Nonconvex Optimization and Its Applications

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Generalized Convexity and Vector Optimization

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Preface

The present lecture note is dedicated to the study of the optimality conditions and the duality results for nonlinear vector optimization problems, in finite and infinite dimensions. The problems include are nonlinear vector optimization problems, symmetric dual problems, continuous-time vector optimization problems, relationships between vector optimization and variational inequality problems.

Nonlinear vector optimization problems arise in several contexts such as in the building and interpretation of economic models; the study of various technological processes; the development of optimal choices in finance; management science; production processes; transportation problems and statistical decisions, etc.

In preparing this lecture note a special effort has been made to obtain a self-contained treatment of the subjects; so we hope that this may be a suitable source for a beginner in this fast growing area of research, a semester graduate course in nonlinear programing, and a good reference book. This book may be useful to theoretical economists, engineers, and applied researchers involved in this area of active research.

The lecture note is divided into eight chapters:

Chapter 1 briefly deals with the notion of nonlinear programing problems with basic notations and preliminaries.

Chapter 2 deals with various concepts of convex sets, convex functions, invex set, invex functions, quasiinvex functions, pseudoinvex functions, type I and generalized type I functions, V-invex functions, and univex functions.

Chapter 3 covers some new type of generalized convex functions, such as Type I univex functions, generalized type I univex functions, nondifferentiable d-type I, nondifferentiable pseudo-d-type I, nondifferentiable quasi d-type I and related functions, and similar concepts for continuous-time case, for nonsmooth continuous-time case, and for n-set functions are introduced.

Chapter 4 deals with the optimality conditions for multiobjective programing problems, nondifferentiable programing problems, minimax fractional programing problems, mathematical programing problems in Banach spaces, in complex spaces, continuous-time programing problems, nonsmooth continuous-time programing problems, and multiobjective fractional subset programing problems under the assumptions of some generalized convexity given in Chap. 3.

In Chap. 5 we give Mond–Weir type and General Mond–Weir type duality results for primal problems given in Chap. 4. Moreover, duality results for nonsmooth programing problems and control problems are also given in Chap. 5.

Chapter 6 deals with second and higher order duality results for minimax programing problems, nondifferentiable minimax programing problems, nondifferentiable mathematical programing problems under assumptions generalized convexity conditions.

Chapter 7 is about symmetric duality results for mathematical programing problems, mixed symmetric duality results for nondifferentiable multiobjective programing problems, minimax mixed integer programing problems, and symmetric duality results for nondifferentiable multiobjective fractional variational problems.

Chapter 8 is about relationships between vector variational-like inequality problems and vector optimization problems under various assumptions of generalized convexity. Such relationships are also studied for nonsmooth vector optimization problems as well. Some characterization of generalized univex functions using generalized monotonicity are also given in this chapter.

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Chapter 1
Introduction

Nonlinear vector optimization (NVO) deals with optimization models with at least one nonlinear function, also called continuous optimization or smooth optimization. A general model is in the following form:

$$\begin{aligned}
&\text{minimize}\, f(x) \\
&\text{subject to} \\
&\qquad x \in X \\
&\qquad g_i(x) \leq 0, \quad i = 1, 2, \ldots, m \\
&\qquad h_j(x) = 0, \quad i = 1, 2, \ldots, k.
\end{aligned}$$

Functions $f : X \to R$, $g : X \to R^m$ and $h : X \to R^k$ are assumed to be continuously differentiable (*i.e.*, smooth functions), and $X \subseteq R^n$ is assumed to be open.

Let $K = \{x : x \in X, g(x) \leq 0, h(x) = 0\}$ denote the set of all feasible solutions of the problem (P).

Linear programming aroused interest in constraints in the form of inequalities and in the theory of linear inequalities and convex sets. The study of Kuhn–Tucker (Kuhn was a student of Tucker and became the principal investigator, worked together on several projects dealing with linear and nonlinear programming problems under generous sponsorship of the Naval Research from 1948 until 1972) appeared in the middle of this interest with a full recognition of such developments.

Kuhn–Tucker (1951) first used the name "Nonlinear Programming." However, the theory of nonlinear programming when the constraints are all in the form of equalities has been known for a long time. The inequality constraints were treated in a fairly satisfactory manner by Karush (1939) in his M.Sc. thesis, at the Department of Mathematics, University of Chicago. A summary of the thesis was published as an appendix to: Kuhn (1976). Karush's work is apparently under the influence of a similar work in the calculus of variations by Valentine (1937). As a struggling graduate student meeting requirements for going on to his Ph.D., the thought of publication never occurred to Karush and he was not encouraged to publish his Master's thesis by his supervisor L.M. Graves. At that time, no one anticipated the future

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interest in these problems and their potential practical applications. The school of classical calculus of variations at Chicago also popularized the theory of optimal control under the name of the "Pontryagin's maximum principle."

It was not the calculus of variations, optimization or control theory that motivated Fritz John, but rather the direct desire to find a method that would help to prove inequalities as they occur in geometry. Next to Karush, but still prior to Kuhn and Tucker, Fritz John (1948) considered the nonlinear programming problem with inequality constraints.

In May of 1948, Dantzig visited von Neumann in Princeton to discuss potential connections between the then very new subject of linear programming and the theory of games. Tucker happened to give Dantzig a lift to the train station for his return trip to Washington DC. On the way, Dantzig gave a short exposition of what linear programming was, using the transportation problem as a simple illustrative example. This sounded like Kirkhoff's Law to Tucker and he made this observation during the ride, but thought little about it until September of 1949.

On leave at Stanford in the fall of 1949, Tucker had a chance to return to the question: what was the relation between linear programming and the Kirkhoff-Maxwell treatment of electrical networks. It was at this point that Tucker (1957) recognized the parallel between Maxwell's potentials and Lagrange multipliers, and identified the underlying optimization problem of minimizing heat loss. Tucker then wrote Gale and Kuhn, inviting them to do a sequel to (Gale et al. 1951). Gale declined, Kuhn accepted and paper developed by correspondence between Stanford and Princeton shifted emphasis from the quadratic case to the general nonlinear programming problem and to properties of *convexity* that imply the *necessary conditions for an optimum are also sufficient.*

A convex nonlinear programing problem can be formulated as:

(P)

$$
\begin{aligned}
&\text{minimize} f(x)\\
&\text{subject to}\\
&\qquad x \in X\\
&\qquad g_i(x) \le 0, \quad i = 1, 2, \ldots, m\\
&\qquad h_j(x) = 0, \quad i = 1, 2, \ldots, k.
\end{aligned}
$$

Functions f, g and h are assumed to be convex.

Nicest among nonlinear programs, useful necessary and sufficient optimality conditions for global minimum are only known for convex programming problems.

The *Fritz John necessary condition* (John 1948) for a feasible point x^* to be optimal for (P) is the existence of $\lambda_0^* \in R$, $\lambda^* \in R^m$ such that

$$
\begin{aligned}
&\lambda_0^* \nabla f(x^*) + {\lambda^*}^T \nabla g(x^*) = 0\\
&{\lambda^*}^T g(x^*) = 0\\
&(\lambda_0^*, \lambda^*) \ge 0,\ (\lambda_0^*, \lambda^*) \ne 0.
\end{aligned}
$$

There are no restrictions on the objective and constraint functions, apart from the differentiability.

However, by imposing a *regularity condition* on the constraint function, the λ_0^* may, without loss of generality, be taken as 1, and we obtain the *Kuhn–Tucker necessary conditions (Kuhn–Tucker 1951)*: there exists $\lambda^* \in R^m$ such that

$$\nabla f(x^*) + \lambda^{*^T} \nabla g(x^*) = 0$$
$$\lambda^{*^T} g(x^*) = 0$$
$$\lambda^* \geq 0,\ \lambda^* \neq 0.$$

There are a variety of regularity conditions, or constraint qualifications, which yield the Kuhn–Tucker necessary conditions. Some requires differentiability with no notion of convexity, and some have an assumption of convexity.

If the functions involved in the problem are convex then the necessary conditions for optimality are also sufficient.

In nonlinear programing, if the model is nonconvex, no efficient algorithm can guarantee finding a global minimum. So, one has to compromise with various types of solution expected. However, for convex programs, every local minimum is a global minimum. For convex programs, any method finding a local minimum will find a global minimum. Moreover, any stationary point is a global minimum in the case of convex programs**.**

1.1 Nonlinear Symmetric Dual Pair of Programming Problems

It is well known that every linear program is symmetric in the sense that the dual of the dual is the original problem. However, this is not the case with a general nonlinear programming problem.

Dantzig et al. (1965) introduced the following problem:

$$\begin{aligned}&\text{minimize}\, f(x,y) - y^T \nabla_y f(x,y)\\&\text{subject to}\\&\qquad \nabla_y f(x,y) \leq 0\\&\qquad x \geq 0,\ y \geq 0,\end{aligned}$$

Its symmetric dual is

$$\begin{aligned}&\text{maximize}\, f(u,v) - u^T \nabla_u f(u,v)\\&\text{subject to}\\&\qquad \nabla_u f(u,v) \leq 0\\&\qquad u \geq 0, v \geq 0.\end{aligned}$$

Mond and Weir (1981) proposed a relaxed version of the problems given by Dantzig et al. (1965). The problem of Mond and Weir has an advantage that one can use more generalized class of convex functions and the objective functions of the primal and dual problems are similar:

$$\begin{aligned}&\text{minimize} f(x,y)\\&\text{subject to}\\&\qquad y^T\nabla_y f(x,y)\leq 0\\&\qquad x\geq 0,\end{aligned}$$

Its symmetric dual is

$$\begin{aligned}&\text{maximize} f(u,v)\\&\text{subject to}\\&\qquad u^T\nabla_u f(u,v)\leq 0\\&\qquad v\geq 0.\end{aligned}$$

These problems have direct connections with two person zero sum games.

The optimization problems discussed above are only finite-dimensional. However, a great deal of optimization theory is concerned with problems involving infinite dimensional case. Two types of problems fitting into this scheme are variational and control problems. Hanson (1964) observed that variational and control problems are continuous-time analogue of finite dimensional nonlinear programming problems. Since then the fields of nonlinear programming and the calculus of variations have to some extent merged together within optimization theory, hence enhancing the potential for continued research in both fields. These types of problems are studied in Sect. 4.6, 4.7, and 5.16–5.19.

1.2 Motivation

Convexity is one of the most frequently used hypotheses in optimization theory. It is usually introduced to give global validity to propositions otherwise only locally true (for convex functions, for instance, any local minimum is also a global minimum) and to obtain sufficient conditions that are generally only necessary, as with the Kuhn–Tucker conditions in nonlinear programing. In microeconomics, convexity plays a fundamental role in general equilibrium theory and in duality results. In particular, in consumer theory, the convexity of preference ensures the existence of a demand function. In game theory, convexity ensures the existence of an equilibrium solution.

Convexity assumptions are often not satisfied in real-world economic models; see Arrow and Intriligator (1981). The necessary KKT conditions imply a maximum under some condition weaker than convexity. It suffices if $-f$ is pseudo-convex and each $-g_i$ is pseudo-concave, or less restrictively if the vector $-(f, g_1, g_2, \ldots, g_m)$ is invex. An economic model where an objective function is to be maximized, subject to constraints on the economic processes involved, leads to a nonlinear programming problem. It is well known that, under some restrictions, a maximum may be described by a Lagrangian function. The Lagrangian has a zero gradient at a maximum point, but this is not enough to imply a maximum, unless additional restrictions, such as concavity or quasi-concavity are imposed, but these often do not hold for many practical problems, so go for invexity, pseudo-invexity, etc.

In the past century, the notion of a convex function has been generalized in various ways, either by an extension to abstract spaces, or by a change in the definition of convexity. One of the more recent generalizations, for instance, is due to Hanson, who introduced *invex* functions in 1981: $f : R^n \to R$ is invex whenever it is differentiable and there exists a function $\eta : R^n \times R^n \to R^n$ such that $f(x) - f(y) \geq \nabla f(x)\, \eta(x, y)$. Many important properties of convex functions are preserved within a wider functional environment, for example, a local minimum is also a global minimum if the function involved is invex.

However, invexity is not the only generalization of convexity. In fact, after the work of Hanson (1981), mathematicians and other practiceners started attempting to further weakening of the concept of invexity. This has finally led to a whole field of research, known as "generalized convexity." It is impossible to collect the entire progress on the subject in one book, as there has been eight international conferences on generalized convexity. However, there is no book on the topic dealing with some generalized convexity and various nonlinear programing problems. This is a desperate need of advanced level students or new researchers in this field.

Chapter 2
Generalized Convex Functions

Convexity is one of the most frequently used hypotheses in optimization theory. It is usually introduced to give global validity to propositions otherwise only locally true, for instance, a local minimum is also a global minimum for a convex function. Moreover, convexity is also used to obtain sufficiency for conditions that are only necessary, as with the classical Fermat theorem or with Kuhn-Tucker conditions in nonlinear programming. In microeconomics, convexity plays a fundamental role in general equilibrium theory and in duality theory. For more applications and historical reference, see, Arrow and Intriligator (1981), Guerraggio and Molho (2004), Islam and Craven (2005). The convexity of sets and the convexity and concavity of functions have been the object of many studies during the past one hundred years. Early contributions to convex analysis were made by Holder (1889), Jensen (1906), and Minkowski (1910, 1911). The importance of convex functions is well known in optimization problems. Convex functions come up in many mathematical models used in economics, engineering, etc. More often, convexity does not appear as a natural property of the various functions and domain encountered in such models. The property of convexity is invariant with respect to certain operations and transformations. However, for many problems encountered in economics and engineering the notion of convexity does no longer suffice. Hence, it is necessary to extend the notion of convexity to the notions of pseudo-convexity, quasi-convexity, etc. We should mention the early work by de de Finetti (1949), Fenchel (1953), Arrow and Enthoven (1961), Mangasarian (1965), Ponstein (1967), and Karamardian (1967). In the recent years, several extensions have been considered for the classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson (1981). Hanson's initial result inspired a great deal of subsequent work which has greatly expanded the role and applications of invexity in nonlinear optimization and other branches of pure and applied sciences.

In this chapter, we shall discuss about various concepts of generalized convex functions introduced in the literature in last thirty years for the purpose of weakening the limitations of convexity in mathematical programming. Hanson (1981) introduced the concept of invexity as a generalization of convexity for scalar constrained optimization problems, and he showed that weak duality and sufficiency of

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the Kuhn-Tucker optimality conditions hold when invexity is required instead of the usual requirement of convexity of the functions involved in the problem.

2.1 Convex and Generalized Convex Functions

Definition 2.1.1. *A subset X of R^n is convex if for every $x_1, x_2 \in X$ and $0 < \lambda < 1$, we have*

$$\lambda x_1 + (1-\lambda) x_2 \in X.$$

Definition 2.1.2. *A function $f : X \to R$ defined on a convex subset X of R^n is convex if for any $x_1, x_2 \in X$ and $0 < \lambda < 1$, we have*

$$f(\lambda x_1 + (1-\lambda) x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2).$$

If we have strict inequality for all $x_1 \neq x_2$ in the above definition, the function is said to be strictly convex.

Historically the first type of generalized convex function was considered by de Finetti (1949) who first introduced the quasiconvex functions (a name given by Fenchel (1953)) after 6 years.

Definition 2.1.3. *A function $f : X \to R$ is quasiconvex on X if*

$$f(x) \le f(y) \Rightarrow f(\lambda x + (1-\lambda) y) \le f(y), \quad \forall x,\ y \in X,\ \forall\, \lambda \in [0,\ 1]$$

or, equivalently, in non-Euclidean form

$$f(\lambda x + (1-\lambda) y) \le \max\{f(x),\ f(y)\}, \quad \forall x,\ y \in X,\ \forall\, \lambda \in [0,\ 1].$$

For further study and characterization of quasiconvex functions, one can see Giorgi et al. (2004).

In the differentiable case, we have the following definition given in Avriel et al. (1988):

Definition 2.1.4. *A function $f : X \to R$ is said to be quasiconvex on X if*

$$f(x) \le f(y) \Rightarrow (x-y)\nabla f(y) \le 0, \quad \forall x,\ y \in X.$$

An important property of a differentiable convex function is that any stationary point is also a global minimum point; however, this useful property is not restricted to differentiable convex functions only. The family of pseudoconvex functions introduced by Mangasarian (1965) and under the name of semiconvex functions by Tuy (1964), strictly includes the family of differentiable convex functions and has the above mentioned property as well.

Definition 2.1.5. *Let $f : X \to R$ be differentiable on the open set $X \subset R^n$; then f is pseudoconvex on X if:*

$$f(x) < f(y) \Rightarrow (x-y)\nabla f(y) < 0, \quad \forall x,\ y \in X$$

or equivalently if

$$(x-y)\nabla f(y) \geq 0 \Rightarrow f(x) \geq f(y), \quad \forall x,\ y \in X.$$

From this definition it appears obvious that, if f is pseudoconvex and $\nabla f(y) = 0$, then y is a global minimum of f over X. Pseudoconvexity plays a key role in obtaining sufficient optimality conditions for a nonlinear programming problem as, if a differentiable objective function can be shown or assumed to be pseudoconvex, then the usual first-order stationary conditions are able to produce a global minimum.

The function $f : X \to R$ is called *pseudoconcave* if $-f$ is pseudoconvex.

Functions that are both pseudoconvex and pseudoconcave are called *pseudolinear*. Pseudolinear functions are particularly important in certain optimization problems, both in scalar and vector cases; see Chew and Choo (1984), Komlosi (1993), Rapcsak (1991), Kaul et al. (1993), and Mishra (1995).

The following result due to Chew and Choo (1984) characterizes the class of pseudolinear functions.

Theorem 2.1.1. *Let $f : X \to R$, where $X \subset R^n$ is an open convex set. Then the following statements are equivalent:*

(i) f is pseudolinear.
(ii) For any $x, y \in X$, it is $(x-y)\nabla f(y) = 0$ if and only if $f(x) = f(y)$.
(iii) There exists a function $p : X \times X \to R_+$ such that

$$f(x) = f(y) + p(x,y) \cdot (x-y)\nabla f(y).$$

The class of pseudolinear functions includes many classes of functions useful for applications, e.g., the class of linear fractional functions (see, e.g., Chew and Choo (1984)).

An example of a pseudolinear function is given by $f(x) = x + x^3$, $x \in R$. More generally, Kortanek and Evans (1967) observed that if f is pseudolinear on the convex set $X \subset R^n$, then the function $F = f(x) + [f(x)]^3$ is also pseudolinear on X.

For characterization of the solution set of a pseudolinear program, one can see Jeyakumar and Yang (1995).

Ponstein (1967) introduced the concept of strictly pseudoconvex functions for differentiable functions.

Definition 2.1.6. *A function $f : X \to R$, differentiable on the open set $X \subset R^n$, is strictly pseudoconvex on X if*

$$f(x) \leq f(y) \Rightarrow (x-y)\nabla f(y) < 0, \quad \forall\, x,\ y \in X,\ x \neq y,$$

or equivalently if

$$(x-y)\nabla f(y) \geq 0 \Rightarrow f(x) > f(y), \quad \forall\, x,\, y \in X,\, x \neq y.$$

The comparison of the definitions of pseudoconvexity and strict pseudoconvexity shows that strict pseudoconvexity implies pseudoconvexity. Ponstein (1967) showed that pseudoconvexity plus strict quasiconvexity implies strict pseudoconvexity and that strict pseudoconvexity implies strict quasiconvexity.

In a minimization problem, if the strict pseudoconvexity of the objective function can be shown or assumed, then the solution to the first-order optimality conditions is a unique global minimum. Many other characterization of strict pseudoconvex functions are given by Diewert et al. (1981).

Convex functions play an important role in optimization theory. The optimization problem:

$$\text{minimize}\, f(x) \;\text{ for } x \in X \subseteq R^n, \quad \text{subject to } \; g(x) \leqq 0,$$

is called a convex program if the functions involved are convex on some subset X of R^n. Convex programs have many useful properties:

1. The set of all feasible solutions is convex.
2. Any local minimum is a global minimum.
3. The Karush–Kuhn–Tucker optimality conditions are sufficient for a minimum.
4. Duality relations hold between the problem and its dual.
5. A minimum is unique if the objective function is strictly convex.

However, for many problems encountered in economics and engineering the notion of convexity does no longer suffice. To meet this demand and the convexity requirement to prove sufficient optimality conditions for a differentiable mathematical programming problem, the notion of invexity was introduced by Hanson (1981) by substituting the linear term $(x-y)$, appearing in the definition of differentiable convex, pseudoconvex and quasiconvex functions, with an arbitrary vector-valued function.

2.2 Invex and Generalized Invex Functions

Definition 2.2.1. *A function $f : X \to R$, X open subset of R^n, is said to be invex on X with respect to η if there exists vector-valued function $\eta : X \times X \to R^n$ such that*

$$f(x) - f(y) \geq \eta^T(x,y)\nabla f(y), \quad \forall x,\, y \in X.$$

The name "invex" was given by Craven (1981) and stands for "invariant convex."

Similarly f is said to be *pseudoinvex* on X with respect to η if there exists vector-valued function $\eta : X \times X \to R^n$ such that

$$\eta^T(x,y)\nabla f(y) \geq 0 \Rightarrow f(x) \geq f(y), \quad \forall x, y \in X.$$

The function $f : X \to R$, X open subset of R^n, is said to be *quasiinvex* on X with respect to η if there exists vector-valued function $\eta : X \times X \to R^n$ such that

$$f(x) \le f(y) \Rightarrow \eta^T(x,y)\nabla f(y) \le 0, \quad \forall x, y \in X.$$

Craven (1981) gave necessary and sufficient conditions for function f to be invex assuming that the functions f and η are twice continuously differentiable.

Ben-Israel and Mond (1986) and Kaul and Kaur (1985) also studied some relationships among the various classes of (generalized) invex functions and (generalized) convex functions. Let us list their results for the sake completion:

(1) A differentiable convex function is also invex, but not conversely, see example, Kaul and Kaur (1985).
(2) A differentiable pseudo-convex function is also pseudo-invex, but not conversely, see example, Kaul and Kaur (1985).
(3) A differentiable quasi-convex function is also quasi-invex, but not conversely, see example, Kaul and Kaur (1985).
(4) Any invex function is also pseudo-invex for the same function $\eta(x,\bar{x})$, but not conversely, see example, Kaul and Kaur (1985).
(5) Any pseudo-invex function is also quasi-invex, but not conversely.

Further insights on these relationships can be deduced by means of the following characterizations of invex functions:

Theorem 2.2.1. *(Ben-Israel and Mond (1986)) Let $f : X \to R$ be differentiable on the open set $X \subset R^n$; then f is invex if and only if every stationary point of f is a global minimum of f over X.*

It is adequate, in order to apply invexity to the study of optimality and duality conditions, to know that a function is invex without identifying an appropriate function $\eta(x,\bar{x})$. However, Theorem 2.2.1 allows us to find a function $\eta(x,\bar{x})$, when $f(x)$ is known to be invex; viz.

$$\eta(x,\bar{x}) = \begin{cases} \dfrac{[f(x) - f(\bar{x})]\nabla f(\bar{x})}{\nabla f(\bar{x})\nabla f(\bar{x})}, & \text{if} \quad \nabla f(\bar{x}) \ne 0 \\ 0, & \text{if} \quad \nabla f(\bar{x}) = 0. \end{cases}$$

Remark 2.2.1. If we consider an invex function f on set $X_0 \subseteq X$, with X_0 not open, it is not true that any local minimum of f on X_0 is also a global minimum. Let us consider the following example.

Example 2.2.1. Let $f(x,y) = y\left(x^2 - 1\right)^2$ and $X_0 = \{(x,y) : (x,y) \in R^2, x \ge -1/2, y \ge 1\}$. Every stationary point of f on X_0 is a global minimum of f on X_0, and therefore f is invex on X_0. The point $(-1/2, 1)$ is a local minimum point of f on X_0, with $f(-1/2,1) = 9/16$, but the global minimum is $f(1,y) = f(-1,y) = 0$.

In order to consider some type of invexity for nondifferentiable functions, Ben-Israel and Mond (1986) and Weir and Mond (1988) introduced the following function:

Definition 2.2.2. *A function $f : X \to R$ is said to be pre-invex on X if there exists a vector function $\eta : X \times X \to R^n$ such that*

$$(y + \lambda \eta (x,\ y)) \in X, \quad \forall \lambda \in [0,\ 1], \quad \forall\, x,\ y \in X$$

and

$$f(y + \lambda \eta (x,\ y)) \leq \lambda f(x) + (1 - \lambda) f(y), \quad \forall \lambda \in [0,\ 1], \quad \forall\, x,\ y \in X.$$

Weir and Mond (1988a) gave the following example of a pre-invex function which is not convex.

Example 2.2.2. $f(x) = -|x|,\ x \in R$. Then f is a pre-invex function with η given as follows:

$$\eta (x,\ y) = \begin{cases} x - y, & \text{if } y \leq 0 \quad \text{and } x \leq 0 \\ x - y, & \text{if } y \geq 0 \quad \text{and } x \geq 0 \\ y - x, & \text{if } y > 0 \quad \text{and } x < 0 \\ y - x, & \text{if } y < 0 \quad \text{and } x > 0. \end{cases}$$

As for convex functions, any local minimum of a pre-invex function is a global minimum and nonnegative linear combinations of pre-invex functions are pre-invex. Pre-invex functions were utilized by Weir and Mond (1988a) to establish proper efficiency results in multiple objective optimization problems.

2.3 Type I and Related Functions

Subsequently, Hanson and Mond (1982) introduced two new classes of functions which are not only sufficient but are also necessary for optimality in primal and dual problems, respectively. Let

$$P = \{x : x \in X, g(x) \leqq 0\} \quad \text{and} \quad D = \{x : (x,y) \in Y\},$$

where $Y = \{(x,y) : x \in X, y \in R^m, \nabla_x f(x) + y^T \nabla_x g(x) = 0; y \geqq 0\}$.

Hanson and Mond (1982) defined:

Definition 2.3.1. *$f(x)$ and $g(x)$ as Type I objective and constraint functions, respectively, with respect to $\eta(x)$ at $\bar{x}$ if there exists an n-dimensional vector function $\eta(x)$ defined for all $x \in P$ such that*

$$f(x) - f(\bar{x}) \geqq [\nabla_x f(\bar{x})]^T \eta (x,\bar{x})$$

and

$$-g(\bar{x}) \geqq [\nabla_x g(\bar{x})]^T \eta (x,\bar{x}),$$

the objective and constraint functions $f(x)$ and $g(x)$ are calledstrictly Type I if we have strict inequalities in the above definition.

Definition 2.3.2. *$f(x)$ and $g(x)$ as Type II objective and constraint functions, respectively, with respect to $\eta(x)$ at $\bar{x}$ if there exists an n-dimensional vector function $\eta(x)$ defined for all $x \in P$ such that*

$$f(\bar{x}) - f(x) \geqq [\nabla_x f(x)]^T \eta(x,\bar{x})$$

and

$$-g(x) \geqq [\nabla_x g(x)]^T \eta(x,\bar{x}).$$

the objective and constraint functions $f(x)$ and $g(x)$ are called strictly Type II if we have strict inequalities in the above definition.

Rueda and Hanson (1988) established the following relations:

1. If $f(x)$ and $g(x)$ are convex objective and constraint functions, respectively, then $f(x)$ and $g(x)$ are Type I, but the converse is not necessarily true, as can be seen from the following example.

Example 2.3.1. The functions $f:\left(0,\frac{\pi}{2}\right) \to R$ and $g:\left(0,\frac{\pi}{2}\right) \to R$ defined by $f(x) = x + \sin x$ and $g(x) = -\sin x$ are Type I functions with respect to $\eta(x) = \left(\frac{2}{\sqrt{3}}\right)\left(\sin x - \frac{1}{2}\right)$ at $\bar{x} = \pi/6$, but $f(x)$ and $g(x)$ are not convex with respect to the same $\eta(x) = \left(\frac{2}{\sqrt{3}}\right)\left(\sin x - \frac{1}{2}\right)$ as can be seen by taking $x = \pi/4$ and $\bar{x} = \pi/6$.

2. If $f(x)$ and $g(x)$ are convex objective and constraint functions, respectively, then $f(x)$ and $g(x)$ are Type II, but the converse is not necessarily true, as can be seen from the following example.

Example 2.3.2. The functions $f:\left(0,\frac{\pi}{2}\right) \to R$ and $g:\left(0,\frac{\pi}{2}\right) \to R$ defined by $f(x) = x + \sin x$ and $g(x) = -\sin x$ are Type II functions with respect to $\eta(x) = \frac{\left(\frac{1}{2}-\sin x\right)}{\cos x}$ at $\bar{x} = \pi/6$, but $f(x)$ and$g(x)$ are not convex with respect to the same $\eta(x) = \frac{\left(\frac{1}{2}-\sin x\right)}{\cos x}$ at $\bar{x} = \pi/6$.

3. If $f(x)$ and $g(x)$ are strictly convex objective and constraint functions, respectively, then $f(x)$ and $g(x)$ are strictly Type I, but the converse is not necessarily true, as can be seen from the following example.

Example 2.3.3. The functions $f:\left(0,\frac{\pi}{2}\right) \to R$ and $g:\left(0,\frac{\pi}{2}\right) \to R$ defined by $f(x) = -x + \cos x$ and $g(x) = -\cos x$ are strictly Type I functions with respect to $\eta(x) = 1 - \left(\frac{2}{\sqrt{2}}\right)\cos x$ at $\bar{x} = \pi/4$, but $f(x)$ and$g(x)$ are not strictly convex with respect to the same $\eta(x) = 1 - \left(\frac{2}{\sqrt{2}}\right)\cos x$.

4. If $f(x)$ and $g(x)$ are strictly convex objective and constraint functions, respectively, then $f(x)$ and $g(x)$ are strictly Type II, but the converse is not necessarily true, as can be seen from the following example.

Example 2.3.4. The functions $f:\left(0,\frac{\pi}{2}\right)\to R$ and $g:\left(0,\frac{\pi}{2}\right)\to R$ defined by $f(x)=-x+\cos x$ and $g(x)=-\cos x$ are strictly Type II functions with respect to $\eta(x)=\frac{\left(\cos x-\frac{\sqrt{2}}{2}\right)}{\sin x}$ at $\bar{x}=\pi/4$, but $f(x)$ and $g(x)$ are not strictly convex with respect to the same $\eta(x)=1-\left(\frac{2}{\sqrt{2}}\right)\cos x$.

Rueda and Hanson (1988) defined:

Definition 2.3.3. *$f(x)$ and $g(x)$ as pseudo-Type I objective and constraint functions, respectively, with respect to $\eta(x)$ at $\bar{x}$ if there exists an n-dimensional vector function $\eta(x)$ defined for all $x\in P$ such that*

$$[\nabla_x f(x)]^T\eta(x,\bar{x})\geqq 0\Rightarrow f(\bar{x})-f(x)\geqq 0$$

and

$$[\nabla_x g(x)]^T\eta(x,\bar{x})\geqq 0\Rightarrow -g(x)\geqq 0.$$

Definition 2.3.4. *$f(x)$ and $g(x)$ as quasi-Type I objective and constraint functions, respectively, with respect to $\eta(x)$ at $\bar{x}$ if there exists an n-dimensional vector function $\eta(x)$ defined for all $x\in P$ such that*

$$f(x)-f(\bar{x})\leqq 0\Rightarrow[\nabla_x f(\bar{x})]^T\eta(x,\bar{x})\leqq 0.$$

and

$$-g(x)\leqq 0\Rightarrow[\nabla_x g(x)]^T\eta(x,\bar{x})\leqq 0.$$

Pseudo-Type II and quasi-Type II objective and constraint functions are defined similarly.

It was shown by Rueda and Hanson (1988) that:

1. Type I objective and constraint functions ⇒ pseudo-Type I objective and constraint functions, but the converse is not necessarily true, as can be seen from the following example.

Example 2.3.5. The functions $f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to R$ and $g:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to R$ defined by $f(x)=-\cos^2 x$ and $g(x)=-\cos x$ are pseudo-Type I functions with respect to $\eta(x)=-\frac{1}{2}+\left(\sqrt{2}/2\right)\cos x$ at $\bar{x}=-\pi/4$, but $f(x)$ and $g(x)$ are not Type I with respect to the same $\eta(x)=-\frac{1}{2}+\left(\sqrt{2}/2\right)\cos x$ as can be seen by taking $x=0$.

2. Type II objective and constraint functions ⇒ pseudo-Type II objective and constraint functions, but the converse is not necessarily true, as can be seen from the following.

Example 2.3.6. The functions $f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to R$ and $g:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to R$ defined by $f(x)=-\cos^2 x$ and $g(x)=-\cos x$ are pseudo-Type II functions with respect to $\eta(x)=\sin x\left(\cos x-\sqrt{2}/2\right)$ at $\bar{x}=-\pi/4$, but $f(x)$ and $g(x)$ are not Type II with respect to the same $\eta(x)=\sin x\left(\cos x-\sqrt{2}/2\right)$ as can be seen by taking $x=\pi/3$.

3. Type I objective and constraint functions $\Rightarrow$ quasi-Type I objective and constraint functions, but the converse is not necessarily true, as can be seen from the following example.

Example 2.3.7. The functions $f : (0,\pi) \to R$ and $g : (0,\pi) \to R$ defined by $f(x) = \sin^3 x$ and $g(x) = -\cos x$ are quasi-Type I functions with respect to $\eta(x) = -1$ at $\bar{x} = \pi/2$, but $f(x)$ and $g(x)$ are not Type I with respect to the same $\eta(x) = -1$ as can be seen by taking $\bar{x} = \pi/4$.

4. Type II objective and constraint functions $\Rightarrow$ quasi-Type II objective and constraint functions, but the converse is not necessarily true, as can be seen from the following example.

Example 2.3.8. The functions $f : (0,\infty) \to R$ and $g : (0,\infty) \to R$ defined by $f(x) = -\frac{1}{x}$ and $g(x) = 1 - x$ are quasi-Type II functions with respect to $\eta(x) = 1 - x$ at $\bar{x} = 1$, but $f(x)$ and $g(x)$ are not Type II with respect to the same $\eta(x) = 1 - x$ as can be seen by taking $x = 2$.

5. Strictly Type I objective and constraint functions $\Rightarrow$ Type I objective and constraint functions, but the converse is not necessarily true, as can be seen from the following example.

Example 2.3.9. The functions $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ and $g : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ defined by $f(x) = -\sin x$ and $g(x) = -\cos x$ are Type I functions with respect to $\eta(x) = \sin x$ at $\bar{x} = 0$, but $f(x)$ and $g(x)$ are not strictly Type I with respect to the same $\eta(x) = \sin x$ at $\bar{x} = 0$.

6. Strictly Type II objective and constraint functions $\Rightarrow$ Type II objective and constraint functions, but the converse is not necessarily true, as can be seen from the following example.

Example 2.3.10. The functions $f : \left(0, \frac{\pi}{2}\right) \to R$ and $g : \left(0, \frac{\pi}{2}\right) \to R$ defined by $f(x) = -\sin x$ and $g(x) = -e^{-x}$ are Type II functions with respect to $\eta(x) = 1$ at $\bar{x} = 0$, but $f(x)$ and$g(x)$ are not strictly Type I with respect to the same $\eta(x)$ at $\bar{x} = 0$.

Kaul et al. (1994) further extended the concepts of Rueda and Hanson (1988) to pseudo-quasi Type I, quasi-pseudo Type I objective and constraint functions as follows.

Definition 2.3.5. *$f(x)$ and $g(x)$ as quasi-pseudo-Type I objective and constraint functions, respectively, with respect to $\eta(x)$ at $\bar{x}$ if there exists an n-dimensional vector function $\eta(x)$ defined for all $x \in P$ such that*

$$f(x) - f(\bar{x}) \leqq 0 \Rightarrow [\nabla_x f(\bar{x})]^T \eta(x,\bar{x}) \leqq 0$$

and

$$[\nabla_x g(x)]^T \eta(x,\bar{x}) \geqq 0 \Rightarrow -g(x) \geqq 0.$$

Definition 2.3.6. *$f(x)$ and $g(x)$ as pseudo-quasi-Type I objective and constraint functions, respectively,with respect to $\eta(x)$ at $\bar{x}$ if there exists an n-dimensional vector function $\eta(x)$ defined for all $x \in P$ such that*

$$[\nabla_x f(\bar{x})]^T \eta(x,\bar{x}) \geqq 0 \Rightarrow f(x) - f(\bar{x}) \geqq 0$$

and

$$-g(x) \leqq 0 \Rightarrow [\nabla_x g(x)]^T \eta(x,\bar{x}) \leqq 0.$$

2.4 Univex and Related Functions

Let f be a differentiable function defined on a nonempty subset X of R^n and let $\phi : R \to R$ and $k : X \times X \to R_+$. For $x,\bar{x} \in X$, we write $k(x,\bar{x}) = \lim_{\lambda \to 0} b(x,\bar{x},\lambda) \geqq 0$. Bector et al. (1992) defined b-invex functions as follows.

Definition 2.4.1. *The function f is said to be B-invex with respect to η and k, at $\bar{x}$ if for all $x \in X$, we have*

$$k(x,\bar{x})[f(x) - f(\bar{x})] \geqq [\nabla_x f(\bar{x})]^T \eta(x,\bar{x}).$$

Bector et al. (1992) further extended this concept to univex functions as follows.

Definition 2.4.2. *The function f is said to be univex with respect to η,ϕ and k, at $\bar{x}$ if for all $x \in X$, we have*

$$k(x,\bar{x})\,\phi[f(x) - f(\bar{x})] \geqq [\nabla_x f(\bar{x})]^T \eta(x,\bar{x}).$$

Definition 2.4.3. *The function f is said to be quasi-univex with respect to η,ϕ and k, at $\bar{x}$ if for all $x \in X$, we have*

$$\phi[f(x) - f(\bar{x})] \leqq 0 \Rightarrow k(x,\bar{x})\,\eta(x,\bar{x})^{\mathrm{T}} \nabla_x f(\bar{x}) \leqq 0.$$

Definition 2.4.4. *The function f is said to be pseudo-univex with respect to η,ϕ and k, at $\bar{x}$ if for all $x \in X$, we have*

$$\eta(x,\bar{x})^{\mathrm{T}} \nabla_x f(\bar{x}) \geqq 0 \Rightarrow k(x,\bar{x})\,\phi[f(x) - f(\bar{x})] \geqq 0.$$

Bector et al. (1992) gave the following relations with some other generalized convex functions existing in the literature.

1. Every B-invex function is univex function with $\phi : R \to R$ defined as $\phi(a) = a, \forall a \in R$, but not conversely.

Example 2.4.1. Let $f : R \to R$ be defined by $f(x) = x^3$, where,

$$\eta(x,\bar{x}) = \begin{cases} x^2 + \bar{x}^2 + x\bar{x}, & x > \bar{x} \\ x - \bar{x}, x & \leqq \bar{x} \end{cases}$$

and

$$k(x,\bar{x}) = \begin{cases} \bar{x}^2/(x-\bar{x}), & x > \bar{x} \\ 0, & x \leqq \bar{x}. \end{cases}$$

Let $\phi : R \to R$ be defined by $\phi(a) = 3a$. The function f is univex but not b-invex, because for $x = 1,\ \bar{x} = 1/2,\ k(x,\bar{x})\,\phi[f(x) - f(\bar{x})] < \eta(x,\bar{x})^{\mathrm{T}} \nabla_x f(\bar{x})$.

2. Every invex function is univex function with $\phi : R \to R$ defined as $\phi(a) = a, \forall a \in R$, and $k(x,\bar{x}) \equiv 1$, but not conversely.

Example 2.4.2. The function considered in above example is univex but not invex, because for $x = -3, \bar{x} = 1, f(x) - f(\bar{x}) < \eta(x,\bar{x})^{\mathrm{T}} \nabla_x f(\bar{x})$.

3. Every convex function is univex function with $\phi : R \to R$ defined as $\phi(a) = a, \forall a \in R, k(x,\bar{x}) \equiv 1$, and $\eta(x,\bar{x}) \equiv x - \bar{x}$, but not conversely.

Example 2.4.3. The function considered in above example is univex but not convex, because for $x = -2, \bar{x} = 1, f(x) - f(\bar{x}) < (x - \bar{x})^{\mathrm{T}} \nabla_x f(\bar{x})$.

4. Every b-vex function is univex function with $\phi : R \to R$ defined as $\phi(a) = a, \forall a \in R$, and $\eta(x,\bar{x}) \equiv x - \bar{x}$, but not conversely.

Example 2.4.4. The function considered in above example is univex but not b-vex, because for $x = \frac{1}{10}, \bar{x} = \frac{1}{100}, k(x,\bar{x})[f(x) - f(\bar{x})] < (x - \bar{x})^{\mathrm{T}} \nabla_x f(\bar{x})$.

Rueda et al. (1995) obtained optimality and duality results for several mathematical programs by combining the concepts of type I functions and univex functions. They combined the Type I and univex functions as follows.

Definition 2.4.5. *The differentiable functions $f(x)$ and $g(x)$ are called Type I univex objective and constraint functions, respectively with respect to $\eta, \phi_0, \phi_1, b_0, b_1$ at $\bar{x} \in X$, if for all $x \in X$, we have*

$$b_0(x,\bar{x})\,\phi_0[f(x) - f(\bar{x})] \geqq \eta(x,\bar{x})^{\mathrm{T}} \nabla_x f(\bar{x})$$

and

$$-b_1(x,\bar{x})\,\phi_1[g(\bar{x})] \geqq \eta(x,\bar{x})^{\mathrm{T}} \nabla_x g(\bar{x}).$$

Rueda et al. (1995) gave examples of functions that are univex but not Type I univex.

Example 2.4.5. The functions $f, g : [1,\infty) \to R$, defined by $f(x) = x^3$ and $g(x) = 1 - x$, are univex at $\bar{x} = 1$ with respect to $b_0 = b_1 = 1, \eta(x,\bar{x}) = x - \bar{x}, \phi_0(a) = 3a, \phi_1(a) = 1$, but g does not satisfy the second inequality of the above definition at $\bar{x} = 1$.

They also pointed out that there are functions which are Type I univex but not univex.

Example 2.4.6. The functions $f, g : [1,\infty) \to R$, defined by $f(x) = -1/x$ and $g(x) = 1 - x$, are Type I univex with respect to $b_0 = b_1 = 1, \eta(x,\bar{x}) = -1/(x - \bar{x})$, $\phi_0(a) = a$, $\phi_1(a) = -a$, at $\bar{x} = 1$, but g is not univex at $\bar{x} = 1$.

Following Rueda et al. (1995), Mishra (1998b) gave several sufficient optimality conditions and duality results for multiobjective programming problems by combining the concepts of Pseudo-quasi-Type I, quasi-pseudo-Type I functions and univex functions.

2.5 *V*-Invex and Related Functions

Jeyakumar and Mond (1992) introduced the notion of V-invexity for a vector function $f = (f_1, f_2, \ldots, f_p)$ and discussed its applications to a class of constrained multiobjective optimization problems. We now give the definitions of Jeyakumar and Mond (1992) as follows.

Definition 2.5.1. *A vector function $f : X \to R^p$ is said to be V-invex if there exist functions $\eta : X \times X \to R^n$ and $\alpha_i : X \times X \to R^+ - \{0\}$ such that for each $x, \bar{x} \in X$ and for $i = 1, 2, \ldots, p$,*

$$f_i(x) - f_i(\bar{x}) \geqq \alpha_i(x,\bar{x}) \nabla f_i(\bar{x}) \eta(x,\bar{x}).$$

for $p = 1$ and $\bar{\eta}(x,\bar{x}) = \alpha_i(x,\bar{x}) \eta(x,\bar{x})$ the above definition reduces to the usual definition of invexity given by Hanson (1981).

Definition 2.5.2. *A vector function $f : X \to R^p$ is said to be V-pseudoinvex if there exist functions $\eta : X \times X \to R^n$ and $\beta_i : X \times X \to R^+ - \{0\}$ such that for each $x, \bar{x} \in X$ and for $i = 1, 2, \ldots, p$,*

$$\sum_{i=1}^{p} \nabla f_i(\bar{x}) \eta(x,\bar{x}) \geqq 0 \Rightarrow \sum_{i=1}^{p} \beta_i(x,\bar{x}) f_i(x) \geqq \sum_{i=1}^{p} \beta_i(x,\bar{x}) f_i(\bar{x}).$$

Definition 2.5.3. *A vector function $f : X \to R^p$ is said to be V-quasiinvex if there exist functions $\eta : X \times X \to R^n$ and $\delta_i : X \times X \to R^+ - \{0\}$ such that for each $x, \bar{x} \in X$ and for $i = 1, 2, \ldots, p$,*

$$\sum_{i=1}^{p} \delta_i(x,\bar{x}) f_i(x) \leqq \sum_{i=1}^{p} \delta_i(x,\bar{x}) f_i(\bar{x}) \Rightarrow \sum_{i=1}^{p} \nabla f_i(\bar{x}) \eta(x,\bar{x}) \leqq 0.$$

It is evident that every V-invex function is both V-pseudo-invex (with $\beta_i(x,\bar{x}) = \frac{1}{\alpha_i(x,\bar{x})}$) and V-quasi-invex (with $\delta_i(x,\bar{x}) = \frac{1}{\alpha_i(x,\bar{x})}$). Also if we set $p = 1$, $\alpha_i(x,\bar{x}) = 1$, $\beta_i(x,\bar{x}) = 1$, $\delta_i(x,\bar{x}) = 1$ and $\eta(x,\bar{x}) = x - \bar{x}$, then the above definitions reduce to those of convexity, pseudo-convexity and quasi-convexity, respectively.

Definition 2.5.4. *A vector optimization problem:*

$$\text{(VP)} \qquad V - \min(f_1, f_2, \ldots, f_p) \quad \textit{subject to } g(x) \leqq 0,$$

where $f_i : X \to R, i = 1,2,\ldots,p$ and $g : X \to R^m$ are differentiable functions on X, is said to be V-invex vector optimization problem if each $f_1, f_2, \ldots, f_p$ and $g_1, g_2, \ldots, g_m$ is a V-invex function.

Note that, invex vector optimization problems are necessarily V-invex, but not conversely. As a simple example, we consider following example from Jeyakumar and Mond (1992).

Example 2.5.1. Consider

$$\min_{x_1,x_2 \in R} \left(\frac{x_1^2}{x_2}, \frac{x_1}{x_2} \right)$$
$$\text{subject to } 1 - x_1 \le 1, 1 - x_2 \le 1.$$

Then it is easy to see that this problem is a V-invex vector optimization problem with $\alpha_1 = \frac{\bar{x}_2}{x_2}$, $\alpha_2 = \frac{\bar{x}_1}{x_1}$, $\beta_1 = 1 = \beta_2$, and $\eta(x,\bar{x}) = x - \bar{x}$; but clearly, the problem does not satisfy the invexity conditions with the same η.

It is also worth noticing that the functions involved in the above problem are invex, but the problem is not necessarily invex.

It is known (see Craven (1981)) that invex problems can be constructed from convex problems by certain nonlinear coordinate transformations. In the following, we see that V-invex functions can be formed from certain nonconvex functions (in particular from convex-concave or linear fractional functions) by coordinate transformations.

Example 2.5.2. Consider function, $h : R^n \to R^p$ defined by $h(x) = (f_1(\phi(x)), \ldots, f_p(\phi(x)))$, where $f_i : R^n \to R, i = 1,2,\ldots,p$, are strongly pseudo-convex functions with real positive functions $\alpha_i, \phi : R^n \to R^n$ is surjective with $\phi'(\bar{x})$ onto for each $\bar{x} \in R^n$. Then, the function h is V-invex.

Example 2.5.3. Consider the composite vector function $h(x) = (f_1(F_1(x)), \ldots, f_p(F_p(x)))$, where for each $i = 1, 2, \ldots, p$, $F_i : X_0 \to R$ is continuously differentiable and pseudolinear with the positive proportional function $\alpha_i(\cdot, \cdot)$, and $f_i : R \to R$ is convex. Then, $h(x)$ is $V-$invex with $\eta(x, y) = x - y$. This follows from the following convex inequality and pseudolinearity conditions:

$$\begin{aligned} f_i(F_i(x)) - f_i(F_i(y)) &\ge f_i'(F_i(y))(F_i(x) - F_i(y)) \\ &= f_i'(F_i(y))\,\alpha_i(x, y)\,F_i'(y)(x - y) \\ &= \alpha_i(x, y)\,(f_i \circ F_i)'(y)(x - y). \end{aligned}$$

For a simple example of a composite vector function, we consider

$$h(x_1, x_2) = \left(e^{x_1/x_2}, \frac{x_1 - x_2}{x_1 + x_2} \right), \quad \text{where} \quad X_0 = \left\{ (x_1, x_2) \in R^2 : x_1 \ge 1, x_2 \ge 1 \right\}.$$

Example 2.5.4. Consider the function $H(x) = (f_1((g_1 \circ \psi)(x)), \ldots, f_p((g_p \circ \psi)(x)))$, where each f_i is pseudolinear on R^n with proportional functions $\alpha_i(x, y), \psi$ is a differentiable mapping from R^n onto R^n such that $\psi'(y)$ is surjective for each $y \in R^n$, and $f_i : R \to R$ is convex for each i. Then H is V-invex.

Jeyakumar and Mond (1992) have shown that the V-invexity is preserved under a smooth convex transformation.

Proposition 2.5.1. *Let $\psi : R \to R$ be differentiable and convex with positive derivative everywhere; let $h : X_0 \to R^p$ be V-invex. Then, the function*

$$h_{\psi}(x) = (\psi(h_1(x)), \ldots, \psi(h_p(x))), \quad x \in X_0$$

is V-invex.

The following very important property of V-invex functions was also established by Jeyakumar and Mond (1992).

Proposition 2.5.2. *Let $f : R^n \to R^p$ be V-invex. Then $y \in R^n$ is a (global) weak minimum of f if and only if there exists $0 \neq \tau \in R^p$, $\tau \geq 0$, $\sum_{i=1}^{p} \tau_i f_i'(y) = 0$.*

By Proposition 2.5.2, one can conclude that for a V-invex vector function every critical point (i.e., $f_i'(y) = 0,\ i = 1, \ldots, p$) is a global weak minimum.

Hanson et al. (2001) extended the (scalarized) generalized type-I invexity into a vector (V-type-I) invexity.

Definition 2.5.5. *The vector problem (VP) is said to be V-type-I at $\bar{x} \in X$ if there exist positive real-valued functions α_i and β_j defined on $X \times X$ and an n-dimensional vector-valued function $\eta : X \times X \to R^n$ such that*

$$f_i(x) - f_i(\bar{x}) \geqq \alpha_i(x,\bar{x}) \nabla f_i(\bar{x}) \eta(x,\bar{x})$$

and

$$-g_j(\bar{x}) \geqq \beta_j(x,\bar{x}) \nabla g_j(\bar{x}) \eta(x,\bar{x}),$$

for every $x \in X$ and for all $i = 1,2,\ldots,p$ and $j = 1,2,\ldots,m$.

Definition 2.5.6. *The vector problem (VP) is said to be quasi-V-type-I at $\bar{x} \in X$ if there exist positive real-valued functions α_i and β_j defined on $X \times X$ and an n-dimensional vector-valued function $\eta : X \times X \to R^n$ such that*

$$\sum_{i=1}^{p} \tau_i \alpha_i(x,\bar{x}) [f_i(x) - f_i(\bar{x})] \leqq 0 \Rightarrow \sum_{\mathrm{i}=1}^{\mathrm{p}} \tau_i \eta(x,\bar{x}) \nabla f_i(\bar{x}) \leqq 0$$

and

$$-\sum_{j=1}^{m} \lambda_j \beta_j(x,\bar{x}) g_j(\bar{x}) \leqq 0 \Rightarrow \sum_{j=1}^{m} \lambda_j \eta(x,\bar{x}) \nabla g_j(\bar{x}) \leqq 0,$$

for every $x \in X$.

Definition 2.5.7. *The vector problem (VP) is said to be pseudo-V-type-I at* $\bar{x} \in X$ *if there exist positive real-valued functions* α_i*and* β_j *defined on* $X \times X$ *and an n-dimensional vector-valued function* $\eta : X \times X \to R^n$ *such that*

$$\sum_{i=1}^{p} \tau_i \eta(x,\bar{x}) \nabla f_i(\bar{x}) \geqq 0 \Rightarrow \sum_{i=1}^{p} \tau_i \alpha_i(x,\bar{x}) [f_i(x) - f_i(\bar{x})] \geqq 0$$

and

$$\sum_{j=1}^{m} \lambda_j \eta(x,\bar{x}) \nabla g_j(\bar{x}) \geqq 0 \Rightarrow -\sum_{j=1}^{m} \lambda_j \beta_j(x,\bar{x}) g_j(\bar{x}) \geqq 0,$$

for every $x \in X$.

Definition 2.5.8. *The vector problem (VP) is said to be quasi-pseudo-V-type-I at* $\bar{x} \in X$ *if there exist positive real-valued functions* α_i *and* β_j *defined on* $X \times X$ *and an n-dimensional vector-valued function* $\eta : X \times X \to R^n$ *such that*

$$\sum_{i=1}^{p} \tau_i \alpha_i(x,\bar{x}) [f_i(x) - f_i(\bar{x})] \leqq 0 \Rightarrow \sum_{i=1}^{p} \tau_i \eta(x,\bar{x}) \nabla f_i(\bar{x}) \leqq 0$$

and

$$\sum_{j=1}^{m} \lambda_j \eta(x,\bar{x}) \nabla g_j(\bar{x}) \geqq 0 \Rightarrow -\sum_{j=1}^{m} \lambda_j \beta_j(x,\bar{x}) g_j(\bar{x}) \geqq 0,$$

for every $x \in X$.

Definition 2.5.9. *The vector problem (VP) is said to be pseudo-quasi-V-type-I at* $\bar{x} \in X$ *if there exist positive real-valued functions* α_i *and* β_j *defined on* $X \times X$ *and an n-dimensional vector-valued function* $\eta : X \times X \to R^n$ *such that*

$$\sum_{i=1}^{p} \tau_i \eta(x,\bar{x}) \nabla f_i(\bar{x}) \geqq 0 \Rightarrow \sum_{i=1}^{p} \tau_i \alpha_i(x,\bar{x}) [f_i(x) - f_i(\bar{x})] \geqq 0$$

and

$$-\sum_{j=1}^{m} \lambda_j \beta_j(x,\bar{x}) g_j(\bar{x}) \leqq 0 \Rightarrow \sum_{j=1}^{m} \lambda_j \eta(x,\bar{x}) \nabla g_j(\bar{x}) \leqq 0,$$

for every $x \in X$.

Nevertheless the study of generalized convexity of a vector function is not yet sufficiently explored and some classes of generalized convexity have been introduced recently. Several attempts have been made by many authors to introduce possibly a most wide class of generalized convex function, which can meet the demand of a real life situation to formulate a nonlinear programming problem and therefore get a best possible solution for the same. Recently, Aghezzaf and Hachimi (2001) introduced a new class of functions, which we shall give in next section.

2.6 Further Generalized Convex Functions

Definition 2.6.1. *f is said to be weak strictly pseudoinvex with respect to η at $\bar{x} \in X$ if there exists a vector function $\eta(x,\bar{x})$ defined on $X \times X$ such that, for all $x \in X$,*

$$f(x) \leq f(\bar{x}) \Rightarrow \nabla f(\bar{x})\eta(x,\bar{x}) < 0.$$

This definition is a slight extension of that of the pseudoinvex functions. This class of functions does not contain the class of invex functions, but does contain the class of strictly pseudoinvex functions.

Every strictly pseudoinvex function is weak strictly pseudoinvex with respect to the same η. However, the converse is not necessarily true, as can be seen from the following example.

Example 2.6.1. The function $f = (f_1, f_2)$ defined on $X = R$, by $f_1(x) = x(x+2)$ and $f_2(x) = x(x+2)^2$ is weak strictly pseudoinvex function with respect to $\eta(x,\bar{x}) = x+2$ at $\bar{x} = 0$, but it s not strictly pseudoinvex with respect to the same $\eta(x,\bar{x})$ at $\bar{x}$ because for $\bar{x} = -2$, we have

$$f(x) \leqq f(\bar{x}) \text{ but } \nabla f(\bar{x})\eta(x,\bar{x}) = 0 \not< 0.$$

Definition 2.6.2. *f is said to be strong pseudoinvex with respect to η at $\bar{x} \in X$ if there exists a vector function $\eta(x,\bar{x})$ defined on $X \times X$ such that, for all $x \in X$,*

$$f(x) \leq f(\bar{x}) \Rightarrow \nabla f(\bar{x})\eta(x,\bar{x}) \leq 0.$$

Instead of the class of weak strictly pseudoinvex, the class of strong pseudoinvex functions does contain the class of invex functions. Also, every weak strictly pseudoinvex function is strong pseudoinvex with respect to the same η. However, the converse is not necessarily true, as can be seen from the following example.

Example 2.6.2. The function $f = (f_1, f_2)$ defined on $X = R$, by $f_1(x) = x^3$ and $f_2(x) = x(x+2)^2$ is strongly pseudoinvex function with respect to $\eta(x,\bar{x}) = x$ at $\bar{x} = 0$, but it is not weak strictly pseudoinvex with respect to the same $\eta(x,\bar{x})$ at $\bar{x}$ because for $\bar{x} = -1$

$$f(x) \leq f(\bar{x}) \text{ but } \nabla f(\bar{x})\eta(x,\bar{x}) = (0,-4)^T \not< 0,$$

also f is not invex with respect to the same η at $\bar{x}$, as can be seen by taking $\bar{x} = -2$.

There exist functions f that are pseudoinvex but not strong pseudoinvex with respect to the same η. Conversely, we can find functions that are strong pseudoinvex, but they are not pseudoinvex with respect to the same η.

Example 2.6.3. The function $f : R \to R^2$, defined by $f_1(x) = x(x-2)^2$ and $f_2(x) = x(x-3)$, is pseudoinvex with respect to $\eta(x,\bar{x}) = x - \bar{x}$ at $\bar{x} = 0$, but it is not weak strictly pseudoinvex with respect to the same $\eta(x,\bar{x})$ when $x = 2$.

Example 2.6.4. The function $f : R \to R^2$, defined by $f_1(x) = x(x-2)$ and $f_2(x) = x^2(x-1)$, is strong pseudoinvex with respect to $\eta(x,\bar{x}) = x - \bar{x}$ at $\bar{x} = 0$, but it is not pseudoinvex with respect to the same $\eta(x,\bar{x})$ at that point.

Remark 2.6.1. If f is both pseudoinvex and quasiinvex with respect to η at $\bar{x} \in X$, then it is strong pseudoinvex function with respect to the same η at $\bar{x}$.

Definition 2.6.3. *f is said to be weak quasiinvex with respect to η at $\bar{x} \in X$ if there exists a vector function $\eta(x,\bar{x})$ defined on $X \times X$ such that, for all $x \in X$,*

$$f(x) \leq f(\bar{x}) \Rightarrow \nabla f(\bar{x})\eta(x,\bar{x}) \leqq 0.$$

Every quasiinvex function is weak quasiinvex with respect to the same η. However, the converse is not necessarily true.

Example 2.6.5. Define a function $f : R \to R^2$, by $f_1(x) = x(x-2)^2$ and $f_2(x) = x^2(x-2)$, then the function is weak quasiinvex with respect to $\eta(x,\bar{x}) = x - \bar{x}$ at $\bar{x} = 0$, but it is not quasiinvex with respect to the same $\eta(x,\bar{x})$ at $\bar{x} = 0$, because $f(x) \leqq f(\bar{x})$ but $\nabla f(\bar{x})\eta(x,\bar{x}) \nleqq 0$, for $x = 2$.

Definition 2.6.4. *f is said to be weak pseudoinvex with respect to η at $\bar{x} \in X$ if there exists a vector function $\eta(x,\bar{x})$ defined on $X \times X$ such that, for all $x \in X$,*

$$f(x) < f(\bar{x}) \Rightarrow \nabla f(\bar{x})\eta(x,\bar{x}) \leq 0.$$

The class of weak pseudoinvex functions does contain the class of invex functions, pseudoinvex functions, strong pseudoinvex functions and strong quasiinvex functions.

Remark 2.6.2. Notice from Examples 2.6.1–2.6.4, that the concepts of weak strictly pseudoinvex, strong pseudoinvex, weak pseudoinvex, and pseudoinvex vector-valued functions are different, in general. However, they coincide in the scalar-valued case.

Definition 2.6.5. *f is said to be strong quasiinvex with respect to η at $\bar{x} \in X$ if there exists a vector function $\eta(x,\bar{x})$ defined on $X \times X$ such that, for all $x \in X$,*

$$f(x) \leqq f(\bar{x}) \Rightarrow \nabla f(\bar{x})\eta(x,\bar{x}) \leq 0.$$

Every strong quasiinvex function s both quasiivex and strong pseudoinvex with respect to the same η.

Aghezzaf and Hachimi (2001) introduced the class of weak prequasiinex functions by generalzing the class of preinvex (Ben-Israel and Mond 1986) and the class prequasiinvex functions (Suneja et al. 1993).

Definition 2.6.6. *We say that f is weak prequasiinvex at $\bar{x} \in X$ with respect to η if X is invex at $\bar{x}$ with respect to η and, for each $x \in X$,*

$$f(x) \leq f(\bar{x}) \Rightarrow f(\bar{x} + \lambda\eta(x,\bar{x})) \leqq f(\bar{x}), \quad 0 < \lambda \leqq 0.$$

Every prequasiinvex function is weak prequasiinvex with respect to the same η. But the converse is not true.

Example 2.6.6. The function $f : R \rightarrow R^2$, defined by $f_1(x) = x(x-2)^2$ and $f_2(x) = x(x-2)$, is weak prequasiivex at $\bar{x} = 0$ with respect to $\eta(x,\bar{x}) = x - \bar{x}$, but it is not prequasiinvex with respect to the same $\eta(x,\bar{x})$ at $\bar{x} = 0$, because $f_1(x) = x(x-2)^2$ is not prequasiinvex at $\bar{x} = 0$ with respect to same $\eta(x,\bar{x})$.

Chapter 3
Generalized Type I and Related Functions

3.1 Generalized Type I Univex Functions

Following Rueda et al. (1995) and Aghezzaf and Hachimi (2001), we define the generalized type I univex problems. In the following definitions, $b_0, b_1 : X \times X \times [0,1] \to R^+$, $b(x,a) = \lim_{\lambda \to 0} b(x,a,\lambda) \geq 0$, and b does not depend on λ if functions are differentiable, $\phi_0, \phi_1 : R \to R$ and $\eta : X \times X \to R^n$ is an n-dimensional vector-valued function.

Consider the following multiobjective programming problem:

$$\text{(VP)} \qquad \begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } g(x) \leqq 0, \\ &\qquad\qquad\quad x \in X, \end{aligned}$$

where $f : X \to R^k$, $g : X \to R^m$, X is a nonempty open subset of R^n.

Definition 3.1.1. *We say the problem (VP) is of weak strictly pseudo type I univex at $a \in X_0$ if there exist a real-valued function b_0, b_1, ϕ_0, ϕ_1 and η such that*

$$b_0(x,a)\phi_0[f(x) - f(a)] \leq 0 \Rightarrow (\nabla f(a))\eta(x,a) < 0,$$
$$-b_1(x,a)\phi_1[g(a)] \leqq 0 \Rightarrow (\nabla g(a))\eta(x,a) \leqq 0,$$

for every $x \in X_0$ and for all $i = 1, \ldots, p$, and $j = 1, \ldots, m$.

If (VP) is weak strictly pseudo type I univex at each $a \in X$, we say (VP) is of weak strictly pseudo type I univex on X.

Remark 3.1.1. If in the above definition we set $b_0(x,a) = 1 = b_1(x,a)$, ϕ_0 and ϕ_1 as identity functions, we get the weak strictly pseudo quasi type I functions defined in Aghezzaf and Hachimi (2001).

There exist functions which are weak strictly pseudoquasi type I univex, see the following example.

Example 3.1.1. The function $f : R^2 \to R^2$ defined by $f(x) = \left(x_1 e^{\sin x_2}, x_2(x_2-1)e^{\cos x_1}\right)$ and $g : R^2 \to R$ defined by $g(x) = 2x_1 + x_2 - 2$ are weak strictly pseudoquasi type I univex with respect to $b_0 = 1 = b_1, \phi_0$, and ϕ_1 are identity function on R and $\eta(x,a) = (x_1 + x_2 - 1, x_2 - x_1)$ at $a = (0,0)$.

Definition 3.1.2. *We say the problem (VP) is of strong pseudoquasi type I univex at $a \in X_0$ if there exist a real-valued function b_0, b_1, ϕ_0, ϕ_1 and η such that*

$$b_0(x,a)\,\phi_0[f(x) - f(a)] \le 0 \Rightarrow (\nabla f(a))\,\eta(x,a) \le 0,$$
$$-b_1(x,a)\,\phi_1[g(a)] \leqq 0 \Rightarrow (\nabla g(a))\,\eta(x,a) \leqq 0,$$

for every $x \in X_0$ and for all $i = 1, \ldots, p$, and $j = 1, \ldots, m$.

If (VP) is of *strong pseudoquasi type I univex at each $a \in X$*, we say (VP) is *of strong pseudoquasi type I univex on X.*

Example 3.1.2. The function $f : R^2 \to R^2$ and $g : R^2 \to R$ defined by $f(x) = \left(x_1(x_1-1)^2, x_2(x_2-1)^2(x_2^2+2)\right)$ and $g(x) = x_1^2 + x_2^2 - 9$ are strong pseudoquasi type I univex with respect to $b_0 = 1 = b_1, \phi_0$, and ϕ_1 are identity function on R and $\eta(x,a) = (x_1 - 1, x_2 - 1)$ at $a = (0,0)$, but (f,g) are not weak strictly pseudoquasi type I univex with respect to same b_0, b_1, ϕ_0, ϕ_1 and η as can be seen by taking $x = (1,-1)$.

Definition 3.1.3. *We say the problem (VP) is of weak quasistrictly pseudo type I univex b_0, b_1, ϕ_0, ϕ_1 and η at $a \in X_0$ if there exist a real-valued function b_0, b_1, ϕ_0, ϕ_1 and η such that*

$$b_0(x,a)\,\phi_0[f(x) - f(a)] \le 0 \Rightarrow (\nabla f(a))\,\eta(x,a) \leqq 0,$$
$$-b_1(x,a)\,\phi_1[g(a)] \leqq 0 \Rightarrow (\nabla g(a))\,\eta(x,a) \le 0,$$

for every $x \in X_0$ and for all $i = 1, \ldots, p$, and $j = 1, \ldots, m$.

If (VP) is of weak quasistrictly pseudo type I univex at each $a \in X$, we say (VP) is *of weak quasistrictly pseudo type I univex on X.*

Example 3.1.3. The function $f : R^2 \to R^2$ and $g : R^2 \to R$ defined by $f(x) = \left(x_1^3\left(x_1^2+1\right), x_2^2(x_2-1)^3\right)$ and $g(x) = \left((2x_1 - 4)\,e^{-x_2^2}, (x_1 + x_2 - 2)\left(x_1^2 + 2x_1 + 4\right)\right)$ are weak quasistrictly pseudo type I univex with respect to $b_0 = 1 = b_1, \phi_0$, and ϕ_1 are identity function on R and $\eta(x,a) = (x_1, x_2(1 - x_2))$ at $a = (0,0)$, but (f,g) are not type I univex with respect to same b_0, b_1, ϕ_0, ϕ_1 and η as can be seen by taking $x = (1,0)$. Type I univex functions are defined in Rueda et al. (1995).

Definition 3.1.4. *We say the problem (VP) is of weak strictly pseudo type I univex with respect to b_0, b_1, ϕ_0, ϕ_1 and η at $a \in X_0$ if there exist a real-valued function b_0, b_1, ϕ_0, ϕ_1 and η such that*

$$b_0(x,a)\,\phi_0[f(x) - f(a)] \le 0 \Rightarrow (\nabla f(a))\,\eta(x,a) < 0,$$
$$-b_1(x,a)\,\phi_1[g(a)] \leqq 0 \Rightarrow (\nabla g(a))\,\eta(x,a) < 0,$$

for every $x \in X_0$ and for all $i = 1, \ldots, p$, and $j = 1, \ldots, m$.

If (VP) is of weak strictly pseudo type I univex at each $a \in X$, we say (VP) is *of weak strictly pseudo type I univex on X.*

3.2 Nondifferentiable *d*–Type I and Related Functions

Following Rueda et al. (1995), we define the generalized d-type I univex functions. In the following definitions, $b_0, b_1 : X \times X \times [0,1] \to R^+$, $\phi_0, \phi_1 : R \to R$ and $\eta : X \times X \to R^n$ is an n-dimensional vector-valued function.

Definition 3.2.1. (f,g) *is said to be d-type-I univex with respect to* b_0, b_1, ϕ_0, ϕ_1 *and* η *at* $u \in X$ *if there exist* b_0, b_1, ϕ_0, ϕ_1 *and* η *such that for all* $x \in X$,

$$b_0(x,u)\,\phi_0[f(x) - f(u)] \geqq f'(u, \eta(x,u))$$

and

$$-b_1(x,u)\,\phi_0[g(u)] \geqq g'(u, \eta(x,u)).$$

Definition 3.2.2. (f,g) *is said to be weak strictly-pseudoquasi d-type-I univex with respect to* b_0, b_1, ϕ_0, ϕ_1 *and* η *at* $u \in X$ *if there exist* b_0, b_1, ϕ_0, ϕ_1 *and* η *such that for all* $x \in X$,

$$b_0(x,u)\,\phi_0[f(x) - f(u)] \leq 0 \Rightarrow f'(u, \eta(x,u)) < 0$$

and

$$-b_1(x,u)\,\phi_1[g(u)] \leqq 0 \Rightarrow g'(u, \eta(x,u)) \leqq 0.$$

Definition 3.2.3. (f,g) *is said to be strong pseudoquasi d-type-I univex with respect to* b_0, b_1, ϕ_0, ϕ_1 *and* η *at* $u \in X$ *if there exist* b_0, b_1, ϕ_0, ϕ_1 *and* η *such that for all* $x \in X$,

$$b_0(x,u)\,\phi_0[f(x) - f(u)] \leq 0 \Rightarrow f'(u, \eta(x,u)) \leq 0$$

and

$$-b_1(x,u)\,\phi_1[g(u)] \leqq 0 \Rightarrow g'(u, \eta(x,u)) \leqq 0.$$

Definition 3.2.4. (f,g) *is said to be weak quasistrictly-pseudo d-type-I univex with respect to* b_0, b_1, ϕ_0, ϕ_1 *and* η *at* $u \in X$ *if there exist* b_0, b_1, ϕ_0, ϕ_1 *and* η *such that for all* $x \in X$,

$$b_0(x,u)\,\phi_0[f(x) - f(u)] \leq 0 \Rightarrow f'(u, \eta(x,u)) \leqq 0$$

and

$$-b_1(x,u)\,\phi_1[g(u)] \leqq 0 \Rightarrow g'(u, \eta(x,u)) \leq 0.$$

Definition 3.2.5. (f,g) *is said to be weak strictly-pseudo d-type-I univex with respect to* b_0, b_1, ϕ_0, ϕ_1 *and* η *at* $u \in X$ *if there exist* b_0, b_1, ϕ_0, ϕ_1 *and* η *such that for all* $x \in X$,

$$b_0(x,u)\,\phi_0[f(x) - f(u)] \leq 0 \Rightarrow f'(u, \eta(x,u)) < 0$$

and

$$-b_1(x,u)\,\phi_1[g(u)] \leqq 0 \Rightarrow g'(u, \eta(x,u)) < 0.$$

Remark 3.2.1. If we take $b_0 = b_1 = 1$ and ϕ_0 and ϕ_1 as the identity functions and if functions f, and g are differentiable functions, then the above definitions reduce to the definitions given in Aghezzaf and Hachimi (2001).

Remark 3.2.2. If we take $b_0 = b_1 = 1$ and ϕ_0 and ϕ_1 as the identity functions, the functions defined in the above definitions extend the ones given in Suneja and Srivastava (1997) to the directionally differentiable form of the functions given in Aghezzaf and Hachimi (2001) and Antczak (2002a).

Remark 3.2.3. If functions are differentiable, then the above definitions are extensions of the ones given in Rueda et al. (1995) and in Mishra (1998b).

For examples of differentiable generalized type functions, one can refer to Aghezzaf and Hachimi (2001).

3.3 Continuous-Time Analogue of Generalized Type I Functions

Let $I = [a,b]$ be a real interval and $\psi : I \times R^n \times R^n \to R$ be a continuously differentiable function. In order to consider $\psi(t,x,\dot{x})$, where $x : I \to R^n$ is differentiable with derivative $\dot{x}$, we denote the partial derivatives of ψ by ψ_t,

$$\psi_x = \left[\frac{\partial \psi}{\partial x^1}, \ldots, \frac{\partial \psi}{\partial x^n}\right], \psi_{\dot{x}} = \left[\frac{\partial \psi}{\partial \dot{x}^1}, \ldots, \frac{\partial \psi}{\partial \dot{x}^n}\right].$$

The partial derivatives of the other functions used will be written similarly. Let $C(I,R^n)$ denote the space of piecewise smooth functions x with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differential operator D is given by

$$u^i = Dx^i \Leftrightarrow x^i(t) = \alpha + \int_a^t u^i(s)\, ds,$$

where α is a given boundary value. Therefore, $D = \frac{d}{dt}$ except at discontinuities.

We consider the following continuous vector optimization problem

(MP)

$$\begin{aligned}
&\text{minnimize} \int_a^b f(t,x,\dot{x})\, dt = \left(\int_a^b f_1(t,x,\dot{x})\, dt, \ldots, \int_a^b f_p(t,x,\dot{x})\, dt\right) \\
&\text{subject to} \qquad x(a) = \alpha, \quad x(b) = \beta, \\
&\qquad\qquad\qquad g(t,x,\dot{x}) \leqq 0, \quad t \in I, \\
&\qquad\qquad\qquad x \in C(I,R^n),
\end{aligned}$$

where $f_i : I \times R^n \times R^n \to R, i \in P = \{1, \ldots, p\}, g : I \times R^n \times R^n \to R^m$ are assumed to be continuously differentiable functions.

Definition 3.3.1. *A pair* (f,g) *is said to be weak strictly pseudoquasi type I at* $u \in C(I,R^n)$ *with respect to* η *if there exists a vector function* $\eta : I \times R^n \times R^n \to R^n$ *with* $\eta(t,x,x) = 0$ *such that for* $\forall x \in K$*,*

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b f(t,u,\dot{u})\,dt$$

$$\Rightarrow \int_a^b \left[\eta(t,x,u)^T f_x(t,u,\dot{u}) + \frac{d}{dt}(\eta(t,x,u))^T f_{\dot{x}}(t,u,\dot{u})\right]dt < 0$$

and

$$-\int_a^b g(t,u,\dot{u})\,dt \leqq 0$$

$$\Rightarrow \int_a^b \left[\eta(t,x,u)^T g_x(t,u,\dot{u}) + \frac{d}{dt}\Big(\eta(t,x,u)^T g_{\dot{x}}(t,u,\dot{u})\Big)\right]dt \leqq 0.$$

Definition 3.3.2. *A pair* (f,g) *is said to be strong pseudoquasi type I at* $u \in C(I,R^n)$ *with respect to* η *if there exists a vector function* $\eta : I \times R^n \times R^n \to R^n$ *with* $\eta(t,x,x) = 0$ *such that for* $\forall x \in K$*,*

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b f(t,u,\dot{u})\,dt$$

$$\Rightarrow \int_a^b \left[\eta(t,x,u)^T f_x(t,u,\dot{u}) + \frac{d}{dt}(\eta(t,x,u))^T f_{\dot{x}}(t,u,\dot{u})\right]dt \leq 0$$

and

$$-\int_a^b g(t,u,\dot{u})\,dt \leqq 0$$

$$\Rightarrow \int_a^b \left[\eta(t,x,u)^T g_x(t,u,\dot{u}) + \frac{d}{dt}\Big(\eta(t,x,u)^T g_{\dot{x}}(t,u,\dot{u})\Big)\right]dt \leqq 0.$$

Definition 3.3.3. *A pair* (f,g) *is said to be weak quasistrictly pseudo type I at* $u \in C(I,R^n)$ *with respect to* η *if there exists a vector function* $\eta : I \times R^n \times R^n \to R^n$ *with* $\eta(t,x,x) = 0$ *such that for* $\forall x \in K$*,*

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b f(t,u,\dot{u})\,dt$$

$$\Rightarrow \int_a^b \left[\eta(t,x,u)^T f_x(t,u,\dot{u}) + \frac{d}{dt}(\eta(t,x,u))^T f_{\dot{x}}(t,u,\dot{u})\right] dt \leqq 0$$

and

$$-\int_a^b g(t,u,\dot{u})dt \leqq 0$$

$$\Rightarrow \int_a^b \left[\eta(t,x,u)^T g_x(t,u,\dot{u}) + \frac{d}{dt}\left(\eta(t,x,u)^T g_{\dot{x}}(t,u,\dot{u})\right)\right] dt \leq 0.$$

Definition 3.3.4. *A pair (f,g) is said to be weak strictly pseudo type I at $u \in C(I,R^n)$ with respect to η if there exists a vector function $\eta : I \times R^n \times R^n \to R^n$ with $\eta(t,x,x) = 0$ such that for $\forall x \in K$,*

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b f(t,u,\dot{u})\,dt$$

$$\Rightarrow \int_a^b \left[\eta(t,x,u)^T f_x(t,u,\dot{u}) + \frac{d}{dt}(\eta(t,x,u))^T f_{\dot{x}}(t,u,\dot{u})\right] dt < 0$$

and

$$-\int_a^b g(t,u,\dot{u})\,dt \leqq 0$$

$$\Rightarrow \int_a^b \left[\eta(t,x,u)^T g_x(t,u,\dot{u}) + \frac{d}{dt}\left(\eta(t,x,u)^T g_{\dot{x}}(t,u,\dot{u})\right)\right] dt < 0.$$

The following remark will be frequently used in the proofs of various theorems throughout Chapter 4 and Chapter 5.

Remark 3.3.1. Let $\psi : I \times R^n \times R^n \to R$ be a continuously differentiable function with respect to each of its arguments. Let $x, \mathrm{u} : I \to R^n$ be differentiable with $x(a) = u(a) = \alpha$ and $x(b) = u(b) = \beta$. Then we have

$$\int_a^b \frac{d}{dt}(\eta(t,x,u))^T \psi_{\dot{x}}(t,u,\dot{u})\,dt = -\int_a^b \eta(t,x,u)^T \frac{d}{dt}(\psi_{\dot{x}}(t,u,\dot{u}))\,dt.$$

3.4 Nondifferentiable Continuous-Time Analogue of Generalized Type I Functions

Let U be a nonempty subset of Z and $\psi : U \to R$ be a locally Lipschitz function on U.

Definition 3.4.1. *The function $\psi : U \to R$ is called invex at $\bar{z} \in U$ if there exists a function $\eta : U \times U \to Z$ such that*

$$\psi(z) - \psi(\bar{z}) \geq \psi^0(\bar{z}; \eta(z, \bar{z})) \text{ for all } z \in U.$$

The function ψ is strictly invex if the above inequality is strict for $z \neq \bar{z}$.

We also need to use an invexity notion in the continuous-time context. Let $U \subset R^n$ be a nonempty subset of R^n and $\bar{x} \in X$. Suppose that a given function $\psi : [0,T] \times U \to R$ is locally Lipschitz throughout $[0,T]$.

Definition 3.4.2. *The function $\psi(t, \cdot)$ is said to be invex at $\bar{x}(t)$ if there exists $\eta : U \times U \to R^n$ such that the function $t \to \eta(x(t), \bar{x}(t))$ is in $L_\infty^n[0,T]$ and*

$$\psi(t, x(t)) - \psi(t, \bar{x}(t)) \geq \psi^0(t, \bar{x}(t); \eta(x(t), \bar{x}(t))) \quad \text{a.e. in } [0,T] \text{ for all } x \in X.$$

We say that ψ is strictly invex if the above inequality is strict for $x(t) \neq \bar{x}(t)$ a.e. in $[0,T]$.

Definition 3.4.3. *The pair of functions $\psi : U \to R$ and $\varphi : U \to R$ is said to be type I at $\bar{z} \in U$ if there exists a function $\eta : U \times U \to Z$ such that*

$$\psi(z) - \psi(\bar{z}) \geq \psi^0(\bar{z}; \eta(z, \bar{z})) \quad \forall z \in U$$

and

$$-\varphi(\bar{z}) \geq \varphi^0(\bar{z}; \eta(z, \bar{z})) \quad \forall z \in U.$$

We also need to use a type I notion in the continuous-time context. Let $U \subset R^n$ be a nonempty subset of R^n and $\bar{x} \in X$. Suppose that a given function $\psi : [0,T] \times U \to R$ is locally Lipschitz throughout $[0,T]$.

Definition 3.4.4. *The pair of functions $\psi(t, \cdot)$ and $\varphi(t, \cdot)$ is said to be type I at $\bar{x}(t)$ if there exists $\eta : U \times U \to R^n$ such that the function $t \to \eta(x(t), \bar{x}(t))$ is in $L_\infty^n[0,T]$ with*

$$\psi(t, x(t)) - \psi(t, \bar{x}(t)) \geq \psi^0(t, \bar{x}(t); \eta(x(t), \bar{x}(t))) \quad \text{a.e. in } [0,T] \quad \forall x \in X$$

and

$$-\psi(t, \bar{x}(t)) \geq \psi^0(t, \bar{x}(t); \eta(x(t), \bar{x}(t))) \quad \text{a.e. in } [0,T] \quad \forall x \in X.$$

3.5 Generalized Convex Functions in Complex Spaces

Let C^n (or R^n) denote an n-dimensional complex (or real) spaces, $C^{m\times n}$ (or $R^{m\times n}$) the collection of $m\times n$ complex matrices (or real) matrices, $R^n_+ = \{x\in R^n : x_i \geq 0 \text{ for all } i=1,2,\ldots,n\}$ the nonnegative orthant of R^n, and $x \geq y$ represent $x-y\in R^n_+$ for $x,y\in R^n$. For $z\in C^n$, let the real vectors $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote real and imaginary parts of each component of z respectively, and write $\bar{z} = \mathrm{Re}(z) - i\mathrm{Im}(z)$ as the conjugate of z. Given a matrix $A = [\mathrm{a_{ij}}] \in C^{m\times n}$, we use $\bar{A} = [\bar{\mathrm{a}}_{\mathrm{ij}}]$ to express its conjugate transpose. The inner product of $x,y\in C^n$ is $\langle x,y\rangle = y^H x$.

A nonempty subset S of C^m is said to be a *polyhedral cone* if there is an integer r and a matrix $K\in C^{r\times m}$ such that $S = \{z\in C^m : \mathrm{Re}(Kz)\geq 0\}$. The dual (also polar) of S is $S^* = \{\omega\in C^m : z\in S \Rightarrow \mathrm{Re}\langle z,\omega\rangle \geq 0\}$. It is clear that $S=S^{**}$ if S is a polyhedral cone. Define the *manifold* $Q = \left\{\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \in C^{2n} : \omega_2 = \bar{\omega}_1\right\}$.

For $\xi = (z,\bar{z})\in S^0$, we define $W(\xi) = \{\varsigma\in W : \mathrm{Re}\phi(\xi,\varsigma) = \sup_{\mu\in W}\mathrm{Re}\phi(\xi,\mu)\}$, and note that $W(\xi)$ is compact and nonempty. For each $\varsigma\in W$, the function $\phi(\cdot,\cdot) : C^{2n}\times C^{2m}\to C$, and $g : C^{2n}\to C^p$ are differentiable with respect to $\xi = (z,\bar{z})$ if

$$\phi(z,\bar{z};\varsigma) - \phi(z_0,\bar{z}_0;\varsigma) = \eta^T(z,z_0)\nabla_z\phi(z_0,\bar{z}_0;\varsigma) + \eta^H(z,z_0)\nabla_{\bar{z}}\phi(z_0,\bar{z}_0;\varsigma) + O(|z-z_0|)$$

and

$$g(z,\bar{z}) - g(z_0,\bar{z}_0) = \eta^T(z,z_0)\nabla_z g(z_0,\bar{z}_0) + \eta^H(z,z_0)\nabla_{\bar{z}} g(z_0,\bar{z}_0) + O(|z-z_0|),$$

where, $\nabla_z\phi$, $\nabla_{\bar{z}}\phi$, $\nabla_z g$, and $\nabla_{\bar{z}} g$ denote, respectively, the vectors of partial derivatives of ϕ and g with respect to z and $\bar{z}$. Further $O(|z-z_0|)/|z-z_0| \to 0$ as $z\to z_0$. Note that with $u\in C^p$

$$\nabla_{\bar{z}} u^H g(z,z_0) \equiv \nabla_z g(z_0,\bar{z}_0)\,\bar{u}.$$

We also need the following definitions, which are extensions of definitions given by Smart (1990) and Smart and Mond (1991).

Definition 3.5.1. *(1) The real part of $\phi(\cdot,\varsigma)$ is said to be invex with respect to R_+ on the manifold $Q \equiv \{(\omega_1,\omega_2)\in C^{2n} : \omega_2 = \omega_1\}$ if there exists a function $\eta : C^n\times C^n\to C^n$ such that*

$$\mathrm{Re}\left[\varphi(z_2,\bar{z}_2;\varsigma) - \varphi(z_1,\bar{z}_1;\varsigma) - \eta^T(z_2,z_1)\nabla_z\varphi(z_1,\bar{z}_1;\varsigma) - \eta^H(z_2,z_1)\nabla_{\bar{z}}\varphi(z_1,\bar{z}_1;\varsigma)\right] \geq 0 \quad \textit{for all } z_1,z_2\in C^n.$$

The function $-g$ is said to be *invex with respect to the polyhedral cone S* if there exists a function $\eta : C^n\times C^n\to C^n$ such that

$$\mathrm{Re}\left\langle u, g(z_2,\bar{z}_2) - g(z_1,\bar{z}_1) - \eta^T(z_2,z_1)\nabla_z g(z_1,\bar{z}_1) - \eta^H(z_2,z_1)\nabla_{\bar{z}} g(z_1,\bar{z}_1)\right\rangle \geq 0, \quad \text{for all } z_1,z_2\in C^n.$$

In the above definition, if the strict inequality holds, the real part of $\phi(\cdot,\varsigma)$ and $-g$ are said to be *strict invex with respect to R_+ and the polyhedral cone S*, respectively.

(2) The real part of $\phi(\cdot,\varsigma)$ is said to be *pseudoinvex* with respect to R_+ on the manifold $Q \equiv \left\{(\omega_1,\omega_2) \in C^{2n} : \omega_2 = \omega_1\right\}$ if there exists a function $\eta : C^n \times C^n \to C^n$ such that

$$\begin{aligned}&\mathrm{Re}\left[\eta^T(z_2,z_1)\nabla_z\varphi(z_1,\bar{z}_1;\varsigma)+\eta^H(z_2,z_1)\nabla_{\bar{z}}\varphi(z_1,\bar{z}_1;\varsigma)\right]\geq 0\\&\qquad\Rightarrow Re\left[\varphi(z_2,\bar{z}_2;\varsigma)-\varphi(z_1,\bar{z}_1;\varsigma)\right]\geq 0,\quad\text{for all}\quad z_1,z_2\in C^n.\end{aligned}$$

The function $-g$ is said to be *pseudoinvex with respect to the polyhedral cone S* if there exists a function $\eta : C^n \times C^n \to C^n$ such that

$$\begin{aligned}&\mathrm{Re}\left\langle u,\eta^T(z_2,z_1)\nabla_z g(z_1,\bar{z}_1)+\eta^H(z_2,z_1)\nabla_{\bar{z}} g(z_1,\bar{z}_1)\right\rangle\geq 0\\&\qquad\Rightarrow \mathrm{Re}\left\langle u,g(z_2,\bar{z}_2)-g(z_1,\bar{z}_1)\right\rangle\geq 0,\quad\text{for all}\quad z_1,z_2\in C^n.\end{aligned}$$

In the above definition, if the strict inequalities hold for all $z_2 \neq z_1$, the real part of $\phi(\cdot,\varsigma)$ and $-g$ are said to be *strict pseudoinvex* with respect ot $\eta$and R_+ and the polyhedral cone S, respectively.

(3) The real part of $\phi(\cdot,\varsigma)$ is said to be *quasiinvex* with respect to R_+ on the manifold $Q \equiv \left\{(\omega_1,\omega_2) \in C^{2n} : \omega_2 = \omega_1\right\}$ if there exists a function $\eta : C^n \times C^n \to C^n$ such that

$$\begin{aligned}&\mathrm{Re}\left[\varphi(z_2,\bar{z}_2;\varsigma)-\varphi(z_1,\bar{z}_1;\varsigma)\right]\leq 0\\&\Rightarrow \mathrm{Re}\left[\eta^T(z_2,z_1)\nabla_z\varphi(z_1,\bar{z}_1;\varsigma)+\eta^H(z_2,z_1)\nabla_{\bar{z}}\varphi(z_1,\bar{z}_1;\varsigma)\right]0,\quad\text{for all}\quad z_1,z_2\in C^n.\end{aligned}$$

The function $-g$ is said to be *quasiinvex* with respect to the polyhedral cone S if there exists a function $\eta : C^n \times C^n \to C^n$ such that

$$\begin{aligned}&\mathrm{Re}\left\langle u,g(z_2,\bar{z}_2)-g(z_1,\bar{z}_1)\right\rangle\leq 0\\&\Rightarrow \mathrm{Re}\left\langle u,\eta^T(z_2,z_1)\nabla_z g(z_1,\bar{z}_1)+\eta^H(z_2,z_1)\nabla_{\bar{z}} g(z_1,\bar{z}_1)\right\rangle\leq 0,\ \text{for all}\ z_1,z_2\in C^n.\end{aligned}$$

3.6 Semilocally Connected Type I Functions

Let $X_0 \subseteq R^n$ be a set and $\eta : X_0 \times X_0 \to R^n$ be a vector application. We say that X_0 is *invex at* $\bar{x} \in X_0$ if $\bar{x} + \lambda\eta(x,\bar{x}) \in X_0$ for any $x \in X_0$ and $\lambda \in [0,1]$. We say that the set X_0 is invex if X_0 is invex at any $x \in X_0$.

We remark that if $\eta(x,\bar{x}) = x - \bar{x}$ for any $x \in X_0$ then X_0 is invex at $\bar{x}$ iff X_0 is a convex set at $\bar{x}$.

Definition 3.6.1. *We say that the set $X_0 \subseteq R^n$ is an η-locally starshaped set at $x,\bar{x} \in X_0$, if for any $x \in X_0$, there exists $0 < a_\eta(x,\bar{x}) \leqq 1$ such that $\bar{x} + \lambda\eta(x,\bar{x}) \in X_0$ for any $\lambda \in [0, a_\eta(x,\bar{x})]$.*

Definition 3.6.2. *(Preda 1996) Let $f : X_0 \to R^n$ be a function, where $X_0 \subseteq R^n$ is an η-locally starshaped set at $\bar{x} \in X_0$. We say that f is*

(a) Semilocally Pre-invex (slpi) at $\bar{x}$ if, corresponding to $\bar{x}$ and each $x \in X_0$, there exists a positive number $d_\eta(x,\bar{x}) \leqq a_\eta(x,\bar{x})$ such that $f(\bar{x}+\lambda\eta(x,\bar{x})) \leqq \lambda f(x)+(1-\lambda)f(\bar{x})$ for $0<\lambda<d_\eta(x,\bar{x})$.
(b) Semilocally quasi-preinvex (slqpi) at $\bar{x}$ if, corresponding to $\bar{x}$ and each $x \in X_0$, there exists a positive number $d_\eta(x,\bar{x}) \leqq a_\eta(x,\bar{x})$ such that $f(x) \leqq f(\bar{x})$ and $0<\lambda<d_\eta(x,\bar{x})$ implies $f(\bar{x}+\lambda\eta(x,\bar{x})) \leqq f(\bar{x})$.

Definition 3.6.3. *Let $f : X_0 \to R^n$ be a function, where $X_0 \subseteq R^n$ is an η-locally starshaped set at $\bar{x} \in X_0$. We say that f is η-semidifferentiable at $\bar{x}$ if $(df)^+(\bar{x},\eta(x,\bar{x}))$ exists for each $\bar{x} \in X_0$, where*

$$(df)^+(\bar{x},\eta(x,\bar{x})) = \lim_{\lambda\to 0+}\frac{1}{\lambda}\left[f(\bar{x}+\lambda\eta(x,\bar{x}))-f(\bar{x})\right]$$

(the right derivative at $\bar{x}$ along the direction $\eta(x,\bar{x})$). If f is η-semidifferentiable at any $\bar{x} \in X_0$, then f is said to be η-semidifferentiable on X_0.

Remark 3.6.1. If $\eta(x,\bar{x}) = x-\bar{x}$, the η-semidifferentiability is the semidifferentiability notion. As is given by Preda (2003), if a function is directionally differentiable, then it is semidifferentiable but the converse is not true.

Definition 3.6.4. *(Preda 1996) We say that f is semilocally pseudo–preinvex (slppi) at $\bar{x}$ if for any $\bar{x} \in X_0$, $(df)^+(\bar{x},\eta(x,\bar{x})) \geqq 0 \Rightarrow f(x) \geqq f(\bar{x})$. If f is slppi at any $\bar{x} \in X_0$, then f is said to be slppi on X_0.*

Definition 3.6.5. *Let X and Y be two subsets of X_0 and $\bar{y} \in Y$. We say that Y is η-locally starshaped at $\bar{y}$ with respect to X if for any $x \in X$ there exists $0<a_\eta(x,\bar{y}) \leqq 1$ such that $\bar{y}+\lambda\eta(x,\bar{y}) \in Y$ for any $0 \leqq \lambda \leqq a_\eta(x,\bar{y})$.*

Definition 3.6.6. *Let Y be η-locally starshaped at $\bar{y}$ with respect to X and f be an η-semidifferentiable function at $\bar{y}$. We say that f is*

(a) slppi at $\bar{y} \in Y$ with respect to X, if for any $x \in X$, $(df)^+(\bar{y},\eta(x,\bar{y})) \geqq 0 \Rightarrow f(x) \geqq f(\bar{y})$.
(b) strictly semilocally pseudo-preinvex (sslppi) at $\bar{y} \in Y$ with respect to X, if for any $x \in X, x \neq y\,(df)^+(\bar{y},\eta(x,\bar{y})) \geqq 0 \Rightarrow f(x) > f(\bar{y})$.

We say that f is *(slppi) sslppi on Y with respect to X*, if f is (slppi) sslppi at any point of Y with respect to X.

Definition 3.6.7. *(Elster and Nehse 1980). A function $f : X_0 \to R^k$ is a convexlike if for any $x,y \in X_0$ and $0 \leqq \lambda \leqq 1$, there is $z \in X_0$ such that*

$$f(z) \leqq \lambda f(x)+(1-\lambda)f(y).$$

Remark 3.6.2. The convex and the peinvex functions are convexlike functions.

We consider the following vector fractional optimization problem:

$$\text{(VFP)} \quad \text{minimize} \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)} \right)$$
$$\text{subject to} \begin{cases} h_j(x) \leqq 0, & j = 1, 2, \ldots, m. \\ x \in X_0, \end{cases}$$

where $X_0 \subseteq R^n$ is a nonempty set and $g_i(x) > 0$ for all $x \in X_0$ and each $i = 1, \ldots, p$. Let $f = (f_1, \ldots, f_p)$, $g = (g_1, \ldots, g_p)$ and $h = (h_1, \ldots, h_m)$.

We put $X = \{x \in X_0 : h_j(x) \leqq 0, j = 1, 2, \ldots, m\}$ for the feasible set of problem (VFP).

Definition 3.6.8. *We say that the problem (VFP) is η-semidifferentiable Type I-preinvex at $\bar{x}$ if for any $\bar{x} \in X_0$, we have*

$$f_i(x) - f_i(\bar{x}) \geqq (df_i)^+ (\bar{x}, \eta(x, \bar{x})), \quad \forall i \in P,$$
$$g_i(x) - g_i(\bar{x}) \leqq (dg_i)^+ (\bar{x}, \eta(x, \bar{x})), \quad \forall i \in P,$$
$$-h_j(\bar{x}) \geqq (dh_j)^+ (\bar{x}, \eta(x, \bar{x})), \quad \forall j \in M.$$

Definition 3.6.9. *We say that the problem (VFP) is η-semidifferentiable pseudo-quasi-Type I-preinvex at $\bar{x}$ if for any $x \in X_0$, we have*

$$(df_i)^+ (\bar{x}, \eta(x, \bar{x})) \geqq 0 \Rightarrow f_i(x) \geqq f_i(\bar{x}), \quad \forall i \in P,$$
$$(dg_i)^+ (\bar{x}, \eta(x, \bar{x})) \geqq 0 \Rightarrow g_i(x) \leqq g_i(\bar{x}) \quad \forall i \in P,$$
$$-h_j(\bar{x}) \leqq 0 \Rightarrow (dh_j)^+ (\bar{x}, \eta(x, \bar{x})) \leqq 0, \quad \forall j \in M.$$

The problem (VFP) is η-*semidifferentiable pseudo-quasi-Type I-preinvex* on X_0 if it is η-semidifferentiable pseudo-quasi-Type I-preinvex at any $\bar{x} \in X_0$.

Definition 3.6.10. *We say that the problem (VFP) is η-semidifferentiable quasi-pseudo-Type I-preinvex at $\bar{x}$ if for any $x \in X_0$, we have*

$$f_i(x) \leqq f_i(\bar{x}) \Rightarrow (df_i)^+ (\bar{x}, \eta(x, \bar{x})) \leqq 0, \quad \forall i \in P,$$
$$g_i(x) \leqq g_i(\bar{x}) \Rightarrow (dg_i)^+ (\bar{x}, \eta(x, \bar{x})) \geqq 0 \quad \forall i \in P,$$
$$(dh_j)^+ (\bar{x}, \eta(x, \bar{x})) \geqq 0 \Rightarrow -h_j(\bar{x}) \geqq 0, \quad \forall j \in M.$$

The problem (VFP) is η-semidifferentiable quasi-pseudo-Type I-preinvex on X_0 if it is η-semidifferentiable pseudo-quasi-Type I-preinvex at any $\bar{x} \in X_0$.

3.7 (Ғ, ρ, σ, θ)-V-Type-I and Related n-Set Functions

Let (X,Λ,μ) be a finite atomless measure space with $L_1(X,\Lambda,\mu)$ separable, and let d be the pseudometric on Λ^n defined by

$$d(R,S)=\left[\sum_{i=1}^{n}\mu^2(R_i\Delta S_i)\right]^{1/2},\quad R=(R_1,R_2,\ldots,R_n),\quad S=(S_1,S_2,\ldots,S_n)\in\Lambda^n,$$

where Δ denotes the symmetric difference; thus , (Λ^n,d) is a pseudometric space. For $h\in L_1(X,\Lambda,\mu)$ and $T\in\Lambda$ with characteristic function $\chi_T\in L_\infty(X,\Lambda,\mu)$, the integral $\int_T h\,d\mu$ will be denoted by $\langle h,\chi_T\rangle$.

Next we recall the notion of differentiability and convexity for n-set functions. They were originally introduced by Morris (1979) for set functions, and subsequently extended by Corley (1987) for n-set functions.

Definition 3.7.1. *A function $F:\Lambda\to R$ is said to be differentiable at S^* if there exists $DF(S^*)\in L_1(X,\Lambda,\mu)$,called the derivative of F at S^*, such that for each $S\in\Lambda$,*

$$F(S)=F(S^*)+\langle DF(S^*),\chi_S-\chi_{S^*}\rangle+V_F(S,S^*),$$

where $V_F(S,S^*)$ is $o(d(S,S^*))$, that is, $\lim_{d(S,S^*)\to 0}V_F(S,S^*)/d(S,S^*)=0$.

Definition 3.7.2. *A function $G:\Lambda^n\to R$ is said to have a partial derivative at $S^*=(S_1^*,S_2^*,\ldots,S_n^*)\in\Lambda^n$ with respect to its i^{th} argument if the function $F(S_i)=G\left(S_1^*,\ldots,S_{i-1,}^*S_i^*,S_{i+1}^*,\ldots,S_n^*\right)$ has derivative $DF(S_i^*),i\in\underline{n}$; in that case, the i^{th} partial derivative of G at S^* is defined to be $D_iG(S^*)=DF(S_i^*),i\in\underline{n}$.*

Definition 3.7.3. *A function $G:\Lambda^n\to R$ is said to be differentiable at S^* if all the partial derivatives $D_iG(S^*),i\in\underline{n}$ exist and*

$$G(S)=G(S^*)+\sum_{i=1}^{n}\left\langle DG_i(S^*),\chi_{S_i}-\chi_{S_i^*}\right\rangle+W_G(S,S^*),$$

where $W_G(S,S^*)$ is $o(d(S,S^*))$, for all $S\in\Lambda^n$.

It was shown by Morris (1979) that for any triplet $(S,T,\lambda)\in\Lambda\times\Lambda\times[0,1]$, there exist sequences $\{S_k\}$ and $\{T_k\}\in\Lambda$ such that

$$\chi_{S_k}\overset{w^*}{\to}\lambda\chi_{S\backslash T}\quad\text{and}\quad\chi_{T_k}\overset{w^*}{\to}\lambda\chi_{T\backslash S}$$

imply

$$\chi_{S_k\cup T_k\cup(S\cap T)}\overset{w^*}{\to}\lambda\chi_S+(1-\lambda)\chi_T,$$

where $\overset{w^*}{\to}$ denotes the weak* convergence of elements in $L_\infty(X,\Lambda,\mu)$ and $S\backslash T$ is the complement of T relative to S. The sequence $\{V_k(\lambda)\}=\{S_k\cup T_k\cup(S\cap T)\}$ satisfying (2.1) and (2.2) is called the Morris sequence associated with (S,T,λ).

It was shown by Corley (1987) and Morris (1979) that if a differentiable function $F : \Lambda \to R$ is *convex*, then

$$F(S) \geqq F(T) + \sum_{i=1}^{n} \langle D_i F(T), \chi_{S_i} - \chi_{T_i} \rangle, \quad \forall S, T \in \Lambda^n.$$

Following the introduction of the notion of convexity for set functions by Morris (1979) and its extension for n-set functions by Corley (1987), various generalizations of convexity for set and n-set functions were proposed by Preda and Stancu-Minasian (1997a).

For predecessors and point-function counterparts of these convexity concepts, readers can refer to the original papers where the extensions to set and n-set functions are discussed. A survey of some advances in the area of generalized convex functions and their role in developing optimality conditions and duality relations for optimization problems is given by Pini and Singh (1997).

For the purpose of formulating and proving various sufficiency criteria and duality results for (P), in this study we shall use a new class of generalized convex n-set functions, called (Ŧ, ρ, σ, θ)-V-Type-I and related nonconvex functions, that will be defined later in this section. This class of functions may be viewed as an n-set version of a combination of three classes of point-functions: Ŧ-convex functions, type-I functions and V-invex functions, which were introduced by Hanson and Mond (1982), Jeyakumar and Mond (1992), and Tanaka and Maruyama (1984).

Let $S, S^* \in \Lambda^n$, let the function $F : \Lambda^n \to R^p$, with components $F_i, i \in \underline{p}$, be differentiable at S^*, let Ŧ$(S,S^*;.) : L_1^n(X, \Lambda, \mu) \to R$ be a sublinear function, and let $\theta : \Lambda^n \times \Lambda^n \to \Lambda^n \times \Lambda^n$ be a function such that $S \neq S^* \Rightarrow \theta(S, S^*) \neq (0,0)$.

Definition 3.7.4. *The pair of functions* (F, G) *are said to be (Ŧ, ρ, σ, θ)-V-type-I at* S^* *if there exist functions* $\alpha_i : \Lambda^n \times \Lambda^n \to R_+ \backslash \{0\}, i \in \underline{p}$, $\beta_j : \Lambda^n \times \Lambda^n \to R_+ \backslash \{0\}, j \in \underline{m}$, $\rho \in R$ *and* $\overline{\rho} \in R$ *such that for each* $S \in \Lambda^n$, $i \in \underline{p}$ *and* $j \in \underline{m}$,

$$F_i(S) - F_i(S^*) \geq Ŧ(S, S^*; \alpha_i(S, S^*) DF_i(S^*)) + \rho d^2(\theta(S, S^*))$$

and

$$-G_j(S^*) \geqq Ŧ(S, S^*; \beta_j(S, S^*) DG_j(S^*)) + \bar{\rho} d^2(\theta(S, S^*)).$$

Definition 3.7.5. *The pair of functions* (F, G) *are said to be (Ŧ, ρ, σ, θ)-V-pseudo-quasi-type-I at* S^* *if there exist functions* $\alpha_i : \Lambda^n \times \Lambda^n \to R_+ \backslash \{0\}, i \in \underline{p}$, $\beta_j : \Lambda^n \times \Lambda^n \to R_+ \backslash \{0\}, j \in \underline{m}$, $\rho \in R$ *and* $\overline{\rho} \in R$ *such that for each* $S \in \Lambda^n$, $i \in \underline{p}$ *and* $j \in \underline{m}$,

$$Ŧ\left(S, S^*; \sum_{i=1}^{p} DF_i(S^*)\right) \geqq -\rho d^2(\theta(S, S^*)) \Rightarrow \sum_{i=1}^{p} \alpha_i(S, S^*) F_i(S) \geq \sum_{i=1}^{p} \alpha_i(S, S^*) F_i(S^*)$$

and

$$-\sum_{j=1}^{m} \beta_j(S, S^*) G_j(S^*) \leqq 0 \Rightarrow Ŧ\left(S, S^*; \sum_{j=1}^{m} DG_j(S^*)\right) \leqq -\overline{\rho} d^2(\theta(S, S^*)).$$

Definition 3.7.6. *The pair of functions* (F,G) *are said to be* ($\digamma$, ρ, σ, θ)*-V-quasi-pseudo-type-I at* S^* *if there exist functions* $\alpha_i : \Lambda^n \times \Lambda^n \to R_+ \setminus \{0\}, i \in \underline{p}$, $\beta_j : \Lambda^n \times \Lambda^n \to R_+ \setminus \{0\}, j \in \underline{m}$, $\rho \in R$ *and* $\overline{\rho} \in R$ *such that for each* $S \in \Lambda^n$, $i \in \underline{p}$ *and* $j \in \underline{m}$,

$$\sum_{i=1}^{p} \alpha_i(S,S^*)F_i(S) \leqq \sum_{i=1}^{p} \alpha_i(S,S^*)F_i(S^*) \Rightarrow \digamma\left(S,S^*;\sum_{i=1}^{p} DF_i(S^*)\right) \leqq -\rho d^2(\theta(S,S^*))$$

and

$$\digamma\left(S,S^*;\sum_{j=1}^{m} Dg_j(S^*)\right) \geqq -\overline{\rho} d^2(\theta(S,S^*)) \Rightarrow \quad -\sum_{j=1}^{m} \beta_j(S,S^*)G_j(S^*) \geqq 0.$$

3.8 Nondifferentiable d-V-Type-I and Related Functions

Consider the following vector optimization problem

$$\begin{aligned} (\text{P}) \qquad & \text{minimize } f(x) \\ & \text{subject to } g(x) \leqq 0, \\ & \quad x \in X, \end{aligned}$$

where $f : X \to R^k$, $g : X \to R^m$, X is a nonempty open subset of R^n, $\eta : X \times X \to R^n$ is a vector function. Through this paper, $f(u,\eta(x,u)) = \lim_{\lambda \to 0^+} \frac{f(u+\lambda\eta(x,u))-f(u)}{\lambda}$. A similar notation is made for $g(u,\eta(x,u))$.

Let $D = \{x \in X : g(x) \leqq 0\}$ be the set of all the feasible solutions for (P) and denote $I = \{1,\dots,k\}, M = \{1,2,\dots,m\}$, $J(x) = \{j \in M : g_j(x) = 0\}$, and $J(x) = \{j \in M : g_j(x) < 0\}$. It is obvious that $J(x) \cup J(x) = M$.

Definition 3.8.1. (f_i, g_j) $i = 1,2,\dots,p$ *and* $j = 1,2\dots,m$ *is said to be d-V-type-I with respect to* η, $\alpha_i(x,u)$ *and* $\beta_j(x,u)$ *at* $u \in X$ *if there exist vector functions* $\eta : X \times X \to R^n, \alpha_i(x,u) X \times X \to R_+$ *and* $\beta_j(x,u) X \times X \to R_+$ *such that for all* $x \in X$,

$$f_i(x) - f_i(u) \geqq \alpha_i(x,u) f_{\mathrm{i}}(u,\eta(x,u))$$

and

$$-g_j(u) \geqq \beta_j(x,u) g_j(u,\eta(x,u)).$$

Definition 3.8.2. (f_i, g_j) $i = 1,2,\dots,p$ *and* $j = 1,2,\dots,m$ *is said to be weak strictly-pseudoquasi d-V-type-I with respect to* $\eta, \alpha_i(x,u)$ *and* $\beta_j(x,u)$ *at* $u \in X$ *if there exist functions* $\eta: X \times X \to R^n$, $\alpha_i(x,u) X \times X \to R_+$ *and* $\beta_j(x,u) X \times X \to R_+$ *such that for all* $x \in X$,

$$\sum_{i=1}^{p} \alpha_i(x,u) f_i(x) \leq \sum_{i=1}^{p} \alpha_i(x,u) f_i(u) \Rightarrow \sum_{i=1}^{p} f_i{}'(u,\eta(x,u)) < 0$$

and

$$-\sum_{j=1}^{m} \beta_j(x,u) g_j(u) \leqq 0 \Rightarrow \sum_{j=1}^{m} g_j{}'(u,\eta(x,u)) \leqq 0.$$

Definition 3.8.3. (f_i, g_j) *$i = 1,2,\ldots,p$ and $j = 1,2,\ldots,m$ is said to be strong pseudo-quasi d-V-type-I with respect to η, $\alpha_i(x,u)$ and $\beta_j(x,u)$ at $u \in X$ if there exist functions $\eta : X \times X \to R^n$, $\alpha_i(x,u) X \times X \to R_+$ and $\beta_j(x,u) X \times X \to R_+$ such that for all $x \in X$,*

$$\sum_{i=1}^{p} \alpha_i(x,u) f_i(x) \leq \sum_{i=1}^{p} \alpha_i(x,u) f_i(u) \Rightarrow \sum_{i=1}^{p} f_i'(u,\eta(x,u)) \leq 0$$

and

$$-\sum_{j=1}^{m} \beta_j(x,u) g_j(u) \leqq 0 \Rightarrow \sum_{j=1}^{m} g_j'(u,\eta(x,u)) \leqq 0.$$

Definition 3.8.4. (f_i, g_j) *$i = 1,2,\ldots,p$ and $j = 1,2,\ldots,m$ is said to be weak quasi-strictly-pseudo d-V-type-I with respect to η, $\alpha_i(x,u)$ and $\beta_j(x,u)$ at $u \in X$ if there exist functions $\eta : X \times X \to R^n$, $\alpha_i(x,u) X \times X \to R_+$ and $\beta_j(x,u) X \times X \to R_+$ such that for all $x \in X$,*

$$\sum_{i=1}^{p} \alpha_i(x,u) f_i(x) \leq \sum_{i=1}^{p} \alpha_i(x,u) f_i(u) \Rightarrow \sum_{i=1}^{p} f_i'(u,\eta(x,u)) \leqq 0$$

and

$$-\sum_{j=1}^{m} \beta_j(x,u) g_j(u) \leqq 0 \Rightarrow \sum_{j=1}^{m} g_j'(u,\eta(x,u)) \leq 0.$$

Definition 3.8.5. (f_i, g_j) *$i = 1,2,\ldots,p$ and $j = 1,2,\ldots,m$ is said to be weak strictly-pseudo d-V-type-I with respect to η, $\alpha_i(x,u)$ and $\beta_j(x,u)$ at $u \in X$ if there exist functions $\eta : X \times X \to R^n$, $\alpha_i(x,u) X \times X \to R_+$ and $\beta_j(x,u) X \times X \to R_+$ such that for all $x \in X$,*

$$\sum_{i=1}^{p} \alpha_i(x,u) f_i(x) \leq \sum_{i=1}^{p} \alpha_i(x,u) f_i(u) \Rightarrow \sum_{i=1}^{p} f_i'(u,\eta(x,u)) < 0$$

and

$$-\sum_{j=1}^{m} \beta_j(x,u) g_j(u) \leqq 0 \Rightarrow \sum_{j=1}^{m} g_j'(u,\eta(x,u)) < 0.$$

Remark 3.8.1. The functions defined above are different from those in Jeyakumar and Mond (1992), Suneja and Srivastava (1997), Aghezzaf and Hachimi (2000), Antczak (2002a), Hanson et al. (2001), and Mishra et al. (2005). For examples of differentiable generalized type functions, one can refer to Aghezzaf and Hachimi (2000).

