assume that the following statements are true:*

1. $f \in C[J^{k+1}, J]$ *is non-decreasing in each of its arguments.*
2. $f(z_1, z_2, \ldots, z_{k+1})$ *is strictly increasing in each of the arguments* $z_{i_1}, z_{i_2}, \ldots, z_{i_l}$ *where* $1 \le i_1 < i_2 < \ldots < i_l \le k+1$, *and where* $i_1, i_2, \ldots, i_l$ *are relatively prime.*
3. *Every point* c *in* J *is an equilibrium point of Eq.(1.1).*

Then every solution of Eq.(1.1) has a finite limit.

PROOF Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1.1). It suffices to show that there exists $\bar{x} \in J$ such that $\lim_{n\to\infty} x_n = \bar{x}$.

We know by Lemma 1.2 that

$$m \le x_n \le M \qquad \text{for all} \qquad n \ge -k$$

where

$$m = \min\{x_{-k}, x_{-k+1}, \ldots, x_0\} \qquad \text{and} \qquad M = \max\{x_{-k}, x_{-k+1}, \ldots, x_0\}.$$

To complete the proof, it suffices to show that

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n.$$

So for the sake of contradiction, suppose that

$$\liminf_{n\to\infty} x_n < \limsup_{n\to\infty} x_n.$$

It follows by Theorem 1.8 that there exists a full limiting sequence $\{L_n\}_{n=-\infty}^{\infty}$ of $\{x_n\}_{n=-k}^{\infty}$ with $L_0 = \liminf_{n\to\infty} x_n$. Note that $L_0 \le L_{-n}$ for every $n = 0, 1, \ldots$.

We claim that there exists an integer $N < 0$ such that

$$L_0 = L_N = L_{N-1} = \cdots = L_{N-k}.$$

Proof of the claim:

Since f is non-decreasing in each of its arguments, we see that

$$L_0 = f(L_0, L_0, \ldots, L_0) \le f(L_{-1}, L_{-2}, \ldots, L_{-k}) = L_0$$

and hence

$$f(L_0, L_0, \ldots, L_0) = f(L_{-1}, L_{-2}, \ldots, L_{-k}).$$

So as f is strictly increasing in each of the arguments $z_{i_1}, z_{i_2}, \ldots, z_{i_l}$, it follows that

$$L_0 = L_{-i_1} = L_{-i_2} = \cdots = L_{-i_l}.$$

Given $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_l \in \{0, 1, \ldots\}$, set

$$n = \tilde{p}_1 i_1 + \tilde{p}_2 i_2 + \cdots + \tilde{p}_l i_l$$

and suppose that

$$L_0 = L_{-n}.$$

Then

$$L_0 = L_{-n} = f(L_{-n-1}, L_{-n-2}, \ldots, L_{-n-k}) \ge f(L_0, L_0, \ldots, L_0) = L_0$$

and so as before,

$$L_0 = L_{-n-i_1} = L_{-n-i_2} = \cdots = L_{-n-i_l}.$$

Thus

$$\begin{aligned} L_0 &= L_{-[(\tilde{p}_1+1)i_1+\tilde{p}_2 i_2+\cdots+\tilde{p}_l i_l]} \\ &= L_{-[\tilde{p}_1 i_1+(\tilde{p}_2+1)i_2+\cdots+\tilde{p}_l i_l]} \\ &\vdots \\ &= L_{-[\tilde{p}_1 i_1+\tilde{p}_2 i_2+\cdots+(\tilde{p}_l+1)i_l]}. \end{aligned}$$

It follows by induction that given $p_1, p_2, \ldots, p_l \in \{0, 1, \ldots\}$,

$$L_0 = L_{-(p_1 i_1+p_2 i_2+\cdots+p_l i_l)}.$$

Now $i_1, i_2, \ldots, i_l$ are relatively prime, and so there exist non-zero integers $\lambda_1, \lambda_2, \ldots, \lambda_l$ such that

$$\lambda_1 i_1 + \lambda_2 i_2 + \cdots + \lambda_l i_l = 1.$$

Set

$$N = -k(|\lambda_1| i_1 + |\lambda_2| i_2 + \cdots + |\lambda_l| i_l).$$

For $0 \leq j \leq k$ and $1 \leq s \leq l$, define

$$p_s^j = k|\lambda_s| + j\lambda_s.$$

Then for $0 \leq j \leq k$ and $1 \leq s \leq l$

$$p_s^j \geq 0$$

is a non-negative integer, and

$$\begin{aligned} -(p_1^j i_1 + p_2^j i_2 + \cdots + p_l^j i_j) &= -(k|\lambda_1| i_1 + j\lambda_1 i_1) - (k|\lambda_2| i_2 + j\lambda_2 i_2) \\ &\quad - \cdots - (k|\lambda_l| i_l + j\lambda_l i_l) \\ &= N - j. \end{aligned}$$

Thus

$$L_0 = L_N = L_{N-1} = \cdots = L_{N-k}$$

and so the claim is true.

Now

$$L_{N+1} = f(L_N, L_{N-1}, \ldots, L_{N-k}) = f(L_0, L_0, \ldots, L_0) = L_0.$$

It follows by induction that

$$L_0 = L_{-1} = L_{-2} = \cdots = L_{-k}.$$

By Theorem 1.8, there exists a subsequence $\{x_{r_i}\}_{i=0}^{\infty}$ of $\{x_n\}_{n=-k}^{\infty}$ such that

$$\lim_{i \to \infty} x_{r_i - j} = L_{-j} = L_0$$

for every $0 \leq j \leq k$. So as $L_0 = \liminf_{n \to \infty} x_n$, there exists a positive integer s such that $r_s \geq 0$ and

$$\max\{x_{r_s}, x_{r_s - 1}, \ldots, x_{r_s - k}\} \leq \frac{1}{2}\left(\limsup_{n \to \infty} x_n + \liminf_{n \to \infty} x_n\right).$$

It follows by Lemma 1.2 that $x_n \leq \frac{1}{2}\left(\limsup_{n\to\infty} x_n + \liminf_{n\to\infty} x_n\right)$ for all $n \geq r_s$, and so

$$\limsup_{n\to\infty} x_n \leq \frac{1}{2}\left(\limsup_{n\to\infty} x_n + \liminf_{n\to\infty} x_n\right) < \limsup_{n\to\infty} x_n.$$

This is a contradiction, and so the proof is complete. ▯

The following convergence result is due to Hautus and Bolis. See [63] and Theorem 2.6.2 in [73].

THEOREM 1.10
Consider the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}) \quad , \quad n = 0, 1, \ldots \tag{1.9}$$

where $F \in C(I^{k+1}, \mathbf{R})$, and where I is an open interval of real numbers. Let $x^* \in I$ be an equilibrium solution of Eq.(1.9). Finally, suppose that F satisfies the following two conditions:

1. *F is non-decreasing in each of its arguments.*

2. *F satisfies the negative feedback property*

$$(u - x^*)[F(u, u, \ldots, u) - u] < 0 \qquad \text{for all} \qquad u \in I - \{x^*\}.$$

Then the equilibrium point x^* is a global attractor of all solutions of Eq.(1.9).

PROOF First note by Condition *2* on F that x^* is the only equilibrium solution of Eq.(1.9).

Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1.9) with $x_0, x_{-1}, \ldots, x_{-k} \in I$. Set

$$m = \min\{x^*, x_0, x_{-1}, \ldots, x_{-k}\} \text{ and } M = \max\{x^*, x_0, x_{-1}, \ldots, x_{-k}\}.$$

Then by Conditions *1* and *2* on F, it follows that

$$m \leq F(m, m, \ldots, m) \leq x_1 \leq F(M, M, \ldots, M) \leq M,$$

and hence by induction on n that $m \leq x_n \leq M$ for all $n \geq -k$. In particular, since $[m, M] \subset I$, we see that $x_n \in I$ for all $n \geq -k$.

Let

$$\lambda = \liminf_{n\to\infty} x_n \qquad \text{and} \qquad \mu = \limsup_{n\to\infty} x_n.$$

It suffices to show $\lambda = x^* = \mu$.

Now $m \leq \lambda \leq \mu \leq M$. Since I is an open interval of real numbers and $[m, M] \subset I$, it follows that there exists $\varepsilon > 0$ such that $[m - \varepsilon, M + \varepsilon\,] \subset I$. Hence there exists $N \geq 0$ such that $\lambda - \varepsilon < x_{n-k}$ for all $n \geq N$. Thus for $n \geq N$, we see that

$$F(\lambda - \varepsilon, \lambda - \varepsilon, \ldots, \lambda - \varepsilon) \leq F(x_n, x_{n-1}, \ldots, x_{n-k}) = x_{n+1}.$$

It follows that $F(\lambda - \varepsilon, \lambda - \varepsilon, \ldots, \lambda - \varepsilon) \leq \lambda$, and hence by the continuity of F that $F(\lambda, \lambda, \ldots, \lambda) \leq \lambda$. Thus we see by Condition *2* on F that $x^* \leq \lambda$. It follows similarly that $\mu \leq x^*$, and so we see that $\lambda = x^* = \mu$. ▯

The following global attractivity results from [78] are very useful in establishing convergence results in a wide spectrum of difference equations.

THEOREM 1.11
Let $g : [a, b] \times [a, b] \to [a, b]$ be a continuous function, where a and b are real numbers with $a < b$, and consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}) \quad , \quad n = 0, 1, \ldots . \tag{1.10}$$

Suppose that g satisfies the following conditions:

1. *$g(x, y)$ is non-decreasing in $x \in [a, b]$ for each fixed $y \in [a, b]$, and $g(x, y)$ is non-decreasing in $y \in [a, b]$ for each fixed $x \in [a, b]$;*
2. *If (m, M) is a solution of the system*

$$m = g(m, m) \qquad \text{and} \qquad M = g(M, M)$$

   then $m = M$.

Then there exists exactly one equilibrium $\bar{x}$ of Eq.(1.10), and every solution of Eq.(1.10) converges to $\bar{x}$.

THEOREM 1.12
Let $g : [a, b] \times [a, b] \to [a, b]$ be a continuous function, where a and b are real numbers with $a < b$, and consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}) \quad , \quad n = 0, 1, \ldots . \tag{1.11}$$

Suppose that g satisfies the following conditions:

1. *$g(x, y)$ is non-increasing in $x \in [a, b]$ for each fixed $y \in [a, b]$, and $g(x, y)$ is non-decreasing in $y \in [a, b]$ for each fixed $x \in [a, b]$;*

2. *If (m, M) is a solution of the system*

$$m = g(M, m) \quad \text{and} \quad M = g(m, M)$$

then $m = M$.

Then there exists exactly one equilibrium $\bar{x}$ of Eq.(1.11), and every solution of Eq.(1.11) converges to $\bar{x}$.

THEOREM 1.13
Let $g : [a, b] \times [a, b] \to [a, b]$ be a continuous function, where a and b are real numbers with $a < b$, and consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}) \quad , \quad n = 0, 1, \ldots . \tag{1.12}$$

Suppose that g satisfies the following conditions:

1. *$g(x, y)$ is non-decreasing in $x \in [a, b]$ for each fixed $y \in [a, b]$, and $g(x, y)$ is non-increasing in $y \in [a, b]$ for each fixed $x \in [a, b]$;*

2. *If (m, M) is a solution of the system*

$$m = g(m, M) \quad \text{and} \quad M = g(M, m)$$

then $m = M$.

Then there exists exactly one equilibrium $\bar{x}$ of Eq.(1.12), and every solution of Eq.(1.12) converges to $\bar{x}$.

THEOREM 1.14
Let $g : [a, b] \times [a, b] \to [a, b]$ be a continuous function, where a and b are real numbers with $a < b$, and consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}) \quad , \quad n = 0, 1, \ldots . \tag{1.13}$$

Suppose that g satisfies the following conditions:

1. *$g(x, y)$ is non-increasing in $x \in [a, b]$ for each fixed $y \in [a, b]$, and $g(x, y)$ is non-increasing in $y \in [a, b]$ for each fixed $x \in [a, b]$;*

2. *If (m, M) is a solution of the system*

$$m = g(M, M) \quad \text{and} \quad M = g(m, m)$$

then $m = M$.

Then there exists exactly one equilibrium $\bar{x}$ of Eq.(1.13), and every solution of Eq.(1.13) converges to $\bar{x}$.

REMARK 1.1 It is interesting to note that Assumption 2 of Theorem 1.11 is equivalent to assuming that Eq.(1.10) has a unique equilibrium in $[a, b]$, while Assumption 2 of Theorem 1.12 is equivalent to assuming that Eq.(1.12) has no solutions which are periodic with prime period 2. ▯

The following result extends and unifies Theorems 1.11, 1.12, 1.13, and 1.14.

THEOREM 1.15
Let $g : [a, b]^{k+1} \to [a, b]$ be a continuous function, where k is a positive integer, and where $[a, b]$ is an interval of real numbers. Consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}, \ldots, x_{n-k}) \quad , \quad n = 0, 1, \ldots . \tag{1.14}$$

Suppose that g satisfies the following conditions:

1. *For each integer i with $1 \leq i \leq k+1$, the function $g(z_1, z_2, \ldots, z_{k+1})$ is weakly monotonic in z_i for fixed $z_1, z_2, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{k+1}$.*

2. *If (m, M) is a solution of the system*

$$m = g(m_1, m_2, \ldots, m_{k+1}) \quad \text{and} \quad M = g(M_1, M_2, \ldots, M_{k+1})$$

then $m = M$, where for each $i = 1, 2, \ldots, k+1$, we set

$$m_i = \begin{cases} m \text{ if } g \text{ is non– decreasing in } z_i \\ \\ M \text{ if } g \text{ is non–increasing in } z_i \end{cases}$$

and

$$M_i = \begin{cases} M \text{ if } g \text{ is non–decreasing in } z_i \\ \\ m \text{ if } g \text{ is non–increasing in } z_i. \end{cases}$$

Then there exists exactly one equilibrium $\bar{x}$ of Eq.(1.14), and every solution of Eq.(1.14) converges to $\bar{x}$.

Theorem 1.15 can be easily adapted to systems. As an example, we give Theorem 1.16. See [79].

THEOREM 1.16
Let $[a,b]$ and $[c,d]$ be intervals of real numbers, and let

$$f_1 : [a,b] \times [c,d] \to [a,b] \qquad \text{and} \qquad f_2 : [a,b] \times [c,d] \to [c,d]$$

be continuous functions. Consider the system of difference equations

$$\begin{array}{l} x_{n+1} = f_1(x_n, y_n) \\ \\ y_{n+1} = f_2(x_n, y_n) \end{array} \quad , \qquad n = 0, 1, \ldots \tag{1.15}$$

with initial condition $(x_0, y_0) \in [a,b] \times [c,d]$. Suppose that the following statements are true:

1. *$f_1(x,y)$ is non-decreasing in x and is non-increasing in y.*
2. *$f_2(x,y)$ is non-increasing in x and is non-decreasing in y.*
3. *If $(m_1, M_1, m_2, M_2) \in [a,b]^2 \times [c,d]^2$ is a solution of the system of equations*

$$\begin{cases} m_1 = f_1(m_1, M_2)\ ,\ M_1 = f_1(M_1, m_2) \\ \\ m_2 = f_2(M_1, m_2)\ ,\ M_2 = f_2(m_1, M_2) \end{cases}$$

then $m_1 = M_1$ and $m_2 = M_2$.

Then there exists exactly one equilibrium point $(\bar{x}, \bar{y})$ of System (1.15), and every solution of System (1.15) converges to $(\bar{x}, \bar{y})$.

PROOF Note that by the Brouwer fixed point theorem, the function

$$T : [a,b] \times [c,d] \to [a,b] \times [c,d]$$

given by

$$T(x,y) = [f_1(x,y), f_2(x,y)]$$

has a fixed point $(\bar{x}, \bar{y})$ which is clearly an equilibrium point of System (1.15), and so it is true that System (1.15) has at least one equilibrium point.

Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System (1.15). It suffices to show that

$$\lim_{n \to \infty} (x_n, y_n) = (\bar{x}, \bar{y}).$$

Set $m_1^0 = a, M_1^0 = b, m_2^0 = c, M_2^0 = d$, and for each $i \geq 0$, let

$$m_1^{i+1} = f_1(m_1^i, M_2^i) \qquad , \qquad M_1^{i+1} = f_1(M_1^i, m_2^i)$$

and

$$m_2^{i+1} = f_2(M_1^i, m_2^i) \quad , \quad M_2^{i+1} = f_2(m_1^i, M_2^i).$$

Then

$$m_1^0 = a \le f_1(m_1^0, M_2^0) \le f_1(M_1^0, m_2^0) \le b = M_1^0$$

and

$$m_2^0 = c \le f_2(M_1^0, m_2^0) \le f_1(m_1^0, M_2^0) \le d = M_1^0$$

and so we see that

$$m_1^0 \le m_1^1 \le M_1^1 \le M_1^0 \quad \text{and} \quad m_2^0 \le m_2^1 \le M_2^1 \le M_2^0.$$

We similarly have

$$m_1^1 = f_1(m_1^0, M_2^0) \le f_1(m_1^1, M_2^1) \le f_1(M_1^1, m_2^1) \le f_1(M_1^0, m_2^0) = M_1^1$$

and

$$m_2^1 = f_2(M_1^0, m_2^0) \le f_2(M_1^1, m_2^1) \le f_2(m_1^1, M_2^1) \le f_2(m_1^0, M_2^0) = M_2^1$$

and so we see that

$$m_1^0 \le m_1^1 \le m_1^2 \le M_1^2 \le M_1^1 \le M_1^0 \text{ and } m_2^0 \le m_2^1 \le m_2^2 \le M_2^2 \le M_2^1 \le M_2^0.$$

Note that

$$m_1^0 = a \le x_n \le b = M_1^0 \quad \text{and} \quad m_2^0 = c \le y_n \le d = M_2^0 \quad \text{for all} \quad n \ge 0.$$

For all $n \ge 0$, we have

$$m_1^1 = f_1(m_1^0, M_2^0) \le f_1(x_n, y_n) \le f_1(M_1^0, m_2^0) = M_1^1$$

and

$$m_2^1 = f_2(M_1^0, m_2^0) \le f_2(x_n, y_n) \le f_2(m_1^0, M_2^0) = M_2^1$$

and so

$$m_1^1 \le x_n \le M_1^1 \quad \text{and} \quad m_2^1 \le y_n \quad \le M_2^1 \quad \text{for all} \quad n \ge 1.$$

For all $n \ge 1$, we have

$$m_1^2 = f_1(m_1^1, M_2^1) \le f_1(x_n, y_n) \le f_1(M_1^1, m_2^1) = M_1^2$$

and

$$m_2^2 = f_2(M_1^1, m_2^1) \le f_2(x_n, y_n) \le f_2(m_1^1, M_2^1) = M_2^2$$

and so

$$m_1^2 \le x_n \le M_1^2 \quad \text{and} \quad m_2^2 \le y_n \quad \le M_2^2 \qquad \text{for all} \qquad n \ge 2.$$

It follows by induction that for $i \ge 0$, the following statements are true:

(i) $a = m_1^0 \le m_1^1 \le \cdots \le m_1^{i-1} \le m_1^i \le M_1^i \le M_1^{i-1} \le \cdots \le M_1^1 \le M_1^0 = b.$

(ii) $c = m_2^0 \le m_2^1 \le \cdots \le m_2^{i-1} \le m_2^i \le M_2^i \le M_2^{i-1} \le \cdots \le M_2^1 \le M_2^0 = d.$

(iii) $m_1^i \le x_n \le M_1^i \qquad \text{for all} \qquad n \ge i.$

(iv) $m_2^i \le y_n \le M_2^i \qquad \text{for all} \qquad n \ge i.$

Set

$$m_1 = \lim_{i\to\infty} m_1^i \quad , \quad M_1 = \lim_{i\to\infty} M_1^i \quad , \quad m_2 = \lim_{i\to\infty} m_2^i \qquad \text{and} \qquad M_2 = \lim_{i\to\infty} M_2^i.$$

Then

$$a \le m_1 \le M_1 \le b \qquad \text{and} \qquad c \le m_2 \le M_2 \le d.$$

By the continuity of f_1 and f_2, we have

$$m_1 = f_1(m_1, M_2) \qquad , \qquad M_1 = f_1(M_1, m_2)$$

$$m_2 = f_2(M_1, m_2) \qquad , \qquad M_2 = f_2(m_1, M_2)$$

and hence

$$m_1 = M_1 \qquad \text{and} \qquad m_2 = M_2$$

from which the proof follows. ∎

Chapter 2

EQUATIONS WITH PERIODIC SOLUTIONS

2.1 Introduction

Our aim in this chapter is to present a wealth of examples of difference equations with the property that every solution of each equation is **periodic with the same period.** These are examples which we believe everyone should be exposed to for enrichment and appreciation of the fascinating world of difference equations and their richness in periodicities. See Chapter 6 for further results on periodic solutions.

A question of fundamental importance is the following. What is it that makes every solution of a difference equation periodic with the same period? Is there an easily verifiable necessary and sufficient condition that can be used to test for this property?

2.2 What Do the Following Equations Have in Common?

$$x_{n+1} = \frac{1}{x_n}, \quad n = 0, 1, \ldots. \tag{2.1}$$

$$x_{n+1} = \frac{1}{x_n x_{n-1}}, \quad n = 0, 1, \ldots. \tag{2.2}$$

$$x_{n+1} = \frac{1}{x_{n-1}}, \quad n = 0, 1, \ldots. \tag{2.3}$$

$$x_{n+1} = \frac{1 + x_n}{x_{n-1}}, \quad n = 0, 1, \ldots. \tag{2.4}$$

$$x_{n+1} = \frac{x_n}{x_{n-1}}, \quad n = 0, 1, \ldots. \tag{2.5}$$

$$x_{n+1} = \frac{1 + x_n + x_{n-1}}{x_{n-2}}, \qquad n = 0, 1, \ldots . \tag{2.6}$$

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1} x_{n-3}}, \qquad n = 0, 1, \ldots \tag{2.7}$$

The answer is that **every solution of each of the above equations is periodic with the same period.**

Every solution of Eq.(2.1) is **periodic with period 2.**

Every solution of Eq.(2.2) is **periodic with period 3.**

Every solution of Eq.(2.3) is **periodic with period 4.**

Every solution of Eq.(2.4) is **periodic with period 5.**

Every solution of Eq.(2.5) is **periodic with period 6.**

Every solution of Eq.(2.6) is **periodic with period 8.**

Every solution of Eq.(2.7) is **periodic with period 10.**

Indeed, if the initial condition of Eq.(2.1) is a non-zero real number denoted by

$$x_0 = \alpha$$

then the solution of Eq.(2.1) is the **period-2 sequence**

$$\alpha, \frac{1}{\alpha}, \ldots .$$

If the initial conditions are non-zero real numbers denoted by

$$x_{-1} = \alpha \qquad \text{and} \qquad x_0 = \beta$$

then the solution of Eq.(2.2) is the **period-3 sequence**

$$\alpha, \beta, \frac{1}{\alpha\beta}, \ldots ;$$

the solution of Eq.(2.3) is the **period-4 sequence**

$$\alpha, \beta, \frac{1}{\alpha}, \frac{1}{\beta}, \ldots ;$$

the solution of Eq.(2.4) is the **period-5 sequence**

$$\alpha, \beta, \frac{1+\beta}{\alpha}, \frac{1+\alpha+\beta}{\alpha\beta}, \frac{1+\alpha}{\beta}, \ldots$$

and the solution of Eq.(2.5) is the **period-6 sequence**

$$\alpha, \beta, \frac{\beta}{\alpha}, \frac{1}{\alpha}, \frac{1}{\beta}, \frac{\alpha}{\beta}, \ldots .$$

If the initial conditions of Eq.(2.6) are the non-zero real numbers

$$x_{-2} = \alpha, \qquad x_{-1} = \beta, \qquad \text{and} \qquad x_0 = \gamma$$

then the solution of Eq.(2.6) is the **period-8 sequence**

$$\alpha, \beta, \gamma, \frac{1+\beta+\gamma}{\alpha}, \frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\alpha\beta},$$

$$\frac{(1+\alpha+\beta)(1+\beta+\gamma)}{\alpha\beta\gamma}, \frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\beta\gamma}, \frac{1+\alpha+\beta}{\gamma}, \ldots .$$

Finally, if the initial conditions of Eq.(2.7) are the non-zero real numbers

$$x_{-3} = \alpha, \qquad x_{-2} = \beta, \qquad x_{-1} = \gamma, \qquad x_0 = \delta$$

then the solution of Eq.(2.7) is the **period-10 sequence**

$$\alpha, \beta, \gamma, \delta, \frac{\beta\delta}{\alpha\gamma}, \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}, \frac{\alpha\gamma}{\beta\delta}, \ldots .$$

What is it that makes every solution of a difference equation periodic with the same period?

Is there an easily verifiable test that we can apply to determine whether or not this is true?

REMARK 2.1

1. It is interesting to note that Eqs.(2.1) and (2.2) follow a pattern. In fact, for every $k \in \{0, 1, \ldots\}$, every solution of the difference equation

$$x_{n+1} = \frac{1}{x_n x_{n-1} \cdots x_{n-k}}, \qquad n = 0, 1, \ldots \tag{2.8}$$

is periodic with period $(k+2)$.

Indeed, if the initial conditions of Eq.(2.8) are the non-zero real numbers $x_{-k}, x_{-k+1}, \ldots, x_0$, then the solution of Eq.(2.8) is the **period-$(k+2)$ sequence**

$$\boldsymbol{x_{-k}, x_{-k+1}, \ldots, x_0, \frac{1}{x_0 x_{-1} \cdots x_{-k}}, \ldots .}$$

2. Note that Eqs.(2.5) and (2.7) also follow a pattern. Given a non-negative integer $k \geq 0$, every solution of the equation

$$x_{n+1} = \frac{x_n x_{n-2} \cdots x_{n-2k}}{x_{n-1} x_{n-3} \cdots x_{n-(2k+1)}} \quad , \qquad n = 0, 1, \ldots \tag{2.9}$$

is periodic with period $(4k+6)$.

Indeed, if the initial conditions of Eq.(2.9) are the non-zero real numbers $x_{-2k-1}, x_{-2k}, \ldots, x_0$, then the solution of Eq.(2.9) is the **period-$(4k+6)$ sequence**

$$x_{-2k-1}, x_{-2k}, \ldots, x_0, \frac{x_0 x_{-2} \cdots x_{-2k}}{x_{-1} x_{-3} \cdots x_{-2k-1}},$$

$$\frac{1}{x_{-2k-1}}, \frac{1}{x_{-2k}}, \ldots, \frac{1}{x_0}, \frac{x_{-1} x_{-3} \cdots x_{-2k-1}}{x_0 x_{-2} \cdots x_{-2k}}, \ldots .$$

3. In view of the periodicities of Eqs.(2.1), (2.4), and (2.6), we might be misled into believing that Eqs.(2.1), (2.4), and (2.6) also follow a pattern and, in particular, that every solution of the difference equation

$$x_{n+1} = \frac{1 + x_n + x_{n-1} + x_{n-2}}{x_{n-3}} \quad , \qquad n = 0, 1, \ldots \tag{2.10}$$

is periodic with period 11. Unfortunately, Eqs.(2.1), (2.4), and (2.6) do not follow any obvious pattern. If $\{x_n\}_{n=-3}^{\infty}$ is the solution of Eq.(2.10) with initial conditions $x_{-3} = x_{-2} = x_{-1} = 1$, we see that the first 12 terms of $\{x_n\}_{n=-3}^{\infty}$ are

$$1,\ 1,\ 1,\ 1,\ 4,\ 7,\ 13,\ 25,\ \frac{23}{2},\ \frac{101}{14},\ \frac{313}{91},\ \frac{29,498}{31,850}$$

and so $\{x_n\}_{n=-3}^{\infty}$ **is not periodic with period 11.** $\square$

2.3 What Do the Following Equations Have in Common?

$$x_{n+1} = \frac{\max\{x_n, 1\}}{x_{n-1}}, \quad n = 0, 1, \ldots \tag{2.11}$$

$$x_{n+1} = \frac{\max\{x_n, 1\}}{x_n x_{n-1}}, \quad n = 0, 1, \ldots \tag{2.12}$$

$$x_{n+1} = \frac{\max\{x_n, 1\}}{x_n^2 x_{n-1}}, \quad n = 0, 1, \ldots \tag{2.13}$$

$$x_{n+1} = \frac{\max\{x_n^2, 1\}}{x_n x_{n-1}}, \quad n = 0, 1, \ldots \tag{2.14}$$

$$x_{n+1} = \frac{\max\{x_n^2, 1\}}{x_n^3 x_{n-1}}, \quad n = 0, 1, \ldots \tag{2.15}$$

The answer is that **every positive solution of each of the the above difference equations is periodic with the same period.**

Every solution of Eq.(2.11) is **periodic with period 5.**

Every solution of Eq.(2.12) is **periodic with period 7.**

Every solution of Eq.(2.13) is **periodic with period 8.**

Every solution of Eq.(2.14) is **periodic with period 9.**

Every solution of Eq.(2.15) is **periodic with period 12.**

For example, the proof that every positive solution of Eq.(2.12) is **periodic with period 7** is evident from the following table. The proof in the other cases follows similarly.

Case 1	Case 2	Case 3	Case 4
$x_{-1}=\alpha\leq 1$ $x_0=\beta\leq 1$	$x_{-1}=\alpha\geq 1$ $x_0=\beta\leq 1$	$x_{-1}=\alpha\leq 1$ $x_0=\beta\geq 1$	$x_{-1}=\alpha\geq 1$ $x_0=\beta\geq 1$
$x_1=\frac{1}{\alpha\beta}$	$x_1=\frac{1}{\alpha\beta}$	$x_1=\frac{1}{\alpha}$	$x_1=\frac{1}{\alpha}$
$x_2=\frac{1}{\beta}$	$x_2=\max\left\{\alpha,\frac{1}{\beta}\right\}$	$x_2=\frac{1}{\beta}$	$x_2=\frac{\alpha}{\beta}$
$x_3=\alpha\beta$	$x_3=\alpha\beta$	$x_3=\alpha\beta$	$x_3=\max\{\alpha,\beta\}$
$x_4=\frac{1}{\alpha}$	$x_4=\frac{1}{\alpha}$	$x_4=\max\left\{\beta,\frac{1}{\alpha}\right\}$	$x_4=\frac{\beta}{\alpha}$
$x_5=\frac{1}{\alpha\beta}$	$x_5=\frac{1}{\beta}$	$x_5=\frac{1}{\alpha\beta}$	$x_5=\frac{1}{\beta}$
$x_6=\alpha$	$x_6=\alpha$	$x_6=\alpha$	$x_6=\alpha$
$x_7=\beta$	$x_7=\beta$	$x_7=\beta$	$x_7=\beta$

Are there any other values of k and l for which every positive solution of the difference equation

$$x_{n+1}=\frac{\max\left\{x_n^k,1\right\}}{x_n^l x_{n-1}} \quad , \qquad n=0,1,\ldots \tag{2.16}$$

is periodic with the same period? What are they? Is there an easily applicable test to determine this? What is it?

2.4 Lyness' Equation

Eq.(2.4), that is, the equation

$$x_{n+1}=\frac{1+x_n}{x_{n-1}} \quad , \qquad n=0,1,\ldots \tag{2.4}$$

was introduced by Lyness in 1942, see [97], while he was working on a problem in number theory. See also [12], [51], [52], [53], [73], [75], [84], [114], and [117].

As we mentioned in Section 2.2, **every positive solution of Eq.(2.4) is periodic with period 5.** Indeed if

$$x_{-1} = \alpha \qquad \text{and} \qquad x_0 = \beta$$

are positive initial conditions, then the solution $\{x_n\}_{n=-1}^{\infty}$ is the **period-5 sequence**

$$\alpha, \beta, \frac{1+\beta}{\alpha}, \frac{1+\alpha+\beta}{\alpha\beta}, \frac{1+\alpha}{\beta}, \dots. \tag{2.17}$$

Eq.(2.4) arises in *frieze patterns*, see [27]. An example of a frieze pattern is the following.

$$\begin{array}{ccccccccccc} \cdots 1 & & 1 & & 1 & & 1 & & 1 \cdots \\ \cdots & 1 & & 3 & & 1 & & 3 & \cdots \\ \cdots 2 & & 2 & & 2 & & 2 & & 2 \cdots \\ \cdots & 3 & & 1 & & 3 & & 1 & \cdots \end{array}$$

The property which defines a frieze pattern is that except for possible borders of zeros and ones, every four adjacent numbers forming a rhombus

$$\begin{array}{ccc} & q & \\ p & & r \\ & s & \end{array}$$

are positive and satisfy the unimodular equation

$$pr - qs = 1.$$

Coxeter, see [28], has shown that every frieze pattern is periodic. For example, the frieze pattern shown below is **periodic with period 5.**

$$\begin{array}{cccccccccccc} \cdots 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\ \cdots & x_1 & & x_3 & & x_5 & & x_2 & & x_4 \cdots \\ \cdots x_5 & & x_2 & & x_4 & & x_1 & & x_3 & & \cdots \\ \cdots & 1 & & 1 & & 1 & & 1 & & 1 \cdots \end{array}$$

If $x_1 = \alpha$ and $x_2 = \beta$ are arbitrary positive numbers, then from the definition of frieze patterns it follows that

$$x_3 = \frac{1+\beta}{\alpha}, \qquad x_4 = \frac{1+\alpha+\beta}{\alpha\beta}, \qquad \text{and} \qquad x_5 = \frac{1+\alpha}{\beta}.$$

Therefore, the above pattern is generated by Eq.(2.4).

REMARK 2.2 It is an amazing fact that Eqs.(2.11) and (2.4) have great similarities. The solution of Eq.(2.11) with positive initial conditions

$$x_{-1} = \alpha \qquad \textit{and} \qquad x_0 = \beta$$

is the period-5 sequence

$$\alpha, \beta, \frac{\max\{1, \beta\}}{\alpha}, \frac{\max\{1, \alpha, \beta\}}{\alpha\beta}, \frac{\max\{1, \alpha\}}{\beta}, \ldots.$$

Compare this with the period-5 solution in (2.17).

What is it that these two equations have in common? Are there other pairs of equations with similar behavior? ☐

In the sequel, the equation

$$x_{n+1} = \frac{a + x_n}{x_{n-1}} \quad , \qquad n = 0, 1, \ldots \tag{2.18}$$

will be referred to as *Lyness' Equation.*

For a given positive number a, what are all the positive periodic solutions that Eq.(2.18) possesses?

Clearly Eq.(2.18) has the unique positive equilibrium point

$$\bar{x} = \frac{1 + \sqrt{1 + 4a}}{2}.$$

In the remainder of this section, we shall establish the following three properties of Lyness' equation:

1. Eq.(2.18) possesses an **invariant**. That is, there exists a non-trivial function $F(x_{n-1}, x_n)$ such that for every solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(2.18),

$$F(x_{n-1}, x_n) = \textit{constant} = F(x_{-1}, x_0) \qquad \text{for all} \qquad n \geq 0.$$

2. Every positive solution of Eq.(2.18) is bounded and persists.

3. No non-trivial solution of Eq.(2.18) has a limit.

THEOREM 2.1
Let $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(2.18). Then*

$$(a + x_{n-1} + x_n)\left(1 + \frac{1}{x_{n-1}}\right)\left(1 + \frac{1}{x_n}\right) = constant \quad for\ all \quad n \geq 0. \quad (2.19)$$

PROOF Indeed for $n \geq 0$,

$$\begin{aligned}
&(a + x_n + x_{n+1})\left(1 + \frac{1}{x_n}\right)\left(1 + \frac{1}{x_{n+1}}\right) \\
&= (a + x_n + \frac{a + x_n}{x_{n-1}})\left(1 + \frac{1}{x_n}\right)\left(1 + \frac{x_{n-1}}{a + x_n}\right) \\
&= \left(\frac{a + x_n}{a + x_n} + \frac{1}{x_{n-1}}\right)\left(1 + \frac{1}{x_n}\right)(a + x_n + x_{n-1}) \\
&= (a + x_{n-1} + x_n)\left(1 + \frac{1}{x_{n-1}}\right)\left(1 + \frac{1}{x_n}\right).
\end{aligned}$$

The proof follows by induction. ▯

The quantity

$$(a + x_{n-1} + x_n)\left(1 + \frac{1}{x_{n-1}}\right)\left(1 + \frac{1}{x_n}\right)$$

is called an *invariant*, or *first integral*, of Eq.(2.18).

The proof of the next theorem follows directly from Theorem 2.1 and will be omitted.

THEOREM 2.2
Every positive solution of Eq.(2.18) is bounded and persists.

The following lemma states that the only eventually constant solution of Eq.(2.18) is the trivial solution.

LEMMA 2.1
Let $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(2.18) which is eventually constant. Then*

$$x_n = \bar{x} \qquad for\ all \qquad n \geq -1.$$

PROOF The proof follows from the fact that Eq.(2.18) has the unique positive equilibrium point $\bar{x}$, as well as the fact that for $n \geq 0$,

$$x_{n-1} = \frac{a + x_n}{x_{n+1}}.$$ ▯

The next lemma describes the nature of the semi-cycles of the non-trivial solutions of Eq.(2.18).

LEMMA 2.2
Let $\{x_n\}_{n=-1}^{\infty}$ be a positive, non-trivial solution of Eq.(2.18). Then the following statements are true:

1. *Every semi-cycle of $\{x_n\}_{n=-1}^{\infty}$ has at most three terms.*

2. *The maximum of every positive semi-cycle occurs in either the first or second term, and the minimum of every negative semi-cycle occurs in either the first or second term.*

3. *$\{x_n\}_{n=-1}^{\infty}$ is strictly oscillatory about $\bar{x}$.*

PROOF We shall first show that every positive semi-cycle of $\{x_n\}_{n=-1}^{\infty}$ has at most three terms, and that the maximum of every positive semi-cycle occurs in either the first or second term. With this in mind, suppose there exists $N \geq 0$ such that $\bar{x} \leq x_{N-1}$ and $\bar{x} \leq x_N$, where at least one of the two inequalities is strict. Then

$$x_{N+1} = \frac{a + x_N}{x_{N-1}} = x_N \left(\frac{\frac{a}{x_N} + 1}{x_{N-1}} \right) < x_N \left(\frac{\frac{a}{\bar{x}} + 1}{\bar{x}} \right) = x_N.$$

It suffices to assume that $\bar{x} \leq x_{N+1}$. Then

$$x_{N+2} = \frac{a + x_{N+1}}{x_N} < \frac{a + x_N}{x_N} < \frac{a + \bar{x}}{\bar{x}} = \bar{x}.$$

The proof that every negative semi-cycle has at most three terms, and that the minimum of every negative semi-cycle occurs in either the first term or the second term is similar and will be omitted.

Finally, we shall show that $\{x_n\}_{n=-1}^{\infty}$ is strictly oscillatory about $\bar{x}$. For the sake of contradiction, suppose that this is not the case. Then it follows by the above that without loss of generality, we may assume that $x_{-1} < \bar{x}$, $x_0 = \bar{x}$, and $x_1 < \bar{x}$. Thus we have

$$\bar{x} > x_1 = \frac{a + x_0}{x_{-1}} = \frac{a + \bar{x}}{x_{-1}} > \frac{a + \bar{x}}{\bar{x}} = \bar{x},$$

which is a contradiction, and the proof is complete. ▯

THEOREM 2.3

No non-trivial solution of Eq.(2.18) has a limit.

PROOF For the sake of contradiction, suppose $\{x_n\}_{n=-1}^{\infty}$ is a non-trivial solution of Eq.(2.18) which converges to a limit. Then as $\bar{x}$ is the unique positive equilibrium point of Eq.(2.18), the limit is clearly $\bar{x}$. Suppose $N \geq 0$. By Theorem 2.1, we see that

$$(a + x_{N-1} + x_N)\left(1 + \frac{1}{x_{N-1}}\right)\left(1 + \frac{1}{x_N}\right) = (a + 2\bar{x})\left(1 + \frac{1}{\bar{x}}\right)^2 .$$

It follows that

$$\begin{aligned}(x_{N-1} + 1)x_N^2 + \left[x_{N-1}^2 - \left(2\bar{x} + 2 + \frac{2a+2}{\bar{x}} + \frac{a}{\bar{x}^2}\right)x_{N-1} + (a+1)\right]x_n \\ + (x_{N-1}^2 + (a+1)x_{N-1} + a) \quad = \quad 0.\end{aligned} \tag{2.20}$$

We view Eq.(2.20) as a quadratic equation in x_N. The discriminant of Eq.(2.20) is

$$\begin{aligned}\mathcal{D}_N = &\left[x_{N-1}^2 - \left(2\bar{x} + 2 + \frac{2a+2}{\bar{x}} + \frac{a}{\bar{x}^2}\right)x_{N-1} + (a+1)\right]^2 \\ &-4(x_{N-1} + 1)(x_{N-1}^2 + (a+1)x_{N-1} + a).\end{aligned}$$

With this in mind, for $x > 0$, set

$$\begin{aligned}D(x) = &\left[x^2 - \left(2\bar{x} + 2 + \frac{2a+2}{\bar{x}} + \frac{a}{\bar{x}^2}\right)x + (a+1)\right]^2 \\ &-4(x+1)(x^2 + (a+1)x + a).\end{aligned}$$

Then

$$\begin{aligned}D'(x) = \; &2\left[x^2 - \left(2\bar{x} + 2 + \frac{2a+2}{\bar{x}} + \frac{a}{\bar{x}^2}\right)x + (a+1)\right] \times \\ &\left[2x - \left(2\bar{x} + 2 + \frac{2a+2}{\bar{x}} + \frac{a}{\bar{x}^2}\right)\right] - 4[x^2 + (a+1)x + a] - 4(x+1)[2x + (a+1)]\end{aligned}$$

and

$$D''(x) = 2\left[2x - \left(2\bar{x} + 2 + \frac{2a+2}{\bar{x}} + \frac{a}{\bar{x}^2}\right)\right]^2$$
$$+4\left[x^2 - \left(2\bar{x} + 2 + \frac{2a+2}{\bar{x}} + \frac{a}{\bar{x}^2}\right)x + (a+1)\right]$$
$$-24x - 8a - 16.$$

It follows by computation that $D(\bar{x}) = 0$, $D'(\bar{x}) = 0$, and $D''(\bar{x}) < 0$. Thus we see that $D(x) < 0$ for x sufficiently close to $\bar{x}$ but with $x \neq \bar{x}$. So as $\{x_n\}_{n=-1}^{\infty}$ is a non-trivial solution of Eq.(2.18) which converges to $\bar{x}$, it follows by Lemma 2.1 that there exists $N \geq 0$ such that $\mathcal{D}_N < 0$. This is a contradiction, and the proof is complete. □

2.5 Todd's Equation

Eq.(2.6), that is, the equation

$$x_{n+1} = \frac{1 + x_n + x_{n-1}}{x_{n-2}} \quad , \qquad n = 0, 1, \ldots$$

is called Todd's Equation. See [98]. As we saw in Section 2.2, **every positive solution of Eq.(2.6) is periodic with period 8.** See [73]. Indeed, if

$$x_{-2} = \alpha,\ x_{-1} = \beta,\ \text{and}\ x_0 = \gamma$$

are given positive initial conditions, then it follows by a computation that the solution $\{x_n\}_{n=-2}^{\infty}$ of Eq.(2.6) is the **period-8 sequence**

$$\alpha, \beta, \gamma, \frac{1+\beta+\gamma}{\alpha}, \frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\alpha\beta}, \frac{(1+\alpha+\beta)(1+\beta+\gamma)}{\alpha\beta\gamma},$$
$$\frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\beta\gamma}, \frac{1+\alpha+\beta}{\gamma}, \ldots.$$

Are there values of a, other than $a = 1$, such that every positive solution of the equation

$$x_{n+1} = \frac{a + x_n + x_{n-1}}{x_{n-2}} \quad , \qquad n = 0, 1, \ldots \tag{2.21}$$

is periodic? Does Eq.(2.21) possess a positive non-periodic solution?

One can see that Eq.(2.21) possesses the invariant

$$(a + x_{n-2} + x_{n-1} + x_n)\left(1 + \frac{1}{x_{n-2}}\right)\left(1 + \frac{1}{x_{n-1}}\right)\left(1 + \frac{1}{x_n}\right) = \text{constant},$$

for all $n \geq 0$ and, in general, for $a \in (0, \infty)$ and $k \in \{0, 1, \ldots\}$, the equation

$$x_{n+1} = \frac{a + x_n + \cdots + x_{n-(k-1)}}{x_{n-k}} \quad , \quad n = 0, 1, \ldots \tag{2.22}$$

possesses the invariant

$$(a + x_{n-k} + \cdots + x_n)\left(1 + \frac{1}{x_{n-k}}\right)\cdots\left(1 + \frac{1}{x_n}\right) = \text{constant} \quad \text{for all} \quad n \geq 0.$$

It follows by using the invariant that every positive solution of Eq.(2.22) is bounded and persists.

2.6 The Gingerbreadman Equation

The **gingerbreadman difference equation** is the piecewise linear difference equation

$$x_{n+1} = |x_n| - x_{n-1} + 1 \quad , \quad n = 0, 1, \ldots \tag{2.23}$$

which was investigated by Devaney, see [32], and was shown to be chaotic in certain regions and stable in others. The name of this equation is due to the fact that the orbits of certain points in the plane fill a region that looks like a "gingerbreadman."

If you use a computer to plot the orbit of the solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(2.23) with initial conditions

$$(x_{-1},\ x_0) = \left(-\frac{1}{10},\ 0\right)$$

the computer may predict that after 100,000 iterations, **the solution is still not periodic**. See [110]. Although a computer may be fooled due to round-off and truncation errors, one can show that the orbit of the solution of Eq.(2.23) with initial condition

$$(x_{-1},\ x_0) = \left(-\frac{1}{10},\ 0\right)$$

is **periodic with period 126**. An easy way to see this is to make the substitution

$$x_n = \frac{1}{10} y_n.$$

Then Eq.(2.23) is transformed into the difference equation

$$y_{n+1} = |y_n| - y_{n-1} + 10 \quad , \quad n = 0, 1, \ldots \tag{2.24}$$

and the initial conditions $(x_{-1},\ x_0) = \left(-\frac{1}{10},\ 0\right)$ of the solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(2.23) are transformed into

$$(y_{-1},\ y_0) = (-1,\ 0).$$

Let $\{y_n\}_{n=-1}^{\infty}$ be the solution of Eq.(2.24) with initial conditions $(y_{-1},\ y_0) = (-1,\ 0)$. Then the values of y_n for $-1 \leq n \leq 126$ are given as follows:

$y_{-1} = -1$	$y_0 = 0$					
$y_1 = 11$	$y_2 = 21$	$y_3 = 20$	$y_4 = 9$	$y_5 = -1$	$y_6 = 2$	$y_7 = 13$
$y_8 = 21$	$y_9 = 18$	$y_{10} = 7$	$y_{11} = -1$	$y_{12} = 4$	$y_{13} = 15$	$y_{14} = 21$
$y_{15} = 16$	$y_{16} = 5$	$y_{17} = -1$	$y_{18} = 6$	$y_{19} = 17$	$y_{20} = 21$	$y_{21} = 14$
$y_{22} = 3$	$y_{23} = -1$	$y_{24} = 8$	$y_{25} = 19$	$y_{26} = 21$	$y_{27} = 12$	$y_{28} = 1$
$y_{29} = -1$	$y_{30} = 10$	$y_{31} = 21$	$y_{32} = 21$	$y_{33} = 10$	$y_{34} = -1$	$y_{35} = 1$
$y_{36} = 12$	$y_{37} = 21$	$y_{38} = 19$	$y_{39} = 8$	$y_{40} = -1$	$y_{41} = 3$	$y_{42} = 14$
$y_{43} = 21$	$y_{44} = 17$	$y_{45} = 6$	$y_{46} = -1$	$y_{47} = 5$	$y_{48} = 16$	$y_{49} = 21$
$y_{50} = 15$	$y_{51} = 4$	$y_{52} = -1$	$y_{53} = 7$	$y_{54} = 18$	$y_{55} = 21$	$y_{56} = 13$
$y_{57} = 2$	$y_{58} = -1$	$y_{59} = 9$	$y_{60} = 20$	$y_{61} = 21$	$y_{62} = 11$	$y_{63} = 0$
$y_{64} = -1$	$y_{65} = 11$	$y_{66} = 22$	$y_{67} = 21$	$y_{68} = 9$	$y_{69} = -2$	$y_{70} = 3$
$y_{71} = 15$	$y_{72} = 22$	$y_{73} = 17$	$y_{74} = 5$	$y_{75} = -2$	$y_{76} = 7$	$y_{77} = 19$
$y_{78} = 22$	$y_{79} = 13$	$y_{80} = 1$	$y_{81} = -2$	$y_{82} = 11$	$y_{83} = 23$	$y_{84} = 22$
$y_{85} = 9$	$y_{86} = -3$	$y_{87} = 4$	$y_{88} = 17$	$y_{89} = 23$	$y_{90} = 16$	$y_{91} = 3$
$y_{92} = -3$	$y_{93} = 10$	$y_{94} = 23$	$y_{95} = 23$	$y_{96} = 10$	$y_{97} = -3$	$y_{98} = 3$
$y_{99} = 16$	$y_{100} = 23$	$y_{101} = 17$	$y_{102} = 4$	$y_{103} = -3$	$y_{104} = 9$	$y_{105} = 22$
$y_{106} = 23$	$y_{107} = 11$	$y_{108} = -2$	$y_{109} = 1$	$y_{110} = 13$	$y_{111} = 22$	$y_{112} = 19$
$y_{113} = 7$	$y_{114} = -2$	$y_{115} = 5$	$y_{116} = 17$	$y_{117} = 22$	$y_{118} = 15$	$y_{119} = 3$
$y_{120} = -2$	$y_{121} = 9$	$y_{122} = 21$	$y_{123} = 22$	$y_{124} = 11$		
$y_{125} = -1$	$y_{126} = 0.$					

Therefore, the sequence $\{y_n\}_{n=-1}^{\infty}$ (and hence also $\{x_n\}_{n=-1}^{\infty}$) is periodic with prime period 126.

It is interesting to note that the gingerbreadman difference equation is a special case of the **max difference equation**

$$x_{n+1} = \frac{\max\left\{x_n^2, A\right\}}{x_n x_{n-1}} \quad , \quad n = 0, 1, \ldots. \tag{2.25}$$

Indeed the change of variables, see [83],

$$x_n = \begin{cases} A^{\frac{1+y_n}{2}} \text{ if } \quad A > 1 \\ \\ e^{\frac{y_n}{2}} \quad \text{if } \quad A = 1 \\ \\ A^{\frac{1-y_n}{2}} \text{ if } 0 < A < 1 \end{cases}$$

reduces Eq.(2.25) to the difference equation

$$y_{n+1} = |y_n| - y_{n-1} + \delta \qquad , \qquad n = 0, 1, \ldots$$

where

$$\delta = \begin{cases} -1 \text{ if } A > 1 \\ \\ \;\; 0 \text{ if } A = 1 \\ \\ \;\; 1 \text{ if } A < 1. \end{cases}$$

To see this, observe that if $\alpha, \beta \in \mathbf{R}$, then

$$\min\{\alpha, \beta\} = \frac{1}{2}(\alpha + \beta - |\,\alpha - \beta\,|) \qquad \text{and} \qquad \max\{\alpha, \beta\} = \frac{1}{2}(\alpha + \beta + |\,\alpha - \beta\,|).$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2.25) and suppose $0 < A < 1$. Then

$$A^{\frac{1-y_{n+1}}{2}} = \frac{\max\left\{A^{1-y_n}, A\right\}}{A^{\frac{2-y_n-y_{n-1}}{2}}} = \frac{A^{\min\{1-y_n,1\}}}{A^{\frac{2-y_n-y_{n-1}}{2}}} = \frac{A^{\frac{1}{2}(2-y_n-|\,y_n\,|)}}{A^{\frac{2-y_n-y_{n-1}}{2}}}$$

$$= A^{\frac{1}{2}(-|\,y_n\,|+y_{n-1})}$$

and so

$$y_{n+1} = |\,y_n\,| - y_{n-1} + 1.$$

The proof in the other cases is similar and will be omitted.

Note that Eq.(2.25) with

$$A \in (0,\ 1)$$

reduces to the gingerbreadman difference equation (2.23).

When

$$A = 1$$

Eq.(2.25) reduces to Eq.(2.14) which by the above change of variables is transformed into the difference equation

$$y_{n+1} = |y_n| - y_{n-1} \quad , \quad n = 0, 1, \ldots . \tag{2.26}$$

Hence **every solution of Eq.(2.26) is periodic with period 9.**

What is the set of initial conditions $(x_{-1},\ x_0) \in (0, \infty) \times (0, \infty)$ through which the solutions of Eq.(2.23) are periodic?

Are there values of A, other than A = 1, for which every solution of Eq.(2.25) is periodic with the same period? What do the solutions of Eq.(2.25) do for values of A not equal to 1?

2.7 The Generalized Lozi Equation

Lozi's map is the system of difference equations

$$\begin{cases} x_{n+1} = 1 - a|x_n| + y_n \\ \\ y_{n+1} = bx_n \end{cases} \quad , \quad n = 0, 1, \ldots$$

introduced by Lozi, see [96], in 1978 as a piecewise linear analogue of the *Hénon map*

$$\begin{cases} x_{n+1} = 1 - ax_n^2 + y_n \\ \\ y_{n+1} = bx_n \end{cases} \quad , \quad n = 0, 1, \ldots .$$

The Hénon map was introduced by the theoretical astronomer Hénon, see [64], in 1976 to illuminate the strange attractor which was observed by the meteorologist Lorenz, see [95], in 1963 in the simple-looking non-linear system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = 10(y - x) \\ \\ \dfrac{dy}{dt} = x(28 - z) - y \\ \\ \dfrac{dz}{dt} = xy - \dfrac{8}{3}z \end{cases}$$

which Lorenz used in his research to model weather patterns.

When Lorenz used Euler's method to integrate this system numerically in his Royal-McBee LGP-30 computer, the solutions of this system exhibited **extremely complicated** behavior. The solutions exhibited **sensitive dependence upon initial conditions** about which Lorenz coined the phrase **butterfly effect**. If a butterfly flaps its wings in Tokyo, Japan, this may cause it to rain in Kingston, Rhode Island four days later. This is bad news for numerical methods, and means that we should be suspicious of what we "see in the computer" until we establish it by a rigorous proof.

The solutions oscillate irregularly, never exactly repeating but always remaining in a bounded region in the (x, y, z) space, and they settle onto a complicated set resembling an owl's mask or a pair of butterfly wings, which we now call a *strange attractor*, strange because its boundary is a fractal (with dimension between 2 and 3). All solutions approach the attractor quite rapidly, and there are no periodic or asymptotically periodic solutions. The term *strange attractor* was coined by Ruelle and Takens, see [113], in 1971.

The Lozi map is the first system for which it was established (by Misiurewicz, see [106], in 1980) that it possesses a strange attractor. For the Hénon map, the existence of a strange attractor was established by Benedicks and Carleson, see [17], in 1991. The existence of a strange attractor for the Lorenz equations was established by Warwick Tucker. See [121] and [122].

By eliminating the variable y_n, Lozi's map reduces to the second-order piecewise linear difference equation

$$x_{n+1} = 1 - a|x_n| + bx_{n-1} \quad , \quad n = 0, 1, \ldots \tag{2.27}$$

where a and b are real numbers.

Several of the equations which we have recently investigated, and which exhibit an interesting periodic character, are of the form

$$x_{n+1} = \frac{\max\{x_n^k, A\}}{x_n^l x_{n-1}^m} \quad , \quad n = 0, 1, \ldots \tag{2.28}$$

where

$$k, l, m \in \mathbf{Z} \text{ and } A, x_{-1}, x_0 \in (0, \infty).$$

Some special cases of this equation were investigated in [13], [67], [82], and [83] and were found to have very interesting dynamics. See also Chapter 6.

As we have seen, when $A = 1$ and $m = 1$, every solution of Eq.(2.28) is **periodic with period**

$$\begin{aligned}
3 &\text{ if } k = 0 \text{ and } l = 1 \\
4 &\text{ if } k = 0 \text{ and } l = 0 \\
5 &\text{ if } k = 1 \text{ and } l = 0 \\
6 &\text{ if } k = 0 \text{ and } l = -1 \\
7 &\text{ if } k = 1 \text{ and } l = 1 \\
8 &\text{ if } k = 1 \text{ and } l = 2 \\
9 &\text{ if } k = 2 \text{ and } l = 2 \\
12 &\text{ if } k = 2 \text{ and } l = 3.
\end{aligned}$$

It follows easily that the change of variables, see [83],

$$x_n = \begin{cases} A^{\frac{1+y_n}{k}} & \text{if } A > 1 \\ e^{\frac{y_n}{k}} & \text{if } A = 1 \\ A^{\frac{1-y_n}{k}} & \text{if } A < 1 \end{cases}$$

together with the observation that

$$\min\{\alpha,\beta\} = \frac{1}{2}[(\alpha+\beta) - |\alpha-\beta|] \qquad \text{and} \qquad \max\{\alpha,\beta\} = \frac{1}{2}[(\alpha+\beta) + |\alpha-\beta|]$$

transforms Eq.(2.28) into the piecewise linear equation

$$y_{n+1} = \frac{k}{2}|y_n| + \left(\frac{k}{2} - l\right) y_n - m y_{n-1} + \delta \quad , \qquad n = 0, 1, \ldots \tag{2.29}$$

where

$$\delta = \begin{cases} k-1-l-m & \text{if } A > 1 \\ 0 & \text{if } A = 1 \\ -(k-1-l-m) & \text{if } A < 1. \end{cases}$$

We call Eq.(2.29) the **generalized Lozi's equation**. See [83].

Are there other values of k and l for which every solution of Eq.(2.29) with m = 1 and $\delta = 0$ is periodic with the same period?

2.8 When Is Every Solution of $x_{n+1} = \dfrac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}$ Periodic with the Same Period?

Consider the non-linear second-order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}} \quad , \qquad n = 0, 1, \ldots \tag{2.30}$$

where the parameters $\alpha, \beta, \gamma, A, B, C$ are non-negative real numbers with $B + C > 0$, and where the initial conditions x_{-1} and x_0 are non-negative real numbers such that the right-hand side of Eq.(2.30) is well defined for all $n \geq 0$.

The following four special examples of Eq.(2.30)

$$x_{n+1} = \frac{1}{x_n}, \qquad n = 0, 1, \ldots \tag{2.31}$$

$$x_{n+1} = \frac{1}{x_{n-1}}, \qquad n = 0, 1, \ldots \tag{2.32}$$

$$x_{n+1} = \frac{1 + x_n}{x_{n-1}}, \qquad n = 0, 1, \ldots \tag{2.33}$$

$$x_{n+1} = \frac{x_n}{x_{n-1}}, \qquad n = 0, 1, \ldots \tag{2.34}$$

are remarkable in the following sense.

Every positive solution of Eq.(2.31) is **periodic with period 2**.
Every positive solution of Eq.(2.32) is **periodic with period 4**.
Every positive solution of Eq.(2.33) is **periodic with period 5**.
Every positive solution of Eq.(2.34) is **periodic with period 6**.

The following result characterizes all possible special cases of equations of the form of Eq.(2.30) with the property that **every solution of the equation is periodic with the same period**. See [78].

THEOREM 2.4
Let $p \geq 2$ be a positive integer, and assume that every positive solution of Eq.(2.30) is periodic with period p. Then the following statements are true:

1. *Suppose that $C > 0$. Then $A = B = \gamma = 0$.*
2. *Suppose that $C = 0$. Then $\gamma(\alpha + \beta) = 0$.*

PROOF Consider the solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(2.30) with

$$x_{-1} = 1 \qquad \text{and} \qquad x_0 \in (0, \infty).$$

Then clearly

$$x_{p-1} = x_{-1} = 1 \qquad \text{and} \qquad x_p = x_0$$

and so by Eq.(2.30)

$$x_0 = x_p = \frac{\alpha + \beta + \gamma x_{p-2}}{A + B + C x_{p-2}}.$$

Thus we see that

$$(A + B)x_0 + (Cx_0 - \gamma)x_{p-2} = \alpha + \beta. \tag{2.35}$$

(a) Assume $C > 0$. Then we claim that

$$A = B = 0. \tag{2.36}$$

Otherwise, $A + B > 0$. So by choosing

$$x_0 > \max\left\{\frac{\alpha+\beta}{A+B}, \frac{\gamma}{C}\right\}$$

we see that Eq.(2.35) is impossible. Hence Eq.(2.36) is true. In addition to Eq.(2.36), we now also claim that

$$\gamma = 0. \tag{2.37}$$

If not, then $\gamma > 0$. So by choosing

$$x_0 < \min\left\{\frac{\alpha+\beta}{A+B}, \frac{\gamma}{C}\right\}$$

we see again that Eq.(2.35) is impossible. Thus Eq.(2.37) also holds.

(b) Assume $C = 0$ and for the sake of contradiction, suppose that $\gamma(\alpha+\beta) > 0$. Then again by choosing x_0 sufficiently small, we see that Eq.(2.35) is impossible. ∎

The following corollary of Theorem 2.4 states that Eqs.(2.31), (2.32), (2.33), and (2.34) are the only special cases of Eq.(2.30) with the property that every positive solution is periodic with the same period. See [78].

COROLLARY 2.1

Let $p \in \{2,3,4,5,6\}$. Assume that $B+C > 0$, and that every positive solution of Eq.(2.30) is periodic with period p. Then up to a change of variables of the form

$$x_n = \lambda y_n$$

Eq.(2.30) reduces to one of the Eqs.(2.31), (2.32), (2.33), and (2.34).

For the more general difference equation

$$x_{n+1} = \frac{\alpha + \alpha_0 x_n + \cdots + \alpha_k x_{n-k}}{A + A_0 x_n + \cdots + A_k x_{m-k}}, \qquad n = 0, 1, \ldots \tag{2.38}$$

with non-negative initial conditions and non-negative parameters, **what are all special equations with the property that every solution is periodic with the same period?** In addition to Eqs.(2.31)-(2.34) and Todd's Eq.(2.6), are there any other **surprises**?

2.9 Period-2 Solutions of $x_{n+1} = \dfrac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}$

Consider the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}} \quad , \qquad n = 0, 1, \ldots \tag{2.39}$$

with non-negative parameters and non-negative initial conditions. To avoid degenerate cases, we shall assume that

$$\alpha + \beta + \gamma, B + C, \beta + B, \gamma + C \in (0, \infty).$$

We also assume that the parameters and initial conditions are chosen in such a way that the denominator of Eq.(2.39) is always positive. See [55] and [56].

There are now 30 equations with positive parameters which are included in Eq.(2.39) as special cases.

For some choices of the non-zero parameters of Eq.(2.39), six of these equations have a multitude of prime period-two solutions, six have a unique two-cycle, and two have one or the other of the above properties, depending upon the particular values of the non-zero parameters. See [56] and [78].

2.9.1 The Case $C = 0$.

In the case $C = 0$, Eq.(2.39) assumes the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n} \quad , \qquad n = 0, 1, \ldots \tag{2.40}$$

and exhibits the following trichotomy character when $B > 0$. See Chapter 5 for a proof.

$$\begin{cases} \gamma < \beta + A \Longrightarrow \text{every solution converges;} \\ \\ \gamma = \beta + A \Longrightarrow \text{every solution converges to a period} - 2 \text{ solution;} \\ \\ \gamma > \beta + A \Longrightarrow \text{there exist unbounded solutions.} \end{cases}$$

When

$$\gamma = \beta + A$$

every prime period-2 solution of Eq.(2.40) is given by

$$B\phi\psi = \alpha + \beta(\phi + \psi) \qquad \text{with} \qquad \phi, \psi \in [0, \infty) \text{ and } \phi \neq \psi.$$

REMARK 2.3 In the sequel, when we say that "every solution of a difference equation converges to a periodic solution with period p," we mean that every solution of the equation converges to a periodic solution of the equation with (not necessarily prime) period p, and that there exist solutions of the equation with prime period p. ▯

2.9.2 The Case $C > 0$

In the case $C > 0$, a necessary condition for Eq.(2.39) to have a prime period-2 solution is

$$\gamma > 0$$

and so we can rewrite Eq.(2.39) in the normalized form

$$x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{A + Bx_n + x_{n-1}} \quad , \quad n = 0, 1, \ldots \tag{2.41}$$

with non-negative parameters and non-negative initial conditions.

2.9.2.1 Subcase (a)

Suppose $\alpha = \beta = 0$.

In this case Eq.(2.41) is the equation

$$x_{n+1} = \frac{x_{n-1}}{A + Bx_n + x_{n-1}} \quad , \quad n = 0, 1, \ldots \tag{2.42}$$

with

$$A \in [0, \infty) \quad \text{and} \quad \mathrm{B} > 0. \tag{2.43}$$

Eq.(2.42) has prime period-2 solutions if and only if

$$A < 1.$$

Furthermore when (2.43) holds, the following statements are true:

(a) When

$$B = 1$$

every prime period-2 solution

$$\ldots, \phi, \psi, \ldots$$

of Eq.(2.42) is given by

$$\phi + \psi = 1 - A \quad \text{with} \quad \phi, \psi \in [0, \infty) \text{ and } \phi \neq \psi.$$

(b) When

$$B \neq 1$$

Eq.(2.42) has the unique period-2 solution

$$\ldots, 0, 1-A, \ldots \tag{2.44}$$

and the solution (2.44) of Eq.(2.42) is locally asymptotically stable when

$$B > 1$$

and is unstable when

$$0 < B < 1.$$

2.9.2.2 Subcase (b)

Suppose

$$\alpha + \beta > 0. \tag{2.45}$$

In this case Eq.(2.41) has prime period-2 solutions if and only if

$$\beta + A < 1, \qquad B > 1, \qquad \text{and} \qquad 4\alpha < (1-\beta-A)[B(1-\beta-A)-(1+3\beta-A)]. \tag{2.46}$$

Furthermore, when Eq.(2.45) and Eq.(2.46) hold, Eq.(2.41) has the unique prime period-2 solution

$$\ldots, \phi, \psi, \ldots \tag{2.47}$$

where the values of ϕ and ψ are the two positive and distinct roots of the quadratic equation

$$t^2 - (1-\beta-A)t + \frac{\alpha + \beta(1-\beta-A)}{B-1} = 0. \tag{2.48}$$

2.9.3 Equations with a Unique Prime Period-Two Solution

It follows from Sections 2.9.1 and 2.9.2 that after some obvious renormalizations, the only equations of the type of Eq.(2.39) with a unique prime period-two solution are the following nine equations with positive parameters. See also [56] and [78].

Case 1 (See [78], p.18)

$$y_{n+1} = \frac{y_{n-1}}{p + y_{n-1}} \quad , \qquad n = 0, 1, \ldots \tag{2.49}$$

Case 2 (See [78], p.60)

$$y_{n+1} = \frac{y_{n-1}}{py_n + y_{n-1}} \quad , \quad n = 0, 1, \dots \tag{2.50}$$

Case 3 (See [76] and [78], p.92)

$$y_{n+1} = \frac{p + y_{n-1}}{qy_n + y_{n-1}} \quad , \quad n = 0, 1, \dots \tag{2.51}$$

Case 4 (See [78], p.113 and [81])

$$y_{n+1} = \frac{py_n + y_{n-1}}{qy_n + y_{n-1}} \quad , \quad n = 0, 1, \dots \tag{2.52}$$

Case 5 (See [78], p.133)

$$y_{n+1} = \frac{y_{n-1}}{p + qy_n + y_{n-1}} \quad , \quad n = 0, 1, \dots \tag{2.53}$$

Case 6 (See [78], p.149)

$$y_{n+1} = \frac{p + qy_{n-1}}{1 + y_n + ry_{n-1}} \quad , \quad n = 0, 1, \dots \tag{2.54}$$

Case 7 (See [78], p.158)

$$y_{n+1} = \frac{py_n + y_{n-1}}{r + qy_n + y_{n-1}} \quad , \quad n = 0, 1, \dots \tag{2.55}$$

Case 8 (See [78], p.175)

$$y_{n+1} = \frac{r + py_n + y_{n-1}}{qy_n + y_{n-1}} \quad , \quad n = 0, 1, \dots \tag{2.56}$$

Case 9 (See [78], p.184)

$$y_{n+1} = \frac{p + qy_n + y_{n-1}}{r + sy_n + y_{n-1}} \quad , \quad n = 0, 1, \dots . \tag{2.57}$$

It is surprising that even the **local asymptotic stability** character of the prime period-2 solutions of Eqs.(2.55), (2.56), and (2.57) has not yet been determined.

2.10 The Riccati Difference Equation

The *Riccati difference equation* is the difference equation

$$z_{n+1} = \frac{a + bz_n}{c + dz_n} \quad , \qquad n = 0, 1, \ldots \tag{2.58}$$

where the parameters $a,\ b,\ c,\ d$ are given real numbers, and the initial condition z_0 is an arbitrary real number.

To avoid degenerate cases, we assume throughout this section without further mention that

$$|a| + |b| \neq 0 \qquad \text{and} \qquad |c| + |d| \neq 0.$$

We shall also assume throughout this section, unless otherwise mentioned, that

$$d \neq 0 \qquad \text{and} \qquad bc - ad \neq 0.$$

Indeed, when $d = 0$, Eq.(2.58) is a linear equation, while if

$$d \neq 0 \qquad \text{and} \qquad bc - ad = 0$$

Eq.(2.58) reduces to the trivial difference equation

$$z_{n+1} = \frac{\frac{bc}{d} + bz_n}{c + dz_n} = \frac{b(c + dz_n)}{d(c + dz_n)} = \frac{b}{d} \quad , \qquad n = 0, 1, \ldots .$$

Suppose $\bar{z}$ is an equilibrium point of Eq.(2.58). Then

$$d\bar{z}^2 + (c - b)\bar{z} - a = 0.$$

Thus we see that Eq.(2.58) has exactly two equilibrium points if $(b-c)^2+4ad > 0$, exactly one equilibrium point if $(b-c)^2+4ad = 0$, and no equilibrium points if $(b - c)^2 + 4ad < 0$.

For the results in this section, see [57].

THEOREM 2.5
The following statements are true:

1. *Eq.(2.58) has a prime period-2 solution if and only if* $b + c = 0$.
2. *Suppose* $b + c = 0$. *Then every solution* $\{z_n\}_{n=0}^{\infty}$ *of Eq.(2.58) with* $z_0 \neq -\dfrac{c}{d}$ *is periodic with period 2.*

Let $\mathcal{G}$ be the set of all initial conditions $z_0 \in \mathbf{R}$ such that the solution $\{z_n\}_{n=0}$ of Eq.(2.58) exists for all $n \geq 0$, and set $\mathcal{F} = \mathbf{R} - \mathcal{G}$.

Thus $\mathcal{F}$ is the set of all $z_0 \in \mathbf{R}$ such that the solution of Eq.(2.58) with initial condition z_0 fails to exist after a finite number of terms. $\mathcal{G}$ is called the *good set* of Eq.(2.58), and $\mathcal{F}$ is called the *forbidden set* of Eq.(2.58).

When $b + c = 0$, the forbidden set of Eq.(2.58) is the singleton

$$\mathcal{F} = \left\{-\frac{c}{d}\right\}$$

while in the degenerate case $d(bc - ad) = 0$, the forbidden set of Eq.(2.58) is the empty set.

Throughout the remainder of this section we shall assume that

$$d \neq 0, \qquad bc - ad \neq 0, \qquad \text{and} \qquad b + c \neq 0. \tag{2.59}$$

The change of variables

$$z_n = \frac{b+c}{d} w_n - \frac{c}{d}$$

transforms Eq.(2.58) into the difference equation with one parameter

$$w_{n+1} = 1 - \frac{\mathcal{R}}{w_n} \quad , \qquad n = 0, 1, \dots \tag{2.60}$$

where the parameter $\mathcal{R}$, which we call the *Riccati number* of Eq.(2.58), is the non-zero real number

$$\mathcal{R} = \frac{bc - ad}{(b+c)^2}$$

and where the initial condition w_0 of Eq.(2.60) is

$$w_0 = \frac{dz_0 + c}{b + c}.$$

We make the further change of variables

$$\begin{cases} w_n = \dfrac{u_{n+1}}{u_n} \text{ for } n = 0, 1, \dots \\ \\ u_0 = 1 \end{cases}$$

which reduces Eq.(2.60) to the second order linear difference equation

$$u_{n+2} - u_{n+1} + \mathcal{R} u_n = 0 \quad , \qquad n = 0, 1, \dots \tag{2.61}$$

with initial conditions

$$u_0 = 1 \qquad \text{and} \qquad u_1 = w_0.$$

The characteristic equation of Eq.(2.61) is

$$\lambda^2 - \lambda + \mathcal{R} = 0 \tag{2.62}$$

with characteristic roots

$$\lambda_1 = \frac{1 - \sqrt{1 - 4\mathcal{R}}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{1 + \sqrt{1 - 4\mathcal{R}}}{2}.$$

THEOREM 2.6

Assume that (2.59) holds, and that $\mathcal{R} < \frac{1}{4}$. Then the forbidden set $\mathcal{F}$ of Eq.(2.58) is given as follows:

$$\mathcal{F} = \left\{ \frac{b+c}{d} \left(\frac{\lambda_1 \lambda_2^n - \lambda_2 \lambda_1^n}{\lambda_2^n - \lambda_1^n} \right) - \frac{c}{d} : n \geq 1 \right\}.$$

For any solution $\{z_n\}_{n=0}^{\infty}$ of Eq.(2.58) with $z_0 \in \mathcal{G}$, we have

$$z_n = \frac{b+c}{d} \left[\frac{c_1 \lambda_1^{n+1} + c_2 \lambda_2^{n+1}}{c_1 \lambda_1^n + c_2 \lambda_2^n} \right] - \frac{c}{d} \qquad \textit{for all} \qquad n = 0, 1, \ldots$$

where

$$c_1 = \frac{\lambda_2(b+c) - (dz_0 + c)}{(b+c)(\lambda_2 - \lambda_1)} \qquad \textit{and} \qquad c_2 = \frac{(dz_0 + c) - \lambda_1(b+c)}{(b+c)(\lambda_2 - \lambda_1)}.$$

REMARK 2.4 Suppose Eq.(2.59) holds, and that $\mathcal{R} < \frac{1}{4}$. Then the following statements are true:

1. $\dfrac{\lambda_1(b+c) - c}{d}$ is the smaller of the two equilibrium points of Eq.(2.58).
2. $\dfrac{\lambda_2(b+c) - c}{d}$ is the larger of the two equilibrium points of Eq.(2.58).

In particular, $\dfrac{\lambda_1(b+c) - c}{d}$ and $\dfrac{\lambda_2(b+c) - c}{d}$ are elements of $\mathcal{G}$. ∎

COROLLARY 2.2

Assume that Eq.(2.59) holds, and that $\mathcal{R} < \frac{1}{4}$. Let $\{z_n\}_{n=0}^{\infty}$ be a solution of Eq.(2.58) with $z_0 \in \mathcal{G} - \left\{ \dfrac{\lambda_1(b+c) - c}{d} \right\}$. Then

$$\lim_{n \to \infty} z_n = \frac{\lambda_2(b+c) - c}{d}.$$

THEOREM 2.7
Assume that Eq.(2.59) holds, and that $\mathcal{R} = \frac{1}{4}$. *Then the forbidden set* $\mathcal{F}$ *of Eq.(2.58) is given as follows:*

$$\mathcal{F} = \left\{ \frac{n(b-c)-(b+c)}{2dn} : n \geq 1 \right\}.$$

For any solution $\{z_n\}_{n=0}^{\infty}$ *of Eq.(2.58) with* $z_0 \in \mathcal{G}$, *we have*

$$z_n = \frac{b+c}{d}\left[\frac{(b+c)+(n+1)(2dz_0+(c-b))}{2(b+c)+2n(2dz_0+(c-b))}\right] - \frac{c}{d} \quad \textit{for all} \quad n \geq 0.$$

REMARK 2.5 Suppose Eq.(2.59) holds, and that $\mathcal{R} = \frac{1}{4}$. Then $\frac{b-c}{2d}$ is the unique equilibrium point of Eq.(2.58). In particular, $\frac{b-c}{2d} \in \mathcal{G}$. ▯

COROLLARY 2.3
Assume that Eq.(2.59) holds, and that $\mathcal{R} = \frac{1}{4}$. *Let* $\{z_n\}_{n=0}^{\infty}$ *be a solution of Eq.(2.58) with* $z_0 \in \mathcal{G}$. *Then*

$$\lim_{n\to\infty} z_n = \frac{b-c}{2d}.$$

THEOREM 2.8
Assume that Eq.(2.59) holds, and that $\mathcal{R} > \frac{1}{4}$. *Let* $\theta \in \left(0, \frac{\pi}{2}\right)$ *be chosen such that*

$$\cos\theta = \frac{1}{2\sqrt{\mathcal{R}}} \quad \textit{and} \quad \sin\theta = \frac{\sqrt{4\mathcal{R}-1}}{2\sqrt{\mathcal{R}}}.$$

Then the forbidden set $\mathcal{F}$ *of Eq.(2.58) is given as follows:*

$$\mathcal{F} = \left\{ \frac{b-c}{2d} - \frac{(b+c)\sqrt{4\mathcal{R}-1}}{2d}\cot n\theta : n \geq 1 \textit{ and } \sin n\theta \neq 0 \right\}.$$

For any solution $\{z_n\}_{n=0}^{\infty}$ *of Eq.(2.58) with* $z_0 \in \mathcal{G}$, *we have*

$$z_n = \frac{b+c}{d}\left[\frac{\sqrt{\mathcal{R}}\left(\cos(n+1)\theta + \frac{2w_0-1}{\sqrt{4\mathcal{R}-1}}\sin(n+1)\theta\right)}{\cos n\theta + \frac{2w_0-1}{\sqrt{4\mathcal{R}-1}}\sin n\theta}\right] - \frac{c}{d} \quad \textit{for all} \quad n \geq 0$$

where $w_0 = \dfrac{dz_0+c}{b+c}$.

COROLLARY 2.4

Assume that Eq.(2.59) holds, and that $\mathcal{R} > \frac{1}{4}$. *Let* $\{z_n\}_{n=0}^{\infty}$ *be a solution of Eq.(2.58) with* $z_0 \in \mathcal{G}$. *Then the following statements are true:*

1. *Suppose that* $\theta = \frac{q}{p}\pi$, *where p and q are positive, relative prime integers. Then* $\{z_n\}_{n=0}^{\infty}$ *is a periodic solution of Eq.(2.58) with prime period p.*
2. *Suppose that* θ *is not a rational multiple of* π. *Then the orbit of* $\{z_n\}_{n=0}^{\infty}$ *is dense in* $\mathbf{R}$.

REMARK 2.6 Let us summarize the asymptotic behavior and the periodic nature of the solutions of the Riccati difference equation Eq.(2.58) with initial condition in the good set $\mathcal{G}$ under the assumption that

$$d \neq 0 \qquad \textit{and} \qquad bc - ad \neq 0.$$

1. Every solution is periodic with period 2 if and only if

$$b = -c.$$

2. When

$$b \neq -c$$

the the following statements are true:

(a) Every solution of Eq.(2.58) has a finite limit if and only if

$$\mathcal{R} \leq \frac{1}{4}.$$

(b) When

$$\mathcal{R} > \frac{1}{4}$$

either every solution of Eq.(2.58) is periodic with the same prime period $p \geq 3$, or else every solution of Eq.(2.58) is dense in the real line $\mathbf{R}$. The precise character of the solutions of Eq.(2.58) depends upon the value of $\theta \in \left(0, \frac{\pi}{2}\right)$, where

$$\cos\theta = \frac{1}{2\sqrt{\mathcal{R}}} \qquad \textit{and} \qquad \sin\theta = \frac{\sqrt{4\mathcal{R} - 1}}{2\sqrt{\mathcal{R}}}.$$

(i) Suppose that $\theta = \frac{q}{p}\pi$, where p and q are positive, relatively prime integers. Then every solution of Eq.(2.58) is periodic with prime period p.

(ii) Suppose that θ is not a rational multiple of π. Then the orbit of every solution of Eq.(2.58) is dense in $\mathbf{R}$. ▯

2.11 Sharkovsky's Theorem

Throughout this monograph, we deal almost exclusively with difference equations of order greater than one. For a first-order difference equation of the form

$$x_{n+1} = F(x_n) \quad , \quad n = 0, 1, \ldots \tag{2.63}$$

where

$$F : I \to I$$

is a continuous function mapping some interval of real numbers I into itself, the most glorified result known about periodic solutions of Eq.(2.63) is known as **Sharkovsky's Theorem**. See [116]. For a good reference for this and other related theorems, read the historical remarks by M. Misiurewicz in [107]. See also [11].

Before stating the theorem, we introduce the so-called *Sharkovsky ordering* of the set of positive integers $\mathbf{N} = \{1, 2, \ldots\}$.

$$3 \prec 5 \prec 7 \prec \cdots \prec 3 \cdot 2 \prec 5 \cdot 2 \prec 7 \cdot 2 \prec \cdots \prec 3 \cdot 2^2 \prec 5 \cdot 2^2 \prec 7 \cdot 2^2 \prec \cdots$$
$$\cdots \prec 2^4 \prec 2^3 \prec 2^2 \prec 2 \prec 1.$$

THEOREM 2.9
(Sharkovsky's Theorem) Let I be an interval of real numbers. Then the following statements are true.

1. *Let $F : I \to I$ be a continuous function, and let k be a positive integer. Suppose there exists a point $p \in I$ which has minimal period k. Let l be a positive integer such that $k \prec l$ in the Sharkovsky ordering. Then there exists a point $q \in I$ such that q has minimal period l.*

2. *Let l be a positive integer. Then there exists a continuous function $G : I \to I$ such that the following statements are true.*

   (a) There exists a point $p \in I$ which has minimal period l.

   (b) If k is a positive real integer such that $k \prec l$ in the Sharkovsky ordering, then there exists no point $q \in I$ having minimal period k.

3. *There exists a continuous function $H : I \to I$ such that for every positive integer n, H has a point $p \in I$ of minimal period 2^n, but there are no points $q \in I$ of any other period.*

2.12 Period 3 Implies Chaos

A special case of Sharkovsky's Theorem is the celebrated theorem of Li and Yorke, **Period 3 Implies Chaos**, see [94], in which it was shown that if I is an interval of real numbers and $F \in C[I, I]$, and if Eq.(2.63) possesses a periodic solution of prime period 3, then Eq.(2.63) possesses solutions of prime period p for every positive integer p. In the theorem, F^2 stands for F composed with itself, F^3 is F composed with F^2, and in general for $n > 1$, F^n (the n^{th} iterate of F) is F composed with F^{n-1}.

THEOREM 2.10
(**Period 3 Implies Chaos**)
Let I be an interval of real numbers, and let $F : I \to \mathbf{R}$ be a continuous function. Assume there is a point $a \in I$ such that

$$F^3(a) \le a < F(a) < F^2(a)$$

or

$$F^3(a) \ge a > F(a) > F^2(a).$$

Then the following statements are true:

1. *For every $k \in \{1, 2, \ldots\}$, there is a point $p_k \in I$ having minimal period k (i.e., $F^k(p_k) = p_k$, and $F^n(p_k) \neq p_k$ for $1 \le n < k$).*
2. *There is an uncountable set of aperiodic points $S \subset I$ which satisfies the following conditions:*

   (a) For every $p, q \in S$ with $p \neq q$,

   $$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0$$

   and

   $$\liminf_{n\to\infty} |F^n(p) - F^n(q)| = 0.$$

   (b) For every point $p \in S$ and every periodic point $q \in I$,

   $$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0.$$

REMARK 2.7 Two comments are in order about Theorem 2.10.

1. Suppose $b \in I$ is a point of minimal period 3. Then the hypothesis about the existence of the point $a \in I$ is automatically satisfied. To see this, first re-label $\{b, F(b), F^2(b)\}$, if necessary, to have $b = \min\{b, F(b), F^2(b)\}$. Then if
$$b < F(b) < F^2(b),$$
take $a = b$ and observe that
$$F^3(a) = a < F(a) < F^2(a)$$
while if
$$b < F^2(b) < F(b),$$
then take $a = F(b)$ and observe that
$$F^3(a) = a > F(a) > F^2(a)$$
and so period 3 really does imply chaos.

2. The second fact to mention is that in practice, it is much easier to find a point a satisfying either
$$F^3(a) = a < F(a) < F^2(a)$$
or
$$F^3(a) = a > F(a) > F^2(a)$$
than it is to find a point b of period 3, and hence the condition on a is extremely practical. ∎

Example 2.1
A simple example of a first-order difference equation which possesses prime period solutions of every period is the "logistic equation"

$$x_{n+1} = 4x_n(1 - x_n) \quad , \quad n = 0, 1, \ldots \tag{2.64}$$

with initial condition $x_0 \in [0, 1]$.

Solution:
Let $\theta \in \left[0, \frac{\pi}{2}\right]$, and take $x_0 = \sin^2 \theta$. Then it follows by induction on n that

$$x_n = \sin^2(2^n\theta) \qquad \text{for} \qquad n = 0, 1, \ldots .$$

In particular when $\theta = \frac{\pi}{9}$,

$$x_0 = \sin^2 \frac{\pi}{9}$$

$$x_1 = \sin^2 \frac{2\pi}{9}$$

$$x_2 = \sin^2 \frac{4\pi}{9}$$

$$x_3 = \sin^2 \frac{8\pi}{9} = x_0.$$

Thus x_0 is a point of period 3, and so by Theorem 2.9, Eq.(2.64) possesses periodic solutions of every period. ∎

2.13 Open Problems and Conjectures

OPEN PROBLEM 2.1
Assume that $f : (0,\infty) \to (0,\infty)$ is a continuous function, and that every positive solution of the equation

$$x_{n+1} = \frac{f(x_n, x_{n-1})}{x_n} \quad , \qquad n = 0, 1, \ldots$$

is periodic with period $p \geq 5$. Find f. See [2], [3], [4], and [103].

CONJECTURE 2.1
Assume that $f \in C^1[(0,\infty) \times (0,\infty), (0,\infty)]$, and that every positive solution of the equation

$$x_{n+1} = \frac{f(x_n, x_{n-1})}{x_{n-2}} \quad , \qquad n = 0, 1, \ldots$$

is periodic with period 8. Show that $f(x, y) = 1 + x + y$. See [2] and [4].

CONJECTURE 2.2
Assume $A \in (0,\infty)$. Show that no positive non-equilibrium solution of the equation

$$x_{n+1} = \frac{\max\{A, x_n, x_{n-1}\}}{x_{n-2}} \quad , \qquad n = 0, 1, \ldots$$

has a limit. Extend and generalize.

OPEN PROBLEM 2.2
Find all values of k and l such that every positive solution of

$$x_{n+1} = \frac{\max\left\{x_n^k, 1\right\}}{x_n^l x_{n-1}} \quad , \quad n = 0, 1, \ldots$$

is periodic. See [1] and [83].

CONJECTURE 2.3
Show that for no value of a other than $a = 1$ is every positive solution of

$$x_{n+1} = \frac{a + x_n}{x_{n-1}} \quad , \quad n = 0, 1, \ldots$$

periodic.

CONJECTURE 2.4
Show that for no value of a other than $a = 1$ is every positive solution of

$$x_{n+1} = \frac{a + x_n + x_{n-1}}{x_{n-2}} \quad , \quad n = 0, 1, \ldots$$

periodic.

CONJECTURE 2.5
Show that for no value of A other than $A = 1$ is every positive solution of

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_n x_{n-1}} \quad , \quad n = 0, 1, \ldots$$

periodic.

CONJECTURE 2.6
Show that for no value of A other than $A = 1$ is every positive solution of

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_n^2 x_{n-1}} \quad , \quad n = 0, 1, \ldots$$

periodic.

CONJECTURE 2.7

Show that for no value of A other than A = 1 is every positive solution of

$$x_{n+1} = \frac{\max\{x_n^2, A\}}{x_n x_{n-1}} \quad , \; n = 0, 1, \ldots$$

periodic.

CONJECTURE 2.8

Show that for no value of A other than A = 1 is every positive solution of

$$x_{n+1} = \frac{\max\{x_n^2, A\}}{x_n^3 x_{n-1}} \quad , \qquad n = 0, 1, \ldots$$

periodic.

OPEN PROBLEM 2.3

Assume $k, A \in [0, \infty)$. Show that every positive solution of the equation

$$x_{n+1} = \frac{\max\{x_n^k, A\}}{x_{n-1}} \quad , \qquad n = 0, 1, \ldots$$

is bounded if and only if $k \in [0, 2)$. See [68].

OPEN PROBLEM 2.4

Assume $k, A \in [0, \infty)$. Show that every positive solution of the equation

$$x_{n+1} = \frac{\max\{x_n^k, A\}}{x_n x_{n-1}} \quad , \; n = 0, 1, \ldots$$

is bounded if and only if $k \in [0, 3)$. See [68].

OPEN PROBLEM 2.5

For the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + C x_{n-1} + D x_{n-2}} \quad , \qquad n = 0, 1, \ldots$$

with non-negative parameters and positive initial conditions, find all special cases with the property that every positive solution is periodic with the same period. Extend this result to higher order equations.

CONJECTURE 2.9

Assume that $a \in (0, \infty)$ and $k \in \{1, 2, \ldots\}$. Show that no positive non-equilibrium solution of the equation

$$x_{n+1} = \frac{a + x_n + x_{n-1} + \cdots + x_{n-(k-1)}}{x_{n-k}} \quad , \qquad n = 0, 1, \ldots$$

has a limit. Hint: You may need to use the fact that this equation possesses the invariant

$$(a+x_n+\cdots+x_{n-(k-1)})\left(1+\frac{1}{x_n}\right)\cdots\left(1+\frac{1}{x_{n-(k-1)}}\right) = \text{constant } \textit{for all } n \geq 0.$$

OPEN PROBLEM 2.6
Find all values of $a \in (0,\infty)$ and $k \in \{1,2,\ldots\}$ for which every positive solution of

$$x_{n+1} = \frac{a + x_n + \cdots + x_{n-(k-1)}}{x_{n-k}}, \qquad n = 0,1,\ldots$$

is periodic and determine the period.

REMARK 2.8 A Generalized Riccati Equation.
Recall that every solution of the Riccati equation

$$x_{n+1} = \frac{x_n}{x_n - 1}, \qquad n = 0,1,\ldots \tag{2.65}$$

with

$$x_0 \neq -1$$

is defined for all $n \geq 0$ and is periodic with period two.

It is interesting to note that Eq.(2.65) is a special case of the difference equation

$$x_{n+1} = \frac{x_n + x_{n-1}\cdots + x_{n-k}}{x_n x_{n-1}\cdots x_{n-k} - 1}, \qquad n = 0,1,\ldots \tag{2.66}$$

with $k \geq 0$. Indeed, one can see that every solution of Eq.(2.66) which is defined for all $n \geq 0$ is periodic with period $(k+2)$. This follows from the observation that

$$x_1 = \frac{x_0 + x_{-1} + \cdots + x_{-k}}{x_0 x_{-1}\cdots x_{-k} - 1}$$

and so by a simple computation

$$x_2 = \frac{\dfrac{x_0 + x_{-1}\cdots + x_{-k}}{x_0 x_{-1}\cdots x_{-k} - 1} + x_0 + \cdots + x_{-k+1}}{\dfrac{x_0 + x_{-1}\cdots + x_{-k}}{x_0\cdots x_{-k} - 1}\cdot x_0 x_{-1}\cdots x_{-k+1} - 1} = x_{-k}.$$

An interesting property of every well-defined solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2.66) is that

$$x_{n+1} + x_n + \cdots + x_{n-k} = x_{n+1}x_n\cdots x_{n-k} \qquad \text{for all} \qquad n \geq 0. \qquad \square$$

OPEN PROBLEM 2.7

Determine the good set $\mathcal{G}$ of Eq.(2.66). That is, find the set $\mathcal{G}$ of all initial conditions $(x_{-k}, \ldots, x_0)$ such that the solution of Eq.(2.66) is well defined for all $n \geq 0$.

