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Graphs of Groups on Surfaces
Interactions and Models

Arthur T. WHITE

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Foreword

Topological graph theory began in the middle of the eighteenth century, with Euler's polyhedral identity; it took heightened interest in the latter part of the nineteenth century, with Heawood's Map-Color Conjecture; and it emerged as a field of study in its own right with the Complete Graph Theorem of Ringel and Youngs in 1968 (completing the proof that Heawood had started). Ringel's books *Färbungsprobleme auf Flachen und Graphen* (VEB Deutscher Verlag der Wissenschaften, Berlin, 1959) and *Map Color Theorem* (Springer-Verlag, Berlin, 1974) are, as the titles suggest, devoted to the interplay between the conjecture and the theorem mentioned above. The first book devoted to topological graph theory as an independent field of study is my *Graphs, Groups and Surfaces* (North-Holland, Amsterdam, 1973; Revised Edition, 1984). Related books include Gross and Tucker's *Topological Graph Theory* (Wiley Interscience, New York, 1987), Bonnington and Little's *The Foundations of Topological Graph Theory* (Springer, New York, 1995), and Mohar and Thomassen's *Graphs on Surfaces* [MT1]. See also Archdeacon's "Topological Graph Theory" [A12], a survey article with 271 references.

Whereas the two editions of *Graphs, Groups and Surfaces* introduced topological graph theory in general, with a particular emphasis on various interactions among the three structures of the title (see Figure 0-1) as well as models of hypergraphs, block designs, and compositions of English church-bell music, the present book will also use suitable imbeddings of graphs of groups on surfaces to model finite fields and finite geometries. The material on change ringing is greatly updated, and introductions to enumerative and random topological graph theory have been added. The unifying concept is that of a Cayley map: the lift, as a branched covering space, of an index-one voltage graph imbedding, for a fixed group and generating set. (The latter consists of one vertex, a directed loop for each generator, and a particular imbedding of the loop digraph. The covering graph is then a Cayley color graph.)

I have attempted to make all this material, with its fascinating interconnections, readily accessible to a beginning graduate (or an advanced undergraduate) student (introductory knowledge of both group theory and topology would be helpful), while at the same time providing the research mathematician with a useful reference book in topological graph theory. The latter aspect will not be comprehensive, however, as the field is not too broad to allow this reasonably. The focus will be on beautiful connections, both elementary and deep, within mathematics that can best be described by the intuitively pleasing device of imbedding graphs of groups on surfaces. Several peripheral (but significant) results are stated without proof. An effort has been made to provide those proofs of theorems which are most indicative of the charm and beauty of the subject and which illustrate the techniques employed. Proofs missing in the text can be supplied by the reader, as part of the problem sets (of the total of 297 problems, 30 have been designated as "difficult" (*) and 9 as "unsolved" (**)), or can be found in the references.

A bibliography is provided, for future reading; items h and n, f and i, k respectively are especially suitable for more extensive treatments of the theories of graphs, of groups, and of surfaces, which are seen interacting in this text.

I thank everyone who read either edition of *Graphs, Groups and Surfaces*, particularly all those who sent me comments and corrections. I especially thank Margo Chapman for preparing the manuscript, and Ramón Figueroa-Centeno for producing the figures and greatly assisting with the preparation, of the present volume. I also thank Jim Laser, Michelle Schultz, Jay Treiman, and Mary Van Popering for their considerable help. Finally I thank Western Michigan University for funding the sabbatical year during which this book was written and the Mathematical Institute and Wolfson College, University of Oxford, for hosting my sabbatical visit.