3.9 Nonsmooth Invex and Related Functions

we recall some notions of nonsmooth analysis. For more details, see Clarke (1983). Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative octant. In the sequel, X be a nonempty open subset of R^n.

Definition 3.9.1. *A function $f : X \to R$ is said to be Lipschitz near $x \in X$ if for some $K > 0$,*

$|f(y) - f(z)| \leq K \|y - z\|$, $\forall y, z$ *within a neighborhood of x.*

We say that f: $X \to R$ is locally Lipschitz on X if it is Lipschitz near any point of X.

Definition 3.9.2. *If $f : X \to R$ is Lipschitz at $x \in X$, the generalized derivative (in the sense of Clarke) of f at $x \in X$ in the direction $v \in R^n$, denoted by $f^0(x; v)$, is given by*

$$f^0(x; v) = \limsup_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda}.$$

Definition 3.9.3. *The Clarke's generalized gradient of f at $x \in X$, denoted by $\partial f(x)$, is defined as follows:*

$$\partial f(x) = \left\{ \xi \in R^n : f^0(x; v) \geq \xi^T v \ \text{for all } v \in R^n \right\}.$$

It follows that, for any $v \in R^n$

$$f^0(x; v) = \max \left\{ \xi^T v : \xi \in \partial f(x) \right\}.$$

Definition 3.9.4. *The nondifferentiable function $f : X \to R$ is invex with respect to $\eta : X \times X \to R^n$ if*

$$f(x) - f(u) \geqq \xi^T \eta(x, u), \quad \forall \xi \in \partial f(u), \forall x, u \in X.$$

Definition 3.9.5. *The nondifferentiable function $f : X \to R$ is strictly-invex with respect to $\eta : X \times X \to R^n$ if*

$$f(x) - f(u) > \xi^T \eta(x, u), \quad \forall \xi \in \partial f(u), \forall x \neq u \in X.$$

Definition 3.9.6. *The nondifferentiable function $f : X \to R$ is pseudo-invex with respect to $\eta : X \times X \to R^n$ if*

$$f(x) - f(u) < 0 \Rightarrow \xi^T \eta(x, u) < 0, \quad \forall \xi \in \partial f(u), \forall x, u \in X.$$

Definition 3.9.7. *Let X be an invex set in R^n with respect to $\eta : X \times X \to R^n$. Then $F : X \to R^n$ is said to be (strictly) pseudo-invex monotone with respect to η on X if for every pair of distinct points $x, y \in X$,*

$$\langle u, \eta(y, x) \rangle \geq 0, \Rightarrow \langle v, \eta(y, x) \rangle \ (>) \geq 0, \quad \forall u \in F(x) \text{ and } v \in F(y).$$

Definition 3.9.8. *The nondifferentiable function $f : X \to R$ is strictly pseudo-invex with respect to $\eta\colon X \times X \to R^n$ if*

$$\xi^T \eta(x,u) \geq 0 \Rightarrow f(x) > f(u), \quad \forall \xi \in \partial f(u), \forall x, u \in X.$$

3.10 Type I and Related Functions in Banach Spaces

Let E, F, and G be three Banach spaces. Consider the following mathematical programming problem:

$$\text{(P)} \qquad \text{minimize}\{f(x) : x \in C, -g(x) \in K\},$$

where f and g are mappings from E into F and G, respectively, and C and K are two subsets of E and G.

The Clarke generalized directional derivative of a locally Lipschitz function from E into IR, as definition 3.9.8 at x in the direction d, denoted by $f^0(x;d)$ (see Clarke (1983), is given by

$$f^0(x;d) = \limsup_{\substack{x \to x^0 \\ t \downarrow 0}} \frac{f(x+td) - f(x^0)}{t}.$$

The Clarke generalized gradient of ϕ at x is given by

$$\partial\phi(\bar{x}) = \{x^* \in E^* : \phi^0(\bar{x},d) \geq < x^*, d >, \forall d \in X\},$$

where E^* denotes the topological dual of E and $< \bullet, \bullet >$ is the duality pairing.

Let C be a nonempty subset of E and consider its distance function, i.e., the function $\delta_C(.) : E \to \text{IR}$ defined by

$$\delta_C(x) = \inf\{||x - c|| : c \in C\}.$$

The distance function is not everywhere differentiable, but is globally Lipschitz.

Let $\bar{x} \in C$. A vector $d \in E$ is said to be *tangent to C* at $\bar{x}$ if

$$\delta_C{}^0(\bar{x}, d) = 0.$$

The set of tangent vectors to C at $\bar{x}$ is a closed convex cone in E, called the (Clarke) *tangent cone to C at $\bar{x}$* and denoted by $T_C(\bar{x})$.

Definition 3.10.1. *A mapping h: $E \to G$ is said to be strongly compact Lipschitzian at $\bar{x} \in E$ if there exist a multifunction R: $E \to$ comp(G) (comp(G) is the set of all norm compact subsets of G) and a function $r : ExE \to IR_+$ satisfying the following conditions:*

(i) $\lim_{x \to \bar{x}, d \to 0} r(x,d) = 0;$

(ii) There exists $\alpha > 0$ *such that* $t^{-1}[h(x+td)-h(x)] \in R(d)+||d||\,r(x,t)B_G$*, for all* $x \in \bar{x}+\alpha B_G$ *and* $t \in (0,\alpha)$*, where* B_G *denotes the closed unit ball around the origin of G ;*

(iii) $R(0)=\{0\}$ *and R is upper semicontinuous.*

Remark 3.10.1. If G is finite-dimensional, then h is strongly compact Lipschitian at $\bar{x}$ if and only if it is locally Lipschitz near $\bar{x}$. If h is strongly compact Lipschitzian, then for all $u^* \in G^*, (u^* \circ f)(x) = < u^*, h(x) >$ is locally Lipschitz.

From now on, let $Q \subset F$ and $K \subset G$ denote pointed closed convex cones with nonempty interior; let Q^*, K^* be their respective dual cones. The cone Q induces a partial order $\leq$ on F given by

$$z' \leq z \quad \text{if } z - z' \in Q$$
$$z' \leq z \quad \text{if } z - z' \in \operatorname{int} Q$$

$z' \geq z$ is the negation of the first one of the above relations and $z' > z$ is the negation of the second one of the above relations. Analogously, K induces a partial order on G.

Phuong et al. (1995) introduced the following notion of *invexity* for a locally Lipschitz real-valued function $\phi : E \to IR$, with respect to a nonempty set $C \subset E$.

Definition 3.10.2. ϕ *is said to be invex at* $x \in C$*, with respect to C, if for every* $y \in C$*, there is* $\eta(y,x) \in T_C(x)$ *such that*

$$\phi(y) - \phi(x) \geq \phi^0(x; \eta(y,x)).$$

ϕ *is invex on C if this inequality holds for every x,* $y \in C$*.*

Following Phuong et al. (1995), Brandao et al. (1999) extended the notion of invexity for functions between Banach spaces in a broader sense, as follows.

Definition 3.10.3. $f : E \to F$ *and* $g : E \to G$ *are invex if* u^* *o f and* v^* *o g are invex in the sense of Definition 3.10.2, for all* $u^* \in Q^*$ *and* $v^* \in K^*$*.*

We extend the notions of type-I function introduced by Hanson and Mond (1987a), pseudotype I, quasitype I functions introduced by Rueda and Hanson (1988) and pseudoquasitype-I and quasipseudotype-I functions introduced by Kaul et al. (1994) to the sense of Phuong et al. (1995), as follows:

Definition 3.10.4. *The locally Lipschitz real-valued functions* f : $E \to IR$ *and g:* $E \to IR$ *are said to be type-I at* $x \in C$*, with respect to C, if for every* $y \in C$*, there is* $\eta(y,x) \in T_C(x)$ *such that*

$$f(y) - f(x) \geq f^0(x; \eta(y,x))$$
$$-g(x) \geq g^0(x; \eta(y,x)).$$

Definition 3.10.5. *(f, g) is said to be quasitype-I at* $x \in C$*, with respect to C, if for every* $y \in C$*, there is* $\eta(y,x) \in T_C(x)$ *such that*

$$f(y) \leq f(x) \Rightarrow f^0(x;\eta(y,x)) \leq 0.$$
$$-g(x) \leq 0 \Rightarrow g^0(x;\eta(y,x)) \leq 0.$$

Definition 3.10.6. (f,g) *is said to be pseudo type I at* $x \in C$, *with respect to* C, *if for every* $y \in C$, *there is* $\eta(y,x) \in T_C(x)$ *such that*

$$f^0(x;\eta(y,x)) \geq 0 \Rightarrow f(y) \geq f(x)$$

$$g^0(x;\eta(y,x)) \geq 0 \Rightarrow -g(x) \leq 0.$$

Definition 3.10.7. (f,g) *is said to be quasipseudo type I at* $x \in C$, *with respect to* C, *if for every* $y \in C$, *there is* $\eta(y,x) \in T_C(x)$ *such that*

$$f(y) \leq f(x) \Rightarrow f^0(x;\eta(y,x)) \leq 0,$$
$$g^0(x;\eta(y,x)) \geq 0 \Rightarrow -g(x) \geq 0.$$

If in the above definition, we have

$$g^0(x;\eta(y,x)) \geq 0 \Rightarrow -g(x) > 0.$$

Then, we say that (f,g) *is quasistrictlypseudo type I at* $x \in C$.

Definition 3.10.8. (f,g) *is said to be pseudoquasi type I at* $x \in C$, *with respect to* C, *if for every* $y \in C$, *there is* $\eta(y,x) \in T_C(x)$ *such that*

$$f^0(x;\eta(y,x)) \geq 0 \Rightarrow f(y) \geq f(x)$$
$$-g(x) \leq 0 \Rightarrow g^0(x;\eta(y,x)) \leq 0.$$

We use the notion of generalized invexity (type I, pseudo type I, quasi type I, etc.) for functions between Banach spaces in a broad sense. Formally, in the following sense, we say $f : E \rightarrow F$ and $g : E \rightarrow G$ are *type I, quasitype I, pseudotype I, quasipseudo-type I, pseudoquasi type I at* $x \in C$ if $u^* \circ f$ and $v^* \circ g$ are type I, quasitype-I, pseudo-type I, quasi-pseudotype I, pseudo quasi type I, in the sense of Definitions 3.10.4, 3.10.5, 3.10.6, 3.10.7, and 3.10.8, respectively, for all $u^* \in Q^*$ and $v^* \in K^*$.

Chapter 4
Optimality Conditions

In this chapter, we study optimality conditions for several mathematical programs involving type-I and other related functions.

Throughout this chapter, the following convention for vectors in R^n will be followed:

$$x > y \text{ if and only if } x_i > y_i,\ i = 1, 2, \ldots, n,$$
$$x \geqq y \text{ if and only if } x_i \geqq y_i,\ i = 1, 2, \ldots, n,$$
$$x \geq y \text{ if and only if } x_i \geqq y_i,\ i = 1, 2, \ldots, n, \text{ but } x \neq y.$$

4.1 Optimality Conditions for Vector Optimization Problems

In this section, we establish some sufficient optimality conditions for an $a \in X_0$ to be an efficient solution of a vector optimization problem under various generalized type I univex functions defined in the previous chapter.

We consider the following vector optimization problem:

$$\text{(VP)} \qquad \text{minimize } f(x) = (f_1(x), \ldots, f_p(x))$$
$$\text{subject to } \quad g(x) \leq 0,$$
$$x \in X \subseteq R^n,$$

where $f : X \to R^p$ and $g : X \to R^m$ are differentiable functions and $X \subseteq R^n$ is an open set. Here the minimization means finding the collection of efficient points defined in previous chapter.

Let X_0 be the set of all feasible solutions of (VP).

Theorem 4.1.1. *(Sufficiency). Suppose that*

(i) $a \in X_0$*;*
(ii) There exist $\tau^0 \in R^p$, $\tau^0 > 0$, *and* $\lambda \in R^m$, $\lambda^0 \geqq 0$, *such that*

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Nonconvex Optimization and Its Applications.

(a) $\tau^0 \nabla f(a) + \lambda^0 \nabla g(a) = 0$,
(b) $\lambda^0 g(a) = 0$,
(c) $\tau^0 e = 1$, *where* $e = (1, \ldots, 1)^T \in R^p$;

(iii) Problem (VP) is strong pseudoquasi type I univex at $a \in X_0$ *with respect to some* b_0, b_1, ϕ_0, ϕ_1 *and* η;
(iv) $u \leq 0 \Rightarrow \phi_0(u) \leq 0$ *and* $u \leqq 0 \Rightarrow \phi_1(u) \leqq 0$,
(v) $b_0(x,a) > 0$, *and* $b_1(x,a) \geqq 0$;

for all feasible x. Then a is an efficient solution to (VP).

Proof. Suppose contrary to the result that a is not an efficient solution to (VP). Then there exists a feasible solution x to (VP) such that

$$f(x) \leq f(a).$$

By conditions (iv), (v) and the above inequality, we have

$$b_0(x,a)\, \phi_0[f(x) - f(a)] \leq 0. \tag{4.1.1}$$

By the feasibility of a, we have

$$-\lambda^0 g(a) \leqq 0.$$

By conditions (iv), (v) and the above inequality, we have

$$-b_1(x,a)\, \phi_1\left[\lambda^0 g(a)\right] \leqq 0. \tag{4.1.2}$$

By inequalities (4.1.1), (4.1.2), and condition (iii), we have

$$(\nabla f(a))\, \eta(x,a) \leq 0, \text{ and } \lambda^0 \nabla g(a)\, \eta(x,a) \leqq 0.$$

Since $\tau^0 > 0$, the above inequalities give

$$\left[\tau^0 \nabla f(a) + \lambda^0 \nabla g(a)\right] \eta(x,a) < 0. \tag{4.1.3}$$

which contradicts condition (ii). This completes the proof. □

Theorem 4.1.2. *(Sufficiency). Suppose that*

(i) $a \in X_0$;
(ii) There exist $\tau^0 \in R^p$, $\tau^0 \geq 0$, *and* $\lambda \in R^m$, $\lambda^0 \geqq 0$, *such that*

(a) $\tau^0 \nabla f(a) + \lambda^0 \nabla g(a) = 0$,
(b) $\lambda^0 g(a) = 0$,
(c) $\tau^0 e = 1$, *where* $e = (1, \ldots, 1)^T \in R^p$;

(iii) Problem (VP) is weak strictly pseudo quasi type I univex at $a \in X_0$ *with respect to some* b_0, b_1, ϕ_0, ϕ_1 *and* η;

(iv) $u \leq 0 \Rightarrow \phi_0(u) \leq 0$ *and* $u \leqq 0 \Rightarrow \phi_1(u) \leqq 0$,
(v) $b_0(x,a) > 0$, *and* $b_1(x,a) \geqq 0$;

for all feasible x. Then a is an efficient solution to (VP).

Proof. Suppose contrary to the result that a is not an efficient solution to (VP). Then there exists a feasible solution x to (VP) such that

$$f(x) \leq f(a).$$

By conditions (iv), (v), and the above inequality, we get (4.1.1). By the feasibility of a, conditions (iv) and (v), give (4.1.2).

By inequalities (4.1.1), (4.1.2), and condition (iii), we have

$$(\nabla f(a))\, \eta(x,a) < 0, \text{ and } \lambda^0 \nabla g(a)\, \eta(x,a) \leqq 0.$$

Since $\tau^0 \geq 0$, the above inequalities give

$$\left[\tau^0 \nabla f(a) + \lambda^0 \nabla g(a)\right] \eta(x,a) < 0.$$

which contradicts condition (ii). This completes the proof. □

Theorem 4.1.3. *(Sufficiency). Suppose that*

(i) $a \in X_0$;
(ii) There exist $\tau^0 \in R^p$, $\tau^0 \geqq 0$, *and* $\lambda \in R^m$, $\lambda^0 \geqq 0$, *such that*

(a) $\tau^0 \nabla f(a) + \lambda^0 \nabla g(a) = 0$,
(b) $\lambda^0 g(a) = 0$,
(c) $\tau^0 e = 1$, *where* $e = (1, \ldots, 1)^T \in R^p$;

(iii) Problem (VP) is weak strictly pseudo type I univex at $a \in X_0$ *with respect to some* b_0, b_1, ϕ_0, ϕ_1, *and* η;
(iv) $u \leq 0 \Rightarrow \phi_0(u) \leq 0$ *and* $u \leqq 0 \Rightarrow \phi_1(u) \leqq 0$,
(v) $b_0(x,a) > 0$ *and* $b_1(x,a) \geqq 0$;

for all feasible x. Then a is an efficient solution to (VP).

Proof. Suppose contrary to the result that a is not an efficient solution to (VP). Then there exists a feasible solution x to (VP) such that

$$f(x) \leq f(a).$$

By conditions (iv), (v), and the above inequality, we get (4.1.1). By the feasibility of a, conditions (iv) and (v), give (4.1.2).
By inequalities (4.1.1), (4.1.2), and condition (iii), we have

$$(\nabla f(a))\, \eta(x,a) < 0, \text{ and } \lambda^0 \nabla g(a)\, \eta(x,a) < 0.$$

Since $\tau^0 \geqq 0$, the above inequalities give

$$\left[\tau^0 \nabla f(a) + \lambda^0 \nabla g(a)\right] \eta(x,a) < 0.$$

which contradicts condition (ii). This completes the proof. □

4.2 Optimality Conditions for Nondifferentiable Vector Optimization Problems

In this section, we establish a Karush–Kuhn–Tucker sufficient optimality condition.

Consider the following vector optimization problem:

$$\begin{aligned} \text{(P)} \quad & \text{minimize } f(x) \\ & \text{subject to } g(x) \leqq 0, \\ & \qquad\qquad x \in X, \end{aligned}$$

where $f : X \to R^k$, $g : X \to R^m$, X is a nonempty open subset of R^n. Let $\eta : X \times X \to R^n$ be a vector function. Throughout this section, $f'(u, \eta(x,u)) = \lim\limits_{\lambda \to 0^+} \frac{f(u+\lambda\eta(x,u)) - f(u)}{\lambda}$.

Let $D = \{x \in X : g(x) \leqq 0\}$ be the set of all the feasible solutions for (P) and denote, $J(x) = \left\{j \in M : g_j(x) = 0\right\}$ and $\tilde{J}(x) = \left\{j \in M : g_j(x) < 0\right\}$. It is obvious that $J(x) \cup \tilde{J}(x) = M$.

The following results from Antczak (2002a) and Weir and Mond (1988) will be needed in the sequel of this section.

Lemma 4.2.1. *If $\bar{x}$ is a locally weak Pareto solution or a weak Pareto efficient solution of (P) and if g_j is continuous at $\bar{x}$ for $j \in \tilde{J}(\bar{x})$, then the following system of inequalities*

$$f'(\bar{x}, \eta(x,\bar{x})) < 0,$$

$$g'_{\tilde{J}(\bar{x})}(\bar{x}, \eta(x,\bar{x})) < 0$$

has no solution for $x \in X$.

Lemma 4.2.2. *Let S be a nonempty set in R^n and $\psi : S \to R^p$ be a preinvex function on S. Then either $\psi(x) < 0$ has a solution $x \in S$, or $\lambda^T \psi(x) \geqq 0$ for all $x \in S$, or some $\lambda \in R^m_+$, but both alternatives are never true.*

Lemma 4.2.3. *(Fritz John necessary optimality condition). Let $\bar{x}$ be a weak Pareto efficient solution for (P). Suppose that g_j is continuous for $j \in \tilde{J}(\bar{x})$, f and g are directionally differentiable at $\bar{x}$ with $f'(\bar{x}, \eta(x,\bar{x}))$, and $g'_{J(\bar{x})}(\bar{x}, \eta(x,\bar{x}))$ pre-invex functions of x on X. Then there exist $\bar{\xi} \in R^k_+$, $\bar{\mu} \in R^m_+$ such that $\left(\bar{x}, \bar{\xi}, \bar{\mu}\right)$ satisfies the following conditions:*

$$\bar{\xi}^T f'(\bar{x}, \eta(x, \bar{x})) + \bar{\mu}^T g'(\bar{x}, \eta(x, \bar{x})) \geqq 0 \ \forall x \in X,$$
$$\bar{\mu}^T g(\bar{x}) = 0,$$
$$g(\bar{x}) \leqq 0.$$

Lemma 4.2.4. *(Karush–Kuhn–Tucker necessary optimality condition). Let $\bar{x}$ be a weak Pareto efficient solution for (P). Suppose that g_j is continuous for $j \in \tilde{J}(\bar{x})$, f and g are directionally differentiable at $\bar{x}$ with $f'(\bar{x}, \eta(x, \bar{x}))$, and $g'_{J(\bar{x})}(\bar{x}, \eta(x, \bar{x}))$ pre-invex functions of x on X. Moreover, suppose that g satisfies the general Slater's constraint qualification at $\bar{x}$. Then there exists $\bar{\mu} \in R^m_+$ such that $(\bar{x}, \bar{\mu})$ satisfies the following conditions:*

$$f'(\bar{x}, \eta(x, \bar{x})) + \overline{\mu}^T g'(\bar{x}, \eta(x, \bar{x})) \geqq 0 \forall x \in X, \tag{4.2.1}$$
$$\overline{\mu}^T g(\overline{x}) = 0, \tag{4.2.2}$$
$$g(\overline{x}) \leqq 0. \tag{4.2.3}$$

Theorem 4.2.1. *Let $\bar{x}$ be a feasible solution for (P) at which conditions (4.2.1)–(4.2.3) are satisfied. Moreover, if any of the following conditions are satisfied:*

(a) $(f, \mu^T g)$ is strong pseudo-quasi d-type-I univex at $\bar{x}$ with respect to some b_0, b_1, ϕ_0, ϕ_1 and η with $b_0 > 0$, $a < 0 \Rightarrow \phi_0(a) < 0$ and $a = 0 \Rightarrow \phi_1(a) \geqq 0$;
(b) $(f, \mu^T g)$ is weak strictly pseudo-quasi d-type-I univex at $\bar{x}$ with respect to some b_0, b_1, ϕ_0, ϕ_1 and η with $b_0 \geq 0$, $a < 0 \Rightarrow \phi_0(a) \leq 0$ and $a = 0 \Rightarrow \phi_1(a) \geqq 0$;
(c) $(f, \mu^T g)$ is weak strictly pseudo d-type-I univex at $\bar{x}$ with respect to some b_0, b_1, ϕ_0, ϕ_1 and η with $b_0 \geq 0, a < 0 \Rightarrow \phi_0(a) \leq 0$ and $a = 0 \Rightarrow \phi_1(a) \geqq 0$.

Then $\bar{x}$ is a weak Pareto efficient solution for (P).

Proof. We proceed by contradiction. Assume that $\bar{x}$ is not a weak Pareto efficient solution of (P). Then there is a feasible solution x of (P) such that

$$f_i(x) < f_i(\bar{x}), \quad \text{for any } i \in \{1, 2, ..., k\}.$$

Since $b_0 > 0$ and $a < 0 \Rightarrow \phi_0(a) < 0$, from the above inequality, we get

$$b_0(x, \bar{x})\, \phi_0[f_i(x) - f_i(\bar{x})] < 0.$$

Since $b_1 \geqq 0$ and $a = 0 \Rightarrow \phi_1(a) \geqq 0$, from (2.2), we get

$$-b_1(x, \overline{x})\, \phi_1\left[\overline{\mu}^T g(\overline{x})\right] \leqq 0.$$

By the generalized univexity type condition in (a) and the above two inequalities, we get

$$f'(\overline{x}, \eta(x, \overline{x})) < 0 \tag{4.2.4}$$

and

$$\overline{\mu}^T g'(\overline{x}, \eta(x, \overline{x})) \leqq 0. \tag{4.2.5}$$

By (4.2.4) and (4.2.5), we get

$$f'(\bar{x}, \eta(x,\bar{x})) + \overline{\mu}^T g'(\bar{x}, \eta(x,\bar{x})) < 0,$$

which contradicts (4.2.1).

For the proof of part (b), we assume that $\bar{x}$ is not a weak Pareto efficient solution of (P). Then there is a feasible solution x of (P) such that

$$f_i(x) < f_i(\bar{x}), \text{ for any } i \in \{1, 2, ..., k\}.$$

Since $b_0 \geq 0$, $a < 0 \Rightarrow \phi_0(a) \leq 0$, from the above inequality, we get

$$b_0(x, \bar{x}) \phi_0 [f_i(x) - f_i(\bar{x})] \leq 0.$$

Since $b_1 \geqq 0$ and $a = 0 \Rightarrow \phi_1(a) \geqq 0$, from (2), we get

$$-b_1(x, \bar{x}) \phi_1 \left[\overline{\mu}^T g(\bar{x})\right] \leqq 0.$$

By the generalized univexity type condition in (b) and the above two inequalities, we get (4.2.4) and (4.2.5). From (4.2.4) and (4.2.5), we get

$$f'(\bar{x}, \eta(x,\bar{x})) + \overline{\mu}^T g'(\bar{x}, \eta(x,\bar{x})) < 0,$$

again a contradiction to (4.2.1).

Assume that $\bar{x}$ is not a weak Pareto efficient solution of (P). Then there is a feasible solution x of (P) such that

$$f_i(x) < f_i(\bar{x}), \text{ for any } i \in \{1, 2, ..., k\}.$$

Since $b_0 \geq 0$ and $a < 0 \Rightarrow \phi_0(a) < 0$, from the above inequality, we get

$$b_0(x, \bar{x}) \phi_0 [f_i(x) - f_i(\bar{x})] \leq 0.$$

Since $b_1 \geqq 0$ and $a = 0 \Rightarrow \phi_1(a) \geqq 0$, from (4.2.2), we get

$$-b_1(x, \bar{x}) \phi_1 \left[\overline{\mu}^T g(\bar{x})\right] \leqq 0.$$

By condition (c) and the above two inequalities, we get

$$f'(\bar{x}, \eta(x,\bar{x})) < 0$$

and

$$\overline{\mu}^T g'(\bar{x}, \eta(x,\bar{x})) < 0.$$

By these two inequalities, we get

$$f'(\bar{x}, \eta(x,\bar{x})) + \overline{\mu}^T g'(\bar{x}, \eta(x,\bar{x})) < 0,$$

which contradicts (4.2.1). This completes the proof. □

Example 4.2.1. Consider function $f = (f_1, f_2)$ defined on $X = R$, by $f_1 = x^2$ and $f_2 = x^3$ and function g defined on $X = R$, by

$$g = \begin{cases} -2x^2, & -1 \leq x < 2 \\ -x^3, & 2 \leq x < 2.5. \end{cases}$$

Clearly, g is not differentiable at $x = 2$, but only directionally differentiable at $x = 2$. The feasible set is nonempty. Let $\eta(x, \bar{x}) = (x - \bar{x})/2$ and $\bar{x} = 0$. We can easily show

(i) If $x \in [-1, 2), -g_1(\bar{x}) = 0$, implies that $g'(\bar{x}, \eta) = 0$.
(ii) The case $x \in [2, 4)$ can be verified similarly. We have

$$f(x) \leqq f(\bar{x}) \Rightarrow \nabla f(\bar{x})\, \eta(x, \bar{x}) = 0, \text{ for all } x.$$

Thus (f, g) is strong pseudo-quasi d-type I at $x = 0$. But, f and g are not d-invex functions at $x = 0$ with respect to the same $\eta(x, \bar{x}) = (x - \bar{x})/2$. Therefore, Theorem 13 of Antczak (2002a) is not applicable. Then, by part (a) of Theorem 4.2.1, $\bar{x}$ is a weak Pareto solution for the vector optimization problem. □

It may be interesting to extend an earlier work of Kim (2006) to the setting of the problems considered above.

4.3 Optimality Conditions for Minimax Fractional Programs

Consider the following minimax fractional programming problem (see; Liu et al. (1997a) and Bector (1994a)):

$$\text{(P)} \qquad \text{minimize } F(x) = \sup_{y \in Y} \frac{f(x, y)}{h(x, y)} \qquad \text{subject to } g(x) \leqq 0, \tag{4.3.1}$$

where Y is a compact subset of R^m, $f(\cdot, \cdot)$, and $h(\cdot, \cdot) : R^n \times R^m \mapsto R$ are differentiable functions with $f(x, y) \geqq 0$ and $h(x, y) > 0$, and $g(\cdot, \cdot) : R^n \mapsto R^p$ is a differentiable function. Denote

$$Y(x) = \left\{ y \in Y : \frac{f(x, y)}{h(x, y)} = \sup_{z \in Y} \frac{f(x, z)}{h(x, z)} \right\}, J = \{1, 2, ..., p\}, J(x) = \{ j \in J : g_j(x) = 0 \}$$

and

$$K = \Big\{ (s, t, \bar{y}) \in N \times R^s_+ \times R^{ms} : 1 \leqq s \leqq n + 1, t = (t_1, ..., t_s) \in R^s_+ \text{ with } \sum_{i=1}^{s} t_i = 1 \text{ and } \bar{y} = (y_1, ..., y_s) \text{ and } y_i \in Y(x), i = 1, ..., s \Big\}.$$

Lemma 4.3.1. *(Chandra and Kumar, 1995). Let x^* be an optimal solution to (P) and let $\nabla g_j(x^*), j \in J(x^*)$ be linearly independent. Then there exist $(s^*, t^*, \bar{y}) \in K$,*

$\nu^* \in R$ and $\mu^* \in R^p_+$ such that

$$\sum_{i=1}^{s*} t_i^* \left\{\nabla f\left(x^*, y_i\right) - \nu \nabla h\left(x^*, y_i\right)\right\} + \nabla \sum_{j=1}^{p} \mu_j^* g_j\left(x^*\right) = 0, \tag{4.3.2}$$

$$f\left(x^*, y_i\right) - \nu^* h\left(x^*, y_i\right) = 0, i = 1, ..., s^*, \tag{4.3.3}$$

$$\sum_{j=1}^{p} \mu_j^* g_j\left(x^*\right) = 0, \tag{4.3.4}$$

$$\mu^* \in R^p_+, t_i^* \geqq 0, \sum_{\mathrm{i}=1}^{\mathrm{s}*} t_i^* = 1,\ y_i \in Y\left(x^*\right), i = 1, ..., s^*. \tag{4.3.5}$$

In order to relax the convexity assumption in the above lemma, we impose the following definitions which are slight modifications of the vector type invexity introduced in Hanson et al. (2001).

Theorem 4.3.1. *(Sufficient Conditions). Assume that $\left(x^*, \mu^*, \nu^*, s^*, t^*, \bar{y}\right)$ satisfies (4.3.2)–(4.3.5). If $\left(\sum_{i=1}^{s*} t_i^* \left(f\left(\cdot, y_i\right) - \nu^* h\left(\cdot, y_i\right)\right), \sum_{j=1}^{p} \mu_j^* g_j\left(\cdot\right)\right)$ is pseudo quasi V-type I at x^* with respect to η, α_i, β_j, t, and μ, then x^* is an optimal solution of (P).*

Proof. From the positivity of β_j, $j = 1, 2, ..., m$ and (4), we have

$$-\sum_{j=1}^{m} \mu_j^* \beta_j\left(x, x^*\right) g_j\left(x^*\right) \leqq 0.$$

By the condition of quasi V-type I, we get

$$\eta^T\left(x, x^*\right) \nabla \sum_{j=1}^{m} \mu_j^* g_j\left(x^*\right) \leqq 0. \tag{4.3.6}$$

From (4.3.6) and (4.3.2), we get

$$\eta^T\left(x, x^*\right) \nabla \sum_{i=1}^{s*} t_i^* \left(f\left(x^*, y_i\right) - \nu^* h\left(x^*, y_i\right)\right) \geqq 0. \tag{4.3.7}$$

By the pseudo V-type I condition, (4.3.7) and (4.3.3), we get

$$\sum_{i=1}^{s*} t_i^* \alpha_i\left(x, x^*\right)\left(f\left(x, y_i\right) - \nu^* h\left(x, y_i\right)\right) \geqq \sum_{i=1}^{s*} t_i^* \alpha_i\left(x, x^*\right)\left(f\left(x^*, y_i\right) - \nu^* h\left(x^*, y_i\right)\right) = 0.$$

That is,

$$\sum_{i=1}^{s*} t_i^* \alpha_i\left(x, x^*\right)\left(f\left(x, y_i\right) - \nu^* h\left(x, y_i\right)\right) \geqq 0.$$

Since $\alpha_i(x, x^*)$ and t_i are positive, there exists an i_0 such that

$$\left(f\left(x, y_{i_0}\right) - v^* h\left(x, y_{i_0}\right)\right) \geqq 0.$$

It follows that

$$\sup_{y \in Y} \frac{f(x, y)}{h(x, y)} \geqq \frac{f\left(x, y_{i_0}\right)}{h\left(x, y_{i_0}\right)} \geqq v^* = \sup_{y \in Y} \frac{f(x^*, y)}{h(x^*, y)}.$$

Thus, the proof is completed. □

Theorem 4.3.2. *Assume that* $(x^*, \mu^*, v^*, s^*, t^*, \bar{y})$ *satisfies (4.3.2)–(4.3.5). If* $\left(\sum_{i=1}^{s*} t_i^* \left(f(\cdot, y_i) - v^* h(\cdot, y_i)\right), \sum_{j=1}^{p} \mu_j^* g_j(\cdot)\right)$ *is quasi strictly pseudo V-type I at* x^* *with respect to* $\eta, \alpha_i, \beta_j, t$, *and* μ, *then* x^* *is an optimal solution of (P).*

Proof. Assume that x^* is not an optimal solution for (P). Then there exists an x feasible for (P) such that

$$\sup_{y \in Y} \frac{f(x,y)}{h(x,y)} \leqq v^* = \sup_{y \in Y} \frac{f(x^*,y)}{h(x^*,y)}.$$

It follows that

$$\sum_{i=1}^{s*} t_i^* \alpha_i(x,x^*) \left(f(x,y_i) - v^* h(x,y_i)\right) \leqq 0.$$

From (4.3.3), we have

$$\sum_{i=1}^{s*} t_i^* \alpha_i(x,x^*) \left(f(x^*,y_i) - v^* h(x^*,y_i)\right) = 0.$$

From the above inequalities, we get

$$\sum_{i=1}^{s*} t_i^* \alpha_i(x,x^*) \left(f(x,y_i) - v^* h(x,y_i)\right) \leqq \sum_{i=1}^{s*} t_i^* \alpha_i(x,x^*) \left(f(x^*,y_i) - v^* h(x^*,y_i)\right).$$

By the quasi V-type I condition and the above inequality, we get

$$\eta^T(x,x^*) \nabla \sum_{i=1}^{s*} t_i^* \left(f(x^*,y_i) - v^* h(x^*,y_i)\right) \leqq 0. \tag{4.3.8}$$

From (4.3.8) and (4.3.2), we get

$$\eta^T(x,x^*) \nabla \sum_{j=1}^{m} \mu_j^* g_j(x^*) \geqq 0.$$

By the strict pseudo V-type I condition and the above inequality, we get

$$\sum_{j=1}^{m} \mu_j^* \beta_j(x,x^*)\, g_j(x^*) < 0.$$

Since β_j is positive for $j = 1,\dots,m$, the above inequality gives

$$\sum_{j=1}^{m} \mu_j^* g_j(x^*) < 0,$$

which contradicts (4.3.4). This completes the proof. □

Theorem 4.3.3. *Assume that* $(x^*, \mu^*, v^*, s^*, t^*, \bar{y})$ *satisfies (4.3.2)–(4.3.5). If* $\left(\sum_{i=1}^{s*} t_i^* (f(\cdot,y_i) - v^* h(\cdot,y_i)),\ \sum_{j=1}^{p} \mu_j^* g_j(\cdot)\right)$ *is semi-strictly quasi V-type I at* x^**, with respect to* η*,* α_i*,* β_j*,* t*, and* μ*. Then,* x^* *is an optimal solution of (P).*

Proof. Assume that x^* is not an optimal solution for (P). Then there exists a feasible x of (P) such that

$$\sup_{y \in Y} \frac{f(x,y)}{h(x,y)} \leqq v^* = \sup_{y \in Y} \frac{f(x^*,y)}{h(x^*,y)}.$$

It follows that

$$\sum_{i=1}^{s*} t_i^* \alpha_i(x,x^*)\,(f(x,y_i) - v^* h(x,y_i)) \leqq 0.$$

From (4.3.3), we have

$$\sum_{i=1}^{s*} t_i^* \alpha_i(x,x^*)\,(f(x^*,y_i) - v^* h(x^*,y_i)) = 0.$$

From the above two inequalities, we get

$$\sum_{i=1}^{s^*} t_i^* \alpha_i(x,x^*)\,(f(x,y_i) - v^* h(x,y_i)) \leqq \sum_{i=1}^{s^*} t_i^* \alpha_i(x,x^*)\,(f(x^*,y_i) - v^* h(x^*,y_i)).$$

By the semi strict quasi V-type I condition and the above inequality, we get

$$\eta^T(x,x^*)\,\nabla \sum_{i=1}^{s^*} t_i^* (f(x^*,y_i) - v^* h(x^*,y_i)) < 0. \tag{4.3.9}$$

Since β_j is positive for $j = 1,\dots,m$, from (4.3.4), we get

$$\sum_{j=1}^{m} \mu_j^* \beta_j(x,x^*)\, g_j(x^*) = 0.$$

By the quasi V-type I condition and the above inequality, we get

$$\eta^T(x,x^*)\nabla\sum_{j=1}^{m}\mu_j^* g_j(x^*) \leqq 0. \tag{4.3.10}$$

From (4.3.9) and (4.3.10), we get a contradiction to (4.3.2). Hence, the conclusion follows. □

Theorem 4.3.4. *Assume that* $(x^*,\mu^*,v^*,s^*,t^*,\bar{y})$ *satisfies (4.3.2)–(4.3.5). If* $\left(\sum_{i=1}^{s*} t_i^* (f(\cdot,y_i)-v^*h(\cdot,y_i)),\ \sum_{j=1}^{p}\mu_j^* g_j(\cdot)\right)$ *is strictly pseudo V-type I at* x^* *with respect to* η, α_i, β_j, t, *and* μ. *Then* x^* *is an optimal solution of (P).*

Proof. Since β_j is positive for $j=1,\ldots,m$, from (4.3.4), we get

$$\sum_{j=1}^{m}\mu_j^*\beta_j(x,x^*)g_j(x^*)=0.$$

By the strict pseudo V-type I condition and the above inequality, we get

$$\eta^T(x,x^*)\nabla\sum_{j=1}^{m}\mu_j^* g_j(x^*) < 0. \tag{4.3.11}$$

From (4.3.11) and (4.3.2), we get

$$\eta^T(x,x^*)\nabla\sum_{i=1}^{s^*}t_i^*(f(x^*,y_i)-v^*h(x^*,y_i))>0.$$

By the strict pseudo V-type I condition and the above inequality, we get

$$\sum_{i=1}^{s^*}t_i^*\alpha_i(x,x^*)(f(x,y_i)-v^*h(x,y_i))>\sum_{i=1}^{s^*}t_i^*\alpha_i(x,x^*)(f(x^*,y_i)-v^*h(x^*,y_i)).$$

From (4.3.3) and the above inequality, we get

$$\sum_{i=1}^{s^*}t_i^*\alpha_i(x,x^*)(f(x,y_i)-v^*h(x,y_i))>0. \tag{4.3.12}$$

Since α_i is positive for $i=1,\ldots,s^*$, from (4.3.12) it follows that there exists an i_0 such that

$$\left(f\left(x,y_{i_0}\right)-v^*h\left(x,y_{i_0}\right)\right)>0.$$

Therefore,

$$\sup_{y\in Y}\frac{f(x^*,y)}{h(x^*,y)}=v^*\leqq\sup_{y\in Y}\frac{f(x,y)}{h(x,y)}.$$

Hence, the result follows. □

4.4 Optimality Conditions for Vector Optimization Problems on Banach Spaces

Consider the generalized vector optimization problem defined by

$$\text{(P)} \qquad \text{minimize } \{f(x) : x \in C,\ -g(x) \in K\},$$

where $f: E \to F$ and $g: E \to G$ are strongly compact Lipschitzian at $x_0 \in E$, $K \subset G$ is a pointed closed convex cone with nonempty interior, and C is a nonempty subset of E.

Let $\Im$ denote the set of all feasible solutions of problem (P), assumed to be nonempty, that is,

$$\Im = \{x \in C : g(x) \leq 0\} \neq \phi.$$

The following proposition is from Abdouni and Thibault (1992).

Proposition 4.4.1. *If* $x_0 \in \Im$ *is a weak efficient solution for (P), then there exists a nonzero pair of vectors* $(u^*, v^*) \in Q^* x\, K^*$ *such that, for some* $k > 0$,

$$0 \in \partial(u^* \, o f + v^* \, o\, g + k\delta_C)(x_0),$$
$$< v^*, g(x_0) >= 0.$$

We adopt the following Slater-type constraint qualification.

In the rest of this section, we suppose that the restriction of (P) satisfies the Slater condition.

Theorem 4.4.1. *(Sufficient Optimality): Suppose that there exist* $x_0 \in \Im$ *and* $u^* \in Q^*$, $u^* \neq 0$, $v^* \in K^*$, *such that, for some* $k > 0$,

$$0 \in \partial(u^* o\, f + v^* o\, g + k\delta_C)(x_0), \tag{4.4.1}$$

$$< v^*, g(x_0) >= 0. \tag{4.4.2}$$

If $(u^* o\, f,\ v^* o\, g)$ *is type-I at* $x_0 \in \Im$ *with respect to C, then* x_0 *is a weak efficient solution of (P).*

Proof. Suppose that x_0 is not a weak efficient solution of (P), then there exists $\hat{x} \in \Im$ such that $f(\hat{x}) - f(x_0) < 0$. Since $u^* \neq 0$, we get

$$\langle u^*, f(\hat{x}) - f(x_0) \rangle < 0. \tag{4.4.3}$$

By the type-I hypothesis on f at x_0, there is $\eta(\hat{x}, x_0) \in T_C(x_0)$, such that

$$(u^* \circ f)^0 (x_0; \eta(\hat{x}, x_0)) \leq \langle u^*, f(\hat{x}, x_0) \rangle.$$

Combining this inequality with (4.4.3), we obtain

$$(u^* \circ f)^0 (x_0; \eta(\hat{x}, x_0)) < 0. \tag{4.4.4}$$

Moreover, the type-I assumption on g at x_0 implies that, for the same $\eta(\hat{x}, x_0) \in T_C(x_0)$, we have

$$(v^* o\ g)^0(x_0; \eta(\hat{x}, x_0)) \leq < v^*, -g(x_0) > .$$

Since $x \in \Im$ and (4.4.2) holds, we get

$$(v^* o\ g)^0(x_0; \eta(\hat{x}, x_0)) \leq 0. \tag{4.4.5}$$

From (4.4.4) and (4.4.5), we get

$$(u^* o\ f)^0(x_0; \eta(\hat{x}, x_0)) + (v^* o\ g)^0(x_0; \eta(\hat{x}, x_0) < 0. \tag{4.4.6}$$

However, from (4.4.1), we get

$$0 \leq (u^* o\ f)^0(x_0; \eta) + (v^* o\ g)^0(x_0; \eta), \qquad \forall \eta \in T_C(x_0),$$

which contradicts (4.4.6). Therefore, x_0 is a weak Pareto optimal solution of (P). □

Theorem 4.4.2. *Suppose that there exist* $x_0 \in \Im$ *and* $u^* \in Q^*$, $u^* \neq 0$, $v^* \in K^*$ *such that, for some* $k > 0$, *(4.4.1) and (4.4.2) of Theorem 4.4.1 hold. If (f, g) is pseudoquasitype I at* x_0*, with respect to C, for the same* η*, then* x_0 *is a weak Pareto optimal solution of (P).*

Proof. Since, $< v^*, g(x_0) >= 0$ and (f, g) is pseudoquasi-type I at x_0, we have

$$(v^* o\ g)^0(x_0; \eta(\hat{x}, x_0)) \leq 0.$$

On using the above inequality in (4.4.1), we get

$$\begin{aligned} &(u^* o\ f)^0(x_0; \eta(\hat{x}, x_0)) \geq 0, \quad \forall \eta \in T_C(x_0) \\ &\Rightarrow < u^*, f(\hat{x}) - f(x_0) > \geq 0 \\ &\Rightarrow f(\hat{x}) - f(x_0) > 0 (\text{because } u^* \neq 0) \end{aligned}$$

Therefore, x_0 is a weak efficient solution of (P). □

The proof of the following theorem is easy and hence omitted.

Theorem 4.4.3. *Suppose that there exist* $x_0 \in \Im$ *and* $u^* \in Q^*$, $u^* \neq 0$, $v^* \in K^*$ *such that, for some* $k > 0$, *(4.4.1) and (4.4.2) of Theorem 4.4.1, hold. If (f, g) is quasistrictly pseudo – type I at* x_0*, with respect C, for same* $\eta \in T_C(x_0)$*, then* x_0 *is a weak efficient solution of (P).*

4.5 Optimality Conditions for Complex Minimax Programs on Complex Spaces

Mathematical programming in the complex space originated from Levinson's (1966) discussion of linear programming problems. See also Abrams (1972), Abrams and Ben-Israel (1969, 1971), Hanson and Mond (1967), Mond (1973), Mond and Craven (1975), and Mond and Hanson (1968b).

Several authors have been interested in the optimality conditions and duality theorems for complex nonlinear programming problems, see Ferrero (1992), Lai et al. (2001), Lai and Liu (2002), Liu (1997, 1999b), Liu et al. (1997b), Mishra (2001a), Mishra and Rueda (2003), Smart (1990), Smart and Mond (1991), Stancu-Minasian (1997), Weir (1992) and Weir and Mond (1984).

In this section, we establish sufficient optimality criteria for the following optimization problem under generalized invex complex functions:

$$
\begin{aligned}
\text{(P)} \qquad & \text{minimize } f(\xi) = \sup_{\varsigma \in W} \operatorname{Re}\phi(\xi,\varsigma) \\
& \text{subject to } \xi \in S^0 = \left\{ \xi \in C^{2n} : -g(\xi) \in S \right\},
\end{aligned}
$$

where $\xi = (z,\bar{z})$, $\varsigma = (\omega,\varpi)$ for $z \in C^n$, $\omega \in C^m$, $\phi(\cdot,\cdot) : C^{2n} \times C^{2m} \to C$ is analytic with respect to ξ, W is a specified compact subset in C^{2m}, S is a polyhedral cone in C^p and $g : C^{2n} \to C^p$ is analytic.

We shall use the following lemma for the problem (P).

Lemma 4.5.1. *(Liu 1999b). Let $\phi(\cdot,\cdot) : C^{2n} \times C^{2m} \to C$ be differentiable with respect to ξ for each $\varsigma \in W, g : C^{2n} \to C^p$ be differentiable with respect to ξ and let $S \subset C^p$ be a polyhedral cone with nonempty interior. Let $\xi^0 = (z_0,\bar{z}_0)$ be a solution to the minimax problem (P). Then there exist a positive integer s, scalars $\lambda_i \geq 0, i = 1,2,\ldots,s$, $0 \neq u \in S^*$, and vectors $\varsigma_i \in W(\xi^0)$, $i = 1,2,\ldots,s$, such that*

$$
\sum_{i=1}^{s} \lambda_i \overline{\nabla_z \phi(\xi^0,\varsigma_i)} + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\xi^0,\varsigma_i) + u^T \overline{\nabla_z g(\xi^0)} + u^H \nabla_{\bar{z}} g(\xi^0), \tag{4.5.1}
$$

$$
\operatorname{Re}\langle u, g(\xi^0) \rangle = 0. \tag{4.5.2}
$$

Lemma 4.5.2. *(Liu 1999b) (Necessary Optimality Conditions). Let $\xi^0 = (z_0,\bar{z}_0)$ be an optimal solution of (P) and let $\phi(\cdot,\cdot) : C^{2n} \times C^{2m} \to C$ be differentiable with respect to ξ for each $\varsigma \in W, g : C^{2n} \to C^p$ be differentiable with respect to ξ and let $S \subset C^p$ be a polyhedral cone with nonempty interior. In addition, we suppose that the following conditions (CQ) holds:*

$$
\text{(CQ)} \qquad u^T \overline{\nabla_z g(\xi^0)} + u^H \nabla_{\bar{z}} g(\xi^0) = 0 \text{ imply } u = 0 \text{ for all } u \in C^p.
$$

Then there exist a positive integer s, scalars $\lambda_i \geq 0$, $i = 1,2,\ldots,s$, $0 \neq u \in S^*$, and vectors $\varsigma_i \in W(\xi^0)$, $i = 1,2,\ldots,s$ such that the relations (2.1) and (2.2) hold and

$$\sum_{i=1}^{s} \lambda_i = 1. \tag{4.5.3}$$

Theorem 4.5.1. *(Sufficient Optimality Conditions). Let $\xi^0 = (z_0, \bar{z}_0) \in S^0$ and assume that there exist a positive integer S, scalars $\lambda_i \geq 0$, $i = 1,2,...,s$, $0 \neq u \in S^*$, and vectors $\varsigma_i \in W\left(\xi^0\right)$, $i = 1,2,...,s$ satisfy conditions (4.5.1)–(4.5.3). If any one of the following conditions hold:*

(a) $\sum_{i=1}^{s} \lambda_i \phi(\cdot,\ \varsigma_i)$ has pseudoinvex real part with respect to η and R^+ on the manifold Q and $g(\cdot)$ is a quasi invex function with respect to the polyhedral cone $S \subset C^p$ on the manifold Q;
(b) $\sum_{i=1}^{s} \lambda_i \phi(\cdot, \varsigma_i)$ has quasiinvex real part with respect to η and R^+ on the manifold Q and $g(\cdot)$ is a strictly pseudoinvex function with respect to the polyhedral cone $S \subset C^p$ on the manifold Q;
(c) $\sum_{i=1}^{s} \lambda_i \phi(\cdot, \varsigma_i) + u^H g(\cdot)$ has pseudoinvex real part with respect to η and R^+ on the manifold Q.

Then $\xi^0 = (z_0,\ \bar{z}_0)$ is an optimal solution of (P).

Proof. Suppose contrary that $\xi^0 = (z_0,\ \bar{z}_0)$ were not an optimal solution of (P). Then there exists a feasible solution $\xi = (z,\ \bar{z}) \in S^0$ such that

$$\sup_{\varsigma \in \mathrm{W}} \mathrm{Re}\phi(\xi, \varsigma) < \sup_{\varsigma \in \mathrm{W}} \mathrm{Re}\phi\left(\xi^0, \varsigma\right).$$

Since $\varsigma_i \in W\left(\xi^0\right)$, for all $i = 1,2,...,s$, we have

$$\mathrm{Re}\phi(\xi, \varsigma_i) < \mathrm{Re}\phi\left(\xi^0, \varsigma_i\right), \text{ for all } i = 1,2,...,s.$$

With $\lambda_i \geq 0$, $i = 1,2,...,s$, and $\sum\limits_{i=1}^{s} \lambda_i = 1$, we have

$$\mathrm{Re}\left[\sum_{i=1}^{s} \lambda_i \phi(\xi, \varsigma_i) - \sum_{i=1}^{\mathrm{s}} \lambda_i \phi\left(\xi^0, \varsigma_i\right)\right] < 0. \tag{4.5.4}$$

Using the pseudoinvexity of $\sum\limits_{i=1}^{s} \lambda_i \phi(\cdot,\ \varsigma_i)$, we get from the inequality (4.5.4), we get

$$\mathrm{Re}\left\langle \eta^T(z, z_0), \sum_{\imath=1}^{s} \lambda_\imath \overline{\nabla_z \varphi(\xi^0, \varsigma_i)} + \sum_{\imath=1}^{s} \lambda_\imath \nabla_{\bar{z}} \varphi\left(\xi^0, \varsigma_i\right) \right\rangle < 0. \tag{4.5.5}$$

Consequently, expressions (4.5.1) and (4.5.5) yields

$$\mathrm{Re}\left\langle \eta^T(z, z_0), u^T \overline{\nabla_z g(\xi^0)} + u^H \nabla_{\bar{z}} g\left(\xi^0\right) \right\rangle > 0.$$

It follows that

$$\operatorname{Re}\left\langle u, \eta^T(z,z_0)\nabla_z g\left(\xi^0\right) + \eta^H(z,z_0)\nabla_{\bar{z}} g\left(\xi^0\right)\right\rangle > 0. \tag{4.5.6}$$

Utilizing the feasibility of ξ for (P), $u \in S^*$, and the equality (4.5.2), we obtain

$$\operatorname{Re}\langle u, g(\xi)\rangle \leq 0 = \operatorname{Re}\left\langle u, g\left(\xi^0\right)\right\rangle. \tag{4.5.7}$$

Using the quasiinvexity of g, we get from the inequality (4.5.7)

$$\operatorname{Re}\left\langle u, \eta^T(z,z_0)\nabla_z g\left(\xi^0\right) + \eta^H(z,z_0)\nabla_{\bar{z}} g\left(\xi^0\right)\right\rangle \leq 0,$$

which contradicts the inequality (4.5.6). Therefore, $\xi^0 \in S^0$ is an optimal solution of (P). □

Hypothesis (b) follows along the same lines as in (a).

If hypothesis (c) holds, from the inequality (4.5.4) and (4.5.7), we get

$$\operatorname{Re}\left[\sum_{\mathrm{i}=1}^{\mathrm{s}} \lambda_{\mathrm{i}}\phi(\xi,\varsigma_i) + u^H g(\xi)\right] < \operatorname{Re}\left[\sum_{\mathrm{i}=1}^{\mathrm{s}} \lambda_{\mathrm{i}}\phi\left(\xi^0,\varsigma_i\right) + u^H g\left(\xi^0\right)\right]. \tag{4.5.8}$$

Using the pseudoinvexity of $\sum_{i=1}^{s} \lambda_i\phi(\cdot,\ \varsigma_i) + u^H g(\cdot)$ and (4.5.8), we get

$$\operatorname{Re}\left\langle \eta^T(z,z_0), \sum_{\mathrm{i}=1}^{\mathrm{s}} \lambda_{\mathrm{i}}\overline{\nabla_z\phi\left(\xi^0,\varsigma_i\right)} + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}}\phi\left(\xi^0,\varsigma_i\right) + u^H\overline{\nabla_z g\left(\xi^0\right)} + u^H\nabla_{\bar{z}} g\left(\xi^0\right)\right\rangle < 0,$$

which contradicts the equality (4.5.1). Therefore, $\xi^0 \in S^0$ is an optimal solution of (P). □

4.6 Optimality Conditions for Continuous-Time Optimization Problems

The Optimization problems discussed in the previous sections are only finite-dimensional. However, a great deal of optimization theory is concerned with problems involving infinite dimensional case. Two types of problems fitting into this scheme are variational problems and control problems. Hanson (1964) observed that variational problems and control problems are continuous-time analogue of finite dimensional nonlinear programming problems. Since then the fields of nonlinear programming problems and the calculus of variations have to some extent merged together within optimization theory, hence enhancing the potential for continued research in both fields. See for example, Mond and Hanson (1967, 1968a, 1968c).

In the last two decades, many authors have been interested in the optimality conditions and duality results of continuous-time programming problems, see for example Bector and Husain (1992), Bhatia and Mehra (1999), Chen (1996, 2000), Kim and Kim (2002), Mishra (1996b), Mishra and Mukherjee (1994a, 1994b 1995a), Mond et al. (1988), Mond and Smart (1989), Mukherjee and Mishra (1994, 1995), Nahak and Nanda (1996, 1997a), and Zalmai (1985, 1990b).

Let $I = [a,\ b]$ be a real interval and $\psi : I \times R^n \times R^n \to R$ be a continuously differentiable function. In order to consider $\psi(t,x,\dot{x})$, where $x : I \to R^n$ is differentiable with derivative x, we denote the partial derivatives of ψ by ψ_t,

$$\psi_x = \left[\frac{\partial \psi}{\partial x^1},\ldots,\frac{\partial \psi}{\partial x^n}\right], \psi_{\dot{x}} = \left[\frac{\partial \psi}{\partial \dot{x}^1},\ldots,\frac{\partial \psi}{\partial \dot{x}^n}\right].$$

The partial derivatives of the other functions used will be written similarly. Let $C(I,R^n)$ denote the space of piecewise smooth functions x with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differential operator D is given by

$$u^i = Dx^i \Leftrightarrow x^i(t) = \alpha + \int_a^t u^i(s)\,ds,$$

in which α is a given boundary value. Therefore, $D = \frac{d}{dt}$ except at discontinuities.

We consider the following continuous vector optimization problem:

$$\text{(MP)} \qquad \text{minimize} \int_a^b f(t,x,\dot{x})\,dt = \left(\int_a^b f_1(t,x,\dot{x})\,dt,\ldots,\int_a^b f_p(t,x,\dot{x})\,dt\right)$$

$$\text{subject to} \qquad x(a) = \alpha, \quad x(b) = \beta,$$
$$g(t,x,\dot{x}) \leqq 0, \quad t \in I,$$
$$x \in C(I,R^n),$$

where $f_i : I \times R^n \times R^n \to R$, $i \in P = \{1,\ldots,p\}$, $g : I \times R^n \times R^n \to R^m$ are assumed to be continuously differentiable functions.

Let K denote the set of all feasible solutions for (MP), that is,

$$K = \{x \in C(I,R^n) : x(a) = \alpha, x(b) = \beta, g(t,x(t),\dot{x}(t)) \leqq 0, t \in I\}.$$

Craven (1993) obtained Kuhn-Tucker type necessary conditions for the above problem and proved that the necessary conditions are also sufficient if the objective functions are pseudoconvex and the constraint functions are quasiconvex.

In relation to (MP), we introduce the following problems $\left(P_k^*\right)$ for each $k = 1,\ldots,p$:

$$(P_k^*) \qquad \text{minimize} \int_a^b f_k(t,x,\dot{x})\,dt$$

$$\text{subject to } x(a)=\alpha, \quad x(b)=\beta,$$

$$\int_a^b f_i(t,x,\dot{x})\,dt \leqq \int_a^b f_i(t,x^*,\dot{x}^*)\,dt, \quad i \in P, i \neq k,$$

$$g(t,x,\dot{x}) \leqq 0, t \in I.$$

The following lemma can be established in the lines of Chankong and Haimes (1983).

Lemma 4.6.1. *x^* is an efficient solution of (MP) if and only if x^* is an optimal solution of $\left(P_k^*\right)$ for each $k=1,\ldots,p$.*

Lemma 4.6.2. *(Chankong and Haimes 1983). For every optimal normal solution of $\left(P_k^*\right)$, for each $k=1,\ldots,p$, there exist real numbers $\lambda_{1k},\ldots,\lambda_{pk}$ with $\lambda_{kk}=1$ and piecewise smooth function $y_k : I \to R^m$ such that*

$$\begin{aligned} f_{kx}(t,x^*,\dot{x}^*) &+ \sum_{\substack{i=1\\ i\neq k}}^{p} \lambda_{ik} f_{ik}(t,x^*,\dot{x}^*) + y_k(t)^T h_k(t,x^*,\dot{x}^*) \\ &= \frac{d}{dt}\left(f_{kx^\bullet}(t,x^*,\dot{x}^*) + \sum_{\substack{i=1\\ i\neq k}}^{p} \lambda_{ik} f_{ix^\bullet}(t,x^*,\dot{x}^*) + y_k(t)^T g_{x^\bullet}(t,x^*,\dot{x}^*) \right), \end{aligned} \tag{4.6.1}$$

$$y_k(t)^T g(t,x^*,\dot{x}^*) = 0, t \in I, \tag{4.6.2}$$

$$y_k(t) \geqq 0,\ t \in I, \tag{4.6.3}$$

$$\lambda_{ik} \geqq 0, i=1,\ldots,p, i \neq k. \tag{4.6.4}$$

Theorem 4.6.1. *(Necessary Optimality Conditions). Let $x^* \in K$ be an efficient solution for (MP), which is assumed to be a normal solution for $\left(P_k^*\right)$ for each $k=1,\ldots,p$. Then there exist $\lambda^* \in R^p$ and a piecewise smooth function $y^* : I \to R^m$ such that*

$$\begin{aligned} \lambda^{*T} f_x(t,x^*,\dot{x}^*) &+ y^*(t)^T g_x(t,x^*,\dot{x}^*) \\ &= \frac{d}{dt}\left(\lambda^{*T} f_{\dot{x}}(t,x^*,\dot{x}^*) + y^*(t)^T g_{\dot{x}}(t,x^*,\dot{x}^*) \right), \end{aligned} \tag{4.6.5}$$

$$y^*(t)^T g(t,x^*,\dot{x}^*) = 0,\ t \in I, \tag{4.6.6}$$

$$y^*(t) \geqq 0,\ t \in I, \tag{4.6.7}$$

$$\lambda^* \geqq 0. \tag{4.6.8}$$

Proof. It follows from Lemma 4.6.1 and Lemma 4.6.2. □

The following theorems present various sufficient optimality conditions.

Theorem 4.6.2. *Let x^* be feasible for (MP). Suppose that there exist $\lambda^* \in R^p$, $\lambda^* > 0$ and piecewise smooth function $y^* : I \to R^m$ such that for $\forall t \in I$, (4.6.5)–(4.6.8) hold. If $\left(\lambda^{*T} f, y^{*T} g\right)$ is strong pseudoquasi type I with respect to η, for $\forall x \in K$, then x^* is an efficient solution for (MP).*

Proof. Assume that x^* is not an efficient solution for (MP). Thus, there exists an $x \in K$ such that

$$\int_a^b f(t,x,\dot{x})\,dt \leq \int_a^b f(t,x^*,\dot{x}^*)\,dt. \tag{4.6.9}$$

In light of (4.6.6), we have

$$-\int_a^b y^{*T} g(t,x^*,\dot{x}^*)\,dt = 0. \tag{4.6.10}$$

From (4.6.9) and (4.6.10) along with that $\left(\lambda^{*T} f,\ y^{*T} g\right)$ is strong pseudoquasi type I with respect to η, we have

$$\int_a^b \left[\eta(t,x,x^*)^T f_x(t,x^*,\dot{x}^*) + \frac{d}{dt}\left(\eta(t,x,x^*)\right)^T f_{\dot{x}}(t,x^*,\dot{x}^*)\right] dt \leq 0 \tag{4.6.11}$$

and

$$\int_a^b \left[\eta(t,x,x^*)^T y^{*T} g_x(t,x^*,\dot{x}^*) + \frac{d}{dt}\left(\eta(t,x,x^*)^T y^{*T} g_{\dot{x}}(t,x^*,\dot{x}^*)\right)\right] dt \leqq 0. \tag{4.6.12}$$

Since $\lambda^* > 0$, from (4.6.11) and (4.6.12), we have

$$\int_a^b \left[\eta(t,x,x^*)^T \lambda^{*T} f_x(t,x^*,\dot{x}^*) + \frac{d}{dt}\left(\eta(t,x,x^*)\right)^T \lambda^{*T} f_{\dot{x}}(t,x^*,\dot{x}^*)\right] dt < 0$$

and

$$\int_a^b \left[\eta(t,x,x^*)^T y^{*T} g_x(t,x^*,\dot{x}^*) + \frac{d}{dt}\left(\eta(t,x,x^*)^T y^{*T} g_{\dot{x}}(t,x^*,\dot{x}^*)\right)\right] dt \leqq 0.$$

By the above two inequalities, we get

$$\int_a^b \Big[\eta\,(t,x,x^*)^T\,\big(\lambda^{*T} f_x\,(t,x^*,\dot{x}^*)+y^{*T} g_x\,(t,x^*,\dot{x}^*)\big)$$
$$+\frac{d}{dt}\,(\eta\,(t,x^*,\dot{x}^*))^T\,\big(\lambda^{*T} f_{\dot{x}}\,(t,x^*,\dot{x}^*)+y^{*T} g_{\dot{x}}\,(t,x^*,\dot{x}^*)\big)\Big]\,dt<0.$$

By the above inequality along with Remark 3.3.1, we get

$$\int_a^b \Big[\eta\,(t,x,x^*)^T\,\big(\lambda^{*T} f_x\,(t,x^*,\dot{x}^*)+y^{*T} g_x\,(t,x^*,\dot{x}^*)$$
$$-\frac{d}{dt}\,\big(\lambda^{*T} f_{\dot{x}}\,(t,x^*,\dot{x}^*)+y^{*T} g_{\dot{x}}\,(t,x^*,\dot{x}^*)\big)\big)\Big]\,dt<0. \tag{4.6.13}$$

which contradicts (4.6.5). This completes the proof. □

Theorem 4.6.3. *Let x^* be feasible for (MP). Suppose that there exist $\lambda^* \in R^p$, $\lambda^* \geqq 0$ and piecewise smooth function $y^* : I \to R^m$ such that for $\forall t \in I$, (4.6.5)–(4.6.8) hold. If $\left(\lambda^{*T} f,\, y^{*T} g\right)$ is weak strictly pseudoquasi type I with respect to η, then x^* is an efficient solution for (MP).*

Proof. Proceeding by contradiction, using the condition on $\left(\lambda^{*T} f,\, y^{*T} g\right)$, (4.6.9) and (4.6.10), we get

$$\int_a^b \Big[\eta\,(t,x,x^*)^T\,\lambda^{*T} f_x\,(t,x^*,\dot{x}^*)+\frac{d}{dt}\,(\eta\,(t,x,x^*))^T\,\lambda^{*T} f_{\dot{x}}\,(t,x^*,\dot{x}^*)\Big]\,dt<0$$

and

$$\int_a^b \Big[\eta\,(t,x,x^*)^T\,y^{*T} g_x\,(t,x^*,\dot{x}^*)+\frac{d}{dt}\Big(\eta\,(t,x,x^*)^T\,y^{*T} g_{\dot{x}}\,(t,x^*,\dot{x}^*)\Big)\Big]\,dt \leqq 0.$$

By the above two inequalities along with Remark 3.3.1, we get (4.6.13), which contradicts (4.6.5). This completes the proof. □

Theorem 4.6.4. *Let x^* be feasible for (MP). Suppose that there exist $\lambda^* \in R^p$, $\lambda^* \geqq 0$ and piecewise smooth function $y^* : I \to R^m$ such that for $\forall t \in I$,(4.6.5)–(4.6.8) hold. If $\left(\lambda^{*T} f,\, y^{*T} g\right)$ is weak strictly pseudo type I with respect to η, then x^* is an efficient solution for (MP).*

Proof. Proceeding by contradiction, using the condition on $\left(\lambda^{*T} f,\, y^{*T} g\right)$, (4.6.9) and (4.6.10), we get

$$\int_a^b \left[\eta(t,x,x^*)^T \lambda^{*T} f_x(t,x^*,\dot{x}^*) + \frac{d}{dt}(\eta(t,x,x^*))^T \lambda^{*T} f_{\dot{x}}(t,x^*,\dot{x}^*) \right] dt < 0$$

and

$$\int_a^b \left[\eta(t,x,x^*)^T y^{*T} g_x(t,x^*,\dot{x}^*) + \frac{d}{dt}\left(\eta(t,x,x^*)^T y^{*T} g_{\dot{x}}(t,x^*,\dot{x}^*) \right) \right] dt < 0.$$

By the above two inequalities along with Remark 3.3.1, we get (4.6.13), which contradicts (4.6.5). This completes the proof. □

In the following theorems, we establish various sufficient optimality conditions for (MP) for a more general class of functions.

Theorem 4.6.5. *Let x^* be feasible for (MP) and let there exist $\lambda^* \in R^p$, $\lambda^* > 0$ and piecewise smooth function $y^* : I \to R^m$ such that for all $t \in I$, (4.6.5)–(4.6.8) hold. Furthermore, if $\left(\lambda^{*T} f, y^{*T} g\right)$ is strong pseudo-quasi type I univex with respect to b_0, b_1, ϕ_0, ϕ_1 and η, with $b_0 > 0$, $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, and $a = 0 \Rightarrow \phi_1(a) \leq 0$, for all $x \in K$, then x^* is an efficient solution for (MP).*

Proof. Assume that x^* is not an efficient solution for (MP). Then there exists an $x \in K$, such that

$$\int_a^b f(t,x,\dot{x})\, dt \leq \int_a^b f(t,x^*,\dot{x}^*)\, dt.$$

By $b_0 > 0$, $a \leq 0 \Rightarrow \phi_0(a) \leq 0$ and the above inequality, we get

$$b_0(x,u)\,\phi_0 \left[\int_a^b f(t,x,\dot{x})\, dt - \int_a^b f(t,x^*,\dot{x}^*)\, dt \right] \leq 0. \tag{4.6.14}$$

In light of (4.6.6), we have

$$-\int_a^b y^{*T} g(t,x^*,\dot{x}^*)\, dt = 0.$$

By $b_1 \geqq 0$, and $a = 0 \Rightarrow \phi_1(a) \leq 0$ and the above equality, we get

$$-b_1(x,u)\,\phi_1 \left[\int_a^b y^{*T} g(t,x^*,\dot{x}^*)\, dt \right] = 0. \tag{4.6.15}$$

From (4.6.14) and (4.6.15) along with the strong pseudo-quasi type-I univexity of $\left(\lambda^{*T} f, y^{*T} g\right)$ with respect to b_0, b_1, ϕ_0, ϕ_1 and η, we have

$$\int_a^b \left[\eta(t,x,x^*)^T f_x(t,x^*,\dot{x}^*) + \frac{d}{dt}(\eta(t,x,x^*))^T f_{\dot{x}}(t,x^*,\dot{x}^*)\right] dt \leq 0 \tag{4.6.16}$$

and

$$\int_a^b \left[\eta(t,x,x^*)^T y^{*T} g_x(t,x^*,\dot{x}^*) + \frac{d}{dt}\left(\eta(t,x,x^*)^T y^{*T} g_{\dot{x}}(t,x^*,\dot{x}^*)\right)\right] dt \leqq 0. \tag{4.6.17}$$

Since $\lambda^* > 0$, from (4.6.16) and (4.6.17), we have

$$\int_a^b \left[\eta(t,x,x^*)^T \lambda^{*T} f_x(t,x^*,\dot{x}^*) + \frac{d}{dt}(\eta(t,x,x^*))^T \lambda^{*T} f_{\dot{x}}(t,x^*,\dot{x}^*)\right] dt < 0$$

and

$$\int_a^b \left[\eta(t,x,x^*)^T y^{*T} g_x(t,x^*,\dot{x}^*) + \frac{d}{dt}\left(\eta(t,x,x^*)^T y^{*T} g_{\dot{x}}(t,x^*,\dot{x}^*)\right)\right] dt \leqq 0.$$

By the above two inequalities, we get

$$\int_a^b \Big[\eta(t,x,x^*)^T \left(\lambda^{*T} f_x(t,x^*,\dot{x}^*) + y^{*T} g_x(t,x^*,\dot{x}^*)\right)$$
$$+ \frac{d}{dt}(\eta(t,x^*,\dot{x}^*))^T \left(\lambda^{*T} f_{\dot{x}}(t,x^*,\dot{x}^*) + y^{*T} g_{\dot{x}}(t,x^*,\dot{x}^*)\right)\Big] dt < 0.$$

The above inequality along with Remark 3.3.1 gives

$$\int_a^b \Big[\eta(t,x,x^*)^T \Big(\lambda^{*T} f_x(t,x^*,\dot{x}^*) + y^{*T} g_x(t,x^*,\dot{x}^*) - \frac{d}{dt}\left(\lambda^{*T} f_{\dot{x}}(t,x^*,\dot{x}^*)\right.$$
$$\left.+ y^{*T} g_{\dot{x}}(t,x^*,\dot{x}^*)\right)\Big)\Big] dt < 0. \tag{4.6.18}$$

which contradicts (4.6.5). This completes the proof. □

Theorem 4.6.6. *Let x^* be feasible for (MP) and let there exist $\lambda^* \in R^p$ $\lambda^* \geqq 0$ and piecewise smooth function $y^* : I \to R^m$ such that for all $t \in I$, (4.6.5)–(4.6.8) hold. Furthermore, if $\left(\lambda^{*T} f, y^{*T} g\right)$ is weak strictly pseudo-quasi type I univex with respect to b_0, b_1, ϕ_0, ϕ_1 and η with $b_0 > 0$, $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, and $a = 0 \Rightarrow \phi_1(a) \leq 0$, for all $x \in K$, then x^* is an efficient solution for (MP).*

Proof. Proceeding by contradiction, using the condition on $\left(\lambda^{*T} f, y^{*T} g\right)$ for (4.6.14) and (4.6.15), we get

$$\int_a^b \left[\eta(t,x,x^*)^T \lambda^{*T} f_x(t,x^*,\dot{x}^*) + \frac{d}{dt}(\eta(t,x,x^*))^T \lambda^{*T} f_{\dot{x}}(t,x^*,\dot{x}^*)\right] dt < 0$$

and

$$\int_a^b \left[\eta(t,x,x^*)^T y^{*T} g_x(t,x^*,\dot{x}^*) + \frac{d}{dt}\left(\eta(t,x,x^*)^T y^{*T} g_{\dot{x}}(t,x^*,\dot{x}^*)\right)\right] dt \leqq 0.$$

By the above two inequalities along with Remark 3.3.1, we get (4.6.18), which contradicts (4.6.5). This completes the proof. □

Theorem 4.6.7. *Let x^* be feasible for (MP) and let there exist $\lambda^* \in R^p$, $\lambda^* \geqq 0$ and piecewise smooth function $y^* : I \to R^m$ such that for any $t \in I$, (4.6.5)–(4.6.8) hold. Furthermore, if $\left(\lambda^{*T} f, y^{*T} g\right)$ is weak strictly pseudo type-I univex with respect to b_0, b_1, ϕ_0, ϕ_1 and η with $b_0 > 0$, $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, and $\mathrm{a} = 0 \Rightarrow \phi_1(a) \leq 0$, for all $x \in K$, then x^* is an efficient solution for (MP).*

Proof. Proceeding by contradiction, using the condition on$\left(\lambda^{*T} f, y^{*T} g\right)$ for (4.6.14) and (4.6.15), we get

$$\int_a^b \left[\eta(t,x,x^*)^T \lambda^{*T} f_x(t,x^*,\dot{x}^*) + \frac{d}{dt}(\eta(t,x,x^*))^T \lambda^{*T} f_{\dot{x}}(t,x^*,\dot{x}^*)\right] dt < 0,$$

$$\int_a^b \left[\eta(t,x,x^*)^T y^{*T} g_x(t,x^*,\dot{x}^*) + \frac{d}{dt}\left(\eta(t,x,x^*)^T y^{*T} g_{\dot{x}}(t,x^*,\dot{x}^*)\right)\right] dt < 0.$$

By the above two inequalities along with Remark 3.3.1, we get (4.6.18), which contradicts (4.6.5). This completes the proof. □

4.7 Optimality Conditions for Nondifferentiable Continuous-Time Optimization Problems

Consider the following continuous-time nondifferentiable vector optimization problem:

$$\text{(CNP)} \qquad \text{minimize } \phi(x) = \int_0^T f(t, x(t))\, dt,$$

$$\text{subject to } g_i(t,x(t)) \leq 0 \qquad \text{a.e. in } [0,T],$$

$$i \in I = \{1,\ldots,m\}, \qquad x \in X$$

where X is an open, nonempty convex subset of the Banach space $L_\infty^n[0,T]$ of all n-dimensional vector-valued Lebesgue measurable functions, which are essentially bounded, defined on the compact interval $[0, T] \subset R$, with the norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty = \max_{1 \le j \le n} \; ess \; \sup \left\{ \left| x_j(t) \right|, 0 \le t \le T \right\},$$

where for each $t \in [0, T]$, $x_j(t)$ is the jth component of $x(t) \in R^n$, ϕ is a real-valued function defined on X, $g(t, x(t)) = \gamma(t) x(t)$, and $f(t, x(t)) = \Gamma(x)(t)$, where γ is a map from X into the normed space $\Lambda_1^m [0, T]$ of all Lebesgue measurable essentially bounded m-dimensional vector functions defined on $[0, T]$, with the norm $\|\cdot\|_1$ defined by

$$\|x\|_1 = \max_{1 \le j \le m} \int_0^T \left| y_j(t) \right| dt,$$

and Γ is a map from X into the normed space $\Lambda_1^1 [0, T]$.

We prove the sufficiency of Fritz John and Karush–Kuhn–Tucker conditions of global optimality for (CNP) in the Lipschitz case, using the notion of Type I functions and generalizations.

Firstly, we give some basic concepts and notations needed in this section. Let Z be a Banach space and $\psi : Z \to R$ be a locally Lipschitz function; i.e., for each $x \in Z$, there exist $\varepsilon > 0$ and a constant $K > 0$, depending on ε, such that

$$|\psi(x_1) - \psi(x_2)| \le K \|x_1 - x_2\| \qquad \forall x_1, x_2 \in x + \varepsilon B,$$

where B is the open unit ball of Z.

*The Clarke generalized directional derivativ*e of ψ at x in the direction of a given $v \in Z$, denoted by $\psi^0(x; v)$, is defined by

$$\psi^0(x; v) = \limsup_{\substack{y \to x \\ s \to 0^+}} \frac{\psi(y + sv) - \psi(y)}{s}.$$

The generalized gradient of ψ *at* x, denoted by $\partial\psi(x)$, is defined by

$$\partial\psi(x) = \left\{ \xi \in Z^* : \langle \xi, v \rangle \le \psi^0(x; v) \quad \forall \mathrm{v} \in \mathrm{Z} \right\}.$$

Here, Z^* denotes the dual space of continuous linear functionals on Z, and $\langle \cdot, \cdot \rangle : Z^* \times Z \to R$ is the duality pairing. For more details, see Clarke (1983).

Let Ω be the set of all feasible solutions to (CNP) (we suppose nonempty), i.e.,

$$\Omega = \{ x \in X : g_i(t, x(t)) \le 0 \text{ a.e. in } [0, T], i \in I \}.$$

Let V be an open convex subset of R^n containing the set

$$\{ x(t) \in R^n : x \in \Omega, t \in [0, T] \}.$$

We assume that f and g_i, $i \in I$, are real functions defined on $[0, T] \times V$. The function $t \to f(t, x(t))$ is assumed to be Lebesgue measurable and integrable for all $x \in X$.

We assume that, given $a \in V$, there exist an $\varepsilon > 0$ and a positive number k such that $\forall t \in [0, T]$, and $\forall x_1, x_2 \in a + \varepsilon B$ (B denotes the unit ball of R^n) we have

$$|f(t, x_1) - f(t, x_2)| \leq k \|x_1 - x_2\|.$$

Similar hypothesis are assumed for g_i, $i \in I$. Hence, $f(t, \cdot)$ and $g_i(t, \cdot)$, $i \in I$, are locally Lipschitz on V throughout $[0, T]$.

We can suppose the Lipschitz constant is (locally) the same for all functions involved.

Now, assume $\bar{x} \in X$ and $h \in L_\infty^n[0, T]$ are given. The continuous Clarke generalized directional derivatives of f and g_i,s are given by

$$f^0(t, \bar{\mathrm{x}}(\mathrm{t}); h(t)) = \Gamma^0(\bar{\mathrm{x}}; h)(\mathrm{t}) = \limsup_{\substack{\mathrm{y} \to \bar{\mathrm{x}} \\ \mathrm{s} \to 0^+}} \frac{\Gamma(\mathrm{y} + \mathrm{sh})(t) - \Gamma(\mathrm{y})(t)}{\mathrm{s}}$$

and

$$g_i^0(t, \bar{\mathrm{x}}(\mathrm{t}); h(t)) = \gamma_i^0(\bar{\mathrm{x}}; h)(\mathrm{t}) = \limsup_{\substack{\mathrm{y} \to \bar{\mathrm{x}} \\ \mathrm{s} \to 0^+}} \frac{\gamma_i(\mathrm{y} + \mathrm{sh})(t) - \gamma_i(\mathrm{y})(t)}{\mathrm{s}}$$

a.e. in $[0, T]$.

In this section we obtain Fritz John and Karush–Kuhn–Tucker global sufficient optimality conditions for (CNP) in the Lipschitz case under type I assumptions on the data of (CNP).

Theorem 4.7.1. *Let $\bar{x} \in \Omega$. Suppose that $(f(t, \cdot), \mathrm{g_i}(t, \cdot))$ are strict type I at $\bar{x}(t)$ (with respect to V) throughout $[0, T]$ for each $\mathrm{i} \in I$, with the same $\eta(x(t), \bar{x}(t))$ for all functions. Suppose further that there exist $\bar{\lambda}_0 \in R, \bar{\lambda} \in L_\infty^m[0, T]$ such that*

$$0 \leq \int_0^T \left[\bar{\lambda}_0 f^0(t, \bar{x}(t); h(t)) + \sum_{i=1}^m \bar{\lambda}_i(t) g_i^0(t, \bar{x}(t); h(t)) \right] dt \quad \forall h \in L_\infty^n[0, T], \tag{4.7.1}$$

$$\bar{\lambda}_0 \geq 0, \quad \bar{\lambda}(t) \geq 0 \quad a.e. \text{ in} [0, T], \tag{4.7.2}$$

$$\left(\bar{\lambda}_0, \bar{\lambda}(t)\right) \neq 0 \qquad a.e. \text{ in} [0, T], \tag{4.7.3}$$

$$\bar{\lambda}_i g_i(t, \bar{x}(t)) = 0 \quad a.e. \text{ in} [0, T], i \in I. \tag{4.7.4}$$

Then $\bar{x}$ is a global optimal solution of (CNP).

Proof. Suppose, to the contrary, that $\bar{x}$ is not optimal for (CNP). Then there exists $\tilde{x} \in \Omega$, $\bar{x} \neq \tilde{x}$, such that

$$\int_0^T f(t,\tilde{x}(t))\,dt < \int_0^T f(t,\bar{x}(t))\,dt. \tag{4.7.5}$$

Since $(f(t,\cdot), g_i(t,\cdot))$ are type I at $\bar{x}(t)$ throughout $[0,T]$ for each $i \in I$, we have the inequalities

$$f(t,\tilde{x}(t)) - f(t,\bar{x}(t)) \geq f^0(t,\bar{x}(t);\eta(x(t),\bar{x}(t))) \quad a.e. \text{ in } [0,T] \tag{4.7.6}$$

and

$$-g_i(t,\bar{x}(t)) > g_i^0(t,\bar{x}(t);\eta(x(t),\bar{x}(t))) \quad a.e. \text{ in } [0,T],\ i \in I, \tag{4.7.7}$$

for some $\eta(\tilde{x}(t),\bar{x}(t))$. Because $\bar{x} \in \Omega$ and $\bar{\lambda}_i \geq 0 \quad a.e.$ in $[0,T], i \in I$, it is clear from (4.7.4) that

$$\bar{\lambda}_i g_i(t,\bar{x}(t)) = 0 \quad a.e. \text{ in } [0,T], i \in I. \tag{4.7.8}$$

From (4.7.2)–(4.7.8), it follows that

$$0 > \int_0^T \left[\bar{\lambda}_0 f^0(t,\bar{x}(t);\eta(\tilde{x}(t),\bar{x}(t))) + \sum_{i=1}^m \bar{\lambda}_i(t) g_i^0(t,\bar{x}(t);\eta(\tilde{x}(t),\bar{x}(t))) \right] dt,$$

which, with $h(t) = \eta(\tilde{x}(t),\bar{x}(t))$, contradicts (4.7.1). Therefore, we conclude that $\bar{x}$ is a global optimal solution of (CNP). □

Remark 4.7.1. From the above proof it is clear that if for each $i \in I, (f(t,\cdot), g_i(t,\cdot))$ are type I, and at least one of these functions, say $(f(t,\cdot), g_k(t,\cdot))$ is strictly type I at $\bar{x}(t)$ throughout $[0,T]$ such that the corresponding multiplier function $\bar{\lambda}_k$ is nonzero on a subset of $[0,T]$ with positive Lebesgue measure, then the assertion of the theorem remains valid.

Theorem 4.7.2. *Let $\bar{x} \in \Omega$. Suppose that $(f(t,\cdot), g_i(t,\cdot))$ are type I at $\bar{x}(t)$ (with respect to V) throughout $[0,T]$ for each $i \in I$, with the same $\eta(x(t),\bar{x}(t))$ for all functions. Suppose further, that there exist $\bar{\lambda} \in L_\infty^m[0,T]$ such that*

$$0 \leq \int_0^T \left[f^0(t,\bar{x}(t);h(t)) + \sum_{i=1}^m \bar{\lambda}_i(t) g_i^0(t,\bar{x}(t);h(t)) \right] dt \quad \forall h \in L_\infty^n[0,T], \tag{4.7.9}$$

$$\bar{\lambda}_i(t) \geq 0 \quad a.e. \text{ in } [0,T], i \in I \tag{4.7.10}$$

$$\bar{\lambda}_i g_i(t,\bar{x}(t)) = 0 \quad a.e. \text{ in } [0,T], i \in I. \tag{4.7.11}$$

Then $\bar{x}$ is a global optimal solution of (CNP).

Proof. Let $x \in \Omega$ be given. It follows from (4.7.11) and the second part of the type I assumption that

$$\bar{\lambda}_i g_i^0(t,\bar{x}(t);\eta(x(t),\bar{x}(t))) \leq 0 \quad a.e. \text{ in } [0,T], i \in I. \tag{4.7.12}$$

Now, setting $h(t) = \eta(x(t), \bar{x}(t))$ in (4.7.9), we get

$$0 \leq \int_0^T \left[f^0(t, \bar{x}(t); \eta(x(t), \bar{x}(t))) + \sum_{i=1}^m \bar{\lambda}_i(t) g_i^0(t, \bar{x}(t); \eta(x(t), \bar{x}(t))) \right] dt. \tag{4.7.13}$$

Combining (4.7.12) and (4.7.13), we obtain

$$\int_0^T \left[f^0(t, \bar{x}(t); \eta(x(t), \bar{x}(t))) \right] dt \geq 0.$$

By the first part of the type I assumption, together with the last inequality, we get

$$\phi(\bar{x}) \leq \phi(x).$$

Hence, because $x \in \Omega$ is arbitrary, we can conclude that $\bar{x}$ is a global optimal solution of (CNP). □

Furthermore, we can obtain global sufficient optimality conditions for (CNP) under generalized invexity and Clarke regularity assumptions. The theorems stated below generalize the smooth case by Zalmai (1990b) and Rojas-Medar and Brandao (1998).

Now, let us recall the notion of Clarke regularity which is assumed to hold throughout this section. Let $U \subset Z$ be a nonempty subset of Z and ψ be a real locally Lipschitz function defined on some open subset of Z containing the set U. We say that ψ is Clarke regular at $x \in U$ if for all $v \in Z$, the usual one-sided directional derivative of ψ at x in the direction of $v \in Z$, denoted by $\psi'(x; v)$, exists and $\psi'(x; v) = \psi^0(x; v)$.

We define the Lagrangian function $L : X \times R \times L_\infty^m[0, T] \to R$ by

$$L(x, \lambda_0; \lambda) = \int_0^T \left[\lambda_0 f(t, x(t)) + \sum_{i=1}^m \lambda_i(t) g_i(t, x(t)) \right] dt.$$

When $\lambda_0 \neq 0$, we can assume that $\lambda_0 = 0$ by normalizing the Lagrange multipliers. In this case we denote $L(x, 1, \lambda)$ by $L(x, \lambda)$.

In the sequel $L'_x(\bar{x}, \lambda_0, \lambda; h)$ denotes the usual directional derivative of $L(\cdot, \lambda_0, \lambda)$ at $\bar{x}$ in the direction $h \in L_\infty^n[0, T]$, and $\partial_x L(x, \lambda_0, \lambda)$ means the generalized gradient of $L(\cdot, \lambda_0, \lambda)$.

Rojas-Medar and Brandao (1998) have pointed out that conditions (4.7.1)–(4.7.4) ((4.7.9)–(4.7.11)) in Theorem 4.7.1 (Theorem 4.7.2) cannot be written in terms of the Clarke generalized gradient of the Lagrangian function, in general. However, under the Clarke regularity assumption, it is possible. In fact, if $(f(t, \cdot), g_i(t, \cdot))$ are Clarke regular, then condition (4.7.1) is equivalent to $L'_x(\bar{x}, \lambda_0, \lambda; h) \geq 0$ for all $h \in L_\infty^n[0, T]$, and therefore, $0 \in \partial_x L(\bar{x}, \bar{\lambda}_0, \bar{\lambda})$. More precisely, we have the following corollaries:

Corollary 4.7.1. *Let $\bar{x} \in \Omega$. Suppose that $(f(t, \cdot), g_i(t, \cdot))$ are strict type I at $\bar{x}(t)$ (with respect to V) throughout $[0, T]$ for each $i \in I$, with the same $\eta(x(t), \bar{x}(t))$ for* *all functions. Suppose further that there exist* $\bar{\lambda}_0 \in R, \bar{\lambda} \in L^m_\infty[0,T]$ *such that*

$$0 \in \partial_x L\left(\bar{x}, \bar{\lambda}_0, \bar{\lambda}\right), \tag{4.7.14}$$

$$\bar{\lambda}_0 \geq 0, \bar{\lambda}(t) \geq 0 \quad a.e. \text{in}\, [0,T], \tag{4.7.15}$$

$$\left(\bar{\lambda}_0, \bar{\lambda}(t)\right) \neq 0 \quad a.e. \text{ in}\, [0,T], \tag{4.7.16}$$

$$\bar{\lambda}_i g_i(t, \bar{x}(t)) = 0 \quad a.e. \text{ in}\, [0,T], i \in I. \tag{4.7.17}$$

Then $\bar{x}$ *is a global optimal solution of (CNP).*

Corollary 4.7.2. *Let* $\bar{x} \in \Omega$. *Suppose that* $(f(t,\cdot), \mathrm{g_i}(t,\cdot))$ *are type I at* $\bar{x}(t)$ *(with respect to V) throughout* $[0,T]$ *for each* $i \in I$*, with the same* $\eta(x(t), \bar{x}(t))$ *for all functions. Suppose further that there exists* $\bar{\lambda} \in L^m_\infty[0,T]$ *such that*

$$0 \in \partial_x L\left(\bar{x}, \bar{\lambda}_0, \bar{\lambda}\right), \tag{4.7.18}$$

$$\bar{\lambda}(t) \geq 0 \quad a.e. \text{ in}\, [0,T], \tag{4.7.19}$$

$$\bar{\lambda}_i g_i(t, \bar{x}(t)) = 0 \quad a.e. \text{ in}\, [0,T], i \in I. \tag{4.7.20}$$

Then $\bar{x}$ *is a global optimal solution of (CNP).*

We establish the following two results on the sufficiency of the Karush–Kuhn–Tucker conditions. The first is obtained under the hypothesis of pseudo-quasi-type I assumption and the second is under quasi-pseudo-type I assumption. These results extend the Propositions 4.3 and 4.4 of Rojas-Medar and Brandao (1998).

Proposition 4.7.1. *Let* $\bar{x} \in \Omega$. *Suppose that* $(\phi(\cdot), \mathrm{g_i}(t,\cdot))$ *are pseudo-quasi-type I at* $\bar{x}(t)$ *(with respect to V) throughout* $[0,T]$ *for each* $i \in I$*, with the same* $\eta(x(t), \bar{x}(t))$ *for all functions. If there exists* $\bar{\lambda} \in L^m_\infty[0,T]$ *such that*

$$0 \in \partial_x L\left(\bar{x}, \bar{\lambda}\right), \tag{4.7.21}$$

$$\bar{\lambda}_i(t) \geq 0 \quad a.e. \text{ in}\, [0,T], i \in I, \tag{4.7.22}$$

$$\bar{\lambda}_i g_i(t, \bar{x}(t)) = 0 \quad a.e. \text{ in}\, [0,T], i \in I. \tag{4.7.23}$$

Then $\bar{x}$ *is a global optimal solution of (CNP).*

Proof. Since, for each $x \in \Omega$, from (4.7.23), we have

$$\bar{\lambda}_i g_i(t, \bar{x}(t)) = 0 \quad a.e. \text{ in}\, [0,T], i \in I.$$

From (4.7.22) and the second part of the Pseudo-quasi-type I assumption, we get

$$\bar{\lambda}_i(t)\, g'_i(t, \bar{x}(t); \eta(x(t), \bar{x}(t))) \leq 0 \quad a.e. \text{ in}\, [0,T], i \in I.$$

Hence, we have

$$\int_0^{\mathrm{T}} \sum_{\mathrm{i} \in 1} \bar{\lambda}_i(t)\, g'_i(t, \bar{x}(t); \eta(x(t), \bar{x}(t))) \leq 0 \quad \forall x \in \Omega. \tag{4.7.24}$$

From (4.7.21), we get

$$0 \leq \int_0^T \left[f'(t,\bar{x}(t);\eta(x(t),\bar{x}(t))) + \sum_{i\in I} \bar{\lambda}_i(t) g_i'(t,\bar{x}(t);\eta(x(t),\bar{x}(t))) \right] dt \quad \forall x \in \Omega. \tag{4.7.25}$$

From (4.7.25) and (4.7.24), we get

$$\int_0^T f'(t,\bar{x}(t);\eta(x(t),\bar{x}(t))) \geq 0 \quad \forall x \in \Omega.$$

By the first part of the Pseudo-quasi-type I assumption and the last inequality, we get

$$\phi(\bar{x}) \leq \phi(x) \quad \forall x \in \Omega.$$

Therefore, $\bar{x}$ is a global optimal solution of (CNP). □

Proposition 4.7.2. *Let* $\bar{x} \in \Omega$. *Suppose that* $\left(\phi(\cdot), \int_0^T \sum_{i=1}^m \bar{\lambda}_i(t) g_i(t,\bar{x}(t))\, dt\right)$ *are pseudo-quasi-type I at* $\bar{x}(t)$ *(with respect to* V*) throughout* $[0,T]$ *for each* $i \in I$*, with the same* $\eta(x(t),\bar{x}(t))$ *for all functions. If there exists* $\bar{\lambda} \in L_\infty^m[0,T]$*, such that* $(\bar{x},\bar{\lambda})$ *satisfies (4.7.21)-(4.7.23). Then* $\bar{x}$ *is a global optimal solution of (CNP).*

Proof. Since, for each $x \in \Omega$, from (4.7.23), we have

$$\int_0^T \sum_{i=1}^m \bar{\lambda}_i(t) g_i(t,x(t))\, dt = 0, \quad \forall x \in \Omega.$$

From the second part of the Pseudo-quasi-type I assumption of the proposition and the last equation, we get

$$\int_0^T \sum_{i=1}^m \bar{\lambda}_i(t) g_i(t,\bar{x}(t);\eta(x(t),\bar{x}(t))) \leq 0 \quad \forall x \in \Omega.$$

The rest of the proof follows by using the same argument as in the proof of the previous proposition. □

4.8 Optimality Conditions for Fractional Optimization Problems with Semilocally Type I Pre-invex Functions

In this section, sufficient optimality conditions are obtained for a nonlinear multiple objective fractional optimization problem involving η-semidifferentiable type-I-preinvex and related functions. Furthermore, a general dual is formulated and duality results are proved under the assumptions of generalized semilocally type-I-preinvex and related functions. Our results generalize the results of Preda (2003) and Stancu-Minasian (2002).

Ewing (1977) introduced semilocally convex functions which were applied by him to derive sufficient optimality conditions for variational and control problems. Such functions have certain important convex type properties, e.g., local minima of semilocally convex functions defined on locally starshaped sets are also global minima, and nonnegative linear combinations of semilocally convex functions are also semilocally convex. Some generalizations of semilocally convex functions and their properties were investigated in Kaul and Kaur (1982a, 1982b), Preda et al. (1996, 2003), Preda et al. (1996), Stancu-Minasian (2002), Suneja and Gupta (1998), Mukherjee and Mishra (1996). Kaul and Kaur (1982b) derived sufficient optimality criteria for a class of nonlinear programming problems by using generalized semilocally functions. Preda and Stancu-Minasian (1997a, 1997b) extended the results of Preda et al. (1996) to the vector optimization problems.

Elster and Nehse (1980) considered a class of convexlike functions and obtained a saddle point optimality conditions for mathematical programs involving such functions.

In this section, we extend the work of Preda (2003) to the case of semilocally type I and related functions. Our results generalize and unify the results obtained in the literature on this topic.

For necessary back ground of definitions, please see Sect. 4.3.6.

Lemma 4.8.1. *Let $f : X_0 \to R^n$ be an η-semidifferentiable function at $\bar{x} \in X_0$. If f is slqpi at $\bar{x}$ and $f(x) \leqq f(\bar{x})$ then $(df)^+ (\bar{x}, \eta(x, \bar{x})) \leqq 0$.*

Lemma 4.8.2. *(Hayashi and Komiya 1982). Let S be a nonempty set in R^n and $\psi : S \to R^k$ be a convexlike function. Then either*

$$\psi(x) < 0 \text{ has a solution } x \in S$$

or

$$\lambda^T \psi(x) \geqq 0 \text{ for all } x \in S,$$

for some $\lambda \in R^k, \lambda \geq 0$, but both alternatives are never true (the symbol T denotes the transpose of a matrix).

Using Lemma 4.8.2 from above instead of Lemma 2.9 from Preda and Stancu-Minasian (1997b), we have that the Theorems 3.4 and 3.5 stated there are still true. Thus, in establishing our results we will use the following version of Theorem 3.5 from Preda and Stancu-Minasian (1997b).

Lemma 4.8.3. *Let $\bar{x} \in X$ be a (local) weak minimum solution for the following problem:*

$$\text{minimize} \left(\varphi_1(x), \varphi_2(x), \ldots, \varphi_p(x)\right)$$

$$\text{subject to} \begin{cases} h_j(x) \leqq 0, & j \in M, \\ x \in X_0, \end{cases}$$

where $\varphi = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_p(x)) : X_0 \to R^p$ *and* $h_1, \ldots, h_m$ *are* η-*semidifferentiable at* $\bar{x}$. *Also, assume that* $h_j\,(j \in N(\bar{x}))$ *is a continuous function at* $\bar{x}$ *and* $(d\varphi)^+(\bar{x}, \eta(x,\bar{x}))$ *and* $(dh)^+(\bar{x}, \eta(x,\bar{x}))$ *are convexlike functions of* $\bar{x}$ *on* X_0. *If h satisfies a regularity condition at* $\bar{x}$ *(see; Preda and Stancu-Minasian 1997b), then there exist* $\lambda^0 \in R^p, u^0 \in R^m$ *such that*

$$\lambda^{0^T}(d\varphi)^+(\bar{x}, \eta(x,\bar{x})) + u^{0^T}(dh)^+(\bar{x}, \eta(x,\bar{x})) \geqq 0 \textit{ for all } x \in X_0,$$
$$u^{0^T} h(\bar{x}) = 0,\ h(\bar{x}) \leqq 0,$$
$$\lambda^{0^T} e = 1,\ \lambda^0 \geq 0,\ u^0 \geqq 0,$$

where $e = (1,1,\ldots,1)^T \in R^p$.

In this section, we consider the following vector fractional optimization problem:

$$\text{(VFP)} \qquad \text{minimize}\left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)}\right)$$
$$\text{subject to}\begin{cases} h_i(x) \leqq 0, & j = 1,2,\ldots,m, \\ x \in X_0, \end{cases}$$

where $X_0 \subseteq R^n$ is a nonempty set and $g_i(x) > 0$ for all $x \in X_0$ and each $i = 1, \ldots, p$. Let $f = (f_1, \ldots, f_p)$, $g = (g_1, \ldots, g_p)$ and $h = (h_1, \ldots, h_m)$.

We put $X = \{x \in X_0 : h_j(x) \leqq 0, j = 1,2,\ldots,m\}$ for the feasible set of problem (VFP). We say that (VFP) satisfies the generalized Slater's constraint qualification (GSCQ) at $\bar{x} \in X$ if h^0 is slppi at $\bar{x}$ and there exists an $\hat{x} \in X$ such that $h^0(\hat{x}) < 0$.

Lemma 4.8.4. *Let* $\bar{x} \in X$ *be a (local) weak minimum solution for (VFP). Further, we assume that* h_j *is continuous at* $\bar{x}$ *for any* $j \in N(\bar{x})$ *and that* f, g, h^0 *are* η-*semidifferentiable at* $\bar{x}$. *Then, the system*

$$\begin{cases} (df)^+(\bar{x}, \eta(x,\bar{x})) < 0, \\ (dg)^+(\bar{x}, \eta(x,\bar{x})) > 0, \\ \left(dh^0\right)^+(\bar{x}, \eta(x,\bar{x})) < 0 \end{cases}$$

has no solution $x \in X_0$.

Lemma 4.8.5. *(Fritz-John Necessary Optimality Condition). Let us suppose that* $h_j\,(j \in N(\bar{x}))$ *is a continuous function at* $\bar{x}$ *and* $(df)^+(\bar{x}, \eta(x,\bar{x}))$, $(dg)^+(\bar{x}, \eta(x,\bar{x}))$ *and* $\left(dh^0\right)^+(\bar{x}, \eta(x,\bar{x}))$ *are convexlike functions of x on* X_0. *If* $\bar{x}$ *is a locally weak minimum solution for (VFP), then there exist* $\lambda^0 \in R^p, u^0 \in R^p, v^0 \in R^m$ *such that*

$$\lambda^{0^T}(df)^+(\bar{x}, \eta(x,\bar{x})) - u^{0^T}(dg)^+(\bar{x}, \eta(x,\bar{x})) + v^{0^T}\left(dh^0\right)^+(\bar{x}, \eta(x,\bar{x}))$$
$$\geqq 0 \text{ for all } x \in X_0,$$

$$v^{0^T} h(\bar{x}) = 0,$$
$$\left(\lambda^0, u^0, v^0\right) \neq 0,\ \left(\lambda^0, u^0, v^0\right) \geqq 0.$$

For each $u = (u_1, \ldots, u_p) \in R_+^p$, where R_+^p denotes the positive orthant of R_+^p, we consider

$$(\mathrm{VFP_u}) \quad \text{minimize}\,(f_1(x) - u_1 g_1(x), \ldots, f_p(x) - u_p g_p(x))$$
$$\text{subject to} \begin{cases} h_j(x) \leqq 0, j \in M, \\ x \in X_0. \end{cases}$$

The following lemma is easy to prove.

Lemma 4.8.6. *If $\bar{x}$ is a (local) weak minimum for (VFP) then $\bar{x}$ is a (local) weak minimum for $(\mathrm{VFP_u^0})$, where $u^0 = f(\bar{x}) / g(\bar{x})$.*

Using this lemma we can get the following Karush–Kuhn–Tucker necessary optimality condition for the problem (VFP).

Lemma 4.8.7. *(Karush–Kuhn–Tucker Necessary Optimality Condition). Let $\bar{x}$ be a locally weak minimum solution for (VFP), let h_j be a continuous at $\bar{x}$ for $j \in N(\bar{x})$ and let $(df_i)^+(\bar{x}, \eta(x, \bar{x}))$, $(dg_i)^+(\bar{x}, \eta(x, \bar{x}))$, $i \in P$ and $\left(dh^0\right)^+(\bar{x}, \eta(x, \bar{x}))$ be convexlike functions of x on X_0. If g satisfies (GSCQ) at $\bar{x}$, then there exist $\lambda^0 \in R_+^p, u^0 \in R_+^p, v^0 \in R^m$ such that*

$$\sum_{i=1}^{p} \lambda_i^0 \left((df)^+(\bar{x}, \eta(x, \bar{x})) - u^{0^T}(dg)^+(\bar{x}, \eta(x, \bar{x})) + v^{0^T}\left(dh^0\right)^+(\bar{x}, \eta(x, \bar{x}))\right) \geqq 0$$

for all $x \in X_0$,

$$v^{0^T} h(\bar{x}) = 0,$$
$$h(\bar{x}) \leqq 0,$$
$$\lambda^{0^T} e = 1,$$
$$\lambda^0 \geq 0, u^0 \geqq 0,\ v^0 \geqq 0,$$

where $e = (1, \ldots, 1)^T \in R^p$.

Remark 4.8.1. In the above theorem we can suppose, for any $i \in P$, that $(df_i)^+(\bar{x}, \eta(x, \bar{x})) - u_i^0 (dg_i)^+(\bar{x}, \eta(x, \bar{x}))$ is convexlike on X_0, where $u_i^0 = f_i(\bar{x}) / g_i(\bar{x})$, instead of considering that $(df_i)^+(\bar{x}, \eta(x, \bar{x}))$ and $(dg_i)^+(\bar{x}, \eta(x, \bar{x}))$, $i \in P$ are convexlike on X_0, for any $i \in P$.

Now using the concept of locally weak optimality, we give some sufficient optimality conditions for the problem (VFP).

Theorem 4.8.1. *Let $\bar{x} \in X$ and (VFP) be η-semilocally type I-preinvex at $\bar{x}$. Further, we assume that there exist $\lambda^0 \in R^p, u^0 \in R^p$ and $v^0 \in R^m$ such that*

$$\sum_{i=1}^{p} \lambda_i^0 \left((df_i)^+ (\bar{x}, \eta(x,\bar{x})) \right) + v^{0^T} (dh)^+ (\bar{x}, \eta(x,\bar{x})) \geqq 0 \text{ for all } x \in X, \tag{4.8.1}$$

$$(dg_i)^+ (\bar{x}, \eta(x,\bar{x})) \leqq 0, \ \forall x \in X, \forall i \in P, \tag{4.8.2}$$

$$v^{0^T} h(\bar{x}) = 0, \tag{4.8.3}$$

$$h(\bar{x}) \leqq 0, \tag{4.8.4}$$

$$\lambda^{0^T} e = 1, \tag{4.8.5}$$

$$\lambda^0 \geq 0, u^0 \geqq 0, v^0 \geqq 0, \tag{4.8.6}$$

where $e = (1, \ldots, 1)^T \in R^p$. *Then* $\bar{x}$ *is a weak minimum solution for (VFP).*

Proof. Suppose that the result does not hold. Hence, there exists $\tilde{x} \in X$ such that

$$\frac{f_i(\tilde{x})}{g_i(\tilde{x})} < \frac{f_i(\bar{x})}{g_i(\bar{x})} \text{ for any } i \in P. \tag{4.8.7}$$

Since (VFP) is η-semilocally type I-preinvex at $\bar{x}$, we get

$$f_i(\tilde{x}) - f_i(\bar{x}) \geqq (df)^+ (\bar{x}, \eta(\tilde{x},\bar{x})), \quad i \in P, \tag{4.8.8}$$

$$g_i(\tilde{x}) - g_i(\bar{x}) \leqq (dg_i)^+ (\bar{x}, \eta(\tilde{x},\bar{x})), \quad i \in P, \tag{4.8.9}$$

$$-h_j(\bar{x}) \geqq (dh_j)^+ (\bar{x}, \eta(\tilde{x},\bar{x})), \quad j \in M. \tag{4.8.10}$$

Multiplying (4.8.8) by $\lambda_i^0 \geqq 0, i \in P, \lambda^0 \in R_+^p$, (3.10) by $v_j^0 \geqq 0, j \in M$, then sumarizing the obtained relations and using (4.8.1), we get

$$\sum_{i=1}^{p} \lambda_i^0 (f_i(\tilde{x}) - f_i(\bar{x})) - \sum_{j=1}^{m} v_j^0 h_j(\bar{x}) \geqq \sum_{i=1}^{p} \lambda_i^0 (df_i)^+ (\bar{x}, \eta(\tilde{x},\bar{x}))$$
$$+ \sum_{J=1}^{m} v_j^0 (dh_j)^+ (\bar{x}, \eta(\tilde{x},\bar{x})) \geqq 0.$$

Hence,

$$\sum_{i=1}^{p} \lambda_i^0 (f_i(\tilde{x}) - f_i(\bar{x})) - \sum_{j=1}^{m} v_j^0 h_j(\bar{x}) \geqq 0. \tag{4.8.11}$$

Since $x \in X, v^0 \geqq 0$, by (4.8.3) and (4.8.11), we get

$$\sum_{i=1}^{p} \lambda_i^0 (f_i(\tilde{x}) - f_i(\bar{x})) \geqq 0. \tag{4.8.12}$$

Using (4.8.5), (4.8.6), and (4.8.12), we obtain that there exists $i_0 \in P$ such that

$$f_{i_0}(\tilde{x}) \geqq f_{i_0}(\bar{x}). \tag{4.8.13}$$

By (4.8.2) and (4.8.9) it follows that

$$g_i(\tilde{x}) \leqq g_i(\bar{x}), i \in P. \tag{4.8.14}$$

Now using (4.8.13), (4.8.14), and $f \geqq 0, g > 0$, we obtain

$$\frac{f_{i_0}(\tilde{x})}{g_{i_0}(\tilde{x})} \geqq \frac{f_{i_0}(\bar{x})}{g_{i_0}(\bar{x})},$$

which is a contradiction to (4.8.7). Thus, the theorem is proved and $\bar{x}$ is a weak minimum solution for (VFP). □

Theorem 4.8.2. *Let $\bar{x} \in X$ and (VFP) is η-semilocally type I-preinvex at $\bar{x}$. Further, we assume that there exists $\lambda^0 \in R^p, u_i^0 = f_i(\bar{x}) / g_i(\bar{x}), i \in P$ and $v^0 \in R^m$ such that*

$$\sum_{i=1}^{p} \lambda_i^0 \left((df_i)^+ (\bar{x}, \eta(x,\bar{x})) - u_i^0 (dg_i)^+ (\bar{x}, \eta(x,\bar{x})) \right) + v^{0^T} (dh)^+ (\bar{x}, \eta(x,\bar{x})) \geqq 0, \quad \forall x \in X, \tag{4.8.15}$$

$$v^{0^T} h(\bar{x}) = 0, \tag{4.8.16}$$

$$h(\bar{x}) \leqq 0, \tag{4.8.17}$$

$$\lambda^{0^T} e = 1, \tag{4.8.18}$$

$$\lambda^0 \geq 0, u^0 \geqq 0, v^0 \geqq 0, \tag{4.8.19}$$

where $e = (1, \ldots, 1)^T \in R^p$. Then $\bar{x}$ is a weak minimum solution for (VFP).