OPEN PROBLEM 2.8

Investigate the good set and the character of the solutions of the difference equation

$$x_{n+1} = \frac{x_n + x_{n-1} + \cdots + x_{n-k}}{x_n x_{n-1} \cdots x_{n-k} + A}, \qquad n = 0, 1, \ldots$$

where A is a real parameter.

CONJECTURE 2.10

Show that the unique prime period-2 solution of Eq.(2.55) is locally asymptotically stable.

CONJECTURE 2.11

Show that the unique prime period-2 solution of Eq.(2.56) is locally asymptotically stable.

CONJECTURE 2.12

Show that the unique prime period-2 solution of Eq.(2.57) is locally asymptotically stable.

Chapter 3

EQUATIONS WITH EVENTUALLY PERIODIC SOLUTIONS

3.1 Introduction

In this chapter we present various examples of difference equations which have the property that every solution is **eventually periodic** with a prescribed period. We believe that these are fascinating examples which need to be brought to the attention of the general mathematical community. See also Chapter 7.

A question of great importance is the following. **What is it that makes all the solutions of a difference equation be eventually periodic with the same period? Is there an easily verifiable necessary and sufficient condition that can be used to test for this property?**

3.2 The Equation $x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A}{x_{n-1}}\right\}$

In this section we discuss the periodic character of solutions of the difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A}{x_{n-1}}\right\} \quad , \qquad n = 0, 1, \ldots \tag{3.1}$$

with positive parameter A and with positive initial conditions. The main result which we establish is the following theorem. See [9].

THEOREM 3.1

Assume $A \in (0, \infty)$*. Then every positive solution of Eq.(3.1) is eventually periodic with period*

$$\begin{cases} 2 \text{ if } A < 1 \\ 3 \text{ if } A = 1 \\ 4 \text{ if } A > 1. \end{cases}$$

REMARK 3.1 Before we present the proof of Theorem 3.1, we make the observation that the periodic character of the solutions of Eq.(3.1) depends upon whether the parameter A is dominated by 1, equals 1, or dominates 1. In all cases it seems that the the dominant term in Eq.(3.1) determines the period of the solutions. We really do not understand what is going on, or why.

For example, when A is dominated by 1 (when $A < 1$), every solution of Eq.(3.1) is eventually periodic with period 2. Note that the "dominating" difference equation

$$x_{n+1} = \frac{1}{x_n} \quad , \quad n = 0, 1, \ldots \tag{3.2}$$

has the property that every solution is periodic with period 2. When $A > 1$, every solution of Eq.(3.1) is eventually periodic with period 4, and every solution of the "dominating" difference equation

$$x_{n+1} = \frac{A}{x_{n-1}} \quad , \quad n = 0, 1, \ldots \tag{3.3}$$

has period 4. Finally, when $A = 1$, every solution of Eq.(3.1) is eventually periodic with period 3, the average of the periods of the solutions of Eqs.(3.2) and (3.3). $\square$

PROOF

(i) Suppose $0 < A < 1$. For each $n \geq -1$, set $x_n = A^{y_n}$. Then Eq.(3.1) is transformed into the difference equation

$$y_{n+1} = \min\{-y_n, 1 - y_{n-1}\} \quad , \quad n = 0, 1, \ldots \tag{3.4}$$

where $y_{-1} = \frac{1}{A} \ln x_{-1}$ and $y_0 = \frac{1}{A} \ln x_0$ are real numbers. It suffices to show that $\{y_n\}_{n=-1}^{\infty}$ is eventually periodic with period 2.

Set

$$\mathbf{S} = \left\{ (\alpha, -\alpha) : -\frac{1}{2} \leq \alpha \leq \frac{1}{2} \right\}.$$

Case 1. Suppose $(y_{-1}, y_0) \in \mathbf{S}$.
Then $\{y_n\}_{n=-1}^{\infty}$ is clearly periodic with period 2.

Case 2. Suppose $(y_{-1}, y_0) \notin \mathbf{S}$.
Note that $\{y_n\}_{n=-1}^{\infty}$ oscillates about zero, and so, without loss of generality, we may assume that $y_{-1} \geq 0$ and $y_0 \leq 0$.

Set

$$\mathbf{L} = \{(\alpha, -\alpha) : \alpha \in \mathbf{R}\}.$$

We claim there exists $N \in \{0, 1, \ldots\}$ such that $(y_N, y_{N+1}) \in \mathbf{L}$.

For the sake of contradiction, suppose this claim is false. Then $\{y_n\}_{n=-1}^{\infty}$ satisfies the difference equation

$$y_{n+1} = 1 - y_{n-1} \quad , \quad n = 0, 1, \ldots \tag{3.5}$$

as well as the difference inequality

$$-y_n > 1 - y_{n-1} \quad , \quad n = 0, 1, \ldots. \tag{3.6}$$

It follows from Eq.(3.5) that $\{y_n\}_{n=-1}^{\infty}$ is periodic with period 4, while it follows from Inequality (3.6) that

$$\lim_{n \to \infty} y_n = -\infty.$$

This is a contradiction, and so we see that there does exist $N \geq 0$ so that $(y_N, y_{N+1}) \in \mathbf{L}$.

The proof follows from Case 1 if $(y_N, y_{N+1}) \in \mathbf{S}$.

So suppose that $(y_N, y_{N+1}) \in \mathbf{L} - \mathbf{S}$. Note that if $y_N < -\frac{1}{2}$, then $y_{N+1} > \frac{1}{2}$ and hence

$$y_{N+2} = \min\{-y_{N+1}, 1 - y_N\} = -y_{N+1}.$$

Hence it follows that without loss of generality we may assume that $y_{-1} > \frac{1}{2}$, and that $(y_{-1}, y_0) \in \mathbf{L}$. The first five terms of $\{y_n\}_{n=-1}^{\infty}$ are

$$y_{-1}, -y_{-1}, 1 - y_{-1}, y_{-1} - 1, 1 - y_{-1}.$$

Thus (y_{-1}, y_0) and (y_2, y_3) are elements of $\mathbf{L}$, and the distance between (y_{-1}, y_0) and (y_2, y_3) is $\sqrt{2}$. Now $y_{-1} > \frac{1}{2}$ and the length of $\mathbf{S}$ is $\sqrt{2}$, and so we see that (y_2, y_3) is closer to $\mathbf{S}$ than (y_{-1}, y_0) is. So as the length of $\mathbf{S}$ is $\sqrt{2}$, it follows that there exists $M \geq 0$ such that $(y_M, y_{M+1}) \in \mathbf{S}$, and the proof follows from Case 1.

(ii) Suppose $A = 1$. It is easy to see that $\{x_n\}_{n=-1}^{2}$ contains two consecutive terms each greater than or equal to 1, and so without loss of generality we may assume that $x_{-1} \geq 1$ and $x_0 \geq 1$. Then $\{x_n\}_{n=-1}^{\infty}$ is either

$$x_{-1}, x_0, \frac{1}{x_{-1}}, x_{-1}, x_{-1}, \frac{1}{x_{-1}}, x_{-1}, \ldots$$

or

$$x_{-1}, x_0, \frac{1}{x_0}, x_0, x_0, \frac{1}{x_0}, x_0, \ldots .$$

Hence in either case $\{x_n\}_{n=1}^{\infty}$ is periodic with period 3.

(iii) Suppose $A > 1$. For each $n \geq -1$, set $x_n = A^{y_n + \frac{1}{2}}$. Then Eq.(3.1) is transformed into the difference equation

$$y_{n+1} = \max\{-1 - y_n, -y_{n-1}\} \quad , \quad n = 0, 1, \ldots \tag{3.7}$$

where $y_{-1} = \dfrac{\ln x_{-1}}{\ln A} - \dfrac{1}{2}$ and $y_0 = \dfrac{\ln x_0}{\ln A} - \dfrac{1}{2}$ are real numbers.

Set

$$\mathbf{B} = \{(\alpha, \beta) : |\alpha| + |\beta| \leq 1\}$$

and

$$\mathbf{T} = \{(\alpha, -\alpha - 1) : \alpha \leq 0\}.$$

It is easy to see that $\{y_n\}_{n=-1}^{\infty}$ is of the form

$$y_{-1}, y_0, -y_{-1}, -y_0, y_{-1}, y_0, \ldots$$

if and only if $(y_{-1}, y_0) \in \mathbf{B}$, and so it suffices to consider the case where $(y_{-1}, y_0) \notin \mathbf{B}$.

It is easy to see that $\{y_n\}_{n=-1}^{3}$ contains two consecutive non-negative terms, and so without loss of generality we may assume that $y_{-1} \geq 0$ and $y_0 \geq 0$. Then since $y_{-1} + y_0 > 1$, we see that either $y_1 = -1 - y_0$ and $y_2 = -1 - y_1$ or $y_1 = -y_{-1}$ and $y_2 = -1 - y_1$. In either case $(y_1, y_2) \in \mathbf{T} - \mathbf{B}$. Finally, suppose that $(y_N, y_{N+1}) \in \mathbf{T} - \mathbf{B}$ for some $N \geq 1$. Clearly $y_N < -1$, and so

$$y_{N+2} = -y_N, y_{N+3} = 1 + y_N, y_{N+4} = -2 - y_N.$$

Note that $(y_{N+3}, y_{N+4}) \in \mathbf{T}$, and that the distance between (y_N, y_{N+1}) and (y_{N+3}, y_{N+4}) is $\sqrt{2}$. Moreover, the point $(y_{N+3}, y_{N+4}) = (1 + y_N, -2 - y_N)$ is closer to $\mathbf{B}$ than the point $(y_N, y_{N+1}) = (y_N, -1 - y_N)$ is. So as the length of $\mathbf{B} \cap \mathbf{T}$ is $\sqrt{2}$, it follows that there exists $n_0 \geq 4$ such that $(y_{n_0}, y_{n_0+1}) \in \mathbf{B} \cap \mathbf{T}$. ∎

3.3 Max Equations with Periodic Coefficients

Recall from Section 3.2 that every positive solution of the difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A}{x_{n-1}}\right\} \quad , \qquad n = 0, 1, \ldots \tag{3.1}$$

where $A \in (0, \infty)$, is eventually periodic with period

$$\begin{cases} 2 \text{ if } A < 1 \\ 3 \text{ if } A = 1 \\ 4 \text{ if } A > 1. \end{cases}$$

The above result was extended in [14], [15], and [54] to the difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right\} \quad , \qquad n = 0, 1, \ldots \tag{3.8}$$

where the coefficient sequence $\{A_n\}_{n=0}^{\infty}$ is periodic, either with period 2 or with period 3.

For positive period-2 coefficients $\{A_n\}_{n=0}^{\infty}$ with

$$A_n = \begin{cases} A_0 \text{ if } n \text{ is even} \\ A_0 \text{ if } n \text{ is even} \end{cases}$$

it was shown in [15] that every positive solution of Eq.(3.8) is eventually periodic with period

$$\begin{cases} 2 \text{ if } A_0 A_1 < 1 \\ 6 \text{ if } A_0 A_1 = 1 \\ 4 \text{ if } A_0 A_1 > 1. \end{cases}$$

For positive period-3 coefficients $\{A_n\}_{n=0}^{\infty}$, the following results were obtained in [14] and [54]:

1. Suppose $A_n \in (0, 1)$ for all $n \geq 0$. Then every positive solution of Eq.(3.8) is eventually periodic with period 2.

2. Suppose $A_n \in (1, \infty)$ for all $n \geq 0$. Then every positive solution of Eq.(3.8) is eventually periodic with period 12.

3. Suppose there exists $i \geq 0$ such that $A_{i+1} < 1 < A_i$. Then Eq.(3.8) has unbounded solutions.

4. In all other cases, every positive solution of Eq.(3.8) is eventually periodic with period 3.

The proofs of the above results are given in Chapter 7.

What is it that determines the periodic nature of solutions of Eq.(3.8) when the coefficient sequence $\{A_n\}_{n=0}^{\infty}$ is a positive constant or a periodic sequence with a given period? Can we extend the predictions to equations with more terms? This is a fascinating problem of paramount difficulty and complexity which may not be possible to resolve in our lifetime.

3.4 The $(3x+1)$ Conjecture

This is the well-known and famous, but still not confirmed, conjecture that every solution of the difference equation

$$x_{n+1} = \begin{cases} \dfrac{3x_n + 1}{2} & \text{if} \quad x_n \quad \text{is odd} \\[2ex] \dfrac{x_n}{2} & \text{if} \quad x_n \quad \text{is even} \end{cases}, \quad n = 0, 1, \ldots . \tag{3.9}$$

with initial condition

$$x_0 \in \{1, 2, \ldots\}$$

is eventually the **two-cycle** $(1,\ 2)$.

On the other hand, it is conjectured that every solution $\{x_n\}_{n=0}^{\infty}$ of Eq.(3.9) with initial condition

$$x_0 \in \{0,\ -1,\ -2, \ldots\},$$

is eventually either the **one-cycle** (0), the **one-cycle** (-1), the **three-cycle** $(-5,\ -7,\ -10)$, or the **eleven-cycle** $(-17,\ -25,\ -37,\ -55,\ -82,\ -41,\ -61,\ -91,\ -136,\ -68,\ -34)$.

The $(3x+1)$ conjecture is also known as the *Collatz Problem*, the *Syracuse Problem*, *Kakutani's Problem*, *Ulam's Problem*, and *Hasse's Algorithm*. According to Paul Erdös, **mathematics is not yet ready for such problems.**

See the interesting paper [93] by J.C. Lagarias for the history of the $(3x+1)$ conjecture, and for a survey on the literature of this problem up until the year 1985. In fact, the following beautiful excerpt comes from [93].

Is the (3x+1) *problem intractably hard*? The problem of settling the (3x+1) problem seems connected to the fact that it is a deterministic process that simulates "random" behavior. We face this dilemma: On the one hand, to the extent that the problem has structure, we can analyze it—yet it is precisely this structure that seems to prevent us from proving that it behaves "randomly." On the other hand, to the extent that the problem is structureless and "random," we have nothing to analyze and consequently cannot rigorously prove anything. Of course there remains the possibility that someone will find some hidden regularity in the (3x+1) problem that allows some of the conjectures about it to be settled.

If the (3x+1) *problem is intractable, why should one bother to study it*? One answer is provided by the following aphorism: "No problem is so intractable that something interesting cannot be said about it. Study of the (3x+1) problem has uncovered a number of interesting phenomena; I believe further study of it may be rewarded by the discovery of other new phenomena."

See also [125] and the references cited therein.

3.5 Periodicities in the Spirit of the $(3x+1)$ Conjecture

Motivated by the $(3x+1)$ conjecture, we looked at the difference equation

$$x_{n+1} = \begin{cases} \dfrac{\alpha x_n + \beta x_{n-1}}{2} \text{ if } & x_n + x_{n-1} \quad \text{is even} \\ \\ \gamma x_n + \delta x_{n-1} \text{ if } & x_n + x_{n-1} \quad \text{is odd} \end{cases}, \quad n = 0, 1, \ldots \tag{3.10}$$

with

$$\alpha, \beta, \gamma, \delta \in \{-1, 1\} \text{ and } x_{-1}, x_0 \in \mathbf{Z} = \{\ldots, -1, 0, 1, \ldots\}$$

and discovered some fascinating results, as well as several open problems and conjectures, in the spirit of the $(3x+1)$ conjecture. For references, see [8] and [26]. Here we state some known results about Eq.(3.10), while the open problems and conjectures are stated at the end of Section 3.6. See also Chapter 8 for the proofs of the results which are stated here.

Note that if $\{x_n\}_{n=-1}^{\infty}$ is a solution of Eq.(3.10), then $\{-x_n\}_{n=-1}^{\infty}$ is also a solution of Eq.(3.10). The change of variables

$$x_{2n-1} = y_{2n-1} \qquad \text{and} \qquad x_{2n} = -y_{2n}$$

reduces the 16 possible cases of Eq.(3.10) to 8, because the study of the solutions of Eq.(3.10) in the case of a given set of parameters $(\alpha, \beta, \gamma, \delta)$ is similar to that of the case of $(-\alpha, \beta, -\gamma, \delta)$.

Given integers $p, q \in Z$, let $\text{gcod}(p, q)$ denote the greatest common odd divisor of p and q.

1. **The equation**

$$x_{n+1} = \begin{cases} \dfrac{x_n + x_{n-1}}{2} & \text{if} \quad x_n + x_{n-1} \quad \text{is even} \\ x_n + x_{n-1} & \text{if} \quad x_n + x_{n-1} \quad \text{is odd} \end{cases}, \quad n = 0, 1, \ldots . \tag{3.11}$$

THEOREM 3.2

See [8]. The following statements are true:

1. *There exist solutions of Eq.(3.11) which are eventually constant, and there exist solutions of Eq.(3.11) which are not eventually constant.*
2. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(3.11) which is not eventually constant. Then either* $\lim_{n\to\infty} x_n = -\infty$ *or* $\lim_{n\to\infty} x_n = \infty$.

2. **The equation**

$$x_{n+1} = \begin{cases} \dfrac{x_n + x_{n-1}}{2} & \text{if} \quad x_n + x_{n-1} \quad \text{is even} \\ x_n - x_{n-1} & \text{if} \quad x_n + x_{n-1} \quad \text{is odd} \end{cases}, \quad n = 0, 1, \ldots . \tag{3.12}$$

THEOREM 3.3

(D. Clark and J.T. Lewis) See [26]. Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(3.12). Suppose that* $x_{-1} \neq x_0$ *and that* $\text{gcod}(x_{-1}, x_0) = 1$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is either eventually the constant* 1, *the constant* -1, *or the six-cycle* $(1,\ 3,\ 2,\ -1,\ -3,\ -2)$.

3. **The equation**

$$x_{n+1} = \begin{cases} \dfrac{x_n + x_{n-1}}{2} & \text{if} \quad x_n + x_{n-1} \quad \text{is even} \\ -x_n + x_{n-1} & \text{if} \quad x_n + x_{n-1} \quad \text{is odd} \end{cases}, \quad n = 0, 1, \ldots . \tag{3.13}$$

THEOREM 3.4

See [8]. Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(3.13). Suppose that* $x_{-1} \neq x_0$ *and that* $\text{gcod}(x_{-1}, x_0) = 1$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is either eventually the constant* 1, *the constant* -1, *the four-cycle* $(1, 2, -1, 3)$, *the four-cycle* $(-1, -2, 1, -3)$, *or the six-cycle* $(1, 0, 1, -1, 0, -1)$.

4. **The equation**

$$x_{n+1} = \begin{cases} \dfrac{x_n - x_{n-1}}{2} & \text{if} \quad x_n + x_{n-1} \quad \text{is even} \\ x_n + x_{n-1} & \text{if} \quad x_n + x_{n-1} \quad \text{is odd} \end{cases}, \quad n = 0, 1, \ldots. \tag{3.14}$$

For Eq.(3.14), see Conjecture 3.6.

5. **The equation**

$$x_{n+1} = \begin{cases} \dfrac{x_n + x_{n-1}}{2} & \text{if} \quad x_n + x_{n-1} \quad \text{is even} \\ -x_n - x_{n-1} & \text{if} \quad x_n + x_{n-1} \quad \text{is odd} \end{cases}, \quad n = 0, 1, \ldots. \tag{3.15}$$

THEOREM 3.5

See [8]. Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(3.15). Suppose that* $x_{-1} \neq x_0$ *and that* $\text{gcod}(x_{-1}, x_0) = 1$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is either eventually the constant* 1, *the constant* -1, *the three-cycle* $(1,\ 0,\ -1)$, *or the three-cycle* $(-1,\ 0,\ 1)$.

6. **The equation**

$$x_{n+1} = \begin{cases} \dfrac{x_n - x_{n-1}}{2} & \text{if} \quad x_n + x_{n-1} \quad \text{is even} \\ x_n - x_{n-1} & \text{if} \quad x_n + x_{n-1} \quad \text{is odd} \end{cases}, \quad n = 0, 1, \ldots. \tag{3.16}$$

THEOREM 3.6

See [8]. Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(3.16). Suppose that* $x_{-1} \neq x_0$ *and that* $\text{gcod}(x_{-1}, x_0) = 1$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is eventually the six-cycle* $(1,\ 1,\ 0,\ -1,\ -1,\ 0)$.

7. **The equation**

$$x_{n+1} = \begin{cases} \dfrac{x_n - x_{n-1}}{2} \text{ if } & x_n + x_{n-1} \quad \text{is even} \\ -x_n + x_{n-1} \text{ if } & x_n + x_{n-1} \quad \text{is odd} \end{cases} , \quad n = 0, 1, \ldots . \tag{3.17}$$

THEOREM 3.7

See [8]. Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(3.17). Suppose that* $x_{-1} \neq x_0$ *and that* $\text{gcod}(x_{-1}, x_0) = 1$ *and* $x_{-1} \neq x_0$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is eventually the eight-cycle* $(1,\ 1,\ 0,\ 1,\ -1,\ -1,\ 0,\ -1)$.

8. **The equation**

$$x_{n+1} = \begin{cases} \dfrac{x_n - x_{n-1}}{2} \text{ if } & x_n + x_{n-1} \quad \text{is even} \\ -x_n - x_{n-1} \text{ if } & x_n + x_{n-1} \quad \text{is odd} \end{cases} , \quad n = 0, 1, \ldots . \tag{3.18}$$

For Eq.(3.18), see Open Problem 3.5.

3.6 Open Problems and Conjectures

We first pose some **open problems and conjectures** about the solutions of the non-autonomous Eq.(3.8) when $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive real numbers. See [16].

OPEN PROBLEM 3.1

Let $\{A_n\}_{n=0}^{\infty}$ *be a periodic sequence of positive real numbers with period* $k \geq 3$. *Find necessary and sufficient conditions for each of the following statements to be true:*

(i) Every positive solution of Eq.(3.8) is bounded.

(ii) Every positive solution of Eq.(3.8) is eventually periodic. In this case, determine all possible such periods.

CONJECTURE 3.1

Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with period $k \geq 2$. Assume that $A_n \in (0,1)$ for all $n \geq 0$. Then every positive solution of Eq.(3.8) is eventually periodic with period 2.

For the cases $k = 2$ and 3, the above conjecture was shown to be true in [15] and [14], respectively.

CONJECTURE 3.2

Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with prime period $k \geq 2$. Assume that $A_n \in (1,\infty)$ for all $n \geq 0$. Then every positive solution of Eq.(3.8) is eventually periodic with period

$$\begin{cases} 2k \text{ if } k \text{ is even} \\ \\ 4k \text{ if } k \text{ is odd.} \end{cases}$$

For the cases $k = 2$ and 3, the above conjecture was shown to be true in [15] and [54], respectively.

CONJECTURE 3.3

Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with prime period $k \geq 3$.

(i) Assume k is not a multiple of 3. Then every positive solution of Eq.(3.8) is eventually constant or eventually periodic with prime period $p \in \{2,\ k,\ 2k,\ 4k\}$, and p is independent of the initial conditions.

(ii) Assume k is a multiple of 3. Then one of the following statements is true:

(a) Every positive solution of Eq.(3.8) is eventually constant or eventually periodic with prime period $p \in \{2,\ k,\ 2k,\ 4k\}$, and p is independent of the initial conditions.

(b) Every positive solution of Eq.(3.8) is eventually constant or unbounded.

In the case $k = 3$, the above conjectures were established in [54].

Motivated by Theorem 3.1, we pose some **open problems and conjectures** about the behavior of the solutions of the difference equation

$$x_{n+1} = \max\left\{\frac{A_0}{x_n}, \frac{A_1}{x_{n-1}}, \cdots, \frac{A_k}{x_{n-k}}\right\} \quad , \quad n = 0, 1, \ldots \tag{3.19}$$

where the parameters $A_0, A_1, \ldots, A_k$ are real numbers and where the initial conditions are nonzero real numbers. See [89] and [120].

CONJECTURE 3.4

Assume that $A_0, A_1, \ldots, A_{k-1} \in [0, \infty)$ and $A_k \in (0, \infty)$. Then every positive solution of Eq.(3.19) is eventually periodic with period

$$p \in \{2,\ 3, \ldots,\ 2(k+1)\}.$$

It is fascinating to observe how the period in the above conjecture is determined by the *"dominant"* parameter among $A_0, A_1, \ldots, A_k$. For example, if for some $j_0 \in \{0, 1, \ldots, k\}$, $A_{j_0} > \max\{A_j : j \neq j_0\}$, then every positive solution of Eq.(3.19) is eventually periodic with period $2(j_0+1)$. Also, if some consecutive string of the parameters $A_0, A_1, \ldots, A_k$ are equal and dominate the remaining ones, then every positive solution of Eq.(3.19) is eventually periodic with period equal to the average of the periods of the *"dominating difference equations."* In particular, it can be easily shown that every positive solution of the difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{1}{x_{n-1}}, \cdots, \frac{1}{x_{n-k}}\right\} \quad , \quad n = 0, 1, \ldots \tag{3.20}$$

is eventually periodic with period $(k+2)$. Note that $(k+2)$ is the **average eventual period** of the $k+1$ difference equations

$$y_{n+1} = \frac{1}{y_{n-j}} \quad , \quad n = 0, 1, \ldots$$

for $j = 0, 1, \ldots, k$.

CONJECTURE 3.5

Assume that $A_0, A_1, \ldots, A_k \in \mathbf{R}$. Then every bounded solution of Eq.(3.19) is eventually periodic.

OPEN PROBLEM 3.2

Assume that $A_0, A_1, \ldots, A_k \in \mathbf{R}$. Obtain necessary and sufficient conditions for every solution of Eq.(3.19) to be unbounded.

OPEN PROBLEM 3.3

Assume that $A_0, A_1, \ldots, A_k \in \mathbf{R}$. *Investigate the periodic character of the solutions of Eq.(3.19).*

We next pose some open problems and conjectures about the solutions of Eq.(3.10) for the various values of α, β, γ, and δ which remain to be studied.

OPEN PROBLEM 3.4

Find all points $(x_{-1},\ x_0) \in \mathbf{Z} \times \mathbf{Z}$ *through which the solution* $\{x_n\}_{n=-1}^{\infty}$ *of Eq.(3.11) is eventually constant.*

CONJECTURE 3.6

See [8]. Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(3.14). Suppose that* $\text{gcod}(x_{-1}, x_0) = 1$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is either eventually the three-cycle* $(0,\ 1,\ 1)$, *the three-cycle* $(0,\ -1,\ -1)$, *or the ten-cycle* $(3,\ 2,\ 5,\ 7,\ 1,\ -3,\ -2,\ -5,\ -7,\ -1)$.

OPEN PROBLEM 3.5

See [8]. Determine the character of the solutions of Eq.(3.18).

Similar problems are of interest for the equation

$$x_{n+1} = \begin{cases} \dfrac{x_n + x_{n-1}}{3} & \text{if 3 divides } x_n + x_{n-1} \\ \\ x_n + x_{n-1} & \text{otherwise} \end{cases}, \quad n = 0, 1, \ldots . \tag{3.21}$$

where $x_{-1}, x_0 \in \mathbf{Z}$.

CONJECTURE 3.7

The following statements are true.

(i) Every positive solution of Eq.(3.21) which is not eventually a 3-cycle converges to ∞.

(ii) Eq.(3.21) has an unbounded solution.

Chapter 4

CONVERGENCE TO PERIODIC SOLUTIONS

4.1 Introduction

In this chapter we present a few simple examples of difference equations with the property that every solution of the equation converges to a periodic solution. Some quite general results of difference equations with the property that their solutions converge to periodic solutions are given in Chapter 5.

What is it that makes every solution of a difference equation converge to a periodic solution with period k?

4.2 A Trichotomy Result for the Equation $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}$

Consider the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n} \quad , \qquad n = 0, 1, \ldots \tag{4.1}$$

where $\alpha \in [0, \infty)$, and where the initial conditions x_{-1} and x_0 are arbitrary positive real numbers.

Clearly, the only equilibrium point of Eq.(4.1) is $\bar{x} = \alpha + 1$.

The main result of this section is that Eq.(4.1) possesses the following period-2 trichotomy. See [7].

THEOREM 4.1
The following statements are true:

1. *Every positive solution of Eq.(4.1) converges to* $\bar{x}$ *if and only if* $\alpha > 1$.
2. *Every positive solution of Eq.(4.1) converges to a period-2 solution of Eq.(4.1) if and only if* $\alpha = 1$.
3. *There exist unbounded positive solutions of Eq.(4.1) if and only if* $0 \leq \alpha < 1$.

REMARK 4.1 It is interesting to note that what we shall actually establish during the proof of Theorem 4.1 is the following:

1. Suppose $\alpha > 1$. Then $\bar{x}$ is a globally asymptotically stable equilibrium point of Eq.(4.1).
2. Suppose $\alpha = 1$. Then every positive solution of Eq.(4.1) converges to a period-2 solution of Eq.(4.1), and there exist positive solutions of Eq.(4.1) which are periodic with prime period 2.
3. Every positive solution of Eq.(4.1) is bounded if and only if $\alpha \geq 1$.
4. Suppose $0 \leq \alpha < 1$. Then $\bar{x}$ is an unstable saddle point equilibrium of Eq.(4.1). Thus by the Stable Manifold Theorem, Eq.(4.1) also has positive solutions which converge to $\bar{x}$, and so, in particular, are bounded.

□

4.2.1 Preliminaries

The linearized equation of Eq.(4.1) about the unique equilibrium point $\bar{x} = \alpha + 1$ is

$$y_{n+1} + \frac{1}{\alpha+1} y_n - \frac{1}{\alpha+1} y_{n-1} = 0 \quad , \quad n = 0, 1, \ldots$$

with characteristic equation

$$\lambda^2 + \frac{1}{\alpha+1}\lambda - \frac{1}{\alpha+1} = 0. \tag{4.2}$$

THEOREM 4.2
The following statements are true:

1. *Suppose $\alpha > 1$. Then $\bar{x}$ is a locally asymptotically stable equilibrium point of Eq.(4.1).*
2. *Suppose $0 \leq \alpha < 1$. Then $\bar{x}$ is an unstable saddle point equilibrium of Eq.(4.1).*

PROOF

(i) Suppose $\alpha > 1$. Then

$$\left| \frac{1}{\alpha+1} \right| + \left| -\frac{1}{\alpha+1} \right| = \frac{2}{\alpha+1} < 2$$

and so it follows by Theorem 1.6 that $\bar{x}$ is a locally asymptotically stable equilibrium point of Eq.(4.1).

(ii) Suppose $0 \leq \alpha < 1$. Let $h : \mathbf{R} \to \mathbf{R}$ be given by

$$h(\lambda) = \lambda^2 + \frac{1}{\alpha+1}\lambda - \frac{1}{\alpha+1}.$$

Then

$$\begin{aligned} h(1) &= 1 > 0 \\ h(0) &= -\frac{1}{\alpha+1} < 0 \\ h(-1) &= \frac{\alpha-1}{\alpha+1} < 0 \\ \lim_{\lambda\to-\infty} h(\lambda) &= \infty. \end{aligned}$$

Thus we see that Eq.(4.2) has the two roots λ_1 and λ_2 with

$$\lambda_1 < -1 < 0 < \lambda_2 < 1$$

from which the result follows. See Section 1.2. ∎

The proofs of the following three lemmas follow from simple computations and will be omitted.

LEMMA 4.1
The following statements are true:

1. *Eq.(4.1) has positive prime period-2 solutions if and only if* $\alpha = 1$.
2. *Suppose* $\alpha = 1$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(4.1). Then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic with period 2 if and only if* $x_{-1} > 1$ *and* $x_0 = \dfrac{x_{-1}}{x_{-1}-1}$.

LEMMA 4.2
Let $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(4.1) which is eventually constant. Then* $\{x_n\}_{n=-1}^{\infty}$ *is the trivial solution*

$$x_n = \alpha + 1 \qquad \text{for} \qquad n = -1, 0, \ldots .$$

LEMMA 4.3
Let $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(4.1), and let* $L > \alpha$. *Then the following statements are true:*

1. $\lim_{n\to\infty} x_{2n} = L$ *if and only if* $\lim_{n\to\infty} x_{2n+1} = \frac{L}{L-1}$.

2. $\lim_{n\to\infty} x_{2n+1} = L$ *if and only if* $\lim_{n\to\infty} x_{2n} = \frac{L}{L-1}$.

4.2.2 Analysis of the Semi-cycles of Eq.(4.1)

LEMMA 4.4
Let $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(4.1) which consists of a single semi-cycle. Then* $\{x_n\}_{n=-1}^{\infty}$ *converges monotonically to* $\bar{x}$.

PROOF Suppose $0 < x_n < \alpha+1$ for all $n \geq -1$. The case where $x_n \geq \alpha+1$ for all $n \geq -1$ is similar and will be omitted. Note that for $n \geq 0$,

$$0 < \alpha + \frac{x_{n-1}}{x_n} = x_{n+1} < \alpha + 1$$

and so

$$0 < x_{n-1} < x_n < \alpha + 1$$

from which the result follows. ∎

LEMMA 4.5
Let $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(4.1) which consists of at least two semi-cycles. Then* $\{x_n\}_{n=-1}^{\infty}$ *oscillates about* $\bar{x}$. *Moreover, with the possible exception of the first semi-cycle, every semi-cycle has length 1, and every term of* $\{x_n\}_{n=-1}^{\infty}$ *is strictly greater than* α, *and with the possible exception of the first two semi-cycles, no term of* $\{x_n\}_{n=-1}^{\infty}$ *is equal to* $\alpha + 1$.

PROOF It suffices to consider the following two cases:

Case 1. Suppose $x_{-1} < \alpha + 1 \leq x_0$. Then

$$x_1 = \alpha + \frac{x_{-1}}{x_0} < \alpha + 1 \quad \text{and} \quad x_2 = \alpha + \frac{x_0}{x_1} > \alpha + 1.$$

Case 2. Suppose $x_0 < \alpha + 1 \leq x_{-1}$. Then

$$x_1 = \alpha + \frac{x_{-1}}{x_0} > \alpha + 1 \quad \text{and} \quad x_2 = \alpha + \frac{x_0}{x_1} < \alpha + 1.$$

∎

The next lemma will be useful in determining the limiting behavior of the positive solutions of Eq.(4.1).

LEMMA 4.6

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(4.1), and let $N \geq 0$ be a non-negative integer. Then the following statements are true:

1. $x_{N+1} > x_{N-1}$ *if and only if* $x_{N-1} + \alpha x_N - x_{N-1}x_N > 0.$
2. $x_{N+1} = x_{N-1}$ *if and only if* $x_{N-1} + \alpha x_N - x_{N-1}x_N = 0.$
3. $x_{N+1} < x_{N-1}$ *if and only if* $x_{N-1} + \alpha x_N - x_{N-1}x_N < 0.$

PROOF The proof follows from the computation

$$x_{N+1} - x_{N-1} = \left(\alpha + \frac{x_{N-1}}{x_N}\right) - x_{N-1} = \frac{\alpha x_N + x_{N-1} - x_{N-1}x_N}{x_N}.$$ ▯

4.2.3 The Case $0 \leq \alpha < 1$

We consider the case where $0 \leq \alpha < 1$, and we show that there exist positive solutions of Eq.(4.1) which are unbounded.

THEOREM 4.3

Let $0 \leq \alpha < 1$, and let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(4.1) such that $0 < x_{-1} \leq 1$ and $x_0 \geq \frac{1}{1-\alpha}$. Then the following statements are true:

1. $\lim_{n\to\infty} x_{2n} = \infty.$
2. $\lim_{n\to\infty} x_{2n+1} = \alpha.$

PROOF Note that $\frac{1}{1-\alpha} > \alpha + 1$, and so $x_0 > \alpha + 1$.

So as

$$x_{2n+1} = \alpha + \frac{x_{2n-1}}{x_{2n}} \qquad \text{for all} \qquad n \geq 0$$

it suffices to show that

$$x_1 \in (\alpha, 1] \qquad \text{and} \qquad x_2 \geq \alpha + x_0.$$

Indeed

$$x_1 = \alpha + \frac{x_{-1}}{x_0} > \alpha.$$

Also

$$x_1 = \alpha + \frac{x_{-1}}{x_0} \le \alpha + \frac{1}{x_0} \le \alpha + (1-\alpha) = 1$$

and so

$$x_1 \in (\alpha, 1].$$

Hence

$$x_2 = \alpha + \frac{x_0}{x_1} \ge \alpha + x_0.$$

∎

4.2.4 The Case $\alpha = 1$

We consider the case where $\alpha = 1$, and we show that every solution of Eq.(4.1) converges to a period-2 solution of Eq.(4.1).

Clearly when $\alpha = 1$, the unique equilibrium point of Eq.(4.1) is

$$\bar{x} = 2.$$

Note also that in this case Eq.(4.1) reduces to

$$x_{n+1} = 1 + \frac{x_{n-1}}{x_n} \quad , \quad n = 0, 1, \ldots . \tag{4.3}$$

THEOREM 4.4
Suppose $\alpha = 1$. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(4.3). Then the following statements are true:

1. *Suppose $\{x_n\}_{n=-1}^{\infty}$ consists of a single semi-cycle. Then $\{x_n\}_{n=-1}^{\infty}$ converges monotonically to $\bar{x} = 2$.*
2. *Suppose $\{x_n\}_{n=-1}^{\infty}$ consists of at least two semi-cycles. Then $\{x_n\}_{n=-1}^{\infty}$ converges to a prime period-2 solution of Eq.(4.3).*

PROOF We know by Lemma 4.4 that if $\{x_n\}_{n=-1}^{\infty}$ consists of a single semi-cycle, then $\{x_n\}_{n=-1}^{\infty}$ converges monotonically to $\bar{x} = 2$. So it suffices to consider the case where $\{x_n\}_{n=-1}^{\infty}$ consists of at least two semi-cycles.

So assume that $\{x_n\}_{n=-1}^{\infty}$ consists of at least two semi-cycles. Then it follows by Lemma 4.5 that $\{x_n\}_{n=-1}^{\infty}$ is oscillatory about $\bar{x} = 2$. It also follows by Lemma 4.5 that without loss of generality we may assume that every semi-cycle of $\{x_n\}_{n=-1}^{\infty}$ has length 1, and that every term of $\{x_n\}_{n=-1}^{\infty}$ is greater than 1. Finally, we may also assume that the first semi-cycle of $\{x_n\}_{n=-1}^{\infty}$ is a negative semi-cycle.

Now observe that for $n \ge 0$

$$x_n + x_{n+1} - x_n x_{n+1} = \frac{x_{n-1} + x_n - x_{n-1}x_n}{x_n}$$

and so by Lemma 4.6, the following three statements are true:

1. Suppose $1 < x_{-1} < x_1 < 2$. Then
$$1 < x_{-1} < x_1 < x_3 < \cdots < 2$$
and
$$2 < x_0 < x_4 < \cdots .$$

2. Suppose $1 < x_{-1} = x_1 < 2$. Then
$$1 < x_0 = x_2 = x_4 < \cdots < 2$$
and
$$2 < x_0 = x_2 = x_4 = \cdots .$$

3. Suppose $1 < x_1 < x_{-1} < 2$. Then
$$1 < \cdots < x_3 < x_1 < x_{-1} < 2$$
and
$$2 < \cdots < x_4 < x_2 < x_0.$$

The proof of the theorem follows from Lemma 4.3 and statements 1, 2, and 3 above. ∎

4.2.5 The Case $\alpha > 1$

We consider the case $\alpha > 1$, and we show that the equilibrium point $\bar{x} = \alpha + 1$ of Eq.(4.1) is globally asymptotically stable. We first give a lemma which will be useful in the sequel.

LEMMA 4.7
Suppose $\alpha > 1$, and let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(4.1). Then
$$\alpha + \frac{\alpha - 1}{\alpha} \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq \frac{\alpha^2}{\alpha - 1}.$$

PROOF It follows by Lemmas 4.4 and 4.5 that we may assume that every semi-cycle of $\{x_n\}_{n=-1}^{\infty}$ has length one, that $\alpha < x_n$ for all $n \geq -1$, and that $\alpha < x_0 < \alpha + 1 < x_{-1}$.

We shall first show that $\limsup_{n\to\infty} x_n \leq \dfrac{\alpha^2}{\alpha - 1}$. Note that for $n \geq 0$,
$$x_{2n+1} < \alpha + \frac{x_{2n-1}}{\alpha}.$$

So as every solution of the difference equation

$$y_{m+1} = \alpha + \frac{1}{\alpha} y_m \qquad , \qquad m = 0, 1, \ldots$$

converges to $\dfrac{\alpha^2}{\alpha - 1}$, it follows that $\limsup_{n\to\infty} x_n \leq \dfrac{\alpha^2}{\alpha - 1}$.

We shall next show that $\alpha + \dfrac{\alpha - 1}{\alpha} \leq \liminf_{n\to\infty} x_n$. Let $\varepsilon > 0$. Clearly there exists $N \geq 1$ such that for all $n \geq N$,

$$x_{2n-1} < \frac{\alpha^2 + \varepsilon}{\alpha - 1}.$$

Let $n \geq N$. Then

$$x_{2n} = \alpha + \frac{x_{2n-2}}{x_{2n-1}} > \alpha + \alpha\left(\frac{\alpha - 1}{\alpha^2 + \varepsilon}\right) = \frac{\alpha^3 + \alpha\varepsilon + \alpha(\alpha - 1)}{\alpha^2 + \varepsilon}$$

and thus

$$\liminf_{n\to\infty} x_n \geq \frac{\alpha^3 + \alpha\varepsilon + \alpha(\alpha - 1)}{\alpha^2 + \varepsilon}.$$

So as ε is arbitrary, we have

$$\liminf_{n\to\infty} x_n \geq \frac{\alpha^3 + \alpha(\alpha - 1)}{\alpha^2} = \alpha + \frac{\alpha - 1}{\alpha}.$$ ▯

THEOREM 4.5
Let $\alpha > 1$. Then $\bar{x} = \alpha + 1$ is a globally asymptotically stable equilibrium point of Eq.(4.1).

PROOF We know by Theorem 4.2 that $\bar{x} = \alpha + 1$ is a locally asymptotically stable equilibrium point of Eq.(4.1).

So let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(4.1). It suffices to show that

$$\lim_{n\to\infty} x_n = \alpha + 1.$$

Let $\varepsilon > 0$, and set

$$a = \alpha \qquad \text{and} \qquad b = \frac{\alpha^2 + \varepsilon}{\alpha - 1}.$$

Then it follows by Lemma 4.7 that without loss of generality we may assume that

$$x_n \in [a, b] \qquad \text{for all} \qquad n \geq -1.$$

For $x, y \in [a, b]$, set

$$g(x, y) = \alpha + \frac{y}{x}.$$

Then g is a continuous function which is decreasing in $x \in [a, b]$ for each fixed $y \in [a, b]$, and g is increasing in $y \in [a, b]$ for each fixed $x \in [a, b]$. Note that

$$g\left(\frac{\alpha^2+\varepsilon}{\alpha-1}, \alpha\right) = \alpha + \alpha\left(\frac{\alpha-1}{\alpha^2+\varepsilon}\right) > \alpha$$

and

$$g\left(\alpha, \frac{\alpha^2+\varepsilon}{\alpha-1}\right) = \alpha + \frac{1}{\alpha}\cdot\frac{\alpha^2+\varepsilon}{\alpha-1} = \frac{\alpha^3+\varepsilon}{\alpha^2-\alpha} < \frac{\alpha^3+\varepsilon\cdot\alpha}{\alpha^2-\alpha} = \frac{\alpha^2+\varepsilon}{\alpha-1}.$$

Hence

$$\alpha < g(x,y) < \frac{\alpha^2+\varepsilon}{\alpha-1} \qquad \text{for all} \qquad x, y \in \left[\alpha, \frac{\alpha^2+\varepsilon}{\alpha-1}\right].$$

Finally, recall by Lemma 4.1 that Eq.(4.1) has no positive prime period-2 solutions, and so by Theorem 1.11,

$$\lim_{n\to\infty} x_n = \alpha + 1.$$

∎

4.3 The Equation $x_{n+1} = 1 + \frac{x_{n-1}}{x_{n-2}}$

THEOREM 4.6

See [108]. *Every positive solution of the difference equation*

$$x_{n+1} = 1 + \frac{x_{n-1}}{x_{n-2}} \quad , \qquad n = 0, 1, \ldots \tag{4.4}$$

converges to a periodic solution of Eq.(4.4) with period-2.

PROOF Let $\{x_n\}_{n=-2}^{\infty}$ be a positive solution of Eq.(4.4). Then for $n \geq 5$

$$x_{n-2} = 1 + \frac{x_{n-4}}{x_{n-5}} = 1 + \frac{x_{n-4}}{x_{n-6}}\cdot\frac{x_{n-6}}{x_{n-5}} = 1 + \frac{1+\frac{x_{n-6}}{x_{n-7}}}{x_{n-6}}\cdot\frac{x_{n-6}}{x_{n-5}}$$

$$= 1 + \left(\frac{1}{x_{n-6}} + \frac{1}{x_{n-7}}\right)\frac{x_{n-6}}{x_{n-5}} = 1 + \frac{1}{x_{n-5}} + \frac{1}{x_{n-7}(x_{n-3}-1)}.$$

and so

$$x_{n+1} = 1 + \frac{x_{n-1}}{1 + \frac{1}{x_{n-5}} + \frac{1}{x_{n-7}(x_{n-3}-1)}}.$$

Let $f : (0,\infty)^4 \to (0,\infty)$ be given by

$$f(z_1,z_2,z_3,z_4) = 1 + \frac{z_1}{1+\frac{1}{z_3}+\frac{1}{z_4(z_2-1)}} \qquad \text{for} \qquad (z_1,z_2,z_3,z_4) \in (0,\infty)^4.$$

Then for $n \geq 3$

$$x_{2n+1} = f(x_{2n-1}, x_{2n-3}, x_{2n-5}, x_{2n-7})$$

and

$$x_{2n+2} = f(x_{2n}, x_{2n-2}, x_{2n-4}, x_{2n-6}).$$

The proof follows by Theorem 1.9. □

The reader should notice the elegant and distinct character of the proofs of Theorems 4.4 and 4.6, and should compare them with the equally elegant but entirely different proof given in Chapter 5 about a quite general rational difference equation.

4.4 The Equation $x_{n+1} = p_n + \frac{x_{n-1}}{x_n}$

Consider the difference equation

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n} \quad , \qquad n = 0,1,\ldots \tag{4.5}$$

where the parameter $\{p_n\}_{n=0}^{\infty}$ is the positive period-2 sequence

$$p_n = \begin{cases} \alpha \text{ if n is even} \\ \beta \text{ if n is odd.} \end{cases}$$

and where the initial conditions x_{-1} and x_0 are positive real numbers.

Note that Eq.(4.5) is Eq.(4.1) with a period-2 coefficient. See [79] and [80].

4.4.1 Decoupling the Even and Odd Terms

Note that for $n \geq 0$,

$$x_{2n+1} = \alpha + \frac{x_{2n-1}}{x_{2n}} \qquad \text{and} \qquad x_{2n+2} = \beta + \frac{x_{2n}}{x_{2n+1}}$$

and so

$$x_{2n+1} > \alpha \qquad \text{and} \qquad x_{2n+2} > \beta.$$

For $n \geq 0$, set

$$y_n = -\alpha + x_{2n-1} \qquad \text{and} \qquad z_n = -\beta + x_{2n}.$$

Then for $n \geq 1$, we have

$$y_n = \frac{x_{2n-3}}{x_{2n-2}} \qquad \text{and} \qquad z_n = \frac{x_{2n-2}}{x_{2n-1}}$$

and so

$$y_{n+1} = \frac{x_{2n-1}}{x_{2n}} = \frac{y_n + \alpha}{\beta + \dfrac{x_{2n-2}}{\alpha + y_n}} = \frac{(y_n+\alpha)^2}{\beta(y_n+\alpha) + x_{2n-2}} = \frac{y_n(y_n+\alpha)^2}{\beta y_n(y_n+\alpha) + (y_{n-1}+\alpha)}$$

and

$$z_{n+1} = \frac{x_{2n}}{x_{2n+1}} = \frac{z_n + \beta}{\alpha + \dfrac{x_{2n-1}}{\beta + z_n}} = \frac{(z_n+\beta)^2}{\alpha(z_n+\beta) + x_{2n-1}} = \frac{z_n(z_n+\beta)^2}{\alpha z_n(z_n+\beta) + (z_{n-1}+\beta)}.$$

Thus

$$y_{n+1} = \frac{y_n(y_n+\alpha)^2}{\beta y_n(y_n+\alpha) + (y_{n-1}+\alpha)} \quad , \quad n = 1, 2, \ldots \tag{4.6}$$

$$z_{n+1} = \frac{z_n(z_n+\beta)^2}{\alpha z_n(z_n+\beta) + (z_{n-1}+\beta)} \quad , \quad n = 1, 2, \ldots \tag{4.7}$$

and

$$y_n > 0 \ \text{ and } \ z_n > 0 \qquad \text{for} \qquad n \geq 1.$$

4.4.2 Local Stability of Eqs.(4.6) and (4.7)

Zero is always an equilibrium point of both Eqs.(4.6) and (4.7).

When

$$\alpha = \beta = 1$$

every non-negative real number $\bar{y} \geq 0$ is an equilibrium point of Eq.(4.6), and every non-negative real number $\bar{z} \geq 0$ is an equilibrium point of Eq.(4.7). It follows by Theorem 4.4 that every solution of Eq.(4.5) converges to a period-2 solution of Eq.(4.5).

When

$$|\alpha - 1| + |\beta - 1| > 0$$

Eq.(4.6), and similarly, Eq.(4.7), has, in addition to the zero equilibrium point, a positive equilibrium point if and only if

$$(\alpha - 1)(\beta - 1) > 0$$

in which case, the positive equilibrium point of Eq.(4.6) is

$$\bar{y} = \frac{\alpha - 1}{\beta - 1}$$

while the the positive equilibrium point of Eq.(4.7)

$$\bar{z} = \frac{\beta - 1}{\alpha - 1}.$$

The linearized equation of Eq.(4.6) about the zero equilibrium point $\tilde{y} = 0$ has the characteristic roots

$$0 \qquad \text{and} \qquad \alpha.$$

Thus the zero equilibrium point $\tilde{y} = 0$ of Eq.(4.6) is a sink when

$$\alpha < 1,$$

is a non-hyperbolic equilibrium point when

$$\alpha = 1,$$

and is an unstable saddle-point equilibrium point when

$$\alpha > 1.$$

Similarly, the zero equilibrium point $\tilde{z} = 0$ of Eq.(4.7) is a sink when

$$\beta < 1,$$

is a non-hyperbolic equilibrium point when

$$\beta = 1,$$

and is an unstable saddle-point equilibrium point when

$$\beta > 1.$$

The linearized equation of Eq.(4.6) about the positive equilibrium point $\bar{y} = \dfrac{\alpha - 1}{\beta - 1}$ is

$$w_{n+1} - \frac{3 - 2\alpha - 2\beta - \alpha\beta + \alpha^2\beta + \alpha\beta^2}{(\alpha\beta - 1)^2} w_n + \frac{(\alpha - 1)(\beta - 1)}{(\alpha\beta - 1)^2} w_{n-1} = 0\,.$$

By applying Theorem 1.3, we see that the positive equilibrium point $\bar{y} = \dfrac{\alpha - 1}{\beta - 1}$ of Eq.(4.6) is locally asymptotically stable when

$$\alpha > 1 \qquad \text{and} \qquad \beta > 1$$

and is an unstable saddle-point when

$$\alpha < 1 \qquad \text{and} \qquad \beta < 1.$$

We similarly see that the positive equilibrium point $\bar{z} = \dfrac{\beta - 1}{\alpha - 1}$ of Eq.(4.7) is locally asymptotically stable when

$$\alpha > 1 \qquad \text{and} \qquad \beta > 1$$

and is an unstable saddle-point when

$$\alpha < 1 \qquad \text{and} \qquad \beta < 1.$$

In summary, we have established the following results for Eqs.(4.6) and (4.7).

THEOREM 4.7
The following statements are true:

1. *The zero equilibrium point of Eq.(4.6) is locally asymptotically stable when*
$$0 < \alpha < 1.$$

2. *The zero equilibrium point of Eq.(4.6) is a non-hyperbolic equilibrium point when*
$$\alpha = 1.$$

3. *The zero equilibrium point of Eq.(4.6) is an unstable saddle-point when*
$$\alpha > 1.$$

4. *Suppose* $(\alpha-1)(\beta-1) > 0$. *Then the positive equilibrium point* $\bar{y} = \dfrac{\alpha - 1}{\beta - 1}$ *of Eq.(4.6) is locally asymptotically stable when*
$$\alpha > 1 \qquad \textit{and} \qquad \beta > 1$$
and is an unstable saddle-point when
$$0 < \alpha < 1 \qquad \textit{and} \qquad 0 < \beta < 1.$$

THEOREM 4.8
The following statements are true:

1. *The zero equilibrium point of Eq.(4.7) is locally asymptotically stable when*
$$0 < \beta < 1.$$

2. *The zero equilibrium point of Eq.(4.7) is a non-hyperbolic equilibrium point when*
$$\beta = 1.$$

3. *The zero equilibrium point of Eq.(4.7) is an unstable saddle-point when*
$$\beta > 1.$$

4. *Suppose* $(\alpha-1)(\beta-1) > 0$. *Then the positive equilibrium point* $\bar{z} = \dfrac{\beta-1}{\alpha-1}$ *of Eq.(4.7) is locally asymptotically stable when*
$$\alpha > 1 \qquad \text{and} \qquad \beta > 1$$
and is an unstable saddle-point when
$$0 < \alpha < 1 \qquad \text{and} \qquad 0 < \beta < 1.$$

4.4.3 Period-2 Solutions of Eq.(4.5)

Let
$$\phi, \psi, \phi, \psi, \ldots$$
be a prime period-2 solution of Eq.(4.5). Then
$$\phi = \alpha + \frac{\phi}{\psi} \qquad \text{and} \qquad \psi = \beta + \frac{\psi}{\phi}$$
with $\phi, \psi \in (0, \infty)$ and $\phi \neq \psi$. Hence
$$\alpha\psi + \phi = \beta\phi + \psi$$
and so
$$(\alpha - 1)\psi = (\beta - 1)\phi.$$
It follows that Eq.(4.5) has a prime period-2 solution if and only if
$$\alpha, \beta \in (0, 1) \qquad \text{or} \qquad \alpha, \beta \in (1, \infty),$$
in which case Eq.(4.5) has the unique prime period-2 solution
$$\frac{\alpha\beta-1}{\beta-1}, \frac{\alpha\beta-1}{\alpha-1}, \frac{\alpha\beta-1}{\beta-1}, \frac{\alpha\beta-1}{\alpha-1}, \ldots . \tag{4.8}$$

4.4.4 Global Asymptotic Stability of the Period-2 Solution

In this section, we show that the prime period-2 Solution (4.8) of Eq.(4.5) is globablly asymptotically stable when $\alpha > 1$ and $\beta > 1$.

LEMMA 4.8
Assume that $\alpha > 1$ and $\beta > 1$. Choose

$$U \geq \frac{\alpha\beta}{\beta - 1} \qquad \text{and} \qquad V \geq \frac{\alpha\beta}{\alpha - 1}$$

and for $(u, v) \in [\alpha, U] \times [\beta, V]$, let $f_1(u, v)$ and $f_2(u, v)$ be given by

$$f_1(u, v) = \alpha + \frac{u}{v}$$

and

$$f_2(u, v) = \beta + \frac{v}{\alpha + \frac{u}{v}}.$$

Then the following statements are true:

1. $f_1 : [\alpha, U] \times [\beta, V] \to [\alpha, U]$ *and* $f_2 : [\alpha, U] \times [\beta, V] \to [\beta, V]$.
2. *If $(m_1, M_1, m_2, M_2) \in [\alpha, U]^2 \times [\beta, V]^2$ is a solution of the system of equations*

$$\begin{cases} m_1 = \alpha + \dfrac{m_1}{M_2}, & M_1 = \alpha + \dfrac{M_1}{m_2} \\ m_2 = \beta + \dfrac{m_2}{\alpha + \dfrac{M_1}{m_2}}, & M_2 = \beta + \dfrac{M_2}{\alpha + \dfrac{m_1}{M_2}} \end{cases} \tag{4.9}$$

then $m_1 = M_1$ and $m_2 = M_2$.

PROOF Note that $\dfrac{\alpha\beta}{\beta - 1} > \alpha$ and $\dfrac{\alpha\beta}{\alpha - 1} > \beta$.

Let $(u, v) \in [\alpha, U] \times [\beta, V]$.

(i) Clearly

$$\alpha < \alpha + \frac{u}{v} \leq \alpha + \frac{U}{\beta} \leq U$$

and

$$\beta < \beta + \frac{v}{\alpha + \frac{u}{v}} \leq \beta + \frac{V}{\alpha + \frac{\alpha}{V}} \leq V$$

and so we see that

$$f_1 : [\alpha, U] \times [\beta, V] \to [\alpha, U] \qquad \text{and} \qquad f_2 : [\alpha, U] \times [\beta, V] \to [\beta, V].$$

(ii) Suppose that $(m_1, M_1, m_2, M_2) \in [\alpha, U]^2 \times [\beta, V]^2$ is a solution of System (4.9). Then in particular, $m_1 = \alpha + \dfrac{m_1}{M_2}$ and $M_1 = \alpha + \dfrac{M_1}{m_2}$ and so

$$\begin{cases} m_1 = \alpha + \dfrac{m_1}{M_2} \ , \ M_1 = \alpha + \dfrac{M_1}{m_2} \\ \\ m_2 = \beta + \dfrac{m_2}{M_1} \ , \ M_2 = \beta + \dfrac{M_2}{m_1} \end{cases}$$

from which it follows by a simple computation that

$$m_1 = M_1 = \frac{\alpha\beta - 1}{\beta - 1} \qquad \text{and} \qquad m_2 = M_2 = \frac{\alpha\beta - 1}{\alpha - 1}.$$

∎

THEOREM 4.9

Assume that $\alpha > 1$ and $\beta > 1$. Then the prime period-2 solution (4.8) of Eq.(4.5) is globally asymptotically stable.

PROOF We know by Theorems 4.7 and 4.8 that Solution (4.8) is a locally asymptotically stable solution of Eq.(4.5).

Thus it suffices to show that Solution (4.8) is a global attractor of Eq.(4.5).

So let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(4.5). It suffices to show that

$$\lim_{n\to\infty} x_{2n-1} = \frac{\alpha\beta - 1}{\beta - 1} \qquad \text{and} \qquad \lim_{n\to\infty} x_{2n} = \frac{\alpha\beta - 1}{\alpha - 1}.$$

For $n \geq 0$, set

$$u_n = x_{2n-1} \qquad \text{and} \qquad v_n = x_{2n}.$$

Then by Eq.(4.5), we see that

$$\left.\begin{aligned} u_{n+1} &= \alpha + \frac{u_n}{v_n} \\ v_{n+1} &= \beta + \frac{v_n}{\alpha + \frac{u_n}{v_n}} \end{aligned}\right\} , \qquad n = 0, 1, \ldots \tag{4.10}$$

Choose U and V so that

$$U \geq \max\left\{x_{-1}, \frac{\alpha\beta - 1}{\beta - 1}\right\} \qquad \text{and} \qquad V \geq \max\left\{x_0, \frac{\alpha\beta - 1}{\alpha - 1}\right\}.$$

The proof now follows by Theorem 1.16 and Lemma 4.8. ∎

4.4.5 Existence of Unbounded Solutions

In this section, we present sufficient conditions for Eq.(4.5) to have unbounded solutions.

THEOREM 4.10
Suppose that either $\alpha \in (0,1)$ or $\beta \in (0,1)$. Then Eq.(4.5) possesses unbounded solutions.

PROOF There are two cases to consider.

Case 1: Suppose $0 < \alpha < 1$. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(4.5) such that

$$0 < x_{-1} \leq 1 \qquad \text{and} \qquad x_0 > \frac{1}{1-\alpha}.$$

Then

$$x_1 = \alpha + \frac{x_{-1}}{x_0} \leq \alpha + \frac{1}{x_0} < \alpha + (1-\alpha) = 1$$

$$x_2 = \beta + \frac{x_0}{x_1} > \beta + x_0 > x_0$$

$$x_3 = \alpha + \frac{x_1}{x_2} < \alpha + \frac{1}{x_2} < \alpha + \frac{1}{x_0} < 1$$

$$x_4 = \beta + \frac{x_2}{x_3} > \beta + x_2 > x_2.$$

It follows by induction that

$$0 < x_{2n+1} < 1 \qquad \text{for all} \qquad n \geq 0$$

and that

$$\{x_{2n}\}_{n=0}^{\infty} \text{ is a strictly monotonically increasing sequence.}$$

We claim that $\{x_{2n}\}_{n=0}^{\infty}$ diverges to infinity. For the sake of contradiction, suppose that this is not the case. Then we see that there exists $L > 0$ such that

$$\lim_{n \to \infty} x_{2n} = L.$$

Hence there exists $N \geq 0$ such that

$$0 < L - x_{2n} < \beta \qquad \text{for all} \qquad n \geq N.$$

In particular, $L > x_{2N+2} = \beta + \dfrac{x_{2N}}{x_{2N+1}} > \beta + x_{2N} > L.$

This is a contradiction, and so we see that the claim is true. In particular, it follows that

$$\lim_{n\to\infty} x_{2n} = \infty \qquad \text{and} \qquad \lim_{n\to\infty} x_{2n+1} = \alpha.$$

Case 2: Suppose $0 < \beta < 1$. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(4.5) such that

$$x_{-1} > \frac{1}{1-\beta} \qquad \text{and} \qquad 0 < x_0 \leq 1.$$

Then

$$\begin{aligned} x_1 &= \alpha + \frac{x_{-1}}{x_0} \geq \alpha + x_{-1} > \quad x_{-1} \\ x_2 &= \beta + \frac{x_0}{x_1} \leq \beta + \frac{1}{x_1} < \beta + \frac{1}{x_{-1}} < 1 \\ x_3 &= \alpha + \frac{x_1}{x_2} > \alpha + x_1 > \quad x_1 \\ x_4 &= \beta + \frac{x_2}{x_3} < \beta + \frac{1}{x_3} < \beta + \frac{1}{x_1} < 1. \end{aligned}$$

It follows similarly to the proof in Case 1 that

$$\lim_{n\to\infty} x_{2n+1} = \infty \qquad \text{and} \qquad \lim_{n\to\infty} x_{2n} = \beta.$$

□

4.4.6 Comparison of Limits

Suppose that either $\alpha, \beta \in (0,1)$ or $\alpha, \beta \in (1, \infty)$. In this section we address the question of whether periodicity in the model is beneficial or deleterious to the equilibrium solution. We shall compare the average value

$$\frac{\dfrac{\alpha\beta - 1}{\beta - 1} + \dfrac{\alpha\beta - 1}{\alpha - 1}}{2} \tag{4.11}$$

of the two terms of the period-2 Solution (4.8) of Eq.(4.5) to the "average" equilibrium

$$\frac{\alpha + \beta}{2} + 1 \tag{4.12}$$

of the associated autonomous equation.

The following identity, the proof of which follows by a simple computation and will be omitted, clearly shows that (4.11) is always greater than (4.12), and so periodicity in the model is always helpful to the equilibrium solution:

$$\frac{\dfrac{\alpha\beta-1}{\beta-1}+\dfrac{\alpha\beta-1}{\alpha-1}}{2}-\left(\frac{\alpha+\beta}{2}+1\right)=\frac{1}{2}\frac{(\alpha-\beta)^2}{(\alpha-1)(\beta-1)}.$$

4.5 The Equation $x_{n+1}=\dfrac{A_0}{x_n}+\dfrac{A_1}{x_{n-1}}+\cdots+\dfrac{A_{k-1}}{x_{n-k+1}}$

Consider the equation

$$x_{n+1}=\frac{A_0}{x_n}+\frac{A_1}{x_{n-1}}+\cdots+\frac{A_{k-1}}{x_{n-k+1}}\quad,\qquad n=0,1,\ldots \tag{4.13}$$

where $A_i\in[0,\infty)$ for $i=0,1,\ldots,k-1$, and where $A=\sum_{i=0}^{k-1}A_i>0$.

In this section we give a detailed description of the global character of all positive solutions of Eq.(4.13). See [40] and [41].

Clearly Eq.(4.13) has the unique equilibrium point $\bar{x}=\sqrt{A}$.

LEMMA 4.9
The equilibrium point $\bar{x}=\sqrt{A}$ of Eq.(4.13) is locally stable.

PROOF Let $\varepsilon>0$. It suffices to show that there exists $\delta>0$ such that if $\{x_n\}_{n=-k+1}^{\infty}$ is a solution of Eq.(4.13) with

$$|x_{-k+1}-\bar{x}|<\delta,|x_{-k+2}-\bar{x}|<\delta,\ldots,|x_0-\bar{x}|<\delta,$$

then

$$|x_n-\bar{x}|<\varepsilon\qquad\text{for all}\qquad n\geq -k+1.$$

Let $f_1,f_2,\ldots,f_k\in C\left[(0,\infty)^k,(0,\infty)\right]$ be given by

$$f_1(x_0,x_{-1},\ldots,x_{-k+1})=\frac{A_0}{x_0}+\frac{A_1}{x_{-1}}+\cdots+\frac{A_{k-1}}{x_{-k+1}}$$

and inductively for $2\leq j\leq k$,

$$f_j(x_0,x_{-1},\ldots,x_{-k+1})=f_1(\tilde{f}_{j-1},\tilde{f}_{j-2},\ldots,\tilde{f}_1,x_0,x_{-1},\ldots,x_{-k+j})$$

[where we have adopted the notation $\tilde{f}_l=f_l(x_0,x_{-1},\ldots,x_{-k+1})$].

Note that $f_j(\bar{x}, \bar{x}, \ldots, \bar{x}) = \bar{x}$ for all $1 \leq j \leq k$. Thus there exists $\delta \in (0, \varepsilon)$ such that if

$$|x_{-k+1} - \bar{x}| < \delta, |x_{-k+2} - \bar{x}| < \delta, \ldots, |x_0 - \bar{x}| < \delta,$$

then

$$|f_j(x_0, x_{-1}, \ldots, x_{-k+1}) - \bar{x}| < \varepsilon \qquad \text{for all} \qquad 1 \leq j \leq k.$$

We claim this is the required $\delta > 0$. With this in mind, let $\{x_n\}_{n=-k+1}^{\infty}$ be a solution of Eq.(4.13) with

$$|x_{-k+1} - \bar{x}| < \delta, |x_{-k+2} - \bar{x}| < \delta, \ldots, |x_0 - \bar{x}| < \delta.$$

Note that for $1 \leq j \leq k$,

$$x_j = f_j(x_0, x_{-1}, \ldots, x_{-k+1})$$

and so

$$|x_j - \bar{x}| < \varepsilon \qquad \text{for all} \qquad -k+1 \leq j \leq k.$$

For $n \geq k$, we have

$$\begin{aligned} x_{n+1} &= \frac{A_0}{x_n} + \frac{A_1}{x_{n-1}} + \cdots + \frac{A_{k-1}}{x_{n-k+1}} \\ &= \frac{A_0}{\frac{A_0}{x_{n-1}} + \cdots + \frac{A_{k-1}}{x_{n-k}}} + \frac{A_1}{\frac{A_0}{x_{n-2}} + \cdots + \frac{A_{k-1}}{x_{n-k-1}}} + \cdots + \frac{A_{k-1}}{\frac{A_0}{x_{n-k}} + \cdots + \frac{A_{k-1}}{x_{n-2k+1}}}. \end{aligned}$$

Let $F : (0, \infty)^{2k} \to (0, \infty)$ be given by

$$\begin{aligned} F(z_1, z_2, \ldots, z_{2k}) = \frac{A_0}{\dfrac{A_0}{z_2} + \cdots + \dfrac{A_{k-1}}{z_{k+1}}} + \frac{A_1}{\dfrac{A_0}{z_3} + \cdots + \dfrac{A_{k-1}}{z_{k+2}}} \\ + \cdots + \frac{A_{k-1}}{\dfrac{A_0}{z_{k+1}} + \cdots + \dfrac{A_{k-1}}{z_{2k}}}. \end{aligned}$$

Then

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-2k+1}) \qquad \text{for} \qquad n = k, k+1, \ldots.$$

It follows by Lemma 1.2 that

$$|x_n - \bar{x}| < \varepsilon \qquad \text{for all} \qquad n \geq -k+1.$$

□

Let J be a non-empty set of positive integers. We denote by $\langle J \rangle$ the greatest common divisor of the elements of J.

LEMMA 4.10

Let J be a non-empty set of positive integers, and set $L = \{i + j : i, j \in J\}$. Then the following statements are true.

1. *Either $\langle L \rangle = \langle J \rangle$ or $\langle L \rangle = 2 \langle J \rangle$.*

2. *Suppose $\langle J \rangle = 1$ and $\langle L \rangle = 2$. Then every element in J is odd.*

PROOF

(i) Without loss of generality, we may assume that $\langle J \rangle = 1$ and that $\langle L \rangle = d > 1$. Let p be any prime divisor of d. Since $\langle J \rangle = 1$, there exists $j \in J$ such that p does not divide j. Note that as $2j \in L$, we have $2j \equiv 0 \pmod{d}$. Therefore, $2j \equiv 0 \pmod{p}$, and so as p does not divide j, we must have $p = 2$. Thus there exists a positive integer $m \geq 1$ such that $d = 2^m$, and so in particular $2j \equiv 0 \pmod{2^m}$. Now $p = 2$ does not divide j, and so it follows that $m = 1$, and hence that $d = 2$.

(ii) For the sake of contradiction, suppose there exists $l \in \{1, 2, \ldots\}$ such that $2l \in J$. Then after rewriting the elements of J, if necessary, we may assume that $J = \{2l, j_2, \ldots, j_q\}$. Hence

$$L = \{4l, 2l + j_2, \ldots, 2l + j_q, \ldots, 2j_q\}.$$

So as $\langle L \rangle = 2$, we see that, in particular, 2 is a common divisor of $j_2, j_3, \ldots, j_q$. Hence $\langle J \rangle \geq 2$. This is a contradiction, and the proof is complete. ∎

We are now ready for the main result of this section.

THEOREM 4.11

See [41]. Let $A_0, A_1, \ldots, A_{k-1}$ be non-negative real numbers, and suppose that the set $J = \{j \geq 1 : A_{j-1} > 0\}$ is not empty. Set $L = \{i + j : i, j \in J\}$, and let

$$p = \begin{cases} 1 & \text{if } \langle L \rangle = \langle J \rangle \\ 2 \langle J \rangle & \text{if } \langle L \rangle \neq \langle J \rangle . \end{cases}$$

Then every positive solution of Eq.(4.13) converges to a periodic solution of Eq.(4.13) with (not necessarily prime) period p. Moreover, there exist solutions of Eq.(4.13) which are periodic with prime period p.

PROOF Let $\{x_n\}_{n=-k+1}^{\infty}$ be a positive solution of Eq.(4.13). We first shall show that $\{x_n\}_{n=-k+1}^{\infty}$ converges to a periodic solution of Eq.(4.13) with period p.

Case 1. Suppose $\langle J \rangle = 1$. Then for $n \geq k$, we have

$$x_{n+1} = \frac{A_0}{x_n} + \frac{A_1}{x_{n-1}} + \cdots + \frac{A_{k-1}}{x_{n-k+1}}$$

$$= \frac{A_0}{\frac{A_0}{x_{n-1}} + \cdots + \frac{A_{k-1}}{x_{n-k}}} + \frac{A_1}{\frac{A_0}{x_{n-2}} + \cdots + \frac{A_{k-1}}{x_{n-k-1}}} + \cdots$$

$$+ \frac{A_{k-1}}{\frac{A_0}{x_{n-k}} + \cdots + \frac{A_{k-1}}{x_{n-2k+1}}}.$$

Let $F : (0, \infty)^{2k} \to (0, \infty)$ be given by

$$F(z_1, z_2, \ldots, z_{2k}) = \frac{A_0}{\frac{A_0}{z_2} + \cdots + \frac{A_{k-1}}{z_{k+1}}} + \frac{A_1}{\frac{A_0}{z_3} + \cdots + \frac{A_{k-1}}{z_{k+2}}}$$

$$+ \cdots + \frac{A_{k-1}}{\frac{A_0}{z_{k+1}} + \cdots + \frac{A_{k-1}}{z_{2k}}}.$$

Then

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-2k+1}) \quad \text{for} \quad n = k, k+1, \ldots. \tag{4.14}$$

It is easy to see that $F(z_1, z_2, \ldots, z_{2k})$ depends exactly on those arguments z_j for which $j \in L$, and that it is strictly increasing in those arguments.

It is also clear that $F(c, c, \ldots, c) = c$ for every $c \in (0, \infty)$. Finally, if we set

$$m = \min\{x_k, x_{k-1}, \ldots, x_1, x_0, x_{-1}, \ldots, x_{-k+1}\}$$

and

$$M = \max\{x_k, x_{k-1}, \ldots, x_1, x_0, x_{-1}, \ldots, x_{-k+1}\}$$

then it follows by Lemma 1.2 that $m \leq x_n \leq M$ for all $n \geq -k+1$.

Suppose $\langle L \rangle = 1$. Then $p = 1$, and so it suffices to show that

$$\lim_{n \to \infty} x_n = \sqrt{A}.$$

It follows by Theorem 1.9 applied to Eq.(4.14) that there exists $a \in [m, M]$ with

$$\lim_{n\to\infty} x_n = a.$$

So as Eq.(4.13) has the unique equilibrium point $\sqrt{A}$, we see that

$$\lim_{n\to\infty} x_n = \sqrt{A}$$

and the proof is complete.

Suppose $\langle L\rangle \neq 1$. Then it follows by Lemma 4.10 that $\langle L\rangle = 2$, and so $p = 2$. Thus we must show that $\{x_n\}_{n=-k+1}^{\infty}$ converges to a solution of Eq.(4.13) with period 2.

Now since $\langle L\rangle = 2$, there exist positive integers $i_1, i_2, \ldots, i_l$ such that

$$L = \{2i_1, 2i_2, \ldots, 2i_l\}.$$

Recall that $F(z_1, z_2, \ldots, z_{2k})$ depends only on those arguments z_q for which $q \in L$. Let $G : (0,\infty)^k \to (0,\infty)$ be given by

$$G(v_1, v_2, \ldots, v_k) = F(1, v_1, 1, v_2, \ldots, 1, v_k).$$

Then

$$x_{n+1} = G(x_{n-1}, x_{n-3}, \ldots, x_{n-2k+1}) \qquad \text{for} \qquad n = k, k+1, \ldots.$$

Thus if we make the substitution $x_{2n} = u_n$ and $x_{2n+1} = v_n$, we see that

$$u_{n+1} = G(u_n, u_{n-1}, \ldots, u_{n-k+1}) \quad \text{and} \quad v_{n+1} = G(v_n, v_{n-1}, \ldots, v_{n-k+1}).$$

Now $G(v_1, v_2, \ldots, v_k)$ depends only upon those arguments v_q for which $2q \in L$, and G is strictly increasing in those arguments. So as $\langle L\rangle = 2$, it follows from Theorem 1.9 that there exist positive real numbers $a, b \in [m, M]$ such that

$$\lim_{n\to\infty} x_{2n} = a \qquad \text{and} \qquad \lim_{n\to\infty} x_{2n+1} = b$$

and so it is true that $\{x_n\}_{n=-k+1}^{\infty}$ converges to a solution of Eq.(4.13) with period 2.

Finally, by Lemma 4.10, we know that all the elements in J are odd; that is, $A_s > 0$ only if s is even. Thus we see that $b = \dfrac{A}{a}$, and that $\{x_n\}_{n=-k+1}^{\infty}$ is a periodic solution of Eq.(4.13) if and only if there exists $a \in (0,\infty)$ such that $x_n = a$ if $n \geq -k+1$ is even, while $x_n = \dfrac{A}{a}$ if $n \geq -k+1$ is odd. In particular, there do exist periodic solutions of Eq.(4.13) with prime period 2, and so the proof is complete.

Case 2. Suppose $\langle J\rangle = q > 1$. Then there exist positive integers $i_1, i_2, \ldots, i_l$ with $\langle\{i_1, i_2, \ldots, i_l\}\rangle = 1$ such that $J = \{qi_1, qi_2, \ldots, qi_l\}$. Hence we see that for $1 \le j \le q$, we have

$$x_{qs+j} = \frac{A_{qi_1-1}}{x_{q(s-i_1)+j}} + \frac{A_{qi_2-1}}{x_{q(s-i_2)+j}} + \cdots + \frac{A_{qi_l-1}}{x_{q(s-i_l)+j}} \quad \text{for} \quad s = 0, 1, \ldots. \tag{4.15}$$

For $1 \le j \le q$, $1 \le r \le l$, and $0 \le s$, set

$$u_s^j = x_{q(s-1)+j} \quad \text{and} \quad \bar{A}_{i_r-1} = A_{qi_r-1}.$$

Then for $1 \le j \le q$, we have

$$u_{s+1}^j = \frac{\bar{A}_{i_1-1}}{u_{s-(i_1-1)}^j} + \frac{\bar{A}_{i_2-1}}{u_{s-(i_2-1)}^j} + \cdots + \frac{\bar{A}_{i_l-1}}{u_{s-(i_l-1)}^j} \quad \text{for} \quad s = k, k+1, \ldots. \tag{4.16}$$

For $i \in \{0, 1, \ldots, k-1\} - \{i_1 - 1, i_2 - 1, \ldots, i_l - 1\}$, set $\bar{A}_i = 0$. Then

$$\bar{A}_0 + \bar{A}_1 + \cdots + \bar{A}_{k-1} = A_0 + A_1 + \cdots + A_{k-1} = A.$$

The corresponding set $\bar{J} = \left\{j : \bar{A}_{j-1} > 0\right\}$ for Eq.(4.16) is

$$\bar{J} = \{i_1, i_2, \ldots, i_l\}.$$

Now $\left\langle \bar{J} \right\rangle = 1$, and so we may apply Case 1 of the current theorem. As $\langle J\rangle = q$, it follows by Lemma 4.10 that either $\langle L\rangle = q$ or $\langle L\rangle = 2q$.

Let us first consider the case $\langle L\rangle = q$. Then $p = 1$, and so it suffices to show that $\lim_{n\to\infty} x_n = \bar{x}$.

Now $\left\langle \bar{L} \right\rangle = 1$, where the set $\bar{L} = \{i + j : i, j \in \bar{J}\}$, and so it follows by Case 1 that for each $1 \le j \le q$, $\lim_{s\to\infty} u_s^j = \bar{x}$, and hence that $\lim_{n\to\infty} x_n = \bar{x}$, as was to be shown.

Next assume that $\langle L\rangle = 2q$. Then $p = 2q$, and so we must show that $\{x_n\}_{n=-k+1}^{\infty}$ converges to a periodic solution of Eq.(4.13) with period $2q$. In this case, $\left\langle \bar{L} \right\rangle = 2$, and so it follows by Case 1 that for each $1 \le j \le q$, there exists $a_j \in (0, \infty)$ such that $\{u_s^j\}_{s=0}^{\infty}$ converges to the period 2 solution

$$a_j, \frac{A}{a_j}, a_j, \frac{A}{a_j}, \ldots$$

of Eq.(4.16). It follows that $\{x_n\}_{n=-k+1}^{\infty}$ converges to the periodic solution

$$a_1, a_2, \ldots, a_q, \frac{A}{a_1}, \frac{A}{a_2}, \ldots, \frac{A}{a_q}, \ldots . \tag{4.17}$$

of Eq.(4.13) with period $2q$. It is also clear that there exist positive solutions of Eq.(4.13) which are periodic with prime period $2q$. The proof is complete. □

The following example shows that the non-trivial periodic solutions of Eq.(4.13) need not be periodic with prime period $2q$.

Example 4.1
Consider the difference equation

$$x_{n+1} = \frac{1}{x_{n-5}} + \frac{1}{x_{n-17}} \quad , \qquad n = 0, 1, \ldots . \tag{4.18}$$

For this difference equation $J = \{6, 18\}$ and $L = \{12, 24, 36\}$, and so every positive periodic solution of Eq.(4.18) is periodic with period 12 and is given as follows:

$$a, b, c, d, e, f, \frac{2}{a}, \frac{2}{b}, \frac{2}{c}, \frac{2}{d}, \frac{2}{e}, \frac{2}{f}, \ldots .$$

In particular if $0 < a < \sqrt{2}$, then

$$a, \sqrt{2}, \frac{2}{a}, \sqrt{2}, a, \sqrt{2}, \frac{2}{a}, \sqrt{2}, a, \sqrt{2}, \frac{2}{a}, \sqrt{2}, \ldots$$

is a positive solution of Eq.(4.18) which is periodic with prime period 4. □

4.6 Convergence of Solutions of Systems to Period-2 Solutions

Consider the system of difference equations

$$\begin{cases} x_{n+1} = \dfrac{a}{x_n} + \dfrac{b}{y_n} \\ \\ y_{n+1} = \dfrac{c}{x_n} + \dfrac{d}{y_n} \end{cases} , \; n = 0, 1, \ldots \tag{4.19}$$

where

$$a, b, c, d \in (0, \infty).$$

We say that a solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of Eq.(4.19) is a *positive solution* if

$$x_n > 0 \text{ and } y_n > 0 \qquad \text{for all} \qquad n \geq 0.$$

The proof of Lemma 4.11 follows by a computation and will be omitted.

LEMMA 4.11
The following statements are true:

1. *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a positive solution of Eq.(4.19). Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is a period-2 solution of Eq.(4.19) if and only if*

$$ay_0^2 + (b - c)x_0y_0 - dx_0^2 = 0.$$

2. *There exist positive prime period-2 solutions of Eq.(4.19).*

The following result was established in [57]. See also [99] and [100].

THEOREM 4.12
See [57]. *Every positive solution of Eq.(4.19) converges to a period-2 solution of Eq.(4.19).*

PROOF Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a positive solution of Eq.(4.19). It suffices to show that

$$\lim_{n\to\infty} x_n \qquad \text{and} \qquad \lim_{n\to\infty} y_n$$

exist and are finite numbers.

Set

$$z_n = \frac{x_n}{y_n} \qquad \text{for} \qquad n = 0, 1, \ldots .$$

Then by dividing the first equation in Eq.(4.19) by the second equation, it follows that $\{z_n\}_{n=0}^{\infty}$ satisfies the Riccati difference equation

$$z_{n+1} = \frac{a + bz_n}{c + dz_n} \quad , \qquad n = 0, 1, \ldots \tag{4.20}$$

with Riccati number

$$\mathcal{R} = \frac{bc - ad}{(b + c)^2} < \frac{1}{4}.$$

It follows by Theorem 2.6 that

$$z_n = \frac{b + c}{d}\left(\frac{c_1\lambda_1^{n+1} + c_2\lambda_2^{n+1}}{c_1\lambda_1^n + c_2\lambda_2^n}\right) \qquad \text{for all} \qquad n \geq 0$$

where

$$\lambda_1 = \frac{1-\sqrt{1-4\mathcal{R}}}{2} \quad , \quad \lambda_2 = \frac{1+\sqrt{1-4\mathcal{R}}}{2}$$

and

$$c_1 = \frac{\lambda_2(b+c)-(dz_0+c)}{(b+c)(\lambda_2-\lambda_1)} \quad , \quad c_2 = \frac{(dz_0+c)-\lambda_1(b+c)}{(b+c)(\lambda_2-\lambda_1)}.$$

Set

$$P_n = a + b\left[\frac{b+c}{d}\left(\frac{c_1\lambda_1^{n+1}+c_2\lambda_2^{n+1}}{c_1\lambda_1^n+c_2\lambda_2^n}\right)-\frac{c}{d}\right] \quad \text{for} \quad n = 0, 1, \ldots .$$

Then

$$x_n x_{n+1} = a + b\frac{x_n}{y_n} = P_n \text{ for } n = 0, 1, \ldots$$

and so

$$x_{2n+2} = \frac{P_{2n+1}}{P_{2n}} x_{2n} \text{ for } n = 0, 1, \ldots$$

and

$$x_{2n+3} = \frac{P_{2n+2}}{P_{2n+1}} x_{2n+1} \text{ for } n = 0, 1, \ldots .$$

Therefore,

$$x_{2n+2} = x_0 \prod_{k=0}^{n} \frac{P_{2k+1}}{P_{2k}} \text{ for } n = 0, 1, \ldots .$$

and

$$x_{2n+3} = x_1 \prod_{k=0}^{n} \frac{P_{2k+2}}{P_{2k+1}} \text{ for } n = 0, 1, \ldots .$$

Set

$$u_k = \frac{P_{2k+1}-P_{2k}}{P_{2k}} \text{ for } k = 0, 1, \ldots$$

and

$$v_k = \frac{P_{2k+2}-P_{2k+1}}{P_{2k+1}} \text{ for } k = 0, 1, \ldots .$$

Then

$$\lim_{n\to\infty} x_{2n+2} = x_0 \prod_{k=0}^{\infty} (1+u_k)$$

and

$$\lim_{n\to\infty} x_{2n+3} = x_1 \prod_{k=0}^{\infty} (1+v_k)\,.$$

It remains to show that these two infinite products converge. We shall establish this for the first infinite product. The proof for the second is similar and will be omitted. To this end observe that

$$|P_{2k+1} - P_{2k}| = \left|\frac{b(b+c)(1-4\mathcal{R})}{d}\right| \left|\frac{1}{\frac{c_1\lambda_2}{c_2}\left(\frac{\lambda_1}{\lambda_2}\right)^{2k+1} + 1 + \frac{c_2\lambda_1}{c_1}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+1}}\right|$$

and because $|\lambda_1| < |\lambda_2|$, it follows by the limit comparison test that the series $\sum_{k=1}^{\infty} |P_{2k+1} - P_{2k}|$ converges absolutely. Note that

$$\lim_{k\to\infty}\left|\frac{1}{P_{2k}}\right| = \left|\frac{1}{a + b\left(\frac{b+c}{d}\lambda_2 - \frac{c}{d}\right)}\right| \in (0,\infty)$$

from which it is now clear by the limit comparison test that the series $\sum_{k=0}^{\infty} u_k$ converges absolutely. ∎

4.7 Open Problems and Conjectures

CONJECTURE 4.1

(A Period-4 Trichotomy) Consider the difference equation

$$x_{n+1} = \frac{p + qx_n + rx_{n-2}}{x_{n-1}} \tag{4.21}$$

with $p \in [0,\infty)$ and $q, r \in (0,\infty)$. Show that the following statements are true:

(i) Every positive solution of Eq.(4.21) converges to the positive equilibrium of Eq.(4.21) if and only if $q > r$.

(ii) Every positive solution of Eq.(4.21) converges to a solution of Eq.(4.21) which is periodic with period 4 if and only if $q = r$.

(iii) Eq.(4.21) has unbounded solutions if and only if $q < r$.

See [21] and [119] where Statement (ii) has been confirmed.

CONJECTURE 4.2
(A Period-6 Trichotomy) Consider the difference equation

$$x_{n+1} = \frac{p + x_n}{qx_{n-1} + x_{n-2}} \quad , \qquad n = 0, 1, \ldots \tag{4.22}$$

with $p, q \in [0, \infty)$. *Show that the following statements are true:*

(i) Every positive solution of Eq.(4.22) converges to the positive equilibrium of Eq.(4.22) if and only if $pq^2 > 1$.

(ii) Every positive solution of Eq.(4.22) converges to a solution of Eq.(4.22) which is periodic with period 6 if and only if $pq^2 = 1$.

(iii) Eq.(4.22) has unbounded solutions if and only if $pq^2 < 1$.

CONJECTURE 4.3
Show that every positive solution of the equation

$$x_{n+1} = \frac{1}{x_n x_{n-1}} + \frac{1}{x_{n-3}x_{n-4}} \quad , \qquad n = 0, 1, \ldots$$

converges to a period-3 solution. See [35].

The above conjecture seems to be a special case of the following more general conjecture.

CONJECTURE 4.4
Show that every positive solution of the difference equation

$$x_{n+1} = \frac{1}{\prod_{i=0}^{k} x_{n-1}} + \frac{1}{\prod_{j=k+2}^{2(k+1)} x_{n-j}} \quad , \qquad n = 0, 1, \ldots \tag{4.23}$$

converges to a periodic solution with period $(k+2)$. *See [35].*
For $k = 0$, *Eq.(4.23) reduces to the equation*

$$x_{n+2} = \frac{1}{x_n} + \frac{1}{x_{n-2}} \quad , \qquad n = 0, 1, \ldots$$

for which the result is known. See [34].

OPEN PROBLEM 4.1
Consider the rational equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Cx_{n-1} + Dx_{n-2}} \quad , \quad n = 0, 1, \ldots \tag{4.24}$$

with non-negative parameters and non-negative initial conditions. For each

$$p \in \{2, 4, 5, 6\}$$

obtain necessary and sufficient conditions on $\alpha, \beta, \gamma, \delta, A, B, C, D$ *so that every solution of Eq.(4.24) converges to a (not necessarily prime) period-p solution of Eq.(4.24).*

OPEN PROBLEM 4.2
Assume that $p, q \in (0, \infty)$. *Investigate the periodic character and the asymptotic behavior of the positive solutions of the difference equation*

$$x_{n+1} = \frac{px_n}{qx_{n-1} + x_{n-2}} \quad , \quad n = 0, 1, \ldots .$$

OPEN PROBLEM 4.3
Assume that $p, q \in (0, \infty)$. *Investigate the periodic character and the asymptotic behavior of the positive solutions of the difference equation*

$$x_{n+1} = \frac{px_{n-1}}{qx_n + x_{n-2}} \quad , \quad n = 0, 1, \ldots .$$

OPEN PROBLEM 4.4
Assume that $p, q \in (0, \infty)$. *Investigate the periodic character and the asymptotic behavior of the positive solutions of the difference equation*

$$x_{n+1} = \frac{px_{n-2}}{qx_n + x_{n-1}} \quad , \quad n = 0, 1, \ldots .$$

OPEN PROBLEM 4.5
Assume that $p, q \in (0, \infty)$. *Investigate the periodic character and the asymptotic behavior of the positive solutions of the difference equation*

$$x_{n+1} = \frac{px_{n-1} + qx_{n-2}}{x_n} \quad , \quad n = 0, 1, \ldots .$$

OPEN PROBLEM 4.6
Assume that $p, q \in (0, \infty)$. *Investigate the periodic character and the asymptotic behavior of the positive solutions of the difference equation*

$$x_{n+1} = \frac{px_n + qx_{n-2}}{x_{n-1}} \quad , \quad n = 0, 1, \ldots .$$

OPEN PROBLEM 4.7

Assume that $p, q \in (0, \infty)$. Investigate the periodic character and the asymptotic behavior of the positive solutions of the difference equation

$$x_{n+1} = \frac{px_n + qx_{n-1}}{x_{n-2}} \quad , \qquad n = 0, 1, \ldots .$$

For the second-order rational difference Eq.(2.30) with non-negative coefficients and positive initial conditions, we offer the following conjecture.

CONJECTURE 4.5

Assume that

$$C > 0.$$

Then every solution of Eq.(2.30) is bounded.

CONJECTURE 4.6

Assume that

$$a_{ij} \in (0, \infty) \qquad \text{for} \qquad i, j \in \{1, 2, \ldots, k\}.$$

Show that every positive solution of the system

$$\begin{cases} x^1_{n+1} = \dfrac{a_{11}}{x^1_n} + \dfrac{a_{12}}{x^2_n} + \cdots + \dfrac{a_{1k}}{x^k_n} \\ x^2_{n+1} = \dfrac{a_{21}}{x^1_n} + \dfrac{a_{22}}{x^2_n} + \cdots + \dfrac{a_{2k}}{x^k_n} \\ \qquad \cdots \\ x^k_{n+1} = \dfrac{a_{k1}}{x^1_n} + \dfrac{a_{k2}}{x^2_n} + \cdots + \dfrac{a_{kk}}{x^k_n} \end{cases} \quad , \qquad n = 0, 1, \ldots \tag{4.25}$$

converges to a solution which is periodic with period 2. See [102] for the case $k = 3$.