A.T.W. Kalamazoo January 2001

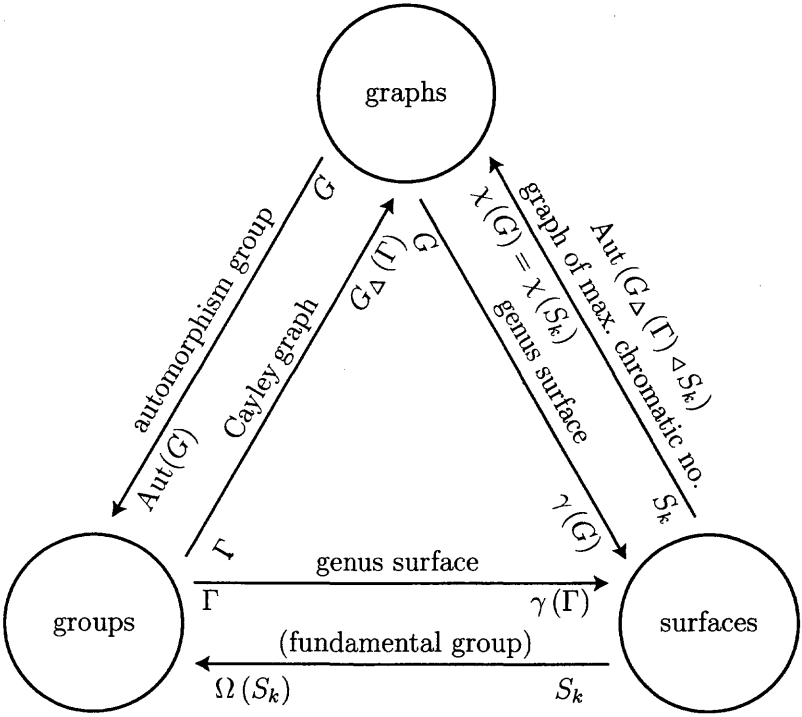
graphs
groups
surfaces
automorphism group
G
Aut(G)
Cayley graph
Γ
G▵(Γ)
G
χ(G) = χ(S_k)
genus surface
γ(G)
graph of max. chromatic no.
Aut(G▵(Γ)▵S_k)
S_k
genus surface
Γ
γ(Γ)
(fundamental group)
Ω(S_k)
S_k

Figure 0-1.

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Contents

CHAPTER 1

HISTORICAL SETTING

In coloring the regions of a map, one must take care to color differently any two countries sharing a common boundary line, so that the two countries can be distinguished. One would think that an economy-minded map-maker would wish to minimize the number of colors to be used for a given map, although there appears to be no historical evidence of any such effort. Nevertheless a conjecture was made, about one and one half centuries ago, to the effect that four colors would always suffice for a map drawn on the sphere, the regions of which were all connected. The first reported mention of this problem (see [BCL1] and [O1]) was by Francis Guthrie, through his brother Frederick and Augustus de Morgan, in 1852. The first written references were by Cayley, in 1878 and 1879. Incorrect "proofs" of the Four Color Conjecture were published soon after by Kempe and Tait. The error in Kempe's "proof" was found by Heawood [H4] in 1890; this error has reappeared in various guises in subsequent years. Ore and Stemple [OS1] showed that any counterexample to the conjecture must involve a map of at least 72 regions. The conjecture continued to provide one of the most famous unsolved problems in mathematics, until Appel and Haken affirmed it in 1976. [AH1].

It is an astonishing fact that several related, seemingly much more difficult, map-coloring problems were completely solved prior to the four color problem. Chief among these is the Heawood Map-coloring Conjecture, which gives the chromatic number for *every* closed 2-manifold *other than the sphere*; we state the orientable case:

$$\chi(S_k) = f(k) = \left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor, \text{ for } k > 0,$$

where k is the genus of the closed orientable 2-manifold S_k. Heawood showed in 1870 [H4] that $\chi(S_k) \leq f(k)$, and in 1891 Heffter [H5] showed the reverse inequality for a possibly infinite set of natural numbers k; almost eight decades passed before it was shown that $\chi(S_k) \geq f(k)$, for all $k > 0$. In 1965 this problem was given the place of honor on the dust jacket for Tietze's *Famous Problems of Mathematics* [T7]. An outline of the major portion of the solution now follows.

The dual of a map drawn on S_k is a pseudograph imbedded in S_k, and it can be shown (see Section 8-4) that $\chi(S_k) \geq f(k)$, for $k > 0$, provided the complete graph K_n has genus given by

$$(*)\ \gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil, \ n \geq 7.$$

Heawood established $(*)$ for $n = 7$ in 1890, and Heffter for $8 \leq n \leq 12$ in 1891; Ringel handled $n = 13$ in 1952. The first major breakthrough occurred in 1954, when Ringel showed $(*)$ for $n \equiv 5 \pmod{12}$. During 1961 - 1965, Ringel treated the residue cases $7, 10$, and $3 \pmod{12}$, while independently Gustin settled the cases 3, 4, and 7. Gustin's method involved the powerful and beautiful idea of quotient graph and quotient manifold, and relies upon the fact that K_n can be regarded as a Cayley color graph for a group presentation; thus graph theory, group theory, and surface topology combined to solve this famous problem of mathematics.