Proof. Suppose that the result does not hold. Then if $\bar{x}$ is not a weak minimum solution for (VFP), we have that there exists $\tilde{x} \in X$ such that

$$\frac{f_i(\tilde{x})}{g_i(\tilde{x})} < \frac{f_i(\bar{x})}{g_i(\bar{x})} \text{ for any } i \in P,$$

that is,

$$f_i(\tilde{x}) < u_i^0 g_i(\tilde{x}) \text{ for any } i \in P. \tag{4.8.20}$$

Since (VFP) is η-semilocally type I-preinvex at $\bar{x}$, we get

$$\begin{aligned} f_i(\tilde{x}) - f_i(\bar{x}) &\geqq (df)^+ (\bar{x}, \eta(\tilde{x},\bar{x})), \quad i \in P, \\ g_i(\tilde{x}) - g_i(\bar{x}) &\leqq (dg_i)^+ (\bar{x}, \eta(\tilde{x},\bar{x})), \quad i \in P, \\ -h_j(\bar{x}) &\geqq (dh_j)^+ (\bar{x}, \eta(\tilde{x},\bar{x})), \quad j \in M. \end{aligned}$$

Using these inequalities (4.8.19) and (4.8.15), we get

$$\sum_{i=1}^{p}\lambda_i^0\left(f_i(\tilde{x})-f_i(\bar{x})\right)-\sum_{i=1}^{p}\lambda_i^0u_i^0\left(g_i(\tilde{x})-g_i(\bar{x})\right)-\sum_{i=1}^{m}v_j^0h_j(\bar{x})$$
$$\geqq\sum_{i=1}^{p}\lambda_i^0\left((df_i)^+\left(\bar{x},\eta(\tilde{x},\bar{x})\right)-u_i^0(dg_i)^+\left(\bar{x},\eta(\tilde{x},\bar{x})\right)\right)$$
$$+\sum_{i=1}^{m}v_j^0(dh_j)^+\left(\bar{x},\eta(\tilde{x},\bar{x})\right)\geqq 0.$$

Therefore,

$$\sum_{i=1}^{p}\lambda_i^0\left(f_i(\tilde{x})-u_i^0g_i(\tilde{x})\right)-\left(f_i(\bar{x})-u_i^0g_i(\bar{x})\right)-\sum_{i=1}^{m}v_j^0h_j(\bar{x})\geqq 0.$$

Since $u_i^0=f_i(\bar{x})/g_i(\bar{x})$, $i\in P$, we obtain

$$\sum_{i=1}^{p}\lambda_i^0\left(f_i(\tilde{x})-u_i^0g_i(\tilde{x})\right)-\sum_{i=1}^{m}v_j^0h_j(\bar{x})\geqq 0.$$

Since $\tilde{x}\in X, v^0\geqq 0$, by (4.8.16) and (4.8.19), we get

$$\sum_{i=1}^{p}\lambda_i^0\left(f_i(\tilde{x})-u_i^0g_i(\tilde{x})\right)\geqq 0. \tag{4.8.21}$$

Since $\lambda_i^0\geqq 0, \lambda^{0^T}e=1$, we obtain that there exists $i_0\in P$ such that

$$f_{i_0}(\tilde{x})-u_{i_0}^0g_{i_0}(\tilde{x})\geqq 0,$$

that is,

$$\frac{f_{i_0}(\tilde{x})}{g_{i_0}(\tilde{x})}\geqq\frac{f_{i_0}(\bar{x})}{g_{i_0}(\bar{x})},$$

which is a contradiction to (4.8.15). Thus, the theorem is proved and $\bar{x}$ is a weak minimum solution for (VFP). □

Theorem 4.8.3. *Let $\bar{x}\in X$, $\lambda^0\in R^p, u_i^0=f_i(\bar{x})/g_i(\bar{x}), i\in P$ and $v^0\in R^m$ be such that the conditions (4.8.15)–(4.8.19) of Theorem 4.8.2 hold. Furthermore, we assume that* $(\mathrm{VFP_u})$ *is η-semilocally pseudo-quasi-type I-preinvex at $\bar{x}$. Then $\bar{x}$ is a weak minimum solution for* $(\mathrm{VFP_u})$.

Proof. Suppose that $\bar{x}$ is not a weak minimum solution for $(\mathrm{VFP_u})$. Then there exists $\tilde{x}\in X$ such that

$$\frac{f_i(\tilde{x})}{g_i(\tilde{x})}<\frac{f_i(\bar{x})}{g_i(\bar{x})}\quad\text{for any } i\in P,$$

that is,

$$f_i(\tilde{x})<u_i^0g_i(\tilde{x})\quad\text{for any } i\in P,$$

which is equivalent to

$$f_i(\tilde{x}) - u_i^0 g_i(\tilde{x}) < f_i(\bar{x}) - u_i^0 g_i(\bar{x}) \quad \text{for any } i \in P.$$

By the η-semilocally pseudo-type I-preinvexity at $\bar{x}$, of $(\mathrm{VFP_u})$, we get

$$(df_i)^+(\bar{x}, \eta(\tilde{x}, \bar{x})) - u_i^0 (dg_i)^+(\bar{x}, \eta(\tilde{x}, \bar{x})) < 0 \quad \text{for any } i \in P.$$

Using $\lambda_i^0 \in R_+^p, \lambda^{0^T} e = 1$, we obtain

$$\sum_{i=1}^{p} \lambda_i^0 \left((df_i)^+(\bar{x}, \eta(\tilde{x}, \bar{x})) - u_i^0 (dg_i)^+(\bar{x}, \eta(\tilde{x}, \bar{x}))\right) < 0. \tag{4.8.22}$$

By the η-semilocally quasi-type I-preinvexity at $\bar{x}$, of (VFP_u) and (4.8.16) and $v^0 \in R_+^m$ we get

$$\sum_{j=1}^{m} v_j^0 (dh_j)^+(\bar{x}, \eta(\tilde{x}, \bar{x})) \leqq 0. \tag{4.8.23}$$

Now, by (4.8.22) and (4.8.23) we obtain

$$\sum_{i=1}^{p} \lambda_i^0 \left((df_i)^+(\bar{x}, \eta(\tilde{x}, \bar{x})) - u_i^0 (dg_i)^+(\bar{x}, \eta(\tilde{x}, \bar{x}))\right) + \sum_{j=1}^{m} v_j^0 (dh_j)^+(\bar{x}, \eta(\tilde{x}, \bar{x})) < 0,$$

which is a contradiction to (4.8.20). Thus, the theorem is proved and $\bar{x}$ is a weak minimum solution for (VFP_u). □

4.9 Optimality Conditions for Vector Fractional Subset Optimization Problems

In this section, we shall use a new class of generalized convex n-set functions, the called ($\mathcal{F}$, ρ, σ, θ)-V-Type-I and related non-convex functions, introduced in Sect. 4.3.7 and then establish a number of parametric and semi-parametric sufficient optimality conditions for the primal problem under the aforesaid assumptions. This work partially extends an earlier work of Zalmai (2002) to a wider class of functions.

Consider the following vector fractional subset optimization problem:

$$\text{(P)} \qquad \text{minimize} \left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \ldots, \frac{F_p(S)}{G_p(S)},\right)$$
$$\text{subject to } H_j(S) \leqq 0,\ j \in \underline{m}, S \in \Lambda^n,$$

where Λ^n is the n-fold product of the σ-agebra Λ of the subsets of a given set X, $F_i, G_i, i \in \underline{p} \equiv \{1, 2, \ldots, p\}$ $H_j(S) \leqq 0,\ j \in \underline{m} \equiv \{1, 2, \ldots, m\}$, are real valued functions defined on Λ^n, and for each $G_i(S) > 0$, for each $i \in \underline{p}$, for all $S \in \Lambda^n$. The necessary background is given in Sect. 4.3.7.

The point-function counterparts of (P) are known in the area of mathematical programming as vector fractional optimization problems. These problems have been the focus of intense interest in the past few years, which has resulted in numerous publications the readers may consult a fairly extensive list of references related to various aspects of fractional programming by Pini and Singh (1997). For more information about general vector optimization problems with point-functions, readers may consult Dinkelbach (1967), Antczak (2002a), Bector et al. (1994b), Britan (1981), Egudo (1989), Hanson and Mond (1987a), Ivanov and Nehse (1985), Mishra (1998b), Mishra and Mukherjee (1995b), Mishra and Mukherjee (1996), Mishra et al. (2005), and Xu (1996).

In the area of subset optimization, vector optimization problems have been investigated by Kim et al. (1998). Much attention has been paid to analysis of optimization problems with set functions, for example, see Chou et al. (1985), Corley (1987), Hsia and Lee (1987), Kim et al. (1998b), Lai and Lin (1989), Lin (1990, 1991a, 1991b, 1992), Mazzoleni (1979), Morris (1979), Preda (1995), Preda and Stancu-Minasian (2001), Rosenmuller and Weidner (1974), Tanaka and Maruyama (1984), and Zalmai (1989, 1990a, 2001).

Notice that, all these results are also applicable, when appropriately specialized, to the following three classes of problems with vector, fractional, and conventional objective functions, which are particular cases of (P):

$$\text{Minimize}_{S\in X}\left(F_1(S),\, F_2(S),\ldots,F_p(S)\right) \tag{P1}$$

$$\text{Minimize}_{S\in X}\frac{F_1(S)}{G_1(S)} \tag{P2}$$

$$\text{Minimize}_{S\in X}F_1(S) \tag{P3}$$

where X(assumed to be nonempty) is the feasible set of (P), that is,

$$X=\left\{S\in\Lambda^n : H_j(S)\leqq 0,\, j\in \underline{m}\right\}.$$

Throughout this section, we shall deal exclusively with efficient solutions of (P). We recall that an $S^*\in\Xi$ is said to be an efficient solution (P) if there is no $S\in\Xi$ such that

$$\left(\frac{F_1(S)}{G_1(S)},\frac{F_2(S)}{G_2(S)},\ldots,\frac{F_p(S)}{G_p(S)}\right)\leq\left(\frac{F_1(S^*)}{G_1(S^*)},\frac{F_2(S^*)}{G_2(S^*)},\ldots,\frac{F_p(S^*)}{G_p(S^*)}\right).$$

In order to derive a set of necessary conditions for (P), we employ a Dinkelbach-type (1967) indirect approach via the following auxiliary problem:

$$(\mathrm{P}\lambda)\qquad \text{Minimize}_{S\in\Xi}\left(\mathrm{F}_1(S)-\lambda_1G_1(S),\ldots,\mathrm{F}_\mathrm{p}(S)-\lambda_pG_p(S)\right),$$

where $\lambda_i, i\in\underline{p}$, are parameters. This problem is equivalent to (P) in the sense that for particular choices of $\lambda_i, i\in\underline{p}$, the two problems have the same set of efficient solutions. This equivalence is stated more precisely in the following lemma whose proof is straightforward, and hence, omitted.

Lemma 4.9.1. *An $S^* \in \Xi$ is an efficient solution of (P) if and only if it is an efficient solution of* $(\mathrm{P}\lambda^*)$ *with* $\lambda_i^* = F_i(S^*)/G_i(S^*), i \in \underline{p}$.

Now applying Theorem 3.23 of Lin (1991b) to $(\mathrm{P}\lambda)$ and using Lemma 4.9.1, we obtain the following necessary efficiency results for (P).

Lemma 4.9.2. *Assume that* $F_i, G_i, i \in \underline{p}$, *and* $H_j, j \in \underline{m}$, *are differentiable at* $S^* \in \Lambda^{\mathrm{n}}$, *and that for each* $i \in \underline{p}$, *there exists* $\hat{S}^i \in \Lambda^n$ *such that*

$$H_j(S^*) + \sum_{k=1}^{n} \left\langle D_k H_j(S^*), \chi_{\hat{S}_k^i} - \chi_k^* \right\rangle < 0, j \in \underline{m},$$

and for each $l \in \underline{p} \setminus \{i\}$,

$$\sum_{k=1}^{n} \left\langle D_k F_l(S^*) - \lambda_l^* D_k G_l(S^*), \chi_{\hat{S}_k^l} - \chi_{S_k^*} \right\rangle < 0.$$

If S^* *is an efficient solution of (P) and* $\lambda_i^* = F_i(S^*)/G_i(S^*)$, $i \in \underline{p}$, *then there exist* $u^* \in U = \left\{ u \in R^p : u > 0, \sum_{i=1}^{p} u_i = 1 \right\}$ *and* $v^* \in R_+^m$ *such that*

$$\sum_{k=1}^{n} \left\langle \sum_{i=1}^{p} u_i^* \left[D_k F_i(S^*) - \lambda_i^* D_k G_i(S^*) \right] + \sum_{j=1}^{m} v_j^* D_k H_j(S^*), \chi_{S_k} - \chi_{S_k^*} \right\rangle \geqq 0, \forall S \in \Lambda^n,$$

$$v_j^* H_j(S^*) = 0, j \in \underline{m}.$$

The above theorem contains two sets of parameters u_i^* and $\lambda_i^*, i \in \underline{p}$. It is possible to eliminate one of these two sets of parameters, and thus, obtain a semi-parametric version of Lemma 4.9.2. Indeed, this can be accomplished by simply replacing λ_i^* by $F_i(S^*)/G_i(S^*), i \in \underline{p}$, and redefining u^* and v^*. For further reference, we state this next theorem.

Lemma 4.9.3. *Assume that* $F_i, G_i, i \in \underline{p}$, *and* $H_j, j \in \underline{m}$, *are differentiable at* $S^* \in \Lambda^{\mathrm{n}}$, *and that for each* $i \in \underline{p}$, *there exists* $\hat{S}^i \in \Lambda^n$ *such that*

$$H_j(S^*) + \sum_{k=1}^{n} \left\langle D_k H_j(S^*), \chi_{\hat{S}_k^i} - \chi_k^* \right\rangle < 0, j \in \underline{m},$$

and for each $l \in \underline{p} \setminus \{i\}$,

$$\sum_{k=1}^{n} \left\langle G_l(S^*) D_k F_l(S^*) - F_l(S^*) D_k G_l(S^*), \chi_{\hat{S}_k^l} - \chi_{S_k^*} \right\rangle < 0.$$

If S^* *is an efficient solution of (P), then there exist* $u^* \in U$ *and* $v^* \in R_+^m$ *such that*

$$\sum_{k=1}^{n}\left\langle \sum_{i=1}^{p} u_i^*\left[G_i(S^*)D_kF_i(S^*)-F_i(S^*)D_kG_i(S^*)\right]+\sum_{j=1}^{m} v_j^*D_kH_j(S^*),\chi_{S_k}-\chi_{S_k^*}\right\rangle$$
$$\geqq 0, \forall S\in\Lambda^n,\ v_j^*H_j(S^*)=0,\ j\in \underline{m}.$$

The form and contents of the necessary efficiency conditions given in Lemma 4.9.3 are used by Zalmai (2002) to derive a number of semi-parametric sufficient efficiency criteria as well as for constructing various duality models for (P).

We shall establish some other parametric sufficient optimality conditions for (P) under various generalized (Ŧ, ρ, σ, θ)-V-type-I assumptions. In order to simplify the statements and proofs of these sufficiency results, we shall introduce along the way some additional notations. For stating our first sufficiency theorem, we use the real-valued functions $\mathrm{A}_i(\cdot;\lambda,u)$ and $\mathrm{B}_j(\cdot,v)$ defined for fixed λ, u and v on Λ^n by

$$\mathrm{A}_i(\cdot;\lambda,u)=u_i\left[F_i(S)-\lambda_iG_i(S)\right],\ i\in \underline{p},$$

and

$$\mathrm{B}_j(\cdot,v)=v_jH_j(S),\ j\in \underline{m}.$$

Theorem 4.9.1. *Let $S^*\in\Xi$ and assume that $F_i, G_i, i\in\underline{p}$, and $H_j, j\in\underline{m}$, are differentiable at $S^*\in\Lambda^{\mathrm{n}}$, and there exist $u^*\in U$ and $v^*\in R_+^{\underline{m}}$ such that*

$$Ŧ\left(S,S^*;\sum_{i=1}^{p}u_i^*\left[DF_i(S^*)-\lambda_i^*DG_i(S^*)\right]+\sum_{j=1}^{m}v_j^*DH_j(S^*)\right)\geqq 0, \forall S\in\Lambda^n, \quad (4.9.1)$$

$$F_i(S^*)-\lambda_i^*G_i(S^*)=0, i\in\underline{p}, \quad (4.9.2)$$

$$v_j^*H_j(S^*)=0,\ j\in\underline{m}, \quad (4.9.3)$$

where Ŧ$(S,S^*;\cdot):L_1^n(X,\Lambda,\mu)\to R$ is a sublinear function. Assume furthermore that any one of the following sets of hypotheses is satisfied:

(a) (i) $(\mathrm{A}_i(\cdot;\lambda^*,u^*),\mathrm{B}_j(\cdot,v^*))\,\forall i\in\underline{p}$ and $j\in\underline{m}$, are (Ŧ, α, β, ρ, σ, θ)-V-pseudo-quasi-type-I at S^*;
(ii) $\rho+\sigma\geqq 0$;

(b) (i) $(\mathrm{A}_i(\cdot;\lambda^*,u^*),\mathrm{B}_j(\cdot,v^*))\,\forall i\in\underline{p}$ and $j\in\underline{m}$, are (Ŧ, α, β, ρ, σ, θ)-V-pseudo-prestrict-quasi-type-I at S^*;
(ii) $\rho+\sigma>0$;

(c) (i) $(\mathrm{A}_i(\cdot;\lambda^*,u^*),\mathrm{B}_j(\cdot,v^*))\,\forall i\in\underline{p}$ and $j\in\underline{m}$, are (Ŧ, α, β, ρ, σ, θ)-V-prestrict-quasi-strict-pseudo-type-I at S^*;
(ii) $\rho+\sigma\geqq 0$.

Then S^* is an efficient solution of (P).

Proof. Let S be an arbitrary feasible solution of (P), then by the sublinearity of Ŧ and (4.9.1) it follows that

$$Ŧ\left(S,S^*;\sum_{i=1}^{p}u_i^*\left[DF_i(S^*)-\lambda_i^*DG_i(S^*)\right]\right)+Ŧ\left(S,S^*;\sum_{j=1}^{m}v_j^*DH_j(S^*)\right)\geqq 0. \quad (4.9.4)$$

(a) Since $v^* \geqq 0, S \in \Xi$ and (4.9.3) holds, it is clear that $v_j^* H_j(S^*) = 0, \forall j \in \underline{m}$, and hence,

$$-\sum_{j=1}^{m} \beta_j(S,S^*) v_j^* H_j(S^*) \leqq 0,$$

which by virtue of second part of (i) implies that

$$\digamma\left(S,S^*;\sum_{j=1}^{m} v_j^* DH_j(S^*)\right) \leqq -\sigma d^2(\theta(S,S^*)). \tag{4.9.5}$$

From (4.9.4) and (4.9.5), we see that

$$\digamma\left(S,S^*;\sum_{i=1}^{p} u_i^*\left[DF_i(S^*) - \lambda_i^* DG_i(S^*)\right]\right) \geqq \sigma d^2(\theta(S,S^*)) \geqq -\rho d^2(\theta(S,S^*)),$$

where the second inequality follows from (ii). By first part of (i), the last inequality implies that

$$\sum_{i=1}^{p} \alpha_i(S,S^*) u_i^*\left[F_i(S) - \lambda_i^* G_i(S)\right] \geqq \sum_{i=1}^{p} \alpha_i(S,S^*) u_i^*\left[F_i(S^*) - \lambda_i^* G_i(S^*)\right],$$

which in view of (4.9.2) becomes

$$\sum_{i=1}^{p} \alpha_i(S,S^*) u_i^*\left[F_i(S) - \lambda_i^* G_i(S)\right] \geqq 0. \tag{4.9.6}$$

Since $\alpha_i(S,S^*) u_i^* > 0$ for each $i \in \underline{p}$, (4.9.6) implies that $(\mathrm{F}_1(S) - \lambda_1^* G_1(S), \ldots, \mathrm{F}_\mathrm{p}(S) - \lambda_p^* G_p(S)) \nleq (0,\ldots,0)$, which in turn implies that

$$\phi(S) \equiv \left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \ldots, \frac{F_p(S)}{G_p(S)}\right) \nleq \lambda^*.$$

Because $\lambda^* = \phi(S^*)$ and $S \in \Xi$ was arbitrary, we conclude that S^* is an efficient solution of (P).

Proofs of parts (b) and (c) are similar to that of part (a). □

Next we discuss some sufficient optimality conditions for mixed type of combinations of the problem functions. For this we need to introduce some additional notations.

Let $\{J_0, J_1, \ldots, J_q\}$ be a partition of the index set $\underline{m}$; thus, $J_r \subset \underline{m}$ for each $r \in \{0,1,\ldots,q\}, J_r \cap J_s = \Phi$ for each $r,s \in \{0,1,\ldots,q\}$ with $r \neq s$, and $\bigcup_{r=0}^{q} J_r = \underline{m}$. In addition we use the real-valued functions $\Gamma_i(\cdot;\lambda,u,v)$ and $\Delta_t(\cdot,v)$ defined for fixed $\lambda, u$and$v$ on Λ^n as follows:

$$\Gamma_i(\cdot;\lambda,u,v) = u_i\left[F_i(S) - \lambda_i G_i(S) + \sum_{j\in J_0} v_j H_j(S)\right], \quad i \in \underline{p},$$
$$\Delta_t(S,v) = \sum_{j\in J_t} v_j H_j(S), \quad t \in \underline{q}.$$

Making use of this notation, we next state some generalized sufficiency conditions for (P).

Theorem 4.9.2. *Let $S^* \in \Xi$ and assume that $F_i, G_i, i \in \underline{p}$, and $H_j, j \in \underline{m}$, are differentiable at $S^* \in \Lambda^n$, and there exist $u^* \in U$ and $v^* \in R_+^m$ such that (4.9.1)–(4.9.3) hold. Assume furthermore, that any one of the following sets of hypotheses is satisfied:*

(d) (i) $(\Gamma_i(\cdot;\lambda^*,u^*,v^*), \Delta_j(\cdot,v^*))$ $\forall i \in \underline{p}$ and $j \in \underline{m}$, are (Ŧ, α, β, ρ, σ, θ)-V-pseudo-quasi-type-I at S^*;
(ii) $\rho + \sigma \geqq 0$;
(e) (i) $(\Gamma_i(\cdot;\lambda^*,u^*,v^*), \Delta_j(\cdot,v^*))$ $\forall i \in \underline{p}$ and $j \in \underline{m}$, are (Ŧ, α, β, ρ, σ, θ)-V-pseudo-prestrict-quasi-type-I at S^*;
(ii) $\rho + \sigma > 0$;
(f) (i) $(\Gamma_i(\cdot;\lambda^*,u^*,v^*), \Delta_j(\cdot,v^*))$ $\forall\ i \in \underline{p}$ and $j \in \underline{m}$, are (Ŧ, α, β, ρ, σ, θ)-V-prestrict-quasi-strict-pseudo-type-I at S^*;
(ii) $\rho + \sigma \geqq 0$.

Then S^* is an efficient solution of (P).

Proof. Let S be an arbitrary feasible solution of (P), then by the sublinearity of Ŧ and (4.9.1) it follows that

$$Ŧ\left(S,S^*;\sum_{i=1}^{p} u_i^*\left[DF_i(S^*) - \lambda_i^* DG_i(S^*)\right] + \sum_{j\in J_0} v_j^* DH_j(S^*)\right) + Ŧ\left(S,S^*;\sum_{t=1}^{q}\sum_{j\in J_t} v_j^* DH_j(S^*)\right) \geqq 0. \tag{4.9.7}$$

(a) Since $v^* \geqq 0, S \in \Xi$ it follows from (4.9.3) that for each $t \in \underline{q}$:

$$-\sum_{t\in J_t} v_t^* H_t(S^*) = 0,$$

and so

$$-\sum_{t=1}^{q} \beta_t(S,S^*)\Delta_t(S^*,v^*) = 0,$$

which by virtue of second part of (i) implies that

$$Ŧ\left(S,S^*;\sum_{t=1}^{q}\sum_{j\in J_t} v_j^* DH_j(S^*)\right) \leqq -\sigma d^2(\theta(S,S^*)). \tag{4.9.8}$$

From (4.9.7) and (4.9.8), we see that

$$Ŧ\left(S,S^*;\sum_{i=1}^{p}u_i^*\left[DF_i\left(S^*\right)-\lambda_i^*DG_i\left(S^*\right)\right]+\sum_{j\in J_0}v_j^*DH_j\left(S^*\right)\right)\geqq\sigma d^2\left(\theta\left(S,S^*\right)\right)$$
$$\geqq-\rho d^2\left(\theta\left(S,S^*\right)\right),$$

where the second inequality follows from (ii). Since $\sum_{i=1}^{p}u_i^*=1$, the above inequality can be expressed as

$$Ŧ\left(S,S^*;\sum_{i=1}^{p}u_i^*\left[DF_i\left(S^*\right)-\lambda_i^*DG_i\left(S^*\right)+\sum_{j\in J_0}v_j^*DH_j\left(S^*\right)\right]\right)\geqq-\rho d^2\left(\theta\left(S,S^*\right)\right),$$

which by virtue of the first part of hypothesis (i) implies that

$$\sum_{i=1}^{p}\alpha_i\left(S,S^*\right)\Gamma_i\left(\mathrm{S},\lambda^*,u^*,v^*\right)\geqq\sum_{i=1}^{p}\alpha_i\left(S,S^*\right)\Gamma_i\left(\mathrm{S}^*,\lambda^*,u^*,v^*\right)=0, \tag{4.9.9}$$

where the equality follows from (4.9.2) and (4.9.3). Since $v_j^*H_j\left(S\right)\leqq 0,\forall j\in \underline{m}$, and $\alpha_i\left(S,S^*\right)>0$, $\forall i\in \underline{p}$, we deduce from (4.9.9) that

$$\sum_{i=1}^{p}\alpha_i\left(S,S^*\right)u_i^*\left[F_i\left(S\right)-\lambda_i^*G_i\left(S\right)\right]\geqq 0,$$

which is precisely (4.9.6). Therefore, the rest of the proof is identical to that of Part (a) of Theorem 4.9.1.

Proofs of parts (e) and (f) are similar to that of part (d). □

Remark 4.9.1. Note that Theorem 4.9.2 contains a number of special cases that can easily be identified by appropriate choices of the partitioning sets $J_0, J_1, \ldots, J_q$.

In the remaining part of this section, we present some additional sets of general parametric sufficient optimality conditions using a variant of the partitioning scheme employed in Theorem 4.9.2.

Let $\{I_0, I_1, \ldots, I_k\}$ be partitions of $\underline{p}$ such that $K=\{0,1,\ldots,k\}\subset Q=\{0,1,\ldots,q\}$, $k<q$, and let the function $\Theta_t\left(\cdot,\lambda^*,u^*,v^*\right):\Lambda^n\to R$ be defined, for fixed λ^*, u^* and v^* by

$$\Theta_t\left(S,\lambda^*,u^*,v^*\right)=\sum_{i\in I_t}u_i^*\left[F_i\left(S\right)-\lambda_i^*G_i\left(S\right)\right]+\sum_{j\in J_t}v_j^*H_j\left(S\right), t\in K.$$

Theorem 4.9.3. *Let $S^*\in\Xi$ and assume that $F_i, G_i, i\in\underline{p}$, and $H_j, j\in\underline{m}$, are differentiable at $S^*\in\Lambda^{\mathrm{n}}$, and there exist $u^*\in U$ and $v^*\in R_+^m$ such that (4.9.1)–(4.9.3) hold. Assume furthermore, that any one of the following sets of hypotheses is satisfied:*

(g) (i) $\left(\Theta_t\left(\cdot,\lambda^*,u^*,v^*\right),\Delta_j\left(\cdot,v^*\right)\right)$ $\forall\, t\in K$ and $j\in\underline{m}$, are (Ŧ, α, β, ρ, σ, θ)-V-strict pseudo-quasi-type-I at S^*;

(ii) $\rho+\sigma\geqq 0$;

(h) (i) $(\Theta_t(\cdot,\lambda^*,u^*,v^*),\Delta_j(\cdot,v^*))\ \forall\ t\in K$ *and* $j\in \underline{m}$*, are* ($\mathcal{F}$, α, β, ρ, σ, θ)*-V-pseudo-prestrict-quasi-type-I at* S^**;*

(ii) $\rho+\sigma>0$*;*

(i) (i) $(\Theta_t(\cdot,\lambda^*,u^*,v^*),\Delta_j(\cdot,v^*))\ \forall\ t\in K$ *and* $j\in \underline{m}$*, are* ($\mathcal{F}$, α, β, ρ, σ, θ)*-V-prestrict-quasi-strict-pseudo-type-I at* S^**;*

(ii) $\rho+\sigma\geqq 0$.

Then S^* *is an efficient solution of (P).*

Proof. Suppose to the contrary that S^* is not an efficient solution of (P). Then there is $S^*\in\Xi$ such that $\left(F_1(\tilde S)/G_1(\tilde S),\ldots,F_p(\tilde S)/G_p(\tilde S)\right)\leq (F_1(S^*)/G_1(S^*),\ldots,F_p(S^*)/G_p(S^*))$, which in view of (4.9.2) implies that $F_i(\tilde S)-\lambda_i^*G_i(\tilde S)\leqq 0,\forall i\in\underline{p}$, with strict inequality holding for at least one index $l\in\underline{p}$. Since $u^*>0$, these inequalities yield

$$\sum_{i\in I_t} u_i^*\left[F_i(\tilde S)-\lambda_i^*G_i(\tilde S)\right]\leqq 0, t\in K. \tag{4.9.10}$$

Since $v^*\geqq 0$ and, $S,S^*\in\Xi$, it follows from (4.9.2), (4.9.3), and (4.9.10) that for each $t\in K$,

$$\begin{aligned}\Theta_t(\tilde S,\lambda^*,u^*,v^*) &= \sum_{i\in I_t} u_i^*\left[F_i(\tilde S)-\lambda_i^*G_i(\tilde S)\right]+\sum_{j\in J_T} v_j^*H_j(\tilde S)\\ &\leqq \sum_{i\in I_t} u_i^*\left[F_i(\tilde S)-\lambda_i^*G_i(\tilde S)\right]\\ &\leqq 0\\ &= \sum_{i\in I_t} u_i^*\left[F_i(S^*)-\lambda_i^*G_i(S^*)\right]+\sum_{j\in J_T} v_j^*H_j(S^*)=\Theta_t(S^*,\lambda^*,u^*,v^*)\end{aligned}$$

and so

$$\sum_{t\in K}\alpha_t(\tilde S,S^*)\Theta_t(\tilde S,\lambda^*,u^*,v^*)<\sum_{t\in K}\alpha_t(\tilde S,S^*)\Theta_t(S^*,\lambda^*,u^*,v^*),$$

which in view of first part of the hypotheses (i) implies that

$$\mathcal{F}\left(\tilde S,S^*;\sum_{i=1}^{p} u_i^*\left[DF_i(S^*)-\lambda_i^*DG_i(S^*)\right]+\sum_{t\in K}\sum_{j\in J_t} v_j^*DH_j(S^*)\right)<-\rho d^2\left(\theta(\tilde S,S^*)\right). \tag{4.9.11}$$

As for each $t\in M\backslash K$, $-\sum_{t\in M\backslash K}\beta_t(\tilde S,S^*)\Delta_t(S^*,v^*)=0$, and hence the second part of the hypotheses g(i) implies that

$$\mathcal{F}\left(\tilde S,S^*;\sum_{t\in M\backslash K}\sum_{j\in J_t} v_j^*DH_j(S^*)\right)\leqq-\sigma d^2\left(\theta(\tilde S,S^*)\right). \tag{4.9.12}$$

Now from (4.9.11), (4.9.12), g(ii), and the sublinearity, we get

$$\digamma\left(\tilde{S},S^*;\sum_{i=1}^{p}u_i^*\left[DF_i(S^*)-\lambda_i^*DG_i(S^*)\right]+\sum_{j=1}^{m}v_j^*DH_j(S^*)\right)$$
$$<-(\rho+\sigma)\,d^2\left(\theta\left(\tilde{S},S^*\right)\right)<0,$$

which contradicts (4.9.1). Hence, S^* is an efficient solution of (P).

The proofs for (h) and (i) are similar to that of part (g). □

Now we present the semi-parametric version of the general parametric sufficient optimality conditions in the present section given above. These sufficient optimality conditions are motivated by forms and features of Lemma 4.9.3.

In the statements and proofs of the sufficiency theorems, we use the functions $\mathrm{B}_j(\cdot,v^*)$ defined above, $\mathrm{E}_i(\cdot,S^*,u^*)$ and $\mathrm{H}_i(\cdot,S^*,u^*,v^*)$ defined for fixed S^*,u^*, and v^* on Λ^n by

$$\mathrm{E}_i(S,S^*,u^*)=u_i^*\left[G_i(S^*)F_i(S)-F_i(S^*)G_i(S)\right]$$

and

$$\mathrm{H}_i(S,S^*,u^*,v^*)=u_i^*\left[G_i(S^*)F_i(S)-F_i(S^*)G_i(S)+\sum_{j\in J_0}v_j^*H_j(S)\right],i\in\underline{p}.$$

Theorem 4.9.4. *Let $S^*\in\Xi$ and assume that $F_i,G_i,i\in\underline{p}$, and $H_j,j\in\underline{m}$, are differentiable at $S^*\in\Lambda^{\mathrm{n}}$, and there exist $u^*\in U$ and $v^*\in R_+^{\underline{m}}$ such that*

$$\digamma\left(S,S^*;\sum_{i=1}^{p}u_i^*\left[G_i(S^*)DF_i(S^*)-F_i(S^*)DG_i(S^*)+\sum_{j\in J_0}v_j^*DH_j(S^*)\right]\right)$$
$$\geqq 0,\forall S\in\Lambda^n, \tag{4.9.13}$$

$$v_j^*H_j(S^*)=0,j\in\underline{m}, \tag{4.9.14}$$

where $\digamma$ is a sublinear function. Assume furthermore that any one of the following set of hypotheses is satisfied:

(j) (i) $(\mathrm{E}_i(\cdot,S^*,u^*),\mathrm{B}_j(\cdot,v^*))$ $\forall i\in\underline{p}$ and $j\in\underline{m}$, are ($\digamma$, α, β, ρ, σ, θ)-V-pseudo-quasi-type-I at S^*;
(ii) $\rho+\sigma\geqq 0$;

(k) (i) $(\mathrm{E}_i(\cdot,S^*,u^*),\mathrm{B}_j(\cdot,v^*))$ $\forall i\in\underline{p}$ and $j\in\underline{m}$, are ($\digamma$, α, β, ρ, σ, θ)-V-pseudo-prestrict-quasi-type-I at S^*;
(ii) $\rho+\sigma>0$;

(l) (i) $(\mathrm{E}_i(\cdot,S^*,u^*),\mathrm{B}_j(\cdot,v^*))$ $\forall i\in\underline{p}$ and $j\in\underline{m}$, are ($\digamma$, α, β, ρ, σ, θ)-V-prestrict-quasi-strict-pseudo-type-I at $\overline{S}^*$;
(ii) $\rho+\sigma\geqq 0$.

Then S^* is an efficient solution of (P).

Proof. Let S be an arbitrary feasible solution of (P), then by the sublinearity of $\digamma$ and (4.1) it follows that

$$\digamma\left(S,S^*;\sum_{i=1}^{p}u_i^*\left[G_i\left(S^*\right)DF_i\left(S^*\right)-F_i\left(S^*\right)DG_i\left(S^*\right)\right]\right)$$
$$+\digamma\left(S,S^*;\sum_{j=1}^{m}v_j^*DH_j\left(S^*\right)\right)\geqq 0. \tag{4.9.15}$$

(j) Combining (4.9.15) and (4.9.5), which is valid for the present case because of our assumption specified in (i) and using (ii), we obtain

$$\digamma\left(S,S^*;\sum_{i=1}^{p}u_i^*\left[G_i\left(S^*\right)DF_i\left(S^*\right)-F_i\left(S^*\right)DG_i\left(S^*\right)\right]\right)\geqq -\rho d^2\left(\theta\left(S,S^*\right)\right),$$

which in the light of the hypotheses implies that

$$\sum_{i=1}^{p}\alpha_i\left(S,S^*\right)u_i^*\left[G_i\left(S^*\right)F_i\left(S\right)-F_i\left(S^*\right)G_i\left(S\right)\right]\geqq\sum_{i=1}^{p}\alpha_i\left(S,S^*\right)u_i^*\left[G_i\left(S^*\right)F_i\left(S^*\right)\right.$$
$$\left.-F_i\left(S^*\right)G_i\left(S^*\right)\right]=0. \tag{4.9.16}$$

Since $\alpha_i\left(S,S^*\right)u_i^*>0$ for each $i\in \underline{p}$, (4.9.16) implies that $\left(G_1\left(S^*\right)\mathrm{F}_1\left(S\right)-\mathrm{F}_1\left(S^*\right)\right.$ $\left.G_1,\left(S\right),\ldots G_p\left(S^*\right)\mathrm{F_p}\left(S\right)-\mathrm{F_p}\left(S^*\right)G_p\left(S\right)\right)\nleq(0,\ldots,0)$, which in turn implies that

$$\phi(S)\equiv\left(\frac{F_1(S)}{G_1(S)},\frac{F_2(S)}{G_2(S)},\ldots,\frac{F_p(S)}{G_p(S)}\right)\nleq\left(\frac{F_1(S^*)}{G_1(S^*)},\frac{F_2(S^*)}{G_2(S^*)},\ldots,\frac{F_p(S^*)}{G_p(S^*)}\right)\equiv\phi(S^*).$$

Hence, we conclude that S^* is an efficient solution of (P).

Proofs of parts (k) and (l) are similar to that of part (j).

The proof of the following theorem is easy, so we state it without any proof. □

Theorem 4.9.5. *Let $S^*\in\Xi$ and assume that $F_i,G_i,i\in\underline{p}$, and $H_j,j\in\underline{m}$, are differentiable at $S^*\in\Lambda^{\mathrm{n}}$, and there exist $u^*\in U$ and $v^*\in R_+^m$ such that (4.9.13) and (4.9.14) hold. Assume furthermore that any one of the following set of hypotheses is satisfied:*

(m) (i) $\left(\mathrm{E}_i\left(\cdot,S^*,u^*\right),\mathrm{B}_j\left(\cdot,v^*\right)\right)$ $\forall i\in\underline{p}$ *and* $j\in\underline{m}$, *are* ($\digamma$, α, β, ρ, σ, θ)*-V-pseudo-quasi-type-I at* S^**;*

(ii) $\rho+\sigma\geqq 0$*;*

(n) (i) $\left(\mathrm{E}_i\left(\cdot,S^*,u^*\right),\mathrm{B}_j\left(\cdot,v^*\right)\right)$ $\forall i\in\underline{p}$ *and* $j\in\underline{m}$, *are* ($\digamma$, α, β, ρ, σ, θ)*-V-pseudo-prestrict-quasi-type-I at* S^**;*

(ii) $\rho+\sigma>0$*;*

(o) (i) $\left(\mathrm{E}_i\left(\cdot,S^*,u^*\right),\mathrm{B}_j\left(\cdot,v^*\right)\right)$ $\forall i\in\underline{p}$ *and* $j\in\underline{m}$, *are* ($\digamma$, α, β, ρ, σ, θ)*-V-prestrict-quasi-strict-pseudo-type-I at* S^**;*

(ii) $\rho+\sigma\geqq 0$.

Then S^* is an efficient solution of (P).

For the next Theorem let the function $\Pi_t(\cdot, S^*, u^*, v^*) : \Lambda^n \to R$ *be defined for fixed* S^*, u^* *and* v^* *by*

$$\Pi_t(S, S^*, u^*, v^*) = \sum_{i \in I_t} u_i^* [G_i(S^*) F_i(S) - F_i(S^*) G_i(S)] + \sum_{j \in J_t} v_j^* H_j(S), \ \forall t \in K.$$

Theorem 4.9.6. *Let* $S^* \in \Xi$ *and assume that* $F_i, G_i, i \in \underline{p}$, *and* $H_j, j \in \underline{m}$, *are differentiable at* $S^* \in \Lambda^{\mathrm{n}}$, *and there exist* $u^* \in U$ *and* $v^* \in R_+^m$ *such that (4.9.1)–(4.9.3) hold. Assume furthermore, that any one of the following sets of hypotheses is satisfied:*

(p) (i) $(\Pi_t(\cdot, S^*, u^*, v^*), \Delta_j(\cdot, v^*))$ $\forall t \in K$ *and* $j \in \underline{m}$, *are (*Ŧ*,* α*,* β*,* ρ*,* σ*,* θ*)-V-pseudo-quasi-type-I at* S^**;*
(ii) $\rho + \sigma \geqq 0$*;*

(q) (i) $(\Theta_t(\cdot, \lambda^*, u^*, v^*), \Delta_j(\cdot, v^*))$ $\forall t \in K$ *and* $j \in \underline{m}$, *are (*Ŧ*,* α*,* β*,* ρ*,* σ*,* θ*)-V-pseudo-prestrict-quasi-type-I at* S^**;*
(ii) $\rho + \sigma > 0$*;*

(r) (i) $(\Theta_t(\cdot, \lambda^*, u^*, v^*), \Delta_j(\cdot, v^*))$ $\forall t \in K$ *and* $j \in \underline{m}$, *are (*Ŧ*,* α*,* β*,* ρ*,* σ*,* θ*)-V-prestrict-quasi-strict-pseudo-type-I at* S^**;*
(ii) $\rho + \sigma \geqq 0$.

Then S^* *is an efficient solution of (P).*

The proof can be given on similar lines to the above theorem.

Chapter 5
Duality Theory

The concept of duality is of fundamental importance in linear programming. Wolfe (1961) used the Kuhn–Tucker conditions to formulate a dual program for a nonlinear optimization problem in the spirit of duality in linear programming, that is, with the aim of defining a problem whose objective value gives lower bound on the optimal value of the original or primal problem and whose optimal solution yields an optimal solution for the primal problem under certain regularity conditions. Wolfe (1961) established the weak duality, that is, with the same convexity conditions as required for the sufficiency of the Kuhn–Tucker conditions, every feasible solution of the dual has an objective value less than or equal to the objective value of every feasible solution of the primal problem.

Generalized convexity plays a crucial role in the study of the duality theory. Mond and Weir (1981) proposed a new type of dual based on the Wolfe dual. The advantage of Mond–Weir dual over the Wolfe dual is that the objective function in Mond-Weir dual is the same as that of the primal problem and that duality results are achieved by means of further relaxation of invexity requirements.

5.1 Mond–Weir Type Duality for Vector Optimization Problems

In this section, we give some weak, strong and converse duality theorems for (VP) studied in Sect. 4.1 and the following Mond–Weir dual suggested by Egudo (1989).

$$\begin{aligned} \text{(MWD)} \qquad & \text{maximize } f(y) \\ & \text{subject to } \tau\nabla f(y) + \lambda\nabla g(y) = 0, \\ & \lambda g(y) \geqq 0, \\ & \lambda \geqq 0,\ \tau \geqq 0 \text{ and } \tau\, e = 1; \end{aligned}$$

where $e = (1, \ldots, 1)^T \in R^p$.

S.K. Mishra et al., *Generalized Convexity and Vector Optimization,*
Nonconvex Optimization and Its Applications.

Let Y^0 be the set of feasible solutions of problem (MWD); i.e.

$$Y^0 = \{(y, \tau, \lambda) : \tau \nabla f(y) + \lambda \nabla g(y) = 0,\ \lambda\ g(y) \geqq 0,\ \tau \in R^p,\ \lambda \in R^m,\ \lambda \geqq 0\}$$

Theorem 5.1.1. *(Weak Duality). Suppose that*

(i) $x \in X_0$;
(ii) $(y, \tau, \lambda) \in Y^0$ *and* $\tau > 0$;
(iii) Problem (VP) is strong pseudo quasi type I univex at y with respect to some $b_0,\ b_1,\ \phi_0,\ \phi_1$ and η;
(iv) $u \leq 0 \Rightarrow \phi_0(u) \leq 0$ *and* $u \leqq 0 \Rightarrow \phi_1(u) \leqq 0$,
(v) $b_0(x, y) > 0$, *and* $b_1(x, y) \geqq 0$.

Then $f(x) \nleq f(y)$.

Proof. Suppose contrary to the result that, i.e.,

$$f(x) \leq f(y).$$

By conditions (iv), (v) and the above inequality, we have

$$b_0(x, y)\,\phi_0[f(x) - f(y)] \leq 0. \tag{5.1.1}$$

By the feasibility of $(y,\ \tau,\ \lambda)$, we have

$$-\lambda^0 g(y) \leqq 0.$$

By conditions (iv), (v) and the above inequality, we have

$$-b_1(x, y)\,\phi_1[\lambda\, g(y)] \leqq 0. \tag{5.1.2}$$

By inequalities (5.1.1), (5.1.2) and condition (iii), we have

$$(\nabla f(y))\ \eta(x, y) \leq 0,\ \text{and}\ \lambda \nabla g(y)\,\eta(x, y) \leqq 0.$$

Since $\tau > 0$, the above inequalities give

$$[\tau \nabla f(y) + \lambda \nabla g(y)]\,\eta(x, y) < 0.$$

which contradicts condition (iia). This completes the proof. □

Theorem 5.1.2. *(Weak Duality). Suppose that*

(i) $x \in X_0$;
(ii) $(y,\ \tau,\ \lambda) \in Y^0$, *and* $\tau^0 \geq 0$;
(iii) The problem (VP) is weak strictly pseudo quasi type I univex at y with respect to some b_0, b_1, ϕ_0, ϕ_1 *and* η;
(iv) $u \leq 0 \Rightarrow \phi_0(u) \leq 0$ *and* $u \leqq 0 \Rightarrow \phi_1(u) \leqq 0$;
(v) $b_0(x, y) > 0$, *and* $b_1(x, y) \geqq 0$.

Then $f(x) \nleq f(y)$.

Proof. Suppose contrary to the result that, i.e.,

$$f(x) \leq f(y).$$

By conditions (iv), (v) and the above inequality, we get (5.1.1). By the feasibility of (y, τ, λ), conditions (iv) and (v), give (5.1.2).

By inequalities (5.1.1), (5.1.2) and condition (iii), we have

$$(\nabla f(y))\,\eta(x,y) < 0, \text{ and } \lambda \nabla g(y)\,\eta(x,y) \leqq 0.$$

Since $\tau^0 \geq 0$, the above inequalities give

$$[\tau \nabla f(y) + \lambda \nabla g(y)]\,\eta(x,y) < 0.$$

which contradicts condition (iia). This completes the proof. □

Theorem 5.1.3. *(Weak Duality). Suppose that*

(i) $x \in X_0$*;*
(ii) $(y, \tau, \lambda) \in Y^0$*;*
(iii) The problem (VP) is weak strictly pseudo type I univex at y with respect to some b_0, b_1, ϕ_0, ϕ_1 *and* η*;*
(iv) $u \leq 0 \Rightarrow \phi_0(u) \leq 0$ *and* $u \leqq 0 \Rightarrow \phi_1(u) \leqq 0$*;*
(v) $b_0(x,y) > 0$*, and* $b_1(x,y) \geqq 0$*.*

Then $f(x) \nleq f(y)$*.*

Proof. Suppose contrary to the result that, i.e.,

$$f(x) \leq f(y).$$

By conditions (iv), (v) and the above inequality, we get (5.1.1). By the feasibility of (y, τ, λ), conditions (iv) and (v), give (5.1.2).

By inequalities (5.1.1), (5.1.2) and condition (iii), we have

$$(\nabla f(y))\,\eta(x,y) < 0, \text{ and } \lambda \nabla g(y)\,\eta(x,y) < 0.$$

Since $\tau \geqq 0$, the above inequalities give

$$[\tau \nabla f(y) + \lambda \nabla g(y)]\,\eta(x,y) < 0.$$

which contradicts condition (iia). This completes the proof. □

Theorem 5.1.4. *(Strong Duality). Let* $\bar{x}$ *be an efficient solution for (VP) and* $\bar{x}$ *satisfies a constraint qualification (Marusciac 1982) for (VP). Then there exist* $\bar{\tau} \in R^p$ *and* $\bar{\lambda} \in R^m$ *such that* $\left(\bar{x}, \bar{\tau}, \bar{\lambda}\right)$ *is feasible for (MWD). If any of the weak duality (Theorems 5.1.1–5.1.3) also holds, then* $\left(\bar{x}, \bar{\tau}, \bar{\lambda}\right)$ *is efficient solution for (MWD).*

Proof. Since $\overline{x}$ is efficient for (VP) and satisfies a constraint qualification for (VP), then from Kuhn–Tucker necessary optimality conditions, we obtain $\overline{\tau} > 0$, and $\overline{\lambda} \geqq 0$ such that

$$\overline{\tau}\nabla f(\overline{x}) + \overline{\lambda}\nabla g(\overline{x}) = 0,$$
$$\overline{\lambda} g(\overline{x}) = 0.$$

The vector $\overline{\tau}$ may be normalized according to $\overline{\tau}e = 1$, $\overline{\tau} > 0$, which gives that the triplet $\left(\overline{x}, \overline{\tau}, \overline{\lambda}\right)$ is feasible for (MWD). The efficiency of $\left(\overline{x}, \overline{\tau}, \overline{\lambda}\right)$ for (MWD) follows from weak duality theorem. This completes the proof. □

5.2 General Mond–Weir Type Duality for Vector Optimization Problems

In this section, we consider a general Mond–Weir type of dual to (VP) studied in Sect. 4.1 and establish weak and strong duality theorems under the weaker type I univexity.

We consider the following general Mond–Weir type dual problem:

$$\text{(GMWD)} \qquad \text{maximize } f(y) + \lambda_{J_0} g_{J_0}(y)\, e$$
$$\text{subject to } \tau\nabla f(y) + \lambda\nabla g(y) = 0, \tag{5.2.1}$$
$$\lambda_{J_t} g_{J_t} \geqq 0,\ 1 \leqq t \leqq r, \tag{5.2.2}$$
$$\lambda \geqq 0,\ \tau \geqq 0 \text{ and } \tau e = 1;$$

where $e = (1, \ldots, 1)^T \in R^p$ and J_t, $0 \leqq t \leqq r$ are partitions of the set M.

Theorem 5.2.1. *(Weak Duality). Suppose that for all feasible x for (VP) and all feasible (y, τ, λ) for (GMWD), we have*

(a) $\tau > 0$, and $\left(f + \lambda_{J_0} g_{J_0}(\cdot)\, e, \lambda_{J_t} g_{J_t}(\cdot)\right)$ is strong pseudoquasi type I univex at y for any t, $1 \leqq t \leqq r$ with respect to b_0, b_1, ϕ_0, ϕ_1 and η with ϕ_0 and ϕ_1 increasing;
(b) $\left(f + \lambda_{J_0} g_{J_0}(\cdot)\, e, \lambda_{J_t} g_{J_t}(\cdot)\right)$ is weak strictly pseudoquasi type I univex at y for any t, $1 \leqq t \leqq r$ with respect to b_0, b_1, ϕ_0, ϕ_1 and η with ϕ_0 and ϕ_1 increasing;
(c) $\left(f + \lambda_{J_0} g_{J_0}(\cdot)\, e, \lambda_{J_t} g_{J_t}(\cdot)\right)$ is weak strictly pseudo type I univex at y for any t, $1 \leqq t \leqq r$ with respect to b_0, b_1, ϕ_0, ϕ_1 and η with ϕ_0 and ϕ_1 increasing.

Then $f(x) \nleq f(y) + \lambda_{J_0} g_{J_0}(y)\, e$.

Proof. Suppose contrary to the result. Then we have

$$f(x) \leq f(y) + \lambda_{J_0} g_{J_0}(y)\, e.$$

Since x is feasible for (VP) and $\lambda \geqq 0$, the above inequality implies that

$$f(x) + \lambda_{J_0} g_{J_0}(x)\, e \leq f(y) + \lambda_{J_0} g_{J_0}(y)\, e. \tag{5.2.3}$$

By the feasibility of (y, τ, λ) inequality (5.2.2) gives

$$-\lambda_{J_t} g_{J_t}(y) \leqq 0, \quad \text{for all } 1 \leqq t \leqq r. \tag{5.2.4}$$

Since ϕ_0 and ϕ_1 are increasing, from (5.2.3) and (5.2.4), we have

$$b_0(x,y)\phi_0((f(x) + \lambda_{J_0} g_{J_0}(x)e) - (f(y) + \lambda_{J_0} g_{J_0}(y)e)) \leq 0, \tag{5.2.5}$$

$$-b_1(x,y)\,\phi_1(\lambda_{J_t} g_{J_t}(y)) \leqq 0, \quad \text{for all } 1 \leqq t \leqq r. \tag{5.2.6}$$

By condition (a), from (5.2.5) and (5.2.6), we have

$$(\nabla f(y) + \lambda_{J_0} \nabla g_{J_0}(y)e)\eta(x,y) \leq 0.$$
$$(\lambda_{J_t} \nabla g_{J_t}(y))\eta(x,y) \leqq 0, \quad \text{for all } 1 \leqq t \leqq r.$$

Since $\tau > 0$, the above inequalities give

$$\left[\tau \nabla f(y) + \sum_{t=0}^{r} \lambda_{J_t} \nabla g_{J_t}(y)\right] \eta(x,y) < 0. \tag{5.2.7}$$

Since J_t, $0 \leqq t \leqq r$ are partitions of the set M, (5.2.7) is equivalent to

$$[\tau \nabla f(y) + \lambda \nabla g(y)]\,\eta(x,y) < 0,$$

which contradicts (5.2.1).

By condition (b), from (5.2.5) and (5.2.6), we have

$$(\nabla f(y) + \lambda_{J_0} \nabla g_{J_0}(y)e)\eta(x,y) < 0,$$
$$(\lambda_{J_t} \nabla g_{J_t}(y))\eta(\mathrm{x},\mathrm{y}) \leqq 0, \quad \text{for all } 1 \leqq t \leqq r.$$

Since $\tau \geq 0$, the above inequalities give (5.2.7), which again contradicts (5.2.1).

By condition (c), from (5.2.5) and (5.2.6), we have

$$(\nabla f(y) + \lambda_{J_0} \nabla g_{J_0}(y)e)\eta(x,y) < 0,$$
$$(\lambda_{J_t} \nabla g_{J_t}(y))\eta(\mathrm{x},\mathrm{y}) < 0, \quad \text{for all } 1 \leqq t \leqq r.$$

Since $\tau \geqq 0$, the above inequalities give (5.2.7), which again contradicts (5.2.1). This completes the proof. □

Theorem 5.2.2. *(Strong Duality). Let $\bar{x}$ be an efficient solution for (VP) and let $\bar{x}$ satisfy a constraint qualification (Maeda 1994) for (VP). Then there exist $\bar{\tau} \in R^p$ and $\bar{\lambda} \in R^m$ such that $\left(\bar{x}, \bar{\tau}, \bar{\lambda}\right)$ is feasible for (GMWD). If any of the weak duality in Theorem 5.2.1 also holds, then $\left(\bar{x}, \bar{\tau}, \bar{\lambda}\right)$ is an efficient solution for (GMWD).*

Proof. Since $\bar{x}$ is efficient for (VP) and satisfies a generalized constraint qualification (Maeda 1994), by Kuhn–Tucker necessary conditions there exist $\bar{\tau} > 0$ and

$\overline{\lambda} \geqq 0$ such that

$$\overline{\tau}\nabla f(\overline{x}) + \overline{\lambda}\nabla g(\overline{x}) = 0,$$
$$\overline{\lambda}_i g_i(\overline{x}) = 0,\ 1 \leqq i \leqq p.$$

The vector $\overline{\tau}$ may be normalized according to $\overline{\tau}e = 1, \overline{\tau} > 0$, which gives that the triplet $\left(\overline{x}, \overline{\tau}, \overline{\lambda}\right)$ is feasible for (GMWD). Efficiency follows from Theorem 5.2.1. This completes the proof. □

5.3 Mond–Weir Duality for Nondifferentiable Vector Optimization Problems

Now, in relation to (P) given in Sect. 4.2, we consider the following dual problem, which is in the format of Mond–Weir (1981):

(MWD) maximize $f(y) = (f_1(y), f_2(y), \ldots, f_k(y))$

subject to $\left(\xi^T f' + \mu^T g'\right)(y, \eta(x,y)) \geqq 0,\ \mathit{for\ all}\ x \in D, \quad (5.3.1)$

$$\mu_j g_j(y) \geqq 0,\ j = 1, \ldots, m, \quad (5.3.2)$$
$$\xi^T e = 1, \quad (5.3.3)$$
$$\xi \in R_+^k, \mu \in R_+^m,$$

where $e = (1, 1, \ldots, 1) \in R^k$.

Let

$$W = \left\{ \begin{array}{l} (y, \xi, \mu) \in X \times R^k \times R^m : \left(\xi^T f' + \mu^T g'\right)(y, \eta(x, y)) \geqq 0, \\ \qquad \mu_j g_j(y) \geqq 0,\ j = 1, \ldots, m, \xi \in R_+^k, \xi^T e = 1, \mu \in R_+^m \end{array} \right\}$$

denote the set of all the feasible solutions of (MWD). Throughout this section functions are same as in Sect. 4.2.

We denote by pr_X W the projection of set W on X.

Theorem 5.3.1. *(Weak Duality). Let x and* (y, ξ, μ) *be feasible solutions for (P) and (MWD), respectively. Moreover, we assume that any one of the following conditions holds:*

(a) $\left(f, \mu^T g\right)$ *is strong pseudo-quasi d−type-I at y with respect to* η *and* $\xi > 0$*;*
(b) $\left(f, \mu^T g\right)$ *is weak strictly pseudo-quasi d−type-I at y with respect to* η*;*
(c) $\left(f, \mu^T g\right)$ *is weak strictly pseudo d−type-I at y with respect to* η *at y on* $D \bigcup pr_X W$*. Then the following can not hold:*

$$f(x) \leq f(y).$$

Proof. We proceed by contradiction. Suppose that

$$f(x) \leq f(y). \quad (5.3.4)$$

Since (y, ξ, μ) is feasible for (MWD), it follows that

$$-\sum_{j=1}^{m} \mu_j g_j(y) \leqq 0. \tag{5.3.5}$$

By condition (a), (5.3.4) and (5.3.5) imply

$$f'(y, \eta(x, y)) \leq 0, \tag{5.3.6}$$

$$\sum_{j=1}^{m} \mu_j g_j'(y, \eta(x, y)) \leqq 0. \tag{5.3.7}$$

Since $\xi > 0$, the above two inequalities give

$$\sum_{i=1}^{k} \xi_i f_i'(y, \eta(x, y)) + \sum_{l=1}^{m} \mu_j g_j'(y, \eta(x, y)) < 0, \tag{5.3.8}$$

which contradicts (5.3.1).

By condition (b), (5.3.4) and (5.3.5) imply

$$f'(y, \eta(x,y)) < 0, \tag{5.3.9}$$

$$-\sum_{j=1}^{m} \mu_j g_j(y) \leqq 0. \tag{5.3.10}$$

Since $\xi \geqq 0$, (5.3.9) and (5.3.10) imply (5.3.8), again a contradiction to (5.3.1).

By condition (c), (5.3.4) and (5.3.5) imply

$$f'(y, \eta(x, y)) < 0, \tag{5.3.11}$$

$$-\sum_{j=1}^{m} \mu_j g_j(y) < 0. \tag{5.3.12}$$

Since $\xi \geqq 0$, (5.3.11) and (5.3.12) imply (5.3.8), again a contradiction to (5.3.1). This completes the proof. □

Theorem 5.3.2. *(Strong Duality). Let $\bar{x}$ be a locally weak Pareto efficient solution or weak Pareto efficient solution for (P) at which the generalized Slater's constraint qualification is satisfied, let f, g be directionally differentiable at $\bar{x}$ with $f'(\bar{x}, \eta(x, \bar{x}))$, and $g'(\bar{x}, \eta(x, \bar{x}))$ preinvex functions on X, and let g_j be continuous for $j \in \hat{J}(\bar{x})$. Then there exists $\bar{\mu} \in R^m_+$ such that $(\bar{x}, 1, \bar{\mu})$ is feasible for (MWD). If the weak duality between (P) and (MWD) in Theorem 5.3.1 holds, then $(\bar{x}, 1, \bar{\mu})$ is a locally weak Pareto efficient solution for (MWD).*

Proof. Since $\bar{x}$ satisfies all the conditions of Lemma 4.2.4, there exists $\bar{\mu} \in R^m_+$, such that $(\bar{x}, \bar{\mu})$ satisfies the conditions (4.2.1)–(4.2.3) of Chap. 4. By conditions (4.2.1)–(4.2.3), we have that $(\bar{x}, 1, \bar{\mu})$ is feasible for (MWD). Also, by the weak duality, it follows that $(\bar{x}, 1, \bar{\mu})$ is a locally weak Pareto efficient solution for (MWD). □

Theorem 5.3.3. *(Converse Duality). Let* $\left(\bar{y}, \bar{\xi}, \bar{\mu}\right)$ *be a weak Pareto efficient solution for (MWD). Moreover, we assume that the hypothesis of Theorem 5.3.1 hold at* $\bar{y}$ *in* $D \bigcup pr_X W$*, then* $\bar{y}$ *is a weak Pareto efficient solution for (P).*

Proof. We proceed by contradiction. Suppose that $\bar{y}$ is not any weak Pareto efficient solution for (P), that is, there exists $\tilde{x} \in D$ such that $f(\tilde{x}) < f(\bar{y})$. Since condition (a) of Theorem 5.3.1 holds, we get

$$\sum_{i=1}^{k} \overline{\xi_i} f_i'(\bar{y}, \eta(\tilde{x}, \bar{y})) < 0. \tag{5.3.13}$$

From the feasibility of $\tilde{x}$ for (P) and $\left(\bar{y}, \bar{\xi}, \bar{\mu}\right)$ for (MWD) respectively, we have

$$-\sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{y}) \leqq 0,$$

which in light of condition (a) of Theorem 5.3.1 yields

$$\sum_{j=1}^{m} \bar{\mu}_j g_j'(\bar{y}, \eta(\tilde{x}, \bar{y})) \leqq 0. \tag{5.3.14}$$

By (5.3.13) and (5.3.14), we get

$$\sum_{i=1}^{k} \overline{\xi_i} f_i'(\bar{y}, \eta(\tilde{x}, \bar{y})) + \sum_{j=1}^{m} \bar{\mu}_j g_j'(\bar{y}, \eta(\tilde{x}, \bar{y})) < 0. \tag{5.3.15}$$

This contradicts the dual constraint (5.3.1).

By condition (b), we get

$$\sum_{i=1}^{k} f_i'(\bar{y}, \eta(\tilde{x}, \bar{y})) < 0$$

and

$$\sum_{j=1}^{m} \bar{\mu}_j g_j'(\bar{y}, \eta(\tilde{x}, \bar{y})) \leqq 0.$$

Since $\overline{\xi_i} \geqq 0$, the above two inequalities imply (5.3.15), again contradiction to (5.3.1).

By condition (c), we have

$$\sum_{i=1}^{k} f_i'(\bar{y}, \eta(\tilde{x}, \bar{y})) < 0$$

and

$$\sum_{j=1}^{m} \overline{\mu}_j g'_j (\overline{y}, \eta (\tilde{x}, \overline{y})) < 0.$$

Since $\overline{\xi_i} \geqq 0$, the above two inequalities imply (5.3.15), again a contradiction to (5.3.1). This completes the proof. □

5.4 General Mond–Weir Duality for Nondifferentiable Vector Optimization Problems

In this section, we shall continue our discussion on duality for (P) given in Sect. 4.2, by considering a general Mond–Weir type dual problem of (P) and proving weak and strong duality theorems under an assumption of the generalized d-invexity introduced in Chap. 3.

We consider the following general Mond–Weir type dual to (P)

$$\text{(GMWD)} \qquad \text{maximize } \phi(y, \xi, \mu) = f(y) + \mu_{J_0}^T g_{J_0}(y) e$$

$$\text{subject to } \left(\xi^T f' + \mu^T g'\right)(y, \eta(x, y)) \geqq 0, \ \textit{for all } x \in D, \quad (5.4.1)$$

$$\mu_{J_t} g_{J_t}(y) \geqq 0, 1 \leqq t \leqq r \quad (5.4.2)$$

$$\xi^T e = 1, \quad (5.4.3)$$

$$\xi \in R_+^k, \mu \in R_+^m,$$

where J_t, $0 \leqq t \leqq r$ are partitions of set M and $e = (1, 1, \ldots, 1) \in R^k$.

Let

$$\tilde{W} = \left\{ \begin{array}{l} (y, \xi, \mu) \in X \times R^k \times R^m : \left(\xi^T f' + \mu^T g'\right)(y, \eta(x, y)) \geqq 0, \\ \qquad \mu_j g_j(y) \geqq 0, j = 1, \ldots, m, \xi \in R_+^k, \xi^T e = 1, \mu \in R_+^m \end{array} \right\}$$

denote the set of all the feasible solutions of (MWD).

Theorem 5.4.1. *(Weak Duality). Let x and (y, ξ, μ) be feasible solutions for (P) and (GMWD) respectively. If any one of the following conditions holds:*

(a) $\xi > 0$, and $\left(f + \mu_{J_0} g_{J_0}, \mu_{J_t} g_{J_t}\right)$ is strong pseudo d-type-I at y in $D \bigcup pr_X \tilde{W}$, with respect to η for any t, $1 \leqq t \leqq r$;

(b) $\left(f + \mu_{J_0} g_{J_0}, \mu_{J_t} g_{J_t}\right)$ is weak strictly pseudo-quasi d-type-I at y in $D \bigcup pr_X \tilde{W}$, with respect to η for any t, $1 \leqq t \leqq r$;

(c) $\left(f + \mu_{J_0} g_{J_0}, \mu_{J_t} g_{J_t}\right)$ is weak strictly pseudo d-type-I at y in $D \bigcup pr_X \tilde{W}$, with respect to η for any t, $1 \leqq t \leqq r$.

Then the following can not hold:

$$f(x) \leq \phi(y, \xi, \mu).$$

Proof. We proceed by contradiction. Suppose that

$$f(x) \leq \phi(y, \xi, \mu). \tag{5.4.4}$$

Since x is feasible for (P) and $\mu \geqq 0$, (5.4.4) implies that

$$f(x) + \mu_{J_0}^T g_{J_0}(x) e \leq f(y) + \mu_{J_0}^T g_{J_0}(y) e. \tag{5.4.5}$$

From (5.4.2), we have

$$-\mu_{J_t}^T g_{J_t} \leqq 0, \text{ for all } 1 \leqq t \leqq r. \tag{5.4.6}$$

By condition (a), from (5.4.5) and (5.4.6), we have

$$(f' + \mu_{J_0} g'_{J_0} e)(y, \eta(x,y)) \leq 0,$$

and

$$\mu_{J_t} g'_{J_t}(y, \eta(x, y)) \leqq 0, \ \forall 1 \leqq t \leqq r.$$

Since $\xi > 0$, the above two inequalities yield

$$\left(\xi^T f' + \sum_{t=0}^{r} \mu_{J_t} g'_{J_t} \right)(y, \eta(x, y)) < 0. \tag{5.4.7}$$

Since $J_0, \ldots, J_r$ are partitions of M, (5.4.7) is equivalent to

$$\left(\xi^T f' + \mu^T g' \right)(y, \eta(x, y)) < 0, \tag{5.4.8}$$

which contradicts dual constraint (5.4.1).

By condition (b), we have

$$\left(f' + \mu_{J_0} g'_{J_0} e \right)(y, \eta(x, y)) < 0,$$

and

$$\mu_{J_t} g'_{J_t}(y, \eta(x,y)) \leqq 0, \ \forall 1 \leqq t \leqq r.$$

Since $\xi \geqq 0$, the above two inequalities yield

$$\left(\xi^T f' + \sum_{t=0}^{r} \mu_{J_t} g'_{J_t} \right)(y, \eta(x, y)) < 0.$$

The above inequality leads to (5.4.8), which contradicts (5.4.1).

By condition (c), we get

$$\left(f' + \mu_{J_0} g'_{J_0} e \right)(y, \eta(x, y)) < 0,$$

and

$$\mu_{J_t} g'_{J_t}(y, \eta(x, y)) < 0, \ \forall 1 \leqq t \leqq r.$$

Since $\xi \geqq 0$, the above two inequalities yield

$$\left(\xi^T f' + \sum_{t=0}^{r} \mu_{J_t} g'_{J_t}\right)(y, \eta(x,y)) < 0.$$

The above inequality leads to (5.4.8), which contradicts (5.4.1). This completes the proof. □

Theorem 5.4.2. *(Strong Duality). Let $\overline{x}$ be a locally weak Pareto efficient solution or weak Pareto efficient solution for (P) at which the generalized Slater's constraint qualification is satisfied, let f and g be directionally differentiable at $\overline{x}$ with $f'(\overline{x}, \eta(x,\overline{x}))$, and $g'(\overline{x}, \eta(x,\overline{x}))$ preinvex functions on X, and let g_j be continuous for $j \in \hat{J}(\overline{x})$. Then, there exists $\overline{\mu} \in R^m_+$ such that $(\overline{x}, 1, \overline{\mu})$ is feasible for (GMWD). Moreover, if the weak duality between (P) and (GMWD) in Theorem 5.4.1 holds, then $(\overline{x}, 1, \overline{\mu})$ is a locally weak Pareto efficient solution or weak Pareto efficient solution for (GMWD).*

Proof. The proof of this theorem is similar to the proof of Theorem 4.2 in Chap. 4. □

5.5 Mond–Weir Duality for Nondifferentiable Vector Optimization Problems with *d*−Univex Functions

In relation to (P) given in Sect. 4.2, we consider the following dual problem which is in the form of Mond–Weir (1981):

$$\text{(MWD)} \quad \text{maximize } f(y) = (f_1(y), f_2(y), \ldots, f_k(y))$$

$$\text{subject to } \left(\xi^T f' + \mu^T g'\right)(y, \eta(x,y)) \geqq 0, \ \textit{for all } x \in D, \tag{5.5.1}$$

$$\mu_j g_j(y) \geqq 0, j = 1, \ldots, m, \tag{5.5.2}$$

$$\xi^T e = 1, \tag{5.5.3}$$

$$\xi \in R^k_+, \mu \in R^m_+,$$

where $e = (1, 1, \ldots, 1) \in R^k$.

Let

$$W = \left\{ \begin{array}{l} (y, \xi, \mu) \in X \times R^k \times R^m : \left(\xi^T f' + \mu^T g'\right)(y, \eta(x,y)) \geqq 0, \\ \qquad \mu_j g_j(y) \geqq 0, j = 1, \ldots, m, \xi \in R^k_+, \xi^T e = 1, \mu \in R^m_+ \end{array} \right\}$$

denote the set of all the feasible solutions of (MWD).

We denote by $\mathrm{pr}_X W$ the projection of set W on X.

Theorem 5.5.1. *(Weak duality). Let x and (y, ξ, μ) be a feasible solution for (P) and (MWD) respectively. Moreover, suppose that one of the following conditions holds:*

(d) *f is strong pseudo d-univex at y on $D \bigcup pr_X W$ with respect to some b_0, ϕ_0 and η with $\xi > 0$, $b_0 > 0$, $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, and $\mu^T g$ is quasi d-univex at y on $D \bigcup pr_X W$ with respect to b_1, ϕ_1 and η with $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$;*

(e) *f is weak strictly pseudod-univex at y on $D \bigcup pr_X W$ with respect to some b_0, ϕ_0 and η with $b_0 \geq 0$, $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, and $\mu^T g$ is quasi d-univex at y on $D \bigcup pr_X W$ with respect to b_1, ϕ_1 and η with $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$;*

(f) *f is weak strictly pseudo d-univex at y on $D \bigcup pr_X W$ with respect to some b_0, ϕ_0 and η with $b_0 \geq 0$, $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, and $\mu^T g$ is quasi d-univex at y on $D \bigcup pr_X W$ with respect to b_1, ϕ_1 and η with $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$.*

Then

$$f(x) \nleq f(y).$$

Proof. We proceed by contradiction. Assume that

$$f(x) \leq f(y).$$

Since $b_0 > 0$, $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, from the above inequality, we get

$$b_0(x,y)\,\phi_0[f(x) - f(y)] \leq 0. \tag{5.5.4}$$

Since x is feasible for (P) and (y, ξ, μ) is feasible for (MWD), it follows that

$$\sum_{j=1}^{m} \mu_j g_j(x) - \sum_{j=1}^{m} \mu_j g_j(y) \leqq 0.$$

Since $b_1 \geqq 0$, $a \leqq 0 \Rightarrow \phi_1(a) \leq 0$, from the above inequality, we get

$$b_1(x,y)\,\phi_1\left[\sum_{j=1}^{m} \mu_j g_j(x) - \sum_{j=1}^{m} \mu_j g_j(y)\right] \leqq 0. \tag{5.5.5}$$

By the generalized d-univex condition in (a), (5.5.4) and (5.5.5) imply

$$f'(y, \eta(x,y)) \leq 0, \tag{5.5.6}$$

and

$$\sum_{j=1}^{m} \mu_j g'_j(y, \eta(x,y)) \leqq 0. \tag{5.5.7}$$

Since $\xi > 0$, from (5.5.6) and (5.5.7), we get

$$\sum_{i=1}^{k} \xi_i f'_i(y, \eta(x,y)) + \sum_{l=1}^{m} \mu_j g'_j(y, \eta(x,y)) < 0, \tag{5.5.8}$$

which contradicts (5.5.1).

For the proof of part (b), again assume that

$$f(x) \leq f(y).$$

Since $b_0 \geq 0$, $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, from the above inequality, we get

$$b_0(x,y)\,\phi_0[f(x)-f(y)] \leq 0. \tag{5.5.9}$$

Since x is feasible for (P) and $(y,\ \xi,\ \mu)$ is feasible for (MWD), it follows that

$$\sum_{j=1}^{m}\mu_j g_j(x) - \sum_{j=1}^{m}\mu_j g_j(y) \leqq 0.$$

Since $b_1 \geqq 0$, $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$, from the above inequality, we get

$$b_1(x,y)\,\phi_1\left[\sum_{j=1}^{m}\mu_j g_j(x) - \sum_{j=1}^{m}\mu_j g_j(y)\right] \leqq 0. \tag{5.5.10}$$

By the generalized d-univex condition in (b), (5.5.9) and (5.5.10) imply

$$f'(y,\eta(x,y)) < 0,$$

and

$$\sum_{j=1}^{m}\mu_j g_j'(y,\eta(x,y)) \leqq 0.$$

Since $\xi \geqq 0$, the above two inequalities imply (5.5.8), again a contradiction to (5.5.1).

For the proof of part (c), proceeding as in part (b), we get (5.5.9) and (5.5.10). By the generalized univexity condition in part (c), (5.5.9) and (5.5.10) imply

$$f'(y,\eta(x,y)) < 0, \tag{5.5.11}$$

and

$$\sum_{j=1}^{m}\mu_j g_j'(y,\eta(x,y)) \leq 0. \tag{5.5.12}$$

Since $\xi \geqq 0$, (5.5.11) and (5.5.12) imply (5.5.8), again a contradiction to (5.5.1). This completes the proof. □

Theorem 5.5.2. *(Strong duality). Let $\bar{x}$ be a locally weak Pareto efficient solution or a weak Pareto efficient solution for (P) at which the generalized Slater's constraint qualification is satisfied. Let $(f,\ g)$ be directionally differentiable at $\bar{x}$ with $f'(\bar{x},\eta(x,\bar{x}))$, and $(\bar{x},\eta(x,\bar{x}))$ preinvex functions on X, and let g_j be continuous for $j \in \hat{J}(\bar{x})$. Then there exists $\bar{\mu} \in R_+^m$ such that $(\bar{x},1,\bar{\mu})$ is feasible for (MWD). If the weak duality between (P) and (MWD) in Theorem 5.5.1 holds, then $(\bar{x},1,\bar{\mu})$ is a locally weak Pareto efficient solution for (MWD).*

Proof. Since $\bar{x}$ satisfies all the conditions of Lemma 4.2.4, there exists $\bar{\mu} \in R_+^m$ such that conditions (4.2.1)–(4.2.3) hold. By (4.2.1)–(4.2.3), we have that $(\bar{x},1,\bar{\mu})$ is feasible for (MWD). By the weak duality, it follows that $(\bar{x},1,\bar{\mu})$ is a locally weak Pareto efficient solution for (MWD). □

Theorem 5.5.3. *(Converse duality). Let* $\left(\overline{y},\overline{\xi},\overline{\mu}\right)$ *be a weak Pareto efficient solution for (MWD). If the hypothesis of Theorem 5.5.1 holds at* $\overline{y}$ *on* $D\bigcup pr_X W$*, then* $\overline{y}$ *is a weak Pareto efficient solution for (P).*

Proof. We proceed by contradiction. Assume that $\overline{y}$ is not any weak Pareto efficient solution for (P), that is, there exists $\tilde{x}\in D$ such that

$$f(\tilde{x})<f(\overline{y}).$$

We know from condition (a) of Theorem 5.5.1 that $b_0>0$, and $a<0\Rightarrow\phi_0(a)<0$, from this and the above inequality, we get

$$b_0(\tilde{x},y)\,\phi_0\left[f(\tilde{x})-f(\overline{y})\right]<0.$$

By the generalized univexity condition (a) in Theorem 5.5.1, we get

$$\sum_{i=1}^{k}\overline{\xi}_i f_i'(\overline{y},\eta(\tilde{x},\overline{y}))<0. \tag{5.5.13}$$

From the feasibility of $\tilde{x}$ for (P), $\left(\overline{y},\overline{\xi},\overline{\mu}\right)$ for (MWD), $b_1\geqq 0$ and $a\leqq 0\Rightarrow\phi_1(a)\leqq 0$, we have

$$b_1(\tilde{x},\overline{y})\,\phi_1\left[\sum_{j=1}^{m}\overline{\mu}_j g_j(\tilde{x})-\sum_{j=1}^{m}\overline{\mu}_j g_j(\overline{y})\right]\leqq 0.$$

The above inequality in light of the generalized d-univexity condition (a) in Theorem 5.5.1 yields

$$\sum_{j=1}^{m}\overline{\mu}_j g_j'(\overline{y},\eta(\tilde{x},\overline{y}))\leqq 0. \tag{5.5.14}$$

By (5.5.13) and (5.5.14), we get

$$\sum_{i=1}^{k}\overline{\xi}_i f_i'(\overline{y},\eta(\tilde{x},\overline{y}))+\sum_{j=1}^{m}\overline{\mu}_j g_j'(\overline{y},\eta(\tilde{x},\overline{y}))<0. \tag{5.5.15}$$

This contradicts the dual constraint (5.5.1).

Similarly by condition (b) in Theorem 5.5.1, we get

$$\sum_{i=1}^{k}f_i'(\overline{y},\eta(\tilde{x},\overline{y}))<0$$

and

$$\sum_{j=1}^{m}\overline{\mu}_j g_j'(\overline{y},\eta(\tilde{x},\overline{y}))\leqq 0.$$

Since $\overline{\xi}_i\geqq 0$, the above two inequalities imply (5.5.15), again contradiction to (5.5.1).

Using condition (c) of Theorem 5.5.1, we have

$$\sum_{i=1}^{k} f_i'\left(\bar{y}, \eta\left(\tilde{x}, \bar{y}\right)\right) < 0$$

and

$$\sum_{j=1}^{m} \overline{\mu}_j g_j'\left(\bar{y}, \eta\left(\tilde{x}, \bar{y}\right)\right) \leq 0.$$

Since $\overline{\xi_i} \geqq 0$, the above two inequalities imply (5.5.15), again a contradiction to (5.5.1). This completes the proof. □

5.6 General Mond–Weir Duality for Nondifferentiable Vector Optimization Problems with $d-$Univex Functions

We shall continue our discussion on duality for (P) given in Sect. 4.2 in the present section by considering a general Mond–Weir type dual problem and proving weak and strong duality theorems under an assumption of the generalized d-univexity introduced in Chap. 3.

We consider the following general Mond–Weir type dual to (P):

(GMWD) maximize $\phi\left(y, \xi, \mu\right) = f\left(y\right) + \mu_{J_0}^T g_{J_0}\left(y\right) e$

$$\text{subject to } \left(\xi^T f' + \mu^T g'\right)\left(y, \eta\left(x, y\right)\right) \geqq 0, \text{ for all } x \in D, \quad (5.6.1)$$

$$\mu_{J_t} g_{J_t}\left(y\right) \geqq 0, 1 \leqq t \leqq r, \quad (5.6.2)$$

$$\xi^T e = 1, \quad (5.6.3)$$

$$\xi \in R_+^k, \mu \in R_+^m,$$

where J_t, $0 \leqq t \leqq r$ are partitions of set M and $e = (1, 1, \ldots, 1) \in R^k$.

Let

$$\tilde{W} = \left\{ \begin{array}{l} (y, \xi, \mu) \in X \times R^k \times R^m : \left(\xi^T f' + \mu^T g'\right)\left(y, \eta\left(x, y\right)\right) \geqq 0, \\ \qquad \mu_j g_j\left(y\right) \geqq 0, j = 1, \ldots, m, \xi \in R_+^k, \xi^T e = 1, \mu \in R_+^m \end{array} \right\}$$

denote the set of all the feasible solutions of (MWD).

Theorem 5.6.1. *(Weak duality). Let x and (y, ξ, μ) be a feasible solution for (P) and (GMWD), respectively. Assume that one of the following conditions holds:*

(d) $\xi > 0$, and $f + \mu_{J_0} g_{J_0}$ is strong pseudo $d-$univex and $\mu_{J_t} g_{J_t}$ is quasi $d-$univex at y on $D \bigcup pr_X W$ with respect to some b_0, b_1, ϕ_0, ϕ_1 and η with $b_0 > 0$, $\xi > 0$, $a \leq 0 \Rightarrow \phi_0\left(a\right) \leq 0$, and $a \leqq 0 \Rightarrow \phi_1\left(a\right) \leqq 0$ for any t, $1 \leqq t \leqq r$.

(e) $f + \mu_{J_0} g_{J_0}$ is weak strictly pseudo d-univex and $\mu_{J_t} g_{J_t}$ is quasi $d-$univex at y on $D \bigcup pr_X W$ with respect to some b_0, b_1, ϕ_0, ϕ_1 and η with $b_0 \geq 0$, $a \leq 0 \Rightarrow \phi_0\left(a\right) \leq 0$, and $a \leqq 0 \Rightarrow \phi_1\left(a\right) \leqq 0$ for any t, $1 \leqq t \leqq r$.

(f) $f + \mu_{J_0} g_{J_0}$ *is weak strictly pseudo d-univex and* $\mu_{J_t} g_{J_t}$ *is strictly quasi* $d-$*univex at* y *on* $D \bigcup pr_X W$ *with respect to some* b_0, b_1, ϕ_0, ϕ_1 *and* η *with* $b_0 \geq 0$, $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, *and* $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$ *for any* t, $1 \leqq t \leqq r$.

Then the following can not hold:

$$f(x) \leq \phi(y, \xi, \mu).$$

Proof. We proceed by contradiction. Suppose that

$$f(x) \leq \phi(y, \xi, \mu). \tag{5.6.4}$$

Since x is feasible for (P) and $\mu \geqq 0$, (5.6.4) implies that

$$f(x) + \mu_{J_0}^T g_{J_0}(x) e \leq f(y) + \mu_{J_0}^T g_{J_0}(y) e.$$

Since $b_0 > 0$, $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, from the above inequality, we get

$$b_0(x,y)\phi_0\left[f(x) + \mu_{J_0}^T g_{J_0}(x) e - f(y) + \mu_{J_0}^T g_{J_0}(y) e\right] \leq 0. \tag{5.6.5}$$

From the feasibility of x for (P) and (5.6.2), we have

$$\mu_{J_t}^T g_{J_t}(x) - \mu_{J_t}^T g_{J_t}(\mathrm{y}) \leqq 0, \text{ for all } 1 \leqq t \leqq r.$$

Since $b_1 \geqq 0$, $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$, from the above inequality, we get

$$b_1(x,y)\phi_1\left[\mu_{J_t}^T g_{J_t}(x) - \mu_{J_t}^T g_{J_t}(y)\right] \leqq 0, \text{ for all } 1 \leqq t \leqq r. \tag{5.6.6}$$

By condition (a), from (5.6.5) and (5.6.6), we have

$$\left(f' + \mu_{J_0} g'_{J_0} e\right)(y, \eta(x,y)) \leq 0,$$

and

$$\mu_{J_t} g'_{J_t}(y, \eta(x,y)) \leqq 0, \ \forall 1 \leqq t \leqq r.$$

Since $\xi > 0$, the above two inequalities yield

$$\left(\xi^T f' + \sum_{t=0}^{r} \mu_{J_t} g'_{J_t}\right)(y, \eta(x,y)) < 0. \tag{5.6.7}$$

Since $J_0, \ldots, J_r$ are partitions of M, (5.6.7) is equivalent to

$$\left(\xi^T f' + \mu^T g'\right)(y, \eta(x,y)) < 0, \tag{5.6.8}$$

which contradicts the dual constraint (5.6.1).

Similarly, by condition (b), we have

$$\left(f' + \mu_{J_0} g'_{J_0} e\right)(y, \eta(x,y)) < 0,$$

and

$$\mu_{J_t} g'_{J_t}(y, \eta(x,y)) \leqq 0,\ \forall 1 \leqq t \leqq r.$$

Since $\xi \geqq 0$, the above two inequalities yield

$$\left(\xi^T f' + \sum_{t=0}^{r} \mu_{J_t} g'_{J_t}\right)(y, \eta(x,y)) < 0.$$

The above inequality leads to (5.6.8), which contradicts (5.6.1).

By condition (c), we get

$$\left(f' + \mu_{J_0} g'_{J_0} e\right)(y, \eta(x,y)) < 0,$$

and

$$\mu_{J_t} g'_{J_t}(y, \eta(x,y)) < 0,\ \forall 1 \leqq t \leqq r.$$

Since $\xi \geqq 0$, the above two inequalities yield

$$\left(\xi^T f' + \sum_{t=0}^{r} \mu_{J_t} g'_{J_t}\right)(y, \eta(x,y)) < 0.$$

The above inequality leads to (5.6.8), which contradicts (5.6.1). This completes the proof. □

Theorem 5.6.2. *(Strong duality). Let $\bar{x}$ be a locally weak Pareto efficient solution or a weak Pareto efficient solution for (P) at which the generalized Slater's constraint qualification is satisfied. Let f and g be directionally differentiable at $\bar{x}$ with $f'(\bar{x}, \eta(x,\bar{x}))$ and $g'(\bar{x}, \eta(x,\bar{x}))$ preinvex functions on X, and let g_j be continuous for $j \in \hat{J}(\bar{x})$. Then, there exists $\bar{\mu} \in R^m_+$ such that $(\bar{x}, 1, \bar{\mu})$ is feasible for (GMWD). Moreover, if the weak duality between (P) and (GMWD) in Theorem 5.6.1 holds, then $(\bar{x}, 1, \bar{\mu})$ is a locally weak Pareto efficient solution for (GMWD).*

Proof. The proof of this theorem is similar to the proof of Theorem 5.5.2 in the previous section. □

5.7 Mond–Weir Duality for Nondifferentiable Vector Optimization Problems with *d*–Type-I Univex Functions

In relation to (P) given in Sect. 4.2, we consider the following dual problem which is in the format of Mond–Weir (1981):

$$\begin{aligned}
\text{(MWD)} \quad & \text{maximize } f(y) = (f_1(y), f_2(y), \ldots, f_k(y)) \\
& \text{subject to } \left(\xi^T f' + \mu^T g'\right)(y, \eta(x,y)) \geqq 0, \ for\ all\ x \in D, && (5.7.1) \\
& \mu_j g_j(y) \geqq 0, \quad j = 1, \ldots, m, && (5.7.2) \\
& \xi^T e = 1, && (5.7.3) \\
& \xi \in R_+^k, \ \mu \in R_+^m,
\end{aligned}$$

where $e = (1, 1, \ldots, 1) \in R^k$.

Let

$$W = \left\{ \begin{array}{r} (y, \xi, \mu) \in X \times R^k \times R^m : \left(\xi^T f' + \mu^T g'\right)(y, \eta(x,y)) \geqq 0, \\ \mu_j g_j(y) \geqq 0, j = 1, \ldots, m, \xi \in R_+^k, \xi^T e = 1, \mu \in R_+^m \end{array} \right\}$$

denote the set of all the feasible solutions of (MWD).

We denote by $\mathrm{pr_X W}$ the projection of set W on X.

Theorem 5.7.1. *(Weak Duality). Let x and (y, ξ, μ) be a feasible solution for (P) and (MWD) respectively. Moreover, suppose that any one of the following conditions holds:*

(g) $\left(f, \mu^T g\right)$ is strong pseudo-quasi d-type-I univex at y on $D \bigcup pr_X W$ with respect to some b_0, b_1, ϕ_0, ϕ_1 and η with $b_0 > 0$, $\xi > 0$, $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, and $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$.

(h) $\left(f, \mu^T g\right)$ is weak strictly pseudo-quasi dz-type-I univex at y on $D \bigcup pr_X W$ with respect to some b_0, b_1,ϕ_0, ϕ_1 and η with $b_0 \geq 0$, $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, and $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$.

(i) $\left(f, \mu^T g\right)$ is weak strictly pseudo d-type-I univex at y on $D \bigcup pr_X W$ with respect to some b_0, b_1, ϕ_0, ϕ_1 and η with $b_0 \geq 0$, $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, and $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$.

Then the following can not hold:

$$f(x) \leq f(y).$$

Proof. We proceed by contradiction. Assume that

$$f(x) \leq f(y).$$

Since $b_0 > 0$, $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, from the above inequality, we get

$$b_0(x, y)\, \phi_0[f(x) - f(y)] \leq 0. \qquad (5.7.4)$$

Since (y, ξ, μ) is feasible for (MWD), it follows that

$$-\sum_{j=1}^{m} \mu_j g_j(y) \leqq 0.$$

Since $b_1 \geqq 0, a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$, from the above inequality, we get

$$-b_1(x,y)\,\phi_1\left[\sum_{j=1}^{m}\mu_j g_j(y)\right] \leqq 0. \tag{5.7.5}$$

By the generalized univexity condition in (a), (5.7.4) and (5.7.5) imply

$$f'(y,\eta(x,y)) \le 0, \tag{5.7.6}$$

and

$$\sum_{j=1}^{m}\mu_j g_j'(y,\eta(x,y)) \leqq 0. \tag{5.7.7}$$

Since $\xi > 0$, from (5.7.6) and (5.7.7), we get

$$\sum_{i=1}^{k}\xi_i f_i'(y,\eta(x,y)) + \sum_{l=1}^{m}\mu_j g_j'(y,\eta(x,y)) < 0, \tag{5.7.8}$$

which contradicts (5.7.1).

For the proof of part (b), again assume that

$$f(x) \le f(y).$$

Since $b_0 \ge 0$, $a \le 0 \Rightarrow \phi_0(a) \le 0$, from the above inequality, we get

$$b_0(x,y)\,\phi_0[f(x) - f(y)] \le 0. \tag{5.7.9}$$

Since (y, ξ, μ) is feasible for (MWD), it follows that

$$-\sum_{j=1}^{m}\mu_j g_j(y) \leqq 0.$$

Since $b_1 \geqq 0$, $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$, from the above inequality, we get

$$-b_1(x,y)\,\phi_1\left[\sum_{j=1}^{m}\mu_j g_j(y)\right] \leqq 0. \tag{5.7.10}$$

By the generalized univexity condition in (b), (5.7.9) and (5.7.10) imply

$$f'(y,\eta(x,y)) < 0,$$

and

$$-\sum_{j=1}^{m}\mu_j g_j(y) \leqq 0.$$

Since $\xi \geqq 0$, the above two inequalities imply (5.7.8), again a contradiction to (5.7.1).

For the proof of part (c), proceeding as in part (b), we get (5.7.9) and (5.7.10). By the generalized univexity condition in part (c), (5.7.9) and (5.7.10) imply

$$f'(y,\eta(x,y)) < 0, \tag{5.7.11}$$

and

$$-\sum_{j=1}^{m} \mu_j g_j(y) < 0. \tag{5.7.12}$$

Since $\xi \geqq 0$, (5.7.11) and (5.7.12) imply (5.7.8), again a contradiction to (5.7.1). This completes the proof. □

Theorem 5.7.2. *(Strong Duality). Let $\overline{x}$ be a locally weak Pareto efficient solution or a weak Pareto efficient solution for (P) at which the generalized Slater's constraint qualification is satisfied. Let (f,g) be directionally differentiable at $\overline{x}$ with $f'(\overline{x},\eta(x,\overline{x}))$, and $g'(\overline{x},\eta(x,\overline{x}))$ preinvex functions on X, and let g_j be continuous for $j \in \tilde{J}(\overline{x})$,. Then there exists $\overline{\mu} \in R^m_+$ such that $(\overline{x},1,\overline{\mu})$ is feasible for (MWD). If the weak duality between (P) and (MWD) in Theorem 5.7.1 holds, then $(\overline{x},1,\overline{\mu})$ is a locally weak Pareto efficient solution for (MWD).*

Proof. Since $\overline{x}$ satisfies all the conditions of Lemma 4.2.4, there exists $\overline{\mu} \in R^m_+$ such that conditions (4.2.1)–(4.2.3) hold. By (4.2.1)–(4.2.3), we have that $(\overline{x},1,\overline{\mu})$ is feasible for (MWD). Also, by the weak duality, it follows that $(\overline{x},1,\overline{\mu})$ is a locally weak Pareto efficient solution for (MWD). □

Theorem 5.7.3. *(Converse Duality). Let $\left(\overline{y},\overline{\xi},\overline{\mu}\right)$ be a weak Pareto efficient solution for (MWD). If the hypothesis of Theorem 5.7.1 holds at $\overline{y}$ in $D \bigcup pr_X W$, then $\overline{y}$ is a weak Pareto efficient solution for (P).*

Proof. We proceed by contradiction. Assume that $\overline{y}$ is not any weak Pareto efficient solution for (P), that is, there exists $\tilde{x} \in D$ such that

$$f(\tilde{x}) < f(\overline{y}).$$

From condition (a) of Theorem 5.7.1, we know that $b_0 > 0$ and $a < 0 \Rightarrow \phi_0(a) < 0$.

By this and the above inequality, we get

$$b_0(\tilde{x},y)\,\phi_0[f(\tilde{x}) - f(\overline{y})] < 0.$$

By the generalized univexity condition (a) in Theorem 5.7.1, we get

$$\sum_{i=1}^{k} \overline{\xi}_i f_i'(\overline{y},\eta(\tilde{x},\overline{y})) < 0. \tag{5.7.13}$$

From the feasibility of $\tilde{x}$ for (P), $\left(\bar{y},\bar{\xi},\bar{\mu}\right)$ for (MWD), $b_1 \geqq 0$ and $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$, we have

$$-b_1(\tilde{x},\bar{y})\,\phi_1\left[\sum_{j=1}^{m}\bar{\mu}_j g_j(\bar{y})\right] \leqq 0.$$

The above inequality in light of the generalized univexity condition (a) in Theorem 5.7.1 yields

$$\sum_{j=1}^{m}\bar{\mu}_j g'_j(\bar{y},\eta(\tilde{x},\bar{y})) \leqq 0. \tag{5.7.14}$$

By (5.7.13) and (5.7.14), we get

$$\sum_{i=1}^{k}\bar{\xi}_i f'_i(\bar{y},\eta(\tilde{x},\bar{y})) + \sum_{j=1}^{m}\bar{\mu}_j g'_j(\bar{y},\eta(\tilde{x},\bar{y})) < 0. \tag{5.7.15}$$

This contradicts the dual constraint (5.7.1).

Similarly by condition (b) in Theorem 5.7.1, we get

$$\sum_{i=1}^{k} f'_i(\bar{y},\eta(\tilde{x},\bar{y})) < 0$$

and

$$\sum_{j=1}^{m}\bar{\mu}_j g'_j(\bar{y},\eta(\tilde{x},\bar{y})) \leqq 0.$$

Since $\bar{\xi}_i \geqq 0$, the above two inequalities imply (5.7.15), again a contradiction to (5.7.1).

Using condition (c) of Theorem 5.7.1, we have

$$\sum_{i=1}^{k} f'_i(\bar{y},\eta(\tilde{x},\bar{y})) < 0$$

and

$$\sum_{j=1}^{m}\bar{\mu}_j g'_j(\bar{y},\eta(\tilde{x},\bar{y})) < 0.$$

Since $\bar{\xi}_i \geqq 0$, the above two inequalities imply (5.7.15), again a contradiction to (5.7.1). This completes the proof. □

5.8 General Mond–Weir Duality for Nondifferentiable Vector Optimization Problems with *d*–Type-I Univex Functions

We continue our discussion on duality for (P) given in Sect. 4.2 in this section by considering a general Mond–Weir type dual problem of (P) and proving weak and strong duality theorems under an assumption of the generalized $d-$univexity.

We consider the following general Mond–Weir type dual to (P):
(GMWD) maximize $\phi(y,\xi,\mu) = f(y) + \mu_{J_0}^T g_{J_0}(y)e$

$$\text{subject to } \left(\xi^T f' + \mu^T g'\right)(y,\eta(x,y)) \geqq 0, \textit{ for all } x \in D, \tag{5.8.1}$$

$$\mu_{J_t} g_{J_t}(y) \geqq 0, 1 \leqq t \leqq r, \tag{5.8.2}$$

$$\xi^T e = 1, \tag{5.8.3}$$

$$\xi \in R_+^k,\ \mu \in R_+^m,$$

where $J_t,\ 0 \leqq t \leqq r$ are partitions of set M and $e = (1,1,...,1) \in R^k$.

Let

$$\tilde{W} = \Big\{ (y,\xi,\mu) \in X \times R^k \times R^m : \left(\xi^T f' + \mu^T g'\right)(y,\eta(x,y)) \geqq 0,$$
$$\mu_j g_j(y) \geqq 0, j = 1,\ldots,m, \xi \in R_+^k, \xi^T e = 1, \mu \in R_+^m \Big\}$$

denote the set of all the feasible solutions of (GMWD).

Theorem 5.8.1. *(Weak Duality). Let x and* (y,ξ,μ) *be a feasible solution for (P) and (GMWD) respectively. If any one of the following conditions holds:*

(g) $\xi > 0$, *and* $\left(f + \mu_{J_0} g_{J_0}, \mu_{J_t} g_{J_t}\right)$ *is strong pseudo-quasi d−type-I univex at y on* $D \cup pr_X W$ *with respect to some* b_0, b_1, ϕ_0, ϕ_1 *and* η *with* $b_0 > 0, \xi > 0, a \leq 0 \Rightarrow \phi_0(a) \leq 0$, *and* $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$ *for any t,* $1 \leqq t \leqq r$.

(h) $\left(f + \mu_{J_0} g_{J_0}, \mu_{J_t} g_{J_t}\right)$ *is weak strictly pseudo-quasi d− type-I univex at y on* $D \cup pr_X W$ *with respect to some* b_0, b_1, ϕ_0, ϕ_1 *and* η *with* $b_0 \geq 0$, $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, *and* $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$ *for any t,* $1 \leqq t \leqq r$.

(i) $\left(f + \mu_{J_0} g_{J_0}, \mu_{J_t} g_{J_t}\right)$ *is weak strictly pseudo d−type-I univex at y on* $D \cup pr_X W$ *with respect to some* b_0, b_1, ϕ_0, ϕ_1 *and* η *with* $b_0 \geq 0$, $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, *and* $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$ *for any t,* $1 \leqq t \leqq r$,
then the following cannot hold:

$$f(x) \leq \phi(y,\xi,\mu).$$

Proof. We proceed by contradiction. Assume that

$$f(x) \leq \phi(y,\xi,\mu). \tag{5.8.4}$$

Since x is feasible for (P) and $\mu \geqq 0$, (5.8.4) implies

$$f(x) + \mu_{J_0}^T g_{J_0}(x)e \leq f(y) + \mu_{J_0}^T g_{J_0}(y)e.$$

Since $b_0 > 0, a \leq 0 \Rightarrow \phi_0(a) \leq 0$, from the above inequality, we get

$$b_0(x,y)\phi_0\left[f(x) + \mu_{J_0}^T g_{J_0}(x)e - f(y) + \mu_{J_0}^T g_{J_0}(y)e\right] \leq 0. \tag{5.8.5}$$

From (5.8.2), we have

$$-\mu_{J_t}^T g_{J_t} \leqq 0, \text{ for all } 1 \leqq t \leqq r.$$

Since $b_1 \geqq 0, a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$, from the above inequality, we get

$$-b_1(x,y)\,\phi_1\left[\mu_{J_t}^T g_{J_t}\right] \leqq 0, \text{ for all } 1 \leqq t \leqq r. \tag{5.8.6}$$

By condition (a), from (5.8.5) and (5.8.6), we have

$$\left(f' + \mu_{J_0} g'_{J_0} e\right)(y, \eta(x,y)) \leq 0,$$

and

$$\mu_{J_t} g'_{J_t}(y, \eta(x,y)) \leqq 0, \quad \forall 1 \leqq t \leqq r.$$

Since $\xi > 0$, the above two inequalities yield

$$\left(\xi^T f' + \sum_{t=0}^{r} \mu_{J_t} g'_{J_t}\right)(y, \eta(x,y)) < 0. \tag{5.8.7}$$

Since $J_0, \dots, J_r$ are partitions of M, (5.8.7) is equivalent to

$$\left(\xi^T f' + \mu^T g'\right)(y, \eta(x,y)) < 0, \tag{5.8.8}$$

which contradicts the dual constraint (5.8.1).

Similarly, by condition (b), we have

$$\left(f' + \mu_{J_0} g'_{J_0} e\right)(y, \eta(x,y)) < 0,$$

and

$$\mu_{J_t} g'_{J_t}(y, \eta(x,y)) \leqq 0, \quad \forall 1 \leqq t \leqq r.$$

Since $\xi \geqq 0$, the above two inequalities yield

$$\left(\xi^T f' + \sum_{t=0}^{r} \mu_{J_t} g'_{J_t}\right)(y, \eta(x,y)) < 0.$$

The above inequality leads to (5.8.8), which contradicts (5.8.1).

By condition (c), we get

$$\left(f' + \mu_{J_0} g'_{J_0} e\right)(y, \eta(x,y)) < 0,$$

and

$$\mu_{J_t} g'_{J_t}(y, \eta(x,y)) < 0, \quad \forall 1 \leqq t \leqq r.$$

Since $\xi \geqq 0$, the above two inequalities yield

$$\left(\xi^T f' + \sum_{t=0}^{r} \mu_{J_t} g'_{J_t}\right)(y, \eta(x,y)) < 0.$$

The above inequality leads to (5.8.8), which contradicts (5.8.1). This completes the proof. □

Theorem 5.8.2. *(Strong Duality). Let $\bar{x}$ be a locally weak Pareto efficient solution or a weak Pareto efficient solution for (P) at which the generalized Slater's constraint qualification is satisfied. Let f and g be directionally differentiable at $\bar{x}$ with $f'(\bar{x},\eta(x,\bar{x}))$ and $g'(\bar{x},\eta(x,\bar{x}))$ preinvex functions on X, and let g_j be continuous for $j \in \tilde{J}(\bar{x})$. Then, there exists $\bar{\mu} \in R^m_+$ such that $(\bar{x},1,\bar{\mu})$ is feasible for (GMWD). Moreover, if the weak duality between (P) and (GMWD) in Theorem 5.8.1 holds, then $(\bar{x},1,\bar{\mu})$ is a locally weak Pareto efficient solution for (GMWD).*

Proof. The proof of this theorem is similar to the proof of Theorem 5.7.2. □

5.9 First Duality Model for Fractional Minimax Programs

In this section, we consider the following dual to (P) considered in Sect. 4.3:

(DI) $\quad \text{maximize}_{(s,t,\bar{y})\in K} \sup_{(z,t,\bar{y})\in H_1(s,t,\bar{y})} \nu$

$$\text{subject to} \sum_{i=1}^{s} t_i \{\nabla f(z,y_i) - \nu \nabla h(z,y_i)\} + \nabla \sum_{j=1}^{m} \mu_j g_j(z) = 0, \tag{5.9.1}$$

$$\sum_{i=1}^{s} t_i \{f(z,y_i) - \nu h(z,y_i)\} \geqq 0, \tag{5.9.2}$$

$$\sum_{j=1}^{m} \mu_j g_j(z) \geqq 0, \tag{5.9.3}$$

$$(s,t,\bar{y}) \in K,$$

where $H_1(s,t,\bar{y})$ denotes the set of all triplets $(z,\mu,\nu) \in R^n \times R^p_+ \times R_+$ satisfying (5.9.1)–(5.9.3). For a triplet$(s,t,\bar{y}) \in K$, if the set $H_1(s,t,\bar{y})$ is empty, then we define the supremum over it to be $-\infty$.

Theorem 5.9.1. *(Weak Duality). Let x be feasible for (P) and let $(z,\mu,\nu,s,t,\bar{y})$ be feasible for (DI). Assume that $t > 0$, $\left(\sum_{i=1}^{s} t_i(f(\cdot,y_i) - \nu h(\cdot,y_i)), \sum_{j=1}^{m} \mu_j g_j(\cdot)\right)$ is pseudo quasi V-type I at z with respect to the same $\eta, \alpha_i, \beta_j, t$ and μ. Then*

$$\sup_{y\in Y} \frac{f(x,y)}{h(x,y)} \nleq \nu.$$

Proof. Suppose contrary to the result, that is, $\sup_{y\in Y} \frac{f(x,y)}{h(x,y)} \leq \nu$ holds. Then there exists an i_0 such that

$$f(x,y_{i_0}) - \nu h(x,y_{i_0}) \leq 0.$$

Since $t > 0$ and $\alpha_i(x,z) > 0$, from the above inequality along with (5.9.2), we get

$$\sum_{i=1}^{s} t_i \alpha_i(x,z)\left(f(x,y_i) - \nu h(x,y_i)\right) < \sum_{i=1}^{s} t_i \alpha_i(x,z)\left(f(z,y_i) - \nu h(z,y_i)\right). \quad (5.9.4)$$

From (5.9.3) and the positivity of $\beta_j(x,z)$, we get

$$-\sum_{j=1}^{m} \mu_j \beta_j(x,z)\, g_j(z) \leqq 0.$$

Using the above inequality and the quasi V-type I condition, we get

$$\eta^T(x,z)\nabla \sum_{j=1}^{m} \mu_j g_j(z) \leqq 0. \quad (5.9.5)$$

From (5.9.5) and (5.9.1), we get

$$\eta^T(x,z)\nabla \sum_{i=1}^{s} t_i\left(f(z,y_i) - \nu h(z,y_i)\right) \geqq 0.$$

By the pseudo V-type I condition on $\sum_{i=1}^{s} t_i\left(f(\cdot,y_i) - \nu h(\cdot,y_i)\right)$ at z and the above inequality, we get

$$\sum_{i=1}^{s} t_i \alpha_i(x,z)\left(f(x,y_i) - \nu h(x,y_i)\right) \geqq \sum_{i=1}^{s} t_i \alpha_i(x,z)\left(f(z,y_i) - \nu h(z,y_i)\right),$$

which contradicts (5.9.4). Hence, the result follows. □

Theorem 5.9.2. *(Weak Duality). Let x be feasible for (P) and let* $(z,\mu,\nu,s,t,\bar{y})$ *be feasible for (DI). Assume that* $\left(\sum_{i=1}^{s} t_i\left(f(\cdot,y_i) - \nu h(\cdot,y_i)\right), \sum_{j=1}^{m} \mu_j g_j(\cdot)\right)$ *is semi strictly V-type I at z with respect to the same* $\eta, \alpha_i, \beta_j, t$ *and* μ*. Then*

$$\sup_{y\in Y} \frac{f(x,y)}{h(x,y)} \nleq \nu.$$

Proof. From (5.9.3) and the positivity of $\beta_j(x,z)$, we get

$$-\sum_{j=1}^{m} \mu_j \beta_j(x,z)\, g_j(z) \leqq 0.$$

By using the above inequality and the quasi V-type I condition, we get

$$\eta^T(x,z)\nabla \sum_{j=1}^{m} \mu_j g_j(z) \leqq 0. \quad (5.9.6)$$

By (5.9.6) and (5.9.1), we get

$$\eta^T(x,z)\nabla\sum_{i=1}^{s}t_i\left(f(z,y_i)-\nu h(z,y_i)\right)\geqq 0.$$

By the semi strict pseudo V-type I condition and the above inequality, we get

$$\sum_{i=1}^{s}t_i\alpha_i(x,z)\left(f(x,y_i)-\nu h(x,y_i)\right)>\sum_{i=1}^{s}t_i\alpha_i(x,z)\left(f(z,y_i)-\nu h(z,y_i)\right). \quad (5.9.7)$$

Now suppose contrary to the result, that is, $\sup_{y\in Y}\frac{f(x,y)}{h(x,y)}\leq\nu$ holds. Then there exists an i_0 such that

$$f\left(x,y_{i_0}\right)-\nu h\left(x,y_{i_0}\right)\leq 0.$$

Since $t_i\geqq 0$ and $\alpha_i(x,z)>0$, from the above inequality along with (5.9.2), we get

$$\sum_{i=1}^{s}t_i\alpha_i(x,z)\left(f(x,y_i)-\nu h(x,y_i)\right)\leqq\sum_{i=1}^{s}t_i\alpha_i(x,z)\left(f(z,y_i)-\nu h(z,y_i)\right),$$

which contradicts (5.9.7). Hence, the result follows. □

Theorem 5.9.3. *(Strong Duality). Assume that x^* is an optimal solution to (P) and $\nabla g_j(x), j\in J(x^*)$ are linearly independent. If in addition the hypothesis of Theorems 5.9.1 or 5.9.2 holds for all feasible solutions $(z,\mu,\nu,s,t,\bar{y})$ of (DI), then the two problems (P) and (DI) have the same extremal values.*

Proof. By Lemma 4.3.1, there exist $(s^*,t^*,\bar{y})\in K, (x^*,\mu^*,\nu^*)\in H_1(s^*,t^*,\bar{y})$ such that $(x^*,\mu^*,\nu^*,s^*,t^*,\bar{y})$ is a feasible solution to (DI). Since $\nu^*=\frac{f(x^*,y_i)}{h(x^*,y_i)}$, the optimality of this feasible solution for (DI) follows from Theorem 5.9.1 or Theorem 5.9.2. □

Theorem 5.9.4. *(Strict Converse Duality). Let $\bar{x}$ be optimal to (P) and let $(z,\mu,\nu,s,t,\bar{y})$ be optimal to (DI). Assume that $\left(\sum_{i=1}^{s}t_i(f(\cdot,y_i)-\nu h(\cdot,y_i)),\sum_{j=1}^{m}\mu_j g_j(\cdot)\right)$ is strictly pseudo quasi V-type at z with respect to the same η,α_i,β_j,t and μ for all $(s,t,\bar{y})\in K,(z,\mu,\nu)\in H_1(s,t,\bar{y})$, and $\nabla g_j(\bar{x}), j\in J(\bar{x})$ are linearly independent. Then $\bar{x}=z$, that is, z is an optimal solution to (P) and $\sup_{y\in Y}\frac{f(z,y)}{h(z,y)}=\nu$.*

Proof. Assume that $\bar{x}\neq z$. From Theorem 5.9.3, we have

$$\sup_{y\in Y}\frac{f(\bar{x},y)}{h(\bar{x},y)}=\nu. \quad (5.9.8)$$

From (5.9.3) and the positivity of $\beta_j(x,z)$, we get

$$-\sum_{j=1}^{m}\mu_j\beta_j(x,z)g_j(z)\leqq 0.$$

By the quasi V-type I condition and the above inequality, we get

$$\eta^T(x,z)\nabla\sum_{j=1}^{m}\mu_j g_j(z) \leqq 0. \tag{5.9.9}$$

From (5.9.9) and (5.9.1), we get

$$\eta^T(x,z)\nabla\sum_{i=1}^{s}t_i\left(f(z,y_i)-\nu h(z,y_i)\right) \geqq 0.$$

By the strict pseudo V-type I condition and the above inequality, we get

$$\sum_{i=1}^{s}t_i\alpha_i(x,z)(f(x,y_i)-\nu h(x,y_i)) > \sum_{i=1}^{s}t_i\alpha_i(x,z)(f(z,y_i)-\nu h(z,y_i)).$$

The above inequality along with (5.9.2) gives

$$\sum_{i=1}^{s}t_i\alpha_i(x,z)(f(x,y_i)-\nu h(x,y_i)) > 0.$$

Since $\alpha_i(x,z) > 0$, there exists an i_0 such that

$$f(x,y_{i_0})-\nu h(x,y_{i_0}) > 0.$$

It follows that

$$\sup_{y\in Y}\frac{f(x,y)}{h(x,y)} \geqq \frac{f(x,y_{i_0})}{h(x,y_{i_0})} > \nu = \sup_{y\in Y}\frac{f(z,y)}{h(z,y)},$$

which contradicts (5.9.8). The proof is completed. □

5.10 Second Duality Model for Fractional Minimax Programs

In order to discuss a parameter free model for (P) considered in Sect. 4.3, we state another version of Lemma 4.3.1. This is done by replacing the parameter ν^* with $f(x^*,y_i)/h(x^*,y_i)$ and by rewriting the multiplier functions associated with inequality constraints. The result of Lemma 4.3.1 can be stated as follows.