OPEN PROBLEM 4.8

It is known that every positive solution of the difference equation

$$x_{n+1} = 1 + \frac{x_{n-1}}{x_n} \quad , \qquad n = 0, 1, \ldots$$

converges to a solution which is periodic with period 2.

Given a positive solution $\{x_n\}_{n=-1}^{\infty}$ of the above equation, determine the limiting period-2 solution explicitly in terms of the initial conditions x_{-1}, x_0.

OPEN PROBLEM 4.9

It is known that every positive solution of the difference equation

$$x_{n+1} = 1 + \frac{x_{n-1}}{x_{n-2}} \quad , \quad n = 0, 1, \ldots$$

converges to a solution which is periodic with period 2.

Given a positive solution $\{x_n\}_{n=-2}^{\infty}$ *of the above equation, determine the limiting period-2 solution explicitly in terms of the initial conditions* x_{-2}, x_{-1}, x_0.

OPEN PROBLEM 4.10

It is known that every positive solution of the difference equation (with $A, B \in (0, \infty)$*)*

$$x_{n+1} = \frac{A}{x_n} + \frac{B}{x_{n-2}} \quad , \quad n = 0, 1, \ldots$$

converges to a solution which is periodic with period 2.

Given a positive solution $\{x_n\}_{n=-2}^{\infty}$ *of the above equation, determine the limiting period-2 solution explicitly in terms of the initial conditions* x_{-2}, x_{-1}, x_0.

OPEN PROBLEM 4.11

It is known that every positive solution of the difference equation

$$x_{n+1} = \frac{x_n + x_{n-2}}{x_{n-1}} \quad , \quad n = 0, 1, \ldots$$

converges to a solution which is periodic with period 4.

Given a positive solution $\{x_n\}_{n=-2}^{\infty}$ *of the above equation, determine the limiting period-4 solution explicitly in terms of the initial conditions* x_{-2}, x_{-1}, x_0.

OPEN PROBLEM 4.12

It is known that every positive solution of the difference equation

$$x_{n+1} = \frac{1 + x_{n-2}}{x_n} \quad , \quad n = 0, 1, \ldots$$

converges to a solution which is periodic with period 5.

Given a positive solution $\{x_n\}_{n=-2}^{\infty}$ *of the above equation, determine the limiting period-5 solution explicitly in terms of the initial conditions* x_{-2}, x_{-1}, x_0.

OPEN PROBLEM 4.13

It has been conjectured that every positive solution of the difference equation

$$x_{n+1} = \frac{1 + x_n}{x_{n-1} + x_{n-2}} \quad , \quad n = 0, 1, \ldots$$

converges to a solution which is periodic with period 6.

Given a positive solution $\{x_n\}_{n=-2}^{\infty}$ *of the above equation, determine the limiting period-6 solution explicitly in terms of the initial conditions* x_{-2}, x_{-1}, x_0.

OPEN PROBLEM 4.14

It is known that every positive solution of the difference equation

$$x_{n+1} = \frac{1 + x_n + x_{n-k}}{x_{n-(k-1)}} \qquad , \qquad n = 0, 1, \ldots$$

converges to a solution which is periodic with period $(2k)$. *See [119].*

Given a positive solution $\{x_n\}_{n=-k}^{\infty}$ *of the above equation, determine the limiting period-*$(2k)$ *solution explicitly in terms of the initial conditions* $x_{-k}, x_{k+1}, \ldots, x_0$.

OPEN PROBLEM 4.15

It is known that every positive solution of the difference equation

$$x_{n+1} = \frac{x_{n-k}}{1 + x_n + \cdots + x_{n-(k-1)}} \qquad , \qquad n = 0, 1, \ldots$$

converges to a solution which is periodic with period $(k+1)$.

Given a positive solution $\{x_n\}_{n=-k}^{\infty}$ *of the above equation, determine the limiting period-*$(k+1)$ *solution explicitly in terms of the initial conditions* $x_{-k}, x_{k+1}, \ldots, x_0$.

OPEN PROBLEM 4.16

It is known that every positive solution of the difference equation

$$x_{n+1} = \frac{1 + x_{n-1} + x_{n-2}}{1 + x_n} \qquad , \qquad n = 0, 1, \ldots$$

converges to a solution which is periodic with period 2.

Given a positive solution $\{x_n\}_{n=-2}^{\infty}$ *of the above equation, determine the limiting period-2 solution explicitly in terms of the initial conditions* x_{-2}, x_{-1}, x_0.

OPEN PROBLEM 4.17

It is known that every positive solution of the difference equation

$$x_{n+1} = \frac{1 + x_{n-1}}{1 + x_n} \qquad , \qquad n = 0, 1, \ldots$$

converges to a solution which is periodic with period 2.

Given a positive solution $\{x_n\}_{n=-2}^{\infty}$ *of the above equation, determine the limiting period-2 solution explicitly in terms of the initial conditions* x_{-1}, x_0.

OPEN PROBLEM 4.18

It is known that every positive solution of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1} + x_{n-2}} \quad , \qquad n = 0, 1, \ldots$$

converges to a solution which is periodic with period 2.

Given a positive solution $\{x_n\}_{n=-2}^{\infty}$ *of the above equation, determine the limiting period-2 solution explicitly in terms of the initial conditions* x_{-2}, x_{-1}, x_0.

OPEN PROBLEM 4.19

It is known that every positive solution of the difference equation

$$x_{n+1} = x_{n-1} e^{-x_n} \quad , \qquad n = 0, 1, \ldots$$

converges to a solution which is periodic with period 2.

Given a positive solution $\{x_n\}_{n=-1}^{\infty}$ *of the above equation, determine the limiting period-2 solution explicitly in terms of the initial conditions* x_{-1}, x_0.

OPEN PROBLEM 4.20

It is known that every positive solution of the system of difference equations (with $r \geq 0$*)*

$$\begin{cases} A_{n+1} = J_n \\ \\ J_{n+1} = A_n e^{r-(A_n+J_n)} \end{cases} , \; n = 0, 1, \ldots$$

converges to a solution which is periodic with period 2.

Given a positive solution $\{(A_n, J_n)\}_{n=0}^{\infty}$ *of the above system, determine the limiting period-2 solution explicitly in terms of the initial condition* (A_0, J_0).

OPEN PROBLEM 4.21

It is known that every positive solution of the system of difference equations (with $A_{ij} > 0$*)*

$$\begin{cases} x_{n+1} = \dfrac{A_{11}}{x_n} + \dfrac{A_{12}}{y_n} \\ \\ y_{n+1} = \dfrac{A_{21}}{x_n} + \dfrac{A_{22}}{y_n} \end{cases} , \; n = 0, 1, \ldots$$

converges to a solution which is periodic with period 2.

Given a positive solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ *of the above system, determine the limiting period-2 solution explicitly in terms of the initial condition* (x_0, y_0).

OPEN PROBLEM 4.22

It has been conjectured that every positive solution of the difference equation

$$x_{n+1} = \frac{1}{x_n x_{n-1}} + \frac{1}{x_{n-3}x_{n-4}} \qquad , \qquad n = 0, 1, \ldots$$

converges to a solution which is periodic with period 3.

Given a positive solution $\{x_n\}_{n=-4}^{\infty}$ *of the above equation, determine the limiting period-3 solution explicitly in terms of the initial conditions* $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$.

OPEN PROBLEM 4.23

Let $A_0, A_1, \ldots, A_{k-1} \geq 0$ *be non-negative real numbers, and suppose that the set* $J = \{j \geq 1 : A_{j-1} > 0\}$ *is not empty. Set* $L = \{i + j : i, j \in J\}$. *Let* $\langle J \rangle$ *denote the greatest common divisor of the elements of J, and let* $\langle L \rangle$ *denote the greatest common divisor of the elements of L. It is known that every positive solution of the difference equation*

$$x_{n+1} = \frac{A_0}{x_n} + \frac{A_1}{x_{n-1}} + \cdots + \frac{A_{k-1}}{x_{n-k+1}} \qquad , \qquad n = 0, 1, \ldots$$

has a finite limit if and only if $\langle J \rangle = \langle L \rangle$, *and that every solution of the above equation converges to a solution which is periodic with period* $2\langle J \rangle$ *if and only if* $\langle J \rangle \neq \langle L \rangle$.

Given a positive solution $\{x_n\}_{n=-k+1}^{\infty}$ *of the above equation when* $\langle J \rangle \neq \langle L \rangle$, *determine the limiting periodic solution explicitly in terms of the initial conditions* $x_{-k+1}, x_{-k+2}, \ldots, x_0$.

Chapter 5

THE EQUATION $x_{n+1} = \dfrac{\alpha + \gamma x_{n-(2k+1)} + \delta x_{n-2l}}{A + x_{n-2l}}$

5.1 Introduction and Preliminaries

In this chapter, we investigate the periodic character, the boundedness nature, and the global asymptotic stability of the rational difference equation

$$x_{n+1} = \frac{\alpha + \gamma x_{n-(2k+1)} + \delta x_{n-2l}}{A + x_{n-2l}}, \quad n = 0, 1, \ldots \tag{5.1}$$

where k and l are non-negative integers, the parameters $\alpha, \gamma, \delta, A$ are non-negative real numbers, and the initial conditions are non-negative real numbers such that the denominator in Eq.(5.1) is always positive. See [60].

Two special cases of Eq.(5.1) are the rational difference equations

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + x_n}, \quad n = 0, 1, \ldots \tag{5.2}$$

and

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2}}{A + x_{n-2}}, \quad n = 0, 1, \ldots \tag{5.3}$$

with non-negative parameters and non-negative initial conditions. The following trichotomy results describe the character of the solutions of Eqs.(5.2) and (5.3). See also Sections 4.2 and 4.3.

THEOREM 5.1
See [49] and [50], or see [78]. *The following statements are true:*

1. *Every solution of Eq.(5.2) has a finite limit if and only if*

$$\gamma < \beta + A. \tag{5.4}$$

2. *Every solution of Eq.(5.2) converges to a period-2 solution if and only if*

$$\gamma = \beta + A. \tag{5.5}$$

3. *Eq.(5.2) has positive unbounded solutions if and only if*

$$\gamma > \beta + A. \tag{5.6}$$

THEOREM 5.2
See [23]. *Assume*

$$\gamma + \delta + A > 0. \tag{5.7}$$

Then the following statements are true:

1. *Every solution of Eq.(5.3) has a finite limit if and only if*

$$\gamma < \delta + A. \tag{5.8}$$

2. *Every solution of Eq.(5.3) converges to a period-two solution if and only if*

$$\gamma = \delta + A. \tag{5.9}$$

3. *Eq.(5.3) has positive unbounded solutions if and only if*

$$\gamma > \delta + A. \tag{5.10}$$

When (5.4) holds, Eq.(5.2) may have a single equilibrium, which may be either zero or positive, or else Eq.(5.2) has two equilibria, one zero and the other positive. When Eq.(5.2) has a single equilibrium and (5.4) holds, the unique equilibrium is globally asymptotically stable. When Eq.(5.2) has two equilibria and (5.4) holds, the zero equilibrium is unstable, and the positive equilibrium is locally asymptotically stable, and is a global attractor with basin of attraction all positive solutions of Eq.(5.2).

When we say in Statement 2 of Theorem 5.1 that every solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(5.2) converges to a period-2 solution, we mean that there exists a period-2 solution

$$\ldots, \phi, \psi, \ldots$$

of Eq.(5.2) such that

$$\lim_{n\to\infty} x_{2n-1} = \phi \qquad \text{and} \qquad \lim_{n\to\infty} x_{2n} = \psi$$

and that these limits ϕ and ψ are not always equal, although they may sometimes be equal. In fact, when (5.5) holds, Eq.(5.2) possesses infinitely many solutions which are periodic with prime period-2.

When (5.6) holds, Eq.(5.2) has a positive equilibrium $\bar{x} > 0$ which is an unstable saddle-point equilibrium. Thus by the Stable Manifold Theorem, Eq.(5.2) also has solutions which converge to $\bar{x}$, and so, in particular, are bounded.

It is also interesting to note that the trichotomy character of the solutions of Eq.(5.2) is a non-linear phenomenon, and is not true when the term x_n is missing from the denominator of Eq.(5.2).

Our goal in this chapter is to extend the above results for Eqs.(5.2) and (5.3) to the quite general Eq.(5.1). In fact we shall show that the positive solutions of Eq.(5.1) exhibit a trichotomy character depending upon how the parameter γ compares with the sum $\delta + A$.

Let d be defined as follows:

$$d = \begin{cases} 2l+1 & \text{if } \gamma = \delta = A = 0 \\ \\ k+1 & \text{if} \quad \alpha = \delta = 0 \quad \text{and } A > 0 \\ \\ \gcd(k+1, 2l+1) & \quad \text{otherwise.} \end{cases}$$

The main result of this chapter is that Eq.(5.1) exhibits the following period-$2d$ trichotomy:

THEOREM 5.3
The following statements are true:

1. *Every positive solution of Eq.(5.1) has a finite limit if and only if*

$$\gamma < \delta + A.$$

2. *Every positive solution of Eq.(5.1) converges to a non-negative periodic solution of Eq.(5.1) with period* $2d$ *(and there exist non-negative periodic solutions of Eq.(5.1) with prime period* $2d$*) if and only if*

$$\gamma = \delta + A.$$

3. *There exist unbounded solutions of Eq.(5.1) if and only if*

$$\gamma > \delta + A.$$

This result generalizes the period-2 trichotomy results in [7], [23], [48], [50], [58], [78], and [108].

REMARK 5.1 Eq.(5.1) always has at least one non-negative equilibrium point. It sometimes has two equilibrium points, one zero and the other positive. It is interesting to note that under the condition $\gamma < \delta + A$, we shall establish the following: When Eq.(5.1) has a single equilibrium point, every positive solution of Eq.(5.1) converges to the equilibrium point. When

Eq.(5.1) has two equilibrium points, one zero and the other positive, every positive solution of Eq.(5.1) converges to the positive equilibrium point. □

The proof of Theorem 5.3 is quite involved and, for the sake of clarity and completeness, we first consider several special cases of Eq.(5.1) whose asymptotic behavior and periodic nature are instrumental in establishing the complete result.

Throughout this chapter, we set

$$\mathcal{K} = \max\{2k+1, 2l\}.$$

5.2 The Equation $x_{n+1} = \dfrac{\alpha + x_{n-(2k+1)}}{1 + x_{n-2l}}$

Consider the difference equation

$$x_{n+1} = \frac{\alpha + x_{n-(2k+1)}}{1 + x_{n-2l}}, \qquad n = 0, 1, \ldots \tag{5.11}$$

where k and l are non-negative integers, α is a positive parameter, and the initial conditions are non-negative real numbers.

The study of the periodic nature of the solutions of Eq.(5.11) will be instrumental in the proof that solutions of Eq.(5.1) converge to periodic solutions of Eq.(5.1).

Here $\alpha > 0$, $\gamma = 1$, $\delta = 0$, and $A = 1$, and so

$$d = \gcd(k+1, 2l+1).$$

In agreement with our goal, in this section we show that every solution of Eq.(5.11) converges to a periodic solution with period $2d$, and that there exist periodic solutions of Eq.(5.11) with prime period $2d$.

The case $k = 0$ and $l = 0$ was investigated in [48], and the case $k = 0$ and $l = 1$ was investigated in [23]. See also [7], [78], and [108].

Note that Eq.(5.11) has the unique positive equilibrium point $\bar{x} = \sqrt{\alpha}$.

Note also that if $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ is a solution of Eq.(5.11) with non-negative initial conditions, then $x_n > 0$ for all $n \geq 1$. For this reason it follows that without loss of generality we need only consider positive solutions of Eq.(5.11).

5.2.1 Preliminaries

The following four lemmas will be useful in this section.

LEMMA 5.1
Let $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ be a positive solution of Eq.(5.11). Choose positive real numbers m and M with $mM = \alpha$, and such that

$$m \le \min\{x_{-\mathcal{K}}, x_{-\mathcal{K}+1}, \dots, x_0\} \quad \textit{and} \quad \max\{x_{-\mathcal{K}}, x_{-\mathcal{K}+1}, \dots, x_0\} \le M.$$

Then

$$m \le x_n \le M \quad \textit{for all} \quad n \ge -\mathcal{K}.$$

PROOF Note that $m = \dfrac{\alpha + m}{1 + M}$ and $M = \dfrac{\alpha + M}{1 + m}$, and so

$$m = \frac{\alpha + m}{1 + M} \le \frac{\alpha + x_{-(2k+1)}}{1 + x_{-2l}} \le \frac{\alpha + M}{1 + m} = M.$$

That is, $m \le x_1 \le M$. It follows by induction that

$$m \le x_n \le M \quad \text{for all} \quad n \ge -\mathcal{K}.$$

∎

REMARK 5.2 Let $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ be a positive solution of Eq.(5.11). Note that by Lemma 5.1,

$$0 < m \le \liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n \le M.$$

In particular, $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$ are positive real numbers. ∎

LEMMA 5.2
Let $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ be a positive solution of Eq.(5.11). Set $I = \liminf_{n\to\infty} x_n$ and $S = \limsup_{n\to\infty} x_n$. Then

$$IS = \alpha.$$

PROOF By Theorem 1.8, there exist full limiting sequences $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ such that

$$I_0 = I \quad \text{and} \quad S_0 = S.$$

Thus

$$S = S_0 = \frac{\alpha + S_{-(2k+2)}}{1 + S_{-(2l+1)}} \le \frac{\alpha + S}{1 + I}$$

and so $S + SI \leq \alpha + S$. That is,

$$SI \leq \alpha.$$

Also,

$$I = I_0 = \frac{\alpha + I_{-(2k+2)}}{1 + I_{-(2l+1)}} \geq \frac{\alpha + I}{1 + S}.$$

Hence we also have $IS \geq \alpha$, and so $IS = \alpha$. ∎

REMARK 5.3 Suppose $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ is a positive solution of Eq.(5.11). Throughout this section, we set

$$I = \liminf_{n\to\infty} x_n \qquad \text{and} \qquad S = \limsup_{n\to\infty} x_n.$$

Note that as $IS = \alpha$, we have

$$I = \frac{\alpha + I}{1 + S} \qquad \textit{and} \qquad S = \frac{\alpha + S}{1 + I}.$$

That is, $I, S, I, S, \ldots$ is a periodic solution of Eq.(5.11) with period-2. Thus if it is not true that $\lim\limits_{n\to\infty} x_n = \bar{x}$, then $I, S, I, S, \ldots$ is a periodic solution of Eq.(5.11) with prime period-2. ∎

LEMMA 5.3

Let $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ be a positive solution of Eq.(5.11), and let $\{L_n\}_{n=-\infty}^{\infty}$ be a full limiting sequence of $\{x_n\}_{n=-\mathcal{K}}^{\infty}$. Suppose j is an integer such that $L_j \in \{I, S\}$. Then the following statements are true:

1. $L_j = L_{j-2(k+1)}$.
2. $\dfrac{\alpha}{L_j} = L_{j-(2l+1)}$.

PROOF Suppose $L_j = I$. The case where $L_j = S$ is similar, and will be omitted. Note that

$$I = L_j = \frac{\alpha + L_{j-2(k+1)}}{1 + L_{j-(2l+1)}}$$

and so

$$I + IL_{j-(2l+1)} = \alpha + L_{j-2(k+1)}.$$

Thus

$$0 \geq I - L_{j-2(k+1)} = \alpha - IL_{j-(2l+1)} = I(S - L_{j-(2l+1)}) \geq 0$$

and hence

$$I = L_{j-2(k+1)} \qquad \text{and} \qquad S = L_{j-(2l+1)}.$$ ∎

LEMMA 5.4

Let $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ be a positive solution of Eq.(5.11), and let $\{L_n\}_{n=-\infty}^{\infty}$ be a full limiting sequence of $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ such that $L_0 = I$. For every pair of non-negative integers $r \geq 0$ and $s \geq 0$, set

$$I_{(r,s)} = L_{-[(2l+1)2r+2(k+1)s]} \qquad \text{and} \qquad S_{(r,s)} = L_{-[(2l+1)(2r+1)+2(k+1)s]}.$$

Then for every pair of non-negative integers $r \geq 0$ and $s \geq 0$,

$$I = I_{(r,s)} \qquad \text{and} \qquad S = S_{(r,s)}.$$

PROOF The proof is by double induction on r and s.

For every integer $s \geq 0$, let $\mathcal{P}(s)$ be the following proposition:

For every integer $r \geq 0$, let $\mathcal{Q}_s(r)$ be the following proposition:

$$I = I_{(r,s)} \qquad \text{and} \qquad S = S_{(r,s)}.$$

We first show that $\mathcal{P}(0)$ is true.

We shall first show that $\mathcal{Q}_0(0)$ is true.
Now $I = L_0$. Hence

$$I = L_{-[(2l+1)2\cdot 0+2(k+1)\cdot 0]} = I_{(0,0)}.$$

Since $I = L_0$, we have by Lemma 5.3 that $S = L_{-(2l+1)}$, and hence that

$$S = L_{-(2l+1)} = L_{-[(2l+1)(2\cdot 0+1)+2(k+1)\cdot 0]} = S_{(0,0)}$$

Thus $\mathcal{Q}_0(0)$ is true.

Suppose $r \geq 0$, and $\mathcal{Q}_0(r)$ is true. We shall show that $\mathcal{Q}_0(r+1)$ is true.
Now, by $\mathcal{Q}_0(r)$,

$$S = L_{-[(2l+1)(2r+1)+2(k+1)\cdot 0]} = L_{-(2l+1)(2r+1)}$$

and so by Lemma 5.3,

$$I = L_{-[(2l+1)(2r+1)+(2l+1)]} = L_{-(2l+1)2(r+1)} = L_{-[(2l+1)2(r+1)+2(k+1)\cdot 0]} = I_{(r+1,0)}.$$

Also, $I = L_{-(2l+1)2(r+1)}$, and so by Lemma 5.3,

$$S = L_{-[(2l+1)2(r+1)+(2l+1)]} = L_{-[(2l+1)2((r+1)+1)+2(k+1)\cdot 0]} = S_{(r+1,0)}.$$

Thus we see that $\mathcal{Q}_0(r+1)$ is true, and so $\mathcal{P}(0)$ is true.

Next suppose that $s \geq 0$ is a non-negative integer, and that $\mathcal{P}(s)$ is true. We shall show that $\mathcal{P}(s+1)$ is true.

We shall first show that $\mathcal{Q}_{s+1}(0)$ is true. Now $\mathcal{Q}_s(0)$ is true, and so

$$I = L_{-[(2l+1)(2\cdot 0)+2(k+1)s]} = L_{-2(k+1)s}$$

and hence by Lemma 5.3, we see that

$$I = L_{-[2(k+1)s+2(k+1)]} = L_{-2(k+1)(s+1)} = I_{(0,s+1)}.$$

In particular, by Lemma 5.3 we see that

$$S = L_{-[2(k+1)(s+1)+(2l+1)]} = L_{-(2l+1)(2\cdot 0+1)-2(k+1)(s+1)} = S_{(0,s+1)}$$

and so $\mathcal{Q}_{s+1}(0)$ is true.

Finally, suppose that $r \geq 0$ is a non-negative integer and that $\mathcal{Q}_{s+1}(r)$ is true. We shall show that $\mathcal{Q}_{s+1}(r+1)$ is true.

Now since $\mathcal{Q}_s(r+1)$ is true, we see that

$$I = L_{-[(2l+1)2(r+1)+2(k+1)s]}$$

and

$$S = L_{-[(2l+1)(2(r+1)+1)+2(k+1)s]}.$$

Hence by Lemma 5.3

$$S = L_{-[(2l+1)(2(r+1)+1)+2(k+1)s+2(k+1)]} = L_{-[(2l+1)(2(r+1)+1)+2(k+1)(s+1)]}$$
$$= S_{(r+1,s+1)}.$$

Also, as $\mathcal{Q}_{s+1}(r)$ is true,

$$S = L_{-[(2l+1)(2r+1)+2(k+1)(s+1)]}$$

and so by Lemma 5.3 we have

$$I = L_{-[(2l+1)(2r+1)+2(k+1)(s+1)+(2l+1)]} = L_{-[(2l+1)(2(r+1))+2(k+1)(s+1)]}$$
$$= I_{(r+1,s+1)}$$

and so $\mathcal{Q}_{s+1}(r+1)$ is true. $\square$

5.2.2 Period-2 Solutions

In this section we consider the case where the greatest common divisor of $k+1$ and $2l+1$ is 1, and we show that every positive solution of Eq.(5.11) converges to a periodic solution of Eq.(5.11) with period-2, and that there exist periodic solutions of Eq.(5.11) with prime period-2.

Note that the periodic solutions of Eq.(5.11) with prime period-2 are the sequences

$$\ldots, \phi, \frac{\alpha}{\phi}, \phi, \frac{\alpha}{\phi}, \ldots$$

with $\phi > 0$ and $\phi \neq \sqrt{\alpha}$.

LEMMA 5.5

Let k and l be non-negative integers such that $d = \gcd(k+1, 2l+1) = 1$. Let $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ be a positive solution of Eq.(5.11), and suppose that $\{L_n\}_{n=-\infty}^{\infty}$ is a full limiting sequence of $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ such that $L_0 \in \{I, S\}$. Then for all $n \in \{\ldots, -1, 0, 1 \ldots\}$,

$$L_{2n} = L_0 \qquad \textit{and} \qquad L_{2n+1} = \frac{\alpha}{L_0}.$$

PROOF Suppose that $L_0 = I$. The proof when $L_0 = S$ is similar and will be omitted.

Claim: There exists integers $N \leq 0$ and $j_0 \geq \mathcal{K} + 2$ such that

$$L_N = I, L_{N-1} = S, L_{N-2} = I, L_{N-3} = S, \ldots, L_{N-j_0+1} = I, L_{N-j_0} = S.$$

Proof of the claim: As $k+1$ and $2l+1$ are relatively prime, there exists $\delta \in \{-1, 1\}$ and positive integers μ and ν such that

$$\mu(k+1) - \nu(2l+1) = \delta.$$

For non-negative integers $r \geq 0$ and $s \geq 0$, recall the definition and properties from Lemma 5.4 of

$$I_{(r,s)} = L_{-[(2l+1)2r+2(k+1)s]} \qquad \text{and} \qquad S_{(r,s)} = L_{-[(2l+1)(2r+1)+2(k+1)s]}.$$

Set $a = \max\{k+1, 2l+1\}$. Now it follows by Lemma 5.4 that for all $r \geq 0$ and $s \geq 0$, we have

$$I_{(r,s)} = I \qquad \text{and} \qquad S_{(r,s)} = S$$

and that

$$\begin{aligned}
L_{-(2l+1)2a\nu} &= I_{(a\nu,0)} \\
L_{-[(2l+1)2a\nu+2\delta]} &= L_{-[(2l+1)2a\nu+2\mu(k+1)-2\nu(2l+1)]} \\
&= L_{-[(2l+1)2(a-1)\nu+2(k+1)\mu]} \\
&= I_{[(a-1)\nu,\mu]} \\
&\vdots \\
L_{-[(2l+1)2a\nu+2(a-1)\delta]} &= L_{-[(2l+1)2a\nu+2(a-1)\mu(k+1)-2(a-1)\nu(2l+1)]} \\
&= L_{-[(2l+1)2\nu+2(k+1)(a-1)\mu]} \\
&= I_{[\nu,(a-1)\mu]} \\
L_{-[(2l+1)2a\nu+2a\delta]} &= L_{-[(2l+1)2a\nu+2a\mu(k+1)-2a\nu(2l+1)]} \\
&= I_{-[(2l+1)2\cdot 0\cdot\nu+2(k+1)a\mu]} \\
&= I_{(0,a\mu)}.
\end{aligned}$$

Thus we see that

$$I = L_{-(2l+1)2a\nu} = L_{-[(2l+1)2a\nu+2\delta]} = \cdots = L_{-[(2l+1)2a\nu+2a\delta]}.$$

It also follows by Lemma 5.4 that

$$\begin{aligned}
L_{-(2l+1)(2a\nu+1)} &= S_{(a\nu,0)} \\
L_{-[(2l+1)(2a\nu+1)+2\delta)]} &= L_{-[(2l+1)(2a\nu+1)+2\mu(k+1)-2\nu(2l+1)]} \\
&= L_{-[(2l+1)(2(a-1)\nu+1)+2(k+1)\mu]} \\
&= S_{[(a-1)\nu,\mu]} \\
&\vdots \\
L_{-[(2l+1)(2a\nu+1)+2(a-1)\delta]} &= L_{-[(2l+1)(2a\nu+1)+2(a-1)\mu-2(a-1)\nu(2l+1)]} \\
&= L_{-[(2l+1)(2\nu+1)+2(k+1)\mu(a-1)]} \\
&= S_{[\nu,(a-1)\mu]} \\
L_{-[(2l+1)(2a\nu+1)+2a\delta]} &= L_{-[(2l+1)(2a\nu+1)+2a\mu(k+1)-2a\nu(2l+1))]} \\
&= L_{-[(2l+1)(2\cdot 0\cdot\nu+1)+2a\mu(k+1)]} \\
&= S_{[0,(k+1)\mu]}
\end{aligned}$$

and so we see that it is also the case that

$$S = L_{-(2l+1)(2a\nu+1)} = L_{-[(2l+1)(2a\nu+1)+2\delta]} = \cdots = L_{-[(2l+1)(2a\nu+1)+2a\delta]}.$$

Suppose $\delta = 1$. Then

$$\begin{aligned}
I &= L_{-(2l+1)2a\nu} \\
I &= L_{-(2l+1)2a\nu-2} \\
&\vdots \\
I &= L_{-(2l+1)2a\nu-2l} \\
S &= L_{-(2l+1)2a\nu-(2l+1)} \\
I &= L_{-(2l+1)2a\nu-2(l+1)} \\
S &= L_{-(2l+1)2a\nu-[(2l+1)+2]} \\
I &= L_{-(2l+1)2a\nu-2(l+2)} \\
S &= L_{-(2l+1)2a\nu-[(2l+1)+4]} \\
&\vdots \\
I &= L_{-(2l+1)2a\nu-2a} \\
S &= L_{-(2l+1)2a\nu-(2a+1)} \\
S &= L_{-(2l+1)2a\nu-(2a+3)} \\
&\vdots \\
S &= L_{-(2l+1)2a\nu-[2a+(2l+1)]}.
\end{aligned}$$

In this case, we take $N = -(2l+1)2a\nu - 2l$ and $j_0 = 2a + 1 - 2l$. Note that

$$j_0 = 2a + 1 - 2l = a + [a - (2l - 1)] \geq \mathcal{K} + 2.$$

Suppose $\delta = -1$. Then

$$\begin{aligned}
I &= L_{-(2l+1)2a\nu+2a} \\
I &= L_{-(2l+1)2a\nu+2(a-1)} \\
&\vdots \\
I &= L_{-(2l+1)2a\nu-2l+2a} \\
S &= L_{-(2l+1)2a\nu-(2l+1)+2a} \\
I &= L_{-(2l+1)2a\nu-2l+2(a-1)} \\
S &= L_{-(2l+1)2a\nu-(2l+1)+2(a-1)} \\
I &= L_{-(2l+1)2a\nu-2l+2(a-2)} \\
&\vdots \\
I &= L_{-(2l+1)2a\nu} \\
S &= L_{-(2l+1)2a\nu-1} \\
S &= L_{-(2l+1)2a\nu-3} \\
&\vdots \\
S &= L_{-(2l+1)2a\nu-(2l+1)+2} \\
S &= L_{-(2l+1)2a\nu-(2l+1)}.
\end{aligned}$$

Here we take $N = -(2l+1)2a\nu - 2l + 2a$, and once again we take $j_0 = 2a+1-2l$. As before, $j_0 \geq \mathcal{K} + 2$.

Thus the claim is correct.

Note that

$$L_{N+1} = \frac{\alpha + L_{N-(2k+1)}}{1 + L_{N-2l}} = \frac{\alpha + S}{1 + I} = S$$

and

$$L_{N+2} = \frac{\alpha + L_{N-2k}}{1 + L_{N-(2l-1)}} = \frac{\alpha + I}{1 + S} = I.$$

It follows by induction, both forwards and backwards, that the proof is complete. ∎

LEMMA 5.6
Let k and l be non-negative integers such that $\gcd(k+1, 2l+1) = 1$, and suppose that $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ is a positive solution of Eq.(5.11). Let $\mathcal{L}_0$ be a limit point of $\{x_n\}_{n=-\mathcal{K}}^{\infty}$. Then $\mathcal{L}_0 \in \{I, S\}$.

PROOF For the sake of contradiction, suppose that $\mathcal{L}_0 \notin \{I, S\}$. Then we have $I < \mathcal{L}_0 < S$, and so, in particular, $I < S$.

Let $\{L_n\}_{n=-\infty}^{\infty}$ be a full limiting sequence of $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ such that $L_0 = \mathcal{L}_0$. It follows by Lemma 5.5 that $L_n \notin \{I, S\}$ for all $n \in \{\ldots, -1, 0, 1 \ldots\}$. Now $S = \frac{\alpha}{I}$ by Lemma 5.2, and so there exists $\varepsilon > 0$ such that

$$L_j \in \left(I + \varepsilon, \frac{\alpha}{I + \varepsilon}\right) \qquad \text{for all} \qquad -\mathcal{K} \le j \le 0.$$

By Theorem 1.8, there exists a subsequence $\{x_{n_i}\}_{n=0}^{\infty}$ of $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ such that

$$\lim_{i\to\infty} x_{n_i+j} = L_j \qquad \text{for all} \qquad -\mathcal{K} \le j.$$

It follows that there exists $N \ge 0$ such that

$$x_{N-\mathcal{K}}, x_{N-(\mathcal{K}-1)}, \ldots, x_N \in \left(I + \varepsilon, \frac{\alpha}{I + \varepsilon}\right)$$

and hence by Lemma 5.1 that

$$I \in \left[I + \varepsilon, \frac{\alpha}{I + \varepsilon}\right].$$

This is a contradiction, and the proof is complete. ∎

THEOREM 5.4

Let k and l be non-negative integers such that $\gcd(k+1, 2l+1) = 1$. *Then every positive solution of Eq.(5.11) converges to a periodic solution of Eq.(5.11) with period-2, and there exist periodic solutions of Eq.(5.11) with prime period-2.*

PROOF Note that the periodic solutions of Eq.(5.11) with period-2 are the sequences

$$\phi, \psi, \phi, \psi, \ldots$$

with $\phi, \psi \in (0, \infty)$ such that $\phi\psi = \alpha$. Thus there do exist periodic solutions of Eq.(5.11) with prime period-2.

Let $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ be a positive solution of Eq.(5.11). It suffices to show that $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ converges to a periodic solution of Eq.(5.11) with period-2.

If $I = S$, then $\lim_{n\to\infty} x_n = \bar{x}$, and the proof is complete. So without loss of generality, suppose $I < S$.

Now $IS = \alpha$ and $I < S$. Thus $0 < I < \sqrt{\alpha} < S$.

By Theorem 1.8, there exists a full limiting sequence $\{L_n\}_{n=-\infty}^{\infty}$ of $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ with $L_0 = I$ and a subsequence $\{x_{n_i}\}_{i=0}^{\infty}$ such that

$$\lim_{i\to\infty} x_{n_i+j} = L_j \qquad \text{for all} \qquad j \geq -\mathcal{K}.$$

It follows by Lemma 5.5 that there exists $i_0 \geq 0$ such that

$$x_{n_{i_0}-2r} < \sqrt{\alpha} \qquad \text{for all} \qquad r \geq 0 \quad \text{such that } 0 \leq 2r \leq \mathcal{K}$$

and

$$x_{n_{i_0}-(2r+1)} > \sqrt{\alpha} \qquad \text{for all} \qquad r \geq 0 \quad \text{such that } 0 \leq 2r+1 \leq \mathcal{K}.$$

Now

$$x_{n_{i_0}+1} = \frac{\alpha + x_{n_{i_0}-(2k+1)}}{1 + x_{n_{i_0}-2l}} > \frac{\alpha + \sqrt{\alpha}}{1 + \sqrt{\alpha}} = \sqrt{\alpha}$$

and

$$x_{n_{i_0}+2} = \frac{\alpha + x_{n_{i_0}-2k}}{1 + x_{n_{i_0}-(2l-1)}} < \frac{\alpha + \sqrt{\alpha}}{1 + \sqrt{\alpha}} = \sqrt{\alpha}.$$

It follows by induction that for all $n \geq 0$,

$$x_{n_{i_0}+2n} < \sqrt{\alpha} \qquad \text{and} \qquad x_{n_{i_0}+2n+1} > \sqrt{\alpha}.$$

Let $\mathcal{L}$ be a limit point of $\{x_{n_{i_0}+2n}\}_{n=0}^{\infty}$. Then $\mathcal{L} \leq \sqrt{\alpha}$. So, as I and S are the only limit points of $\{x_n\}_{n=\mathcal{K}}^{\infty}$ and $I < \sqrt{\alpha} < S$, we see that $\mathcal{L} = I$. Thus

$\{x_{n_{i_0}+2n}\}_{n=0}^{\infty}$ is a bounded sequence with the single limit point I, and so it follows that

$$\lim_{n\to\infty} x_{n_{i_0}+2n} = I.$$

We similarly have $\lim_{n\to\infty} x_{n_{i_0}+2n+1} = S$, and so the proof is complete. ∎

5.2.3 Period-2d Solutions

We are now ready for the main result of this section.

THEOREM 5.5
Let k and l be non-negative integers such that $\gcd(k+1, 2l+1) = d$. Then every positive solution of Eq.(5.11) converges to a periodic solution of Eq.(5.11) with period 2d, and there exist periodic solutions of Eq.(5.11) with prime period 2d.

PROOF Let $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ be a positive solution of Eq.(5.11). We shall show that $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ converges to a periodic solution of Eq.(5.11) with period $2d$.

There exist non-negative integers k' and l' with $\gcd(k'+1, 2l'+1) = 1$ such that

$$k+1 = d(k'+1) \qquad \text{and} \qquad 2l+1 = d(2l'+1).$$

Hence for all $0 \le j \le d-1$ and $m \ge 0$,

$$\begin{aligned} x_{d(m+1)+j} &= \frac{\alpha + x_{d(m+1)+j-1-(2k+1)}}{1 + x_{d(m+1)+j-1-2l}} = \frac{\alpha + x_{d(m+1)+j-2(k+1)}}{1 + x_{d(m+1)+j-(2l+1)}} \\ &= \frac{\alpha + x_{d[(m+1)-2(k'+1)]+j}}{1 + x_{d[(m+1)-(2l'+1)]+j}} = \frac{\alpha + x_{d[m-(2k'+1)]+j}}{1 + x_{d[m-2l']+j}}. \end{aligned}$$

Set $\mathcal{K}' = \max\{k'+1, 2l'+1\}$, and for each $j = 0, 1, \ldots, d-1$, let

$$y_m^{(j)} = x_{dm+j} \qquad \text{for} \qquad m \ge -\mathcal{K}.$$

Then for each $j = 0, 1, \ldots, d-1$, $\{y_m^{(j)}\}_{m=-\mathcal{K}'}^{\infty}$ is a solution of the difference equation

$$y_{m+1} = \frac{\alpha + y_{m-(2k'+1)}}{1 + y_{m-2l'}} \quad , \qquad m = 0, 1, \ldots. \tag{5.12}$$

So as $k'+1$ and $2l'+1$ are relatively prime, it follows by Theorem 5.4 that $\{y_m^{(j)}\}_{m=-\mathcal{K}'}^{\infty}$ converges to a periodic solution of Eq.(5.12) with period-2, from which the proof follows.

Finally, observe that the periodic solutions of Eq.(5.11) with period $2d$ are the sequences

$$\phi_0, \phi_1, \dots, \phi_{d-1}, \psi_0, \psi_1, \dots, \psi_{d-1}, \phi_0, \phi_1, \dots, \phi_{d-1}, \psi_0, \psi_1, \dots, \psi_{d-1}, \dots$$

where for $0 \leq j \leq d-1$, we have

$$\phi_j, \psi_j \in (0, \infty) \qquad \text{and} \qquad \phi_j \psi_j = \alpha.$$

Thus there do exist periodic solutions of Eq.(5.11) with prime period $2d$. □

REMARK 5.4 Let k and l be non-negative integers such that $\gcd(k+1, 2l+1) = d$. It is interesting to note that it follows from the proof of Theorem 5.5 that there exists a periodic solution of Eq.(5.11) with prime period P if and only if $P = 2d'$ for some divisor d' of d. □

5.3 The Equation $x_{n+1} = \dfrac{\alpha + x_{n-(2k+1)}}{A + x_{n-2l}}$

In this section we study the global behavior of the difference equation

$$x_{n+1} = \frac{\alpha + x_{n-(2k+1)}}{A + x_{n-2l}} \quad , \qquad n = 0, 1, \dots \tag{5.13}$$

where k and l are non-negative integers, α and A are non-negative parameters, and the initial conditions are non-negative real numbers chosen such that the denominator in Eq.(5.13) is always positive. The study of the asymptotic character of the solutions of Eq.(5.13) will be instrumental in the proof of Theorem 5.3.

The case $k = 0$ and $l = 1$ was investigated in [10] and [23].

5.3.1 Local Stability Character of the Equilibrium Point

Eq.(5.13) has the non-negative equilibrium point

$$\bar{x} = \frac{(1-A) + \sqrt{(1-A)^2 + 4\alpha}}{2}.$$

$\bar{x}$ is the only equilibrium point of Eq.(5.13) when α is positive, or when α is zero and $A \geq 1$.

When $\alpha = 0$ and $0 < A < 1$, then in addition to the positive equilibrium point $\bar{x}$, zero is also an equilibrium point of Eq.(5.13).

THEOREM 5.6
Suppose $\alpha = 0$ and $0 < A < 1$. Then the zero equilibrium point $\tilde{x} = 0$ is an unstable equilibrium point of Eq.(5.13).

PROOF The linearized equation of Eq.(5.13) about $\tilde{x} = 0$ is

$$z_{n+1} - \frac{1}{A} z_{n-(2k+1)} + 0 \cdot z_{n-2l} = 0 \quad , \quad n = 0, 1, \ldots$$

with characteristic equation

$$\lambda^{\mathcal{K}+1} - \frac{1}{A}\lambda^{\mathcal{K}-(2k+1)} + 0 \cdot \lambda^{\mathcal{K}-2l} = 0. \tag{5.14}$$

The result follows by Theorem 1.1. ∎

THEOREM 5.7
The following statements are true:

1. *Suppose $1 < A$. Then $\bar{x}$ is a locally asymptotically stable equilibrium point of Eq.(5.13).*
2. *Suppose $0 \leq A < 1$. Then $\bar{x}$ is an unstable equilibrium point of Eq.(5.13) whose characteristic equation has at least one root with modulus less than 1 and at least one root with modulus greater than 1.*

PROOF The linearized equation of Eq.(5.13) about $\bar{x}$ is

$$z_{n+1} - \frac{1}{A + \bar{x}} z_{n-(2k+1)} + \frac{\bar{x}}{A + \bar{x}} z_{n-2l} = 0 \quad , \quad n = 0, 1, \ldots$$

with characteristic equation

$$\lambda^{\mathcal{K}+1} - \frac{1}{A + \bar{x}}\lambda^{\mathcal{K}-(2k+1)} + \frac{\bar{x}}{A + \bar{x}}\lambda^{\mathcal{K}-2l} = 0. \tag{5.15}$$

It follows by Theorem 1.6 that $\bar{x}$ is a locally asymptotically stable equilibrium point of Eq.(5.13) when $1 < A$.

When $0 \leq A < 1$, it is easy to see that Eq.(5.15) has a root in $(-\infty, -1)$. On the other hand, the product of the roots of Eq.(5.15) has modulus less than 1, and so Eq.(5.15) also has a root with modulus less than 1. ∎

5.3.2 The Case $\alpha = A = 0$

THEOREM 5.8
Suppose $\alpha = A = 0$. Then there exist positive solutions of Eq.(5.13) which are neither bounded nor persist.

PROOF The substitution $x_n = e^{z_n}$ transforms Eq.(5.13) into the linear equation

$$z_{n+1} + z_{n-2l} - z_{n-(2k+1)} = 0 \quad , \quad n = 0, 1, \dots \tag{5.16}$$

with characteristic equation

$$\lambda^{\mathcal{K}+1} + \lambda^{\mathcal{K}-2l} - \lambda^{\mathcal{K}-(2k+1)} = 0. \tag{5.17}$$

It is easy to see that Eq.(5.17) has a negative root $\lambda_1 < -1$, from which the result follows. ∎

5.3.3 The Case $0 < \alpha$ and $1 < A$

THEOREM 5.9
Suppose $0 < \alpha$ and $1 < A$. Then the equilibrium point $\bar{x}$ is a globally asymptotically stable equilibrium point of Eq.(5.13) with basin of attraction the positive solutions of Eq.(5.13).

PROOF We know by Theorem 5.7 that $\bar{x}$ is a locally asymptotically stable equilibrium point of Eq.(5.13), and so it suffices to show that $\bar{x}$ is a global attractor of Eq.(5.13). So let $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ be a positive solution of Eq.(5.13). It suffices to show that $\lim_{n\to\infty} x_n = \bar{x}$.

Set $I = \liminf_{n\to\infty} x_n$ and $S = \limsup_{n\to\infty} x_n$. Note that for $n \geq 0$

$$x_{n+1} = \frac{\alpha + x_{n-(2k+1)}}{A + x_{n-2l}} < \frac{\alpha}{A} + \frac{1}{A} x_{n-(2k+1)}$$

and so as $1 < A$, it follows by Theorem 1.7 that there exists $B > 0$ such that $0 < x_n \leq B$ for all $n \geq -\mathcal{K}$. Hence for $n \geq 0$ we also have

$$x_{n+1} = \frac{\alpha + x_{n-(2k+1)}}{A + x_{n-2l}} > \frac{\alpha}{A + B}$$

and so $0 < I \leq S < \infty$. It suffices to show that $I = S$.

By Theorem 1.8, there exist solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of the difference equation

$$z_{n+1} = \frac{\alpha + z_{n-(2k+1)}}{A + z_{n-2l}} \quad , \quad n \in \{\ldots, -1, 0, 1, \ldots\}$$

with $I_0 = I$ and $S_0 = S$ such that for all $n \in \{\ldots, -1, 0, 1, \ldots\}$, we have

$$I \leq I_n \leq S \quad \text{and} \quad I \leq S_n \leq S.$$

Note that

$$I = I_0 = \frac{\alpha + I_{-(2k+2)}}{A + I_{-(2l+1)}} \geq \frac{\alpha + I}{A + S} \quad \text{and} \quad S = S_0 = \frac{\alpha + S_{-(2k+1)}}{A + S_{-(2l+1)}} \leq \frac{\alpha + S}{A + I}.$$

Hence

$$\alpha + I - AI \leq IS \leq \alpha + S - AS$$

and thus

$$(A-1)S \leq (A-1)I.$$

So as $1 < A$, we have $I = S$, as was to be shown. ∎

5.3.4 The Case $0 < \alpha$ and $0 \leq A < 1$

In this section we show that when $0 < \alpha$ and $0 \leq A < 1$, there exist positive solutions of Eq.(5.13) which are neither bounded nor persist.

The proof of Lemma 5.7 is straightforward and will be omitted.

LEMMA 5.7
Suppose $0 < \alpha$ *and* $0 \leq A < 1$. *Then the following statements are true:*

1. *Suppose* $(1 - A) + \dfrac{\alpha}{1 - A} < x$. *Then* $0 < \dfrac{\alpha}{x - (1 - A)} < 1 - A$.

2. *Suppose* $1 - A < x$ *and* $\dfrac{\alpha}{x - (1 - A)} < y$. *Then* $\dfrac{\alpha + y}{A + x} < y$.

THEOREM 5.10
Suppose $0 < \alpha$ *and* $0 \leq A < 1$. *Then there exist positive solutions of Eq.(5.13) which are neither bounded nor persist.*

PROOF Recall that $\mathcal{K} = \max\{2k + 1, 2l\}$.

Let r be the largest non-negative integer such that $0 \leq 2r + 1 \leq \mathcal{K}$, let s be the largest non-negative integer such that $0 \leq 2s \leq \mathcal{K}$, and let $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ be a solution of Eq.(5.13) such that

$$(1-A) + \frac{\alpha}{1-A} < x_{-2r-1} < x_{-2r+1} < \cdots < x_{-3} < x_{-1}$$

and

$$\frac{\alpha}{x_{-2r-1} - (1-A)} < x_0 < x_{-2} < \cdots < x_{-2s+2} < x_{-2s} < (1-A).$$

Note that such a choice of initial conditions is possible by Lemma 5.7.

We claim that for each integer j with $0 \le j \le k$,

$\{x_{2j-2k-1+n(2k+2)}\}_{n=0}^{\infty}$ is a monotonically increasing subsequence of $\{x_n\}_{n=-\mathcal{K}}^{\infty}$

and

$\{x_{2j-2k+n(2k+2)}\}_{n=0}^{\infty}$ is a monotonically decreasing subsequence of $\{x_n\}_{n=-\mathcal{K}}^{\infty}$.

Proof of the claim:

Case 1. Suppose $0 \le k < l$. Then

$$x_1 = \frac{\alpha + x_{-2k-1}}{A + x_{-2l}} > \alpha + x_{-2k-1} > x_{-2k-1}$$

because $0 < A + x_{-2l} < 1$

$$x_2 = \frac{\alpha + x_{-2k}}{A + x_{-2l+1}} < x_{-2k}$$

because $(1-A) + \dfrac{\alpha}{1-A} < x_{-2l+1}$

and $\dfrac{\alpha}{x_{-2l+1} - (1-A)} < x_{-2k}$

If $k \ge 1$, then

$$x_3 = \frac{\alpha + x_{-2k+1}}{A + x_{-2l+2}} > \alpha + x_{-2k+1} > x_{-2k+1}$$

because $0 < A + x_{-2l+2} < 1$

$$x_4 = \frac{\alpha + x_{-2k+2}}{A + x_{-2l+3}} < x_{-2k+2}$$

$$\text{because} \quad (1-A) + \frac{\alpha}{1-A} < x_{-2l+3}$$

$$\text{and} \quad \frac{\alpha}{x_{-2l+3} - (1-A)} < x_{-2k+2}$$

$$\vdots$$

$$x_{2k-1} = \frac{\alpha + x_{-3}}{A + x_{2k-2l-2}} > \alpha + x_{-3} > x_{-3}$$

$$\text{because} \quad 0 < A + x_{2k-2l-2} < 1$$

$$x_{2k} = \frac{\alpha + x_{-2}}{A + x_{2k-2l-1}} < x_{-2}$$

$$\text{because} \quad (1-A) + \frac{\alpha}{1-A} < x_{2k-2l-1}$$

$$\text{and} \quad \frac{\alpha}{x_{2k-2l-1} - (1-A)} < x_{-2}.$$

Thus in any event, for $k \geq 0$ we have

$$x_{2k+1} = \frac{\alpha + x_{-1}}{A + x_{2k-2l}} > \alpha + x_{-1} > x_{-1}$$

$$\text{because} \quad 0 < A + x_{2k-2l} < 1$$

$$x_{2k+2} = \frac{\alpha + x_0}{A + x_{2k-2l+1}} < x_0$$

$$\text{because} \quad (1-A) + \frac{\alpha}{1-A} < x_{2k-2l+1}$$

$$\text{and} \quad \frac{\alpha}{x_{2k-2l+1} - (1-A)} < x_0.$$

It follows that

$$x_{2k+3} = \frac{\alpha + x_1}{A + x_{2k-2l+2}} > \frac{\alpha + x_{-2k-1}}{A + x_{-2l}} = x_1$$

$$x_{2k+4} = \frac{\alpha + x_2}{A + x_{2k-2l+3}} < \frac{\alpha + x_{-2k}}{A + x_{-2l+1}} = x_2$$

$$\vdots$$

$$x_{4k+3} = \frac{\alpha + x_{2k+1}}{A + x_{4k-2l+2}} > \frac{\alpha + x_{-1}}{A + x_{2k-2l}} = x_{2k+1} > x_{-1}$$

$$x_{4k+4} = \frac{\alpha + x_{2k+2}}{A + x_{4k-2l+3}} < \frac{\alpha + x_0}{A + x_{2k-2l+1}} = x_{2k+2} < x_0.$$

The proof of the claim in this case follows by induction.

Case 2. Suppose $0 = l \leq k$.

$$x_1 = \frac{\alpha + x_{-2k-1}}{A + x_0} > \alpha + x_{-2k-1} > x_{-2k-1}$$

because $0 < A + x_0 < 1$

$$x_2 = \frac{\alpha + x_{-2k}}{A + x_1} < x_{-2k}$$

because $(1 - A) + \dfrac{\alpha}{1 - A} < x_{-2k-1} < x_1$

and $\dfrac{\alpha}{x_1 - (1 - A)} < \dfrac{\alpha}{x_{-2k-1} - (1 - A)} < x_{-2k}.$

If $k \geq 1$,

$$x_3 = \frac{\alpha + x_{-2k+1}}{A + x_2} > \alpha + x_{-2k+1} > x_{-2k+1}$$

because $0 < A + x_2 < A + x_{-2k} < 1$

$$x_4 = \frac{\alpha + x_{-2k+2}}{A + x_3} < x_{-2k+2}$$

because $(1 - A) + \dfrac{\alpha}{1 - A} < x_{-2k+1} < x_3$

and $\dfrac{\alpha}{x_3 - (1 - A)} < \dfrac{\alpha}{x_{-2k+1} - (1 - A)} < x_{-2k+2}.$

If $k \geq 2$,

$$x_5 = \frac{\alpha + x_{-2k+3}}{A + x_4} > \alpha + x_{-2k+3} > x_{-2k+3}$$

because $0 < A + x_4 < A + x_{-2k+2} < 1$

$$x_6 = \frac{\alpha + x_{-2k+4}}{A + x_5} < x_{-2k+4}$$

because $(1 - A) + \dfrac{\alpha}{1 - A} < x_{-2k+3} < x_5$

and $\dfrac{\alpha}{x_5 - (1 - A)} < \dfrac{\alpha}{x_{-2k+3} - (1 - A)} < x_{-2k+4}$

$$\vdots$$

$$x_{2k-1} = \frac{\alpha + x_{-3}}{A + x_{2k-2}} > \alpha + x_{-3} > x_{-3}$$

because $0 < A + x_{2k-2} < A + x_{-4} < 1$

$$x_{2k} = \frac{\alpha + x_{-2}}{A + x_{2k-1}} < x_{-2}$$

because

$(1 - A) + \dfrac{\alpha}{1 - A} < x_{-3} < x_{2k-1}$

and $\dfrac{\alpha}{x_{2k-1} - (1 - A)} < \dfrac{\alpha}{x_{-3} - (1 - A)} < x_{-2}.$

$$x_{2k+1} = \frac{\alpha + x_{-1}}{A + x_{2k}} > \alpha + x_{-1} > x_{-1}$$

because $0 < A + x_{2k} < A + x_{-2} < 1$

$$x_{2k+2} = \frac{\alpha + x_0}{A + x_{2k+1}} < x_0$$

because $(1 - A) + \dfrac{\alpha}{1 - A} < x_{-1} < x_{2k+1}$

and $\dfrac{\alpha}{x_{2k+1} + (1 - A)} < \dfrac{\alpha}{x_{-1} - (1 - A)} < x_0.$

Thus in any event, for $k \geq 0$ we have

$$x_{2k+3} = \frac{\alpha + x_1}{A + x_{2k+2}} > \frac{\alpha + x_{-2k-1}}{A + x_0} = x_1$$

$$x_{2k+4} = \frac{\alpha + x_2}{A + x_{2k+3}} < \frac{\alpha + x_{-2k}}{A + x_1} = x_2$$

$$\vdots$$

$$x_{4k+3} = \frac{\alpha + x_{2k+1}}{A + x_{4k+2}} > \frac{\alpha + x_{-1}}{A + x_{2k}} = x_{2k+1} > x_{-1}$$

$$x_{4k+4} = \frac{\alpha + x_{2k+2}}{A + x_{4k+3}} < \frac{\alpha + x_0}{A + x_{2k+1}} = x_{2k+2} < x_0.$$

The proof of the claim in this case follows by induction.

Case 3. Suppose $0 < l \leq k$.

There exist integers m and i with $m \geq 1$ and $0 \leq i < l$ such that

$$k = ml + i.$$

We have

$$x_1 = \frac{\alpha + x_{-2k-1}}{A + x_{-2l}} > \alpha + x_{-2k-1} > x_{-2k-1}$$

because $0 < A + x_{-2l} < 1$

$$x_2 = \frac{\alpha + x_{-2k}}{A + x_{-2l+1}} < x_{-2k}$$

because $(1 - A) + \dfrac{\alpha}{1 - A} < x_{-2l+1}$

and $\dfrac{\alpha}{x_{-2l+1} - (1 - A)} < x_{-2k}.$

$$\vdots$$

$$x_{2l-1} = \frac{\alpha + x_{2l-2k-3}}{A + x_{-2}} > \alpha + x_{2l-2k-3} > x_{2l-2k-3}$$

because $0 < A + x_{-2} < 1$

$$x_{2l} = \frac{\alpha + x_{2l-2k-2}}{A + x_{-1}} < x_{2l-2k-2}$$

because $(1-A) + \frac{\alpha}{1-A} < x_{-1}$

and $\frac{\alpha}{x_{-1} - (1-A)} < x_{2l-2k-2}$

$$x_{2l+1} = \frac{\alpha + x_{2l-2k-1}}{A + x_0} > \alpha + x_{2l-2k-1} > x_{2l-2k-1}$$

because $0 < A + x_0 < 1$

$$x_{2l+2} = \frac{\alpha + x_{2l-2k}}{A + x_1} < x_{2l-2k}$$

because $(1-A) + \frac{\alpha}{1-A} < x_{-2k-1} < x_1$

and $\frac{\alpha}{x_1 - (1-A)} < \frac{\alpha}{x_{-2k-1} - (1-A)} < x_{2l-2k}$

$$\vdots$$

$$x_{2(m-1)l+1} = \frac{\alpha + x_{2(m-1)l-2k-1}}{A + x_{2(m-2)l}} > \alpha + x_{2(m-1)l-2k-1} > x_{2(m-1)l-2k-1}$$

because

$$0 < A + x_{2(m-2)l} < 1 \quad \text{if} \quad 1 \leq m \leq 2$$

while if $m \geq 3$, then

$$4l + 2i + 2 \leq 6l \leq 2ml + 2i = 2k$$

and so

$$0 < A + x_{2(m-2)l} = A + x_{2k-4l-2i} < A + x_{-4l-2i-2} \leq A + x_{-2k} < 1.$$

Also,

$$x_{2(m-1)l+2} = \frac{\alpha + x_{2(m-1)l-2k}}{A + x_{2(m-2)l+1}} < x_{2(m-2)l-2k}$$

because if $1 \le m \le 2$, then

$$(1-A) + \frac{\alpha}{1-A} < x_{-2k-1} < x_{2(m-2)l+1}$$

and

$$\frac{\alpha}{x_{2(m-2)l+1} - (1-A)} < \frac{\alpha}{x_{-2k-1} - (1-A)} < x_{2(m-1)l-2k}$$

while if $m \ge 3$, then

$$(1-A) + \frac{\alpha}{1-A} < x_{-4l-2i-1} < x_{2k-4l-2i+1} = x_{2(m-2)l+1}$$

and

$$\frac{\alpha}{x_{2(m-2)l+1} - (1-A)} < \frac{\alpha}{x_{-4l-2i-1} - (1-A)} < x_{2(m-1)l-2k}$$

$$\vdots$$

$$x_{2ml-1} = \frac{\alpha + x_{2ml-2k-3}}{A + x_{2(m-1)l-2}} > \alpha + x_{2ml-2k-3} > x_{2ml-2k-3}$$

because if $m = 1$, or if $m = 2$ and $l = 1$, then

$$0 < A + x_{2(m-1)l-2} < 1$$

while otherwise

$$0 < A + x_{2(m-1)l-2} < A + x_{2(m-1)l-2k-4} = A + x_{-2l-2i-4} < 1$$

$$x_{2ml} = \frac{\alpha + x_{2ml-2k-2}}{A + x_{2(m-1)l-1}} < x_{2ml-2k-2}$$

because if $m = 1$, then

$$(1-A) + \frac{\alpha}{1-A} < x_{-1} = x_{2(m-1)l-1}$$

and

$$\frac{\alpha}{x_{2(m-1)l-1} - (1-A)} = \frac{\alpha}{x_{-1} - (1-A)} < x_{2ml-2k-2}$$

while if $m > 1$, then

$$(1-A) + \frac{\alpha}{1-A} < x_{-2l-2i-3} < x_{2k-2l-2i-1} = x_{2(m-1)l-1}$$

and

$$\frac{\alpha}{x_{2(m-1)l-1} - (1-A)} < \frac{\alpha}{x_{-2l-2i-3} - (1-A)} < x_{-2i-2} = x_{2ml-2k-2}.$$

In any event,

$$x_{2ml+1} = \frac{\alpha + x_{2ml-2k-1}}{A + x_{2(m-1)l}} > \alpha + x_{2ml-2k-1} > x_{2ml-2k-1}$$

because if $m = 1$, then

$$0 < A + x_{2(m-1)l} < 1$$

while if $m \geq 2$, then

$$0 < A + x_{2(m-1)l} < A + x_{2(m-1)l-2k-2} < 1$$

$$x_{2ml+2} = \frac{\alpha + x_{2ml-2k}}{A + x_{2(m-1)l+1}} < x_{2ml-2k}$$

because

$$1 - A + \frac{\alpha}{1-A} < x_{-2l-2i-1} < x_{2k-2l-2i+1} = x_{2(m-1)l+1}$$

and

$$\frac{\alpha}{x_{2(m-1)l+1} - (1-A)} < \frac{\alpha}{x_{-2l-2i1} - (1-A)} < x_{2ml-2k}$$

$$\vdots$$

$$x_{2k+1} = \frac{\alpha + x_{-1}}{A + x_{2k-2l}} > \alpha + x_{-1} > x_{-1}$$

because if $k = l$, then

$$0 < A + x_{2k-2l} = A + x_0 < 1$$

while if $k > l$, then

$$0 < A + x_{2k-2l} < A + x_{-2l-2} < 1$$

$$x_{2k+2} = \frac{\alpha + x_0}{A + x_{2k-2l+1}} < x_0$$

$$\text{because} \quad (1-A) + \frac{\alpha}{1-A} < x_{-2l-1} < x_{2k-2l+1}$$

$$\text{and} \quad \frac{\alpha}{x_{2k-2l-1} - (1-A)} < \frac{\alpha}{x_{-2l-1} - (1-A)} < x_0.$$

Thus

$$x_{2k+3} = \frac{\alpha + x_1}{A + x_{2k-2l+2}} > \frac{\alpha + x_{-2k-1}}{A + x_{-2l}} = x_1$$

$$x_{2k+4} = \frac{\alpha + x_2}{A + x_{2k-2l+3}} < \frac{\alpha + x_{-2k}}{A + x_{-2l+1}} = x_2.$$

The proof of the claim in this case follows by induction.

Thus we see that the claim is true.

It follows by induction from the claim that for each non-negative integer j with $0 \leq j \leq k$,

$$x_{(2j+1)+n(2k+2)} > (n+1)\alpha + x_{(2j+1)-(2k+2)} \quad \text{for} \quad n \geq 0.$$

Hence we see that

$$\lim_{n\to\infty} x_{2n+1} = \infty$$

and thus that

$$\lim_{n\to\infty} x_{2n+2} = 0.$$

□

5.3.5 The Case $\alpha = 0$ and $A > 0$

When $\alpha = 0$ and $A > 0$, Eq.(5.13) reduces to the equation

$$x_{n+1} = \frac{x_{n-(2k+1)}}{A + x_{n-2l}} \quad , \quad n = 0, 1, \ldots. \tag{5.18}$$

In agreement with our goal, in this section we shall establish the following result.

THEOREM 5.11
The following statements are true:

1. *Suppose* $A > 1$. *Then the zero equilibrium of Eq.(5.18) is globally asymptotically stable.*
2. *Suppose* $A = 1$. *Then every non-negative solution of Eq.(5.18) converges to a non-negative periodic solution of Eq.(5.18) with period* $2(k+1)$, *and there exist non-negative periodic solutions of Eq.(5.18) with prime period* $2(k+1)$.
3. *Suppose* $0 < A < 1$. *Then there exist solutions of Eq.(5.18) which are neither bounded nor persist.*

PROOF It follows by Theorem 5.7 that the zero equilibrium point of Eq.(5.18) is locally asymptotically stable when $A > 1$.

We consider the cases $A > 1$ and $A = 1$ together.

Let $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ be a non-negative solution of Eq.(5.18). For $n \geq 0$, we have

$$x_{n+1} = \frac{x_{n-(2k+1)}}{A + x_{n-2l}} \leq \frac{1}{A} x_{n-(2k+1)} \leq x_{n-(2k+1)}.$$

Thus

$$\begin{array}{ccccccccc}
x_{-(2k+1)} & \geq & x_1 & \geq & x_{1+(2k+2)} & \geq & x_{1+(2k+2)2} & \geq & \cdots \\
x_{1-(2k+1)} & \geq & x_2 & \geq & x_{2+(2k+2)} & \geq & x_{2+(2k+2)2} & \geq & \cdots \\
 & \vdots & & & & & & & \\
x_{2k-(2k+1)} & \geq & x_{2k+1} & \geq & x_{(2k+1)+(2k+2)} & \geq & x_{(2k+1)+2(2k+2)2} & \geq & \cdots \\
x_{(2k+1)-(2k+1)} & \geq & x_{2k+2} & \geq & x_{(2k+2)+(2k+2)} & \geq & x_{(2k_2)_(2k+2)2} & \geq &
\end{array}$$

and so we see that for each $0 \leq j \leq 2k+1$, the sequence $\{x_{j+(2k+2)n}\}_{n=-1}^{\infty}$ is a decreasing subsequence of $\{x_n\}_{n=-\mathcal{K}}^{\infty}$. It follows that there exists a periodic solution $\{\mathcal{L}_n\}_{n=-\infty}^{\infty}$ with period $2(k+1)$ of the difference equation

$$L_{n+1} = \frac{L_{n-(2k+1)}}{A + L_{n-2l}} \quad , \qquad n \in \{\ldots, -1, 0, 1, \ldots\} \tag{5.19}$$

such that

$$\lim_{n \to \infty} x_{j+(2k+2)n} = \mathcal{L}_j \qquad \text{for} \qquad j \geq 0.$$

Hence

$$\mathcal{L}_0 = \mathcal{L}_{2k+2} = \frac{\mathcal{L}_0}{A + \mathcal{L}_{2k+1-2l}}$$

and so we see that either

$$\mathcal{L}_0 = 0$$

or

$$\mathcal{L}_0 > 0 \quad \text{and} \quad A + \mathcal{L}_{2k+1-2l} = 1.$$

Suppose $A > 1$. Then it follows that $\mathcal{L}_0 = 0$. We similarly have

$$\mathcal{L}_1 = \mathcal{L}_2 = \cdots = \mathcal{L}_{2k+1}$$

and thus

$$\lim_{n\to\infty} x_n = 0$$

as was to be shown.

Suppose $A = 1$. Let $\phi_0, \phi_1, \ldots, \phi_{2k+1}$ be distinct positive real numbers. It follows that the sequence

$$\ldots, \phi_0, 0, \phi_1, 0, \ldots, \phi_{2k+1}, 0, \phi_0, 0, \phi_1, 0, \ldots, \phi_{2k+1}, 0, \ldots$$

is a periodic solution of Eq.(5.19) with prime period $2(k+1)$, from which the proof follows.

Finally, consider the case $0 < A < 1$.

Eq.(5.18) has the unique positive equilibrium point $\bar{x} = 1 - A$.

Let r be the largest non-negative integer such that $0 \le 2r + 1 \le \mathcal{K}$, and let s be the largest non-negative integer such that $0 \le 2s \le \mathcal{K}$.

Let $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ be a solution of Eq.(5.18) such that the initial conditions are as follows:

$$0 < x_{-(2r+1)}, x_{-(2r-1)}, \ldots, x_{-1} \le 1 - A$$

and

$$1 - A < x_{-2s}, x_{-(2s-2)}, \ldots, x_0.$$

Observe that

$$0 < x_1 = \frac{x_{-(2k+1)}}{A + x_{-2l}} < \frac{x_{-(2k+1)}}{A + (1 - A)} = x_{-(2k+1)} \le 1 - A$$

and

$$x_2 = \frac{x_{1-(2k+1)}}{A + x_{1-2l}} \ge \frac{x_{1-(2k+1)}}{A + (1 - A)} = x_{1-(2k+1)} > 1 - A.$$

It follows by induction that for all $n \geq 0$ and $0 \leq j \leq k$, we have

$$x_{2j+(2k+2)(n+1)} > x_{2j+(2k+2)n} > 1 - A$$

and

$$0 < x_{(2j+1)+(2k+2)(n+1)} < x_{(2j+1)+(2k+2)n} \leq 1 - A.$$

Hence for each $0 \leq j \leq k$

$$\lim_{n\to\infty} x_{(2j+1)+(2k+2)n} = \mathcal{L}_{2j+1} \in [0, 1-A) \text{ and } \lim_{n\to\infty} x_{2j+(2k+2)n} = \mathcal{L}_{2j} \in (1-A, \infty].$$

We claim that for each $0 \leq j \leq k$, $\mathcal{L}_{2j+1} = 0$.

For the sake of contradiction, suppose there exists $j \in \{0, 1, \ldots, k\}$ with

$$\mathcal{L}_{2j+1} \in (0, 1-A).$$

Then

$$\mathcal{L}_{2j+1} = \lim_{n\to\infty} x_{(2j+1)+(2k+2)+(2k+2)n} = \lim_{n\to\infty} \frac{x_{(2j+1)+(2k+2)n}}{A + x_{2j+(2k+2)+(2k+2)n-2l}}.$$

So, as

$$\lim_{n\to\infty} x_{(2j+1)+(2k+2)n} = \mathcal{L}_{2j+1} \in (0, 1-A)$$

and $\{x_{2j+(2k+2)+(2k+2)n}\}_{n=0}^{\infty}$ is a positive, strictly increasing sequence bounded from below by $1 - A$, it follows that

$$\lim_{n\to\infty} x_{2j+(2k+2)+(2k+2)n-2l} = \mathcal{L}_{2j-2l} \in (1 - A, \infty)$$

and

$$\mathcal{L}_{2j+1} = \frac{\mathcal{L}_{2j+1}}{A + \mathcal{L}_{2j-2l}}.$$

So, as $\mathcal{L}_{2j+1} > 0$, we see that

$$1 < A + \mathcal{L}_{2j-2l} = 1$$

which is a contradiction.

Thus it is true that for each $0 \leq j \leq k$, $\mathcal{L}_{2j+1} = 0$, and so

$$\lim_{n\to\infty} x_{2n+1} = 0.$$

We now claim that for each $0 \leq j \leq k$, $\mathcal{L}_{2j} = \infty$.

For the sake of contradiction, suppose there exists $j \in \{0, 1, \ldots, k\}$ with

$$\mathcal{L}_{2j} \in (1 - A, \infty).$$

Then

$$\mathcal{L}_{2j} = \lim_{n\to\infty} x_{2j+(2k+2)+(2k+2)n} = \lim_{n\to\infty} \frac{x_{2j+(2k+2)n}}{A + x_{2j+(2k+1)+(2k+2)n-2l}} = \frac{\mathcal{L}_{2j}}{A}$$

and so

$$A = 1$$

which is a contradiction. Hence

$$\lim_{n\to\infty} x_{2n} = \infty$$

and the proof is complete. ▯

5.3.6 The Trichotomy Result for Eq.(5.13)

The following theorem is the main result of this section.

THEOREM 5.12
Let α and A be non-negative real numbers. Set $d = k+1$ if $\alpha = 0$, and set $d = \gcd(k+1, 2l+1)$ if $\alpha > 0$. Then the following statements are true:

1. *Suppose $A > 1$. Then every solution of Eq.(5.13) converges.*
2. *Suppose $A = 1$. Then every solution of Eq.(5.13) converges to a periodic solution of Eq.(5.13) with period $2d$, and there exist periodic solutions of Eq.(5.20) with prime period $2d$.*
3. *Suppose $0 \le A < 1$. Then there exist unbounded solutions of Eq.(5.13).*

PROOF The proof follows by Theorems 5.5, 5.8, 5.9, 5.10, and 5.11. ▯

5.4 The Equation $x_{n+1} = \dfrac{\gamma x_{n-(2k+1)} + x_{n-2l}}{A + x_{n-2l}}$

In this section we study the global behavior of the difference equation

$$x_{n+1} = \frac{\gamma x_{n-(2k+1)} + x_{n-2l}}{A + x_{n-2l}}, \qquad n = 0, 1, \ldots \tag{5.20}$$

where k and l are non-negative integers, $\gamma > 0$ is a positive parameter, A is a non-negative parameter, and the initial conditions are non-negative real numbers chosen such that $0 < A + x_{n-2l}$ for all $n \ge 0$.

The case $k = 0$ and $l = 1$ was investigated in [23] and [59].

5.4.1 Attracting Intervals of Eq.(5.20)

We first establish the fact that every non-negative solution of Eq.(5.20) eventually enters and remains in the interval $[0, 1)$ when $0 < \gamma < A$, and that every positive solution of Eq.(5.20) eventually enters and remains in the interval $(1, \infty)$ when $0 \leq A < \gamma$.

The following lemma, the proof of which is straightforward and will be omitted, gives three identities that will be useful in the study of Eq.(5.20).

LEMMA 5.8

Suppose $\gamma > 0$ and $A > 0$ are positive real numbers, and let $\{x_n\}_{n=-K}^{\infty}$ be a non-negative solution of Eq.(5.20). Then the following statements are true:

1. $x_{n+1} - 1 = \dfrac{\gamma \left[x_{n-(2k+1)} - \frac{A}{\gamma}\right]}{A + y_{n-2l}}$ *for all* $n \geq 0$.

2. $x_{n+1} - \dfrac{\gamma}{A} = \dfrac{\left(1 - \frac{\gamma}{A}\right) x_{n-2l} + \gamma[x_{n-(2k+1)} - 1]}{A + x_{n-2l}}$ *for all* $n \geq 0$.

3. $x_{n+1} - x_{n-(2k+1)} = \dfrac{(\gamma - A)x_{n-(2k+1)} + [1 - x_{n-(2k+1)}]x_{n-2l}}{A + x_{n-2l}}$ *for all* $n \geq 0$.

LEMMA 5.9

Suppose that $0 < \gamma < A$. Let $\{x_n\}_{n=-K}^{\infty}$ be a non-negative solution of Eq.(5.20), and suppose that there exists a non-negative integer $N \geq 0$ such that

$$x_N \leq 1.$$

Then

$$x_{N+(2k+2)n} < 1 \quad \textit{for all} \quad n \geq 1.$$

PROOF Observe that

$$x_{N+(2k+2)} = \frac{\gamma x_N + x_{N+(2k+1)-2l}}{A + x_{N+(2k+1)-2l}} < \frac{A + x_{N+(2k+1)-2l}}{A + x_{N+(2k+1)-2l}} = 1.$$

The proof follows by induction. ∎

THEOREM 5.13

Suppose that $0 < \gamma < A$. Let $\{x_n\}_{n=-K}^{\infty}$ be a non-negative solution of Eq.(5.20). Then there exists $N \geq 0$ such that

$$x_n < 1 \quad \textit{for all} \quad n \geq N.$$

PROOF Let $M \geq 0$. It follows by Lemma 5.9 that it suffices to show that there exists $n_M \geq 0$ such that $x_{M+(2k+2)n_M} \leq 1$. For the sake of contradiction, suppose that

$$x_{M+(2k+2)n} > 1 \qquad \text{for all} \qquad n \geq 0.$$

Then, in particular, $x_M > 1$. Hence by Lemma 5.8.2, we see that $x_{M+(2k+2)} > \dfrac{\gamma}{A}$, and so by Lemma 5.8.1, we have $x_M > \dfrac{\gamma}{A} > 1$. It follows by Lemma 5.8.2 that

$$x_{M+(2k+2)n} > \frac{A}{\gamma} > 1 \qquad \text{for all} \qquad n \geq 0.$$

We claim that

$$\frac{A}{\gamma} < x_{M+(2k+2)(n+1)} < x_{M+(2k+2)n} \qquad \text{for all} \qquad n \geq 0.$$

Indeed, by Lemma 5.8.3, given $n \geq 0$,

$$x_M > x_{M+(2k+2)} > \cdots > x_{M+(2k+2)n}$$

and the proof of the claim follows by induction.

Thus there exists $L \geq \dfrac{\gamma}{A} > 1$ such that

$$\lim_{n\to\infty} x_{M+(2k+2)n} = L.$$

Now for $n \geq 0$, we have

$$x_{M+(2k+2)(n+1)} = \frac{\gamma x_{M+(2k+2)n} + x_{M+(2k+2)n+(2k+1)-2l}}{A + x_{M+(2k+2)n+(2k+1)-2l}}$$

from which it follows that

$$x_{M+(2k+2)n+(2k+1)-2l} = \frac{\gamma x_{M+(2k+2)n} - A x_{M+(2k+2)(n+1)}}{x_{M+(2k+2)(n+1)} - 1}.$$

Hence as $L > 1$ and $\gamma < A$, we have

$$\lim_{n\to\infty} x_{M+(2k+2)n+(2k+1)-2l} = \frac{L(\gamma - A)}{L - 1} < 0,$$

which is a contradiction, and the proof is complete. ∎

LEMMA 5.10

Suppose that $0 \leq A < \gamma$. Let $\{x_n\}_{n=-K}^{\infty}$ be a positive solution of Eq.(5.20), and suppose that there exists a non-negative integer $N \geq 0$ such that

$$x_N \geq 1.$$

Then

$$x_{N+(2k+2)n} > 1 \qquad \textit{for all} \qquad n \geq 1.$$

PROOF Observe that

$$x_{N+(2k+2)} = \frac{\gamma x_N + x_{N+(2k+1)-2l}}{A + x_{N+(2k+1)-2l}} > \frac{\gamma + x_{N+(2k+1)-2l}}{\gamma + x_{N+(2k+1)-2l}} = 1.$$

The proof follows by induction. ∎

THEOREM 5.14
Suppose that $0 \leq A < \gamma$. Let $\{x_n\}_{n=-K}^{\infty}$ be a positive solution of Eq.(5.20). Then there exists $N \geq 0$ such that

$$x_n > 1 \qquad \textit{for all} \qquad n \geq N.$$

PROOF We first consider the case $A = 0$.

In this case, Eq.(5.20) reduces to the equation

$$x_{n+1} = 1 + \frac{\gamma x_{n-(2k+1)}}{x_{n-2l}} \quad , \qquad n = 0, 1, \ldots \tag{5.21}$$

and so the result is clear.

We next consider the case $A > 0$.

Let $M \geq 0$. It follows by Lemma 5.10 that it suffices to show that there exists $n_M \geq 0$ such that $x_{M+(2k+2)n_M} \geq 1$. For the sake of contradiction, suppose that

$$x_{M+(2k+2)n} < 1 \qquad \text{for all} \qquad n \geq 0.$$

In particular, $0 < x_M < 1$. We also have $x_{M+(2k+2)} < 1$, and so by Lemma 5.8.1, we see that $x_M < \frac{A}{\gamma} < 1$. It follows by Lemma 5.8.1 that

$$x_{M+(2k+2)n} < \frac{A}{\gamma} < 1 \qquad \text{for all} \qquad n \geq 0.$$

We claim that

$$0 < x_{M+(2k+2)n} < x_{M+(2k+2)(n+1)} < \frac{A}{\gamma} \qquad \text{for all} \qquad n \geq 0.$$

Indeed, by Lemma 5.8.3, given $n \geq 0$,

$$0 < x_M < x_{M+(2k+2)} < \cdots < x_{M+(2k+2)n}$$

and the proof of the claim follows by induction.

Thus there exists $L \in \left(0, \dfrac{q}{p}\right]$ such that

$$\lim_{n\to\infty} x_{M+(2k+2)n} = L.$$

Now for $n \geq 0$, we have

$$x_{M+(2k+2)(n+1)} = \frac{\gamma x_{M+(2k+2)n} + x_{M+(2k+2)n+(2k+1)-2l}}{A + y_{M+(2k+2)n+(2k+1)-2l}}$$

from which it follows that

$$A_{M+(2k+2)n+(2k+1)-2l} = \frac{\gamma x_{M+(2k+2)n} - Ax_{M+(2k+2)(n+1)}}{x_{M+(2k+2)(n+1)} - 1}.$$

Thus as $0 < L \leq \dfrac{A}{\gamma} < 1$ and $A < \gamma$, we see that

$$\lim_{n\to\infty} x_{M+(2k+2)n+(2k+1)-2l} = \frac{L(\gamma - A)}{L - 1} < 0,$$

which is a contradiction, and the proof is complete. ▯

5.4.2 Stability Character of the Equilibrium Points

$\tilde{x} = 0$ is an equilibrium point of Eq.(5.20) if and only if $A > 0$. Furthermore, when $\gamma + 1 \leq A$, then $\tilde{x} = 0$ is the only equilibrium point of Eq.(5.20). However, when $\gamma + 1 > A \geq 0$, then Eq.(5.20) has the unique positive equilibrium point $\bar{x} = \gamma + 1 - A$.

5.4.2.1 The Stability Character of the Zero Equilibrium $\tilde{x} = 0$ of Eq.(5.20)

The following theorem describes the local stability character of the zero equilibrium point of Eq.(5.20).

THEOREM 5.15
Let $0 < \gamma$ and $0 < A$. Then the following statements are true:

1. *Suppose $A > \gamma + 1$. Then $\tilde{x} = 0$ is a locally asymptotically stable equilibrium point of Eq.(5.20).*

2. *Suppose $A = \gamma + 1$. Then $\tilde{x} = 0$ is a locally stable equilibrium point of Eq.(5.20).*

3. *Suppose $0 < A < \gamma + 1$. Then $\tilde{y} = 0$ is an unstable equilibrium point of Eq.(5.20).*

PROOF The linearized equation of Eq.(5.20) about $\tilde{x} = 0$ is

$$z_{n+1} - \frac{\gamma}{A} z_{n-(2k+1)} - \frac{1}{A} z_{n-2l} \quad , \qquad n = 0, 1, \ldots$$

with characteristic equation

$$\lambda^{\mathcal{K}+1} - \frac{\gamma}{A} \lambda^{\mathcal{K}-(2k+1)} - \frac{1}{A} \lambda^{\mathcal{K}-2l} = 0.$$

It follows by Theorem 1.6 that $\tilde{x} = 0$ is a locally asymptotically stable equilibrium point of Eq.(5.20) when $\gamma + 1 < A$.

Let $c : \mathbf{R} \to \mathbf{R}$ be given by

$$c(\lambda) = \lambda^{\mathcal{K}+1} - \frac{\gamma}{A} \lambda^{\mathcal{K}-(2k+1)} - \frac{1}{A} \lambda^{\mathcal{K}-2l}.$$

Note that if $0 < A < \gamma+1$, then $c(1) = \dfrac{A - (\gamma + 1)}{A} < 0$, and so as $\lim_{\lambda \to \infty} c(\lambda) = \infty$, we see that c has a positive root greater than 1. It follows by Theorem 1.1 that $\tilde{y} = 0$ is an unstable equilibrium point of Eq.(5.20) when $0 < A < \gamma + 1$.

Finally, suppose $A = \gamma + 1$. We shall show that $\tilde{x} = 0$ is a locally stable equilibrium point of Eq.(5.20). Let $\varepsilon > 0$, and suppose that $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ is a non-negative solution of Eq.(5.20) such that

$$0 \le x_{-\mathcal{K}} < \varepsilon, 0 \le x_{-\mathcal{K}+1} < \varepsilon, \ldots, 0 \le x_0 < \varepsilon.$$

It suffices to show that $0 \le x_1 < \varepsilon$. Now

$$0 \le x_1 = \frac{\gamma x_{-(2k+1)} + x_{-2l}}{(\gamma + 1) + x_{-2l}} \le \frac{\gamma x_{-(2k+1)} + x_{-2l}}{\gamma + 1} < \frac{\gamma \varepsilon + \varepsilon}{\gamma + 1} = \varepsilon. \qquad \square$$

THEOREM 5.16

Let γ and A be positive real numbers such that $\gamma + 1 \le A$. Then $\tilde{x} = 0$ is a globally asymptotically stable equilibrium point of Eq.(5.20).

PROOF We know by Theorem 5.15 that $\bar{x} = 0$ is a locally stable equilibrium point of Eq.(5.20), and so it suffices to show that $\tilde{x} = 0$ is a global attractor of Eq.(5.20). So let $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ be a non-negative solution of Eq.(5.20). It suffices to show that $\lim_{n \to \infty} x_n = 0$. We distinguish between the following two cases.

Suppose $A > \gamma + 1$.

Observe that for all $n \ge 0$, we have

$$x_{n+1} = \frac{\gamma x_{n-(2k+1)} + x_{n-2l}}{A + x_{n-2l}} \le \frac{\gamma}{A} x_{n-(2k+1)} + \frac{1}{A} x_{n-2l}.$$

Let $\{z_n\}_{n=-\mathcal{K}}^{\infty}$ be the solution of the difference equation

$$z_{n+1} = \frac{\gamma}{A} z_{n-(2k+1)} + \frac{1}{A} z_{n-2l} \quad , \quad n = 0, 1, \ldots \tag{5.22}$$

with initial conditions $z_{-\mathcal{K}} = x_{-\mathcal{K}}, z_{-\mathcal{K}+1} = x_{-\mathcal{K}+1}, \ldots, z_0 = x_0$. It follows by Theorem 1.7 that

$$x_n \leq z_n \quad \text{for all} \quad n \geq -\mathcal{K}.$$

Now by Theorem 1.6, we see that

$$\lim_{n\to\infty} z_n = 0$$

and thus that it is also the case that

$$\lim_{n\to\infty} x_n = 0.$$

Suppose $A = \gamma + 1$.

Now $A = \gamma + 1 > \gamma$, and so by Theorem 5.13, there exists $N \geq o$ such that

$$x_n < 1 \quad \text{for all} \quad n \geq N.$$

Thus we see that

$$S = \limsup_{n\to\infty} x_n \leq 1 < \frac{\gamma + 1}{\gamma}.$$

It follows by Theorem 1.8 that there exists a solution $\{L_n\}_{n=-\infty}^{\infty}$ of the difference equation

$$z_{n+1} = \frac{\gamma z_{n-(2k+1)} + z_{n-2l}}{A + z_{n-2l}} \quad , \quad n \in \{\ldots, -1, 0, 1, \ldots\}$$

with $L_0 = S$ such that

$$0 \leq L_n \leq S \quad \text{for all} \quad n \in \{\ldots, -1, 0, 1, \ldots\}.$$

Note that as $S < \dfrac{\gamma + 1}{\gamma}$, we have

$$S = L_0 = \frac{\gamma L_{-(2k+2)} + L_{-(2l+1)}}{(\gamma + 1) + L_{-(2l+1)}} \leq \frac{\gamma S + L_{-(2l+2)}}{(\gamma + 1) + L_{-(2l+1)}} \leq \frac{(\gamma + 1)S}{(\gamma + 1) + S}.$$

This implies that

$$(\gamma + 1)S + S^2 \leq (\gamma + 1)S$$

and so $S^2 \leq 0$. Hence $S = 0$, and so the proof is complete. □

5.4.2.2 The Stability Character of the Positive Equilibrium $\bar{x}$ of Eq.(5.20)

Recall that there exists a positive equilibrium point $\bar{x}$ of Eq.(5.20) if and only if $0 \le A < \gamma + 1$, and that if $0 \le A < \gamma + 1$, then the positive equilibrium point $\bar{x}$ of Eq.(5.20) is

$$\bar{x} = \gamma + 1 - A.$$

The following theorem describes the local stability character of the positive equilibrium point $\bar{x}$ of Eq.(5.20).

THEOREM 5.17
Suppose $0 < \gamma$ and $0 \le A < \gamma + 1$. Then the following statements are true:

1. *The positive equilibrium point $\bar{x}$ of Eq.(5.20) is locally asymptotically stable when*
$$\gamma - 1 < A < \gamma + 1.$$

2. *The positive equilibrium point $\bar{x}$ of Eq.(5.20) is an unstable equilibrium point whose characteristic equation has at least one root with modulus less than 1 and at least one root with modulus greater than 1 when*
$$0 \le A < \gamma - 1.$$

PROOF The linearized equation of Eq.(5.20) about the positive equilibrium $\bar{x}$ is

$$z_{n+1} - \frac{\gamma}{\gamma+1} z_{n-(2k+1)} + \frac{\gamma - A}{\gamma+1} z_{n-2l} = 0 \quad , \quad n = 0, 1, \ldots \tag{5.23}$$

with characteristic equation

$$\lambda^{\mathcal{K}+1} - \frac{\gamma}{\gamma+1} \lambda^{\mathcal{K}-(2k+1)} + \frac{\gamma - A}{\gamma+1} \lambda^{\mathcal{K}-2l} = 0.$$

Suppose $\gamma - 1 < A < \gamma + 1$.
Then it follows by Theorem 1.6 that $\bar{x}$ is a locally asymptotically stable equilibrium point of Eq.(5.20).

Suppose $0 \le A < \gamma - 1$.
Let $c : \mathbf{R} \to \mathbf{R}$ be given by

$$c(\lambda) = \lambda^{\mathcal{K}+1} - \frac{\gamma}{\gamma+1} \lambda^{\mathcal{K}-(2k+1)} + \frac{\gamma - A}{\gamma+1} \lambda^{\mathcal{K}-2l}.$$

Then $|c(0)| < 1$ and $\lim_{\lambda \to -\infty} c(-1)c(\lambda) = -\infty$. It follows that there exists a root λ_1 with $|\lambda_1| < 1$ and a second root λ_2 with $\lambda_2 < -1$, and so the proof is complete. ∎

THEOREM 5.18

Let $0 < \gamma$ and $0 \leq A$, and suppose that $\gamma - 1 < A < \gamma + 1$. Then the positive equilibrium point $\bar{x}$ of Eq.(5.20) is globally asymptotically stable with basin of attraction the eventually positive solutions of Eq.(5.20).

PROOF We know by Theorem 5.17 that $\bar{x}$ is a locally asymptotically stable equilibrium point of Eq.(5.20). Let $\{x_n\}_{n=-K}^{\infty}$ be a positive solution of Eq.(5.20). It suffices to show that

$$\lim_{n \to \infty} x_n = \bar{x}.$$

The proof of Theorem 5.18 is a direct result of the following five lemmas.

Let $g : (0, \infty)^2 \to (0, \infty)$ be given by

$$g(u, v) = \frac{\gamma u + v}{A + v}.$$

Then

$$x_{n+1} = g(x_{n-(2k+1)}, x_{n-2l}) \qquad \text{for all} \qquad n \geq 0.$$

The proof of Lemma 5.11 follows by a simple computation and will be omitted.

LEMMA 5.11

Suppose γ and A are positive real numbers. Let $u, v \in (0, \infty)$. Then the following statements are true:

1. $\frac{\partial g}{\partial u}(u, v) > 0.$

2. $\frac{\partial g}{\partial v}(u, v) = \frac{\gamma \left(\frac{A}{\gamma} - u\right)}{(A + v)^2}.$

LEMMA 5.12

Suppose γ and A are positive real numbers such that $\gamma < A < \gamma + 1$. Let $m \in (0, \bar{x}]$, and suppose that $u, v \in [m, 1]$. Then $g(u, v) \in [m, 1]$.

PROOF Note that as $\gamma < A < \gamma + 1$, we have $\bar{x} = \gamma + 1 - A \in (0, 1)$.