Lemma 5.10.1. *Let x^* be an optimal solution to (P) and let $\nabla g_j(x^*), j\in J(x^*)$ be linearly independent. Then there exist $(s^*,t^*,\bar{y})\in K$ and $\mu^*\in R_+^p$ such that*

$$\sum_{i=1}^{s*}t_i^*\{h(x^*,y_i)\nabla f(x^*,y_i)-f(x^*,y_i)\nabla h(x^*,y_i)\}+\nabla\sum_{j=1}^{p}\mu_j^* g_j(x^*)=0, \tag{5.10.1}$$

$$\sum_{j=1}^{p}\mu_j^* g_j(x^*)=0, \tag{5.10.2}$$

$$\mu^*\in R_+^p, t_i^*\geqq 0, \sum_{\text{i}=1}^{\text{s}*}t_i^*=1, y_i\in Y(x^*), i=1,\ldots,s^*.$$

Now we consider the dual (DII) given by Liu and Wu (1998), to the problem (P) considered in Sect. 4.3.

(DII) maximize$_{(s,t,\bar{y})\in K}$ sup$_{(z,\mu y)\in H_2(s,t,\bar{y})} F(z)$

$$\text{subject to } \sum_{i=1}^{s} t_i\{h(z,y_i)\nabla f(z,y_i) - f(z,y_i)\nabla h(z,y_i)\} + \nabla \sum_{j=1}^{p} \mu_j g_j(z) = 0, \quad (5.10.3)$$

$$\sum_{j=1}^{p} \mu_j g_j(z) \geqq 0, \quad (5.10.4)$$

$$F(z) = \sup_{y\in Y} \frac{f(z,y)}{h(z,y)}, \quad (5.10.5)$$

where $y_i \in Y(z)$, $H_2(s,t,\bar{y})$ denotes the set of all $(z,\mu) \in R^m \times R^p_+$ satisfying (5.10.3)–(5.10.5). If the set $H_2(s,t,\bar{y})$ is empty, then we define the supremum over it to be $-\infty$. Throughout this section, we denote $\psi_1(\cdot)$ as $\sum_{i=1}^{s} t_i\{h(z,y_i)f(\cdot,y_i) - f(z,y_i)$ $h(\cdot,y_i)\}$.

Theorem 5.10.1. *(Weak Duality). Let x be feasible for (P) and let $(z,\mu,s,t,\bar{y})$ be feasible for (DII). Assume that $\left(\psi_1(\cdot), \sum_{j=1}^{m} \mu_j g_j(\cdot)\right)$ is pseudo quasi V-type I with respect to the same $\eta, \alpha_i, \beta_j, t$ and μ. Then*

$$\sup_{y\in Y} \frac{f(x,y)}{h(x,y)} \not\leq F(z).$$

Proof. From 5.10.4 and the positivity of $\beta_j(x,z)$, we get

$$-\sum_{j=1}^{m} \mu_j \beta_j(x,z) g_j(z) \leqq 0.$$

By the quasi V-type I condition and the above inequality, we get

$$\eta^T(x,z)\nabla \sum_{j=1}^{m} \mu_j g_j(z) \leqq 0. \quad (5.10.6)$$

From 5.10.6 and 5.10.3, we get

$$\eta^T(x,z)\nabla\psi_1(z) \geqq 0.$$

By the pseudo V-type I condition, the above inequality gives

$$h(z,y_{i_0})f(x,y_{i_0}) - f(z,y_{i_0})h(\cdot,y_{i_0}) \geqq 0.$$

It follows that

$$\sup_{y\in Y}\frac{f(x,y)}{h(x,y)} \geqq \frac{f(x,y_{i_0})}{h(x,y_{i_0})} \geqq \frac{f(z,y_{i_0})}{h(z,y_{i_0})}.$$

Since $y_{i_0} \in Y(z)$, we have

$$F(z) = \frac{f(z,y_{i_0})}{h(x,y_{i_0})}.$$

This completes the proof. □

Similarly, one can establish strong and strict converse duality theorems for (P) and (DII), so we state without proof.

Theorem 5.10.2. *(Strong Duality). Assume that x^* is an optimal solution to (P) and $\nabla g_j(x), j \in J(x^*)$ are linearly independent. If in addition the hypothesis of Theorem 5.10.1 holds for all feasible solutions $(z,\mu,s,t,\bar{y})$ of (DII), then the two problems (P) and (DII) have the same extremal values.*

Theorem 5.10.3. *(Strict Converse Duality). Let $\bar{x}$ be optimal to (P) and let $(z,\mu,v,s,t,\bar{y})$ be optimal to (DII). Assume that $\left(\psi_1(\cdot), \sum_{j=1}^{m}\mu_j g_j(\cdot)\right)$ is pseudo quasi V-type I with respect to the same $\eta, \alpha_i, \beta_j, t$ and μ at z and $\nabla g_j(\bar{x}), j \in J(\bar{x})$ are linearly independent. Then $\bar{x} = z$, that is, z is an optimal solution to (P).*

5.11 Third Duality Model for Fractional Minimax Programs

Following Liu and Wu (1998), based on (4.3.2) and (4.3.3), we get

$$\nabla\sum_{i=1}^{s*} t_i^* f(x^*,y_i) - \frac{f(x^*,y_i)}{h(x^*,y_i)}\nabla\sum_{i=1}^{s*} t_i^* h(x^*,y_i) + \nabla\sum_{j=1}^{p}\mu_j^* g_j(x^*) = 0, \text{ for all } i = 1,\dots,s^*.$$

Multiplying the above equations respectively by $t_i h(x^*,y_i), i = 1,\dots,s^*$, and adding them, we get

$$\nabla\sum_{i=1}^{s*} t_i^* h(x^*,y_i)\nabla\left[\sum_{i=1}^{s*} t_i^* f(x^*,y_i) + \sum_{j=1}^{p}\mu_j^* g_j(x^*)\right] - \sum_{i=1}^{s*} t_i^* f(x^*,y_i)\nabla\sum_{i=1}^{s*} t_i^* h(x^*,y_i) = 0.$$

From the above equation and 4.3.4, we get the following lemma from Liu and Wu (1998).

Lemma 5.11.1. *Let x^* be an optimal solution to (P) and let $\nabla g_j(x^*), j \in J(x^*)$ be linearly independent. Then there exist $(s^*,t^*,\bar{y}) \in K$ and $\mu^* \in R_+^p$ such that*

$$\nabla \left(\frac{\sum_{i=1}^{s^*} t_i^* f(x^*, y_i) + \sum_{j=1}^{p} \mu_j^* g_j(x^*)}{\sum_{i=1}^{s^*} t_i^* h(x^*, y_i)} \right) = 0, \tag{5.11.1}$$

$$\sum_{j=1}^{p} \mu_j^* g_j(x^*) = 0, \tag{5.11.2}$$

$$\mu^* \in R_+^p, t_i^* \geqq 0, \sum_{i=1}^{s^*} t_i^* = 1, y_i \in Y(x^*), i = 1, \ldots, s^*.$$

In this section, we consider the following parameter free dual problem for (P):

$$\text{(DIII)} \quad \text{maximize}_{(s,t,\bar{y}) \in K} \sup_{(z,\mu) \in H_3(s,t,\bar{y})} \left(\frac{\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j=1}^{p} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i h(z, y_i)} \right)$$

$$\text{subject to} \quad \nabla \left(\frac{\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j=1}^{p} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i h(z, y_i)} \right) = 0, \tag{5.11.3}$$

where $H_3(s,t,\bar{y})$ denotes the set of all $(z,\mu) \in R^m \times R_+^p$ satisfying 5.11.3, If the set $H_3(s,t,\bar{y})$ is empty, then we define the supremum over it to be $-\infty$. Throughout this section for the sake of simplicity, we denote by $\psi_2(\cdot)$ as

$$\sum_{i=1}^{s} t_i h(z, y_i) \left[\sum_{i=1}^{s} t_i f(\cdot, y_i) + \sum_{j=1}^{p} \mu_j g_j(\cdot) \right] - \left[\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j=1}^{p} \mu_j g_j(z) \right] \sum_{i=1}^{s} t_i h(\cdot, y_i).$$

We shall give weak, strong and converse duality theorems without any proof as they can be proved in light of Theorems 5.9.1 – 5.10.1 proved in the previous sections and Theorems 5.2 and 5.3 of Liu and Wu (1998).

Theorem 5.11.1. *(Weak Duality). Let x be feasible for (P) and let $(z,\mu,s,t,\bar{y})$ be feasible for (DIII). Assume that $\left(\psi_1(\cdot), \sum_{j=1}^{m} \mu_j g_j(\cdot) \right)$ is pseudo quasi V-type I with respect to the same $\eta, \alpha_i, \beta_j, t$ and μ at z. Then*

$$\sup_{y \in Y} \frac{f(x,y)}{h(x,y)} \nleq \left(\frac{\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j=1}^{p} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i h(z, y_i)} \right).$$

Theorem 5.11.2. *(Strong Duality). Assume that x^* is an optimal solution to (P) and $\nabla g_j(x^*), j \in J(x^*)$ are linearly independent. If in addition the hypothesis of*

Theorem 5.11.1 holds for any feasible solution $(z,\mu,s,t,\bar{y})$ *of (DIII), then the two problems (P) and (DII) have the same extremal values.*

Theorem 5.11.3. *(Converse Duality). Let* $\bar{x}$ *be optimal to (P) and let* $(z,\mu,v,s,t,\bar{y})$ *be optimal to (DIII). Assume that* $\left(\psi_1(\cdot), \sum_{j=1}^{m} \mu_j g_j(\cdot)\right)$ *is pseudo quasi V-type I with respect to the same* η,α_i,β_j,t *and* μ *at* z *and* $\nabla g_j(\bar{x}), j \in J(\bar{x})$ *are linearly independent. Then* $\bar{x} = z$*, that is,* z *is an optimal solution to (P).*

5.12 Mond–Weir Duality for Nondifferentiable Vector Optimization Problems

Nondifferentiable optimization problems have been studied by many authors, see for example, Antczak (2002b), Brandao et al. (1999), Craven (1989), Giorgi and Guerraggio (1996), Giorgi et al. (2004), Kim and Lee (2001), Lee (1994), Minami (1983), Mishra (1996a, 1996c, 1997b), Mishra and Giorgi (2000), Mishra et al. (2004) and Mishra and Mukherjee (1995b, 1996), The purpose of this section is to give some duality results for a nondifferentiable vector optimization problem using the Clarke derivatives.

Consider the following pair of vector optimization problems:

$$\text{(VP)} \qquad \text{minimize} f(x) = (f_1(x), \ldots, f_p(x))$$
$$\text{subject to } g_j(x) \leqq 0, j = 1, \ldots, m,$$

$$\text{(VD)} \qquad \text{maximize} f(x) = (f_1(u), \ldots, f_p(u))$$
$$\text{subject to } 0 \in \sum_{i=1}^{p} \mu_i \partial^c f_i(u) + \sum_{j=1}^{m} \lambda_j \partial^c g_j(u), \tag{5.12.1}$$
$$\lambda_j g_j(u) \geqq 0, j = 1, \ldots, m, \tag{5.12.2}$$
$$(\mu_1, \ldots, \mu_p, \lambda_1, \ldots, \lambda_m) \geq 0, \tag{5.12.3}$$

where $f_i : X \to R, i = 1, \ldots, p$, and $g_j : X \to R, j = 1, \ldots, m$, are locally Lipschitz functions.

Remark 5.12.1. In formulating (VD), we do not use the Kuhn–Tucher type necessary optimality condition (see Clarke 1983).

It is worth noticing that in the proof of the following duality theorems, we do not require a constraint qualification as in Kim and Lee (2001), and the class of functions used in this section are wider enough.

We establish the following weak and strong duality theorems between (VP) and (VD).

Theorem 5.12.1. *(Weak Duality). Suppose that* (f,g) *is weak strictly pseudo type I with respect to* η*. Then, for any (VP)-feasible solution* x *and any (VD)-feasible solution* (u,μ,λ)*, it follows that* $f(x) \nless f(u)$*.*

Proof. Suppose that there exists a (VP)-feasible solution x and a (VD)-feasible solution (u,μ,λ) such that

$$f_i(x) < f_i(u), \forall i = 1,\ldots,p. \tag{5.12.4}$$

By the condition on f and (5.12.4), we get

$$\langle \xi_i, \eta(x,u)\rangle < 0, \text{ for any } \xi_i \in \partial^c f_i(u), i = 1,\ldots,p. \tag{5.12.5}$$

We consider two cases.

Case (I): $\lambda = 0$. From (5.12.3) and (5.12.5), we have

$$\sum_{i=1}^{p} \langle \mu_i \xi_i, \eta(x,u)\rangle < 0, \text{ for any } \xi_i \in \partial^c f_i(u).$$

This contradicts 5.12.1.

Case (II): $\lambda \neq 0$. Let $M = \{j : \lambda_j > 0\}$. From (5.12.2), we get

$$-g_j(u) \leqq 0, j \in M.$$

By the condition and the above inequality, we get

$$\langle \varsigma_j, \eta(x,u)\rangle < 0, j \in M \text{ and } \varsigma_j \in \partial^c g_j(u).$$

Since $\lambda_j = 0$, for any $j \notin M$, we have

$$\sum_{j=1}^{m} \langle \lambda_j \varsigma_j, \eta(x,u)\rangle < 0, j \in \{1,\ldots,m\} \text{ and any } \varsigma_j \in \partial^c g_j(u).$$

On the other hand, (5.12.5) implies that

$$\sum_{i=1}^{p} \langle \mu_i \xi_i, \eta(x,u)\rangle \leqq 0, \text{ for any } \xi_i \in \partial^c f_i(u).$$

By the above two inequalities, we get

$$\left\langle \sum_{i=1}^{p} \mu_i \xi_i + \sum_{j=1}^{m} \lambda_j \varsigma_j, \eta(x,u) \right\rangle < 0, \text{ for any } \xi_i \in \partial^c f_i(u) \text{ and } \varsigma_j \in \partial^c g_j(u).$$

This contradicts (5.12.1). The proof is completes. □

We can further extend the above weak duality theorem to the class of type I functions.

Theorem 5.12.2. *(Weak Duality). Suppose that* (f, g) *is weak strictly pseudo type I univex with respect to* $\eta, b_0 > 0, b_1 \geqq 0, \phi_0$ *and* ϕ_1 *are increasing. Then, for any (VP)-feasible solution* x *and any (VD)-feasible solution* (u, μ, λ)*, it follows that* $f(x) \nless f(u)$.

Proof. Suppose that there exist a (VP)-feasible solution x and a (VD)-feasible solution (u, μ, λ) such that

$$f_i(x) < f_i(u), \forall i = 1, \dots, p. \tag{5.12.6}$$

By the condition on ϕ_0, $b_0 > 0$ and (5.12.6), we get

$$b_0(x, u)\, \phi_0 [f_i(x) - f_i(u)] < 0, i = 1, \dots, p.$$

By the condition on f, from the above inequality, we get

$$\langle \xi_i, \eta(x, u) \rangle < 0, \text{ for any } \xi_i \in \partial^c f_i(u), i = 1, \dots, p. \tag{5.12.7}$$

We consider two cases.

Case (I): $\lambda = 0$. From (5.12.3) and (5.12.7), we have

$$\sum_{i=1}^{p} \langle \mu_i \xi_i, \eta(x, u) \rangle < 0, \text{ for any } \xi_i \in \partial^c f_i(u).$$

This contradicts (5.12.1).

Case (II): $\lambda \neq 0$. Let $M = \{ j : \lambda_j > 0 \}$. From (5.12.2), we get

$$-g_j(u) \leqq 0, j \in M.$$

By the condition on ϕ_1 and $b_1 \geqq 0$, from the above inequality, we get

$$-b_1(x, u)\, \phi_1 [g_j(u)] \leqq 0, j \in M.$$

By the condition on g and the above inequality, we get

$$\langle \varsigma_j, \eta(x, u) \rangle < 0, j \in M \text{ and } \varsigma_j \in \partial^c g_j(u).$$

Since $\lambda_j = 0$, for any $j \notin M$, we have

$$\sum_{j=1}^{m} \langle \lambda_j \varsigma_j, \eta(x, u) \rangle < 0, j \in \{1, \dots, m\} \text{ and any } \varsigma_j \in \partial^c g_j(u).$$

On the other hand, (5.12.7) implies that

$$\sum_{i=1}^{p} \langle \mu_i \xi_i, \eta(x, u) \rangle \leqq 0, \text{ for any } \xi_i \in \partial^c f_i(u).$$

By above two inequalities, we get

$$\left\langle \sum_{i=1}^{p} \mu_i \xi_i + \sum_{j=1}^{m} \lambda_j \varsigma_j, \eta(x,u) \right\rangle < 0, \text{ for any } \xi_{\mathrm{i}} \in \partial^c f_i(u) \text{ and } \varsigma_j \in \partial^c g_j(u).$$

This contradicts (5.12.1). This completes the proof. □

Theorem 5.12.3. *(Strong Duality). Let $\overline{x}$ be a weakly efficient solution for (VP). Then, there exist $\overline{\mu} \in R^p$ and $\overline{\lambda} \in R^m$ such that $\left(\overline{x}, \overline{\mu}, \overline{\lambda}\right)$ is a (VD)-feasible solution and their objective values are equal. Moreover, if (f,g) is weak strictly pseudo type I with respect to η, then $\left(\overline{x}, \overline{\mu}, \overline{\lambda}\right)$ is a weakly efficient solution for (VD).*

The proof is similar to the proof of Theorem 2.2 in Kim and Lee (2001) in the light of the above Theorem 5.12.1.

Theorem 5.12.4. *(Strong Duality). Let $\overline{x}$ be a weakly efficient solution for (VP). Then, there exist $\overline{\mu} \in R^p$ and $\overline{\lambda} \in R^m$ such that $\left(\overline{x}, \overline{\mu}, \overline{\lambda}\right)$ is a (VD)-feasible solution and their objective values are equal. Moreover, if (f,g) is weak strictly pseudo type I univex with respect to η, b_0, b_1, ϕ_0 and ϕ_1, then $\left(\overline{x}, \overline{\mu}, \overline{\lambda}\right)$ is a weakly efficient solution for (VD).*

The proof is similar to the proof of Theorem 2.2 in Kim and Lee (2001) in the light of the above Theorem 5.12.2.

Now, we give an example illustrating Theorems 5.12.1 and 5.12.3.

Example 5.12.1. Let $f_1(x) = x, f_2(x) = x(x+1)$ and $g(x) = x^2 - 1$, where $x \in R$. Then, f_1, f_2 and g are locally Lipschitz and $\partial^c f_i(x) = \{f_i'(x)\}, i = 1,2$, and $\partial^{\mathrm{c}} g(x) = \{g'(x)\}$, where $f_{\mathrm{i}}', i = 1,2$, and g' are the derivatives of $f_i, i = 1,2$, and g, respectively. Consider the vector optimization problem:

$$\text{(VP)} \qquad \text{minimize } (f_1(x), f_2(x)) \\ \text{subject to } x \in X = \{x \in R : g(x) \leqq 0\}$$

Its Mond–Weir dual problem is

$$\text{(VD)} \qquad \text{maximize } (f_1(u), f_2(u)) \\ \text{subject to } (u, \mu_1, \mu_2, \lambda) \in \Omega,$$

where

$$\Omega = \left\{ \begin{array}{r} (u, \mu_1, \mu_2, \lambda) \in R^4 : 0 = \mu_1 f_1'(u) + \mu_2 f_2'(u) + \lambda g'(u), \lambda g(u) \geqq 0, \\ (\mu_1, \mu_2, \lambda) \geqq 0, (\mu_1, \mu_2, \lambda) \neq 0 \end{array} \right\}.$$

Then, (f,g) is weak strictly pseudo type I with $\eta(x,u) = x - u - 1$. Let

$\Gamma = \{u \in R : \exists (\mu_1, \mu_2, \lambda) \in R^3, \text{s.t.} (u, \mu_1, \mu_2, \lambda) \in \Omega\}$.Then, $X = [-1, 1]$ and

$\Gamma = (-\infty, 0]$

For any $x \in X$ and any $(u, \mu_1, \mu_2, \lambda) \in \Omega$,

$$(f_1(x), f_2(x)) \not< (f_1(u), f_2(u))$$

equivalently, for any $x \in [-1, 1]$ and any $u \in (-\infty, 0]$,

$$(x, x(x+1)) \not< (u, u(u+1)), \tag{5.12.8}$$

So, the weak duality in Theorem 5.12.1 between (VP) and (VD) holds. Moreover, $[-1, 0]$ is the set of all weakly efficient solutions of (VP).

Since $\Gamma = (-\infty, 0]$, for any $u \in [-1, 0]$, there exist $(\mu_1^u, \mu_2^u, \lambda^u) \in R^3$ and $(u, \mu_1^u, \mu_2^u, \lambda^u) \in \Omega$. By (5.12.8), for any $u \in [-1, 0], (u, \mu_1^u, \mu_2^u, \lambda^u)$ is a weakly efficient solution of (VD). Thus, the strong duality in Theorem 5.12.3 holds between (VP) and (VD).

Example 5.12.2. Let $f_1(x) = x, f_2(x) = x(x+1)$ and $g(x) = x^2 - 1$, where $x \in R$. Then, f_1, f_2 and g are locally Lipschitz and $\partial^c f_i(x) = \{f_i'(x)\}, i = 1, 2$, and $\partial^c g(x) = \{g'(x)\}$, where $f_i', i = 1, 2$, and g' are the derivatives of $f_i, i = 1, 2$, and g, respectively.

Consider the vector optimization problem:

$$\text{(VP)} \qquad \begin{array}{l} \text{minimize } (f_1(x), f_2(x)) \\ \text{subject to } x \in X = \{x \in R : g(x) \leqq 0\}, \end{array}$$

and its Mond–Weir dual problem

$$\text{(VD)} \qquad \begin{array}{l} \text{maximize } (f_1(u), f_2(u)) \\ \text{subject to } (u, \mu_1, \mu_2, \lambda) \in \Omega, \end{array}$$

where

$$\Omega = \{(u, \mu_1, \mu_2, \lambda) \in R^4 : 0 = \mu_1 f_1'(u) + \mu_2 f_2'(u) + \lambda g'(u), \lambda g(u) \geqq 0, (\mu_1, \mu_2, \lambda) \geqq 0, (\mu_1, \mu_2, \lambda) \neq 0\}.$$

Then, (f, g) is weak strictly pseudo type I univex with ϕ_0, ϕ_1 identity functions on R, $\eta(x, u) = x - u - 1$, and $b_0 = b_1 = 1$. Let

$$\Gamma = \{u \in R : \exists (\mu_1, \mu_2, \lambda) \in R^3, \text{s.t.} (u, \mu_1, \mu_2, \lambda) \in \Omega\}.$$
$$\text{Then}, X = [-1, 1] \text{ and } \Gamma = (-\infty, 0]$$

For any $x \in X$ and any $(u, \mu_1, \mu_2, \lambda) \in \Omega$,

$$(f_1(x), f_2(x)) \not< (f_1(u), f_2(u)).$$

equivalently, for any $x \in [-1, 1]$ and any $u \in (-\infty, 0]$,

$$(x, x(x+1)) \not< (u, u(u+1)). \tag{5.12.9}$$

So, weak duality Theorem 5.12.2 between (VP) and (VD) holds. Moreover, $[-1,0]$ is the solution of all weakly efficient solution of (VP).

Since $\Gamma = (-\infty, 0]$, for any $u \in [-1,0]$, $\exists(\mu_1, \mu_2, \lambda) \in R^3$ and $(u, \mu_1, \mu_2, \lambda) \in \Omega$. So by (5.12.9), for any $u \in [-1,0]$, $(u, \mu_1, \mu_2, \lambda)$ is a weakly efficient solution of (VD). Thus, the strong duality Theorem 5.12.4 holds between (VP) and (VD).

5.13 Duality for Vector Optimization Problems on Banach Spaces

We consider the following dual of problem (P):

$$\begin{aligned} \text{(D)} \qquad & \text{maximize } \mathrm{f(w)} \\ & \text{subject to } \mathrm{w \in C, u^* \in Q^*, u^* \neq 0, v^* \in K^*}, \\ & \qquad \mathrm{< v^*, g(w) \geq 0, 0g \in \partial(u^* of + v^* og + k\delta_C)(w)}. \end{aligned}$$

In this section, we provide weak and strong duality relations between Problems (P) and (D).

Theorem 5.13.1. *(Weak Duality). Let x and* $(\mathrm{w, u^*, v^*})$ *be feasible solutions for problems (P) and (D), respectively. Suppose that (f, g) is type – I at w with respect to C, for the same* η*. Then,*

$$\mathrm{f(x) < f(w)}.$$

Proof. Contrary to the result, suppose that there are feasible solutions $\hat{x}$ and $(\mathrm{w, u^*, v^*})$ for problems (P) and (D), respectively, such that $\mathrm{f(\hat{x}) - f(w) < 0}$.

Since, $\mathrm{u^*} \neq$, we obtain

$$\mathrm{< u^*, f(\hat{x}) - -f(w) >< 0}.$$

By the first part of the assumption, there is $\eta(\hat{x}, \mathrm{w}) \in \mathrm{T_C(w)}$ such that

$$\mathrm{(u^* of)^0(w, \eta(\hat{x}, w)) \leq < u^*, f(\hat{x}) - -f(w) >}.$$

Hence,

$$\mathrm{(u^* of)^0(w, \eta(\hat{x}, w)) < 0}. \tag{5.13.1}$$

Since $\mathrm{< v^*, g(w) > \geq 0}$, we get

$$\mathrm{< v^*, g(w) > \leq 0}$$

$$\Rightarrow \mathrm{(v^* o\, g)^0(w, \eta(\hat{x}, w)) \leq 0} \tag{5.13.2}$$

by using the second part of the hypothesis of type – I function.

Then, from (5.13.1) and (5.13.2), we have

$$\mathrm{(u^* o\, f)^0(w, \eta(x, w)) + (v^* o\, g)^0(w, \eta(x, w)) < 0}. \tag{5.13.3}$$

On the other hand, since $0 \in \partial(\mathrm{u}^*\mathrm{o}\ \mathrm{f} + \mathrm{v}^*\mathrm{o}\ \mathrm{g} + \mathrm{k}\delta_\mathrm{C})(\mathrm{w})$ we have

$$0 \leq (\mathrm{u}^*\mathrm{o}\,\mathrm{f})^0(\mathrm{w};\eta) + (\mathrm{v}^*\mathrm{o}\,\mathrm{g})^0(\mathrm{w},\eta), \forall \eta \in \mathrm{T_C}(\mathrm{w}),$$

which contradicts (5.13.3) Therefore, $\mathrm{f(x)} < \mathrm{f(w)}$. □

Theorem 5.13.2. *(Weak Duality). Let x and* $(w,\mathrm{u}^*,\ v^*)$ *be feasible solutions for Problems (P) and (D), respectively. Suppose that (f, g) is pseudo – quasi – type I at w with respect to C, for the same* η. *Then,* $\mathrm{f(x)} < \mathrm{f(w)}$.

Proof. Contrary to the conclusion, suppose that there are feasible solutions $\hat{x}$ and (w, u^*, v^*) for problems (P) and (D), respectively such that $\mathrm{f}(\hat{x}) - \mathrm{f(w)} < 0$.

Since $\mathrm{u}^* \neq 0$, we obtain

$$< \mathrm{u}^*, \mathrm{f}(\hat{x}) - \mathrm{f(w)} >< 0.$$

By the first part of the assumption on f at w, there is $\eta(\hat{x}, \mathrm{w}) \in \mathrm{T_C}(\mathrm{w})$ such that

$$(\mathrm{u}^*\mathrm{o}\ \mathrm{f})^0(\mathrm{w},\eta(\hat{x},\mathrm{w})) \leq < \mathrm{u}^*, \mathrm{f}(\hat{x}) - \mathrm{f(w)} > .$$

Hence

$$(\mathrm{u}^*\mathrm{o}\ \mathrm{f})^0(\mathrm{w},\eta(\hat{x},\mathrm{w})) < 0. \tag{5.13.4}$$

Since, $- < (\mathrm{v}^*, \mathrm{g(w)} > \leq 0$, we have

$$(\mathrm{v}^*\mathrm{o}\ \mathrm{g})^0(\mathrm{w},\eta(\hat{x},\mathrm{w})) \leq 0. \tag{5.13.5}$$

Adding (5.13.4) and (5.13.5), we get

$$(\mathrm{u}^*\mathrm{o}\ \mathrm{f})^0(\mathrm{w},\eta(\hat{x},\mathrm{w})) + (\mathrm{v}^*\mathrm{o}\ \mathrm{g})^0(\mathrm{w},\eta(\hat{x},\mathrm{w})) < 0. \tag{5.13.6}$$

On the other hand, since $0 \in \partial(\mathrm{u}^*\mathrm{o}\ \mathrm{f} + \mathrm{v}^*\mathrm{o}\ \mathrm{g} + \mathrm{k}\delta_\mathrm{C})(\mathrm{w})$, we have

$$0 \leq (\mathrm{u}^*\mathrm{o}\ \mathrm{f})^0(\mathrm{w};\eta) + (\mathrm{v}^*\mathrm{o}\ \mathrm{g})^0(\mathrm{w},\eta), \forall \eta \in \mathrm{T_C}(\mathrm{w}),$$

which contradicts (5.13.6). Therefore, $\mathrm{f(x)} < \mathrm{f(w)}$. □

It is not hard to prove the following theorem.

Theorem 5.13.3. *(Weak Duality). Let x and (w,* u^*, v^**) be feasible solutions for Problems (P) and (D), respectively. Suppose that (f, g) is quasistrictly–pseudo–type I at w, with respect to C, for the same* η. *Then,* $\mathrm{f(x)} < \mathrm{f(w)}$.

Theorem 5.13.4. *(Strong Duality). Suppose that (f, g) is type I at any feasible point x of (P), with respect to C, and assume that Problem (P) satisfies the Slater condition. If* $\mathrm{x_0}$ *is a weak efficient solution of (P), then there exists (*$\bar{\mathrm{u}}^*$, $\bar{\mathrm{v}}^*$*)* $\in \mathrm{Q^*xK^*}$ *such that* $< \bar{\mathrm{v}}^*, \mathrm{g(x_0)} >= 0$, $(\mathrm{x_0}, \bar{\mathrm{u}}^*, \bar{\mathrm{v}}^*$*) is a weak efficient solution for (D), and the objective values of the two problems are same.*

Proof. Since the Slater condition is satisfied, from Proposition 4.4.1, it follows that there exist $\bar{u}^*$, $\bar{v}^*$ such that $< \bar{v}^*, g(x_0) >= 0$ and (x_0, $\bar{u}^*$, $\bar{v}^*$) is feasible for (D). Suppose that (x_0, u^*, v^*) is not any weak efficient solution for (D). So there exists a feasible point (x, u^*, v^*) for (D) such that $f(x) > f(x_0)$, which contradicts Theorem 5.13.1. Hence, (x_0, $\bar{u}^*$, v^*) is a weak efficient solution for (D). It is obvious that the objective function values of (P) and (D) are equal at their respective weak efficient solutions. □

Theorem 5.13.5. *(Strong Duality). Suppose that (f, g) is pseudo–quasi type I at any feasible solution x of (P), with respect to C, and assume that (P) satisfies the Slater condition. If* x_0 *is a weak efficient solution of (P), then there exists* ($\bar{u}^*$, $\bar{v}^*$) $\in Q^* \times K^*$ *such that* $< \bar{v}^*, g(x_0) >= 0$, (x_0, $\bar{u}^*$, $\bar{v}^*$) *is a weak efficient solution for (D), and the optimal objective values of the two problems are same.*

Proof. The proof of this theorem is similar to that of Theorem 5.13.4, except that here we invoke Theorem 5.13.2. □

In the proof of the following strong duality theorem, we need Theorem 5.13.3. The rest of the proof is similar to the proof of Theorem 5.13.4.

Theorem 5.13.6. *(Strong Duality). Suppose that (f, g) is quasi-strictly– pseudo-type I at any feasible point x of (P), with respect to C, and assume that (P) satisfies the Slater condition. If* x_0 *is a weak efficient solution of (P), then there exists* ($\bar{u}^*$, $\bar{v}^*$) $\in Q^* \times K^*$ *such that* $< \bar{v}^*, g(x_0) >= 0$, (x_0, u^*, $\bar{v}^*$) *is a weak efficient solution for (D), and the optimal objective values of the two problems are same.*

5.14 First Dual Model for Complex Minimax Programs

In this section and onwards, for $\xi = (z_1, \bar{z}_1) \in C^{2n}$, we let

$$Y(\xi) = \left\{ \begin{array}{l} (s, \lambda, \upsilon) \in N \times R_+^s \times C^{2ms} : \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \in R_+^s \text{ with } \sum_{i=1}^{s} \lambda_i = 1, \text{and} \\ \upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_s) \text{ with } \upsilon_i \in W(\xi), i = 1, 2, \ldots, s \end{array} \right\}.$$

By the optimality conditions of the preceding section, we will show that the following formulation is a dual problem to the minimax complex (P) discussed in Sect. 4.5.

$$\text{(DI)} \qquad \text{maximize}_{(s,\lambda,\varsigma)\in Y(\xi)} \sup_{(\xi,u,\bar{u},t)\in X(s,\lambda,\varsigma)} t$$

where $X(s,\lambda,\varsigma)$ denotes the set of all $(\xi,u.\bar{u},t) \in C^{2n} \times C^p \times C^p \times R$ to satisfy

$$\sum_{i=1}^{s} \lambda_i \overline{\nabla_z \phi(\xi,\varsigma_i)} + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\xi,\varsigma_i) + u^T \overline{\nabla_z g(\xi)} + u^H \nabla_{\bar{z}} g(\xi) = 0, \tag{5.14.1}$$

$$\sum_{i=1}^{s} \lambda_i [\mathrm{Re}\phi(\xi,\varsigma_i) - t] \geq 0, \tag{5.14.2}$$

$$\mathrm{Re}\langle u, g(\xi)\rangle \geq 0, \tag{5.14.3}$$

$$(s,\lambda,\varsigma) \in Y(\xi), \tag{5.14.4}$$

$$0 \neq u \in S^*. \tag{5.14.5}$$

We define the supremum over $X(s,\lambda,\varsigma)$ to be $-\infty$ if for a triplet $(s,\lambda,\varsigma) \in Y(\xi)$ the set $X(s,\lambda,\varsigma) = \phi$.

Then, we can derive the following weak duality theorem for (P) and (DI).

Theorem 5.14.1. *(Weak Duality). Let $\xi = (z,\bar{z}) \in S^0$ be a feasible solution of (P) and $(s,\lambda,\varsigma,\xi,u,\bar{u},t)$ be a feasible solution of (DI). If any one of the following holds:*

(a) $\sum_{i=1}^{s} \lambda_i \phi(\cdot,\varsigma_i)$ has pseudo-invex real part with respect to η and R^+ on the manifold Q and $g(\cdot)$ is a quasi-invex function with respect to the polyhedral cone $S \subset C^p$ on the manifold Q;

(b) $\sum_{i=1}^{s} \lambda_i \phi(\cdot,\varsigma_i)$ has quasi-invex real part with respect to η and R^+ on the manifold Q and $g(\cdot)$ is a strictly pseudo-invex function with respect to the polyhedral cone $S \subset C^p$ on the manifold Q;

(c) $\sum_{i=1}^{s} \lambda_i \phi(\cdot,\varsigma_i) + u^H g(\cdot)$ has pseudo-invex real part with respect to η and R^+ on the manifold Q,

then

$$\sup_{\varsigma \in W} \mathrm{Re}\phi(\xi,\varsigma) \geq t\,.$$

Proof. Suppose contrary that

$$\sup_{\varsigma \in W} \mathrm{Re}\phi(\xi,\varsigma) < t\,.$$

Then, we have

$$\mathrm{Re}\phi(\xi,\varsigma) < t \text{ for all } \varsigma \in \mathrm{W}\,.$$

It follows that

$$\mathrm{Re}[\lambda_i \phi(\xi,\varsigma_i)] \leq \lambda_i t \text{ for all } i = 1,2,\ldots,s, \tag{5.14.6}$$

with at least one strict inequality since $\lambda \neq 0$.

From the inequalities (5.14.2) and (5.14.6), we have

$$\sum_{i=1}^{s} \mathrm{Re}[\lambda_i \phi(\xi,\varsigma_i)] < \sum_{i=1}^{s} \lambda_i t \leq \sum_{i=1}^{s} \mathrm{Re}[\lambda_i \phi(\xi,\varsigma_i)]. \tag{5.14.7}$$

If hypothesis (a) holds, using the pseudo-invexity of $\sum_{i=1}^{s} \lambda_i \phi(\cdot, \varsigma_i)$ and inequality (5.14.7), we get

$$\operatorname{Re}\left\langle \eta(z, z_1), \sum_{i=1}^{s} \lambda_i \overline{\nabla_z \phi(\xi, \varsigma_i)} + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\xi, \varsigma_i) \right\rangle < 0. \tag{5.14.8}$$

From (5.14.8) and (5.14.1), we get

$$\operatorname{Re}\left\langle \eta(z, z_1), u^T \overline{\nabla_z g(\xi)} + u^H \nabla_{\bar{z}} g(\xi) \right\rangle > 0.$$

It follows that

$$\operatorname{Re}\left\langle u, \eta^T(z, z_1) \nabla_z g(\xi) + \eta^H(z, z_1) \nabla_{\bar{z}} g(\xi) \right\rangle > 0. \tag{5.14.9}$$

Utilizing the feasibility of ξ for (P), $u \in S^*$, and the inequality (5.14.3), we get

$$\operatorname{Re}\langle u, g(\varsigma)\rangle \leq 0 \leq \operatorname{Re}\langle u, g(\xi)\rangle. \tag{5.14.10}$$

Using the quasi-invexity of g and inequality (5.14.10), we get

$$\operatorname{Re}\left\langle u, \eta^T(z, z_1) \nabla_z g(\xi) + \eta^H(z, z_1) \nabla_{\bar{z}} g(\xi) \right\rangle \leq 0,$$

which contradicts the inequality (5.14.9). Hence, the result holds.

Hypothesis (b) follows along with the same lines as (a).

If hypothesis (c) holds, from the inequality (5.14.7) and (5.14.10), we get

$$\operatorname{Re}\left[\sum_{i=1}^{s} \lambda_i \phi(\xi, \varsigma_i) + u^H g(\xi)\right] < \operatorname{Re}\left[\sum_{i=1}^{s} \lambda_i \phi(\xi, \varsigma_i) + u^H g(\xi)\right]. \tag{5.14.11}$$

Using the pseudo-invexity of $\sum_{i=1}^{s} \lambda_i \phi(\cdot, \varsigma_i) + u^H g(\cdot)$ and (5.14.11), we get

$$\operatorname{Re}\left\langle \eta(z, z_1), \sum_{i=1}^{s} \lambda_i \overline{\nabla_z \phi(\xi, \varsigma_i)} + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\xi, \varsigma_i) + u^T \overline{\nabla_z g(\xi)} + u^H \nabla_{\bar{z}} g(\xi) \right\rangle < 0,$$

which contradicts the inequality (5.14.1). Hence the proof is complete. □

Theorem 5.14.2. *(Strong Duality). Let ξ^0 be an optimal solution of the problem (P) and the condition (CQ) as defined in Lemma 4.5.2 is satisfied at ξ^0. Then there exist $(s, \lambda, \varsigma) \in Y(\xi^0)$ and $(\xi, u.\bar{u}, t) \in X(s, \lambda, \varsigma)$ such that $(s, \lambda, \varsigma, \xi^0, u, \bar{u}, t)$ is a feasible solution of (DI). If the hypothesis of Theorem 5.14.1 is also satisfied, then $(s, \lambda, \varsigma, \xi^0, u, \bar{u}, t)$ is an optimal solution of (DI), and the two problems (P) and (DI) have the same optimal value.*

Proof. Since ξ^0 is an optimal solution of (P) and the condition (CQ) is satisfied, then Lemma 4.5.2 guarantees the existence of a positive s, scalars $\lambda_i \geq 0, i = 1,2,\ldots,s, 0 \neq u \in S^*$, and vectors $\varsigma_i \in W(\xi^0) = \{\varsigma \in W : \mathrm{Re}\phi(\xi^0,\varsigma) = \sup_{\mu\in W}$ $\mathrm{Re}\phi(\xi^0,\mu)\}, i = 1,2,\ldots,s$ such that

$$\sum_{i=1}^{s} \lambda_i \overline{\nabla_z \phi\left(\xi^0,\varsigma_i\right)} + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi\left(\xi^0,\varsigma_i\right) + u^T \overline{\nabla_z g\left(\xi^0\right)} + u^H \nabla_{\bar{z}} g\left(\xi^0\right) = 0,$$
$$\mathrm{Re}\left\langle u, g\left(\xi^0\right)\right\rangle = 0,$$

and $t = \mathrm{Re}\phi\left(\xi^0,\varsigma_i\right)$, i $= 1,2,\ldots,s$. Thus, $\left(s,\lambda,\varsigma,\xi^0,u,\bar{u},t\right)$ is a feasible solution of (DI). The optimality of $\left(s,\lambda,\varsigma,\xi^0,u,\bar{u},t\right)$ for (DI) follows from Theorem 5.14.1. □

Theorem 5.14.3. *(Strict Converse Duality). Let $\hat{\xi}$ and $\left(\hat{s},\hat{\lambda},\hat{\varsigma},\hat{\xi},\hat{u},\hat{\bar{u}},\hat{t}\right)$ be optimal solution of (P) and (DI) respectively, and assume that the assumptions of Theorem 5.14.2 are fulfilled. If $\sum_{i=1}^{\hat{s}} \hat{\lambda}_i \phi\left(\cdot,\hat{\varsigma}_i\right)$ has strictly pseudo-invex real part with respect to η and R_+ and g is quasi-invex with respect to the polyhedral cone S, then $\hat{\xi} = \hat{\varsigma}$; that is, $\hat{\varsigma}$ is an optimal solution of (P).*

Proof. We assume that $\left(\hat{z},\hat{\bar{z}}\right) = \hat{\xi} \neq \hat{\varsigma} = \left(\hat{z}_1,\hat{\bar{z}}_1\right)$ for getting a contradiction. From Theorem 5.14.2, we know that

$$\sup_{\nu\in W} \mathrm{Re}\phi(\hat{\xi},\nu) = \hat{t}. \tag{5.14.12}$$

Utilizing the feasibility of $\hat{\xi}$ for (P), $\hat{u} \in S^*$, and the inequality (5.14.3), we have

$$\mathrm{Re}\langle \hat{u}, g(\hat{\xi})\rangle \leq 0 \leq \mathrm{Re}\langle \hat{u}, g(\hat{\varsigma})\rangle.$$

Using the quasi-invexity of g, we get from the above inequality

$$\mathrm{Re}\left\langle \hat{u}, \eta^T\left(\hat{z},\hat{z}_1\right)\nabla_z g\left(\hat{\xi}\right) + \eta^H\left(\hat{z},\hat{z}_1\right)\nabla_{\bar{z}} g\left(\hat{\xi}\right)\right\rangle \leq 0. \tag{5.14.13}$$

From relation (5.14.13) and (5.14.1), we obtain

$$\mathrm{Re}\left\langle \eta\left(\hat{z},\hat{z}_1\right), \sum_{i=1}^{\hat{s}} \hat{\lambda}_i \overline{\nabla_z \phi\left(\hat{\xi},\hat{\nu}_i\right)} + \sum_{i=1}^{\hat{s}} \hat{\lambda}_i \nabla_{\bar{z}} \phi\left(\hat{\xi},\hat{\nu}_i\right)\right\rangle \geq 0. \tag{5.14.14}$$

Using the strict pseudo-invexity of $\sum_{i=1}^{\hat{s}} \hat{\lambda}_i \phi\left(\cdot,\hat{\nu}_i\right)$, the inequalities (5.14.14) and (5.14.2), we get

$$\sum_{i=1}^{\hat{s}} \mathrm{Re}\left[\hat{\lambda}_i \phi\left(\hat{\xi},\hat{\nu}_i\right)\right] > \sum_{i=1}^{\hat{s}} Re\left[\hat{\lambda}_i \phi\left(\hat{\varsigma},\hat{\nu}_i\right)\right] \sum_{i=1}^{\hat{s}} \hat{\lambda}_i \hat{t}.$$

Therefore, there exists a certain i_0, such that

$$\mathrm{Re}\phi\left(\hat{\xi},\hat{\nu}_{i_0}\right) > \hat{t}.$$

It follows that

$$\sup_{\nu\in W}\mathrm{Re}\phi\left(\hat{\xi},\nu\right) \geq \mathrm{Re}\phi\left(\hat{\xi},\hat{\nu}_{i_0}\right) > \hat{t},$$

which contradicts (5.14.12). Therefore, we conclude that $\hat{\xi} = \hat{\varsigma}$. Hence the proof is complete. □

5.15 Second Dual Model for Complex Minimax Programs

We shall continue our discussion of duality model for (P) considered in Sect. 4.5 by showing that the following problem (DII) is also a dual problem for (P).

$$\text{(DII)} \qquad \text{maximize}_{(s,\lambda,\varsigma)\in Y(\xi)} \sup_{(\xi,u,\bar{u},t)\in X(s,\lambda,\varsigma)} f(\xi)$$

where $X(s,\lambda,\varsigma)$ denotes the set of all $(\xi,u.\bar{u}) \in C^{2n}\times C^p\times C^p$ to satisfy

$$\sum_{i=1}^{s}\lambda_i\overline{\nabla_z\phi(\xi,\varsigma_i)} + \sum_{i=1}^{s}\lambda_i\nabla_{\bar{z}}\phi(\xi,\varsigma_i) + u^T\overline{\nabla_z g(\xi)} + u^H\nabla_{\bar{z}}g(\xi) = 0, \tag{5.15.1}$$

$$\mathrm{Re}\langle u, g(\xi)\rangle \geq 0, \tag{5.15.2}$$

$$f(\xi) = \sup_{\nu\in W}\mathrm{Re}\phi(\xi,\nu), \tag{5.15.3}$$

$$(s,\lambda,\varsigma) \in Y(\xi), \tag{5.15.4}$$

$$0 \neq u \in S^*. \tag{5.15.5}$$

We define the supremum over $X(s,\lambda,\varsigma)$ to be $-\infty$ if for a triplet $(s,\lambda,\varsigma)\in Y(\xi)$ the set $X(s,\lambda,\varsigma) = \phi$.

Now we establish the following weak, strong and strict converse duality theorem for (P) and (DII).

Theorem 5.15.1. *(Weak Duality). Let $\varsigma = (z,\bar{z}) \in S^0$ be a feasible solution of (P) and $(s,\lambda,\nu,\xi,u,\bar{u},t)$ be a feasible solution of (DII). If any one of the following holds:*

(a) $\sum_{i=1}^{s}\lambda_i\phi(\cdot,\nu_i)$ has pseudo-invex real part with respect to η and R^+ on the manifold Q and $g(\cdot)$ is a quasi-invex function with respect to the polyhedral cone $S\subset C^p$ on the manifold Q;

(b) $\sum_{i=1}^{s}\lambda_i\phi(\cdot,\nu_i)$ has quasi-invex real part with respect to η and R^+ on the manifold Q and $g(\cdot)$ is a strictly pseudo-invex function with respect to the polyhedral cone $S\subset C^p$ on the manifold Q;

(c) $\sum_{i=1}^{s} \lambda_i \phi(\cdot, \nu_i) + u^H g(\cdot)$ *has pseudoinvex real part with respect to* η *and* R^+ *on the manifold Q,*
then

$$f(\varsigma) \geq f(\xi).$$

Proof. Suppose contrary to the result, we then have

$$f(\varsigma) < f(\xi);$$

that is,

$$\sup_{\nu \in W} \mathrm{Re}\phi(\varsigma, \nu) < \sup_{\nu \in W} \mathrm{Re}\phi(\xi, \nu).$$

Then, we have

$$\mathrm{Re}\phi(\varsigma, \nu) < \sup_{\nu \in W} \mathrm{Re}\phi(\xi, \nu) \text{ for all } \nu \in \mathrm{W}. \tag{5.15.6}$$

Since $\nu_i \in W(\xi)$ for all $i = 1, 2, \ldots, s$, we obtain

$$\sup_{\nu \in W} \mathrm{Re}\phi(\xi, \nu) = \mathrm{Re}\phi(\xi, \nu_i), \text{ for all } i = 1, 2, \ldots, s. \tag{5.15.7}$$

From relation (5.15.6) and (5.15.7), we obtain

$$\mathrm{Re}\phi(\varsigma, \nu) < \mathrm{Re}\phi(\xi, \nu_i), \text{ for all } i = 1, 2, \ldots, s, \nu \in W.$$

It follows that

$$\mathrm{Re}[\lambda_i \phi(\varsigma, \nu_i)] \leq \mathrm{Re}[\lambda_i \phi(\xi, \nu_i)] \text{ for all i} = 1, 2, \ldots, s,$$

with at least one strict inequality since $\lambda \neq 0$.

Thus, we have

$$\sum_{i=1}^{s} \mathrm{Re}[\lambda_i \phi(\varsigma, \nu_i)] < \sum_{i=1}^{s} \mathrm{Re}[\lambda_i \phi(\xi, \nu_i)]. \tag{5.15.8}$$

Using the pseudo-invexity of the real part of $\sum_{i=1}^{s} \lambda_i \phi(\varsigma, \nu_i)$ and the above inequality, we get

$$\mathrm{Re}\left\langle \eta(z, z_1), \sum_{i=1}^{s} \lambda_i \overline{\nabla_z \phi(\xi, \nu_i)} + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\xi, \nu_i) \right\rangle < 0. \tag{5.15.9}$$

From inequalities (5.15.9) and (5.15.1), we get

$$\mathrm{Re}\langle \eta(z, z_1), u^T \overline{\nabla_z g(\xi)} + u^H \nabla_{\bar{z}} g(\xi) \rangle > 0. \tag{5.15.10}$$

Thus, we have

$$\mathrm{Re}\langle u, \eta^T(z, z_1) \nabla_z g(\xi) + \eta^H(z, z_1) \nabla_{\bar{z}} g(\xi) \rangle > 0. \tag{5.15.11}$$

By the feasibility of ς for (P), $u \in S^*$, and the inequality (5.2), we get

$$\text{Re}\langle u, g(\varsigma)\rangle \leq 0 \leq \text{Re}\langle u, g(\xi)\rangle. \tag{5.15.12}$$

Using the quasi-invexity of g and the inequality (5.15.12) we get

$$\text{Re}\langle u, \eta^T(z,z_1)\nabla_z g(\xi) + \eta^H(z,z_1)\nabla_{\bar{z}} g(\xi)\rangle \leq 0,$$

which contradicts the inequality (5.15.11) Hence the result is true.

Hypothesis (b) follows along with the same lines as (a).

If hypothesis (c) holds, from the inequalities (5.15.8)and (5.15.12), we get

$$\sum_{i=1}^{s} \text{Re}\left[\lambda_i \phi(\varsigma, v_i) + u^H g(\varsigma)\right] < \sum_{i=1}^{s} \text{Re}\left[\lambda_i \phi(\xi, v_i) + u^H g(\xi)\right]. \tag{5.15.13}$$

By the pseudo-invexity of $\sum\limits_{i=1}^{s} \lambda_i \phi(\cdot, v_i) + u^H g(\cdot)$ and the above inequality, we get

$$\text{Re}\left\langle \eta(z,z_1), \sum_{i=1}^{s} \lambda_i \overline{\nabla_z \phi(\xi, v_i)} + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\xi, v_i) + u^T \overline{\nabla_z g(\xi)} + u^H \nabla_{\bar{z}} g(\xi) \right\rangle < 0,$$

which contradicts the inequality (5.15.1). Hence the result of Theorem holds. □

Theorem 5.15.2. *(Strong Duality). Let ς^0 be an optimal solution of the problem (P) and the condition (CQ) as defined in Lemma 4.5.2 is satisfied at ς^0. Then there exist $(s,\lambda,v) \in Y(\varsigma^0)$ and $(\varsigma^0, u.\bar{u}) \in X(s,\lambda,v)$ such that $(s,\lambda,v,\varsigma^0,u,\bar{u})$ is a feasible solution of (DII). If the hypothesis of Theorem 5.15.1 is also satisfied, then $(s,\lambda,v,\varsigma^0,u,\bar{u})$ is an optimal solution of (DII), and the two problems (P) and (DII) have the same optimal value.*

Proof. By Lemma 4.5.2, there exist $(s,\lambda,v) \in Y(\varsigma^0)$ and $(\varsigma^0, u.\bar{u}) \in X(s,\lambda,v)$ such that $(s,\lambda,v,\varsigma^0,u,\bar{u})$ is a feasible solution of (DII). Since (P) and (DII) have the same objective function, the optimality of $(s,\lambda,v,\varsigma^0,u,\bar{u})$ for (DII) follows from Theorem 5.15.1. □

Theorem 5.15.3. *(Strict Converse Duality). Let $\hat{\varsigma}$ and $\left(\hat{s},\hat{\lambda},\hat{v},\hat{\varsigma},\hat{u},\hat{\bar{u}}\right)$ be an optimal solution of (P) and (DII) respectively, and assume that the assumptions of Theorem 5.15.2 are fulfilled. If $\sum\limits_{i=1}^{\hat{s}} \hat{\lambda}_i \phi(\cdot, \hat{v}_i)$ has strictly pseudo-invex real part with respect to η and R_+ and g is quasi-invex with respect to the polyhedral cone S, then $\hat{\varsigma} = \hat{\xi}$; that is, $\hat{\xi}$ is an optimal solution of (P).*

Proof. We assume that $\left(\hat{z},\hat{\bar{z}}\right) = \hat{\varsigma} \neq \hat{\xi} = \left(\hat{z}_1,\hat{\bar{z}}_1\right)$ and would like to reach a contradiction. From Theorem 5.15.2, we know that

$$\sup_{v \in W} \text{Re}\phi(\hat{\varsigma}, v) = \sup_{v \in W} \text{Re}\phi\left(\hat{\xi}, v\right). \tag{5.15.14}$$

Utilizing the feasibility of $\hat{\varsigma}$ for (P), $\hat{u} \in S^*$, and the inequality (5.15.2), we have

$$\mathrm{Re}\langle \hat{u}, g(\hat{\varsigma})\rangle \leq 0 \leq \mathrm{Re}\left\langle \hat{u}, g\left(\hat{\xi}\right)\right\rangle .$$

Using the quasi-invexity of g, we get from the above inequality

$$\mathrm{Re}\left\langle \hat{u}, \eta^T(\hat{z},\hat{z}_1)\nabla_z g\left(\hat{\xi}\right) + \eta^H(\hat{z},\hat{z}_1)\nabla_{\bar{z}} g\left(\hat{\xi}\right)\right\rangle \leq 0. \tag{5.15.15}$$

From relation (5.15.15) and (5.15.1), we obtain

$$\mathrm{Re}\left\langle \eta(\hat{z},\hat{z}_1), \sum_{i=1}^{\hat{s}} \hat{\lambda}_i \overline{\nabla_z \phi\left(\hat{\xi},\hat{v}_i\right)} + \sum_{i=1}^{\hat{s}} \hat{\lambda}_i \nabla_{\bar{z}}\phi\left(\hat{\xi},\hat{v}_i\right)\right\rangle \geq 0. \tag{5.15.16}$$

Using the strict pseudo-invexity of $\sum_{i=1}^{\hat{s}} \hat{\lambda}_i\phi(\cdot,\hat{v}_i)$ and the above inequalities, we get

$$\sum_{i=1}^{\hat{s}} \mathrm{Re}\left[\hat{\lambda}_i\phi(\hat{\varsigma},\hat{v}_i)\right] > \sum_{i=1}^{\hat{s}} \mathrm{Re}\left[\hat{\lambda}_i\phi\left(\hat{\xi},\hat{v}_i\right)\right].$$

Therefore, there exists a certain i_0, such that

$$\mathrm{Re}\phi\left(\hat{\varsigma},\hat{v}_{i_0}\right) > \mathrm{Re}\phi\left(\hat{\xi},\hat{v}_{i_0}\right).$$

It follows that

$$\sup_{v\in W}\mathrm{Re}\phi(\hat{\varsigma},v) \geq \mathrm{Re}\phi\left(\hat{\varsigma},\hat{v}_{i_0}\right) > \mathrm{Re}\phi\left(\hat{\xi},\hat{v}_{i_0}\right) = \sup_{v\in W}\mathrm{Re}\phi\left(\hat{\xi},v\right),$$

which contradicts (5.15.14). Therefore, we conclude that $\hat{\xi} = \hat{\varsigma}$. Hence the proof is complete. □

5.16 Mond–Weir Duality for Continuous–Time Vector Optimization Problems

The Mond–Weir type dual problem associated to (MP) considered in Sect. 4.6 is given by

(MWD) maximize $\displaystyle\int_a^b f(t,u,\dot{u})\,dt = \left(\int_a^b f_1(t,u,\dot{u})\,dt, \ldots, \int_a^b f_p(t,u,\dot{u})\,dt\right)$

subject to

$$u(a) = \alpha,\ u(b) = \beta, \tag{5.16.1}$$

$$\lambda^T f_x(t,u,\dot{u}) + y(t)^T g_x(t,u,\dot{u}) = \frac{d}{dt}\left(\lambda^T f_{\dot{x}}(t,u,\dot{u}) + y(t)^T g_{\dot{x}}(t,u,\dot{u})\right), \tag{5.16.2}$$

$$y(t)^T\, g(t,u,\dot{u}) \geqq 0,\ t \in I, \tag{5.16.3}$$

$$y(t) \geqq 0,\ t \in I, \tag{5.16.4}$$

$$\lambda \in R^p, \lambda \geqq 0, \lambda^{\mathrm{T}} e = 1, \mathrm{e} = (1,\ldots,1) \in R^p. \tag{5.16.5}$$

Theorem 5.16.1. *(Weak Duality). Let x be a feasible solution for (MP) and* (u,λ,y) *be feasible for (MWD). Let either of the following conditions hold:*

(i) $\lambda > 0$ *and* $\left(\lambda^T f, y^T g\right)$ *is strong pseudo-quasi type I with respect to* η*;*
(ii) $\left(\lambda^T f, y^T g\right)$ *is weak strictly pseudo-quasi type I with respect to* η*;*
(iii) $\left(\lambda^T f, y^T g\right)$ *is weak strictly pseudo type I with respect to* η*,*

then

$$\int_a^b f(t,x,\dot{x})dt \nleq \int_a^b f(t,u,\dot{u})\,dt.$$

Proof. Let

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b f(t,u,\dot{u})\,dt. \tag{5.16.6}$$

Since (u,λ,y) is feasible for (MWD), it follows that

$$-\int_a^b y^T g(t,u,\dot{u})\,dt \leqq 0. \tag{5.16.7}$$

By the condition (i), (5.16.6) and (5.16.7) imply

$$\int_a^b \left[\eta(t,x,u)^T\, \lambda^T f_x(t,u,\dot{u}) + \frac{d}{dt}(\eta(t,x,u))^T\, \lambda^T f_{\dot{x}}(t,u,\dot{u})\right] dt < 0,$$

$$\int_a^b \left[\eta(t,x,u)^T\, \lambda^T g_x(t,u,\dot{u}) + \frac{d}{dt}(\eta(t,x,u))^T\, \lambda^T g_{\dot{x}}(t,u,\dot{u})\right] dt \leqq 0,$$

By the above two inequalities, we get

$$\int_a^b \left[\eta(t,x,u)^T\, \lambda^T f_x(t,u,\dot{u}) + \frac{d}{dt}(\eta(t,x,u))^T\, \lambda^T f_{\dot{x}}(t,u,\dot{u})\right] dt$$
$$+ \int_a^b \left[\eta(t,x,u)^T\, \lambda^T g_x(t,u,\dot{u}) + \frac{d}{dt}(\eta(t,x,u))^T\, \lambda^T g_{\dot{x}}(t,u,\dot{u})\right] dt < 0,$$

The above inequality in light of Remark 3.3.1 is

$$\int_a^b \left[\eta(t,x,u)^T \left(\lambda^T f_x(t,u,\dot{u}) + y^T g_x(t,u,\dot{u}) - \frac{d}{dt}\left(\lambda^T f_{\dot{x}}(t,u,\dot{u}) + y^T g_{\dot{x}}(t,u,\dot{u})\right)\right)\right] dt < 0.$$

This contradicts (5.16.2).
By the condition (ii), (5.16.6) and (5.16.7) imply

$$\int_a^b \left[\eta(t,x,u)^T \lambda^T f_x(t,u,\dot{u}) + \frac{d}{dt}(\eta(t,x,u))^T \lambda^T f_{\dot{x}}(t,u,\dot{u})\right] dt < 0,$$

$$\int_a^b \left[\eta(t,x,u)^T \lambda^T g_x(t,u,\dot{u}) + \frac{d}{dt}(\eta(t,x,u))^T \lambda^T g_{\dot{x}}(t,u,\dot{u})\right] dt \leqq 0.$$

By the above two inequalities in light of Remark 3.3.1, we get

$$\int_a^b \left[\eta(t,x,u)^T \left(\lambda^T f_x(t,u,\dot{u}) + y^T g_x(t,u,\dot{u}) - \frac{d}{dt}\left(\lambda^T f_{\dot{x}}(t,u,\dot{u}) + y^T g_{\dot{x}}(t,u,\dot{u})\right)\right)\right] dt < 0,$$

which contradicts (5.16.2).

By the condition (iii), (5.16.6) and (5.16.7) imply

$$\int_a^b \left[\eta(t,x,u)^T \lambda^T f_x(t,u,\dot{u}) + \frac{d}{dt}(\eta(t,x,u))^T \lambda^T f_{\dot{x}}(t,u,\dot{u})\right] dt < 0,$$

$$\int_a^b \left[\eta(t,x,u)^T \lambda^T g_x(t,u,\dot{u}) + \frac{d}{dt}(\eta(t,x,u))^T \lambda^T g_{\dot{x}}(t,u,\dot{u})\right] dt < 0.$$

By the above two inequalities in light of Remark 3.3.1, we get

$$\int_a^b \left[\eta(t,x,u)^T \left(\lambda^T f_x(t,u,\dot{u}) + y^T g_x(t,u,\dot{u}) - \frac{d}{dt}\left(\lambda^T f_{\dot{x}}(t,u,\dot{u}) + y^T g_{\dot{x}}(t,u,\dot{u})\right)\right)\right] dt < 0,$$

which contradicts (5.16.2). The proof is complete. □

Theorem 5.16.2. *(Strong Duality). Let x^* be an efficient solution for (MP). Assume that x^* is normal for each $\left(P_k^*\right)$, $k = 1,\ldots,p$. Then there exist $\lambda^* \in R^p$ and a piecewise smooth function $y^* : I \to R^m$ such that (x^*, λ^*, y^*) is feasible for (MWD). Furthermore, if for each feasible (u, λ, y) of (MWD) any of the conditions of Theorem 5.16.1 holds, then (x^*, λ^*, y^*) is an efficient solution for (MWD).*

Proof. Since x^* is efficient solution for (MP), it follows from Theorem 4.6.1 that there exist $\lambda^* \in R^p$ and a piecewise smooth function $y^* : I \to R^m$ such that

(4.6.5)–(4.6.8) hold. Moreover, $x^* \in K$, hence the feasibility of (x^*, λ^*, y^*) for (MWD) follows and efficiency follows from weak duality Theorem 5.16.1. □

5.17 General Mond–Weir Duality for Continuous–Time Vector Optimization Problems

In this section, we consider general Mond–Weir type dual to (MP) considered in Sect. 4.6, and establish weak and strong duality theorems.

$$\text{(GMWD)} \quad \text{maximize} \int_a^b \left[f(t,u,\dot{u}) + y_{J_0}^T g(t,u,\dot{u})\right] \mathrm{dt}$$

$$= \left(\int_a^b \left[f_1(t,u,\dot{u}) + y_{J_0}^T g_{J_0}(t,u,\dot{u})\right] dt, \ldots, \int_a^b \left[f_p(t,u,\dot{u}) + y_{J_0}^T g_{J_0}(t,u,\dot{u})\right] dt \right)$$

$$\text{subject to } u(a) = \alpha,\ u(b) = \beta \tag{5.17.1}$$

$$\lambda^T f_x(t,u,\dot{u}) + y(t)^T g_x(t,u,\dot{u}) = \frac{d}{dt}(\lambda^T f_{\dot{x}}(t,u,\dot{u}) + y(t)^T g_{\dot{x}}(t,u,\dot{u})), \tag{5.17.2}$$

$$y(t)_{J_t}^T g_{J_t}(t,u,\dot{u}) \geqq 0,\ 1 \leqq t \leqq r, \tag{5.17.3}$$

$$y(t) \geqq 0,\ t \in I, \tag{5.17.4}$$

$$\lambda \in R^p, \lambda \geqq 0,\ \lambda^{\mathrm{T}} e = 1, \mathrm{e} = (1, \ldots, 1) \in R^p. \tag{5.17.5}$$

where $\mathrm{J_t}$, $J_t, 0 \leqq t \leqq r$ are partitions of the set M.

Theorem 5.17.1. *(Weak Duality). Let x be a feasible solution for (MP) and let (u, λ, y) be feasible for (GMWD). Let either of the following conditions holds:*

(i) $\lambda > 0$ and $\left(\lambda^T f + y_{J_0}^T g_{J_0}, y_{J_t}^T g_{J_t}\right)$ is strong pseudo-quasi type I with respect to η for any $t, 1 \leqq t \leqq r$;

(ii) $\left(\lambda^T f + y_{J_0}^T g_{J_0}, y_{J_t}^T g_{J_t}\right)$ is weak strictly pseudo-quasi type I with respect to η for any $t, 1 \leqq t \leqq r$;

(iii) $\left(\lambda^T f + y_{J_0}^T g_{J_0}, y_{J_t}^T g_{J_t}\right)$ is weak strictly pseudo type I with respect to η for any $t, 1 \leqq t \leqq r$. Then

$$\int_a^b f(t,x,\dot{x}) dt \nleq \int_a^b \left[f(t,u,\dot{u}) + y_{J_0}^T g(t,u,\dot{u})\right] \mathrm{dt}. \tag{5.17.6}$$

Proof. Let

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b \left[f(t,u,\dot{u}) + y_{J_0}^T g(t,u,\dot{u})\right]\mathrm{dt}.$$

Since x is feasible for (MP) and $y(t) \geqq 0$, (5.17.6) implies

$$\int_a^b \left[f(t,x,\dot{x}) + y_{J_0}^T g(t,x,\dot{x})\right]\mathrm{dt} \leq \int_a^b \left[f(t,u,\dot{u}) + y_{J_0}^T g(t,u,\dot{u})\right]\mathrm{dt}. \tag{5.17.7}$$

Since (u,λ,y) is feasible for (MWD), it follows that

$$-\int_a^b y^T g(t,u,\dot{u})\,dt \leqq 0. \tag{5.17.8}$$

By the condition (i), (5.17.7) and (5.17.8) imply

$$\int_a^b \Big[\eta(t,x,u)^T \left(f_x(t,x,u) + y_{J_0}^T g_{xJ_0}(t,x,u)\right) + \frac{d}{dt}\left(\eta(t,x,u)\right)^T \left(f_{\dot{x}}(t,x,u) + y_{J_0}^T g_{\dot{x}J_0}(t,x,u)\right)\Big]\,dt \leq 0,$$

and

$$\int_a^b \Big[\eta(t,x,u)^T \left(y_{J_t}^T g_{xJ_t}(t,x,u)\right) + \frac{d}{dt}\left(\eta(t,x,u)\right)^T \left(y_{J_t}^T g_{\dot{x}J_t}(t,x,u)\right)\Big]\,dt \leqq 0, \quad \forall 1 \leqq t \leqq r.$$

Since $\lambda > 0$, the above two inequalities give

$$\int_a^b \left[\eta(t,x,u)^T \left(\lambda^T f_x(t,x,u) + \sum_{t=0}^{r} y_{J_t}^T g_{xJ_t}(t,x,u)\right) + \frac{d}{dt}\left(\eta(t,x,u)\right)^T \left(\lambda^T f_{\dot{x}}(t,x,u) + \sum_{t=0}^{r} y^T g_{\dot{x}J_t}(t,x,u)\right)\right] dt < 0.$$

Since $J_0, J_1, \ldots, J_r$ are partitions of M, the above inequality is equivalent to

$$\int_a^b \Big[\eta(t,x,u)^T \left(\lambda^T f_x(t,u,\dot{u}) + y^T g_x(t,x,u)\right) + \frac{d}{dt}\left(\eta(t,x,u)\right)^T \left(\lambda^T f_{\dot{x}}(t,x,u) + y^T g_{\dot{x}}(t,x,u)\right)\Big]\,dt < 0.$$

The above inequality in light of Remark 3.3.1 is

$$\int_a^b \left[\eta(t,x,u)^T\left(\lambda^T f_x(t,u,\dot{u}) + y^T g_x(t,u,\dot{u}) - \frac{d}{dt}\left(\lambda^T f_{\dot{x}}(t,u,\dot{u}) + y^T g_{\dot{x}}(t,u,\dot{u})\right)\right)\right] dt < 0.$$

This contradicts (5.17.2).

By the condition (ii), (5.17.7) and (5.17.8) imply

$$\int_a^b \Big[\eta(t,x,u)^T\left(f_x(t,x,u) + y_{J_0}^T g_{xJ_0}(t,x,u)\right) + \frac{d}{dt}(\eta(t,x,u))^T\left(f_{\dot{x}}(t,x,u) + y_{J_0}^T g_{\dot{x}J_0}(t,x,u)\right)\Big] dt < 0,$$

and

$$\int_a^b \Big[\eta(t,x,u)^T\left(y_{J_t}^T g_{xJ_t}(t,x,u)\right) + \frac{d}{dt}(\eta(t,x,u))^T\left(y_{J_t}^T g_{\dot{x}J_t}(t,x,u)\right)\Big] dt \leqq 0, \quad \forall 1 \leqq t \leqq r.$$

Since $\lambda \geqq 0$, the above two inequalities give

$$\int_a^b \Bigg[\eta(t,x,u)^T\left(\lambda^T f_x(t,x,u) + \sum_{t=0}^{r} y_{J_t}^T g_{xJ_t}(t,x,u)\right) + \frac{d}{dt}(\eta(t,x,u))^T\left(\lambda^T f_{\dot{x}}(t,x,u) + \sum_{t=0}^{r} y^T g_{\dot{x}J_t}(t,x,u)\right)\Bigg] dt < 0.$$

Since $J_0, J_1, \ldots, J_r$ are partitions of M, the above inequality is equivalent to

$$\int_a^b \Big[\eta(t,x,u)^T\left(\lambda^T f_x(t,u,\dot{u}) + y^T g_x(t,x,u)\right) + \frac{d}{dt}(\eta(t,x,u))^T \left(\lambda^T f_{\dot{x}}(t,x,u) + y^T g_{\dot{x}}(t,x,u)\right)\Big] dt < 0.$$

The above inequality in light of Remark 3.3.1 is

$$\int_a^b \left[\eta(t,x,u)^T\left(\lambda^T f_x(t,u,\dot{u}) + y^T g_x(t,u,\dot{u}) - \frac{d}{dt}\left(\lambda^T f_{\dot{x}}(t,u,\dot{u}) + y^T g_{\dot{x}}(t,u,\dot{u})\right)\right)\right] dt < 0.$$

This contradicts (5.17.2).

By the condition (iii), (5.17.7) and (5.17.8) imply

$$\int_a^b \Big[\eta(t,x,u)^T \left(f_x(t,x,u) + y_{J_0}^T g_{xJ_0}(t,x,u)\right)$$
$$d + \frac{d}{dt}(\eta(t,x,u))^T \left(f_{\dot{x}}(t,x,u) + {y_{J_0}}^T g_{\dot{x}J_0}(t,x,u)\right)\Big]\, dt < 0,$$

and

$$\int_a^b \Big[\eta(t,x,u)^T \left(y_{J_t}^T g_{xJ_t}(t,x,u)\right)$$
$$+\frac{d}{dt}(\eta(t,x,u))^T \left(y_{J_t}^T g_{\dot{x}J_t}(t,x,u)\right)\Big]\, dt < 0, \forall 1 \leqq t \leqq r.$$

Since $\lambda \geqq 0$, the above two inequalities give

$$\int_a^b \left[\eta(t,x,u)^T \left(\lambda^T f_x(t,x,u) + \sum_{t=0}^{r} y_{J_t}^T g_{xJ_t}(t,x,u)\right)\right.$$
$$\left. +\frac{d}{dt}(\eta(t,x,u))^T \left(\lambda^T f_{\dot{x}}(t,x,u) + \sum_{t=0}^{r} y^T g_{\dot{x}J_t}(t,x,u)\right)\right] dt < 0.$$

Since $J_0, J_1, \ldots, J_r$ are partitions of M, the above inequality is equivalent to

$$\int_a^b \Big[\eta(t,x,u)^T \left(\lambda^T f_x(t,u,\dot{u}) + y^T g_x(t,x,u)\right) + \frac{d}{dt}(\eta(t,x,u))^T$$
$$\left(\lambda^T f_{\dot{x}}(t,x,u) + y^T g_{\dot{x}}(t,x,u)\right)\Big]\, dt < 0.$$

The above inequality in light of Remark 3.3.1 is

$$\int_a^b \left[\eta(t,x,u)^T \left(\lambda^T f_x(t,u,\dot{u}) + y^T g_x(t,u,\dot{u}) - \frac{d}{dt}\left(\lambda^T f_{\dot{x}}(t,u,\dot{u}) + y^T g_{\dot{x}}(t,u,\dot{u})\right)\right)\right] dt < 0.$$

This contradicts (5.17.2). The proof is complete. □

5.18 Duality for Nondifferentiable Continuous–Time Optimization Problems

In this section, we introduce the following two duals to the problem (CNP) considered in Sect. 4.7.

(1) Wolfe dual

$$\text{(WCD)} \qquad \text{maximize } \varphi(\mathrm{t}) = \left(\int_0^T [f(t,u(t)) + \lambda(t) g(t,u(t))] dt \right)$$

subject to

$$0 \leq \int_0^T \left[f^0(t,u(t);h(t)) + \sum_{i=1}^m \lambda_i(t) g_i^0(t,u(t);h(t)) \right] dt \forall h \in L_\infty^n[0,T], \tag{5.18.1}$$

$$\lambda(t) \geq 0, \text{a.e.in}\,[0,T], \tag{5.18.2}$$

$$u \in X.$$

Let W_1 denote the set of all feasible solutions of (WCD).

(2) Mond–Weir dual

$$\text{(MWCD)} \qquad \text{maximize } \psi(t) = \int_0^T f(t,u(t))\, dt$$

subject to

$$0 \leq \int_0^T \left[f^0(t,u(t);h(t)) + \sum_{i=1}^m \lambda_i(t) g_i^0(t,u(t);h(t)) \right] dt \qquad \forall h \in L_\infty^n[0,T], \tag{5.18.3}$$

$$\lambda(t) g(t,u(t)) \geq 0, \text{ a.e.in}\,[0,T] \tag{5.18.4}$$

$$\lambda(t) \geq 0, \text{ a.e.in}\,[0,T], \tag{5.18.5}$$

$$u \in X.$$

Let W_2 denote the set of all feasible solutions of (MWCD).

Maximization in (WCD) and (MWCD) means obtaining efficient solutions of the problems. Recall from Sect. 4.7 that Ω is the set of all feasible solutions to (CNP).

Theorem 5.18.1. *(Weak Duality). Assume that for all $x \in \Omega$ and for all $(u,\lambda) \in W_1$, and $(f(\cdot), \lambda(t) g(\cdot))$ are type I with respect to the same η then, $\phi(t) \nleq \varphi(t)$.*

Proof. Suppose, to the contrary, that $\phi(t) \leq \varphi(t)$. Then there exist $x \in \Omega$ and $(u,\lambda) \in W_1$, such that

$$\int_0^T f(t,x(t))\, dt < \int_0^T f(t,u(t) + \lambda(t) g(t,u(t)))\, dt. \tag{5.18.6}$$

Since $(f(t,\cdot), \mathrm{g_i}(t,\cdot))$ are type I at $\bar{x}(t)$ throughout $[0,T]$ for each $i \in I$, we have the inequalities

$$f(t,x(t)) - f(t,u(t)) \geq f^0(t,u(t);\eta(x(t),u(t))) \quad a.e.\text{in}[0,T], \tag{5.18.7}$$

and

$$-g_i(t,u(t)) \geq g_i^0(t,\bar{x}(t);\eta(x(t),u(t))) \quad a.e.\text{in}[0,T], \quad i \in I, \tag{5.18.8}$$

for some $\eta(x(t),u(t))$. Because $x \in \Omega$, $(u,\lambda) \in W_1$ and $\bar{\lambda}_i \geq 0$ $a.e.$ in $[0,T], i \in I$, it is clear from (5.18.6)–(5.18.8) that

$$0 > \int_0^T \left[f^0(t,u(t);\eta(x(t),u(t))) + \sum_{i=1}^m \lambda_i(t) g_i^0(t,u(t);\eta(x(t),u(t)))\right] dt,$$

which, with $h(t) = \eta(x(t),\ u(t))$, contradicts (5.18.1). Therefore, we conclude that $\phi(t) \nleq \varphi(t)$. □

Theorem 5.18.2. *(Weak Duality). Assume that for all $x \in \Omega$ and for all $(u,\lambda) \in W_2$, and $(f(\cdot), \lambda(t) g(\cdot))$ are pseudo-quasi-type I with respect to the same η then, $\phi(t) \nleq \psi(t)$.*

Proof. Since for each $x \in \Omega$, from (5.18.4), we have

$$-\lambda_i(t) g_i(t,u(t)) \leq 0 \quad a.e.\text{in}[0,T], i \in I.$$

From (5.18.5) and the second part of the Pseudo-quasi-type I assumption, we get

$$\lambda_i(t) g_i^0(t,u(t);\eta(x(t),u(t))) \leq 0, \quad a.e.\text{in}[0,T], i \in I.$$

Hence, we have

$$\int_0^{\mathrm{T}} \sum_{\mathrm{i} \in \mathrm{I}} \lambda_i(t) g_i^0(t,u(t);\eta(x(t),u(t))) \leq 0, \qquad \forall u \in W_2. \tag{5.18.9}$$

From (5.18.3), we get

$$0 \leq \int_0^{\mathrm{T}} \left[f^0(t,u(t);\eta(x(t),u(t))) + \sum_{i=1}^m \lambda_i(t) g_i^0(t,u(t);\eta(x(t),u(t)))\right] dt, \forall u \in W_2. \tag{5.18.10}$$

From (5.18.9) and (5.18.10), we get

$$\int_0^T f^0(t,u(t);\eta(x(t),u(t))) \geq 0, \quad \forall u \in W_2.$$

By the first part of the Pseudo-quasi-type I assumption and the last inequality, we get

$$\phi(\mathrm{x}) \leq \psi(\mathrm{u}), \quad \forall u \in W_2.$$

□

Theorem 5.18.3. *(Strong Duality). Let x^* be an efficient solution for (WCD) and let $(f(t,\cdot), g(t,\cdot))$ be uniformly Lipschitz. Further, if the constraint qualification in Cravens (1995) holds at x^*, then there exists λ such that (x^*,λ) is feasible for (WCD). Moreover, if the weak duality in Theorem 5.18.1 holds, then (x^*,λ) is efficient for (WCD).*

Proof. The feasibility of (x^*,λ) for (WCD) follows from Theorem 4.7.2. The efficiency of (x^*,λ) for (WCD) follows from the weak duality in Theorem 5.18.1. □

5.19 Duality for Vector Control Problems

Hanson (1964) observed that variational problems and control problems are continuous–time analogue of finite dimensional nonlinear programming problems. Since then the fields of nonlinear programming and the calculus of variations have to some extent merged together within optimization theory. Several authors have been interested in these problems, see for example, Bhatia and Kumar (1995), Chen (2002), Craven (1978, 1995), Kim et al. (1993), Ledzewicz-Kowalwski (1985), Mishra and Mukherjee (1999), Mond and Hansen (1968a), Mond and Smart (1988), Nahak and Nanda (1997b) and Zhian (2001).

Consider the vector control problem:

$$\text{(VCP)} \qquad \text{minimize} \int_{t_0}^{t_f} f(t,x,u)\,dt = \left(\int_{t_0}^{t_f} f_1(t,x,u)\,dt, \ldots, \int_{t_0}^{t_f} f_p(t,x,u)\,dt \right)$$

$$\text{subject to } x(t_0) = \alpha, x\left(t_f\right) = \beta, \tag{5.19.1}$$

$$\dot{x} = h(t,x,u), \quad t \in I, \tag{5.19.2}$$

$$g(t,x,u) = (g_1(t,x,u), \ldots, .g_l(t,x,u))^T \leq 0, t \in I. \tag{5.19.3}$$

For any partition $\{\Sigma, \Sigma'\}$ of $\{1,2,\ldots,l\}$, i.e., $\Sigma \cup \Sigma' = \{1,2,\ldots,l\}$ and $\Sigma \cap \Sigma' = \phi$, we propose two types of general duals for (VCP).

(VCD1)

$$\text{maximize} \left(\int_{t_0}^{t_f} \left[f_1(t,y,v) + \mu(t)_\Sigma^T g_\Sigma(t,y,v) \right] dt, \ldots, \right.$$

$$\left. \int_{t_0}^{t_f} \left[f_p(t,y,v) + \mu(t)_\Sigma^T g_\Sigma(t,y,v) \right] dt \right)$$

subject to $y(t_0) = \alpha, y(t_f) = \beta,$ (5.19.4)

$$\sum_{i=1}^{p} \lambda_i f_{iy}(t,y,v) + g_y(t,y,v)\,\mu(t) + h_y(t,y,v)\,\gamma(t) + \dot{\gamma}(t) = 0,\ t \in I, \tag{5.19.5}$$

$$\sum_{i=1}^{p} \lambda_i f_{iv}(t,y,v) + g_v(t,y,v)\,\mu(t) + h_v(t,y,v)\,\gamma(t) = 0,\ t \in I, \tag{5.19.6}$$

$$\int_{t_0}^{t_f} \gamma(t)^T\,[h(t,y,v) - \dot{y}]\,dt \geq 0, \tag{5.19.7}$$

$$\int_{t_0}^{t_f} \mu(t)_{\Sigma'}^T\, g_{\Sigma'}(t,y,v)\,dt \geq 0, \tag{5.19.8}$$

$$\mu(t) \geq 0, \quad t \in I, \tag{5.19.9}$$

$$\lambda_i \geq 0,\ i = 1,2,\ldots,p, \sum_{i=1}^{p} \lambda_i = 1, \tag{5.19.10}$$

where $\mu(t)_{\Sigma}$ denotes the $|\Sigma|$ column vector function with the component indices in Σ, and similar notations have the same meanings.

(VCD2)

$$\max\left(\int_{t_0}^{t_f}\left\{f_1(t,y,v) + \mu(t)_{\Sigma}^T g_{\Sigma}(t,y,v) + \gamma(t)^T\,[h(t,y,v) - \dot{y}]\right\}dt, \ldots,\right.$$
$$\left.\int_{t_0}^{t_f}\left\{f_p(t,y,v) + \mu(t)_{\Sigma}^T g_{\Sigma}(t,y,v) + \gamma(t)^T\,[h(t,y,v) - \dot{y}]\right\}dt\right)$$

subject to $y(t_0) = \alpha, y(t_f) = \beta,$

$$\sum_{i=1}^{p} \lambda_i f_{iy}(t,y,v) + g_y(t,y,v)\,\mu(t) + h_y(t,y,v)\,\gamma(t) + \dot{\gamma}(t) = 0,\ t \in I,$$

$$\sum_{i=1}^{p} \lambda_i f_{iv}(t,y,v) + g_v(t,y,v)\,\mu(t) + h_v(t,y,v)\,\gamma(t) = 0,\ t \in I,$$

$$\int_{t_0}^{t_f} \mu(t)_{\Sigma'}^T\, g_{\Sigma'}(t,y,v)\,dt \geq 0,$$

$$\mu(t) \geq 0,\ t\ \in I,$$

$$\lambda_i \geq 0,\ i = 1,2,\ldots,p,\ \sum_{i=1}^{p} \lambda_i = 1.$$

Theorem 5.19.1. *(Weak Duality). Assume that for any feasible* $(\bar{x},\bar{u})$ *for (VCP) and any feasible* $(\bar{y},\bar{v},\bar{\lambda},\bar{\mu},\bar{\gamma})$ *for (VCD1),* $\int_{t_0}^{t_f}[f_i(t,y,v)+\bar{\mu}(t)_{\Sigma}^{T}g_{\Sigma}(t,y,v)]dt$ *is strictly quasi-univex at* $(\bar{y},\bar{v})$ *on* $X\times U$ *with respect to* η, ξ, b_0 *and* ϕ_0, $\int_{t_0}^{t_f}\bar{\gamma}(t)^T[h(t,y,v)-\dot{y}]dt$ *is quasi-univex at* $(\bar{y},\bar{v})$ *on* $X\times U$ *with respect to* η, ξ, b_1 *and* ϕ_1 *and* $\int_{t_0}^{t_f}\bar{\mu}(t)_{\Sigma'}^{T}g_{\Sigma'}(\mathrm{t,y,v})dt$ *is quasi-univex at* $(\bar{y},\bar{v})$ *on* $X\times U$ *with respect to* η, ξ, b_2 *and* ϕ_2 *with* $b_0>0$, *and* $\mathrm{a}\leq 0\Rightarrow\phi_r(a)\leq 0, r=0,1,2$, *then the following can not hold simultaneously,*

$$\int_{t_0}^{t_f} f_i(t,\bar{x},\bar{u})\,dt \leq \int_{t_0}^{t_f}\left[f_i(t,\bar{y},\bar{v})+\bar{\mu}(t)_{\Sigma}^{T}g_{\Sigma}(t,\bar{y},\bar{v})\right]dt,\quad \forall i\in\{1,2,\ldots,p\}, \tag{5.19.11}$$

$$\int_{t_0}^{t_f} f_j(t,\bar{x},\bar{u})\,dt < \int_{t_0}^{t_f}\left[f_j(t,\bar{y},\bar{v})+\bar{\mu}(t)_{\Sigma}^{T}g_{\Sigma}(t,\bar{y},\bar{v})\right]dt,\quad \text{for some } j\in\{1,2,\ldots,p\}. \tag{5.19.12}$$

Proof. Suppose to the contrary that (5.19.11) and (5.19.12) hold for some feasible $(\bar{x},\bar{u})$ for (VCP) and some feasible $\left(\bar{y},\bar{v},\bar{\lambda},\bar{\mu},\bar{\gamma}\right)$ for (VCD1), then by (2.3), (2.9) and (5.19.11)

$$\int_{t_0}^{t_f}\left[f_i(t,\bar{x},\bar{u})+\bar{\mu}(t)_{\Sigma}^{T}g_{\Sigma}(t,\bar{x},\bar{u})\right]dt \leq \int_{t_0}^{t_f}\left[f_i(t,\bar{y},\bar{v})+\bar{\mu}(t)_{\Sigma}^{T}g_{\Sigma}(t,\bar{y},\bar{v})\right]dt,$$
$$\forall i\in\{1,2,\ldots,p\}.$$

By $b_0>0$, and $\mathrm{a}\leq 0\Rightarrow\phi_0(a)\leq 0$, we get

$$b_0(\bar{x},\bar{y})\,\phi_0\left[\int_{t_0}^{t_f}\left[f_i(t,\bar{x},\bar{u})+\bar{\mu}(t)_{\Sigma}^{T}g_{\Sigma}(t,\bar{x},\bar{u})\right]dt\right]$$
$$\leq b_0(\bar{x},\bar{y})\,\phi_0\left[\int_{t_0}^{t_f}\left[f_i(t,\bar{y},\bar{v})+\bar{\mu}(t)_{\Sigma}^{T}g_{\Sigma}(t,\ \bar{y},\bar{v})\right]dt\right],\quad \forall i\in\{1,2,\ldots,p\}.$$

By the strictly quasi-univexity of $\int_{t_0}^{t_f}\left[f_i(t,y,v)+\bar{\mu}(t)_{\Sigma}^{T}g_{\Sigma}(t,y,v)\right]dt$, when $(\bar{x},\bar{u})\neq(\bar{y},\bar{v})$,

$$\int_{t_0}^{t_f} \left\{\eta^T(t,\bar{x},\bar{u},\bar{y},\bar{v})\left[f_{iy}(t,\bar{y},\bar{v}) + g_{\Sigma y}(t,\bar{y},\bar{v})\,\bar{\mu}(t)_\Sigma\right]\right.$$
$$\left. +\xi^{\mathrm{T}}(t,\bar{x},\bar{u},\bar{y},\bar{v})\left[f_{iv}(t,\bar{y},\bar{v}) + g_{\Sigma v}(t,\bar{y},\bar{v})\,\bar{\mu}(t)_\Sigma\right]\right\} dt < 0.$$

Multiplying each inequality in the above inequality by $\bar{\lambda}_i \geq 0, i = 1,2,\ldots,p$, and then adding the resulting inequalities, we get

$$\int_{t_0}^{t_f} \left\{\eta^T(t,\bar{x},\bar{u},\bar{y},\bar{v})\left[\sum_{i=1}^{p} \bar{\lambda}_i f_{iy}(t,\bar{y},\bar{v}) + g_{\Sigma y}(t,\bar{y},\bar{v})\,\bar{\mu}(t)_\Sigma\right]\right.$$
$$\left. + \xi^{\mathrm{T}}(t,\bar{x},\bar{u},\bar{y},\bar{v})\left[\sum_{i=1}^{p} \bar{\lambda}_i f_{iv}(t,\bar{y},\bar{v}) + g_{\Sigma v}(t,\bar{y},\bar{v})\,\bar{\mu}(t)_\Sigma\right]\right\} dt < 0. \quad (5.19.13)$$

By (5.19.2) and (5.19.7), we get

$$\int_{t_0}^{t_f} \bar{\gamma}(t)^T \left[h(t,\bar{x},\bar{u}) - \dot{\bar{x}}\right] dt \leq \int_{t_0}^{t_f} \bar{\gamma}(t)^T \left[h(t,\bar{y},\bar{v}) - \dot{\bar{y}}\right] dt.$$

By $b_1 \geq 0$, and a $\leq 0 \Rightarrow \phi_1(a) \leq 0$, we get

$$b_1(\bar{x},\bar{y})\,\phi_1\left[\int_{t_0}^{t_f} \bar{\gamma}(t)^T \left[h(t,\bar{x},\bar{u}) - \dot{\bar{x}}\right] dt\right] \leq b_1(\bar{x},\bar{y})\,\phi_1\left[\int_{t_0}^{t_f} \bar{\gamma}(t)^T \left[h(t,\bar{y},\bar{v}) - \dot{\bar{y}}\right] dt\right].$$

The quasi-univexity of $\int_{t_0}^{t_f} \bar{\gamma}(t)^T \left[h(t,y,v) - \dot{y}\right] dt$ implies that

$$\int_{t_0}^{t_f} \left[\eta^T(t,\bar{x},\bar{u},\bar{y},\bar{v})\, h_y(t,\bar{y},\bar{v})\, \bar{\gamma}(t) - \frac{d\eta^T}{dt}\bar{\gamma}(t) + \xi^T(t,\bar{x},\bar{u},\bar{y},\bar{v})\, h_v(t,\bar{y},\bar{v})\, \bar{\gamma}(t)\right] dt \leq 0. \quad (5.19.14)$$

By integrating $\frac{d}{dt}\eta(t,\bar{x},\bar{u},\bar{y},\bar{v})\,\bar{\gamma}(t)$ from t_0 to t_f by parts and applying the boundary conditions (5.19.1), we get

$$-\int_{t_0}^{t_f} \frac{d\eta^T}{dt}\bar{\gamma}(t)\,dt = \int_{t_0}^{t_f} \eta^T \dot{\bar{\gamma}}(t)\,dt. \quad (5.19.15)$$

Using (5.19.15) in (5.19.14), we get

$$\int_{t_0}^{t_f} \left[\eta^T(t,\bar{x},\bar{u},\bar{y},\bar{v})\left[h_y(t,\bar{y},\bar{v})\,\bar{\gamma}(t)+\dot{\bar{\gamma}}(t)\right]+\xi^T(t,\bar{x},\bar{u},\bar{y},\bar{v})\,h_v(t,\bar{y},\bar{v})\,\bar{\gamma}(t)\right]dt \leq 0. \tag{5.19.16}$$

By (5.19.3), (5.19.8) and (5.19.9), we get

$$\int_{t_0}^{t_f} \overline{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(\mathrm{t},\overline{\mathrm{x}},\overline{\mathrm{u}})\,dt \leq \int_{t_0}^{t_f} \overline{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(\mathrm{t},\overline{\mathrm{y}},\overline{\mathrm{v}})\,dt.$$

By $b_2 \geq 0$, and a $\leq 0 \Rightarrow \phi_2(a) \leq 0$, from the above inequality, we get

$$b_2(\overline{x},\overline{y})\,\phi_2\left[\int_{t_0}^{t_f} \overline{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(t,\overline{x},\overline{u})\,dt\right] \leq b_2(\overline{x},\overline{y})\,\phi_2\left[\int_{t_0}^{t_f} \overline{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(t,\overline{y},\overline{v})\,dt\right].$$

It follows from the quasi-univexity of $\int_{t_0}^{t_f} \overline{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(\mathrm{t},\mathrm{y},\mathrm{v})\,dt$ that

$$\int_{t_0}^{t_f} \left[\eta^T(t,\overline{x},\overline{u},\overline{y},\overline{v})\,g_{\Sigma' y}(t,\overline{y},\overline{v})\,\overline{\mu}(t)_{\Sigma'}+\xi^T(t,\overline{x},\overline{u},\overline{y},\overline{v})\,g_{\Sigma' v}(t,\overline{y},\overline{v})\,\overline{\mu}(t)_{\Sigma'}\right]dt \leq 0. \tag{5.19.17}$$

By (5.19.13), (5.19.16) and (5.19.17), we get

$$\begin{aligned}
&\int_{t_0}^{t_f} \Bigg\{\eta^T(t,\overline{x},\overline{u},\overline{y},\overline{v})\Bigg[\sum_{i=1}^{p}\overline{\lambda}_i f_{iy}(t,\overline{y},\overline{v})+g_{\Sigma y}(t,\overline{y},\overline{v})\,\overline{\mu}(t)_{\Sigma}\\
&\qquad +\mathrm{g}_{\Sigma' y}(t,\overline{y},\overline{v})\,\overline{\mu}(t)_{\Sigma'}+h_y(t,\overline{y},\overline{v})\,\overline{\gamma}(t)+\dot{\overline{\gamma}}(t)\Bigg]\\
&\qquad +\xi^T(t,\overline{x},\overline{u},\overline{y},\overline{v})\Bigg[\sum_{i=1}^{p}\overline{\lambda}_i f_{iv}(t,\overline{y},\overline{v})+g_{\Sigma v}(t,\overline{y},\overline{v})\,\overline{\mu}(t)_{\Sigma}\\
&\qquad +g_{\Sigma' v}(t,\overline{y},\overline{v})\,\overline{\mu}(t)_{\Sigma'}+h_v(t,\overline{y},\overline{v})\,\overline{\gamma}(t)\Bigg]\Bigg\}dt\\
&=\int_{t_0}^{t_f} \Bigg\{\eta^T(t,\overline{x},\overline{u},\overline{y},\overline{v})\Bigg[\sum_{i=1}^{p}\overline{\lambda}_i f_{iy}(t,\overline{y},\overline{v})+g_y(t,\overline{y},\overline{v})\,\overline{\mu}(t)\ +h_y(t,\overline{y},\overline{v})\,\overline{\gamma}(t)+\dot{\overline{\gamma}}(t)\Bigg]\\
&\quad +\xi^T(t,\overline{x},\overline{u},\overline{y},\overline{v})\Bigg[\sum_{i=1}^{p}\overline{\lambda}_i f_{iv}(t,\overline{y},\overline{v})+g_v(t,\overline{y},\overline{v})\,\overline{\mu}(t)+h_v(t,\overline{y},\overline{v})\,\overline{\gamma}(t)\Bigg]\Bigg\}dt<0,
\end{aligned}$$

which contradicts (5.19.5) and (5.19.6). □

Theorem 5.19.2. *(Weak Duality). Assume that for any feasible* $(\bar{x},\bar{u})$ *for (VCP) and any feasible* $(\bar{y},\bar{v},\bar{\lambda},\bar{\mu},\bar{\gamma})$ *for (VCD1), one of the functionals* $\int_{t_0}^{t_f}[f_i(t,y,v)+\bar{\mu}(t)_{\Sigma}^{T}g_{\Sigma}(t,y,v)]dt$, $\int_{t_0}^{t_f}\bar{\gamma}(t)^T[h(t,y,v)-\dot{y}]\,dt$ *or* $\int_{t_0}^{t_f}\bar{\mu}(t)_{\Sigma'}^{T}g_{\Sigma'}(\mathrm{t,y,v})\,dt$ *is strictly quasi-univex at* $(\bar{y},\bar{v})$ *on* $X\times U$ *with respect to* η,ξ, b_r *and* ϕ_r, *for* $r=0,1,2$ *with* $b_r>0$, *and the other two are quasi-univex at* $(\bar{y},\bar{v})$ *on* $X\times U$ *with respect to* η, ξ, b_r *and* ϕ_r, *for remaining r, and* $\mathrm{a}\le 0\Rightarrow\phi_r(a)\le 0, r=0,1,2$, *then (5.19.11) and (5.19.12) can not hold simultaneously.*

Proof. Suppose to the contrary that (5.19.11) and (5.19.12) hold for some feasible $(\bar{x},\bar{u})$ for (VCP) and some feasible $\left(\bar{y},\bar{v},\bar{\lambda},\bar{\mu},\bar{\gamma}\right)$ for (VCD1). Multiplying each inequality of (5.19.11) by $\bar{\lambda}_i\ge 0, i=1,2,...,p$, and then adding the resulting inequalities, we get

$$\int_{t_0}^{t_f}\sum_{i=1}^{p}\bar{\lambda}_i f_i(t,\bar{x},\bar{u})\,dt\le\int_{t_0}^{t_f}\left[\sum_{i=1}^{p}\bar{\lambda}_i f_i(t,\bar{y},\bar{v})+\bar{\mu}(t)_{\Sigma}^{T}g_{\Sigma}(t,\bar{y},\bar{v})\right]dt.$$

By (5.19.3) and (5.19.9), we get

$$\int_{t_0}^{t_f}\left[\sum_{i=1}^{p}\bar{\lambda}_i f_i(t,\bar{x},\bar{u})+\bar{\mu}(t)_{\Sigma}^{T}g_{\Sigma}(t,\bar{x},\bar{u})\right]dt\le\int_{t_0}^{t_f}\left[\sum_{i=1}^{p}\bar{\lambda}_i f_i(t,\bar{y},\bar{v})+\bar{\mu}(t)_{\Sigma}^{T}g_{\Sigma}(t,\bar{y},\bar{v})\right]dt.$$

The remaining part of the proof is similar to the proof of Theorem 5.19.1. □

Theorem 5.19.3. *(Weak Duality). Assume that for any feasible* $(\bar{x},\bar{u})$ *for (VCP) and any feasible* $(\bar{y},\bar{v},\bar{\lambda},\bar{\mu},\bar{\gamma})$ *for (VCD1),* $\int_{t_0}^{t_f}\left\{\sum_{i=1}^{p}\bar{\lambda}_i f_i(t,y,v)+\bar{\mu}(t)^T g(t,y,v)+\bar{\gamma}(t)^T[h(t,y,v)-\dot{y}]\right\}dt$ *is strictly pseudo-univex at* $(\bar{y},\bar{v})$ *on* $X\times U$ *with respect to* η,ξ, b_0 *and* ϕ_0 *with* $b_0>0$ *and* $\phi_0(a)>0\Rightarrow \mathrm{a}>0$, *then (5.19.11) and (5.19.12) cannot hold simultaneously.*

Proof. Multiply (5.19.5) from left by η^T and integrate from t_0 to t_f on both sides,

$$\int_{t_0}^{t_f}\eta^T(t,\bar{x},\bar{u},\bar{y},\bar{v})\left[\sum_{i=1}^{p}\bar{\lambda}_i f_{iy}(t,\bar{y},\bar{v})+g_y(t,\bar{y},\bar{v})\bar{\mu}(t)+h_y(t,\bar{y},\bar{v})\bar{\gamma}(t)+\dot{\bar{\gamma}}(t)\right]dt=0.$$

Integrate $\int_{t_0}^{t_f}\eta^T\dot{\bar{\gamma}}(t)\,dt$ by parts,

$$\int_{t_0}^{t_f}\left\{\eta^T\left[\sum_{i=1}^{p}\bar{\lambda}_i f_{iy}(t,\bar{y},\bar{v})+g_y(t,\bar{y},\bar{v})\bar{\mu}(t)+h_y(t,\bar{y},\bar{v})\bar{\gamma}(t)\right]-\frac{d\eta^T}{dt}\bar{\gamma}(t)\right\}dt=0,$$

So we have

$$\int_{t_0}^{t_f} \left\{ \eta^T \left[\sum_{i=1}^{p} \overline{\lambda}_i f_{iy}(t,\overline{y},\overline{v}) + g_y(t,\overline{y},\overline{v})\overline{\mu}(t) + h_y(t,\overline{y},\overline{v})\overline{\gamma}(t) \right] + \frac{d\eta^T}{dt} \left[\sum_{i=1}^{p} \overline{\lambda}_i f_{i\dot{y}}(t,\overline{y},\overline{v}) + g_{\dot{y}}(t,\overline{y},\overline{v})\overline{\mu}(t) + \left[h(t,\overline{y},\overline{v}) - \dot{\overline{y}} \right]_{\dot{y}} \overline{\gamma}(t) \right] \right\} dt = 0. \tag{5.19.18}$$

Multiply (5.19.6) from left by ξ^T, and integrate from t_0 to t_f, we get

$$\int_{t_0}^{t_f} \xi^T \left[\sum_{i=1}^{p} \overline{\lambda}_i f_{iv}(t,\overline{y},\overline{v}) + g_v(t,\overline{y},\overline{v})\overline{\mu}(t) + h_v(t,\overline{y},\overline{v})\overline{\gamma}(t) \right] dt = 0. \tag{5.19.19}$$

By (5.19.18) and (5.19.19), we get

$$\int_{t_0}^{t_f} \left\{ \eta^T \left[\sum_{i=1}^{p} \overline{\lambda}_i f_{iy}(t,\overline{y},\overline{v}) + g_y(t,\overline{y},\overline{v})\overline{\mu}(t) + h_y(t,\overline{y},\overline{v})\overline{\gamma}(t) \right] + \frac{d\eta^T}{dt} \left[\sum_{i=1}^{p} \overline{\lambda}_i f_{i\dot{y}}(t,\overline{y},\overline{v}) + g_{\dot{y}}(t,\overline{y},\overline{v})\overline{\mu}(t) + \left[h(t,\overline{y},\overline{v}) - \dot{\overline{y}} \right]_{\dot{y}} \overline{\gamma}(t) \right] + \xi^T \left[\sum_{i=1}^{p} \overline{\lambda}_i f_{iv}(t,\overline{y},\overline{v}) + g_v(t,\overline{y},\overline{v})\overline{\mu}(t) + h_v(t,\overline{y},\overline{v})\overline{\gamma}(t) \right] \right\} dt = 0. \tag{5.19.20}$$

By the strictly pseudo-univexity of

$$\int_{t_0}^{t_f} \left\{ \sum_{i=1}^{p} \overline{\lambda}_i f_i(t,y,v) + \overline{\mu}(t)^T g(t,y,v) + \overline{\gamma}(t)^T [h(t,y,v) - \dot{y}] \right\} dt, \text{we get}$$

$$b_0(\overline{x},\overline{y})\phi_0 \left[\int_{t_0}^{t_f} [\sum_{i=1}^{p} \overline{\lambda}_i f_i(t,\overline{x},\overline{u}) + \overline{\mu}(t)^T g(t,\overline{x},\overline{u}) + \overline{\gamma}(t)^T [h(t,\overline{x},\overline{u}) - \dot{\overline{x}}]] dt \right] > b_0(\overline{x},\overline{y})\phi_0 \left[\int_{t_0}^{t_f} [\sum_{i=1}^{p} \overline{\lambda}_i f_i(t,\overline{y},\overline{v}) + \overline{\mu}(t)^T g(t,\overline{y},\overline{v}) + \overline{\gamma}(t)^T [h(t,\overline{y},\overline{v}) - \dot{\overline{y}}]] dt \right].$$

By $b_0 > 0$ and $\phi_0(a) > 0 \Rightarrow$ a > 0, we get

$$\int_{t_0}^{t_f} \left[\sum_{i=1}^{p} \overline{\lambda}_i f_i(t,\overline{x},\overline{u}) + \overline{\mu}(t)^T g(t,\overline{x},\overline{u}) + \overline{\gamma}(t)^T \left[h(t,\overline{x},\overline{u}) - \dot{\overline{x}} \right] \right] dt$$
$$> \int_{t_0}^{t_f} \left[\sum_{i=1}^{p} \overline{\lambda}_i f_i(t,\overline{y},\overline{v}) + \overline{\mu}(t)^T g(t,\overline{y},\overline{v}) + \overline{\gamma}(t)^T \left[h(t,\overline{y},\overline{v}) - \dot{\overline{y}} \right] \right] dt.$$

By (5.19.2), (5.19.3), (5.19.7) and (5.19.8), we get

$$\int_{t_0}^{t_f} \sum_{i=1}^{p} \overline{\lambda}_i f_i(t,\overline{x},\overline{u})\, dt > \int_{t_0}^{t_f} \left[\sum_{i=1}^{p} \overline{\lambda}_i f_i(t,\overline{y},\overline{v}) + \overline{\mu}(t)_{\Sigma}^T g_{\Sigma}(t,\overline{y},\overline{v}) \right] dt.$$

It follows that (5.19.11) and (5.19.12) cannot hold simultaneously. □

Theorem 5.19.4. *(Weak Duality). Assume that for any feasible $(\bar{x},\bar{u})$ for (VCP) and any feasible $(\bar{y},\bar{v},\bar{\lambda},\bar{\mu},\bar{\gamma})$ for (VCD1), $\int_{t_0}^{t_f} \left[f_i(t,y,v) + \bar{\mu}(t)_{\Sigma}^T g_{\Sigma}(t,y,v) \right] dt$ is strictly pseudo-univex at $(\bar{y},\bar{v})$ on $X \times U$ with respect to η, ξ, b_0 and ϕ_0, $\int_{t_0}^{t_f} \bar{\gamma}(t)^T$ $[h(t,y,v) - \dot{y}]\, dt$ is quasi-univex at $(\bar{y},\bar{v})$ on $X \times U$ with respect to η, ξ, b_1 and ϕ_1 and $\int_{t_0}^{t_f} \bar{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(t,y,v)\, dt$ is quasi-univex at $(\bar{y},\bar{v})$ on $X \times U$ with respect to η, ξ, b_2 and ϕ_2 with $b_0 > 0$, and $a \leq 0 \Rightarrow \phi_r(a) \leq 0, r = 0,1,2$, then (5.19.11) and (5.19.12) cannot hold simultaneously.*

Proof. As in the proof of the above theorem, we obtain (5.19.20). From (5.19.20), we get

$$\int_{t_0}^{t_f} \left\{ \eta^T \left[\sum_{i=1}^{p} \bar{\lambda}_i f_{iy}(t,\bar{y},\bar{v}) + g_{\Sigma_y}(t,\bar{y},\bar{v})\, \bar{\mu}(t)_{\Sigma} \right] \right.$$
$$+ \frac{d\eta^T}{dt} \left[\sum_{i=1}^{p} \bar{\lambda}_i f_{i\dot{y}}(t,\bar{y},\bar{v}) + g_{\Sigma_{\dot{y}}}(t,\bar{y},\bar{v})\, \bar{\mu}(t)_{\Sigma} \right]$$
$$\left. + \xi^T \left[\sum_{i=1}^{p} \bar{\lambda}_i f_{iv}(t,\bar{y},\bar{v}) + g_{\Sigma v}(t,\bar{y},\bar{v})\, \bar{\mu}(t)_{\Sigma} \right] \right\} dt$$
$$= -\int_{t_0}^{t_f} \left[\eta^T g_{\Sigma'_y}(t,\bar{y},\bar{v})\, \bar{\mu}(t)_{\Sigma'} + \frac{d\eta^T}{dt} g_{\Sigma'_{\dot{y}}}(t,\bar{y},\bar{v})\, \bar{\mu}(t)_{\Sigma'} \right.$$ □

Further, it will be interesting to extend the work of the present section to the type of problem considered by Arana-Jimenez et al. (2008).