Now, $\gamma < A$ and $u, v \in [m, 1]$, and so by Lemma 5.11

$$0 < g(u,v) = \frac{\gamma u + v}{A + v} \leq \frac{\gamma + v}{A + v} \leq \frac{\gamma + 1}{A + 1} < 1.$$

As $m \leq \bar{x} = \gamma + 1 - A$, it follows that $m \leq g(m, m)$. Finally, $\gamma < A$ and $u, v \in [m, 1]$, and so again by Lemma 5.11, we see that

$$g(m, m) \leq g(u, v)$$

and thus the proof is complete. ∎

LEMMA 5.13
Let γ and A be positive real numbers such that $\gamma < A < \gamma + 1$. Then

$$\lim_{n\to\infty} x_n = \bar{x}.$$

PROOF By Theorem 5.13, there exists $N \geq 0$ such that $0 < x_n < 1$ for all $n \geq N$. Let $m = \min\{\bar{x}, x_N, x_{N+1}, \ldots x_{N+K}\}$. It follows by Lemma 5.12 that

$$0 < m \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq 1 < \frac{A}{\gamma}.$$

Set $I = \liminf_{n\to\infty} y_n$ and $S = \limsup_{n\to\infty} y_n$. It suffices to show that $I = S$. By Theorem 1.8, there exist solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of the difference equation

$$z_{n+1} = \frac{\gamma z_{n-(2k+1)} + z_{n-2l}}{A + z_{n-2l}} \quad , \quad n \in \{\ldots, -1, 0, 1, \ldots\}$$

with $I_0 = I$ and $S_0 = S$ such that for all $n \in \{\ldots, -1, 0, 1, \ldots\}$,

$$I \leq I_n \leq S \quad \text{and} \quad I \leq S_n \leq S.$$

Thus by Lemma 5.11

$$I = I_0 = \frac{\gamma I_{-(2k+2)} + I_{-(2l+1)}}{A + I_{-(2l+1)}} \geq \frac{\gamma I + I_{-(2l+1)}}{A + I_{-(2l+1)}} \geq \frac{\gamma I + I}{A + I} = \frac{(\gamma+1)I}{A + I}$$

and

$$S = S_0 = \frac{\gamma S_{-(2k+2)} + S_{-(2l+1)}}{A + S_{-(2l+1)}} \leq \frac{\gamma S + S_{-(2l+1)}}{A + S_{-(2l+1)}} \leq \frac{\gamma S + S}{A + S} = \frac{(\gamma+1)S}{A + S}.$$

Hence

$$S \leq \gamma + 1 - A \leq I$$

and the proof is complete. ∎

LEMMA 5.14
Let γ and A be positive real numbers such that $A = \gamma$. Then

$$\lim_{n\to\infty} x_n = \bar{x}.$$

PROOF Since $\gamma = A$, we see that $\bar{x} = 1$. Now by Lemma 5.8 we have

$$x_{n+1} - 1 = \frac{\gamma\left(x_{n-(2k+1)} - 1\right)}{\gamma + x_{n-2l}} \quad \text{for all} \quad n \geq 0$$

and

$$x_{n+1} - x_{n-(2k+1)} = \frac{(1 - x_{n-(2k+1)})x_{n-2l}}{\gamma + x_{n-2l}} \quad \text{for all} \quad n \geq 0$$

from which it follows that $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ converges to a positive periodic solution $\{z_n\}_{n=-\mathcal{K}}^{\infty}$ of Eq.(5.20) with period $2k+2$.

Suppose $0 \leq j \leq 2k+1$. Then

$$z_{j-(2k+1)} = z_{j+1} = \frac{\gamma z_{j-(2k+1)} + z_{j-2l}}{\gamma + z_{j-2l}}$$

and thus

$$\gamma z_{j-(2k+1)} + z_{j-2l} z_{j-(2k+1)} = \gamma z_{j-(2k+1)} + z_{j-2l}.$$

Hence

$$z_{j-2l} z_{j-(2k+1)} = z_{j-2l}.$$

It follows that $z_{j+1} = z_{j-(2k+1)} = 1$, and so we see that

$$\lim_{n\to\infty} x_n = 1.$$

∎

LEMMA 5.15
Let $0 < \gamma$ and $0 \leq A$, and suppose that $\gamma - 1 < A < \gamma$. Then

$$\limsup_{n\to\infty} x_n \leq \frac{1}{A + 1 - \gamma}.$$

PROOF By Theorem 5.14, there exists $N \geq 0$ such that

$$x_n > 1 \quad \text{for all} \quad n \geq N.$$

Note that for $n \geq N + \mathcal{K}$, it follows by Lemma 5.11 that

$$x_{n+1} = \frac{\gamma + x_{n-(2k+1)}}{A + x_{n-2l}} < \frac{\gamma x_{n-(2k+1)} + 1}{A+1} = \frac{\gamma}{A+1} x_{n-(2k+1)} + \frac{1}{A+1}.$$

Let $\{z_n\}_{n=N}^{\infty}$ be the solution of the difference equation

$$z_{n+1} = \frac{\gamma}{A+1} z_{n-(2k+1)} + \frac{1}{A+1} \quad , \qquad n = 0, 1, \ldots \tag{5.24}$$

with initial conditions $z_N = x_N, z_{N+1} = x_{N+1}, \ldots, z_{N+\mathcal{K}} = x_{N+\mathcal{K}}$.
It follows by Theorem 1.7 that

$$x_n \leq z_n \qquad \text{for all} \qquad n \geq N.$$

Hence

$$\limsup_{n\to\infty} x_n \leq \lim_{n\to\infty} z_n = \frac{1}{A+1-\gamma}$$

and the proof is complete. ▯

LEMMA 5.16
Let $0 < \gamma$ and $0 \leq A$, and suppose that $\gamma - 1 < A < \gamma$. Then

$$\lim_{n\to\infty} x_n = \bar{x}.$$

PROOF By Theorem 5.14 and Lemma 5.15, we have

$$1 \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq \frac{1}{A+1-\gamma}.$$

Set $I = \liminf_{n\to\infty} x_n$ and $S = \limsup_{n\to\infty} x_n$. It follows by Theorem 1.8 that there exist solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of the difference equation

$$z_{n+1} = \frac{\gamma z_{n-(2k+1)} + z_{n-2l}}{A + z_{n-2l}} \quad , \qquad n \in \{\ldots, -1, 0, 1, \ldots\}$$

with $I_0 = I$ and $S_0 = S$ such that for all $n \in \{\ldots, -1, 0, 1, \ldots\}$,

$$I \leq I_n \leq S \qquad \text{and} \qquad I \leq S_n \leq S.$$

Since $0 \leq \dfrac{A}{\gamma} < 1$, we see by Lemma 5.11 that

$$I = I_0 = \frac{\gamma I_{-(2k+2)} + I_{-(2l+1)}}{A + I_{-(2l+1)}} \geq \frac{\gamma I + S}{A+S}$$

and

$$S = S_0 = \frac{\gamma S_{-(2k+2)} + S_{-(2l+1)}}{A + S_{-(2l+1)}} \leq \frac{\gamma S + I}{A+I}.$$

Hence

$$AI + IS \geq \gamma I + S$$

and

$$AS + IS \leq \gamma S + I.$$

Thus we have

$$\gamma I + S - AI \leq IS \leq \gamma S + I - AS$$

and hence

$$[A - (\gamma - 1)]S \leq [A - (\gamma - 1)]I.$$

Therefore, $I = S$, and the proof is complete. ∎

The proof of Theorem 5.18 is now complete. ∎

5.4.3 Periodic Solutions of Eq.(5.20)

THEOREM 5.19

Suppose that $\gamma = A + 1$. Then every eventually positive solution of Eq.(5.20) converges to a positive periodic solution of Eq.(5.20) with period $2 \cdot \gcd(k + 1, 2l + 1)$, and there exist positive periodic solutions of Eq.(5.20) with prime period $2 \cdot \gcd(k + 1, 2l + 1)$.

PROOF Since $\gamma = A + 1 > A$, we see by Theorem 5.14 that every positive solution of Eq.(5.20) eventually enters and remains in the interval $(1, \infty)$. Hence the change of variables

$$x_n = 1 + \gamma z_n$$

reduces Eq.(5.20) to the equation

$$z_{n+1} = \frac{\frac{1}{\gamma^2} + z_{n-(2k+1)}}{1 + z_{n-2l}} \quad , \qquad n = 0, 1, \ldots$$

with positive initial conditions. The proof follows by Theorem 5.5. ∎

5.4.4 Existence of Unbounded Solutions When $\gamma > A + 1$

In this section, we show that there exist unbounded solutions of Eq.(5.20) when $\gamma > A + 1$.

THEOREM 5.20

Suppose $\gamma > A + 1$. Then there exist solutions of Eq.(5.20) which are not bounded.

PROOF Let r be the largest non-negative integer such that $0 \leq 2r+1 \leq \mathcal{K}$, and let s be the largest non-negative integer such that $0 \leq 2s \leq \mathcal{K}$.

Let $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ be a solution of Eq.(5.20) such that the initial conditions are as follows:

$$1 < x_{-(2r+1)}, x_{-(2r-1)}, \ldots, x_{-1} \leq \gamma - A$$

and

$$\frac{(\gamma - A)^2}{\gamma - 1 - A} \leq x_{-2s}, x_{-(2s-2)}, \ldots, x_0.$$

Then it suffices to show

$$\lim_{n\to\infty} x_{2n} = \infty \qquad \text{and} \qquad \lim_{n\to\infty} x_{2n+1} = 1.$$

Note that

$$0 \leq \frac{A}{\gamma} < 1 < \frac{(\gamma - A)^2}{\gamma - 1 - A}.$$

It follows by Lemma 5.10 that

$$x_n > 1 \qquad \text{for all} \qquad n \geq \mathcal{K}.$$

Set $m = \min\{k, l\}$, and note that

$$2m - (2k+1) \leq 2k - 2k - 1 = -1 \qquad \text{and} \qquad 2m - 2l \leq 2l - 2l = 0$$

and so for $0 \leq j \leq m-1$, it follows by Lemma 5.11 that

$$1 < x_{2j+1} = \frac{\gamma x_{2j-(2k+1)} + x_{2j-2l}}{A + x_{2j-2l}} \leq \frac{\gamma(\gamma - A) + \frac{(\gamma-A)^2}{\gamma-1-A}}{A + \frac{(\gamma-A)^2}{\gamma-1-A}} = (\gamma-A)\left(\frac{\gamma^2 - \gamma A - A}{\gamma^2 - \gamma A - A}\right)$$

$$= \gamma - A$$

and

$$x_{2j+2} = \frac{\gamma x_{2j-2k} + x_{2j-(2l-1)}}{A + x_{2j-(2l-1)}} \geq \frac{\gamma x_{2j-2k} + (\gamma - A)}{A + \gamma - A} > \frac{\gamma x_{2j-2k} + 1}{\gamma} = x_{2j-2k} + \frac{1}{\gamma}.$$

Note also that we have

$$x_{2m+1} = \frac{\gamma x_{2m-(2k+1)} + x_{2m-2l}}{A + x_{2m-2l}} \leq \frac{\gamma(\gamma - A) + \frac{(\gamma-A)^2}{\gamma-1-A}}{A + \frac{(\gamma-A)^2}{\gamma-1-A}} = (\gamma-A)\left(\frac{\gamma^2 - \gamma A - A}{\gamma^2 - \gamma A - A}\right) = \gamma - A$$

and since $2m - (2l-1) \leq 2m+1$,

$$x_{2m+2} = \frac{\gamma x_{2m-2k} + x_{2m-(2l-1)}}{A + x_{2m-(2l-1)}} \geq \frac{\gamma x_{2m-2k} + (\gamma - A)}{A + \gamma - A} > \frac{\gamma x_{2m-2k} + 1}{\gamma} = x_{2m-2k} + \frac{1}{\gamma}.$$

It follows by induction that

$$\lim_{n\to\infty} x_{2n} = \infty$$

from which the proof follows. ∎

5.4.5 The Trichotomy Result for Eq.(5.20)

The following theorem is the main result of this section.

THEOREM 5.21
Let γ be a positive real number and A a non-negative real number. Then the following statements are true:

1. *Suppose that $\gamma + 1 \leq A$. Then every solution of Eq.(5.20) converges to $\tilde{x} = 0$.*
2. *Suppose $\gamma - 1 < A < \gamma + 1$. Then every eventually positive solution of Eq.(5.20) converges to $\bar{x} = \gamma + 1 - A$..*
3. *Suppose $A = \gamma - 1$. Then every eventually positive solution of Eq.(5.20) converges to a positive periodic solution of Eq.(5.20) with period $2 \cdot \gcd(k+1, 2l+1)$, and there exist positive periodic solutions of Eq.(5.20) with prime period $2 \cdot \gcd(k+1, 2l+1)$.*
4. *Suppose $0 \leq A < \gamma - 1$. Then there exist solutions of Eq.(5.20) which are not bounded.*

PROOF The proof follows by Theorems 5.16, 5.18, 5.19, and 5.20. □

5.5 The Remaining Cases of Eq.(5.1)

In this section we establish Theorem 5.3 for the remaining cases of Eq.(5.1).

Recall that $\alpha + \gamma + \delta > 0$.

5.5.1 The Case $\gamma = \delta = A = 0$

Suppose $\gamma = \delta = A = 0$. Then Eq.(5.1) reduces to the equation

$$x_{n+1} = \frac{\alpha}{x_{n-2l}} \quad , \quad n = 0, 1, \ldots \tag{5.25}$$

every positive solution of which is periodic with period $2(2l+1)$, and there exist positive periodic solutions of Eq.(5.1) with prime period $2(2l+1)$.

5.5.2 The Case $\gamma = 0$ and $\delta + A > 0$

Suppose $\gamma = 0$ and $\delta + A > 0$. Then Eq.(5.1) reduces to the equation

$$x_{n+1} = \frac{\alpha + \delta x_{n-2l}}{A + x_{n-2l}} \quad , \quad n = 0, 1, \ldots . \tag{5.26}$$

which decomposes into $2l+1$ equations, each of which is essentially a Riccati equation with Riccati number

$$\mathcal{R} = \frac{\delta A - \alpha}{(\delta + A)^2} < \frac{1}{4}$$

and so it follows that every positive solution of Eq.(5.26) converges to the equilibrium point

$$\bar{x} = \frac{\delta - A + \sqrt{(\delta - A)^2 + 4\alpha}}{2}.$$

See [57].

5.5.3 The Case $\gamma > 0$

5.5.3.1 The Case $\alpha = \delta = A = 0$

Suppose $\gamma > 0$ and $\alpha = \delta = A = 0$. Then Eq.(5.1) reduces to the equation

$$x_{n+1} = \frac{x_{n-(2k+1)}}{x_{n-2l}} \quad , \quad n = 0, 1, \dots . \tag{5.27}$$

The substitution $x_n = e^{z_n}$ in Eq.(5.27) yields the equation

$$z_{n+1} + z_{n-2l} - z_{n-(2k+1)} = 0 \quad , \quad n = 0, 1, \dots \tag{5.28}$$

with characteristic equation

$$\lambda^{\mathcal{K}+1} + \lambda^{\mathcal{K}-2l} - \lambda^{\mathcal{K}-(2k+1)} = 0. \tag{5.29}$$

Since Eq.(5.29) has a negative real root $\lambda_1 < -1$, it follows that there exist positive solutions of Eq.(5.27) which are neither bounded nor persist.

5.5.3.2 The Case $\alpha = \delta = 0$ and $A > 0$

Suppose γ and A are positive real numbers and $\alpha = \delta = 0$. Then Eq.(5.1) reduces to the equation

$$x_{n+1} = \frac{\gamma x_{n-(2k+1)}}{A + x_{n-2l}} \quad , \quad n = 0, 1, \dots \tag{5.30}$$

The change of variables $x_n = \gamma y_n$ transforms Eq.(5.30) into the equation

$$y_{n+1} = \frac{y_{n-(2k+1)}}{\frac{A}{\gamma} + y_{n-2l}} \quad , n = 0, 1, \dots \tag{5.31}$$

which was treated in Theorem 5.11.

5.5.3.3 The Case $\alpha = A = 0$ and $\delta > 0$

Suppose γ and δ are positive and $\alpha = A = 0$. Then Eq.(5.1) reduces to the equation

$$x_{n+1} = \frac{\gamma x_{n-(2k+1)} + \delta x_{n-2l}}{x_{n-2l}} \quad , \quad n = 0, 1, \dots \tag{5.32}$$

which was treated in Theorem 5.21 (with $A = 0$).

5.5.3.4 The Case $\delta = A = 0$ and $\alpha > 0$

Suppose γ and α are positive and $\delta = A = 0$. Then Eq.(5.1) reduces to the equation

$$x_{n+1} = \frac{\alpha + \gamma x_{n-(2k+1)}}{x_{n-2l}} \quad , \quad n = 0, 1, \dots \tag{5.33}$$

which was treated in Theorem 5.10 (with $A = 0$).

5.5.3.5 The Case $\alpha = 0$ and $\delta, A \in (0, \infty)$

Suppose γ, δ, A are positive and $\alpha = 0$. Then Eq.(5.1) reduces to the equation

$$x_{n+1} = \frac{\gamma x_{n-(2k+1)} + \delta x_{n-2l}}{A + x_{n-2l}} \quad , \quad n = 0, 1, \dots \tag{5.34}$$

which was treated in Theorem 5.21.

5.5.3.6 The Case $\delta = 0$ and $\alpha, A \in (0, \infty)$

Suppose γ, α, A are positive and $\delta = 0$. Then Eq.(5.1) reduces to the equation

$$x_{n+1} = \frac{\alpha + \gamma x_{n-(2k+1)}}{A + x_{n-2l}} \quad , \quad n = 0, 1, \dots \tag{5.35}$$

which was treated in Theorem 5.12.

5.5.3.7 The Case $A = 0$ and $\alpha, \delta \in (0, \infty)$

Suppose $A = 0$ and α, γ, δ are positive. Then Eq.(5.1) reduces to the equation

$$x_{n+1} = \frac{\alpha + \gamma x_{n-(2k+1)} + \delta x_{n-2l}}{x_{n-2l}} \quad , \quad n = 0, 1, \dots . \tag{5.36}$$

The change of variables $x_n = \delta y_n$ transforms Eq.(5.36) into

$$y_{n+1} = \frac{\frac{\alpha}{\delta^2} + \frac{\gamma}{\delta} y_{n-(2k+1)} + y_{n-2l}}{y_{n-2l}} \quad , \quad n = 0, 1, \dots . \tag{5.37}$$

Therefore, given $n \geq 0$, we have

$$y_{n+1} - 1 = \frac{\frac{\alpha}{\delta^2} + \frac{\gamma}{\delta} y_{n-(2k+1)} y_{n-2l}}{>} 0$$

and so we see that $y_{n+1} > 1$. We next make the change of variables

$$y_n = z_n + 1 \qquad \text{for} \qquad n \geq 1,$$

which reduces Eq.(5.37) to

$$z_{n+1} = \frac{\left(\frac{\alpha}{\delta^2} + \frac{\gamma}{\delta}\right) + \frac{\gamma}{\delta} z_{n-(2k+1)}}{1 + z_{n-2l}} \quad , \qquad n = 1, 2, \ldots . \tag{5.38}$$

Finally, we make the change of variables $z_n = \frac{\gamma}{\delta} w_n$, which transforms Eq.(5.38) into the equation

$$w_{n+1} = \frac{\left(\frac{\alpha}{\gamma^2} + \frac{\delta}{\gamma}\right) + w_{n-(2k+1)}}{\frac{\delta}{\gamma} + w_{n-2l}} \quad , \qquad n = 1, 2, \ldots \tag{5.39}$$

which was treated in Theorem 5.12.

5.5.3.8 The Case $\alpha, \delta, A \in (0, \infty)$

Let $\{x_n\}_{n=-\mathcal{K}}^{\infty}$ be a solution of Eq.(5.1). Note that without loss of generality, we may assume that $x_n > 0$ for all $n \geq -\mathcal{K}$.

Case 1. We first consider the case $\alpha \leq (\gamma + \delta)A$.

Let r be the positive root of the equation

$$t^2 + (\gamma + \delta - A)t - \alpha = 0.$$

Let $h : \mathbf{R} \to \mathbf{R}$ be given by

$$h(t) = t^2 + (\gamma + \delta - A)t - \alpha.$$

Then as

$$h(A) = A^2 + (\gamma + \delta - A)A - \alpha \geq 0$$

it follows that $r \leq A$.

For $n \geq -\mathcal{K}$, set

$$w_n = \frac{x_n + r}{\delta + r}.$$

Then it follows that

$$w_{n+1} = \frac{\frac{\gamma}{\delta + r} w_{n-(2k+1)} + w_{n-2l}}{\frac{A-r}{\delta+r} + w_{n-2l}} \qquad \text{for all} \qquad n = 0, 1, \ldots .$$

Let

$$\mathcal{P} = \frac{\gamma}{\delta + r} \qquad \text{and} \qquad \mathcal{Q} = \frac{A - r}{\delta + r}.$$

Then $\{w_n\}_{n=-\mathcal{K}}^{\infty}$ is a positive solution of the difference equation

$$z_{n+1} = \frac{\mathcal{P} z_{n-(2k+1)} + z_{n-2l}}{\mathcal{Q} + z_{n-2l}} \quad , \quad n = 0, 1, \ldots$$

where $\mathcal{P} > 0$ and $\mathcal{Q} \geq 0$.

Consider the difference equation

$$y_{n+1} = \frac{p y_{n-(2k+1)} + y_{n-2l}}{q + y_{n-2l}} \quad , \quad n = 0, 1, \ldots \tag{5.40}$$

where $p > 0$, $q \geq 0$, and the initial conditions are non-negative.

It follows by Theorem 5.21 in Section 5.4 that the following statements are true:

1. Suppose that $p + 1 \leq q$. Then every non-negative solution of Eq.(5.40) converges to $\tilde{y} = 0$.
2. Suppose $p - 1 < q < p + 1$. Then every eventually positive solution of Eq.(5.40) converges to $\bar{y} = p + 1 - q$.
3. Suppose $p - 1 = q$. Then every eventually positive solution of Eq.(5.40) converges to a periodic solution of Eq.(5.40) with period $2d$, and there exist periodic solutions of Eq.(5.40) with prime period $2d$.
4. Suppose $0 \leq q < p - 1$. Then there exist positive unbounded solutions of Eq.(5.40).

Note that

$$0 \leq \mathcal{Q} < \mathcal{P} + 1 \quad \text{if and only if} \quad \frac{A - r}{\delta + r} < \frac{\gamma}{\delta + r} + 1$$

$$\text{if and only if} \quad \frac{A - \gamma - \delta}{2} < r.$$

Now,

$$h\left(\frac{A - \gamma - \delta}{2}\right) = \left(\frac{A - \gamma - \delta}{2}\right)^2 + (\gamma + \delta - A)\left(\frac{A - \gamma - \delta}{2}\right) - \alpha$$

$$= -\left(\frac{A - \gamma - \delta}{2}\right)^2 - \alpha < 0$$

and so we see that it is always the case that

$$0 \leq \mathcal{Q} < \mathcal{P} + 1.$$

We also have

$$\mathcal{P} - 1 < \mathcal{Q} \qquad \text{if and only if} \qquad \frac{\gamma}{\delta + r} - 1 < \frac{A - r}{\delta + r}$$

and so we see that

$$\mathcal{P} - 1 < \mathcal{Q} \qquad \text{if and only if} \qquad \gamma < \delta + A.$$

It follows similarly to the above that

$$\mathcal{P} - 1 = \mathcal{Q} \qquad \text{if and only if} \qquad \gamma = \delta + A$$

and

$$0 \leq \mathcal{Q} < \mathcal{P} - 1 \qquad \text{if and only if} \qquad \gamma > \delta + a.$$

Case 2. We next consider the case $\alpha > (\gamma + \delta)A$.

For $n \geq 0$, we have

$$x_{n+1} - \delta = \frac{(\alpha - \delta A) + \gamma x_{n-(2k+1)}}{A + x_{n-2l}}$$

and so

$$x_n > \delta \qquad \text{for all} \qquad n \geq 1.$$

For $n \geq -\mathcal{K}$, set

$$w_n = x_n - \delta.$$

Then it follows that

$$w_{n+1} = \frac{\frac{\alpha + \gamma\delta - A\delta}{\gamma^2} + w_{n-(2k+1)}}{\frac{A+\delta}{\gamma} + w_{n-2l}} \qquad \text{for all} \qquad n = 0, 1, \ldots .$$

Let

$$\mathcal{P} = \frac{\alpha + \gamma\delta - A\delta}{\gamma^2} \qquad \mathcal{Q} = \frac{A + \delta}{\gamma}.$$

Then $\{w_n\}_{n=-\mathcal{K}}^{\infty}$ is a positive solution of the difference equation

$$w_{n+1} = \frac{\mathcal{P} + w_{n-(2k+1)}}{\mathcal{Q} + w_{n-2l}} \qquad , \qquad n = 0, 1, \ldots .$$

where $\mathcal{P} > 0$ and $\mathcal{Q} > 0$.

Consider the difference equation

$$y_{n+1} = \frac{p + y_{n-(2k+1)}}{q + y_{n-2l}} \quad , \qquad n = 0, 1, \ldots \tag{5.41}$$

where $p > 0$, $q > 0$, and the initial conditions are positive. Note that Eq.(5.41) has the unique equilibrium point $\bar{y} = \dfrac{1 - q + \sqrt{(1-q)^2 + 4p}}{2}$.

It follows by Theorem 5.12 in Section 5.3 that the following statements are true:

1. Suppose $q > 1$. Then every solution of Eq.(5.41) converges to $\bar{y}$.
2. Suppose $q = 1$. Then every solution of Eq.(5.41) converges to a periodic solution of Eq.(5.41) with period $2d$, and there exist periodic solutions of Eq.(5.41) with prime period $2d$.
3. Suppose $0 \leq q < 1$. Then there exist positive unbounded solutions of Eq.(5.41).

The proof of Theorem 5.3 is now complete.

5.6 Open Problems and Conjectures

In many sections throughout this book we have discussed the periodic nature of the solutions of rational difference equations of the form

$$x_{n+1} = \frac{a + a_0 x_n + \cdots + a_k x_{n-k}}{b + b_0 x_n + \cdots + b_k x_{n-k}} \quad , \qquad n = 0, 1, \ldots \tag{5.42}$$

with non-negative parameters and non-negative initial conditions.

A thorough investigation of the periodic character of the solutions of Eq.(5.42) for all values of the parameters and all possible initial conditions is a problem of paramount difficulty whose complete solution we may not see in our lifetime.

Eq.(5.42) offers a rich potential for serious research investigation. Throughout this book we have posed several open problems and conjectures dealing with various special cases of Eq.(5.42) which we believe will result in some deep and beautiful results on the long-term character of solutions of such equations.

When is every solution of Eq.(5.42) periodic?

When does every solution of Eq.(5.42) converge to a periodic solution?

When is every non-equilibrium solution of Eq.(5.42) periodic with prime period $p > 1$?

When does Eq.(5.42) possess a unique p-cycle?

When does Eq.(5.42) possess a finite number of periodic solutions with distinct prime periods $p > 1$?

Can Eq.(5.42) possess a p-cycle and a q-cycle when p and q are relatively prime integers greater than one?

When Eq.(5.42) possesses a unique p-cycle, when does every solution of Eq.(5.42) converge to the equilibrium or the p-cycle?

What is it that makes every solution of Eq.(5.42) converge to a periodic solution?

Recall the trichotomy results, Theorems 5.1 and 5.2, respectively, for Eqs.(5.2) and (5.3).

Another trichotomy result was recently established for the equation

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{A + Bx_n + Dx_{n-2}} \quad , \qquad n = 0, 1, \ldots . \tag{5.43}$$

See [24]. More precisely, the following result is true when $\gamma + A + B > 0$:

1. Every solution of Eq.(5.43) has a finite limit if and only if
$$\gamma < A.$$
2. Every solution of Eq.(5.43) converges to a period-two solution of Eq.(5.43) if and only if $\gamma = A$.
3. There exist positive unbounded solutions of Eq.(5.43) if and only if $\gamma > A$.

These equations are special cases of the third-order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Dx_{n-2}} \quad , \qquad n = 0, 1, \ldots \tag{5.44}$$

with non-negative parameters and with non-negative initial conditions. See [20].

One can see that Eq.(5.44) possesses prime period-2 solutions if and only if

$$\gamma = \beta + \delta + A. \tag{5.45}$$

OPEN PROBLEM 5.1

Obtain necessary and sufficient conditions on $\alpha, \beta, \gamma, \delta, A, B,$ and D so that Eq.(5.44) possesses the following trichotomy:

(i) *Every solution of Eq.(5.44) has a finite limit if and only if*

$$\gamma < \beta + \delta + A.$$

(ii) *Every solution of Eq.(5.44) converges to a (not necessarily prime) period-two solution of Eq.(5.44) if and only if*

$$\gamma = \beta + \delta + A.$$

(iii) *Eq.(5.44) has positive unbounded solutions if and only if*

$$\gamma > \beta + \delta + A.$$

CONJECTURE 5.1

Show that when

$$\gamma > \beta + \delta + A$$

Eq.(5.44) has both bounded and unbounded solutions, and that every bounded solution of Eq.(5.44) converges to an equilibrium point of Eq.(5.44).

OPEN PROBLEM 5.2

In the special case of Eq.(5.44) where

$$\gamma = 0 \qquad \text{and} \qquad \beta + \delta + A > 0$$

under what additional conditions on $\alpha, \beta, \delta, A, B,$ and D is it true that every solution of the equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \delta x_{n-2}}{A + B x_n + D x_{n-2}} \quad , \qquad n = 0, 1, \ldots \tag{5.46}$$

has a finite limit?

Note that in the special case where

$$\gamma = 0 \qquad \text{and} \qquad \beta + \delta + A = 0$$

Eq.(5.44) reduces to

$$x_{n+1} = \frac{\alpha}{B x_n + D x_{n-2}} \quad , \qquad n = 0, 1, \ldots \tag{5.47}$$

for which Statement (ii) of Open Problem 5.1 is true if and only if

$$B > 0.$$

In the special case of Eq.(5.46) where

$$\beta = A = D = 0 \qquad \text{and} \qquad \alpha > 0$$

a necessary and sufficient condition for period-2 trichotomy is $\delta = 0$. When $\delta > 0$, the resulting equation is of the form

$$y_{n+1} = \frac{p + y_{n-2}}{y_n} \quad , \qquad n = 0, 1, \ldots \tag{5.48}$$

with $p > 0$, for which it was shown in [21] that every positive solution of Eq.(5.48) converges to a period-5 solution if and only if $p = 1$. See also [18]. In [18], the following results were also established about the positive solutions of Eq.(5.48).

1. Every positive solution of Eq.(5.48) converges to the equilibrium if $p \geq 2$.

2. Eq.(5.48) has positive unbounded solutions if and only if $p < 1$.

CONJECTURE 5.2

Show that when

$$1 < p < 2$$

every positive bounded solution of Eq.(5.48) converges to the equilibrium of Eq.(5.48).

OPEN PROBLEM 5.3

Assume that

$$\gamma = \beta + \delta + A \quad , \quad A > 0 \qquad \text{and} \qquad B + D > 0.$$

Is it true that every solution of Eq.(5.44) converges to a period-2 solution?

In light of the above discussion in this section, the remaining cases of Eq.(5.44) are the following 28 equations with positive parameters. For each of these equations, the question of paramount importance is whether the trichotomy result, and in particular the result on period-2 convergence, is true. When the trichotomy result fails to hold, what is the region of parameters where local asymptotic stability of the equilibrium implies global asymptotic stability? See [20] and [22].

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{x_n}, \quad n = 0, 1, \dots \tag{5.49}$$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{x_{n-2}}, \quad n = 0, 1, \dots \tag{5.50}$$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + x_n}, \quad n = 0, 1, \dots \tag{5.51}$$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + x_{n-2}}, \quad n = 0, 1, \dots \tag{5.52}$$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{B x_n + D x_{n-2}}, \quad n = 0, 1, \dots \tag{5.53}$$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + D x_{n-2}}, \quad n = 0, 1, \dots \tag{5.54}$$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{x_{n-2}}, \quad n = 0, 1, \dots \tag{5.55}$$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + x_{n-2}}, \quad n = 0, 1, \dots \tag{5.56}$$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{B x_n + D x_{n-2}}, \quad n = 0, 1, \dots \tag{5.57}$$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + B x_n + D x_{n-2}}, \quad n = 0, 1, \dots \tag{5.58}$$

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2}}{x_n}, \quad n = 0, 1, \dots \tag{5.59}$$

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2}}{A + x_n}, \quad n = 0, 1, \dots \tag{5.60}$$

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2}}{B x_n + D x_{n-2}}, \quad n = 0, 1, \dots \tag{5.61}$$

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + D x_{n-2}}, \quad n = 0, 1, \dots \tag{5.62}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{x_n}, \quad n = 0, 1, \dots \tag{5.63}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{x_{n-2}}, \quad n = 0, 1, \dots \tag{5.64}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + x_n}, \quad n = 0, 1, \ldots \tag{5.65}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + x_{n-2}}, \quad n = 0, 1, \ldots \tag{5.66}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{B x_n + D x_{n-2}}, \quad n = 0, 1, \ldots \tag{5.67}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + D x_{n-2}}, \quad n = 0, 1, \ldots \tag{5.68}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1}}{x_{n-2}}, \quad n = 0, 1, \ldots \tag{5.69}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1}}{A + x_{n-2}}, \quad n = 0, 1, \ldots \tag{5.70}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1}}{B x_n + D x_{n-2}}, \quad n = 0, 1, \ldots \tag{5.71}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1}}{A + B x_n + D x_{n-2}}, \quad n = 0, 1, \ldots \tag{5.72}$$

$$x_{n+1} = \frac{\gamma x_{n-1} + \delta x_{n-2}}{x_n}, \quad n = 0, 1, \ldots \tag{5.73}$$

$$x_{n+1} = \frac{\gamma x_{n-1} + \delta x_{n-2}}{A + x_n}, \quad n = 0, 1, \ldots \tag{5.74}$$

$$x_{n+1} = \frac{\gamma x_{n-1} + \delta x_{n-2}}{B x_n + D x_{n-2}}, \quad n = 0, 1, \ldots \tag{5.75}$$

$$x_{n+1} = \frac{\gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + D x_{n-2}}, \quad n = 0, 1, \ldots \tag{5.76}$$

OPEN PROBLEM 5.4

For those cases among Eqs.(5.49) through (5.76) where

$$\gamma < \beta + \delta + A$$

is not a necessary and sufficient condition for the local asymptotic stability of the equilibrium of Eq.(5.44), determine the region of local asymptotic stability, and in particular the value $\gamma^\star \geq 0$ *(if it exists) such that the equilibrium of the equation is locally asymptotically stable for*

$$\gamma^\star < \gamma < \beta + \delta + A.$$

Investigate the (chaotic) character of the solutions when

$$0 < \gamma < \gamma^\star$$

and the trichotomy behavior in the interval $[\gamma^\star, \infty)$.

OPEN PROBLEM 5.5

When does the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + x_n + x_{n-2}} \quad , \qquad n = 0, 1, \ldots \tag{5.77}$$

possess the following period-2 trichotomy?

(i) *Every solution of Eq.(5.77) has a finite limit if and only if*

$$\gamma < \beta + \delta + A.$$

(ii) *Every solution of Eq.(5.77) converges to a period-2 solution if and only if*

$$\gamma = \beta + \delta + A.$$

(iii) *Eq.(5.77) has positive unbounded solutions if and only if*

$$\gamma > \beta + \delta + A.$$

CONJECTURE 5.3

Show that except for Eqs.(5.2), (5.3), and (5.43), no other special case of Eq.(5.44) may possess the trichotomy character described in Open Problem 5.1.

OPEN PROBLEM 5.6

Consider the rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + C x_{n-1} + D x_{n-2}} \quad , \qquad n = 0, 1, \ldots \tag{5.78}$$

with non-zero parameters and non-zero initial conditions. Let p *be a given positive integer. Obtain necessary and sufficient conditions on* p *and the parameters of Eq.(5.78) so that every solution of Eq.(5.78) converges to a periodic solution of with period* p.

When $p \in \{4, 5, 6\}$, the following are special cases of Eq.(5.78) where every solution converges to a periodic solution with period p.

Every positive solution of

$$x_{n+1} = \frac{\alpha + x_n + x_{n-2}}{x_{n-1}} \quad , \qquad n = 0, 1, \ldots \tag{5.79}$$

converges to a period-4 solution.

Every positive solution of

$$x_{n+1} = \frac{1 + x_{n-2}}{x_n} \quad , \qquad n = 0, 1, \ldots \tag{5.80}$$

converges to a period-5 solution.

Every positive solution of

$$x_{n+1} = \frac{1 + x_n}{x_{n-1} + x_{n-2}} \quad , \qquad n = 0, 1, \ldots \tag{5.81}$$

converges to a period-6 solution.

OPEN PROBLEM 5.7

For $p \geq 7$, are there any non-trivial examples of Eq.(5.78) where every non-negative solution converges to a periodic solution with period p?

OPEN PROBLEM 5.8

Assume that k, l, m are non-negative integers. Investigate the asymptotic behavior, the periodic nature, and the boundedness character of the solutions of the difference equation

$$x_{n+1} = \frac{\alpha + \gamma x_{n-(2k+1)}}{A + B x_{n-2l} + x_{n-2m}} \quad , \qquad n = 0, 1, \ldots$$

with non-negative parameters and non-negative initial conditions.

For the case $k = l = 0$ and $m = 1$, see [24].

A period-2 convergence result and the existence of unbounded solutions were recently established for the equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + x_n} \quad , \qquad n = 0, 1, \ldots . \tag{5.82}$$

See [22]. More precisely, the following results were established.

THEOREM 5.22
Assume that

$$\gamma = \beta + \delta + A \qquad \text{and} \qquad \beta + A > 0.$$

Then every solution of Eq.(5.82) converges to a (not necessarily prime) period-2 solution of Eq.(5.82).

THEOREM 5.23
Assume that

$$\gamma > \beta + \delta + A.$$

Then Eq.(5.82) has unbounded solutions. More precisely, let k be a number chosen such that

$$0 < k < \gamma - \beta - \delta - A$$

and let $\{x_n\}_{n=-2}^{\infty}$ be any solution of Eq.(5.82) with

$$x_{-2}, x_0 \in (0, \gamma - A) \qquad \text{and} \qquad x_{-1} > \frac{\alpha + \gamma(\gamma - A)}{k}.$$

Then

$$\lim_{n\to\infty} x_{2n+1} = \infty \qquad \text{and} \qquad \lim_{n\to\infty} x_{2n} = \frac{\beta\gamma + \delta A}{\gamma - \delta}.$$

An obvious problem which was not investigated in [22] is the following:

OPEN PROBLEM 5.9
Investigate the character of the solutions of Eq.(5.82) when

$$\gamma < \beta + \delta + A.$$

Chapter 6

MAX EQUATIONS WITH PERIODIC SOLUTIONS

6.1 Introduction

In this chapter we investigate the periodic character and the boundedness nature of the solutions of some autonomous difference equations of the form

$$x_{n+1} = \frac{\max\left\{x_n^k, A\right\}}{x_n^l x_{n-1}} \quad , \qquad n = 0, 1, \ldots \tag{6.1}$$

where A is a positive real number, k and l are natural numbers, and where the initial conditions x_{-1} and x_0 are arbitrary positive real numbers.

As we saw in Section 2.7, these equations can be transformed into piecewise linear difference equations of the form

$$y_{n+1} = \frac{k}{2}|y_n| + \left(\frac{k}{2} - l\right) y_n - y_{n-1} + \delta \quad , \qquad n = 0, 1, \ldots$$

where

$$\delta = \begin{cases} k - l - 2 & \text{if } A > 1 \\ 0 & \text{if } A = 1 \\ -(k - l - 2) & \text{if } A < 1. \end{cases}$$

Eq.(6.1) was investigated in [43] and [67] when $k = 1$ and $l = 0$, in [45] when $k = 1$ and $l = 1$, in [32] when $k = 2$ and $l = 1$, and in [69] when $k = 1$ and $l = 2$. See also [13], [29], [31], [67], [68], [83], and [117].

6.2 The Max Equation $x_{n+1} = \dfrac{\max\{x_n, A\}}{x_{n-1}}$

Consider the difference equation

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_{n-1}} \quad , \qquad n = 0, 1, \ldots \tag{6.2}$$

where the parameter A is a positive real number and the initial conditions x_{-1}, x_0 are arbitrary positive constants. See [43] and [67].

The study of this equation was motivated by a problem in [42]. Given a positive integer ν, it was shown in [42] that every solution of

$$a_{n+\nu} = \max\{a_{n+1}, a_{n+2}, \ldots, a_{n+\nu-1}, 0\} - a_n \quad , \quad n = 0, 1, \ldots \tag{6.3}$$

with monotonic initial conditions $\{a_0, a_1, \ldots, a_{\nu-1}\}$ is periodic with period $3\nu - 1$. In particular, it follows that for $\nu = 2$, every solution of

$$a_{n+2} = \max\{a_{n+1}, 0\} - a_n \quad , \quad n = 0, 1, \ldots \tag{6.4}$$

is periodic with period 5. The change of variables

$$x_{n-1} = e^{a_n} \quad \text{for} \quad n \geq 0$$

reduces Eq.(6.4) to Eq.(6.2) with $A = 1$. This is the max variant of Lyness' equation

$$x_{n+1} = \frac{x_n + A}{x_{n-1}} \quad , \quad n = 0, 1, \ldots \tag{6.5}$$

every solution of which is periodic with period 5 if and only if $A = 1$. Eq.(6.5) has been thoroughly investigated, and a wealth of information is known about its solutions. See Section 2.4, p.138 of [73], [74], and the references cited within.

6.2.1 Boundedness and Persistence of Solutions

The following result shows that Eq.(6.2) possesses the invariant

$$\max\left\{1, \frac{1}{x_{n-1}}\right\} \max\left\{1, \frac{1}{x_n}\right\} \max\{A, x_{n-1}, x_n\} = \text{constant} \quad \text{for all} \quad n \geq 0.$$

This invariant is an indispensable tool for showing, among other things, that every positive solution of Eq.(6.2) is bounded and persists.

LEMMA 6.1
Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(6.2), and for $n \geq 0$, set

$$I_n = \max\left\{1, \frac{1}{x_{n-1}}\right\} \max\left\{1, \frac{1}{x_n}\right\} \max\{A, x_{n-1}x_n\}.$$

Then $I_n = I_0$ for all $n \geq 0$.

PROOF Observe that for $n \geq 0$,

$$\begin{aligned}
I_{n+1} &= \max\left\{1, \frac{1}{x_n}\right\} \max\left\{1, \frac{1}{x_{n+1}}\right\} \max\{A, x_n, x_{n+1}\} \\
&= \max\left\{1, \frac{1}{x_n}\right\} \max\left\{1, \frac{x_{n-1}}{\max\{x_n, A\}}\right\} \max\left\{\max\{x_n, A\}, \frac{\max\{x_n, A\}}{x_{n-1}}\right\} \\
&= \max\left\{1, \frac{1}{x_n}\right\} \frac{\max\{\max\{A, x_n\}, x_{n-1}\}}{\max\{x_n, A\}} \max\{x_n, A\} \max\left\{1, \frac{1}{x_{n-1}}\right\} \\
&= \max\left\{1, \frac{1}{x_n}\right\} \max\{x_n, A, x_{n-1}\} \max\left\{1, \frac{1}{x_{n-1}}\right\} \\
&= I_n
\end{aligned}$$

from which the result follows. ▯

THEOREM 6.1
Every solution of Eq.(6.2) is bounded and persists.

PROOF The proof follows immediately from Lemma 6.1. ▯

6.2.2 Oscillation of Solutions

The proofs of the next two lemmas follow directly by computation.

LEMMA 6.2
Eq.(6.2) has a unique positive equilibrium point $\bar{x}$. Furthermore,

$$\bar{x} = \begin{cases} 1 & \text{if} \quad A \leq 1 \\ \sqrt{A} & \text{if} \quad A > 1. \end{cases}$$

LEMMA 6.3
Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(6.2), and suppose that there exists $N \geq -1$ such that $x_n = \bar{x}$ for all $n \geq N$. Then $x_n = \bar{x}$ for all $n \geq -1$.

In the following lemma, we describe the semi-cycle character of the solutions of Eq.(6.2).

LEMMA 6.4

Let $\{x_n\}_{n=-1}^{\infty}$ be a non-trivial solution of Eq.(6.2). Then the following statements are true:

1. *With the possible exception of the first semi-cycle, a positive semi-cycle has at least two terms and the second term is greater than $\bar{x}$.*
2. *Every positive semi-cycle has at most four terms, and if a positive semi-cycle $x_{N-1}, x_N, x_{N+1}, x_{N+2}$ does have four terms, then $A \leq 1$ and*
$$x_{N-1} = x_{N+2} = \bar{x} < x_N = x_{N+1}.$$
3. *If $A \geq 1$, then every negative semi-cycle has at most two terms. Furthermore, if a negative semi-cycle has exactly one term and is preceded by a positive semi-cycle with at least two terms, then the term immediately preceding it is $\bar{x}$, and the term immediately following it is also $\bar{x}$.*
4. *If $A < 1$, then every negative semi-cycle has at most three terms. Every negative semi-cycle which follows a positive semi-cycle has exactly two terms, and the second term is greater than or equal to the first term.*
5. *With the possible exception of the first semi-cycle, a positive semi-cycle has the property that the maximum term occurs in either the first or second term, and the terms in the positive semi-cycle after the maximum are non-increasing.*

PROOF

(i) Suppose $x_{N-1} < \bar{x} \leq x_N$. Then $x_{N+1} = \dfrac{\max\{x_N, A\}}{x_{N-1}} > \dfrac{\max\{\bar{x}, A\}}{\bar{x}} = \bar{x}$.

(ii) Suppose $x_{N-1}, x_N, x_{N+1}, x_{N+2} \in [\bar{x}, \infty)$.

We claim that $A \leq 1$.

For the sake of contradiction, suppose $A > 1$. Then $\bar{x} = \sqrt{A} > 1$. Now
$$x_{N+1} = \frac{\max\{x_N, A\}}{x_{N-1}}.$$

Suppose $\max\{x_N, A\} = x_N$.

Then in particular, $x_N \geq A$. Now $x_{N+1} = \dfrac{x_N}{x_{N-1}}$, and so
$$x_{N+2} = \frac{\max\{x_{N+1}, A\}}{x_N} = \frac{\max\left\{\frac{x_N}{x_{N-1}}, A\right\}}{x_N} = \frac{\max\{x_N, x_{N-1}A\}}{x_{N-1}x_N}.$$

Suppose $\max\{x_N, x_{N-1}A\} = x_N$. Then

$$x_{N+2} = \frac{x_N}{x_{N-1}x_N} = \frac{1}{x_{N-1}} \leq \frac{1}{\sqrt{A}} < 1 < \bar{x},$$

which is impossible. Thus $\max\{x_N, x_{N-1}A\} = x_{N-1}A$, and so

$$\sqrt{A} \leq x_{N+2} = \frac{x_{N-1}A}{x_{N-1}x_N} = \frac{A}{x_N} \leq \sqrt{A}.$$

Hence $x_N = \sqrt{A}$, which is also impossible because we have assumed that $x_N \geq A > \sqrt{A}$.

Thus we must have $\max\{x_N, A\} = A$.

Hence

$$\sqrt{A} \leq x_{N+1} = \frac{\max\{x_N, A\}}{x_{N-1}} = \frac{A}{x_{N-1}} \leq \sqrt{A}$$

and so $x_{N-1} = x_{N+1} = \sqrt{A}$. Therefore, as $\{x_n\}_{n=-1}^{\infty}$ is a non-trivial solution of Eq.(6.2), we see that $x_N > \sqrt{A}$. Hence

$$x_{N+2} = \frac{\max\{x_{N+1}, A\}}{x_N} = \frac{\max\left\{\sqrt{A}, A\right\}}{x_N} = \frac{A}{x_N} < \sqrt{A} = \bar{x},$$

which is also impossible.

Thus we see that the claim is true, and so $A \leq 1$. In particular,

$$x_{N-1}, x_N, x_{N+1}, x_{N+2} \in [1, \infty).$$

Therefore,

$$1 \leq x_{N+2} = \frac{\max\{x_{N+1}, 1\}}{x_N} = \frac{x_{N+1}}{x_N} = \frac{\frac{\max\{x_N,1\}}{x_{N-1}}}{x_N} = \frac{\left(\frac{x_N}{x_{N-1}}\right)}{x_N} = \frac{1}{x_{N-1}} \leq 1$$

and so we see that

$$x_{N-1} = x_{N+2} = 1 = \bar{x}.$$

Hence

$$x_{N+1} = \frac{\max\{x_N, 1\}}{x_{N-1}} = \frac{x_N}{x_{N-1}} = x_N$$

and so, as $\{x_n\}_{n=-1}^{\infty}$ is a non-trivial solution of Eq.(6.2), it follows that

$$x_{N-1} = x_{N+2} = \bar{x} < x_N = x_{N+1}.$$

(iii) Suppose $A \geq 1$, and that $x_{N-1}, x_N \in (0, \bar{x})$. Then

$$x_{N+1} = \frac{\max\{x_N, A\}}{x_{N-1}} = \frac{A}{x_{N-1}} > \sqrt{A} = \bar{x}$$

and so x_{N+1} is the first term of a positive semi-cycle.

Suppose $x_{N-1}, x_N, x_{N+2} \in [\bar{x}, \infty)$ and $x_{N+1} < \bar{x}$. Then

$$\sqrt{A} \leq x_{N+2} = \frac{\max\{x_{N+1}, A\}}{x_N} \leq \frac{A}{\sqrt{A}} = \sqrt{A}$$

and so

$$x_{N+2} = \sqrt{A}.$$

In particular,

$$\sqrt{A} = x_{N+2} = \frac{\max\{x_{N+1}, A\}}{x_N} = \frac{A}{x_N}$$

and thus

$$x_N = \sqrt{A}.$$

(iv) Suppose $A < 1$.

We first suppose that $x_{N-1}, x_N \in (0, 1)$. Then

$$x_{N+2} = \frac{\max\{x_{N+1}, A\}}{x_N} = \frac{\max\{x_N, A, x_{N-1}A\}}{x_{N-1}x_N}$$

$$= \frac{\max\{x_N, A\}}{x_{N-1}x_N} \geq \frac{x_N}{x_{N-1}x_N} = \frac{1}{x_{N-1}} > 1 = \bar{x}$$

and so every negative semi-cycle has at most three terms.

We next suppose $x_N < 1 \leq x_{N-1}$. Then

$$x_{N+1} = \frac{\max\{x_N, A\}}{x_{N-1}} < \frac{1}{x_{N-1}} \leq 1.$$

Also,

$$x_{N+1} = \frac{\max\{x_N, A\}}{x_{N-1}} \geq \frac{\max\{x_N, A\}}{1} \geq x_N$$

and

$$x_{N+2} = \frac{\max\{x_{N+1}, A\}}{x_N} \geq \frac{x_{N+1}}{x_N} \geq \frac{x_N}{x_N} = 1$$

and so every negative semi-cycle preceded by a positive semi-cycle has exactly two terms, and the second term is greater than or equal to the first term.

(v) Suppose $x_{N-1} < \bar{x} \leq x_N$.

By Statements (i) and (ii) of this theorem, it suffices to assume that x_N is the first term in a positive semi-cycle of length either two or three, and to show that

$$x_{N+2} \leq x_{N+1}.$$

Case 1 Suppose $A \leq 1$.

Then $\bar{x} = 1$. Thus $x_{N-1} < 1 \leq x_N$, and so

$$x_{N+1} = \frac{\max\{x_N, A\}}{x_{N-1}} > \frac{x_N}{1} = x_N$$

and

$$x_{N+2} = \frac{\max\{x_{N+1}, A\}}{x_N} = \frac{x_{N+1}}{x_N} \leq x_{N+1}.$$

Case 2 Suppose $A > 1$.

Then $\bar{x} = \sqrt{A}$, and so $x_{N-1} < \sqrt{A} \leq x_N$.

Case 2 (a) Suppose $x_N \geq A$.

Then

$$x_{N+1} = \frac{\max\{x_N, A\}}{x_{N-1}} = \frac{x_N}{x_{N-1}}$$

and so

$$x_{N+2} = \frac{\max\{x_{N+1}, A\}}{x_N} = \frac{\max\{x_N, x_{N-1}A\}}{x_{N-1}x_N}.$$

Note that if $x_{N-1}A \geq x_N$, then as $x_N \geq A$, we have

$$x_{N+2} = \frac{x_{N-1}A}{x_{N-1}x_N} = \frac{A}{x_N} \leq 1 < \sqrt{A} < x_{N+1}$$

and so it suffices to consider the case $x_{N-1}A < x_N$. Hence

$$x_{N+2} = \frac{x_N}{x_{N-1}x_N} = \frac{1}{x_{N-1}} < \frac{x_N}{x_{N-1}} = x_{N+1}.$$

Case 2 (b) Suppose $\sqrt{A} \leq x_N < A$.

Then

$$x_{N+1} = \frac{\max\{x_N, A\}}{x_{N-1}} = \frac{A}{x_{N-1}}$$

and so

$$\begin{aligned} x_{N+2} &= \frac{\max\{x_{N+1}, A\}}{x_N} = \frac{\max\{A, Ax_{N-1}\}}{x_{N-1}x_N} \\ &= \frac{\max\{A, Ax_{N-1}\}}{x_N A} x_{N+1} = \max\left\{\frac{1}{x_N}, \frac{x_{N-1}}{x_N}\right\} x_{N+1} \\ &< x_{N+1}. \end{aligned}$$

□

The next theorem follows immediately from Lemma 6.4.

THEOREM 6.2
Every non-trivial solution of Eq.(6.2) is strictly oscillatory about $\bar{x}$.

6.2.3 Periodicity of Solutions of Eq.(6.2)

THEOREM 6.3
Suppose $A = 1$. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(6.2). Then $\{x_n\}_{n=-1}^{\infty}$ is periodic with period 5. In fact, after re-labeling $\{x_n\}_{n=-1}^{\infty}$, if necessary, $\{x_n\}_{n=-1}^{\infty}$ is of the following form:

$$x_{-1},\ x_0,\ x_1 = \frac{1}{x_{-1}},\ x_2 = \frac{1}{x_{-1}x_0},\ x_3 = \frac{1}{x_0},\ x_4 = x_{-1},\ x_5 = x_0, \dots. \tag{6.6}$$

PROOF Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of Eq.(6.2). Then

$$y_1 = \frac{\max\{y_0, 1\}}{y_{-1}}.$$

Case 1 Suppose $0 < y_0 \leq 1$. Then

$$y_1 = \frac{1}{y_{-1}}$$

and

$$y_2 = \frac{\max\left\{\frac{1}{y_{-1}}, 1\right\}}{y_0}.$$

Case 1(a) Suppose $1 \leq y_{-1}$.
Then $0 < y_0 \leq 1 \leq y_{-1}$, and so

$$y_2 = \frac{1}{y_0} \quad , \quad y_3 = \frac{\max\left\{\frac{1}{y_0}, 1\right\}}{\frac{1}{y_{-1}}} = \frac{y_{-1}}{y_0}$$

$$y_4 = \frac{\max\left\{\frac{y_{-1}}{y_0}, 1\right\}}{\frac{1}{y_0}} = y_{-1} \quad , \quad y_5 = \frac{\max\{y_{-1}, 1\}}{\frac{y_{-1}}{y_0}} = y_0.$$

Therefore, if $\{x_n\}_{n=-1}^{\infty}$ is the solution of Eq.(6.2) with $x_{-1} = y_0$ and $x_0 = y_1$, then $\{x_n\}_{n=-1}^{\infty}$ satisfies (6.6).

Case 1(b) Suppose $0 < y_{-1} < 1$.
Then $0 < y_{-1} < 1$ and $0 \le y_0 \le 1$. Hence

$$y_2 = \frac{1}{y_{-1}y_0} \quad , \quad y_3 = \frac{\max\left\{\frac{1}{y_{-1}y_0}, 1\right\}}{\frac{1}{y_{-1}}} = \frac{1}{y_0}$$

$$y_4 = \frac{\max\left\{\frac{1}{y_0}, 1\right\}}{\frac{1}{y_{-1}y_0}} = y_{-1} \quad , \quad y_5 = \frac{\max\{y_{-1}, 1\}}{\frac{1}{y_0}} = y_0.$$

Therefore, if $\{x_n\}_{n=-1}^{\infty}$ is the solution of Eq.(6.2) with $x_{-1} = y_{-1}$ and $x_0 = y_0$, then $\{x_n\}_{n=-1}^{\infty}$ satisfies (6.6).

Case 2 Suppose $1 < y_0$. Then

$$y_1 = \frac{y_0}{y_{-1}}$$

and

$$y_2 = \frac{\max\left\{\frac{y_0}{y_{-1}}, 1\right\}}{y_0}.$$

Case 2(a) Suppose $y_0 \le y_{-1}$.
Then $1 < y_0 \le y_{-1}$. Hence

$$y_2 = \frac{1}{y_0} \quad , \quad y_3 = \frac{\max\left\{\frac{1}{y_0}, 1\right\}}{\frac{y_0}{y_{-1}}} = \frac{y_{-1}}{y_0}$$

$$y_4 = \frac{\max\left\{\frac{y_{-1}}{y_0}, 1\right\}}{\frac{1}{y_0}} = y_{-1} \quad , \quad y_5 = \frac{\max\{x_{-1}, 1\}}{\frac{y_{-1}}{y_0}} = y_0.$$

Therefore, if $\{x_n\}_{n=-1}^{\infty}$ is the solution of Eq.(6.2) with $x_{-1} = y_1$ and $x_0 = y_2$, then $\{x_n\}_{n=-1}^{\infty}$ satisfies (6.6).

Case 2(b) Suppose $0 < y_{-1} < y_0$.
Then $0 < y_{-1} < y_0$ and $1 < y_0$. We have

$$y_2 = \frac{1}{y_{-1}}$$

and

$$y_3 = \frac{\max\left\{\frac{1}{y_{-1}}, 1\right\}}{\frac{y_0}{y_{-1}}}.$$

Case 2(b)(i) Suppose $1 \leq y_{-1}$.
Then $1 \leq y_{-1} < y_0$, and so

$$y_3 = \frac{y_{-1}}{y_0}, \quad y_4 = \frac{\max\left\{\frac{y_{-1}}{y_0}, 1\right\}}{\frac{1}{y_{-1}}} = y_{-1}, \quad y_5 = \frac{\max\{x_{-1}, 1\}}{\frac{y_{-1}}{y_0}} = y_0.$$

Therefore, if $\{x_n\}_{n=-1}^{\infty}$ is the solution of Eq.(6.2) with $x_{-1} = y_2$ and $x_0 = y_3$, then $\{x_n\}_{n=-1}^{\infty}$ satisfies (6.6).

Case 2(b)(ii) Suppose $0 < y_{-1} < 1$.
Then $0 < y_{-1} < 1 < y_0$, and so

$$y_3 = \frac{1}{y_0}, \quad y_4 = \frac{\max\left\{\frac{1}{y_0}, 1\right\}}{\frac{1}{y_{-1}}} = y_{-1}, \quad y_5 = \frac{\max\{y_{-1}, 1\}}{\frac{1}{y_0}} = y_0.$$

Therefore, if $\{x_n\}_{n=-1}^{\infty}$ is the solution of Eq.(6.2) with $x_{-1} = y_3$ and $x_0 = y_4$, then $\{x_n\}_{n=-1}^{\infty}$ satisfies (6.6). ▯

REMARK 6.1 Suppose $A = 1$. Let $\{y_n\}_{n=-1}^{\infty}$ be a non-trivial solution of Eq.(6.2). Then it is of some interest to note that the re-labeling process in the proof of Theorem 6.3 is as follows: regardless of the choice of initial conditions $(y_{-1}, y_0) \in (0,\infty) \times (0,\infty)$ of $\{y_n\}_{n=-1}^{\infty}$, there exists an integer N with $-1 \leq N \leq 3$ such that

$$y_N \in (0,1],\ y_{N+1} \in (0,1],\ y_{-1}y_0 < 1,\ y_{N+2} \in [1,\infty),\ y_{N+3} \in (1,\infty)$$

$$y_{N+4} \in [1,\infty)$$

and x_{-1} and x_0 are chosen such that

$$x_{-1} = y_N \quad \textit{and} \quad x_0 = y_{N+1}.$$

▯

If $A > 1$, set

$$\mathcal{P}_4 = \{(u,v)\} : 1 \leq u \leq A \quad \textit{and} \quad 1 \leq v \leq A\}$$

$$\mathcal{CP}_4 = (0,\infty) \times (0,\infty) - \mathcal{P}_4$$

while if $0 < A < 1$, set

$$\mathcal{P}_6 = \left\{(u,v) : A \leq u \leq \frac{1}{A},\ A \leq v \leq \frac{1}{A},\ \textit{and}\ Au \leq v \leq \frac{1}{A}u\right\}$$

$$\mathcal{CP}_6 = (0,\infty) \times (0,\infty) - \mathcal{P}_6.$$

Note that if $A > 1$, then

$$(\bar{x}, \bar{x}) = (\sqrt{A}, \sqrt{A}) \text{ is an interior point of } \mathcal{P}_4,$$

while if $0 < A < 1$, then

$$(\bar{x}, \bar{x}) = (1, 1) \text{ is an interior point of } \mathcal{P}_6.$$

The next two theorems state that Eq.(6.2) has an invariant rectangle of solutions with period 4 if $A > 1$, while Eq.(6.2) has an invariant hexagon of solutions with period 6 if $A < 1$.

THEOREM 6.4

Suppose $A > 1$. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(6.2). Then the following statements are true:

1. *Suppose $(x_{-1}, x_0) \in \mathcal{P}_4$. Then $(x_n, x_{n+1}) \in \mathcal{P}_4$ for all $n \geq 0$. Moreover, $\{x_n\}_{n=-1}^{\infty}$ is periodic with period 4 and is given by*

$$x_{-1}, x_0, \frac{A}{x_{-1}}, \frac{A}{x_0}, x_{-1}, x_0, \ldots .$$

2. *Suppose $(x_{-1}, x_0) \in \mathcal{CP}_4$. Then $(x_n, x_{n+1}) \in \mathcal{CP}_4$ for all $n \geq 0$. Furthermore there exist sub-sequences $\{x_{m_i}\}_{i=0}^{\infty}$ and $\{x_{M_i}\}_{i=0}^{\infty}$ of $\{x_n\}_{n=-1}^{\infty}$ such that*

$$x_{m_i} < 1 \quad and \quad x_{M_i} > A \qquad for\ all \qquad i \geq 0. \tag{6.7}$$

PROOF

(i) Suppose that $(x_{-1}, x_0) \in \mathcal{P}_4$. Then

$$x_1 = \frac{\max\{x_0, A\}}{x_{-1}} = \frac{A}{x_{-1}} \in [1, A] \ , \ x_2 = \frac{\max\left\{\frac{A}{x_{-1}}, A\right\}}{x_0} = \frac{A}{x_0} \in [1, A]$$

$$x_3 = \frac{\max\left\{\frac{A}{x_0}, A\right\}}{\frac{A}{x_{-1}}} = x_{-1} \in [1, A] \ , \ x_4 = \frac{\max\{x_{-1}, A\}}{\frac{A}{x_0}} = x_0 \in [1, A]$$

and so we see that $\{x_n\}_{n=-1}^{\infty}$ is periodic with period 4, and that $(x_n, x_{n+1}) \in \mathcal{P}_4$ for all $n \geq -1$.

(ii) Suppose that $(x_{-1}, x_0) \in \mathcal{CP}_4$.

We shall first show that $(x_n, x_{n+1}) \in \mathcal{CP}_4$ for all $n \geq -1$. For the sake of contradiction, suppose that there exists an integer $N \geq 0$ such that $(x_N, x_{N+1}) \in \mathcal{P}_4$. Then

$$x_{N+1} = \frac{\max\{x_N, A\}}{x_{N-1}} = \frac{A}{x_{N-1}}$$

and so

$$x_{N-1} = \frac{A}{x_{N+1}} \in [1, A,].$$

Hence $(x_{N-1}, x_N) \in \mathcal{P}_4$. It follows by induction that $(x_{-1}, x_0) \in \mathcal{P}_4$, which is a contradiction.

Thus we see that it is true that $(x_n, x_{n+1}) \in \mathcal{CP}_4$ for all $n \geq -1$.

To complete the proof, we must show that there exist sub-sequences $\{x_{m_i}\}_{i=0}^{\infty}$ and $\{x_{M_i}\}_{i=0}^{\infty}$ of $\{x_n\}_{n=-1}^{\infty}$ such that (6.7) holds.

We shall first prove the existence of the sub-sequence $\{x_{m_i}\}_{i=0}^{\infty}$. For the sake of contradiction, suppose that there exists an integer $n_0 \geq -1$ such that

$$x_n \geq 1 \qquad \text{for all} \qquad n \geq n_0.$$

Then for $n \geq n_0$, we have

$$x_{n+2} = \frac{\max\{x_{n+1}, A\}}{x_n} \leq \max\{x_{n+1}, A\}. \tag{6.8}$$

Suppose there exists $n \geq n_0$ such that $x_n \leq A$. Then by (6.8), we see that $x_{n+1} \leq A$, and so it follows by Statement (i) of this theorem that $(x_{-1}, x_0) \in \mathcal{P}_4$. This is impossible, and so we see that $x_n > A$ for all $n \geq n_0$. This is also impossible since $\{x_n\}_{n=-1}^{\infty}$ is strictly oscillatory about $\bar{x} = \sqrt{A}$, and the proof is complete.

We shall next prove the existence of the sequence $\{x_{M_i}\}_{i=0}^{\infty}$. For the sake of contradiction, suppose that there exists an integer $n_0 \geq -1$ such that

$$x_n \leq A \qquad \text{for all} \qquad n \geq n_0.$$

Then, given $n \geq n_0$, we see that

$$x_{n+2} = \frac{\max\{x_{n+1}, A\}}{x_n} = \frac{A}{x_n} \geq 1.$$

It follows that $(x_{n_0+1}, x_{n_0+2}) \in \mathcal{P}_4$, and hence by by Statement (i) of this theorem that $(x_{-1}, x_0) \in \mathcal{P}_4$. This is a contradiction, and the proof is complete. □

THEOREM 6.5

Suppose $0 < A < 1$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(6.2). Then the following statements are true:*

1. *Suppose* $(x_{-1}, x_0) \in \mathcal{P}_6$. *Then* $(x_n, x_{n+1}) \in \mathcal{P}_6$ *for all* $n \geq 0$. *Moreover,* $\{x_n\}_{n=-1}^{\infty}$ *is periodic with period 6 and is given by*

$$x_{-1}, x_0, \frac{x_0}{x_{-1}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{x_{-1}}{x_0}, x_{-1}, x_0, \ldots .$$

2. *Suppose* $(x_{-1}, x_0) \in \mathcal{CP}_6$. *Then* $(x_n, x_{n+1}) \in \mathcal{CP}_6$ *for all* $n \geq 0$.

PROOF

(i) Suppose $(x_{-1}, x_0) \in \mathcal{P}_6$.

It suffices to show that $x_1 = \dfrac{x_0}{x_{-1}}$, and that $(x_0, x_1) \in \mathcal{P}_6$. Now

$$x_1 = \frac{\max\{x_0, A\}}{x_{-1}} = \frac{x_0}{x_{-1}}.$$

Note that

$$A \leq \frac{x_0}{x_{-1}} \quad \text{since } Ax_{-1} \leq x_0$$

$$\frac{x_0}{x_{-1}} \leq \frac{1}{A} \quad \text{since } Ax_0 \leq x_{-1}$$

$$Ax_0 \leq \frac{x_0}{x_{-1}} \quad \text{since} \quad x_{-1} \leq \frac{1}{A}$$

$$\frac{x_0}{x_{-1}} \leq \frac{1}{A}x_0 \quad \text{since} \quad A \leq x_{-1}$$

and so $(x_0, x_1) \in \mathcal{P}_6$.

(ii) Suppose $(x_{-1}, x_0) \in \mathcal{CP}_6$.

We shall show that $(x_n, x_{n+1}) \in \mathcal{CP}_6$ for all $n \geq -1$. For the sake of contradiction, suppose there exists an integer $N \geq 0$ such that $(x_N, x_{N+1}) \in \mathcal{P}_6$. Then

$$x_{N+1} = \frac{\max\{x_N, A\}}{x_{N-1}} = \frac{x_N}{x_{N-1}}$$

and so

$$x_{N-1} = \frac{x_N}{x_{N+1}}.$$

Now by Statement 1 of this theorem,

$$x_{N+1} = x_{N+7} = \frac{x_{N+6}}{x_{N+5}} = \frac{x_N}{x_{N+5}}$$

and hence

$$x_{N+5} = \frac{x_N}{x_{N+1}} = x_{N-1}.$$

Thus $(x_{N-1}, x_N) = (x_{N+5}, x_{N+6}) \in \mathcal{P}_6$. It follows by induction that $(x_{-1}, x_0) \in \mathcal{P}_6$. This is a contradiction, and the proof is complete. ▯

The proof of Corollary 6.1 follows directly from Theorems 6.3, 6.4, and 6.5.

COROLLARY 6.1

The equilibrium $\bar{x}$ of Eq.(6.2) is stable but is not locally asymptotically stable.

REMARK 6.2 It is interesting to note that the proof of Corollary 6.1 was accomplished by using only elementary methods. It is known that the equilibrium of Lyness' Eq.(2.4) is also stable but is not locally asymptotically stable. The proof of this, however, is quite subtle, and uses, among other things, the powerful and deep technique known as KAM Theory. See [75]. ▯

THEOREM 6.6

No non-trivial solution of Eq.(6.2) has a limit.

PROOF By Theorems 6.3, 6.4, and 6.5, we need only consider the case where $A < 1$, and where $\{x_n\}_{n=-1}^{\infty}$ is a solution of Eq.(6.2) with $(x_{-1}, x_0) \in \mathcal{CP}_6$.

So assume $A < 1$, and let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(6.2) with $(x_{-1}, x_0) \in \mathcal{CP}_6$. It suffices to show that $\{x_n\}_{n=-1}^{\infty}$ has no limit.

By Lemma 6.4, it follows that without loss of generality we may assume that in each positive semi-cycle of $\{x_n\}_{n=-1}^{\infty}$, the maximum term of $\{x_n\}_{n=-1}^{\infty}$ is greater than 1. Let x_{M_i} be the maximum term in the i^{th} positive semi-cycle.

Set

$$I_0 = \max\left\{1, \frac{1}{x_{-1}}\right\} \max\left\{1, \frac{1}{x_0}\right\} \max\{1, x_{-1}, x_0\}.$$

Then for $i \geq 0$, it follows by Lemma 6.4 that

$$x_{M_i} > 1 \qquad \text{and} \qquad x_{M_1-1} = \frac{\max\{x_{M_i}, 1\}}{x_{M_i+1}} = \frac{x_{M_i}}{x_{M_i+1}} \geq 1$$

and so it follows by Lemma 6.1 that

$$x_{M_i} = I_{M_i} = I_0.$$

Thus we see that $I_0 > 1$, and that

$$\limsup_{n\to\infty} x_n = I_0.$$

Now since $\{x_n\}_{n=-1}^{\infty}$ is strictly oscillating about $\bar{x} = 1$, we see that

$$\liminf_{n\to\infty} x_n \leq 1$$

and the proof is complete. ▯

6.3 The Max Equation $x_{n+1} = \dfrac{\max\{x_n, A\}}{x_n x_{n-1}}$

Consider the difference equation

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_n x_{n-1}} \quad , \qquad n = 0, 1, \dots \tag{6.9}$$

where A is a real parameter and the initial conditions x_{-1}, x_0 are non-zero real numbers. In this section, we study the asymptotic behavior, the oscillatory character, and the periodic nature of the solutions of Eq 6.9. See [45].

Note that when $A = 0$, every non-trivial solution of 6.9 is positive, and is periodic with prime period four.

When $A > 0$, the change of variables

$$x_n = \begin{cases} A^{1+y_n} & \text{if } A > 1 \\ e^{y_n} & \text{if } A = 1 \\ A^{-1+y_n} & \text{if } A < 1 \end{cases}$$

reduces Eq.(6.9) to the piecewise linear difference equation

$$y_{n+1} = \frac{1}{2}|y_n| - \frac{1}{2}y_n - y_{n-1} + \delta \quad , \qquad n = 0, 1, \ldots \qquad (6.10)$$

where

$$\delta = \begin{cases} -2 \text{ if } A > 1 \\ \\ 0 \ \text{ if } A = 1 \\ \\ 2 \ \text{ if } A < 1 \end{cases}$$

which is a special case of Lozi's map. See [96].

6.3.1 The Case Where A is Positive

In Section 6.3.1, we consider the case where A is positive. The proofs of the following two lemmas follow directly by computation.

LEMMA 6.5
Suppose $A > 0$. Then Eq.(6.9) has a unique equilibrium point $\bar{x}$. Furthermore,

$$\bar{x} = \begin{cases} 1 \quad \text{if } A \leq 1 \\ \\ \sqrt[3]{A} \text{ if } A > 1. \end{cases}$$

LEMMA 6.6
Suppose $A > 0$. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(6.9), and suppose there exists $N \geq -1$ such that $x_n = \bar{x}$ for all $n \geq N$. Then

$$x_n = \bar{x} \qquad \text{for all} \qquad n \geq -1.$$

REMARK 6.3 Suppose $A > 0$. Then Lemma 6.6 states that the only eventually trivial solution of Eq.(6.9) is the trivial solution itself. ▯

6.3.1.1 The Case $A = 1$

THEOREM 6.7
Suppose $A = 1$. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive, non-trivial solution of Eq.(6.9). Then $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period 7.

PROOF Let $\alpha = x_{-1}$ and $\beta = x_0$. The proof of the theorem follows from the fact that the first nine terms of $\{x_n\}_{n=-1}^{\infty}$ are given as follows:

Case 1 $\alpha \geq 1,\ \beta \geq 1,$ and $\alpha \leq \beta$. $\{x_n\}_{n=-1}^{7} = \left\{\alpha, \beta, \frac{1}{\alpha}, \frac{\alpha}{\beta}, \beta, \frac{\beta}{\alpha}, \frac{1}{\beta}, \alpha, \beta\right\}.$

Case 2 $\alpha \geq 1,\ \beta \geq 1,$ and $\alpha > \beta$. $\{x_n\}_{n=-1}^{7} = \left\{\alpha, \beta, \frac{1}{\alpha}, \frac{\alpha}{\beta}, \alpha, \frac{\beta}{\alpha}, \frac{1}{\beta}, \alpha, \beta\right\}.$

Case 3 $\alpha \geq 1,\ \beta < 1,$ and $\alpha\beta < 1$. $\{x_n\}_{n=-1}^{7} = \left\{\alpha, \beta, \frac{1}{\alpha\beta}, \frac{1}{\beta}, \alpha\beta, \frac{1}{\alpha}, \frac{1}{\beta}, \alpha, \beta\right\}.$

Case 4 $\alpha > 1,\ \beta < 1,$ and $\alpha\beta \geq 1$. $\{x_n\}_{n=-1}^{7} = \left\{\alpha, \beta, \frac{1}{\alpha\beta}, \alpha, \alpha\beta, \frac{1}{\alpha}, \frac{1}{\beta}, \alpha, \beta\right\}.$

Case 5 $\alpha < 1,\ \beta \geq 1,$ and $\alpha\beta \geq 1$. $\{x_n\}_{n=-1}^{7} = \left\{\alpha, \beta, \frac{1}{\alpha}, \frac{1}{\beta}, \alpha\beta, \beta, \frac{1}{\alpha\beta}, \alpha, \beta\right\}.$

Case 6 $\alpha < 1,\ \beta \geq 1,$ and $\alpha\beta < 1$. $\{x_n\}_{n=-1}^{7} = \left\{\alpha, \beta, \frac{1}{\alpha}, \frac{1}{\beta}, \alpha\beta, \frac{1}{\alpha}, \frac{1}{\alpha\beta}, \alpha, \beta\right\}.$

Case 7 $\alpha < 1$ and $\beta < 1$. $\{x_n\}_{n=-1}^{7} = \left\{\alpha, \beta, \frac{1}{\alpha\beta}, \frac{1}{\beta}, \alpha\beta, \frac{1}{\alpha}, \frac{1}{\alpha\beta}, \alpha, \beta\right\}.$

The proof is complete. ∎

6.3.1.2 Boundedness and Persistence of Solutions

In Theorem 6.8, we show that Eq.(6.9) possesses an invariant, and we use this invariant to prove that every positive solution of Eq.(6.9) is bounded and persists.

THEOREM 6.8
Suppose $A > 0$, *and let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(6.9). For each* $n \geq 0$, *set*

$$I_n = \max\{A, x_{n-1}, x_n\} \max\left\{1, \frac{1}{x_n x_{n-1}}\right\}.$$

Then

$$I_n = I_0 \quad \textit{for all} \quad n \geq 0.$$

PROOF Let $n \geq 0$. Then

$$
\begin{aligned}
I_{n+1} &= \max\{A, x_n, x_{n+1}\} \max\left\{1, \frac{1}{x_{n+1}x_n}\right\} \\
&= \max\left\{A, x_n, \frac{\max\{x_n, A\}}{x_n x_{n-1}}\right\} \max\left\{1, \frac{1}{\frac{\max\{x_n A\}}{x_{n-1}}}\right\} \\
&= \max\left\{\max\{x_n, A\}, \frac{\max\{x_n, A\}}{x_n x_{n-1}}\right\} \max\left\{1, \frac{x_{n-1}}{\max\{x_n, A\}}\right\} \\
&= \max\left\{\max\{x_n, A\}, \frac{\max\{x_n, A\}}{x_n x_{n-1}}\right\} \frac{\max\{x_{n-1}, x_n, A\}}{\max\{x_n, A\}} \\
&= \max\left\{1, \frac{1}{x_n x_{n-1}}\right\} \max\{x_{n-1}, x_n, A\} \\
&= I_n.
\end{aligned}
$$

□

COROLLARY 6.2
Suppose $A > 0$, and let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(6.9). Then $\{x_n\}_{n=-1}^{\infty}$ is bounded and persists. Moreover,

$$x_n \in \left[\frac{1}{I_0}, I_0\right] \qquad \textit{for all} \qquad n \geq 1.$$

PROOF Suppose $n \geq 0$. Then

$$\max\{x_n, A\} \max\{A, x_{n-1}, x_n\} \max\left\{1, \frac{1}{x_n x_{n-1}}\right\} \geq x_n \cdot x_{n-1} \cdot 1$$

and so

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_n x_{n-1}} \geq \frac{1}{\max\{A, x_{n-1}, x_n\} \max\left\{1, \frac{1}{x_n x_{n-1}}\right\}} = \frac{1}{I_n} = \frac{1}{I_0}.$$

Also,

$$x_{n+1} \leq \max\{A, x_n x_{n+1}\} \max\left\{1, \frac{1}{x_{n+1} x_n}\right\} = I_{n+1} = I_0. \square$$

6.3.1.3 Invariant Regions

Set
$$R_1 = \left\{(x,y) : x, y \in \left[\frac{1}{A}, A\right] \text{ and } xy \geq 1\right\}$$
and
$$R_2 = \left\{(x,y) : x, y \in \left[A, \frac{1}{A}\right]\right\}.$$

THEOREM 6.9
The following statements are true:

1. *Suppose $A > 1$. Then R_1 is an invariant region. Furthermore, if $\{x_n\}_{n=-1}^{\infty}$ is a non-trivial solution of Eq.(6.9) with $(x_{-1}, x_0) \in R_1$, then $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period 3.*

2. *Suppose $0 < A < 1$. Then R_2 is an invariant region. Furthermore, if $\{x_n\}_{n=-1}^{\infty}$ is a non-trivial solution of Eq.(6.9) with $(x_{-1}, x_0) \in R_2$, then $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period 4.*

PROOF
(i) Suppose $A > 1$, and let $(x_{-1}, x_0) \in R_1$. Then
$$x_1 = \frac{\max\{x_0, A\}}{x_0 x_{-1}} = \frac{A}{x_0 x_{-1}} \in R_1.$$
We also have
$$x_0 x_1 = \frac{A}{x_{-1}} \geq \frac{A}{A} = 1$$
and so $(x_0, x_1) \in R_1$. It follows similarly that
$$x_2 = \frac{A}{x_1 x_0} = x_{-1} \qquad \text{and} \qquad x_3 = \frac{A}{x_2 x_1} = x_0$$
from which the proof is immediate.

(ii) Suppose $0 < A < 1$, and let $(x_{-1}, x_0) \in R_2$. Then
$$x_1 = \frac{\max\{x_0, A\}}{x_0 x_{-1}} = \frac{1}{x_{-1}} \in \left[A, \frac{1}{A}\right].$$
In particular, $(x_0, x_1) \in R_2$. It follows similarly that
$$x_2 = \frac{1}{x_0} \in \left[A, \frac{1}{A}\right], \quad x_3 = \frac{1}{x_1} = x_{-1} \in \left[A, \frac{1}{A}\right], \quad x_4 = \frac{1}{x_2} = x_0 \in \left[A, \frac{1}{A}\right]$$
and the proof is complete. ∎

REMARK 6.4 Note that if $A > 1$, then $(\bar{x}, \bar{x}) = (\sqrt[3]{A}, \sqrt[3]{A})$ is an interior point of R_1, while if $0 < A < 1$, then $(\bar{x}, \bar{x}) = (1, 1)$ is an interior point of R_2. The proof of the following corollary follows directly from this fact, together with the closed form of the solutions of Eq.(6.9) given in the proofs of Theorems 6.7 and 6.9. □

COROLLARY 6.3

Suppose $A > 0$. Then the equilibrium $\bar{x}$ of Eq.(6.9) is stable, but is not locally asymptotically stable.

6.3.1.4 The Case $A > 1$

REMARK 6.5 Suppose $A > 1$, and that $\{x_n\}_{n=-1}^{\infty}$ is a positive solution of Eq.(6.9) with $(x_{-1}, x_0) \notin R_1$. The following two lemmas characterize $\{x_n\}_{n=-1}^{\infty}$ by giving a partial closed form solution. □

Set

$$R = \{(x, y) : 0 < x < A \text{ and } 0 < y \leq A^{-1}\}.$$

LEMMA 6.7

Suppose $A > 1$. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(6.9) such that $(x_{N-1}, x_N) \in R$ for some non-negative integer $N \geq 0$. Then there exists a positive integer $l > 0$ such that

$$A^{-1-2l} < x_N \leq A^{1-2l}.$$

Furthermore, for $k = 0, 1, \ldots, l-1$,

$$\begin{aligned}
x_{N+7k-1} &= \frac{x_{N-1}}{A^{2k}} < A^{1-2k} \\
x_{N+7k} &= A^{2k} x_N \leq \frac{1}{A} \\
x_{N+7k+1} &= \frac{A}{x_{N-1} x_N} > A \\
x_{N+7k+2} &= \frac{1}{A^{2k} x_N} \geq A
\end{aligned}$$

$$x_{N+7k+3} = \frac{x_{N-1}x_N}{A} < \frac{1}{A}$$

$$x_{N+7k+4} = \frac{A^{2k+2}}{x_{N-1}} > A$$

$$x_{N+7k+5} = \frac{A}{x_{N-1}x_N} > A.$$

Moreover,

$$x_{N+7l-1} = \frac{x_{N-1}}{A^{2l}} < \frac{1}{A}$$

and

$$x_{N+7l} = A^{2l}x_N \in \left(\frac{1}{A}, A\right].$$

PROOF The proof is by induction and will be omitted. ▯

Set

$$C = \{(x, y) : 0 < x < A\ ,\ A^{-1} < y \leq A\ ,\ \text{and } xy < 1\}.$$

REMARK 6.6 Note that the solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(6.9) in Lemma 6.7 has the property that $(x_{N+7l-1}, x_{N+7l}) \in C$; that is, $\{x_n\}_{n=-1}^{\infty}$ also satisfies the hypotheses of Lemma 6.8. ▯

LEMMA 6.8

Suppose $A > 1$. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(6.9) such that $(x_{N-1}, x_N) \in C$ for some non-negative integer $N \geq 0$. Then there exists a positive integer $m > 0$ such that

$$\frac{1}{A} < x_N(x_{N-1}x_N)^{m-1} \leq A \qquad \text{and} \qquad x_N(x_{N-1}x_N)^m \leq \frac{1}{A}.$$

Furthermore, for $k = 0, 1, \ldots, m-1$,

$$x_{N+3k} = (x_{N-1}x_N)^k x_N \in \left(\frac{1}{A}, A\right]$$

$$x_{N+3k+1} = \frac{A}{x_{N-1}x_N} > A$$

$$x_{N+3k+2} = \frac{1}{(x_{N-1}x_N)^k x_N} \in \left[\frac{1}{A}, A\right).$$

Moreover

$$x_{N+3m-1} = \frac{x_{N-1}}{(x_{N-1}x_N)^m} \in \left[\frac{1}{A}, A\right)$$

and

$$x_{N+3m} = (x_{N-1}x_N)^m \leq \frac{1}{A}.$$

PROOF The proof is by induction and will be omitted. ▯

REMARK 6.7 Note that the solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(6.9) in Lemma 6.8 has the property that $(x_{N+3m-1}, x_{N+3m}) \in R$; that is, $\{x_n\}_{n=-1}^{\infty}$ also satisfies the hypotheses of Lemma 6.7. ▯

Define the following sets:

$$S = \{(x,y) \in \mathbf{R}^+ \times \mathbf{R}^+ : x \leq A^{-1} \text{ and } xy > 1\}$$

$$T = \{(x,y) \in \mathbf{R}^+ \times \mathbf{R}^+ : A \leq y < x\}$$

$$U = \{(x,y) \in \mathbf{R}^+ \times \mathbf{R}^+ : x \geq A \text{ and } xy < 1\}$$

$$V = \{(x,y) \in \mathbf{R}^+ \times \mathbf{R}^+ : y > A \text{ and } xy \leq 1\}$$

$$W = \{(x,y) \in \mathbf{R}^+ \times \mathbf{R}^+ : A < x \leq y\}$$

$$Z = \{(x,y) \in \mathbf{R}^+ \times \mathbf{R}^+ : y < A^{-1} \text{ and } xy \geq 1\}$$

$$D = \{(x,y) \in \mathbf{R}^+ \times \mathbf{R}^+ : A^{-1} < x \leq A \text{ and } y > A\}$$

$$E = \{(x,y) \in \mathbf{R}^+ \times \mathbf{R}^+ : x > A \text{ and } A^{-1} \leq y < A\}.$$

The proof of the next lemma follows directly from Lemmas 6.7 and 6.8.

LEMMA 6.9

Suppose $A > 1$. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(6.9) such that $(x_{N-1}, x_N) \in R$ for some non-negative integer $N \geq 0$. Then there exist positive integers $l > 0$ and $m > 0$ such that

$$A^{-1-2l} < x_N \leq A^{1-2l}$$

and

$$\frac{1}{A} < x_N(x_{N-1}x_N)^{m-1} \leq A \quad \text{and} \quad x_N(x_{N-1}x_N)^m \leq \frac{1}{A}.$$

Furthermore, for $k = 0, 1, \ldots, l-1$,

$$\begin{aligned}(x_{N+7k-1}, x_{N+7k}) &\in R\\ (x_{N+7k}, x_{N+7k+1}) &\in S\\ (x_{N+7k+1}, x_{N+7k+2}) &\in T\\ (x_{N+7k+2}, x_{N+7k+3}) &\in U\\ (x_{N+7k+3}, x_{N+7k+4}) &\in V\\ (x_{N+7k+4}, x_{N+7k+5}) &\in W\\ (x_{N+7k+5}, x_{N+7k+6}) &\in Z\end{aligned}$$

while for $k = 0, 1, \ldots, m-1$,

$$\begin{aligned}(x_{N+7l+3k-1}, x_{N+7l+3k}) &\in C\\ (x_{N+7l+3k}, x_{N+7l+3k+1}) &\in D\\ (x_{N+7l+3k+1}, x_{N+7l+3k+2}) &\in E\\ (x_{N+7l+3k+2}, x_{N+7l+3k+3}) &\in R.\end{aligned}$$

6.3.1.5 The Case $0 < A < 1$

REMARK 6.8 Suppose $0 < A < 1$, and that $\{x_n\}_{n=-1}^{\infty}$ is a positive solution of Eq.(6.9) with $(x_{-1}, x_0) \notin R_2$. The following two lemmas characterize $\{x_n\}_{n=-1}^{\infty}$ by giving a partial closed form solution. ▯

Set

$$F = \{(x, y) : 0 < x < A\ ,\ A^{-1} \leq y\ ,\ \text{and } xy < 1\}.$$

LEMMA 6.10

Suppose $0 < A < 1$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(6.9) such that* $(x_{N-1}, x_N) \in F$ *for some non-negative integer* $N \geq 0$. *Then there exists a positive integer* $1 > 0$ *such that*

$$A^{1-2l} < x_N \leq A^{-1-2l}.$$

Furthermore, for $k = 0, 1, \ldots,$

$$\begin{array}{lclcl}
x_{N+7k-1} & = & x_{N-1} & < & A \\
x_{N+7k} & = & A^{2k}x_N & \geq & \frac{1}{A} > A \\
x_{N+7k+1} & = & \frac{1}{x_{N-1}} & > & \frac{1}{A} > A \\
x_{N+7k+2} & = & \frac{1}{A^{2k}x_N} & \leq & A \\
x_{N+7k+3} & = & A^{2k+1}x_{N-1}x_N & < & A^{2k+1} \leq A \\
x_{N+7k+4} & = & \frac{1}{x_{N-1}} & > & \frac{1}{A} > A \\
x_{N+7k+5} & = & \frac{1}{A^{2k+1}x_{N-1}x_N} & > & A^{-2k-1} \geq \frac{1}{A}.
\end{array}$$

Moreover,

$$x_{N+7l-1} = x_{N-1} < A$$

and

$$x_{N+7l} = A^{2l}x_N \in \left[A, \frac{1}{A}\right).$$

PROOF The proof is by induction and will be omitted. ∎

Set

$$M = \{(x,y) \in \mathbf{R}^+ \times \mathbf{R}^+ : x < A \text{ and } A \leq y \leq A^{-1}\}.$$

REMARK 6.9 Note that the solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(6.9) in Lemma 6.10 has the property that $(x_{N+7l-1}, x_{N+7l}) \in M$; that is, $\{x_n\}_{n=-1}^{\infty}$ also satisfies the hypotheses of Lemma 6.11. ∎

LEMMA 6.11

Suppose $0 < A < 1$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(6.9) such that* $(x_{N-1}, x_N) \in M$ *for some non-negative integer* $N \geq 0$. *Then there exists a positive integer* $m > 0$ *such that*

$$A \leq \left(\frac{A}{x_{N-1}}\right)^{m-1} x_N < \frac{1}{A} \quad \text{and} \quad \left(\frac{A}{x_{N-1}}\right)^{m} x_N \geq \frac{1}{A}.$$

Furthermore, for $k = 0, 1, \ldots, m-1$,

$$x_{N+4k} = \frac{A^k}{x_{N-1}^k} x_N \in \left[A, \frac{1}{A}\right)$$

$$x_{N+4k+1} = \frac{1}{x_{N-1}} > \frac{1}{A} > A$$

$$x_{N+4k+2} = \frac{x_{N-1}^k}{A^k x_N} \in \left(A, \frac{1}{A}\right]$$

$$x_{N+4k+3} = x_{N-1} < A.$$

Moreover,

$$x_{N+4m-1} = x_{N-1} < A$$

and

$$x_{N+4m} = \frac{A^m}{x_{N-1}^m} \geq \frac{1}{A}.$$

REMARK 6.10 Note that the solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(6.9) in Lemma 6.11 has the property that $(x_{N+4m-1}, x_{N+4m}) \in F$; that is, $\{x_n\}_{n=-1}^{\infty}$ also satisfies the hypotheses of Lemma 6.10. □

Define the following sets:

$$G = \{(x, y) \in \mathbf{R}^+ \times \mathbf{R}^+ : A^{-1} \leq x < y\}$$

$$H = \{(x, y) \in \mathbf{R}^+ \times \mathbf{R}^+ : xy > 1 \text{ and } y \leq A\}$$

$$I = \{(x, y) \in \mathbf{R}^+ \times \mathbf{R}^+ : x \leq A \text{ and } y < A\}$$

$$J = \{(x, y) \in \mathbf{R}^+ \times \mathbf{R}^+ : x < A \text{ and } xy > 1\}$$

$$K = \{(x, y) \in \mathbf{R}^+ \times \mathbf{R}^+ : A^{-1} < y \leq x\}$$

$$L = \{(x, y) \in \mathbf{R}^+ \times \mathbf{R}^+ : A^{-1} < x \text{ and } xy \leq 1\}$$

$$O = \{(x, y) \in \mathbf{R}^+ \times \mathbf{R}^+ : A \leq x < A^{-1} \text{ and } y > A^{-1}\}$$

$$P = \{(x, y) \in \mathbf{R}^+ \times \mathbf{R}^+ : x > A^{-1} \text{ and } A < y \leq A^{-1}\}$$

$$Q = \{(x, y) \in \mathbf{R}^+ \times \mathbf{R}^+ : A < x \leq A^{-1} \text{ and } y < A\}.$$

The next result follows directly from Lemmas 6.10 and 6.11.

LEMMA 6.12

Suppose $0 < A < 1$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(6.9) such that* $(x_{N-1}, x_N) \in F$ *for some non-negative integer* $N \geq 0$. *Then there exist positive integers* $l > 0$ *and* $m > 0$ *such that*

$$A^{1-2l} \leq x_N < A^{-1-2l}$$

and

$$A \leq x_N \left(\frac{A}{x_{N-1}}\right)^{m-1} < \frac{1}{A} \qquad \textit{and} \qquad x_N \left(\frac{A}{x_{N-1}}\right)^{m} \geq \frac{1}{A}.$$

Furthermore, for $k = 0, 1, \ldots, l-1$,

$$\begin{aligned}
(x_{N+7k-1}, x_{N+7k}) &\in F \\
(x_{N+7k}, x_{N+7k+1}) &\in G \\
(x_{N+7k+1}, x_{N+7k+2}) &\in H \\
(x_{N+7k+2}, x_{N+7k+3}) &\in I \\
(x_{N+7k+3}, x_{N+7k+4}) &\in J \\
(x_{N+7k+4}, x_{N+7k+5}) &\in K \\
(x_{N+7k+5}, x_{N+7k+6}) &\in L
\end{aligned}$$

while for $k = 0, 1, \ldots, m-1$,

$$\begin{aligned}
(x_{N+7l+4k-1}, x_{N+7l+4k}) &\in M \\
(x_{N+7l+4k}, x_{N+7l+4k+1}) &\in O \\
(x_{N+7l+4k+1}, x_{N+7l+4k+2}) &\in P \\
(x_{N+7l+4k+2}, x_{N+7l+4k+3}) &\in Q \\
(x_{N+7l+4k+3}, x_{N+7l+4k+4}) &\in F.
\end{aligned}$$

The proof of the following lemma is straightforward and will be omitted.

LEMMA 6.13
Assume that $A > 0$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive, non-trivial solution of Eq.(6.9). Then the following statements are true:*

1. *Suppose* $0 < A \leq 1$. *Then every positive semi-cycle of* $\{x_n\}_{n=-1}^{\infty}$ *has at most three terms. With the possible exception of the first positive semi-cycle of* $\{x_n\}_{n=-1}^{\infty}$, *every positive semi-cycle of* $\{x_n\}_{n=-1}^{\infty}$ *has at least two terms. Furthermore, if a positive semi-cycle of* $\{x_n\}_{n=-1}^{\infty}$ *has three terms, then the first and third terms in the Deni-cycle are each equal to the equilibrium* $\bar{x} = 1$.

2. *Suppose* $A > 1$. *Then every positive semi-cycle of* $\{x_n\}_{n=-1}^{\infty}$ *has at most two terms. Moreover, with the possible exception of the case where* $x_{-1} = \bar{x}$ *and* $x_0 < \bar{x}$, *every positive semi-cycle has a term strictly greater than* $\bar{x}$.

3. *Every negative semi-cycle of* $\{x_n\}_{n=-1}^{\infty}$ *has at most two terms.*

As an immediate consequence of Lemma 6.13, we have the following result.

THEOREM 6.10
Assume $A > 0$. *Then every non-trivial solution of Eq.(6.9) is strictly oscillatory about* $\bar{x}$.

Recall by Theorem 6.8 that every solution of Eq.(6.9) is bounded from above by a bound I_0 given in terms of its initial conditions. Our next result shows that the solutions outside of the invariant regions actually achieve this upper bound.

THEOREM 6.11
Assume $A > 0$ *is a positive real number not equal to one. Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(6.9). Suppose that the initial conditions* (x_{-1}, x_0) *of* $\{x_n\}_{n=-1}^{\infty}$ *are as follows:*

$$(x_{-1}, x_0) \notin R_1 \text{ if } \quad A > 1$$

and

$$(x_{-1}, x_0) \notin R_2 \text{ if } 0 < A < 1$$

Then with the possible exception of the first positive semi-cycle, every positive semi-cycle of $\{x_n\}_{n=-1}^{\infty}$ *contains a term equal to* I_0.

PROOF We consider the case $A > 1$. The case $0 < A < 1$ is similar and will be omitted.

Let x_m be the maximum term in a positive semi-cycle, where $m \geq 0$. It suffices to show that

$$x_m = I_0.$$

Recall that the positive semi-cycle of which x_m is a member has at most two terms.

Case 1 Suppose $x_{m+1} \geq \sqrt[3]{A}$. Then

$$x_m x_{m+1} \geq x_{m+1}^2 \geq A^{\frac{2}{3}} > 1$$

and so $x_m > A$ since $(x_m, x_{m+1}) \notin R_1$. Hence

$$I_0 = I_{m+1} = \max\{A, x_m, x_{m+1}\} \max\left\{1, \frac{1}{x_m x_{m+1}}\right\} = x_m.$$

Case 2 Suppose $x_{m-1} \geq \sqrt[3]{A}$. Then

$$x_{m-1} x_m \geq x_{m-1}^2 \geq A^{\frac{2}{3}} > 1$$

and so $x_m > A$ since $(x_{m-1}, x_m) \notin R_1$. Thus

$$I_0 = I_m = \max\{A, x_{m-1}, x_m\} \max\left\{1, \frac{1}{x_{m-1} x_m}\right\} = x_m.$$

Case 3 Suppose $x_{m-1}, x_{m+1} < \sqrt[3]{A}$.

We claim that $x_m > A$.

For the sake of contradiction, suppose $x_m \leq A$. Now it follows by Lemma 6.13 that

$$x_m > \sqrt[3]{A}.$$

Thus

$$\sqrt[3]{A} > x_{m+1} = \frac{\max\{x_m, A\}}{x_m x_{m-1}} = \frac{A}{x_m x_{m-1}}$$

and hence

$$x_m x_{m-1} > A^{\frac{2}{3}} > 1.$$

So, as $(x_{m-1}, x_m) \notin R_1$, we must have $x_{m-1} < \frac{1}{A}$. So

$$1 = \frac{1}{A} A > x_{m-1} x_m > 1,$$

which is a contradiction.

Thus it is true that

$$x_m > A.$$

Thus

$$\sqrt[3]{A} > x_{m+1} = \frac{\max\{x_m, A\}}{x_m x_{m-1}} = \frac{x_m}{x_m x_{m-1}} = \frac{1}{x_{m-1}}$$

and so

$$x_{m-1} \in \left(A^{-\frac{1}{3}}, A^{\frac{1}{3}}\right).$$

Hence

$$x_{m-1}x_m > A^{-\frac{1}{3}}A = A^{\frac{2}{3}} > 1$$

and so

$$I_0 = I_m = \max\{A, x_{m-1}, x_m\} \max\left\{1, \frac{1}{x_m x_{m-1}}\right\} = x_m.$$ ▯

The following theorem shows that there are periodic solutions of Eq.(6.9) other than those in the invariant regions R_1 and R_2.

THEOREM 6.12
Let $k \geq 2$ be an integer. Then the following statements are true:

1. *Suppose $A > 1$, and the initial conditions of $\{x_n\}_{n=-1}^{\infty}$ are chosen such that*

$$x_{-1} = A^k \qquad \textit{and} \qquad x_0 \in \left[\frac{1}{A}, A\right].$$

Then $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period

$$\begin{cases} 7k-1 \ \textit{if} \ k \ \textit{is even} \\ \dfrac{7k-1}{2} \ \textit{if} \ k \ \textit{is odd}. \end{cases}$$

2. *Suppose $0 < A < 1$, and the initial conditions of $\{x_n\}_{n=-1}^{\infty}$ are chosen such that*

$$x_{-1} \in \left[A^k, A^{k-1}\right] \qquad \textit{and} \qquad x_0 = A^k.$$

Then $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period

$$\begin{cases} 7k+1 \ \textit{if} \ k \ \textit{is even} \\ \dfrac{7k+1}{2} \ \textit{if} \ k \ \textit{is odd}. \end{cases}$$

PROOF We give the proof for the case $A > 1$ and $k = 2l + 1$ is an odd integer greater than or equal to 3. The proof for the other cases is similar and will be omitted.

$$
\begin{aligned}
x_{-1} &= A^{2l+1} \\
x_0 &\in \left[\frac{1}{A}, A\right] \\
x_1 &= \frac{\max\{x_0, A\}}{x_0 A^{2l+1}} = \frac{A}{x_0 A^{2l+1}} \\
x_2 &= \frac{\max\left\{\frac{A}{x_0 A^{2l+1}}, A\right\}}{\frac{A}{x_0 A^{2l+1}} x_0} = A^{2l+1} \\
x_3 &= \frac{\max\left\{A^{2l+1}, A\right\}}{A^{2l+1} \frac{A}{x_0 A^{2l+1}}} = \frac{x_0 A^{2l+1}}{A}
\end{aligned}
$$

and

$$
\begin{aligned}
x_4 &= \frac{\max\left\{\frac{x_0 A^{2l+1}}{A}, A\right\}}{\frac{x_0 A^{2l+1}}{A} A^{2l+1}} = \frac{1}{A^{2l+1}} \\
x_5 &= \frac{\max\left\{\frac{1}{A^{2l+1}}, A\right\}}{\frac{1}{A^{2l+1}} \frac{x_0 A^{2l+1}}{A}} = \frac{A^2}{x_0} \\
x_6 &= \frac{\max\left\{\frac{A^2}{x_0}, A\right\}}{\frac{A^2}{x_0} \frac{1}{A^{2l+1}}} = A^{2l+1} \\
x_7 &= \frac{\max\left\{A^{2l+1}, A\right\}}{A^{2l+1} \frac{A^2}{x_0}} = \frac{x_0}{A^2} \\
x_8 &= \frac{\max\left\{\frac{x_0}{A^2}, A\right\}}{\frac{x_0}{A^2} A^{2l+1}} = \frac{A^3}{x_0 A^{2l+1}} \\
x_9 &= \frac{\max\left\{\frac{A^3}{x_0 A^{2l+1}}, A\right\}}{\frac{A^3}{x_0 A^{2l+1}} \frac{x_0}{A^2}} = A^{2l+1} \\
x_{10} &= \frac{\max\left\{A^{2l+1}, A\right\}}{A^{2l+1} \frac{A^3}{x_0 A^{2l+1}}} = \frac{x_0 A^{2l+1}}{A^3}.
\end{aligned}
$$

Thus we see that if $l = 1$, then $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period 10. In general for $l \geq 1$, it follows by induction that

$$
\begin{aligned}
x_{-1} &= A^{2l+1} \\
x_0 &\in \left[\frac{1}{A}, A\right] \\
x_1 &= \frac{\max\{x_0, A\}}{x_0 A^{2l+1}} = \frac{A}{x_0 A^{2l+1}} \\
x_2 &= \frac{\max\left\{\frac{A}{x_0 A^{2l+1}}, A\right\}}{\frac{A}{x_0 A^{2l+1}} x_0} = A^{2l+1} \\
x_3 &= \frac{\max\left\{A^{2l+1}, A\right\}}{A^{2l+1}\frac{A}{x_0 A^{2l+1}}} = \frac{x_0 A^{2l+1}}{A}
\end{aligned}
$$

and for $0 \le i \le l-1$

$$
\begin{aligned}
x_{4+7i} &= \frac{1}{A^{2l+1}} \\
x_{5+7i} &= \frac{A^{2i+2}}{x_0} \\
x_{6+7i} &= A^{2l+1} \\
x_{7+7i} &= \frac{x_0}{A^{2i+2}} \\
x_{8+7i} &= \frac{A^{2i+3}}{x_0 A^{2l+1}} \\
x_{9+7i} &= A^{2l+1} \\
x_{10+7i} &= \frac{x_0 A^{2l+1}}{A^{2i+3}}
\end{aligned}
$$

from which the proof follows. ∎

6.3.1.6 The Third Quadrant

In this section we see that the period-7 phenomenon is unique to the first quadrant. The proof of the next lemma follows by computation and will be omitted.

LEMMA 6.14

Let $A > 0$, and suppose that $\{x_n\}_{n=-1}^{\infty}$ is a solution of Eq.(6.9). Then the following statements are true:

1. *Suppose that $x_{-1} < 0$ and $x_0 < 0$. Then $x_1 > 0$, $x_2 < 0$, and $x_3 < 0$.*
2. *Suppose that $x_{-1} < 0$ and $x_0 > 0$. Then $x_1 < 0$, $x_2 < 0$, and $x_3 > 0$.*
3. *Suppose that $x_{-1} > 0$ and $x_0 < 0$. Then $x_1 < 0$, $x_2 > 0$, and $x_3 < 0$.*

From Lemma 6.14, we see that if we wish to study those solutions of Eq.(6.9) for which one or both of the initial conditions are negative, it suffices to consider the case where both initial conditions are negative.

The following theorem implies that some solutions of Eq.(6.9) have the property that one sub-sequence diverges to negative infinity, while a second sub-sequence converges to zero.

THEOREM 6.13

Suppose that $A > 0$. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(6.9) with $x_{-1} < 0$ and $x_0 < 0$ such that $x_{-1}x_0 < 1$. Then for each $k \geq 0$, the following statements are true:

$$\begin{aligned} x_{3k} &= (x_{-1}x_0)^k x_0 \\ x_{3k+1} &= \frac{A}{x_{-1}x_0} \\ x_{3k+2} &= \frac{1}{(x_{-1}x_0)^k x_0}. \end{aligned}$$

PROOF The result clearly holds for $k = 0$. So suppose that $k > 0$ and that the result holds for $k - 1$. It suffices to show that the result holds for k. Now,

$$x_{3k} = \frac{\max\left\{\dfrac{1}{(x_{-1}x_0)^{k-1}x_0}, A\right\}}{\dfrac{1}{(x_{-1}x_0)^{k-1}x_0}\dfrac{A}{x_{-1}x_0}} = (x_{-1}x_0)^k x_0$$

$$x_{3k+1} = \frac{\max\left\{(x_{-1}x_0)^k x_0, A\right\}}{(x_{-1}x_0)^k x_0 \dfrac{1}{(x_{-1}x_0)^{k-1}x_0}} = \frac{A}{x_{-1}x_0}$$

$$x_{3k+2} = \frac{\max\left\{\dfrac{A}{x_{-1}x_0}, A\right\}}{\dfrac{A}{x_{-1}x_0}(x_{-1}x_0)^k x_0} = \frac{1}{(x_{-1}x_0)^k x_0}.$$ ▯

The next result shows that all other solutions of Eq.(6.9) which begin in the third quadrant are periodic with prime period 3.

THEOREM 6.14
Suppose that $A > 0$. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(6.9) with $x_{-1} < 0$ and $x_0 < 0$ such that $x_{-1}x_0 \geq 1$. Then $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period 3.

PROOF As the only equilibrium point of Eq.(6.9) is positive, it suffices to show that $\{x_n\}_{n=-1}^{\infty}$ is periodic with period 3. Now,

$$x_1 = \frac{\max\{x_0, A\}}{x_0 x_{-1}} = \frac{A}{x_0 x_{-1}}$$

$$x_2 = \frac{\max\left\{\frac{A}{x_0 x_{-1}}, A\right\}}{\frac{A}{x_0 x_{-1}} x_0} = x_{-1}$$

$$x_3 = \frac{\max\{x_{-1}, A\}}{x_{-1}\frac{A}{x_0 x_{-1}}} = x_0$$

and the proof is complete. ▯

6.3.2 The Case Where A is Negative

Throughout the remainder of this section, we shall assume that A is negative. One can see that when $A < 0$ and $x_{-1}, x_0 \in (-\infty, 0)$, the change of variables

$$x_n = -\frac{1}{y_n} \quad \text{and} \quad B = -\frac{1}{A}$$

reduces Eq.(6.9) to the equation

$$y_{n+1} = \frac{\max\{y_n, B\}}{y_n y_{n-1}}, \quad n = 0, 1, \ldots$$

with parameter $B > 0$ and initial conditions $y_{-1}, y_0 > 0$. Thus the behavior of the solutions of Eq.(6.9) with negative initial conditions can be easily deduced from our previous investigations.

6.3.2.1 Solutions Outside the Third Quadrant

Note that Eq.(6.9) has the unique positive equilibrium point $\bar{x} = 1$.

THEOREM 6.15
Assume that $A < 0$, and that $\{x_n\}_{n=-1}^{\infty}$ is a non-trivial solution of Eq.(6.9) with $x_{-1}, x_0 \in (0, \infty)$. Then $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period 4.

PROOF As $x_{-1}, x_0 > 0$, it follows by a simple computation that $\{x_n\}_{n=-1}^{\infty}$ is given by

$$x_{-1}, x_0, \frac{1}{x_{-1}}, \frac{1}{x_0}, x_{-1}, x_0, \ldots$$

from which the proof follows. ▯

When $A < 0$ and $\{x_n\}_{n=-1}^{\infty}$ is a solution of Eq.(6.9) with $x_{-1}x_0 < 0$, we see that $\{(x_{n-1}, x_n) : n \geq 0\}$ alternates between the second and fourth quadrant. Hence without loss of generality, we shall assume that

$$x_{-1} < 0 \qquad \text{and} \qquad x_0 > 0.$$

The proof of the next theorem follows by simple computation, and will be omitted.

THEOREM 6.16
Assume that $A < 0$, and let $\{x_n\}_{n=-1}^{\infty}$ be a non-trivial solution of Eq.(6.9) with $x_{-1} < 0$ and $x_0 > 0$. Then the following statements are true:

1. *Assume that $x_{-1} = -1$.*

   (a) *Suppose $A \leq -1$ and $x_0 = 1$. Then $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period 2.*

   (b) *Suppose $A \leq -1$ and $x_0 \neq 1$. Then $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period 4.*

   (c) *Suppose $A > -1$. Then $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period 4.*

2. *Assume that $x_{-1} \neq -1$.*

   (a) *Suppose $\min\left\{x_{-1}, \frac{1}{x_{-1}}\right\} \geq A$. Then $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period 4.*

   (b) *Suppose $\min\left\{x_{-1}, \frac{1}{x_{-1}}\right\} < A$. Then $\{x_n\}_{n=-1}^{\infty}$ is neither bounded, nor does it persist.*

6.4 The Max Equation $x_{n+1} = \dfrac{\max\{x_n^2, A\}}{x_n x_{n-1}}$

As we saw in Section 2.7, the max equation

$$x_{n+1} = \frac{\max\{x_n^2, A\}}{x_n x_{n-1}} \quad , \quad n = 0, 1, \ldots \tag{6.11}$$

can be transformed into the difference equation

$$y_{n+1} = |y_n| - y_{n-1} + \delta \quad , \quad n = 0, 1, \ldots \tag{6.12}$$

where

$$\delta = \begin{cases} -1 \text{ if } A > 1 \\ 0 \text{ if } A = 1 \\ 1 \text{ if } A < 1. \end{cases}$$

The case $\delta = 0$, or equivalently, the case $A = 1$, was investigated in [29] where it was shown that every solution of the difference equation

$$y_{n+1} = |y_n| - y_{n-1} \quad , \quad n = 0, 1, \ldots \tag{6.13}$$

and so also every positive solution of

$$x_{n+1} = \frac{\max\{x_n^2, 1\}}{x_n x_{n-1}} \quad , \quad n = 0, 1, \ldots \tag{6.14}$$

is periodic with period 9.

The case $\delta = 1$, or equivalently the case $A < 1$, was investigated in [32]. This is the so-called *gingerbreadman equation*

$$y_{n+1} = |y_n| - y_{n-1} + 1 \quad , \quad n = 0, 1, \ldots \tag{6.15}$$

and it was shown in [32] that Eq.(6.15) was chaotic in certain regions of the plane, and was stable in other regions of the plane. It was also shown in [32] that every solution of Eq.(6.15) is bounded.

The case $\delta = -1$, or equivalently the case $A > 1$, has not yet been investigated.

Some of the interesting questions we wish to ask about Eq.(6.11) are the following:

Does Eq.(6.11) possess an invariant?

Can we give a straightforward direct and analytic proof that every positive solution of Eq.(6.11) is bounded?

6.5 The Max Equation $x_{n+1} = \dfrac{\max\{x_n, A\}}{x_n^2 x_{n-1}}$

An interesting feature of this equation is the fact that every solution of the equation is periodic with period 8 if and only if $A = 1$. We first discuss how this equation was discovered. The non-autonomous difference equation of the Lyness-type

$$y_{n+1} = \frac{\max\{a_n y_n, b_n\}}{y_{n-1}} \quad , \quad n = 0, 1, \ldots \tag{6.16}$$

was investigated in [69], with

$$a_n = \begin{cases} a_0 \text{ if n is even} \\ a_1 \text{ if n is odd} \end{cases} \quad \text{and} \quad b_n = \begin{cases} b_0 \text{ if n is even} \\ b_1 \text{ if n is odd} \end{cases}$$

and $a_0, a_1, b_0, b_1 \in [0, \infty)$ with $a_0 + b_0 > 0$ and $a_1 + b_1 > 0$, and where the initial conditions y_{-1} and y_0 are arbitrary positive real numbers.

From Eq.(6.16) with $b_0 = 0$ and $a_1 > 0$, we have

$$y_{2n+1} = \frac{a_0 y_{2n}}{y_{2n-1}} \quad , \quad n = 0, 1, \ldots \tag{6.17}$$

and

$$y_{2n+2} = \frac{\max\{a_1 y_{2n+1}, b_1\}}{y_{2n}} \quad , \quad n = 0, 1, \ldots . \tag{6.18}$$

After substituting Eq.(6.17) into Eq.(6.18) and making the change of variables

$$y_{2n+1} = \sqrt[3]{a_0^2 a_1}\, x_n \quad \text{for} \quad n = -1, 0, \ldots$$

we see that

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_n^2 x_{n-1}} \quad , \quad n = 0, 1, \ldots \tag{6.19}$$

where $A = \dfrac{b_1}{\sqrt[3]{a_0^2 a_1^4}}$, and where $x_{-1}, x_0 \in (0, \infty)$.

In Theorem 6.17, we show that Eq.(6.19) possesses an invariant.

THEOREM 6.17

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(6.19), and for $n \geq 0$, set

$$I_n = \max\left\{\frac{1}{x_{n-1}}, \frac{A}{x_{n-1} x_n}, \frac{1}{x_n}, x_{n-1} x_n\right\}.$$

Then $I_n = I_0$ for all $n \geq 0$.

PROOF Observe that for $n \geq 0$,

$$I_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A}{x_n x_{n+1}}, \frac{1}{x_{n+1}}, x_n x_{n+1}\right\}$$

$$= \max\left\{\frac{1}{x_n}, \frac{A}{x_n \frac{\max\{x_n, A\}}{x_n^2 x_{n-1}}}, \frac{1}{\frac{\max\{x_n, A\}}{x_n^2 x_{n-1}}}, x_n \frac{\max\{x_n, A\}}{x_n^2 x_{n-1}}\right\}$$

$$= \max\left\{\frac{1}{x_n}, \frac{A x_n x_{n-1}}{\max\{x_n, A\}}, \frac{x_n^2 x_{n-1}}{\max\{x_n, A\}}, \frac{\max\{x_n, A\}}{x_n x_{n-1}}\right\}.$$

Case 1 Suppose $x_n > A$. Then

$$I_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A x_n x_{n-1}}{x_n}, \frac{x_n^2 x_{n-1}}{x_n}, \frac{x_n}{x_n x_{n-1}}\right\}$$

$$= \max\left\{\frac{1}{x_n}, A x_{n-1}, x_n x_{n-1}, \frac{1}{x_{n-1}}\right\}$$

$$= \max\left\{\frac{1}{x_n}, x_n x_{n-1}, \frac{1}{x_{n-1}}\right\}$$

$$= \max\left\{\frac{1}{x_{n-1}}, \frac{A}{x_{n-1} x_n}, \frac{1}{x_n}, x_{n-1} x_n\right\} = I_n.$$

Case 2 Suppose $0 < x_n \leq A$. Then

$$I_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A x_n x_{n-1}}{A}, \frac{x_n^2 x_{n-1}}{A}, \frac{A}{x_n x_{n-1}}\right\}$$

$$= \max\left\{\frac{1}{x_n}, x_n x_{n-1}, \frac{x_n^2 x_{n-1}}{A}, \frac{A}{x_n x_{n-1}}\right\}.$$

$$= \max\left\{\frac{1}{x_n}, x_n x_{n-1}, \frac{A}{x_n x_{n-1}}\right\}$$

$$= \max\left\{\frac{1}{x_{n-1}}, \frac{A}{x_{n-1} x_n}, \frac{1}{x_n}, x_{n-1} x_n\right\} = I_n. \qquad \square$$

The proof of Theorem 6.18 follows directly from Theorem 6.17 and will be omitted.

THEOREM 6.18
Every positive solution of Eq.(6.19) is bounded and persists.

The following remark is easily proved.

REMARK 6.11 The following statements are true:

1. Every positive solution of Eq.(6.19) is periodic with period 3 if $A = 0$.
2. Every positive solution of Eq.(6.19) is periodic with period 8 if $A = 1$.

$\square$

When $A \notin \{0, 1\}$, we pose the following Conjecture:

CONJECTURE 6.1
Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(6.19) with $x_{-1} = 1$ and $x_0 = A^k$, where $k \in \{0, 1, \ldots\}$ and $A \in (0, 1) \cup (1, \infty)$. Then the following statements are true:

(i) Suppose $0 < A < 1$. Then $\{x_n\}_{n=-1}^{\infty}$ is periodic with period 3 for $k = 1$ and is periodic with period $8k + 1$ for $k = 2, 3, \ldots$. When $k = 0$, $\{x_n\}_{n=-1}^{\infty}$ is identically equal to 1.

(ii) Suppose $A > 1$. Then $\{x_n\}_{n=-1}^{\infty}$ is periodic with period 7 for $k = 0$, while for $k = 1, 2, \ldots$, $\{x_n\}_{n=-1}^{\infty}$ is periodic with period

$$\begin{cases} 8k - 1 & \textit{if } k \textit{ or } k-1 \textit{ is a multiple of } 3 \\ \dfrac{8k-1}{3} & \textit{if } \quad k-2 \textit{ is a multiple of } 3. \end{cases}$$

6.6 The Max Equation $x_{n+1} = \dfrac{\max\{x_n, A_n\}}{x_{n-1}}$

Consider the difference equation

$$x_{n+1} = \frac{\max\{x_n, A_n\}}{x_{n-1}} \quad , \qquad n = 0, 1, \ldots \tag{6.20}$$

where

$$A_n = \begin{cases} A_0 \text{ if } n \text{ is even} \\ A_1 \text{ if } n \text{ is odd} \end{cases}$$

with $A_0, A_1 \in (0, \infty)$ and with $x_{-1}, x_0 \in (0, \infty)$. See [69].

Our first result shows that Eq.(6.20) possesses an invariant.

THEOREM 6.19
Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(6.20), and for each $n \geq 0$, set

$$I_n = \max\left\{A_{n+1}, \frac{A_n}{x_{n-1}}\right\} \max\left\{A_n, \frac{A_{n+1}}{x_n}\right\} \max\{A_n A_{n+1}, A_n x_{n-1}, A_{n+1} x_n\}.$$

Then

$$I_n = I_0 \quad \textit{for all} \quad n \geq 0.$$

PROOF Suppose $n \geq 0$ is an integer. Then

$$\begin{aligned}
I_{n+1} &= \max\left\{A_{n+2}, \frac{A_{n+1}}{x_n}\right\} \max\left\{A_{n+1}, \frac{A_{n+2}}{x_{n+1}}\right\} \times \\
&\quad \times \max\{A_{n+1}A_{n+2}, A_{n+1}x_n, A_{n+2}x_{n+1}\} \\
&= \max\left\{A_n, \frac{A_{n+1}}{x_n}\right\} \max\left\{A_{n+1}, \frac{A_n x_{n-1}}{\max\{x_n, A_n\}}\right\} \times \\
&\quad \times \max\left\{A_{n+1}A_n, A_{n+1}x_n, \frac{A_n \max\{x_n, A_n\}}{x_{n-1}}\right\} \\
&= \max\left\{A_n, \frac{A_{n+1}}{x_n}\right\} \max\left\{\frac{A_{n+1}\max\{x_n, A_n\}}{\max\{x_n, A_n\}}, \frac{A_n x_{n-1}}{\max\{x_n, A_n\}}\right\} \times \\
&\quad \times \max\left\{A_{n+1}\max\{A_n, x_n\}, \frac{A_n \max\{x_n A_n\}}{x_{n-1}}\right\} \\
&= \max\left\{A_n, \frac{A_{n+1}}{x_n}\right\} \max\{A_{n+1}x_n, A_{n+1}A_n, A_n x_{n-1}\} \times \\
&= \max\left\{A_n, \frac{A_{n+1}}{x_n}\right\} \max\{A_{n+1}x_n, A_{n+1}A_n, A_n x_{n-1}\} \times \\
&\quad \times \max\left\{A_{n+1}, \frac{A_n}{x_{n-1}}\right\} = I_n.
\end{aligned}$$

∎

The proof of the following result is a direct consequence of Theorem 6.19 and will be omitted.

THEOREM 6.20
Every positive solution of Eq.(6.20) is bounded and persists.

6.7 The Max Equation $x_{n+1} = \dfrac{\max\{x_n, A_n\}}{x_n x_{n-1}}$

Consider the difference equation

$$x_{n+1} = \frac{\max\{x_n, A_n\}}{x_n x_{n-1}} \quad , \qquad n = 0, 1, \ldots \tag{6.21}$$

where $\{A_n\}_{n=0}^{\infty}$ is a sequence of positive real numbers, and where the initial conditions x_{-1} and x_0 are arbitrary positive real numbers.

The case where the sequence $\{A_n\}_{n=0}^{\infty}$ is constant was investigated in Section 6.3.

6.7.1 Boundedness and Persistence of Solutions of Eq.(6.21)

In this section we show that under appropriate hypotheses, Eq.(6.21) possesses an invariant, or more generally, an *energy function*, which can be used to show that all solutions of Eq.(6.21) are bounded and persist.

LEMMA 6.15

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(6.21), and let $\{P_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers. For each integer $n \geq 1$, set

$$E_n = \max\left\{\frac{P_{n-1}}{x_{n-1}}, \frac{P_n}{x_n}, P_{n+1}x_{n-1}, P_{n+2}x_n, \frac{P_n A_{n-1}}{x_{n-1}x_n}\right\}.$$

Then for all $n \geq 1$,

$$E_{n+1} = \max\left\{\frac{P_{n+3}}{x_{n-1}}, \frac{P_n}{x_n}, P_{n+1}x_{n-1}, P_{n+2}x_n, \frac{P_{n+3}A_n}{x_{n-1}x_n}\right\}.$$

PROOF Suppose $n \geq 1$ is a positive integer. Then

$$E_{n+1} = \max\left\{\frac{P_n}{x_n}, \frac{P_{n+1}x_{n-1}x_n}{\max\{x_n, A_n\}}, P_{n+2}x_n, P_{n+3}\frac{\max\{x_n, A_n\}}{x_n x_{n-1}}, \frac{P_{n+1}x_{n-1}A_n}{\max\{x_n, A_n\}}\right\}$$

$$= \max\left\{\frac{P_n}{x_n}, \frac{P_{n+1}x_{n-1}}{\max\{x_n, A_n\}} \cdot \max\{x_n, A_n\}, P_{n+2}x_n, \frac{P_{n+3}x_n}{x_{n-1}x_n}, \frac{P_{n+3}A_n}{x_{n-1}x_n}\right\}.$$

∎

The proof of Theorem 6.21 is a direct consequence of Lemma 6.15.

THEOREM 6.21
Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers such that

$$P_{n+3} \leq P_{n-1} \quad \textit{for } n \geq 1$$

$$P_{n+3}A_n \leq P_n A_{n-1} \textit{ for } n \geq 1$$

and

$$P \leq P_n \quad \textit{for } n \geq 0$$

for some $P \in (0, \infty)$. Then the following statements are true:

1. *$E_{n+1} \leq E_n$ for all $n \geq 1$.*
2. *Every positive solution of Eq.(6.21) is bounded and persists.*

COROLLARY 6.4
Let $A \in (0, \infty)$ be a positive real number, and assume that one of the following conditions is true:

1. *$A \leq A_{n+4} \leq A_n$ for $n \geq 0$.*
2. *$A_n \leq A_{n+4}$ for $n \geq 0$ and $A_n^2 \leq A_{n-1}A_{n+3}$ for $n \geq 1$.*
3. *$A \leq A_n \leq A_{n+1}$ for $n \geq 0$.*

Then every positive solution of Eq.(6.21) is bounded and persists.

PROOF The proof of Statement 1 follows from Theorem 6.21 with $P = A$, and with $P_n = A_n$ for $n \geq 0$.

The proof of Statement 2 follows from Theorem 6.21 with $P = A_0$, and with $P_n = \dfrac{1}{A_n}$ for $n \geq 0$.

The proof of Statement 3 follows from Theorem 6.21 with $P_n = \dfrac{1}{A_{n-3}}$ for $n \geq 4$. ∎

Lemma 6.16 states that Eq.(6.21) has an invariant when $\{A_n\}_{n=0}^{\infty}$ is a sequence of positive real numbers which is periodic with period 4.

LEMMA 6.16

Assume that $\{A_n\}_{n=0}^{\infty}$ *is a sequence of positive real numbers which is periodic with period 4, and let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(6.21). For each* $n \geq 0$, *set*

$$I_n = \max\left\{\frac{A_{n-1}}{x_{n-1}}, \frac{A_n}{x_n}, A_{n+1}x_{n-1}, A_{n+2}x_n, \frac{A_{n-1}A_n}{x_{n-1}x_n}\right\}.$$

Then $I_n = I_0$ *for all* $n \geq 0$.

PROOF Suppose $n \geq 0$ is an integer. Then

$$\begin{aligned}
I_{n+1} &= \max\left\{\frac{A_n}{x_n}, \frac{A_{n+1}}{x_{n+1}}, A_{n+2}x_n, A_{n+3}x_{n+1}, \frac{A_nA_{n+1}}{x_nx_{n+1}}\right\} \\
&= \max\left\{\frac{A_n}{x_n}, \frac{A_{n+1}x_nx_{n-1}}{\max\{x_n, A_n\}}, A_{n+2}x_n, \frac{A_{n-1}\max\{x_n, A_n\}}{x_nx_{n-1}}, \frac{A_nA_{n+1}x_nx_{n-1}}{x_n\max\{x_nA_n\}}\right\} \\
&= \max\left\{\frac{A_n}{x_n}, \frac{A_{n+1}x_nx_{n-1}}{\max\{x_n, A_n\}}, A_{n+2}x_n, \frac{A_{n-1}\max\{x_n, A_n\}}{x_nx_{n-1}}, \frac{A_nA_{n+1}x_{n-1}}{\max\{x_nA_n\}}\right\} \\
&= \max\left\{\frac{A_n}{x_n}, \frac{A_{n+1}x_{n-1}\max\{x_n, A_n\}}{\max\{x_n, A_n\}}, A_{n+2}x_n, \frac{A_{n-1}\max\{x_n, A_n\}}{x_nx_{n-1}}\right\} \\
&= \max\left\{\frac{A_n}{x_n}, A_{n+1}x_{n-1}, A_{n+2}x_n, \frac{A_{n-1}}{x_{n-1}}, \frac{A_{n-1}A_n}{x_nx_{n-1}}\right\} \\
&= I_n.
\end{aligned}$$

∎

COROLLARY 6.5

Assume that $\{A_n\}_{n=0}^{\infty}$ *is a sequence of positive real numbers which is periodic with period 4. Then every positive solution of Eq.(6.21) is bounded and persists.*

6.7.2 Periodicity

In this section we show that under appropriate conditions, every solution of Eq.(6.21) with initial conditions in certain regions is periodic with the same period.

THEOREM 6.22
Assume that for some $A \in (0,1)$

$$0 < A_n \leq A \qquad \textit{for all} \qquad n \geq 0.$$

Set

$$\mathcal{I} = \left[A, \frac{1}{A}\right] - \{1\}.$$

Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(6.21) with* $x_{-1}, x_0 \in \mathcal{I}$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic with prime period-4.*

PROOF Note that for all $n \geq 0$

$$x_{-1}, x_0, \frac{1}{x_{-1}}, \frac{1}{x_0} \in \left[A, \frac{1}{A}\right] \subset \left[A_n, \frac{1}{A}\right]$$

and so

$$x_1 = \frac{\max\{x_0, A_0\}}{x_0 x_{-1}} = \frac{x_0}{x_0 x_{-1}} = \frac{1}{x_{-1}}$$

$$x_2 = \frac{\max\{x_1, A_1\}}{x_1 x_0} = \frac{x_1}{x_1 x_0} = \frac{1}{x_0}$$

from which the proof follows. ∎

THEOREM 6.23
Assume that $\{A_n\}_{n=0}^{\infty}$ *is a period-2 sequence of positive real numbers such that*

$$A_0, A_1 \in (1, \infty).$$

Let

$$p = \min\left\{A_1, \frac{A_0^2}{A_1}\right\} \quad , \quad q = \min\left\{A_0, \frac{A_1^2}{A_0}\right\} \quad , \quad r = \max\left\{\frac{A_0}{A_1}, \frac{A_1}{A_0}\right\}$$

and set

$$\mathcal{S} = \{(x, y) : 0 < x \leq p,\ 0 < y \leq q,\ \textit{and}\ xy \geq r\}.$$

Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(6.21) with* $(x_{-1}, x_0) \in \mathcal{S}$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic with period-6.*

PROOF

$$x_1 = \frac{\max\{x_0, A_0\}}{x_0 x_{-1}} = \frac{A_0}{x_0 x_{-1}} \text{ because } x_0 \le A_0$$

$$x_2 = \frac{\max\left\{\frac{A_0}{x_0 x_{-1}}, A_1\right\}}{\frac{A_0}{x_0 x_{-1}} x_0} = \frac{A_1}{A_0} x_{-1} \text{ because } x_{-1} x_0 \ge \frac{A_0}{A_1}$$

$$x_3 = \frac{\max\left\{\frac{A_1}{A_0} x_{-1}, A_0\right\}}{\frac{A_1}{A_0} x_{-1} \frac{A_0}{x_0 x_{-1}}} = \frac{A_0}{A_1} x_0 \text{ because } x_{-1} \le \frac{A_0^2}{A_1}$$

$$x_4 = \frac{\max\left\{\frac{A_0}{A_1} x_0, A_1\right\}}{\frac{A_0}{A_1} x_0 \frac{A_1}{A_0} x_{-1}} = \frac{A_1}{x_0 x_{-1}} \text{ because } x_0 \le \frac{A_1^2}{A_0}$$

$$x_5 = \frac{\max\left\{\frac{A_1}{x_0 x_{-1}}, A_0\right\}}{\frac{A_1}{x_0 x_{-1}} \frac{A_0}{A_1} x_0} = x_{-1} \text{ because } x_{-1} x_0 \ge \frac{A_1}{A_0}$$

$$x_6 = \frac{\max\{x_{-1}, A_0\}}{x_{-1} \frac{A_1}{x_0 x_{-1}}} = x_0 \text{ because } x_{-1} \le A_1.$$

∎

THEOREM 6.24
Assume that for some $A \in (0, 1]$, we have

$$A_n = \begin{cases} \dfrac{1}{A} & \text{if } n \text{ is even} \\ A & \text{if } n \text{ is odd} \end{cases}$$

and let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(6.21). Then the following statements are true:

1. *Suppose $1 = A = x_{-1} = x_0$. Then $\{x_n\}_{n=-1}^{\infty}$ is identically equal to the equilibrium point $\bar{x} = 1$.*
2. *Suppose $A = 1$ and $(x_{-1}, x_0) \ne (1, 1)$. Then $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period 7.*
3. *Suppose $A \ne 1$ and $x_{-1}, x_0 \in (0, \infty) - \left(A \frac{1}{A}\right)$. Then $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period 14.*

PROOF We consider the case $x_{-1}, x_0 \in (0, A]$. The other cases are similar and will be omitted. Now,

$$
\begin{aligned}
x_1 &= \frac{\max\left\{x_0, \frac{1}{A}\right\}}{x_0 x_{-1}} = \frac{1}{A x_0 x_{-1}} \geq \frac{1}{A} \\
x_2 &= \frac{\max\{x_1, A\}}{x_1 x_0} = \frac{1}{x_0} \geq \frac{1}{A} \\
x_3 &= \frac{\max\left\{x_2, \frac{1}{A}\right\}}{x_2 x_1} = \frac{1}{x_1} = A x_0 x_{-1} \leq A \\
x_4 &= \frac{\max\{x_3, A\}}{x_3 x_2} = \frac{A}{A x_{-1}} = \frac{1}{x_{-1}} \geq \frac{1}{A} \\
x_5 &= \frac{\max\left\{x_4, \frac{1}{A}\right\}}{x_4 x_3} = \frac{1}{x_3} = \frac{1}{A x_0 x_{-1}} \geq \frac{1}{A} \geq A \\
x_6 &= \frac{\max\{x_5, A\}}{x_5 x_4} = \frac{1}{x_4} = x_{-1} \leq A \leq \frac{1}{A} \\
x_7 &= \frac{\max\left\{x_6, \frac{1}{A}\right\}}{x_6 x_5} = \frac{1}{A \frac{1}{A x_0}} = x_0 \leq A.
\end{aligned}
$$

Clearly this solution is constantly equal to one if $1 = A = x_{-1} = x_0$ and is periodic with prime period 7 if $A = 1$ and $(x_{-1}, x_0) \neq (1, 1)$.

Thus it remains to consider the case $0 < A < 1$.

$$
\begin{aligned}
x_8 &= \frac{\max\{x_7, A\}}{x_7 x_6} = \frac{A}{x_0 x_{-1}} \geq \frac{A}{A^2} = \frac{1}{A} \\
x_9 &= \frac{\max\left\{x_8, \frac{1}{A}\right\}}{x_8 x_7} = \frac{1}{x_7} = \frac{1}{x_0} \geq \frac{1}{A} \geq A \\
x_{10} &= \frac{\max\{x_9, A\}}{x_9 x_8} = \frac{1}{x_8} = \frac{x_{-1} x_0}{A} \leq \frac{A^2}{A} = A \leq \frac{1}{A} \\
x_{11} &= \frac{\max\left\{x_{10}, \frac{1}{A}\right\}}{x_{10} x_9} = \frac{\frac{1}{A}}{\frac{x_{-1} x_0}{A} \frac{1}{x_0}} = \frac{1}{x_{-1}} \geq \frac{1}{A} \geq A \\
x_{12} &= \frac{\max\{x_{11}, A\}}{x_{11} x_{10}} = \frac{1}{x_{10}} = \frac{A}{x_{-1} x_0} \geq \frac{1}{A}
\end{aligned}
$$

$$x_{13} = \frac{\max\left\{x_{12}, \frac{1}{A}\right\}}{x_{12}x_{11}} = \frac{1}{x_{11}} = x_{-1} \leq A$$

$$x_{14} = \frac{\max\{x_{13}, A\}}{x_{13}x_{12}} = \frac{A}{x_{-1}\frac{A}{x_{-1}x_0}} = x_0$$

from which the result follows. ∎

6.8 Open Problems and Conjectures

OPEN PROBLEM 6.1
Assume that $A \in (0, \infty)$, and that r_1 and r_2 are positive rational numbers. Investigate the periodic nature of the solution of the difference equation

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_{n-1}} \quad , \quad n = 0, 1, \ldots$$

with initial conditions $x_{-1} = A^{r_1}$ and $x_0 = A^{r_2}$.

OPEN PROBLEM 6.2
Assume that $A \in (0, \infty)$, and that r_1 and r_2 are positive rational numbers. Investigate the periodic nature of the solution of the difference equation

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_n x_{n-1}} \quad , \quad n = 0, 1, \ldots$$

with initial conditions $x_{-1} = A^{r_1}$ and $x_0 = A^{r_2}$.

OPEN PROBLEM 6.3
Assume that $A \in (0, \infty)$, and that r_1 and r_2 are positive rational numbers. Investigate the periodic nature of the solution of the difference equation

$$x_{n+1} = \frac{\max\{x_n^2, A\}}{x_n x_{n-1}} \quad , \quad n = 0, 1, \ldots$$

with initial conditions $x_{-1} = A^{r_1}$ and $x_0 = A^{r_2}$.

OPEN PROBLEM 6.4
Assume that $A \in (0, \infty)$, and that r_1 and r_2 are positive rational numbers. Investigate the periodic nature of the solution of the difference equation

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_n^2 x_{n-1}} \quad , \qquad n = 0, 1, \ldots$$

with initial conditions $x_{-1} = A^{r_1}$ and $x_0 = A^{r_2}$.

OPEN PROBLEM 6.5
Assume that $A \in (0, \infty)$, k and l are natural numbers, and that r_1 and r_2 are positive rational numbers. Investigate the periodic nature of the solution of the difference equation

$$x_{n+1} = \frac{\max\{x_n^k, A\}}{x_n^l x_{n-1}} \quad , \qquad n = 0, 1, \ldots$$

with initial conditions $x_{-1} = A^{r_1}$ and $x_0 = A^{r_2}$.

CONJECTURE 6.2
Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of the initial value problem

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_{n-1}} \quad , \qquad n = 0, 1, \ldots$$

$$x_{-1} = 1 \text{ and } x_0 = A^{\frac{k}{m}}$$

where k and m are positive, relatively prime integers such that either

$$k = m = 1 \qquad \text{or} \qquad 0 < m < k.$$

Then the following statements are true:

(i) If $0 < A < 1$, then $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period $5k + m$.

(ii) If $A > 1$, then $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period $5k - m$.

REMARK 6.12 The following lemma, the proof of which follows by computation and will be omitted, shows that a consequence of Conjecture 6.2 being true is that in the case $0 < A < 1$, the equation in Conjecture 6.2 possesses periodic solutions of all orders greater than 114. ▯

LEMMA 6.17

Let n be a positive integer $n \notin P = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 12, 13, 14, 15, 18, 19, 20, 22, 24, 25, 30, 32, 33, 34, 42, 44, 48, 54, 55, 78, 80, 84, 114\}$. Then there exist positive integers k and m such that

$$\begin{cases} n = 5k + m \qquad , (k, m) = 1 \qquad \textit{and} \\ \textit{either} \qquad 0 < m < k \qquad \textit{or} \qquad k = m = 1 \end{cases}$$

CONJECTURE 6.3

Assume that $A \in (0, \infty)$, and that k is a rational number. Show that every positive solution of the equation

$$x_{n+1} = \frac{\max\{x_n^k, A\}}{x_{n-1}} \qquad , \qquad n = 0, 1, \ldots$$

is periodic if and only if $0 \leq k \leq 1$. See [68].

CONJECTURE 6.4

Assume that $A \in (0, \infty)$, and that k is a rational number. Show that every positive solution of the equation

$$x_{n+1} = \frac{\max\{x_n^k, A\}}{x_n x_{n-1}} \qquad , \qquad n = 0, 1, \ldots$$

is periodic if and only if $0 \leq k \leq 3$. See [68].

OPEN PROBLEM 6.6

Let r and s be real numbers. Obtain necessary and sufficient conditions so that every positive solution of the equation

$$x_{n+1} = \frac{\max\{x_n^r, x_n^s\}}{x_{n-1}} \qquad , \qquad n = 0, 1, \ldots$$

is periodic. See [65].

Chapter 7

MAX EQUATIONS WITH PERIODIC COEFFICIENTS

7.1 Introduction

Our aim in this chapter is to investigate the periodic character and the boundedness behavior of the solutions of the difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right\} \quad , \qquad n = 0, 1, \ldots \tag{7.1}$$

where $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive real numbers, and where the initial conditions x_{-1} and x_0 are arbitrary positive real numbers.

Eq.(7.1) has been investigated in [9] when $\{A_n\}_{n=0}^{\infty}$ is a constant sequence, in [14] and [15] when $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence with period 2, and in [54] when $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence with period 3. In Sections 7.2 and 7.3 we present these known results about Eq.(7.1), and in Section 7.4 we discuss some open problems and conjectures about Eq.(7.1) when $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive real numbers with period greater than or equal to 4, or simply an arbitrary sequence of positive real numbers.

7.2 The Case Where $\{A_n\}$ is a Period-2 Sequence

In this section we study the equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right\} \quad , \qquad n = 0, 1, \ldots \tag{7.2}$$

where the initial conditions x_{-1} and x_0 are positive, and where $\{A_n\}_{n=0}^{\infty}$ is a positive periodic sequence of period 2. See [15]. The case where $A_0 = A_1$ was studied in [9].

The following lemma plays a key role in our analysis of the long-term behavior of the positive solutions of Eq.(7.2).

LEMMA 7.1
Assume $A_0, A_1 \in (0, \infty)$, *and let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(7.2). Then the following statements are true.*

1. *Assume* $A_0 x_0 \leq x_{-1}$. *Then* $x_0 x_1 = 1$.
2. *Assume* $x_{-1} < A_0 x_0$ *and* $A_0 A_1 \leq x_{-1} x_0$. *Then* $x_1 x_2 = 1$.
3. *Assume* $x_{-1} < A_0 x_0$, $x_{-1} x_0 < A_0 A_1$, *and* $A_1 x_{-1} \leq x_0$. *Then* $x_2 x_3 = 1$.
4. *Assume* $x_{-1} < A_0 x_0$, $x_{-1} x_0 < A_0 A_1$, $x_0 < A_1 x_{-1}$, *and* $x_{-1} x_0 \leq 1$. *Then* $x_3 x_4 = 1$.
5. *Assume* $x_{-1} < A_0 x_0$, $x_{-1} x_0 < A_0 A_1$, $x_0 < A_1 x_{-1}$, *and* $1 < x_{-1} x_0$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic with period 4.*

PROOF

(i) Assume $A_0 x_0 \leq x_{-1}$. Then

$$x_1 = \max\left\{\frac{1}{x_0}, \frac{A_0}{x_{-1}}\right\} = \frac{1}{x_0}.$$

(ii) Assume $x_{-1} < A_0 x_0$ and $A_0 A_1 \leq x_{-1} x_0$. Then

$$x_1 = \max\left\{\frac{1}{x_0}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}}$$

and

$$x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{x_{-1}}{A_0}, \frac{A_1}{x_0}\right\} = \frac{x_{-1}}{A_0}.$$

(iii) Assume $x_{-1} < A_0 x_0$, $x_{-1} x_0 < A_0 A_1$, and $A_1 x_{-1} \leq x_0$. Then

$$x_1 = \frac{A_0}{x_{-1}}, \quad x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{x_{-1}}{A_0}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0},$$

and

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_0}{x_1}\right\} = \max\left\{\frac{x_0}{A_1}, x_{-1}\right\} = \frac{x_0}{A_1}.$$

(iv) Assume $x_{-1} < A_0x_0$, $x_{-1}x_0 < A_0A_1$, $x_0 < A_1x_{-1}$, and $x_{-1}x_0 \leq 1$. Then

$$x_1 = \frac{A_0}{x_{-1}}, \quad x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{x_{-1}}{A_0}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0},$$

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_0}{x_1}\right\} = \max\left\{\frac{x_0}{A_1}, x_{-1}\right\} = x_{-1},$$

and

$$x_4 = \max\left\{\frac{1}{x_3}, \frac{A_1}{x_2}\right\} = \max\left\{\frac{1}{x_{-1}}, x_0\right\} = \frac{1}{x_{-1}}.$$

(v) Assume $x_{-1} < A_0x_0$, $x_{-1}x_0 < A_0A_1$, $x_0 < A_1x_{-1}$, and $1 < x_{-1}x_0$. Then $\{x_n\}_{n=-1}^{\infty}$ is periodic with period 4.

$$x_1 = \frac{A_0}{x_{-1}}, \quad x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{x_{-1}}{A_0}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0},$$

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_0}{x_1}\right\} = \max\left\{\frac{x_0}{A_1}, x_{-1}\right\} = x_{-1},$$

and

$$x_4 = \max\left\{\frac{1}{x_3}, \frac{A_1}{x_2}\right\} = \max\left\{\frac{1}{x_{-1}}, x_0\right\} = x_0,$$

from which the proof follows, since minus 1 and 3 are both odd. ∎

REMARK 7.1 Hypothesis (v). in Lemma 7.1 implies that $A_0A_1 > 1$, and so is vacuous if $A_0A_1 \leq 1$. However in the case where $A_0A_1 > 1$, Hypothesis (v). can be satisfied. For example, let $A_0 = \frac{1}{2}$, $A_1 = 10$, $x_{-1} = 1$, and $x_0 = 3$. Then

$$x_1 = \max\left\{\frac{1}{x_0}, \frac{A_0}{x_{-1}}\right\} = \max\left\{\frac{1}{3}, \frac{1}{2}\right\} = \frac{1}{2}$$

$$x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{2, \frac{10}{3}\right\} = \frac{10}{3}$$

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_0}{x_1}\right\} = \max\left\{\frac{3}{10}, 1\right\} = 1$$

$$x_4 = \max\left\{\frac{1}{x_3}, \frac{A_1}{x_2}\right\} = \max\{1, 3\} = 3$$

and it follows that $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period 4, and, moreover, that $x_{n-1}x_n \neq 1$ for all $n \geq 0$. ∎

7.2.1 The Case Where $0 < A_0A_1 < 1$

It was shown in [9] that if $A_0 = A_1 \in (0,1)$, then every positive solution of Eq.(7.2) is eventually periodic with period 2. The extension of this result to Eq.(7.2) is as follows.

THEOREM 7.1
Assume $A_0, A_1 \in (0,\infty)$ and $A_0A_1 \in (0,1)$. Then every positive solution of Eq.(7.2) is eventually periodic with period 2.

PROOF Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.2). We shall show that $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period 2.

It follows by Lemma 7.1 that without loss of generality, we may assume that $x_{-1}x_0 = 1$.

The proof will be given in the following three lemmas.

LEMMA 7.2
Suppose $x_{-1} \in \left(0, \sqrt{A_0}\right)$ and $x_0 = \dfrac{1}{x_{-1}}$. Then $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period 2.

PROOF Note that

$$x_1 = \max\left\{\frac{1}{x_0}, \frac{A_0}{x_{-1}}\right\} = \max\left\{x_{-1}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$

$$x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{x_{-1}}{A_0}, A_1x_{-1}\right\} = \frac{x_{-1}}{A_0},$$

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_0}{x_1}\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_{-1}\right\} = \frac{A_0}{x_{-1}}.$$

Choose $k \in \{0,1,\ldots\}$ such that

$$A_0(A_0A_1)^{k+1} \le x_{-1}^2 < A_0(A_0A_1)^k.$$

Suppose $k = 0$. Then $x_4 = \max\left\{\dfrac{x_{-1}}{A_0}, \dfrac{(A_0A_1)}{x_{-1}}\right\} = \dfrac{x_{-1}}{A_0}$, and the proof is complete, since 1 and 3 are both odd.

Suppose $k \ge 1$.
Let m be the largest integer less than or equal to $\dfrac{k-1}{2}$. Note that if k is odd,

then $k = 2m + 1$, while if k is even, then $k = 2m + 2$.

Claim: If $0 \leq j \leq m$, then the following equalities are true.

$$x_{6j+1} = \frac{A_0 \left(A_0 A_1\right)^j}{x_{-1}}, \quad x_{6j+2} = \frac{x_{-1}}{A_0 \left(A_0 A_1\right)^j}, \quad x_{6j+3} = \frac{A_0 \left(A_0 A_1\right)^j}{x_{-1}}$$

and

$$x_{6j+4} = \frac{\left(A_0 A_1\right)^{j+1}}{x_{-1}}, \quad x_{6j+5} = \frac{x_{-1}}{\left(A_0 A_1\right)^{j+1}}, \quad x_{6j+6} = \frac{\left(A_0 A_1\right)^{j+1}}{x_{-1}}.$$

Moreover, if k is even, then

$$x_{3k+1} = \frac{A_0 \left(A_0 A_1\right)^{m+1}}{x_{-1}}, \quad x_{3k+2} = \frac{x_{-1}}{A_0 \left(A_0 A_1\right)^{m+1}},$$

and $x_{3k+3} = \dfrac{A_0 \left(A_0 A_1\right)^{m+1}}{x_{-1}}$.

Proof of the claim:

$$x_4 = \max\left\{\frac{1}{x_3}, \frac{A_1}{x_2}\right\} = \max\left\{\frac{x_{-1}}{A_0}, \frac{\left(A_0 A_1\right)}{x_{-1}}\right\} = \frac{\left(A_0 A_1\right)}{x_{-1}},$$

$$x_5 = \max\left\{\frac{1}{x_4}, \frac{A_0}{x_3}\right\} = \max\left\{\frac{x_{-1}}{\left(A_0 A_1\right)}, x_{-1}\right\} = \frac{x_{-1}}{\left(A_0 A_1\right)},$$

$$x_6 = \max\left\{\frac{1}{x_5}, \frac{A_1}{x_4}\right\} = \max\left\{\frac{\left(A_0 A_1\right)}{x_{-1}}, \frac{x_{-1}}{A_0}\right\} = \frac{\left(A_0 A_{-1}\right)}{x_{-1}}.$$

Suppose $1 \leq j \leq m$, and that the claim is true for $j - 1$. Then

$$\begin{aligned}
x_{6j+1} &= \max\left\{\frac{x_{-1}}{(A_0A_1)^j}, \frac{A_0 (A_0A_1)^j}{x_{-1}}\right\} = \frac{A_0 (A_0A_1)^j}{x_{-1}}, \\
x_{6j+2} &= \max\left\{\frac{x_{-1}}{A_0 (A_0A_1)^j}, \frac{x_{-1}}{A_0 (A_0A_1)^{j-1}}\right\} = \frac{x_{-1}}{A_0 (A_0A_1)^j}, \\
x_{6j+3} &= \max\left\{\frac{A_0 (A_0A_1)^j}{x_{-1}}, \frac{x_{-1}}{(A_0A_1)^j}\right\} = \frac{A_0 (A_0A_1)^j}{x_{-1}}, \\
x_{6j+4} &= \max\left\{\frac{x_{-1}}{A_0 (A_0A_1)^j}, \frac{(A_0A_1)^{j+1}}{x_{-1}}\right\} = \frac{(A_0A_1)^{j+1}}{x_{-1}}, \\
x_{6j+5} &= \max\left\{\frac{x_{-1}}{(A_0A_1)^{j+1}}, \frac{x_{-1}}{(A_0A_1)^j}\right\} = \frac{x_{-1}}{(A_0A_1)^{j+1}}, \\
x_{6j+6} &= \max\left\{\frac{(A_0A_1)^{j+1}}{x_{-1}}, \frac{x_{-1}}{A_0 (A_0A_1)^j}\right\} = \frac{(A_0A_1)^{j+1}}{x_{-1}}.
\end{aligned}$$

Finally, if k is even, then (recall that $3k+1 = 6m+7$)

$$\begin{aligned}
x_{3k+1} &= \max\left\{\frac{x_{-1}}{(A_0A_1)^{m+1}}, \frac{A_0 (A_0A_1)^{m+1}}{x_{-1}}\right\} = \frac{A_0 (A_0A_1)^{m+1}}{x_{-1}}, \\
x_{3k+2} &= \max\left\{\frac{x_{-1}}{A_0 (A_0A_1)^{m+1}}, \frac{x_{-1}}{A_0 (A_0A_1)^m}\right\} = \frac{x_{-1}}{A_0 (A_0A_1)^{m+1}}, \\
x_{3k+3} &= \max\left\{\frac{A_0 (A_0A_1)^{m+1}}{,} \frac{x_{-1}}{(A_0A_1)^{m+1}}\right\} = \frac{A_0 (A_0A_1)^{m+1}}{x_{-1}},
\end{aligned}$$

and the proof of the claim is complete.

Case 1 Suppose $k = 2m+1$. Then $3k+1 = 6m+4$, and so

$$x_{3k+1} = \frac{(A_0A_1)^{m+1}}{x_{-1}}, \quad x_{3k+2} = \frac{x_{-1}}{(A_0A_1)^{m+1}}$$

$$x_{3k+3} = \frac{(A_0A_1)^{m+1}}{x_{-1}},$$

$$x_{3k+4} = \max\left\{\frac{x_{-1}}{(A_0A_1)^{m+1}}, \frac{A_0 (A_0A_1)^{m+1}}{x_{-1}}\right\} = \frac{x_{-1}}{(A_0A_1)^{m+1}},$$

and the proof is complete, since $3k+1$ and $3k+3$ are both even.

Case 2 Suppose $k = 2m+2$. Then

$$x_{3k+1} = \frac{A_0\left(A_0A_1\right)^{m+1}}{x_{-1}}, \quad x_{3k+2} = \frac{x_{-1}}{A_0\left(A_0A_1\right)^{m+1}},$$

$$x_{3k+3} = \frac{A_0\left(A_0A_1\right)^{m+1}}{x_{-1}},$$

$$x_{3k+4} = \max\left\{\frac{x_{-1}}{A_0\left(A_0A_1\right)^{m+1}}, \frac{A_1\left(A_0A_1\right)^{m+1}}{x_{-1}}\right\}$$

$$= \frac{x_{-1}}{A_0\left(A_0A_1\right)^{m+1}}$$

and the proof is complete, since $3k+1$ and $3k+3$ are both odd. ▯

LEMMA 7.3

Suppose $x_{-1} \in \left[\sqrt{A_0}, \frac{1}{\sqrt{A_1}}\right]$ *and* $x_0 = \frac{1}{x_{-1}}$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic with period 2.*

PROOF Since $x_0 = \frac{1}{x_{-1}}$, we see that

$$x_1 = \max\left\{\frac{1}{x_0}, \frac{A_0}{x_{-1}}\right\} = \max\left\{x_{-1}, \frac{A_0}{x_{-1}}\right\} = x_{-1}$$

and

$$x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{1}{x_{-1}}, A_1x_{-1}\right\} = \frac{1}{x_{-1}}$$

from which it follows that $\{x_n\}_{n=-1}^{\infty}$ is periodic with period 2. ▯

LEMMA 7.4

Assume that $x_{-1} \in \left(\frac{1}{\sqrt{A_1}}, \infty\right)$ *and* $x_0 = \frac{1}{x_{-1}}$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is eventually periodic with period 2.*

PROOF

Case 1 Suppose $A_0 \leq A_1$. Then, in particular, we have $A_0 < 1$.

Suppose $A_1 \leq 1$. Then $1 < x_{-1}$. Also $A_0 x_0 < x_0 = \frac{1}{x_{-1}} < x_{-1}$, and so it follows by Lemma 7.1 that $x_0 x_1 = 1$. Now, $x_0 = \frac{1}{x_{-1}} < \sqrt{A_1} \leq \frac{1}{\sqrt{A_1}}$, and so the result follows by applying Lemmas 7.2 and 7.3 to the solution $\{\tilde{x}_n\}_{n=-1}^{\infty}$ of Eq.(7.2), where $\tilde{x}_{-1} = x_0$ and $\tilde{x}_0 = x_1$.

So, we are left with the case $A_1 > 1$. We shall show that $\{x_n\}_{n=-1}^{\infty}$ becomes periodic with period 2 at the $(3k+2)^{nd}$ term, and that $x_{3k+2} = \frac{1}{x_{3k+1}}$. Note that

$$A_0 < \frac{1}{A_1} < x_{-1}^2,$$

and so

$$x_1 = \max\left\{\frac{1}{x_0}, \frac{A_0}{x_{-1}}\right\} = \max\left\{x_{-1}, \frac{A_0}{x_{-1}}\right\} = x_{-1},$$

$$x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{1}{x_{-1}}, A_1 x_{-1}\right\} = A_1 x_{-1},$$

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_0}{x_1}\right\} = \max\left\{\frac{1}{A_1 x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{1}{A_1 x_{-1}},$$

$$x_4 = \max\left\{\frac{1}{x_3}, \frac{A_1}{x_2}\right\} = \max\left\{A_1 x_{-1}, \frac{1}{x_{-1}}\right\} = A_1 x_{-1}.$$

Choose $k \in \{0, 1, \ldots\}$ such that

$$(A_0 A_1)^{k+1} A_1 x_{-1}^2 \leq 1 < (A_0 A_1)^k A_1 x_{-1}^2.$$

Suppose $k = 0$. Then $x_5 = \max\left\{\frac{1}{A_1 x_{-1}}, (A_0 A_1) x_{-1}\right\} = \frac{1}{A_1 x_{-1}}$, and the proof is complete, since 2 and 4 are both even.

Suppose $k \geq 1$.
Let m be the largest integer less than or equal to $\frac{k-1}{2}$. Note that if k is odd, then $k = 2m+1$, while if k is even, then $k = 2m+2$.

Claim: If $0 \leq j \leq m$, then the following equalities are true.

$$x_{6j+2} = (A_0 A_1)^j A_1 x_{-1}, \;\; x_{6j+3} = \frac{1}{(A_0 A_1)^j A_1 x_{-1}},$$

$$x_{6j+4} = (A_0 A_1)^j A_1 x_{-1},$$

and

$$x_{6j+5} = (A_0A_1)^{j+1}\, x_{-1}, \;\; x_{6j+6} = \frac{1}{(A_0A_1)^{j+1}\, x_{-1}},$$

$$x_{6j+7} = (A_0A_1)^{j+1}\, x_{-1}.$$

Moreover, if k is even, then

$$x_{3k+2} = (A_0A_1)^{m+1}\, A_1x_{-1}, \;\; x_{3k+3} = \frac{1}{(A_0A_1)^{m+1}\, A_1x_{-1}},$$

$$x_{3k+4} = (A_0A_1)^{m+1}\, A_1x_{-1}.$$

Proof of the claim:

$$x_5 = \max\left\{\frac{1}{x_4}, \frac{A_0}{x_3}\right\} = \max\left\{\frac{1}{A_1x_{-1}}, (A_0A_1)\, x_{-1}\right\} = (A_0A_1)\, x_{-1},$$

$$x_6 = \max\left\{\frac{1}{x_5}, \frac{A_1}{x_4}\right\} = \max\left\{\frac{1}{(A_0A_1)\, x_{-1}}, \frac{1}{x_{-1}}\right\} = \frac{1}{(A_0A_1)\, x_{-1}},$$

$$x_7 = \max\left\{\frac{1}{x_6}, \frac{A_0}{x_5}\right\} = \max\left\{(A_0A_1)\, x_{-1}, \frac{1}{A_1x_{-1}}\right\} = (A_0A_1)\, x_{-1}.$$

Suppose $1 \le j \le m$, and that the claim is true for $j-1$. Then

$$x_{6j+2} = \max\left\{\frac{1}{(A_0A_1)^j\, x_{-1}}, (A_0A_1)^j\, A_1x_{-1}\right\} = (A_0A_1)^j\, A_1x_{-1},$$

$$x_{6j+3} = \max\left\{\frac{1}{(A_0A_1)^j\, A_1x_{-1}}, \frac{1}{(A_0A_1)^{j-1}\, A_1x_{-1}}\right\} = \frac{1}{(A_0A_1)^j\, A_1x_{-1}},$$

$$x_{6j+4} = \max\left\{(A_0A_1)^j\, A_1x_{-1}, \frac{1}{(A_0A_1)^j\, x_{-1}}\right\} = (A_0A_1)^j\, A_1x_{-1},$$

$$x_{6j+5} = \max\left\{\frac{1}{(A_0A_1)^j\, A_1x_{-1}}, (A_0A_1)^{j+1}\, x_{-1}\right\} = (A_0A_1)^{j+1}\, x_{-1},$$

$$x_{6j+6} = \max\left\{\frac{1}{(A_0A_1)^{j+1}\, x_{-1}}, \frac{1}{(A_0A_1)^j\, x_{-1}}\right\} = \frac{1}{(A_0A_1)^{j+1}\, x_{-1}},$$

$$x_{6j+7} = \max\left\{(A_0A_1)^{j+1}\,x_{-1}, \frac{1}{(A_0A_1)^{j}\,A_1x_{-1}}\right\} = (A_0A_1)^{j+1}\,x_{-1}.$$

Finally, if k is even, then (recall that $3k+2 = 6m+8$)

$$\begin{aligned}
x_{3k+2} &= \max\left\{\frac{1}{(A_0A_1)^{m+1}\,x_{-1}}, (A_0A_1)^{m+1}\,A_1x_{-1}\right\}\\
&= (A_0A_1)^{m+1}\,A_1x_{-1},\\
x_{3k+3} &= \max\left\{\frac{1}{(A_0A_1)^{m+1}\,A_1x_{-1}}, \frac{1}{(A_0A_1)^{m}\,A_1x_{-1}}\right\}\\
&= \frac{1}{(A_0A_1)^{m+1}\,A_1x_{-1}},\\
x_{3k+4} &= \max\left\{(A_0A_1)^{m+1}\,A_{-1}x_{-1}, (A_0A_1)^{m}\,A_1x_{-1}\right\}\\
&= (A_0A_1)^{m+1}\,A_1x_{-1},
\end{aligned}$$

and the proof of the claim is complete.

Case 1(a) Suppose $k = 2m+1$. Then $3k+2 = 6m+5$, and so

$$x_{3k+2} = (A_0A_1)^{m+1}x_{-1}, \qquad x_{3k+3} = \frac{1}{(A_0A_1)^{m+1}\,x_{-1}},$$

$$x_{3k+4} = (A_0A_1)^{m+1}x_{-1}, \qquad x_{3k+5} = \frac{1}{(A_0A_1)^{m+1}A_1x_{-1}}$$

and the proof is complete, since $3k+2$ and $3k+4$ are both odd.

Case 1(b) Suppose $k = 2m+2$. Then $3k+2 = 6m+6$, and so

$$x_{3k+2} = (A_0A_1)^{m+1}\,A_1x_{-1}, \qquad x_{3k+3} = \frac{1}{(A_0A_1)^{m+1}\,A_1x_{-1}},$$

$$\begin{aligned}
x_{3k+4} &= (A_0A_1)^{m+1}\,A_1x_{-1},\\
x_{3k+5} &= \max\left\{\frac{1}{(A_0A_1)^{m+1}\,A_1x_{-1}}, (A_0A_1)^{m+2}\,x_{-1}\right\}\\
&= \frac{1}{(A_0A_1)^{m+1}\,A_1x_{-1}},
\end{aligned}$$

and the proof is complete, since $3k+2$ and $3k+4$ are both even.

Case 2. Suppose $A_1 < A_0$. Then as $1 < A_1 x_{-1}^2$ and $A_0 A_1 < 1$, we see $A_0 < x_{-1}^2$. It follows by Lemma 7.1 that $x_0 x_1 = 1$. The proof follows by applying Case 1 to the solution $\{\tilde{x}_n\}_{n=-1}^{\infty}$ of Eq.(7.2), where $\tilde{x}_{-1} = x_0$ and $\tilde{x}_0 = x_1$. ▯

The proof of Theorem 7.1 is complete. ▯

7.2.2 The Case Where $A_0 A_1 = 1$

It was shown in [9] that if $A_0 = A_1 = 1$, then every positive solution of Eq.(7.2) is eventually periodic with period 3. The extension of this result to Eq.(7.2) is as follows.

THEOREM 7.2
Assume $A_0, A_1 \in (0, \infty)$ and $A_0 A_1 = 1$. Then every positive solution of Eq.(7.2) is eventually periodic with period 6.

PROOF Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.2). We shall show that $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period 6.

It follows by Lemma 7.1 that without loss of generality, we may assume that $x_{-1} x_0 = 1$.

Suppose $x_{-1} \in \left(0, \sqrt{A_0}\right)$ and $x_0 = \dfrac{1}{x_{-1}}$. Then

$$x_1 = \max\left\{\frac{1}{x_0}, \frac{A_0}{x_{-1}}\right\} = \max\left\{x_{-1}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$

$$x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{x_{-1}}{A_0}, \frac{x_{-1}}{A_0}\right\} = \frac{x_{-1}}{A_0},$$

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_0}{x_1}\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_{-1}\right\} = \frac{A_0}{x_{-1}},$$

$$x_4 = \max\left\{\frac{1}{x_3}, \frac{A_1}{x_2}\right\} = \max\left\{\frac{x_{-1}}{A_0}, \frac{1}{x_{-1}}\right\} = \frac{1}{x_{-1}},$$

$$x_5 = \max\left\{\frac{1}{x_4}, \frac{A_0}{x_3}\right\} = \max\left\{x_{-1}, x_{-1}\right\} = x_{-1},$$

$$x_6 = \max\left\{\frac{1}{x_5}, \frac{A_1}{x_4}\right\} = \max\left\{\frac{1}{x_{-1}}, \frac{x_{-1}}{A_0}\right\} = \frac{1}{x_{-1}}$$

and the proof is complete, since minus 1 and 5 are both odd.

Suppose $x_{-1} \in \left[\sqrt{A_0}, \infty\right)$. Then

$$x_1 = \max\left\{\frac{1}{x_0}, \frac{A_0}{x_{-1}}\right\} = \max\left\{x_{-1}, \frac{A_0}{x_{-1}}\right\} = x_{-1},$$

$$x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{1}{x_{-1}}, \frac{x_{-1}}{A_0}\right\} = \frac{x_{-1}}{A_0},$$

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_0}{x_1}\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$

$$x_4 = \max\left\{\frac{1}{x_3}, \frac{A_1}{x_2}\right\} = \max\left\{\frac{x_{-1}}{A_0}, \frac{1}{x_{-1}}\right\} = \frac{x_{-1}}{A_0},$$

$$x_5 = \max\left\{\frac{1}{x_4}, \frac{A_0}{x_3}\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_{-1}\right\} = x_{-1},$$

$$x_6 = \max\left\{\frac{1}{x_5}, \frac{A_1}{x_4}\right\} = \max\left\{\frac{1}{x_{-1}}, \frac{1}{x_{-1}}\right\} = \frac{1}{x_{-1}},$$

from which the result follows, as minus 1 and 5 are both odd. ▯

7.2.3 The Case Where $1 < A_0 A_1$

It was shown in [9] that if $A_0 = A_1 \in (1, \infty)$, then every positive solution of Eq.(7.2) is eventually periodic with period 4. The extension of this result to Eq.(7.2) is as follows.

THEOREM 7.3
Assume $A_0, A_1 \in (0, \infty)$ and $A_0 A_1 \in (1, \infty)$. Then every positive solution of Eq.(7.2) is eventually periodic with period 4.

PROOF Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.2). We shall show that $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period 4.

It follows by Lemma 7.1 that without loss of generality, we may assume that $x_{-1} x_0 = 1$.

The proof will be given in the following three lemmas.

LEMMA 7.5
Suppose $x_{-1} \in \left(0, \frac{1}{\sqrt{A_1}}\right)$ and $x_0 = \frac{1}{x_{-1}}$. Then $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period 4.

PROOF Note that $A_1 x_{-1}^2 < 1$ and $x_{-1}^2 < A_0$. Note also that

$$x_1 = \max\left\{\frac{1}{x_0}, \frac{A_0}{x_{-1}}\right\} = \max\left\{x_{-1}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$

$$x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{x_{-1}}{A_0}, A_1 x_{-1}\right\} = A_1 x_{-1},$$

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_0}{x_1}\right\} = \max\left\{\frac{1}{A_1 x_{-1}}, x_{-1}\right\} = \frac{1}{A_1 x_{-1}},$$

$$x_4 = \max\left\{\frac{1}{x_3}, \frac{A_1}{x_2}\right\} = \max\left\{A_1 x_{-1}, \frac{1}{x_{-1}}\right\} = \frac{1}{x_{-1}},$$

$$x_5 = \max\left\{\frac{1}{x_4}, \frac{A_0}{x_3}\right\} = \max\{x_{-1}, (A_0 A_1) x_{-1}\} = (A_0 A_1) x_{-1}.$$

Choose $k \in \{0, 1, \ldots\}$ such that

$$(A_0 A_1)^k A_1 x_{-1}^2 \leq 1 < (A_0 A_1)^{k+1} A_1 x_{-1}^2.$$

Suppose $k = 0$. Then

$$x_6 = \max\left\{\frac{1}{x_5}, \frac{A_1}{x_4}\right\} = \max\left\{\frac{1}{(A_0 A_1) x_{-1}}, A_1 x_{-1}\right\} = A_1 x_{-1},$$

$$x_7 = \max\left\{\frac{1}{x_6}, \frac{A_0}{x_5}\right\} = \max\left\{\frac{1}{A_1 x_{-1}}, \frac{1}{x_{-1}}\right\} = \frac{1}{A_1 x_{-1}},$$

and the proof is complete, since 2 and 6 are both even.

Suppose $k \geq 1$.

Let m be the largest integer less than or equal to $\frac{k-1}{2}$. Note that if k is odd, then $k = 2m + 1$, while if k is even, then $k = 2m + 2$.

Claim: If $0 \leq j \leq m$, then the following equalities are true.

$$x_{6j+3} = \frac{1}{(A_0 A_1)^j A_1 x_{-1}}, \quad x_{6j+4} = \frac{1}{(A_0 A_1)^j x_{-1}},$$

$$x_{6j+5} = (A_0 A_1)^{j+1} x_{-1},$$

and

$$x_{6j+6} = \frac{1}{(A_0 A_1)^{j+1} x_{-1}}, \qquad x_{6j+7} = \frac{1}{(A_0 A_1)^j A_1 x_{-1}},$$

$$x_{6j+8} = (A_0 A_1)^{j+1} A_1 x_{-1}.$$

Moreover, if k is even, then

$$x_{3k+3} = \frac{1}{(A_0A_1)^{m+1}A_1x_{-1}}, \quad x_{3k+4} = \frac{1}{(A_0A_1)^{m+1}x_{-1}},$$

$$x_{3k+5} = (A_0A_1)^{m+2}x_{-1}.$$

Proof of the claim:

$$x_6 = \max\left\{\frac{1}{x_5}, \frac{A_1}{x_4}\right\} = \max\left\{\frac{1}{(A_0A_1)x_{-1}}, A_1x_{-1}\right\} = \frac{1}{(A_0A_1)x_{-1}},$$

$$x_7 = \max\left\{\frac{1}{x_6}, \frac{A_0}{x_5}\right\} = \max\left\{(A_0A_1)x_{-1}, \frac{1}{A_1x_{-1}}\right\} = \frac{1}{A_1x_{-1}},$$

$$x_8 = \max\left\{\frac{1}{x_7}, \frac{A_1}{x_6}\right\} = \max\{A_1x_{-1}, (A_0A_1)x_{-1}\} = (A_0A_1)x_{-1}.$$

Suppose $1 \le j \le m$, and that the claim is true for $j-1$. Then

$$x_{6j+3} = \max\left\{\frac{1}{(A_0A_1)^jA_1x_{-1}}, (A_0A_1)^jx_{-1}\right\} = \frac{1}{(A_0A_1)^jA_1x_{-1}},$$

$$x_{6j+4} = \max\left\{(A_0A_1)^jA_1x_{-1}, \frac{1}{(A_0A_1)^jx_{-1}}\right\} = \frac{1}{(A_0A_1)^jx_{-1}},$$

$$x_{6j+5} = \max\left\{(A_0A_1)^jx_{-1}, (A_0A_1)^{j+1}x_{-1}\right\} = (A_0A_1)^{j+1}x_{-1},$$

$$x_{6j+6} = \max\left\{\frac{1}{(A_0A_1)^{j+1}x_{-1}}, (A_0A_1)^jA_1x_{-1}\right\} = \frac{1}{(A_0A_1)^{j+1}x_{-1}},$$

$$x_{6j+7} = \max\left\{(A_0A_1)^{j+1}x_{-1}, \frac{1}{(A_0A_1)^jA_1x_{-1}}\right\} = \frac{1}{(A_0A_1)^jA_1x_{-1}},$$

$$x_{6j+8} = \max\left\{(A_0A_1)^jA_1x_{-1}, (A_0A_1)^{j+1}A_1x_{-1}\right\} = (A_0A_1)^{j+1}A_1x_{-1}.$$

Finally, if k is even, then (recall that $3k+3 = 6m+9$)

$$x_{3k+3} = \max\left\{\frac{1}{(A_0A_1)^{m+1}A_1x_{-1}}, (A_0A_1)^{m+1}x_{-1}\right\} = \frac{1}{(A_0A_1)^{m+1}A_1x_{-1}},$$

$$x_{3k+4} = \max\left\{(A_0A_1)^{m+1}A_1x_{-1}, \frac{1}{(A_0A_1)^{m+1}x_{-1}}\right\} = \frac{1}{(A_0A_1)^{m+1}x_{-1}},$$

$$x_{3k+5} = \max\left\{(A_0A_1)^{m+1}x_{-1}, (A_0A_1)^{m+2}x_{-1}\right\} = (A_0A_1)^{m+2}x_{-1}$$

and the proof of the claim is complete.

Case 1 Suppose $k = 2m+1$. Then $3k+2 = 6m+5$, and so

$$x_{3k+2} = (A_0A_1)^{m+1}x_{-1}, \qquad x_{3k+3} = \frac{1}{(A_0A_1)^{m+1}x_{-1}},$$

$$x_{3k+4} = \frac{1}{(A_0A_1)^{m}A_1x_{-1}}, \qquad x_{3k+5} = (A_0A_1)^{m+1}A_1x_{-1},$$

$$\begin{aligned} x_{3k+6} &= \max\left\{\frac{1}{(A_0A_1)^{m+1}A_1x_{-1}}, (A_0A_1)^{m+1}x_{-1}\right\} \\ &= (A_0A_1)^{m+1}x_{-1}, \\ x_{3k+7} &= \max\left\{\frac{1}{(A_0A_1)^{m+1}x_{-1}}, \frac{1}{(A_0A_1)^{m+1}x_{-1}}\right\} \\ &= \frac{1}{(A_0A_1)^{m+1}x_{-1}}, \end{aligned}$$

and the proof is complete, since $3k+2$ and $3k+6$ are both odd.

Case 2. Suppose $k = 2m+2$. Then $3k+2 = 6m+8$, and so

$$x_{3k+2} = (A_0A_1)^{m+1}A_1x_{-1},$$

$$x_{3k+3} = \frac{1}{(A_0A_1)^{m+1}A_1x_{-1}}, \qquad x_{3k+4} = \frac{1}{(A_0A_1)^{m+1}x_{-1}},$$

$$x_{3k+5} = (A_0A_1)^{m+2}x_{-1},$$

$$\begin{aligned} x_{3k+6} &= \max\left\{\frac{1}{(A_0A_1)^{m+2}x_{-1}}, (A_0A_1)^{m+1}A_1x_{-1}\right\} \\ &= (A_0A_1)^{m+1}A_1x_{-1}, \\ x_{3k+7} &= \max\left\{\frac{1}{(A_0A_1)^{m+1}A_1x_{-1}}, \frac{1}{(A_0A_1)^{m+1}A_1x_{-1}}\right\} \\ &= \frac{1}{(A_0A_1)^{m+1}A_1x_{-1}} \end{aligned}$$

and the proof is complete, since $3k+2$ and $3k+6$ are both even. ▯

LEMMA 7.6
Suppose $x_{-1} \in \left[\dfrac{1}{\sqrt{A_1}}, \sqrt{A_0}\right]$ *and* $x_0 = \dfrac{1}{x_{-1}}$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic with period 4.*

PROOF Note that $1 \le A_1 x_1^2$ and $x_{-1}^2 \le A_0$. Since $x_0 = \dfrac{1}{x_{-1}}$, we see that

$$x_1 = \max\left\{\frac{1}{x_0}, \frac{A_0}{x_{-1}}\right\} = \max\left\{x_{-1}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$

$$x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{x_{-1}}{A_0}, A_1 x_{-1}\right\} = A_1 x_{-1},$$

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_0}{x_1}\right\} = \max\left\{\frac{1}{A_1 x_{-1}}, x_{-1}\right\} = x_{-1},$$

$$x_4 = \max\left\{\frac{1}{x_3}, \frac{A_1}{x_2}\right\} = \max\left\{\frac{1}{x_{-1}}, \frac{1}{x_{-1}}\right\} = \frac{1}{x_{-1}} = x_0.$$

So as minus 1 and 3 are both odd, we see that $\{x_n\}_{n=-1}^{\infty}$ is periodic with period 4. ∎

LEMMA 7.7
Assume that $x_{-1} \in \left(\sqrt{A_0}, \infty\right)$ *and* $x_0 = \dfrac{1}{x_{-1}}$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is eventually periodic with period 4.*

PROOF Note that $\dfrac{1}{A_1} < A_0 < x_{-1}^2$, and so in particular, $1 < A_1 x_{-1}^2$. Note also that

$$x_1 = \max\left\{\frac{1}{x_0}, \frac{A_0}{x_{-1}}\right\} = \max\left\{x_{-1}, \frac{A_0}{x_{-1}}\right\} = x_{-1},$$

$$x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{1}{x_{-1}}, A_1 x_{-1}\right\} = A_1 x_{-1},$$

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_0}{x_1}\right\} = \max\left\{\frac{1}{A_1 x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$

$$x_4 = \max\left\{\frac{1}{x_3}, \frac{A_1}{x_2}\right\} = \max\left\{\frac{x_{-1}}{A_0}, \frac{1}{x_{-1}}\right\} = \frac{x_{-1}}{A_0},$$

$$x_5 = \max\left\{\frac{1}{x_4}, \frac{A_0}{x_3}\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_{-1}\right\} = x_{-1},$$

$$x_6 = \max\left\{\frac{1}{x_5}, \frac{A_1}{x_4}\right\} = \max\left\{\frac{1}{x_{-1}}, \frac{(A_0 A_1)}{x_{-1}}\right\} = \frac{(A_0 A_1)}{x_{-1}}.$$

Choose $k \in \{0, 1, \ldots\}$ such that

$$(A_0A_1)^k A_0 \le x_{-1}^2 < (A_0A_1)^{k+1} A_0.$$

Suppose $k = 0$. Then

$$x_7 = \max\left\{\frac{1}{x_6}, \frac{A_0}{x_5}\right\} = \max\left\{\frac{x_{-1}}{(A_0A_1)}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$

$$x_8 = \max\left\{\frac{1}{x_7}, \frac{A_1}{x_6}\right\} = \max\left\{\frac{x_{-1}}{A_0}, \frac{A_0}{x_{-1}}\right\} = \frac{x_{-1}}{A_0},$$

and the proof is complete, since 3 and 7 are both odd.

Suppose $k \ge 1$.

Let m be the largest integer less than or equal to $\dfrac{k-1}{2}$. Note that if k is odd, then $k = 2m + 1$, while if k is even, then $k = 2m + 2$.

Claim: If $0 \le j \le m$, then the following equalities are true.

$$x_{6j+4} = \frac{x_{-1}}{A_0(A_0A_1)^j}, \quad x_{6j+5} = \frac{x_{-1}}{(A_0A_1)^j}, \quad x_{6j+6} = \frac{(A_0A_1)^{j+1}}{x_{-1}},$$

and

$$x_{6j+7} = \frac{x_{-1}}{(A_0A_1)^{j+1}}, \quad x_{6j+8} = \frac{x_{-1}}{A_0(A_0A_1)^j}, \quad x_{6j+9} = \frac{A_0(A_0A_1)^{j+1}}{x_{-1}}.$$

Moreover, if $k = 2m + 2$, then

$$x_{3k+4} = \frac{x_{-1}}{A_0(A_0A_1)^{m+1}}, \quad x_{3k+5} = \frac{x_{-1}}{(A_0A_1)^{m+1}}, \quad x_{3k+6} = \frac{(A_0A_1)^{m+2}}{x_{-1}}.$$

Proof of the claim:

$$x_7 = \max\left\{\frac{1}{x_6}, \frac{A_0}{x_5}\right\} = \max\left\{\frac{x_{-1}}{(A_0A_1)}, \frac{A_0}{x_{-1}}\right\} = \frac{x_{-1}}{(A_0A_1)},$$

$$x_8 = \max\left\{\frac{1}{x_7}, \frac{A_1}{x_6}\right\} = \max\left\{\frac{(A_0A_1)}{x_{-1}}, \frac{x_{-1}}{A_0}\right\} = \frac{x_{-1}}{A_0},$$

$$x_9 = \max\left\{\frac{1}{x_8}, \frac{A_0}{x_7}\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_0(A_0A_1)}{x_{-1}}\right\} = \frac{A_0(A_0A_1)}{x_{-1}}.$$

Suppose $1 \le j \le m$, and that the claim is true for $j - 1$. Then

$$x_{6j+4} = \max\left\{\frac{x_{-1}}{A_0(A_0A_1)^j}, \frac{(A_0A_1)^j}{x_{-1}}\right\} = \frac{x_{-1}}{A_0(A_0A_1)^j},$$

$$x_{6j+5} = \left\{ \frac{A_0(A_0A_1)^j}{x_{-1}}, \frac{x_{-1}}{(A_0A_1)^j} \right\} = \frac{x_{-1}}{(A_0A_1)^j},$$

$$x_{6j+6} = \max\left\{ \frac{(A_0A_1)^j}{x_{-1}}, \frac{(A_0A_1)^{j+1}}{x_{-1}} \right\} = \frac{(A_0A_1)^{j+1}}{x_{-1}},$$

$$x_{6j+7} = \max\left\{ \frac{x_{-1}}{(A_0A_1)^{j+1}}, \frac{A_0(A_0A_1)^j}{x_{-1}} \right\} = \frac{x_{-1}}{(A_0A_1)^{j+1}},$$

$$x_{6j+8} = \max\left\{ \frac{(A_0A_1)^{j+1}}{x_{-1}}, \frac{x_{-1}}{A_0(A_0A_1)^j} \right\} = \frac{x_{-1}}{A_0(A_0A_1)^j},$$

$$x_{6j+9} = \max\left\{ \frac{A_0(A_0A_1)^j}{x_{-1}}, \frac{A_0(A_0A_1)^{j+1}}{x_{-1}} \right\} = \frac{A_0(A_0A_1)^{j+1}}{x_{-1}}.$$

Finally, if k is even, then (recall that $3k+4 = 6m+10$)

$$x_{3k+4} = \max\left\{ \frac{x_{-1}}{A_0(A_0A_1)^{m+1}}, \frac{(A_0A_1)^{m+1}}{x_{-1}} \right\} = \frac{x_{-1}}{A_0(A_0A_1)^{m+1}},$$

$$x_{3k+5} = \max\left\{ \frac{A_0(A_0A_1)^{m+1}}{x_{-1}}, \frac{x_{-1}}{(A_0A_1)^{m+1}} \right\} = \frac{x_{-1}}{(A_0A_1)^{m+1}},$$

$$x_{3k+6} = \max\left\{ \frac{(A_0A_1)^{m+1}}{x_{-1}}, \frac{(A_0A_1)^{m+2}}{x_{-1}} \right\} = \frac{(A_0A_1)^{m+2}}{x_{-1}},$$

and the proof of the claim is complete.