5.20 Duality for Vector Fractional Subset Optimization Problems

In this section, we shall establish certain duality theorems for three parametric and three semi-parametric dual models to the primal problem discussed in Sect. 4.9.

Consider the following vector fractional subset optimization problem given in Sect. 4.9.

$$\text{(P)} \qquad \text{minimize} \left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \ldots, \frac{F_p(S)}{G_p(S)}, \right)$$
$$\text{subject to } H_j(S) \leqq 0, j \in \underline{m}, S \in \Lambda^n,$$

where Λ^n is the n-fold product of the σ-algebra Λ of the subsets of a given set X, $F_i, G_i, i \in \underline{p} \equiv \{1,2,\ldots,p\}$ $\quad H_j(S) \leqq 0,\ j \in \underline{m} \equiv \{1,2,\ldots,m\}$, are real valued functions defined on Λ^n, and for each $G_i(S) > 0,$ for each$i \in \underline{p}$, for all $S \in \Lambda^n$.

The necessary preliminaries and definitions are already given in Sect. 3.7.

Dual Model I

We consider the following dual problem for (P):

$$\text{(DI)} \qquad \text{maximize } \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$$
$$\text{subject to}$$

$$Ŧ\left(S, T; \sum_{i=1}^{p} u_i \left[DF_i(T) - \lambda_i DG_i(T) \right] + \sum_{j=1}^{m} v_j DH_j(T) \right) \geqq 0, \forall S \in \Lambda^n, \qquad (5.20.1)$$

$$u_i \lfloor F_i(T) - \lambda_i G_i(T) \rfloor \geqq 0,\ i \in \underline{p}, \qquad (5.20.2)$$

$$v_j H_j(T) \geqq 0,\ j \in \underline{m}, \qquad (5.20.3)$$

$$T \in \Lambda^{\mathrm{n}},\ \lambda \in R_+,\ u \in U,\ v \in R_+^m,$$

where $Ŧ(S,T;\cdot) : L_1^n(X,\Lambda,\mu) \to R$ is a sublinear function. Throughout our discussion, we assume that the functions $F_i, G_i, i \in \underline{p}$, and $H_j, j \in \underline{m}$, are differentiable on Λ^n. We shall introduce along the way some additional notations. For stating our first duality theorem, we use the real-valued functions $\mathrm{A}_i(\cdot;\lambda,u)$ and $\mathrm{B}_j(\cdot,v)$ defined for fixed λ, u and v on Λ^n by

$$\mathrm{A}_i(\cdot;\lambda,u) = u_i \left[F_i(S) - \lambda_i G_i(S) \right],\ i \in \underline{p},$$

and

$$\mathrm{B}_j(\cdot,v) = v_j H_j(S),\ j \in \underline{m}.$$

Theorem 5.20.1. *Let S and (T, λ, u, v) be an arbitrary feasible solution of (P) and (DI), respectively, and assume that any one of the following sets of hypotheses is satisfied:*

(a) (i) $(\mathrm{A}_i(\cdot;\lambda,u),\mathrm{B}_j(\cdot,v))$ $\forall i \in \underline{p}$ *and* $j \in \underline{m}$, *are* $(\digamma, \alpha,\beta, \rho, \sigma, \theta)$*-V-pseudo-quasi-type-I at* T;
(ii) $\rho+\sigma \geqq 0$;

(b) (i) $(\mathrm{A}_i(\cdot;\lambda,u),\mathrm{B}_j(\cdot,v))$ $\forall i \in \underline{p}$ *and* $j \in \underline{m}$, *are* $(\digamma, \alpha,\beta, \rho, \sigma, \theta)$*-V-pseudo-prestrict-quasi-type-I at* T;
(ii) $\rho+\sigma > 0$;

(c) (i) $(\mathrm{A}_i(\cdot;\lambda,u),\mathrm{B}_j(\cdot,v))$ $\forall i \in \underline{p}$ *and* $j \in \underline{m}$, *are* $(\digamma, \alpha,\beta, \rho, \sigma, \theta)$*-V-prestrict-quasi-strict-pseudo-type-I at* T;
(ii) $\rho+\sigma \geqq 0$.

Then,

$$\phi(S) \equiv \left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \ldots, \frac{F_p(S)}{G_p(S)}\right) \nleq \lambda.$$

Proof. Let S be an arbitrary feasible solution of (P), then by the sublinearity of $\digamma$ and (5.20.1) it follows that

$$\digamma\left(S,T;\sum_{i=1}^{p} u_i\left[DF_i(T)-\lambda_i DG_i(T)\right]\right)+\digamma\left(S,T;\sum_{j=1}^{m} v_j DH_j(T)\right) \geqq 0. \quad (5.20.4)$$

(a) From (5.20.3) that $-v_j H_j(T) \leqq 0$, and hence,

$$-\sum_{j=1}^{m} \beta_j(S,T) v_j H_j(T) \leqq 0,$$

which by virtue of second part of (i) implies that

$$\digamma\left(S,T;\sum_{j=1}^{m} v_j DH_j(T)\right) \leqq -\sigma d^2(\theta(S,T)). \quad (5.20.5)$$

From (5.20.4) and (5.20.5), we see that

$$\digamma\left(S,T;\sum_{i=1}^{p} u_i\left[DF_i(T)-\lambda_i DG_i(T)\right]\right) \geqq \sigma d^2(\theta(S,T)) \geqq -\rho d^2(\theta(S,T)),$$

where the second inequality follows from (ii). By first part of (i), the last inequality implies that

$$\sum_{i=1}^{p} \alpha_i(S,T) u_i\left[F_i(S)-\lambda_i G_i(S)\right] \geqq \sum_{i=1}^{p} \alpha_i(S,T) u_i\left[F_i(T)-\lambda_i G_i(T)\right]$$

which in view of (5.20.2) becomes

$$\sum_{i=1}^{p} \alpha_i(S,T) u_i\left[F_i(S)-\lambda_i G_i(S)\right] \geqq 0. \quad (5.20.6)$$

Since $u_i \alpha_i (S,T) > 0$ for each $i \in \underline{p}$, (5.20.6) implies that $(F_1(S) - \lambda_1 G_1(S), \ldots, F_p(S) - \lambda_p G_p(S)) \nleq (0,\ldots,0)$, which in turn implies that

$$\phi(S) \equiv \left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \ldots, \frac{F_p(S)}{G_p(S)} \right) \nleq \lambda.$$

The proofs for part (b) and (c) are similar to that of part (a). □

Theorem 5.20.2. *(Strong Duality). Let S^* be a regular efficient solution of (P), let $\digamma(S,S^*;DF(S^*)) = \sum_{k=1}^{n} \left\langle D_k F(S^*), \chi_{S_k} - \chi_{S_k^*} \right\rangle$ for any differentiable function $F : \Lambda^n \to R$ and $S \in \Lambda^n$, and assume that ant one of the three sets of hypotheses specified in Theorem 5.20.1 holds for all feasible solutions of (DI). Then there exist $u^* \in U$ and $v^* \in R_+^m$ such that (S^*, u^*, v^*) is an efficient solution of (DI) and the objective values of (P) and (DI) are same.*

Proof. By Lemma 4.9.2, there exist $u^* \in U$ and $v^* \in R_+^m$ such that (S^*, u^*, v^*) is an feasible solution of (DI). That it is an efficient solution follows from Theorem 5.20.1. □

Dual Model II

We shall formulate a relatively more general parametric duality model than the Dual Model I by making use of the partitioning scheme introduced as follows:

Let $\{J_0, J_1, \ldots, J_q\}$ be a partition of the index set $\underline{m}$. Thus, $J_r \subset \underline{m}$ for each $r \in \{0,1,\ldots,q\}$, $J_r \cap J_s = \Phi$ for each $r,s \in \{0,1,\ldots,q\}$ with $r \neq s$, and $\bigcup_{r=0}^{q} J_r = \underline{m}$.

The duality model considered in this section is in the form:

(DII) maximize $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$

subject to

$$\digamma\left(S,T; \sum_{i=1}^{p} u_i \left[DF_i(T) - \lambda_i DG_i(T) \right] + \sum_{j=1}^{m} v_j DH_j(T) \right) \geqq 0, \forall S \in \Lambda^n, \quad (5.20.7)$$

$$u_i \left[F_i(T) - \lambda_i G_i(T) + \sum_{j \in J_0} v_j H_j(T) \right] \geqq 0,\ i \in \underline{p}, \quad (5.20.8)$$

$$\sum_{j \in J_t} v_j H_j(T) \geqq 0,\ t \in \underline{m} \quad (5.20.9)$$

$$T \in \Lambda^{\mathrm{n}}, \quad \lambda \in R_+^p, \quad u \in U, \quad v \in R_+^m,$$

where $\digamma(S,T;\cdot) : L_1^n(X,\Lambda,\mu) \to R$ is a sublinear function.

We will show that (DII) is a dual problem for (P) by establishing weak and strong duality theorems. In this section, we also use the notations

$$\Gamma_i(\cdot;\lambda,u,v)=u_i\left[F_i(S)-\lambda_i G_i(S)+\sum_{j\in J_0}v_jH_j(S)\right],i\in \underline{p}$$

and

$$\Delta_t(S,v)=\sum_{j\in J_t}v_jH_j(S),\ t\in\underline{m}.$$

Theorem 5.20.3. *(Weak Duality). Let S and* (T,λ,u,v) *be an arbitrary feasible solution of (P) and (DII), respectively. Assume that any one of the following three sets of hypotheses is satisfied:*

(a) (i) $(\Gamma_i(\cdot;\lambda,u,v),\ \Delta_j(\cdot,v))$ $\forall i\in\underline{p}$ *and* $j\in\underline{m}$, *are (*$\digamma$, α, β, ρ, σ, θ*)-V-pseudo-quasi-type-I at T;*
(ii) $\rho+\sigma\geqq 0$*;*
(b) (i) $(\Gamma_i(\cdot;\lambda,u,v),\ \Delta_j(\cdot,v))$ $\forall i\in\underline{p}$ *and* $j\in\underline{m}$, *are (*$\digamma$, α, β, ρ, σ, θ*)-V-pseudo-prestrict-quasi-type-I at T;*
(ii) $\rho+\sigma>0$*;*
(c) (i) $(\Gamma_i(\cdot;\lambda,u,v),\ \Delta_j(\cdot,v))$ $\forall i\in\underline{p}$ *and* $j\in\underline{m}$, *are (*$\digamma$, α, β, ρ, σ, θ*)-V-prestrict-quasi-strict-pseudo-type-I at* $\overline{T}$*;*
(ii) $\rho+\sigma\geqq 0$.

Then,

$$\phi(S)\equiv\left(\frac{F_1(S)}{G_1(S)},\frac{F_2(S)}{G_2(S)},\ldots,\frac{F_p(S)}{G_p(S)}\right)\nleq\lambda.$$

Proof. Let S be an arbitrary feasible solution of (P), then by the sublinearity of $\digamma$ and (5.20.7) it follows that

$$\digamma\left(S,T;\sum_{i=1}^{p}u_i\left[DF_i(T)-\lambda_i DG_i(T)\right]+\sum_{j\in J_0}v_jDH_j(T)\right)$$
$$+\digamma\left(S,T;\sum_{t=1}^{m}\sum_{j\in J_t}v_jDH_j(T)\right)\geqq 0. \tag{5.20.10}$$

(a) Since $v\geqq 0, S\in\Xi$ it follows from (5.20.9) that for each $t\in\underline{m}$:

$$-\sum_{t\in J_t}v_tH_t(T)=-\Delta_t(T,v)\leqq 0,$$

and

$$-\sum_{t=1}^{q}\beta_t(S,T)\Delta_t(T,v)\leqq 0,$$

which by virtue of second part of (i) implies that

$$\digamma\left(S,T;\sum_{t=1}^{q}\sum_{j\in J_t}v_jDH_j(T)\right)\leqq-\sigma d^2(\theta(S,T)). \tag{5.20.11}$$

From (5.20.10) and (5.20.11), we see that

$$\digamma\left(S,T;\sum_{i=1}^{p}u_i\left[DF_i\left(T\right)-\lambda_i DG_i\left(T\right)\right]+\sum_{j\in J_0}v_j DH_j\left(T\right)\right)$$
$$\geqq \sigma d^2\left(\theta\left(S,T\right)\right)\geqq -\rho d^2\left(\theta\left(S,T\right)\right),$$

where the second inequality follows from (ii). By virtue of the first part of hypothesis (i), the above inequality implies that

$$\sum_{i=1}^{p}\alpha_i\left(S,T\right)\Gamma_i\left(\mathrm{S},\lambda,u,v\right)\geqq\sum_{i=1}^{p}\alpha_i\left(S,T\right)\Gamma_i\left(\mathrm{T},\lambda,u,v\right). \tag{5.20.12}$$

Since $\alpha_i\left(S,T\right)>0,\ u_i\geqq 0\ \forall\ i\ \in \underline{p}$, and (5.20.8) holds, we deduce from (5.20.12) that

$$\sum_{i=1}^{p}\alpha_i\left(S,T\right)\Gamma_i\left(\mathrm{S},\lambda,u,v\right)\geqq 0,$$

which simplifies to

$$\sum_{i=1}^{p}\alpha_i\left(S,T\right)u_i\left[F_i\left(S\right)-\lambda_i G_i\left(S\right)\right]\geqq 0.$$

which is precisely (5.20.6). Therefore, the rest of the proof is identical to that of Part (a) of Theorem 5.20.1.

The proofs of parts (b) and (c) are similar to that of part (a). □

Remark 5.20.1. Note that Theorem 5.20.3 contains a number of special cases that can easily be identified by appropriate choices of the partitioning sets $J_0,J_1,\ldots,J_q$.

Theorem 5.20.4. *(Strong Duality). Let S^* be a regular efficient solution of (P), let $\digamma(S,S^*;DF\left(S^*\right))=\sum_{k=1}^{n}\left\langle D_kF\left(S^*\right),\chi_{S_k}-\chi_{S_k^*}\right\rangle$ for any differentiable function $F:\Lambda^n\rightarrow R$ and $S\in\Lambda^n$. Assume that ant one of the three sets of hypotheses specified in Theorem 5.20.3 holds for all feasible solutions of (DII). Then there exist $u^*\in U$ and $v^*\in R_+^m$ such that $\left(S^*,u^*,v^*\right)$ is an efficient solution of (DII) and the objective values of (P) and (DII) are same.*

Proof. By Lemma 4.9.2, there exist $u^*\in U$ and $v^*\in R_+^m$ such that $\left(S^*,u^*,v^*\right)$ is an feasible solution of (DII). That it is an efficient solution follows from Theorem 5.20.3. □

Dual Model III

We present another general parametric duality model for (P). It is again based on the partitioning scheme employed in the previous section. The dual model can be given as follows:

(DIII) maximize $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_p)$
subject to

$$\mathcal{F}\left(S, T; \sum_{i=1}^{p} u_i \left[DF_i(T) - \lambda_i DG_i(T)\right] + \sum_{j=1}^{m} v_j DH_j(T)\right) \geqq 0, \forall S \in \Lambda^n, \tag{5.20.13}$$

$$F_i(T) - \lambda_i G_i(T) \geqq 0, i \in \underline{p}, \tag{5.20.14}$$

$$\sum_{j \in J_t} v_j H_j(T) \geqq 0,\ t \in \underline{m} \cup \{0\}, \tag{5.20.15}$$

$$T \in \Lambda^{\mathrm{n}},\ \lambda \in R_+^p,\ u \in U, v \in R_+^m,$$

where $\mathcal{F}(S, T; \cdot) : L_1^n(X, \Lambda, \mu) \to R$ is a sublinear function.

We will show that (DIII) is a dual problem for (P) by establishing weak and strong duality theorems. Let $\{I_0, I_1, \ldots, I_k\}$ be partitions of $\underline{p}$ such that $K = \{0, 1, \ldots, k\} \subset Q = \{0, 1, \ldots, q\}, k < q$, and let the function $\Theta_t(\cdot, \overline{\lambda, u}, v) : \Lambda^n \to R$ be defined, for fixed λ, u and v by

$$\Theta_t(S, \lambda, u, v) = \sum_{i \in I_t} u_i \left[F_i(S) - \lambda_i G_i(S)\right] + \sum_{j \in J_t} v_j H_j(S), t \in K.$$

and $\Delta_t(S, v) = \sum_{j \in J_t} v_j H_j(S), t \in \underline{m}$.

Theorem 5.20.5. *(Weak Duality). Let S and (T, λ, u, v) be an arbitrary feasible solution of (P) and (DIII), respectively. Assume that any one of the following three sets of hypotheses is satisfied:*

(a) (i) $(\Theta_t(\cdot, \lambda, u, v), \Delta_j(\cdot, v))$ $\forall t \in K$ and $j \in \underline{m}$, are ($\mathcal{F}$, α, β, ρ, σ, θ)-V-strict pseudo-quasi-type-I at T;
(ii) $\rho + \sigma \geqq 0$;
(b) (i) $(\Theta_t(\cdot, \lambda, u, v), \Delta_j(\cdot, v))$ $\forall t \in K$ and $j \in \underline{m}$, are ($\mathcal{F}$, α, β, ρ, σ, θ)-V-pseudo-prestrict-quasi-type-I at T;
(ii) $\rho + \sigma > 0$;
(c) (i) $(\Theta_t(\cdot, \lambda, u, v), \Delta_j(\cdot, v))$ $\forall t \in K$ and $j \in \underline{m}$, are ($\mathcal{F}$, α, β, ρ, σ, θ)-V-prestrict-quasi-strict-pseudo-type-I at T;
(ii) $\rho + \sigma \geqq 0$.

Then,

$$\phi(S) \equiv \left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \ldots, \frac{F_p(S)}{G_p(S)}\right) \nleq \lambda.$$

Proof. Suppose to the contrary that $\phi(S) \le \lambda$. This implies that $F_i(S) - \lambda_i G_i(S) \leqq 0$, $\forall i \in \underline{p}$, with strict inequality holding for at least one $l \in \underline{p}$. From these inequalities, non-negativity of v, primal feasibility of S, and (5.20.14) it is easily seen that for each $t \in K$,

$$\begin{aligned}\Theta_t(S,\lambda,u,v) &= \sum_{i\in I_t} u_i\left[F_i(S)-\lambda_i G_i(S)\right]+\sum_{j\in J_t} v_j H_j(S)\\ &\leqq \sum_{i\in I_t} u_i\left[F_i(S)-\lambda_i G_i(S)\right]\\ &\leqq 0\\ &= \sum_{i\in I_t} u_i\left[F_i(S)-\lambda_i G_i(S)\right]+\sum_{j\in J_t} v_j H_j(S)=\Theta_t(S,\lambda,u,v)\end{aligned}$$

and hence,

$$\sum_{t\in K}\alpha_t(S,T)\Theta_t(S,\lambda,u,v)<\sum_{t\in K}\alpha_t(S,T)\Theta_t(T,\lambda,u,v),$$

which in view of first part of the hypotheses (i) implies that

$$\digamma\left(S,T;\sum_{i=1}^{p}u_i\left[DF_i(T)-\lambda_i DG_i(T)\right]+\sum_{t\in K}\sum_{j\in J_t}v_j DH_j(T)\right)<-\rho d^2(\theta(S,T)). \tag{5.20.16}$$

As for each $t\in M\backslash K, -\sum_{t\in M\backslash K}\beta_t(S,T)\Delta_t(S,v)\leqq 0$, and hence, the second part of the hypotheses a(i) implies that

$$\digamma\left(S,T;\sum_{t\in M\backslash K}\sum_{j\in J_t}v_j DH_j(T)\right)\leqq -\sigma d^2(\theta(S,T)). \tag{5.20.17}$$

Now from (5.20.16), (5.20.17), a(ii) and the sublinearity, we get

$$\digamma\left(S,T;\sum_{i=1}^{p}u_i[DF_i(T)-\lambda_i DG_i(T)]+\sum_{j=1}^{m}v_j DH_j(T)\right)<-(\rho+\sigma)d^2(\theta(S,T))<0,$$

which contradicts (5.20.13). Hence,

$$\phi(S)\equiv\left(\frac{F_1(S)}{G_1(S)},\frac{F_2(S)}{G_2(S)},\dots,\frac{F_p(S)}{G_p(S)}\right)\nleq\lambda.$$

The proofs of parts (b) and (c) are similar to that of part (a). □

Theorem 5.20.6. *(Strong Duality). Let S^* be a regular efficient solution of (P), let $\digamma(S,S^*;DF(S^*))=\sum_{k=1}^{n}\left\langle D_kF(S^*),\chi_{S_k}-\chi_{S_k^*}\right\rangle$ for any differentiable function $F:\Lambda^n\to R$ and $S\in\Lambda^n$. Assume that any one of the three sets of hypotheses specified in Theorem 5.20.5 holds for all feasible solutions of (DIII). Then there exist $u^*\in U$ and $v^*\in R_+^m$ such that (S^*u^*,v^*) is an efficient solution of (DIII) and the objective values of (P) and (DIII) are same.*

Proof. By Lemma 4.9.2, there exist $u^* \in U$ and $v^* \in R_+^m$ such that (S^*, u^*, v^*) is an feasible solution of (DIII). That it is an efficient solution follows from Theorem 5.20.5. □

Dual Model IV

We will investigate the following duality model for (P), which may be written as the semi-parametric counterpart of (DI):

$$\text{(DIV)} \qquad \text{maximize} \left(\frac{F_1(T)}{G_1(T)}, \ldots, \frac{F_p(T)}{G_p(T)} \right)$$

subject to

$$\digamma\left(S, T; \sum_{i=1}^{p} u_i \left[G_i(T) DF_i(T) - F_i(T) DG_i(T) \right] + \sum_{j=1}^{m} v_j DH_j(T) \right) \geqq 0, \forall S \in \Lambda^n, \tag{5.20.18}$$

$$v_j H_j(T) \geqq 0,\ t \in \underline{m}, \tag{5.20.19}$$

$$T \in \Lambda^{\mathrm{n}},\ u \in U, v \in R_+^m,$$

where $\digamma(S,T;\cdot) : L_1^n(X,\Lambda,\mu) \to R$ is a sublinear function. In the remaining part of this section, we assume that $G_i(T) > 0$ and $F_i(T) \geqq 0\ i \in \underline{p}$, for all T and u such that (T,u,v) is a feasible solution of the dual problem under consideration. In addition, in the statements and proofs of theorems to follow in this section, we use the notations, $\mathrm{E}_i(\cdot,T,u)$, $\mathrm{B}_j(\cdot,v)$ and $L_i(\cdot,T,u,v)$ defined for fixed S, u, and v on Λ^n by

$$\mathrm{E}_i(S,T,u) = u_i \left[G_i(T) F_i(S) - F_i(T) G_i(S) \right] \forall i \in \underline{p},$$

$$\mathrm{B}_i(S,v) = v_j H_j(S), j \in \underline{m},$$

and

$$L_i(S,T,u,v) = u_i \left[G_i(T) F_i(S) - F_i(T) G_i(S) + \sum_{j \in J_0} v_j H_j(S) \right],\ i \in \underline{p}.$$

Now we can establish weak, strong and strict converse duality theorem for (P) and (DIV).

Theorem 5.20.7. *(Weak Duality). Let S and (T,u,v) be an arbitrary feasible solution for (P) and (DIV), respectively. Assume that any one of the following three sets of hypotheses is satisfied:*

(a) (i) $(\mathrm{E}_i, B_j), \forall i \in \underline{p}$ and $\forall j \in \underline{m}$, *are* ($\digamma$, α, β, ρ, σ, θ)*-V-pseudo-quasi-type-I at T;*

(ii) $\rho + \sigma \geqq 0$*;*

(b) (i) $(\mathrm{E}_i(\cdot,T,u),\mathrm{B}_j(\cdot,v))\,\forall i\in\underline{p}$ *and* $j\in\underline{m}$, *are* $(\mathcal{F},\alpha,\beta,\rho,\sigma,\theta)$*-V-pseudo-prestrict-quasi-type-I at* T*;*

(ii) $\rho+\sigma>0$*;*

(c) (i) $(\mathrm{E}_i(\cdot,T,u),\mathrm{B}_j(\cdot,v))\,\forall i\in\underline{p}$ *and* $j\in\underline{m}$, *are* $(\mathcal{F},\alpha,\beta,\rho,\sigma,\theta)$*-V-prestrict-quasi-strict-pseudo-type-I at* T*;*

(ii) $\rho+\sigma\geqq 0$.

Then,

$$\left(\frac{F_1(S)}{G_1(S)},\frac{F_2(S)}{G_2(S)},\dots,\frac{F_p(S)}{G_p(S)}\right)\not\leq\left(\frac{F_1(T)}{G_1(T)},\frac{F_2(T)}{G_2(T)},\dots,\frac{F_p(T)}{G_p(T)}\right).$$

Proof. Let S be an arbitrary feasible solution of (P), then by the sublinearity of $\mathcal{F}$ and (6.1) it follows that

$$\left(S,T;\sum_{i=1}^{p}u_i\left[G_i(T)DF_i(T)-F_i(T)DG_i(T)\right]\right)+\mathcal{F}\left(S,T;\sum_{j=1}^{m}v_jDH_j(T)\right)\geqq 0. \tag{5.20.20}$$

Following as in the proof of Theorem 5.20.1, from the second part of the assumption a(i) and (5.20.2), we get

$$\mathcal{F}\left(S,T;\sum_{i=1}^{p}u_i\left[G_i(T)DF_i(T)-F_i(T)DG_i(T)\right]\right)\geqq-\rho\,d^2(\theta(S,T)),$$

which in the light of the hypotheses implies that

$$\sum_{i=1}^{p}\alpha_i(S,T)u_i\left[G_i(T)F_i(S)-F_i(T)G_i(S)\right]$$
$$\geqq\sum_{i=1}^{p}\alpha_i(S,T)u_i\left[G_i(T)F_i(T)-F_i(T)G_i(T)\right]=0. \tag{5.20.21}$$

Since $\alpha_i(S,T)u_i>0$ for each $i\in\underline{p}$, (5.20.21) implies that

$$(G_1(T)\mathrm{F}_1(S)-\mathrm{F}_1(T)G_1(S),\dots,G_p(T)\mathrm{F}_p(S)-\mathrm{F}_p(T)G_p(S))\not\leq(0,\dots,0),$$

which in turn implies that

$$\left(\frac{F_1(S)}{G_1(S)},\frac{F_2(S)}{G_2(S)},\dots,\frac{F_p(S)}{G_p(S)}\right)\not\leq\left(\frac{F_1(T)}{G_1(T)},\frac{F_2(T)}{G_2(T)},\dots,\frac{F_p(T)}{G_p(T)}\right).$$

The proofs of parts (b) and (c) are similar to that of part (a). □

Theorem 5.20.8. *(Strong Duality). Let* S^* *be a regular efficient solution of (P), let* $\mathcal{F}(S,S^*;DF(S^*))=\sum_{k=1}^{n}\left\langle D_kF(S^*),\chi_{S_k}-\chi_{S_k^*}\right\rangle$ *for any differentiable function* $F:\Lambda^n\to R$ *and* $S\in\Lambda^n$*. Assume that any one of the three sets of hypotheses specified in Theorem 5.20.7 holds for all feasible solutions of (DIV). Then there exist* $u^*\in U$

and $v^* \in R_+^m$ such that (S^*, u^*, v^*) is an efficient solution of (DIV) and the objective values of (P) and (DIV) are same.

Proof. By Lemma 4.9.3, there exist $u^* \in U$ and $v^* \in R_+^m$ such that (S^*, u^*, v^*) is an feasible solution of (DIV). That it is an efficient solution follows from Theorem 5.20.7. □

Dual Model V

We will present a more general semi-parametric duality model for (P):

$$\text{(DV)} \quad \text{maximize} \left(\frac{F_1(T) + \sum_{j \in J_0} v_j H_j(T)}{G_1(T)}, \ldots, \frac{F_p(T) + \sum_{j \in J_0} v_j H_j(T)}{G_p(T)} \right)$$

subject to

$$\digamma\left(S, T; \sum_{i=1}^{p} u_i \left[G_i(T) \left[DF_i(T) + \sum_{j \in J_0} v_j DH_j(T) \right] - \left[F_i(T) + \Delta_0(T, v) \right] DG_i(T) \right] + \sum_{j \in \underline{m} \setminus J_0} v_j DH_j(T) \right) \geqq 0, \forall S \in \Lambda^n, \tag{5.20.22}$$

$$\sum_{j \in J_t} v_j H_j(T) \geqq 0, t \in \underline{m} \cup \{0\}, \tag{5.20.23}$$

$$T \in \Lambda^n, \ u \in U, v \in R_+^m,$$

where $\digamma(S, T; \cdot) : L_1^n(X, \Lambda, \mu) \to R$ is a sublinear function. In addition, in the statements and proofs of theorems to follow in this section, we use the following notation defined for fixed S, u, and v on Λ^n by:

$$\Pi_i(S,T,u,v) = u_i \left[G_i(T) \left\{ F_i(S) + \sum_{j \in J_0} v_j DH_j(S) \right\} - \{F_i(T) + \Delta_0(T,v)\} G_i(S) \right], \forall_i \in \underline{p}.$$

Theorem 5.20.9. *(Weak Duality). Let S and (T, u, v) be an arbitrary feasible solution for (P) and (DV), respectively. Assume that any one of the following three sets of hypotheses is satisfied:*

(a) (i) $(\Pi_i(\cdot, T, v), \Delta_j(\cdot, v)), \forall i \in \underline{p}$, and $\forall j \in \underline{m}$ are $(\digamma, \alpha, \beta, \rho, \sigma, \theta)$-V-pseudo-quasi-type-I at T;
(ii) $\rho + \sigma \geqq 0$;

(b) (i) $(\Pi_i(\cdot,T,v),\Delta_j(\cdot,v)), \forall i \in \underline{p}$ and $\forall j \in \underline{m}$ *are* $(\digamma, \alpha, \beta, \rho, \sigma, \theta)$*-V-pseudo-prestrict-quasi-type-I at* T*;*

(ii) $\rho+\sigma>0$*;*

(c) (i) $(\Pi_i(\cdot,T,v),\Delta_j(\cdot,v)), \forall i \in \underline{p}$, and $\forall j \in \underline{m}$ *are* $(\digamma, \alpha, \beta, \rho, \sigma, \theta)$*-V-prestrict-quasi-strict-pseudo-type-I at* T*;*

(ii) $\rho+\sigma \geqq 0$.

Then,

$$\left(\frac{F_1(S)}{G_1(S)},\frac{F_2(S)}{G_2(S)},\ldots,\frac{F_p(S)}{G_p(S)}\right) \nleq \left(\frac{F_1(T)+\sum_{j\in J_0} v_jH_j(T)}{G_1(T)},\frac{F_2(T)+\sum_{j\in J_0} v_jH_j(T)}{G_2(T)},\ldots,\frac{F_p(T)+\sum_{j\in J_0} v_jH_j(T)}{G_p(T)}\right).$$

Proof. (a) Let S be an arbitrary feasible solution of (P), then by the sublinearity of $\digamma$ and (5.20.22), it follows that

$$\digamma\left(S,T;\sum_{i=1}^{p}u_i\left[G_i(T)\left\{DF_i(T)+\sum_{j\in J_0}v_jDH_j(T)\right\}-\{F_i(T)+\Delta_0(T,v)\}DG_i(T)\right]\right) + \digamma\left(S,T;\sum_{t=1}^{m}\sum_{j\in J_t}v_jDH_j(T)\right) \geqq 0. \tag{5.20.24}$$

Following as in the proof of Theorem 5.20.1, from the second part of the assumption a(i) and (5.20.24), we get

$$\digamma\left(S,T;\sum_{i=1}^{p}u_i\left[G_i(T)\left\{DF_i(T)+\sum_{j\in J_0}v_jDH_j(T)\right\}-\{F_i(T)+\Delta_0(T,v)\}DG_i(T)\right]\right) \geqq -\rho\, d^2(\theta(S,T)),$$

which in the light of the hypotheses implies that

$$\sum_{i=1}^{p}\alpha_i(S,T)\Pi_i(S,T,u,v) \geqq \sum_{i=1}^{p}\alpha_i(S,T)\Pi_i(T,T,u,v)=0. \tag{5.20.25}$$

The equality holds due to the fact that $\Pi_i(T,T,u,v)=0$. Since $\alpha_i(S,T)\,u_i>0$ for each $i\in \underline{p,}$ (7.4) implies that

$$G_1(T)F_1(S)-[F_1(T)+\Delta_0(T,v)]G_1(S),\ldots,G_p(T)F_p(S)-\left[F_p(T)+\Delta_0(T,v)\right] G_p(S) \nleq (0,\ldots,0),$$

which in turn implies that

$$\left(\frac{F_1(S)}{G_1(S)},\frac{F_2(S)}{G_2(S)},\ldots,\frac{F_p(S)}{G_p(S)}\right) \nleq \left(\frac{F_1(T)+\sum_{j\in J_0} v_jH_j(T)}{G_1(T)},\frac{F_2(T)+\sum_{j\in J_0} v_jH_j(T)}{G_2(T)},\ldots,\right.$$
$$\left.\frac{F_p(T)+\sum_{j\in J_0} v_jH_j(T)}{G_p(T)}\right).$$

The proofs of parts (b) and (c) are similar to that of part (a). □

Theorem 5.20.10. *(Strong Duality). Let S^* be a regular efficient solution of (P) and let $\digamma(S,S^*;DF(S^*)) = \sum_{k=1}^{n}\left\langle D_kF(S^*),\chi_{S_k}-\chi_{S_k^*}\right\rangle$ for any differentiable function $F:\Lambda^n \to R$ and $S\in\Lambda^n$. Assume that ant one of the three sets of hypotheses specified in Theorem 5.20.9 holds for all feasible solutions of (DV). Then there exist $u^*\in U$ and $v^*\in R_+^m$ such that (S^*,u^*,v^*) is an efficient solution of (DV) and the objective values of (P) and (DV) are same.*

Proof. By Lemma 4.9.3, there exist $u^*\in U$ and $v^*\in R_+^m$ such that (S^*,u^*,v^*) is an feasible solution of (DV), using the arguments as in the proof of Theorem 9.2 from Zalmai (2002). That (S^*,u^*,v^*) is an efficient solution follows from Theorem 5.20.9. □

Dual Model VI

Finally, we will discuss another general duality model for (P) which may be viewed as the semi-parametric version of (DIII). It can be stated as follows:

$$\text{(DVI)} \quad \text{maximize}\left(\frac{F_1(T)}{G_1(T)},\frac{F_p(T)}{G_p(T)}\right)$$

subject to

$$\digamma\left(S,T;\sum_{i=1}^{p}u_i\left[G_i(T)DF_i(T)-F_i(T)DG_i(T)\right]+\sum_{j=1}^{m}v_jDH_j(T)\right)\geqq 0,\ \forall\, S\in\Lambda^n, \tag{5.20.26}$$

$$\sum_{j\in J_t} v_jH_j(T)\geqq 0,\ t\in \underline{m}\cup\{0\},T\in\Lambda^{\mathrm{n}}, \tag{5.20.27}$$

$$u\in U, v\in R_+^m,$$

where $\digamma(S,T;\cdot):L_1^n(X,\Lambda,\mu)\to R$ is a sublinear function.

We shall show that (DVI) is a dual problem to (P) by proving weak and strong duality theorems.

Theorem 5.20.11. *(Weak Duality). Let S and* (T, u, v) *be an arbitrary feasible solution for (P) and (DVI), respectively. Assume that any one of the following three sets of hypotheses is satisfied:*

(d) (i) $(\Pi_i(\cdot, T, v), \Delta_j(\cdot, v)), \forall i \in \{1, \ldots, k\}$ *and* $\forall j \in \{k+1, \ldots, m\}$ *are (Ŧ, α, β, ρ, σ, θ)-V-pseudo-quasi-type-I at T;*

(ii) $\rho + \sigma \geqq 0$*;*

(e) (i) $(\Pi_i(\cdot, T, v), \Delta_j(\cdot, v)), \forall i \in \{1, \ldots, k\}$ *and* $\forall j \in \{k+1, \ldots, m\}$ *are (Ŧ, α, β, ρ, σ, θ)-V-pseudo-prestrict-quasi-type-I at T;*

(ii) $\rho + \sigma > 0$*;*

(f) (i) $(\Pi_i(\cdot, T, v), \Delta_j(\cdot, v)), \forall i \in \{1, \ldots, k\}$ *and* $\forall j \in \{k+1, \ldots, m\}$ *are (Ŧ, α, β, ρ, σ, θ)-V-prestrict-quasi-strict-pseudo-type-I at T;*

(ii) $\rho + \sigma \geqq 0$.

Then,

$$\left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \ldots, \frac{F_p(S)}{G_p(S)}\right) \nleq \left(\frac{F_1(T)}{G_1(T)}, \frac{F_2(T)}{G_2(T)}, \ldots, \frac{F_p(T)}{G_p(T)}\right).$$

Proof. The proof can be done following the discussions above in this section and the proof of the Theorem 10.1 in Zalmai (2002). □

Theorem 5.20.12. *(Strong Duality). Let* S^* *be a regular efficient solution of (P) and let* $Ŧ(S, S^*; DF(S^*)) = \sum_{k=1}^{n} \left\langle D_k F(S^*), \chi_{S_k} - \chi_{S_k^*} \right\rangle$ *for any differentiable function* $F: \Lambda^n \to R$ *and* $S \in \Lambda^n$*. Assume that ant one of the three sets of hypotheses specified in Theorem 5.20.11 holds for all feasible solutions of (DVI). Then there exist* $u^* \in U$ *and* $v^* \in R_+^m$ *such that* (S^*, u^*, v^*) *is an efficient solution of (DVI) and the objective values of (P) and (DVI) are same.*

Proof. By Lemma 4.9.3, there exist $u^* \in U$ and $v^* \in R_+^m$ such that (S^*, u^*, v^*) is an feasible solution of (DVI). That it is an efficient solution follows from Theorem 5.20.11. □

Theorems 5.20.1–5.20.12 can be further extended to the class of functions introduced by Hachimi and Aghezzaf (2004). Furthermore, it might be interesting to see if the work can be extended to the class of (p, r)-invex functions introduced by Antczak (2001). Moreover, the second order and higher order duality results for the class of n-set functions are still open.

Chapter 6
Second and Higher Order Duality

The purpose of this chapter is to show that the duality theory, which has evolved with the traditional (first order) duals and convexity assumptions, can be developed further in two ways: one is in a more general setting of a modified dual (namely, a second order and a higher order dual), the other is in the generalized convexity. The benefit of doing this not only that results obtained by these kinds of duals under generalized convexity extend some well-known classical results of (first order) duality for convex optimization problems, but also that higher order duality can provide a lower bound to the infimum of a primal optimization problem when it is difficult to find a feasible solution for the first order dual.

6.1 Second Order Duality for Nonlinear Optimization Problems

Consider the nonlinear optimization problem

$$\text{(P)} \qquad \text{minimize } f(x) \quad \text{subject to } g(x) \geq 0,$$

where f and g are twice differentiable functions from R^n to R and R^m, respectively.
Mangasarian (1975) formulated the following second order dual to (P).

$$\begin{aligned} \text{(MD2)} \qquad & \text{maximize } f(u) - y^T g(u) - \frac{1}{2} p^T \nabla^2 \left[f(u) - y^T g(u)\right] p \\ & \text{subject to } \nabla \left[f(u) - y^T g(u)\right] + \nabla^2 \left[f(u) - y^T g(u)\right] = 0, \\ & \qquad y \geq 0. \end{aligned}$$

Mangasarian (1975) established duality theorems under somewhat complicated assumptions. Mond (1974a) gave simpler conditions than those of Mangasarian. A different form of second order duality was given by Mond and Weir (1981–1983). Following Mond and Weir (1981–1983), Egudo and Hanson (1993) extended the work of Mond (1974a) by defining a class of functions called second order invexity. Subsequently, Hanson (1993) defined second order type-I functions and Mishra (1997a) defined second order pseudo-type-I, second order quasi-type-I, second order quasi-pseudo-type-I and second order pseudo-quasi-type-I functions. Second order duality results for several nonlinear optimization problems were obtained by Mishra (1997a). Mond (1974b) considered the following class of nondifferentiable optimization problem:

$$\text{(P)} \qquad \text{minimize } f(x) + (x^T Bx)^{1/2}$$
$$\text{subject to } g(x) \geq 0, \tag{6.1.1}$$

where f and g are differentiable functions from R^n to R and R^m, respectively, and B is an $n \times n$ symmetric positive semidefinite matrix.

Zhang and Mond (1997) introduced a generalized dual to the problem (P) and established duality results under pseudo-invexity and quasi-invexity of the objective and constraint functions involved in the problem (P) and further extended these results to the second-order case.

In this section, we consider the Zhang and Mond (1997) type general dual to (P) and establish the duality theorems under the assumption of type I functions and its generalizations. A general second-order dual to (P) will also be presented as an extension of the dual studied in Mishra (1997a). These results extend the work of Zhang and Mond (1997) to the type I functions and its generalizations.

Let

$$Z_0 = \{z : z^T \nabla g_i(x_0) \geq 0\, (\forall i \in Q_0) \quad \textit{and} \quad z^T \nabla f(x_0) + z^T Bx_0/(x_0^T Bx_0)^{1/2} < 0$$
$$\textit{if}\, x_0^T Bx_0 > 0, z^T \nabla f(x_0) + (z^T Bz)^{1/2} < 0\, \textit{if}\, x_0^T Bx_0 = 0\},$$

where $Q_0 = \{i : g_i(x_0) = 0\}$. Mond (1974b) gave the following necessary conditions for x_0 to be an optimal solution to (P).

Lemma 6.1.1. *If x_0 is an optimal solution of (P) and the corresponding set Z_0 is empty, then there exist $y \in R^m$, $y \geq 0$ and $w \in R^n$ such that*

$$y^T g(x_0) = 0,\, \nabla y^T g(x_0) = \nabla f(x_0) + Bw,\, w^T Bw \leq 1,\, (x_0^T Bx_0)^{1/2} = x_0^T Bw.$$

We shall make use of the generalized Schwarz inequality from Riesz and Nagy (1955):

$$(x^T Bw) \leq (x^T Bx)^{1/2} (w^T Bw)^{1/2}. \tag{6.1.2}$$

Note that in (6.1.2) equality holds if, for $\lambda \geq 0$, $Bx = \lambda Bw$.

Now we present a more general nondifferentiable second-order dual to (P).

Zhang and Mond (1997) presented the second-order dual to (P) as the following problem:

$$\text{(2GD)}\quad \text{maximize}\quad f(u) - \sum_{i\in I_0} y_i g_i(u) + u^T Bw - \frac{1}{2} p^T \left[\nabla^2 f(u) - \nabla^2 \sum_{i\in I_0} y_i g_i(u)\right] p$$

$$\text{subject to}\quad \nabla f(u) - \nabla y^T g(u) + Bw + [\nabla^2 f(u)]p - [\nabla^2 y^T g(u)]p = 0,$$

$$\sum_{i\in I_\alpha} y_i g_i(u) - \frac{1}{2} p^T [\nabla^2 \sum_{i\in I_\alpha} y_i g_i(u)]p \le 0, \quad \alpha = 1,2,\ldots,s,$$

$$w^T Bw \le 1,$$

$$y \ge 0.$$

By introducing two additional vectors q and $r \in R^n$, as in Hanson (1993) and Mishra (1997a), we formulate the following second-order dual:

$$\text{(2MGD)}\quad \text{maximize}\, f(u) - \sum_{i\in I_0} y_i g_i(u) + u^T Bw - \frac{1}{2} q^T \left[\nabla^2 f(u) - \nabla^2 \sum_{i\in I_0} y_i g_i(u)\right] r$$

$$\text{subject to}\ \nabla f(u) - \nabla y^T g(u) + Bw + [\nabla^2 f(u)]p - [\nabla^2 y^T g(u)]p = 0, \qquad (6.1.3)$$

$$\sum_{i\in I_\alpha} y_i g_i(u) - \frac{1}{2} q^T [\nabla^2 \sum_{i\in I_\alpha} y_i g_i(u)]r \le 0,\ \alpha = 1,2,\ldots,s,$$

$$w^T Bw \le 1,$$

$$y \ge 0.$$

Let $K_1 = \{u/(u,y,w,p,q,r)$ be the set of all feasible solutions for (2MGD)$\}$.

Definition 6.1.1. *(Mishra 1997a). For $i = 1,2,\ldots,m$, $(f,-g_i)$ is said to be second-order pseudo-quasi type I at $u \in K_1$ with respect to the functions $\eta(x,u)$, $p(x,u)$, $q(x,u)$ and $r(x,u)$ if for all $x \in K_1$*

$$\eta(x,u)^T \lfloor \nabla f(u) + [\nabla^2 f(u)]p(x,u) \rfloor \ge 0$$

$$\Rightarrow\ f(x) \ge f(u) - \frac{1}{2} q(x,u)^T [\nabla^2 f(u)]r(x,u),$$

and

$$g_i(u) \le \frac{1}{2} q(x,u)^T [\nabla^2 g_i(u)]r(x,u)$$

$$\Rightarrow \eta(x,u)^T \left[\nabla g_i(u) + [\nabla^2 g_i(u)]p(x,u)\right] \ge 0,\ i = 1,2,\ldots,m.$$

Theorem 6.1.1. *(Weak Duality). Let x be feasible for (P) and (u,y,w,p,q,r) be feasible for (2MGD). If* $(f(.) - \sum_{i\in I_0} y_i g_i(.) + (.)^T Bw, -\sum_{i\in I_\alpha} y_i g_i(.))$, $\alpha = 1,2,\ldots,s$, *is second-order pseudo-quasi type I at u with respect to* η, p, q, *and r, then*

$$\inf(\mathrm{P}) \geq \sup(\mathrm{2MGD}).$$

Proof. Since x is feasible for (P) and (u,y,w,p,q,r) is feasible for (2MGD), from the second part of the hypothesis, we have

$$\eta(x,u)^T \left[\nabla \sum_{i\in I_\alpha} y_i g_i(u) + \left[\nabla^2 \sum_{i\in I_\alpha} y_i g_i(u)\right] p\right] \geq 0,\ \alpha = 1,2,\ldots,s.$$

Hence, $$\eta(x,u)^T \left[\nabla \sum_{i\in M\backslash I_0} y_i g_i(u) + \left[\nabla^2 \sum_{i\in M\backslash I_0} y_i g_i(u)\right] p\right] \geq 0. \tag{6.1.4}$$

From (6.1.4) and (6.1.3), we have

$$\eta(x,u)^T \left[\nabla f(u) + \left[\nabla^2 f(u)\right] p - \nabla \sum_{i\in I_0} y_i g_i(u) - \left[\nabla^2 \sum_{i\in I_0} y_i g_i(u)\right] p + Bw\right] \geq 0.$$

By the first part of the hypothesis, we have

$$\begin{aligned} &f(x) - \sum_{i\in I_0} y_i g_i(x) + x^T Bw \\ &\geq f(u) - \sum_{i\in I_0} y_i g_i(u) + u^T Bw - \frac{1}{2} q^T \nabla^2 \left[f(u) - \sum_{i\in I_0} y_i g_i(u) + u^T Bw\right] r. \end{aligned}$$

Thus, from $y \geq 0$, $g(x) \geq 0$, we have

$$\begin{aligned} f(x) + x^T Bw \geq{}& f(u) - \sum_{i\in I_0} y_i g_i(u) + u^T Bw \\ &- \frac{1}{2} q^T \nabla^2 \left[f(u) - \sum_{i\in I_0} y_i g_i(u) + u^T Bw\right] r. \end{aligned} \tag{6.1.5}$$

Since $w^T Bw \leq 1$, it follows from the generalized Schwarz inequality, (6.1.2) and (6.1.5) that

$$f(x) + (x^T Bx)^{1/2} \geq f(u) - \sum_{i\in I_0} y_i g_i(u) + u^T Bw - \frac{1}{2} q^T \nabla^2 \left[f(u) - \sum_{i\in I_0} y_i g_i(u)\right] r.$$

□

Theorem 6.1.2. *(Strong Duality). If* x_0 *is an optimal solution of (P) and the corresponding set* Z_0 *is empty, then there exist* $y \in R^m$ *and* $w \in R^n$ *such that* $(x_0, y, w, p = q = r = 0)$ *is feasible for (2MGD) and the objective values of (P) and (2MGD) are* *equal. If for any feasible (u,y,w,p,q,r),* $\left(f(.) - \sum_{i \in I_0} y_i g_i(.) + (.)^T Bw, - \sum_{i \in I_\alpha} y_i g_i(.)\right)$, $\alpha = 1,2,\ldots,s$, *is second-order pseudo-quasi type I at u with respect to* η, p, q, *and r, then* $(x_0, y, w, p = q = r = 0)$ *is an optimal solution for (2MGD).*

Proof. Since x_0 is an optimal solution of (P) and the corresponding set Z_0 is empty, from Lemma 6.1.1, there exist $y \in R^m$ and $w \in R^n$ such that

$$y^T g(x_0) = 0, \nabla y^T g(x_0) = \nabla f(x_0) + Bw, w^T Bw \leq 1, (x_0^T Bx_0)^{1/2} = x_0^T Bw, y \geq 0.$$

So, $(x_0, y, w, p = q = r = 0)$ is feasible for (2MGD) and the corresponding values of (P) and (2MGD) are equal. If $\left(f(.) - \sum_{i \in I_0} y_i g_i(.) + (.)^T Bw, - \sum_{i \in I_\alpha} y_i g_i(.)\right), \alpha = 1,2,\ldots,s$, is second-order pseudo-quasi type I with respect to η, p, q, and r, then from Theorem 6.1.1, $(x_0, y, w, p = q = r = 0)$ is an optimal solution for (2MGD). □

6.2 Second Order Duality for Minimax Programs

We will apply the optimality conditions of minimax programming to formulate a general second order Mond–Weir dual to the minimax program involving second order pseudo-b-type I, second order quasi-b-type I, second order pseudo-quasi-b-type I and second order quasi-pseudo-b-type I functions. We also establish weak, strong and strict converse duality theorems in this section.

We consider the following minimax optimization problem:

$$\text{(P)} \qquad \text{minimize } f(x) = \sup_{y \in Y} \phi(x, y)$$
$$\text{subject to } g(x) \leqq 0, \tag{6.2.1}$$

where Y is a compact subset of $R^m, \phi(\cdot,\cdot) : R^n \times R^m \mapsto R$ is twice differentiable function in $x \in R^n$, and $g(\cdot) : R^n \mapsto R^p$ is twice differentiable function in $x \in R^n$.

Schmitendorf (1977) established some necessary and sufficient optimality conditions for (P) under the conditions of convexity. Tanimoto (1981) applied the optimality conditions of Mond (1974a) to define a first order dual problem and derived the duality theorems for convex minimax optimization problems considered by Schmitendorf. Weir (1992) relaxed convexity assumptions in the sufficient optimality of Mond (1974a) and also employed the optimality conditions to construct several first order dual problems for (P) which involve pseudoconvex and quasiconvex functions, and established weak and strong duality results. There are many other authors investigated the optimality and first order duality theorems for minimax optimization problems. For details, one can refer to Lai and Lee (2002a), Lai et al. (1999), Liu (1999a), Liu and Wu (1998) and Mishra (1995, 1998a).

By introducing an additional vector $p \in R^n$, Mangasarian (1975) was the first to formulate the second order dual to nonlinear optimization problem. Instead of imposing explicit condition on p, Mond (1974a) included p in a second order type convexity. Mishra and Rueda (2000) introduced the concepts of higher order type I, higher order pseudo type I and higher order quasi type I functions and established various higher order duality results involving these functions. Furthermore, higher order duality results are obtained for nondifferentiable optimization problems by Mishra and Rueda (2002). Zhang and Mond (1996) introduced the concepts of second order B-invex, second order pseudo B-invex and second order quasi B-invex functions, and constructed general second order Mond–Weir dual to a nonlinear optimization problem and proved duality theorems. Liu (1999b) established a second order duality theorem for minimax optimization problem using the concepts of the second order B-invex and related functions introduced by Zhang and Mond (1996).

In this section, we will establish second and higher order duality theorems for minimax optimization problem considered above.

Throughout this section, let R^n be the n-dimensional Euclidean space and R^n_+ be its non-negative orthant.

Let

$$Y(x) = \left\{ y \in Y : \phi(x,y) = \sup_{z \in Y} \phi(x,z) \right\}, \quad J = \{1,2,...,p\},$$

$$J(x) = \left\{ j \in J : g_j(x) = 0 \right\}$$

and

$$K = \Big\{ (s,t,\bar{y}) \in N \times R^s_+ \times R^{ms} : 1 \le s \le n+1, \quad t = (t_1,...,t_s) \in R^s_+ \text{ with } \sum_{i=1}^{s} t_i = 1 \text{ and } \bar{y} = (y_1,...,y_s) \text{ and } y_i \in Y(x), i = 1,...,s \Big\}.$$

Schmitendorf (1977) established the following necessary conditions for (P):

Lemma 6.2.1. *(Necessary Conditions). Let x^* be an optimal solution to (P) and $\nabla g_j(x^*), j \in J(x^*)$ be linearly independent. Then there exist $(s^*, t^*, \bar{y}^*) \in K$ and $\mu^* \in R^p_+$ such that*

$$\sum_{i=1}^{s*} t_i^* \nabla \phi(x^*, y_i^*) + \nabla \sum_{j=1}^{p} \mu_j^* g_j(x^*) = 0, \tag{6.2.2}$$

$$\mu_j^* g_j(x^*) = 0, \tag{6.2.3}$$

$$\mu^* \in R^p_+, t_i^* \ge 0, \sum_{i=1}^{s*} t_i^* = 1, y_i \in Y(x^*), i = 1,...,s^*. \tag{6.2.4}$$

Definition 6.2.1. (f,g) *is said to be second order type I at $\bar{x} \in X$ with respect to η and b if there exists a vector function $\eta : X \times X \to R^n$ such that, for all $x \in X, p \in R^n, y_i \in Y(x), i = 1,2,...,s$ and $j = 1,...,m$*

$$f(x,y_i) - f(\bar{x},\mathrm{y_i}) + \frac{1}{2}p^T\nabla^2 f(\bar{x},y_i)\,p \geq \eta^T(x,\bar{x})\left[\nabla f(\bar{x},y_i) + \nabla^2 f(\bar{x},y_i)\,p\right] \tag{6.2.5}$$

and

$$-g_j(\bar{x}) + \frac{1}{2}p^T\nabla^2 g_j(\bar{x})\,p \geq \eta^T(x,\bar{x})\left[\nabla g_j(\bar{x}) + \nabla^2 g_j(\bar{x})\,p\right]. \tag{6.2.6}$$

Definition 6.2.2. (f,g) *is said to be second order-quasi type I at* $\bar{x} \in X$ *with respect to* η *if there exists a vector function* $\eta : X \times X \to R^n$ *such that, for all* $x \in X, p \in R^n, y_i \in Y(x), i = 1,2,\ldots,s$ *and* $j = 1,\ldots,m$

$$f(x,y_i) - f(\bar{x},\mathrm{y_i}) + \frac{1}{2}p^T\nabla^2 f(\bar{x},y_i)\,p \leq 0 \Rightarrow \eta^T(x,\bar{x})\left[\nabla f(\bar{x},y_i) + \nabla^2 f(\bar{x},y_i)\,p\right] \leq 0 \tag{6.2.7}$$

and

$$-g_j(\bar{x}) + \frac{1}{2}p^T\nabla^2 g_j(\bar{x})\,p \leq 0 \Rightarrow \eta^T(x,\bar{x})\left[\nabla g_j(\bar{x}) + \nabla^2 g_j(\bar{x})\,p\right] \leq 0. \tag{6.2.8}$$

If the second (implied) inequality in (6.2.7) is strict when $x \neq \bar{x}$, *then* (f,g) *is said to be semi strictly quasi type I at* $\bar{x} \in X$.

Definition 6.2.3. (f,g) *is said to be second order-pseudo-type I at* $\bar{x} \in X$ *with respect to* η *if there exists a vector function* $\eta : X \times X \to R^n$ *such that, for all* $x \in X, p \in R^n, y_i \in Y(x), i = 1,2,\ldots,s$ *and* $j = 1,\ldots,m$

$$\eta^T(x,\bar{x})\left[\nabla f(\bar{x},y_i) + \nabla^2 f(\bar{x},y_i)\,p\right] \geq 0 \Rightarrow f(x,y_i) - f(\bar{x},\mathrm{y_i}) + \frac{1}{2}p^T\nabla^2 f(\bar{x},y_i)\,p \geq 0 \tag{6.2.9}$$

and

$$\eta^T(x,\bar{x})\left[\nabla g_j(\bar{x}) + \nabla^2 g_j(\bar{x})\,p\right] \geq 0 \Rightarrow -g_j(\bar{x}) + \frac{1}{2}p^T\nabla^2 g_j(\bar{x})\,p \geq 0. \tag{6.2.10}$$

If the second (implied) inequality in (6.2.9) (or (6.2.10)) is strict, then (f,g) *is said to be semi strictly pseudo type I in* f *(or in* g*) at* $\bar{x} \in X$. *If the second (implied) inequalities in (6.2.9) and (6.2.10) both are strict, then* (f,g) *is said to be strictly pseudo type I at* $\bar{x} \in X$.

Definition 6.2.4. (f,g) *is said to be second order quasi-pseudo-type I at* $\bar{x} \in X$ *with respect to* η *if there exists a vector function* $\eta : X \times X \to R^n$ *such that, for all* $x \in X, p \in R^n$, $y_i \in Y(x), i = 1,2,\ldots,s$ *and* $j = 1,\ldots,m$

$$f(x,y_i) - f(\bar{x},\mathrm{y_i}) + \frac{1}{2}p^T\nabla^2 f(\bar{x},y_i)\,p \leq 0 \Rightarrow \eta^T(x,\bar{x})\left[\nabla f(\bar{x},y_i) + \nabla^2 f(\bar{x},y_i)\,p\right] \leq 0 \tag{6.2.11}$$

and

$$\eta^T(x,\bar{x})\left[\nabla g_j(\bar{x}) + \nabla^2 g_j(\bar{x})\,p\right] \geq 0 \Rightarrow -g_j(\bar{x}) + \frac{1}{2}p^T\nabla^2 g_j(\bar{x})\,p \geq 0. \tag{6.2.12}$$

If the second (implied) inequality in (6.2.11) (or (6.2.12)) is strict, then (f,g) is said to be *semi strictly pseudo type I* in f (in g) at $\bar{x} \in X$. If the second (implied) inequalities in (6.2.11) and (6.2.12) both are strict, then (f,g) is said to be *strictly pseudo-type I* at $\bar{x} \in X$. If the second (implied) inequality in (6.2.12) is strict, (f,g) is said to be *quasi strictly pseudo-type I* at $\bar{x} \in X$.

Definition 6.2.5. *(f,g) is said to be Second order pseudo-quasi-type I at $\bar{x} \in X$ with respect to η if there exists a vector function $\eta : X \times X \to R^n$ such that, for all $x \in X, p \in R^n$, $y_i \in Y(x), i = 1,2,...,s$ and $j = 1,...,m$*

$$\eta^T(x,\bar{x})\left[\nabla f(\bar{x},y_i) + \nabla^2 f(\bar{x},y_i)p\right] \geq 0 \Rightarrow f(x,y_i) - f(\bar{x},\mathrm{y_i}) + \frac{1}{2}p^T\nabla^2 f(\bar{x},y_i)p \geq 0 \tag{6.2.13}$$

and

$$-g_j(\bar{x}) + \frac{1}{2}p^T\nabla^2 g_j(\bar{x})p \leq 0 \Rightarrow \eta^T(x,\bar{x})\left[\nabla g_j(\bar{x}) + \nabla^2 g_j(\bar{x})p\right] \leq 0. \tag{6.2.14}$$

If the second (implied) inequality in (6.2.13) is strict, then (f,g) is said to be strictly pseudo quasi-type I at $\bar{x} \in X$.

Making use of the optimality conditions, we consider the general second order Mond–Weir dual (D) to the minimax optimization problem (P) as follows:

(DI) $\quad \text{maximize}_{(s,t,\bar{y})\in K} \sup_{(z,t,\bar{y})\in H_1(s,t,\bar{y})} \varphi(z,\mu) - \frac{1}{2}p^T\nabla^2\varphi(z,\mu)p$

subject to

$$\sum_{i=1}^{s} t_i\nabla\phi(z,y_i) + \nabla\sum_{j=1}^{p}\mu_j g_j(z) + \nabla^2\left[\sum_{i=1}^{s} t_i\phi(z,y_i)\right]p + \nabla^2\left[\sum_{j=1}^{p}\mu_j g_j(z)\right]p = 0, \tag{6.2.15}$$

$$\sum_{j\in J_\alpha}^{p}\mu_j g_j(z) - \frac{1}{2}p^T\nabla^2\sum_{\mathrm{j}\in \mathrm{J}_\alpha}\mu_j g_j(z)p \geq 0, \quad \alpha = 1,2,...,k, \tag{6.2.16}$$

$$\varphi(z,\mu) = \sum_{i=1}^{s} t_i\phi(z,y_i) + \sum_{j\in J_0}\mu_j g_j(z)$$

where $J_\alpha \subseteq P = \{1,2,...,p\}, \alpha = 0,1,2,...,k$ with $J_\alpha \cap J_\beta = \Phi$, if $\alpha \neq \beta$ and $\cup_{\alpha=0}^{k} J_\alpha = P$, $H_1(s,t,\bar{y})$ denotes the set of all triplets $(z,\mu,v) \in R^n \times R_+^p \times R_+$. If for a triplet $(s,t,\bar{y}) \in K$ the set $H_1(s,t,\bar{y})$ is empty, then we define the supremum over it to be $-\infty$.

Theorem 6.2.1. *(Weak Duality). Let x be feasible for (P) and let $(z,\mu,s,t,\bar{y},p)$ be feasible for (D). Assume that $\left(\sum_{i=1}^{s} t_i\phi(\cdot,y_i) + \sum_{j\in J_0}\mu_j g_j(\cdot), \sum_{j\in J_\alpha}\mu_j g_j(\cdot)\right)$ is second order pseudo-quasi-type I at z with respect to the same η. Then*

$$\sup_{y\in Y}\phi(x,y)\geq\varphi(z,\mu)-\frac{1}{2}p^T\nabla^2\varphi(z,\mu)\,p.$$

Proof. Suppose contrary to the result, that is, $\sup_{y\in Y}\phi(x,y)<\varphi(z,\mu)-\frac{1}{2}p^T\nabla^2$ $\varphi(z,\mu)\,p$ holds. Thus, we have

$$\phi(x,y)<\sum_{i=1}^{s}t_i\phi(z,y_i)+\sum_{j\in J_0}\mu_jg_j(z)-\frac{1}{2}p^T\nabla^2\left[\sum_{i=1}^{s}t_i\phi(z,y_i)+\sum_{j\in J_0}\mu_jg_j(z)\right]p\text{ for all }y\in Y.$$

Since $\sum_{i=1}^{s}t_i=1$, we have

$$\sum_{i=1}^{s}t_i\phi(x,y_i)<\sum_{i=1}^{s}t_i\phi(z,y_i)+\sum_{j\in J_0}\mu_jg_j(z)-\frac{1}{2}p^T\nabla^2\left[\sum_{i=1}^{s}t_i\phi(z,y_i)+\sum_{j\in J_0}\mu_jg_j(z)\right]p.$$

By the first part of the second order pseudo-quasi-type I assumption, we get

$$\eta^T(x,z)\left\{\nabla\left[\sum_{i=1}^{s}t_i\phi(z,y_i)+\sum_{j\in J_0}\mu_jg_j(z)\right]+\nabla^2\left[\sum_{i=1}^{s}t_i\phi(z,y_i)+\sum_{j\in J_0}\mu_jg_j(z)\right]p\right\}<0. \tag{6.2.17}$$

From (6.2.17) and (6.2.15), we get

$$\eta^T(x,z)\left\{\nabla\sum_{j\in PJ_0}\mu_jg_j(z)+\nabla^2\sum_{j\in PJ_0}\mu_jg_j(z)p\right\}>0.$$

Since $\cup_{\alpha=0}^{k}J_\alpha=P$, we have

$$\eta^T(x,z)\left\{\nabla\sum_{\alpha=1}^{k}\sum_{j\in PJ_0}\mu_jg_j(z)+\nabla^2\sum_{\alpha=1}^{k}\sum_{j\in PJ_0}\mu_jg_j(z)p\right\}>0. \tag{6.2.18}$$

From inequality (6.2.16) and the second part of the second order pseudo-quasi-type I assumption, we get

$$\eta^T(x,z)\left\{\nabla\sum_{j\in J_\alpha}\mu_jg_j(z)+\nabla^2\sum_{j\in J_\alpha}\mu_jg_j(z)p\right\}\leq 0,\alpha=1,2,...,k.$$

Thus, we have

$$\eta^T(x,z)\left\{\nabla\sum_{\alpha=1}^{k}\sum_{j\in J_\alpha}\mu_jg_j(z)+\nabla^2\sum_{\alpha=1}^{k}\sum_{j\in J_\alpha}\mu_jg_j(z)p\right\}\leq 0,$$

which contradicts the inequality (6.2.18). The proof is completed. □

Theorem 6.2.2. *(Strong Duality). Assume that x^* is an optimal solution to (P) and $\nabla g_j(x^*), j \in J(x^*)$ is linearly independent. Then there exist $(s^*, t^*, \bar{y}^*) \in K$, $(x^*, \mu^*, p^* = 0) \in H_1(s^*, t^*, \bar{y}^*)$ such that $(x^*, \mu^*, s^*, t^*, \bar{y}^*, p^* = 0)$ is feasible for (D). If in addition the hypothesis of the Theorem 6.2.1 holds for any feasible solution $(z, \mu, s, t, \bar{y}, p)$ of (D), then $(x^*, \mu^*, s^*, t^*, \bar{y}^*, p^* = 0)$ is an optimal solution of (D), and problems (P) and (D) have the same extremal values.*

Proof. By Lemma 6.2.1 there exist $(s^*, t^*, \bar{y}^*) \in K, (x^*, \mu^*, p^* = 0) \in H_1(s^*, t^*, \bar{y}^*)$ such that $(x^*, \mu^*, s^*, t^*, \bar{y})$ is feasible for (D). Optimality of $(x^*, \mu^*, s^*, t^*, \bar{y}^*, p^* = 0)$ for (D) follows from weak duality Theorem 6.2.1. □

Theorem 6.2.3. *(Strict Converse Duality). Let $\bar{x}$ be an optimal solution for (P) and $(x^*, \mu^*, s^*, t^*, \bar{y}^*, p^* = 0)$ be an optimal solution for (D). Assume that $\left(\sum_{i=1}^{s} t_i^* \phi(\cdot, y_i^*) + \sum_{j \in J_0} \mu_j^* g_j(\cdot), \sum_{j \in J_\alpha} \mu_j g_j(\cdot)\right)$ is second order pseudo-quasi-type I at x^* with respect to the same η for all $(z, \mu, s, t, \bar{y}, p)$ and $\nabla g_j(\bar{x}), j \in J(\bar{x})$ is linearly independent. Then $\bar{x} = x^*$, that is, x^* is an optimal solution for (P).*

Proof. Assume that $\bar{x} \neq x^*$ for getting a contradiction. From Theorem 6.2.2, we have,

$$\phi(\bar{x}, y) \leq \sum_{i=1}^{s*} t_i^* \phi(x^*, y_i^*) + \sum_{j \in J_0} \mu_j^* g_j(x^*) - \frac{1}{2} p^{*T} \nabla^2 \left[\sum_{i=1}^{s*} t_i^* \phi(x^*, y_i^*) + \sum_{j \in J_0} \mu_j^* g_j(x^*)\right] p^*$$

for all $y \in Y$. From $\sum_{i=1}^{s} t_i^* = 1$ and $y_i^* \in Y(x^*), i = 1, 2, ..., s^*$, we have

$$\sum_{i=1}^{s*} t_i^* \phi(\bar{x}, y_i^*) \leq \sum_{i=1}^{s*} t_i^* \phi(x^*, y_i^*) + \sum_{j \in J_0} \mu_j^* g_j(x^*) - \frac{1}{2} p^{*T} \nabla^2 \left[\sum_{i=1}^{s*} t_i^* \phi(x^*, y_i^*) + \sum_{j \in J_0} \mu_j^* g_j(x^*)\right] p^*.$$

From the feasibility of $\bar{x}$ and $\mu^* \geq 0$, we have

$$\begin{aligned}\sum_{i=1}^{s*} t_i^* \phi(\bar{x}, y_i^*) + \sum_{j \in J_0} \mu_j^* g_j(\bar{x}) \geq & \sum_{i=1}^{s*} t_i^* \phi(x^*, y_i^*) + \sum_{j \in J_0} \mu_j^* g_j(x^*) \\ & - \frac{1}{2} p^{*T} \nabla^2 \left[\sum_{i=1}^{s*} t_i^* \phi(x^*, y_i^*) + \sum_{j \in J_0} \mu_j^* g_j(x^*)\right] p^*.\end{aligned} \tag{6.2.19}$$

On the other hand, from the duality constraint and second part of the second order pseudo-quasi-type I assumption, we get

$$\eta^T(\bar{x}, x^*) \left\{\nabla \sum_{j \in J_\alpha} \mu_j^* g_j(x^*) + \nabla^2 \sum_{j \in J_\alpha} \mu_j^* g_j(x^*) p^*\right\} \leq 0, \alpha = 1, 2, ..., k..$$

Thus, we have

$$\eta^T(\bar{x},x^*)\left\{\nabla\sum_{\alpha=1}^{k}\sum_{j\in J_\alpha}\mu_j^* g_j(x^*)+\nabla^2\sum_{\alpha=1}^{k}\sum_{j\in J_\alpha}\mu_j^* g_j(x^*)p^*\right\}\leq 0.$$

Since $\cup_{\alpha=0}^{k} J_\alpha = P$, we have

$$\eta^T(\bar{x},x^*)\left\{\nabla\sum_{\alpha=1}^{k}\sum_{j\in J_\alpha}\mu_j^* g_j(x^*)+\nabla^2\sum_{\alpha=1}^{k}\sum_{j\in J_\alpha}\mu_j^* g_j(x^*)p^*\right\}\leq 0. \tag{6.2.20}$$

From (6.2.20) and (6.2.15), we have

$$\eta^T(\bar{x},x^*)\left\{\nabla\left[\sum_{i=1}^{s*}t_i^*\phi(x^*,y_i^*)+\sum_{j\in J_0}\mu_j^* g_j(x^*)\right]\right.$$
$$\left.+\nabla^2\left[\sum_{i=1}^{s*}t_i^*\phi(x^*,y_i^*)+\sum_{j\in J_0}\mu_j^* g_j(x^*)\right]p^*\right\}\geq 0.$$

By the first part of second order pseudo-quasi-type I assumption, we get

$$\sum_{i=1}^{s*}t_i^*\phi(\bar{x},y_i^*)+\sum_{j\in J_0}\mu_j^* g_j(\bar{x})\geq\sum_{i=1}^{s*}t_i^*\phi(x^*,y_i^*)+\sum_{j\in J_0}\mu_j^* g_j(x^*)$$
$$-\frac{1}{2}p^{*T}\nabla^2\left[\sum_{i=1}^{s*}t_i^*\phi(x^*,y_i^*)+\sum_{j\in J_0}\mu_j^* g_j(x^*)\right]p^* \tag{6.2.21}$$

which contradicts (6.2.19). □

Remark 6.2.1. A question arises as to whether the second order duality results developed in this section hold for the following minimax fractional optimization problem:

$$\text{minimize } F(x)=\sup_{y\in Y}\frac{f(x,y)}{h(x,y)}$$
$$\text{subject to } g(x)\leqq 0,$$

where Y is a compact subset of R^m, and the following complex minimax optimization problem:

$$\text{minimize } f(\xi)=\sup_{\nu\in W} Re\phi(\xi,\nu)$$
$$\text{subject to } \xi\in S^0=\left\{\xi\in C^{2n}:-g(\xi)\in S\right\}$$

where $\xi=(z,\bar{z}),\nu=(\omega,\varpi)$ for $z\in C^n,\omega\in C^m,\phi(\cdot,\cdot):C^{2n}\times C^{2m}\to C$ is analytic with respect to ξ,W is a specified compact subset in C^{2m},S is a polyhedral cone in C^p, and $g:C^{2n}\to C^p$ is analytic.

Another question also arises as to whether the second order duality results developed in this section can be extended to the case of higher order duality. Moreover, is it possible to obtain higher order duality results for the two problems posed in the Remark 6.2.1 above in the present section?

6.3 Second Order Duality for Nondifferentiable Minimax Programs

In this section, we consider the general second order Mond–Weir dual to the nondifferentiable minimax optimization problem:

$$\text{(P)} \qquad \text{minimize } f(x) = \sup_{y\in Y} \phi(x,y) + \langle x, Ax\rangle^{1/2}$$
$$\text{subject to } g(x) \leq 0, \tag{6.3.1}$$

where Y is a compact subset of R^m, $\phi(\cdot,\cdot): R^n \times R^m \mapsto R$ is a twice differentiable function in $x \in R^n$, $g(\cdot): R^n \mapsto R^r$ is a twice differentiable function in $x \in R^n$, and A is an $n \times n$ positive semi-definite (symmetric) matrix.

The second order nondifferentiable dual to the above problem (P) is given as follows:

$$\text{(D)} \qquad \text{maximize}_{(s,t,\bar{y})\in K} \sup_{(z,u,\mu,p)\in H(s,t,\bar{y})} \varphi(z,u,\mu) - \frac{1}{2}p^T \nabla^2 \varphi(z,u,\mu)\, p$$

subject to

$$\sum_{i=1}^{s} t_i \nabla\phi(z,y_i) + Au + \nabla \sum_{j=1}^{r} \mu_j g_j(z) + \nabla^2 \left[\sum_{i=1}^{s} t_i \phi(z,y_i)\right] p + \nabla^2 \left[\sum_{j=1}^{r} \mu_j g_j(z)\right] p = 0, \tag{6.3.2}$$

$$\sum_{j\in J_\alpha} \mu_j g_j(z) - \frac{1}{2} p^T \nabla^2 \sum_{j\in J_\alpha} \mu_j g_j(z)\, p \geq 0, \quad \alpha = 1,2,\ldots,k, \tag{6.3.3}$$

$$\langle u, Au\rangle \leq 1, \langle z, Az\rangle^{1/2} = \langle z, Au\rangle, \tag{6.3.4}$$

$$\varphi(z,u,\mu) = \sum_{i=1}^{s} t_i \phi(z,y_i) + \langle z, Au\rangle + \sum_{j\in J_0} \mu_j g_j(z)$$

where $J_\alpha \subseteq P = \{1,2,\ldots,r\}, \alpha = 0,1,2,\ldots,k$, with $J_\alpha \cap J_\beta = \phi$, if $\alpha \neq \beta, \cup_{\alpha=0}^{k} J_\alpha = P$, and $H(s,t,\bar{y})$ denotes the set of all $(z,u,\mu,p) \in R^n \times R^n \times R_+^r \times R_+$. If for a triplet $(s,t,\bar{y}) \in K$ the set $H(s,t,\bar{y})$ is empty, then we define the supremum over it to be $-\infty$.

The following necessary conditions for (P) are a particular case of the necessary conditions given in Lai et al. (1999) which will be needed in the sequel.

Lemma 6.3.1. *(Necessary Conditions). Let x^* be an optimal solution of (P) satisfying $\langle x^*, Ax^*\rangle^{1/2} > 0$ and let $\nabla g_j(x^*), j \in J(x^*)$, be linearly independent. Then there exist $(s^*,t^*,y^*) \in K, u \in R^n$ and $\mu^* \in R_+^r$ such that*

$$\sum_{i=1}^{s*} t_i^* \nabla \phi\left(x^*, y_i^*\right) + Au + \nabla \sum_{j=1}^{r} \mu_j^* g_j\left(x^*\right) = 0,$$

$$\mu_j^* g_j\left(x^*\right) = 0,$$

$$\mu^* \in R_+^r,\, t_i^* \geq 0, \sum_{\mathrm{i}=1}^{\mathrm{s}^*} t_i^* = 1,\, y_i^* \in Y\left(x^*\right),\, i = 1, ..., s^*,$$

$$\langle u, Au \rangle \leq 1, \langle x^*, Au \rangle = \langle x^*, Ax^* \rangle^{1/2}.$$

Theorem 6.3.1. *(Weak Duality). Let x be feasible to (P) and let $(z, u, \mu, s, t, \bar{y}, p)$ be feasible to (D). If for any feasible $(x, z, u, \mu, s, t, \bar{y}, p)$ there exists a function η : $R^n \times R^n \to R^n$ such that*

$$\eta^T(x,z)\left\{\nabla \sum_{i=1}^{s} t_i \phi\left(z, y_i\right) + Au + \nabla \sum_{j \in J_0} \mu_j g_j(z) + \nabla^2 \left[\sum_{i=1}^{s} t_i \phi\left(z, y_i\right) + \sum_{j \in J_0} \mu_j g_j(z)\right] p\right\} \geq 0$$
$$\Rightarrow \sum_{i=1}^{s} t_i \phi\left(x, y_i\right) + \langle x, Au \rangle + \sum_{j \in J_0} \mu_j g_j(x) - \left\{\sum_{i=1}^{s} t_i \phi\left(z, y_i\right) + \langle z, Au \rangle + \sum_{j \in J_0} \mu_j g_j(z)\right\}$$
$$- \frac{1}{2} p^T \nabla^2 \left[\sum_{i=1}^{s} t_i \phi\left(z, y_i\right) + \sum_{j \in J_0} \mu_j g_j(z)\right] p \geq 0$$

and

$$\sum_{j \in J_\alpha} \mu_j g_j(z) - \frac{1}{2} p^T \nabla^2 \sum_{j \in J_\alpha} \mu_j g_j(z)\, p \geq 0$$
$$\Rightarrow \eta^T(x,z)\left\{\nabla \sum_{j \in J_\alpha} \mu_j g_j(z) + \nabla^2 \sum_{j \in J_\alpha} \mu_j g_j(z) p\right\} \leq 0, \alpha = 1, 2, ..., k.$$

Then

$$\sup_{y \in Y} \phi(x, y) + \langle x, Au \rangle^{1/2} \geq \varphi(z, u, \mu) - \frac{1}{2} p^T \nabla^2 \varphi(z, u, \mu)\, p.$$

Proof. Suppose, contrary to the result, that $\sup_{y \in Y} \phi(x,y) + \langle x, Au \rangle\, 1/2 < \varphi(z, u, \mu) - \frac{1}{2} p^T \nabla^2 \varphi(z, u, \mu) p$ holds. Thus, we have $\phi(x,y) + \langle x, Au \rangle^{1/2} < \sum_{i=1}^{s} t_i \phi(z, y_i) + \langle z, Au \rangle + \sum_{j \in J_0} \mu_j g_j(z) - \frac{1}{2} p^T \nabla^2 \left[\sum_{i=1}^{s} t_i \phi(z, y_i) + \sum_{j \in J_0} \mu_j g_j(z)\right] p$ for all $y \in Y$. Since $\sum_{i=1}^{s} t_i = 1$, we have

$$\sum_{i=1}^{s} t_i \phi\left(x, y_i\right) + \langle x, Au \rangle^{1/2} < \sum_{i=1}^{s} t_i \phi\left(z, y_i\right) + \langle z, Au \rangle + \sum_{j \in J_0} \mu_j g_j(z)$$
$$- \frac{1}{2} p^T \nabla^2 \left[\sum_{i=1}^{s} t_i \phi\left(z, y_i\right) + \sum_{j \in J_0} \mu_j g_j(z)\right] p.$$

By (6.3.1) and $\mu \in R_{+}^{r}$, we have

$$\sum_{i=1}^{s} t_i \phi(x,y_i) + \langle x, Au\rangle^{1/2} + \sum_{j \in J_0} \mu_j g_j(x) < \sum_{i=1}^{s} t_i \phi(z,y_i) + \langle z, Au\rangle + \sum_{j \in J_0} \mu_j g_j(z) - \frac{1}{2} p^T \nabla^2 \left[\sum_{i=1}^{s} t_i \phi(z,y_i) + \sum_{j \in J_0} \mu_j g_j(z) \right] p.$$

Using the generalized Schwarz inequality $\langle x, Au\rangle^{1/2} \leq \langle x, Ax\rangle^{1/2} \langle u, Au\rangle^{1/2}$ and $\langle u, Au\rangle^{1/2} \leq 1$, we get

$$\sum_{i=1}^{s} t_i \phi(x,y_i) + \langle x, Au\rangle + \sum_{j \in J_0} \mu_j g_j(x) < \sum_{i=1}^{s} t_i \phi(z,y_i) + \langle z, Au\rangle + \sum_{j \in J_0} \mu_j g_j(z) - \frac{1}{2} p^T \nabla^2 \left[\sum_{i=1}^{s} t_i \phi(z,y_i) + \sum_{j \in J_0} \mu_j g_j(z) \right] p.$$

By the first part of the second order pseudo-quasi type I assumption, we get

$$\eta^T(x,z) \left\{ \nabla \sum_{i=1}^{s} t_i \phi(z,y_i) + Au + \nabla \sum_{j \in J_0} \mu_j g_j(z) + \nabla^2 \left[\sum_{i=1}^{s} t_i \phi(z,y_i) + \sum_{j \in J_0} \mu_j g_j(z) \right] p \right\} < 0. \tag{6.3.5}$$

From (6.3.2) and (6.3.5), we get

$$\eta^T(x,z) \left\{ \nabla \sum_{j \in P \setminus J_0} \mu_j g_j(z) + \nabla^2 \sum_{j \in P \setminus J_0} \mu_j g_j(z) p \right\} > 0.$$

Since $\cup_{\alpha=0}^{k} J_\alpha = P$, we have

$$\eta^T(x,z) \left\{ \nabla \sum_{\alpha=1}^{k} \sum_{j \in J_\alpha} \mu_j g_j(z) + \nabla^2 \sum_{\alpha=1}^{k} \sum_{j \in J_\alpha} \mu_j g_j(z) p \right\} > 0. \tag{6.3.6}$$

From (6.3.3) and the second part of the second order pseudo-quasi type I assumption, we get

$$\eta^T(x,z) \left\{ \nabla \sum_{j \in J_\alpha} \mu_j g_j(z) + \nabla^2 \sum_{j \in J_\alpha} \mu_j g_j(z) p \right\} \leq 0, \alpha = 1, 2, ..., k.$$

Thus, we have

$$\eta^T(x,z) \left\{ \nabla \sum_{\alpha=1}^{k} \sum_{j \in J_\alpha} \mu_j g_j(z) + \nabla^2 \sum_{\alpha=1}^{k} \sum_{j \in J_\alpha} \mu_j g_j(z) p \right\} \leq 0,$$

which contradicts (6.3.6). □

Theorem 6.3.2. *(Strong Duality). Assume that x^* is an optimal solution to (P) and $\nabla g_j(x^*), j \in J(x^*)$, is linearly independent. Then there exist $(s^*, t^*, \bar{y}^*) \in K$, $(x^*, u^*, \mu^*, p^* = 0) \in H(s^*, t^*, \bar{y}^*)$ such that $(x^*, u^*, \mu^*, s^*, t^*, \bar{y}^*, p^* = 0)$ is feasible for (D). If, in addition, the hypothesis of Theorem 6.3.1 holds for any feasible solution $(z, u, \mu, s, t, \bar{y}, p)$ of (D), then $(x^*, u^*, \mu^*, s^*, t^*, \bar{y}^*, p^* = 0)$ is an optimal solution of (D) and problems (P) and (D) have the same extremal values.*

Proof. By Lemma 6.3.1 there exists $(s^*, t^*, \bar{y}^*) \in K, (x^*, u^*, \mu^*, p^* = 0) \in H(s^*, t^*, \bar{y}^*)$ such that $(x^*, u^*, \mu^*, s^*, t^*, \bar{y})$ is feasible for (D). The optimality of $(x^*, u^*, \mu^*, s^*, t^*, \bar{y}^*, p^* = 0)$ for (D) follows from Theorem 6.3.1. □

Theorem 6.3.3. *(Strict Converse Duality). Let $\bar{x}$ be optimal to (P) and let $(x^*, u^*, \mu^*, s^*, t^*, \bar{y}^*, p^* = 0)$ be optimal to (D). Assume that $\left(\sum_{i=1}^{s} t_i^* \phi(\cdot, y_i^*) + \langle ., Au^* \rangle + \sum_{j \in J_0} \mu_j^* g_j(\cdot), \sum_{j \in J_\alpha} \mu_j^* g_j(\cdot)\right)$ is second order strictly pseudo-quasi type I at x^* with respect to the same η for any feasible solution, and $\nabla g_j(\bar{x}), j \in J(\bar{x})$, is linearly independent. Then $\bar{x} = x^*$, that is, x^* is an optimal solution to (P).*

Proof. We assume that $\bar{x} \neq x^*$ and prove a contradiction. From Theorem 6.3.2, we have,

$$\phi(\bar{x}, y) + \langle \bar{x}, Au^* \rangle^{1/2} \leq \sum_{i=1}^{s*} t_i^* \phi(x^*, y_i^*) + \langle x^*, Au^* \rangle + \sum_{j \in J_0} \mu_j^* g_j(x^*) - \frac{1}{2} p^* T \nabla^2 \left[\sum_{i=1}^{s*} t_i^* \phi(x^*, y_i^*) + \sum_{j \in J_0} \mu_j^* g_j(x^*) \right] p^*$$

for all $y \in Y$.

From $\sum_{i=1}^{s} t_i^* = 1$ and $y_i^* \in Y(x^*), i = 1, 2, ..., s^*$, we have

$$\sum_{i=1}^{s*} t_i^* \phi(\bar{x}, y_i^*) + \langle \bar{x}, Au^* \rangle^{1/2} \leq \sum_{i=1}^{s*} t_i^* \phi(x^*, y_i^*) + \langle x^*, Au^* \rangle + \sum_{j \in J_0} \mu_j^* g_j(x^*) - \frac{1}{2} p^{*T} \nabla^2 \left[\sum_{i=1}^{s*} t_i^* \phi(x^*, y_i^*) + \sum_{j \in J_0} \mu_j^* g_j(x^*) \right] p^*.$$

From the feasibility of $\bar{x}$ and $\mu^* \geq 0$, we have

$$\sum_{i=1}^{s*} t_i^* \phi(\bar{x}, y_i^*) + \langle \bar{x}, Au^* \rangle^{1/2} + \sum_{j \in J_0} \mu_j^* g_j(\bar{x}) \leq \sum_{i=1}^{s*} t_i^* \phi(x^*, y_i^*) + \langle x^*, Au^* \rangle + \sum_{j \in J_0} \mu_j^* g_j(x^*) - \frac{1}{2} p^{*T} \nabla^2 \left[\sum_{i=1}^{s*} t_i^* \phi(x^*, y_i^*) + \sum_{j \in J_0} \mu_j^* g_j(x^*) \right] p^*.$$

Using the generalized Schwarz inequality $\langle \bar{x}, Au^* \rangle^{1/2} \leq \langle \bar{x}, A\bar{x} \rangle^{1/2} \langle \bar{u}, A\bar{u} \rangle^{1/2}$ and $\langle \bar{u}, A\bar{u} \rangle^{1/2} \leq 1$, we get

$$\sum_{i=1}^{s*} t_i^* \phi(\bar{x}, y_i^*) + \langle \bar{x}, Au^* \rangle + \sum_{j \in J_0} \mu_j^* g_j(\bar{x}) \leq \sum_{i=1}^{s*} t_i^* \phi(x^*, y_i^*) + \langle x^*, Au \rangle^* + \sum_{j \in J_0} \mu_j^* g_j(x^*) - \frac{1}{2} p^{*T} \nabla^2 \left[\sum_{i=1}^{s*} t_i^* \phi(x^*, y_i^*) + \sum_{j \in J_0} \mu_j^* g_j(x^*) \right] p^*. \tag{6.3.7}$$

On the other hand, from the duality constraint and the second part of the second order strictly pseudo-quasi type I assumption, we get

$$\eta^T(\bar{x}, x^*) \left\{ \nabla \sum_{j \in J_\alpha} \mu_j^* g_j(x^*) + \nabla^2 \sum_{j \in J_\alpha} \mu_j^* g_j(x^*) p^* \right\} \leq 0, \quad \alpha = 1, 2, \ldots, k.$$

Therefore, we have

$$\eta^T(\bar{x}, x^*) \left\{ \nabla \sum_{\alpha=1}^{k} \sum_{j \in J_\alpha} \mu_j^* g_j(x^*) + \nabla^2 \sum_{\alpha=1}^{k} \sum_{j \in J_\alpha} \mu_j^* g_j(x^*) p^* \right\} \leq 0.$$

Since $\cup_{\alpha=0}^{k} J_\alpha = P$, we have

$$\eta^T(\bar{x}, x^*) \left\{ \nabla \sum_{j \in P \setminus J_0} \mu_j^* g_j(x^*) + \nabla^2 \sum_{j \in P \setminus J_0} \mu_j^* g_j(x^*) p^* \right\} \leq 0. \tag{6.3.8}$$

From (6.3.2) and (6.3.8), we have

$$\eta^T(\bar{x}, x^*) \left\{ \nabla \sum_{i=1}^{s*} t_i^* \phi(x^*, y_i^*) + Au^* + \nabla \sum_{j \in J_0} \mu_j^* g_j(x^*) + \nabla^2 \left[\sum_{i=1}^{s*} t_i^* \phi(x^*, y_i^*) + \sum_{j \in J_0} \mu_j^* g_j(x^*) \right] p^* \right\} \geq 0.$$

By the first part of the second order strictly pseudo-quasi type I assumption, we get

$$\sum_{i=1}^{s*} t_i^* \phi(\bar{x}, y_i^*) + \langle \bar{x}, Au^* \rangle + \sum_{j \in J_0} \mu_j^* g_j(\bar{x}) > \sum_{i=1}^{s*} t_i^* \phi(x^*, y_i^*) + \langle x^*, Au^* \rangle + \sum_{j \in J_0} \mu_j^* g_j(x^*) - \frac{1}{2} p^{*T} \nabla^2 \left[\sum_{i=1}^{s*} t_i^* \phi(x^*, y_i^*) + \sum_{j \in J_0} \mu_j^* g_j(x^*) \right] p^*,$$

which contradicts (6.3.7). □

Remark 6.3.1. We conclude this section with some ideas for future development in the research area. A question arises as to whether the second order duality results developed in this section hold for the following minimax fractional optimization problem:

$$\text{minimize } F(x) = \sup_{y \in Y} \frac{f(x,y)}{h(x,y)}$$
$$\text{subject to } g(x) \leq 0,$$

where Y is a compact subset of R^m, and the following complex minimax optimization problem:

$$\text{minimize } f(\xi) = \sup_{\nu \in W} Re\phi(\xi,\nu)$$
$$\text{subject to } \xi \in S^0 = \left\{\xi \in C^{2n} : -g(\xi) \in S\right\}$$

where $\xi = (z,\bar{z}), \nu = (\omega,\overline{\omega})$ for $z \in C^n, \omega \in C^m, \phi(\cdot,\cdot) : C^{2n} \times C^{2m} \to C$ is analytic with respect to ξ, W is a specified compact subset in C^{2m}, S is a polyhedral cone in C^p, and $g : C^{2n} \to C^p$ is analytic.

Other questions also arise as to whether the second order duality results developed in this section can be extended to the case of higher order duality and whether it is possible to obtain higher order duality results for the two optimization problems posed in Remark 6.3.1.

6.4 Higher Order Duality for Nonlinear Optimization Problems

Consider the following nonlinear optimization problem:

$$\text{(P)} \qquad \text{minimize } f(x)$$
$$\text{subject to } g(x) \geq 0, \tag{6.4.1}$$

where $f : R^n \to R$ and $g : R^n \to R^m$ are twice differentiable functions.

The Mangasarian second-order dual given by Mangasarian (1975) is

$$\text{(MD)} \qquad \text{maximize } f(u) - y^T g(u) - \frac{1}{2} p^T \nabla^2 [f(u) - y^T g(u)]\, p$$
$$\text{subject to } \nabla[f(u) - y^T g(u)] + \nabla^2 [f(u) - y^T g(u)] = 0,$$
$$y \geq 0.$$

Mangasarian (1975) formulated the following higher-order dual by introducing two differentiable functions $h\colon R^n \times R^n \to R$ and $k\colon R^n \times R^n \to R^m$.

$$\text{(HD1)} \qquad \text{maximize } f(u) + h(u,p) - y^T g(u) - y^T k(u,p)$$
$$\text{subject to } \nabla_p h(u,p) = \nabla_p (y^T k(u,p)), \tag{6.4.2}$$
$$y \geq 0, \tag{6.4.3}$$

where $\nabla_p h(u,p)$ denotes the $n\times 1$ gradient of h with respect to p and $\nabla_p(y^T k(u,p))$ denotes the $n\times 1$ gradients of $y^T k$ with respect to p.

If

$$h(u,p)=p^T\nabla f(u)+\frac{1}{2}p^T\nabla^2 f(u)p \text{ and } k(u,p)=p^T\nabla g(u)+\frac{1}{2}p^T\nabla^2 g(u)p$$

then (HD1) becomes (MD).

Mond and Zhang (1998) obtained duality results for various higher-order dual problems under higher-order invexity assumptions. They also considered the following higher-order dual to (P):

$$\text{(HD)} \qquad \text{maximize } f(u)+h(u,p)-p^T\nabla_p h(u,p)$$

$$\text{subject to } \nabla_p h(u,p)=\nabla_p(y^T k(u,p)), \tag{6.4.4}$$

$$y_i g_i(u)+y_i k_i(u,p)-p^T\nabla_p(y_i k_i(u,p))\le 0,\ i=1,2,\ldots,m, \tag{6.4.5}$$

$$y\ge 0. \tag{6.4.6}$$

In this section, we will give more general invexity-type conditions, such as higher-order type I, higher-order pseudo type I, and higher-order quasi type I conditions and establish various duality results under these conditions.

Mond and Zhang (1998) proved duality results between (P) and (HD) assuming that there exists a function $\eta: R^n\times R^n\to R^n$ such that

$$f(x)-f(u)\ge\alpha(x,u)\nabla_p h(u,p)\eta(x,u)+h(u,p)-p^T(\nabla_p h(u,p)), \tag{6.4.7}$$

and

$$g_i(x)-g_i(u)\le\beta_i(x,u)\nabla_p k_i(u,p)\eta(x,u)+k_i(u,p)-p^T(\nabla_p k_i(u,p)),\ i=1,2,\ldots,m, \tag{6.4.8}$$

where $\alpha: R^n\times R^n\to R_+\backslash\{0\}$, and $\beta_i: R^n\times R^n\to R_+\backslash\{0\}$, $i=1,2,\ldots,m$, are positive functions.

Combining the concept of type I functions and conditions (6.4.7) and (6.4.8) when $h(u,p)=p^T\nabla f(u)$, and $k_i(u,p)=p^T\nabla g_i(u)$, $i=1,2,\ldots,m$, we say that $(f,-g_i)$, $i=1,2,\ldots,m$, is V-type I at the point u with respect to functions η, α, and β_i,

$$\text{if} \qquad f(x)-f(u)\ge\alpha(x,u)\nabla f(u)\eta(x,u),$$

and

$$-g_i(u)\le\beta_i(x,u)\nabla g_i(u)\eta(x,u),\ i=1,2,\ldots,m.$$

Mond and Zhang (1995) extended the notion of V-invexity to the second-order case and established duality theorems under generalized second-order V-invexity conditions.

If $(f,-g_i)$, $i=1,2,\ldots,m$, satisfies conditions (6.4.7) and (6.4.8) with $h(u,p)=p^T\nabla f(u)+\frac{1}{2}p^T\nabla^2 f(u)p$ and $k_i(u,p)=p^T\nabla g_i(u)+\frac{1}{2}p^T\nabla^2 g_i(u)p$, then $(f,-g_i)$ is said to be second-order V-type I.

Theorem 6.4.1. *(Weak Duality). Let x be feasible for (P) and let (u,y,p) feasible for (HD1). If, for any feasible (x,u,y,p), there exists a function $\eta : R^n \times R^n \to R^n$ such that*

$$f(x) - f(u) \geq \eta(x,u)^T \nabla_p h(u,p) + h(u,p) - p^T(\nabla_p h(u,p)) \tag{6.4.9}$$

and

$$-g_i(u) \leq \eta(x,u)^T \nabla_p k_i(u,p) + k_i(u,p) - p^T(\nabla_p k_i(u,p)),\ i = 1, 2, \ldots, m, \tag{6.4.10}$$

then

$$\text{infimum}\,(\text{P}) \geq \text{supremum}\,(\text{HD1}).$$

Proof. By (6.4.9), (6.4.2), (6.4.3) and (6.4.10), we have

$$\begin{aligned}
&f(x) - f(u) - h(u,p) + y^T g(u) + y^T k(u,p) \\
&\quad \geq \eta(x,u)^T \nabla_p h(u,p) - p^T(\nabla_p h(u,p)) + y^T g(u) + y^T k(u,p), \\
&\quad = \eta(x,u)^T \nabla_p (y^T k(u,p)) - p^T(\nabla_p y^T k(u,p)) + y^T g(u) + y^T k(u,p) \\
&\quad \geq 0.
\end{aligned}$$

The proof is completed. □

The following strong duality theorem is similar to Theorem 2 of Mond and Zhang (1998).

Theorem 6.4.2. *(Strong Duality). Let x_0 be a local or global optimal solution of (P) at which a constraint qualification is satisfied, and let*

$$h(x_0,0) = 0, \quad k(x_0,0) = 0, \quad \nabla_p h(x_0,0) = \nabla f(x_0), \quad \nabla_p k(x_0,0) = \nabla g(x_0). \tag{6.4.11}$$

Then there exists $y \in R^m$ such that $(x_0, y, p = 0)$ is feasible for (HD1), and the corresponding objective values of (P) and (HD1) are equal. If (6.4.9) and (6.4.10) are satisfied for any feasible (x,u,y,p), then x_0 and $(x_0, y, p = 0)$ are a global optimal solution for (P) and (HD1), respectively.

Remark 6.4.1. If $h(u,p) = p^T \nabla f(u)$ and $k_i(u,p) = p^T \nabla g_i(u), i = 1,2,\ldots,m$, then (6.4.9) and (6.4.10) become the conditions given by Hanson and Mond (1987) to define a type I function. If $h(u,p) = p^T f(u) + \frac{1}{2} p^T \nabla^2 f(u) p$, and $k_i(u,p) = p^T \nabla g_i(u) + \frac{1}{2} p^T \nabla^2 g_i(u) p$, $i = 1,2,\ldots,m$, then (6.4.9) and (6.4.10) become the second-order type I conditions given by Hanson (1993) when $p = q = r$.

6.5 Mond–Weir Higher Order Duality for Nonlinear Optimization Problems

In this section, we shall establish higher order duality results for (P) and (HD) given in the previous section

Theorem 6.5.1. *(Weak Duality). Let x be feasible for (P) and let (u,y,p) be feasible for (HD). If, for all feasible (x,u,y,p), there exists a function $\eta : R^n \times R^n \to R^n$ such that*

$$f(x) - f(u) \geq \alpha(x,u)\nabla_p h(u,p)\eta(x,u) + h(u,p) - p^T(\nabla_p h(u,p)) \tag{6.5.1}$$

and

$$-g_i(u) \leq \beta_i(x,u)\nabla_p k_i(u,p)\eta(x,u) + k_i(u,p) - p^T(\nabla_p k_i(u,p)),\ i = 1, 2, \ldots, m, \tag{6.5.2}$$

where $\alpha : R^n \times R^n \to R_+ \backslash \{0\}$, and $\beta_i : R^n \times R^n \to R_+ \backslash \{0\}, i = 1,2,\ldots,m$, are positive functions, then

$$\text{infimum(P)} \geq \text{supremum(HD)}.$$

Proof. Since x is feasible for (P) and (u,y,p) is feasible for (HD), we have

$$-y_i g_i(u) - y_i k_i(u,p) + p^T \nabla_p (y_i k_i(u,p)) \geq 0,\ i = 1,2,\ldots,m.$$

By (6.5.2) and $y_i \geq 0$, we obtain

$$\beta_i(x,u)\nabla_p(y_i k_i(u,p))\eta(x,u) \geq 0,\ i = 1,2,\ldots,m.$$

Since $\beta_i(x,u) > 0$, we have

$$\nabla_p(y_i k_i(u,p))\eta(x,u) \geq 0,\ i = 1,2,\ldots,m,$$

hence,

$$\nabla_p(y^T k(u,p))\eta(x,u) \geq 0. \tag{6.5.3}$$

By (6.5.1), (6.4.4), (6.5.3) and $\alpha(x,u) > 0$, we have t

$$\begin{aligned} f(x) - f(u) - h(u,p) + p^T \nabla_p h(u,p) & \\ &\geq \alpha(x,u)\nabla_p h(u,p)\eta(x,u) \\ &= \alpha(x,u)\nabla_p(y^T k(u,p))\eta(x,u), \\ &\geq 0. \end{aligned}$$

The proof is completed. □

Theorem 6.5.2. *(Strong Duality). Let x_0 be local or global optimal solution of (P) at which a constraint qualification is satisfied, and let conditions (6.4.11) be satisfied. Then there exists $y \in R^m$ such that $(x_0, y, p = 0)$ is feasible for (HD) and the corresponding objective values of (P) and (HD) are equal. If (6.5.1) and (6.5.2) are also satisfied for all feasible (x,u,y,p), then x_0 and $(x_0, y, p = 0)$ are a global optimal solution for (P) and (HD), respectively.*

Proof. It follows on the lines of the proof of Theorem 5 of Mond and Zhang (1998). □

Remark 6.5.1. If $h(u,p) = p^T\nabla f(u)$, and $k_i(u,p) = p^T\nabla g_i(u), i = 1,2,\ldots,m$, then $(f,-g_i), i = 1,2,\ldots,m$, satisfying conditions (6.5.1) and (6.5.2), is V-type I, and the higher-order dual (HD) reduces to the Mond–Weir dual:

$$\text{(D)} \qquad \begin{aligned} &\text{maximize } f(u) \\ &\text{subject to } \nabla f(u) - \nabla y^T g(u) = 0 \\ &\qquad\qquad y_i g_i(u) \leq 0, i = 1,2,\ldots,m, \\ &\qquad\qquad y \geq 0. \end{aligned}$$

If

$$h(u,p) = p^T\nabla f(u) + \frac{1}{2}p^T\nabla^2 f(u)p \text{ and } k_i(u,p) = p^T\nabla g_i(u) + \frac{1}{2}p^T\nabla^2 g_i(u)p,$$

$i = 1,2,\ldots,m$, then $(f,-g_i)$, $i = 1,2,\ldots,m$, satisfying conditions (6.5.1) and (6.5.2), is second-order V-type I, and the higher-order dual (HD) reduces to the second-order Mond–Weir dual:

$$\text{(2D)} \qquad \begin{aligned} &\text{maximize } f(u) - \frac{1}{2}p^T\nabla^2 f(u)p \\ &\text{subject to } \nabla f(u) + \nabla^2 f(u)p = \nabla y^T g(u) + \nabla^2 y^T g(u)p, \\ &\qquad\qquad y_i g_i(u) - \frac{1}{2}p^T\nabla^2 y_i g_i(u)p \leq 0, \, i = 1, 2, \ldots, m, \\ &\qquad\qquad y \geq 0. \end{aligned}$$

The conditions (6.4.9) and (6.4.10) are some special cases of the conditions (6.5.1) and (6.5.2), where $\alpha(x,u) = 1$, and $\beta_i(x,u) = 1$, $i = 1,2,\ldots,m$.

We can also show that (HD) is a dual to (P) under some weaker conditions.

Theorem 6.5.3. *(Weak Duality). Let x be feasible for (P) and let (u,y,p) be feasible for (HD). If, for any feasible (x,u,y,p), there exists a function $\eta : R^n \times R^n \to R^n$ such that*

$$\eta(x,u)^T\nabla_p h(u,p) \geq 0 \Rightarrow f(x) - f(u) - h(u,p) + p^T\nabla_p h(u,p) \geq 0 \qquad (6.5.4)$$

and

$$-\sum_{i=1}^{m}\phi_i(x,u)\{y_ig_i(u)+y_ik_i(u,p)-p^T\nabla_p(y_ik_i(u,p))\}\geq 0$$
$$\Rightarrow \eta(x,u)^T\nabla_p(y^Tk(u,p))\geq 0, \tag{6.5.5}$$

where $\phi_i{:}R^n\times R^n\to R_+\backslash\{0\}$, $i=1,2,\ldots,m$, *are positive functions, then*

$$\text{infimum(P)}\geq\text{supremum(HD)}.$$

Proof. Since x is feasible for (P) and (u,y,p) is feasible for (HD), by (6.4.1), (6.4.5) and (6.4.6), we have

$$-y_ig_i(u)-y_ik_i(u,p)+p^T\nabla_p(y_ik_i(u,p))\geq 0,\ i=1,2,\ldots,m.$$

Since $\phi_i(x,u)>0$, it follows that

$$-\sum_{i=1}^{m}\phi_i(x,u)\{y_ig_i(u)+y_ik_i(u,p)-p^T\nabla_p(y_ik_i(u,p))\}\geq 0,$$

and by (6.5.5), we obtain

$$\eta(x,u)^T\nabla_p(y^Tk(u,p))\geq 0.$$

Using (6.4.4), it follows that

$$\eta(x,u)^T\nabla_ph(u,p)\geq 0.$$

Therefore, by (6.5.4), we have

$$f(x)\geq f(u)+h(u,p)-p^T\nabla_ph(u,p).$$

The proof is completed. □

Remark 6.5.2. If $h(u,p)=p^T\nabla f(u)$, and $k_i(u,p)=p^T\nabla g_i(u)$, $i=1,2,\ldots,m$, then (6.5.4) becomes the condition for f to be pseudo-type I (See, Rueda and Hanson 1988), and if $\phi=1$, (6.5.5) becomes the condition for $-g$ to be quasi-type I (see, Rueda and Hanson 1988). If

$$h(u,p)=p^T\nabla f(u)+\frac{1}{2}p^T\nabla^2 f(u)p$$

and

$$k_i(u,p)=p^T\nabla g_i(u)p,\ i=1,2,\ldots,m,$$

then (6.5.4) becomes the condition for f to be second-order pseudo-type I (see, Mishra 1997a). If $\phi=1$, (6.5.5) becomes the condition for $-y^T g$ to be second-order quasi-type I (see, Mishra 1997a).

Strong duality between (P) and (HD) holds if (6.5.1) and (6.5.2) are replaced by (6.5.4) and (6.5.5), respectively.

Theorem 6.5.4. *(Strict Converse Duality). Let x^0 be an optimal solution of (P) at which a constraint qualification is satisfied. Let condition (2.3) be satisfied at x^0, and let conditions (6.5.4) and (6.5.5) be satisfied for all feasible (x,u,y,p). If (x^*,y^*,p^*) is an optimal solution of (HD), and if, for all $x \neq x^*$*

$$\eta(x,x^*)^T \nabla_p h(x^* p^*) \geq 0 \Rightarrow f(x) - f(x^*) - h(x^*,p^*) + p^{*T} \nabla_p h(x^*,p^*) > 0, \quad (6.5.6)$$

then $x^0 = x^*$, i.e., x^* solves (P) and

$$f(x^0) = f(x^*) + h(x^*,p^*) - p^{*T} \nabla_p h(x^*,p^*).$$

Proof. We suppose that $x^0 \neq x^*$ and exhibit a contradiction. Since x^0 is an optimal solution of (P) at which a constraint qualification is satisfied, it follows by strong duality, that there exists $y^0 \in R^m$ such that $(x^0,y^0,p=0)$ solves (HD) and the corresponding objective values of (P) and (HD) are equal. Therefore,

$$f(x^0) = f(x^*) + h(x^*,p^*) - p^{*T} \nabla_p h(x^*,p^*). \quad (6.5.7)$$

Since x^0 is feasible for (P) and (x^*,y^*,p^*) is feasible for (HD), we have

$$-y_i g_i(x^*) - y_i^* k_i(x^*,p^*) + p^{*T} \nabla_p (y_{i_i}^* k_i(x^*,p^*)) \geq 0, \ i = 1,2,\ldots,m,$$

Since $\phi_i(x^0,x^*) > 0$, it follows that

$$-\sum_{i=1}^{m} \phi_i(x^0,x^*)\{y_i^* g_i(x^*) + y_i^* k_i(x^*,p^*) - p^{*T} \nabla_p (y_i^* k_i(x^*,p^*))\} \geq 0.$$

By (6.5.5), we obtain

$$\eta(x^0,x^*)^T \nabla_p (y^{*T} k(x^*,p^*)) \geq 0,$$

and by (6.4.4), we have

$$\eta(x^0,x^*)^T \nabla_p h(x^*,p^*) \geq 0.$$

From (6.5.6), it follows that

$$f(x^0) - f(x^*) - h(x^*,p^*) + p^{*T} \nabla_p h(x^*,p^*) > 0,$$

which is a contradiction to (6.5.7). This completes the proof. □

6.6 General Mond–Weir Higher Order Duality for Nonlinear Optimization Problems

In this section, we consider the following general Mond–Weir type higher-order dual to (P) as given by Mond and Zhang (1998).

$$\text{(M} - \text{WHD) maximize} \quad f(u)+h(u,p)-p^T\nabla_p h(u,p)-\sum_{i\in I_0} y_i g_i(u)-\sum_{i\in I_0} y_i k_i(u,p) + p^T\nabla_p[\sum_{i\in I_0} y_i k_i(u,p)]$$

$$\text{subject to} \quad \nabla_p h(u,p)=\nabla_p(y^T k(u,p))$$

$$\sum_{i\in I_\alpha} y_i g_i(u)+\sum_{i\in I_\alpha} y_i k_i(u,p)-p^T\nabla_p[\sum_{i\in I_\alpha} y_i k_i(u,p)]\leq 0,\ \alpha=1,2,\ldots,r,$$

$$y\geq 0,$$

where $I_\alpha \subseteq M=\{1,2,\ldots,m\}, \alpha=0,1,2,\ldots,r$ with $\bigcup_{\alpha=0}^{r} I_\alpha = M$ and $I_\alpha \bigcap I_\beta = \emptyset$, if $\alpha\neq\beta$.

Mond and Zhang (1998) showed that (M-WHD) is a dual to (P) under the following conditions:

$$\eta(x,u)^T\left[\nabla_p h(u,p)-\nabla_p\Big(\sum_{i\in I_0} y_i k_i(u,p)\Big)\right]\geq 0$$
$$\Rightarrow f(x)-\sum_{i\in I_0} y_i g_i(x)-(f(u)-\sum_{i\in I_0} y_i g_i(u))-(h(u,p)-\sum_{i\in I_0} y_i k_i(u,p))$$
$$+p^T\left[\nabla_p h(u,p)-\nabla_p\Big(\sum_{i\in I_0} y_i k_i(u,p)\Big)\right]\geq 0 \tag{6.6.1}$$

and

$$\sum_{i\in I_\alpha} y_i g_i(x)-\sum_{i\in I_\alpha} y_i g_i(u)-\sum_{i\in I_\alpha} y_i k_i(u,p)+p^T\nabla_p\Big(\sum_{i\in I_\alpha} y_i k_i(u,p)\Big)\geq 0$$
$$\Rightarrow \eta(x,u)^T\nabla_p\Big(\sum_{i\in I_\alpha} y_i k_i(u,p)\Big)\geq 0,\ \alpha=1,2,\ldots,r. \tag{6.6.2}$$

We can generalize (6.6.1) and (6.6.2) under which (M-WHD) is a dual to (P), to generalized type I conditions, i.e., pseudo-type I and quasi-type I conditions. Since the proof follows along the lines of the one in Mond and Zhang (1998), we give the theorem without proof.

Theorem 6.6.1. *(Weak Duality). Let x be feasible for (P) and let (u,y,p) be feasible for (M-WHD). If for any feasible (x,u,y,p),*

$$\eta(x,u)^T\left[\nabla_p h(u,p)-\nabla_p\left(\sum_{i\in I_0} y_i k_i(u,p)\right)\right]\geq 0$$
$$\Rightarrow f(x)-\sum_{i\in I_0} y_i g_i(x)-\left(f(u)-\sum_{i\in I_0} y_i g_i(u)\right)-\left(h(u,p)-\sum_{i\in I_0} y_i k_i(u,p)\right)$$
$$+p^T\left[\nabla_p h(u,p)-\nabla_p\left(\sum_{i\in I_0} y_i k_i(u,p)\right)\right]\geq 0 \tag{6.6.3}$$

and

$$-\sum_{i\in I_\alpha} y_i g_i(u)-\sum_{i\in I_\alpha} y_i k_i(u,p)+p^T\nabla_p\left(\sum_{i\in I_\alpha} y_i k_i(u,p)\right)\geq 0$$
$$\Rightarrow \eta(x,u)^T\nabla_p\left(\sum_{i\in I_\alpha} y_i k_i(u,p)\right)\geq 0,\ \alpha=1,2,\ldots,r, \tag{6.6.4}$$

then

$$\text{infimum (P)} \geq \text{supremum(M}-\text{WHD)}.$$

Remark 6.6.1. If $I_0=\emptyset$, and $I_i=\{i\}, i=1,2,\ldots,m,(r=m)$, then (M-WHD) becomes (HD) and the conditions (6.6.3) and (6.6.4) reduce to the conditions (6.5.4) and (6.5.5), respectively.

6.7 Mangasarian Type Higher Order Duality for Nondifferentiable Optimization Problems

In this section, we consider several higher-order duals to a nondifferentiable optimization problem and establish duality theorems under the higher-order generalized invexity conditions introduced in an earlier work by Mishra and Rueda (2000).

Mond (1974b) considered the following nondifferentiable optimization problem:

$$\text{(NDP)} \qquad \text{minimize } f(x)+(x^TBx)^{1/2}$$
$$\text{subject to } g(x)\geq 0, \tag{6.7.1}$$

where f and g are twice differentiable functions from R^n to R and R^m, respectively, and B is an $n\times n$ positive semi-definite (symmetric) matrix.

Let x_0 satisfy (6.7.1). Mond (1974b) defined the set

$$Z_0=\{z: z^T\nabla g_i(x_0)\geq 0\,(\forall\ i\in Q_0) \quad \textit{and}$$
$$z^T\nabla f(x_0)+z^TBx_0/(x_0^TBx_0)^{1/2}<0 \quad \textit{if} \quad x_0^TBx_0>0,$$
$$z^T\nabla f(x_0)+(z^TBz)^{1/2}<0 \quad \textit{if} \quad x_0^TBx_0=0\},$$

where $Q_0=\{i: g_i(x_0)=0\}$, and established the following necessary conditions for x_0 to be an optimal solution to (NDP).

Proposition 6.7.1. *If x_0 is an optimal solution of (NDP) and the corresponding set Z_0 is empty, then there exist $y \in R^m$, $y \geq 0$ and $w \in R^n$ such that*

$$y^T g(x_0) = 0, \quad \nabla y^T g(x_0) = \nabla f(x_0) + Bw, \quad w^T Bw \leq 1, \quad (x_0^T Bx_0)^{1/2} = x_0^T Bw.$$

We shall make use of the generalized Schwarz inequality from Reisz and Nagy (1955):

$$(x^T Bw) \leq (x^T Bx)^{1/2} (w^T Bw)^{1/2}. \tag{6.7.2}$$

The second-order Mangasarian type (Mangasarian 1975) and Mond–Weir type (1981–1983) duals to (NDP) were given by Bector and Chandra (1997) as the following problems:

$$\begin{aligned} \text{(ND2MD)} \qquad & \text{maximize } f(u) - y^T g(u) + u^T Bw - \frac{1}{2} p^T \nabla^2 [f(u) - y^T g(u)]\, p \\ & \text{subject to } \nabla f(u) - \nabla y^T g(u) + Bw + \nabla^2 f(u) p - \nabla^2 y^T g(u) p = 0, \\ & \qquad w^T Bw \leq 1, y \geq 0, \end{aligned}$$

where $u, w, p \in R^n$ and $y \in R^m$;

$$\begin{aligned} \text{(ND2D)} \qquad & \text{maximize } f(u) + u^T Bw - \frac{1}{2} p^T \nabla^2 f(u)\, p \\ & \text{subject to } \nabla f(u) - \nabla y^T g(u) + Bw + \nabla^2 f(u) p - \nabla^2 y^T g(u) p = 0, \\ & \qquad y^T g(u) - \frac{1}{2} p^T \nabla^2 y^T g(u) p \leq 0, \\ & \qquad w^T Bw \leq 1, \ y \geq 0. \end{aligned}$$

Using the second-order convexity condition Bector and Chandra (1997) established duality theorems between (NDP) and (ND2MD) and (ND2D), respectively.

The Mangasarian type (1975) and Mond–Weir type (1981–1983) higher-order dual to (NDP) were given by Zhang (1998) as follows:

$$\begin{aligned} \text{(NDHMD)} \quad & \text{maximize } f(u) + h(u,p) + (u+p)^T Bw - y^T g(u) - y^T k(u,p) \\ & \text{subject to } \nabla_p h(u,p) + Bw = \nabla_p (y^T k(u,p)) \\ & \qquad w^T Bw \leq 1, \ y \geq 0, \end{aligned} \tag{6.7.3}$$

where $u, w, p \in R^n$ and $y \in R^m$;

$$\begin{aligned} \text{(NDHD)} \quad & \text{maximize } f(u) + h(u,p) + u^T Bw - p^T \nabla_p h(u,p) \\ & \text{subject to } \nabla_p h(u,p) + Bw = \nabla_p (y^T k(u,p)), \\ & \qquad y^T g(u) + y^T k(u,p) - p^T \nabla_p (y^T k(u,p)) \leq 0, \\ & \qquad w^T Bw \leq 1, \ y \geq 0. \end{aligned}$$

Duality results have been established under higher-order invexity and generalized higher-order invexity assumptions between (NDP) and (NDHMD) and (NDHD), as in Zhang (1998).

Definition 6.7.1. *The function f and the constraint functions $g_i, i = 1,2,\ldots,m$, are said to be higher-order type I at u with respect to a function η if, for all x, the following inequalities hold:*

$$f(x)+x^T Bw - f(u) - u^T Bw \geq \eta(x,u)^T [\nabla_p h(u,p) + Bw] + h(u,p) - p^T (\nabla_p h(u,p))$$

and

$$-g_i(u) \leq \eta(x,u)^T \nabla_p k_i(u,p) + k_i(u,p) - p^T (\nabla_p k_i(u,p)),\ i = 1,2,\ldots,m.$$

Definition 6.7.2. *The objective function f and the constraint functions $g_i, i = 1, 2,\ldots, m$, are said to be higher-order pseudo-quasi type I at u with respect to a function η if, for all x, the following implications hold:*

$$\eta(x,u)^T [\nabla_p h(u,p) + Bw] \geq 0$$
$$\Rightarrow f(x) + x^T Bw - f(u) - h(u,p) - u^T Bw + p^T \nabla_p h(u,p) \geq 0$$

and

$$-g_i(u) \geq k_i(u,p) - p^T \nabla_p k_i(u,p)$$
$$\Rightarrow \eta(x,u)^T \nabla_p k_i(u,p) \geq 0,\quad i = 1, 2,\ldots, m.$$

In this section, we will establish some duality results between (NDP) and (NDHMD). The following Theorem generalizes Theorem 4.4.1 in Zhang (1998) to higher-order type I functions.

Theorem 6.7.1. *(Weak Duality). Let x be feasible for (NDP) and let (u,y,w,p) be feasible for (NDHMD). If, for all feasible (x,u,y,w,p), there exists a function $\eta : R^n \times R^n \to R^n$ such that*

$$f(x)+x^T Bw - f(u) - u^T Bw \geq \eta(x,u)^T [\nabla_p h(u,p) + Bw] + h(u,p) - p^T (\nabla_p h(u,p)) \tag{6.7.4}$$

and

$$-g_i(u) \leq \eta(x,u)^T \nabla_p k_i(u,p) + k_i(u,p) - p^T (\nabla_p k_i(u,p)),\quad i = 1,2,\ldots,m, \tag{6.7.5}$$

then

$$\text{infimum(NDP)} \geq \text{supremum(NDHMD)}.$$

Proof.

$$
\begin{aligned}
&f(x)+x^T Bw - f(u) - h(u,p) - (u+p)^T Bw + y^T g(u) + y^T k(u,p)\\
&\quad \geq \eta(x,u)^T [\nabla_p h(u,p) + Bw] - p^T [\nabla_p h(u,p) + Bw] + y^T g(u) + y^T k(u,p)\\
&\quad = \eta(x,u)^T [\nabla_p (y^T k(u,p))] - p^T [\nabla_p y^T k(u,p)] + y^T g(u) + y^T k(u,p) \geq 0.
\end{aligned}
$$

The first inequality follows from (6.7.4), the equality follows from (6.7.3), and the second inequality follows from (6.7.5) and $y \geq 0$.

Since $w^T Bw \leq 1$, by the generalized Schwarz inequality (6.7.2), it follows that

$$f(x) + (x^T Bx)^{1/2} \geq f(u) + h(u,p) + (u+p)^T Bw - y^T g(u) - y^T k(u,p).$$

The proof is completed. □

Theorem 6.7.2. *(Strong Duality). Let x_0 be a local or global optimal solution of (NDP) with corresponding set Z_0 empty and*

$$h(x_0,0) = 0,\ k(x_0,0) = 0,\ \nabla_p h(x_0,0) = \nabla f(x_0),\ \nabla_p k(x_0,0) = \nabla g(x_0). \tag{6.7.6}$$

Then there exist $y \in R^m$ and $w \in R^n$ such that $(x_0, y, w, p = 0)$ is feasible for (NDHMD) and the corresponding objective values of (NDP) and (NDHMD) are equal. If (6.7.4) and (6.7.5) are also satisfied at x_0 for any feasible (x,u,y,w,p), then x_0 and $(x_0, y, w, p = 0)$ are a global optimal solution for (NDP) and (NDHMD), respectively.

Proof. Since x_0 is an optimal solution to (NDP) and the corresponding set Z_0 is empty, then from Proposition 6.7.1, there exist $y \in R^m$ and $w \in R^n$ such that

$$y^T g(x_0) = 0,\ \nabla y^T g(x_0) = \nabla f(x_0) + Bw,\ w^T Bw \leq 1,\ (x_0^T Bx_0)^{1/2} = x_0^T Bw,\ y \geq 0.$$

Then, using (6.7.6), we have that $(x_0, y, w, p = 0)$ is feasible for (NDHMD) and the corresponding objective values of (NDP) and (NDHMD) are equal.

If (6.7.4) and (6.7.5) are also satisfied, then from Theorem 6.7.1, $(x_0, y, w, p = 0)$ is an optimal solution for (NDHMD). This completes the proof. □

Now we turn to show that weak duality between (NDP) and (NDHMD) holds under weaker higher-order type I conditions than those given in Theorem 6.7.1. The following theorem is a generalization of Theorem 4.4.3 in Zhang (1998). □

Theorem 6.7.3. *(Weak Duality). Let x be feasible for (NDP) and let (u,y,w,p) be feasible for (NDHMD). If, for all feasible (x,u,y,w,p), there exists a function $\eta : R^n \times R^n \to R^n$ such that*

$$\eta(x,u)^T [\nabla_p h(u,p) + Bw - \nabla_p (y^T k(u,p))] \geq 0,$$

$$
\begin{aligned}
\Rightarrow\ & f(x) + x^T Bw - (f(u) + u^T Bw - y^T g(u))\\
& - (h(u,p) - y^T k(u,p)) + p^T [\nabla_p h(u,p) - \nabla_p y^T k(u,p)] \geq 0,
\end{aligned}
\tag{6.7.7}
$$

then

$$\text{infimum(NDP)} \geq \text{supremum(NDHMD)}.$$

Proof. From $\nabla_p h(u,p) + Bw = \nabla_p (y^T k(u,p))$, we have

$$\eta(x,u)^T [\nabla_p h(u,p) + Bw - \nabla_p (y^T k(u,p))] = 0.$$

Hence, by (6.7.7), we have

$$\begin{aligned} &f(x) + x^T Bw - (f(u) + u^T Bw - y^T g(u)) \\ &- (h(u,p) - y^T k(u,p)) + p^T [\nabla_p h(u,p) - \nabla_p y^T k(u,p)] \geq 0. \end{aligned}$$

Since (u,y,w,p) is feasible for (NDHMD), we get

$$f(x) + x^T Bw \geq f(u) + (u+p)^T Bw - y^T g(u) + h(u,p) - y^T k(u,p).$$

Then, by $w^T Bw \leq 1$ and the generalized Schwarz inequality (6.7.2) it follows that

$$f(x) + (x^T Bx)^{1/2} \geq f(u) + (u+p)^T Bw - y^T g(u) + h(u,p) - y^T k(u,p).$$

The proof is completed. □

Remark 6.7.1. If $h(u,p) = p^T \nabla f(u) + \frac{1}{2} p^T \nabla^2 f(u) p$, and $k_i(u,p) = p^T \nabla g_i(u) + \frac{1}{2} p^T \nabla^2 g_i(u) p$, $i = 1,2,\ldots,m$, then the conditions (6.7.4) and (6.7.5) are sufficient for $(f(.) + .^T Bw, -g_i(.))$ to be second-order type I.

Remark 6.7.2. The following example given in Mishra (1997a) shows that condition (6.7.7) is weaker than (6.7.4) and (6.7.5).

Consider $f(x_1,x_2) = x_1x_2 + x_1 + x_2, g_1(x_1,x_2) = x_2 - x_1^3, g_2(x_1,x_2) = 1 - x_1x_2$, $g_3(x_1,x_2) = x_1 - x_2^2, B \equiv 0, p = 0$, and h and k defined as in Remark 6.7.1. It is not hard to verify that condition (6.7.7) is satisfied at $(0,0)$ for any η, while conditions (6.7.4) and (6.7.5) are satisfied only when the components of η are non-negative.

6.8 Mond–Weir Type Higher Order Duality for Nondifferentiable Optimization Problems

We consider a Mond–Weir type higher-order dual to (NDP) as in Zhang (1998).

$$\begin{aligned} \text{(NDHD)} \quad \text{maximize} \quad & f(u) + h(u,p) + u^T Bw - p^T \nabla_p h(u,p) \\ \text{subject to} \quad & \nabla_p h(u,p) + Bw = \nabla_p (y^T k(u,p)), \qquad (6.8.1) \\ & y^T g(u) + y^T k(u,p) - p^T \nabla_p (y^T k(u,p)) \leq 0, \qquad (6.8.2) \\ & w^T Bw \leq 1, y \geq 0. \end{aligned}$$

The following theorem is a generalization of Theorem 4.4.4 in Zhang (1998) to higher-order type I functions.

Theorem 6.8.1. *(Weak Duality). Let x be feasible for (NDP) and let (u,y,w,p) be feasible for (NDHD). If, for all feasible (x,u,y,w,p), $f(.)+.^T Bw$ and $g(.)$ satisfy the conditions (6.7.4) and (6.7.5) of Theorem 6.7.1, respectively, then*

$$\text{infimum (NDP)} \geq \text{supremum (NDHD)}.$$

Proof. By (6.7.4), (6.8.1), (6.7.5), (6.8.2) and $y \geq 0$, we have

$$\begin{aligned} f(x)+x^T Bw-f(u)-h(u,p)-u^T Bw+p^T \nabla_p h(u,p) \\ &\geq \eta(x,u)^T [\nabla_p h(u,p)+Bw] \\ &= \eta(x,u)^T [\nabla_p (y^T k(u,p))] \\ &\geq -y^T g(u)-y^T k(u,p)+p^T (\nabla_p (y^T k(u,p))) \\ &\geq 0. \end{aligned}$$

Since $w^T Bw \leq 1$, by the generalized Schwarz inequality (6.7.2), it follows that

$$f(x)+(x^T Bx)^{1/2} \geq f(u)+h(u,p)+u^T Bw-p^T \nabla_p h(u,p).$$

The proof is completed. □

The following strong duality theorem follows along the lines of Theorem 6.7.2.

Theorem 6.8.2. *(Strong Duality). Let x_0 be a local or global optimal solution of (NDP) with corresponding set Z_0 empty and let condition (6.7.6) be satisfied. Then there exist $y \in R^m$ and $w \in R^n$ such that $(x_0, y, w, p=0)$ is feasible for (NDHD) and the corresponding objective values of (NDP) and (NDHD) are equal. If (6.7.4) and (6.7.5) are also satisfied at x_0 for all feasible (x,u,y,w,p), then x_0 and $(x_0, y, w, p=0)$ are a global optimal solution for (NDP) and (NDHD), respectively.*

Weaker conditions under which (NDHD) is a dual to (NDP) can also be obtained. The following is a generalization of Theorem 4.4.6 in Zhang (1998) to higher-order pseudo-quasi type I functions.

Theorem 6.8.3. *(Weak Duality). Let x be feasible for (NDP) and let (u,y,w,p) be feasible for (NDHD). If, for any feasible (x,u,y,w,p), there exists a function $\eta: R^n \times R^n \to R^n$ such that*

$$\begin{aligned} &\eta(x,u)^T [\nabla_p h(u,p)+Bw] \geq 0 \\ &\quad \Rightarrow f(x)+x^T Bw-f(u)-h(u,p)-u^T Bw+p^T \nabla_p h(u,p) \geq 0 \end{aligned} \qquad (6.8.3)$$

and

$$-y^T g(u) \geq y^T k(u,p) - p^T \nabla_p y^T k(u,p) \Rightarrow \eta(x,u)^T \left[\nabla_p (y^T k(u,p))\right] \geq 0, \quad (6.8.4)$$

then

$$\text{infimum(NDP)} \geq \text{supremum(NDHD)}.$$

Proof. Since (u,y,w,p) is feasible for (NDHD), by (6.8.1)–(6.8.4), we have

$$\begin{aligned} -y^T g(u) &\geq y^T k(u,p) - p^T \nabla_p y^T k(u,p) \\ &\Rightarrow \eta(x,u)^T \left[\nabla_p (y^T k(u,p))\right] \geq 0 \\ &\Rightarrow \eta(x,u)^T \left[\nabla_p h(u,p) + Bw\right] \geq 0 \\ &\Rightarrow f(x) + x^T Bw \geq f(u) + h(u,p) + u^T Bw - p^T \nabla_p h(u,p). \end{aligned}$$

Since $w^T Bw \leq 1$, by the generalized Schwarz inequality (6.7.2), it follows that

$$f(x) + (x^T Bx)^{1/2} \geq f(u) + h(u,p) + u^T Bw - p^T \nabla_p h(u,p).$$

The proof is completed. □

Remark 6.8.1. If $h(u,p) = p^T \nabla f(u) + \frac{1}{2} p^T \nabla^2 f(u) p$, and $k_i(u,p) = p^T \nabla_{g_i}(u) + \frac{1}{2} p^T \nabla^2 g_i(u) p, i = 1,2,\ldots,m$, then (NDHD) becomes (ND2D).

6.9 General Mond–Weir Type Higher Order Duality for Nondifferentiable Optimization Problems

In this section, we consider the following general higher-order dual to (NDP).

$$\begin{aligned} \text{(NDHGD)} \quad & \text{maximize } f(u) + h(u,p) + u^T Bw - p^T \nabla_p h(u,p) - \sum_{i \in I_0} y_i g_i(u) \\ & \qquad - \sum_{i \in I_0} y_i k_i(u,p) + p^T \nabla_p \Big[\sum_{i \in I_0} y_i k_i(u,p) \Big] \\ & \text{subject to } \nabla_p h(u,p) + Bw = \nabla_p (y^T k(u,p)) \\ & \qquad \sum_{i \in I_\alpha} y_i g_i(u) + \sum_{i \in I_\alpha} y_i k_i(u,p) - p^T \Big[\nabla_p \Big(\sum_{i \in I_\alpha} y_i k_i(u,p) \Big) \Big] \leq 0, \\ & \qquad \alpha = 1,2,\ldots,r, \\ & \qquad w^T Bw \leq 1, \; y \geq 0, \end{aligned}$$

where $I_\alpha \subseteq M = \{1,2,\ldots,m\}, \alpha = 0,1,2,\ldots,r,$ with $\bigcup_{\alpha=0}^{r} I_\alpha = M$ and $I_\alpha \cap I_\beta = \emptyset$ if $\alpha \neq \beta$.

Theorem 6.9.1. *(Weak Duality). Let x be feasible for (NDP) and let* (u,y,w,p) *be feasible for (NDHGD). If, for any feasible* (x,u,y,w,p)*, there exists a function* $\eta: R^n \times R^n \to R^n$ *such that*

$$\eta(x,u)^T\Big[\nabla_p h(u,p) + Bw - \nabla_p\Big(\sum_{i\in I_0} y_i k_i(u,p)\Big)\Big] \geq 0$$
$$\Rightarrow f(x) + x^T Bw - \Big(f(u) + u^T Bw - \sum_{i\in I_0} y_i g_i(u)\Big)$$
$$-(h(u,p) - \sum_{i\in I_0} y_i k_i(u,p)) + p^T\Big[\nabla_p h(u,p) - \nabla_p\Big(\sum_{i\in I_0} y_i k_i(u,p)\Big)\Big] \geq 0 \tag{6.9.1}$$

and

$$-\sum_{i\in I_\alpha} y_i g_i(u) - \sum_{i\in I_\alpha} y_i k_i(u,p) + p^T\Big[\nabla_p\Big(\sum_{i\in I_\alpha} y_i k_i(u,p)\Big)\Big] \geq 0$$
$$\Rightarrow \eta(x,u)^T\Big[\nabla_p\Big(\sum_{i\in I_\alpha} y_i k_i(u,p)\Big)\Big] \geq 0, \quad \alpha = 1,2,\ldots,r, \tag{6.9.2}$$

then

$$\text{infimum(NDP)} \geq \text{supremum(NDHGD)}.$$

Proof. Since (u,y,w,p) is feasible for (NDHGD), we have, for all $\alpha = 0,1,2,\ldots,r$,

$$-\sum_{i\in I_\alpha} y_i g_i(u) - \sum_{i\in I_\alpha} y_i k_i(u,p) + p^T\Big[\nabla_p(\sum_{i\in I_\alpha} y_i k_i(u,p))\Big] \geq 0,$$

by (4.2), it

$$\Rightarrow \eta(x,u)^T[\nabla_p(\sum_{i\in I_\alpha} y_i k_i(u,p))] \geq 0,$$
$$\Rightarrow \eta(x,u)^T[\nabla_p(\sum_{i\in M\setminus I_0} y_i k_i(u,p))] \geq 0,$$

Since (u,y,w,p) is feasible for (NDHGD), it

$$\Rightarrow \eta(x,u)^T[\nabla_p h(u,p) + Bw - \nabla_p(\sum_{i\in I_0} y_i k_i(u,p))] \geq 0,$$
$$\Rightarrow f(x) + x^T Bw - (f(u) + u^T Bw - \sum_{i\in I_0} y_i g_i(u))$$
$$-(h(u,p) - \sum_{i\in I_0} y_i k_i(u,p)) + p^T[\nabla_p h(u,p) - \nabla_p(\sum_{i\in I_0} y_i k_i(u,p))] \geq 0.$$

Since $w^T Bw \leq 1$, by the generalized Schwarz inequality (6.7.2), it follows that

$$f(x) + (x^T Bx)^{1/2} \geq f(u) + u^T Bw - \sum_{i\in I_0} y_i g_i(u)$$
$$+ h(u,p) - (\sum_{i\in I_0} y_i k_i(u,p)) + p^T [\nabla_p h(u,p) - \nabla_p(\sum_{i\in I_0} y_i k_i(u,p))].$$

The proof is completed. □

The proof of the following strong duality theorem follows along the lines of Theorem 6.7.2, therefore we state the result but omit the proof.

Theorem 6.9.2. *(Strong Duality). Let x_0 be a local or global optimal solution of (NDP) with corresponding set Z_0 empty and let condition (6.7.6) be satisfied. Then there exist $y \in R^m$ and $w \in R^n$ such that $(x_0, y, w, p = 0)$ is feasible for (NDHGD) and the corresponding objective values of (NDP) and (NDHGD) are equal. If (6.9.1) and (6.9.2) are also satisfied at x_0 for all feasible (x,u,y,w,p), then x_0 and $(x_0, y, w, p = 0)$ are a global optimal solution for (NDP) and (NDHGD), respectively.*

Remark 6.9.1. If $I_0 = M$, then (NDHGD) becomes (NDHMD) and the conditions (6.9.1) and (6.9.2) of Theorem 6.9.1 reduce to the condition (6.7.7) of Theorem 6.7.3. If $I_0 = \emptyset$ and $I_\alpha = M$ for some $\alpha \in \{1,2,\ldots,r\}$, then (NDHGD) becomes (NDHD) and the conditions (6.9.1) and (6.9.2) reduce to the conditions (6.8.3) and (6.8.4), respectively, of Theorem 6.8.3.

Theorem 6.9.3. *(Strict Converse Duality). Let x_0 be an optimal solution of (NDP) with the corresponding set Z_0 empty. Let conditions (6.7.6) be satisfied at x_0, and let conditions (6.9.1) and (6.9.2) of Theorem 6.9.1 be satisfied for all feasible (x,u,y,w,p). If (x^*, y^*, w^*, p^*) is an optimal solution of (NDHGD) and if, for all $x \neq x^*$*

$$\eta(x,x^*)^T [\nabla_p h(x^*,p^*) + Bw^* - \nabla_p(\sum_{i\in I_0} y_i^* k_i(x^*,p^*))] \geq 0$$
$$\Rightarrow f(x) + x^T Bw^* - (f(x^*) + x^{*T} Bw^* - \sum_{i\in I_0} y_i^* g_i(x^*))$$
$$- (h(x^*,p^*) - \sum_{i\in I_0} y_i^* k_i(x^*,p^*)) + p^{*T} [\nabla_p h(x^*,p^*)$$
$$- \nabla_p(\sum_{i\in I_0} y_i^* k_i(x^*,p^*))] > 0$$

then $x_0 = x^*$, i.e., x^* solves (NDP) and

$$f(x_0) + (x_0^T Bx_0)^{1/2} = f(x^*) + h(x^*,p^*) + x^{*T} Bw^* - p^{*T} \nabla_p h(x^*,p^*) - \sum_{i\in I_0} y_i^* g_i(x^*)$$
$$- \sum_{i\in I_0} y_i^* k_i(x^*,p^*) + p^{*T} \nabla_p [\sum_{i\in I_0} y_i^* k_i(x^*,p^*)].$$

Chapter 7
Symmetric Duality

Dorn (1960) introduced symmetric duality as a program and its dual to be symmetric if the dual of the dual is the original problem. A linear program and its dual are symmetric in this sense. However, this is not the case in nonlinear programs in general. Following Dorn (1960) many authors have contributed to symmetric duality, see Dantzig et al. (1965), Bazaraa and Goode (1973), Chandra et al. (1985), Cottle (1963), Hou and Yang (2001), Kim et al. (1998a), Mond (1965), Mond and Weir (1981), Mond et al. (1987), Nanda and Das (1996), Devi (1998), Mishra (2000a, 2000b, 2001b), Mishra and Wang (2005) and Weir and Mond (1988b).

Balas (1991) generalized symmetric duality of Dantzig et al. (1965) by constraining some of the primal and dual variables to belong to arbitrary sets and thus introduced formulated a distinct pair of minimax symmetric dual programs on the lines of Mond and Weir (1981). Following Balas (1991) many results have appeared in the literature see for example, Kim et al. (1998a), Mishra (2000a, 2000b), Mishra and Wang (2005).

In this chapter, we formulate various pairs of higher-order multiobjective symmetric dual models and establish higher-order symmetric duality results under higher-order invexity and higher-order incavity assumptions on the functions involved. Results of Suneja et al. (2003), Hou and Yang (2001), Mishra (2000b) and Mond and Schechter (1996) are special cases of the results obtained in this chapter.

7.1 Higher Order Symmetric Duality

Throughout this section, we denote by R^n the n-dimensional Euclidean space and R^n_+ the non-negative orthant of R^n, respectively.

Let C be a compact convex set in R^n. The support function of C is defined by

$$s(x|C) = \max\{x^T y : y \in C\}.$$

A support function, being convex and everywhere finite, has a subdifferential (see Clarke 1983), that is, there exists $z \in R^n$ such that

$$s(y|C) \geq s(x|C) + z^T(y-x) \forall y \in C.$$

The subdifferential of $s(x|C)$ is given by

$$\partial s(x|C) = \{z \in C : z^T x = s(x|C)\}.$$

For any set $D \subset R^n$, the normal cone to D at a point $x \in D$ is defined by

$$N_D(x) = \{y \in R^n : y^T(z-x) \leq 0 \ \forall z \in D\}.$$

It is obvious that for a compact convex set C, $y \in N_C(x)$ if and only if $s(y|C) = x^T y$, or equivalently, $x \in \partial s(y|C)$.

For a real-valued twice differentiable function $g(x,y)$ defined on an open set in $R^n \times R^m$, we denote by $\nabla_x g(\bar{x},\bar{y})$ the gradient of g with respect to x at $(\bar{x},\bar{y})$, $\nabla_{xx} g(\bar{x},\bar{y})$ the Hessian matrix with respect to x at $(\bar{x},\bar{y})$. $\nabla_y g(\bar{x},\bar{y})$, $\nabla_{xy} g(\bar{x},\bar{y})$ and $\nabla_{yy} g(\bar{x},\bar{y})$ can be similarly defined.

Definition 7.1.1. *A function f is said to higher-order-invex at $u \in X$ with respect to $\eta : R^n \times R^n \to R^n$ and $h : X \times R^n \to R$, if for all $(x,p) \in X \times R^n$*

$$f(x) - f(u) \geq \eta(x,u)^T [\nabla_u f(u) + \nabla_p h(u,p)] + h(u,p) - p^T \nabla_p h(u,p).$$

Definition 7.1.2. *A function f is said to higher-order-pseudo-invex at $u \in X$ with respect to $\eta : R^n \times R^n \to R^n$ and $h : X \times R^n \to R$, if for all $(x,p) \in X \times R^n$*

$$\eta(x,u)^T [\nabla_u f(u) + \nabla_p h(u,p)] \geq 0 \Rightarrow f(x) - f(u) - h(u,p) + p^T \nabla_p h(u,p) \geq 0.$$

Consider the following vector optimization problem:

(MP) minimize $f(x)$
subject to $x \in X_0$,

where f is a k-dimensional vector function defined on R^n and $X_0 \subseteq R^n$.

Definition 7.1.3. *A feasible point x^* is said to be a weak minimum (efficient solution) of (MP) if there does not exist any feasible x such that $f(x) < (\leq) f(x^*)$.*

Definition 7.1.4. *A feasible point x^* is said to be a properly efficient solutionof (MP) if it is an efficient solution of (MP) and if there exists a scalar $M > 0$ such that for each i and $x \in X_0$ satisfying $f_i(x) < f_i(x*)$ we have*

$$\frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x*)} \leq M$$

for some j, satisfying $f_j(x) > f_j(x^*)$.

We consider the following higher-order symmetric duality problem:

$$\text{(SHP)} \quad \text{minimize } f(x,y) + s(x|C) - y^T z + h(x,y,p) - p^T \nabla_p h(x,y,p)$$

$$\text{subject to } \nabla_y f(x,y) - z + \nabla_p h(x,y,p) \leq 0, \tag{7.1.1}$$

$$y^T [\nabla_y f(x,y) - z + \nabla_p h(x,y,p)] \geq 0, \tag{7.1.2}$$

$$z \in D, \tag{7.1.3}$$

$$\text{(SHD)} \quad \text{maximize } f(u,v) - s(v|D) + u^T w + g(u,v,q) - q^T \nabla_q g(u,v,q)$$

$$\text{subject to } \nabla_u f(u,v) + w + \nabla_q g(u,v,q) \geq 0, \tag{7.1.4}$$

$$u^T [\nabla_u f(u,v) + w + \nabla_q g(u,v,q)] \leq 0, \tag{7.1.5}$$

$$w \in C, \tag{7.1.6}$$

where C and D are two compact convex sets in R^n and R^m, respectively, and $f : R^n \times R^m \to R$; both $h : R^n \times R^m \times R^m \to R$; and $g : R^n \times R^m \times R^n \to R$ are differentiable functions.

Theorem 7.1.1. *(Weak Duality). For each feasible solution (x,y,z,p) of (SHP) and for each feasible solution (u,v,w,q) of (SHD), if $f(.,v) + .^T w$ is higher-order-η_1 -invex in the first variable at u with respect to η_1 and $g(u,v,q)$ and $-[f(x.,.) - .^T z]$ is higher-order-η_2 -invex in the second variable at y with respect to η_2 and $-h(x,y,p)$ and*

$$\eta_1(x,u) + u \geq 0 \tag{7.1.7}$$

$$\eta_2(v,y) + y \geq 0. \tag{7.1.8}$$

Then

$$\inf(SHP) \geq \sup(SHD).$$

Proof. Since (u,v,w,q) is feasible for (SHD), from (7.1.4) and (7.1.7), it follows that

$$[\eta_1(x,u) + u]^T [\nabla_u f(u,v) + w + \nabla_q g(u,v,q)] \geq 0.$$

From (7.1.5), we get

$$\eta_1(x,u)^T [\nabla_u f(u,v) + w + \nabla_q g(u,v,q)] \geq 0. \tag{7.1.9}$$

Since (x,y,z,p) is feasible for (SHP), from (7.1.1) and (7.1.8), it follows that

$$[\eta_2(v,y) + y]^T [\nabla_y f(x,y) - z + \nabla_p h(x,y,p)] \leq 0.$$

Using (7.1.2), we get

$$\eta_2(v,y)^T [\nabla_y f(x,y) - z + \nabla_p h(x,y,p)] \leq 0. \tag{7.1.10}$$

By the higher-order-η_1-invexity of $f(.,v)+.^T w$ with respect to η_1 and $g(u,v,q)$, and from (7.1.9), we get

$$[f(x,v)+x^T w]-[f(u,v)+u^T w]\geq g(u,v,q)-q^T\nabla_q g(u,v,q),$$

that is,

$$f(x,v)\geq f(u,v)-x^T w-+u^T w+g(u,v,q)-q^T\nabla_q g(u,v,q). \tag{7.1.11}$$

On the other hand, by the higher-order-η_2 -invexity of $-[f(x.,.).^T z]$ with respect to η_2 and $-h(x,y,p)$ and from (7.1.10), we get

$$-[f(x,v)-v^T z]+[f(x,y)-y^T z]\geq -h(x,y,p)+p^T\nabla_p h(x,y,p),$$

that is,

$$f(x,v)\leq f(x,y)+v^T z-y^T z+h(x,y,p)-p^T\nabla_p h(x,y,p). \tag{7.1.12}$$

From (7.1.11) and (7.1.12), we obtain

$$\begin{aligned} f(u,v)-x^T w-+u^T w+g(u,v,q)-q^T\nabla_q g(u,v,q)& \\ \leq f(x,y)+v^T z-y^T z+h(x,y,p)-p^T\nabla_p h(x,y,p).& \end{aligned} \tag{7.1.13}$$

Noting that $x^T w\leq s(x|C)$ and $v^T z\leq s(v|D)$, (7.1.13) yields

$$\begin{aligned} f(u,v)-s(v|D)-+u^T w+g(u,v,q)-q^T\nabla_q g(u,v,q)& \\ \leq f(x,y)+s(x|C)-y^T z+h(x,y,p)-p^T\nabla_p h(x,y,p),& \end{aligned}$$

that is, $\sup(SHD)\leq\inf(SHP)$. This completes the proof. □

Theorem 7.1.2. *(Weak Duality). For each feasible solution* (x,y,z,p) *of (SHP) and for each feasible solution* (u,v,w,q) *of (SHD), if* $f(.,v)+.^T w$ *is higher-order-*η_1 *– pseudo-invex in the first variable at u with respect to* η_1 *and* $g(u,v,q)$ *and* $-\left[f(x.,.)-.^T z\right]$ *is higher-order-*η_2 *–pseudo-invex in the second variable at y with respect to* η_2 *and* $-h(x,y,p)$ *and*

$$\begin{aligned} \eta_1(x,u)+u\geq 0\\ \eta_2(v,y)+y\geq 0, \end{aligned}$$

then

$$\inf(SHP)\geq\sup(SHD).$$

Proof. Because $f(.,v)+.^T w$ is higher-order-η_1 – pseudo-invex in the first variable at u with respect to η_1 and $k(u,v,q)$, from (7.1.9), we get (7.1.11). Since $-\left[f(x.,.)-.^T z\right]$ is higher-order-η_2 – pseudo-invex in the second variable at y with respect to η_2 and $-h(x,y,p)$, from (7.1.10), we get (7.1.12). From (7.1.11) and (7.1.12), we arrive at the result as in Theorem 7.1.1. □

Theorem 7.1.3. *(Strong Duality). Let* $(\bar{x},\bar{y},\bar{z},\bar{p})$ *be a local or global optimal solution of (SHP). Assume that* $f: R^n \times R^n \to R$ *is differentiable at* $(\bar{x},\bar{y}), h: R^n \times R^n \times R^n \to R$ *is twice differentiable at* $(\bar{x},\bar{y},\bar{p})$*, and* $g: R^n \times R^n \times R^n \to R$ *is differentiable at* $(\bar{x},\bar{y},\bar{q})$*. If h and g satisfy the conditions:*

(1) $h(\bar{x},\bar{y},\bar{p}=0)=0$, $g(\bar{x},\bar{y},\bar{p}=0)=0$, $\nabla_x h(\bar{x},\bar{y},\bar{p}=0)=\nabla_q g(\bar{x},\bar{y},\bar{p}=0)$*,*
(2) the Hessian matrix $\nabla_{pp} h(\bar{x},\bar{y},\bar{p})$ *is positive definite;*
(3) $\nabla_y f(\bar{x},\bar{y})-\bar{z}+\nabla_p h(\bar{x},\bar{y},\bar{p})\neq 0$*, and for each* $\bar{p}\neq 0$*,*

$$p^T[\nabla_y f(\bar{x},\bar{y})-\bar{z}+\nabla_p h(\bar{x},\bar{y},\bar{p})]\neq 0,$$

then (i) $\bar{p}=0$*, (ii) there exists* $\bar{w}\in C$ *such that* $(\bar{x},\bar{y},\bar{w},\bar{p}=0)$ *is a feasible solution of (SHD) and the corresponding objective values of (SHP) and (SHD) are equal. Furthermore, if the hypotheses of Theorem 7.1.1 or 7.1.2 are satisfied for all feasible solutions* (x,y,z,p) *of (SHP) and* (u,v,w,q) *of (SHD), then* $(\bar{x},\bar{y},\bar{w},\bar{p}=0)$ *is a global optimal solution of (SHD).*

Proof. The proof follows similar to the proof of Theorem 2 of Yang et al. (2003b) in light of the formulation of the problems in this section and the Theorem 7.1.1 or Theorem 7.1.2 above. □

Similarly, we have the following converse duality.

Theorem 7.1.4. *(Converse Duality). Let* $(\bar{u},\bar{v},\bar{w},\bar{q})$ *be a local or global optimal solution of (SHD),* $f: R^n \times R^n \to R$ *is differentiable at* $(\bar{u},\bar{v}), h: R^n \times R^n \times R^n \to R$ *is twice differentiable at* $(\bar{u},\bar{v},\bar{q}), g: R^n \times R^n \times R^n \to R$ *is differentiable at* $(\bar{u},\bar{v},\bar{q})$*. If h and g satisfy the conditions:*

(1) $h(\bar{u},\bar{v},0)=0, g(\bar{u},\bar{v},0)=0, \nabla_p h(\bar{u},\bar{v},0)=\nabla_u g(\bar{u},\bar{v},0)$*;*
(2) the Hessian matrix $\nabla_{qq} g(\bar{u},\bar{v},\bar{q})$ *is positive definite;*
(3) $\nabla_x f(\bar{u},\bar{v})-\bar{w}+\nabla_q g(\bar{u},\bar{v},\bar{q})\neq 0$*, and for each* $\bar{q}\neq 0$*,*

$$q^T\,[\nabla_x f(\bar{u},\bar{v})-\bar{w}+\nabla_q g(\bar{u},\bar{v},\bar{q})]\neq 0,$$

then (i) $\bar{q}=0$*; (ii) there exists* $\bar{z}\in D$ *such that* $(\bar{u},\bar{v},\bar{w},\bar{q}=0)$ *is a feasible solution of (SHP) and the corresponding objective values of (SHP) and (SHD) are equal. Furthermore, if the hypotheses of Theorem 7.1.1 or Theorem 7.1.2 are satisfied for all feasible solutions* (x,y,z,p) *of (SHP) and* (u,v,w,q) *of (SHD), then* $(\bar{u},\bar{v},\bar{w},\bar{q}=0)$*, is a global optimal solution of (SHP).*

7.2 Mond–Weir Type Higher Order Symmetric Duality

Consider the following pair of higher-order vector optimization problems:

$$\text{(SP)} \qquad \text{minimize } F(x,y,p)=(F_1(x,y,p),F_2(x,y,p),\ldots,F_k(x,y,p))$$

subject to

$$\sum_{i=1}^{k} \lambda_i[\nabla_y f_i(x,y) + \nabla_p h_i(x,y,p)] \leqq 0, \tag{7.2.1}$$

$$y^T \sum_{i=1}^{k} \lambda_i[\nabla_y f_i(x,y) + \nabla_p h_i(x,y,p)] \geq 0, \tag{7.2.2}$$

$$\lambda > 0,$$

(SD) maximize $G(u,v,q) = (G_1(u,v,q), G_2(u,v,q), \ldots, G_k(u,v,q))$

subject to

$$\sum_{i=1}^{k} \lambda_i[\nabla_x f_i(u,v) + \nabla_q g_i(u,v,q)] \geqq 0, \tag{7.2.3}$$

$$u^T \sum_{i=1}^{k} \lambda_i[\nabla_x f_i(u,v) + \nabla_q g_i(u,v,q)] \leq 0, \tag{7.2.4}$$

$$\lambda > 0,$$

where

$$F_i(x,y,p) = f_i(x,y) + h_i(x,y,p) - h^T \nabla_p h_i(x,y,p),$$
$$G_i(u,v,q) = f_i(u,v) + g_i(u,v,q) - q^T \nabla_q g_i(u,v,q),$$

$f_i : R^n \times R^m \to R, i = 1, 2, \ldots, k$ are differentiable functions; and $h, g : R^n \times R^m \times R^n \to R$ are differentiable functions.

Theorem 7.2.1. *(Weak Duality). For each feasible solution (x,y,p) of (SP) and for each feasible solution (u,v,q) of (SD), let either of the following conditions hold:*

(a) for $i = 1, 2, \ldots, k$, f_i is higher-order invex with respect to η_1 in the first variable at u and $-f_i$ is higher-order invex with respect to η_2 in the second variable at y.
(b) $\sum_{i=1}^{k} \lambda_i f_i$ is higher-order pseudo-invex with respect to η_1 in the first variable at u and $-\sum_{i=1}^{k} \lambda_i f_i$ is higher-order pseudo-invex with respect to η_2 in the second variable at y, and

$$\eta_1(x,u) + u \geqq 0, \tag{7.2.5}$$
$$\eta_2(v,y) + y \geqq 0. \tag{7.2.6}$$

Then

$$F(x,y,p) \nleq G(u,v,q).$$

Proof. Since (u,v,q) is feasible for (SD), from (7.2.3) and (7.2.5), we get

$$[\eta_1(x,u) + u]^T \sum_{i=1}^{k} \lambda_i[\nabla_x f_i(u,v) + \nabla_q g_i(u,v,q)] \geq 0.$$

Using (7.2.4), we get

$$\eta_1^T(x,u) \sum_{i=1}^{k} \lambda_i[\nabla_x f_i(u,v) + \nabla_q g_i(u,v,q)] \geq 0. \tag{7.2.7}$$

Since (x,y,p) is feasible for (SP), from (7.2.1) and (7.2.6), we get

$$[\eta_2(v,y)+y]^T\sum_{i=1}^{k}\lambda_i[\nabla_y f_i(x,y)+\nabla_p h_i(x,y,p)]\leq 0.$$

Using (7.2.2), we get

$$\eta_2^T(v,y)\sum_{i=1}^{k}\lambda_i[\nabla_y f_i(x,y)+\nabla_p h_i(x,y,p)]\leq 0. \tag{7.2.8}$$

(a) Since f_i is higher-order invex with respect to η_1 in the first variable at u we have for $i=1,2,\ldots,k$,

$$f_i(x,v)-f_i(u,v)\geq \eta_1^T(x,u)[\nabla_q g_i(u,v,q)+\nabla_x f_i(u,v)]+g_i(u,v,q)-q^T\nabla_q g_i(u,v,q).$$

Since $\lambda_i>0, i=1,2,\ldots,k$, using (7.2.7), we get

$$\sum_{i=1}^{k}\lambda_i[f_i(x,v)-f_i(u,v)-g_i(u,v,q)+q^T\nabla_q g_i(u,v,q)]\geq 0. \tag{7.2.9}$$

Since $-f_i$ is higher-order invex with respect to η_2 in the second variable at y, we have for $i=1,2,\ldots,k$,

$$-f_i(x,v)+f_i(x,y)\geq \eta_2^T(v,y)[-\nabla_y f_i(x,y)-\nabla_p h_i(x,y,p)]-h_i(x,y,p)$$
$$+p^T\nabla_p h_i(x,y,p).$$

Since $\lambda_i>0$, $i=1,2,\ldots,k$, using (7.2.8), we have

$$-\sum_{i=1}^{k}\lambda_i[f_i(x,v)-f_i(x,y)+p^T\nabla_p h_i(x,y,p)-h_i(x,y,p)]\geq 0. \tag{7.2.10}$$

Adding (7.2.9) and (7.2.10), we get

$$\sum_{i=1}^{k}\lambda_i[f_i(x,y)+h_i(x,y,p)-p^T\nabla_p h_i(x,y,p)]$$
$$\geq\sum_{i=1}^{k}\lambda_i[f_i(u,v)+g_i(u,v,q)-q^T\nabla_q g_i(u,v,q)].$$

Hence,

$$F(x,y,p)\nleq G(u,v,q).$$

(b) Since $\sum_{i=1}^{k}\lambda_i f_i$ is higher-order pseudo-invex with respect to η_1 in the first variable, from (7.2.7) we get (7.2.9). Since $-\sum_{i=1}^{k}\lambda_i f_i$ is higher-order pseudo-invex with respect to η_2 in the second variable, from (7.2.8) we get (7.2.10). Adding (7.2.9) and (7.2.10), we obtain the result as in part (a). □

Theorem 7.2.2. *(Strong Duality). Let $(\bar{x},\bar{y},\bar{p})$ be a weak minimum of (SP). Fix $\lambda = \bar{\lambda}$ in (SD) and suppose that*

(a) $h_i(\bar{x},\bar{y},0) = 0 = g_i(\bar{x},\bar{y},0), \quad \nabla_p h_i(\bar{x},\bar{y},0) = \nabla_q g_i(\bar{x},\bar{y},0) = 0,$

(b) the Hessian matrix $\sum_{i=1}^{k} \nabla_{py} h_i(\bar{x},\bar{y},\bar{p})$ *is positive definite,*

(c) $\sum_{i=1}^{k} \lambda_i[\nabla_y f_i(\bar{x},\bar{y}) + \nabla_p h_i(x,y,p)] \neq 0$ *and for each* $p \neq 0$,

$$p^T[\nabla_y f_i(\bar{x},\bar{y}) + \nabla_p h_i(\bar{x},\bar{y},\bar{p})] \neq 0,$$

and
(d) $\nabla_p f_i$ *is non-singular for* $i = 1,2,\ldots,k$.

Then (i) $\bar{p} = 0$; (ii) $(\bar{x},\bar{y},\bar{p} = 0)$ is a feasible solution of (SD), and the corresponding objective values of (SP) and (SD) are equal. Furthermore, if the hypothesis of Theorem 7.2.1 is satisfied for all feasible solutions (x,y,p) of (SP) and (u,v,q) of (SD), then $(\bar{x},\bar{y},\bar{p} = 0)$ is a global optimal solution for (SD).

Proof. Since $(\bar{x},\bar{y},\bar{p})$ is a weak minimum of (SP), by Fritz–John optimality conditions, there exist $\alpha \in R^k, \beta \in R^m, \gamma \in R$ and $\delta \in R^k$, such that

$$\sum_{i=1}^{k} \alpha_i[\nabla_x f_i - \nabla_{px} f_i(\bar{p})] + \sum_{i=1}^{k} \bar{\lambda}_i[\nabla_{yx} f_i + \nabla_{px} f_i(\bar{p})](\beta - \gamma\bar{y}) = 0, \tag{7.2.11}$$

$$\begin{aligned}\sum_{i=1}^{k} \alpha_i[\nabla_y f_i - \nabla_{py} f_i(\bar{p})] + \sum_{i=1}^{k} \bar{\lambda}_i[\nabla_{yy} f_i + \nabla_{py} f_i(\bar{p})](\beta - \gamma\bar{y}) \\ - \gamma\sum_{i=1}^{k} \bar{\lambda}_i[\nabla_y f_i - \nabla_p f_i(\bar{p})] = 0,\end{aligned} \tag{7.2.12}$$

$$(\beta - \gamma\bar{y})^T(\nabla_y f_i + \nabla_p f_i(\bar{p})) - \delta_i = 0, \quad i = 1,2,\ldots,k, \tag{7.2.13}$$

$$[(\beta - \gamma\bar{y})\bar{\lambda}_i - \alpha_i\bar{p}]^T\nabla_p f_i = 0, \quad i = 1,2,\ldots,k, \tag{7.2.14}$$

$$\beta^T\sum_{i=1}^{k} \bar{\lambda}_i[\nabla_y f_i + \nabla_p f_i(\bar{p})] = 0, \tag{7.2.15}$$

$$\gamma\bar{y}\sum_{i=1}^{k} \bar{\lambda}_i[\nabla_y f_i + \nabla_p f_i(\bar{p})] = 0, \tag{7.2.16}$$

$$\delta^T\bar{\lambda} = 0, \tag{7.2.17}$$

$$(\alpha,\beta,\gamma,\delta) \geqq 0, \quad (\alpha,\beta,\gamma,\delta) \neq 0. \tag{7.2.18}$$

Since $\bar{\lambda} > 0$, from (7.2.17), we get

$$\delta = 0.$$

Therefore, from (7.2.13), we get

$$(\beta - \gamma\bar{y})^T(\nabla_y f_i + \nabla_p f_i(\bar{p})) = 0, \quad i = 1,2,\ldots,k. \tag{7.2.19}$$

Since $\nabla_p f_i$ is non-singular for $i = 1,2,\ldots,k$, from (7.2.14), we get

$$(\beta - \gamma\bar{y})\bar{\lambda}_i = \alpha_i \bar{p}, \quad i = 1,2,\ldots,k. \tag{7.2.20}$$

From (7.2.12), we get

$$\sum_{i=1}^{k}(\alpha_i - \gamma\bar{\lambda}_i)\nabla_y f_i + \sum_{i=1}^{k}\bar{\lambda}_i \nabla_p f_i(\beta - \gamma\bar{y} - \gamma\bar{p}))$$
$$-\sum_{i=1}^{k}\nabla_{py} f_i(\bar{p})[(\beta - \gamma\bar{y})\bar{\lambda}_i - \alpha_i \bar{p}] = 0.$$

Using (7.2.20), we get

$$\sum_{i=1}^{k}(\alpha_i - \gamma\bar{\lambda}_i)[\nabla_y f_i + \nabla_p f_i(\bar{p})] + \sum_{i=1}^{k}\bar{\lambda}_i \nabla_{py} f_i(\bar{p})(\beta - \gamma\bar{y}) = 0. \tag{7.2.21}$$

Premultiplying by $(\beta - \gamma\bar{y})^T$ and using (7.2.19), we get

$$(\beta - \gamma\bar{y})^T \sum_{i=1}^{k}\bar{\lambda}_i [\nabla_{py} f_i(\bar{p})](\beta - \gamma\bar{y}) = 0.$$

Using the fact that $\sum_{i=1}^{k}\bar{\lambda}_i \nabla_{py} f_i(\bar{p})$ is positive definite, we get

$$\beta = \gamma\bar{y}. \tag{7.2.22}$$

Using (7.2.22) in (7.2.21), we get

$$\sum_{i=1}^{k}(\alpha_i - \gamma\bar{\lambda}_i)[\nabla_y f_i + \nabla_p f_i(\bar{p})] = 0.$$

By condition (b), we get

$$\alpha_i = \gamma\bar{\lambda}_i, i = 1,2,\ldots,k. \tag{7.2.23}$$

If $\gamma = 0$, from (7.2.22) and (7.2.23), it follows that $\beta = 0 = \alpha$, which contradicts (7.2.18). Hence, $\gamma > 0$. Since $\bar{\lambda}_i > 0, i = 1,2,\ldots,k$, from (7.2.23), we have $\alpha_i > 0$, $i = 1,2,\ldots,k$. Using (7.2.22) in (7.2.20), we have $\alpha_i \bar{p} = 0, i = 1,2,\ldots,k$, and hence, $\bar{p} = 0$. Using (7.2.22), and the fact that $\bar{p} = 0$ in (7.2.11), it follows that

$$\sum_{i=1}^{k}\alpha_i \nabla_x f_i = 0,$$

which by (7.2.23) gives

$$\sum_{i=1}^{k}\bar{\lambda}_i \nabla_x f_i = 0,$$

and hence, we have

$$\bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i \nabla_x f_i = 0.$$

Thus, it follows that $(\bar{x}, \bar{y}, \bar{p} = 0)$ is a feasible solution of (SD) and

$$F(\bar{x}, \bar{y}, \bar{p}) = G(\bar{x}, \bar{y}, \bar{q}). \tag{7.2.24}$$

If $(\bar{x}, \bar{y}, \bar{q})$ is not efficient for (SD), then there exists a feasible solution (u, v, q) of (SD) such that

$$G(\bar{x}, \bar{y}, \bar{p}) \leq G(u, v, q),$$

which by (7.2.24) gives

$$F(\bar{x}, \bar{y}, \bar{p}) \leq G(u, v, q),$$

which is a contradiction to Theorem 7.2.1.

If $(\bar{x}, \bar{y}, \bar{q})$ is not properly efficient for (SD), then for some feasible (u, v, q) of (SD) and some i,

$$f_i(u, v) - q^T \nabla_q f_i(u, v, q) > f_i(\bar{x}, \bar{y}) - \bar{p}^T \nabla_p f_i(\bar{x}, \bar{y}, \bar{p})$$

and

$$\begin{aligned} f_i(u, v) - q^T \nabla_q f_i(u, v, q) - f_i(\bar{x}, \bar{y}) + \bar{p}^T \nabla_p f_i(\bar{x}, \bar{y}, \bar{p}) \\ > M[f_j(\bar{x}, \bar{y}) - \bar{p}^T \nabla_p f_j(\bar{x}, \bar{y}, \bar{p}) - f_j(u, v) + q^T \nabla_q f_j(u, v, q)] \end{aligned}$$

for all $M > 0$ and all j satisfying

$$f_j(\bar{x}, \bar{y}) - \bar{p}^T \nabla_p f_j(\bar{x}, \bar{y}, \bar{p}) > f_j(u, v) - q^T \nabla_q f_j(u, v, q).$$

This means that $f_i(u, v) - q^T \nabla_q f_i(u, v, q) - f_i(\bar{x}, \bar{y}) + \bar{p}^T \nabla_p f_i(\bar{x}, \bar{y}, \bar{p})$ can be made arbitrarily large. Therefore, for any $\bar{\lambda} > 0$,

$$\sum_{i=1}^{k} \bar{\lambda}_i [f_i(u, v) - q^T \nabla_q f_i(u, v, q)] > \sum_{i=1}^{k} \bar{\lambda}_i [f_i(\bar{x}, \bar{y}) - \bar{p}^T \nabla_p f_i(\bar{x}, \bar{y}, \bar{p})],$$

which again contradicts Theorem 7.2.1. □

Theorem 7.2.3. *(Converse Duality). Let $(\bar{u}, \bar{v}, \bar{p} = 0)$ be a weak minimum of (SD). Fix $\lambda = \bar{\lambda}$ in (SD) and suppose that*

(a) $h_i(\bar{x}, \bar{y}, 0) = 0 = g_i(\bar{x}, \bar{y}, 0)$, $\nabla_p h_i(\bar{x}, \bar{y}, 0) = \nabla_q g_i(\bar{x}, \bar{y}, 0) = 0$,

(b) the Hessian matrix $\sum_{i=1}^{k} \nabla_{py} h_i(\bar{x}, \bar{y}, \bar{p})$ is positive definite,

(c) $\sum_{i=1}^{k} \lambda_i [\nabla_y f_i(\bar{x}, \bar{y}) + \nabla_p h_i(x, y, p)] \neq 0$ and for each $p \neq 0$,

$$p^T [\nabla_y f_i(\bar{x}, \bar{y}) + \nabla_p h_i(\bar{x}, \bar{y}, \bar{p})] \neq 0,$$

and
(d) $\nabla_p f_i$ *is non-singular for* $i = 1, 2, \ldots, k$.

Then $(\bar{u}, \bar{v}, \bar{p} = 0)$ *is feasible for (SP) and* $F(\bar{u}, \bar{v}, \bar{p}) = G(\bar{u}, \bar{v}, \bar{q})$.

Proof. The proof is similar to that of Theorem 7.2.2. □

7.3 Self Duality

An optimization problem is said to be self-dual if it is formally identical to its dual, that is, the dual can be recast in the form of the primal. If we assume the functions f_i to be skew-symmetric, that is

$$f_i(x, y) = -f_i(y, x), \quad i = 1, 2, \ldots, k,$$

then we can show that the problems (SP) and (SD) are self-dual. By recasting the dual problem (SD) as a minimization problem, we have

$$\begin{aligned}
&\text{minimize } -G(u, v, q) = (-G_1(u, v, q), -G_2(u, v, q), \ldots, -G_k(u, v, q)) \\
&\text{subject to } -\sum_{i=1}^{k} \lambda_i [\nabla_x f_i(u, v) + \nabla_q g_i(u, v, q)] \leqq 0, \\
&\qquad -u^T \sum_{i=1}^{k} \lambda_i [\nabla_x f_i(u, v) + \nabla_q g_i(u, v, q)] \geq 0, \\
&\qquad \lambda > 0,
\end{aligned}$$

where $G_i(u, v, q) = f_i(u, v) + g_i(u, v, q) - q^T \nabla_q g_i(u, v, q)$.

Since f_i is skew-symmetric,

$$\nabla_x f_i(u, v) = -\nabla_y f_i(v, u), \nabla_q g_i(u, v, q) = -\nabla_p h_i(v, u, p)$$

for each $i = 1, 2, \ldots, k$. Therefore, the dual problem (SD) can be rewritten as

$$\begin{aligned}
&\text{minimize } H(v, u, q) = (H_1(v, u, q), H_2(v, u, q), \ldots, H_k(v, u, q)) \\
&\text{subject to } \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(v, u) + \nabla_p h_i(v, u, p)] \leqq 0, \\
&\qquad u^T \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(v, u) + \nabla_p h_i(v, u, p)] \geq 0, \\
&\qquad \lambda > 0,
\end{aligned}$$

where $H_i(v, u, q) = f_i(v, u) + h_i(v, u, p) - p^T \nabla_p h_i(v, u, p)$.

This shows that the dual problem (SD) is identical to (SP). Hence, if (u, v, q) is feasible for (SD), then (v, u, q) is feasible for (SP) and conversely.

Theorem 7.3.1. *(Self Duality). Let $f_i, i = 1,2,\ldots,k$, be skew-symmetric. Then (SP) is self-dual. If (SP) and (SD) are dual problems, and $(\bar{x},\bar{y},\bar{p})$ is a joint optimal solution, then so is $(\bar{y},\bar{x},\bar{p})$ and*

$$F(\bar{x},\bar{y},\bar{p}) = G(\bar{y},\bar{x},\bar{p}) = 0.$$

Proof. It follows along the lines of the corresponding result by Weir and Mond (1988b). □

7.4 Higher Order Vector Nondifferentiable Symmetric Duality

Throughout this section, we denote by R^n the n-dimensional Euclidean space and R^n_+ the non-negative orthant of R^n, respectively. Let x and $y \in R^n$. We denote $x \geqq y \Leftrightarrow x_i \geqq y_i, i = 1,2,\ldots,n$; $x \geq y \Leftrightarrow x \geqq y$ and $x \neq y$; $x > y \Leftrightarrow x_i > y_i, i = 1,2,\ldots,n$; $x \not> y$ is the negation of $x > y$. Let $X \subseteq R^n$.

Definition 7.4.1. *A functional $F : X \times X \times R^n \to R$ is said to be sublinear if for any $x,u \in X$, any $\alpha, \alpha_1, \alpha_2 \in R^n$ and any $a \in R, a \geq 0$,*

$$F(x,u,\alpha_1+\alpha_2) \leqq F(x,u,\alpha_1) + F(x,u,\alpha_2)$$

and

$$F(x,u,a\alpha) = aF(x,u,\alpha).$$

Let F be a sublinear functional, $f : X \to R$ be a twice differentiable function at $u \in X$.

Definition 7.4.2. *Suppose that $h : X \times R^n \to R$ is a differentiable function, and F is sublinear with respect to third variable. The function f is said to higher-order-F-convex at $u \in X$ with respect to $h : X \times R^n \to R$, if for all $(x,p) \in X \times R^n$*

$$f_i(x) - f_i(u) \geqq \eta(x,u)^T F(x,u;\nabla_u f_i(u) + \nabla_p h_i(u,p)) + h_i(u,p) - p^T \nabla_p h_i(u,p), \quad \forall i \in \{1,2,\ldots,k\}.$$

If for all $(x,p) \in X \times R^n$

$$F(x,u;\nabla_u f_i(u) + \nabla_p h_i(u,p)) \geqq 0 \Rightarrow f_i(x) - f_i(u) - h_i(u,p) + p^T \nabla_p h_i(u,p) \geqq 0, \quad \forall i \in \{1,2,\ldots,k\}$$

then f is said to be higher-order F–pseudo-convex at $u \in X$ with respect to h.

The following notations are for the vector case of the notions given in Sect. 7.7.1.

Let $C_i, \forall i \in \{1,2,\ldots,k\}$, be compact convex sets in R^n. The support function of C_i is defined by

$$s(x|C_i) = \max\{x^T y : y \in C_i\}, \forall i \in \{1,2,\ldots,k\}.$$

A support function, being convex and everywhere finite, has a subdifferential (see Clarke 1983), that is, there exists $z \in R^n$ such that

$$s(y|C_i) \geq s(x|C_i) + z^T(y-x) \forall y \in C_i, \forall i \in \{1,2,\ldots,k\}.$$

The subdifferential of $s(x|C_i)$ is given by

$$\partial s(x|C_i) = \{z \in C_i : z^T x = s(x|C_i)\}, \forall i \in \{1,2,\ldots,k\}.$$

For any set $D_i \subset R^n, \forall i \in \{1,2,\ldots,k\}$, the normal cone to D_i at a point $x \in D_i$ is defined by

$$N_{Di}(x) = \{y \in R^n : y^T(z-x) \leq 0 \ \forall z \in D_i\}, \forall i \in \{1,2,\ldots,k\}.$$

It is obvious that for compact convex sets $C_i, y \in N_{Ci}(x)$ if and only if $s(y|C_i) = x^T y$, or equivalently, $x \in \partial s(y|C_i), \forall i \in \{1,2,\ldots,k\}$.

For a real-valued twice differentiable function $g(x,y)$ defined on an open set in $R^n \times R^m$, denote by $\nabla_x g(\bar{x},\bar{y})$ the gradient of g with respect to x at $(\bar{x},\bar{y}), \nabla_{xx} g(\bar{x},\bar{y})$ the Hessian matrix with respect to x at $(\bar{x},\bar{y})$. Similarly, $\nabla_y g(\bar{x},\bar{y})$, $\nabla_{xy} g(\bar{x},\bar{y})$ and $\nabla_{yy} g(\bar{x},\bar{y})$ are also defined.

We consider the following optimization problems:

$$\begin{aligned}(\text{MSHP})\ \text{minimize}\ \phi(x,y,p) = &(f_1(x,y) + s(x|C_1) - y^T z_1 + h_1(x,y,p) \\ &- p^T \nabla_p h_1(x,y,p), \ldots, f_k(x,y) + s(x|C_k) - y^T z_k \\ &+ h_k(x,y,p) - p^T \nabla_p h_k(x,y,p))\end{aligned}$$

$$\text{subject to} \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x,y) - z_i + \nabla_p h_i(x,y,p)] \geqq 0, \tag{7.4.1}$$

$$y^T \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x,y) - z_i + \nabla_p h_i(x,y,p)] \geqq 0, \tag{7.4.2}$$

$$\lambda > 0, z_i \in D_i, i = 1,2,\ldots,k. \tag{7.4.3}$$

$$\begin{aligned}(\text{MSHD})\quad \text{maximize}\ \psi(u,v,q) = &(f_1(u,v) - s(v|D_1) + u^T w_1 + g_1(u,v,q) \\ &- q^T \nabla_q g_1(u,v,q), \ldots, f_k(u,v) - s(v|D_k) \\ &+ u^T w_k + g_k(u,v,q) - q^T \nabla_q g_k(u,v,q))\end{aligned}$$

$$\text{subject to} \sum_{i=1}^{k} \lambda_i [\nabla_u f_i(u,v) + w_i + \nabla_q g_i(u,v,q)] \geqq 0, \tag{7.4.4}$$

$$u^T \sum_{i=1}^{k} \lambda_i [\nabla_u f_i(u,v) + w_i + \nabla_q g_i(u,v,q)] \leqq 0, \tag{7.4.5}$$

$$\lambda > 0, w_i \in C_i, i = 1,2,\ldots,k. \tag{7.4.6}$$

where C_i and D_i, $i = 1,2,\ldots,k$ are compact convex sets in R^n and R^m, respectively and $f_i : R^n \times R^m \to R$; both $h_i : R^n \times R^m \times R^m \to R$; and $g_i : R^n \times R^m \times R^n \to R$, $i = 1,2,\ldots,k$ are differentiable functions.

Theorem 7.4.1. *(Weak Duality). For each feasible solution (x,y,λ,z,p) of (MSHP) and for each feasible solution (u,ν,λ,w,q) of (MSHD), let either of the following conditions hold:*

(a) For $i = 1,2,\ldots,k$, $f_i(.,\nu) + .^T w_i$ is higher-order-G – convex in the first variable at u with respect to $g_i(u,\nu,q)$ and $-[f_i(x.,.) - .^T z_i]$ is higher-order-F – convex in the second variable at y with respect to $-h_i(x,y,p)$;

(b) $\sum_{i=1}^k \lambda_i[f_i(.,\nu) + .^T w_i]$ is higher-order-G – convex in the first variable at u with respect to $g_i(u,\nu,q)$ and $-\sum_{i=1}^k \lambda_i[f_i(x.,.) - .^T z_i]$ is higher-order-F – convex in the second variable at y with respect to $-h_i(x,y,p)$ and the sublinear functions $G: R^n \times R^m \times R^n \to R$ and $F: R^n \times R^m \times R^n \to R$ satisfy

$$F(x,y;a) + a^T y \geqq 0, \forall a \in R^n_+ \tag{7.4.7}$$

$$G(u,v;a) + a^T u \geqq 0, \forall a \in R^n_+. \tag{7.4.8}$$

Then

$$\phi(x,y,p) \nleq \psi(u,v,q).$$

Proof. Since (u,v,λ,w,q) is feasible for (MSHD), from (7.4.4) and (7.4.8), it follows that

$$G\left(u,v;\sum_{i=1}^k \lambda_i[\nabla_u f_i(u,v) + w_i + \nabla_q g_i(u,v,q)]\right) + \sum_{i=1}^k \lambda_i[\nabla_u f_i(u,v) + w_i + \nabla_q g_i(u,v,q)]^T u \geqq 0.$$

From (7.4.5), we get

$$G\left(u,v;\sum_{i=1}^k \lambda_i[\nabla_u f_i(u,v) + w_i + \nabla_q g_i(u,v,q)]\right) \geq 0. \tag{7.4.9}$$

Since (x,y,λ,z,p) is feasible for (MSHP), from (7.4.1) and (7.4.7), it follows that

$$F\left(x,y;-\sum_{i=1}^k \lambda_i[\nabla_y f_i(x,y) - z_i + \nabla_p h_i(x,y,p)]\right) - \sum_{i=1}^k \lambda_i[\nabla_y f_i(x,y) - z_i + \nabla_p h_i(x,y,p)]^T y \geqq 0.$$

Using (7.4.2), we get

$$F\left(x,y;-\sum_{i=1}^{k}\lambda_i[\nabla_y f_i(x,y)-z_i+\nabla_p h_i(x,y,p)]\right)\geq 0. \tag{7.4.10}$$

(a) By the higher-order-G – convexity of $f_i(.,v)+.^T w_i$ with respect to $g_i(u,v,q)$, for $i=1,2,\ldots,k$, and from (7.4.9), we get

$$f_i(x,v)\geq f_i(u,v)-x^T w_i+u^T w_i+g_i(u,v,q)-q^T\nabla_q g_i(u,v,q).$$

Since $\lambda_i>0, i=1,2,\ldots,k$, we get

$$\sum_{i=1}^{k}\lambda_i[f_i(x,v)-f_i(u,v)+x^T w_i-u^T w_i-g_i(u,v,q)+q^T\nabla_q g_i(u,v,q)]\geq 0. \tag{7.4.11}$$

On the other hand, by the higher-order-F – convexity of $-[f_i(x.,.)-.^T z_i]$ with respect to $-h_i(x,y,p)$ for $i=1,2,\ldots,k$, from (7.4.10), we get

$$f_i(x,v)\leq f_i(x,y)+v^T z_i-y^T z_i+h_i(x,y,p)-p^T\nabla_p h_i(x,y,p).$$

Since $\lambda_i>0, i=1,2,\ldots,k$, we get

$$\sum_{i=1}^{k}\lambda_i[f_i(x,v)-f_i(x,y)-v^T z_i+y^T z_i-h_i(x,y,p)+p^T\nabla_p h_i(x,y,p)]\leq 0. \tag{7.4.12}$$

From (7.4.11) and (7.4.12), we obtain

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_i[f_i(x,y)+x^T w_i-y^T z_i+h_i(x,y,p)-p^T\nabla_p h_i(x,y,p)]\\&\quad\geq\sum_{i=1}^{k}\lambda_i[f_i(u,v)-v^T z_i+u^T w_i+g_i(u,v,q)-q^T\nabla_q g_i(u,v,q)].\end{aligned} \tag{7.4.13}$$

Note that $x^T w_i\leq s(x|C_i)$ and $v^T z_i\leq s(v|D_i)$, for $i=1,2,\ldots,k$, (7.4.13) yields

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_i[f_i(x,y)+s(x|C_i)-y^T z_i+h_i(x,y,p)-p^T\nabla_p h_i(x,y,p)]\\&\quad\geq\sum_{i=1}^{k}\lambda_i[f_i(u,v)-s(v|D_i)+u^T w_i+g_i(u,v,q)-q^T\nabla_q g_i(u,v,q)].\end{aligned}$$

Hence, $\phi(x,y,p)\nleq\psi(u,v,q)$.

(b) Since $\sum_{i=1}^{k}\lambda_i[f_i(.,v)+.^T w_i]$ is higher-order-G – convex in the first variable at u with respect to $g_i(u,v,q)$ from (7.4.9), we get (7.4.11). Because $-\sum_{i=1}^{k}\lambda_i[f_i(x.,.)$ $-.^T z_i]$ is higher-order-F – convex in the second variable at y with respect to $-h_i(x,y,p)$, from (7.4.10), we get (7.4.12). From (7.4.11) and (7.4.12), we arrive at the result as in part (a). □

7.5 Minimax Mixed Integer Optimization Problems

Let U and V be two arbitrary sets of integers in R^n and R^m, respectively. Throughout this section, x and y are integer variables, i.e., the components of x and y are integers. Suppose that the first $n_1(0 \leq n_1 \leq n)$ components of x belong to U and that the first $m_1(0 \leq m_1 \leq m)$ components of y belong to V. Thus, we can write $(x,y) = (x^1,x^2,y^1,y^2)$, where $x^1 = (x_1,x_2,\ldots,x_{n_1})$ and $y^1 = (y_1,y_2,\ldots,y_{m_1})$ with x^2 and y^2 being the remaining components of x and y, respectively.

The following concepts of separability (see Balas 1991) will be needed in the sequel.

Definition 7.5.1. *Let $z^1,z^2,\ldots z^r$ be elements of an arbitrary vector space. A vector function $\theta(z^1,z^2,\ldots z^r)$ is called additively separable with respect to z^1 if there exist vector functions $\varphi(z^1)$ (independent of $z^2,\ldots z^r$) and $\psi(z^2,\ldots z^r)$ (independent of z^1) such that $\theta(z^1,z^2,\ldots z^r) = \varphi(z^1) + \psi(z^2,\ldots z^r)$.*

We consider the following pair of non-differentiable minmax mixed integer higher order symmetric primal and dual programs:

$$\text{(SHMP)} \quad \max_{x^1}\min_{x^2,y,z} f(x,y) + s(x^2|C) - y^{2^T} z + h(x,y,p) - p^T\nabla_p h(x,y,p)$$

$$\text{subject to } \nabla_{y^2} f(x,y) - z + \nabla_p h(x,y,p) \leq 0,$$
$$y^{2^T}[\nabla_{y^2} f(x,y) - z + \nabla_p h(x,y,p)] \geq 0,$$
$$x^2 \geq 0, z \in D,$$
$$x^1 \in U, y^1 \in V,$$
$$p \in R^{m-m_1}.$$

$$\text{(SHMD)} \quad \min_{v^1}\max_{u,v^2,w} f(u,v) - s(v^2|D) - u^{2^T} w + g(u,v,q) - q^T\nabla_q g(u,v,q)$$

$$\text{subject to } \nabla_{u^2} f(u,v) + w + \nabla_q g(u,v,q) \geq 0,$$
$$u^{2^T}[\nabla_{u^2} f(u,v) + w + \nabla_q g(u,v,q)] \leq 0,$$
$$u^2 \geq 0, w \in C,$$
$$u^1 \in U, v^1 \in V,$$
$$q \in R^{n-n_1}.$$

Theorem 7.5.1. *(Symmetric Duality). Let $(\bar{x},\bar{y},\bar{z},\bar{p})$ be an optimal solution f (SHMP). Suppose that the following conditions are satisfied:*

(i) $f(x,y)$ is additively separable with respect to x^1 or y^1;

(ii) for any feasible solution (x,y,z,p) for (SHMP) and any feasible solution (u,v,w,q) for (SHMD), $f(u,v) + (u^2)^T w$ is higher-order-η_1 – invex in the first variable at u^2 with respect to η_1 and $g(u,v,q)$ with $q \in R^{n-n_1}$ for each (u^1,v) and $-[f(x,y) - (y^2)^T z]$ is higher-order-η_2 – invex in the second variable at y^2 with respect to η_2 and $-h(x,y,p)$ with $p \in R^{m-m_1}$ for each (x,y^1);

(iii) $f(x,y)$ *is*
(iv) $\nabla_{y^2y^2} f(\bar{x},\bar{y})$ *is non-singular*
(v) the vector $\overline{p}^T \nabla_p f(\overline{x},\overline{y},\overline{p}) = 0 \Rightarrow \overline{p} = 0$; *and*
(vi) for any (x,y,u,v) *feasible for (SHMP) and (SHMD),*

$$\eta_1(x^2,u^2) + (u^2)^T a \geq 0 \forall a \in R_+^{n-n_1},$$
$$\eta_2(v^2,y^2) + (y^2)^T b \geq 0 \forall b \in R_+^{m-m_1}.$$

Then, there exists $\overline{w}$ *such that* $(\overline{x},\overline{y},\overline{w},\overline{q} = 0)$ *is optimal for (SHMD) and the two optimal objective values are equal.*

Proof. The proof follows along similar lines as that given in the proof of Theorem 5 from Mishra (2001b) in light of the formulation of the problems above and Theorem 1 from Gulati et al. (1997) by using Theorem 7.2.1 or Theorem 7.2.2 and Theorem 7.2.3. □

Remark 7.5.1. The above theorem holds for more general convexity conditions as well, namely, if $f(u,v) + (u^2)^T w$ is higher-order-η_1 – pseudo-invex in the first variable at u^2 with respect to η_1 and $g(u,v,q)$ with $q \in R^{n-n_1}$ for each (u^1,v) and $-[f(x,y) - (y^2)^T z]$ is higher-order-η_2 – pseudo-invex in the second variable at y^2 with respect to η_2 and $-h(x,y,p)$ with $p \in R^{m-m_1}$ for each (x,y^1); then the result of Theorem 7.5.1 remains true.

7.6 Mixed Symmetric Duality in Nondifferentiable Vector Optimization Problems

In this section, we introduce two models of mixed symmetric duality for a class of nondifferentiable vector optimization problems. The first model is a vector case of the model given by Yang et al. (2003a). However, the second model is new. Mixed symmetric duality for this model has not been given so far by any other author. The advantage of the second model over the first one is that it allows further weakening of convexity on the functions involved. We establish weak and strong duality theorems for these two models and discuss several special cases of these models. The results of Yang et al. (2003a) as well as that of Bector et al. (1999) are particular cases of the results obtained in the present section.

Let $f(x,y)$ be real valued twice differentiable function defined on $R^n \times R^m$. Let $\nabla_x f(\bar{x},\overline{y})$ and $\nabla_y f(\bar{x},\overline{y})$ denote the partial derivatives of $f(x,y)$ with respect to x and y at $(\overline{x},\overline{y})$. The symbols $\nabla_{xy} f(\overline{x},\overline{y})$, $\nabla_{yx} f(\overline{x},\overline{y})$ and $\nabla_{y^2} f(\overline{x},\overline{y})$ can be defined similarly.

Consider the following vector optimization problem:

$$\text{(VP)} \qquad \text{minimize } (f_1(x), f_2(x), \ldots, f_p(x))$$
$$\text{subject to } h(x) \leqq 0,$$

where $f_i : R^n \to R, i = 1,2,\ldots,p$ and $h : R^n \to R^m$.

Let us denote the feasible region of problem (VP) by $X_0 = \{x \in R^n : h_j(x) = 0, j = 1,2,\ldots,m\}$. For problem (VP), an efficient solution and a properly efficient solution are as in Sect. 7.7.1. The concept of support function and the concept of sublinearity from Sect. 7.7.1 will also be needed in this section.

Definition 7.6.1. *Let $X \subset R^n, Y \subset R^m$ and $F : X \times Y \times R^n \to R$ be sublinear with respect to its third argument. $f(\cdot,y)$ is said to be F – convex at $\overline{x} \in X$, for fixed $y \in Y$, if*

$$f(x,y) - f(\overline{x},y) \geq F(x,\overline{x};\nabla_x f(\overline{x},y)), \forall x \in X.$$

Definition 7.6.2. *Let $X \subset R^n, Y \subset R^m$ and $F : X \times Y \times R^n \to R$ be sublinear with respect to its third argument. $f(x,\cdot)$ is said to be F – concave at $\overline{y} \in Y$, for fixed $x \in X$, if*

$$f(x,\overline{y}) - f(x,y) \geq F(y,\overline{y};-\nabla_y f(x,\overline{y})), \forall y \in Y.$$

Definition 7.6.3. *Let $X \subset R^n, Y \subset R^m$ and $F : X \times Y \times R^n \to R$ be sublinear with respect to its third argument. $f(\cdot,y)$ is said to be F – pseudoconvex at $\overline{x} \in X$, for fixed $y \in Y$, if*

$$F(x,\overline{x};\nabla_x f(\overline{x},y)) \geq 0 \Rightarrow f(x,y) \geq f(\bar{x},y), \forall x \in X.$$

Definition 7.6.4. *Let $X \subset R^n, Y \subset R^m$ and $F : X \times Y \times R^n \to R$ be sublinear with respect to its third argument. $f(x,\cdot)$ is said to be F – pseudoconcave at $\overline{y} \in Y$, for fixed $x \in X$, if*

$$F(y,\overline{y};\nabla_y f(x,\overline{y})) \geq 0 \Rightarrow f(x,\overline{y}) \geq f(x,y), \forall y \in Y.$$

For $N = \{1,2,\ldots,n\}$ and $M = \{1,2,\ldots,m\}$ let $J_1 \subset N, K_1 \subset M$ and $J_2 = N \backslash J_1$ and $K_2 = M \backslash K_1$. Let $|J_1|$ denote the number of elements in the set J_1. The numbers $|J_2|, |K_1|$ and $|K_2|$ are defined similarly. Notice that if $J_1 = \Phi$, then $J_2 = N$, that is, $|J_1| = 0$ and $|J_2| = n$. Hence, $R^{|J_1|}$ is zero dimensional Euclidean space and $R^{|J_2|}$ is n-dimensional Euclidean space. It is clear that any $x \in R^n$ can be written as $x = (x^1,x^2), x^1 \in R^{|J_1|}, x^2 \in R^{|J_2|}$. Similarly, any $y \in R^m$ can be written as $y = (y^1,y^2), y^1 \in R^{|K_1|}, y^2 \in R^{|K_2|}$. Let $f : R^{|J_1|} \times R^{|K_1|} \to R^l$ and $g : R^{|J_2|} \times R^{|K_2|} \to R^l$ be twice differentiable functions and $e = (1,1,\ldots,1)^T \in R^l$.

Now we can introduce the following two pairs of nondifferentiable vector optimization problems and discuss their duality theorems under some mild assumptions of generalized convexity.

First Model

Primal Problem (MP1)

$$\text{minimize } H(x^1,x^2,y^1,y^2,z^1,z^2,\lambda) = (H_1(x^1,x^2,y^1,y^2,z^1,z^2,\lambda),\ldots,$$
$$H_l(x^1,x^2,y^1,y^2,z^1,z^2,\lambda))$$

$$\text{subject to } (x^1,x^2,y^1,y^2,z^1,z^2,\lambda) \in R^{|J_1|} \times R^{|J_2|} \times R^{|K_1|} \times R^{|K_2|} \times R^{|K_1|} \times R^{|K_2|} \times R^l_+$$

$$\sum_{i=1}^{l} \lambda_i[\nabla_{y^1} f_i(x^1,y^1) - z_i^1] \leq 0 \tag{7.6.1}$$

$$\sum_{i=1}^{l} \lambda_i[\nabla_{y^2} g_i(x^2,y^2) - z_i^2] \leq 0 \tag{7.6.2}$$

$$(y^2)^T \sum_{i=1}^{l} \lambda_i[\nabla_{y^2} g_i(x^2,y^2) - z_i^2] \geq 0 \tag{7.6.3}$$

$$(x^1,x^2) \geq 0, \tag{7.6.4}$$

$$z_i^1 \in D_i^1, \text{ and } z_i^2 \in D_i^2, i = 1,2,\ldots,l \tag{7.6.5}$$

$$\lambda > 0, \sum_{i=1}^{l} \lambda_i = 1 \tag{7.6.6}$$

Dual Problem (MD1)

$$\text{maximize } G(u^1,u^2,v^1,v^2,w^1,w^2,\lambda) = (G_1(u^1,u^2,v^1,v^2,w^1,w^2,\lambda),\ldots,$$
$$G_l(u^1,u^2,v^1,v^2,w^1,w^2,\lambda))$$

$$\text{subject to } (u^1,u^2,v^1,v^2,w^1,w^2,\lambda) \in R^{|J_1|} \times R^{|J_2|} \times R^{|K_1|} \times R^{|K_2|} \times R^{|K_1|} \times R^{|K_2|} \times R^l_+$$

$$\sum_{i=1}^{l} \lambda_i[\nabla_{x^1} f_i(u^1,v^1) + w_i^1] \geq 0 \tag{7.6.7}$$

$$\sum_{i=1}^{l} \lambda_i[\nabla_{x^2} g_i(u^2,v^2) + w_i^2] \geq 0 \tag{7.6.8}$$

$$(u^2)^T \sum_{i=1}^{l} \lambda_i[\nabla_{x^2} g_i(u^2,v^2) + w_i^2] \leq 0 \tag{7.6.9}$$

$$(v^1,v^2) \geq 0, \tag{7.6.10}$$

$$w_i^1 \in C_i^1, \text{ and } w_i^2 \in C_i^2, i = 1,2,\ldots,l \tag{7.6.11}$$

$$\lambda > 0, \sum_{i=1}^{l} \lambda_i = 1 \tag{7.6.12}$$

where

$$H_i(x^1,x^2,y^1,y^2,z,\lambda) = f_i(x^1,y^1) + g_i(x^2,y^2) + s(x^1|C_i^1) + s(x^2|C_i^2) - (y^1)^T \nabla_{y^1} f_i(x^1,y^1) - (y^2)^T z_i^2$$

$$G_i(u^1,u^2,v^1,v^2,w,\lambda) = f_i(u^1,v^1) + g_i(u^2,v^2) - s(v^1|D_i^1) - s(v^2|D_i^2) - (u^1)^T \nabla_{x^1} f_i(u^1,v^1) + (u^2)^T w_i^2$$

and C_i^1 is a compact and convex subsets of $R^{|J_1|}$ for $i = 1,2,\ldots l$, and C_i^2 is a compact and convex subsets of $R^{|J_2|}$ for $i = 1,2,\ldots,l$, similarly, D_i^1 is a compact and convex subsets of $R^{|K_1|}$ for $i = 1,2,\ldots,l$ and D_i^2 is a compact and convex subsets of $R^{|K_2|}$ for $i = 1,2,\ldots,l$.

Second Model

Primal Problem (MP2)

$$\text{minimize } H^*(x^1,x^2,y^1,y^2,z^1,z^2,\lambda) = (H_1^*(x^1,x^2,y^1,y^2,z^1,z^2,\lambda),\ldots, H_l^*(x^1,x^2,y^1,y^2,z^1,z^2,\lambda))$$

$$\text{subject to } (x^1,x^2,y^1,y^2,z^1,z^2,\lambda) \in R^{|J_1|} \times R^{|J_2|} \times R^{|K_1|} \times R^{|K_2|} \times R^{|K_1|} \times R^{|K_2|} \times R_+^l$$

$$\sum_{i=1}^{l} \lambda_i[\nabla_{y^1} f_i(x^1,y^1) - z_i^1] \leq 0 \tag{7.6.13}$$

$$\sum_{i=1}^{l} \lambda_i[\nabla_{y^2} g_i(x^2,y^2) - z_i^2] \leq 0 \tag{7.6.14}$$

$$(y^1)^T \sum_{i=1}^{l} \lambda_i[\nabla_{y^1} f_i(x^1,y^1) - z_i^1] \geq 0 \tag{7.6.15}$$

$$(y^2)^T \sum_{i=1}^{l} \lambda_i[\nabla_{y^2} g_i(x^2,y^2) - z_i^2] \geq 0 \tag{7.6.16}$$

$$(x^1,x^2) \geq 0, \tag{7.6.17}$$

$$z_i^1 \in D_i^1, \text{ and } z_i^2 \in D_i^2, i = 1,2,\ldots,l \tag{7.6.18}$$

$$\lambda > 0, \sum_{i=1}^{l} \lambda_i = 1 \tag{7.6.19}$$

Dual Problem (MD2)

$$\text{maximize } G^*(u^1,u^2,v^1,v^2,w^1,w^2,\lambda) = (G_1^*(u^1,u^2,v^1,v^2,w^1,w^2,\lambda),\ldots, G_l^*(u^1,u^2,v^1,v^2,w^1,w^2,\lambda))$$

$$\text{subject to } (u^1,u^2,v^1,v^2,w^1,w^2,\lambda) \in R^{|J_1|} \times R^{|J_2|} \times R^{|K_1|} \times R^{|K_2|} \times R^{|K_1|} \times R^{|K_2|} \times R_+^l$$

$$\sum_{i=1}^{l} \lambda_i [\nabla_{x^1} f_i(u^1,v^1) + w_i^1] \geq 0 \tag{7.6.20}$$

$$\sum_{i=1}^{l} \lambda_i [\nabla_{x^2} g_i(u^2,v^2) + w_i^2] \geq 0 \tag{7.6.21}$$

$$(u^1)^T \sum_{i=1}^{l} \lambda_i [\nabla_{x^1} f_i(u^1,v^1) + w_i^1] \leq 0 \tag{7.6.22}$$

$$(u^2)^T \sum_{i=1}^{l} \lambda_i [\nabla_{x^2} g_i(u^2,v^2) + w_i^2] \leq 0 \tag{7.6.23}$$

$$(v^1,v^2) \geq 0, \tag{7.6.24}$$

$$w_i^1 \in C_i^1, \text{ and } w_i^2 \in C_i^2, i = 1,2,\ldots,l \tag{7.6.25}$$

$$\lambda > 0, \sum_{i=1}^{l} \lambda_i = 1 \tag{7.6.26}$$

where

$$\begin{aligned} H_i^*(x^1,x^2,y^1,y^2,z,\lambda) &= f_i(x^1,y^1) + g_i(x^2,y^2) + s(x^1|C_i^1) + s(x^2|C_i^2) \\ &\quad - (y^1)^T z_i^1 - (y^2)^T z_i^2 \\ G_i^*(u^1,u^2,v^1,v^2,w,\lambda) &= f_i(u^1,v^1) + g_i(u^2,v^2) - s(v^1|D_i^1) - s(v^2|D_i^2) \\ &\quad - (u^1)^T w_i^1 + (u^2)^T w_i^2 \end{aligned}$$

and C_i^1 is a compact and convex subsets of $R^{|J_1|}$ for $i = 1,2,\ldots,l$ and C_i^2 is a compact and convex subsets of $R^{|J_2|}$ for $i = 1,2,\ldots,l$, similarly, D_i^1 is a compact and convex subsets of $R^{|K_1|}$ for $i = 1,2,\ldots,l$ and D_i^2 is a compact and convex subsets of $R^{|K_2|}$ for $i = 1,2,\ldots,l$.

For the first model, we can prove the following weak duality theorem.

Theorem 7.6.1. *(Weak Duality). Let $(x^1,x^2,y^1,y^2,z^1,z^2,\lambda)$ be feasible for (MP1) and $(u^1,u^2,v^1,v^2,w^1,w^2,\lambda)$ be feasible for (MD1). Suppose that for $i = 1,2,\ldots,l$, $f_i(\cdot,y^1)$ is F_1- convex for fixed y^1, $f_i(x^1,\cdot)$ is F_2- concave for fixed x^1, $g_i(\cdot,y^2)$ $+ \cdot^T w_i^2$ is G_1- convex for fixed y^2 and $g_i(y^2,\cdot) - \cdot^T z_i^2$ is G_2- concave for fixed x^2, and the following conditions are satisfied:*

(i) $F_1(x^1,u^1;\nabla_{x^1} f_i(u^1,v^1)) + (u^1)^T \nabla_{x^1} f_i(u^1,v^1) + +(x^1)^T w_i^1 \geq 0$;
(ii) $G_1(x^2,u^2;\nabla_{x^2} g_i(u^2,v^2) + w_i^2) + (u^2)^T (\nabla_{x^2} g_i(u^2,v^2) + w_i^2) \geq 0$;
(iii) $F_2(y^1,v^1;\nabla_{y^1} f_i(x^1,y^1)) + (y^1)^T \nabla_{y^1} f_i(x^1,y^1) - (v^1)^T z_i^1 \leq 0$;
(iv) $G_2(y^2,v^2;\nabla_{y^2} g_i(x^2,y^2) - z_i^2) + (y^2)^T (\nabla_{y^2} g_i(x^2,y^2) - z_i^2) \leq 0$.

Then

$$H_i(x^1,x^2,y^1,y^2,z,\lambda) \nleq G_i(u^1,u^2,v^1,v^2,w,\lambda).$$

Proof. Suppose that $(x^1,x^2,y^1,y^2,z^1,z^2,\lambda)$ is feasible for (MP1) and $(u^1,u^2,v^1,v^2,w^1,w^2,\lambda)$ is feasible for (MD1). By the F_1- convexity of $f_i(\cdot,v^1)$ and the

F_2- concavity of $f_i(x^1,\cdot)$, for $i=1,2,\ldots,l$, we have

$$f_i(x^1,v^1)-f_i(u^1,v^1)\geq F_1\left(x^1,u^1;\nabla_{x^1}f_i(u^1,v^1)\right),\text{ for } i=1,2,\ldots,l$$

and

$$f_i(x^1,v^1)-f_i(x^1,y^1)\leq F_2(v^1,y^1;\nabla_{y^1}f_i(x^1,y^1)),\text{ for } i=1,2,\ldots,l.$$

Rearranging the above two inequalities, and by using the conditions (i) and (iii), we obtain

$$\begin{aligned}&f_i(x^1,y^1)-f_i(u^1,v^1)\\&\quad\geq-(u^1)^T\nabla_{x^1}f_i(u^1,v^1)-(x^1)^Tw_i^1+(y^1)^T\nabla_{y^1}f_i(x^1,y^1)-(v^1)^Tz_i^1,\end{aligned}$$

for $i=1,2,\ldots,l$.

Using $(v^1)^Tz_i^1\leq s(v^1|D_i^1)$ and $(x^1)^Tw_i^1\leq s(x^1|C_i^1)$, for $i=1,2,\ldots,l$, we have

$$\begin{aligned}&f_i(x^1,y^1)+s(x^1|C_i^1)-(y^1)^T\nabla_{y^1}f_i(x^1,y^1)\\&\quad\geq f_i(u^1,v^1)-s(v^1|D_i^1)-(u^1)^T\nabla_{x^1}f_i(u^1,v^1),\end{aligned}$$

for $i=1,2,\ldots,l$.

Because of (7.6.6) and (7.6.12), the above inequalities yield

$$\begin{aligned}&\sum_{i=1}^{l}\lambda_i[f_i(x^1,y^1)+s(x^1|C_i^1)-(y^1)^T\nabla_{y^1}f_i(x^1,y^1)]\\&\quad\geq\sum_{i=1}^{l}\lambda_i[f_i(u^1,v^1)-s(v^1|D_i^1)-(u^1)^T\nabla_{x^1}f_i(u^1,v^1)].\end{aligned}\tag{7.6.27}$$

By the G_1- convexity of $g_i(\cdot,v^2)+\cdot^Tw_i^2$, and condition (ii), we get

$$\begin{aligned}[g_i(x^2,v^2)+(x^2)^Tw_i^2]-[g_i(u^2,v^2)+(u^2)^Tw_i^2]&\geq G_i(x^2,u^2;\nabla_{x^2}g_i(u^2,v^2)+w_i^2)\\&\geq-(u^2)^T(\nabla_{x^2}g_i(u^2,v^2)+w_i^2).\end{aligned}\tag{7.6.28}$$

Using (7.6.6), (7.6.12) and (7.6.28), we get

$$\begin{aligned}&\sum_{i=1}^{l}\lambda_i[g_i(x^2,v^2)+(x^2)^Tw_i^2]-\sum_{i=1}^{l}\lambda_i[g_i(u^2,v^2)+(u^2)^Tw_i^2]\\&\quad\geq-(u^2)^T\sum_{i=1}^{l}\lambda_i(\nabla_{x^2}g_i(u^2,v^2)+w_i^2).\end{aligned}\tag{7.6.29}$$

From (7.6.29) and (7.6.9), we get

$$\sum_{i=1}^{l}\lambda_i[g_i(x^2,v^2)+(x^2)^Tw_i^2]-\sum_{i=1}^{l}\lambda_i[g_i(u^2,v^2)+(u^2)^Tw_i^2]\geq0.\tag{7.6.30}$$

Similarly, by G_2- concavity of $g_i(x^2,\cdot)-\cdot^T z_i^2$ and condition (iv), we get

$$[g_i(x^2,v^2)-(v^2)^T z_i^2]-[g_i(x^2,y^2)-(y^2)^T z_i^2]\leq G_2(y^2,v^2;\nabla_{y^2}g_i(x^2,y^2)-z_i^2)$$
$$\leq -(y^2)^T[\nabla_{y^2}g_i(x^2,y^2)-z_i^2].$$

Using (7.6.6), (7.6.12) and (7.6.3) the above inequality yields

$$\sum_{i=1}^{l}\lambda_i[g_i(x^2,v^2)-(v^2)^T z_i^2]-\sum_{i=1}^{l}\lambda_i[g_i(x^2,y^2)-(y^2)^T z_i^2]\leq 0. \qquad (7.6.31)$$

Rearranging (7.6.30) and (7.6.31), we get

$$\sum_{i=1}^{l}\lambda_i[g_i(x^2,y^2)-g_i(u^2,v^2)+(x^2)^T w_i^2-(u^2)^T w_i^2-(y^2)^T z_i^2+(v^2)^T z_i^2]\geq 0.$$

Using $(x^2)^T w_i^2\leq s(x^2|C_i^2)$ and $(v^2)^T z_i^2\leq s(v^2|D_i^2)$, for $i=1,2,\ldots,l$, we have

$$\sum_{i=1}^{l}\lambda_i[g_i(x^2,y^2)+s(x^2|C_i^2)-(y^2)^T z_i^2-g_i(u^2,v^2)+s(v^2|D_i^2)-(u^2)^T w_i^2]\geq 0. \qquad (7.6.32)$$

Finally, from (7.6.6), (7.6.12), (7.6.27) and (7.6.32), we have

$$H(x^1,x^2,y^1,y^2,z^1,z^2,\lambda)\nleq G(u^1,u^2,v^1,v^2,w^1,w^2,\lambda). \qquad \square$$

Corollary 7.6.1. *Let* $\left(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{z^1},\overline{z^2},\overline{\lambda}\right)$ *be feasible for (MP1) and let* $\left(\overline{u^1},\overline{u^2},\overline{v^1},\overline{v^2},\overline{w^1},\overline{w^2},\overline{\lambda}\right)$ *be feasible for (MD1) with the corresponding objective function values being equal. If the convexity and concavity assumptions and conditions (i)-(iv) of Theorem 7.6.1 are satisfied, then* $\left(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{z^1},\overline{z^2},\overline{\lambda}\right)$ *and* $\left(\overline{u^1},\overline{u^2},\overline{v^1},\overline{v^2},\overline{w^1},\overline{w^2},\overline{\lambda}\right)$ *are an efficient solution for (MP1) and (MD1), respectively.*

Theorem 7.6.2. *(Weak Duality). Let* $(x^1,x^2,y^1,y^2,z^1,z^2,\lambda)$ *be feasible for (MP2) and* $(u^1,u^2,v^1,v^2,w^1,w^2,\lambda)$ *be feasible for (MD2). Suppose that for* $i=1,2,\ldots,l$, $f_i(\cdot,y^1)+\cdot^T w_i^1$ *is* F_1- *convex for fixed* y^1, $f_i(x^1,\cdot)-\cdot^T z_i^1$ *is* F_2- *concave for fixed* x^1, $g_i(\cdot,y^2)+\cdot^T w_i^2$ *is* G_1- *convex for fixed* y^2 *and* $g_i(y^2,\cdot)-\cdot^T z_i^2$ *is* G_2- *concave for fixed* x^2, *and the following conditions are satisfied:*

(i) $F_1(x^1,u^1;a)+(u^1)^T a\geq 0$, if $a\geq 0$;
(ii) $G_1(x^2,u^2;b)+(u^2)^T b\geq 0$, if $b\geq 0$;
(iii) $F_2(y^1,v^1;c)+(y^1)^T c\leq 0$, if $c\leq 0$; *and*
(iv) $G_2(y^2,v^2;d)+(y^2)^T d\leq 0$, if $d\leq 0$.

Then

$$H_i^*(x^1,x^2,y^1,y^2,z,\lambda)\nleq G_i^*(u^1,u^2,v^1,v^2,w,\lambda).$$

Proof. Suppose $(x^1,x^2,y^1,y^2,z^1,z^2,\lambda)$ be feasible for (MP2) and $(u^1,u^2,v^1,v^2,w^1,w^2,\lambda)$ be feasible for (MD2). Then using the F_1- convexity of $f_i(\cdot,y^1)+\cdot^T w_i^1$ and F_2- concavity of $f_i(x^1,\cdot)-\cdot^T z_i^1$, for $i=1,2,\ldots,l$, we have

$$f_i(x^1,v^1)+(x^1)^T w_i^1 - f_i(u^1,v^1)-(u^1)^T w_i^1 \geq F_1(x^1,u^1;\nabla_{x^1} f_i(u^1,v^1)+w_i^1),$$

and

$$f_i(x^1,v^1)-(v^1)^T z_i^1 - f_i(x^1,y^1)+(y^1)^T z_i^1 \leq F_2(v^1,y^1;\nabla_{y^1} f_i(x^1,y^1)-z_i^1).$$

From (7.6.19), (7.6.26), the sublinearity of F_1 and F_2 and the above inequalities, we get

$$\sum_{i=1}^{l} \lambda_i \left[f_i(x^1,v^1)+(x^1)^T w_i^1 - f_i(u^1,v^1)-(u^1)^T w_i^1\right] \\ \geq F_1\left(x^1,u^1;\sum_{i=1}^{l}\lambda_i[\nabla_{x^1} f_i(u^1,v^1)+w_i^1]\right),$$

and

$$\sum_{i=1}^{l} \lambda_i [f_i(x^1,v^1)-(v^1)^T z_i^1 - f_i(x^1,y^1)+(y^1)^T z_i^1] \\ \leq F_2\left(v^1,y^1;\sum_{i=1}^{l}\lambda_i[\nabla_{y^1} f_i(x^1,y^1)-z_i^1]\right).$$

From constraints (7.6.13), (7.6.20), conditions (i), (iii) and the above inequalities, we get

$$\sum_{i=1}^{l} \lambda_i [f_i(x^1,v^1)+(x^1)^T w_i^1 - f_i(u^1,v^1)-(u^1)^T w_i^1] \geq 0,$$

$$\sum_{i=1}^{l} \lambda_i [f_i(x^1,v^1)-(v^1)^T z_i^1 - f_i(x^1,y^1)+(y^1)^T z_i^1] \leq 0.$$

Rearranging the above two inequalities, we obtain

$$\sum_{i=1}^{l} \lambda_i [f_i(x^1,y^1)-f_i(u^1,v^1)+(x^1)^T w_i^1 - (u^1)^T w_i^1 + (v^1)^T z_i^1 - (y^1)^T z_i^1] \geq 0.$$

Using $(v^1)^T z_i^1 \leq s(v^1|D_i^1)$ and $(x^1)^T w_i^1 \leq s(x^1|C_i^1)$, for $i=1,2,\ldots,l$, we have

$$\sum_{i=1}^{l} \lambda_i [f_i(x^1,y^1)-f_i(u^1,v^1)+s(x^1|C_i^1)-(u^1)^T w_i^1 + s(v^1|D_i^1)-(y^1)^T z_i^1] \geq 0. \tag{7.6.33}$$

Following as in the proof of Theorem 7.6.1, we get

$$\sum_{i=1}^{l} \lambda_i \left[g_i(x^2,y^2)+s(x^2|C_i^2)-(y^2)^T z_i^2 - g_i(u^2,v^2)+s(v^2|D_i^2)-(u^2)^T w_i^2\right] \geq 0. \tag{7.6.34}$$

Finally, from (7.6.19), (7.6.26), (7.6.33) and (7.6.34), we have

$$H^*(x^1,x^2,y^1,y^2,z^1,z^2,\lambda) \nleq G^*\left(u^1,u^2,v^1,v^2,w^1,w^2,\lambda\right). \qquad \square$$

Corollary 7.6.2. *Let* $(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{z^1},\overline{z^2},\overline{\lambda})$ *be feasible for (MP2) and let* $(\overline{u^1},\overline{u^2},\overline{v^1},\overline{v^2},\overline{w^1},\overline{w^2},\overline{\lambda})$ *be feasible for (MD2) with the corresponding objective function values being equal. Let the convexity and concavity assumptions and conditions (i)–(iv) in Theorem 7.6.2 are satisfied, then* $(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{z^1},\overline{z^2},\overline{\lambda})$ *and* $(\overline{u^1},\overline{u^2},\overline{v^1},\overline{v^2},\overline{w^1},\overline{w^2},\overline{\lambda})$ *are an efficient solution for (MP2) and (MD2), respectively.*

Remark 7.6.1. Theorems 7.6.1 and 7.6.2 can be established for more general classes of functions such as F_1- pseudoconvexity and F_2- pseudoconcavity, and G_1- pseudoconvexity and G_2- pseudoconcavity on the functions involved in the corresponding theorems. The proofs will follow the same lines as that of Theorems 7.6.1 and 7.6.2.

Strong duality theorems for the given models can be established on the lines of the proof of Theorem 2 of Yang et al. (2003a) in light of the discussions given above in this section.

Theorem 7.6.3. *(Strong Duality). Let* $(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{z^1},\overline{z^2},\overline{\lambda})$ *be a properly efficient solution for (MP1). Let* $\overline{\lambda}=\lambda$ *be fixed in (MD1) and the Hessian matrices* $\nabla^2_{x^1}\lambda^T f(\overline{x^1},\overline{y^1})$ *and* $\nabla^2_{y^2}\lambda^T g(\overline{x^2},\overline{y^2})$ *be either positive definite or negative definite and* $\nabla_{x^1}\overline{\lambda}^T g(\overline{x^2},\overline{y^2}) \neq \bar{\lambda}^T\overline{z^2}$*. Also let the set* $\{\nabla_{y^2}g_1 - z_1^2,\ldots,\nabla_{y^2}g_l - z_l^2\}$ *is linearly independent. If the generalized convexity hypotheses and conditions (i)-(iv) of Theorem 7.6.1 are satisfied, then* $(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{z^1},\overline{z^2},\overline{\lambda})$ *is a properly efficient solution for (MD1).*

Theorem 7.6.4. *(Strong Duality). Let* $(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{z^1},\overline{z^2},\overline{\lambda})$ *be a properly efficient solution for (MP2). Let* $\overline{\lambda}=\lambda$ *be fixed in (MD2). Suppose that the Hessian matrix* $\nabla^2_{x^1}f_i(\overline{x^1},\overline{y^1})$ *is positive definite for* $i=1,2,\ldots,l$ *and* $\sum_{i=1}^l \lambda_i[\nabla_{y^1}f_i - \bar{z}_i^1] \geq 0$; *and* $\nabla^2_{y^2}g_i(\overline{x^2},\overline{y^2})$ *is positive definite for* $i=1,2,\ldots,l$ *and* $\sum_{i=1}^l \lambda_i[\nabla_{y^2}g_i - \bar{z}_i^2] \geq 0$; *or* $\nabla^2_{x^1}f_i(\overline{x^1},\overline{y^1})$ *is negative definite for* $i=1,2,\ldots,l$ *and* $\sum_{i=1}^l \lambda_i[\nabla_{y^1}f_i - \bar{z}_i^1] \leq 0$; *and* $\nabla^2_{y^2}g_i(\overline{x^2},\overline{y^2})$ *is negative definite for* $i=1,2,\ldots,l$ *and* $\sum_{i=1}^l \lambda_i[\nabla_{y^2}g_i - \bar{z}_i^2] \leq 0$*. Also suppose that the sets* $\{\nabla_{y^1}f_1 - \bar{z}_1^2,\ldots,\nabla_{y^1}f_l - \bar{z}_l^2\}$ *and* $\{\nabla_{y^2}g_1 - \bar{z}_1^2,\ldots,\nabla_{y^2}g_l - \bar{z}_l^2\}$ *are linearly independent. If the generalized convexity hypotheses and conditions (i)–(iv) of Theorem 7.6.2 are satisfied, then* $(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{z^1},\overline{z^2},\overline{\lambda})$ *is a properly efficient solution for (MD2).*

Remark 7.6.2. We consider some special cases of problems (MP1), (MD1), (MP2) and (MD2) by choosing particular forms of compact convex sets, and the number of objective and constraint functions.

(i) If $C_i^1 = C_i^2 = D_i^1 = D_i^2 = \{0\}, i=1,2,\ldots,l$, then (MP1) and (MD1) reduce to the pair of problems studied in Bector et al. (1999).

(ii) If $l = 1$, then (MP1) and (MD1) reduce to the pair of problems studied in Yang et al. (2003a), and (MP2) and (MD2) become an extension of problems studied in Yang et al. (2003a).
(iii) If $|J_2| = 0, |K_2| = 0$ and $l = 1$, then (MP1) and (MD1) reduce to the pair of problems (P) and (D) of Mond and Schechter (1996), and (MP2) and (MD2) reduce to the pair of problems (P1) and (D1) of Mond and Schechter (1996).
(iv) If $|J_2| = 0, |K_2| = 0$, then (MP1) and (MD1) become multiobjective extension of the pair of problems (P) and (D) of Mond and Schechter (1996), and (MP2) and (MD2) becomes the multiobjective extension of the pair of problems (P1) and (D1) of Mond and Schechter (1996).
(v) If $l = 1$, then (MP2) and (MD2) are an extension of the pair of problems studied in Yang et al. (2003a).

These results can be extended to second and higher order case as well as to other classes of generalized convexity.

7.7 Mond–Weir Type Mixed Symmetric First and Second Order Duality in Nondifferentiable Optimization Problems

In this section, we introduce two models of mixed symmetric duality for a class of non-differentiable multiobjective programming problems with multiple arguments. The first model is Mond–Weir type mixed symmetric dual model for a class of non-differentiable mathematical programming problems and the second model is second order case of the first model. Mixed symmetric duality for this model has not been given so far by any other author. The advantage of the first model over the model given by Yang et al. (2003) is that it allows further weakening of convexity on the functions involved. Furthermore, Mangasarian (1975) and Mond (1974a) have indicated possible computational advantages of the second order duals over the first order duals. We establish weak duality theorems for these two models under generalized pseudo-convexity and generalized second order pseudo-convexity assumptions and discuss several special cases of these models. The results of Hou and Yang (2001), Mishra (2000a, b, 2001b), Mond and Schechter (1996), Nanda and Das (1996), as well as Yang et al. (2003b) are particular cases of the results obtained in the present paper.

The concepts of sublinearity and support functions from previous sections will also be needed in the present section.