Case 1 Suppose $k = 2m+1$. Then $3k+3 = 6m+6$, and so

$$x_{3k+3} = \frac{(A_0A_1)^{m+1}}{x_{-1}}, \quad x_{3k+4} = \frac{x_{-1}}{(A_0A_1)^{m+1}}, \quad x_{3k+5} = \frac{x_{-1}}{A_0(A_0A_1)^m},$$

$$x_{3k+6} = \frac{A_0(A_0A_1)^{m+1}}{x_{-1}},$$

$$x_{3k+7} = \max\left\{ \frac{x_{-1}}{A_0(A_0A_1)^{m+1}}, \frac{(A_0A_1)^{m+1}}{x_{-1}} \right\} = \frac{(A_0A_1)^{m+1}}{x_{-1}},$$

$$x_{3k+8} = \max\left\{ \frac{x_{-1}}{(A_0A_1)^{m+1}}, \frac{x_{-1}}{(A_0A_1)^{m+1}} \right\} = \frac{x_{-1}}{(A_0A_1)^{m+1}},$$

and the proof is complete, since $3k+3$ and $3k+7$ are both even.

Case 2 Suppose $k = 2m + 2$. Then $3k + 3 = 6m + 9$, and so

$$x_{3k+3} = \frac{A_0(A_0A_1)^{m+1}}{x_{-1}}, \quad x_{3k+4} = \frac{x_{-1}}{A_0(A_0A_1)^{m+1}},$$

$$x_{3k+5} = \frac{x_{-1}}{(A_0A_1)^{m+1}}, \qquad x_{3k+6} = \frac{(A_0A_1)^{m+2}}{x_{-1}},$$

$$x_{3k+7} = \max\left\{\frac{x_{-1}}{(A_0A_1)^{m+2}}, \frac{A_0(A_0A_1)^{m+1}}{x_{-1}}\right\} = \frac{A_0(A_0A_1)^{m+1}}{x_{-1}},$$

$$x_{3k+8} = \max\left\{\frac{x_{-1}}{A_0(A_0A_1)^{m+1}}, \frac{x_{-1}}{A_0(A_0A_1)^{m+1}}\right\} = \frac{x_{-1}}{A_0(A_0A_1)^{m+1}},$$

and the proof is complete, since $3k + 3$ and $3k + 7$ are both odd. ▯

The proof of Theorem 7.3 is complete. ▯

7.3 Period-3 Coefficients

In this section we study the difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right\} \tag{7.3}$$

where $\{A_n\}_{n=0}^{\infty}$ is a sequence of positive real numbers which is periodic with prime period 3, and we prove the following results.

1. If $A_n \in (0, 1)$ for all $n \geq 0$, then every positive solution of Eq.(7.3) is eventually periodic with period 2.

2. If $A_n \in (1, \infty)$ for all $n \geq 0$, then every positive solution of Eq.(7.3) is eventually periodic with period 12.

3. If $A_{i+1} < 1 < A_i$ for some $i \in \{0, 1, 2\}$, then every positive solution of Eq.(7.3) is unbounded.

4. In all other cases, every positive solution of Eq.(7.3) is eventually periodic with period 3.

7.3.1 Eventually Periodic Solutions with Period 2

Assume that $\{A_n\}_{n=0}^{\infty}$ is a sequence of positive real numbers which is periodic with prime period 3 such that $A_n \in (0,1)$ for all $n \geq 0$. It was shown in [14] that in this case, every positive solution of Eq.(7.3) is eventually periodic with period 2.

The case where the sequence $\{A_n\}_{n=0}^{\infty}$ is identically equal to a positive constant was investigated in [9], and the case where $\{A_n\}_{n=0}^{\infty}$ is a sequence of positive real numbers which is periodic with prime period 2 was investigated in [15].

Observe that Eq.(7.3) has the unique equilibrium point $\bar{x} = 1$.

LEMMA 7.8
Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers which is periodic with prime period 3 such that $A_n \in (0,1)$ for all $n \geq 0$. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3) which is not eventually constant. Then $x_n \neq 1$ for all $n \geq 1$.

PROOF For the sake of contradiction, suppose there exists $N \geq 0$ such that $x_{N+1} = 1$. Then

$$1 = x_{N+1} = \max\left\{\frac{1}{x_N}, \frac{A_N}{x_{N-1}}\right\}$$

and so we see that $x_N > 1$ since $\{x_n\}_{n=-1}^{\infty}$ is not eventually constant. Thus

$$x_{N+2} = \left\{\frac{1}{x_{N+1}}, \frac{A_{N+1}}{x_N}\right\} = \max\left\{1, \frac{A_{N+1}}{x_N}\right\} = 1,$$

which is a contradiction. ∎

LEMMA 7.9
Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers which is periodic with prime period 3 such that $A_n \in (0,1)$ for all $n \geq 0$. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3), and let $m \geq 0$ be a non-negative integer. Then one of the following statements is true:

1. $x_{m-1}x_m = 1$.
2. $x_m x_{m+1} = 1$.
3. $x_{m+1}x_{m+2} = 1$.
4. $x_{m+2}x_{m+3} = 1$.

PROOF Suppose $x_{m-1}x_m \neq 1$, $x_m x_{m+1} \neq 1$, and $x_{m+1}x_{m+2} \neq 1$. It suffices to show that $x_{m+2}x_{m+3} = 1$. So for the sake of contradiction, suppose $x_{m+2}x_{m+3} \neq 1$. Since

$$x_{m+1} = \max\left\{\frac{1}{x_m}, \frac{A_m}{x_{m-1}}\right\} \quad \text{and} \quad x_m x_{m+1} \neq 1$$

we see that

$$x_{m+1} = \frac{A_m}{x_{m-1}}$$

and that

$$\frac{x_{m-1}}{x_m} < A_m.$$

Similarly,

$$x_{m+2} = \max\left\{\frac{1}{x_{m+1}}, \frac{A_{m+1}}{x_m}\right\} = \max\left\{\frac{x_{m-1}}{A_m}, \frac{A_{m+1}}{x_m}\right\}$$

and so

$$x_{m+2} = \frac{A_{m+1}}{x_m} \quad \text{and} \quad x_{m-1}x_m < A_m A_{m+1}.$$

Thus

$$x_{m+3} = \max\left\{\frac{1}{x_{m+2}}, \frac{A_{m+2}}{x_{m+1}}\right\} = \max\left\{\frac{x_m}{A_{m+1}}, \frac{A_{m+2}x_{m-1}}{A_m}\right\} = \frac{A_{m+2}x_{m-1}}{A_m}$$

and

$$\frac{x_m}{x_{m-1}} < \frac{A_{m+1}A_{m+2}}{A_m}.$$

Hence

$$\frac{x_{m-1}}{x_m} > \frac{A_m}{A_{m+1}A_{m+2}} > A_m,$$

which is a contradiction. ∎

7.3.1.1 Analysis of the Semi-cycles of Eq.(7.3)

In this section we give some results about the semi-cycles of the solutions of Eq.(7.3) which shall be useful in the sequel.

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Recall the following definitions.

A positive semi-cycle of $\{x_n\}_{n=-1}^{\infty}$ consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all greater than or equal to $\bar{x} = 1$, with $l \geq -1$ and $m \leq \infty$, such that

$$\text{either } l = -1 \quad \text{or} \quad l > -1 \text{ and } x_{l-1} < 1$$

and

$$\text{either } m = \infty \quad \text{or} \quad m < \infty \text{ and } x_{m+1} < 1.$$

A negative semi-cycle of $\{x_n\}_{n=-1}^{\infty}$ consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all less than $\bar{x} = 1$, with $l \geq -1$ and $m \leq \infty$, such that

$$\text{either } l = -1 \quad \text{or} \quad l > -1 \text{ and } x_{l-1} \geq 1$$

and

$$\text{either } m = \infty \quad \text{or} \quad m < \infty \text{ and } x_{m+1} \geq 1.$$

$\{x_n\}_{n=-1}^{\infty}$ is called non-oscillatory if there exists $N \geq -1$ such that either

$$x_n > 1 \quad \text{for all } n \geq N$$

or

$$x_n < 1 \quad \text{for all } n \geq N.$$

$\{x_n\}_{n=-1}^{\infty}$ is called oscillatory if it is not non-oscillatory.

LEMMA 7.10

Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers which is periodic with prime period 3 such that $A_n \in (0,1)$ for all $n \geq 0$. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3) which is not eventually constant. Then the following statements are true:

1. *$\{x_n\}_{n=-1}^{\infty}$ oscillates about the positive equilibrium point $\bar{x} = 1$ of Eq.(7.3).*

2. *With the possible exception of the first negative semi-cycle, every negative semi-cycle of $\{x_n\}_{n=-1}^{\infty}$ has length equal to 1.*

3. *Let $n \geq 1$ be such that $x_{n-2} \geq 1$ and $x_{n-1} < 1$. Then $x_n = \dfrac{1}{x_{n-1}}$.*

4. *Every positive semi-cycle of $\{x_n\}_{n=-1}^{\infty}$ has length at most 2.*

PROOF Recall by Lemma 7.8 that $x_n \neq 1$ for all $n \geq 1$.

(i) Statement 1 follows from Lemma 7.9.

(ii) Suppose there exists $N \geq 0$ such that $x_{N-1} \geq 1$ and $x_N < 1$. Then

$$x_{N+1} = \max\left\{\frac{1}{x_N}, \frac{A_N}{x_{N-1}}\right\} = \frac{1}{x_N} > 1.$$

(iii) $x_n = \max\left\{\dfrac{1}{x_{n-1}}, \dfrac{A_{n-1}}{x_{n-2}}\right\} = \dfrac{1}{x_{n-1}}.$

(iv) Suppose there exists $N \geq 0$ such that $x_{N-1} \geq 1$ and $x_N \geq 1$. Then

$$x_{N+1} = \max\left\{\frac{1}{x_N}, \frac{A_N}{x_{N-1}}\right\} \leq 1.$$

Now we know by Lemma 7.8 that $x_{N+1} \neq 1$, and so $x_{N+1} < 1$. ∎

LEMMA 7.11

Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers which is periodic with prime period 3 such that $A_n \in (0, 1)$ for all $n \geq 0$. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3) which is not eventually constant. Then the following statements are true:

1. *With the possible exception of the first positive semi-cycle, every positive semi-cycle of $\{x_n\}_{n=-1}^{\infty}$ has a strict maximum which occurs in the first term of the positive semi-cycle.*
2. *With the possible exception of the first positive semi-cycle, the first term of every positive semi-cycle is less than or equal to the last term of the preceding positive semi-cycle.*

PROOF It suffices to assume there exists $n \geq 2$ such that

$$x_{n-2} > 1, \quad x_{n-1} < 1, \quad x_n > 1$$

and to show that

$$x_{n+1} < x_n \leq x_{n-2}.$$

We know by Lemma 7.10 that $x_n = \dfrac{1}{x_{n-1}}$.

We shall first show that $x_n \leq x_{n-2}$. Note that $x_{n-1} = \max\left\{\dfrac{1}{x_{n-2}}, \dfrac{A_{n-2}}{x_{n-3}}\right\}$.

Case 1 Suppose $x_{n-1} = \dfrac{1}{x_{n-2}}$.

Then $x_n = \dfrac{1}{x_{n-1}} = x_{n-2}$.

Case 2 Suppose $x_{n-1} \neq \dfrac{1}{x_{n-2}}$.

Then $x_{n-1} = \dfrac{A_{n-2}}{x_{n-3}}$ and $\dfrac{1}{x_{n-2}} < \dfrac{A_{n-2}}{x_{n-3}}$

from which it follows that $x_n = \dfrac{1}{x_{n-1}} = \dfrac{x_{n-3}}{A_{n-2}} < x_{n-2}$.

We shall next show that $x_{n+1} < x_n$. Now $x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right\}$.

Case 1 Suppose $x_{n+1} = \frac{1}{x_n}$.

Then $x_{n+1} < 1$, and so $x_{n+1} < 1 < x_n$.

Case 2 Suppose $x_{n+1} \neq \frac{1}{x_n}$.

Then $x_{n+1} = \frac{A_n}{x_{n-1}} = A_n x_n < x_n$. ∎

7.3.1.2 The Main Result

Here we show that every positive solution of Eq.(7.3) is eventually periodic with period 2.

LEMMA 7.12
Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers which is periodic with prime period 3 such that $A_n \in (0,1)$ for all $n \geq 0$. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Suppose there exists $N \geq 0$ such that the following statements are true:

1. $x_{N-1}x_N = 1$.
2. $\max\left\{\sqrt{A_0}, \sqrt{A_1}, \sqrt{A_2}\right\} \leq x_N \leq \min\left\{\frac{1}{\sqrt{A_0}}, \frac{1}{\sqrt{A_1}}, \frac{1}{\sqrt{A_2}}\right\}$.

Then $\{x_n\}_{n=N-1}^{\infty}$ is periodic with period 2.

PROOF Since the least common multiple of 2 and 3 is 6, it is clear that it suffices to show that $\{x_n\}_{n=N-1}^{n=N+4}$ is periodic with period 2. Now,

$$x_{N-1} = \frac{1}{x_N}.$$

With this in mind, we make the following computations, from which the proof follows.

$$x_{N+1} = \max\left\{\frac{1}{x_N}, \frac{A_N}{x_{N-1}}\right\} = \max\left\{\frac{1}{x_N}, A_N x_N\right\} = \frac{1}{x_N}$$

$$x_{N+2} = \max\left\{\frac{1}{x_{N+1}}, \frac{A_{N+1}}{x_N}\right\} = \max\left\{x_N, \frac{A_{N+1}}{x_N}\right\} = x_N$$

$$x_{N+3} = \max\left\{\frac{1}{x_{N+2}}, \frac{A_{N+2}}{x_{N+1}}\right\} = \max\left\{\frac{1}{x_N}, A_{N+2}x_N\right\} = \frac{1}{x_N}$$

$$x_{N+4} = \max\left\{\frac{1}{x_{N+3}}, \frac{A_{N+3}}{x_{N+2}}\right\} = \max\left\{x_N, \frac{A_N}{x_N}\right\} \qquad = x_N \qquad \square$$

We are now almost ready for Theorem 7.4 which is the main result of this section. The proof of Theorem 7.4 is by contradiction and requires Lemma 7.13, which is given next. The reader should be alert to the fact that Lemma 7.13 is given for technical reasons only, and, in fact, in view of Theorem 7.4, has contradictory hypotheses.

LEMMA 7.13

Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers which is periodic with prime period 3 such that $A_n \in (0,1)$ for all $n \geq 0$. Suppose there exists a positive solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(7.3) which is not eventually periodic with period 2. Then there exists a subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$ such that for each $k \geq 0$, the following four statements are true:

1. $x_{n_k-1} < 1$.
2. $x_{n_k} = \dfrac{1}{x_{n_k-1}}$.
3. $x_{n_k+1} = A_{n_k}x_{n_k} \neq \dfrac{1}{x_{n_k}}$.
4. $x_{n_k+2} = \dfrac{1}{x_{n_k+1}}$.

PROOF Suppose that the lemma is false. Then there exists $M \geq 1$ such that for every $n \geq M$, it is false that each of the following four statements is true.

(i) $x_{n-1} < 1$.

(ii) $x_n = \dfrac{1}{x_{n-1}}$.

(iii) $x_{n+1} = A_n x_n \neq \dfrac{1}{x_n}$.

(iv) $x_{n+2} = \dfrac{1}{x_{n+1}}$.

Claim: Suppose $N \geq M$ such that $x_{N-2} > 1$ and $x_{N-1} < 1$. Then

$$x_N = \frac{1}{x_{N-1}} \qquad \text{and} \qquad x_{N+1} = \frac{1}{x_N}.$$

Proof of the claim:

As usual,

$$x_N = \max\left\{\frac{1}{x_{N-1}}, \frac{A_{N-1}}{x_{N-2}}\right\} = \frac{1}{x_{N-1}},$$

and so in order to complete the proof of the claim, we need only show that

$$x_{N+1} = \frac{1}{x_N}.$$

Now,

$$x_{N+1} = \max\left\{\frac{1}{x_N}, \frac{A_N}{x_{N-1}}\right\} = \max\left\{\frac{1}{x_N}, A_N x_N\right\}.$$

It suffices to consider the case $x_{N+1} = A_N x_N$.

Note that

$$x_{N+2} = \max\left\{\frac{1}{x_{N+1}}, \frac{A_{N+1}}{x_N}\right\} = \max\left\{\frac{1}{x_{N+1}}, \frac{A_N A_{N+1}}{x_{N+1}}\right\} = \frac{1}{x_{N+1}}$$

and so it follows (since it must be the case that Statement (iii) above is false) that $x_{N+1} = \dfrac{1}{x_N}$.

Thus we see that the claim is true.

Now by Lemma 7.10, there exists $N \geq M$ such that $x_{N-2} > 1$ and $x_{N-1} < 1$. It follows by the claim that $\{x_n\}_{n=N-2}^{\infty}$ is periodic with period 2. This contradicts our assumption that $\{x_n\}_{n=N-2}^{\infty}$ is not eventually periodic with period 2, and so the proof of the lemma is complete. ∎

THEOREM 7.4

Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers which is periodic with prime period 3 such that $A_n \in (0,1)$ for all $n \geq 0$. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period 2.

PROOF It follows by Lemma 7.11 that the terms of the positive semicycles of $\{x_n\}_{n=-1}^{\infty}$ form an eventually non-increasing subsequence of $\{x_n\}_{n=-1}^{\infty}$ which we shall denote by $\{x_{n_k}\}_{k=0}^{\infty}$. Thus there exists $L \geq 1$ such that

$$\lim_{k\to\infty} x_{n_k} = L$$

and there exists $N \geq 0$ such that

$$x_{n_k} \geq L \quad \text{for all} \quad k \geq N.$$

Claim: There exists $K \geq N$ such that $x_{n_K} = L$.

Proof of the claim: For the sake of contradiction, suppose that $x_{n_k} > L$ for all $k \geq N$.

Since $\{x_n\}_{n=-1}^{\infty}$ is not eventually periodic with period 2, it follows by Lemma 7.12 that

$$L \geq \min\left\{\frac{1}{\sqrt{A_0}}, \frac{1}{\sqrt{A_1}}, \frac{1}{\sqrt{A_2}}\right\}.$$

Re-label $\{0, 1, 2\}$ as $\{i_0, i_1, i_2\}$ such that

$$\frac{1}{\sqrt{A_{i_0}}} \leq \frac{1}{\sqrt{A_{i_1}}} \leq \frac{1}{\sqrt{A_{i_2}}}.$$

Note that $A_{i_0} = \max\{A_0, A_1, A_2\}$. Let ε be the smallest positive element of the set

$$\left\{\frac{(1 - A_{i_0})L}{A_{i_0}} \; , \; \left|L - \frac{1}{\sqrt{A_0}}\right| \; , \; \left|L - \frac{1}{\sqrt{A_1}}\right| \; , \; \left|L - \frac{1}{\sqrt{A_2}}\right|\right\}.$$

Clearly there exists $K \geq N$ such that for $k \geq K$,

$$L < x_{n_k} < L + \varepsilon.$$

It follows by Lemma 7.13 that we may also assume that

$$x_{n_K - 1} < 1 \quad , \quad x_{n_K} = \frac{1}{x_{n_K - 1}} \quad , \quad x_{n_K + 1} = A_{n_K} x_{n_K} \neq \frac{1}{x_{n_K}} \quad ,$$

$$\text{and} \quad x_{n_K + 2} = \frac{1}{x_{n_K + 1}}.$$

We claim $x_{n_K+1} < 1$. For the sake of contradiction, suppose that $x_{n_K+1} \geq 1$. Then x_{n_K+1} is itself a term in a positive semi-cycle, and so $x_{n_K+1} > L$. But we also have

$$x_{n_K+1} = A_{n_K} x_{n_K} < A_{i_0}(L+\varepsilon) = A_{i_0}L + A_{i_0}\varepsilon \leq A_{i_0}L + A_{i_0}\frac{(1 - A_{i_0})L}{A_{i_0}} = L,$$

which is impossible.

So it is true that $x_{n_K+1} < 1$, and hence that $x_{n_K+2} > L$.

Note that

$$\frac{1}{x_{n_K}} \neq A_{n_K} x_{n_K} = x_{n_K+1} = \max\left\{\frac{1}{x_{n_K}}, \frac{A_{n_K}}{x_{n_K-1}} =\right\} = \max\left\{\frac{1}{x_{n_K}}, A_{n_K} x_{n_K}\right\},$$

and so it follows that $A_{n_K} x_{n_K} > \dfrac{1}{x_{n_K}}$. That is,

$$x_{n_K} > \frac{1}{\sqrt{A_{n_K}}}. \tag{7.4}$$

Claim (a).

$$A_{n_K} L \geq \frac{1}{L}. \tag{7.5}$$

The proof of Claim (a) is a consequence of the following three cases.

(a) Suppose $\dfrac{1}{\sqrt{A_{i_0}}} \leq L < \dfrac{1}{\sqrt{A_{i_1}}}$.
We claim that

$$A_{n_K} = A_{i_0}.$$

For the sake of contradiction, suppose that $A_{n_K} \neq A_{i_0}$. Then there exists $j \in \{1, 2\}$ such that $A_{n_K} = A_{i_j} < A_{i_0}$. Thus by (7.4),

$$x_{n_K} > \frac{1}{\sqrt{A_{n_K}}} = \frac{1}{\sqrt{A_{i_j}}} > L$$

and so

$$x_{n_K} - L = \left(x_{n_K} - \frac{1}{\sqrt{A_{i_j}}}\right) + \left(\frac{1}{\sqrt{A_{i_j}}} - L\right) > \frac{1}{\sqrt{A_{i_j}}} - L \geq \varepsilon,$$

which is impossible.

Thus it is true that $A_{n_K} = A_{i_0}$, and so $L^2 \geq \dfrac{1}{A_{i_0}} = \dfrac{1}{A_{n_K}}$.

(b) Suppose $\dfrac{1}{\sqrt{A_{i_1}}} \leq L < \dfrac{1}{\sqrt{A_{i_2}}}$.
We claim that

$$A_{n_K} \neq A_{i_2}.$$

For the sake of contradiction, suppose that $A_{n_K} = A_{i_2}$. Then by (7.4) we have

$$x_{n_K} > \frac{1}{\sqrt{A_{n_K}}} = \frac{1}{\sqrt{A_{i_2}}} > L$$

and so

$$x_{n_K} - L = \left(x_{n_K} - \frac{1}{\sqrt{A_{i_2}}} \right) + \left(\frac{1}{\sqrt{A_{i_2}}} - L \right) > \frac{1}{\sqrt{A_{i_2}}} - L \geq \varepsilon,$$

which is impossible.

Thus it is true that $A_{n_K} \neq A_{i_2}$, and hence that $L^2 \geq \dfrac{1}{A_{n_K}}$.

(c) Suppose $\dfrac{1}{\sqrt{A_{i_2}}} \leq L$.

Then clearly $L^2 \geq \dfrac{1}{A_{n_K}}$.

Thus the proof of Claim (a) is complete.

Hence by (7.5),

$$x_{n_K+1} = A_{n_K} x_{n_K} > A_{n_K} L \geq \frac{1}{L}$$

and so

$$L > \frac{1}{x_{n_K+1}} = x_{n_K+2} = x_{n_{(K+1)}} > L,$$

which is a contradiction.

Thus the claim is true, and so there does exist $K \geq N$ such that $x_{n_K} = L$.

Then because $\{x_{n_k}\}_{k=0}^{\infty}$ is non-increasing, we must have

$$x_{n_k} = L \quad \text{for all} \quad k \geq K.$$

It follows by Lemma 7.11 that each positive semi-cycle of $\{x_n\}_{n=n_K}^{\infty}$ consists of a single term which is equal to L. Hence for $k \geq K$ we have

$$x_{n_k} = L \geq 1 \quad , \quad x_{n_k+1} < 1 \quad , \quad x_{n_k+2} = x_{n_{(k+1)}} = L \geq 1$$

and so

$$L = x_{n_k+2} = \max\left\{ \frac{1}{x_{n_k+1}}, \frac{A_{n_k+1}}{x_{n_k}} \right\} = \frac{1}{x_{n_k+1}}.$$

Thus we see that $\{x_n\}_{n=n_K}^{\infty}$ is periodic with period 2. ∎

7.3.2 Eventually Periodic Solutions with Period 12

Throughout this section we assume that $\{A_n\}_{n=0}^{\infty}$ is a sequence of positive real numbers which is periodic with prime period 3 such that

$$A_n \in (1, \infty) \qquad \text{for all} \qquad n \geq 0$$

and we show that every positive solution of Eq.(7.3) is eventually periodic with prime period 12. The following lemmas will be useful in the sequel.

LEMMA 7.14

Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers which is periodic with prime period 3 such that $A_n \in (1, \infty)$ for all $n \geq 0$. Then every positive solution of the difference equation

$$x_{n+1} = \frac{A_n}{x_{n-1}} \qquad , \qquad n = 0, 1, \ldots$$

is periodic with prime period 12.

PROOF The proof follows easily by computation. ∎

LEMMA 7.15

Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers which is periodic with prime period 3 such that $A_n \in (1, \infty)$ for all $n \geq 0$, and let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Suppose there exists $N \in \{0, 1, 2, \ldots\}$ such that

$$x_N = \frac{1}{x_{N-1}} \qquad \text{and} \qquad x_{N+1} = \frac{1}{x_N}. \tag{7.6}$$

Then

$$x_{N+2} = \frac{A_{N+1}}{x_N}.$$

PROOF For the sake of contradiction, suppose that $x_{N+2} = \dfrac{1}{x_{N+1}}$. Then by (7.6),

$$x_{N-1} = x_{N+1} = \max\left\{\frac{1}{x_N}, \frac{A_N}{x_{N-1}}\right\} = \max\left\{x_{N-1}, \frac{A_N}{x_{N-1}}\right\}$$

and so we see that

$$A_N \leq x_{N-1}^2. \tag{7.7}$$

We similarly have

$$\frac{1}{x_{N-1}} = \frac{1}{x_{N+1}} = x_{N+2} = \max\left\{\frac{1}{x_{N+1}}, \frac{A_{N+1}}{x_N}\right\} = \max\left\{\frac{1}{x_{N-1}}, A_{N+1}x_{N-1}\right\}$$

and hence

$$x_{N-1}^2 \leq \frac{1}{A_{N+1}}. \tag{7.8}$$

It follows by (7.7) and (7.8) that $1 < x_{N-1} < 1$, which is a contradiction. ▯

REMARK 7.2 Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). In order to prove that $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period 12, it follows by Lemmas 7.14 and 7.15 that without loss of generality, we may assume that

$$x_{-1} = \frac{1}{x_0} \quad \textit{and} \quad x_1 = A_0 x_0.$$

Observe that in this case,

$$A_0 x_0 = x_1 = \max\left\{\frac{1}{x_0}, \frac{A_0}{x_{-1}}\right\} = \max\left\{\frac{1}{x_0}, A_0 x_0\right\}$$

and so we see that

$$1 \leq A_0 x_0^2.$$

Moreover,

$$x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{1}{A_0 x_0}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0}.$$

▯

LEMMA 7.16

Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with prime period 3 such that $A_n \in (1, \infty)$ for all $n \geq 0$, and suppose there exists $m \in \{0, 1, 2\}$ such that

$$1 < A_{m+1} \leq A_{m+2} \leq A_m.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3) such that

$$x_{-1} = \frac{1}{x_0} \quad \textit{and} \quad x_1 = A_0 x_0.$$

Then there exists $p \in \{0, 1, 2, \ldots\}$ with $p \equiv m \pmod 3$ such that

$$x_p = \frac{1}{x_{p-1}} \quad \textit{and} \quad x_{p+1} = A_m x_p.$$

PROOF By Remark 7.2, we know that

$$A_0 x_0^2 \geq 1 \quad \text{and} \quad x_2 = \frac{A_1}{x_0}.$$

The proof will follow from the following three cases.

Case 1 Suppose $m = 0$. Then

$$1 < A_1 \le A_2 \le A_0.$$

Note that

$$x_0 = \frac{1}{x_{-1}} \quad \text{and} \quad x_1 = A_0 x_0$$

and so the result follows with $p = 0$.

Case 2 Suppose $m = 1$. Then

$$1 < A_2 \le A_0 \le A_1. \tag{7.9}$$

So as $\dfrac{A_2}{A_1} < 1$, there clearly exists $k \in \{0, 1, 2, \ldots\}$ such that

$$A_1^{2k-1} A_2 \le A_0 x_0^2 < A_1^{2k+1} A_2. \tag{7.10}$$

Note that if $k = 0$, we have

$$\frac{A_2}{A_1} \le A_0 x_0^2 < A_1 A_2$$

and so

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_2}{x_1}\right\} = \max\left\{\frac{x_0}{A_1}, \frac{A_2}{A_0 x_0}\right\} = \frac{A_2}{A_0 x_0}$$

$$x_4 = \max\left\{\frac{1}{x_3}, \frac{A_0}{x_2}\right\} = \max\left\{\frac{A_0 x_0}{A_2}, \frac{A_0 x_0}{A_1}\right\} = \frac{A_0 x_0}{A_2} = \frac{1}{x_3}$$

$$x_5 = \max\left\{\frac{1}{x_4}, \frac{A_1}{x_3}\right\} = \max\left\{\frac{A_2}{A_0 x_0}, \frac{A_0 A_1 x_0}{A_2}\right\} = \frac{A_0 A_1 x_0}{A_2} = A_1 x_4$$

and thus the result follows with $p = 4$.

So suppose $k \ge 1$. It follows by induction from (7.9) and (7.10) that for all $0 \le n \le k-1$,

$$x_{3n-1} = \frac{A_1^n}{x_0} \quad , \quad x_{3n} = \frac{x_0}{A_1^n} \quad , \quad x_{3n+1} = \frac{A_0 x_0}{A_1^n}$$

while

$$x_{3k-1} = \frac{A_1^k}{x_0} \quad \text{and} \quad x_{3k} = \frac{x_0}{A_1^k}.$$

Note that

$$x_{3k+1} = \max\left\{\frac{1}{x_{3k}}, \frac{A_{3k}}{x_{3k-1}}\right\} = \max\left\{\frac{A_1^k}{x_0}, \frac{A_0 x_0}{A_1^k}\right\}.$$

We first suppose that $x_{3k+1} = \dfrac{A_1^k}{x_0}$. Then

$$x_{3k+1} = \frac{1}{x_{3k}}.$$

Moreover,

$$A_2 x_0^2 \leq A_0 x_0^2 < A_1^{2k+1} A_2,$$

and so

$$x_{3k+2} = \max\left\{\frac{1}{x_{3k+1}}, \frac{A_{3k+1}}{x_{3k}}\right\} = \left\{\frac{x_0}{A_1^k}, \frac{A_1^{k+1}}{x_0}\right\} = \frac{A_1^{k+1}}{x_0} = A_1 x_{3k+1}$$

and thus the result follows with $p = 3k+1$.

We next suppose that $x_{3k+1} = \dfrac{A_0 x_0}{A_1^k}$. It follows by (7.9) and (7.10) that

$$x_{3k+2} = \max\left\{\frac{1}{x_{3k+1}}, \frac{A_{3k+1}}{x_{3k}}\right\} = \max\left\{\frac{A_1^k}{A_0 x_0}, \frac{A_1^{k+1}}{x_0}\right\} = \frac{A_1^{k+1}}{x_0}$$

$$x_{3k+3} = \max\left\{\frac{1}{x_{3k+2}}, \frac{A_{3k+2}}{x_{3k+1}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1}}, \frac{A_1^k A_2}{A_0 x_0}\right\} = \frac{A_1^k A_2}{A_0 x_0}$$

$$x_{3k+4} = \max\left\{\frac{1}{x_{3k+3}}, \frac{A_{3k+3}}{x_{3k+2}}\right\} = \max\left\{\frac{A_0 x_0}{A_1^k A_2}, \frac{A_0 x_0}{A_1^{k+1}}\right\} = \frac{A_0 x_0}{A_1^k A_2}$$

$$= \frac{1}{x_{3k+3}}$$

$$x_{3k+5} = \max\left\{\frac{1}{x_{3k+4}}, \frac{A_{3k+4}}{x_{3k+3}}\right\} = \max\left\{\frac{A_1^k A_2}{A_0 x_0}, \frac{A_0 x_0}{A_1^{k-1} A_2}\right\} = \frac{A_0 x_0}{A_1^{k-1} A_2}$$

$$= A_1 x_{3k+4}$$

and so the result follows with $p = 3k+4$.

Case 3 Suppose $m = 2$. Then

$$1 < A_0 \le A_1 \le A_2. \tag{7.11}$$

We first suppose that $A_0 x_0^2 \le \dfrac{A_2}{A_1}$. Then

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_2}{x_1}\right\} = \max\left\{\frac{x_0}{A_1}, \frac{A_2}{A_0 x_0}\right\} = \frac{A_2}{A_0 x_0}$$

$$x_4 = \max\left\{\frac{1}{x_3}, \frac{A_3}{x_2}\right\} = \max\left\{\frac{A_0 x_0}{A_2}, \frac{A_0 x_0}{A_1}\right\} = \frac{A_0 x_0}{A_1}$$

$$x_5 = \max\left\{\frac{1}{x_4}, \frac{A_4}{x_3}\right\} = \max\left\{\frac{A_1}{A_0 x_0}, \frac{A_0 A_1 x_0}{A_2}\right\} = \frac{A_1}{A_0 x_0} = \frac{1}{x_4}$$

$$x_6 = \max\left\{\frac{1}{x_5}, \frac{A_5}{x_4}\right\} = \max\left\{\frac{A_0 x_0}{A_1}, \frac{A_1 A_2}{A_0 x_0}\right\} = \frac{A_1 A_2}{A_0 x_0} = A_2 x_5$$

and so the result follows with $p = 5$.

We next suppose that $A_0 x_0^2 > \dfrac{A_2}{A_1}$. There clearly exists $k \in \{0, 1, 2, \ldots\}$ such that

$$A_1^{2k-1} A_2 < A_0 x_0^2 \le A_1^{2k+1} A_2. \tag{7.12}$$

It follows by induction from (7.11) and (7.12) that for all $0 \le n \le k$,

$$x_{3n-1} = \frac{A_1^n}{x_0} \quad , \quad x_{3n} = \frac{x_0}{A_1^n} \quad , \quad x_{3n+1} = \frac{A_0 x_0}{A_1^n}$$

while

$$x_{3k+2} = \frac{A_1^{k+1}}{x_0} \quad , \quad x_{3k+3} = \frac{A_1^k A_2}{A_0 x_0} \quad , \quad x_{3k+4} = \frac{A_0 x_0}{A_1^{k+1}}.$$

Note that

$$x_{3k+5} = \max\left\{\frac{1}{x_{3k+4}}, \frac{A_{3k+4}}{x_{3k+3}}\right\} = \max\left\{\frac{A_1^{k+1}}{A_0 x_0}, \frac{A_0 x_0}{A_1^{k-1} A_2}\right\}.$$

We first suppose that $x_{3k+5} = \dfrac{A_1^{k+1}}{A_0 x_0}$. Then $x_{3k+5} = \dfrac{1}{x_4}$, and

$$x_{3k+6} = \max\left\{\frac{1}{x_{3k+5}}, \frac{A_{3k+5}}{x_{3k+4}}\right\} = \max\left\{\frac{A_0x_0}{A_1^{k+1}}, \frac{A_2A_1^{k+1}}{A_0x_0}\right\} = \frac{A_2A_1^{k+1}}{A_0x_0}$$

$$= A_2x_{3k+5}.$$

The result follows with $p = 3k+5$.

Next suppose that $x_{3k+5} = \dfrac{A_0x_0}{A_1^{k-1}A_2}$. Then

$$x_{3k+6} = \max\left\{\frac{1}{x_{3k+5}}, \frac{A_{3k+5}}{x_{3k+4}}\right\} = \max\left\{\frac{A_1^{k-1}A_2}{A_0x_0}, \frac{A_2A_1^{k+1}}{A_0x_0}\right\} = \frac{A_2A_1^{k+1}}{A_0x_0}$$

$$x_{3k+7} = \max\left\{\frac{1}{x_{3k+6}}, \frac{A_{3k+6}}{x_{3k+5}}\right\} = \max\left\{\frac{A_0x_0}{A_2A_1^{k+1}}, \frac{A_2A_1^{k-1}}{x_0}\right\} = \frac{A_2A_1^{k-1}}{A_0x_0}$$

$$x_{3k+8} = \max\left\{\frac{1}{x_{3k+7}}, \frac{A_{3k+7}}{x_{3k+6}}\right\} = \max\left\{\frac{x_0}{A_2A_1^{k-1}}, \frac{A_0x_0}{A_2A_1^{k}}\right\} = \frac{x_0}{A_2A_1^{k-1}}$$
$$= \frac{1}{x_{3k+7}}$$

$$x_{3k+9} = \max\left\{\frac{1}{x_{3k+8}}, \frac{A_{3k+8}}{x_{3k+7}}\right\} = \max\left\{\frac{A_2A_1^{k-1}}{x_0}, \frac{x_0}{A_1^{k-1}}\right\} = \frac{x_0}{A_1^{k-1}}$$
$$= A_2x_{3k+8}.$$

The result follows with $p = 3k+8$. ∎

LEMMA 7.17

Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with prime period 3 such that $A_n \in (1,\infty)$ for all $n \geq 0$, and suppose there exists $m \in \{0,1,2\}$ such that

$$1 < A_{m+2} \leq A_{m+1} \leq A_m.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3) such that

$$x_{-1} = \frac{1}{x_0} \quad \textit{and} \quad x_1 = A_0x_0.$$

Then there exists $q \in \{0,1,2,\ldots\}$ with $q \equiv (m+1) \pmod 3$ such that

$$x_q = \frac{1}{x_{q-1}} \quad \textit{and} \quad x_{q+1} = A_{m+1}x_q.$$

PROOF By Remark 7.2, we know that

$$A_0x_0^2 \geq 1 \quad \text{and} \quad x_2 = \frac{A_1}{x_0}.$$

The proof will follow from the following three cases.

Case 1 Suppose $m = 0$. Then

$$1 < A_2 \leq A_1 \leq A_0.$$

The proof is identical to the proof of Case 2 in Lemma 7.16.

Case 2 Suppose $m = 1$. Then

$$1 < A_0 \leq A_2 \leq A_1.$$

We shall show that there exists $q \in \{0, 1, 2, \ldots\}$ with $q \equiv 2 \pmod 3$ such that

$$x_q = \frac{1}{x_{q-1}} \quad \text{and} \quad x_{q+1} = \frac{A_2}{x_q}.$$

Since $A_0x_0^2 \geq 1$, there exists $k \in \{0, 1, 2, \ldots\}$ such that

$$A_1^{2k-1}A_2 \leq A_0x_0^2 < A_1^{2k+1}A_2.$$

If $k = 0$ and $A_2^2 \geq A_0^2A_1x_0^2$, the result follows with $q = 5$.

If $k \geq 1$, $A_1^{2k} < A_0x_0^2$, and $A_1^{2k-1}A_2^2 \geq A_0^2x_0^2$, then the result follows with $q = 3k + 5$.

In all other cases, by direct computation and induction it can be shown by a proof which is identical to that of the proof of Case 2 in Lemma 7.16 that there exists $p \in \{0, 1, 2, \ldots\}$ with $p \equiv 1 \pmod 3$ such that

$$x_p = \frac{1}{x_{p-1}} \quad \text{and} \quad x_{p+1} = A_1x_p.$$

Therefore $A_1x_p^2 \geq 1$, and so there exists $l \in \{0, 1, 2, \ldots\}$ such that

$$A_2^{2l-1}A_0 \leq A_1x_p^2 < A_2^{2l+1}A_0.$$

Suppose $l = 0$. Then we have $\frac{A_0}{A_2} \leq A_1x_p^2 < A_2A_0$. Hence

$$x_{p+3} = \frac{A_0}{A_1x_p}, \quad x_{p+4} = \frac{A_1x_p}{A_0} = \frac{1}{x_{p+3}}, \quad \text{and} \quad x_{p+5} = \frac{A_2A_1x_p}{A_0} = A_2x_{p+4}.$$

As $p \equiv 1 \pmod 3$, the result follows with $q = p + 4 \equiv 2 \pmod 3$.

Thus it suffices to consider the case $l \geq 1$. By a direct computation when $l = 1$, and by induction when $l \geq 2$, it follows that for all $0 \leq n \leq l - 1$,

$$x_{3n+p-1} = \frac{A_2^n}{x_p}, \quad x_{3n+p} = \frac{x_p}{A_2^n}, \quad x_{3n+p+1} = \frac{A_1 x_p}{A_2^n}$$

and that

$$x_{3l+p-1} = \frac{A_2^l}{x_p} \quad \text{and} \quad x_{3l+p} = \frac{x_p}{A_2^l}.$$

Note that

$$x_{3l+p+1} = \max\left\{\frac{1}{x_{3l+p}}, \frac{A_{3l+p}}{x_{3l+p-1}}\right\} = \max\left\{\frac{A_2^l}{x_p}, \frac{A_1 x_p}{A_2^l}\right\}.$$

We first suppose that $x_{3l+p+1} = \dfrac{A_2^l}{x_p}$. Then $x_{3l+p+1} = \dfrac{1}{x_{3l+p}}$ and

$$x_{3l+p+2} = \max\left\{\frac{1}{x_{3l+p+1}}, \frac{A_{3l+p+1}}{x_{3l+p}}\right\} = \max\left\{\frac{x_p}{A_2^l}, \frac{A_2^{l+1}}{x_p}\right\} = \frac{A_2^{l+1}}{x_p}$$
$$= A_2 x_{3l+p+1}.$$

As $p \equiv 1 \pmod 3$, the result follows with $q = 3l + p + 1 \equiv 2 \pmod 3$.

Next suppose that $x_{3l+p+1} = \dfrac{A_1 x_p}{A_2^l}$. Then

$$x_{3l+p+2} = \max\left\{\frac{1}{x_{3l+p+1}}, \frac{A_{3l+p+1}}{x_{3l+p}}\right\} = \max\left\{\frac{A_2^l}{A_1 x_p}, \frac{A_2^{l+1}}{x_p}\right\} = \frac{A_2^{l+1}}{x_p}$$

$$x_{3l+p+3} = \max\left\{\frac{1}{x_{3l+p+2}}, \frac{A_{3l+p+2}}{x_{3l+p+1}}\right\} = \max\left\{\frac{x_p}{A_2^{l+1}}, \frac{A_0 A_2^l}{A_1 x_p}\right\} = \frac{A_0 A_2^l}{A_1 x_p}$$

$$x_{3l+p+4} = \max\left\{\frac{1}{x_{3l+p+3}}, \frac{A_{3l+p+3}}{x_{3l+p+2}}\right\} = \max\left\{\frac{A_1 x_p}{A_0 A_2^l}, \frac{A_1 x_p}{A_2^{l+1}}\right\} = \frac{A_1 x_p}{A_0 A_2^{l-1}}$$
$$= \frac{1}{x_{3l+p+3}}$$

$$x_{3l+p+5} = \max\left\{\frac{1}{x_{3l+p+4}}, \frac{A_{3l+p+4}}{x_{3l+p+3}}\right\} = \max\left\{\frac{A_0 A_2^l}{A_1 x_p}, \frac{A_1 x_p}{A_0 A_2^{l-1}}\right\} = \frac{A_1 x_p}{A_0 A_2^{l-1}}$$
$$= A_2 x_{3l+p+4}.$$

As $p \equiv 1 \pmod 3$, the result follows with $q = 3l + p + 1 \equiv 2 \pmod 3$.

Case 3 Suppose $m = 2$. Then

$$1 < A_1 \leq A_0 \leq A_2.$$

The proof is identical to that of the proof of Case 1 in Lemma 7.16. □

We are now ready for the main result of this section.

THEOREM 7.5
Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with prime period 3 such that $A_n \in (1, \infty)$ for all $n \geq 0$. Then every positive solution of Eq.(7.3) is eventually periodic with period 12.

PROOF Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). We shall show that $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period 12. By Remark 7.2, it suffices to consider the case where

$$x_{-1} = \frac{1}{x_0} \quad \text{and} \quad x_1 = A_0 x_0,$$

in which case it follows that

$$A_0 x_0^2 \geq 1 \quad \text{and} \quad x_2 = \frac{A_1}{x_0}.$$

Let $m \in \{0, 1, 2\}$. It suffices to consider the following cases.

Case 1 Suppose

$$1 < A_{m+1} \leq A_{m+2} \leq A_m. \tag{7.13}$$

By Lemma 7.16, there exists $p \in \{0, 1, 2, \ldots\}$ with $p \equiv m \pmod 3$ such that

$$x_p = \frac{1}{x_{p-1}} \quad \textit{and} \quad x_{p+1} = A_m x_p.$$

We first suppose that $A_m x_p^2 \geq \dfrac{A_{m+2}}{A_{m+1}}$. There clearly exists $k \in \{0, 1, 2, \ldots\}$ such that

$$A_{m+1}^{2k-1} A_{m+2} \leq A_m x_p^2 < A_{m+1}^{2k+1} A_{m+2}. \tag{7.14}$$

By direct computation when $k = 0$, and by induction when $k \geq 1$, it follows from (7.13) and (7.14) that for all $0 \leq n \leq k$,

$$x_{3n+p-1} = \frac{A_{m+1}^{n}}{x_p}, \; x_{3n+p} = \frac{x_p}{A_{m+1}^{n}}, \; x_{3n+p+1} = \frac{A_m x_p}{A_{m+1}^{n}}$$

and that

$$x_{3k+p+2} = \frac{A_{m+1}^{k+1}}{x_p}, \; x_{3k+p+3} = \frac{A_{m+2}A_{m+1}^{k}}{A_m x_p}, \; x_{3k+p+4} = \frac{A_m x_p}{A_{m+1}^{k+1}}$$

$$x_{3k+p+5} = \frac{A_m x_p}{A_{m+2}A_{m+1}^{k-1}}, \; x_{3k+p+6} = \frac{A_{m+2}A_{m+1}^{k+1}}{A_m x_p}, \; x_{3k+p+7} = \frac{A_{m+2}A_{m+1}^{k-1}}{x_p}$$

$$x_{3k+p+8} = \frac{A_m x_p}{A_{m+2}A_{m+1}^{k}}, \; x_{3k+p+9} = \frac{x_p}{A_{m+1}^{k-1}}, \; x_{3k+p+10} = \frac{A_{m+2}A_{m+1}^{k}}{x_p}$$

and

$$x_{3k+p+11} = \frac{A_{m+1}^{k}}{x_p} = x_{3k+p-1} \quad \text{and} \quad x_{3k+p+12} = \frac{x_p}{A_{m+1}^{k}} = x_{3k+p}.$$

Thus we see that $\{x_n\}_{n=3k+p-1}^{\infty}$ is periodic with period 12.

Next suppose that $A_m x_p^2 < \dfrac{A_{m+2}}{A_{m+1}}$. Then there exists $l \in \{1, 2, \ldots\}$ such that

$$\frac{A_{m+2}}{A_{m+1}^{2l+1}} \leq A_m x_p^2 < \frac{A_{m+2}}{A_{m+1}^{2l-1}}.$$

By a direct computation when $l = 1$, and by induction when $l \geq 2$, it follows that for all $4 \leq n \leq l+3$,

$$x_{3n+p-1} = \frac{A_{m+1}^{n-3}A_m x_p}{A_{m+2}}, \; x_{3n+p} = \frac{A_{m+2}}{A_{m+1}^{n-3}A_m x_p}, \; x_{3n+p+1} = \frac{A_{m+2}}{A_{m+1}^{n-3}x_p}$$

and that

$$x_{3l+p+11} = \frac{A_{m+1}^{l+1}A_m x_p}{A_{m+2}}, \; x_{3l+p+12} = A_{m+1}^{l}x_p, \; x_{3l+p+13} = \frac{A_{m+2}}{A_{m+1}^{l+1}x_p}$$

$$x_{3l+p+14} = \frac{1}{A_{m+1}^{l-1}x_p}, \; x_{3l+p+15} = A_{m+1}^{l+1}x_p, \; x_{3l+p+16} = A_{m+1}^{l-1}A_m x_p$$

$$x_{3l+p+17} = \frac{1}{A_{m+1}^{l}x_p}, \; x_{3l+p+18} = \frac{A_{m+2}}{A_{m+1}^{l-1}A_m x_p}, \; x_{3l+p+19} = A_{m+1}^{l}A_m x_p$$

and

$$x_{3l+p+20} = \frac{A_{m+1}^{l} A_m x_p}{A_{m+2}} = x_{3l+p+8} \quad \text{and} \quad x_{3l+p+21} = \frac{A_{m+2}}{A_{m+1}^{l} A_m x_p} = x_{3l+p+9}.$$

Thus we see that $\{x_n\}_{n=3l+p+8}^{\infty}$ is periodic with period 12.

Case 2 Suppose that

$$1 < A_{m+1} \leq A_{m+2} \leq A_m. \tag{7.15}$$

By Lemma 7.17, there exists $p \in \{0, 1, 2, \ldots\}$ with $p \equiv (m+1) \pmod 3$ such that

$$x_q = \frac{1}{x_{q-1}} \quad \textit{and} \quad x_{q+1} = A_{m+1} x_q.$$

The proof will follow from the following cases.

Case 2(a) Suppose that $A_{m+1} x_q^2 \geq \dfrac{A_m}{A_{m+2}}$. There clearly exists $k \in \{0, 1, 2, \ldots\}$ such that

$$A_{m+2}^{2k-1} A_m \leq A_{m+1} x_q^2 < A_{m+2}^{2k+1} A_m. \tag{7.16}$$

By direct computation when $k = 0$, and by induction when $k \geq 1$, it follows from (7.15) and (7.16) that for all $0 \leq n \leq k$,

$$x_{3n+q-1} = \frac{A_{m+2}^{n}}{x_q}, \; x_{3n+q} = \frac{x_q}{A_{m+2}^{n}}, \; x_{3n+q+1} = \frac{A_{m+1} x_q}{A_{m+2}^{n}}$$

and that

$$x_{3k+q+2} = \frac{A_{m+2}^{k+1}}{x_q}, \; x_{3k+q+3} = \frac{A_m A_{m+2}^{k}}{A_{m+1} x_q}, \; x_{3k+q+4} = \frac{A_{m+1} x_q}{A_{m+2}^{k+1}}$$

$$x_{3k+q+5} = \frac{A_{m+1} x_q}{A_m A_{m+2}^{k-1}}, \; x_{3k+q+6} = \frac{A_m A_{m+2}^{k+1}}{A_{m+1} x_q}, \; x_{3k+q+7} = \frac{A_m A_{m+2}^{k-1}}{x_q}$$

$$x_{3k+q+8} = \frac{A_{m+1} x_q}{A_m A_{m+2}^{k}}, \; x_{3k+q+9} = \frac{x_q}{A_{m+2}^{k-1}}, \; x_{3k+q+10} = \frac{A_m A_{m+2}^{k}}{x_q}$$

and

$$x_{3k+q+11} = \frac{A_{m+2}^{k}}{x_q} = x_{3k+q-1} \quad \text{and} \quad x_{3k+q+12} = \frac{x_q}{A_{m+2}^{k}} = x_{3k+q}.$$

Thus we see that $\{x_n\}_{n=3k+q-1}^{\infty}$ is periodic with period 12.

Case 2(b) Suppose that $\frac{A_m}{A_{m+1}} \leq A_{m+1}x_q^2 < \frac{A_m}{A_{m+2}}$. Then there exists $l \in \{1, 2, \ldots\}$ such that

$$\frac{A_m}{A_{m+2}^{2l+1}} \leq A_{m+1}x_q^2 < \frac{A_m}{A_{m+2}^{2l-1}}.$$

By direct computation when $l = 1$, and by induction when $l \geq 2$, it follows that for all $4 \leq n \leq l + 3$,

$$x_{3n+q-1} = \frac{A_{m+2}^{n-3}A_{m+1}x_q}{A_m}\ ,\ x_{3n+q} = \frac{A_m}{A_{m+2}^{n-3}A_{m+1}x_q}\ ,\ x_{3n+q+1} = \frac{A_m}{A_{m+2}^{n-3}x_q}$$

and that

$$x_{3l+q+11} = \frac{A_{m+2}^{l+1}A_{m+1}x_q}{A_m}\ ,\ x_{3l+q+12} = A_{m+2}^{l}x_q$$

$$x_{3l+q+13} = \frac{A_m}{A_{m+2}^{l+1}x_q}\ ,\ x_{3l+q+14} = \frac{1}{A_{m+2}^{l-1}x_q}$$

$$x_{3l+q+15} = A_{m+2}^{l+1}x_q\ ,\ x_{3l+q+16} = A_{m+2}^{l-1}A_{m+1}x_q$$

$$x_{3l+q+17} = \frac{1}{A_{m+2}^{l}x_q}\ ,\ x_{3l+q+18} = \frac{A_m}{A_{m+2}^{l-1}A_{m+1}x_q}$$

$$x_{3l+q+19} = A_{m+2}^{l}A_{m+1}x_q$$

and

$$x_{3l+q+20} = \frac{A_{m+2}^{l}A_{m+1}x_q}{A_m} = x_{3l+q+8}$$

and

$$x_{3l+q+21} = \frac{A_m}{A_{m+2}^{l}A_{m+1}x_q} = x_{3l+q+9}.$$

Thus we see that $\{x_n\}_{n=3l+q+8}^{\infty}$ is periodic with period 12.

Case 2(c) Suppose that $x_q^2 < \frac{A_m}{A_{m+1}^2}$. Then $x_q^2 < \frac{A_m A_{m+2}^2}{A_{m+1}^2}$, and so follows that there exists $r \in \{0, 1, 2, \ldots\}$ such that

$$\frac{A_m A_{m+2}^2}{A_{m+1}^{2r+4}} \leq x_q^2 < \frac{A_m A_{m+2}^2}{A_{m+1}^{2r+2}}.$$

By direct computation when $r = 0$, and by induction when $r \geq 1$, it follows that for all $0 \leq n \leq r$,

$$x_{3n+q+4} = \frac{A_{m+1}^{n+1} x_q}{A_{m+2}}, \; x_{3n+q+5} = \frac{A_{m+2}}{A_{m+1}^{n+1} x_q}, \; x_{3n+q+6} = \frac{A_m A_{m+2}}{A_{m+1}^{n+1} x_q}$$

and

$$x_{3r+7} = \frac{A_{m+1}^{r+2} x_q}{A_{m+2}}.$$

Thus we see that

$$x_{3r+q+8} = \max\left\{\frac{1}{x_{3r+q+7}}, \frac{A_{m+2}}{x_{3r+q+6}}\right\} = \max\left\{\frac{A_{m+2}}{A_{m+1}^{r+2} x_q}, \frac{A_{m+1}^{r+1} x_q}{A_m}\right\}.$$

Case 2(c)(i) Suppose that $x_q^2 < \frac{A_m A_{m+2}}{A_{m+1}^{2r+3}}$. Now,

$$\frac{A_m A_{m+2}^3}{A_{m+1}^{2r+5}} \leq \frac{A_m A_{m+2}^2}{A_{m+1}^{2r+4}} \leq x_q^2$$

and so there exists $s \in \{1, 2, \ldots\}$ such that

$$\frac{A_m A_{m+2}^{2s+1}}{A_m^{2r+5}} \leq x_q^2 < \frac{A_m A_{m+2}^{2s+3}}{A_{m+1}^{2r+5}}.$$

By direct computation when $s = 1$, and by induction when $s \geq 2$, it follows that for all $0 \leq n \leq s$,

$$x_{3(n+r)+q+8} = \frac{A_{m+2}^{n+1}}{A_{m+1}^{r+2} x_q}, \; x_{3(n+r)+q+9} = \frac{A_{m+1}^{r+2} x_q}{A_{m+2}^{n+1}}$$

and

$$x_{3(n+r)+q+10} = \frac{A_{m+1}^{r+3} x_q}{A_{m+2}^{n+1}}.$$

It follows by computation from the above that $\{x_n\}_{n=3(r+s)+q+8}^{\infty}$ is periodic with period 12.

Case 2(c)(ii) Suppose that

$$x_q^2 \geq \frac{A_m A_{m+2}}{A_{m+1}^{2r+3}} \quad \text{and} \quad x_q^2 > \frac{A_m}{A_{m+2} A_{m+1}^{2r+1}}.$$

Then it follows by computation that $\{x_n\}_{n=3(r+s)+q+8}$ is periodic with period 12.

Case 2(c)(iii) Suppose that

$$x_q^2 \geq \frac{A_m A_{m+2}}{A_{m+1}^{2r+1}} \quad \text{and} \quad x_q^2 \leq \frac{A_m}{A_{m+2}A_{m+1}^{2r+1}}.$$

Then there exists $s \in \{1, 2, \ldots\}$ such that

$$\frac{A_m}{A_{m+2}^{2s+1}A_{m+1}^{2r+1}} < x_q^2 \leq \frac{A_m}{A_{m+2}^{2s-1}A_{m+1}^{2r+1}}.$$

By direct computation when $s = 1$, and by induction when $s \geq 2$, we see that for all $0 \leq n \leq s$,

$$x_{3(n+r)+q+8} = \frac{A_{m+2}^n A_{m+1}^{r+1} x_q}{A_m}$$

$$x_{3(n+r)+q+9} = \frac{A_m}{A_{m+2}^n A_{m+1}^{r+1} x_q}$$

and

$$x_{3(n+r)+q+10} = \frac{A_m}{A_{m+2}^n A_{m+1}^{r} x_q}.$$

It follows by computation from the above that $\{x_n\}_{n=3(r+s)+q+8}^{\infty}$ is periodic with period 12. ∎

7.3.3 Unbounded Solutions

In this section, we assume that $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive real numbers with prime period 3 such that for some $i \in \{0, 1, 2\}$,

$$A_{i+1} < 1 < A_i$$

and we prove that every positive solution of Eq.(7.3) is unbounded. We first establish some useful lemmas.

LEMMA 7.18

Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with prime period 3. Suppose there exists $i \in \{0, 1, 2\}$ such that $A_i \geq 1$. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Suppose there exists $N \geq 0$ such that $x_{N+1} = \dfrac{1}{x_N}$. Then exactly one of the following two statements is true:

1. *$A_i = 1$, $A_j \leq 1$ for $j \in \{1, 2, 3\} - \{i\}$ and $x_n = 1$ for all $n \geq N$.*
2. *There exists a positive integer $k \geq 1$ such that*

$$x_{N+k+1} = \frac{A_{N+k}}{x_{N+k-1}} > \frac{1}{x_{N+k}}.$$

PROOF Consider the case $A_1 = 1$ and $x_2 = \dfrac{1}{x_1}$. The proofs in the other cases are similar and will be omitted.

In this case, we have $N = 1$. It suffices to assume that

$$x_{1+k+1} = \frac{1}{x_{1+k}} \qquad \text{for} \qquad 1 \leq k \leq 6.$$

Now,

(i) $x_1 = \dfrac{1}{x_2} = x_3.$

(ii) $\dfrac{1}{x_1} = \dfrac{1}{x_3} = x_4 = \max\left\{\dfrac{1}{x_3}, \dfrac{A_3}{x_2}\right\} = \max\left\{\dfrac{1}{x_1}, A_0 x_1\right\}$
and thus $1 \geq A_0 x_1^2$.

(iii) $x_1 = \dfrac{1}{x_4} = x_5 = \max\left\{\dfrac{1}{x_4}, \dfrac{A_4}{x_3}\right\} = \max\left\{x_1, \dfrac{A_1}{x_1}\right\}$
and hence $x_1^2 \geq A_1$. So as $A_1 \geq 1$, it follows that $x_0^2 \geq 1$, and thus $A_0 \leq 1$.

(iv) $\dfrac{1}{x_1} = \dfrac{1}{x_5} = x_6 = \max\left\{\dfrac{1}{x_5}, \dfrac{A_5}{x_4}\right\} = \max\left\{\dfrac{1}{x_1}, A_2 x_1\right\}$
and thus $1 \geq A_2 x_1^2$. In particular, $A_2 \leq 1$.

(v) $x_1 = \dfrac{1}{x_6} = x_7.$

(vi) $x_8 = \max\left\{\dfrac{1}{x_7}, \dfrac{A_7}{x_6}\right\} = \max\left\{\dfrac{1}{x_1}, A_1 x_1\right\} = A_1 x_1 = \dfrac{A_7}{x_6}$

from which the proof follows. ∎

LEMMA 7.19
Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with prime period 3. Suppose there exists $i \in \{0, 1, 2\}$ such that $A_i \geq 1$, and that it is not the case that $A_0, A_1, A_2 \in (1, \infty)$. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Suppose there exists $N \geq 0$ such that $x_{N+1} = \dfrac{A_N}{x_{N-1}}$. Then there exists a positive integer $k \geq 1$ such that

$$x_{N+k+1} = \frac{1}{x_{N+k}}.$$

PROOF Consider the case $A_1 \geq 1$ and $x_1 = \dfrac{A_0}{x_{-1}}$. The proofs in the other cases are similar and will be omitted.

In this case, we have $N = 0$. It suffices to assume that

$$x_{k+1} = \frac{A_k}{x_{k-1}} \qquad \text{for} \qquad 1 \leq k \leq 12.$$

Now,

(i) $\dfrac{A_0}{x_{-1}} = x_1.$

(ii) $\dfrac{A_1}{x_0} = x_2.$

(iii) $\dfrac{A_2 x_{-1}}{A_0} = \dfrac{A_2}{x_1} = x_3.$

(iv) $\dfrac{A_0 x_0}{A_1} = \dfrac{A_3}{x_2} = x_4.$

(v) $\dfrac{A_0 A_1}{A_2 x_{-1}} = \dfrac{A_4}{x_3} = x_5 = \max\left\{\dfrac{1}{x_4}, \dfrac{A_4}{x_3}\right\} = \max\left\{\dfrac{A_1}{A_0 x_0}, \dfrac{A_0 A_1}{A_2 x_{-1}}\right\}.$

and so we see that $A_2 x_{-1} \leq A_0^2 x_0$.

(vi) $\dfrac{A_1 A_2}{A_0 x_0} = \dfrac{A_5}{x_4} = x_6.$

(vii) $\dfrac{A_2 x_{-1}}{A_1} = \dfrac{A_6}{x_5} = x_7 = \max\left\{\dfrac{1}{x_6}, \dfrac{A_6}{x_5}\right\} = \max\left\{\dfrac{A_0 x_0}{A_1 A_2}, \dfrac{A_2 x_{-1}}{A_1}\right\}$

and so we see that $A_0 x_0 \leq A_2^2 x_{-1}$.

(viii) $\dfrac{A_0 x_0}{A_2} = \dfrac{A_7}{x_6} = x_8.$

(ix) $\dfrac{A_1}{x_{-1}} = \dfrac{A_8}{x_7} = x_9.$

(x) $\dfrac{A_2}{x_0} = \dfrac{A_9}{x_8} = x_{10}.$

(xi) $x_{-1} = \dfrac{A_{10}}{x_9} = x_{11} = \max\left\{\dfrac{1}{x_{10}}, \dfrac{A_{10}}{x_9}\right\} = \max\left\{\dfrac{x_0}{A_2}, x_{-1}\right\}$

and so we see that $x_0 \leq A_2 x_{-1}$. Thus $x_0 \leq A_2 x_{-1} \leq A_0^2 x_0$, and so

$$A_0 \geq 1.$$

(xii) $x_0 = \dfrac{A_{11}}{x_{10}} = x_{12}.$

(xiii) $\dfrac{A_0}{x_{-1}} = \dfrac{A_{12}}{x_{11}} = x_{13} = \max\left\{\dfrac{1}{x_{12}}, \dfrac{A_{12}}{x_{11}}\right\} = \max\left\{\dfrac{1}{x_0}, \dfrac{A_0}{x_{-1}}\right\}$

and thus $x_{-1} \leq A_0 x_0$. So $A_0 x_0 \leq A_2^2 x_{-1} \leq A_2^2 A_0 x_0$, and hence

$$A_2 \geq 1.$$

It follows that without loss of generality, we may assume that $A_1 = 1$. Hence

(i) $\dfrac{A_0}{x_{-1}} = x_1.$

(ii) $\dfrac{1}{x_0} = \dfrac{A_1}{x_0} = x_2.$

(iii) $\dfrac{A_2 x_{-1}}{A_0} = \dfrac{A_2}{x_1} = x_3.$

(iv) $A_0 x_0 = \dfrac{A_3}{x_2} = x_4.$

(v) $\dfrac{A_0}{A_2 x_{-1}} = \dfrac{A_4}{x_3} = x_5.$

(vi) $$\frac{A_2}{A_0 x_0} = \frac{A_5}{x_4} = x_6 = \max\left\{\frac{1}{x_5}, \frac{A_5}{x_4}\right\} = \max\left\{\frac{A_2 x_{-1}}{A_0}, \frac{A_2}{A_0 x_0}\right\}$$

and so we see that

$$x_{-1}x_0 \leq 1.$$

(vii) $$A_2 x_{-1} = \frac{A_6}{x_5} = x_7.$$

(viii) $$\frac{A_0 x_0}{A_2} = \frac{A_7}{x_6} = x_8.$$

(ix) $$\frac{1}{x_{-1}} = \frac{A_8}{x_7} = x_9.$$

(x) $$\frac{A_2}{x_0} = \frac{A_9}{x_8} = x_{10}.$$

(xi) $$x_{-1} = \frac{A_{10}}{x_9} = x_{11}.$$

(xii) $$x_0 = \frac{A_{11}}{x_{10}} = x_{12} = \max\left\{\frac{1}{x_{11}}, \frac{A_{11}}{x_{10}}\right\} = \max\left\{\frac{1}{x_{-1}}, x_0\right\}$$

and so we also have $1 \leq x_{-1}x_0$. Hence $x_{-1}x_0 = 1$; that is,

$$x_0 = \frac{1}{x_{-1}}.$$

In particular,

$$x_{0+11+1} = x_{12} = x_0 = \frac{1}{x_{-1}} = \frac{1}{x_{11}} = \frac{1}{x_{0+11}}$$

and the proof is complete. ∎

REMARK 7.3 Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). In view of Lemmas 7.18 and 7.19, without loss of generality, for the remainder of this section we shall assume that

$$x_{-1} = \frac{1}{x_0} \qquad \text{and} \qquad x_1 = A_0 x_0 > \frac{1}{x_0}$$

from which it follows that

$$A_0 x_0^2 > 1.$$

□

LEMMA 7.20

Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with prime period 3 such that

$$A_1 < 1 < A_0.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then there exists $p \in \{0, 1, 2, \ldots\}$ with with $p \equiv 0 \pmod 3$ such that

$$x_p = \frac{1}{x_{p-1}} \quad \textit{and} \quad x_{p+1} = A_0 x_p.$$

PROOF The proof is an immediate consequence of Remark 7.3 and will be omitted. □

LEMMA 7.21

Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with prime period 3 such that

$$A_2 < 1 \leq A_1, \quad A_0 A_1 < 1, \quad \textit{and} \quad A_2 \geq A_0^2 x_0^2.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then there exists $p \in \{0, 1, 2, \ldots\}$ with with $p \equiv 1 \pmod 3$ such that

$$x_p = \frac{1}{x_{p-1}} \quad \textit{and} \quad x_{p+1} = A_1 x_p.$$

PROOF By Remark 7.3, we have $x_{-1} = \dfrac{1}{x_0}$ and $x_1 = A_0 x_0$.

Note that as $A_2 < 1 \leq A_1$ and $A_0 A_1 < 1$, we see that it is also the case that $A_0 A_2 < 1$.

Case 1 Suppose

$$A_2^2 \leq A_0^2 A_1 x_0^2.$$

Then

$$x_2 = \frac{1}{A_0 x_0}, \quad x_3 = \frac{A_2}{A_0 x_0}, \quad x_4 = \frac{A_0 x_0}{A_2} = \frac{1}{x_3}, \quad \text{and} \quad x_5 = \frac{A_0 A_1 x_0}{A_2} = A_1 x_4.$$

The result follows with $p = 4$.

Case 2 Suppose

$$A_0^2 A_1 x_0^2 < A_2^2.$$

Clearly there exists $k \in \{1, 2, \ldots\}$ such that

$$A_2^{2k+2} \leq A_0^2 A_1 x_0^2 < A_2^{2k}. \tag{7.17}$$

By direct computation when $k = 1$, and by induction when $k \geq 2$, it follows from (7.17) that for all $1 \leq n \leq k$,

$$x_{3n-1} = \frac{A_2^{n-1}}{A_0 x_0}, \quad x_{3n} = \frac{A_2^n}{A_0 x_0}, \quad x_{3n+1} = \frac{A_0 x_0}{A_2^n}.$$

First suppose that $A_2^{2k+1} \leq A_0^2 x_0^2$. Then

$$x_{3k+2} = \frac{A_2^k}{A_0 x_0} \quad \text{and} \quad x_{3k+3} = \frac{A_0 x_0}{A_2^k}$$

and so

$$x_{3k+4} = \frac{A_2^k}{A_0 x_0} = \frac{1}{x_{3k+3}} \quad \text{and} \quad x_{3k+5} = \frac{A_1 A_2^k}{A_0 x_0} = A_1 x_{3k+4}.$$

The result follows with $p = 3k + 4$.

Next suppose that $A_0^2 x_0^2 < A_2^{2k+1}$. Then

$$x_{3k+2} = \frac{A_2^k}{A_0 x_0} \quad \text{and} \quad x_{3k+3} = \frac{A_2^{k+1}}{A_0 x_0}$$

and so

$$x_{3k+4} = \frac{A_0 x_0}{A_2^{k+1}} = \frac{1}{x_{3k+1}} \quad \text{and} x_{3k+5} = \frac{A_0 A_1 x_0}{A_2^{k+1}} = A_1 x_{3k+4}.$$

The result again follows with $p = 3k + 4$. ∎

LEMMA 7.22

Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with prime period 3 such that

$$A_2 < 1 \leq A_1, \qquad A_0 A_1 < 1, \qquad \textit{and} \qquad A_2 < A_0^2 x_0^2.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then there exists $p \in \{0, 1, 2, \ldots\}$ with with $p \equiv 1 \pmod 3$ such that

$$x_p = \frac{1}{x_{p-1}} \quad \textit{and} \quad x_{p+1} = A_1 x_p.$$

PROOF By Remark 7.3, we have $x_{-1} = \dfrac{1}{x_0}$ and $x_1 = A_0 x_0^2 > 1$.

Note that $A_0 < 1$ since $A_0 A_1 < 1 \leq A_1$. It follows by computation that

$$x_2 = \frac{1}{A_0 x_0} \quad \text{and} \quad x_3 = A_0 x_0.$$

Since $A_2 < A_0^2 x_0^2$ and $A_0 < 1 < A_0 x_0^2$, we see that there exist $k, l \in \{1, 2, \ldots\}$ such that

$$A_0^{2k+2} x_0^2 \leq A_2 < A_0^{2k} x_0^2$$

and

$$A_0^{2l+1} x_0^2 \leq 1 < A_0^{2l-1} x_0^2.$$

We consider the following two cases.

Case 1 Suppose $k + 1 \leq l$. It follows by induction that for all $0 \leq n \leq k$,

$$x_{3n-1} = \frac{1}{A_0^n x_0}, \quad x_{3n} = A_0^n x_0, \quad x_{3n+1} = A_0^{n+1} x_0$$

and

$$x_{3k+2} = \frac{1}{A_0^{k+1} x_0}, \quad x_{3k+3} = \frac{A_2}{A_0^{k+1} x_0}, \quad x_{3k+4} = \frac{A_0^{k+1} x_0}{A_2}.$$

First suppose that $1 \leq A_0^{2k+2} A_1 x_0^2$. Then result follows with $p = 3k + 4$. Next suppose that $A_0^{2k+2} A_1 x_0^2 < 1$. Then there exist $r, s \in \{1, 2, \ldots\}$ such that the following two inequalities are true:

$$A_2^{2r} \leq A_0^{2k+2} A_1 x_0^2 < A_2^{2r-2}$$

and

$$A_2^{2s+1} \leq A_0^{2k+2} x_0^2 < A_2^{2s-1}.$$

If $r \leq s$, the result follows with $p = 3(k + r) + 1$.

If $s < r$, the result follows with $p = 3(k + s) + 4$.

Case 2 Suppose $l \leq k$. By direct computation when $l = 1$, and by induction when $2 \leq l$, it follows that for all $0 \leq n \leq l - 1$,

$$x_{3n-1} = \frac{1}{A_0^n x_0}, \quad x_{3n} = A_0^n x_0, \quad x_{3n+1} = A_0^{n+1} x_0$$

and

$$x_{3l-1} = \frac{1}{A_0^l x_0}, \quad x_{3l} = A_0^l x_0, \quad x_{3l+1} = \frac{1}{A_0^l x_0}.$$

First suppose that $A_0^{2l} x_0^2 \leq A_1$. Then the result follows with $p = 3l + 1$.

Next suppose that $A_1 < A_0^{2l}x_0^2$. Then there exist $r, s \in \{1, 2, \ldots\}$ such that the following inequalities are true:

$$A_0^{2l}A_2^{2r}x_0^2 \le A_1 < A_0^{2l}A_2^{2r-2}x_0^2$$

and

$$A_0^{2l}A_2^{2s-1}x_0^2 \le 1 < A_0^{2l}A_2^{2r-3}x_0^2.$$

If $r < s$, then the result follows with $p = 3(l + r) + 1$.

If $s \le r$, then the result follows with $p = 3(l + s) + 1$. ▯

LEMMA 7.23

Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with prime period 3 such that

$$A_2 < 1 < A_1 \qquad \text{and} \qquad 1 \le A_0A_1.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then there exists $p \in \{0, 1, 2, \ldots\}$ with with $p \equiv 1 \pmod 3$ such that

$$x_p = \frac{1}{x_{p-1}} \quad \text{and} \quad x_{p+1} = A_1x_p.$$

PROOF By Remark 7.3, we have $x_{-1} = \dfrac{1}{x_0}$, $x_1 = A_0x_0$, and $A_0x_0^2 > 1$.

Note also that as $A_0x_0^2 > 1$, there exists $k \in \{0, 1, 2, \ldots\}$ such that

$$A_1^{2k-1}A_2 < A_0x_0^2 \le A_1^{2k+1}A_2.$$

Suppose $k = 0$. Then $A_0x_0^2 \le A_1A_2$.

$$x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{1}{A_0x_0}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0}$$

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_2}{x_1}\right\} = \max\left\{\frac{x_0}{A_1}, \frac{A_2}{A_0x_0}\right\} = \frac{A_2}{A_0x_0}$$

$$x_4 = \max\left\{\frac{1}{x_3}, \frac{A_3}{x_2}\right\} = \max\left\{\frac{A_0x_0}{A_2}, \frac{A_0x_0}{A_1}\right\} = \frac{A_0x_0}{A_2} = \frac{1}{x_3}$$

$$x_5 = \max\left\{\frac{1}{x_4}, \frac{A_4}{x_3}\right\} = \max\left\{\frac{A_2}{A_0x_0}, \frac{A_0A_1x_0}{A_2}\right\} = \frac{A_0A_1x_0}{A_2} = A_1x_4$$

and the proof is complete.

So suppose $1 \le k$. We shall consider the following two cases.

Case 1 Suppose

$$A_0x_0^2 < A_1^{2k}.$$

Clearly there exists $1 \leq l \leq k$ such that

$$A_1^{2k-2l} \leq A_0x_0^2 < A_1^{2k-2l+2}.$$

By direct computation when $k = 1$, and by induction when $k \geq 2$, it follows that for all $0 \leq n \leq k - l$,

$$x_{3n-1} = \frac{A_1^n}{x_0}, \quad x_{3n} = \frac{x_0}{A_1^n}, \quad x_{3n+1} = \frac{A_0x_0}{A_1^n}.$$

Then

$$x_{3(k-l)+2} = \frac{A_1^{k-l+1}}{x_0}, \quad x_{3(k-l)+3} = \frac{x_0}{A_1^{k-l+1}}$$

and

$$x_{3(k-l)+4} = \frac{A_1^{k-l+1}}{x_0} = \frac{1}{x_{3(k-l)+3}} \text{ and } x_{3(k-l)+5} = \frac{A_1^{k-l+2}}{x_0} = A_1x_{3(k-l)+4}.$$

The result follows with $p = 3(k - l) + 4$.

Case 2 Suppose that

$$A_1^{2k} \leq A_0x_0^2.$$

It follows by induction that for all $0 \leq n \leq k$,

$$x_{3n-1} = \frac{A_1^n}{x_0}, \quad x_{3n} = \frac{x_0}{A_1^n}, \quad x_{3n+1} = \frac{A_0x_0}{A_1^n}$$

$$x_{3k+2} = \frac{A_1^{k+1}}{x_0}, \quad x_{3k+3} = \frac{A_1^kA_2}{A_0x_0}$$

and

$$x_{3k+4} = \frac{A_0x_0}{A_1^kA_2} = \frac{1}{x_{3k+3}} \quad \text{and} \quad x_{3k+5} = \frac{A_0x_0}{A_1^{k-1}A_2} = A_1x_{3k+4}.$$

The result follows with $p = 3k + 4$. ∎

LEMMA 7.24

Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with prime period 3 such that

$$A_0 < 1 < A_2.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then there exists $p \in \{0, 1, 2, \ldots\}$ with $p \equiv 2 \pmod 3$ such that

$$x_p = \frac{1}{x_{p-1}} \quad \textit{and} \quad x_{p+1} = A_1 x_p.$$

PROOF The proof is similar to those of Lemmas 7.17, 7.21, 7.22, and 7.23 and will be omitted. ∎

LEMMA 7.25

Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with prime period 3 such that for some $i \in \{0, 1, 2\}$,

$$A_{i+1} < 1 < A_i \quad \textit{and} \quad A_i A_{i+1} < 1.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=-1}^{\infty}$ is unbounded.

PROOF In view of Lemmas 7.20, 7.21, 7.22, 7.23, and 7.24, it suffices to consider the case

$$i = 0, \quad x_{-1} = \frac{1}{x_0}, \quad \text{and} \quad x_1 = A_0 x_0.$$

We consider the following two cases.

Case 1 Suppose

$$A_2 \leq A_0^2 x_0^2.$$

It follows by induction that for all $n \geq 0$,

$$x_{3n-1} = \frac{1}{A_0^n x_0}, \quad x_{3n} = A_0^n x_0, \quad x_{3n+1} = A_0^{n+1} x_0$$

and so $\{x_n\}_{n=-1}^{\infty}$ is unbounded.

Case 2 Suppose

$$A_0^2 x_0^2 < A_2.$$

Then there exists $k \in \{1, 2, \ldots\}$ such that

$$A_0^{2k} x_0^2 < A_2 \leq A_0^{2k+2} x_0^2.$$

By direct computation when $k = 1$, and by induction when $k \geq 2$, it follows that for all $1 \leq n \leq k$,

$$x_{3n-1} = \frac{1}{A_0^n x_0}, \quad x_{3n} = \frac{A_2}{A_0^n x_0}, \quad x_{3n+1} = A_0^{n+1} x_0$$

and so $\{x_n\}_{n=-1}^{\infty}$ is unbounded. ∎

LEMMA 7.26

Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with prime period 3 such that for some $i \in \{0, 1, 2\}$,

$$A_{i+1} < 1 < A_i \quad \textit{and} \quad 1 \leq A_i A_{i+1}.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=-1}^{\infty}$ is unbounded.

PROOF In view of Lemmas 7.20, 7.21, 7.22, 7.23, and 7.24, it suffices to consider the case

$$i = 0, \quad x_{-1} = \frac{1}{x_0}, \quad \text{and} \quad x_1 = A_0 x_0.$$

We consider the following two cases.

Case 1 Suppose

$$A_1 A_2 \leq A_0 x_0^2.$$

It follows by induction that for all $n \geq 0$,

$$x_{3n-1} = \frac{A_1^n}{x_0}, \quad x_{3n} = \frac{x_0}{A_1^n}, \quad x_{3n+1} = \frac{A_0 x_0}{A_1^n}$$

and so $\{x_n\}_{n=-1}^{\infty}$ is unbounded.

Case 2 Suppose

$$A_0 x_0^2 < A_1 A_2.$$

Then as $A_1 < 1 < A_0 x_0^2$, it follows that $1 < A_2$. Thus

$$x_2 = \frac{A_1}{x_0}, \quad x_3 = \frac{A_2}{A_0 x_0}, \quad x_4 = \frac{A_0 x_0}{A_1}$$

and

$$x_5 = \max\left\{\frac{1}{x_4}, \frac{A_1}{x_3}\right\} = \max\left\{\frac{A_1}{A_0 x_0}, \frac{A_0 A_1 x_0}{A_2}\right\}.$$

The proof will follow from the following two sub-cases.

Case 2(a) Suppose

$$A_2 \leq A_0^2 x_0^2.$$

It follows by induction that for all $n \geq 2$,

$$x_{3n-1} = \frac{A_0 A_1^{n-1} x_0}{A_2}, \quad x_{3n} = \frac{A_2}{A_0 A_1^{n-1} x_0}, \quad x_{3n+1} = \frac{A_2}{A_1^{n-1} x_0}$$

and so $\{x_n\}_{n=-1}^{\infty}$ is unbounded.

Case 2(b) Suppose

$$A_0^2 x_0^2 < A_2.$$

We know that $A_0 x_0^2 < A_1 A_2$ and $1 < A_0$, and so it follows that there exists $k \in \{1, 2, \ldots\}$ such that

$$A_0^{2k-1} x_0^2 < A_1 A_2 \leq A_0^{2k+1} x_0^2. \tag{7.18}$$

Since $1 \leq A_0 A_1$ and $A_1 < A_0$, it follows that exactly one of the following two inequalities is true.

$$A_0^{2k-2} x_0^2 < A_1^2 A_2 \leq A_0^{2k} x_0^2 \tag{7.19}$$

$$A_0^{2k} x_0^2 < A_1^2 A_2 \leq A_0^{2k+2} x_0^2 \tag{7.20}$$

First suppose that (7.19) holds. When $k = 1$, it follows by induction that for all $n \geq 2$,

$$x_{3n-1} = \frac{A_1^{n-1}}{A_0 x_0}, \quad x_{3n} = \frac{A_0 x_0}{A_1^{n-1}}, \quad x_{3n+1} = \frac{A_0^2 x_0}{A_1^{n-1}}$$

and so $\{x_n\}_{n=-1}^{\infty}$ is unbounded.

When $k \geq 2$, by direct computation when $k = 2$, and by induction when $k \geq 3$, it follows from (7.18) and (7.19) that for all $2 \leq n \leq k$,

$$x_{3n-1} = \frac{A_1}{A_0^{n-1} x_0}, \quad x_{3n} = \frac{A_1 A_2}{A_0^{n-1} x_0}, \quad x_{3n+1} = \frac{A_0^n x_0}{A_1}.$$

It then follows by induction that for all $n \geq k + 1$,

$$x_{3n-1} = \frac{A_1^{n-k}}{A_0^k x_0}, \quad x_{3n} = \frac{A_0^k x_0}{A_0^{n-k}}, \quad x_{3n+1} = \frac{A_0^{k+1} x_0}{A_1^{n-k}}$$

and so $\{x_n\}_{n=-1}^{\infty}$ is unbounded.

Next suppose that (7.20) holds. By direct computation when $k = 1$, and by induction when $k \geq 2$, it follows from (7.18) and (7.20) that for all $2 \leq n \leq k+1$,

$$x_{3n-1} = \frac{A_1}{A_0^{n-1} x_0}, \quad x_{3n} = \frac{A_1 A_2}{A_0^{n-1} x_0}, \quad x_{3n+1} = \frac{A_0^n x_0}{A_1}.$$

It then follows by induction that for all $n \geq k+2$,

$$x_{3n-1} = \frac{A_0^k A_1^{n-(k+2)} x_0}{A_2}, \quad x_{3n} = \frac{A_2}{A_0^k A_1^{n-(k+2)} x_0}$$

and

$$x_{3n+1} = \frac{A_2}{A_0^{k-1} A_1^{n-(k+2)} x_0}$$

and so $\{x_n\}_{n=-1}^{\infty}$ is unbounded. ∎

The following theorem is the main result of this section.

THEOREM 7.6
Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with prime period 3 such that for some $i \in \{0,1,2\}$,

$$A_{i+1} < 1 < A_i.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=-1}^{\infty}$ is unbounded.

7.3.4 Eventually Periodic Solutions with Period 3

Throughout this section we assume that $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive real numbers with prime period 3, and we examine those cases which have not been dealt with in the previous sections. That is, we assume that one of the following cases holds

$$\left.\begin{array}{l} (a)\ A_0 < 1\ ,\ A_2 < 1 \text{ and } A_1 = 1 \\ (b)\ A_0 < 1\ ,\ A_2 = 1 \text{ and } A_1 > 0 \\ (c)\ A_0 = 1\ ,\ A_2 < 1 \text{ and } A_1 \leq 1 \\ (d)\ A_0 = 1\ ,\ A_2 = 1 \text{ and } A_1 > 0 \\ (e)\ A_0 > 1\ ,\ A_2 > 0 \text{ and } A_1 = 1 \\ (f)\ A_0 > 1\ ,\ A_2 = 1 \text{ and } A_1 > 1 \end{array}\right\} \qquad (7.21)$$

and we show that every solution of Eq.(7.3) is eventually periodic with period 3.

REMARK 7.4 Let $\{A_n\}_{n=-1}^{\infty}$ be a periodic sequence sequence of positive real numbers with prime period 3 such that (7.21) holds, and let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). In view of Lemmas 7.18 and 7.19, without loss of generality, for the remainder of this section we shall assume that

$$x_{-1} = \frac{1}{x_0} \quad \text{and} \quad x_1 = A_0 x_0 > \frac{1}{x_0}$$

from which it follows that

$$A_0 x_0^2 > 1.$$

□

LEMMA 7.27

Let $\{A_n\}_{n=-1}^{\infty}$ be a periodic sequence sequence of positive real numbers with prime period 3 such that

$$A_0 < 1 = A_2 \quad \textit{and} \quad A_0 A_1 < 1.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period-3.

PROOF It follows by Remark 7.4 that we may assume that

$$x_{-1} = \frac{1}{x_0} \quad , \quad x_1 = A_0 x_0 > \frac{1}{x_0} \quad \text{and} \quad A_0 x_0^2 > 1.$$

Thus there exists $k \in \{0, 1, \ldots\}$ such that either

$$A_0^{2k+2} x_0^2 < 1 \leq A_0^{2k+1} x_0^2 \quad \text{or} \quad A_0^{2k+3} x_0^2 < 1 \leq A_0^{2k+2} x_0^2.$$

(i) Suppose

$$A_0^{2k+2} x_0^2 < 1 \leq A_0^{2k+1} x_0^2.$$

By direct computation when $k = 0$, and by induction when $k \geq 1$, it follows that for all $0 \leq n \leq k$,

$$x_{3n-1} = \frac{1}{A_0^n x_0} \quad , \quad x_{3n} = A_0^n x_0 \quad \text{and} \quad x_{3n+1} = A_0^{n+1} x_0.$$

In particular,

$$x_{3k} = A_0^k x_0 \quad \text{and} \quad x_{3k+1} = A_0^{k+1} x_0$$

and so

$$x_{3k+2} = \max\left\{\frac{1}{x_{3k+1}}, \frac{A_{3k+1}}{x_{3k}}\right\} = \max\left\{\frac{1}{A_0^{k+1}x_0}, \frac{A_1}{A_0^k x_0}\right\} = \frac{1}{A_0^{k+1}x_0}$$

$$x_{3k+3} = \max\left\{\frac{1}{x_{3k+2}}, \frac{A_{3k+2}}{x_{3k+1}}\right\} = \max\left\{A_0^{k+1}x_0, \frac{1}{A_0^{k+1}x_0}\right\} = \frac{1}{A_0^{k+1}x_0}$$

$$x_{3k+4} = \max\left\{\frac{1}{x_{3k+3}}, \frac{A_{3k+3}}{x_{3k+2}}\right\} = \max\left\{A_0^{k+1}x_0, A_0^{k+2}x_0\right\} = A_0^{k+1}x_0$$

$$x_{3k+5} = \max\left\{\frac{1}{x_{3k+4}}, \frac{A_{3k+4}}{x_{3k+3}}\right\} = \max\left\{\frac{1}{A_0^{k+1}x_0}, A_0^{k+1}x_0\right\}.$$

Case 1 Suppose $A_0^{2k+2}A_1x_0^2 \leq 1$. Then $x_{3k+5} = \dfrac{1}{A_0^{k+1}x_0}$, and so $x_{3k+1} = x_{3k+4}$ and $x_{3k+2} = x_{3k+5}$. It follows from the fact that $\{A_n\}_{n=0}^{\infty}$ is periodic with period-3 that $\{x_n\}_{n=3k+1}^{\infty}$ is periodic with period-3.

Case 2 Suppose $A_0^{2k+2}A_1x_0^2 > 1$. In particular, $A_1 > 1$. Now $x_{3k+5} = A_0^{k+1}A_1x_0$, and thus

$$x_{3k+6} = \max\left\{\frac{1}{x_{3k+5}}, \frac{A_{3k+5}}{x_{3k+4}}\right\} = \max\left\{\frac{1}{A_0^{k+1}A_1x_0}, \frac{1}{A_0^{k+1}x_0}\right\} = \frac{1}{A_0^{k+1}x_0}$$

$$x_{3k+7} = \max\left\{\frac{1}{x_{3k+6}}, \frac{A_{3k+6}}{x_{3k+5}}\right\} = \max\left\{A_0^{k+1}x_0, \frac{1}{A_0^{k}A_1x_0}\right\} = A_0^{k+1}x_0$$

and so $\{x_n\}_{n=3k+3}^{\infty}$ is periodic with period-3.

(ii) Suppose

$$A_0^{2k+3}x_0^2 < 1 \leq A_0^{2k+2}x_0^2.$$

By direct computation when $k = 0$, and by induction when $k \geq 1$, it follows that for all $0 \leq n \leq k$,

$$x_{3n+1} = A_0^{n+1}x_0 \qquad , x_{3n+2} = \frac{1}{A_0^{n+1}x_0} \qquad x_{3n+3} = A_0^{n+1}x_0.$$

In particular,

$$x_{3k+2} = \frac{1}{A_0^{k+1}x_0} \qquad \text{and} \qquad x_{3k+3} = A_0^{k+1}x_0.$$

Case 1 Suppose $A_0^{2k+2}x_0^2 \leq A_1$.
Then

$$x_{3k+4} = \max\left\{\frac{1}{x_{3k+3}}, \frac{A_{3k+3}}{x_{3k+2}}\right\} = \max\left\{\frac{1}{A_0^{k+1}x_0}, A_0^{k+2}x_0\right\} = \frac{1}{A_0^{k+1}x_0}$$

$$x_{3k+5} = \max\left\{\frac{1}{x_{3k+4}}, \frac{A_{3k+4}}{x_{3k+3}}\right\} = \max\left\{A_0^{k+1}x_0, \frac{A_1}{A_0^{k+1}x_0}\right\} = \frac{A_1}{A_0^{k+1}x_0}$$

$$x_{3k+6} = \max\left\{\frac{1}{x_{3k+5}}, \frac{A_{3k+5}}{x_{3k+4}}\right\} = \max\left\{\frac{A_0^{k+1}x_0}{A_1}, A_0^{k+1}x_0\right\} = A_0^{k+1}x_0$$

$$x_{3k+7} = \max\left\{\frac{1}{x_{3k+6}}, \frac{A_{3k+6}}{x_{3k+5}}\right\} = \max\left\{\frac{1}{A_0^{k+1}x_0}, \frac{A_0^{k+2}x_0}{A_1}\right\} = \frac{1}{A_0^{k+1}x_0}$$

and so we see that $\{x_n\}_{n=3k+3}^{\infty}$ is periodic with period-3.

Case 2 Suppose $A_1 < A_0^{2k+2}x_0$.
Then

$$x_{3k+4} = \max\left\{\frac{1}{x_{3k+3}}, \frac{A_{3k+3}}{x_{3k+2}}\right\} = \max\left\{\frac{1}{A_0^{k+1}x_0}, A_0^{k+2}x_0\right\} = \frac{1}{A_0^{k+1}x_0}$$

$$x_{3k+5} = \max\left\{\frac{1}{x_{3k+4}}, \frac{A_{3k+4}}{x_{3k+3}}\right\} = \max\left\{A_0^{k+1}x_0, \frac{A_1}{A_0^{k+1}x_0}\right\} = A_0^{k+1}x_0$$

$$x_{3k+6} = \max\left\{\frac{1}{x_{3k+5}}, \frac{A_{3k+5}}{x_{3k+4}}\right\} = \max\left\{\frac{1}{A_0^{k+1}x_0}, A_0^{k+1}x_0\right\} = A_0^{k+1}x_0$$

$$x_{3k+7} = \max\left\{\frac{1}{x_{3k+6}}, \frac{A_{3k+6}}{x_{3k+5}}\right\} = \max\left\{\frac{1}{A_0^{k+1}x_0}, \frac{1}{A_0^{k}x_0}\right\} = \frac{1}{A_0^{k+1}x_0}$$

and so $\{x_n\}_{n=3k+3}^{\infty}$ is periodic with period-3. ▯

LEMMA 7.28

Let $\{A_n\}_{n=-1}^{\infty}$ be a periodic sequence sequence of positive real numbers with prime period 3 such that

$$A_0 < 1 = A_2 \qquad \text{and} \qquad A_0A_1 \geq 1.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period-3.

PROOF It follows by Remark 7.4 that we may assume that

$$x_{-1} = \frac{1}{x_0} \quad , \quad x_1 = A_0 x_0 > \frac{1}{x_0} \quad \text{and} \quad A_0 x_0^2 > 1.$$

Now $A_0 < 1$ and $A_0 x_0^2 > 1$, and so we see that $x_0 > 1$. Since $A_1 > 1$ and $A_0 x_0^2 > 1$, it follows that there exists $k \in \{0, 1, \ldots\}$ such that either

$$A_1^{2k} \leq A_0 x_0^2 < A_1^{2k+1} \quad \text{or} \quad A_1^{2k+1} \leq A_0 x_0^2 < A_1^{2k+2}.$$

Note that in either case, $A_0 x_0^2 \leq (A_0 A_1) A_1^{2k+2}$, and so

$$x_0^2 \leq A_1^{2k+3}.$$

Since $x_0 > 1$, it follows by direct computation when $k = 0$, and by induction when $k \geq 1$, that for all $0 \leq n \leq k$, we have

$$x_{3n-1} = \frac{A_1^n}{x_0} \quad , \quad x_{3n} = \frac{x_0}{A_1^n} \quad \text{and} \quad x_{3n+1} = \frac{A_0 x_0}{A_1^n}.$$

Case 1 Suppose $A_1^{2k} \leq A_0 x_0^2 < A_1^{2k+1}$.
Then

$$x_{3k+2} = \max\left\{\frac{1}{x_{3k+1}}, \frac{A_{3k+1}}{x_{3k}}\right\} = \max\left\{\frac{A_1^k}{A_0 x_0}, \frac{A_1^{k+1}}{x_0}\right\} = \frac{A_1^{k+1}}{x_0}$$

$$x_{3k+3} = \max\left\{\frac{1}{x_{3k+2}}, \frac{A_{3k+2}}{x_{3k+1}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1}}, \frac{A_1^k}{A_0 x_0}\right\} = \frac{A_1^k}{A_0 x_0}$$

$$x_{3k+4} = \max\left\{\frac{1}{x_{3k+3}}, \frac{A_{3k+3}}{x_{3k+2}}\right\} = \max\left\{\frac{A_0 x_0}{A_1^k}, \frac{A_0 x_0}{A_1^{k+1}}\right\} = \frac{A_0 x_0}{A_1^k}$$

$$x_{3k+5} = \max\left\{\frac{1}{x_{3k+4}}, \frac{A_{3k+4}}{x_{3k+3}}\right\} = \max\left\{\frac{A_1^k}{A_0 x_0}, \frac{A_0 x_0}{A_1^{k-1}}\right\} = \frac{A_0 x_0}{A_1^{k-1}}$$

$$x_{3k+6} = \max\left\{\frac{1}{x_{3k+5}}, \frac{A_{3k+5}}{x_{3k+4}}\right\} = \max\left\{\frac{A_1^{k-1}}{A_0 x_0}, \frac{A_1^k}{A_0 x_0}\right\} = \frac{A_1^k}{A_0 x_0}$$

$$x_{3k+7} = \max\left\{\frac{1}{x_{3k+6}}, \frac{A_{3k+6}}{x_{3k+5}}\right\} = \max\left\{\frac{A_0 x_0}{A_1^k}, \frac{A_1^{k-1}}{x_0}\right\} = \frac{A_0 x_0}{A_1^k}$$

and so we see that $\{x_n\}_{n=3k+3}^{\infty}$ is periodic with period-3.

Case 2 Suppose $A_1^{2k+1} \leq A_0 x_0^2 < A_1^{2k+2}$.
Then

$$x_{3k+2} = \max\left\{\frac{1}{x_{3k+1}}, \frac{A_{3k+1}}{x_{3k}}\right\} = \max\left\{\frac{A_1^k}{A_0 x_0}, \frac{A_1^{k+1}}{x_0}\right\} = \frac{A_1^{k+1}}{x_0}$$

$$x_{3k+3} = \max\left\{\frac{1}{x_{3k+2}}, \frac{A_{3k+2}}{x_{3k+1}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1}}, \frac{A_1^k}{A_0 x_0}\right\} = \frac{x_0}{A_1^{k+1}}$$

$$x_{3k+4} = \max\left\{\frac{1}{x_{3k+3}}, \frac{A_{3k+3}}{x_{3k+2}}\right\} = \max\left\{\frac{A_1^{k+1}}{x_0}, \frac{A_0 x_0}{A_1^{k+1}}\right\} = \frac{A_1^{k+1}}{x_0}$$

$$x_{3k+5} = \max\left\{\frac{1}{x_{3k+4}}, \frac{A_{3k+4}}{x_{3k+3}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1}}, \frac{A_1^{k+2}}{x_0}\right\} = \frac{A_1^{k+2}}{x_0}$$

$$x_{3k+6} = \max\left\{\frac{1}{x_{3k+5}}, \frac{A_{3k+5}}{x_{3k+4}}\right\} = \max\left\{\frac{x_0}{A_1^{k+2}}, \frac{x_0}{A_1^{k+1}}\right\} = \frac{x_0}{A_1^{k+1}}$$

$$x_{3k+7} = \max\left\{\frac{1}{x_{3k+6}}, \frac{A_{3k+6}}{x_{3k+5}}\right\} = \max\left\{\frac{A_1^{k+1}}{x_0}, \frac{A_0 x_0}{A_1^{k+2}}\right\} = \frac{A_1^{k+1}}{x_0}$$

and so we see that $\{x_n\}_{n=3k+3}^{\infty}$ is periodic with period-3. ▯

The proofs of the next three lemmas are similar to those of Lemmas 7.27 and 7.28 and will be omitted.

LEMMA 7.29
Let $\{A_n\}_{n=-1}^{\infty}$ be a periodic sequence sequence of positive real numbers with prime period 3 such that

$$A_1 < 1 = A_0.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period-3.

LEMMA 7.30
Let $\{A_n\}_{n=-1}^{\infty}$ be a periodic sequence sequence of positive real numbers with prime period 3 such that

$$A_2 < 1 = A_1 \quad \text{and} \quad A_0 < 1.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period-3.

LEMMA 7.31

Let $\{A_n\}_{n=-1}^{\infty}$ be a periodic sequence sequence of positive real numbers with prime period 3 such that

$$A_2 < 1 = A_1 \quad \text{and} \quad A_0 \geq 1.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period-3.

LEMMA 7.32

Let $\{A_n\}_{n=-1}^{\infty}$ be a periodic sequence sequence of positive real numbers with prime period 3 such that

$$A_0 = 1 < A_2 \quad \text{and} \quad A_1 \leq 1.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period-three.

PROOF It follows by Remark 7.4 that we may assume that

$$x_{-1} = \frac{1}{x_0} \quad , \quad x_1 = x_0 > \frac{1}{x_0} \quad \text{and} \quad x_0^2 > 1.$$

Case 1 Suppose that

$$x_0^2 \leq A_2.$$

Then

$$x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{1}{x_0}, \frac{A_1}{x_0}\right\} = \frac{1}{x_0}$$

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_2}{x_1}\right\} = \max\left\{x_0, \frac{A_2}{x_0}\right\} = \frac{A_2}{x_0}$$

$$x_4 = \max\left\{\frac{1}{x_3}, \frac{A_3}{x_2}\right\} = \max\left\{\frac{x_0}{A_2}, x_0\right\} = x_0$$

$$x_5 = \max\left\{\frac{1}{x_4}, \frac{A_4}{x_3}\right\} = \max\left\{\frac{1}{x_0}, \frac{A_1 x_0}{A_2}\right\} = \frac{1}{x_0}$$

and so $\{x_n\}_{n=1}^{\infty}$ is periodic with period-3.

Case 2 Suppose that

$$x_0^2 > A_2.$$

Then

$$x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{1}{x_0}, \frac{A_1}{x_0}\right\} = \frac{1}{x_0}$$

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_2}{x_1}\right\} = \max\left\{x_0, \frac{A_2}{x_0}\right\}$$

and so $\{x_n\}_{n=-1}^{\infty}$ is periodic with period-3. ∎

LEMMA 7.33

Let $\{A_n\}_{n=-1}^{\infty}$ be a periodic sequence sequence of positive real numbers with prime period 3 such that

$$A_0 = 1 < A_1 \leq A_2.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period-3.

PROOF It follows by Remark 7.4 that we may assume that

$$x_{-1} = \frac{1}{x_0} \quad , \quad x_1 = A_0 x_0 > \frac{1}{x_0} \quad \text{and} \quad A_0 x_0^2 > 1.$$

Case 1 Suppose $x_0^2 \geq A_1 A_2$.

There exists $k \in \{0, 1, \ldots\}$ such that either

$$A_1^{2k+1} A_2 \leq x_0^2 < A_1^{2k+2} A_2 \quad \text{or} \quad A_1^{2k+2} A_2 \leq x_0^2 < A_1^{2k+3} A_2.$$

By direct computation when $k = 0$, and by induction when $k \geq 1$, it follows that for all $0 \leq n \leq k$,

$$x_{3n-1} = \frac{A_1^n}{x_0} \quad , \quad x_{3n} = \frac{x_0}{A_1^n} \quad \text{and} \quad x_{3n+1} = \frac{x_0}{A_1^n}.$$

Hence

$$x_{3k+2} = \max\left\{\frac{1}{x_{3k+1}}, \frac{A_{3k+1}}{x_{3k}}\right\} = \max\left\{\frac{A_1^k}{x_0}, \frac{A_1^{k+1}}{x_0}\right\} = \frac{A_1^{k+1}}{x_0}$$

$$x_{3k+3} = \max\left\{\frac{1}{x_{3k+2}}, \frac{A_{3k+2}}{x_{3k+1}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1}}, \frac{A_1^k A_2}{x_0}\right\} = \frac{x_0}{A_1^{k+1}}$$

$$x_{3k+4} = \max\left\{\frac{1}{x_{3k+3}}, \frac{A_{3k+3}}{x_{3k+2}}\right\} = \max\left\{\frac{A_1^{k+1}}{x_0}, \frac{x_0}{A_1^{k+1}}\right\} = \frac{x_0}{A_1^{k+1}}$$

$$x_{3k+5} = \max\left\{\frac{1}{x_{3k+4}}, \frac{A_{3k+4}}{x_{3k+3}}\right\} = \max\left\{\frac{A_1^{k+1}}{x_0}, \frac{A_1^{k+2}}{x_0}\right\} = \frac{A_1^{k+2}}{x_0}$$

$$x_{3k+6} = \max\left\{\frac{1}{x_{3k+5}}, \frac{A_{3k+5}}{x_{3k+4}}\right\} = \max\left\{\frac{x_0}{A_1^{k+2}}, \frac{A_1^{k+1} A_2}{x_0}\right\} = \frac{A_1^{k+1} A_2}{x_0}$$

$$x_{3k+7} = \max\left\{\frac{1}{x_{3k+6}}, \frac{A_{3k+6}}{x_{3k+5}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1} A_2}, \frac{x_0}{A_1^{k+2}}\right\} = \frac{x_0}{A_1^{k+2}}$$

$$x_{3k+8} = \max\left\{\frac{1}{x_{3k+7}}, \frac{A_{3k+7}}{x_{3k+6}}\right\} = \max\left\{\frac{A_1^{k+2}}{x_0}, \frac{x_0}{A_1^k A_2}\right\}.$$

Case 1(a) Suppose that

$$A_1^{2k+1} A_2 \le x_0^2 < A_1^{2k+2} A_2.$$

Then

$$x_{3k+8} = \frac{A_1^{k+2}}{x_0}$$

$$x_{3k+9} = \max\left\{\frac{1}{x_{3k+8}}, \frac{A_{3k+8}}{x_{3k+7}}\right\} = \max\left\{\frac{x_0}{A_1^{k+2}}, \frac{A_1^{k+2} A_2}{x_0}\right\} = \frac{A_1^{k+2} A_2}{x_0}$$

$$x_{3k+10} = \max\left\{\frac{1}{x_{3k+9}}, \frac{A_{3k+9}}{x_{3k+8}}\right\} = \max\left\{\frac{x_0}{A_1^{k+2} A_2}, \frac{x_0}{A_1^{k+2}}\right\} = \frac{x_0}{A_1^{k+2}}$$

$$x_{3k+11} = \max\left\{\frac{1}{x_{3k+10}}, \frac{A_{3k+10}}{x_{3k+9}}\right\} = \max\left\{\frac{A_1^{k+2}}{x_0}, \frac{x_0}{A_1^{k+1} A_2}\right\} = \frac{A_1^{k+2}}{x_0}$$

and so $\{x_n\}_{n=3k+7}^{\infty}$ is periodic with period-three.

Case 1(b) Suppose that

$$A_1^{2k+2}A_2 \le x_0^2 < A_1^{2k+3}A_2.$$

Then

$$x_{3k+8} = \frac{x_0}{A_1^k A_2}$$

and so

$$x_{3k+9} = \max\left\{\frac{1}{x_{3k+8}}, \frac{A_{3k+8}}{x_{3k+7}}\right\} = \max\left\{\frac{A_1^k A_2}{x_0}, \frac{A_1^{k+2}A_2}{x_0}\right\} = \frac{A_1^{k+2}A_2}{x_0}$$

$$x_{3k+10} = \max\left\{\frac{1}{x_{3k+9}}, \frac{A_{3k+9}}{x_{3k+8}}\right\} = \max\left\{\frac{x_0}{A_1^{k+2}A_2}, \frac{A_1^k A_2}{x_0}\right\} = \frac{x_0}{A_1^{k+2}A_2}$$

$$x_{3k+11} = \max\left\{\frac{1}{x_{3k+10}}, \frac{A_{3k+10}}{x_{3k+9}}\right\} = \max\left\{\frac{A_1^{k+2}A_2}{x_0}, \frac{x_0}{A_1^{k+1}A_2}\right\} = \frac{A_1^{k+2}A_2}{x_0}$$

$$x_{3k+12} = \max\left\{\frac{1}{x_{3k+11}}, \frac{A_{3k+11}}{x_{3k+10}}\right\} = \max\left\{\frac{x_0}{A_1^{k+2}A_2}, \frac{A_1^{k+2}A_2^2}{x_0}\right\} = \frac{A_1^{k+2}A_2^2}{x_0}.$$

$$x_{3k+13} = \max\left\{\frac{1}{x_{3k+12}}, \frac{A_{3k+12}}{x_{3k+11}}\right\} = \max\left\{\frac{x_0}{A_1^{k+2}A_2^2}, \frac{x_0}{A_1^{k+2}A_2}\right\} = \frac{x_0}{A_1^{k+2}A_2}$$

$$x_{3k+14} = \max\left\{\frac{1}{x_{3k+13}}, \frac{A_{3k+13}}{x_{3k+12}}\right\} = \max\left\{\frac{A_1^{k+2}A_2}{x_0}, \frac{x_0}{A_1^{k+1}A_2^2}\right\} = \frac{A_1^{k+2}A_2}{x_0}$$

and so $\{x_n\}_{3k+10}^{\infty}$ is periodic with period-3.

Case 2 Suppose $x_0^2 < A_1 A_2$.

Then

$$x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{1}{x_0}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0}$$

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_2}{x_1}\right\} = \max\left\{\frac{x_0}{A_1}, \frac{A_2}{x_0}\right\} = \frac{A_2}{x_0}$$

$$x_4 = \max\left\{\frac{1}{x_3}, \frac{A_3}{x_2}\right\} = \max\left\{\frac{x_0}{A_2}, \frac{x_0}{A_1}\right\} = \frac{x_0}{A_1}$$

$$x_5 = \max\left\{\frac{1}{x_4}, \frac{A_4}{x_3}\right\} = \max\left\{\frac{A_1}{x_0}, \frac{A_1 x_0}{A_2}\right\}.$$

Case 2(a) Suppose that

$$x_0^2 \geq A_2.$$

Then

$$x_5 = \frac{A_1 x_0}{A_2}$$

and hence

$$x_6 = \max\left\{\frac{1}{x_5}, \frac{A_5}{x_4}\right\} = \max\left\{\frac{A_2}{A_1 x_0}, \frac{A_1 A_2}{x_0}\right\} = \frac{A_1 A_2}{x_0}$$

$$x_7 = \max\left\{\frac{1}{x_6}, \frac{A_6}{x_5}\right\} = \max\left\{\frac{x_0}{A_1 A_2}, \frac{A_2}{A_1 x_0}\right\} = \frac{A_2}{A_1 x_0}$$

$$x_8 = \max\left\{\frac{1}{x_7}, \frac{A_7}{x_6}\right\} = \max\left\{\frac{A_1 x_0}{A_2}, \frac{x_0}{A_2}\right\} = \frac{A_1 x_0}{A_2}$$

$$x_9 = \max\left\{\frac{1}{x_8}, \frac{A_8}{x_7}\right\} = \max\left\{\frac{A_2}{A_1 x_0}, A_1 x_0\right\} = A_1 x_0$$

$$x_{10} = \max\left\{\frac{1}{x_9}, \frac{A_9}{x_8}\right\} = \max\left\{\frac{1}{A_1 x_0}, \frac{A_2}{A_1 x_0}\right\} = \frac{A_2}{A_1 x_0}$$

$$x_{11} = \max\left\{\frac{1}{x_{10}}, \frac{A_{10}}{x_9}\right\} = \max\left\{\frac{A_1 x_0}{A_2}, \frac{1}{x_0}\right\} = \frac{A_1 x_0}{A_2}$$

and so $\{x_n\}_{n=7}^{\infty}$ is periodic with period-3.

Case 2(b) Suppose that

$$x_0^2 < A_2.$$

Then

$$x_5 = \frac{A_1}{x_0}$$

$$x_6 = \max\left\{\frac{1}{x_5}, \frac{A_5}{x_4}\right\} = \max\left\{\frac{x_0}{A_1}, \frac{A_1A_2}{x_0}\right\} = \frac{A_1A_2}{x_0}$$

$$x_7 = \max\left\{\frac{1}{x_6}, \frac{A_6}{x_5}\right\} = \max\left\{\frac{1}{x_6}, \frac{A_6}{x_5}\right\} = \max\left\{\frac{A_1}{x_0}, \frac{x_0}{A_2}\right\} = \frac{x_0}{A_1}$$

$$x_8 = \max\left\{\frac{1}{x_7}, \frac{A_7}{x_6}\right\} = \max\left\{\frac{A_1}{x_0} A_2\right\} = \frac{A_1}{x_0}$$

and so $\{x_n\}_{n=4}^{\infty}$ is periodic with period-3. ∎

LEMMA 7.34
Let $\{A_n\}_{n=-1}^{\infty}$ be a periodic sequence sequence of positive real numbers with prime period 3 such that

$$A_0 = 1 < A_2 < A_1.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period-3.

PROOF It follows by Remark 7.4 that we may assume that

$$x_{-1} = \frac{1}{x_0} \quad , \quad x_1 = x_0 > \frac{1}{x_0} \quad \text{and} \quad x_0^2 > 1.$$

We have

$$x_{-1} = \frac{1}{x_0}$$

$$x_0 = x_0$$

$$x_1 = x_0$$

$$x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{1}{x_0}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0}$$

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_2}{x_1}\right\} = \max\left\{\frac{x_0}{A_1}, \frac{A_2}{x_0}\right\}.$$

Case 1 Suppose that

$$A_1A_2 \le x_0^2.$$

There exists $k \in \{0, 1, \ldots\}$ such that either

$$A_1^{2k+1} A_2 \leq x_0^2 < A_1^{2k+2} A_2 \qquad \text{or} \qquad A_1^{2k+2} A_2 \leq x_0^2 < A_1^{2k+3} A_2.$$

By direct computation when $k = 0$, and by induction when $k \geq 1$, it follows that for all $0 \leq n \leq k$, we have

$$x_{3n-1} = \frac{A_1^n}{x_0} \quad , \quad x_{3n} = \frac{x_0}{A_1^n} \quad \text{and} \quad x_{3n+1} = \frac{x_0}{A_1^n}.$$

Thus

$$x_{3k+2} = \max\left\{\frac{1}{x_{3k+1}}, \frac{A_{3k+1}}{x_{3k}}\right\} = \max\left\{\frac{A_1^k}{x_0}, \frac{A_1^{k+1}}{x_0}\right\} = \frac{A_1^{k+1}}{x_0}$$

$$x_{3k+3} = \max\left\{\frac{1}{x_{3k+2}}, \frac{A_{3k+2}}{x_{3k+1}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1}}, \frac{A_1^k A_2}{x_0}\right\} = \frac{x_0}{A_1^{k+1}}$$

$$x_{3k+4} = \max\left\{\frac{1}{x_{3k+3}}, \frac{A_{3k+3}}{x_{3k+2}}\right\} = \max\left\{\frac{A_1^{k+1}}{x_0}, \frac{x_0}{A_1^{k+1}}\right\}.$$

Case 1(a) Consider the case

$$A_1^{2k+1} A_2 \leq x_0^2 < A_1^{2k+2} A_2.$$

Case 1(a)(i) Suppose that

$$A_1^{2k+1} A_2 \leq x_0^2 < A_1^{2k+2}.$$

There exists $l \in \{0, 1, \ldots\}$ such that either

$$A_2^{2l+1} x_0^2 \leq A_1^{2k+3} < A_2^{2l+2} x_0^2 \qquad \text{or} \qquad A_2^{2l+2} x_0^2 \leq A_1^{2k+3} < A_2^{2l+3} x_0^2.$$

It follows by induction that for all $0 \leq n \leq l$,

$$x_{3k+4+3n} = \frac{A_1^{k+1}}{A_2^n x_0} \quad , \quad x_{3k+5+3n} = \frac{A_1^{k+2}}{A_2^n x_0} \quad \text{and} \quad x_{3k+6+3n} = \frac{A_2^{n+1} x_0}{A_1^{k+1}}.$$

In particular,

$$x_{3k+4+3l} = \frac{A_1^{k+1}}{A_2^l x_0} \quad , \quad x_{3k+5+3l} = \frac{A_1^{k+2}}{A_2^l x_0} \quad \text{and} \quad x_{3k+6+3l} = \frac{A_2^{l+1} x_0}{A_1^{k+1}}.$$

Hence

$$x_{3k+7+3l} = \max\left\{\frac{1}{x_{3k+6+3l}}, \frac{A_{3k+6+3l}}{x_{3k+5+3l}}\right\} = \max\left\{\frac{A_1^{k+1}}{A_2^{l+1}x_0}, \frac{A_2^l x_0}{A_1^{k+2}}\right\}.$$

Case 1(a)(i)(1) Suppose that

$$A_2^{2l+1}x_0^2 \le A_1^{2k+3} < A_2^{2l+2}x_0^2.$$

Then

$$x_{3k+7+3l} = \frac{A_1^{k+1}}{A_2^{l+1}x_0}.$$

Thus

$$x_{3k+8+3l} = \max\left\{\frac{1}{x_{3k+7+3l}}, \frac{A_{3k+7+3l}}{x_{3k+6+3l}}\right\} = \max\left\{\frac{A_2^{l+1}x_0}{A_1^{k+1}}, \frac{A_1^{k+2}}{A_2^{l+1}x_0}\right\} = \frac{A_2^{l+1}x_0}{A_1^{k+1}}$$

$$x_{3k+9+3l} = \max\left\{\frac{1}{x_{3k+8+3l}}, \frac{A_{3k+8+3l}}{x_{3k+7+3l}}\right\} = \max\left\{\frac{A_1^{k+1}}{A_2^{l+1}x_0}, \frac{A_2^{l+2}x_0}{A_1^{k+1}}\right\} = \frac{A_2^{l+2}x_0}{A_1^{k+1}}$$

$$x_{3k+10+3l} = \max\left\{\frac{1}{x_{3k+9+3l}}, \frac{A_{3k+9+3l}}{x_{3k+8+3l}}\right\} = \max\left\{\frac{A_1^{k+1}}{A_2^{l+2}x_0}, \frac{A_1^{k+1}}{A_2^{l+1}x_0}\right\} = \frac{A_1^{k+1}}{A_2^{l+1}x_0}$$

$$x_{3k+11+3l} = \max\left\{\frac{1}{x_{3k+10+3l}}, \frac{A_{3k+10+3l}}{x_{3k+9+3l}}\right\} = \max\left\{\frac{A_2^{l+1}x_0}{A_1^{k+1}}, \frac{A_1^{k+2}}{A_2^{l+2}x_0}\right\} = \frac{A_2^{l+1}x_0}{A_1^{k+1}}$$

and so $\{x_n\}_{n=3k+7+3l}^{\infty}$ is periodic with period-3.

Case 1(a)(i)(2) Suppose that

$$A_2^{2l}x_0^2 \le A_1^{2k+3} < A_2^{2l+1}x_0^2.$$

Then

$$x_{3k+7+3l} = \frac{A_2^l x_0}{A_1^{k+2}}$$

and so

$$x_{3k+8+3l} = \max\left\{\frac{1}{x_{3k+7+3l}}, \frac{A_{3k+7+3l}}{x_{3k+6+3l}}\right\} = \max\left\{\frac{A_1^{k+2}}{A_2^l x_0}, \frac{A_1^{k+2}}{A_2^{l+1} x_0}\right\} = \frac{A_1^{k+2}}{A_2^l x_0}$$

$$x_{3k+9+3l} = \max\left\{\frac{1}{x_{3k+8+3l}}, \frac{A_{3k+8+3l}}{x_{3k+7+3l}}\right\} = \max\left\{\frac{A_2^l x_0}{A_1^{k+2}}, \frac{A_1^{k+2}}{A_2^{l-1} x_0}\right\} = \frac{A_1^{k+2} x_0}{A_2^{l-1}}$$

$$x_{3k+10+3l} = \max\left\{\frac{1}{x_{3k+9+3l}}, \frac{A_{3k+9+3l}}{x_{3k+8+3l}}\right\} = \max\left\{\frac{A_2^{l-1}}{A_1^{k+2} x_0}, \frac{A_2^l x_0}{A_1^{k+2}}\right\} = \frac{A_2^l x_0}{A_1^{k+2}}$$

$$x_{3k+11+3l} = \max\left\{\frac{1}{x_{3k+10+3l}}, \frac{A_{3k+10+3l}}{x_{3k+9+3l}}\right\} = \max\left\{\frac{A_1^{k+2}}{A_2^l x_0}, \frac{A_2^{l-1}}{A_1^{k+1} x_0}\right\} = \frac{A_1^{k+2}}{A_2^l x_0}$$

and so $\{x_n\}_{n=3k+7+3l}^{\infty}$ is periodic with period-3.

Case 1(a)(ii) Suppose that

$$A_1^{2k+2} \leq x_0^2 < A_1^{2k+2} A_2.$$

There exists $l \in \{0, 1, \ldots\}$ such that either

$$A_1^{2k+1} A_2^{2l+1} \leq x_0^2 < A_1^{2k+1} A_2^{2l+2} \qquad \text{or} \qquad A_1^{2k+1} A_2^{2l+2} \leq x_0^2 < A_1^{2k+1} A_2^{2l+3}.$$

It follows by induction that for all $0 \leq n \leq l$,

$$x_{3k+4+3n} = \frac{x_0}{A_1^{k+1} A_2^n} \quad , \quad x_{3k+5+3n} = \frac{x_0}{A_1^k A_2^n} \quad \text{and} \quad x_{3k+6+3n} = \frac{A_1^{k+1} A_2^{n+1}}{x_0}.$$

In particular,

$$x_{3k+4+3l} = \frac{x_0}{A_1^{k+1} A_2^l} \quad , \quad x_{3k+5+3l} = \frac{x_0}{A_1^k A_2^l} \quad \text{and} \quad x_{3k+6+3l} = \frac{A_1^{k+1} A_2^{l+1}}{x_0}.$$

Thus

$$x_{3k+7+3l} = \max\left\{\frac{1}{x_{3k+6+3l}}, \frac{A_{3k+6+3l}}{x_{3k+5+3l}}\right\} = \left\{\frac{x_0}{A_1^{k+1} A_2^{l+1}}, \frac{A_1^k A_2^l}{x_0}\right\} = \frac{x_0}{A_1^{k+1} A_2^{l+1}}$$

and

$$x_{3k+8+3l} = \left\{\frac{1}{x_{3k+7+3l}}, \frac{A_{3k+7+3l}}{x_{3k+6+3l}}\right\} = \left\{\frac{A_1^{k+1} A_2^{l+1}}{x_0}, \frac{x_0}{A_1^k A_2^{l+1}}\right\}.$$

Case 1(a)(ii)(1) Suppose that

$$A_1^{2k+1}A_2^{2l+1} \leq x_0^2 < A_1^{2k+1}A_2^{2l+2}.$$

Then

$$x_{3k+8+3l} = \frac{A_1^{k+1}A_2^{l+1}}{x_0}$$

and

$$\begin{aligned}
x_{3k+9+3l} &= \max\left\{\frac{1}{x_{3k+8+3l}}, \frac{A_{3k+8+3l}}{x_{3k+7+3l}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1}A_2^{l+1}}, \frac{A_1^{k+1}A_2^{l+2}}{x_0}\right\} \\
&= \frac{A_1^{k+1}A_2^{l+2}}{x_0} \\
x_{3k+10+3l} &= \max\left\{\frac{1}{x_{3k+9+3l}}, \frac{A_{3k+9+3l}}{x_{3k+8+3l}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1}A_2^{l+2}}, \frac{x_0}{A_1^{k+1}A_2^{l+1}}\right\} \\
&= \frac{x_0}{A_1^{k+1}A_2^{l+1}} \\
x_{3k+11+3l} &= \max\left\{\frac{1}{x_{3k+10+3l}}, \frac{A_{3k+10+3l}}{x_{3k+9+3l}}\right\} = \max\left\{\frac{A_1^{k+1}A_2^{l+1}}{x_0}, \frac{x_0}{A_1^{k+1}A_2^{l+1}}\right\} \\
&= \frac{A_1^{k+1}A_2^{l+1}}{x_0}
\end{aligned}$$

and so $\{x_n\}_{n=3k+7+3l}^{\infty}$ is periodic with period-3.

Case 1(a)(ii)(2) Suppose that

$$A_1^{2k+1}A_2^{2l+2} \leq x_0^2 < A_1^{2k+1}A_2^{2l+3}.$$

Then

$$x_{3k+8+3l} = \frac{x_0}{A_1^{k}A_2^{l+1}}$$

and so

$$x_{3k+9+3l} = \max\left\{\frac{1}{x_{3k+8+3l}}, \frac{A_{3k+8+3l}}{x_{3k+7+3l}}\right\} = \max\left\{\frac{A_1^k A_2^{l+1}}{x_0}, \frac{A_1^{k+1} A_2^{l+2}}{x_0}\right\}$$

$$= \frac{A_1^{k+1} A_2^{l+2}}{x_0}$$

$$x_{3k+10+3l} = \max\left\{\frac{1}{x_{3k+9+3l}}, \frac{A_{3k+9+3l}}{x_{3k+8+3l}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1} A_2^{l+2}}, \frac{A_1^k A_2^{l+1}}{x_0}\right\}$$

$$= \frac{A_1^k A_2^{l+1}}{x_0}$$

$$x_{3k+11+3l} = \max\left\{\frac{1}{x_{3k+10+3l}}, \frac{A_{3k+10+3l}}{x_{3k+9+3l}}\right\} = \max\left\{\frac{x_0}{A_1^k A_2^{l+1}}, \frac{x_0}{A_1^k A_2^{l+2}}\right\}$$

$$= \frac{x_0}{A_1^k A_2^{l+1}}$$

$$x_{3k+12+3l} = \max\left\{\frac{1}{x_{3k+11+3l}}, \frac{A_{3k+11+3l}}{x_{3k+10+3l}}\right\} = \max\left\{\frac{A_1^k A_2^{l+1}}{x_0}, \frac{x_0}{A_1^k A_2^l}\right\}$$

$$= \frac{x_0}{A_1^k A_2^l}$$

$$x_{3k+13+3l} = \max\left\{\frac{1}{x_{3k+12+3l}}, \frac{A_{3k+12+3l}}{x_{3k+11+3l}}\right\} = \max\left\{\frac{A_1^k A_2^l}{x_0}, \frac{A_1^k A_2^{l+1}}{x_0}\right\}$$

$$= \frac{A_1^k A_2^{l+1}}{x_0}$$

$$x_{3k+14+3l} = \max\left\{\frac{1}{x_{3k+13+3l}}, \frac{A_{3k+13+3l}}{x_{3k+12+3l}}\right\} = \max\left\{\frac{x_0}{A_1^k A_2^{l+1}}, \frac{A_1^{k+1} A_2^l}{x_0}\right\}$$

$$= \frac{x_0}{A_1^k A_2^{l+1}}$$

and so $\{x_n\}_{n=3k+10+3l}^{\infty}$ is periodic with period-3.

Case 1(b) Suppose that

$$A_1^{2k+2} A_2 \leq x_0^2 < A_1^{2k+3} A_2.$$

Then

$$x_{3k+4} = \frac{x_0}{A_1^{k+1}}$$

and so

$$x_{3k+5} = \max\left\{\frac{1}{x_{3k+4}}, \frac{A_{3k+4}}{x_{3k+3}}\right\} = \max\left\{\frac{A_1^{k+1}}{x_0}, \frac{A_1^{k+2}}{x_0}\right\} = \frac{A_1^{k+2}}{x_0}$$

$$x_{3k+6} = \max\left\{\frac{1}{x_{3k+5}}, \frac{A_{3k+5}}{x_{3k+4}}\right\} = \max\left\{\frac{x_0}{A_1^{k+2}}, \frac{A_1^{k+1}A_2}{x_0}\right\} = \frac{A_1^{k+1}A_2}{x_0}$$

$$x_{3k+7} = \max\left\{\frac{1}{x_{3k+6}}, \frac{A_{3k+6}}{x_{3k+5}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1}A_2}, \frac{x_0}{A_1^{k+2}}\right\} = \frac{x_0}{A_1^{k+1}A_2}$$

$$x_{3k+8} = \max\left\{\frac{1}{x_{3k+7}}, \frac{A_{3k+7}}{x_{3k+6}}\right\} = \max\left\{\frac{A_1^{k+1}A_2}{x_0}, \frac{x_0}{A_1^{k}A_2}\right\} = \frac{x_0}{A_1^{k}A_2}$$

$$x_{3k+9} = \max\left\{\frac{1}{x_{3k+8}}, \frac{A_{3k+8}}{x_{3k+7}}\right\} = \max\left\{\frac{A_1^{k}A_2}{x_0}, \frac{A_1^{k+1}A_2^2}{x_0}\right\} = \frac{A_1^{k+1}A_2^2}{x_0}$$

$$x_{3k+10} = \max\left\{\frac{1}{x_{3k+9}}, \frac{A_{3k+9}}{x_{3k+8}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1}A_2^2}, \frac{A_1^{k}A_2}{x_0}\right\}.$$

Case 1(b)(i) Suppose that

$$A_1^{2k+2}A_2 \le x_0^2 < A_1^{2k+3}.$$

There exists $l \in \{1, 2, \ldots\}$ such that either

$$A_1^{2k+1}A_2^{2l} \le x_0^2 < A_1^{2k+1}A_2^{2l+1} \quad \text{or} \quad A_1^{2k+1}A_2^{2l+1} \le x_0^2 < A_1^{2k+1}A_2^{2l+2}.$$

It follows by induction that for all $0 \le n \le l$,

$$x_{3k+4+3n} = \frac{x_0}{A_1^{k+1}A_2^n}, \quad x_{3k+5+3n} = \frac{x_0}{A_1^{k}A_2^n} \quad \text{and} \quad x_{3k+6+3n} = \frac{A_1^{k+1}A_2^{n+1}}{x_0}.$$

In particular,

$$x_{3k+4+3l} = \frac{x_0}{A_1^{k+1}A_2^l}, \quad x_{3k+5+3l} = \frac{x_0}{A_1^{k}A_2^l} \quad \text{and} \quad x_{3k+6+3l} = \frac{A_1^{k+1}A_2^{l+1}}{x_0}.$$

Thus

$$x_{3k+7+3l} = \max\left\{\frac{1}{x_{3k+6+3l}}, \frac{A_{3k+6+3l}}{x_{3k+5+3l}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1}A_2^{l+1}}, \frac{A_1^{k}A_2^{l}}{x_0}\right\}.$$

Case 1(b)(i)(1) Suppose that

$$A_1^{2k+1}A_2^{2l} \le x_0^2 < A_1^{2k+1}A_2^{2l+1}.$$

Then

$$x_{3k+7+3l} = \frac{A_1^k A_2^l}{x_0}$$

and so

$$x_{3k+8+3l} = \max\left\{\frac{1}{x_{3k+7+3l}}, \frac{A_{3k+7+3l}}{x_{3k+6+3l}}\right\} = \max\left\{\frac{x_0}{A_1^k A_2^l}, \frac{x_0}{A_1^k A_2^{l+1}}\right\}$$

$$= \frac{x_0}{A_1^k A_2^l}$$

$$x_{3k+9+3l} = \max\left\{\frac{1}{x_{3k+8+3l}}, \frac{A_{3k+8+3l}}{x_{3k+7+3l}}\right\} = \max\left\{\frac{A_1^k A_2^l}{x_0}, \frac{x_0}{A_1^k A_2^{l-1}}\right\}$$

$$= \frac{x_0}{A_1^k A_2^{l-1}}$$

$$x_{3k+10+3l} = \max\left\{\frac{1}{x_{3k+9+3l}}, \frac{A_{3k+9+3l}}{x_{3k+8+3l}}\right\} = \max\left\{\frac{A_1^k A_2^{l-1}}{x_0}, \frac{A_1^k A_2^l}{x_0}\right\}$$

$$= \frac{A_1^k A_2^l}{x_0}$$

$$x_{3k+11+3l} = \max\left\{\frac{1}{x_{3k+10+3l}}, \frac{A_{3k+10+3l}}{x_{3k+9+3l}}\right\} = \max\left\{\frac{x_0}{A_1^k A_2^l}, \frac{A_1^{k+1} A_2^{l-1}}{x_0}\right\}$$

$$= \frac{x_0}{A_1^k A_2^l}$$

and so $\{x_n\}_{n=3k+7+3l}^{\infty}$ is periodic with period-3.

Case 1(b)(i)(2) Suppose that

$$A_1^{2k+1}A_2^{2l+1} \le x_0^2 < A_1^{2k+1}A_2^{2l+2}.$$

Then

$$x_{3k+7+3l} = \frac{x_0}{A_1^{k+1} A_2^{l+1}}$$

and so

$$x_{3k+8+3l} = \max\left\{\frac{1}{x_{3k+7+3l}}, \frac{A_{3k+7+3l}}{x_{3k+6+3l}}\right\} = \max\left\{\frac{A_1^{k+1}A_2^{l+1}}{x_0}, \frac{x_0}{A_1^k A_2^{l+1}}\right\}$$

$$= \frac{A_1^{k+1}A_2^{l+1}}{x_0}$$

$$x_{3k+9+3l} = \max\left\{\frac{1}{x_{3k+8+3l}}, \frac{A_{3k+8+3l}}{x_{3k+7+3l}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1}A_2^{l+1}}, \frac{A_1^{k+1}A_2^{l+2}}{x_0}\right\}$$

$$= \frac{A_1^{k+1}A_2^{l+2}}{x_0}$$

$$x_{3k+10+3l} = \max\left\{\frac{1}{x_{3k+9+3l}}, \frac{A_{3k+9+3l}}{x_{3k+8+3l}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1}A_2^{l+2}}, \frac{x_0}{A_1^{k+1}A_2^{l+1}}\right\}$$

$$= \frac{x_0}{A_1^{k+1}A_2^{l+1}}$$

$$x_{3k+11+3l} = \max\left\{\frac{1}{x_{3k+10+3l}}, \frac{A_{3k+10+3l}}{x_{3k+9+3l}}\right\} = \max\left\{\frac{A_1^{k+1}A_2^{l+1}}{x_0}, \frac{x_0}{A_1^k A_2^{l+2}}\right\}$$

$$= \frac{A_1^{k+1}A_2^{l+1}}{x_0}$$

and so $\{x_n|_{n=3k+7+3l}^{\infty}$ is periodic with period-3.

Case 1(b)(ii) Suppose that

$$A_1^{2k+3} \leq x_0^2 < A_1^{2k+3}A_2.$$

There exists $l \in \{1, 2, \ldots\}$ such that either

$$A_1^{2k+1}A_2^{2l} \leq x_0^2 < A_1^{2k+1}A_2^{2l+1} \qquad \text{or} \qquad A_1^{2k+1}A_2^{2l+1} \leq x_0^2 < A_1^{2k+1}A_2^{2l+2}.$$

For $0 \leq n \leq l$,

$$x_{3k+4+3n} = \frac{x_0}{A_1^{k+1}A_2^n}, \quad x_{3k+5+3n} = \frac{x_0}{A_1^k A_2^n} \text{ and } x_{3k+6+3n} = \frac{A_2^{k+1}A_2^{n+1}}{x_0}.$$

In particular,

$$x_{3k+4+3l} = \frac{x_0}{A_1^{k+1}A_2^l}, \quad x_{3k+5+3l} = \frac{x_0}{A_1^k A_2^l} \text{ and } x_{3k+6+3l} = \frac{A_1^{k+1}A_2^{l+1}}{x_0}.$$

Thus

$$x_{3k+7+3l} = \max\left\{\frac{1}{x_{3k+6}}, \frac{A_{3k+6}}{x_{3k+5}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1}A_2^{l+1}}, \frac{A_1^k A_2^l}{x_0}\right\}.$$

Case 1(b)(ii)(1) Suppose that

$$A_1^{2k+1}A_2^{2l} \le x_0^2 < A_1^{2k+1}A_2^{2l+1}.$$

Then

$$x_{3k+7+3l} = \frac{A_1^k A_2^l}{x_0}$$

and so

$$x_{3k+8+3l} = \max\left\{\frac{1}{x_{3k+7+3l}}, \frac{A_{3k+7+3l}}{x_{3k+6+3l}}\right\} = \max\left\{\frac{x_0}{A_1^k A_2^l}, \frac{x_0}{A_1^k A_2^{l+1}}\right\}$$

$$= \frac{x_0}{A_1^k A_2^l}$$

$$x_{3k+9+3l} = \max\left\{\frac{1}{x_{3k+8+3l}}, \frac{A_{3k+8+3l}}{x_{3k+7+3l}}\right\} = \max\left\{\frac{A_1^k A_2^l}{x_0}, \frac{x_0}{A_1^k A_2^{l-1}}\right\}$$

$$= \frac{x_0}{A_1^k A_2^{l-1}}$$

$$x_{3k+10+3l} = \max\left\{\frac{1}{x_{3k+9+3l}}, \frac{A_{3k+9+3l}}{x_{3k+8+3l}}\right\} = \max\left\{\frac{A_1^k A_2^{l-1}}{x_0}, \frac{A_1^k A_2^l}{x_0}\right\}$$

$$= \frac{A_1^k A_2^l}{x_0}$$

$$x_{3k+11+3l} = \max\left\{\frac{1}{x_{3k+10+3l}}, \frac{A_{3k+10+3l}}{x_{3k+9+3l}}\right\} = \max\left\{\frac{x_0}{A_1^k A_2^l}, \frac{A_1^{k+1} A_2^{l-1}}{x_0}\right\}$$

$$= \frac{x_0}{A_1^k A_2^l}$$

and so $\{x_n\}_{n=3k+7+3l}^{\infty}$ is periodic with period-3.

Case 1(b)(ii)(2) Suppose that

$$A_1^{2k+1}A_2^{2l+1} \le x_0^2 < A_1^{2k+1}A_2^{2l+2}.$$

Then

$$x_{3k+7+3l} = \frac{x_0}{A_1^{k+1}A_2^{l+1}}$$

and so

$$x_{3k+8+3l} = \max\left\{\frac{1}{x_{3k+7+3l}}, \frac{A_{3k+7+3l}}{x_{3k+6+3l}}\right\} = \max\left\{\frac{A_1^{k+1}A_2^{l+1}}{x_0}, \frac{x_0}{A_1^{k}A_2^{l+1}}\right\}$$

$$= \frac{A_1^{k+1}A_2^{l+1}}{x_0}$$

$$x_{3k+9+3l} = \max\left\{\frac{1}{x_{3k+8+3l}}, \frac{A_{3k+8+3l}}{x_{3k+7+3l}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1}A_2^{l+1}}, \frac{A_1^{k+1}A_2^{l+2}}{x_0}\right\}$$

$$= \frac{x_0}{A_1^{k}A_2^{l-1}}$$

$$x_{3k+10+3l} = \max\left\{\frac{1}{x_{3k+9+3l}}, \frac{A_{3k+9+3l}}{x_{3k+8+3l}}\right\} = \max\left\{\frac{A_1^{k}A_2^{l-1}}{x_0}, \frac{x_0}{A_1^{k+1}A_2^{l+1}}\right\}$$

$$= \frac{x_0}{A_1^{k+1}A_2^{l+1}}$$

$$x_{3k+11+3l} = \max\left\{\frac{1}{x_{3k+10+3l}}, \frac{A_{3k+10+3l}}{x_{3k+9+3l}}\right\} = \max\left\{\frac{A_1^{k+1}A_2^{l+1}}{x_0}, \frac{A_1^{k+1}A_2^{l-1}}{x_0}\right\}$$

$$= \frac{A_1^{k+1}A_2^{l+1}}{x_0}$$

and so $\{x_n\}_{n=3k+7+3l}^{\infty}$ is periodic with period-3.

Case 2 Suppose that

$$x_0^2 < A_1A_2.$$

Then

$$x_3 = \frac{A_2}{x_0}$$

and so

$$x_4 = \max\left\{\frac{1}{x_3}, \frac{A_3}{x_2}\right\} = \max\left\{\frac{x_0}{A_2}, \frac{x_0}{A_1}\right\} = \frac{x_0}{A_2}$$

$$x_5 = \max\left\{\frac{1}{x_4}, \frac{A_4}{x_3}\right\} = \max\left\{\frac{A_2}{x_0}, \frac{A_1x_0}{A_2}\right\}.$$

Case 2(a) Suppose that

$$A_2^2 \le A_1 x_0^2.$$

There exists $l \in \{1, 2, \ldots\}$ such that either

$$A_2^{2l} \le A_1 x_0^2 < A_2^{2l+1} \qquad \text{or} \qquad A_2^{2l+1} \le x_0^2 < A_2^{2l+2}.$$

It follows by induction that for all $0 \le n \le l-1$,

$$x_{3n+2} = \frac{A_1 x_0}{A_2^n} \quad , \quad x_{3n+3} = \frac{A_2^{n+1}}{x_0} \quad \text{and} \quad x_{3n+4} = \frac{x_0}{A_2^{n+1}}.$$

In particular,

$$x_{3l-1} = \frac{A_1 x_0}{A_2^l} \quad , x_{3l} = \frac{A_2^{l+1}}{x_0} \quad \text{and} \quad x_{3l+1} = \frac{x_0}{A_2^l}.$$

Case 2(a)(i) Suppose that

$$A_2^{2l} \le A_1 x_0^2 < A_2^{2l+1}.$$

$$x_{3l+2} = \max\left\{\frac{1}{x_{3l+1}}, \frac{A_{3l+1}}{x_{3l}}\right\} = \max\left\{\frac{A_2^l}{x_0}, \frac{A_1 x_0}{A_2^l}\right\} = \frac{A_1 x_0}{A_2^l}$$

$$x_{3l+3} = \max\left\{\frac{1}{x_{3l+2}}, \frac{A_{3l+2}}{x_{3l+1}}\right\} = \max\left\{\frac{A_2^l}{A_1 x_0}, \frac{A_2^{l+1}}{x_0}\right\} = \frac{A_2^{l+1}}{x_0}$$

$$x_{3l+4} = \max\left\{\frac{1}{x_{3l+3}}, \frac{A_{3l+3}}{x_{3l+2}}\right\} = \max\left\{\frac{x_0}{A_2^{l+1}}, \frac{A_2^l}{A_1 x_0}\right\} = \frac{A_2^l}{A_1 x_0}$$

$$x_{3l+5} = \max\left\{\frac{1}{x_{3l+4}}, \frac{A_{3l+4}}{x_{3l+3}}\right\} = \max\left\{\frac{A_1 x_0}{A_2^l}, \frac{A_1 x_0}{A_2^{l+1}}\right\} = \frac{A_1 x_0}{A_2^l}$$

$$x_{3l+6} = \max\left\{\frac{1}{x_{3l+5}}, \frac{A_{3l+5}}{x_{3l+4}}\right\} = \max\left\{\frac{A_2^l}{A_1 x_0}, \frac{A_1 x_0}{A_2^{l-1}}\right\} = \frac{A_1 x_0}{A_2^{l-1}}$$

$$x_{3l+7} = \max\left\{\frac{1}{x_{3l+6}}, \frac{A_{3l+6}}{x_{3l+5}}\right\} = \max\left\{\frac{A_2^{l-1}}{A_1 x_0}, \frac{A_2^l}{A_1 x_0}\right\} = \frac{A_2^l}{A_1 x_0}$$

$$x_{3l+8} = \max\left\{\frac{1}{x_{3l+7}}, \frac{A_{3l+7}}{x_{3l+6}}\right\} = \max\left\{\frac{A_1 x_0}{A_2^l}, \frac{A_2^{l-1}}{x_0}\right\} = \frac{A_1 x_0}{A_2^l}$$

and so $\{x_n\}_{n=3l+4}^{\infty}$ is periodic with period-3.

Case 2(a)(ii) Suppose that

$$A_2^{2l+1} \leq A_1 x_0^2 < A_2^{2l+2}.$$

$$x_{3l+2} = \max\left\{\frac{1}{x_{3l+1}}, \frac{A_{3l+1}}{x_{3l}}\right\} = \max\left\{\frac{A_2^l}{x_0}, \frac{A_1 x_0}{A_2^l}\right\} = \frac{A_1 x_0}{A_2^l}$$

$$x_{3l+3} = \max\left\{\frac{1}{x_{3l+2}}, \frac{A_{3l+2}}{x_{3l+1}}\right\} = \max\left\{\frac{A_2^l}{A_1 x_0}, \frac{A_2^{l+1}}{x_0}\right\} = \frac{A_2^{l+1}}{x_0}$$

$$x_{3l+4} = \max\left\{\frac{1}{x_{3l+3}}, \frac{A_{3l+3}}{x_{3l+2}}\right\} = \max\left\{\frac{x_0}{A_2^{l+1}}, \frac{A_2^l}{A_1 x_0}\right\} = \frac{x_0}{A_2^{l+1}}$$

$$x_{3l+5} = \max\left\{\frac{1}{x_{3l+4}}, \frac{A_{3l+4}}{x_{3l+3}}\right\} = \max\left\{\frac{A_2^{l+1}}{x_0}, \frac{A_1 x_0}{A_2^{l+1}}\right\} = \frac{A_2^{l+1}}{x_0}$$

$$x_{3l+6} = \max\left\{\frac{1}{x_{3l+5}}, \frac{A_{3l+5}}{x_{3l+4}}\right\} = \max\left\{\frac{x_0}{A_2^{l+1}}, \frac{A_2^{l+2}}{x_0}\right\} = \frac{A_2^{l+2}}{x_0}$$

$$x_{3l+7} = \max\left\{\frac{1}{x_{3l+6}}, \frac{A_{3l+6}}{x_{3l+5}}\right\} = \max\left\{\frac{x_0}{A_2^{l+2}}, \frac{x_0}{A_2^{l+1}}\right\} = \frac{x_0}{A_2^{l+1}}$$

$$x_{3l+8} = \max\left\{\frac{1}{x_{3l+7}}, \frac{A_{3l+7}}{x_{3l+6}}\right\} = \max\left\{\frac{A_2^{l+1}}{x_0}, \frac{A_1 x_0}{A_2^{l+2}}\right\} = \frac{A_2^{l+1}}{x_0}$$

and so $\{x_n\}_{n=3l+4}^{\infty}$ is periodic with period-3.

Case 2(b) Suppose that

$$x_0^2 < A_2^2.$$

Then

$$x_5 = \frac{A_2}{x_0} = \frac{1}{x_4}$$

and so

$$x_6 = \max\left\{\frac{1}{x_5}, \frac{A_5}{x_4}\right\} = \max\left\{\frac{x_0}{A_2}, \frac{A_2^2}{x_0}\right\} = \frac{A_2^2}{x_0}$$

$$x_7 = \max\left\{\frac{1}{x_6}, \frac{A_6}{x_5}\right\} = \max\left\{\frac{x_0}{A_2^2}, \frac{A_1 x_0}{A_2^2}\right\} = \frac{x_0}{A_2}$$

$$x_8 = \max\left\{\frac{1}{x_7}, \frac{A_7}{x_6}\right\} = \max\left\{\frac{A_2}{x_0}, \frac{A_1 x_0}{A_2^2}\right\} = \frac{A_2}{x_0}$$

and so $\{x_n\}_{n=4}^{\infty}$ is periodic with period-3. The proof is complete. ▯

The proofs of the next three lemmas are similar to those of Lemmas 7.32, 7.33, and 7.34 and will be omitted.

LEMMA 7.35
Let $\{A_n\}_{n=-1}^{\infty}$ be a periodic sequence sequence of positive real numbers with prime period 3 such that

$$A_1 = 1 < A_0.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period-3.

LEMMA 7.36
Let $\{A_n\}_{n=-1}^{\infty}$ be a periodic sequence sequence of positive real numbers with prime period 3 such that

$$A_2 = 1 < A_1 \qquad \text{and} \qquad A_0 A_1 < 1.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=3k}^{\infty}$ is periodic with period-3.

LEMMA 7.37
Let $\{A_n\}_{n=-1}^{\infty}$ be a periodic sequence sequence of positive real numbers with prime period 3 such that

$$A_2 = 1 < A_1 \qquad \text{and} \qquad A_0 A_1 \geq 1.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period-3.

LEMMA 7.38
Let $\{A_n\}_{n=-1}^{\infty}$ be a periodic sequence sequence of positive real numbers with prime period 3 such that

$$A_0 = A_2 = 1 \qquad \text{and} \qquad A_1 < 1.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=-1}^{\infty}$ is periodic with period-3.

PROOF It follows by Remark 7.4 that we may assume that

$$x_{-1} = \frac{1}{x_0} \quad , \quad x_1 = x_0 > \frac{1}{x_0} \quad \text{and} \quad x_0^2 > 1.$$

We have

$$x_2 = \max\left\{\frac{1}{x_1}, \frac{A_1}{x_0}\right\} = \max\left\{\frac{1}{x_0}, \frac{A_1}{x_0}\right\} = \frac{1}{x_0}$$

$$x_3 = \max\left\{\frac{1}{x_2}, \frac{A_2}{x_1}\right\} = \max\left\{x_0, \frac{1}{x_0}\right\} = x_0$$

and so $\{x_n\}_{n=-1}^{\infty}$ is periodic with period-3. ▯

LEMMA 7.39

Let $\{A_n\}_{n=-1}^{\infty}$ be a periodic sequence sequence of positive real numbers with prime period 3 such that

$$A_0 = A_2 = 1 \qquad \textit{and} \qquad A_1 > 1.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period-3.

PROOF It follows by Remark 7.4 that we may assume that

$$x_{-1} = \frac{1}{x_0} \quad , \quad x_1 = x_0 > \frac{1}{x_0} \quad \text{and} \quad x_0^2 > 1.$$

Since $A_1 > 1$ and $x_0^2 > 1$, there exists $k \in \{0, 1, \ldots\}$ such that either

$$A_1^{2k} \le x_0^2 < A_1^{2k+1} \qquad \text{or} \qquad A_1^{2k+1} \le x_0^2 < A_1^{2k+2}.$$

Case 1 Suppose that

$$A_1^{2k} \le x_0^2 < A_1^{2k+1}.$$

By direct computation when $k = 0$, and by induction when $k \ge 1$, it follows that for all $0 \le n \le k$, we have

$$x_{3n-1} = \frac{A_1^n}{x_0} \quad , \quad x_{3n} = \frac{x_0}{A_1^n} \quad \text{and} \quad x_{3n+1} = \frac{x_0}{A_1^n}.$$

In particular,

$$x_{3k-1} = \frac{A_1^k}{x_0} \quad , \quad x_{3k} = \frac{x_0}{A_1^k} \quad \text{and} \quad x_{3k+1} = \frac{x_0}{A_1^k}.$$

Hence

$$x_{3k+2} = \max\left\{\frac{1}{x_{3k+1}}, \frac{A_{3k+1}}{x_{3k}}\right\} = \max\left\{\frac{A_1^k}{x_0}, \frac{A_1^{k+1}}{x_0}\right\} = \frac{A_1^{k+1}}{x_0}$$

$$x_{3k+3} = \max\left\{\frac{1}{x_{3k+2}}, \frac{A_{3k+2}}{x_{3k+1}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1}}, \frac{A_1^k}{x_0}\right\} = \frac{A_1^k}{x_0}$$

$$x_{3k+4} = \max\left\{\frac{1}{x_{3k+3}}, \frac{A_{3k+3}}{x_{3k+2}}\right\} = \max\left\{\frac{x_0}{A_1^k}, \frac{x_0}{A_1^{k+1}}\right\} = \frac{x_0}{A_1^k}$$

$$x_{3k+5} = \max\left\{\frac{1}{x_{3k+4}}, \frac{A_{3k+4}}{x_{3k+3}}\right\} = \max\left\{\frac{A_1^k}{x_0}, \frac{x_0}{A_1^{k-1}}\right\} = \frac{x_0}{A_1^{k-1}}$$

$$x_{3k+6} = \max\left\{\frac{1}{x_{3k+5}}, \frac{A_{3k+5}}{x_{3k+4}}\right\} = \max\left\{\frac{A_1^{k-1}}{x_0}, \frac{A_1^k}{x_0}\right\} = \frac{A_1^k}{x_0}$$

$$x_{3k+7} = \max\left\{\frac{1}{x_{3k+6}}, \frac{A_{3k+6}}{x_{3k+5}}\right\} = \max\left\{\frac{x_0}{A_1^k}, \frac{A_1^{k-1}}{x_0}\right\} = \frac{x_0}{A_1^k}$$

and so $\{x_n\}_{n=3k+3}^{\infty}$ is periodic with period-3.

Case 2 Suppose

$$A_1^{2k+1} \le x_0^2 < A_1^{2k+2}.$$

By direct computation when $k = 0$, and by induction when $k \ge 1$, it follows that for all $0 \le n \le k$, we have

$$x_{3n-1} = \frac{A_1^n}{x_0} \quad , \quad x_{3n} = \frac{x_0}{A_1^n} \quad \text{and} \quad x_{3n+1} = \frac{x_0}{A_1^n}.$$

In particular,

$$x_{3k-1} = \frac{A_1^k}{x_0} \quad , \quad x_{3k} = \frac{x_0}{A_1^k} \quad \text{and} \quad x_{3k+1} = \frac{x_0}{A_1^k}.$$

Thus

$$x_{3k+2} = \max\left\{\frac{1}{x_{3k+1}}, \frac{A_{3k+1}}{x_{3k}}\right\} = \max\left\{\frac{A_1^k}{x_0}, \frac{A_1^{k+1}}{x_0}\right\} = \frac{A_1^{k+1}}{x_0}$$

$$x_{3k+3} = \max\left\{\frac{1}{x_{3k+2}}, \frac{A_{3k+2}}{x_{3k+1}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1}}, \frac{A_1^k}{x_0}\right\} = \frac{x_0}{A_1^{k+1}}$$

$$x_{3k+4} = \max\left\{\frac{1}{x_{3k+3}}, \frac{A_{3k+3}}{x_{3k+2}}\right\} = \max\left\{\frac{A_1^{k+1}}{x_0}, \frac{x_0}{A_1^{k+1}}\right\} = \frac{A_1^{k+1}}{x_0}$$

$$x_{3k+5} = \max\left\{\frac{1}{x_{3k+4}}, \frac{A_{3k+4}}{x_{3k+3}}\right\} = \max\left\{\frac{x_0}{A_1^{k+1}}, \frac{A_1^{k+2}}{x_0}\right\} = \frac{A_1^{k+2}}{x_0}$$

$$x_{3k+6} = \max\left\{\frac{1}{x_{3k+5}}, \frac{A_{3k+5}}{x_{3k+4}}\right\} = \max\left\{\frac{x_0}{A_1^{k+2}}, \frac{x_0}{A_1^{k+1}}\right\} = \frac{x_0}{A_1^{k+1}}$$

$$x_{3k+7} = \max\left\{\frac{1}{x_{3k+6}}, \frac{A_{3k+6}}{x_{3k+5}}\right\} = \max\left\{\frac{A_1^{k+1}}{x_0}, \frac{x_0}{A_1^{k+2}}\right\} = \frac{A_1^{k+1}}{x_0}$$

and so $\{x_n\}_{n=3k+3}^{\infty}$ is periodic with period-3. ∎

The proofs of the next three lemmas are similar to those of Lemmas 7.38 and 7.39 and will be omitted.

LEMMA 7.40

Let $\{A_n\}_{n=-1}^{\infty}$ be a periodic sequence sequence of positive real numbers with prime period 3 such that

$$A_0 = A_1 = 1.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period-3.

LEMMA 7.41

Let $\{A_n\}_{n=-1}^{\infty}$ be a periodic sequence sequence of positive real numbers with prime period 3 such that

$$A_1 = A_2 = 1 \qquad \textit{and} \qquad A_0 < 1.$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(7.3). Then $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic with period-3.

LEMMA 7.42
Let $\{A_n\}_{n=-1}^{\infty}$ *be a periodic sequence sequence of positive real numbers with prime period 3 such that*

$$A_1 = A_2 = 1 \qquad \text{and} \qquad A_0 > 1.$$

Let $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(7.3). Then* $\{x_n\}_{n=-1}^{\infty}$ *is eventually periodic with period-3.*

The following theorem, the proof of which follows directly from Lemmas 7.27-7.42, is the main result of this section.

THEOREM 7.7
Let $\{A_n\}_{n=-1}^{\infty}$ *be a periodic sequence sequence of positive real numbers with prime period 3 which satisfies (7.21). Then every positive solution of Eq.(7.3) is eventually periodic with period-3.*

7.4 Open Problems and Conjectures

Several open problems and conjectures on max equations with periodic coefficients were posed in Section 3.6. Hopefully the proofs which we presented in this chapter will improve the reader's ability to handle the simple-looking but quite sophisticated problems mentioned in Section 3.6.

The following result was established in [124].

THEOREM 7.8
Consider the difference equation

$$x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{B}{x_{n-1}}, \frac{C}{x_{n-2}}\right\} \quad , \qquad n = 0, 1, \ldots \tag{7.22}$$

where

$$A, B, C, x_{-2}, x_{-1}, x_0 \in (0, \infty).$$

Then every solution of Eq.(7.22) is eventually periodic with period $p \in \{2, 3, 4, 5, 6\}$. *More precisely, every solution is eventually periodic with period*

1. 2 *if* $A > \max\{B, C\}$;
2. 3 *if* $A = B > C$;
3. 4 *if either* $B > \max\{A, C\}$ *or* $A = C > B$ *or* $A = B = C$;
4. 5 *if* $B = C > A$;
5. 6 *if* $C > \max\{A, B\}$.

OPEN PROBLEM 7.1

Extend and generalize the above result for Eq.(7.22) to similar difference equations of order 4 and higher.

OPEN PROBLEM 7.2

Extend Theorem 7.8 to difference equations where the parameters are replaced by period-2 sequences of positive real numbers. Extend and generalize.

Chapter 8

EQUATIONS IN THE SPIRIT OF THE $(3x+1)$ CONJECTURE

8.1 Introduction

We are all familiar with the (3x+1) conjecture, also called the Collatz Problem, which was introduced in Section 3.4, and which states that **every solution of the difference equation**

$$x_{n+1} = \begin{cases} \dfrac{3x_n+1}{2} & \text{if } x_n \text{ is even} \\ \\ \dfrac{x_n}{2} & \text{if } x_n \text{ is odd} \end{cases} \quad , \quad n = 0, 1, \ldots$$

with

$$x_0 \in \{1, 2, \ldots\}$$

is eventually the two cycle

$$1, 2, 1, 2, \ldots .$$

In this chapter we investigate the periodic character and the boundedness nature of the solutions of the following sixteen piecewise linear difference equations which were mentioned in Section 3.5.

$$x_{n+1} = \begin{cases} \dfrac{\alpha x_n + \beta x_{n-1}}{2} & \text{if } \ x_n + x_{n-1} \ \text{ is even} \\ \\ \gamma x_n + \delta x_{n-1} & \text{if } \ x_n + x_{n-1} \ \text{ is odd} \end{cases} \quad , \quad n = 0, 1, \ldots \qquad (C)$$

where $x_{-1}, x_0 \in \mathbf{Z} = \{\ldots, -1, 0, 1, \ldots\}$ and $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$.

The case $\alpha = \beta = \gamma = 1$ and $\delta = -1$ was investigated in [26]. We denoted the above equation by (C) to indicate that this is a Collatz-type difference equation.

8.2 Preliminaries

The following lemmas will be useful in the sequel. The proofs of the first four lemmas are simple computations and are left to the reader.

LEMMA 8.1
The following statements are true:

1. *The trivial solution* $\bar{x} = 0$ *is an equilibrium solution of Eq.(C).*
2. *The only solution of Eq.(C) which is eventually equal to zero is the trivial solution* $\bar{x} = 0$.
3. *Any odd multiple of a solution of Eq.(C) is also a solution of Eq.(C). In particular, the negative of a solution of Eq.(C) is also a solution of Eq.(C).*

The following lemma reduces the number of equations to be studied from sixteen to eight.

LEMMA 8.2
(Duality Lemma)
Let $\{x_n\}_{n=-1}^{\infty}$ *be a sequence of integers. For each* $n \geq 0$, *set* $y_{2n-1} = x_{2n-1}$ *and* $y_{2n} = -x_{2n}$. *Then*

$\{x_n\}_{n=-1}^{\infty}$ *is a solution of Eq.(C)*

if and only if

$\{y_n\}_{n=-1}^{\infty}$ *is a solution of the equation*

$$y_{n+1} = \begin{cases} \dfrac{-\alpha y_n + \beta y_{n-1}}{2} & \text{if } \ y_n + y_{n-1} \ \text{ is even} \\ \\ -\gamma y_n + \delta y_{n-1} & \text{if } \ y_n + y_{n-1} \ \text{ is odd} \end{cases}, \quad n = 0, 1, \ldots \tag{C^*}$$

Given two integers m and n, we let $\mathrm{gcod}(m, n)$ denote the greatest common odd divisor of m and n.

LEMMA 8.3
Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(C). Then*

$$\mathrm{gcod}(x_{n-1}, x_n) = \mathrm{gcod}(x_{-1}, x_0) \ \text{ for all } \ n = 0, 1, \ldots .$$

An *even semi-cycle* of a solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(C) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all even integers, with $l \geq -1$ and $m \leq \infty$, such that

$$\text{either } l = -1, \text{ or } l > -1 \text{ and } x_{l-1} \text{ is odd}$$

and

$$\text{either } m = \infty, \text{ or } m < \infty \text{ and } x_{m+1} \text{ is odd.}$$

An *odd semi-cycle* of a solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(C) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all odd integers, with $l \geq -1$ and $m \leq \infty$, such that

$$\text{either } l = -1, \text{ or } l > -1 \text{ and } x_{l-1} \text{ is even}$$

and

$$\text{either } m = \infty, \text{ or } m < \infty \text{ and } x_{m+1} \text{ is even.}$$

LEMMA 8.4
Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(C). Then the following statements are true.

1. *Except for possibly the first semi-cycle, every even semi-cycle has exactly one term.*
2. *Except for possibly the first semi-cycle, every odd semi-cycle has at least two terms.*

LEMMA 8.5
Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(C). Suppose there exists $N \geq -1$ such that either x_n is even for all $n \geq N$, or x_n is odd for all $n \geq N$. Then the following statements are true.

1. *Suppose $\alpha = \beta = 1$. Then $x_n = x_N$ for all $n \geq N$.*
   (a) *If x_N is even, then $x_n = x_N$ for all $n \geq -1$.*
   (b) *Suppose $N \geq 0$, x_{N-1} is even, and x_N is odd. Then the following statements are true:*
      (i) *if $\gamma = 1$, then $x_{N-1} = 0$.*
      (ii) *if $\gamma = -1$, then $x_{N-1} = \dfrac{2x_N}{\delta}$.*
2. *Suppose $\beta = -1$. Then $x_n = 0$ for all $n \geq -1$.*

PROOF Clearly $x_n + x_{n-1}$ is even for all $n \geq N+1$, and so for $n \geq N+1$,

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{2}.$$

Case 1 Suppose $\alpha = \beta = 1$. Then

$$x_{n+1} = \frac{x_n + x_{n-1}}{2} \quad \text{for} \quad n \geq N+1$$

and so there exist constants c_1 and c_2 such that

$$x_n = c_1 \left(-\frac{1}{2}\right)^n + c_2 \quad \text{for} \quad n \geq N.$$

So, as $\{x_n\}_{n=-1}^{\infty}$ is an integer valued sequence, we see that $c_1 = 0$, and hence that

$$x_n = x_N \quad \text{for} \quad n \geq N.$$

1(a) Suppose x_N is even and $N \geq 0$. It suffices to show that x_{N-1} is also even. If x_{N-1} were odd, then

$$x_{N+1} = \gamma x_N + \delta x_{N-1}$$

which is impossible, since x_{N+1} and x_N are even, and x_{N-1} was assumed to be odd.

1(b) Suppose $N \geq 0$, x_{N-1} is even, and x_N is odd.

1(b)(i) Suppose $\gamma = 1$. Then

$$x_N = x_{N+1} = x_N + \delta x_{N-1}$$

and so we see that $x_{N-1} = 0$.

1(b)(ii) Suppose $\gamma = -1$. Then

$$x_N = x_{N+1} = -x_N + \delta x_{N-1}$$

and so we see that

$$x_{N-1} = \frac{2x_N}{\delta}.$$

Case 2. Suppose $\beta = -1$. Then

$$\lim_{n \to \infty} x_n = 0.$$

So as $\{x_n\}_{n=-1}^{\infty}$ is an integer valued sequence, it follows by Lemma 8.1 that $x_n = 0$ for all $n \geq 0$. ∎

LEMMA 8.6

Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(C). Suppose $|x_{-1}| \leq x_1$, $|x_0| \leq x_1$, and either $\alpha x_0 < x_1$ or $\beta x_{-1} < x_1$. Then $x_1 = \gamma x_0 + \delta x_{-1}$, x_1 is odd, $\gamma x_0 \geq 0$, and $\delta x_{-1} \geq 0$.

LEMMA 8.7

Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(C). Suppose $\alpha = 1$. Then there exists $n_0 \geq 0$ with $x_{n_0-1}x_{n_0} \geq 0$.

PROOF It clearly suffices to consider the case when $\{x_n\}_{n=-1}^{\infty}$ is not eventually constant. If $\gamma \cdot \delta = 1$, the result is true by Lemmas 8.1 and 8.6. So suppose $\gamma \cdot \delta = -1$.

Case 1. Suppose $\gamma = 1$ and $\delta = -1$.
For the sake of contradiction, suppose there exists no $n \geq 0$ such that

$$x_{n-1}x_n \geq 0.$$

Then for all $n \geq 0$, we have $x_{n-1}x_n < 0$.

It follows that $x_{n-1} + x_n$ is even for all $n \geq 0$.

Hence by Lemma 8.5, $\{x_n\}_{n=-1}^{\infty}$ is eventually constant.

This is a contradiction.

Case 2. Suppose $\gamma = -1$ and $\delta = 1$.
Then by Lemmas 8.1 and 8.4, it follows that we may assume that $x_{-1} > 0$ is odd, and that $x_0 < 0$ is even. So

$x_1 = -x_0 + x_{-1} > 0$ is odd , $x_2 = 2x_0 - x_{-1} < 0$ is odd, and $x_3 = \frac{1}{2}(x_2 + \beta x_1)$.

If $\beta = -1$, then

$$x_3 = \frac{1}{2}(x_2 - x_1) < 0,$$

while if $\beta = 1$, then

$$x_3 = \frac{1}{2}(x_2 + x_1) = \frac{1}{2}x_0 < 0.$$

∎

8.3 Boundedness of Solutions

The following lemma is due to Clark and Lewis. See [26].

LEMMA 8.8

Suppose $\alpha = \beta = \gamma = 1$ *and* $\delta = -1$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(C). Finally, suppose that* $x_{-1}x_0 \geq 0$. *Then* $|x_n| \leq \max\{|x_{-1}|, |x_0|\}$ *for all* $n \geq -1$.

PROOF By Statement 3 of Lemma 8.1, it suffices to consider the case where $x_{-1} \geq 0$ and $x_0 \geq 0$.

Let $M = \max\{x_{-1}, x_0\}$. It suffices to establish the following claim.

Claim: There exists $N \geq 1$ such that the following statements are true:

1. $|x_n| \leq M$ for all $-1 \leq n \leq N$;
2. $x_{N-1}x_N \geq 0$.

There are four cases to consider.

1. Suppose $x_{-1} + x_0$ is even.

   Then $0 \leq x_1 = \frac{1}{2}(x_{-1} + x_0) \leq M$, and so the claim is true with $N = 1$.

2. Suppose $x_{-1} < x_0$ and $x_{-1} + x_0$ is odd.

   Then $0 < x_1 = x_0 - x_{-1} \leq x_0 = M$, and so the claim is true with $N = 1$.

3. Suppose $x_{-1} > x_0$, x_{-1} is odd, and x_0 is even.

   Then $-M \leq x_1 = x_0 - x_{-1} < 0$ and $x_2 = -x_{-1} = -M$, and so the claim is true with $N = 2$.

4. Suppose $x_{-1} > x_0$, x_{-1} is even, and x_0 is odd.

   Then there exist $k, l \in \{0, 1, \ldots\}$ such that $x_{-1} = (2k+1) + (2l+1)$ and $x_0 = (2k+1)$. In this case, $M = x_{-1} = (2k+1) + (2l+1)$.

   Suppose $k - l = 0$. Then $x_{-1} = 2(2k+1)$ and $x_0 = (2k+1)$. Hence $x_1 = -(2k+1)$, $x_0 = 0$, and so the claim is true with $N = 1$.

   So it suffices to consider the case where $k - l \neq 0$.

   Note that $0 > x_1 = -(2l+1) > -M$, and $|x_2| = |k - l| < M$. So if $x_2 \leq 0$, then the claim is true with $N = 2$. So suppose $x_2 > 0$.

   Then $x_2 = k - l > 0$, and thus we see that $0 > x_1 = -(2l+1) > -\frac{1}{2}M$, and $0 < x_2 < \frac{1}{2}M$.

   Suppose now that there exists $n_0 \geq 1$ such that the following statements are true:

   (a) $(-1)^n x_n > 0$ for $1 \leq n \leq n_0$;
   (b) x_n is odd for $1 \leq n \leq n_0$;
   (c) $|x_n| < \frac{1}{2}M$ for $1 \leq n \leq n_0 + 1$.

For example, this is true for $n_0 = 1$. It follows by Statement 1(b)(i) of Lemma 8.5 that the proof of the claim follows from the fact that one of the following three statements is true:

(i) $(-1)^{n_0} x_{n_0+1} \geq 0$, $|x_{n_0+1}| < \frac{1}{2}M$, and $x_{n_0} x_{n_0+1} \geq 0$;

(ii) $(-1)^{n_0+1} x_{n_0+1} > 0$, x_{n_0+1} is even, $|x_{n_0+2}| = |x_{n_0+1} - x_{n_0}| \leq |x_{n_0+1}| + |x_{n_0}| < M$, and $x_{n_0+1} x_{n_0+2} > 0$;

(iii) $(-1)^{n_0+1} x_{n_0+1} > 0$, x_{n_0+1} is odd, and $|x_{n_0+2}| = \frac{1}{2}|x_{n_0+1} + x_{n_0}| < \frac{1}{2}M$. ∎

LEMMA 8.9
Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(C). Suppose* $\alpha = \gamma = 1$ *and* $\delta = -1$. *Then* $|x_n| \leq |x_{-1}| + |x_0|$ *for all* $n \geq -1$.

PROOF We shall prove by induction that for every $n \geq 0$,

$$|x_{n-1}|, |x_n|, |x_n - x_{n-1}| \in [0, |x_{-1}| + |x_0|].$$

The claim is clearly true for $n = 0$. So suppose $n \geq 0$ and that

$$|x_{n-1}|, |x_n|, |x_n - x_{n-1}| \in [0, |x_{-1}| + |x_0|].$$

We shall show that

$$|x_n|, |x_{n+1}|, |x_{n+1} - x_n| \in [0, |x_{-1}| + |x_0|].$$

Now $|x_n| \in [0, |x_{-1}|+|x_0|]$ by the inductive hypothesis. Recall that if x_n+x_{n-1} is even, then

$$|x_{n+1}| = \left| \frac{x_n + \beta x_{n-1}}{2} \right| \leq \frac{1}{2}|x_n| + \frac{1}{2}|x_{n-1}|,$$

while if $x_n + x_{n-1}$ is odd, then

$$|x_{n+1}| = |x_n - x_{n-1}|,$$

and so in either case,

$$|x_{n+1}| \leq |x_0| + |x_{-1}|.$$

It remains to show that $|x_{n+1} - x_n| \leq |x_0| + |x_{-1}|$. If $x_n + x_{n-1}$ is even, then

$$|x_{n+1} - x_n| = \left| \frac{x_n + \beta x_{n-1}}{2} - x_n \right| = \left| \frac{-x_n + \beta x_{n-1}}{2} \right|$$

$$\leq \frac{1}{2}|x_n| + \frac{1}{2}|x_{n-1}|,$$

while if $x_n + x_{n-1}$ is odd, then

$$|x_{n+1} - x_n| = |(x_n - x_{n-1}) - x_n| = |-x_{n-1}| = |x_{n-1}|,$$

and so in either case, $|x_{n+1} - x_n| \leq |x_0| + |x_{-1}|$. ∎

LEMMA 8.10

Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(C). Suppose $\alpha = \delta = 1$, and $\gamma = -1$. Then $|x_n| \leq |x_{-1}| + 2|x_0|$ for all $n \geq -1$.

PROOF For each $n \geq 0$, let $P(n)$ be the following proposition:

(i) if x_{n-1} is odd and x_n i s odd, then $|x_{n-1}|, |x_n| \leq |x_{-1}| + 2|x_0|$.

(ii) if x_{n-1} is even and x_n is odd, then $|x_{n-1}|, |x_n|, |-x_{n-1} + x_n| \leq |x_{-1}| + 2|x_0|$.

(iii) if x_{n-1} is odd and x_n is even, then $|x_{n-1}|, |x_n|, |-x_{n-1} + x_n|, |-x_{n-1} + 2x_n| \leq |x_{-1}| + 2|x_0|$.

We shall show by induction that $P(n)$ is true for all $n \geq 0$. Clearly $P(0)$ is true. Suppose n is a nonnegative integer and $P(k)$ is true for $k = 0, 1, \ldots n$. It suffices to show that $P(n+1)$ is true.

Case 1. Suppose x_{n-1} and x_n are both odd. Then $|x_{n-1}|, |x_n| \leq |x_{-1}| + 2|x_0|$, and

$$x_{n+1} = \frac{x_n + \beta x_{n-1}}{2}.$$

Note that

$$|x_{n+1}| = \left|\frac{x_n + \beta x_{n-1}}{2}\right| \leq \frac{1}{2}(|x_n| + |x_{n-1}|) \leq |x_{-1}| + 2|x_0|.$$

Case 1(a) Suppose that x_{n+1} is odd.
Then x_n and x_{n+1} are both odd.
Thus $P(n+1)$ is the statement $|x_n|, |x_{n+1}| \leq |x_{-1}| + |2|x_0|$, and nothing remains to be shown.

Case 1(b) Suppose x_{n+1} is even.
Then x_n is odd and x_{n+1} is even, and so $P(n+1)$ is the statement $|x_n|, |x_{n+1}|, |-x_n + x_{n+1}|, |-x_n + 2x_{n+1}| \leq |x_{-1}| + 2|x_0|$.
It remains to show $|-x_n + x_{n+1}|, |-x_n + 2x_{n+1}| \leq |x_{-1}| + 2|x_0|$.
Now,

$$|-x_n + x_{n+1}| = \left|-x_n + \left(\frac{x_n + \beta x_{n-1}}{2}\right)\right| \leq \frac{|x_n|}{2} + \frac{|x_{n-1}|}{2} \leq |x_{-1}| + 2|x_0|,$$

and

$$|-x_n + 2x_{n+1}| = |-x_n + (x_n + \beta x_{n-1})| = |x_{n-1}| \leq |x_{-1}| + 2|x_0|.$$

Case 2. Suppose x_{n-1} is even and x_n is odd. Then

$$|x_{n-1}|, |x_n|, |-x_{n-1}+x_n| \leq |x_{-1}| + 2|x_0|,$$

and $x_{n+1} = -x_n + x_{n-1}$. In particular, x_n is odd and x_{n+1} is odd. Thus

$P(n+1)$ is the statement $|x_n|, |x_{n+1}| \leq |x_{-1}|+2|x_0|$, which follows from the fact that

$$|x_{n+1}| = |-x_n + x_{n-1}| = |-x_{n-1}+x_n| \leq |x_{-1}| + 2|x_0|.$$

Case 3. Suppose x_{n-1} is odd and x_n is even.

Then

$$|x_{n-1}|, |x_n|, |-x_{n-1}+x_n|, |-x_{n-1}+2x_n| \leq |x_{-1}| + 2|x_0|,$$

and $x_{n+1} = -x_n + x_{n-1}$. In particular, x_n is even and x_{n+1} is odd. Hence $P(n+1)$ is the statement $|x_n|, |x_{n+1}|, |-x_n+x_{n+1}| \leq |x_{-1}|+2|x_0|$, and so it remains to show that $|-x_n+x_{n+1}| \leq |x_{-1}|+2|x_0|$.

But

$$|-x_n+x_{n+1}| = |-x_n - x_n + x_{n-1}| = |x_{n-1} - 2x_n| = |-x_{n-1}+2x_n|$$

$$\leq |x_{-1}| + 2|x_0|.$$

∎

LEMMA 8.11

Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(C) such that every even semi-cycle has length 1. Suppose* $\alpha = \beta = 1$*, and* $\gamma = \delta = -1$*. Then the following statements are true.*

1. *If* $n \geq -1$ *and* x_n *is even, then* $|x_n| \leq |x_{-1}| + |x_0|$.
2. *If* $n \geq 0$, x_{n-1} *is even, and* x_n *is odd, then* $|x_n| \leq 2\left(|x_{-1}| + |x_0|\right)$.
3. *If* $n \geq 0$, x_{n-1} *is odd, and* x_n *is odd, then* $|x_n| \leq |x_{-1}| + |x_0|$.

PROOF Clearly $|x_{-1}| \leq |x_{-1}| + |x_0|$ and $|x_0| \leq |x_{-1}| + |x_0|$. For each $n \geq 1$, let $P(n)$ be the following proposition:

(i) if $n \geq -1$ and x_n is even, then $|x_n| \leq |x_{-1}| + |x_0|$.

(ii) if $n \geq 0$, x_{n-1} is even, and x_n is odd, then $|x_n| \leq 2\left(|x_{-1}| + |x_0|\right)$ and $|x_{n-1}+x_n| \leq |x_{-1}| + |x_0|$.

(iii) if $n \geq 0$, x_{n-1} is odd, and x_n is odd, then $|x_n| \leq |x_{-1}| + |x_0|$ and $|x_{n-1}+x_n| \leq 2(|x_{-1}| + |x_0|)$.

The proof will be by induction on $n \geq 1$.

To show:$P(1)$ is true. Suppose x_1 is even. Then x_0 and x_{-1} are both odd, and $|x_1| = \frac{1}{2}|x_0 + x_{-1}| \leq |x_{-1}| + |x_0|$.

Suppose x_1 is odd.

(i) Suppose x_0 is even.

Then

x_{-1} is odd, and $x_1 = -x_0 - x_{-1}$.

Thus

$|x_1| \leq |x_{-1}|+|x_0| \leq 2(|x_{-1}|+|x_0|)$, and $|x_0+x_1| = |-x_{-1}| \leq |x_{-1}|+|x_0|$.

(ii) Suppose x_0 is odd.

Then

$|x_1| \leq |x_0 + x_{-1}| \leq |x_{-1}| + |x_0|$, and $|x_0 + x_1| \leq \frac{3}{2}|x_0| + |x_{-1}| \leq 2(|x_0| + |x_{-1}|)$.

Thus $P(1)$ is true.

Suppose $n \geq 1$ and $P(1), P(2), \ldots, P(n)$ are all true. To show: $P(n+1)$ is true.

(i) Suppose x_{n+1} is even.

We must show $|x_{n+1}| \leq |x_{-1}| + |x_0|$. We know x_n and x_{n-1} are both odd.

Thus by the inductive hypothesis,

$|x_n+x_{n-1}| \leq 2(|x_{-1}|+|x_0|)$, and so $|x_{n+1}| = \frac{1}{2}|x_n+x_{n-1}| \leq |x_{-1}|+|x_0|$.

(ii) Suppose x_{n+1} is odd and x_n is even.

We must show

$|x_{n+1}| \leq 2(|x_{-1}| + |x_0|)$ and $|x_n + x_{n+1}| \leq |x_{-1}| + |x_0|$.

Clearly x_{n-1} and x_{n-2} are both odd. Since x_{n-1} and x_{n-2} are both odd, we have

$|x_{n-1}| \leq |x_{-1}| + |x_0|$.

Since x_n is even, we also have that

$|x_n| \leq |x_{-1}| + |x_0|$.

Hence

$|x_{n+1}| = | - x_n - x_{n-1}| \leq 2(|x_{-1}| + |x_0|)$

and

$|x_{n+1} + x_n| = | - x_{n-1}| \leq |x_{-1}| + |x_0|$.

(iii) Suppose x_{n+1} is odd and x_n is odd.

We must show that

$|x_{n+1}| \leq |x_{-1}| + |x_0|$ and $|x_n + x_{n+1}| \leq 2(|x_{-1}| + |x_0|)$.

(a) Suppose x_{n-1} is even.

Then

$x_{n+1} = -x_n - x_{n-1}$.

We have $|x_{n-1} + x_n| \leq |x_{-1}| + |x_0|$

and so

$|x_{n+1}| = |x_n + x_{n-1}| \leq |x_{-1}| + |x_0|$.

Also,

$|x_n + x_{n+1}| = |-x_{n-1}| \leq |x_{-1}| + |x_0| \leq 2(|x_{-1}| + |x_0|)$.

(b) Suppose x_{n-1} is odd.

Then

$x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$.

(1) Suppose x_{n-2} is even.
Then
$|x_{n-1}| \leq 2(|x_{-1}| + |x_0|)$.
Also
$|x_n| \leq |x_{-1}| + |x_0|$ and $|x_{n-1} + x_n| \leq 2(|x_{-1}| + |x_0|)$.
Thus
$|x_{n+1}| = \frac{1}{2}|x_n + x_{n-1}| \leq |x_{-1}| + |x_0|$
and
$|x_n + x_{n+1}| = |x_n + \frac{1}{2}(x_n + x_{n-1})| \leq |x_n| + \frac{1}{2}|x_{n-1} + x_n| \leq 2(|x_{-1}| + |x_0|)$.

(2) Suppose x_{n-2} is odd.
Now
x_{n-2} and x_{n-1} are odd, and so $|x_{n-1}| \leq |x_{-1}| + |x_0|$.
Also,
x_{n-1} and x_n are odd, and so $|x_n| \leq |x_{-1}| + |x_0|$.
Thus
$|x_{n+1}| = \frac{1}{2}|x_n + x_{n-1}| \leq \frac{1}{2}(|x_n| + |x_{n-1}|) \leq |x_{-1}| + |x_0|$
and
$|x_n + x_{n+1}| = |x_n + \frac{1}{2}(x_n + x_{n-1})| \leq \frac{3}{2}|x_n| + \frac{1}{2}|x_{n-1}| \leq 2(|x_{-1}| + |x_0|)$.

□

8.4 The Equations

8.4.1 Eq.(1)

Eq.(1) Consider the ΔE

$$x_{n+1} = \begin{cases} \dfrac{x_n + x_{n-1}}{2} & \text{if } \; x_n + x_{n-1} \; \text{ is even} \\ \\ x_n + x_{n-1} & \text{if } \; x_n + x_{n-1} \; \text{ is odd} \end{cases} , \quad n = 0, 1, \ldots \tag{8.1}$$

where $x_{-1}, x_0 \in \mathbf{Z}$.

Clearly every integer is an equilibrium solution of Eq.(8.1).

THEOREM 8.1
The following statements are true:

1. *There exist solutions of Eq.(8.1) which are eventually constant, and there exist solutions of Eq.(8.1) which are not eventually constant.*

2. *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(8.1) which is not eventually constant. Then either $\lim_{n\to\infty} x_n = -\infty$ or $\lim_{n\to\infty} x_n = \infty$.*

PROOF

(i) Statement 1 follows from the fact that every integer is an equilibrium solution, and from Lemma 8.5. In particular, if $x_{-1} \cdot x_0 > 0$ and $x_{-1} \neq x_0$, then $\{x_n\}_{n=-1}^{\infty}$ is not eventually constant.

(ii) Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(8.1) which is not eventually constant. Clearly there exists $N \geq -1$ such that either $x_n > 0$ for all $n \geq N$ or $x_n < 0$ for all $n \geq N$. From the above and Lemma 8.5, it follows without loss of generality that we may assume that x_{-1} is an odd positive integer and that x_0 is an even positive integer. Then the solution $\{x_n\}_{n=-1}^{\infty}$ consists of an odd semi-cycle O_0 followed by an even semi-cycle E_1 followed by an odd semi-cycle O_1, etc. After O_0, every odd semi-cycle O_n has at least two terms. Every even semi-cycle E_n has exactly one term, which in an abuse of notation we shall also refer to as E_n. For each integer $n \geq 0$, the first two terms of O_n are each the sum of E_n with a positive odd integer, and hence are strictly greater than E_n, while every other term (if any) of O_n is the average of two odd integers each strictly greater than E_n, and hence is strictly greater than E_n. Since E_{n+1} is the average of two elements of O_n, we see that $E_n < E_{n+1}$. It follows that $\lim_{n\to\infty} x_n = \infty$. ∎

Eq.(1*) Consider the ΔE

$$x_{n+1} = \begin{cases} \dfrac{-x_n + x_{n-1}}{2} & \text{if } \; x_n + x_{n-1} \; \text{ is even} \\ \\ -x_n + x_{n-1} & \text{if } \; x_n + x_{n-1} \; \text{ is odd} \end{cases}, \quad n = 0, 1, \ldots \tag{1*}$$

where $x_{-1}, x_0 \in \mathbf{Z}$.