Definition 7.7.1. *Let $X \subset R^n, Y \subset R^m$ and $F : X \times Y \times R^n \to R$ be sublinear with respect to its third argument. $f(\cdot, y)$ is said to be second order F-convex at $\bar{x} \in X$, with respect to $p \in R^n$, for fixed $y \in Y$, if*

$$f(x,y) - f(\bar{x},y) + \frac{1}{2}p^T \nabla_{xx} f(\bar{x},y)p \geq F(x,\bar{x}; \nabla_x f(\bar{x},y) + \nabla_{xx} f(\bar{x},y)p), \forall x \in X.$$

f is said to be second order F-concave at $\bar{x} \in X$, with respect to $p \in R^n$, for fixed $y \in Y$, if $-f$ is second order F-convex at $\bar{x} \in X$, with respect to $p \in R^n$.

Definition 7.7.2. *Let $X \subset R^n, Y \subset R^m$ and $F : X \times Y \times R^n \to R$ be sublinear with respect to its third argument. $f(x,\cdot)$ is said to be second order F–pseudo-convex at $\bar{x} \in X$, with respect to $p \in R^n$, for fixed $y \in Y$, if*

$$F(x,\bar{x};\nabla_x f(\bar{x},y)+\nabla_{xx} f(\bar{x},y)p) \geq 0 \Rightarrow f(x,y) \geq f(\bar{x},y)+\frac{1}{2}p^T\nabla_{xx} f(\bar{x},y)p, \forall x \in X.$$

f is said to be second order F – pseudo-concave at $\bar{x} \in X$, with respect to $p \in R^n$, for fixed $y \in Y$, if $-f$ is second order F – pseudo-convex at $\bar{x} \in X$, with respect to $p \in R^n$.

Remark 7.7.1. (i) The second order F - pseudo-convexity reduces to the F – pseudo-convexity introduced by Hanson and Mond (1987b) when $p = 0$.

(ii) For $F(x,\bar{x};a) = \eta(x,\bar{x})^T a$, where $\eta : X \times X \to R^n$, the second order F – convexity reduces to the second order invexity introduced by Hanson (1993), and second order F – pseudo-convexity reduces to the second order pseudo-invexity introduced by Yang (1995).

(iii) For $F(x,\bar{x};a) = \eta(x,\bar{x})^T a$, and $p = 0$, where $\eta : X \times X \to R^n$, the second order F – convexity reduces to the invexity introduced by Hanson (1993), and second order F – pseudo-convexity reduces to the pseudo-invexity introduced by Kaul and Kaur (1985).

Now we state two Mond–Weir type mixed symmetric dual pairs and establish duality theorems under generalized convexity assumptions. The advantage of these models are that they allow further weakening of the convexity assumptions and the advantage of the second order dual may be used to give a tighter bound than the first order dual for the value of the primal objective function, one can see Mishra (1997a).

First Order Model

Primal problem

(NMP) minimize $f(x^1,y^1)+g(x^2,y^2)+s(x^1|C^1)+s(x^2|C^2)-(y^1)^T z^1-(y^2)^T z^2$

subject to $(x^1,x^2,y^1,y^2,z^1,z^2) \in R^{|J_1|} \times R^{|J_2|} \times R^{|K_1|} \times R^{|K_2|} \times R^{|K_1|} \times R^{|K_2|}$

$$\nabla_{y^1} f(x^1,y^1)-z^1 \leq 0, \tag{7.7.1}$$

$$\nabla_{y^2} g(x^2,y^2)-z^2 \leq 0, \tag{7.7.2}$$

$$(y^1)^T[\nabla_{y^1} f(x^1,y^1)-z^1] \geq 0, \tag{7.7.3}$$

$$(y^2)^T[\nabla_{y^2} g(x^2,y^2)-z^2] \geq 0, \tag{7.7.4}$$

$$z^1 \in D^1, z^2 \in D^2. \tag{7.7.5}$$

Dual Problem (NMD)

$$\text{maximize } f(u^1,v^1)+g(u^2,v^2)-s\left(v^1\left|D^1\right.\right)-s\left(v^2\left|D^2\right.\right)+\left(u^1\right)^T w^1+\left(u^2\right)^T w^2$$
$$\text{subject to } \left(u^1,u^2,v^1,v^2,w^1,w^2\right)\in R^{|J_1|}\times R^{|J_2|}\times R^{|K_1|}\times R^{|K_2|}\times R^{|J_1|}\times R^{|J_2|}$$

$$\nabla_{x^1} f\left(u^1,v^1\right)+w^1\geq 0, \tag{7.7.6}$$
$$\nabla_{x^2} g\left(u^2,v^2\right)+w^2\geq 0, \tag{7.7.7}$$
$$\left(u^1\right)^T\left[\nabla_{x^1} f\left(u^1,v^1\right)+w^1\right]\leq 0, \tag{7.7.8}$$
$$\left(u^2\right)^T\left[\nabla_{x^2} g\left(u^2,v^2\right)+w^2\right]\leq 0, \tag{7.7.9}$$
$$w^1\in C^1,\text{and} w^2\in C^2, \tag{7.7.10}$$

where C^1 is a compact and convex subsets of $R^{|J_1|}$ and C^2 is a compact and convex subsets of $R^{|J_2|}$, similarly, D^1 is a compact and convex subsets of $R^{|K_1|}$ and D^2 is a compact and convex subsets of $R^{|K_2|}$.

The following model is Mond–Weir type *second order mixed symmetric dual* model for a class of nondifferentiable optimization problems:

Second Order Model

Primal Problem (SNP)

$$\text{minimize } f(x^1,y^1)+g\left(x^2,y^2\right)+s(x^1|C^1)+s\left(x^2\left|C^2\right.\right)-(y^1)^T z^1-\left(y^2\right)^T z^2$$
$$-\frac{1}{2}\left(p^1\right)^T\nabla_{y^1y^1} f(x^1,y^1)p^1-\frac{1}{2}\left(p^2\right)^T\nabla_{y^2y^2} g\left(x^2,y^2\right)p^2$$

subject to

$$\left(x^1,x^2,y^1,y^2,z^1,z^2,p^1,p^2\right)\in R^{|J_1|}\times R^{|J_2|}\times R^{|K_1|}\times R^{|K_2|}\times R^{|K_1|}\times R^{|K_2|}\times R^{|K_1|}\times R^{|K_2|}$$

$$\left[\nabla_{y^1} f(x^1,y^1)-z^1+\nabla_{y^1y^1} f(x^1,y^1)p^1\right]\leq 0, \tag{7.7.11}$$
$$\left[\nabla_{y^2} g\left(x^2,y^2\right)-z^2+\nabla_{y^2y^2} g\left(x^2,y^2\right)p^2\right]\leq 0, \tag{7.7.12}$$
$$(y^1)^T\left[\nabla_{y^1} f(x^1,y^1)-z^1+\nabla_{y^1y^1} f(x^1,y^1)p^1\right]\geq 0, \tag{7.7.13}$$
$$\left(y^2\right)^T\left[\nabla_{y^2} g\left(x^2,y^2\right)-z^2+\nabla_{y^2y^2} g\left(x^2,y^2\right)p^2\right]\geq 0, \tag{7.7.14}$$
$$z^1\in D^1,\ \text{and } z^2\in D^2, \tag{7.7.15}$$

Dual Problem (SND)

$$\text{maximize } f\left(u^1,v^1\right)+g\left(u^2,v^2\right)-s\left(v^1\left|D^1\right.\right)-s\left(v^2\left|D^2\right.\right)+\left(u^1\right)^T w^1+\left(u^2\right)^T w^2$$
$$-\frac{1}{2}\left(q^1\right)^T \nabla_{y^2y^2} f\left(u^1,v^1\right) q^1-\frac{1}{2}\left(q^2\right)^T \nabla_{y^2y^2} g\left(u^2,v^2\right) q^2$$

subject to

$$\left(u^1,u^2,v^1,v^2,w^1,w^2,q^1,q^2\right)\in R^{|J_1|}\times R^{|J_2|}\times R^{|K_1|}\times R^{|K_2|}\times R^{|J_1|}\times R^{|J_2|}$$
$$\times R^{|J_1|}\times R^{|J_2|}$$

$$\left[\nabla_{x^1} f\left(u^1,v^1\right)+w^1+\nabla_{x^1x^1} f\left(u^1,v^1\right) q^1\right]\geq 0, \tag{7.7.16}$$
$$\left[\nabla_{x^2} g\left(u^2,v^2\right)+w^2+\nabla_{x^2x^2} g\left(u^2,v^2\right) q^2\right]\geq 0, \tag{7.7.17}$$
$$\left(u^1\right)^T\left[\nabla_{x^1} f\left(u^1,v^1\right)+w^1+\nabla_{x^1x^1} f\left(u^1,v^1\right) q^1\right]\leq 0, \tag{7.7.18}$$
$$\left(u^2\right)^T\left[\nabla_{x^2} g\left(u^2,v^2\right)+w^2+\nabla_{x^2x^2} g\left(u^2,v^2\right) q^2\right]\leq 0, \tag{7.7.19}$$
$$w^1\in C^1, \text{and} w^2\in C^2, \tag{7.7.20}$$

where C^1 is a compact and convex subsets of $R^{|J_1|}$ and C^2 is a compact and convex subsets of $R^{|J_2|}$, similarly, D^1 is a compact and convex subsets of $R^{|K_1|}$ and D^2 is a compact and convex subsets of $R^{|K_2|}$.

Now we establish duality theorems for the pair of problems (NMP) and (NMD) as well as (SNP) and (SND) under the F – pseudo-convexity and second order F – pseudo-convexity assumptions.

Theorem 7.7.1. *(Weak Duality). Let* $\left(x^1,x^2,y^1,y^2,z^1,z^2\right)$ *be feasible for (NMP) and* $\left(u^1,u^2,v^1,v^2,w^1,w^2\right)$ *be feasible for (NMD). Suppose that* $f\left(\cdot,v^1\right)+\cdot^T w^1$ *is* F_1 *– pseudo-convex for fixed* v^1*,* $f\left(x^1,\cdot\right)-\cdot^T z^1$ *is* F_2 *– pseudo-concave for fixed* x^1*,* $g\left(\cdot,y^2\right)+\cdot^T w^2$ *is* G_1 *– pseudo-convex for fixed* v^2 *and* $g(y^2,\cdot)-\cdot^T z^2$ *is* G_2 *– pseudo-concave for fixed* x^2*, and the following conditions are satisfied:*

(i) $F_1\left(x^1,u^1;\nabla_{x^1} f\left(u^1,v^1\right)+w^1\right)+\left(u^1\right)^T\left(\nabla_{x^1} f\left(u^1,v^1\right)+w^1\right)\geq 0$;

(ii) $G_1\left(x^2,u^2;\nabla_{x^2} g\left(u^2,v^2\right)+w^2\right)+\left(u^2\right)^T\left(\nabla_{x^2} g\left(u^2,v^2\right)+w^2\right)\geq 0$;

(iii) $F_2\left(y^1,v^1;\nabla_{y^1} f(x^1,y^1)-z^1\right)+(y^1)^T\left(\nabla_{y^1} f(x^1,y^1)-z^1\right)\leq 0$; and

(iv) $G_2\left(y^2,v^2;\nabla_{y^2} g\left(x^2,y^2\right)-z^2\right)+\left(y^2\right)^T\left(\nabla_{y^2} g\left(x^2,y^2\right)-z^2\right)\leq 0$.

Then

$$\inf(\text{NMP})\geq\sup(\text{NMD}).$$

Proof. Suppose that $\left(x^1,x^2,y^1,y^2,z^1,z^2\right)$ is feasible for (NMP) and $\left(u^1,u^2,v^1,v^2,\right.$ $\left.w^1,w^2\right)$ is feasible for (NMD). By the dual constraint (7.7.6), we have $\nabla_{x^1} f\left(u^1,v^1\right)$ $+w^1\geq 0$, and by condition (i), we get

$$F_1\left(x^1,u^1;\nabla_{x^1} f\left(u^1,v^1\right)+w^1\right)\geq-\left(u^1\right)^T\left[\nabla_{x^1} f(u^1,v^1)+w^1\right]\geq 0, (\text{using } (7.7.8)).$$

Then by the F_1 – pseudo-convexity of $f\left(\cdot,v^1\right)+\cdot^T w^1$, we get

$$f\left(x^1,v^1\right)+\left(x^1\right)^T w^1 \geq f(u^1,v^1)+\left(u^1\right)^T w^1. \tag{7.7.21}$$

Similarly, by using the constraint (7.7.1), condition (iii) and constraint (7.7.3), we get

$$F_2\left(y^1,v^1;\nabla_{y^1} f(x^1,y^1)-z^1\right) \leq -(y^1)^T\left[\nabla_{y^1} f(x^1,y^1)-y^1\right] \leq 0.$$

Then by the F_2 – pseudo-concavity of $f\left(x^1,\cdot\right)-\cdot^T z^1$, we get

$$f\left(x^1,v^1\right)-\left(v^1\right)^T z^1 \leq f(x^1,y^1)-(y^1)^T z^1. \tag{7.7.22}$$

Now rearranging (7.7.21) and (7.7.22), we get

$$f(x^1,y^1)+\left(x^1\right)^T w^1-(y^1)^T z^1 \geq f\left(u^1,v^1\right)+\left(u^1\right)^T w^1-\left(v^1\right)^T z^1.$$

Using $\left(v^1\right)^T z^1 \leq s\left(v^1\left|D^1\right.\right)$ and $\left(x^1\right)^T w^1 \leq s(x^1|C^1)$, we have

$$f(x^1,y^1)+s(x^1|C^1)-(y^1)^T z^1 \geq f\left(u^1,v^1\right)-s\left(v^1\left|D^1\right.\right)+\left(u^1\right)^T w^1. \tag{7.7.23}$$

Similarly, using constraints (7.7.7), condition (ii), constraint (7.7.9) and G_1 – pseudo-convexity of the function $g\left(\cdot,y^2\right)+\cdot^T w^2$ and constraint (7.7.2), condition (iv), constraint (7.7.4) and G_2 – pseudo-concavity of $g\left(y^2,\cdot\right)-\cdot^T z^2$ and using $\left(x^2\right)^T w^2 \leq s\left(x^2\left|C^2\right.\right)$ and $\left(v^2\right)^T z^2 \leq s\left(v^2\left|D^2\right.\right)$, finally rearranging the resultants, we get

$$g\left(x^2,y^2\right)+s\left(x^2\left|C^2\right.\right)-\left(y^2\right)^T z^2 \geq g\left(u^2,v^2\right)-s\left(v^2\left|D^2\right.\right)+\left(u^2\right)^T w^2. \tag{7.7.24}$$

Finally, from (7.7.23) and (7.7.24), we have

$$\begin{aligned} & f(x^1,y^1)+g\left(x^2,y^2\right)+s(x^1|C^1)+s\left(x^2\left|C^2\right.\right)-(y^1)^T z^1-\left(y^2\right)^T z^2 \\ & \quad \geq f\left(u^1,v^1\right)+g\left(u^2,v^2\right)-s\left(v^1\left|D^1\right.\right)-s\left(v^2\left|D^2\right.\right)+\left(u^1\right)^T w^1+\left(u^2\right)^T w^2. \end{aligned}$$

That is, $\inf(\text{NMP}) \geq \sup(\text{NMD})$ □

The weak duality for the pair (SNP) and (SND) is established in the following theorem.

Theorem 7.7.2. *(Weak Duality). Let* $(x^1,x^2,y^1,y^2,z^1,z^2,p^1,p^2)$ *be feasible for (SNP) and* $(u^1,u^2,v^1,v^2,w^1,w^2,q^1,q^2)$ *be feasible for (NMD). Suppose there exist sub-linear functionals* F_1,F_2, G_1 *and* G_2 *atisfying:*

(i) $F_1\left(x^1,u^1;\nabla_{x^1} f\left(u^1,v^1\right)+w^1+\nabla_{x^1x^1} f\left(u^1,v^1\right)q^1\right)$
$\quad +\left(u^1\right)^T\left(\nabla_{x^1} f\left(u^1,v^1\right)+w^1+\nabla_{x^1x^1} f\left(u^1,v^1\right)q^1\right) \geq 0;$

(ii) $G_1\left(x^2,u^2;\nabla_{x^2} g\left(u^2,v^2\right)+w^2+\nabla_{x^2x^2} g\left(u^2,v^2\right)q^2\right)$
$\quad +\left(u^2\right)^T\left(\nabla_{x^2} g\left(u^2,v^2\right)+w^2+\nabla_{x^2x^2} g\left(u^2,v^2\right)q^2\right) \geq 0;$

(iii) $F_2\left(y^1, v^1; \nabla_{y^1} f(x^1,y^1) - z^1 + \nabla_{y^1y^1} f(x^1,y^1)p^1\right)$
$+ (y^1)^T \left(\nabla_{y^1} f(x^1,y^1) - z^1 + \nabla_{y^1y^1} f(x^1,y^1)p^1\right) \leq 0;$

and

(iv) $G_2\left(y^2, v^2; \nabla_{y^2} g\left(x^2,y^2\right) - z^2 + \nabla_{y^2y^2} g\left(x^2,y^2\right) p^2\right)$
$+ \left(y^2\right)^T \left(\nabla_{y^2} g\left(x^2,y^2\right) - z^2 + \nabla_{y^2y^2} g\left(x^2,y^2\right) p^2\right) \leq 0.$

Furthermore, assume that $f\left(\cdot, v^1\right) + \cdot^T w^1$ *is second order* F_1 *– pseudo-convex for fixed* v^1, $f\left(x^1, \cdot\right) - \cdot^T z^1$ *is second order* F_2 *– pseudo-concave for fixed* x^1, $g\left(\cdot, y^2\right) + \cdot^T w^2$ *is second order* G_1 *– pseudo-convex for fixed* v^2 *and* $g(y^2, \cdot) - \cdot^T z^2$ *is second order* G_2 *– pseudo-concave for fixed* x^2*, with respect to* q^1, p^1, q^2 *and* p^2*, respectively. Then*

$$\inf(\text{SNP}) \geq \sup(\text{SND}).$$

Proof. Suppose $\left(x^1, x^2, y^1, y^2, z^1, z^2, p^1, p^2\right)$ be feasible for (SNP) and $\left(u^1, u^2, v^1, v^2, w^1, w^2, q^1, q^2\right)$ be feasible for (NMD). By the dual constraint (7.7.16), we have $[\nabla_{x^1} f(u^1,v^1) + w^1 + \nabla_{x^1x^1} f(u^1,v^1)q^1] \geq 0$, and by condition (i) and (7.7.18), we get

$$\begin{aligned} &F_1\left(x^1, u^1; \nabla_{x^1} f\left(u^1,v^1\right) + w^1 + \nabla_{x^1x^1} f\left(u^1,v^1\right) q^1\right) \\ &\quad \geq - \left(u^1\right)^T \left(\nabla_{x^1} f\left(u^1,v^1\right) + w^1 + \nabla_{x^1x^1} f\left(u^1,v^1\right) q^1\right) \geq 0, \end{aligned}$$

Then by the second order F_1 – pseudo-convexity of $f\left(\cdot, v^1\right) + \cdot^T w^1$, we get

$$f\left(x^1,v^1\right) + \left(x^1\right)^T w^1 \geq f(u^1,v^1) + \left(u^1\right)^T w^1 - \frac{1}{2}\left(q^1\right)^T \nabla_{x^1x^1} f\left(u^1,v^1\right) q^1. \quad (7.7.25)$$

Similarly, by using the constraint (7.7.11), condition (iii) and constraint (7.7.13), we get

$$\begin{aligned} &F_2\left(y^1, v^1; \nabla_{y^1} f(x^1,y^1) - z^1 + \nabla_{y^1y^1} f(x^1,y^1)p^1\right) \\ &\quad \leq -(y^1)^T \left[\nabla_{y^1} f(x^1,y^1) - y^1 + \nabla_{y^1y^1} f(x^1,y^1)p^1\right] \leq 0. \end{aligned}$$

Then by the second order F_2 – pseudo-concavity of $f\left(x^1, \cdot\right) - \cdot^T z^1$, we get

$$f\left(x^1,v^1\right) - \left(v^1\right)^T z^1 \leq f(x^1,y^1) - (y^1)^T z^1 - \frac{1}{2}\left(p^1\right)^T \nabla_{y^1y^1} f(x^1,y^1)p^1. \quad (7.7.26)$$

Now rearranging (7.7.25) and (7.7.26), we get

$$\begin{aligned} &f(x^1,y^1) + \left(x^1\right)^T w^1 - (y^1)^T z^1 - \frac{1}{2}\left(p^1\right)^T \nabla_{y^1y^1} f(x^1,y^1)p^1 \\ &\quad \geq f\left(u^1,v^1\right) + \left(u^1\right)^T w^1 - (v^1)^T z^1 - \frac{1}{2}\left(q^1\right)^T \nabla_{x^1x^1} f\left(u^1,v^1\right) q^1. \end{aligned}$$

Using $\left(v^1\right)^T z^1 \leq s\left(v^1 \left|D^1\right.\right)$ and $\left(x^1\right)^T w^1 \leq s(x^1|C^1)$, we have

$$\begin{aligned} & f(x^1,y^1)+s(x^1|C^1)-(y^1)^T z^1-\frac{1}{2}\left(p^1\right)^T \nabla_{y^1y^1} f(x^1,y^1)p^1 \\ & \quad \geq f\left(u^1,v^1\right)-s\left(v^1\left|D^1\right.\right)+\left(u^1\right)^T w^1-\frac{1}{2}\left(q^1\right)^T \nabla_{x^1x^1} f\left(u^1,v^1\right)q^1. \end{aligned} \tag{7.7.27}$$

Similarly, using constraints (7.7.17), condition (ii), constraint (7.7.19) and second order G_1 – pseudo-convexity of the function $g\left(\cdot,y^2\right)+\cdot^T w^2$ and constraint (7.7.12), condition (iv), constraint (7.7.14) and second order G_2 – pseudo-concavity of $g(y^2,\cdot)-\cdot^T z^2$ and using $\left(x^2\right)^T w^2 \leq s\left(x^2\left|C^2\right.\right)$ and $\left(v^2\right)^T z^2 \leq s\left(v^2\left|D^2\right.\right)$, finally rearranging the resultants, we get

$$\begin{aligned} & g\left(x^2,y^2\right)+s\left(x^2\left|C^2\right.\right)-\left(y^2\right)^T z^2-\frac{1}{2}\left(p^2\right)^T \nabla_{y^2y^2} g(x^2,y^2)p^2 \\ & \quad \geq g\left(u^2,v^2\right)-s\left(v^2\left|D^2\right.\right)+\left(u^2\right)^T w^2-\frac{1}{2}\left(q^2\right)^T \nabla_{x^2x^2} g\left(u^2,v^2\right)q^2. \end{aligned} \tag{7.7.28}$$

Finally, from (7.7.27) and (7.7.28), we have

$$\begin{aligned} & f(x^1,y^1)+g\left(x^2,y^2\right)+s(x^1|C^1)+s\left(x^2\left|C^2\right.\right)-(y^1)^T z^1-\left(y^2\right)^T z^2 \\ & \qquad -\frac{1}{2}\left(p^1\right)^T \nabla_{y^1y^1} f(x^1,y^1)p^1-\frac{1}{2}\left(p^2\right)^T \nabla_{y^2y^2} g\left(x^2,y^2\right)p^2 \\ & \qquad \quad \geq f\left(u^1,v^1\right)+g\left(u^2,v^2\right)-s\left(v^1\left|D^1\right.\right)-s\left(v^2\left|D^2\right.\right)+\left(u^1\right)^T w^1+\left(u^2\right)^T w^2 \\ & \qquad -\frac{1}{2}\left(q^1\right)^T \nabla_{x^1x^1} f(u^1,v^1)q^1-\frac{1}{2}\left(q^2\right)^T \nabla_{x^2x^2} g\left(u^2,v^2\right)q^2. \end{aligned}$$

That is,

$$\inf(\text{SNP}) \geq \sup(\text{SND}). \qquad \square$$

Theorem 7.7.3. *(Strong Duality). Let* $\left(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{z^1},\overline{z^2}\right)$ *be an optimal solution for (NMP). Suppose that the Hessian matrix* $\nabla^2_{x^1} f\left(\overline{x^1},\overline{y^1}\right)$ *is positive definite and* $\nabla_{y^1} f-\bar{z}^1 \geq 0$*; and* $\nabla^2_{y^2} g\left(\overline{x^2},\overline{y^2}\right)$ *is positive definite and* $\nabla_{y^2} g-\bar{z}^2 \geq 0$*; or* $\nabla^2_{x^1} f\left(\overline{x^1},\overline{y^1}\right)$ *is negative definite and* $\nabla_{y^1} f-\bar{z}^1 \leq 0$*; and* $\nabla^2_{y^2} g\left(\overline{x^2},\overline{y^2}\right)$ *is negative definite and* $\nabla_{y^2} g-\bar{z}^2 \leq 0$*. If the generalized convexity hypotheses and conditions (i)-(iv) of Theorem 7.7.1 are satisfied, then* $\left(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{z^1},\overline{z^2}\right)$ *is an optimal solution for (NMD).*

Proof. The proof of this theorem can be established in light of the above Theorem 7.7.1. $\square$

Theorem 7.7.4. *(Strong Duality). Let* $\left(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{z^1},\overline{z^2},\overline{p^1},\overline{p^2}\right)$ *be an optimal solution for (SNP) such that*

$$\nabla_{y^1} f\left(\overline{x^1},\overline{y^1}\right) + \nabla_{y^1y^1} f\left(\overline{x^1},\overline{y^1}\right)\overline{p^1} \neq \overline{z^1};$$

and

$$\nabla_{y^2} g\left(\overline{x^2},\overline{y^2}\right) + \nabla_{y^2y^2} g\left(\overline{x^2},\overline{y^2}\right)\overline{p^2} \neq \overline{z^2}.$$

Suppose that the Hessian matrix $\nabla^2_{x^1} f\left(\overline{x^1},\overline{y^1}\right)$ *is positive definite and* $\left(\overline{p^1}\right)^T \left[\nabla_{y^1} f - \bar{z}^1\right] \geq 0$*; and* $\nabla^2_{y^2} g\left(\overline{x^2},\overline{y^2}\right)$ *is positive definite and* $\left(\overline{p^2}\right)^T \left[\nabla_{y^2} g - \bar{z}^2\right] \geq 0$*; or* $\nabla^2_{x^1} f\left(\overline{x^1},\overline{y^1}\right)$ *is negative definite and* $\left(\overline{p^1}\right)^T \left[\nabla_{y^1} f - \bar{z}^1\right] \leq 0$*; and* $\nabla^2_{y^2} g\left(\overline{x^2},\overline{y^2}\right)$ *is negative definite and* $\left(\overline{p^2}\right)^T \left[\nabla_{y^2} g - \bar{z}^2\right] \leq 0$*. If the generalized convexity hypotheses and conditions (i)-(iv) of Theorem 7.7.2 are satisfied, then* $\left(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{z^1},\overline{z^2},\overline{p^1},\overline{p^2}\right)$ *is an optimal solution for (SND).*

Proof. The proof of this theorem can be established on the lines of the proof of strong duality Theorem 3.2 given by Hou and Yang (2001) in light of the above Theorem 7.7.2. □

Remark 7.7.2. We consider some special cases of our problems (NMP) and (NMD) as well as (SNP) and (SND) by choosing particular forms of the compact sets involved in the problems.

(vi) If $C^1 = C^2 = D^1 = D^2 = \{0\}$, then (NMP) and (NMD) reduce to the pair of problems studied in Chandra et al. (1999).

(vii) If $|J_2| = 0, |K_2| = 0$, then (NMP) and (NMD) reduce to the pair of problems (P1) and (D1) of Mond and Schechter (1996).

(viii) If $|J_2| = 0, |K_2| = 0$, then the second order dual pairs (SNP) and (SND) reduce to the pair of problems studied by Hou and Yang (2001).

(ix) If $|J_2| = 0, |K_2| = 0$, and $C^1 = C^2 = D^1 = D^2 = \{0\}$,then (SNP) and (SND) become pair of problems (MP) and (MD) studied by Mishra (2000b).

The results discussed in this section can be extended to the *higher order* case as well as to other generalized convexity assumptions. Some other possible extensions are as follows.

(x) These results can be extended to the case of multi-objective problems.

(xi) These results can be extended to the case of continuous-time problems as well.

(xii) In light of the results established by Mishra and Rueda (2003), the results of this section can further be extended to the case of complex spaces also.

7.8 Second Order Mixed Symmetric Duality in Nondifferentiable Vector Optimization Problems

In this section, a pair of Mond–Weir type second order mixed symmetric dual, is presented for a class of nondifferentiable vector optimization problems. We establish duality theorems for the new pair of dual models under second order generalized convexity assumptions. This mixed second order dual formulation unifies the two existing second order symmetric dual formulations in the literature. Several results including almost every recent work on symmetric duality are obtained as special cases of the results established in the present paper.

Consider the following vector optimization problem:

$$\text{(VP)} \qquad \text{minimize } \left(f_1(x), f_2(x), \ldots, f_p(x)\right)$$
$$\text{subject to } x \in X_0,$$

where $f_i : R^n \to R$, $i = 1, 2, \ldots, p$ and $X_0 \subseteq R^n$.

For problem (VP), an efficient solution and a properly efficient solution are as in Sect. 7.1.

For $N = \{1, 2, \ldots, n\}$ and $M = \{1, 2, \ldots, m\}$ let $J_1 \subset N, K_1 \subset M$ and $J_2 = N \backslash J_1$ and $K_2 = M \backslash K_1$. Let $|J_1|$ denote the number of elements in the set J_1. The numbers $|J_2|$, $|K_1|$ and $|K_2|$ are defined similarly. Notice that if $J_1 = \Phi$, then $J_2 = N$, that is, $|J_1| = 0$ and $|J_2| = n$. Hence, $R^{|J_1|}$ is zero dimensional Euclidean space and $R^{|J_2|}$ is n—dimensional Euclidean space. It is clear that any $x \in R^n$ can be written as $x = \left(x^1, x^2\right), x^1 \in R^{|J_1|}, x^2 \in R^{|J_2|}$. Similarly, any $y \in R^m$ can be written as $y = \left(y^1, y^2\right), y^1 \in R^{|K_1|}, y^2 \in R^{|K_2|}$. Let $f : R^{|J_1|} \times R^{|K_1|} \to R^l$ and $g : R^{|J_2|} \times R^{|K_2|} \to R^l$ be twice differentiable functions and $e = (1, 1, \ldots, 1)^T \in R^l$.

Now we can introduce the following pair of non-differentiable multi-objective programs and discuss duality theorems under some mild assumptions of generalized convexity.

Primal Problem (SMP)

$$\begin{aligned} &\text{minimize } H\left(x^1, x^2, y^1, y^2, z^1, z^2, p^1, p^2 \lambda\right) \\ &\quad = \left(H_1\left(x^1, x^2, y^1, y^2, z^1, z^2, p^1, p^2 \lambda\right), \ldots, H_l\left(x^1, x^2, y^1, y^2, z^1, z^2, p^1, p^2 \lambda\right)\right) \end{aligned}$$

subject to

$$\begin{aligned} \left(x^1, x^2, y^1, y^2, z^1, z^2, p^1, p^2, \lambda\right) \in R^{|J_1|} \times R^{|J_2|} \times R^{|K_1|} \times R^{|K_2|} \times R^{|K_1|} \times R^{|K_2|} \\ \times R^{|K_1|} \times R^{|K_2|} \times R^l_+ \end{aligned}$$

$$\sum_{i=1}^{l} \lambda_i \left[\nabla_{y^1} f_i(x^1, y^1) - z_i^1 + \nabla_{y^1 y^1} f_i(x^1, y^1) p_i^1\right] \leq 0 \tag{7.8.1}$$

$$\sum_{i=1}^{l} \lambda_i \left[\nabla_{y^2} g_i(x^2, y^2) - z_i^2 + \nabla_{y^2 y^2} g_i(x^2, y^2) p_i^2\right] \leq 0 \tag{7.8.2}$$

$$(y^1)^T \sum_{i=1}^{l} \lambda_i \left[\nabla_{y^1} f_i(x^1,y^1) - z_i^1 + \nabla_{y^1y^1} f_i(x^1,y^1) p_i^1\right] \geq 0 \tag{7.8.3}$$

$$(y^2)^T \sum_{i=1}^{l} \lambda_i \left[\nabla_{y^2} g_i(x^2,y^2) - z_i^2 + \nabla_{y^2y^2} g_i(x^2,y^2) p_i^2\right] \geq 0 \tag{7.8.4}$$

$$\left(x^1,x^2\right) \geq 0, \tag{7.8.5}$$

$$z_i^1 \in D_i^1, \text{ and } z_i^2 \in D_i^2, i = 1,2,\ldots,l \tag{7.8.6}$$

$$\lambda > 0, \sum_{i=1}^{l} \lambda_i = 1 \tag{7.8.7}$$

Dual Problem (SMD)

$$\begin{aligned}&\text{maximize } G\left(u^1,u^2,v^1,v^2,w^1,w^2,q^1,q^2,\lambda\right)\\ &\quad = \left(G_1\left(u^1,u^2,v^1,v^2,w^1,w^2,q^1,q^2,\lambda\right),\ldots,G_l\left(u^1,u^2,v^1,v^2,w^1,w^2,q^1,q^2,\lambda\right)\right)\end{aligned}$$

subject to

$$\begin{aligned}\left(u^1,u^2,v^1,v^2,w^1,w^2,q^1,q^2,\lambda\right) \in R^{|J_1|} \times R^{|J_2|} \times R^{|K_1|} \times R^{|K_2|} \times R^{|J_1|} \times R^{|J_2|}\\ \times R^{|J_1|} \times R^{|J_2|} \times R_+^l\end{aligned}$$

$$\sum_{i=1}^{l} \lambda_i \left[\nabla_{x^1} f_i\left(u^1,v^1\right) + w_i^1 + \nabla_{x^1x^1} f_i\left(u^1,v^1\right) q_i^1\right] \geq 0 \tag{7.8.8}$$

$$\sum_{i=1}^{l} \lambda_i \left[\nabla_{x^2} g_i\left(u^2,v^2\right) + w_i^2 + \nabla_{x^2x^2} g_i\left(u^2,v^2\right) q_i^2\right] \geq 0 \tag{7.8.9}$$

$$(u^1)^T \sum_{i=1}^{l} \lambda_i \left[\nabla_{x^1} f_i\left(u^1,v^1\right) + w_i^1 + \nabla_{x^1x^1} f_i\left(u^1,v^1\right) q_i^1\right] \leq 0 \tag{7.8.10}$$

$$(u^2)^T \sum_{i=1}^{l} \lambda_i \left[\nabla_{x^2} g_i(u^2,v^2) + w_i^2 + \nabla_{x^2x^2} g_i(u^2,v^2) q_i^2\right] \leq 0 \tag{7.8.11}$$

$$\left(v^1,v^2\right) \geq 0, \tag{7.8.12}$$

$$w_i^1 \in C_i^1, \text{ and } w_i^2 \in C_i^2, i = 1,2,\ldots,l \tag{7.8.13}$$

$$\lambda > 0, \sum_{i=1}^{l} \lambda_i = 1 \tag{7.8.14}$$

where

$$H_i^*\left(x^1,x^2,y^1,y^2,z^1,z^2,p^1,p^2,\lambda\right)$$
$$= f_i(x^1,y^1)+g_i\left(x^2,y^2\right)+s\left(x^1\left|C_i^1\right.\right)+s\left(x^2\left|C_i^2\right.\right)-(y^1)^T z_i^1$$
$$-(y^2)^T z_i^2-\frac{1}{2}\left(p_i^1\right)^T\nabla_{y^1y^1}f_i(x^1,y^1)p_i^1-\frac{1}{2}\left(q_i^2\right)^T\nabla_{y^2y^2}g_i\left(x^2,y^2\right)q_i^2$$

$$G_i^*\left(u^1,u^2,v^1,v^2,w^1,w^2,q^1,q^2,\lambda\right)$$
$$= f_i(u^1,v^1)+g_i(u^2,v^2)-s\left(v^1\left|D_i^1\right.\right)-s\left(v^2\left|D_i^2\right.\right)-\left(u^1\right)^T w_i^1$$
$$+(u^2)^T w_i^2-\frac{1}{2}\left(q_i^1\right)^T\nabla_{x^1x^1}f_i(u^1,v^1)q_i^1-\frac{1}{2}\left(q_i^2\right)^T\nabla_{x^2x^2}g_i\left(u^2,v^2\right)q_i^2$$

and C_i^1 is a compact and convex subsets of $R^{|J_1|}$ for $i=1,2,\ldots,l$ and C_i^2 is a compact and convex subsets of $R^{|J_2|}$ for $i=1,2,\ldots,l$, similarly, D_i^1 is a compact and convex subsets of $R^{|K_1|}$ for $i=1,2,\ldots,l$ and D_i^2 is a compact and convex subsets of $R^{|K_2|}$ for $i=1,2,\ldots,l$.

For the first model, we can prove the following weak duality theorem.

Theorem 7.8.1. *(Weak Duality). Let* $\left(x^1,x^2,y^1,y^2,z^1,z^2,p^1,p^2,\lambda\right)$ *be feasible for (SMP) and* $\left(u^1,u^2,v^1,v^2,w^1,w^2,q^1,q^2,\lambda\right)$ *be feasible for (SMD). Suppose there exist sublinear functionals* F_1*,* F_2*,* G_1 *and* G_2 *satisfying the following conditions are satisfied:*

(i) $F_1(x^1,u^1;a)+(u^1)^T a\geq 0$; if $a\geq 0$;
(ii) $G_1\left(x^2,u^2;b\right)+(u^2)^T b\geq 0$; if $b\geq 0$;
(iii) $F_2\left(y^1,v^1;c\right)+\left(y^1\right)^T c\leq 0$; if $c\leq 0$;
(iv) $G_2\left(y^2,v^2;d\right)+(y^2)^T d\leq 0$; if $d\leq 0$.

Furthermore, assume that for $i=1,2,\ldots,l$*, assume that* $f_i\left(\cdot,v^1\right)+\cdot^T w_i^1$ *is second order* F_1 *– convex for fixed* v^1*, with respect to* $q_i^1\in R^{|J_1|}$*,* $f_i(x^1,\cdot)-\cdot^T z_i^1$ *is second order* F_2 *– concave for fixed* x^1*, with respect to* $p_i^1\in R^{|K_1|}$*,* $g_i\left(\cdot,v^2\right)+\cdot^T w_i^2$ *is second order* G_1 *– convex for fixed* v^2 *with respect to* $q_i^2\in R^{|J_2|}$ *and* $g_i\left(x^2,\cdot\right)-\cdot^T z_i^2$ *is second order* G_2 *– concave for fixed* x^2*, with respect to* $p_i^2\in R^{|K_2|}$*. Then* $H_i^*\left(x^1,x^2,y^1,y^2,z^1,z^2,p^1,p^2,\lambda\right)\nleq G_i^*\left(u^1,u^2,v^1,v^2,w^1,w^2,q^1,q^2,\lambda\right)$.

Proof. Assume that the result is not true, that is,

$$H_i^*\left(x^1,x^2,y^1,y^2,z^1,z^2,p^1,p^2,\lambda\right)\leq G_i^*\left(u^1,u^2,v^1,v^2,w^1,w^2,q^1,q^2,\lambda\right).$$

Then, since $\lambda > 0$, we have

$$\sum_{i=1}^{l} \lambda_i \left[f_i\left(x^1, y^1\right) + g_i\left(x^2, y^2\right) + s\left(x^1 | C_i^1\right) + s(x^2|C_i^2) - (y^1)^T z_i^1 - (y^2)^T z_i^2 \right.$$
$$\left. -\frac{1}{2}\left(p_i^1\right)^T \nabla_{y^1 y^1} f_i(x^1, y^1) p_i^1 - \frac{1}{2}\left(p_i^2\right)^T \nabla_{y^2 y^2} g_i(x^2, y^2) p_i^2 \right]$$
$$< \sum_{i=1}^{l} \lambda_i \left[f_i(u^1, v^1) + g_i(u^2, v^2) - s\left(v^1 \left| D_i^1\right.\right) - s\left(v^2 \left| D_i^2\right.\right) + (u^1)^T w_i^1 \right.$$
$$\left. + (u^2)^T w_i^2 - \frac{1}{2}\left(q_i^1\right)^T \nabla_{x^1 x^1} f_i\left(u^1, v^1\right) q_i^1 - \frac{1}{2}\left(q_i^2\right)^T \nabla_{x^2 x^2} g_i\left(u^2, v^2\right) q_i^2 \right] \tag{7.8.15}$$

By the second order F_1 – convexity of $f_i\left(\cdot, v^1\right) + \cdot^T w_i^1$ at x^1 with respect to $q_i^1 \in R^{|J_1|}$, we have

$$f_i\left(x^1, v^1\right) + (x^1)^T w_i^1 - f_i\left(u^1, v^1\right) - (u^1)^T w_i^1 + \frac{1}{2}(q_i^1)^T \nabla_{u^1 u^1} f_i\left(u^1, v^1\right) q_i^1$$
$$> F_1\left(x^1, u^1; \nabla_{u^1} f_i\left(u^1, v^1\right) + w_i^1 + \nabla_{u^1 u^1} f_i\left(u^1, v^1\right) q_i^1\right).$$

for $i = 1, 2, \ldots, l$

From $\lambda > 0$ and sublinearity of F_1, we have

$$\sum_{i=1}^{l} \lambda_i \left[f_i\left(x^1, v^1\right) + (x^1)^T w_i^1 - f_i\left(u^1, v^1\right) - (u^1)^T w_i^1 + \frac{1}{2}(q_i^1)^T \nabla_{u^1 u^1} f_i\left(u^1, v^1\right) q_i^1 \right]$$
$$\geq F_1\left(x^1, u^1; \sum_{i=1}^{l} \lambda_i \left[\nabla_{u^1} f_i\left(u^1, v^1\right) + w_i^1 + \nabla_{u^1 u^1} f_i\left(u^1, v^1\right) q_i^1\right]\right). \tag{7.8.16}$$

By the duality constraint (7.8.8), it follows that

$$a = \sum_{i=1}^{l} \lambda_i \left[\nabla_{x^1} f_i\left(u^1, v^1\right) + w_i^1 + \nabla_{x^1 x^1} f_i\left(u^1, v^1\right) q_i^1\right] \in \mathrm{R}_+^{|J_1|}.$$

Thus, by condition (i) given in the Theorem, we have

$$F_1(x^1, u^1; a) + (u^1)^T a \geq 0,$$

that is,

$$F_1\left(x^1, u^1; \sum_{i=1}^{l} \lambda_i \left[\nabla_{u^1} f_i\left(u^1, v^1\right) + w_i^1 + \nabla_{u^1 u^1} f_i\left(u^1, v^1\right) q_i^1\right]\right)$$
$$\geq -(u^1)^T \sum_{i=1}^{l} \lambda_i \left[\nabla_{u^1} f_i\left(u^1, v^1\right) + w_i^1 + \nabla_{u^1 u^1} f_i\left(u^1, v^1\right) q_i^1\right]. \tag{7.8.17}$$

From (7.8.10), (7.8.16) and (7.8.17), we get

$$\sum_{i=1}^{l} \lambda_i \Big[f_i(x^1,v^1) + (x^1)^T w_i^1 - f_i(u^1,v^1) - (u^1)^T w_i^1 + \frac{1}{2}(q_i^1)^T \nabla_{u^1u^1} f_i(u^1,v^1) q_i^1 \Big] \geq 0. \tag{7.8.18}$$

By second order F_2 – concavity of $f_i(x^1,\cdot) - \cdot^T z_i^1$ for fixed x^1, with respect to $p_i^1 \in R^{|K_1|}$, we have

$$\begin{aligned} & - f_i(x^1,v^1) + (v^1)^T z_i^1 + f_i(x^1,y^1) - (y^1)^T z_i^1 - \frac{1}{2}\left(p_i^1\right)^T \nabla_{y^1y^1} f_i(x^1,y^1) p_i^1 \\ & \quad \geq F_2 \left(v^1, y^1; -\nabla_{y^1} f_i\left(x^1,y^1\right) + z_i^1 - \nabla_{y^1y^1} f_i(x^1,y^1) p_i^1 \right). \end{aligned}$$

From $\lambda > 0$ and the sublinearity of F_2, we have

$$\begin{aligned} & \sum_{i=1}^{l} \lambda_i \left[f_i\left(x^1,y^1\right) - (y^1)^T z_i^1 + (v^1)^T z_i^1 - f_i(x^1,v^1) - \frac{1}{2}\left(p_i^1\right)^T \nabla_{y^1y^1} f_i(x^1,y^1) p_i^1 \right] \\ & \quad \geq F_2 \left(v^1, y^1; \sum_{i=1}^{l} \lambda_i \left[-\nabla_{y^1} f_i(x^1,y^1) + z_i^1 - \nabla_{y^1y^1} f_i(x^1,y^1) p_i^1 \right] \right). \end{aligned} \tag{7.8.19}$$

By the primal constraint (7.8.1), it follows that

$$c = -\sum_{i=1}^{l} \lambda_i \left[\nabla_{y^1} f_i(x^1,y^1) - z_i^1 + \nabla_{y^1y^1} f_i(x^1,y^1) p_i^1 \right] \in R_+^{|K_1|}.$$

Thus, by condition (iii) given in the Theorem, we have

$$F_2\left(y^1, v^1; c\right) + (y^1)^T c \leq 0,$$

that is,

$$\begin{aligned} & F_2 \left(v^1, y^1; -\sum_{i=1}^{l} \lambda_i \left[\nabla_{y^1} f_i(x^1,y^1) - z_i^1 + \nabla_{y^1y^1} f_i(x^1,y^1) p_i^1 \right] \right) \\ & \quad \geq (y^1)^T \sum_{i=1}^{l} \lambda_i \left[\nabla_{y^1} f_i(x^1,y^1) - z_i^1 + \nabla_{y^1y^1} f_i(x^1,y^1) p_i^1 \right]. \end{aligned} \tag{7.8.20}$$

From (7.8.3), (7.8.19) and (7.8.20), we get

$$\sum_{i=1}^{l} \lambda_i \left[f_i\left(x^1,y^1\right) - (y^1)^T z_i^1 + (v^1)^T z_i^1 - f_i(x^1,v^1) - \frac{1}{2}\left(p_i^1\right)^T \nabla_{y^1y^1} f_i(x^1,y^1) p_i^1 \right] \geq 0. \tag{7.8.21}$$

Using $(x^1)^T w_i^1 \leq s(x^1|C_i^1)$ and $(v^1)^T z_i^1 \leq s(v^1|D_i^1)$, it follows from (7.8.18) and (7.8.21), that

$$\sum_{i=1}^{l}\lambda_i\left[f_i(x^1,y^1)+s\left(x^1\left|C_i^1\right.\right)-\left(y^1\right)^T z_i^1-\frac{1}{2}\left(p_i^1\right)^T\nabla_{y^1y^1}f_i(x^1,y^1)p_i^1\right]$$
$$\geq\sum_{i=1}^{l}\lambda_i\left[f_i(u^1,v^1)-s\left(v^1\left|D_i^1\right.\right)+(u^1)^T w_i^1-\frac{1}{2}\left(q_i^1\right)^T\nabla_{u^1u^1}f_i(u^1,v^1)q_i^1\right]. \tag{7.8.22}$$

Similarly, using remaining hypotheses and conditions given in the theorem and using constraints of the primal and dual problems for functions $g_i\left(\cdot,v^2\right)+\cdot^T w_i^2$ and $g_i\left(x^2,\cdot\right)-\cdot^T z_i^2$, we get

$$\sum_{i=1}^{l}\lambda_i\left[g_i(x^2,y^2)+s(x^2|C_i^2)-\left(y^2\right)^T z_i^2-\frac{1}{2}\left(p_i^2\right)^T\nabla_{y^2y^2}g_i(x^2,y^2)p_i^2\right]$$
$$\geq\sum_{i=1}^{l}\lambda_i\left[g_i\left(u^2,v^2\right)-s\left(v^2\left|D_i^2\right.\right)+\left(u^2\right)^T w_i^2-\frac{1}{2}\left(q_i^2\right)^T\nabla_{u^2u^2}f_i(u^2,v^2)q_i^2\right]. \tag{7.8.23}$$

From (7.8.22) and (7.8.23), we get

$$\sum_{i=1}^{l}\lambda_i\left[f_i(x^1,y^1)+g_i\left(x^2,y^2\right)+s\left(x^1\left|C_i^1\right.\right)+s(x^2|C_i^2)-\left(y^1\right)^T z_i^1-(y^2)^T z_i^2\right.$$
$$\left.-\frac{1}{2}\left(p_i^1\right)^T\nabla_{y^1y^1}f_i(x^1,y^1)p_i^1-\frac{1}{2}\left(p_i^2\right)^T\nabla_{y^2y^2}g_i(x^2,y^2)p_i^2\right]$$
$$\geq\sum_{i=1}^{l}\lambda_i\left[f_i\left(u^1,v^1\right)+g_i(u^2,v^2)-s\left(v^1\left|D_i^1\right.\right)-s\left(v^2\left|D_i^2\right.\right)+(u^1)^T w_i^1\right.$$
$$\left.+(u^2)^T w_i^2-\frac{1}{2}\left(q_i^1\right)^T\nabla_{u^1u^1}f_i(u^1,v^1)q_i^1-\frac{1}{2}\left(q_i^2\right)^T\nabla_{u^2u^2}g_i\left(u^2,v^2\right)q_i^2\right],$$

which is a contradiction to (7.8.15).

Hence $H_i^*\left(x^1,x^2,y^1,y^2,z^1,z^2,p^1,p^2,\lambda\right)\nleq G_i^*\left(u^1,u^2,v^1,v^2,w^1,w^2,q^1,q^2,\lambda\right)$. □

Theorem 7.8.2. *(Weak Duality). Let* $\left(x^1,x^2,y^1,y^2,z^1,z^2,p^1,p^2,\lambda\right)$ *be feasible for (SMP) and* $\left(u^1,u^2,v^1,v^2,w^1,w^2,q^1,q^2,\lambda\right)$ *be feasible for (SMD). Suppose there exist sublinear functionals* F_1, F_2, G_1 *and* G_2 *satisfying the following conditions are satisfied:*

(i) $F_1(x^1,u^1;a)+(u^1)^T a\geq 0$; if $a\geq 0$;
(ii) $G_1\left(x^2,u^2;b\right)+(u^2)^T b\geq 0$; if $b\geq 0$;
(iii) $F_2\left(y^1,v^1;c\right)+\left(y^1\right)^T c\leq 0$; if $c\leq 0$;
(iv) $G_2\left(y^2,v^2;d\right)+\left(y^2\right)^T d\leq 0$; if $d\leq 0$.

Furthermore, assume that for $i=1,2,\ldots,l$, *assume that* $f_i\left(\cdot,v^1\right)+\cdot^T w_i^1$ *is second order* F_1 *– pseudo-convex for fixed* v^1, *with respect to* $q_i^1\in R^{|J_1|}$, $f_i(x^1,\cdot)-\cdot^T z_i^1$ *is second order* F_2 *– pseudo-concave for fixed* x^1, *with respect to* $p_i^1\in R^{|K_1|}$, $g_i(\cdot,v^2)+\cdot^T w_i^2$ *is second order* G_1 *– pseudo-convex for fixed* v^2 *with respect to* $q_i^2\in R^{|J_2|}$

and $g_i\left(x^2,\cdot\right)-\cdot^T z_i^2$ *is second order* G_2 *– pseudo-concave for fixed* x^2*, with respect to* $p_i^2 \in R^{|K_2|}$*. Then* $H_i^*\left(x^1,x^2,y^1,y^2,z^1,z^2,p^1,p^2,\lambda\right) \nleq G_i^*\left(u^1,u^2,v^1,v^2,w^1,w^2,q^1,q^2,\lambda\right)$.

Proof. Assume that the result is not true, that is,

$$H_i^*\left(x^1,x^2,y^1,y^2,z^1,z^2,p^1,p^2,\lambda\right) \leq G_i^*\left(u^1,u^2,v^1,v^2,w^1,w^2,q^1,q^2,\lambda\right).$$

Then, since $\lambda > 0$, we have

$$\begin{aligned}\sum_{i=1}^{l}\lambda_i\Big[&f_i\left(x^1,y^1\right)+g_i\left(x^2,y^2\right)+s\left(x^1|C_i^1\right)+s\left(x^2\left|C_i^2\right.\right)-\left(y^1\right)^T z_i^1-(y^2)^T z_i^2\\&-\frac{1}{2}\left(p_i^1\right)^T\nabla_{y^1y^1}f_i(x^1,y^1)p_i^1-\frac{1}{2}\left(p_i^2\right)^T\nabla_{y^2y^2}g_i(x^2,y^2)p_i^2\Big]\\&<\sum_{i=1}^{l}\lambda_i\Big[f_i\left(u^1,v^1\right)+g_i(u^2,v^2)-s\left(v^1\left|D_i^1\right.\right)-s\left(v^2\left|D_i^2\right.\right)+(u^1)^T w_i^1\\&\quad+(u^2)^T w_i^2-\frac{1}{2}(q_i^1)^T\nabla_{x^1x^1}f_i(u^1,v^1)q_i^1-\frac{1}{2}\left(q_i^2\right)^T\nabla_{x^2x^2}g_i\left(u^2,v^2\right)q_i^2\Big]\end{aligned} \tag{7.8.24}$$

By the duality constraint (7.8.8), it follows that

$$a=\sum_{i=1}^{l}\lambda_i\left[\nabla_{x^1}f_i\left(u^1,v^1\right)+w_i^1+\nabla_{x^1x^1}f_i\left(u^1,v^1\right)q_i^1\right]\in \mathrm{R}_+^{|\mathrm{J}_1|}.$$

Thus, by condition (i) given in the theorem, we have

$$F_1(x^1,u^1;a)+(u^1)^T a \geq 0,$$

that is,

$$\begin{aligned}&F_1\left(x^1,u^1;\sum_{i=1}^{l}\lambda_i\left[\nabla_{u^1}f_i(u^1,v^1)+w_i^1+\nabla_{u^1u^1}f_i\left(u^1,v^1\right)q_i^1\right]\right)\\&\quad\geq-(u^1)^T\sum_{i=1}^{l}\lambda_i\left[\nabla_{u^1}f_i(u^1,v^1)+w_i^1+\nabla_{u^1u^1}f_i(u^1,v^1)q_i^1\right].\end{aligned} \tag{7.8.25}$$

From (7.8.10) and (7.8.25), we get

$$F_1\left(x^1,u^1;\sum_{i=1}^{l}\lambda_i\left[\nabla_{u^1}f_i(u^1,v^1)+w_i^1+\nabla_{u^1u^1}f_i(u^1,v^1)q_i^1\right]\right)\geq 0. \tag{7.8.26}$$

From (7.8.26) and second order F_1 – pseudo-convexity of $f_i\left(\cdot,v^1\right)+\cdot^T w_i^1$ for fixed v^1, with respect to $q_i^1 \in R^{|J_1|}$, we get

$$\sum_{i=1}^{l}\lambda_i\Big[f_i(x^1,v^1)+(x^1)^T w_i^1-f_i(u^1,v^1)-(u^1)^T w_i^1+\frac{1}{2}(q_i^1)^T\nabla_{u^1u^1}f_i(u^1,v^1)q_i^1\Big]\geq 0. \tag{7.8.27}$$

By primal constraint (7.8.1), it follows that

$$c=-\sum_{i=1}^{l}\lambda_i\left[\nabla_{y^1}f_i(x^1,y^1)-z_i^1+\nabla_{y^1y^1}f_i(x^1,y^1)p_i^1\right]\in R_+^{|K_1|}.$$

Thus, by condition (iii) given in the Theorem, we have

$$F_2\left(y^1,v^1;c\right)+\left(y^1\right)^T c\leq 0,$$

that is,

$$\begin{aligned}F_2&\left(v^1,y^1;-\sum_{i=1}^{l}\lambda_i\left[\nabla y^1 f_i(x^1,y^1)-z_i^1+\nabla_{y^1y^1}f_i(x^1,y^1)p_i^1\right]\right)\\&\geq (y^1)^T\sum_{i=1}^{l}\lambda_i\left[\nabla_{y^1}f_i(x^1,y^1)-z_i^1+\nabla_{y^1y^1}f_i(x^1,y^1)p_i^1\right].\end{aligned} \tag{7.8.28}$$

From (7.8.3) and (7.8.28), we get

$$F_2\left(v^1,y^1;-\sum_{i=1}^{l}\lambda_i\left[\nabla_{y^1}f_i(x^1,y^1)-z_i^1+\nabla_{y^1y^1}f_i(x^1,y^1)p_i^1\right]\right)\geq 0. \tag{7.8.29}$$

From (7.8.29) and the second order F_2 – pseudo-concavity of $f_i(x^1,\cdot)-\cdot^T z_i^1$ for fixed x^1, with respect to $p_i^1\in R^{|K_1|}$, we have

$$\sum_{i=1}^{l}\lambda_i\left[f_i(x^1,y^1)-\left(y^1\right)^T z_i^1+(v^1)^T z_i^1-f_i(x^1,v^1)-\frac{1}{2}\left(p_i^1\right)^T\nabla_{y^1y^1}f_i(x^1,y^1)p_i^1\right]\geq 0. \tag{7.8.30}$$

Using $(x^1)^T w_i^1\leq s\left(x^1|C_i^1\right)$ and $(v^1)^T z_i^1\leq s\left(v^1\left|D_i^1\right.\right)$, it follows from (7.8.27) and (7.8.30), that

$$\begin{aligned}&\sum_{i=1}^{l}\lambda_i\left[f_i(x^1,y^1)+s\left(x^1\left|C_i^1\right.\right)-(y^1)^T z_i^1-\frac{1}{2}\left(p_i^1\right)^T\nabla_{y^1y^1}f_i(x^1,y^1)p_i^1\right]\\&\quad\geq\sum_{i=1}^{l}\lambda_i\left[f_i(u^1,v^1)-s\left(v^1\left|D_i^1\right.\right)+(u^1)^T w_i^1-\frac{1}{2}\left(q_i^1\right)^T\nabla_{u^1u^1}f_i(u^1,v^1)q_i^1\right].\end{aligned} \tag{7.8.31}$$

□

Similarly, using remaining hypotheses and conditions given in the Theorem and using constraints of the primal and dual problems for functions $g_i(\cdot,v^2)+\cdot^T w_i^2$ and $g_i\left(x^2,\cdot\right)-\cdot^T z_i^2$, we get

$$\sum_{i=1}^{l}\lambda_i\left[g_i(x^2,y^2)+s(x^2|C_i^2)-(y^2)^T z_i^2-\frac{1}{2}\left(p_i^2\right)^T\nabla_{y^2y^2}g_i(x^2,y^2)p_i^2\right]$$
$$\geq\sum_{i=1}^{l}\lambda_i\left[g_i(u^2,v^2)-s\left(v^2\left|D_i^2\right.\right)+(u^2)^T w_i^2-\frac{1}{2}\left(q_i^2\right)^T\nabla_{u^2u^2}f_i(u^2,v^2)q_i^2\right]. \tag{7.8.32}$$

From (7.8.31) and (7.8.32), we get

$$\sum_{i=1}^{l}\lambda_i\left[f_i(x^1,y^1)+g_i(x^2,y^2)+s(x^1|C_i^1)+s(x^2|C_i^2)-\left(y^1\right)^T z_i^1-(y^2)^T z_i^2\right.$$
$$\left.-\frac{1}{2}\left(p_i^1\right)^T\nabla_{y^1y^1}f_i(x^1,y^1)p_i^1-\frac{1}{2}\left(p_i^2\right)^T\nabla_{y^2y^2}g_i(x^2,y^2)p_i^2\right]$$
$$\geq\sum_{i=1}^{l}\lambda_i\left[f_i(u^1,v^1)+g_i(u^2,v^2)-s\left(v^1\left|D_i^1\right.\right)-s\left(v^2\left|D_i^2\right.\right)+(u^1)^T w_i^1\right.$$
$$\left.+(u^2)^T w_i^2-\frac{1}{2}\left(q_i^1\right)^T\nabla_{u^1u^1}f_i(u^1,v^1)q_i^1-\frac{1}{2}\left(q_i^2\right)^T\nabla_{u^2u^2}g_i(u^2,v^2)q_i^2\right],$$

which is a contradiction to (7.8.24).

Hence $H_i^*\left(x^1,x^2,y^1,y^2,z^1,z^2,p^1,p^2,\lambda\right)\nleq G_i^*\left(u^1,u^2,v^1,v^2,w^1,w^2,q^1,q^2,\lambda\right)$. □

Theorem 7.8.3. *(Strong Duality). Let* $\left(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{z^1},\overline{z^2},\overline{p^1},\overline{p^2},\overline{\lambda}\right)$ *be a properly efficient solution for (SMP). Let* $\overline{\lambda}=\lambda$ *be fixed in (SMD). Suppose that the Hessian matrix* $\nabla^2_{y^1}f_i\left(\overline{x^1},\overline{y^1}\right)$ *is positive definite for* $i=1,2,\ldots,l$ *and* $\sum_{i=1}^{l}\lambda_i(p_i^1)^T\left[\nabla_{y^1}f_i-\bar{z}_i^1\right]\geq 0$*; and* $\nabla^2_{y^2}g_i\left(\overline{x^2},\overline{y^2}\right)$ *is positive definite for* $i=1,2,\ldots,l$ *and* $\sum_{i=1}^{l}\lambda_i(p_i^2)^T\left[\nabla_{y^2}g_i-\bar{z}_i^2\right]\geq 0$*; or* $\nabla^2_{y^1}f_i\left(\overline{x^1},\overline{y^1}\right)$ *is negative definite for* $i=1,2,\ldots,l$ *and* $\sum_{i=1}^{l}\lambda_i(p_i^1)^T\left[\nabla_{y^1}f_i-\bar{z}_i^1\right]\leq 0$*; and* $\nabla^2_{y^2}g_i\left(\overline{x^2},\overline{y^2}\right)$ *is negative definite for* $i=1,2,\ldots,l$ *and* $\sum_{i=1}^{l}\lambda_i(p_i^2)^T\left[\nabla_{y^2}g_i-\bar{z}_i^2\right]\leq 0$*. Also suppose that the sets* $\{\nabla_{y^1}f_1-\bar{z}_1^2+\nabla_{y^1y^1}f_1\bar{p}_1^1,\ldots,\nabla_{y^1}f_l-\bar{z}_l^2+\nabla_{y^1y^1}f_l\bar{p}_l^1\}$ *and* $\left\{\nabla_{y^2}g_1-\bar{z}_1^2+\nabla_{y^2y^2}g_1\bar{p}_1^2,\ldots,\nabla_{y^2}g_l-\bar{z}_l^2+\nabla_{y^2y^2}g_l\bar{p}_l^2\right\}$ *are linearly independent. Then there exists* $w_i^1\in C_i^1$ *and* $w_i^2\in C_i^2$ *such that* $\left(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{w^1},\overline{w^2},\overline{q^1}=\overline{q^2=0},\overline{\lambda}\right)$ *is feasible for (SMD) and* $H_i^*(x^1,x^2,y^1,y^2,z^1,z^2,p^1,p^2,\lambda)=G_i^*\left(u^1,u^2,v^1,v^2,w^1,w^2,q^1,q^2,\lambda\right)$*. Moreover, if the generalized convexity hypotheses and conditions (i)–(iv) of Theorem 7.8.1 or 7.8.2 are satisfied, then* $\left(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{w^1},\overline{w^2},\overline{q^1},\overline{q^2},\overline{\lambda}\right)$ *is a properly efficient solution for (SMD).*

Proof. The proof follows the lines of the proof of Theorem 2 in Yang et al. (2005) in light of the discussions above in this section. □

Remark 7.8.1. . The converse duality theorem can also be established for the problems considered in this section.

Remark 7.8.2. We can obtain some special cases of the problem (SMP) and (SMD) by choosing particular forms of the sublinear functionals and the compact convex sets involved in the problems, as follows:

(xiii) If $|J_2| = |K_2| = 0$, then (SMP) and (SMD) reduce to the problems (P) and (D) studied by Yang et al. (2005).
(xiv) If $|J_2| = |K_2| = 0$, and $C_i^1 = C_i^2 = D_i^1 = D_i^2 = \{0\}, i = 1,2,\ldots,l$, then (SMP) and (SMD) reduce to the problems (P) and (D) studied by Suneja et al. (2003).
(xv) If $C_i^1 = C_i^2 = D_i^1 = D_i^2 = \{0\}, i = 1,2,\ldots,l$, and $l = 1$, then (SMP) and (SMD) reduce to the problems (MP) and (MD) studied by Mishra (2000b).
(xvi) If $|J_2| = |K_2| = 0$, and $l = 1$, then (SMP) and (SMD) reduce to the pair of problems studied by Hou and Yang (2001).
(xvii) If $|J_2| = |K_2| = 0, \quad l = 1$, and $p^1 = p^2 = q^1 = q^2 = 0$ then (SMP) and (SMD) reduce to the problems (P1) and (D1) studied by Mond and Schechter (1996).
(xviii) If $l = 1$, $p^1 = p^2 = q^1 = q^2 = 0$ and $\nabla_2 f(x^1, y^1) = z^1$, then (SMP) and (SMD) reduce to the problems (MP) and (MD) studied by Yang et al. (2003a).
(xix) If $p_i^1 = p_i^2 = q_i^1 = q_i^2 = 0$ and $C_i^1 = C_i^2 = D_i^1 = D_i^2 = \{0\}, i = 1,2,\ldots,l$, and then (SMP) and (SMD) reduce to a general form of the problems (P1) and (D1) studied by Bector et al. (1999).

The results obtained in the present section can be extended to the class of functions introduced by Antczak (2001) and Aghezaaf and Hachimi (2001). Moreover, these results can be further extended to higher order case as an extension of the results of Chen (2004), to the case of complex functions and to the case of continuous-time problems as well.

7.9 Symmetric Duality for a Class of Nondifferentiable Vector Fractional Variational problems

It is well known due to the works of Schaible (1976a, 1995) that duality results for convex optimization do not apply to fractional programs in general. Duality concepts for such problems had to be defined separately Weir (1991), Yang et al. (2002a, 2002b).

Recently, Kim et al. (2004) introduced a symmetric dual for vector fractional variational problems which is different from the one proposed by Chen (2004). Kim et al. (2004) established weak, strong, converse and self-duality theorems under invexity assumptions.

In this section, we focus on symmetric duality for a class of nondifferentiable fractional variational problems. We introduced a symmetric dual pair for a class of nondifferentiable vector fractional variational problems. We establish duality and self-duality theorems under certain invexity assumptions. The results obtained in this paper extend the very recent results established by Kim et al. (2004) to the nondifferentiable case and also extend an earlier work of Yang et al. (2002b) to the dynamic case. Moreover, these results also include, as special cases, the symmetric duality results of Mond and Schechter (1996), Weir and Mond (1988b) and others.

Let $I = [a,b]$ be a real interval, $f : I \times R^n \times R^n \times R^m \times R^m \to \times R^k$ and $g : I \times R^n \times R^n \times R^m \times R^m \to \times R^k$. Consider the vector-valued function $f(t,x,\dot{x},y,\dot{y})$, where $t \in I, x$ and y are functions of t with $x(t) \in R^n$ and $y(t) \in R^m$ and $\dot{x}$ and $\dot{y}$ denote the derivatives of x and y, respectively, with respect to t.

Assume that f has continuous fourth-order partial derivatives with respect to $x, \dot{x}, y$ and $\dot{y}$. Let f_x and $f_{\dot{x}}$ denote the $k \times n$ matrices of first-order partial derivatives with respect to x and $\dot{x}$; i.e.,

$$f_{ix} = \left(\frac{\partial f_i}{\partial x_1}, \ldots, \frac{\partial f_i}{\partial x_n},\right) \text{ and } f_{i\dot{x}} = \left(\frac{\partial f_i}{\partial \dot{x}_1}, \ldots, \frac{\partial f_i}{\partial \dot{x}_n},\right), i = 1,2,\ldots,k.$$

Similarly, f_y and $f_{\dot{y}}$ denote the $k \times m$ matrices of first-order partial derivatives with respect to $y$and$\dot{y}$.

We consider the following vector fractional variational problem:

$$\text{(FVP)} \quad \text{minmize } \frac{\int_a^b f(t,x(t),\dot{x}(t))\,dt}{\int_a^b g(t,x(t),\dot{x}(t))\,dt} = \left(\frac{\int_a^b f_1(t,x(t),\dot{x}(t))\,dt}{\int_a^b g_1(t,x(t),\dot{x}(t))\,dt}, \ldots, \frac{\int_a^b f_k(t,x(t),\dot{x}(t))\,dt}{\int_a^b g_k(t,x(t),\dot{x}(t))\,dt}\right)$$

$$\text{subject to } x(a) = \alpha, \qquad x(b) = \beta,$$
$$h(t,x(t),\dot{x}(t)) \leqq 0,$$

where $h : I \times R^n \times R^n \to R^l$.

Assume that $g_i(t,x,\dot{x}) > 0$ and $f_i(t,x,\dot{x}) \geqq 0$ for all $i = 1,2,\ldots,k$. Let X denote the set of all feasible solutions of (FVP).

Definition 7.9.1. *A point $x^* \in X$ is said to be an efficient (Pareto optimal) solution of (FVP) if for all $x \in X$,*

$$\frac{\int_a^b f(t,x,\dot{x})\,dt}{\int_a^b g(t,x,\dot{x})\,dt} \nleq \frac{\int_a^b f(t,x^*,\dot{x}^*)\,dt}{\int_a^b g(t,x^*,\dot{x}^*)\,dt}.$$

Definition 7.9.2. *A point $x^* \in X$ is said to be a properly efficient solution of (FVP) if is efficient for (FVP) and if there exists a scalar $M > 0$ such that, for all $i \in \{1,2,\ldots,k\}$,*

$$\frac{\int_a^b f_i(t,x^*,\dot{x}^*)\,dt}{\int_a^b g_i(t,x^*,\dot{x}^*)\,dt} - \frac{\int_a^b f_i(t,x,\dot{x})\,dt}{\int_a^b g_i(t,x,\dot{x})\,dt} \leqq M\left(\frac{\int_a^b f_j(t,x,\dot{x})\,dt}{\int_a^b g_j(t,x,\dot{x})\,dt} - \frac{\int_a^b f_j(t,x^*,\dot{x}^*)\,dt}{\int_a^b g_j(t,x^*,\dot{x}^*)\,dt}\right)$$

for some j, such that

$$\frac{\int_a^b f_j(t,x,\dot{x})\,dt}{\int_a^b g_j(t,x,\dot{x})\,dt} > \frac{\int_a^b f_j(t,x^*,\dot{x}^*)\,dt}{\int_a^b g_j(t,x^*,\dot{x}^*)\,dt}$$

whenever $x \in X$, *and*

$$\frac{\int_a^b f_i(t,x,\dot{x})\,dt}{\int_a^b g_i(t,x,\dot{x})\,dt} < \frac{\int_a^b f_i(t,x^*,\dot{x}^*)\,dt}{\int_a^b g_i(t,x^*,\dot{x}^*)\,dt}.$$

Definition 7.9.3. *A point* $x^* \in X$ *is said to be a weakly efficient solution of (FVP) if there exists no other feasible point* x *for which*

$$\frac{\int_a^b f(t,x^*,\dot{x}^*)\,dt}{\int_a^b g(t,x^*,\dot{x}^*)\,dt} > \frac{\int_a^b f(t,x,\dot{x})\,dt}{\int_a^b g(t,x,\dot{x})\,dt}.$$

Now we recall the invexity for continuous case as follows:

Definition 7.9.4. *The vector of functionals* $\int_a^b f = \left(\int_a^b f_1,\ldots,\int_a^b f_k\right)$ *is said to be invex in* x *and* $\dot{x}$ *if for each* $y : [a,b] \to R^m$, *with* $\dot{y}$ *piecewise smooth, there exists a function* $\eta : [a,b] \times R^n \times R^n \times R^n \times R^n \to R^n$ *such that* $\forall\, i = 1,2,\ldots,k$,

$$\int_a^b \{f_i(t,x,\dot{x},y,\dot{y}) - f_i(t,u,\dot{u},y,\dot{y})\}\,dt \geqq \int_a^b \eta(t,x,\dot{x},u,\dot{u})^T \left[f_i(t,u,\dot{u},y,\dot{y}) - \frac{d}{dt} f_{i\dot{x}}(t,u,\dot{u},y,\dot{y})\right] dt$$

for all $x : [a,b] \to R^n, u : [a,b] \to R^n$, *where* $(\dot{x}(t),\dot{u}(t))$ *is piecewise smooth on* $[a,b]$.

Definition 7.9.5. *The vector of functionals* $-\int_a^b f$ *is said to be invex in* y *and* $\dot{y}$ *if for each* $x : [a,b] \to R^n$, *with* $\dot{x}$ *piecewise smooth, there exists a function* $\xi : [a,b] \times R^m \times R^m \times R^m \times R^m \to R^m$ *such that* $\forall\, i = 1,2,\ldots,k$,

$$-\int_a^b \{f_i(t,x,\dot{x},v,\dot{v}) - f_i(t,x,\dot{x},y,\dot{y})\}\,dt \geqq -\int_a^b \xi(t,v,\dot{v},y,\dot{y})^T \left[f_{iy}(t,x,\dot{x},y,\dot{y}) - \frac{d}{dt} f_{i\dot{y}}(t,x,\dot{x},y,\dot{y})\right] dt$$

for all $v : [a,b] \to R^m, y : [a,b] \to R^m$, *where* $(\dot{v}(t),\dot{y}(t))$ *is piecewise smooth on* $[a,b]$.

In the sequel, we will write $\eta(x,u)$ for $\eta(t,x,\dot{x},u,\dot{u})$ and $\xi(v,y)$ for $\xi(t,v,\dot{v},y,\dot{y})$.

We consider the problem of finding functions $x : [a,b] \to R^n$, and $y : [a,b] \to R^m$, where $(\dot{x}(t),\dot{y}(t))$ is piecewise smooth on $[a,b]$, to solve the following pair o symmetric dual vector nondifferentiable fractional variational problems introduced as follows:

$$\text{(MNFP)}\quad \text{minimize } \frac{\int_a^b \left\{ f\left(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\right) + s(x(t)C) - y(t)^T z \right\} dt}{\int_a^b \left\{ g\left(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\right) - s(x(t)|E) + y(t)^T r \right\} dt}$$

$$= \left(\frac{\int_a^b \left\{ f_1\left(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\right) + s\left(x(t)\,|C_1\right) - y(t)^T z_1 \right\} dt}{\int_a^b \left\{ g_1\left(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\right) - s\left(x(t)|E_1\right) + y(t)^T r_1 \right\} dt}, \ldots \right.$$

$$\left. \frac{\int_a^b \left\{ f_k\left(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\right) + s\left(x(t)\,|C_k\right) - y(t)^T z_k \right\} dt}{\int_a^b \left\{ g_k\left(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\right) - s\left(x(t)\,|E_k\right) + y(t)^T r_k \right\} dt} \right)$$

subject to $x(a) = 0 = x(b), \quad y(a) = 0 = y(b),$

$$\dot{x}(a) = 0 = \dot{x}(b), \quad \dot{y}(a) = 0 = \dot{y}(b),$$

$$\sum_{i=1}^{k} \tau_i \left\{ \left[f_{iy} - Df_{i\dot{y}} - z_i\right] G_i(x,y) - \left[g_{iy} - Dg_{i\dot{y}} + r_i\right] F_i(x,y) \right\} \leqq 0,$$

$$\int_a^b y(t)^T \sum_{i=1}^{k} \tau_i \left\{ \left[f_{iy} - Df_{i\dot{y}} - z_i\right] G_i(x,y) - \left[g_{iy} - Dg_{i\dot{y}} + r_i\right] F_i(x,y) \right\} dt \geqq 0,$$

$$\tau > 0, \quad \tau^T e = 1, \quad t \in I$$

$$z_i \in D_i, \quad r_\mathrm{i} \in F_i, \qquad i = 1,2,\ldots,k.$$

$$\text{(MNFD)}\quad \text{maximize } \frac{\int_a^b \left\{ f\left(t,u(t),\dot{u}(t),v(t),\dot{v}(t)\right) - s\left(v(t)\,|D\right) + u(t)^T w \right\} dt}{\int_a^b \left\{ g\left(t,u(t),\dot{u}(t),v(t),\dot{v}(t)\right) + s\left(v(t)\,|H\right) - u(t)^T s \right\} dt}$$

$$= \left(\frac{\int_a^b \left\{ f_1\left(t,u(t),\dot{u}(t),v(t),\dot{v}(t)\right) - s\left(v(t)\,|D_1\right) + u(t)^T w_1 \right\} dt}{\int_a^b \left\{ g_1\left(t,u(t),\dot{u}(t),v(t),\dot{v}(t)\right) + s\left(v(t)\,|H_1\right) - u(t)^T s_1 \right\} dt}, \ldots \right.$$

$$\left. \frac{\int_a^b \left\{ f_k\left(t,u(t),\dot{u}(t),v(t),\dot{v}(t)\right) - s\left(v(t)\,|D_k\right) + u(t)^T w_k \right\} dt}{\int_a^b \left\{ g_k\left(t,u(t),\dot{u}(t),v(t),\dot{v}(t)\right) + s\left(v(t)\,|H_k\right) - u(t)^T s_k \right\} dt} \right)$$

subject to $u(a) = 0 = u(b), \quad v(a) = 0 = v(b),$

$$\dot{u}(a) = 0 = \dot{u}(b), \quad \dot{v}(a) = 0 = \dot{v}(b),$$

$$\sum_{i=1}^{k} \tau_i \left\{ \left[f_{iu} - Df_{i\dot{u}} + w_i\right] G_i(u,v) - \left[g_{iu} - Dg_{i\dot{u}} - s_i\right] F_i(u,v) \right\} \geqq 0,$$

$$\int_a^b u(t)^T \sum_{i=1}^{k} \tau_i \{ [f_{iu} - Df_{i\dot{u}} + w_i] G_i(u,v) - [g_{iu} - Dg_{i\dot{u}} - s_i] F_i(u,v) \} \leqq 0,$$

$$\tau > 0, \quad \tau^T e = 1, \quad t \in I,$$

$$w_i \in C_i, \quad s_\mathrm{i} \in E_i, \quad i = 1,2,\ldots,k,.$$

where $f_i : I \times R^n \times R^n \times R^m \times R^m \to R_+$ and $g_i : I \times R^n \times R^n \times R^m \times R^m \to R_+ \backslash \{0\}$ are continuously differentiable functions and

$$F_i(x,y) = \int_a^b \left\{ f_i\left(t,x,\dot{x},y,\dot{y}\right) + s\left(x(t)\,|C_i\right) - y(t)^T z_i \right\} dt;$$

$$G_i(x,y) = \int_a^b \left\{ g_i\left(t,x,\dot{x},y,\dot{y}\right) - s\left(x(t)\,|E_i\right) + y(t)^T r_i \right\} dt;$$

$$F_i^*(u,v) = \int_a^b \left\{ f_i\left(t,u,\dot{u},v,\dot{v}\right) - s(v(t)|D_i) + u(t)^T w_i \right\} dt;$$

and

$$G_i^*(u,v) = \int_a^b \left\{ g_i\left(t,u,\dot{u},v,\dot{v}\right) + s(v(t|H_i) - u(t)^T s_i \right\} dt.$$

In the above problems (MNFP) and (MNFD), the numerators are nonnegative and denominators are positive; the differential operator D is given by

$$y = Dx \Leftrightarrow x(t) = \alpha + \int_a^t y(s)ds,$$

and $x(a) = \alpha,\ x(b) = \beta$ are given boundary values; thus $D = d/dt$ except at discontinuities. Let

$$f_x = f_x\left(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\right),$$
$$f_{\dot{x}} = f_{\dot{x}}\left(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\right)\ etc.$$

All the above statements for F_i, G_i, F_i^* and G_i^* will be assumed to hold for subsequent results. It is to be noted that

$$Df_{i\dot{y}} = f_{i\dot{y}t} + f_{i\dot{y}y}\dot{y} + f_{i\dot{y}\dot{y}}\ddot{y} + f_{i\dot{y}x}\dot{x} + f_{i\dot{y}\dot{x}}\ddot{x}$$

and consequently

$$\frac{\partial}{\partial y}Df_{i\dot{y}} = Df_{i\dot{y}y}, \frac{\partial}{\partial \dot{y}}Df_{i\dot{y}} = Df_{i\dot{y}\dot{y}} + f_{i\dot{y}y}, \frac{\partial}{\partial \ddot{y}}Df_{i\dot{y}} = f_{i\dot{y}\dot{y}},$$

$$\frac{\partial}{\partial x}Df_{i\dot{y}} = Df_{i\dot{y}x}, \frac{\partial}{\partial \dot{x}}Df_{i\dot{y}} = Df_{i\dot{y}\dot{x}} + f_{i\dot{y}x}, \frac{\partial}{\partial \ddot{x}}Df_{i\dot{y}} = f_{i\dot{y}\dot{x}}.$$

In order to simplify the notations we introduce

$$p_i = \frac{F_i(x,y)}{G_i(x,y)} = \frac{\int_a^b \left\{ f_i\left(t,x,\dot{x},y,\dot{y}\right) + s\left(x(t)\,|C_i\right) - y(t)^T z_i \right\} dt}{\int_a^b \left\{ g_i\left(t,x,\dot{x},y,\dot{y}\right) - s\left(x(t)\,|E_i\right) + y(t)^T r_i \right\} dt}$$

and

$$q_i = \frac{F_i^*(u,v)}{G_i^*(u,v)} = \frac{\int_a^b \left\{ f_i\left(t,u,\dot{u},v,\dot{v}\right) - s\left(v(t)\,|D_i\right) + u(t)^T w_i \right\} dt;}{\int_a^b \left\{ g_i\left(t,u,\dot{u},v,\dot{v}\right) + s\left(v(t)\,|H_i\right) - u(t)^T s_i \right\} dt.}$$

and express problems (MNFP) and (MNFD) equivalent as follows:

(EMSP) minimize $p = (p_1, \ldots, p_k)^T$

subject to

$$x(a) = 0 = x(b), \quad y(a) = 0 = y(b), \tag{7.9.1}$$

$$\dot{x}(a) = 0 = \dot{x}(b), \quad \dot{y}(a) = 0 = \dot{y}(b), \tag{7.9.2}$$

$$\int_a^b \left\{ f_i(t, x, \dot{x}, y, \dot{y}) + s(x|C_i) - y^T z_i \right\} dt - p_i \int_a^b \left\{ g_i(t, x, \dot{x}, y, \dot{y}) - s(x|E_i) + y^T r_i \right\} dt = 0; \tag{7.9.3}$$

$$\sum_{i=1}^k \tau_i \left\{ [f_{iy} - Df_{i\dot{y}} - z_i] - p_i [g_{iy} - Dg_{i\dot{y}} + r_i] \right\} \leqq 0, \ t \in I, \tag{7.9.4}$$

$$\int_a^b y(t)^T \sum_{i=1}^k \tau_i \left\{ [f_{iy} - Df_{i\dot{y}} - z_i] - p_i [g_{iy} - Dg_{i\dot{y}} + r_i] \right\} \geqq 0, \ t \in I, \tag{7.9.5}$$

$$\tau > 0, \tau^T e = 1, \ t \in I, \tag{7.9.6}$$

$$z_i \in D_i, r_{\mathrm{i}} \in H_i, \ i = 1, 2, \ldots, k. \tag{7.9.7}$$

(EMSD) maximize $q = (q_1, \ldots, q_k)^T$

subject to

$$u(a) = 0 = u(b), \quad v(a) = 0 = v(b), \tag{7.9.8}$$

$$\dot{u}(a) = 0 = \dot{u}(b), \quad \dot{v}(a) = 0 = \dot{v}(b), \tag{7.9.9}$$

$$\int_a^b \left\{ f_i(t, u, \dot{u}, v, \dot{v}) - s(v|D_i) + u^T w_i \right\} dt - q_i \int_a^b \left\{ g_i(t, u, \dot{u}, v, \dot{v}) + s(v|H_i) - u^T s_i \right\} dt = 0; \tag{7.9.10}$$

$$\sum_{i=1}^k \tau_i \left\{ [f_{iu} - Df_{i\dot{u}} + w_i] - q_i [g_{iu} - Dg_{i\dot{u}} - s_i] \right\} \geqq 0, \ t \in I \tag{7.9.11}$$

$$\int_a^b u(t)^T \sum_{i=1}^k \tau_i \left\{ [f_{iu} - Df_{i\dot{u}} + w_i] - q_i [g_{iu} - Dg_{i\dot{u}} - s_i] \right\} \leqq 0, \ t \in I \tag{7.9.12}$$

$$\tau > 0, \tau^T e = 1, \ t \in I, \tag{7.9.13}$$

$$w_i \in C_i, s_{\mathrm{i}} \in E_i, \ i = 1, 2, \ldots, k.. \tag{7.9.14}$$

In the above problems (EMSP) and (EMSD), it is to be noted that p and q are also nonnegative.

Now we state duality theorems for problems (EMSP) and (EMSD) which lead to corresponding relations between (MNFP) and (MNFD). We establish weak, strong and converse duality as well as self-duality relations between (EMSP) and (EMSD).

Theorem 7.9.1. *(Weak Duality). Let* $(x(t), y(t), p, \tau, z_1, z_2, \ldots, z_k, r_1, r_2, \ldots, r_k)$ *be feasible for (EMSP) and let* $(u(t), v(t), q, \tau, w_1, w_2, \ldots, w_k, s_1, s_2, \ldots, s_k)$ *be feasible for (EMSD). Assume that* $\int_a^b \left(f_i + \cdot^T w_i\right) dt$ *and* $-\int_a^b \left(g_i - \cdot^T s_i\right) dt$ *are invex in* x *and* $\dot{x}$ *with respect to* $\eta(x,u)$*, and* $-\int_a^b \left(f_i - \cdot^T z_i\right) dt$ *and* $\int_a^b \left(g_i + \cdot^T r_i\right) dt$ *are invex in* y *and* $\dot{y}$*, with respect to* $\xi(v,y)$*, and* $\eta(x,u) + u(t) \geqq 0$ *and* $\xi(v,y) + y(t) \geqq 0$*,* $\forall t \in I$*, except possibly at corners of* $(\dot{x}(t), \dot{y}(t))$ *or* $(\dot{u}(t), \dot{v}(t))$*. Then one has* $p \nleq q$*.*

Proof. Since $\int_a^b \left(f_i + \cdot^T w_i\right) dt$ and $-\int_a^b \left(g_i - \cdot^T s_i\right) dt$ are invex in x and $\dot{x}$ with respect to $\eta(x,u)$, we have

$$
\begin{aligned}
&\int_a^b \left[\left\{f_i(t,x(t),\dot{x}(t),v(t),\dot{v}(t)) + x^T w_i\right\} - q_i\left\{g_i(t,x(t),\dot{x}(t),v(t),\dot{v}(t)) - x^T s_i\right\}\right] dt \\
&- \int_a^b [\{f_i(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) + u^T w_i\} - q_i\{g_i(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) - u^T s_i\}]dt \\
&\geqq \int_a^b \eta(x,u)^T [\{f_{ix}(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) + w_i\} \\
&\quad - q_i\{g_{ix}(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) - s_i\} - D\{f_{i\dot{x}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) + w_i\} \\
&\quad - q_i\{g_{i\dot{x}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) - s_i\}]dt \\
&= \int_a^b \eta(x,u)^T [\{(f_{ix}(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) + w_i) \\
&\quad - D(f_{i\dot{x}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) + w_i)\} \\
&\quad - q_i\{(g_{ix}(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) - s_i) - D(g_{i\dot{x}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) - s_i)\}]dt.
\end{aligned}
$$

From (7.9.6), (7.9.11) and (7.9.12) with $\eta(x,u) + u(t) \geqq 0$, we obtain

$$
\begin{aligned}
&\sum_{i=1}^k \tau_i \int_a^b [\{f_i(t,x(t),\dot{x}(t),v(t),\dot{v}(t)) + x^T w_i\} \\
&\quad - q_i\{g_i(t,x(t),\dot{x}(t),v(t),\dot{v}(t)) - x^T s_i\}]dt \\
&\qquad \geqq \sum_{i=1}^k \tau_i \int_a^b [\{f_i(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) + u^T w_i\} \\
&\qquad\quad - q_i\{g_i(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) - u^T s_i\}]dt.
\end{aligned}
\tag{7.9.15}
$$

Since $x^T s_i \leq s(x|E_i), s_i \in E_i$ and $x^T w_i \leq s(x|C_i), w_i \in C_i$, (7.9.15) can be written as

$$\begin{aligned}\sum_{i=1}^{k} \tau_i \int_a^b &[\{f_i(t,x(t),\dot{x}(t),v(t),\dot{v}(t)) + s(x|C_i)\}\\ &- q_i\{g_i(t,x(t),\dot{x}(t),v(t),\dot{v}(t)) - s(x|E_i)\}]dt\\ &\geqq \sum_{i=1}^{k} \tau_i \int_a^b [\{f_i(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) + u^T w_i\}\\ &- q_i\{g_i(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) - u^T s_i\}]dt.\end{aligned} \tag{7.9.16}$$

By the invexity of $-\int_a^b \left(f_i - \cdot^T z_i\right) dt$ and $\int_a^b \left(g_i + \cdot^T r_i\right) dt$ are invex in y and $\dot{y}$, with respect to $\xi(v,y)$, for fixed x, we have

$$\begin{aligned}&\int_a^b [\{f_i(t,x(t),\dot{x}(t),v(t),\dot{v}(t)) - v^T z_i\} - p_i\{g_i(t,x(t),\dot{x}(t),v(t),\dot{v}(t)) + v^T r_i\}]dt\\ &- \int_a^b [\{f_i(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) - y^T z_i\} - p_i\{g_i(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) + y^T r_i\}]dt\\ &\leqq \int_a^b \xi(v,y)^T [\{(f_{iy}(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) - z_i)\\ &- D(f_{i\dot{y}}(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) - z_i)\}\\ &- p_i\{(g_{iy}(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) + r_i) - D(g_{i\dot{y}}(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) + r_i)\}]dt.\end{aligned}$$

From (7.9.4), (7.9.5) and (7.9.13) along with $\xi(v,y) + y(t) \geqq 0, \forall t \in I$, we obtain

$$\begin{aligned}&\sum_{i=1}^{k} \tau_i \int_a^b [\{f_i(t,x(t),\dot{x}(t),v(t),\dot{v}(t)) - v^T z_i\} - p_i\{g_i(t,x(t),\dot{x}(t),v(t),\dot{v}(t)) + v^T r_i\}]dt\\ &\leqq \sum_{i=1}^{k} \tau_i \int_a^b [\{f_i(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) - y^T z_i\}\\ &- p_i\{g_i(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) + y^T r_i\}]dt.\end{aligned} \tag{7.9.17}$$

Since $v^T r_i \leq s(v|H_i)$, $r_i \in H_i$ and $v^T z_i \leq s(v|D_i)$, $z_i \in D_i$, (7.9.17) can be written as

$$\begin{aligned}\sum_{i=1}^{k} \tau_i \int_a^b &[\{f_i(t,x(t),\dot{x}(t),v(t),\dot{v}(t)) - s(v|D_i)\}\\ &- p_i\{g_i(t,x(t),\dot{x}(t),v(t),\dot{v}(t)) + s(v|H_i)\}]dt\\ &\leqq \sum_{i=1}^{k} \tau_i \int_a^b [\{f_i(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) - y^T z_i\}\\ &- p_i\{g_i(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) + y^T r_i\}]dt.\end{aligned} \tag{7.9.18}$$

From (7.9.16) and (7.9.18), we get

$$\sum_{i=1}^{k} \tau_i \int_a^b (p_i - q_i)\, g_i(t,x,\dot{x},v,\dot{v})\,dt$$
$$\geqq \sum_{i=1}^{k} \tau_i \Big[\int_a^b \{ f_i(t,u,\dot{u},v,\dot{v}) - s(v|D_i) + u^T w_i \}\,dt$$
$$- q_i \int_a^b \{ g_i(t,u,\dot{u},v,\dot{v}) + s(v|H_i) - u^T s_i \}\,dt \Big] \tag{7.9.19}$$
$$- \sum_{i=1}^{k} \tau_i \Big[\int_a^b \{ f_i(t,x,\dot{x},y,\dot{y}) + s(x|C_i) - y^T z_i \}\,dt$$
$$- p_i \int_a^b \{ g_i(t,x,\dot{x},y,\dot{y}) - s(x|E_i) + y^T r_i \}\,dt \Big].$$

From (7.9.3) and (7.9.10), (7.9.19) yields

$$\sum_{i=1}^{k} \tau_i (p_i - q_i) \int_a^b g_i(t,x,\dot{x},v,\dot{v})\,dt \geqq 0. \tag{7.9.20}$$

If for some $i, p_i < q_i$ and $\forall j \neq i, p_i \leqq q_i$, then $\int_a^b g_i(t,x,\dot{x},v,\dot{v})\,dt > 0, i = 1,2,\ldots,k$, implies that

$$\sum_{i=1}^{k} \tau_i (p_i - q_i) \int_a^b g_i(t,x,\dot{x},v,\dot{v})\,dt < 0,$$

which contradicts (7.9.20). Hence $p \nleq q$. □

Theorem 7.9.2. *(Weak Duality). Let* $(x(t), y(t), p, \tau, z_1, z_2, \ldots, z_k, r_1, r_2, \ldots, r_k)$ *be feasible for (EMSP) and let* $(u(t), v(t), q, \tau, w_1, w_2, \ldots, w_k, s_1, s_2, \ldots, s_k)$ *be feasible for (EMSD). Assume that* $\sum_{i=1}^{k} \tau_i \int_a^b \{ (f_i + \cdot^T w_i) - q_i (g_i - \cdot^T s_i) \}\,dt$ *is pseudo-invex in x and* $\dot{x}$ *with respect to* $\eta(x,u)$*, and* $-\sum_{i=1}^{k} \tau_i \int_a^b \{ (f_i - \cdot^T z_i) - p_i (g_i + \cdot^T r_i) \}\,dt$ *is pseudo-invex in y and* $\dot{y}$*, with respect to* $\xi(v,y)$*, with* $\eta(x,u) + u(t) \geqq 0$ *and* $\xi(v,y) + y(t) \geqq 0, \forall t \in I$*, except possibly at corners of* $(\dot{x}(t), \dot{y}(t))$ *or* $(\dot{u}(t), \dot{v}(t))$*. Then one has* $p \nleq q$*.*

Proof. Using the condition $\eta(x,u) + u(t) \geqq 0 \,\forall\, t \in I$, and duality constraint (7.9.12), we get

$$\int_a^b \eta(x,u)^T \sum_{i=1}^{k} \tau_i \{ [f_{iu} - Df_{i\dot{u}} + w_i] - q_i [g_{iu} - Dg_{i\dot{u}} - s_i] \}\,dt$$
$$= - \int_a^b u(t)^T \sum_{i=1}^{k} \tau_i \{ [f_{iu} - Df_{i\dot{u}} + w_i] - q_i [g_{iu} - Dg_{i\dot{u}} - s_i] \}\,dt \geqq 0.$$

Since $\sum_{i=1}^{k}\tau_i\int_a^b\left\{\left(f_i+\cdot^T w_i\right)-q_i\left(g_i-\cdot^T s_i\right)\right\}dt$ is pseudo-invex with respect to $\eta(x,u)$, it follows that

$$\begin{aligned}
&\sum_{i=1}^{k}\tau_i\int_a^b\Big[\left\{f_i\left(t,x(t),\dot{x}(t),v(t),\dot{v}(t)\right)+x^T w_i\right\}\\
&\quad-q_i\left\{g_i\left(t,x(t),\dot{x}(t),v(t),\dot{v}(t)\right)-x^T s_i\right\}\Big]dt\\
&\qquad\geqq\sum_{i=1}^{k}\tau_i\int_a^b\Big[\left\{f_i\left(t,u(t),\dot{u}(t),v(t),\dot{v}(t)\right)+u^T w_i\right\}\\
&\qquad\quad-q_i\left\{g_i\left(t,u(t),\dot{u}(t),v(t),\dot{v}(t)\right)-u^T s_i\right\}\Big]dt. \qquad (7.9.21)
\end{aligned}$$

Since $x^T s_i\leq s\left(x\left|E_i\right.\right),s_i\in E_i$ and $x^T w_i\leq s\left(x\left|C_i\right.\right),w_i\in C_i$, (7.9.21) can be written as

$$\begin{aligned}
&\sum_{i=1}^{k}\tau_i\int_a^b[\{f_i(t,x(t),\dot{x}(t),v(t),\dot{v}(t))+s(x|C_i)\}\\
&\quad-q_i\{g_i(t,x(t),\dot{x}(t),v(t),\dot{v}(t))-s(x|E_i)\}]dt\\
&\qquad\geqq\sum_{i=1}^{k}\tau_i\int_a^b[\{f_i(t,u(t),\dot{u}(t),v(t),\dot{v}(t))+u^T w_i\}\\
&\qquad\quad-q_i\{g_i(t,u(t),\dot{u}(t),v(t),\dot{v}(t))-u^T s_i\}]dt. \qquad (7.9.22)
\end{aligned}$$

By $\xi(v,y)+y(t)\geqq 0,\forall t\in I$, and primal constraint (7.9.5), we get

$$\begin{aligned}
&\int_a^b\xi(x,u)^T\sum_{i=1}^{k}\tau_i\left\{\left[f_{iy}-Df_{i\dot{y}}-z_i\right]-p_i\left[g_{iy}-Dg_{i\dot{y}}+r_i\right]\right\}dt\\
&\quad=-\int_a^b y(t)^T\sum_{i=1}^{k}\tau_i\left\{\left[f_{iy}-Df_{i\dot{y}}-z_i\right]-p_i\left[g_{iy}-Dg_{i\dot{y}}+r_i\right]\right\}dt\leqq 0.
\end{aligned}$$

By the pseudo-invexity of $-\sum_{i=1}^{k}\tau_i\int_a^b\left\{\left(f_i-\cdot^T z_i\right)-p_i\left(g_i+\cdot^T r_i\right)\right\}dt$ with respect to $\xi(v,y)$, we get

$$\begin{aligned}
&\sum_{i=1}^{k}\tau_i\int_a^b[\left\{f_i\left(t,x(t),\dot{x}(t),v(t),\dot{v}(t)\right)-v^T z_i\right\}\\
&\quad-p_i\left\{g_i\left(t,x(t),\dot{x}(t),v(t),\dot{v}(t)\right)+v^T r_i\right\}]dt\\
&\qquad\leqq\sum_{i=1}^{k}\tau_i\int_a^b[\left\{f_i\left(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\right)-y^T z_i\right\}\\
&\qquad\quad-p_i\left\{g_i\left(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\right)+y^T r_i\right\}]dt. \qquad (7.9.23)
\end{aligned}$$

Since $v^T r_i \leq s(v|H_i), r_i \in H_i$ and $v^T z_i \leq s(v|D_i)$, $z_i \in D_i$, (7.9.23) can be written as

$$\begin{aligned}
&\sum_{i=1}^{k} \tau_i \int_a^b [\{f_i(t,x(t),\dot{x}(t),v(t),\dot{v}(t)) - s(v|D_i)\} \\
&\quad - p_i\{g_i(t,x(t),\dot{x}(t),v(t),\dot{v}(t)) + s(v|H_i)\}]dt \\
&\qquad \leqq \sum_{i=1}^{k} \tau_i \int_a^b [\{f_i(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) - y^T z_i\} \\
&\qquad\quad - p_i\{g_i(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) + y^T r_i\}]dt. \qquad (7.9.24)
\end{aligned}$$

From (7.9.22) and (7.9.24), we get

$$\begin{aligned}
&\sum_{i=1}^{k} \tau_i \int_a^b (p_i - q_i)\, g_i(t,x,\dot{x},v,\dot{v})\, dt \\
&\quad \geqq \sum_{i=1}^{k} \tau_i \Big[\int_a^b \{f_i(t,u,\dot{u},v,\dot{v}) - s(v|D_i) + u^T w_i\}\, dt \\
&\quad\quad - q_i \int_a^b \{g_i(t,u,\dot{u},v,\dot{v}) + s(v|H_i) - u^T s_i\}\, dt\Big] \\
&\quad\quad - \sum_{i=1}^{k} \tau_i \Big[\int_a^b \{f_i(t,x,\dot{x},y,\dot{y}) + s(x|C_i) - y^T z_i\}\, dt \\
&\quad\quad - p_i \int_a^b \{g_i(t,x,\dot{x},y,\dot{y}) - s(x|E_i) + y^T r_i\}\, dt\Big]. \qquad (7.9.25)
\end{aligned}$$

From (7.9.3) and (7.9.10), (7.9.25) yields

$$\sum_{i=1}^{k} \tau_i (p_i - q_i) \int_a^b g_i(t,x,\dot{x},v,\dot{v})\, dt \geqq 0. \qquad (7.9.26)$$

If for some $i, p_i < q_i$ and $\forall j \neq i, p_i \leqq q_i$, then $\int_a^b g_i(t,x,\dot{x},v,\dot{v})\, dt > 0, i = 1,2,\ldots,k$, implies that

$$\sum_{i=1}^{k} \tau_i (p_i - q_i) \int_a^b g_i(t,x,\dot{x},v,\dot{v})\, dt < 0,$$

which contradicts (7.9.26). Hence $p \nleq q$. □

The following Theorems 7.9.3 and 7.9.4 can be established on the lines of the proofs of Theorems 3.2 and 3.3 given by Kim et al. (2004) in the light of the discussions given above in this section.

Theorem 7.9.3. *(Strong Duality) Let* $(\bar{x}(t),\bar{y}(t),\bar{p},\bar{\tau},\bar{z}_1,\bar{z}_2,\ldots,\bar{z}_k,\bar{r}_1,\bar{r}_2,\ldots,\bar{r}_k)$ *be a properly efficient solution for (EMSP) and fix* $\tau=\bar{\tau}$ *in (EMSD), and define*

$$\bar{p}_i=\frac{\int_a^b\left\{f_i\left(t,\bar{x},\dot{\bar{x}},\bar{y},\dot{\bar{y}}\right)+s\left(\bar{x}(t)\,|C_i\right)-\bar{y}(t)^T\bar{z}_i\right\}dt}{\int_a^b\left\{g_i\left(t,\bar{x},\dot{\bar{x}},\bar{y},\dot{\bar{y}}\right)-s\left(\bar{x}(t)\,|E_i\right)+\bar{y}(t)^T\bar{r}_i\right\}dt},i=1,2,\ldots,k$$

Suppose that all the conditions in Theorem 7.9.1 or Theorem 7.9.2 are fulfilled. Furthermore, assume that

$$\text{(I)}\ \sum_{i=1}^k\bar{\tau}_i\int_a^b\psi(t)^T\Big[\left\{\left[(f_{iyy}-z_i)-\bar{p}_i\left(g_{iyy}+r_i\right)\right]-D\left[(f_{i\dot{y}y}-z_i)-\bar{p}_i\left(g_{i\dot{y}y}+r_i\right)\right]\right\}$$
$$-D\left\{\left[(f_{iy\dot{y}}-z_i-Df_{i\dot{y}\dot{y}}-f_{i\dot{y}y})-\bar{p}_i\left(g_{iy\dot{y}}+r_i-Dg_{i\dot{y}\dot{y}}-g_{i\dot{y}y}\right)\right]\right\}$$
$$+D^2\left\{-\left[(f_{i\dot{y}\dot{y}}-z_i)-\bar{p}_i\left(g_{i\dot{y}\dot{y}}+r_i\right)\right]\right\}\Big]\,\psi(t)^T dt=0$$

implies that $\psi(t)=0,\forall t\in I$*, and*

$$\text{(II)}\quad\left[\int_a^b\left\{(f_{1y}-z_1)-\bar{p}_1\left(g_{1y}+r_1\right)\right\}dt,\ldots,\int_a^b\left\{\left(f_{ky}-z_k\right)-\bar{p}_k\left(g_{ky}+r_k\right)\right\}dt\right]$$

is linearly independent. Then there exist $\bar{w}_i\in R^n,\bar{s}_i\in R^m,i=1,2,\ldots,k$ *such that* $(\bar{x}(t),\bar{y}(t),\bar{p},\bar{\tau},\bar{w}_1,\bar{w}_2,\ldots,\bar{w}_k,\bar{s}_1,\bar{s}_2,\ldots,\bar{s}_k)$ *is a properly efficient solution of (EMSD).*

Theorem 7.9.4. *(Converse Duality). Let* $(\bar{x}(t),\bar{y}(t),\bar{p},\bar{\tau},\bar{z}_1,\bar{z}_2,\ldots,\bar{z}_k,\bar{r}_1,\bar{r}_2,\ldots,\bar{r}_k)$ *be a properly efficient solution for (EMSD) and fix* $\tau=\bar{\tau}$ *in (EMSP), and define* $\bar{p}_i$ *as in Theorem 7.9.3. Suppose that all the conditions in Theorem 7.9.1 or 7.9.2 are fulfilled. Furthermore, assume that (I) and (II) of Theorem 7.9.3 are satisfied. Then there exist* $\bar{w}_i\in R^n,\bar{s}_i\in R^m,i=1,2,\ldots,k$ *such that* $(\bar{x}(t),\bar{y}(t),\bar{p},\bar{\tau},\bar{w}_1,\bar{w}_2,\ldots,\bar{w}_k,$ $\bar{s}_1,\bar{s}_2,\ldots,\bar{s}_k)$ *is a properly efficient solution of (EMSP).*

Remark 7.9.1. (i) If the time dependence of problems (MNFP) and (MNFD) is removed and the functions involved are considered to have domain $R^n\times R^m$, we obtain the symmetric dual fractional pair given by

$$\text{(SNMFP)}\quad\text{minimize}\left(\frac{f_1(x,y)+s\left(x\,|C_1\right)-y^Tz_1}{g_1(x,y)-s\left(x|E_1\right)+y^Tr_1},\ldots,\frac{f_k(x,y)+s\left(x\,|C_k\right)-y^Tz_k}{g_k(x,y)-s\left(x\,|E_k\right)+y^Tr_k}\right)$$

subject to

$$\sum_{i=1}^k\tau_i\left[\nabla_yf_i(x,y)-z_i-\frac{f_i(x,y)+s(x|C_i)-y^Tz_i}{g_i(x,y)-s(x|E_i)+y^Tr_i}(\nabla_yg_i(x,y)+r_i)\right]\leq 0,$$

$$y^T\sum_{i=1}^k\tau_i\left[\nabla_yf_i(x,y)-z_i-\frac{f_i(x,y)+s(x|C_i)-y^Tz_i}{g_i(x,y)-s(x|E_i)+y^Tr_i}(\nabla_yg_i(x,y)+r_i)\right]\geq 0,$$

$$z_i\in D_i,\ r_i\in F_i,\ 1\leq i\leq k,$$

$$\tau>0,\ \tau^Te=1,\ x\geq 0$$

$$\text{(SNMFD)} \quad \text{maximize} \left(\frac{f_1(u,v) - s(v|D_1) + u^T w_1}{g_1(u,v) + s(v|F_1) - u^T t_1}, \ldots, \frac{f_k(u,v) - s(v|D_k) + u^T w_k}{g_k(u,v) + s(v|F_k) - u^T t_k} \right)$$

subject to

$$\sum_{i=1}^{k} \tau_i \left[\nabla_u f_i(u,v) + w_i - \frac{f_i(u,v) - s(v|D_i) + u^T w_i}{g_i(u,v) + s(v|F_i) - u^T t_i} (\nabla_u g_i(u,v) - t_i) \right] \geq 0,$$

$$u^T \sum_{i=1}^{k} \tau_i \left[\nabla_u f_i(u,v) + w_i - \frac{f_i(u,v) - s(v|D_i) + u^T w_i}{g_i(u,v) + s(v|F_i) - u^T t_i} (\nabla_u g_i(u,v) - t_i) \right] \leq 0,$$

$$w_i \in C_i, \; t_i \in E_i, \; 1 \leq i \leq k,$$

$$\tau > 0, \; \tau^T e = 1, \; v \geq 0.$$

The pair of problems (SNMFP) and (SNMFD) obtained above is exactly the pair of problems (FP) and (FD) considered by Yang et al. (2002b).

(ii) If we set $k = 1$, and our problems are time independent, we get the following pair of problems:

$$\text{(SNFP)} \quad \text{minimize} \left(\frac{f(x,y) + s(x|C) - y^T z}{g(x,y) - s(x|E) + y^T r} \right)$$

$$\text{subject to} \left[\nabla_y f(x,y) - z - \frac{f(x,y) + s(x|C) - y^T z}{g(x,y) - s(x|E) + y^T r} (\nabla_y g(x,y) + r) \right] \leq 0,$$

$$y^T \left[\nabla_y f(x,y) - z - \frac{f(x,y) + s(x|C) - y^T z}{g(x,y) - s(x|E) + y^T r} (\nabla_y g(x,y) + r) \right] \geq 0,$$

$$z \in D, \; r \in F,$$

$$x \geq 0.$$

$$\text{(SNFD)} \quad \text{maximize} \left(\frac{f(u,v) - s(v|D) + u^T w}{g(u,v) + s(v|F) - u^T t} \right)$$

$$\text{subject to} \left[\nabla_u f(u,v) + w - \frac{f(u,v) - s(v|D) + u^T w}{g(u,v) + s(v|F) - u^T t} (\nabla_u g(u,v) - t) \right] \geq 0,$$

$$u^T \left[\nabla_u f(u,v) + w - \frac{f(u,v) - s(v|D) + u^T w}{g(u,v) + s(v|F) - u^T t} (\nabla_u g(u,v) - t) \right] \leq 0,$$

$$w \in C, \; t \in E,$$

$$v \geq 0.$$

The pair of problems (SNFP) and (SNFD) are exactly the pair of problems (FP) and (FD) considered by Yang et al. (2002a).

(iii) If we remove the nondifferentiable terms of the problems discussed in this section, we get the problems discussed in Sect. 7.4 of Kim et al. (2004).

Chapter 8
Vector Variational-like Inequality Problems

8.1 Relationships Between Vector Variational-Like Inequalities and Vector Optimization Problems

In this section, we will establish some relationships between vector variational-like inequalities and vector optimization problems under the assumptions of $\alpha-$ invex functions. We will identify the vector critical points, the weakly efficient solutions and the solutions of the weak vector variational-like inequality problems, under pseudo-$\alpha-$ invexity assumptions. These conditions are more general than those of existing ones in the literature. In particular, this work extends an earlier work of Ruiz-Garzon et al. (2004) to a wider class of functions, namely the pseudo-$\alpha-$ invex functions introduced recently in Noor (2004a).

The concept of a vector variational inequality was introduced by Giannessi (1980). Since it has shown applications to a wide range of problems in various disciplines in the natural and social sciences, vector variational inequality problems have been generalized in various directions; in particular, vector variational-like inequality problems, see Gianessi (2000), Noor (1990, 1994a,b 1995, 2004a), Yang (1993, 1997). Several authors Chen and Cheng (1998), Ruiz-Garzon et al. (2003, 2004) have discussed relationships between vector variational inequalities and vector optimization problems under some convexity or generalized convexity assumptions. However, as it can be expected, such rather rigid conditions are not always met in applications.

The role of generalized monotonicity of the operator in vector variational inequality problems corresponds to the role of generalized convexity of the objective function in the optimization problem. In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson (1981). Weir and Mond (1988) and Noor (1990) have studied some basic properties of the preinvex functions and their role in optimization and variational-like inequality problems. Noor (2004b) has pointed out that the concept of invexity plays exactly the same role in variational-like inequality problems as the classical convexity plays in variational

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inequality problems, and has shown that the variational-like inequality problems are well-defined in the setting of invexity.