Clearly $\bar{x} = 0$ is the only equilibrium solution of Eq.(1*).

COROLLARY 8.1
The following statements are true:

1. *Eq.(1*) possesses bounded solutions, and Eq.(1*) possesses unbounded solutions.*
2. *Every bounded solution of Eq.(1*) is eventually a two-cycle* $(a, -a)$.
3. *Every unbounded solution of Eq.(1*) consists of two subsequences, one of which diverges to* ∞ *and the other of which diverges to* $-\infty$.

8.4.2 Eq.(2)

The following theorem is due to D. Clark and J.T. Lewis. A proof is given for the sake of completeness.

Eq.(2) Consider the ΔE

$$x_{n+1} = \begin{cases} \dfrac{x_n + x_{n-1}}{2} & \text{if } \; x_n + x_{n-1} \; \text{ is even} \\ \\ x_n - x_{n-1} & \text{if } \; x_n + x_{n-1} \; \text{ is odd} \end{cases}, \quad n = 0, 1, \ldots \tag{8.2}$$

where $x_{-1}, x_0 \in \mathbf{Z}$.

Clearly every real number $\bar{x} \in \mathbf{Z}$ is an equilibrium solution of Eq.(8.2).

THEOREM 8.2
Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(8.2). Suppose that* $\gcd(x_{-1}, x_0) = 1$ *and* $x_{-1} \neq x_0$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is either eventually the constant* 1, *the constant* -1, *or the six-cycle* $(-2, 1, 3, 2, -1, -3)$.

PROOF If $\{x_n\}_{n=-1}^{\infty}$ is eventually constant, it follows by Lemmas 8.3 and 8.5 that $\{x_n\}_{n=-1}^{\infty}$ is eventually either the constant 1 or the constant -1.

So suppose $\{x_n\}_{n=-1}^{\infty}$ is not eventually constant. By Lemmas 8.7 and 8.8, we know that $\{x_n\}_{n=-1}^{\infty}$ is a bounded, integer valued solution of Eq.(8.2), and hence is eventually periodic. So without loss of generality, we assume $\{x_n\}_{n=-1}^{\infty}$ is periodic. Hence there exists an integer $M > 0$ such that $-M \leq x_n \leq M$ for all $n \geq -1$. Because the negative of a solution of Eq.(8.2) is also a solution of Eq.(8.2), without loss of generality we may also assume that $x_0 < x_1 = M$.

Suppose $x_{-1} = 0$. Then as $\gcd(x_{-1}, x_0) = 1$, we see that $x_0 = 1$, and so $\{x_n\}_{n=-1}^{\infty} = (0, 1, 1, \ldots)$. This is impossible because we are assuming that $\{x_n\}_{n=-1}^{\infty}$ is not eventually constant.

Suppose $x_0 = 0$. Then $x_1 = 1$, and so it follows that

$$\{x_n\}_{n=-1}^{\infty} = (-1, 0, 1, 1, \ldots)$$

which also contradicts the assumption that $\{x_n\}_{n=-1}^{\infty}$ is not eventually constant.

Thus we see that $x_{-1} \cdot x_0 \neq 0$. Since $\{x_n\}_{n=-1}^{\infty}$ is not eventually constant and $x_0 < x_1 = M$, it follows easily by Lemma 8.6 that $M = x_1 = x_0 - x_{-1}$ is odd, $-M < x_{-1} < 0$, and $0 < x_0 < M$.

Claim: The following statements are true.

(i) x_{-1} is even, and $-M < x_{-1} < 0$;

(ii) x_0 is odd, and $0 < x_0 < M$;

(iii) $x_1 = M$ is odd;

(iv) x_2 is even, and $0 < x_2 < M$;

(v) x_3 is odd, and $-M < x_3 < 0$;

(vi) $x_4 = -M$;

We shall first show that x_{-1} is even and x_0 is odd.

For the sake of contradiction, suppose that x_{-1} is odd and x_0 is even. Then
$x_1 = x_0 - x_{-1}$ is odd, $x_2 = -x_{-1} > 0$ is odd, and so

$$x_3 = \frac{1}{2}(x_2 + x_1) = \frac{1}{2}x_0 - x_{-1} > 0.$$

It follows by Lemma 8.8 that $x_1 = x_0 - x_{-1} \le \max\{|x_2|, |x_3|\} = \frac{1}{2}x_0 - x_{-1}$. Thus
$x_0 = 0$. This contradicts the fact that $x_{-1}x_0 \neq 0$.

So we see that it is true that x_{-1} is even, and x_0 is odd.

Thus there exist integers $p \in \{1, 2, \ldots\}$ and $k, l \in \{0, 1, \ldots\}$ such that $x_{-1} = -2^p(2k+1)$ and $x_0 = (2l+1)$.

We claim $p = 1$.

For the sake of contradiction, suppose that $p > 1$.
Then
$x_1 = (2l+1) + 2^p(2k+1)$, $x_2 = \frac{1}{2}(x_1 + x_0) = (2l+1) + 2^{p-1}(2k+1)$, and $x_3 = (2l+1) + (2^{p-1} + 2^{p-2})(2k+1)$.
It follows by Lemma 8.6 that
$(2l+1)+2^p(2k+1) = x_1 \le \max\{|x_2|, |x_3|\} = (2l+1)+(2^{p-1}+2^{p-2})(2k+1)$, and so $(2k+1) = 0$. This is a contradiction, and so we see that it is true that $p = 1$.

Hence $x_{-1} = -2(2k+1) \in (-M, 0)$, $x_0 = (2l+1) \in (0, M)$, $x_1 = (2l+1) + 2(2k+1) = M$, $x_2 = (2l+1) + (2k+1) \in (0, M)$, $x_3 = -(2k+1) \in (-M, 0)$, $x_4 = -(2l+1) - 2(2k+1) = -M$, and so the claim is true.

It follows by the claim applied to $-x_4 = M$ that x_5 is even. Thus

$$\begin{array}{lll}
x_{-1} &= -2(2k+1) & \in (-M, 0), \\
x_0 &= (2l+1) & \in (0, M), \\
x_1 &= (2l+1) + 2(2k+1) & = M, \\
x_2 &= (2l+1) + (2k+1) & \in (0, M), \\
x_3 &= -(2k+1) & \in (-M, 0), \\
x_4 &= -(2l+1) - 2(2k+1) & = -M, \\
x_5 &= -\frac{1}{2}\left[(2l+1) + 3(2k+1)\right] & \in (-M, 0), \\
x_6 &= \frac{1}{2}\left[(2l+1) + (2k+1)\right] & \in (0, M), \\
x_7 &= (2l+1) + 2(2k+1) & = M, \\
x_8 &= \dfrac{\frac{3}{2}(2l+1) + \frac{5}{2}(2k+1)}{2} & = \dfrac{2x_1 + x_2}{4}.
\end{array}$$

It follows that given $i \in \{1, 2, \ldots\}$,

$$x_{6i+2} = \frac{x_2 + (2^{2i-1} + 2^{2i-3} + \cdots 2^3 + 2^1)x_1}{2^{2i}}.$$

It also follows that $x_n = x_1$ if and only if $n = 6i+1$ for some $i \in \{0, 1, \ldots\}$. In particular, there exists $i \in \{1, 2, \ldots\}$ such that $\{x_n\}_{n=-1}^{\infty}$ is periodic of prime period $P = 6i$. Thus

$$x_2 = x_{6i+2} = \frac{x_2 + (2^{2i-1} + 2^{2i-3} + \cdots + 2^3 + 2^1)x_1}{2^{2i}},$$

and so

$$\left(2^{2i} - 1\right) x_2 = \left(2^{2i-1} + 2^{2i-3} + \cdots + 2^3 + 2^1\right) x_1.$$

Thus

$$\left(2^{2i-1} + 2^{2i-2} + \cdots + 2^1 + 2^0\right) x_2 = \left(2^{2i-1} + 2^{2i-3} + \cdots + 2^3 + 2^1\right) x_1,$$

and hence

$$\left(2^{2i-1} + 2^{2i-3} + \cdots + 2^3 + 2^1\right) (x_1 - x_2) = \left(2^{2i-2} + 2^{2i-4} + \cdots + 2^2 + 2^0\right) x_2.$$

That is,

$$\left(2^{2i-1} + 2^{2i-3} + \cdots + 2^3 + 2^1\right) (2k+1) = \left(2^{2i-2} + 2^{2i-4} + \cdots + 2^2 + 2^0\right) (2l+2k+2).$$

Thus

$$\left(2^{i} + 2^{2i-2} + \cdots + 2^4 + 2^2\right) k + \left(2^{2i-1} + 2^{2i-3} + \cdots + 2^3 + 2^1\right)$$

$$= \left(2^{2i-1} + 2^{2i-3} + \cdots + 2^3 + 2^1\right) (l + k)$$

$$+ \left(2^{2i-1} + 2^{2i-3} + \cdots + 2^3 + 2^1\right).$$

It follows that

$$\left(2^{2i} - 2^{2i-1} + \cdots + 2^2 - 2^1\right) k = \left(2^{2i-1} + 2^{2i-3} + \cdots + 2^3 + 2^1\right) l.$$

Hence we see that

$$\left(2^{2i-1} + 2^{2i-3} + \cdots + 2^3 + 2^1\right) k = \left(2^{2i-1} + 2^{2i-3} + \cdots + 2^3 + 2^1\right) l,$$

from which it follows that $k = l$. So by the gcod condition, $k = l = 0$. Thus we have the cycle $(-2, 1, 3, 2, -1, -3)$. ∎

Eq.(2*) Consider the ΔE

$$x_{n+1} = \begin{cases} \dfrac{-x_n + x_{n-1}}{2} & \text{if} \quad x_n + x_{n-1} \quad \text{is even} \\ -x_n - x_{n-1} & \text{if} \quad x_n + x_{n-1} \quad \text{is odd} \end{cases}, \quad n = 0, 1, \ldots \tag{2*}$$

where $x_{-1}, x_0 \in \mathbf{Z}$.

Clearly $\bar{x} = 0$ is the only equilibrium solution of Eq.(2*).

COROLLARY 8.2

Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(2*), and suppose that* $\text{gcod}(x_{-1}, x_0) = 1$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is either eventually the two-cycle* $(1, -1)$, *the three-cycle* $(2, 1, -3)$, *or the three-cycle* $(-2, -1, 3)$.

8.4.3 Eq.(3)

Eq.(3) Consider the ΔE

$$x_{n+1} = \begin{cases} \dfrac{x_n + x_{n-1}}{2} & \text{if } \ x_n + x_{n-1} \ \text{ is even} \\ \\ -x_n + x_{n-1} & \text{if } \ x_n + x_{n-1} \ \text{ is odd} \end{cases}, \quad n = 0, 1, \ldots \tag{8.3}$$

where $x_{-1}, x_0 \in \mathbf{Z}$.

Clearly every real number $\bar{x} \in \mathbf{Z}$ is an equilibrium solution of Eq.(8.3).

THEOREM 8.3

Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(8.3). Suppose* $\text{gcod}(x_{-1}, x_0) = 1$ *and* $x_{-1} \neq x_0$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is either eventually the constant* 1, *the constant* -1, *the four-cycle* $(2, -1, 3, 1)$, *the four-cycle* $(-2, 1, -3, -1)$, *or the six-cycle* $(1, 0, 1, -1, 0, -1)$.

PROOF If $\{x_n\}_{n=-1}^{\infty}$ is eventually constant, it follows by Lemmas 8.3 and 8.5 that $\{x_n\}_{n=-1}^{\infty}$ is either eventually the constant 1 or the constant -1.

So suppose $\{x_n\}_{n=-1}^{\infty}$ is not eventually constant. By Lemma 8.10, we know that $\{x_n\}_{n=-1}^{\infty}$ is a bounded, integer valued solution of Eq.(8.3), and hence is eventually periodic. So without loss of generality, we assume $\{x_n\}_{n=-1}^{\infty}$ is periodic. It follows that there exists an integer $M > 0$ such that $-M \leq x_n \leq M$ for all $n \geq -1$. Because the negative of a solution of Eq.(8.3) is also a solution of Eq.(8.3), without loss of generality we may also assume that $x_0 < x_1 = M$.

Suppose $x_{-1} = 0$. Then as $\text{gcod}(x_{-1}, x_0) = 1$, we see that $x_0 = -1$, and that $\{x_n\}_{n=-1}^{\infty} = (0, -1, 1, 0, 1, -1, 0, -1, 1, 0, 1, -1, \ldots)$.

Suppose $x_0 = 0$. Then $x_1 = 1$, and so it follows that

$$\{x_n\}_{n=-1}^{\infty} = (1, 0, 1, -1, 0, -1, 1, 0, 1, -1, 0, -1, \ldots).$$

Suppose $x_{-1} \cdot x_0 \neq 0$.

It follows by Lemma 8.6 that $x_{-1} > 0$, $x_0 < 0$, and that $M = x_1 = -x_0 + x_{-1}$ is odd. We claim that x_{-1} is even and x_0 is odd. For the sake of contradiction, suppose there exist $p \in \{1, 2, \ldots\}$ and $k, l \in \{0, 1, \ldots\}$ such that $x_{-1} = 2l + 1$ and $x_0 = -2^p(2k+1)$. Then $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$x_{-1} = 2l + 1$

$x_0 = -2^p(2k+1)$

$M = x_1 = 2^p(2k+1) + (2l+1)$

$x_2 = -2^{p+1}(2k+1) - (2l+1)$

This is impossible, because $M < 2^{p+1}(2k+1) + (2l+1)$.

Hence there do exist $p \in \{1, 2, \ldots\}$ and $k, l \in \{0, 1, \ldots\}$ such that $x_{-1} = 2^p(2k+1)$ and $x_0 = -(2l+1)$.

We claim $p = 1$. For the sake of contradiction, suppose $p > 1$. Then $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$x_{-1} = 2^p(2k+1)$

$x_0 = -(2l+1)$

$M = x_1 = (2l+1) + 2^p(2k+1)$

$x_2 = 2^{p-1}(2k+1)$

$x_3 = (2l+1) + (2^p - 2^{p-1})(2k+1) = (2l+1) + 2^{p-1}(2k+1)$

$x_4 = -(2l+1)$

The next term is $x_5 = 2^{p-2}(2k+1)$, and so by Lemma 8.10

$$M = (2l+1)+2^p(2k+1) \le (2l+1)+2\cdot 2^{p-2}(2k+1) = (2l+1)+2^{p-1}(2k+1),$$

which is impossible. Thus it is true that $p = 1$. So $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$x_{-1} = 2(2k+1)$

$x_0 = -(2l+1)$

$M = x_1 = (2l+1) + 2(2k+1)$

$x_2 = (2k+1)$

The next term is $x_3 = l + 3k + 2 = (l+k+1) + (2k+1)$. We claim $l+k+1$ is odd. For the sake of contradiction, suppose that $l+k+1$ is even. Then $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$x_{-1} = 2(2k+1)$

$x_0 = -(2l+1)$

$M = x_1 = (2l+1) + 2(2k+1)$

$x_2 = (2k+1)$

$x_3 = (l+k+1) + (2k+1)$

The next term is $x_4 = \frac{1}{2}(l+k+1) + (2k+1)$. If x_4 were even, then we would have $x_5 = \frac{1}{2}(l+k+1)$, and so

$$M = (2l+1)+2(2k+1) \le \left[\frac{1}{2}(l+k+1) + (2k+1)\right] + 2\cdot\frac{1}{2}(l+k+1) = \frac{3l}{2}+\frac{7k}{2}+\frac{5}{2},$$

which is impossible. Thus it must be that x_4 is odd, and hence that

$$x_5 = \frac{3(l+k+1)}{4} + (2k+1).$$

If x_5 were even, then we would have $x_6 = -\dfrac{l+k+1}{4}$, and so

$$M = (2l+1) + 2(2k+1) \le \left[\frac{3(l+k+1)}{4} + (2k+1)\right] + 2\cdot\frac{l+k+1}{4},$$

which is impossible. Thus x_5 is odd.

Suppose now that there exists $N \geq 1$ such that the following statements are true for all $0 \leq n \leq N$:

(i) $x_{4+n} = \left[\frac{2}{3} - \frac{1}{6}\left(-\frac{1}{2}\right)^n\right](l+k+1) + (2k+1)$

(ii) x_{4+n} is odd.

Then we claim that

$$x_{4+(N+1)} = \left[\frac{2}{3} - \frac{1}{6}\left(-\frac{1}{2}\right)^{N+1}\right](l+k+1) + (2k+1)$$

and that $x_{4+(N+1)}$ is odd.

Proof of the claim: Note that

$$\begin{aligned}
x_{4+(N+1)} &= \frac{x_{4+N} + x_{4+N-1}}{2} \\
&= \frac{1}{2}\left[\frac{2}{3} - \frac{1}{6}\left(-\frac{1}{2}\right)^N + \frac{2}{3} - \frac{1}{6}\left(-\frac{1}{2}\right)^{N-1}\right](l+k+1) + (2k+1) \\
&= \left[\frac{2}{3} - \frac{1}{6}(-1)^{N-1}\left(-\left(\frac{1}{2}\right)^{N+1} + \left(\frac{1}{2}\right)^N\right)\right](l+k+1) + (2k+1) \\
&= \left[\frac{2}{3} - \frac{1}{6}(-1)^{N-1}\left(\frac{1}{2}\right)^{N+1}\right](l+k+1) + (2k+1) \\
&= \left[\frac{2}{3} - \frac{1}{6}\left(-\frac{1}{2}\right)^{N+1}\right](l+k+1) + (2k+1)
\end{aligned}$$

and so we need only show that $x_{4+(N+1)}$ is odd. So for the sake of contradiction, suppose that $x_{4+(N+1)}$ is even.

Then

$$
\begin{aligned}
x_{4+(N+2)} &= -x_{4+(N+1)} + x_{4N} \\
&= \left[\frac{1}{6}\left(-\frac{1}{2}\right)^{N+1} - \frac{1}{6}\left(-\frac{1}{2}\right)^{N}\right](l+k+1) \\
&= (-1)^{N+1}\cdot\frac{1}{6}\left[\left(\frac{1}{2}\right)^{N+1} + \left(\frac{1}{2}\right)^{N}\right](l+k+1) \\
&= (-1)^{N+1}\cdot\frac{1}{6}\left[\left(\frac{1}{2}\right)^{N+1} + 2\cdot\left(\frac{1}{2}\right)^{N+1}\right](l+k+1) \\
&= (-1)^{N+1}\cdot\left(\frac{1}{2}\right)^{N+2}(l+k+1).
\end{aligned}
$$

Then as $N \geq 1$,

$$
\begin{aligned}
M &= (2l+1) + 2(2k+1) \\
&\leq |x_{4+(N+1)}| + 2|x_{4+(N+2)}| \\
&= \left(\left[\frac{2}{3} - \frac{1}{6}\left(-\frac{1}{2}\right)^{N+1}\right](l+k+1) + (2k+1)\right) + \left(\frac{1}{2}\right)^{N+1}(l+k+1) \\
&= \left[\frac{2}{3} + \left(1 + (-1)^{N}\cdot\frac{1}{6}\right)\left(\frac{1}{2}\right)^{N+1}\right](l+k+1) + (2k+1) \\
&\leq \left[\frac{2}{3} + \left(\frac{7}{6}\right)\left(\frac{1}{2}\right)^{2}\right](l+k+1) + (2k+1) \quad = \quad \frac{23}{24}(l+k+1) + (2k+1) \\
&< (l+k+1) + (2k+1).
\end{aligned}
$$

This is impossible, and so the claim is true.

But this means that $\{x_n\}_{n=-1}^{\infty}$ is odd for $n \geq 0$, and so by Lemma 8.5, $\{x_n\}_{n=-1}^{\infty}$ is constant for $n \geq 0$. This contradicts the assumption that $\{x_n\}_{n=-1}^{\infty}$ is not eventually constant.

Thus it is the case that $l+k+1$ is odd. So $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$x_{-1} = 2(2k+1)$

$x_0 = -(2l+1)$

$M = x_1 = (2l+1) + 2(2k+1) = 2(l-k) + 3(2k+1)$

$x_2 = (2k+1)$

$x_3 = (l-k) + 2(2k+1)$

$x_4 = -(l-k) - (2k+1)$

$M = x_5 = 2(l-k) + 3(2k+1)$

$x_6 = \frac{1}{2}(l-k) + (2k+1)$

---

and so in particular, we see that for all $n \geq 0$

---

x_{4n-1} is even

---

x_{4n} is odd

$M = x_{4n+1} = 2(l-k) + 3(2k+1)$

x_{4n+2} is odd

.

It follows easily by induction that for $n \geq 1$,

$$x_{4n-1} = \frac{\frac{1}{3}(4^n - 1)}{4^{n-1}}(l-k) + 2(2k+1)$$

$$x_{4n} = -\frac{\frac{1}{6}(4^n + 2)}{4^{n-1}}(l-k) - (2k+1)$$

$$M = x_{4n+1} = 2(l-k) + 3(2k+1)$$

$$x_{4n+2} = \frac{\frac{1}{6}(4^n - 1)}{4^{n-1}}(l-k) + (2k+1)$$

So in particular, for $n \geq 1$,

$$x_{4n-1} = \frac{\frac{1}{3}(4^n - 1)}{4^{n-1}}(l-k) + 2(2k+1) = \frac{4}{3}(l-k) - \frac{1}{3 \cdot 4^{n-1}}(l-k) + (2k+1).$$

Hence

$$\lim_{n \to \infty} x_{4n-1} = \frac{4}{3}(l-k) + (2k+1)$$

is an integer, and thus for all $n \geq 1$,

$$\frac{1}{3 \cdot 4^{n-1}}(l-k)$$

is also an integer, from which we see that $l - k = 0$. So as $\gcd(x_{-1}, x_0) = 1$, it follows that $\{x_n\}_{n=-1}^{\infty} = (2, -1, 3, 1, 2, -1, 3, 1, \ldots)$. ∎

Eq.(3*) Consider the ΔE

$$x_{n+1} = \begin{cases} \dfrac{-x_n + x_{n-1}}{2} & \text{if } \; x_n + x_{n-1} \; \text{ is even} \\ x_n + x_{n-1} & \text{if } \; x_n + x_{n-1} \; \text{ is odd} \end{cases}, \quad n = 0, 1, \ldots \tag{3*}$$

where $x_{-1}, x_0 \in \mathbf{Z}$.
Clearly $\bar{x} = 0$ is the only equilibrium solution of Eq.(3*).

COROLLARY 8.3
Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.*(3*)*, and suppose that* $\gcd(x_{-1}, x_0) = 1$*. Then* $\{x_n\}_{n=-1}^{\infty}$ *is either eventually the two-cycle* $(1, -1)$*, the three-cycle* $(1, 0, 1)$*, the three-cycle* $(-1, 0, -1)$*, the four cycle* $(2, 1, 3, -1)$*, or the four-cycle* $(-2, -1, -3, 1)$*.*

REMARK 8.1 Near the end of the proof of Theorem 8.3, we had established that for $n \geq 3$, $\{x_n\}_{n=-1}^{\infty}$ was given (arranged in even and odd semi-cycles) as follows:

$$x_3 = (l-k) + 2(2k+1)$$

$$x_4 = -(l-k) - (2k+1)$$

$$x_5 = 2(l-k) + 3(2k+1)$$

$$x_6 = \frac{1}{2}(l-k) + (2k+1)$$

$$x_7 = \frac{5}{4}(l-k) + (2k+1)$$

$$x_8 = -\frac{3}{4}(l-k) - (2k+1)$$

$$x_9 = 2(l-k) + 3(2k+1)$$

$$x_{10} = \frac{5}{8}(l-k) + (2k+1)$$

$$x_{11} = \frac{21}{16}(l-k) + (2k+1)$$

$$x_{12} = -\frac{11}{16}(l-k) - (2k+1)$$

$$x_{13} = 2(l-k) + 3(2k+1)$$

$$x_{14} = \frac{21}{32}(l-k) + (2k+1)$$

$$x_{15} = \frac{85}{64}(l-k) + (2k+1)$$

$$x_{16} = -\frac{43}{64}(l-k) - (2k+1)$$

$$x_{17} = 2(l-k) + 3(2k+1)$$

$$x_{18} = \frac{85}{128}(l-k) + (2k+1)$$

Thus a reasonable guess for a closed form solution for $\{x_n\}_{n=-1}^{\infty}$ is (for $n \geq 1$)

$$x_{4n-1} = \frac{a_n}{4^{n-1}}(l-k) + 2(2k+1)$$

$$x_{4n} = -\frac{b_n}{4^{n-1}}(l-k) - (2k+1)$$

$$x_{4n+1} = 2(l-k) + 3(2k+1)$$

$$x_{4n+2} = -\frac{a_n}{2 \cdot 4^{n-1}}(l-k) - (2k+1)$$

where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are solutions, respectively, of the IVPs

$$\begin{cases} a_{n+1} = 4a_n + 1 \\ a_1 = 1 \end{cases}, \quad n = 0, 1, \ldots$$

$$\begin{cases} b_{n+1} = 4b_n - 1 \\ b_1 = 1 \end{cases}, \quad n = 0, 1, \ldots$$

It follows that the corresponding solutions are (for $n \geq 1$)

$$a_n = \frac{1}{3}\left(4^n - 1\right), \ b_n = \frac{1}{6}\left(4^n + 2\right)$$

and so our trial for a closed form solution for $\{x_n\}_{n=-1}^{\infty}$ is (for $n \geq 1$)

$$x_{4n-1} = \frac{\frac{1}{3}\left(4^n - 1\right)}{4^{n-1}}(l-k) + 2(2k+1)$$

$$x_{4n} = -\frac{\frac{1}{6}\left(4^n + 2\right)}{4^{n-1}}(l-k) - (2k+1)$$

$$x_{4n+1} = 2(l-k) + 3(2k+1)$$

$$x_{4n+2} = -\frac{\frac{1}{3}\left(4^n - 1\right)}{2 \cdot 4^{n-1}}(l-k) - (2k+1) = -\frac{\frac{1}{6}\left(4^n - 1\right)}{4^{n-1}}(l-k) - (2k+1) \quad \Box$$

8.4.4 Eq.(4)

Eq.(4) Consider the ΔE

$$x_{n+1} = \begin{cases} \dfrac{x_n + x_{n-1}}{2} & \text{if } \ x_n + x_{n-1} \ \text{ is even} \\ -x_n - x_{n-1} & \text{if } \ x_n + x_{n-1} \ \text{ is odd} \end{cases}, \quad n = 0, 1, \ldots \tag{8.4}$$

where $x_{-1}, x_0 \in \mathbf{Z}$.

Clearly every real number $\bar{x} \in \mathbf{Z}$ is an equilibrium solution of Eq.(8.4).

THEOREM 8.4

Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(8.4). Suppose* $\gcd(x_{-1}, x_0) = 1$ *and* $x_{-1} \neq x_0$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is either eventually the constant* 1, *the constant* -1, *the three-cycle* $(-1, 0, 1)$, *or the three-cycle* $(1, 0, -1)$.

PROOF If $\{x_n\}_{n=-1}^{\infty}$ is eventually constant, it follows by Lemmas 8.3 and 8.5 that $\{x_n\}_{n=-1}^{\infty}$ is either eventually the constant 1 or the constant -1.

So suppose $\{x_n\}_{n=-1}^{\infty}$ is not eventually constant. By Lemma 8.11, we know that $\{x_n\}_{n=-1}^{\infty}$ is a bounded, integer valued sequence, and hence is eventually periodic. So without loss of generality, we assume $\{x_n\}_{n=-1}^{\infty}$ is periodic. Hence there exists an integer $M > 0$ such that $-M \leq x_n \leq M$ for all $n \geq -1$. Without loss of generality, we may assume that $x_0 < x_1 = M$. It follows by Lemma 8.6 that $x_{-1} \leq 0$, $x_0 \leq 0$, and that $M = x_1 = -x_{-1} - x_0 > 0$ is odd.

Case 1 Suppose x_{-1} is even and x_0 is odd. We claim that $x_{-1} = 0$. For the sake of contradiction, suppose $x_{-1} \neq 0$.

Then there exist $k \in \{1, 2, \ldots\}$ and $m, l \in \{0, 1, \ldots\}$ such that

$$x_{-1} = -2^k(2m+1) \quad \text{and} \quad x_0 = -(2l+1).$$

It follows that $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$-2^k(2m+1)$

$-(2l+1)$

$M = (2l+1) + 2^k(2m+1).$

The next term is $2^{k-1}(2m+1)$. Suppose $k \geq 2$. Then $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$x_{-1} = -2^k(2m+1)$

$x_0 = -(2l+1)$

$M = x_1 = (2l+1) + 2^k(2m+1)$

$x_2 = 2^{k-1}(2m+1)$

$x_3 = -(2l+1) - (2^k + 2^{k-1})(2m+1)$

But then $|x_3| = (2l+1) + (2^k + 2^{k-1})(2m+1) > M$. This is impossible. Hence $k = 1$. So $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$-2(2m+1)$

$-(2l+1)$

$M = (2l+1) + 2(2m+1)$

$(2m+1)$

---

and so

$$x_3 = \frac{3(2m+1) + (2l+1)}{2} = 3m + l + 2 = (2m+1) + (m+l+1).$$

Now there exist integers $p, q \geq 0$ such that $(m+l+1) = 2^p(2q+1)$.

We claim $p = 0$. For the sake of contradiction, suppose $p > 0$.

We first suppose that p is even. Then there exists an integer $r > 0$ such that $p = 2r$. So $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$-2(2m+1)$

$-(2l+1)$

$M=(2l+1)+2(2m+1)$

$(2m+1)$

$(2m+1)+2^{2r}(2q+1)$

$(2m+1)+2^{2r-1}(2q+1)$

$(2m+1)+(2^{2r-1}+2^{2r-2})(2q+1)$

$(2m+1)+(2^{2r-1}+2^{2r-3})(2q+1)$

$(2m+1)+(2^{2r-1}+2^{2r-3}+2^{2r-4})(2q+1)$

$(2m+1)+(2^{2r-1}+2^{2r-3}+2^{2r-5})(2q+1)$

$\vdots$

$(2m+1)+(2^{2r-1}+2^{2r-3}+\cdots+2^{3}+2^{2})(2q+1)$

$(2m+1)+(2^{2r-1}+2^{2r-3}+\cdots+2^{3}+2^{1})(2q+1)$

$(2m+1)+(2^{2r-1}+2^{2r-3}+\cdots+2^{3}+2^{1}+2^{0})(2q+1)$

$-2(2m+1)-(2^{2r}+2^{2r-2}+\cdots+2^{4}+2^{2}+2^{0})(2q+1)$

$(2m+1)+(2^{2r}-2^{2r-1}+2^{2r-2}-2^{2r-3}+\cdots+2^{2}-2^{1})(2q+1)$

The next term is

$$-\tfrac{1}{2}\left[(2m+1)+\left[(2^{2r-1}+2^{2r-3}+\cdots+2^{1})+2^{0}\right](2q+1)\right]$$
$$=-\left[m+(2^{2r-2}+2^{2r-4}+\cdots 2^{2}+2^{1})(2q+1)-q\,\right].$$

But

$$\begin{aligned}
&|(2m+1)+(2^{2r}-2^{2r-1}+2^{2r-2}-2^{2r-3}+\cdots+2^{2}-2^{1})(2q+1)| \\
&\qquad +|-\left[m+(2^{2r-2}+2^{2r-4}+\cdots 2^{2}+2^{1})(2q+1)-q\right]| \\
&=(2m+1)+m+[2^{2r}+(-2^{2r-1}+2^{2r-2}+2^{2r-2})+(-2^{2r-3}+2^{2r-4}+2^{2r-4}) \\
&\qquad +\cdots+(-2^{3}+2^{2}+2^{2})+(-2^{1}+2^{1})](2q+1)-q \\
&\le (2m+1)+m+2^{2r}(2q+1)=(2m+1)+m+(m+l+1)=2(2m+1)+l \\
&< 2(2m+1)+(2l+1)=M.
\end{aligned}$$

This is impossible by Statement 3 of Lemma 8.11, since $x_1 = M$ is odd, and by assumption, x_0 is also odd.

We next suppose that p is odd. Then there exists an integer $r \geq 0$ such that $p = (2r+1)$. So $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$-2(2m+1)$

$-(2l+1)$

$M = 2(2m+1) + (2l+1)$

$(2m+1)$

$(2m+1) + 2^{2r+1}(2q+1)$

$(2m+1) + 2^{2r}(2q+1)$

$(2m+1) + (2^{2r} + 2^{2r-1})(2q+1)$

$(2m+1) + (2^{2r} + 2^{2r-2})(2q+1)$

$(2m+1) + (2^{2r} + 2^{2r-2} + 2^{2r-3})(2q+1)$

$(2m+1) + (2^{2r} + 2^{2r-2} + 2^{2r-4})(2q+1)$

$\vdots$

$(2m+1) + (2^{2r} + 2^{2r-2} + \cdots + 2^4 + 2^3)(2q+1)$

$(2m+1) + (2^{2r} + 2^{2r-2} + \cdots + 2^4 + 2^2)(2q+1)$

$(2m+1) + (2^{2r} + 2^{2r-2} + \cdots + 2^4 + 2^2 + 2^1)(2q+1)$

$(2m+1) + (2^{2r} + 2^{2r-2} + \cdots + 2^4 + 2^2 + 2^0)(2q+1)$

$-2(2m+1) - (2^{2r+1} + 2^{2r-1} + \cdots + 2^5 + 2^3 + 2^1 + 2^0)(2q+1)$

$(2m+1) + (2^{2r+1} - 2^{2r} + 2^{2r-1} - 2^{2r-2} + \cdots + 2^5 - 2^4 + 2^3 - 2^2 + 2^1)(2q+1)$

It follows that $x_{\text{next}} = \frac{1}{2}[-(2m+1)+\left[(-2^{2r} - 2^{2r-2} - \cdots - 2^4 - 2^2) - 2^0)\right](2q+1)]$. It is not clear whether x_{next} is even or odd, but in either event,

$$\begin{aligned}|x_{1+\text{next}}| &\leq (2m+1) + (2^{2r+1} - 2^{2r} + 2^{2r-1} - 2^{2r-2} + \cdots + 2^3 - 2^2 + 2^1)(2q+1) \\ &\quad -\tfrac{1}{2}(2m+1) - (2^{2r-1} + 2^{2r-3} + \cdots + 2^1 + \tfrac{1}{2})(2q+1) \\ &= \tfrac{1}{2}(2m+1) + (2^{2r+1} - 2^{2r} - 2^{2r-2} - \cdots - 2^2 - \tfrac{1}{2})(2q+1),\end{aligned}$$

and so

$$\begin{aligned}|x_{\text{next}}| + |x_{1+\text{next}}| &\leq \tfrac{1}{2}(2m+1) + (2^{2r-1} + 2^{2r-3} + \cdots + 2^1 + \tfrac{1}{2})(2q+1) \\ &\quad +\tfrac{1}{2}(2m+1) + (2^{2r+1} - 2^{2r} - 2^{2r-2} - \cdots - 2^2 - \tfrac{1}{2})(2q+1) \\ &\leq (2m+1) + 2^{2r+1}(2q+1) = (2m+1) + (m+l+1) \\ &< 2(2m+1) + (2l+1) = M.\end{aligned}$$

This is impossible by Statement 3 of Lemma 8.11, since $x_1 = M$ is odd, and by assumption, x_0 is also odd.

Thus we see that $p = 0$, and so $m + l + 1 = (2q + 1)$. In particular, x_3 is even, and so $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$-2(2m+1)$

$-(2l+1)$

$M = 2(2m+1) + (2l+1)$

$(2m+1)$

$(2m+1) + (m+l+1)$

$-2(2m+1) - (m+l+1)$

$(2m+1)$

Thus we see that

$$x_{\text{next}} = \frac{1}{2}[-(2m+1) - (m+l+1)] = -\frac{1}{2}[(2m+1) + (2q+1)] = -(m+q+1)$$

and so

$$\begin{aligned}|x_{-1+\text{next}}| + |x_{\text{next}}| &= (2m+1) + (m+q+1) = 3m+q+2 \\ &= 3m + \left(\frac{m+l}{2}\right) + 2 = \frac{7m}{2} + \frac{l}{2} + 2 \\ &< 4m + 2l + 3 = 2(2m+1) + (2l+1) = M.\end{aligned}$$

This is impossible by Lemma 8.11 (iii), since $x_1 = M$ is odd, and by assumption, x_0 is also odd.

Hence if x_{-1} is even and x_0 is odd, we must have $x_{-1} = 0$, and so the solution $\{x_n\}_{n=-1}^{\infty}$ is the three cycle $(0, -1, 1)$.

Case 2. Suppose x_{-1} is odd and x_0 is even. We claim that $x_0 = 0$. For the sake of contradiction, suppose that $x_0 \neq 0$. Then there exist integers $k \in \{1, 2, \ldots\}$ and $m, l \in \{0, 1, \ldots\}$ such that $x_{-1} = -(2l+1)$ and $x_0 = -2^k(2m+1)$. It follows that $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$-(2l+1)$

$-2^k(2m+1)$

$M = (2l+1) + 2^k(2m+1)$

$-(2l+1)$

$2^{k-1}(2m+1)$

$(2l+1) - 2^{k-1}(2m+1)$

$-(2l+1)$

$-2^{k-2}(2m+1)$

$(2l+1) + 2^{k-2}(2m+1)$

$-(2l+1)$
$2^{k-3}(2m+1)$
$\vdots$ $(-1)^{k-1}2^2(2m+1)$
$(2l+1)-(-1)^{k-1}2^2(2m+1)$ $-(2l+1)$
$(-1)^k 2(2m+1)$
$(2l+1)-(-1)^k 2(2m+1)$ $-(2l+1)$ $(-1)^{k+1}(2m+1)$

Recall by Statement 2 of Lemma 8.11 that we must have

$$M \leq 2\left|-(2l+1)\right| + 2\left|(-1)^{k+1}(2m+1)\right|.$$

Note that

$$M = (2l+1) + 2^k(2m+1) \leq 2(2l+1) + 2(2m+1)$$

if and only if

$$2l + 1 + 2^{k+1}m + 2^k \leq 4l + 2 + 4m + 2$$

if and only if

$$(2^{k+1} - 4)m \leq 2l - (2^k - 3)$$

and so we see that

either $k = 1$
or
$k > 1$ and $m < \dfrac{1}{2^k - 2} l < l.$

Suppose $k = 1$. It follows that $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$-(2l+1)$

$-2(2m+1)$

$M = (2l+1) + 2(2m+1)$

$-(2l+1)$

$(2m+1)$

The next term is $m - l$, and so

$$(2l+1) + 2(2m+1) \leq 2(2m+1) + 2|m-l|.$$

Thus we see that $2l + 1 \leq 2|m-l|$. If $m - l \leq 0$, then we would have $2l + 1 \leq 2l - 2m$ which is impossible. Hence

$$m - l > 0.$$

So $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$-(2l+1)$

$-2(2m+1)$

$M = (2l+1) + 2(2m+1)$

$-(2l+1)$

$(2m+1)$

---

and the next term is $m - l > 0$. Now there exist integers $p, q \geq 0$ such that

$$(m - l) = 2^p(2q + 1).$$

Suppose $p = 0$. Then $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semicycles) as follows:

$-(2l + 1)$

$-2(2m + 1)$

$M = (2l + 1) + 2(2m + 1)$

$-(2l + 1)$

$(2m + 1)$

$(2q + 1)$

Suppose $p > 0$. Then $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semicycles) as follows:

$-(2l + 1)$

$-2(2m + 1)$

$M = (2l + 1) + 2(2m + 1)$

$-(2l + 1)$

$(2m + 1)$

$2^p(2q + 1)$

$-(2m + 1) - 2^p(2q + 1)$

$(2m+1)$

$-2^{p-1}(2q+1)$

$-(2m+1)+2^{p-1}(2q+1)$

$(2m+1)$

$\vdots$

$-(2m+1)+(-1)^{p-1}2^2(2q+1)$

$(2m+1)$

$(-1)^{p-1}2(2q+1)$

$-(2m+1)-(-1)^{p-1}2(2q+1)$

$(2m+1)$

$(-1)^p(2q+1)$

Thus whether $p=0$ or $p>0$, $\{x_n\}_{n=-1}^{\infty}$ eventually has the two consecutive odd terms

---

$(2m+1)$

$(-1)^p(2q+1)$

The next term is

$$\frac{(2m+1)+(-1)^p(2q+1)}{2} = \frac{2^p(2m+1)+(-1)^p(m-l)}{2^{p+1}}.$$

Note that

$$2\left|(-1)^p(2q+1)\right| + 2\left|\frac{2^p(2m+1)+(-1)^p(m-l)}{2^{p+1}}\right|$$

$$= 2(2q+1) + \frac{2^p(2m+1)+(-1)^p(m-l)}{2^p}$$

$$= \frac{2\cdot 2^p(2q+1)+2^p(2m+1)+(-1)^p(m-l)}{2^p}$$

$$= \frac{2(m-l)+2^p(2m+1)+(-1)^p(m-l)}{2^p}$$

$$= \frac{[2+2^{p+1}+(-1)^p]m-[2+(-1)^p]l+2^p}{2^p}$$

and

$$\frac{[2+2^{p+1}+(-1)^p]m-[2+(-1)^p]l+2^p}{2^p} < (2l+1)+2(2m+1) = M$$

if and only if

$$[2+2^{p+1}+(-1)^p]m-[2+(-1)^p]l+2^p < 2^{p+1}l+2^{p+2}m+3\cdot 2^p,$$

which is true. This is impossible.

Thus we see that

$$k \geq 2 \quad \text{and} \quad m < \frac{1}{2^k-2}l < l.$$

Hence $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$-(2l+1)$

$-2^k(2m+1)$

$M = (2l+1) + 2^k(2m+1)$

$-(2l+1)$

$2^{k-1}(2m+1)$

$\vdots$

$(-1)^k 2(2m+1)$

$(2l+1)-(-1)^k 2(2m+1)$

$-(2l+1)$

$(-1)^{k+1}(2m+1)$

Suppose k is odd. Then the next term is $m-l$ which is negative.
Note that

$$M = (2l+1) + 2^k(2m+1) \leq 2(2m+1) + 2(l-m)$$

if and only if

$$2l + 1 + 2^{k+1}m + 2^k \leq 4m + 2 + 2l - 2m$$

if and only if

$$(2^{k+1} - 2)m \leq 1 - 2^k$$

which is impossible.

Suppose k is even. Then the next term is $-(m+l+1)$ which is negative. Note that

$$M = (2l+1) + 2^k(2m+1) \leq 2(2m+1) + 2(m+l+1)$$

if and only if

$$2l + 1 + 2^{k+1}m + 2^k \leq 4m + 2 + 2m + 2l + 2$$

if and only if $(2^{k+1} - 6)m + (2^k - 3) \leq 0$

which is also impossible.

Thus it is true that $x_0 = 0$, and so $\{x_n\}_{n=-1}^{\infty}$ is the three cycle $(-1, 0, 1)$. □

Eq.(4*) Consider the ΔE

$$x_{n+1} = \begin{cases} \dfrac{-x_n + x_{n-1}}{2} & \text{if} \quad x_n + x_{n-1} \quad \text{is even} \\ \\ x_n - x_{n-1} & \text{if} \quad x_n + x_{n-1} \quad \text{is odd} \end{cases}, \quad n = 0, 1, \ldots \tag{4*}$$

where $x_{-1}, x_0 \in \mathbf{Z}$.

Clearly $\bar{x} = 0$ is the only equilibrium solution of Eq.(4*).

COROLLARY 8.4
Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(4*), and suppose that* $\gcd(x_{-1}, x_0) = 1$*. Then* $\{x_n\}_{n=-1}^{\infty}$ *is either eventually the two-cycle* $(1, -1)$ *or the six-cycle* $(-1, 0, 1, 1, 0, -1)$*.*

8.4.5 Eq.(5)

Eq.(5) Consider the ΔE

$$x_{n+1} = \begin{cases} \dfrac{x_n - x_{n-1}}{2} & \text{if} \quad x_n + x_{n-1} \quad \text{is even} \\ \\ x_n + x_{n-1} & \text{if} \quad x_n + x_{n-1} \quad \text{is odd} \end{cases}, \quad n = 0, 1, \ldots \tag{8.5}$$

where $x_{-1}, x_0 \in \mathbf{Z}$.

Clearly $\bar{x} = 0$ is the only equilibrium solution of Eq.(8.5).

CONJECTURE 8.1
Suppose $\gcd(x_{-1}, x_0) = 1$*. Then every solution of Eq.(8.5) is either eventually the three-cycle* $(0, 1, 1)$*, the three-cycle* $(0, -1, -1)$*, or the ten-cycle* $(2, 5, 7, 1, -3, -2, -5, -7, -1, 3)$*.*

Eq.(5*) Consider the ΔE

$$x_{n+1} = \begin{cases} \dfrac{-x_n - x_{n-1}}{2} & \text{if} \quad x_n + x_{n-1} \quad \text{is even} \\ \\ -x_n + x_{n-1} & \text{if} \quad x_n + x_{n-1} \quad \text{is odd} \end{cases}, \quad n = 0, 1, \ldots \tag{5*}$$

where $x_{-1}, x_0 \in \mathbf{Z}$.

Clearly $\bar{x} = 0$ is the only equilibrium solution of Eq.(5*).

CONJECTURE 8.2
Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(5*), and suppose that* $\text{gcod}(x_{-1}, x_0) = 1$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is either eventually the five-cycle* $(2, -5, 7, -1, -3)$*, the five-cycle* $(-2, 5, -7, 1, 3)$*, or the six-cycle* $(-1, 1, 0, 1, -1, 0)$.

8.4.6 Eq.(6)

Eq.(6) Consider the ΔE

$$x_{n+1} = \begin{cases} \dfrac{x_n - x_{n-1}}{2} & \text{if } \; x_n + x_{n-1} \; \text{ is even} \\ \\ x_n - x_{n-1} & \text{if } \; x_n + x_{n-1} \; \text{ is odd} \end{cases}, \quad n = 0, 1, \ldots \tag{8.6}$$

where $x_{-1}, x_0 \in \mathbf{Z}$.

Clearly $\bar{x} = 0$ is the only equilibrium solution of Eq.(8.6).

THEOREM 8.5
Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(8.6). Suppose that* $\text{gcod}(x_{-1}, x_0) = 1$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is eventually the six-cycle* $(-1, 0, 1, 1, 0, -1)$.

PROOF Since $\text{gcod}(x_{-1}, x_0) = 1$, it follows that $\{x_n\}_{n=-1}^{\infty}$ is not eventually constant. By Lemma 8.9, $\{x_n\}_{n=-1}^{\infty}$ is a bounded, integer valued sequence, and hence is eventually periodic. So without loss of generality, we assume $\{x_n\}_{n=-1}^{\infty}$ is periodic. Hence there exists an integer $M > 0$ such that $-M \leq x_n \leq M$ for all $n \geq -1$. Without loss of generality, we may assume that $x_0 < x_1 = M$. It follows by Lemma 8.6 that $x_{-1} \leq 0$, $x_0 \geq 0$, and that $x_1 = x_0 - x_{-1}$ is odd.

We claim that $x_{-1}x_0 = 0$.

For the sake of contradiction, suppose that $x_{-1}x_0 \neq 0$. For each $n \geq 0$, let $P(n)$ be the following proposition:

(i) $x_{-1+3n} = (-1)^n \dfrac{x_{-1}}{2^n}$ and x_{-1+3n} is even;

(ii) $x_{3n} = (-1)^n \left(x_0 - \dfrac{2^n - 1}{2^n} \cdot x_{-1} \right)$;

(iii) $x_{1+3n} = (-1)^n \left(x_0 - x_{-1} \right)$.

We shall show by induction that $P(n)$ is true for all $n \geq 0$.

We shall first show that $P(0)$ is true. It suffices to show that x_{-1} is even. For the sake of contradiction, suppose that x_{-1} is odd. Then x_{-1} is odd, x_0 is even, x_1 is odd, $x_2 = x_1 - x_0 = -x_{-1}$ is odd, and so $x_3 = \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(-x_{-1} - x_0 + x_{-1}) = -\frac{1}{2}x_0$. Hence by Lemma 8.9, $0 < x_1 = x_0 - x_{-1} \leq |x_2| + |x_3| = -x_{-1} + \frac{1}{2}x_0$, and so we see that $x_0 = 0$. This contradicts our hypothesis that $x_{-1}x_0 \neq 0$. Thus we see that x_{-1} is even, and so $P(0)$ is true.

We next suppose that $n \geq 0$ and $P(0)$ is true, and we shall show that $P(n+1)$ is true. Since x_{-1+3n} is even, we see that

$$\begin{aligned} x_{-1+3(n+1)} = x_{2+3n} &= \frac{x_{1+3n} - x_{3n}}{2} \\ &= \frac{1}{2}\left[(-1)^n(x_0 - x_{-1}) - (-1)^n\left(x_0 - \frac{2^n - 1}{2^n}x_{-1}\right)\right] \\ &= \frac{1}{2}(-1)^{n+1}\left(x_{-1} - x_0 + x_0 - \frac{2^n - 1}{2^n}x_{-1}\right) \\ &= (-1)^{n+1}\frac{x_{-1}}{2^{n+1}}. \end{aligned}$$

We shall next show that $x_{-1+3(n+1)}$ is even. So for the sake of contradiction, suppose that $x_{-1+3(n+1)}$ is odd. Then

x_{-1+3n} is even , $x_{3n} = (-1)^n\left(x_0 - \frac{2^n - 1}{2^n}x_{-1}\right)$ is odd,

$x_{1+3n} = (-1)^n(x_0 - x_{-1})$ is odd, and $x_{-1+3(n+1)} = (-1)^{n+1}\frac{x_{-1}}{2^{n+1}}$ is odd.
It follows that

$$\begin{aligned} x_{3(n+1)} &= \frac{x_{-1+3(n+1)} - x_{-2+3(n+1)}}{2} = \frac{x_{-1+3(n+1)} - x_{1+3n}}{2} \\ &= \frac{1}{2}\left[(-1)^{n+1}\frac{x_{-1}}{2^{n+1}} - (-1)^n(x_0 - x_{-1})\right) = \frac{1}{2}(-1)^{n+1}\left(\frac{x_{-1}}{2^{n+1}} + x_0 - x_{-1}\right] \\ &= \frac{(-1)^{n+1}}{2}\left(x_0 - \frac{2^{n+1} - 1}{2^{n+1}}\cdot x_{-1}\right). \end{aligned}$$

Thus by Lemma 8.9,

$$\begin{aligned} 0 < x_1 = x_0 - x_{-1} &\leq |x_{-1+3(n+1)}| + |x_{3(n+1)}| \\ &= -\frac{x_{-1}}{2^{n+1}} + \frac{1}{2}\left(x_0 - \frac{2^{n+1} - 1}{2^{n+1}}\cdot x_{-1}\right) = -x_{-1} + \frac{1}{2}x_0, \end{aligned}$$

and so we see that $x_0 = 0$. This contradicts our assumption that $x_{-1}x_0 \neq 0$. Thus it is true that $x_{-1+3(n+1)}$ is even. The rest of the proof that $P(n+1)$ is true is a simple computation and is left to the reader.

So in particular for $n \geq 0$, $x_{-1+3n} = (-1)^n \frac{x_{-1}}{2^n}$, from which we see that $x_{-1} = 0$. This contradicts our assumption that $x_{-1}x_0 \neq 0$.

Thus the claim that $x_{-1}x_0 = 0$ is true. So as $\text{gcod}(x_{-1}, x_0) = 1$, it follows that $\{x_n\}_{n=-1}^{\infty}$ is the six-cycle $(-1, 0, 1, 1, 0, -1)$. □

Eq.(6*) Consider the ΔE

$$x_{n+1} = \begin{cases} \dfrac{-x_n - x_{n-1}}{2} & \text{if} \quad x_n + x_{n-1} \quad \text{is even} \\ -x_n - x_{n-1} & \text{if} \quad x_n + x_{n-1} \quad \text{is odd} \end{cases}, \quad n = 0, 1, \ldots \tag{6*}$$

where $x_{-1}, x_0 \in \mathbf{Z}$.

Clearly $\bar{x} = 0$ is the only equilibrium solution of Eq.(6*).

COROLLARY 8.5
Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(6*), and suppose that* $\text{gcod}(x_{-1}, x_0) = 1$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is eventually either the three cycle* $(1, 0, -1)$ *or the three cycle* $(-1, 0, 1)$.

8.4.7 Eq.(7)

Eq.(7) Consider the ΔE

$$x_{n+1} = \begin{cases} \dfrac{x_n - x_{n-1}}{2} & \text{if} \quad x_n + x_{n-1} \quad \text{is even} \\ -x_n + x_{n-1} & \text{if} \quad x_n + x_{n-1} \quad \text{is odd} \end{cases}, \quad n = 0, 1, \ldots \tag{8.7}$$

where $x_{-1}, x_0 \in \mathbf{Z}$.

Clearly $\bar{x} = 0$ is the only equilibrium solution of Eq.(8.7).

THEOREM 8.6
Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(8.7). Suppose* $\text{gcod}(x_{-1}, x_0) = 1$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is eventually the eight-cycle* $(0, -1, 1, 1, 0, 1, -1, -1)$.

PROOF Since $\gcd(x_{-1}, x_0) = 1$, it follows by Lemma 8.1 that $\{x_n\}_{n=-1}^{\infty}$ is not eventually constant. By Lemma 8.10, we know that $\{x_n\}_{n=-1}^{\infty}$ is a bounded, integer valued sequence, and hence is eventually periodic. So without loss of generality, we assume that $\{x_n\}_{n=-1}^{\infty}$ is periodic. Hence there exists a constant $M > 0$ such that $-M \leq x_n \leq M$ for all $n \geq -1$. Without loss of generality, we may assume that $x_0 < x_1 = M$. It follows by Lemma 8.6 that $x_1 = -x_0 + x_{-1}$ is odd, $x_0 \leq 0$, and $x_{-1} \geq 0$,

We claim that $x_{-1}x_0 = 0$. For the sake of contradiction, suppose $x_{-1}x_0 \neq 0$.

We shall first show that x_0 is odd. For the sake of contradiction, suppose that x_0 is even. Then there exist $k \in \{0, 1, \ldots\}$ and $l \in \{1, 2, \ldots\}$ such that $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$x_{-1} = (2k + 1)$

$x_0 = -2l$

$x_1 = 2l + (2k + 1)$

$x_2 = -4l - (2k + 1).$

It follows that $l = 0$, which is a contradiction.

Thus x_0 is odd.

So there exist $k, l \in \{0, 1, \ldots\}$ and $q \in \{1, 2, \ldots\}$ such that $x_{-1} = 2^q(2k + 1)$ and $x_0 = -(2l + 1)$.

We claim that $q = 1$. For the sake of contradiction, suppose $q > 1$. Then there exists $p \in \{0, 1, \ldots\}$ such that $q \in \{4p + 2, 4p + 3, 4p + 4, 4p + 5\}$.

Suppose $q = 4p + 2$. Then $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$2^{4p+2}(2k + 1)$

$-(2l+1)$

$(2l+1)+2^{4p+2}(2k+1)$

$(2l+1)+2^{4p+1}(2k+1)$

$-2^{4p}(2k+1)$

$(2l+1)+(2^{4p+2}-2^{4p})(2k+1)$

$-(2l+1)-2^{4p+2}(2k+1)$

$-(2l+1)+(-2^{4p+2}+2^{4p-1})(2k+1)$

$2^{4p-2}(2k+1)$

$-(2l+1)+(-2^{4p+2}+2^{4p-2})(2k+1)$

$(2l+1)+2^{4p+2}(2k+1)$

$(2l+1)+(2^{4p+2}-2^{4p-3})(2k+1)$

$-2^{4p-4}(2k+1)$

$\vdots$

$2^2(2k+1)$

$-(2l+1)+(-2^{4p+2}+2^2)(2k+1)$

$(2l+1)+2^{4p+2}(2k+1)$

$(2l+1)+(2^{4p+2}-2^1)(2k+1)$

$-2^0(2k+1).$

The next term is

$$\begin{aligned} x_{\text{next}} &= \tfrac{1}{2}[-(2k+1)-(2l+1)+(2-2^{4p+2})(2k+1)] \\ &= \tfrac{1}{2}(-2k-1-2l-1+4k+2-2^{4p+3}k-2^{4p+2}) \\ &= \tfrac{1}{2}(2k-2^{4p+3}k-2l-2^{4p+2}) \\ &= (1-2^{4p+2})k-l-2^{4p+1}. \end{aligned}$$

Suppose x_{next} is odd. Then $x_{1+\text{next}} = \frac{1}{2}[-l+(-2^{4p+2}+1)k-2^{4p+1}+(2k+1)]$, and so

$$\begin{aligned} (2l+1)+2^{4p+2}(2k+1) &\le |x_{\text{next}}|+2|x_{1+\text{next}}| \\ &= [l+(2^{4p+2}-1)k+2^{4p+1}] \\ &\quad +[l+(2^{4p+2}-1)k+2^{4p+1}-(2k+1)] \\ &= 2l+(2^{4p+2}-2)\cdot 2k+(2^{4p+2}-1) \end{aligned}$$

which is impossible.

Suppose x_{next} is even. Then

$$\begin{aligned} x_{1+\text{next}} &= l+(2^{4p+2}-1)k+2^{4p+1}-(2k+1) = l+(2^{4p+2}-3)k+(2^{4p+1}-1), \\ x_{2+\text{next}} &= -2l+(3-2^{4p+2}-2^{4p+2}+1)k+(-2^{4p+2}+1) \\ &= -2l+(4-2^{4p+3})k+(-2^{4p+2}+1), \end{aligned}$$

and

$$\begin{aligned} x_{3+\text{next}} &= \tfrac{1}{2}[-3l+(7-2^{4p+3}-2^{4p+2})k+(-2^{4p+2}+1-2^{4p+1}+1)] \\ &= -\tfrac{3}{2}l+(\tfrac{7}{2}-2^{4p+2}-2^{4p+1})k+(1-2^{4p+1}-2^{4p}) \\ &= \tfrac{3}{2}(k-l)+(2-2^{4p+2}-2^{4p+1})k+(1-2^{4p+1}-2^{4p}). \end{aligned}$$

Suppose $x_{3+\text{next}}$ is even. Then

$$\begin{aligned} x_{4+\text{next}} &= \tfrac{3}{2}(l-k) + (2^{4p+2} + 2^{4p+1} - 2)k + (2^{4p+1} + 2^{4p} - 1) - 2l + (4 - 2^{4p+3})k \\ &\quad +(-2^{4p+2} + 1) \\ &= -\tfrac{1}{2}(l+k) + (-2^{4p+3} + 2^{4p+2} + 2^{4p+1} + 1)k + (-2^{4p+2} + 2^{4p+1} + 2^{4p}), \end{aligned}$$

$$\begin{aligned} x_{5+\text{next}} &= \tfrac{1}{2}(l+k) + (2^{4p+3} - 2^{4p+2} - 2^{4p+1} - 1)k + (2^{4p+2} - 2^{4p+1} - 2^{4p}) \\ &\quad +\tfrac{3}{2}(k-l) + (2 - 2^{4p+2} - 2^{4p+1})k + (1 - 2^{4p+1} - 2^{4p}) \\ &= -l + (-2^{4p+2} + 3)k + (1 - 2^{4p+1}), \end{aligned}$$

and

$$\begin{aligned} x_{6+\text{next}} &= \tfrac{1}{2}[-l + (-2^{4p+2} + 3)k + (1 - 2^{4p+1}) \\ &\quad +\tfrac{1}{2}(l+k) + (2^{4p+3} - 2^{4p+2} - 2^{4p+1} - 1)k + (2^{4p+2} - 2^{4p+1} - 2^{4p})] \\ &= \tfrac{1}{2}[\tfrac{1}{2}(k-l) + (-2^{4p+1} + 2)k + (1 - 2^{4p})]. \end{aligned}$$

Now $|x_1| \le |x_{5+\text{next}}| + 2|x_{6+\text{next}}| = -x_{5+\text{next}} + 2|x_{6+\text{next}}|$.

Suppose $x_{6+\text{next}} \le 0$. Then

$$\begin{aligned} (2l+1) + 2^{4p+2}(2k+1) &\le l + (2^{4p+2} - 3)k + (2^{4p+1} - 1) \\ &\quad +\tfrac{1}{2}(l-k) + (2^{4p+1} - 2)k + (2^{4p} - 1) \\ &= \tfrac{3}{2}l + (2^{4p+2} + 2^{4p+1} - \tfrac{11}{2})k + (2^{4p+1} + 2^{4p} - 2). \end{aligned}$$

This is impossible.

Suppose $x_{6+next} > 0$. Then $p = 0$, and so the inequality

$$(2l+1)+2^{4p+2}(2k+1) \le l+(2^{4p+2}-3)k+(2^{4p+1}-1)+\frac{1}{2}(k-l)+(2-2^{4p+1})k+(1-2^{4p})$$

reduces to the inequality

$$(2l+1) + 4(2k+1) \le l + k + 1 + \frac{1}{2}(k-l)$$

which is also impossible.

Suppose $x_{3+\text{next}}$ is odd. Then

$$x_{4+\text{next}} = \tfrac{1}{2}[\tfrac{3}{2}(k-l) + (2 - 2^{4p+2} - 2^{4p+1})k + (1 - 2^{4p+1} - 2^{4p}) \\ +2l + (2^{4p+3} - 4)k + (2^{4p+2} - 1)]$$

$$= \tfrac{1}{2}[\tfrac{1}{2}(k+l) + (2^{4p+3} - 2^{4p+2} - 2^{4p+1} - 1)k + (2^{4p+2} - 2^{4p+1} - 2^{4p})],$$

and so (since $x_{3+\text{next}} < 0$),

$$\begin{aligned}(2l+1) + 2^{4p+2}(2k+1) &\le |x_{3+\text{next}}| + 2|x_{4+\text{next}}| = -x_{3+\text{next}} + 2x_{4+\text{next}} \\ &= \tfrac{3}{2}(l-k) + (2^{4p+2} + 2^{4p+1} - 2)k + (2^{4p+1} + 2^{4p} - 1) \\ &+ \tfrac{1}{2}(k+l) + (2^{4p+3} - 2^{4p+2} - 2^{4p+1} - 1)k \\ &+ (2^{4p+2} - 2^{4p+1} - 2^{4p}) \\ &= 2l + (2^{4p+3} - 4)k + (2^{4p+2} - 1)\end{aligned}$$

which is impossible. Thus $q \neq 4p+2$.

Suppose $q = 4p+3$. Then $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$2^{4p+3}(2k+1)$

$-(2l+1)$

$(2l+1) + 2^{4p+3}(2k+1)$

$(2l+1) + 2^{4p+2}(2k+1)$

$-2^{4p+1}(2k+1)$

$(2l+1) + (2^{4p+3} - 2^{4p+1})(2k+1)$

$-(2l+1) - 2^{4p+3}(2k+1)$

$-(2l+1) + (-2^{4p+3} + 2^{4p})(2k+1)$

$2^{4p-1}(2k+1)$

$-(2l+1)+(-2^{4p+3}+2^{4p-1})(2k+1)$

$(2l+1)+2^{4p+3}(2k+1)$

$(2l+1)+(2^{4p+3}-2^{4p-2})(2k+1)$

$-2^{4p-3}(2k+1)$

$\vdots$

$2^{7}(2k+1)$

$-(2l+1)+(-2^{4p+3}+2^{7})(2k+1)$

$(2l+1)+2^{4p+3}(2k+1)$

$(2l+1)+(2^{4p+3}-2^{6})(2k+1)$

$-2^{5}(2k+1)$

$(2l+1)+(2^{4p+3}-2^{5})(2k+1)$

$-(2l+1)-2^{4p+3}(2k+1)$

$-(2l+1)+(-2^{4p+3}+2^{4})(2k+1)$

$2^{3}(2k+1)$

$-(2l+1)+(-2^{4p+3}+2^{3})(2k+1)$

$(2l+1)+2^{4p+3}$

$(2l+1)+(2^{4p+3}-2^{2})(2k+1)$

$-2^1(2k+1)$
$(2l+1)+(2^{4p+3}-2^1)(2k+1)$ $-(2l+1)-2^{4p+3}(2k+1)$
$-(2l+1)+(-2^{4p+3}+1)(2k+1)$
$-(2k+1)$ $-(2l+1)+(-2^{4p+3}+2^1)(2k+1)$

The next term is

$$\begin{aligned} x_{\text{next}} &= \tfrac{1}{2}[-(2l+1)-2^{4p+3}(2k+1)+3(2k+1)] \\ &= \tfrac{1}{2}[-2l-1-2^{4p+4}k-2^{4p+3}+6k+3] \\ &= -l-(2^{4p+3}-3)k-(2^{4p+2}-1). \end{aligned}$$

Suppose x_{next} is odd. Then

$$\begin{aligned} x_{1+\text{next}} &= \tfrac{1}{2}[(-l-(2^{4p+3}-3)k-(2^{4p+2}-1))+((2l+1)+(2^{4p+3}-2)(2k+1))] \\ &= \tfrac{1}{2}[(-l-2^{4p+3}k+3k-2^{4p+2}+1)+(2l+1+2^{4p+4}k-4k+2^{4p+3}-2)] \\ &= \tfrac{1}{2}[l+(2^{4p+3}-1)k+2^{4p+2}] \end{aligned}$$

and so

$$\begin{aligned} (2l+1)+2^{4p+3}(2k+1) &\leq |x_{\text{next}}|+2|x_{1+\text{next}}| \\ &= (l+(2^{4p+3}-3)k+(2^{4p+2}-1)) \\ &\quad +(l+(2^{4p+3}-1)k+2^{4p+2}) \\ &= 2l+(2^{4p+4}-4)k+(2^{4p+3}-1) \end{aligned}$$

which is impossible.

Suppose x_{next} is even.

$$\begin{aligned} x_{1+\text{next}} &= (l + (2^{4p+3} - 3)k + (2^{4p+2} - 1)) + (-(2l+1) + (-2^{4p+3} + 2)(2k+1)) \\ &= l + 2^{4p+3}k - 3k + 2^{4p+2} - 1 - 2l - 1 - 2^{4p+4}k + 4k - 2^{4p+3} + 2 \\ &= -l - (2^{4p+3} - 1)k - 2^{4p+2}, \end{aligned}$$

and

$$\begin{aligned} x_{2+\text{next}} &= (l + 2^{4p+3}k - k + 2^{4p+2}) + (-l - 2^{4p+3}k + 3k - 2^{4p+2} + 1) \\ &= 2k + 1. \end{aligned}$$

Hence

$$\begin{aligned} (2l+1) + 2^{4p+3}(2k+1) &\le |x_{1+\text{next}}| + 2|x_{2+\text{next}}| \\ &= (l + (2^{4p+3} - 1)k + 2^{4p+2}) + (4k+2) \\ &= l + (2^{4p+3} + 3)k + (2^{4p+2} + 2) \end{aligned}$$

which is impossible. Thus $q \neq 4p+3$.

Suppose $q = 4p+4$. Then $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$2^{2p+4}(2k+1)$

$-(2l+1)$

$(2l+1) + 2^{4p+4}(2k+1)$

$(2l+1) + 2^{4p+3}(2k+1)$

$-2^{4p+2}(2k+1)$

$(2l+1) + (2^{4p+4} - 2^{4p+2})(2k+1)$

$-(2l+1) - 2^{4p+4}(2k+1)$

$-(2l+1) + (-2^{4p+4} + 2^{4p+1})(2k+1)$

$2^{4p}(2k+1)$

$-(2l+1) + (-2^{4p+4} + 2^{4p})(2k+1)$

$(2l+1) + 2^{4p+4}(2k+1)$

$(2l+1) + (2^{4p+4} - 2^{4p-1})(2k+1)$

$-2^{4p-2}(2l+1)$

$(2l+1) + (2^{4p+4} - 2^{4p-2})(2k+1)$

$-(2l+1) - 2^{4p+4}(2k+1)$

$-(2l+1) + (-2^{4p+4} + 2^{4p-3})(2k+1)$

$2^{4p-4}(2k+1)$

$\vdots$

$2^{8}(2k+1)$

$-(2l+1) + (-2^{4p+4} + 2^{8})(2k+1)$

$(2l+1) + 2^{4p+4}(2k+1)$

$(2l+1) + (2^{4p+4} - 2^{7})(2k+1)$

$-2^{6}(2k+1)$

$(2l+1)+(2^{4p+4}-2^6)(2k+1)$

$-(2l+1)-2^{4p+4}(2k+1)$

$-(2l+1)+(-2^{4p+4}+2^5)(2k+1)$

$2^4(2k+1)$

$-(2l+1)+(-2^{4p+4}+2^4)(2k+1)$

$(2l+1)+2^{4p+4}(2k+1)$

$(2l+1)+(2^{4p+4}-2^3)(2k+1)$

$-2^2(2k+1)$

$(2l+1)+(2^{4p+4}-2^2)(2k+1)$

$-(2l+1)-2^{4p+4}(2k+1)$

$-(2l+1)+(-2^{4p+4}+2^1)(2k+1)$

$(2k+1)$.

The next term is

$$\begin{aligned} x_{\text{next}} &= \tfrac{1}{2}[(2k+1)+((2l+1)+(2^{4p+4}-2)(2k+1))] \\ &= \tfrac{1}{2}(2k+1+2l+1+2^{4p+5}k-4k+2^{4p+4}-2) \\ &= l+(2^{4p+4}-1)k+2^{4p+3}. \end{aligned}$$

Suppose x_{next} is odd. Then $x_{1+\text{next}} = \frac{1}{2}(l+(2^{4p+4}-1)k+2^{4p+3}-(2k+1))$, and so

$$\begin{aligned} (2l+1)+2^{4p+4}(2k+1) \le\ & (l+(2^{4p+4}-1)k+2^{4p+3}) \\ & +(l+(2^{4p+4}-1)k+2^{4p+3}-(2k+1)) \\ =\ & 2l+(2^{4p+5}-2)k+(2^{4p+4}-1) \end{aligned}$$

which is impossible.

Suppose x_{next} is even. Then

$$x_{1+\text{next}} = [-l + (-2^{4p+4} + 1)k - 2^{4p+3}] + (2k+1)$$
$$= -l - (2^{4p+4} - 3)k - (2^{4p+3} - 1),$$

$$x_{2+\text{next}} = [l + (2^{4p+4} - 3)k + (2^{4p+3} - 1)] + [l + (2^{4p+4} - 1)k + 2^{4p+3}]$$
$$= 2l + (2^{4p+5} - 4)k + (2^{4p+4} - 1),$$

and

$$x_{3+\text{next}} = \tfrac{1}{2}[(2l + (2^{4p+5} - 4)k + (2^{4p+4} - 1)) + (l + (2^{4p+4} - 3)k + (2^{4p+3} - 1))]$$
$$= \tfrac{1}{2}[3l + (2^{4p+5} + 2^{4p+4} - 7)k + (2^{4p+4} + 2^{4p+3} - 2)].$$

Suppose $x_{3+\text{next}}$ is odd. Then

$$x_{4+\text{next}} = \frac{1}{2}[\frac{1}{2}[3l + (2^{4p+5} + 2^{4p+4} - 7)k + (2^{4p+4} + 2^{4p+3} - 2)] \\ -(2l + (2^{4p+5} - 4)k + (2^{4p+4} - 1)],$$

$$= \frac{1}{2}\left[-\frac{1}{2}l - (2^{4p+3} - \frac{1}{2})k - 2^{4p+2}\right]$$

and so

$$(2l+1) + 2^{4p+4}(2k+1) \le \tfrac{3}{2}l + (2^{4p+4} + 2^{4p+3} - \tfrac{7}{2})k + (2^{4p+3} + 2^{4p+2} - 1) \\ +\tfrac{1}{2}l + (2^{4p+3} - \tfrac{1}{2})k + 2^{4p+2}$$

$$= 2l + (2^{4p+5} - 4)k + (2^{4p+4} - 1),$$

which is impossible.

Suppose $x_{3+\text{next}}$ is even.

$$x_{4+\text{next}} = -\tfrac{3}{2}l - (2^{4p+4} + 2^{4p+3} - \tfrac{7}{2})k - (2^{4p+3} + 2^{4p+2} - 1) \\ +2l + (2^{4p+5} - 4)k + (2^{4p+4} - 1)$$

$$= \tfrac{1}{2}l + (2^{4p+3} - \tfrac{1}{2})k + 2^{4p+2},$$

$$x_{5+\text{next}} = -\tfrac{1}{2}l - (2^{4p+3} - \tfrac{1}{2})k - 2^{4p+2} \\ +\tfrac{3}{2}l + (2^{4p+4} + 2^{4p+3} - \tfrac{7}{2})k + (2^{4p+3} + 2^{4p+2} - 1)$$

$$= l + (2^{4p+4} - 3)k + (2^{4p+3} - 1),$$

$$x_{6+\text{next}} = \frac{1}{2}\left[\left(l + (2^{4p+4} - 3)k + (2^{4p+3} - 1)\right) - \left(\frac{1}{2}l + (2^{4p+3} - \frac{1}{2})k + 2^{4p+2}\right)\right]$$
$$= \frac{1}{2}\left[\frac{1}{2}l + (2^{4p+3} - \frac{5}{2})k + (2^{4p+2} - 1)\right]$$

and so

$$(2l+1) + 2^{4p+4}(2k+1) \leq \begin{array}{l} l + (2^{4p+4} - 3)k + (2^{4p+3} - 1) \\ +\frac{1}{2}l + (2^{4p+3} - \frac{5}{2})k + (2^{4p+2} - 1), \end{array}$$

which is impossible. Hence $q \neq 4p + 4$.

Suppose $q = 4p + 5$. Then $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

2^{4p+5}

$-(2l+1)$

$(2l+1) + 2^{4p+5}(2k+1)$

$(2l+1) + 2^{4p+4}(2k+1)$

$-2^{4p+3}(2k+1)$

$(2l+1) + (2^{4p+5} - 2^{4p+3})(2k+1)$

$-(2l+1) - 2^{4p+5}(2k+1)$

$-(2l+1) + (-2^{4p+5} + 2^{4p+2})(2k+1)$

$2^{4p+1}(2k+1)$

$-(2l+1) + (-2^{4p+5} + 2^{4p+1})(2k+1)$

$(2l+1) + 2^{4p+5}(2k+1)$

$(2l+1)+(2^{4p+5}-2^{4p})(2k+1)$

$-2^{4p-1}(2k+1)$

$\vdots$

$2^1(2k+1)$

$-(2l+1)+(-2^{4p+5}+2^1)(2k+1)$

$(2l+1)+2^{4p+5}(2k+1)$

$(2l+1)+(2^{4p+5}-2^0)(2k+1)$

$(2k+1)$

$(2l+1)+(2^{4p+5}-2^1)(2k+1).$

The next term is

$$\begin{aligned} x_{\text{next}} &= \tfrac{1}{2}[(2l+1)+(2^{4p+5}-2^1)(2k+1)-(2k+1)] \\ &= \tfrac{1}{2}(2l+1+2^{4p+6}k-4k+2^{4p+5}-2-2k-1) \\ &= \tfrac{1}{2}[2l+(2^{4p+6}-6)k+(2^{4p+5}-2)] \\ &= l+(2^{4p+5}-3)k+(2^{4p+4}-1). \end{aligned}$$

Suppose x_{next} is odd.

$$\begin{aligned} x_{1+\text{next}} &= \tfrac{1}{2}[(l+(2^{4p+5}-3)k+(2^{4p+4}-1))-((2l+1)+(2^{4p+5}-2)(2k+1))] \\ &= \tfrac{1}{2}(l+2^{4p+5}k-3k+2^{4p+4}-1-2l-1-2^{4p+6}k+4k-2^{4p+5}+2) \\ &= \tfrac{1}{2}(-l-2^{4p+5}k+k-2^{4p+4}) \\ &= -\tfrac{1}{2}[l+(2^{4p+5}-1)k+2^{4p+4}], \end{aligned}$$

and so

$$(2l+1)+2^{4p+5}(2k+1) \leq (l+(2^{4p+5}-3)k+(2^{4p+4}-1))+(l+(2^{4p+5}-1)k+2^{4p+4})$$

which is impossible.

Suppose x_{next} is even. Then

$$\begin{aligned} x_{1+\text{next}} &= -l + (-2^{4p+5}+3)k + (-2^{4p+4}+1) + (2l+1) + (2^{4p+5}-2)(2k+1) \\ &= -l - 2^{4p+5}k + 3k - 2^{4p+4} + 1 + 2l + 1 + 2^{4p+6}k - 4k + 2^{4p+5} - 2 \\ &= l + (2^{4p+5}-1)k + 2^{4p+4}, \end{aligned}$$

$$\begin{aligned} x_{2+\text{next}} &= -[l + (2^{4p+5}-1)k + 2^{4p+4}] + [l + (2^{4p+5}-3)k + (2^{4p+4}-1)] \\ &= -l - 2^{4p+5}k + k - 2^{4p+4} + l + 2^{4p+5}k - 3k + 2^{4p+4} - 1 \\ &= -(2k+1) \end{aligned}$$

and so

$$(2l+1) + 2^{4p+5}(2k+1) \leq (l + (2^{4p+5}-1)k + 2^{4p+4}) + 2(2k+1)$$

which is impossible.

Thus $q = 1$. Hence $\{x_n\}_{n=-1}^{\infty}$ is given (arranged in even and odd semi-cycles) as follows:

$2(2k+1)$
$-(2l+1)$ $(2l+1)+2(2k+1)$
$(2l+1)+(2k+1)$

$(2k+1)$

$(2l+1)$.

The next term is $x_{\text{next}} = l - k$.

Suppose $l - k \leq 0$. Then

$$(2l+1) + 2(2k+1) \leq (2l+1) + 2(k-l) = (2k+1)$$

which is impossible.

Hence $l - k > 0$.
So
$(2l+1)+2(2k+1) \leq (2l+1)+2(l-k)$ from which it follows that $3k+1 \leq l$.

Suppose x_{next} is even. Then

$$x_{1+\text{next}} = -(l-k) + (2l+1) = l+k+1,$$

$$x_{2+\text{next}} = -(l+k+1) + (l-k) = -(2k+1),$$

$$x_{3+\text{next}} = \tfrac{1}{2}[-(2k+1) - (l+k+1)] = -\tfrac{1}{2}(3k+l+2),$$

and so $(2l+1) + 2(2k+1) \leq (2k+1) + (3k+l+2)$.
Thus

$$2l + 1 + 4k + 2 \leq 5k + l + 3$$

from which it follows that $l \leq k$. This is impossible.

Hence x_{next} is odd, and so $x_{1+next} = \frac{1}{2}[(l-k) - (2l+1)] = -\frac{1}{2}[l+k+1]$. Thus

$$(2l+1) + 2(2k+1) \leq (l-k) + (l+k+1) = (2l+1).$$

This is impossible.

Thus our claim $x_{-1}x_0 = 0$ is true. So as $x_{-1} \geq 0$, $x_0 \leq 0$, $1 = \text{gcod}(x_{-1}, x_0) = \text{gcod}(x_0, x_1)$, and $x_1 = M$ is odd, it follows that $\{x_n\}_{n=-1}^{\infty}$ is the eight-cycle $(0, -1, 1, 1, 0, 1, -1, -1)$. ∎

Eq.(7*) Consider the ΔE

$$x_{n+1} = \begin{cases} \dfrac{-x_n - x_{n-1}}{2} & \text{if} \quad x_n + x_{n-1} \quad \text{is even} \\ \\ x_n + x_{n-1} & \text{if} \quad x_n + x_{n-1} \quad \text{is odd} \end{cases}, \quad n = 0, 1, \ldots \tag{7*}$$

where $x_{-1}, x_0 \in \mathbf{Z}$. Clearly $\bar{x} = 0$ is the only equilibrium solution of Eq.(7*).

COROLLARY 8.6
Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(7*), and suppose that* $\gcd(x_{-1}, x_0) = 1$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is eventually the eight-cycle* $(0, -1, -1, 1, 0, 1, 1, -1)$.

8.4.8 Eq.(8)

Eq.(8) Consider the ΔE

$$x_{n+1} = \begin{cases} \dfrac{x_n - x_{n-1}}{2} & \text{if} \quad x_n + x_{n-1} \quad \text{is even} \\ \\ -x_n - x_{n-1} & \text{if} \quad x_n + x_{n-1} \quad \text{is odd} \end{cases}, \quad n = 0, 1, \ldots \tag{8.8}$$

where $x_{-1}, x_0 \in \mathbf{Z}$.

Clearly $\bar{x} = 0$ is the only equilibrium solution of Eq.(8.8).

OPEN PROBLEM 8.1
Determine the character of the solutions of Eq.(8.8).

Eq.(8*) Consider the ΔE

$$x_{n+1} = \begin{cases} \dfrac{-x_n - x_{n-1}}{2} & \text{if} \quad x_n + x_{n-1} \quad \text{is even} \\ \\ x_n - x_{n-1} & \text{if} \quad x_n + x_{n-1} \quad \text{is odd} \end{cases}, \quad n = 0, 1, \ldots \tag{8*}$$

where $x_{-1}, x_0 \in \mathbf{Z}$.

Clearly $\bar{x} = 0$ is the only equilibrium solution of Eq.(8*).

CONJECTURE 8.3
Let $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(8*), and suppose that* $\gcd(x_{-1}, x_0) = 1$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is eventually either the four-cycle* $(1, 0, -1, -1)$, *the four-cycle* $(-1, 0, 1, 1)$, *or the six-cycle* $(1, -2, -3, -1, 2, 3)$.

8.5 Open Problems and Conjectures

In addition to the conjectures stated in Sections 8.4.5 and 8.4.8, we pose the following open problems and conjectures:

OPEN PROBLEM 8.2
Consider Eq.(8.2). Determine the set of all initial conditions $(x_{-1}, x_0) \in \mathbf{Z}\times\mathbf{Z}$ *with* $\mathrm{gcod}(x_{-1}, x_0) = 1$ *such that each of the following statements is true:*

(i) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the constant 1;*

(ii) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the constant minus 1;*

(iii) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the six-cycle* $(-2, 1, 3, 2, -1, -3)$.

OPEN PROBLEM 8.3
Consider Eq.(2^*)*. Determine the set of all initial conditions* $(x_{-1}, x_0) \in \mathbf{Z}\times\mathbf{Z}$ *with* $\mathrm{gcod}(x_{-1}, x_0) = 1$ *such that each of the following statements is true:*

(i) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the two-cycle* $(1, -1)$;

(ii) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the three-cycle* $(2, 1, -3)$;

(iii) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the three-cycle* $(-2, -1, 3)$.

OPEN PROBLEM 8.4
Consider Eq.(8.3). Determine the set of all initial conditions $(x_{-1}, x_0) \in \mathbf{Z}\times\mathbf{Z}$ *with* $\mathrm{gcod}(x_{-1}, x_0) = 1$ *such that each of the following statements is true:*

(i) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the constant 1;*

(ii) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the constant minus 1;*

(iii) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the four-cycle* $(2, -1, 3, 1)$.

(iv) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the four-cycle* $(-2, 1, -3, -1)$.

(v) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the six-cycle* $(1, 0, 1, -1, 0, -1)$.

OPEN PROBLEM 8.5
Consider Eq.(3^*)*. Determine the set of all initial conditions* $(x_{-1}, x_0) \in \mathbf{Z}\times\mathbf{Z}$ *with* $\mathrm{gcod}(x_{-1}, x_0) = 1$ *such that each of the following statements is true:*

(i) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the two-cycle* $(1, -1)$;

(ii) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the three-cycle* $(1,0,1)$;

(iii) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the three-cycle* $(-1,0,-1)$.

(iv) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the three-cycle* $(2,1,3,-1)$.

(v) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the three-cycle* $(-2,-1,-3,1)$.

OPEN PROBLEM 8.6

Consider Eq.(8.4). Determine the set of all initial conditions $(x_{-1}, x_0) \in \mathbf{Z}\times\mathbf{Z}$ *with* $\text{gcod}(x_{-1}, x_0) = 1$ *such that each of the following statements is true:*

(i) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the constant 1;*

(ii) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the constant minus 1;*

(iii) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the three-cycle* $(-1,0,1)$.

(iv) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the three-cycle* $(1,0,-1)$.

OPEN PROBLEM 8.7

Consider Eq.(4^*)*. Determine the set of all initial conditions* $(x_{-1}, x_0) \in \mathbf{Z}\times\mathbf{Z}$ *with* $\text{gcod}(x_{-1}, x_0) = 1$ *such that each of the following statements is true:*

(i) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the two-cycle* $(1,-1)$;

(ii) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the six-cycle* $(-1,0,1,1,0,-1)$;

OPEN PROBLEM 8.8

Consider Eq.(8.6). Determine the set of all initial conditions $(x_{-1}, x_0) \in \mathbf{Z}\times\mathbf{Z}$ *with* $\text{gcod}(x_{-1}, x_0) = 1$ *such that* $\{x_n\}_{n=-1}^{\infty}$ *is eventually the six-cycle* $(-1,0,1,1,0,-1)$.

OPEN PROBLEM 8.9

Consider Eq.(6^*)*. Determine the set of all initial conditions* $(x_{-1}, x_0) \in \mathbf{Z}\times\mathbf{Z}$ *with* $\text{gcod}(x_{-1}, x_0) = 1$ *such that each of the following statements is true:*

(i) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the three-cycle* $(1,0,-1)$;

(ii) $\{x_n\}_{n=-1}^{\infty}$ *is eventually the three-cycle* $(-1,0,1)$;

OPEN PROBLEM 8.10

Consider Eq.(8.7). Determine the set of all initial conditions $(x_{-1}, x_0) \in \mathbf{Z}\times\mathbf{Z}$ *with* $\text{gcod}(x_{-1}, x_0) = 1$ *such that* $\{x_n\}_{n=-1}^{\infty}$ *is eventually the eight-cycle* $(0,-1,0,1,1,0,1,-1,-1)$.

OPEN PROBLEM 8.11

Consider Eq.(7). Determine the set of all initial conditions* $(x_{-1}, x_0) \in \mathbf{Z} \times \mathbf{Z}$ *with* $\gcd(x_{-1}, x_0) = 1$ *such that* $\{x_n\}_{n=-1}^{\infty}$ *is eventually the eight-cycle* $(0, -1, -1, 1, 0, 1, 1, -1)$.

Motivated by the results in Section 3.5, we present here some conjectures on the following eight difference equations where the parameter α is an integer greater than 2. See [46].

$$x_{n+1} = \begin{cases} \dfrac{x_n + x_{n-1}}{\alpha} & \text{if} \quad \alpha \,|\, x_n + x_{n-1} \\ x_n - x_{n-1} & \text{otherwise} \end{cases}, \quad n = 0, 1, \ldots \tag{8.9}$$

$$x_{n+1} = \begin{cases} \dfrac{x_n - x_{n-1}}{\alpha} & \text{if} \quad \alpha \,|\, x_n - x_{n-1} \\ x_n - x_{n-1} & \text{otherwise} \end{cases}, \quad n = 0, 1, \ldots \tag{8.10}$$

$$x_{n+1} = \begin{cases} \dfrac{x_n + x_{n-1}}{\alpha} & \text{if} \quad \alpha \,|\, x_n + x_{n-1} \\ x_n + x_{n-1} & \text{otherwise} \end{cases}, \quad n = 0, 1, \ldots \tag{8.11}$$

$$x_{n+1} = \begin{cases} \dfrac{x_n + x_{n-1}}{\alpha} & \text{if} \quad \alpha \,|\, x_n + x_{n-1} \\ -x_n + x_{n-1} & \text{otherwise} \end{cases}, \quad n = 0, 1, \ldots \tag{8.12}$$

$$x_{n+1} = \begin{cases} \dfrac{x_n - x_{n-1}}{\alpha} & \text{if} \quad \alpha \,|\, x_n - x_{n-1} \\ -x_n + x_{n-1} & \text{otherwise} \end{cases}, \quad n = 0, 1, \ldots \tag{8.13}$$

$$x_{n+1} = \begin{cases} \dfrac{x_n + x_{n-1}}{\alpha} & \text{if} \quad \alpha \,|\, x_n + x_{n-1} \\ -x_n - x_{n-1} & \text{otherwise} \end{cases}, \quad n = 0, 1, \ldots \tag{8.14}$$

$$x_{n+1} = \begin{cases} \dfrac{x_n - x_{n-1}}{\alpha} & \text{if} \quad \alpha \,|\, x_n - x_{n-1} \\ -x_n - x_{n-1} & \text{otherwise} \end{cases}, \quad n = 0, 1, \ldots \tag{8.15}$$

$$x_{n+1} = \begin{cases} \dfrac{x_n - x_{n-1}}{\alpha} & \text{if} \quad \alpha \,|\, x_n - x_{n-1} \\ x_n + x_{n-1} & \text{otherwise} \end{cases}, \quad n = 0, 1, \ldots \tag{8.16}$$

The following two theorems were established in [44].

THEOREM 8.7
The following statements are true:

1. *Every non-trivial solution of Eq.(8.9) is eventually periodic with prime period 6.*
2. *Every non-trivial solution of Eq.(8.10) is eventually periodic with prime period 6.*

Let F_N denote the Nth Fibonacci number. That is, $F_{-1} = 1$, $F_0 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 0$.

THEOREM 8.8
Let $\alpha \geq 4$. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(8.12), and suppose there exist integers $N_0 \geq 0$ and $N \geq -1$ such that $x_{N_0-1} = F_{N+1}$ and $x_{N_0} = F_N$. Finally, suppose that one of the following statements is true:

1. *α does not divide F_{N+2}.*
2. *$\dfrac{F_{M+2}}{\alpha} = F_{M-1}$ for all $0 \leq M \leq N$.*

Then $\{x_n\}_{n=-1}^{\infty}$ is eventually the six-cycle $(1, 0, 1, -1, 0, -1)$.

CONJECTURE 8.4
The following statements are true:

(i) *If* $\alpha = 3$, *every non-trivial solution of Eq.(8.11) is either eventually a three-cycle or is unbounded.*

(ii) *If* $\alpha = 5$,*every non-trivial solution of Eq.(8.11) is either eventually a six-cycle or is unbounded.*

(iii) *For* $\alpha = 4$ *and for all other values of* $\alpha > 5$, *every non-trivial solution of Eq.(8.11) is unbounded.*

CONJECTURE 8.5
The following statements are true:

(i) *Suppose* $\alpha = 3$. *Then every non-trivial solution of Eq.(8.12) is eventually periodic with period 6.*

(ii) *Suppose* $\alpha \geq 4$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a non-trivial solutions of Eq.(8.12) which is not of the type mentioned in Theorem 8.8. Then* $\{x_n\}_{n=-1}^{\infty}$ *is unbounded.*

CONJECTURE 8.6
The following statements are true:

(i) *Suppose* $\alpha = 3$. *Then every non-trivial solution of Eq.(8.13) is eventually periodic with period 6.*

(ii) *Suppose* $\alpha = 4$. *Then every non-trivial solution of Eq.(8.13) is eventually periodic with period 20.*

(iii) *Suppose* $\alpha \geq 5$. *Then every non-trivial solution of Eq.(8.13) is unbounded.*

CONJECTURE 8.7
Every non-trivial solution of Eq.(8.14) is eventually periodic with prime period 3.

CONJECTURE 8.8
Every non-trivial solution of Eq.(8.15) is eventually periodic with prime period 3.

CONJECTURE 8.9

The following statements are true:

(i) *Suppose* $\alpha = 3$. *Then every non-trivial solution of Eq.(8.16) is either eventually a 12-cycle or is unbounded.*

(ii) *Suppose* $\alpha = 4$. *Then every non-trivial solution of Eq.(8.16) is either eventually a 24-cycle or is unbounded.*

(iii) *Suppose* $\alpha \geq 7$. *Then every non-trivial solution of Eq.(8.16) is either eventually an 18-cycle or is unbounded.*

(iv) *For all other values of* $\alpha > 4$ *with* $\alpha \neq 7$, *every non-trivial solution of Eq.(8.16) is unbounded.*

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