Recently, Ruiz-Garzon et al. (2004) established relationships between vector variational-like inequality and optimization problems, under the assumptions of pseudo-invexity. However, Ruiz-Garzon et al. (2004) have obtained some results without invexity assumption on the underlying set while discussing variational-like inequality problems.

In this section, we establish various relationships between generalized vector variational-like inequality problems and vector optimization problems under the assumption of pseudo-α– invex functions.

The following convention for equalities and inequalities will be used throughout the paper. If $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in R^n$, we denote

$$x \leqq y \text{ iff } x_i \leq y_i \ \forall i = 1, 2, \ldots, n;$$
$$x \leq y \text{ iff } x_i \leq y_i \ \forall i = 1, 2, \ldots, n; \text{ with } x \neq y;$$
$$x < y \text{ iff } x_i < y_i \ \forall i = 1, 2, \ldots, n; \text{ and } x \not< y \text{ is the negation of } x < y.$$

Let X be a nonempty subset of $R^n, \eta \colon X \times X \to R^n$ be a continuous map and $\alpha\colon X \times X \to R_{+\backslash\{0\}}$ be a bifunction.

Definition 8.1.1. *A subset X is said to be an $\alpha-$ invex set, if there $\eta\colon X \times X \to R^n, \alpha(x,u) X \times X \to R_+$ such that*

$$u + \lambda \alpha(x,u)\eta(x,u) \in X, \qquad \forall x, u \in X, \quad \lambda \in [0,1].$$

Remark 8.1.1. (i) If $\alpha(y,x) = 1$, then the set X is called the invex (η– connected) set, see Mohan and Neogy 1995).
(ii) If $\eta(y,x) = y - x$ and $0 < \alpha(y,x) < 1$, then the set X is called the star-shaped.
(iii) If $\alpha(y,x) = 1$ and $\eta(y,x) = y - x$, then the set X is called the convex set.

It is well known that the $\alpha-$ *invex* set may not be convex sets, see Noor (2004a).

Definition 8.1.2. *The function f on the α– invex set is said to be $\alpha-$ preinvex function, if there exist $\eta\colon X \times X \to R^n, \alpha(x,u) X \times X \to R_+$ such that*

$$f(u + \lambda\alpha(x,u)\eta(x,u)) \leq (1-\lambda) f(u) + \lambda f(x), \quad \forall x, u \in X, \quad \lambda \in [0,1].$$

From now onward we assume that the set X is a nonempty, closed and α– invex set with respect to $\alpha(\cdot,\cdot)$ and $\eta(\cdot,\cdot)$, unless otherwise specified.

Definition 8.1.3. *Let $f : X \subset R^n \to R^p$ be a differentiable function with a $p \times n$ matrix as its Jacobian. The function f is said to be*

(a) α– invex if and only if there exists a functions $\alpha\colon X \times X \to R_{+\backslash\{0\}}$ and $\eta\colon X \times X \to R^n$, such that

$$f(y) - f(x) \geqq \langle \alpha(y,x) \nabla f(x), \eta(y,x) \rangle, \quad \forall x, y \in X;$$

(b) *strictly* α *-invex if and only if there exists a functions* $\alpha\colon X \times X \to R_{+\setminus\{0\}}$ *and* $\eta\colon X \times X \to R^n$*, such that*

$$f(y) - f(x) > \langle \alpha(y,x)\nabla f(x), \eta(y,x)\rangle, \quad \forall x,\ y \in X,\ x \neq y;$$

(c) *pseudo-*$\alpha-$ *invex if and only if there exists a functions* $\alpha\colon X \times X \to R_{+\setminus\{0\}}$ *and* $\eta\colon X \times X \to R^n$*, such that*

$$f(y) < f(x) \Rightarrow \langle \alpha(y,x)\nabla f(x), \eta(y,x)\rangle < 0, \quad \forall x,\ y \in X.$$

Let $X \subseteq R^n$ be an $\alpha-$ invex nonempty subset of R^n and two continuous maps F: $X \to R^n$ and $\eta\colon X \times X \to R^n$ and $\alpha\colon X \times X \to R_{+\setminus\{0\}}$ be a bifunction.

The variational-like inequality problem (VLIP), is to find a point $\bar{x} \in X$, such that $\eta(y,x)^T F(\bar{x}) \geq 0, \forall y \in X$.

A vector variational-like inequality problem (VVLIP), is to find a point $\bar{x} \in X$, such that there exists no $y \in X$, such that $F(\bar{x})\eta(y,\bar{x}) < 0$.

A weak vector variational-like inequality problem (WVVLIP), is to find a point $\bar{x} \in X$, such that there exists no $y \in X$, such that $F(\bar{x})\eta(y,\bar{x}) < 0$.

We consider the following generalized forms of vector variational-like inequality problems:

(GVVLIP) A generalized vector variational-like inequality problems, is to find a point $\bar{x} \in X$, such that there exists no $y \in X$, such that $\langle \alpha(y,\bar{x})F(\bar{x}), \eta(y,\bar{x})\rangle \leq 0$.

(GWVVLIP) A generalized weak vector variational-like inequality problems, is to find a point $\bar{x} \in X$, such that there exists no $y \in X$, such that $\langle \alpha(y,\bar{x})F(\bar{x}), \eta(y,\bar{x})\rangle < 0$.

Remark 8.1.2. Notice that if $\alpha(y,\bar{x}) = 1$, then the (GVVLIP) and (GWVVLIP) reduce to the (VVLIP) and (WVVLIP) studied in Garzon et al. (2004).

It is well known that in multiobjective optimization problems, the objective functions are conflicting in nature and can not be combined into a single objective. In this sense we must understand the concept of efficient solutions.

Let $f\colon R^n \to R^p$, the vector optimization problem (VOP) is to find the *efficient points* for

$$\text{(VOP)} \qquad V - \min f(x) \quad \text{subject to } x \in X.$$

Definition 8.1.4. *A point* $\bar{x} \in X$ *is said to be efficient (Pareto), if there exists no* $y \in X$ *such that* $f(y) \leq f(\bar{x})$.

Definition 8.1.5. *A point* $\bar{x} \in X$ *is said to be weakly efficient, if there exists no* $y \in X$ *such that* $f(y) < f(\bar{x})$.

Using the concept of pseudo-$\alpha-$invex functions, we shall extend the results given by Ruiz-Garzon et al. (2004) for pseudo-invex functions. In fact, these results also extend earlier works of Kazmi (1996) and Yang and Goh (1997).

In the following theorem we establish that under $\alpha-$ invexity assumptions the solutions of the generalized vector variational-like inequality problem (GVVLIP) are efficient solutions to (VOP).

Theorem 8.1.1. *Let $f\colon X \subset R^n \to R^p$ be differentiable function on X. If $F = \nabla f$, f is $\alpha-$ invex with respect to α and η and $\bar{x}$ solves the generalized vector variational-like inequality problem (GVVLIP) with respect to the same α and η, then $\bar{x}$ is an efficient point to the vector optimization problem (VOP).*

Proof. Suppose $\bar{x}$ is not an efficient point to (VOP), then there exists a $y \in X$ such that $f(y) - f(\bar{x}) \leq 0$.

Since f is $\alpha-$ invex with respect to α and η, we have ensured that $\exists y \in X$, such that

$$\langle \alpha(y,\bar{x}) \nabla f(\bar{x}), \eta(y,\bar{x}) \rangle \leq 0;$$

therefore $\bar{x}$ cannot be a solution to the generalized vector variational-like inequality problem (GVVLIP). This contradiction leads to the result. □

Theorem 8.1.2. *Let $f\colon X \subset R^n \to R^p$ be differentiable function on X. If $F = \nabla f$, $-f$ is strictly-$\alpha-$ invex with respect to α and η. If $\bar{x}$ is a weakly efficient solution to the vector optimization problem (VOP) then $\bar{x}$ also solves the generalized vector variational-like inequality problem (GVVLIP).*

Proof. Suppose that $\bar{x}$ is a weakly efficient solution to (VOP), but does not solve the (GVVLIP). Then there exists a $y \in X$ such that $\langle \alpha(y,\bar{x}) \nabla f(\bar{x}), \eta(y,\bar{x}) \rangle \leq 0$.

By the strict-$\alpha-$ invexity of $-f$ with respect to α and η, we have

$$f(y) - f(\bar{x}) < \langle \alpha(y,\bar{x}) \nabla f(\bar{x}), \eta(y,\bar{x}) \rangle \leq 0;$$

therefore $\exists y \in X$ such that $f(y) < f(\bar{x})$,

which contradicts $\bar{x}$ being a weakly efficient solution to the (VOP). □

Since every efficient solution to (VOP) is a weakly efficient solution to (VOP), from the above theorem, we can get the following result.

Corollary 8.1.1. *Let $f\colon X \subset R^n \to R^p$ be differentiable function on X. If $F = \nabla f$, $-f$ is strictly-$\alpha-$ invex with respect to α and η. If $\bar{x}$ is an efficient solution to the vector optimization problem (VOP) then $\bar{x}$ also solves the generalized vector variational-like inequality problem (GVVLIP).*

Theorem 8.1.3. *Let $f\colon X \subset R^n \to R^p$ be differentiable function on X. If $F = \nabla f$. If $\bar{x}$ is a weakly efficient solution to the vector optimization problem (VOP) then $\bar{x}$ also solves the generalized weak vector variational-like inequality problem (GWVVLIP).*

If f is a pseudo-$\alpha-$ invex function with respect to α and η . If $\bar{x}$ also solves the generalized weak vector variational-like inequality problem (GVVLIP) with respect to the same α and η. Then $\bar{x}$ is a weakly efficient solution to the vector optimization problem (VOP).

Proof. For $(\Rightarrow)$ Let $\bar{x}$ be a weakly efficient solution to the (VOP), since X is an $\alpha-$ invex set, we have that $\nexists y \in X$, such that $f(\bar{x} + t\alpha(y,\bar{x})\eta(y,\bar{x})) - f(\bar{x}) < 0, 0 < t < 1$.

Dividing the above inequality by t and taking the limit as $t \to 0$, we get to $\nexists y \in X$, such that $\langle \alpha(y,\bar{x}) \nabla f(\bar{x}), \eta(y,\bar{x}) \rangle < 0$.

For ($\Leftarrow$) If $\bar{x}$ is not a weakly efficient solution to (VOP), then $\exists y \in X$, such that

$$f(y) < f(\bar{x}).$$

By pseudo-α– invexity of f with respect to α and η, we have ensured that $\exists y \in X$, such that $\langle \alpha(y,\bar{x}) \nabla f(\bar{x}), \eta(y,\bar{x}) \rangle < 0$. This contradicts the fact that $\bar{x}$ is a solution to the (GWVVLIP). □

Theorem 8.1.4. *Let $f\colon X \subset R^n \to R^p$ be differentiable function on X. If $F = \nabla f$, f is strictly-α– invex with respect to α and η. If $\bar{x}$ is a weakly efficient solution to the vector optimization problem (VOP) then $\bar{x}$ is an efficient solution to (VOP).*

Proof. Suppose that $\bar{x}$ is a weakly efficient solution to the (VOP), but not an efficient solution to (VOP). Then, there exists $\exists y \in X$, such that $f(y) \leq f(\bar{x})$.

By the strict-α– invexity of f with respect to the same α and η, we have

$$f(y) - f(\bar{x}) > \langle \alpha(y,\bar{x}) \nabla f(\bar{x}), \eta(y,\bar{x}) \rangle,$$

which is to say, $\exists y \in X$, such that $\langle \alpha(y,\bar{x}) \nabla f(\bar{x}), \eta(y,\bar{x}) \rangle < 0$; therefore, $\bar{x}$ does not solve the (GWVVLIP). This contradiction arises from the first part of Theorem 8.1.3. □

In the sequel we need the following definition from Osuna et al. (1998).

Definition 8.1.6. *A feasible solution $\bar{x} \in X$ is said to be a vector critical point for the problem (VOP) if there exists a vector $\lambda \in R^p$ with $\lambda \geq 0$ such that $\lambda^T \nabla f(\bar{x}) = 0$.*

It should be noticed that scalar stationary points are those whose vector gradients are zero. For vector problems, the vector critical points are those such that there exists a non-negative linear combination of the gradient vectors of each component of objective function, valued at that point, equal to zero.

The following theorem is extension to the context of pseudo-$\alpha-$ invexity of Theorem 2.2 from Osuna et al. (1998) for pseudo-invex case.

Theorem 8.1.5. *All vector critical points are weakly efficient solutions if and only if the vector function f is pseudo-$\alpha-$ invex on X.*

Proof. The proof follows from the proof of Theorem 2.2 of Osuna et al. (1998) and the discussion as above in this section. □

In light of Theorem 8.1.3 and Theorem 8.1.5 we could relate the vector critical points to the solutions of the weak vector variational-like inequality problem (GWVVLIP), with the following result:

Corollary 8.1.2. *Suppose that $F = \nabla f$. If the objective function is pseudo-$\alpha-$ invex with respect to α and η, then the vector critical points, the weakly efficient points and the solutions of the generalized weak vector variational-like inequality problem (GWVVLIP) are equivalent.*

8.2 On Relationships Between Vector Variational Inequalities and Vector Optimization Problems with Pseudo-Univexity

In this section, we establish some relationships between vector optimization problems and vector variational-like inequality problems. We identify the vector critical points, the weakly efficient points and the solutions of the weak vector variational-like inequality problems, under pseudo-univexity assumptions. These conditions are more general than those of existing ones in the literature and even more general than given in the previous Section. In particular, this work extends the results of the previous section as well as an earlier work of Ruiz-Garzon et al. (2004) to a wider class of functions, namely the pseudo-univex functions.

Bector et al. (1992) have introduced the concept of pre-univex functions, univex functions and pseudo-univex functions as a generalization of invex (Hanson 1981) functions. Mishra (1998) have established several sufficient optimality and duality results for a nonlinear programming problems under generalized univexity assumptions. Mishra and Giorgi (2000) have obtained optimality and duality results under nonsmooth setting for semi-univex functions. Mishra et al. (2005) have introduced four new classes of d-univex functions and established Karush–Kuhn–Tucker sufficient optimality conditions and various duality theorems for non-differentiable multiobjective programing problems. Rueda et al. (1995) have established optimality and duality for several programs under a combination of univex and type-I functions.

In this section, we establish various relationships between vector variational-like inequality problems and vector optimization problems under the assumption of pseudo-univex functions.

The convention for equalities and inequalities will be same as in the previous section.

Let X be a nonempty subset of R^n and $\eta\colon X\times X\to R^n$ be a continuous map.

Definition 8.2.1. *(Weir and Mond 1988). Let $u\in X$. The set X is said to be invex at u with respect to the function $\eta\colon X\times X\to R^n$, if for all $u,v\in X, t\in[0,1]$, we have, $u+t\eta(v,u)\in X$.*

The set X is said to be invex with respect to $\eta\colon X\times X\to R^n$, if X is invex at each point of the set X. The invex set X is also called $\eta-$ connected set.

Definition 8.2.2. *(Ruiz et al. 2004). Let X be a nonempty open and invex subset of R^n and $f\colon X\subset R^n\to R^p$ be a differentiable function with a $p\times n$ matrix as its Jacobian. The function f is said to be*

(d) invex if and only if there exists a function $\eta\colon X\times X\to R^n$, such that

$$f(y)\geqq f(x)+\nabla f(x)\,\eta(y,x),\ \forall x,\ y\in X;$$

(e) strictly invex if and only if there exists a function $\eta\colon X\times X\to R^n$, such that

$$f(y)>f(x)+\nabla f(x)\,\eta(y,x),\ \forall x,\ y\in X,\ x\neq y;$$

(f) pseudo-invex if and only if there exists a function $\eta: X \times X \to R^n$*, such that*

$$f(y) < f(x) \Rightarrow \nabla f(x)\,\eta(y,x) < 0,\ \forall x,\ y \in X.$$

Definition 8.2.3. *(Bector et al. 1992). Let* X *be a nonempty open and invex subset of* R^n *and* $f: X \subset R^n \to R^p$ *be a differentiable function with a* $p \times n$ *matrix as its Jacobian. The function* f *is said to be*

(a) univex with respect to η*,* ϕ *and* k *if and only if there exist a functions* $\eta: X \times X \to R^n, \phi: R \to R$*, and* $k: X \times X \to R_+$*, such that*

$$k(y,x)\,\phi\,[f(y) - f(x)] \geqq \nabla f(x)\,\eta(y,x),\ \forall x,\ y \in X;$$

(b) strictly univex with respect to η *,*ϕ *and* k *if and only if there exists a function* $\eta: X \times X \to R^n$*,* $\phi: R \to R$*, and* $k: X \times X \to R_+$*, such that*

$$k(y,x)\,\phi\,[f(y) - f(x)] > \nabla f(x)\,\eta(y,x),\ \forall x,\ y \in X, x \neq y;$$

(c) pseudo-univex with respect to η, ϕ *and* k *if and only if there exists a function* $\eta: X \times X \to R^n, \phi: R \to R$*, and* $k: X \times X \to R_+$*, such that*

$$k(y,x)\,\phi\,[f(y) - f(x)] < 0 \Rightarrow \nabla f(x)\,\eta(y,x) < 0,\ \forall x,\ y \in X.$$

It should be noticed that the class of univex functions are wider than that of the invex functions, as can be seen from the following example from Bector et al. (1991).

Example 8.2.1. Let $f: R \to R$ defined by $f(y) = y^3$, where,

$$k(y,x) = \begin{cases} \frac{x^2}{y-x}, & y > x, \\ 0, & y \leqq x, \end{cases}$$

$$\eta(y,x) = \begin{cases} y^2 + x^2 + xy, & y > x, \\ y - x, & y \leqq x, \end{cases}$$

Let $\phi: R \to R$ be defined by $\phi(a) = 3a$. The function f is univex, but not invex, because for $y = -3, x = 1, f(y) - f(x) < \nabla f(x)\,\eta(y,x)$.

Example 8.2.2. (Bector et al. 1992). In this example, we point out that the class of pseudo-univex functions are even wider than that of univex functions. Let $f: \left]0, \frac{\pi}{2}\right[\to R$ be defined by $f(y) = \cos y,\ \eta(y,x) = x - y$,

$$k(y,x) = \begin{cases} 0, & y \geqq x, \\ xy, & y < x, \end{cases}$$

and $\phi: R \to R$ be defined by $\phi(a) = 2a$. The function f is pseudo-univex, but not univex, because for $y = \frac{\pi}{3}, x = \frac{\pi}{6}, \nabla f(x)\,\eta(y,x) > k(y,x)\,\phi\,[f(y) - f(x)]$.

Let X be a nonempty invex subset of R^n and two continuous maps $F\colon X \to R^n$ and $\eta\colon X \times X \to R^n$. The variational-like inequality problem (VLIP), is to find a point $\bar{x} \in X$, such that $\eta(y,x)^T F(\bar{x}) \geq 0$, $\forall y \in X$.

Consider the following problems as in Sect. 8.1:

A vector variational-like inequality problem (VVLIP), is to find a point $\bar{x} \in X$, such that there exists no $y \in X$, such that $F(\bar{x})\,\eta(y,\bar{x}) \leq 0$.

A weak vector variational-like inequality problem (WVVLIP), is to find a point $\bar{x} \in X$, such that there exists no $y \in X$, such that $F(\bar{x})\,\eta(y,\bar{x}) < 0$.

It is well known that in multi-objective optimization problems, the objective functions are conflicting in nature and can not be combined into a single objective. In this sense we must understand the concept of efficient solutions.

Let X be a nonempty invex subset of R^n and $f\colon R^n \to R^p$, the vector optimization problem (VOP) is to find the efficient points for

$$\text{(VOP)} \qquad V - \min\ f(x)$$
$$\text{subject to } x \in X.$$

Definition 8.2.4. *A point $\bar{x} \in X$ is said to be efficient (Pareto), if there exists no $y \in X$ such that $f(y) \leq f(\bar{x})$.*

Definition 8.2.5. *A point $\bar{x} \in X$ is said to be weakly efficient, if there exists no $y \in X$ such that $f(y) < f(\bar{x})$.*

In this Section, using the concept of pseudo-univex functions, we shall extend the results given by Ruiz-Garzon et al. (2004). In fact, these results also extend earlier works Kazmi (1996) and Yang and Goh (1997).

In the following theorem we establish that under univexity assumptions, the solution of the vector variational-like inequality problem (VVLIP) are efficient solutions to (VOP).

Theorem 8.2.1. *Let X be a nonempty invex subset of R^n and $f\colon X \subset R^n \to R^p$ be differentiable function on X. If $F = \nabla f$, f is univex with respect to η, ϕ and k with $\phi(a) \leq 0$, whenever, $a \leq 0$, and $\bar{x}$ solves the vector variational-like inequality problem (VVLIP) with respect to the same η, ϕ and k, then $\bar{x}$ is an efficient point to the vector optimization problem (VOP).*

Proof. Suppose $\bar{x}$ is not an efficient point to (VOP), then there exists a $y \in X$ such that $f(y) - f(\bar{x}) \leq 0$. Since, $\phi(a) \leq 0$, whenever, $a \leq 0$, and we get

$$k(y,x)\,\phi\,[f(y) - f(\bar{x})] \leq 0.$$

Since f is univex with respect to η, ϕ and k, we have ensured that $\exists y \in X$, such that

$$\nabla f(\bar{x})\,\eta(y,\bar{x}) \leq 0;$$

therefore $\bar{x}$ cannot be a solution to the vector variational-like inequality problem (VVLIP). This contradiction leads to the result. □

Theorem 8.2.2. *Let X be a nonempty invex subset of R^n and $f\colon X \subset R^n \to R^p$ be differentiable function on X. If $F = \nabla f, -f$ is strictly-univex with respect to η, ϕ, and k with $\phi(a) < 0$, whenever, $a < 0$. If $\bar{x}$ is a weakly efficient solution to the vector optimization problem (VOP) then $\bar{x}$ also solves the vector variational-like inequality problem (VVLIP).*

Proof. Suppose that $\bar{x}$ is a weakly efficient solution to (VOP), but does not solve the (VVLIP). Then there exists a $y \in X$ such that $\nabla f(\bar{x})\,\eta(y,\bar{x}) \leq 0$.

By the strict univexity of $-f$ with respect to η, ϕ and k, we have

$$k(y,x)\,\phi[f(y) - f(\bar{x})] < \nabla f(\bar{x})\,\eta(y,\bar{x}) \leq 0.$$

Since, $\phi(a) < 0$, whenever, $a < 0$, and we get

$$f(y) - f(\bar{x}) < 0,$$

which contradicts $\bar{x}$ being a weakly efficient solution to the (VOP). □

Since every efficient solution to (VOP) is a weakly efficient solution to (VOP), from the above theorem, we can get the following result.

Corollary 8.2.1. *Let X be a nonempty invex subset of R^n and $f\colon X \subset R^n \to R^p$ be differentiable function on X. If $F = \nabla f, -f$ is strictly-univex with respect to η, ϕ and k with $\phi(a) < 0$, whenever, $a < 0$. If $\bar{x}$ is an efficient solution to the vector optimization problem (VOP) then $\bar{x}$ also solves the vector variational-like inequality problem (VVLIP).*

Theorem 8.2.3. *Let X be a nonempty open invex subset of R^n and $f\colon X \subset R^n \to R^p$ be differentiable function on X. If $F = \nabla f$. If $\bar{x}$ is a weakly efficient solution to the vector optimization problem (VOP) then $\bar{x}$ also solves the weak vector variational-like inequality problem (WVVLIP).*

If f is a pseudo-univex function with respect to η, ϕ and k with $\phi(a) < 0$, whenever, $a < 0$. If $\bar{x}$ also solves the weak vector variational-like inequality problem (VVLIP) with respect to the same η then $\bar{x}$ is a weakly efficient solution to the vector optimization problem (VOP).

Proof. For $(\Rightarrow)$ Let $\bar{x}$ be a weakly efficient solution to the (VOP), since X is an invex set, we have that $\nexists y \in X$, such that $f(\bar{x} + t\eta(y,\bar{x})) - f(\bar{x}) < 0, 0 < t < 1$.

Dividing the above inequality by t and taking the limit as $t \to 0$, we get to $\nexists y \in X$, such that $\nabla f(\bar{x})\,\eta(y,\bar{x}) < 0$.

For $(\Leftarrow)$ If $\bar{x}$ is not a weakly efficient solution to (VOP), then $\exists y \in X$, such that $f(y) < f(\bar{x})$. Since, $\phi(a) < 0$, whenever, $a < 0$, and we get

$$k(y,x)\,\phi[f(y) - f(\bar{x})] < 0.$$

By pseudo-univexity of f with respect to η, ϕ and k, we have ensured that $\exists y \in X$, such that $\nabla f(\bar{x})\,\eta(y,\bar{x}) < 0$. This contradicts the fact that $\bar{x}$ is a solution to the (WVVLIP). □

Theorem 8.2.4. *Let X be a nonempty open invex subset of R^n and $f\colon X \subset R^n \to R^p$ be differentiable function on X. If $F = \nabla f$, f is strictly-univex with respect to η, ϕ and k with $\phi(a) \leq 0$, whenever, $a \leq 0$. If $\bar{x}$ is a weakly efficient solution to the vector optimization problem (VOP) then $\bar{x}$ is an efficient solution to (VOP).*

Proof. Suppose that $\bar{x}$ is a weakly efficient solution to the (VOP), but not an efficient solution to (VOP). Then, there exists $\exists y \in X$, such that $f(y) \leq f(\bar{x})$. Since $\phi(a) \leq 0$, whenever, $a \leq 0$, we get

$$k(y,x)\phi[f(y) - f(\bar{x})] \leq 0.$$

By the strict-univexity of f with respect to the same η, ϕ and k, we have

$$0 \geq k(y,x)\phi[f(y) - f(\bar{x})] > \nabla f(\bar{x})\eta(y,\bar{x})$$

which is to say, $\exists y \in X$, such that $\nabla f(\bar{x})\eta(y,\bar{x}) < 0$; therefore, $\bar{x}$ does not solve the (WVVLIP). This contradiction arises from the first part of Theorem 8.2.3. □

The following theorem is extension to the context of pseudo-univexity of Theorem 2.2 from Osuna et al. (1998) for pseudo-invex case.

Theorem 8.2.5. *All vector critical points are weakly efficient solutions if and only if the vector function f is pseudo-univex on X.*

Proof. The proof follows from the proof of Theorem 2.2 from Osuna et al. (1998) and the discussion as above in this section. □

In light of Theorem 8.2.3 and Theorem 8.2.5 we could relate the vector critical points to the solutions of the weak vector variational-like inequality problem (WVVLIP), with the following result:

Corollary 8.2.2. *Suppose that X is an open invex set and $F = \nabla f$. If the objective function is pseudo-univex with respect to η, ϕ and k then the vector critical points, the weakly efficient points and the solutions of the weak vector variational-like inequality problem (WVVLIP) are equivalent.*

8.3 Relationship Between Vector Variational-Like Inequalities and Nondifferentiable Vector Optimization Problems

In this section, we establish some relationships between vector variational-like inequality and non-smooth vector optimization problems under the assumptions of $\alpha-$ invex non-smooth functions. We identify the vector critical points, the weakly efficient points and the solutions of the weak vector variational-like inequality problems, under non-smooth pseudo-$\alpha-$ invexity assumptions. These conditions are more general than those of existing ones in the literature. In particular, this work extends an earlier work of Ruiz-Garzon et al. (2004) to a wider class of functions, namely the non-smooth pseudo-$\alpha-$ invex functions. Moreover, this work extends an earlier work of Mishra and Noor (2005) to non-differentiable case.

Assumptions and conventions are same as in previous sections.

Definition 8.3.1. *A function $f: X \to R$ is said to be Lipschitz near $x \in X$ if for some $K > 0$,*

$$|f(y) - f(z)| \leq K\|y - z\|, \ \forall y, z \text{ within a neighbourhood of } x.$$

We say that $f: X \to R$ is locally Lipschitz on X if it is Lipschitz near any point of X.

Definition 8.3.2. *If $f: X \to R$ is Lipschitz at $x \in X$, the generalized derivative (in the sense of Clarke) of f at $x \in X$ in the direction $v \in R^n$, denoted by $f^0(x;v)$, is given by*

$$f^0(x;v) = \limsup_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda}.$$

Definition 8.3.3. *The Clarke's generalized gradient of f at $x \in X$, denoted by $\partial f(x)$, is defined as follows:*

$$\partial f(x) = \left\{\xi \in R^n : f^0(x;v) \geq \xi^T v \text{ for all } v \in R^n\right\}.$$

It follows that, for any $v \in R^n$

$$f^0(x;v) = \max\left\{\xi^T v : \xi \in \partial f(x)\right\}.$$

These definitions and properties can be extended to a locally Lipschitz vector-valued function $f: X \to R^p$. Denote by f_i, $i = 1, 2, \ldots, p$ the components of f. The Clarke generalized gradient of f at $x \in X$ is the set $\partial f(x) = \partial f_1(x) \times \partial f_2(x) \times \ldots \times \partial f_p(x)$.

Let X be a nonempty subset of $R^n, \eta: X \times X \to R^n$ be a continuous map and $\alpha: X \times X \to R_{+\setminus\{0\}}$ be a bifunction.

Remark 8.3.1(iv) If $\alpha(y,x) = 1$, then the set X is called the invex ($\eta-$ connected) set, see Noor (2004a) and Ruiz-Garzon et al. (2004).

(v) If $\eta(y,x) = y - x$, and $0 < \alpha(y,x) < 1$, then the set X is called the star-shaped.

(vi) If $\alpha(y,x) = 1$ and $\eta(y,x) = y - x$, then the set X is called the convex set.

It is well known that the $\alpha - invex$ set may not be convex sets, see Noor (2004a).

From now onward we assume that the set X is a nonempty, closed and $\alpha-$ invex set with respect to $\alpha(\cdot,\cdot)$ and $\eta(\cdot,\cdot)$, unless otherwise specified.

Definition 8.3.4. *The non-differentiable function $f: X \subset R^n \to R^p$ is:*

(a) $\alpha-$ invex if and only if there exists a functions $\alpha: X \times X \to R_{+\setminus\{0\}}$ and $\eta: X \times X \to R^n$, such that

$$f(y) - f(x) \geqq \langle \alpha(y,x)\xi, \eta(y,x)\rangle, \ \forall \xi \in \partial f(u), \forall x, u \in X;$$

(b) strictly $\alpha-$ *invex if and only if there exists a functions* $\alpha\colon X\times X\to R_{+\backslash\{0\}}$ *and* $\eta\colon X\times X\to R^n$*, such that*

$$f(y)-f(x)>\langle\alpha(y,x)\xi,\eta(y,x)\rangle,\ \forall\xi\in\partial f(u),\ \forall x,u\in X, x\neq y;$$

(c) pseudo- $\alpha-$ *invex if and only if there exists a functions* $\alpha\colon X\times X\to R_{+\backslash\{0\}}$ *and* $\eta\colon X\times X\to R^n$*, such that*

$$f(y)<f(x)\Rightarrow\langle\alpha(y,x)\xi,\eta(y,x)\rangle<0,\forall\xi\in\partial f(u),\forall x,y\in X.$$

We consider the following nondifferentiable vector optimization problem:

$$\text{(NVOP)}\quad \text{Min}\, f(x)=(f_1(x),\ldots,f_p(x))$$
$$\text{subject to}\ x\in X,$$

where $f_i\colon X\to R, i=1,2,\ldots,p$ are non-differentiable locally Lipschitz functions.

In vector optimization problems, multiple objectives are usually non-commensurable and cannot be combined into a single objective. Moreover, often the objectives conflict with each other. Consequently, the concept of optimality for single-objective optimization problems cannot be applied directly to vector optimization. In this regard the concept of efficient solutions is more useful for vector optimization problems.

The following problems are more general than the ones given in the above sections.

(GVVLIP) A vector variational-like inequality problem for non-smooth case, is to find a point $y\in X$, and for any $\xi\in\partial f(y)$, there exists no $x\in X$, such that $\langle\alpha(x,y)\xi,\eta(x,y)\rangle\leq 0$,

(GWVVLIP) A weak vector variational-like inequality problem, is to find a point $y\in X$, and for any $\xi\in\partial f(y)$, there exists no $x\in X$, such that $\langle\alpha(x,y)\xi,\eta(x,y)\rangle<0$.

In this section, using the tools of non-smooth analysis and the concept of non-differentiable vector pseudo-$\alpha-$ invexity, we shall extend the results given by Ruiz-Garzon et al. (2004) and the results of Sect. 8.1 to the nondifferentiable case (also see; Mishra and Noor 2005).

Theorem 8.3.1. *Let* $f\colon X\to R^p$ *be locally Lipschitz and* $\alpha-$ *invex function with respect to* α *and* η*. If* $y\in X$ *solves the generalized vector variational-like inequality problem (GVVLIP) with respect to the same* α *and* η*, then y is an efficient solution to the nondifferentiable vector optimization problem (NVOP).*

Proof. Suppose that y is not an efficient solution to (NVOP). Thus, there exists a $x\in X$ such that $f(x)-f(y)\leq 0$. Since f is $\alpha-$ invex with respect to α and η, we have ensured that $\exists x\in X$ such that

$$\langle\alpha(x,y)\xi,\eta(x,y)\rangle\leq 0,\forall\xi\in\partial f(y);$$

therefore, y cannot be a solution to the generalized vector variational-like inequality problem (GVVLIP). This contradiction leads to the result. □

In order to see the converse of the above theorem, we must impose stronger conditions, as can be observed in the following two theorems.

Theorem 8.3.2. *Let $f: X \to R^p$ be locally Lipschitz and $-f$ is strictly-$\alpha-$ invex with respect to α and η. If $y \in X$ is a weak efficient solution for (NVOP), then y solves the generalized vector variational-like inequality problem (GVVLIP).*

Proof. Suppose that y does not solve (GVVLIP). Thus, there exists a point $x \in X$ such that $\langle \alpha(x,y)\xi, \eta(x,y)\rangle \leq 0, \forall \xi \in \partial f(y)$. By the strict-$\alpha$– invexity of the non-smooth function $-f$ with respect to α and η, we have

$$f(x) - f(y) < \langle \alpha(x,y)\xi, \eta(x,y)\rangle \leq 0, \forall \xi \in \partial f(y);$$

therefore, there exists a $x \in X$ such that $f(x) - f(y) < 0$, which contradicts $y \in X$ being a weakly efficient solution of (NVOP). This contradiction leads to the result. □

As every efficient solution is also a weakly efficient solution to (NVOP), the following result is trivial to prove.

Corollary 8.3.1. *Let $f: X \to R^p$ be locally Lipschitz and $-f$ is strictly-$\alpha-$ invex with respect to α and η. If $y \in X$ is an efficient solution for (NVOP), then, y also solves the vector variational-like inequality problem (VVLIP).*

Theorem 8.3.3. *If $y \in X$ is a weakly efficient solution for (NVOP), then y solves the weak vector variational-like inequality problem (WVVLIP).*

Proof. Let $y \in X$ be a weakly efficient solution for (NVOP). Since X is an $\alpha-$ invex set, we have that $\nexists x \in X$, such that $f(y + t\alpha(x,y)\eta(x,y)) - f(y) < 0, 0 < t < 1$. Dividing the above inequality by t and taking the limit as t tends to zero, we get to $\nexists x \in X$ such that $\langle \alpha(x,y)\xi, \eta(x,y)\rangle < 0$, $\forall \xi \in \partial f(y)$. □

Theorem 8.3.4. *If f is a locally Lipschitz and pseudo-$\alpha-$ invex with respect to α and η and y solves the weak vector variational-like inequality problem (WVVLIP) with respect to the same α and η, then y is a weakly efficient solution to (NVOP).*

Proof. Suppose that y is not a weakly efficient solution to (NVOP). Thus, there exists a $x \in X$, such that $f(x) < f(y)$. By the pseudo-$\alpha-$ invexity of f with respect to α and η, we have that, there exists $x \in X$ such that $\langle \alpha(x,y)\xi, \eta(x,y)\rangle < 0$, $\forall \xi \in \partial f(y)$. This contradicts the fact that y is a solution to (WVVLIP). □

Theorem 8.3.5. *If f is a locally Lipschitz and strictly-$\alpha-$ invex with respect to α and η and y is a weak efficient solution of the (NVOP), then y is an efficient solution to the (NVOP).*

Proof. Suppose that y is a weakly efficient solution of (NVOP), but not an efficient solution to (NVOP). Hence, there exists $x \in X$ such that $f(x) \leq f(y)$. By the strict-$\alpha-$ invexity of the non-smooth function f with respect to α and η we have that

$$0 \geq f(x) - f(y) > \langle \xi \alpha(x,y), \eta(x,y) \rangle, \; \forall \xi \in \partial f(y).$$

That is to say, there exists $x \in X$ such that $\langle \alpha(x,y)\xi, \eta(x,y) \rangle < 0$, $\forall \xi \in \partial f(y)$. Therefore, y does not solve (WVVLIP). Then, by Theorem 8.3.4 we get a contradiction. Hence, y is an efficient solution to the (NVOP). □

The following definition is a simple extension of the concept of vectorial critical point given for the differentiable case given in Definition 8.1. (see Osuna et al. 1998) to the non-smooth case.

Definition 8.3.5. *A feasible solution $y \in X$ is said to be a vector critical point for (VOP) if there exists a vector $\lambda \in R^p$ with $\lambda \geq 0$ such that $\langle \lambda, \xi \rangle = 0$, $\forall \xi \in \partial f(y)$.*

Lemma 8.3.1. *Let $y \in X$ be a vector critical point for (NVOP), and let f is pseudo-$\alpha-$ invex on X with respect to α and η. Then, $y \in X$ is a weakly efficient solution to (NVOP).*

Proof. The proof is obvious using the pseudo-$\alpha-$ invexity of the nondifferentiable function f and the definition of vector critical point for (NVOP). □

Theorem 8.3.6. *All vector critical points are weakly efficient solutions to (NVOP) if and only if f is pseudo-$\alpha-$ invex on X.*

Proof. The proof follows the lines of the proof of Theorem 2.2 from Osuna et al. (1998) in light of the discussion above in this section. □

We can relate the vector critical points to the solutions of the weak vector variational-like inequality problems (WVVLIP) using Theorems 8.3.4 and 8.3.6.

Corollary 8.3.2. *If the objective function f is locally Lipschitz and pseudo-$\alpha-$ invex with respect to α and η, then the vector critical points, the weakly efficient points and the solutions of the weak vector variational-like inequality problem (WVVLIP) are equivalent.*

The results given in the present section can be extended to the class of nondifferentiable pseudo-univex functions. It will be interesting to extend these results for the class of non-smooth r-invex introduced by Antczak (2002b) and (p, r)-invex functions introduced by Antczak (2001).

8.4 Characterization of Generalized Univex functions

Karamardian and Schaible (1990) proved that generalized convexity of functions is equivalent to monotonicity of its gradient functions. It was pointed out that the role of generalized montonicity in variational inequality problems corresponds to the role of generalized convexity of objective functions in mathematical programing. Variational inequalities arise in models for a wide class of Mathematical, Physical, Economic and Social problems. Invexity was introduced as an extension of differentiable convex function by Hanson (1981). Hanson's work inspired a great deal of subsequent work that has greatly expanded the role and applications of invexity in nonlinear optimization and other branches of pure and applied sciences. Furthermore, Noor (1990, 1994, 1995) has studied the role of invex functions in variational-like inequality problems a natural extension of variational inequality problems. Noor (1994) has established that the minimum of invex functions on the invex sets in normed spaces can be characterized by variational-like (prevariational) inequality problems. Thus extended the idea of Karamardian and Schaible (1990) to invex functions and variational-like inequality problems, that is, the concept of invexity plays exactly the same role in variational-like inequality problems as the convexity plays in variational inequality problems. Yang et al. (2005) established relationships between the generalized monotonicities of gradient and generalized invexity of a function.

Invex functions and invex monotonicities are interesting topics in the study of generalized convexity. Generalized invexity and generalized monotonicities are studied in Ruiz-Garzon et al. (2003). Recently, Yang et al. (2005) established that (1) if the gradient of a function is (strictly) pseudo-monotone, then the function is (strictly) pseudo-invex; (2) if the gradient of a function is quasi-monotone, then the function is quasi-invex; and (3) if the gradient of a function is strong pseudo-monotone, then the function is strong pseudo-invex.

However, it is to be noticed that univex functions are more general than that of invex functions. In this section, we established that (1) if the gradient of a function is (strictly) pseudo-monotone, then the function is (strictly) pseudo-univex; (2) if the gradient of a function is quasi-monotone, then the function is quasi-univex; and (3) if the gradient of a function is strong pseudo-monotone, then the function is strong pseudo-univex. The results established in Yang et al. (2005) are particular case of the results obtained in this section.

Let X be a nonempty subset of R^n and $\eta\colon X \times X \to R^n$ be a continuous map.

The set X is said to be invex with respect to $\eta\colon X \times X \to R^n$, if X is invex at each point of the set X. The invex set X is also called η-connected set.

Definition 8.4.1. *(Bector et al. 1992). Let X be a nonempty open and invex subset of R^n and $f\colon X \subset R^n \to R^p$ be a differentiable function with a $p \times n$ matrix as its Jacobian. The function f is said to be*

(a) univex with respect to η, ϕ and k if and only if there exist a functions $\eta\colon X \times X \to R^n$, $\phi\colon R \to R$, and $k\colon X \times X \to R_+$, such that

$$k(y,x)\phi[f(y)-f(x)] \geqq \nabla f(x)\eta(y,x),\ \forall x,y \in X;$$

(b) strictly univex with respect to η, ϕ and k if and only if there exists a function $\eta\colon X\times X \to R^n$, $\phi\colon R\to R$, and $k\colon X\times X \to R_+$, such that

$$k(y,x)\phi[f(y)-f(x)] > \nabla f(x)\eta(y,x),\ \forall x,\ y \in X,\ x \neq y;$$

(c) pseudo-univex with respect to η, ϕ and k if and only if there exists a function $\eta\colon X\times X \to R^n$, $\phi\colon R\to R$, and $k\colon X\times X \to R_+$, such that

$$k(y,x)\phi[f(y)-f(x)] < 0 \Rightarrow \nabla f(x)\,\eta\,(y,x) < 0,\ \forall x,\ y \in X.$$

It was discussed in Sect. 8.2, that the class of univex functions are wider than that of the invex functions, and the class of pseudo-univex functions are even wider than that of univex functions.

Definition 8.4.2. *(Yang* et al. *2005). Let X be an invex set in R^n with respect to η: $X\times X \to R^n$. Then $F\colon X \to R^n$ is said to be (strictly) pseudo-invex monotone with respect to η on X if for every pair of distinct points $x,y \in X$,*

$$\eta\,(y,x)^T\,F\,(x) \geq 0 \Rightarrow \eta\,(y,x)^T\,F\,(x)\,(>) \geq 0.$$

Condition C. (Mohan and Neogy 1995). Let $\eta\colon X\times X \to R^n$. Then for any $x,y \in R^n$ for any $\lambda \in [0,1]$, $\eta\,(y, y+\eta\,(x,y)) = -\lambda\,\eta\,(x,y)$ and $\eta\,(x, y+\eta\,(x,y)) = (1-\lambda)\,\eta\,(x,y)$.

Remark 8.4.1. From Condition C, we have $\eta\,(y+\eta\,(x,y)\,,y) = \bar{\lambda}\,\eta\,(x,y)$.

It was noticed by Yang et al. (2005) that Condition C is different from that η is linear in first argument and skew-symmetric.

Example 8.4.1. Let

$$\eta\,(x,y) = \begin{cases} x-y & \text{if} \quad x\geq 0,\ y\geq 0; \\ x-y & \text{if} \quad x\leq 0,\ y\leq 0; \\ -2-y & \text{if} \quad x>0,\ y\leq 0; \\ 2-y & \text{if} \quad x\leq 0,\ y<0. \end{cases}$$

Then, it is obvious to see that η satisfies Condition C. However, η is neither linear in the first argument nor skew-symmetric.

Theorem 8.4.1. *Suppose that*

1. X is an open invex set of R^n with respect to η;
2. η satisfies Condition C;

1. for each $x \neq y$, $k\,(x,y)\,\phi\,[f\,(x)-f\,(y)] < 0$ implies

$$\eta^T\,(x,y)\,\nabla f\,\left(y+\bar{\lambda}\eta\,(x,y)\right) < 0 \textit{ for some } \bar{\lambda} \in (0,1) \textit{ and } k\,(y,x) > 0;$$

2. *∇f is pseudo-invex monotone with respect to η on X.*

Then f is a pseudo-univex function with respect to η, ϕ and k on X.

Proof. Let $x, y \in X$, $x \neq y$ be such that

$$\eta^T(x, y)\nabla f(y) \geq 0. \tag{8.4.1}$$

We need to show that $k(x, y)\phi[f(x) - f(y)] \geq 0$.

Assume the contrary, that is,

$$k(x, y)\phi[f(x) - f(y)] < 0. \tag{8.4.2}$$

By inequality (8.4.2) and hypotheses 3, we have

$$\eta^T(x, y)\nabla f\left(y + \bar{\lambda}\eta(x, y)\right) < 0, \text{ for some } \bar{\lambda} \in (0, 1). \tag{8.4.3}$$

From Condition C and (8.4.3), we get

$$\eta^T\left(y + \bar{\lambda}\eta(x, y), y\right)\nabla f\left(y + \bar{\lambda}\eta(x, y)\right) < 0, \text{ for some } \bar{\lambda} \in (0, 1). \tag{8.4.4}$$

Since ∇f is pseudo-invex monotone with respect to η on X, from (8.4.4), we get

$$\eta^T\left(y + \bar{\lambda}\eta(x, y), y\right)\nabla f(y) < 0, \text{ for some } \bar{\lambda} \in (0, 1). \tag{8.4.5}$$

Since $\bar{\lambda} \in (0, 1)$ from (8.4.4) and (8.4.5), we get

$$\eta(x, y)^T\nabla f(y) < 0,$$

which is a contradiction to (8.4.1). Hence, f is pseudo-univex with respect to η, ϕ, and k. □

Theorem 8.4.2. *Suppose that*

1. *X is an open invex set of R^n with respect to η;*
2. *η satisfies Condition C;*
3. *for each $x \neq y$, $k(x, y)\phi[f(x) - f(y)] \leq 0$ implies*

$$\eta^T(x, y)\nabla f\left(y + \bar{\lambda}\eta(x, y)\right) \leq 0 \text{ for some } \bar{\lambda} \in (0, 1) \text{ and } k(y, x) > 0;$$

4. *∇f is strictly pseudo-invex monotone with respect to η on X.*

Then f is a strictly pseudo-univex function with respect to η, ϕ and k on X.

Proof. Let $x, y \in X$, $x \neq y$ be such that

$$\eta^T(x, y)\nabla f(y) \geq 0. \tag{8.4.6}$$

We need to show that $k(x, y)\phi[f(x) - f(y)] > 0$.

Assume the contrary, that is,

$$k(x,y)\,\phi\,[f(x)-f(y)] \leq 0 \tag{8.4.7}$$

By inequality (8.4.7) and hypotheses 3, we have

$$\eta^T(x,y)\,\nabla f(y+\bar{\lambda}\eta(x,y)) \leq 0, \text{for some } \bar{\lambda} \in (0,1). \tag{8.4.8}$$

From Condition C, we know that

$$\eta^T(y,y+\bar{\lambda}\eta(x,y)) = -\bar{\lambda}\eta(x,y). \tag{8.4.9}$$

It follows from (8.4.8) and (8.4.9) that

$$\eta^T(y,y+\bar{\lambda}\eta(x,y))\nabla f(y+\bar{\lambda}\eta(x,y)) \geq 0, \text{ for some } \bar{\lambda} \in (0,1). \tag{8.4.10}$$

Since ∇f is strictly pseudo-invex monotone with respect to η on X, from (8.4.10), we get

$$\eta^T(y,y+\bar{\lambda}\,\eta(x,y))\nabla f(y) > 0. \tag{8.4.11}$$

Since $\bar{\lambda} \in (0,1)$ from Condition C and (8.4.11), we get

$$\eta(x,y)^T\,\nabla f(y) < 0,$$

which is a contradiction to (8.4.6). Hence, f is strictly pseudo-univex with respect to η, ϕ, and k. □

Now we establish relationship between quasi-univex functions and quasi-invex monotonicity. That is; if the gradient of a function is quasi-invex monotone, then the function is quasi-univex.

Definition 8.4.3. *Let X be an invex set in R^n with respect to $\eta: X \times X \to R^n$. Then $F: X \to R^n$ is said to be quasi-invex monotone with respect to η on X if for every pair of distinct points $x, y \in X$,*

$$\eta(y,x)^T F(x) > 0 \Rightarrow \eta(y,x)^T F(x) \geq 0.$$

Definition 8.4.4. *(Bector et al. 1992). Let X be a nonempty open and invex subset of R^n and $f: X \subset R^n \to R^p$ be a differentiable function with a $p \times n$ matrix as its Jacobian. The function f is said to be quasi-univex with respect to η, ϕ* and *k if there exist functions $\eta: X \times X \to R^n$, $\phi: R \to R$, and $k: X \times X \to R_+$, such that*

$$k(y,x)\phi\,[f(y)-f(x)] \leq 0 \Rightarrow \eta(y,x)^T\nabla f(x) \leq 0,\ \forall x,y \in X.$$

Theorem 8.4.3. *Suppose that*

1. *X is an open invex set of R^n with respect to η;*
2. *η satisfies Condition C;*
3. *for each $x \neq y$, $k(x,y)\,\phi[f(x)-f(y)] \leq 0$ implies*

$$\eta(x,y)^T\nabla f(y+\bar{\lambda}\,\eta(x,y)) < 0 \textit{ for some } \bar{\lambda} \in (0,1);$$

4. ∇f *is quasi-invex monotone with respect to* η *on* X.

Then f *is quasi-univex function with respect to* η, ϕ *and* k *on* X.

Proof. Assume that f is not quasi-univex with respect to the same η, ϕ and k. Then, there exist $x, y \in X$, such that

$$k\,(x, y)\,\phi\,[f\,(x) - f\,(y)] \leq 0, \tag{8.4.12}$$

but,

$$\eta\,(x, y)^T\,\nabla f\,(y) > 0. \tag{8.4.13}$$

By inequality (8.4.12) and hypotheses 3, we have

$$\eta\,(x, y)^T\,\nabla f\,\left(y + \bar{\lambda}\,\eta\,(x, y)\right) < 0, \text{ for some } \bar{\lambda} \in (0, 1)\,. \tag{8.4.14}$$

From Condition C, we know that

$$\eta^T\,\left(yy + \bar{\lambda}\,\eta\,(x, y)\right) = -\bar{\lambda}\,\eta\,(x, y)\,. \tag{8.4.15}$$

it follows from (8.4.14) and (8.4.15) that

$$\eta^T\,\left(y, y + \bar{\lambda}\,\eta\,(x, y)\right)\nabla f\,\left(y + \bar{\lambda}\,\eta\,(x, y)\right) > 0 \text{ for some } \bar{\lambda} \in (0, 1)\,. \tag{8.4.16}$$

Since ∇f is quasi-invex monotone with respect to η on X, from (8.4.16), we get

$$\eta^T\,\left(y, y + \bar{\lambda}\,\eta\,(x, y)\right)\nabla f\,(y) \geq 0\,. \tag{8.4.17}$$

Since $\bar{\lambda} \in (0, 1)$ from Condition C and (8.4.17), we get

$$\eta\,(x, y)^T\,\nabla f\,(y) \leq 0,$$

which is a contradiction to (8.4.13). Hence, f is quasi-univex with respect to η, ϕ and k.

Finally, we establish relationships between strong pseudo-invex monotonicity and strong pseudo-univexity for differentiable functions.

Definition 8.4.5. *Let* X *be an invex set in* R^n *with respect to* $\eta\colon X \times X \to R^n$. *Then* $F\colon X \to R^n$ *is said to be strong pseudo-invex monotone with respect to* η *on* X *if there exists a scalar* $\beta > 0$, *such that for every pair of distinct points* $x, y \in X$,

$$\eta\,(y, x)^T\,F\,(x) \geq 0 \Rightarrow \eta\,(y, x)^T\,F\,(y) \geq \beta\,\|\eta\,(y, x)\|\,.$$

Definition 8.4.6. *The non-differentiable function* $f\colon X \to R$ *is strong pseudo-univex with respect to* η, ϕ *and* k *if and only if there exists a function* $\eta\colon X \times X \to R^n$, $\phi\colon R \to R$, *and* $k\colon X \times X \to R_+$, *and a scalar* $\alpha > 0$ *such that for every pair of distinct points* $x, u \in X$, *such that*

$$\eta\,(x, u)^T\,\nabla f\,(u) \geq 0 \Rightarrow k\,(x, u)\,\phi\,[f\,(x) - f\,(u)] \geq \alpha\,\|\eta\,(x, u)\|\,.$$

Theorem 8.4.4. *Suppose that*

1. *X is an open invex set of R^n with respect to η;*
2. *η satisfies Condition C;*
3. *$f(x+\eta(y,x)) \le f(y)$, and $f(y) - f(x) \ge \beta\|\eta(y,x)\| \Rightarrow \phi[f(y)-f(x)] \ge \beta\|\eta(y,x)\|$*

$$\forall x, y \in X;$$

4. *∇f is continuous strong pseudo-invex monotone with respect to η on X.*

Then f is strong pseudo-univex function with respect to η, ϕ and k on X.

Proof. Let $x, y \in X$ be such that

$$\eta(y,x)^T \nabla f(x) \ge 0.$$

By Condition C and hypotheses 1, we have

$$\eta(x+\lambda\eta(y,x), x)^T \nabla f(x) \ge 0.$$

By the strong pseudo-invex monotonicity of ∇f with respect to η, there exists a scalar $\beta > 0$

$$\eta(x+\lambda\eta(y,x), x)^T \nabla f(x+\lambda\eta(y,x)) \ge \beta\|\eta x+\lambda\eta(y,x)\|.$$

Again from Condition C and $\lambda \in (0, 1]$

$$\eta(y,x)^T \nabla f(x+\lambda\eta(y,x)) \ge \beta\|\eta(y,x)\|. \tag{8.4.18}$$

Let $g(\lambda) = f(x+\lambda\eta(y,x))$ It follows from (8.4.18) that

$$g'(\lambda) \ge \beta\|\eta(y,x)\|, \forall\lambda \in (0,1].$$

Integrating the last expression between 0 and 1, we get

$$g(1) - g(0) \ge \beta\|\eta(y,x)\|.$$

That is,

$$f(x+\eta(y,x)) - f(x) \ge \beta\|\eta(y,x)\|.$$

By hypotheses 3, and since $k(y,x) > 0$, we get

$$k(y,x)\phi[f(y)-f(x)] \ge \beta\|\eta(y,x)\|.$$

Thus, f is strong pseudo-univex function with respect to η, ϕ and k on X. □

In this section, we have established some characterizations of pseudo-univex functions, strictly pseudo-univex functions, quasi-univex functions and strongly pseudo-univex functions and hence established some new relationships between generalized invex-monotonicity and generalized univexity of differentiable functions. These results can be extended to the non-differentiable case and hence will extend the results of Fan et al. (2003) as a by product.

8.5 Characterization of Nondifferentiable Generalized Invex Functions

Yang et al. (2005) established relationships between the generalized monotonicities of gradient and generalized invexity of a function. Fan et al. (2003) established relationships between convexity of non-differentiable functions and monotonicity of set-valued mappings. However, a very little has been done on the relationships of generalized invexity of non-differentiable functions and generalized invex monotonicity of set-valued mappings.

In this section, we establish under some appropriate conditions that (1) if the Clarke's sub-differential of a function is (strictly) pseudo-monotone, then the function is (strictly) pseudo-invex; (2) if the Clarke's sub-differential of a function is quasi-monotone, then the function is quasi-invex; and (3) if the Clarke's sub-differential of a function is quasi-monotone then the function is quasi-invex. The results given in Ruiz-Garzion et al. (2003) and Yang et al. (2005) are special cases of the results established in this section.

Let R^n be the $n-$ dimensional Euclidean space and R^n_+ be its non-negative octant. In the sequel X be a non-empty open subset of R^n. Definitions 8.3.1–8.3.3 will be needed in this section as well.

Definition 8.5.1. *Let $u \in X$, the set X is said to be invex at u with respect to $\eta\colon X \times X \to R^n$ if, for all $x, u \in X$, $t \in [0, 1]$, $u + t\eta(x,u) \in X$*

Definition 8.5.2. *The non-differentiable function $f\colon X \to R$ is invex with respect to $\eta\colon X \times X \to R^n$ if*

$$f(x) - f(u) \geqq \xi^T \eta(x,u), \quad \forall \xi \in \partial f(u), \forall x, \in X.$$

Definition 8.5.3. *The non-differentiable function $f\colon X \to R$ is strictly-invex with respect to $\eta\colon X \times X \to R^n$ if*

$$f(x) - f(u) > \xi^T \eta(x,u), \quad \forall \xi \in \partial f(u), \forall x \vee u \in X.$$

Definition 8.5.4. *The non-differentiable function $f\colon X \to R$ is pseudo-invex with respect to $\eta\colon X \times X \to R^n$ if*

$$f(x) - f(u) < 0 \Rightarrow \xi^T \eta(x,u) < 0, \quad \forall \xi \in \partial f(u), \forall x u \in X.$$

Definition 8.5.5. *Let X be an invex set in R^n with respect to $\eta\colon X \times X \to R^n$. Then $F\colon X \to R^n$ is said to be (strictly) pseudo-invex monotone with respect to η on X if for every pair of distinct points $x, y \in X$,*

$$\langle u, \eta(y,x)\rangle \geq 0 \Rightarrow \langle v, \eta(y,x)\rangle\,(>) \geq 0, \forall u \in F(x) \text{ and} v \in F(y).$$

Definition 8.5.6. *The non-differentiable function f: $X \to R$ is strictly pseudo-invex with respect to η: $X \times X \to R^n$ if*

$$\xi^T \eta(x,u) \geq 0 \Rightarrow f(x) > f(u), \quad \forall \xi \in \partial f(u), \forall x, u \in X.$$

Condition C. (Mohan and Neogy 1995). Let η: $X \times X \to R^n$. Then for any $x, y \in R^n$ for any $\lambda \in [0, 1]$, $\eta(yy + \eta(x,y)) = -\lambda\,\eta(x,y)$ and $\eta(x, y + \eta(x,y)) = (1-\lambda)\,\eta(x,y)$.

Remark 8.5.1. From Condition C, we have $\eta(y + \eta(x,y), y) = \bar{\lambda}\,\eta(x,y)$.

It was noticed by Yang et al. (2005) that Condition C is different from that η is linear in first argument and skew-symmetric for example see Example 8.4.1 in the previous section.

Theorem 8.5.1. *Suppose that*

1. *X is an open invex set of R^n with respect to η;*
2. *η satisfies Condition C;*
3. *for each $x \neq y$, $f(y) > f(x)$ implies*

$$\eta^T(x,y) f^0\left(y + \bar{\lambda}\,\eta(x,y)\right) < 0 \textit{ for some } \bar{\lambda} \in (0,1);$$

4. *the set-valued map ∂f is pseudo-invex monotone with respect to η on X.*

Then f is a pseudo-invex function with respect to η on X.

Proof. Let $x, y \in X$, $x \neq y$ be such that

$$\eta^T(x,y) f^0(y) \geq 0. \tag{8.5.1}$$

We need to show that $f(x) \geq f(y)$.

Assume the contrary, that is,

$$f(x) \geq, f(y). \tag{8.5.2}$$

By inequality (8.5.2) and hypotheses 3, we have

$$\eta^T(x,y) f^0\left(y + \bar{\lambda}\eta(x,y)\right) < 0, \text{ for some } \bar{\lambda} \in (0,1). \tag{8.5.3}$$

From Condition C and (8.5.3), we get

$$\eta^T\left(y + \bar{\lambda}\eta(x,y), y\right) f^0\left(y + \bar{\lambda}\,\eta(x,y)\right) < 0, \text{ for some } \bar{\lambda} \in (0,1). \tag{8.5.4}$$

Since ∂f is pseudo-invex monotone with respect to η on X, from (8.5.4), we get

$$\eta^T\left(y + \bar{\lambda}\eta(x,y), y\right) f^0(y) < 0 \text{ for some } \bar{\lambda} \in (0,1). \tag{8.5.5}$$

Since $\bar{\lambda} \in (0, 1)$ from (8.5.4) and (8.5.5), we get

$$\eta^T (x, y)\, \xi < 0, \forall \xi \in \partial f (y),$$

which is a contradiction to (8.5.1). Hence, f is pseudo-invex with respect to η. □

Theorem 8.5.2. *Suppose that*

1. *X is an open invex set of R^n with respect to η;*
2. *η satisfies Condition C;*
3. *for each $x \neq y$, $f (y) \geq f (x)$ implies*

$$\eta^T (x, y)\, f^0 \left(y + \bar{\lambda}\, \eta (x, y)\right) \leq 0 \text{ for some } \bar{\lambda} \in (0, 1);$$

4. *the set-valued map ∂f is strictly pseudo-invex monotone with respect to η on X.*

Then f is strictly pseudo-invex function with respect to η on X.

Proof. Let $x, y \in X$, $x \neq y$ be such that

$$\eta^T (x, y)\, f^0 (y) \geq 0. \tag{8.5.6}$$

We need to show that $f (x) > f (y)$.

Assume the contrary, that is,

$$f (x) \leq f (y). \tag{8.5.7}$$

By inequality (8.5.7) and hypotheses 3, we have

$$\eta^T (x, y)\, f^0 \left(y + \bar{\lambda}\, \eta (x, y)\right) \leq 0, \text{ for some } \bar{\lambda} \in (0, 1). \tag{8.5.8}$$

From Condition C, we know that

$$\eta^T \left(y, y + \bar{\lambda} \eta (x, y)\right) = -\bar{\lambda}\, \eta (x, y). \tag{8.5.9}$$

It follows from (8.5.8) and (8.5.9) that

$$\eta^T \left(y, y + \bar{\lambda}\, \eta (x, y)\right) f^0 \left(y + \bar{\lambda}\, \eta (x, y)\right) \geq 0 \text{ for some } \bar{\lambda} \in (0, 1). \tag{8.5.10}$$

Since ∂f is strictly pseudo-invex monotone with respect to η on X, from (8.5.10), we get

$$\eta^T \left(y, y + \bar{\lambda}\, \eta (x, y)\right) f^0 (y) > 0. \tag{8.5.11}$$

Since $\bar{\lambda} \in (0, 1)$ from Condition C and (8.5.11), we get

$$\eta^T (x, y)\, \xi < 0\, \forall\, \xi \in \partial f (y),$$

which is a contradiction to (8.5.6). Hence, f is strictly pseudo-invex with respect to η. □

Definition 8.5.7. *Let X be an invex set in R^n with respect to $\eta: X \times X \to R^n$. Then $F: X \to R^n$ is said to be quasi-invex monotone with respect to η on X if for every pair of distinct points $x, y \in X$,*

$$\langle u, \eta(y,x)\rangle > 0 \Rightarrow \langle v, \eta(y,x)\rangle \geq 0, \forall u \in F(x) \text{ and } v \in F(y).$$

Definition 8.5.8. *The non-differentiable function $f: X \to R$ is quasi-invex with respect to $\eta: X \times X \to R^n$ if*

$$f(x) \leq f(u) \Rightarrow \xi^T \eta(x,u) \leq 0, \quad \forall \xi \in \partial f(u), \forall x, u \in X.$$

Theorem 8.5.3. *Suppose that*

1. *X is an open invex set of R^n with respect to η;*
2. *η satisfies Condition C;*
3. *for each $x \neq y$, $f(y) \geq f(x)$ implies*

$$\eta^T(x,y) f^0\left(y + \bar{\lambda}\eta(x,y)\right) < 0 \textit{ for some } \bar{\lambda} \in (0,1);$$

4. *the set-valued map ∂f is quasi-invex monotone with respect to η on X.*

Then f is quasi-invex function with respect to η on X.

Proof. Assume that f is not quasi-invex with respect to the same η. Then, there exist $x, y \in X$ such that

$$f(y) \leq f(x) \tag{8.5.12}$$

but,

$$\eta^T(x,y) f^0(y) \geq 0. \tag{8.5.13}$$

By inequality (8.5.13) and hypotheses 3, we have

$$\eta^T(x,y) f^0\left(y + \bar{\lambda}\eta(x,y)\right) < 0, \text{ for some } \bar{\lambda} \in (0,1). \tag{8.5.14}$$

From Condition C, we know that

$$\eta^T\left(y, y + \bar{\lambda}\eta(x,y)\right) = -\bar{\lambda}\eta(x,y). \tag{8.5.15}$$

It follows from (8.5.14) and (8.5.15) that

$$\eta^T\left(y, y + \bar{\lambda}\eta(x,y)\right) f^0\left(y + \bar{\lambda}\eta(x,y)\right) > 0, \text{ for some } \bar{\lambda} \in (0,1). \tag{8.5.16}$$

Since ∂f is quasi-invex monotone with respect to η on X, from (8.5.16), we get

$$\eta^T\left(y, y + \bar{\lambda}\eta(x,y)\right) f^0(y) \geq 0. \tag{8.5.17}$$

Since $\bar{\lambda} \in (0, 1)$ from Condition C and (8.5.17), we get

$$\eta^T(x, y)\,\xi \leq 0, \forall\, \xi \in \partial f(y),$$

which is a contradiction to (8.5.13). Hence, f is quasi-invex with respect to η. □

We can extend the results of Sect. 8.4 to the nondifferentiable case as well. Further, it will be interesting to extend the results of this chapter to the class of generalized invex functions given by Fan (2007).

References

Abrams, R. A. (1972). Nonlinear programming in complex space: sufficient conditions and duality, *Journal of Mathematical Analysis and Applications* 38, 619–632.

Abrams, R. A. and A. Ben-Israel (1969). A duality theorem for complex quadratic programming, *Journal of Optimization Theory and Applications* 4, 244–252.

Abrams, R. A. and A. Ben-Israel (1971). Nonlinear programming in complex space: necessary conditions, *SIAM Journal on Control* 9, 606–620.

Aghezzaf, B. and M. Hachimi (2000). Generalized invexity and duality in multiobjective programming problems, *Journal of Global Optimization* 18, 91–101.

Aghezzaf, B. and M. Hachimi (2001). Sufficient optimality conditions and duality in multiobjective optimization involving generalized convexity, *Numerical Functional Analysis and Optimization* 22, 775–788.

Antczak, T. (2001). (p, r)-invex sets and functions, *Journal of Mathematical Analysis and Applications* 263, 355–379.

Antczak, T. (2002a). Multiobjective programming under d-invexity, *European Journal of Operational Research* 137, 28–36.

Antczak, T. (2002b). Lipschitz r-invex functions and nonsmooth programming, *Numerical Functional Analysis and Optimization* 23, 265–283.

Arana-Jimenez, M., R. Osuna-Gomez, A. Rufian-Lizana, and G. Ruiz-Garzon (2008). KT-invex control problem, *Applied Mathematics and Computation* 197, 489–496.

Arrow, K. J. and A. C. Enthoven (1961). Quasiconcave programming, *Econometrica* 29, 779–800.

Arrow, K. J. and M. D. Intriligator (1981). *Handbook of Mathematical Economics*, Vol. 1, North Holland, Amsterdam.

Aubin, J. P. (1993). *Optima and Equilibria – An Introduction to Nonlinear Analysis*, Springer-Verlag, Berlin.

Avriel, M., W. E. Diewert, S. Schaible, and I. Zang (1988). *Generalized Concavity*, Mathematical Concepts and Methods in Science and Engineering, Vol. 36, Plenum Press, New York.

Balas, E. (1991). Minimax and duality for linear and nonlinear mixed-integer programming. In: J. Abadie (Ed.), *Integer and Nonlinear Programming*, North-Holland, Amsterdam.

Bazaraa, M. S. and J. J. Goode (1973). On symmetric duality in nonlinear programming, *Operations Research* 21, 1–9.

Bazaraa, M. S., H. D. Sherali, and C. M. Shetty (1991). *Nonlinear Programming: Theory and Algorithms*, Wiley, New York.

Bector, C. R. and B. L. Bhatia (1985). Sufficient optimality and duality for a minimax problem, *Utilitas Mathematica* 27, 229–247.

Bector, C. R. and S. Chandra (1997). Second order duality with nondifferentiable functions, Working Paper, Department of Mathematics, Indian Institute of Technology, New Delhi.

Bector, C. R. and I. Husain (1992). Duality for multiobjective variational problems, *Journal of Mathematical Analysis and Applications* 166, 214–229.

Bector, C. R., S. Chandra, and I. Husain (1991). Second order duality for a minimax programming problem, *Opsearch* 28, 249–263.

Bector, C. R., S. K. Suneja, and S. Gupta (1992). Univex functions and univex nonlinear programming. In: *Proceedings of the Administrative Sciences Association of Canada*, 115–124.

Bector, C. R., S. Chandra, and V. Kumar (1994a). Duality for minimax programming involving V-invex functions, *Optimization* 30, 93–103.

Bector, C. R., S. Chandra, S. Gupta, and S. K. Suneja (1994b). Univex sets, functions and univex nonlinear programming. In: Generalized Convexity (Pecs 1992) 3–18, Lecture Notes in Economics and Mathematical Systems, 405, Springer, Berlin, 1–18.

Bector C. R., S. Chandra, and Abha (1999). On mixed symmetric duality in multiobjective programming, *Opsearch* 36, 399–407.

Ben-Israel, B. and B. Mond (1986). What is invexity, *Journal of Australian Mathematical Society* 28 B, 1–9.

Bhatia, D. and P. Kumar (1995). Multiobjective control problems with generalized invexity, *Journal of Mathematical Analysis and Applications* 189, 676–692.

Bhatia, D. and A. Mehra (1999). Optimality conditions and duality for multiobjective variational problems with generalized b-invexity, *Journal of Mathematical Analysis and Applications* 234, 234–360.

Bitran, G. (1981). Duality in nonlinear multiple criteria optimization problems, *Journal of Optimization Theory and Applications* 35, 367–406.

Brandao, A. J. V., M. A. Rojas-Medar, and G. N. Silva (1999). Optimality conditions for Pareto nonsmooth nonconvex programming in Banach spaces, *Journal of Optimization Theory and Applications* 103, 65–73.

Chandra, S. and V. Kumar (1995). Duality in fractional minimax programming, *Journal of the Australian Mathematical Society, Series* A 58, 376–386.

Chandra, S., B. D. Craven, and B. Mond (1985). Symmetric dual fractional programming, *Zeitschrift fur Operations Research* 29, 59–64.

Chandra, S., I. Husain, and Abha (1999). On mixed symmetric duality in mathematical programming, *Opsearch* 36, 165–171.

Chankong, V. and Y. Y. Haimes (1983). *Multiobjective Decision Making: Theory and Methodology*, North-Holland, New York.

Chen, X. (2004). Higher-order symmetric duality in non-differentiable multi-objective programming problems, *Journal of Mathematical Analysis and Applications* 290, 423–435.

Chen, G. Y. and G. M. Cheng (1998). Vector variational inequality and vector optimization. In: *Lecture Notes in Economics and Mathematical Systems*, Vol. 285, Springer, Berlin, 408–416.

Chen, G. Y. and B. D. Craven (1994). Existence and continuity of solutions for vector optimization, *Journal of Optimization Theory and Applications* 81, 459–468.

Chen, X. H. (1996). Duality for multiobjective variational problems with invexity, *Journal of Mathematical Analysis and Applications* 203, 236–253.

Chen, X. H. (2000). Symmetric duality for the multiobjective fractional variational problem with partial-invexity, *Journal of Mathematical Analysis and Applications* 245, 105–123.

Chen, X. H. (2002). Duality for a class of multiobjective control problems, *Journal of Mathematical Analysis and Applications* 267, 377–394.

Chew, K. L. and E. U. Choo (1984). Pseudolinearity and efficiency, *Mathematical Programming* 28, 226–239.

Chou, J. H., W. S. Hsia, and T. Y. Lee (1985). On multiple objective programming problems with set functions, *Journal of Mathematical Analysis and Applications* 105, 383–394.

Chou, J. H., W. S. Hsia, and T. Y. Lee (1986). Epigraphs of convex set functions, *Journal of Mathematical Analysis and Applications* 118, 247 – 254.

Clarke, F. H. (1983). *Optimization and Nonsmooth Analysis*, Wiley, New York.

Coladas, L., Z. Li, and S. Wang (1994). Optimality conditions for multiobjective and nonsmooth minimization in abstract spaces, *Bulletin of the Australian Mathematical Society* 50, 205–218.

Corley, H. W. (1987). Optimization theory for n-set functions, *Journal of Mathematical Analysis and Applications* 127, 193–205.

Cottle, R. W. (1963). Symmetric dual quadratic programs, *Quarterly of Appllied Mathematics* 21, 237–243.

Craven, B. D. (1978). *Mathematical Programming and Control Theory*, Chapman and Hall, London.

Craven, B. D. (1981). Invex functions and constrained local minima, *Bulletin of the Australian Mathematical Society* 24, 357–366.

Craven, B. D. (1989). Nonsmooth multiobjective programming, *Numerical Functional Analysis and Optimization* 10, 49–64.

Craven, B. D. (1993). On continuous programming with generalized convexity, *Asia Pacific Journal Operational Research* 10, 219–232.

Craven, B. D. (1995). *Control and Optimization*, Chapman and Hall, New York.

Dafermos, S. (1990). Exchange price equilibrium and variational inequalities, *Mathematical Programming* 46, 391–402.

Dantzig, G. B., E. Esenberg, and R. W. Cottle (1965). Symmetric dual nonlinear programs, *Pacific Journal of Mathematics* 15, 809–812.

De Finetti, B. (1949). Sulle stratification converse, *Annali di Mathematics Pura ed Applicata* 30, 173–183.

Devi, G. (1998). Symmetric duality for nonlinear programming problem involving $\eta-$convex functions, *European Journal of Operational Research* 104, 615–621.

Diewert, W. E., M. Avriel, and I. Zang (1981). Nine kinds of quasi-concavity and concavity, *Journal of Economic Theory* 25, 397–420.

Dinkelbach, W. (1967). On nonlinear fractional programming, *Management Science* 13, 492–498.

Dorn, W. S. (1960). A symmetric dual nonlinear program, *Journal of the Operations Research Society of Japan* 2, 93–97.

Egudo, R. R. (1989). Efficiency and generalized convex duality for multiobjective programs, *Journal of Mathematical Analysis and Applications* 138, 84–94.

Egudo, R. R. and M. A. Hanson (1987). Multi-objective duality with invexity, *Journal of Mathematical Analysis and Applications* 126, 469–477.

Egudo, R. R. and M. A. Hanson (1993). Second order duality in multiobjective programming, *Opsearch* 30, 223–230.

El Abdouni, B. and L. Thibault (1992). Lagrange multipliers for Pareto nonsmooth programming problems in Banach spaces, *Optimization* 26, 277–285.

Elster, K.H. and R. Nehse (1980). *Optimality Conditions for Some Nonconvex Problems*, Springer-Verlag, New York.

Ewing, G. M. (1977). Sufficient conditions for global minima of suitable convex functionals from variational and control theory, *SIAM Review* 19/2, 202–220.

Fan, L. (2007). Generalized invexity of nonsmooth functions, *Nonlinear Analysis*, doi:10.1016/j.na.2007.10.047

Fan, L., S. Liu, and S. Gao (2003). Generalized monotonicity and generalized convexity of non-differentiable functions, *Journal of Mathematical Analysis and Applications* 279, 276–289.

Fang, Y. P. and N. J. Huang (2003). Variational-like inequalities with generalized monotone mappings in Banach spaces, *Journal of Optimization Theory and Applications* 118, 327–338.

Fenchel, W. (1953). *Convex Cones, Sets and Functions*, Lecture Notes, Princeton University, Armed Services Technical Information Agency, AD Number 22695.

Ferrero, O. (1992). On nonlinear programming in complex space, *Journal of Mathematical Analysis and Applications* 164, 399–416.

Gale D., H. W. Tucker, and A. W. Kuhn (1951). Linear programming and the theory of games. In: T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, Wiley, New York, 317–329.

Gianessi, F. (1980). Theorems of alternative, quadratic programs and complementarity problems. In: R. W. Cottle, F. Giannessi, and J. -L. Lions (eds.), *Variational Inequality and Complementarity Problems*, Wiley, New York, 51–186.

Gianessi, F. (2000). *Vector Variational Inequalities and Vector Equilibria: Mathematical Theories*, Kluwer, London.

Giannessi, F., A. Maugeri, and P. M. Pardalos (2001). *Equilibrium Problems: Nonsmooth Optimization and Variational inequality Models*, Kluwer, Dordrecht.

Giorgi, G. and A. Guerraggio (1996). Various types of non-smooth invex functions, *Journal of Information and Optimization Science* 17, 137–150.

Giorgi, G., A. Guerraggio, and J. Thierfelder (2004). *Mathematics of Optimization: Smooth and Nonsmooth Case*, Elsevier Science B. V., Amsterdam.

Guerraggio, A. and E. Molho (2004). The origin of quasi-concavity: a development between mathematics and economics, *Historia Mathematica* 31, 62–75.

Gulati, T. R. and I. Ahmad (1997). Second order symmetric duality for nonlinear minimax mixed integer programs, *European Journal of Operational Research* 101, 122–129.

Hachimi, M. and B. Aghezzaf (2004). Sufficiency and duality in differentiable multiobjective programming involving generalized type I functions, *Journal of Mathematical Analysis and Applications* 296, 382–392.

Hanson, M. A. (1964). Bounds for functionally convex optimal control problems, *Journal of Mathematical Analysis and Applications* 8, 84–89.

Hanson, M. A. (1981). On sufficiency of the Kuhn–Tucker conditions, *Journal of Mathematical Analysis and Applications* 80, 545–550.

Hanson, M. A. (1993). Second order invexity and duality in mathematical programming, *Opsearch* 30, 313–320.

Hanson, M. A. and B. Mond (1967). Duality for nonlinear programming in complex space, *Journal of Mathematical Analysis and Applications* 28, 52–58.

Hanson, M. A. and B. Mond (1982). Further generalization of convexity in mathematical programming, *Journal of Information and Optimization Science* 3, 25–32.

Hanson, M. A. and B. Mond (1987a). Necessary and sufficient conditions in constrained optimization, *Mathematical Programming* 37, 51–58.

Hanson, M. A. and B. Mond (1987b). Convex transformable programming problems and invexity, *Journal of Information and Optimization Sciences* 8, 201–207.

Hanson, M. A., R. Pini, and C. Singh (2001). Multiobjective programming under generalized type I invexity, *Journal of Mathematical Analysis and Applications* 261, 562–577.

Hayashi, M. and H. Komiya (1982). Perfect duality for convexlike programs, *Journal of Optimization Theory and Applications* 38, 179–189.

Holder, O. (1889). Uber einen Mittlwertsatz, Nachr. Ges. Wiss. Goettingen.

Hou, S. H. and X. M. Yang (2001). On second order symmetric duality in non-differentiable programming, *Journal of Mathematical Analysis and Applications* 255, 491–498.

Hsia, W. S. and T. Y. Lee (1987). Proper D solution of multiobjective programming problems with set functions, *Journal of Optimization Theory and Applications* 53, 247–258.

Islam, S. M. N. and B. D. Craven (2005). Some extensions of nonconvex economic modeling: invexity, quasimax and new stability conditions, *Journal of Optimization Theory and Applications* 125, 315–330.

Ivanov, E. H. and R. Nehse (1985). Some results on dual vector optimization problems, *Optimization* 16, 505–517.

Jensen, J. L. W. (1906). Sur les functions convexes et les inegalites entre les valeurs moyennes, *Acta Mathematica* 30, 175–193.

Jeyakumar, V. and B. Mond (1992). On generalized convex mathematical programming, *Journal of Australian Mathematical Society Ser.* B 34, 43–53.

Jeyakumar, V. and X. Q. Yang (1995). On characterizing the solution sets of pseudolinear programs, *Journal of Optimization Theory and Applications* 87, 747–755.

John F. (1948). Extremum problems with inequalities as subsidiary conditions. In: *Studies and Essays*, Inter-science, New York, 187–204.

Karamardian, S. (1967). Duality in mathematical programming, *Journal of Mathematical Analysis and Applications* 20, 344–358.

Karamardian, S. and S. Schaible (1990). Seven kinds of monotone maps, *Journal of Optimization Theory and Applications* 66, 37–46.

Karush W. (1939). *Minima of Functions of several Variables with Inequalities as Side Conditions*, M.Sc. Thesis, Department of Mathematics, University of Chicago.

Kaul, R. N. and S. Kaur (1982a). Generalization of convex and related functions, *European Journal of Operational Research* 9, 369–377.

Kaul, R. N. and S. Kaur (1982b). Sufficient optimality conditions using generalized convex functions, *Opsearch* 19, 212–224.

Kaul, R. N. and S. Kaur (1985). Optimality criteria in nonlinear programming involving nonconvex functions, *Journal of Mathematical Analysis and Applications* 105, 104–112.

Kaul, R. N., S. K. Suneja, and C. S. Lalitha (1993). Duality in pseudolinear multiobjective fractional programming, *Indian Journal of Pure and Applied Mathematics* 24, 279–290.

Kaul, R. N., S. K. Suneja, and M. K. Srivastava (1994). Optimality criteria and duality in multiple objective optimization involving generalized invexity, *Journal of Optimization Theory and Applications* 80, 465–482.

Kazmi, K. R. (1996). Existence of solutions for vector optimization, *Applied Mathematics Letters* 9, 19–22.

Kim, D. S. (2006). Nonsmooth multiobjective fractional programming with generalized invexity, *Taiwanese Journal of Mathematics* 10, 467–478.

Kim, D. S. and A. L. Kim (2002). Optimality and duality for nondifferentiable multiobjective variational problems, *Journal of Mathematical Analysis and Applications* 274, 255–278.

Kim, D. S. and W. J. Lee (1998a). Symmetric duality for multiobjective variational problems with invexity, *Journal of Mathematical Analysis and Applications* 218, 34–48.

Kim, M. H. and G. M. Lee (2001). On duality theorems for nonsmooth Lipschitz optimization problems, *Journal of Optimization Theory and Applications* 110, 669–675.

Kim, D. S., G. M. Lee, J. Y. Park, and K. H. Son (1993). Control problems with generalized invexity, *Math. Japon*. 38, 263–269.

Kim, D.S., Y. B. Yun, and W. J. Lee (1998a). Multi-objective symmetric duality with cone constraints, *European Journal of Operational Research* 107, 686–691.

Kim, D. S., C. L. Jo, and G. M. Lee (1998b). Optimality and duality for multiobjective fractional programming involving n-set functions, *Journal of Mathematical Analysis and Applications* 224, 1–13.

Kim, D. S., W. J. Lee, and S. Schaible (2004). Symmetric duality for invex multiobjective fractional variational problems, *Journal of Mathematical Analysis and Applications* 289, 505–521.

Kinderlehrer, D. and G. Stampacchia (1980). *An Introduction to Variational Inequalities and Their Applications*, Academic Press, London.

Komlosi, S. (1993). First and second order characterizations of pseudolinear functions, *European Journal of Operational Research* 67, 278–286.

Kortanek, K. O. and J. P. Evans (1967). Pseudoconcave programming and Lagrange regularity, *Operations Research* 15, 882–891.

Kuhn H. W. (1976). Nonlinear programming: a historical view, In: R.W. Cottle, C. E. Lemke (eds.) *Nonlinear Programming*, SIAM-AMS Proceedings 9, 1–26.

Kuhn H. W. and A. W. Tucker (1951). Nonlinear programming, In: J. Neyman (ed.): *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, Berkeley, 481–492.

Lai, H. C. and J. C. Lee (2002). On duality theorems for a nondifferentiable minimax fractional programming, *Journal of Computational and Applied Mathematics* 146, 115–126.

Lai, H. C. and L. J. Lin (1989). Optimality for set functions with values in ordered vector spaces, *Journal of Optimization Theory and Applications* 63, 371–389.

Lai, H. C. and J. C. Liu (2002). Complex fractional programming involving generalized quasi/pseudo convex functions, *Zeitschrift fur Angewandte Mathematik und Mechanik (ZAMM)* 82, 159–166.

Lai, H. C., J. C. Liu, and K. Tanaka (1999). Necessary and sufficient conditions for minimax fractional programming, *Journal of Mathematical Analysis and Applications* 230, 311–328.

Lai, H. C., J. C. Lee, and J. C. Liu (2001). Duality for fractional complex programming with generalized convexity, *Journal of Nonlinear and Convex Analysis* 2, 175–191.
Ledzewicz-Kowalwski, U. (1985). A necessary condition for a problem of optimal control with equality and inequality constraints, *Control and Cybernetics* 14, 351–360.
Lee, G. M. (1994). Nonsmooth invexity in multiobjective programming, *Journal of Information and Optimization Sciences* 15, 127–136.
Levinson, N. (1966). Linear programming in complex space, *Journal of Mathematical Analysis and Applications* 14, 44–62.
Lin, L. J. (1990). Optimality of differentiable vector–valued n-set functions, *Journal of Mathematical Analysis and Applications* 149, 255–270.
Lin, L. J. (1991a). Duality theorems of vector valued n-set functions, *Computers and Mathematics with Applications* 21, 165–175.
Lin, L. J. (1991b). On the optimality conditions of vector-valued n-set functions, *Journal of Mathematical Analysis and Applications* 161, 367–387.
Lin, L. J. (1992). On optimality of differentiable nonconvex n-set functions, *Journal of Mathematical Analysis and Applications* 168, 351–366.
Liu, J. C. (1997). Sufficient criteria and duality in complex nonlinear programming involving pseudoinvex functions, *Optimization* 39, 123–135.
Liu, J. C. (1999a). Second order duality for minimax programming, *Utilitas Mathematia*. 56, 53–63.
Liu, J. C. (1999b). Complex minimax programming, *Utilitas Mathematica* 55, 79–96.
Liu, J. C. and C. S. Wu (1998). On minimax fractional optimality conditions with invexity, *Journal of Mathematical Analysis and Applications* 219, 21–35.
Liu, J. C., C. S. Wu, and R. L. Sheu (1997a). Duality for fractional minimax programming, *Optimization* 41, 117–133.
Liu, J. C., C. C. Lin, and R. L. Sheu (1997b). Optimality and duality for complex nondifferentiable fractional programming, *Journal of Mathematical Analysis and Applications* 210, 804–824.
Luo, H. Z. and Z. K. Xu (2004). On characterization of prequasi-invex functions, *Journal of Optimization Theory and Applications* 120, 429–439.
Maeda, T. (1994). Constraint qualifications in multiobjective optimization problems: differentiable case. *Journal of Optimization Theory and Applications* 80, 483–500.
Mancino, O. G. and G. Stampacchia (1972). Convex programming and variational inequalities, *Journal of Optimization Theory and Applications* 9, 3–23.
Mangasarian, O. L. (1965). Pseudo-convex functions, *SIAM Journal on Control* 3, 281–290.
Mangasarian, O. L. (1969). *Nonlinear Programming*, McGraw-Hill, New York.
Mangasarian, O. L. (1975). Second and higher-order duality in nonlinear programming, *Journal of Mathematical Analysis and Applications* 51, 607–620.
Martin, D. H. (1985). The essence of invexity, *Journal Optimization Theory and Applications* 47, 65–76.
Marusciac, I. (1982). On Fritz John type optimality criterion in multiobjective optimization, *L'Analyse Numerique et la Theorie de l'Approximation* 11, 109–114.
Mastroeni, G. (1999). Some remarks on the role of generalized convexity in the theory of variational inequalities, In: G. Giorgi and F. Rossi (eds.) *Generalized Convexity and Optimization for Economic and Financial decisions*, Pitagora Editrice Bologna, 271–281.
Mazzoleni, P. (1979). On constrained optimization for convex set functions, In: A. Prekop (ed.), *Survey of Mathematical Programming*, North-Holland, Amsterdam, 273–290.
Minami, M. (1983). Weak Pareto optimal necessary conditions in a nondifferential multiobjective program on a Banach space, *Journal of Optimization Theory and Applications* 41, 451–461.
Minkowski, H. (1910). *Geometrie der Zahlen*, Teubner, Leipzig.
Minkowski, H. (1911). Theorie der konvexen Korper, insbesondere Begrundung ihres Oberflachenbegriffs, Gesammelte Abhandlungen II, Teubner, Leipzig.
Mishra, S. K. (1995). Pseudolinear fractional minimax programming, *Indian Journal of Pure and Applied Mathematics* 26, 763–772.

Mishra, S. K. (1996a). On sufficiency and duality for generalized quasiconvex nonsmooth programs, *Optimization* 38, 223–235.
Mishra, S. K. (1996b). Generalized proper efficiency and duality for a class of nondifferentiable multiobjective variational problems with V-invexity, *Journal of Mathematical Analysis and Applications* 202, 53–71.
Mishra, S. K. (1996c). Lagrange multipliers saddle points and scalarizations in composite multiobjective nonsmooth programming, *Optimization* 38, 93–105.
Mishra, S. K. (1997a). Second order generalized invexity and duality in mathematical programming, *Optimization* 42, 51–69.
Mishra, S. K. (1997b). On sufficiency and duality in nonsmooth multiobjective programming, *Opsearch* 34, 221–231.
Mishra, S. K. (1998a). Generalized pseudoconvex minimax programming, *Opsearch* 35, 32–44.
Mishra, S. K. (1998b). On multiple objective optimization with generalized univexity, *Journal of Mathematical Analysis and Applications* 224, 131–148.
Mishra, S. K. (2000a). Multiobjective second order symmetric duality with cone constraints, *European Journal of Operational Research* 126, 675–682.
Mishra, S. K. (2000b). Second order symmetric duality in mathematical programming with F-convexity, *European Journal of Operational Research* 127, 507–518.
Mishra, S. K. (2001a). Pseudoconvex complex minimax programming, *Indian Journal of Pure and Applied Mathematics* 32, 205–213.
Mishra, S. K. (2001b). Second order symmetric duality in mathematical programming, In: M. L. Agarwal amd K. Sen (eds.) *Recent Developments in Operational Research*, Narosa Publishing House, New Delhi, 261–272.
Mishra, S. K and G. Giorgi (2000). Optimality and duality with generalized semi-univexity, *Opsearch* 37, 340–350.
Mishra, S. K. and R. N. Mukherjee (1994a). Duality for multiobjective fractional variational problems, *Journal of Mathematical Analysis and Applications* 186, 711–725.
Mishra, S. K. and R. N. Mukherjee (1994b). On efficiency and duality for multiobjective variational problems, *Journal of Mathematical Analysis and Applications*187, 40–54.
Mishra, S. K. and R. N. Mukherjee (1995a). Generalized continuous nondifferentiable fractional programming problems with invexity, *Journal of Mathematical Analysis and Applications* 195, 191–213.
Mishra, S. K. and R. N. Mukherjee (1995b). Generalized convex composite multiobjective nonsmooth programming and conditional proper efficiency, *Optimization* 34, 53–66.
Mishra, S. K. and R. N. Mukherjee (1996). On generalized convex multiobjecive nonsmooth programming, *Journal of the Australian Mathematical Society* B 38, 140–148.
Mishra, S. K. and R. N. Mukherjee (1997). Constrained vector valued ratio games and generalized subdifferentiable multiobjective fractional minmax programming, *Opsearch* 34, 1–15.
Mishra, S. K. and R. N. Mukherjee (1999). Multiobjective control problems with V-invexity, *Journal of Mathematical Analysis and Applications* 235, 1–12.
Mishra, S. K. and M. A. Noor (2005). On vector variational-like inequality problems, *Journal of Mathematical Analysis and Applications* 311, 69–75.
Mishra, S. K. and N. G. Rueda (2000). Higher order generalized invexity and duality in mathematical programming, *Journal of Mathematical Analysis and Applications* 247, 173–182.
Mishra, S. K. and N. G. Rueda (2002). Higher order generalized invexity and duality in nondifferentiable mathematical programming, *Journal of Mathematical Analysis and Applications* 272, 496–506.
Mishra, S. K. and N. G. Rueda (2003). Symmetric duality for mathematical programming in complex spaces with F-convexity, *Journal of Mathematical Analysis and Applications* 284, 250–265.
Mishra, S. K, and S. Y. Wang (2005). Second order symmetric duality for nonlinear multiobjective mixed integer programming, *European Journal of Operational Research* 161, 673–682.
Mishra, S.K., G. Giorgi, and S.Y. Wang (2004). Duality in vector optimization in Banach spaces with generalized convexity, *Journal of Global Optimization* 29, 415–424.

Mishra, S. K., S. Y. Wang, and K. K. Lai (2005). Nondifferentiable multiobjective programming under generalized d-univexity, *European Journal of Operational Research* 160, 218–226.

Mohan, S. R. and S. K. Neogy (1995). On invex sets and preinvex functions, *Journal of Mathematical Analysis and Applications* 189, 901–908.

Mond, B. (1965). A symmetric dual theorem for nonlinear programs, *Quarterly Journal of Applied Mathematics* 23, 265–269.

Mond, B. (1973). Nonlinear complex programming, *Journal of Mathematical Analysis and Applications* 43, 633–641.

Mond, B. (1974a). Second order duality for nonlinear programs, *Opsearch* 11, 90–99.

Mond, B. (1974b). A class of non-differentiable mathematical programming problems, *Journal of Mathematical Analysis and Applications* 46, 169–174.

Mond, B. and B. D. Craven (1975). A class of nondifferentiable complex programming problems, *Journal Mathematics Operations and Statistics* 6, 581–591.

Mond, B. and M. A. Hanson (1967). Duality for variational problems, *Journal of Mathematical Analysis and Applications* 18, 355–364.

Mond, B. and M. A. Hanson (1968a). Duality for control problems, *SIAM Journal of Control* 6, 114–120.

Mond, B. and M. A. Hanson (1968b). Symmetric duality for quadratic programming in complex space, *Journal of Mathematical Analysis and Applications* 23, 284–293.

Mond, B. and M. A. Hanson (1968c). Symmetric duality for variational problems, *Journal of Mathematical Analysis and Applications* 23, 161–172.

Mond, B. and M. Schechter (1996). Non-differentiable symmetric duality, *Bulletin of the Australian Mathematical Society* 53, 177–188.

Mond, B. and I. Smart (1988). Duality and sufficiency in control problems with invexity, *Journal of Mathematical Analysis and Applications* 136, 325–333.

Mond, B. and I. Smart (1989). Duality with invexity for a class of nondifferentiable static and continuous programming problems, *Journal of Mathematical Analysis and Applications* 141, 373–388.

Mond, B. and T. Weir (1981). Generalized concavity and duality, In: S. Schaible and W. T. Ziemba, (eds.), *Generalized Concavity Optimization and Economics,* Academic Press, New York, 263–280.

Mond, B. and T. Weir (1981–1983). Generalized convexity and higher-order duality, *Journal of Mathematical Sciences* 16–18, 74–94.

Mond, B. and J. Zhang (1995). Duality for multi-objective programming involving second-order V-invex functions, In: B. M. Glover and V. Jeyakumar (eds.), *Proceedings of Optimization Mini Conference,* University of New South Wales, Sydney, 89–100.

Mond, B. and J. Zhang (1998). Higher order invexity and duality in mathematical programming, In: J. P. Crouzeix et al. (eds.) *Generalized Convexity, Generalized Monotonicity: Recent Results,* Kluwer, Dordrecht, 357–372.

Mond, B., S. Chandra, and M. V. D. Prasad (1987). Symmetric dual non-differentiable fractional programs, *Indian Journal of Management and Systems* 13, 1–10.

Mond, B., S. Chandra, and I. Husain (1988). Duality for variational problems with invexity, *Journal of Mathematical Analysis and Applications* 134, 322–328.

Morris, R. J. T. (1979). Optimal Constrained Selection of a Measurable Subset, *Journal of Mathematical Analysis and Applications* 70, 546–562.

Mukherjee, R. N. and S. K. Mishra (1994). Sufficiency optimality criteria and duality for multiobjective variational problems with V–invexity, *Indian Journal of Pure and Applied Mathematics* 25, 801–813.

Mukherjee, R. N. and S. K. Mishra (1995). Generalized invexity and duality in multiple objective variational problems, *Journal of Mathematical Analysis and Applications* 195, 307–322.

Mukherjee, R. N. and S. K. Mishra (1996). Multiobjective programming with semilocally convex functions, *Journal of Mathematical Analysis and Applications* 199, 409–424.

Nahak, C. and S. Nanda (1996). Duality for multiobjective variational problems with invexity, *Optimization* 36, 235–258.

Nahak, C. and S. Nanda (1997a). Duality for multiobjective variational problems with pseudoinvexity, *Optimization* 41, 361–382.

Nahak, C. and S. Nanda (1997b). On efficientand duality for multiobjective variational control problems with (F,ρ)-convexity, *Journal of Mathematical Analysis and Applications* 209, 415–434.

Nahak, C. and S. Nanda (2000). Symmetric duality with pseudo-invexity in variational problems, *European Journal of Operational Research* 122, 145–150.

Nanda, S. and L. N. Das (1996). Pseudo-invexity and duality in nonlinear programming, *European Journal of Operational Research* 88, 572–577.

Noor, M. A. (1990). Preinvex functions and variational inequalities, *Journal of Natural Geometry* 9, 63–76.

Noor, M. A. (1994a). Monotone variational-like inequalities, *Communications on Applied Nonlinear Analysis* 1, 47–62.

Noor, M. A. (1994b). Variational-like inequalities, *Optimization* 30, 323–330.

Noor, M. A. (1995). Nonconvex functions and variational inequalities, *Journal of Optimization Theory and Applications* 87, 615–630.

Noor, M. A. (2004a). On generalized preinvex functions and monotonicities, *Journal of Inequalities in Pure and Applied Mathematics* 5(4), Article 110.

Noor, M. A. (2004b). Invex equilibrium problems, *Journal of Mathematical Analysis and Applications* 302, 463–475.

Noor, M. A. (2004c). Generalized mixed quasi-variational-like inequalities, *Applied Mathematics Computation* 156, 145–158.

Osuna, R., A. Rufian, and G. Ruiz (1998). Invex functions and generalized convexity in multiobjective programming, *Journal of Optimization Theory and Applications* 98, 651–661.

Parida, J., M. Sahoo, and A. Kumar (1989). A variational-like inequality problems, *Bulletin of the Australian Mathematical Society* 39, 225–231.

Pini, R. and C. Singh (1997). A survey of recent [1985–1995] advances in generalized convexity with applications to duality theory and optimality conditions, *Optimization* 39, 311–360.

Phuong, T. D., P. H. Sach, and N. D. Yen (1995). Strict lower semicontinuity of the level sets and invexity of a locally Lipschitz function, *Journal of Optimization Theory and Applications* 87, 579–594.

Ponstein, J. (1967). Seven types of convexity, *Society for Industrial and Applied Mathematics Review* 9, 115–119.

Preda, V. (1991). On minimax programming problems containing n-set functions, *Optimization* 22, 527–537.

Preda, V. (1992). On efficiency and duality for multi-objective programs, *Journal of Mathematical Analysis and Applications* 166, 365–377.

Preda, V. (1995). On duality of multiobjective fractional measurable subset selection problems, *Journal of Mathematical Analysis and Applications* 196, 514–525.

Preda, V. (1996). Optimality conditions and duality in multiple objective programming involving semilocally convex and related functions, *Optimization* 36, 219–230.

Preda, V. (2003). Optimality and duality in fractional multiple objective programming involving semilocally preinvex and related functions, *Journal of Mathematical Analysis and Applications* 288, 365–382.

Preda, V. and I. M. Stancu-Minasian (1997a). Mond–Weir duality for multiobjective mathematical programming with n-set functions, *Analele Universitati Bucuresti, Matematica- Informatica* 46, 89–97.

Preda, V. and I. M. Stancu-Minasian (1997b). Duality in multiple objective programming involving semilocally preinvex and related functions, *Glas. Mat.* 32, 153–165.

Preda, V. and I. M. Stancu-Minasian (2001). Optimality and Wolfe duality for multiobjective programming problems involving n-set functions, In: N. Hadjisavvas, J. E. Martinez-Legaz, and J.- P. Penot (eds.), *Generalized Convexity and Generalized Monotonicity*, Springer, Berlin, 349–361.

Preda, V., I. M. Stancu-Minasian, and A. Batatorescu (1996). Optimality and duality in nonlinear programming involving semilocally preinvex and related functions, *Journal of Information and Optimization Science* 17, 585–596.

Rapcsak, T. (1991). On pseudolinear functions, *European Journal of Operational Research* 50, 353–360.

Riesz, F. and B. Sz. Nagy (1955). *Functional Analysis*, Frederick Ungar Publishing, New York.

Rojas-Medar, M. A. and A. J. V. Brandao (1998). Nonsmooth continuous-time optimization problems: sufficient conditions, *Journal of Mathematical Analysis and Applications* 227, 305–318.

Rosenmuller, J. and H. G. Weidner (1974). Extreme convex set functions with finite carries: general theory. *Discrete Mathematics* 10, 343–382.

Rueda, N. G. and M. A. Hanson (1988). Optimality criteria in mathematical programming involving generalized invexity, *Journal of Mathematical Analysis and Applications* 130, 375–385.

Rueda, N. G., M. A. Hanson, and C. Singh (1995). Optimality and duality with generalized convexity, *Journal of Optimization Theory and Applications* 86, 491–500.

Ruiz-Garzion G., R. Osuna-Gomez, and A. Ruffian-Lizana (2003). Generalized invex monotonicity, *European Journal of Operational Research* 144, 501–512.

Ruiz-Garzion, G., R. Osuna-Gomez, and A. Rufian-Lizan (2004). Relationships between vector variational-like inequality and optimization problems, *European Journal of Operational Research* 157, 113–119.

Schaible, S. (1976a). Duality in fractional programming; a unified approach, *Operations Research* 24, 452–461.

Schaible, S. (1976b). Fractional programming: I, duality, *Management Science* 22, 858–867.

Schaible, S. (1995). Fractional programming, In: R. Horst, P.M. Pardalos (eds.), *Handbook of Global Optimization*, Kluwer, Dordrecht, 495–608.

Schaible, S. and W. T. Ziemba (1981). *Generalized Concavity in Optimization and Economics*, Academic Press, New York.

Schmitendorf, W. E. (1977). Necessary conditions and sufficient conditions for static minimax problems, *Journal of Mathematical Analysis and Applications* 57, 683–693.

Smart, I. (1990). *Invex Functions and Their Application to Mathematical Programming*, Ph.D. Thesis, La Trobe University, Bundoora, Victoria, Australia.

Smart, I. and B. Mond (1990). Symmetric duality with invexity in variational problems, *Journal of Mathematical Analysis and Applications* 152, 536–545.

Smart, I. and B. Mond (1991). Complex nonlinear programming: duality with invexity and equivalent real programs, *Journal of Optimization Theory and Applications* 69, 469–488.

Stancu-Minasian, I. M. (1997). *Fractional Programming: Theory, Methods and Applications, Mathematics and Its Application Vol. 409*, Kluwer, Dordrecht.

Stancu-Minasian, I. M. (2002). Optimality and duality in fractional programming involving semilocally preinvex and related functions, *Journal of Information Optimization Science* 23 , 185–201.

Suneja, S.K. and S. Gupta (1998). Duality in multiobjective nonlinear programming involving semilocally convex and related functions, *European Journal of Operational Research* 107, 675–685.

Suneja, S. K. and M. K. Srivastava (1997). Optimality and duality in nondifferentiable multiobjective optimization involving d-type I and related functions, *Journal of Mathematical Analysis and Applications* 206, 465–479.

Suneja, S. K., C. S. Lalitha, and S. Khurana (2003). Second order symmetric duality in multiobjective programming, *European Journal of Operational Research* 144, 492–500.

Suneja, S. K., C. Singh, and C. R. Bector (1993). Generalization of preinvex and b-vex functions, *Journal of Optimization Theory and Applications* 76, 577–587.

Tanaka, K. and Y. Maruyama (1984). The multiobjective optimization problems of set functions, *Journal of Information and Optimization Sciences* 5, 293–306.

Tanimoto, S. (1981). Duality for a class of nondifferentiable mathematical programming problems, *Journal of Mathematical Analysis and Applications* 79, 286–294.

Tanino, T. and Y. Sawaragi (1979). Duality theory in multi-objective programming, *Journal of Optimization Theory and Applications* 27, 509–529.

Tucker, A. W. (1957). Linear and nonlinear programming, *Operations Research* 5, 244–257.

Tuy, H. (1964). Sur les inegalities lineaires, *Colloquium Mathamatics* 13, 107–123.

Valentine F.A. (1937). The problem of Lagrange with differential inequalities as added side conditions, *Contributions to the Calculus of Variations* (1933–1937), University of Chicago Press, Chicago.

Valentine, F. A. (1964). *Convex Sets*, McGraw-Hill, New York.

Weir, T. (1991). Symmetric dual multiobjective fractional programming, *Journal of Australian Mathematical Soceity* Series A 50, 67–74.

Weir, T. (1992). Pseudoconvex minimax programming, *Utilitas Mathamatics*. 42, 234–240.

Weir, T. and B. Mond (1984). Generalized convexity and duality for complex programming problem, *Cahiers de C. E. R. O*. 26, 137–142.

Weir, T. and B. Mond (1988a). Preinvex functions in multiobjective optimization, *Journal of Mathematical Analysis and applications* 136, 29–38.

Weir, T. and B. Mond (1988b). Symmetric and self duality in multiple objective programming, *Asia Pacific Journal of Operational Research* 5, 124–133.

Wolfe, P. (1961). A duality theorem for non-linear programming, *Quarterly of Applied mathematics* 19, 239–244.

Xu, Z. (1996). Mixed type duality in multiobjective programming problems, *Journal of Mathematical Analysis and Applications* 198, 621–635.

Yadav, S. R. and R. N. Mukherjee (1990). Duality for fractional minimax programming problems, *Journal of the Australian Mathematics Society Series* B 31, 484–492.

Yang, X. M. (1995). Second order symmetric duality for nonlinear programs, *Opsearch* 32, 205–209.

Yang, X. M., X. Q. Yang, and K. L. Teo (2001). Characterization and applications of prequasi-invex functions, *Journal of Optimization Theory and Applications* 110, 645–668.

Yang, X. M., K. L. Teo, and X. Q. Yang (2000). Duality for a class of non-differentiable multi-objective programming problems, *Journal of Mathematical Analysis and Applications* 252, 999–1005.

Yang, X. M., K. L. Teo, and X. Q. Yang (2002a). Symmetric duality for a class of nonlinear fractional programming problems, *Journal of Mathematical Analysis and Applications* 271, 7–15.

Yang, X. M., S. Y. Wang, and X. T. Deng (2002b). Symmetric duality for a class of multiobjective fractional programming problems, *Journal of Mathematical Anaysis and Applications* 274, 279–295.

Yang, X. M., K. L. Teo, and X. Q. Yang (2003a). Mixed symmetric duality in non-differentiable mathematical programming, *Indian Journal of Pure and Applied Mathematics* 34, 805–815.

Yang, X. M., X. Q. Yang, and K. L. Teo (2003b). Non-differentiable second order symmetric duality in mathematical programming with F-convexity, *European Journal of Operational Research* 144, 554–559.

Yang, X. M., K. L. Teo, and X. Q. Yang (2004). Higher-order generalized convexity and duality in non-differentiable multi-objective mathematical programming, *Journal of Mathematical Analysis and Applications* 297, 48–55.

Yang, X. M., X. Q. Yang, and K. L. Teo (2005). Criteria for generalized invex monotonicities, *European Journal of Operational Research* 164, 115–119.

Yang, X. M., X. Q. Yang, K. L. Teo, and S. H. Hou (2005). Second order symmetric duality in non-differentiable multi-objective programming with F-convexity, *European Journal of Operational Research* 164, 406–416.

Yang, X. Q. (1993). Generalized convex functions and vector variational inequalities, *Journal of Optimization Theory and applications* 79, 563–580.

Yang, X. Q. (1997). Vector variational inequality and vector pseudolinear optimization, *Journal of Optimization Theory and Applications* 95, 729–734.

Yang, X. Q. and G. Y. Chen (1992). A class of nonconvex functions and prevariational inequalities, *Journal of Mathematical Analysis and Applications* 169, 359–373.

Yang, X. Q. and C. J. Goh (1997). On vector variational inequalities: application to vector equilibria, *Journal of Optimization Theory and Applications* 95, 431–443.

Ye, Q. K. and Y. P. Zheng (1991). *Variational Methods and Applications*, National Defence Industry Press, Beijing.

Ye, Y. L. (1991). d-invexity and optimality conditions, *Journal of Mathematical Analysis and Applications* 162, 242–249.

Zalmai, G. J. (1985). Sufficient optimality conditions in continuous-time nonlinear programming, *Journal of Mathematical Analysis and Applications* 111, 130–147.

Zalmai, G. J. (1987). Optimality criteria and duality for a class of minimax programming problems with generalized invexity conditions, *Utilitas Mathematics* 32, 35–57.

Zalmai, G. J. (1989). Optimality conditions and duality for constrained measurable subset selection problems with minmax objective functions, *Optimization* 20, 377–395.

Zalmai, G. J. (1990a). Sufficiency criteria and duality for nonlinear programs involving n-set functions, *Journal of Mathematical Analysis and Applications* 149, 322–338.

Zalmai, G. J. (1990b). Generalized sufficient criteria in continuous-time programming with application to a class of variational-type inequalities, *Journal of Mathematical Analysis and Applications* 153, 331–355.

Zalmai, G. J. (1991). Optimality conditions and duality for multiobjective measurable subset selection problems, *Optimization* 22, 221–238.

Zalmai, G. J. (2001). Semiparametric sufficient efficiency conditions and duality models for multiobjective fractional subset programming problems with generalized $(\mathcal{F}, \rho, \theta)$-convex functions, *Southeast Asian Bulletin of Mathematics* 25, 529–563.

Zalmai, G. J. (2002). Efficiency conditions and duality models for multiobjective fractional subset programming problems with generalized $(\mathcal{F}, \alpha, \rho, \theta)$-V-convex functions, *Computers & Mathematics with Applications* 43, 1489–1520.

Zhang, J. (1998). *Generalized Convexity and Higher Order Duality for Mathematical Programming Problems*, Ph.D. Thesis, La Trobe University, Australia.

Zhang, J. (1999). Higher-order convexity and duality in multi-objective programming problems, In: A. Eberhard, R. Hill, D. Ralph, and B. M. Glover (eds.), *Progress in Optimization, Contributions from Australasia*, Kluwer, Dordrecht, pp. 101–116.

Zhang, J. and B. Mond (1996). Second order B-invexity and duality in mathematical programming, *Utilitas Mathematics* 50, 19–31.

Zhang, J. and B. Mond (1997). Second-order duality for multi-objective non-linear programming involving generalized convexity, In: B. M. Glover, B. D. Craven, and D. Ralph (eds.), *Proceedings of the Optimization Mini Conference III*, University of Ballarat, Ballarat, 79–95.

Zhian, L. (2001). Duality for a class of multiobjective control problems with generalized invexity, *Journal of Mathematical Analysis and Applications* 256, 446–461.

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