In 1965, Terry, Welch, and Youngs announced their solution to case 0. Gustin, Ringel, and Youngs finished the remaining residue cases (mod 12), except for the isolated values $n = 18, 20$, and 23; their work was announced in 1968 [RY1]. In 1969, Jean Mayer (a Professor of French Literature) [M4] eliminated the last three obstinate graphs by *ad hoc* techniques.

Much of the work of Ringel, Terry, Welch, and Youngs was made possible by Gustin's theory of quotient graphs and quotient manifolds; this theory was developed and modified by Youngs, who also introduced the theory of vortices [Y3]. The theory is considerably more general than was needed to prove the Heawood Map-coloring Theorem, and was unified and developed in more generality by Jacques [J3], in 1969. Jacques' results are accessible in Chapter 9 of [W15], together with many applications to other imbedding problems in graph theory. This was a focal point of that text, and it illustrates vividly the fruitful interaction among graphs, groups and surfaces.

We continue this development through the theory of voltage graphs (introduced by Gross [G4] and by Gross and Alpert ([GA1], [GA2])) and by extending to nonorientable imbeddings. We consider the related structures of block designs and hypergraph imbeddings. In map automorphism groups we study groups acting on graphs of groups on surfaces. Cayley maps dominate our considerations. They allow concrete models of finite geometries and finite fields, as well as for finite groups, and we study them also in contexts of enumerative and of topological graph theory. Finally, in studying change ringing, we use graphs of groups on surfaces to compose pieces of music for English church bells.

The conjunctions of graph theory, group theory, and surface topology described above are foreshadowed, in this text, by several pairwise interactions among these three disciplines. The Heawood Map-coloring

Theorem is proved by finding, for each surface, a graph of largest chromatic number that can be drawn on that surface. Equivalently (as it turns out) we find, for each complete graph, the surface of smallest genus in which it can be drawn. The extension of this latter problem to arbitrary graphs is natural; the solution is particularly elegant for graphs which are the Cayley color graphs of a group. We are led in turn to the problem of finding, for a given group, a surface of minimum genus which represents the group in some way.

Dyck [D7] (see also Burnside [B21], Chapters 18 and 19) considered maps, on surfaces, that are transformed into themselves in accordance with the fixed group Γ, acting transitively on the regions of the map. Any such map gives an upper bound for the parameter $\gamma(\Gamma)$ discussed in Chapter 7 of this text, as a "dual" formed in terms of Burnside's white regions gives a Cayley color graph for Γ. (Cayley [C4] defined his color graphs as complete symmetric digraphs, corresponding to the choice Γ less the identity element as a generating set for Γ; it is sensible to extend his definition to any generating set for the group in question.) Brahana [B18, B19] studied groups represented by regular maps on surfaces; these maps correspond to presentations on two generators, one of which is of order two. In this context the group acts transitively on the edges of the map, and again an upper bound for $\gamma(\Gamma)$ is obtained. In Chapter 7, we regard Γ as acting transitively on the vertices of the map induced by imbedding a Cayley color graph $C_\Delta(\Gamma)$ for Γ in a surface; in Chapter 4, we show that the automorphism group of $C_\Delta(\Gamma)$ is isomorphic to Γ, independent of the generating set Δ selected for Γ, so that in this sense $C_\Delta(\Gamma)$ provides a "picture" of Γ. But more: many properties of Γ, such as commutivity, normality of certain subgroups, the entire multiplication table, can be "seen" from the picture provided by $C_\Delta(\Gamma)$. Thus it is natural to seek the simplest surface on which to draw this picture; this is given by the parameter $\gamma(\Gamma)$.

This point of view may give a surface of lower genus for a given group than the other two approaches listed above; for example, the group $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_4$ is toroidal for Dyck (or Burnside) and for Brahana, yet $\gamma(\mathbb{Z}_2 \times \mathbb{Z}_4) = 0$.

There is one correspondence depicted in Figure 0-1 which we discuss only briefly in this text: to every surface S_k there corresponds a unique group, $\Omega(S_k)$, called the *fundamental group* of the surface; the groups $\Omega(S_k)$ have been completely determined – they are given by $2k$ generators $a_1, b_1, \ldots, a_k, b_k$ and the single defining relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_k b_k a_k^{-1} b_k^{-1} = e$$

(see, for example, [S11].) Each of the other five correspondences illustrated in Figure 0-1 (where the inner triangle commutes, for proper choice of Δ) is germane, as outlined above, to the conjunction of graph theory, group theory, and surface topology described in this introduction and which we now begin to develop.

CHAPTER 2

A BRIEF INTRODUCTION TO GRAPH THEORY

In this chapter we introduce basic terminology from the theory of graphs that will be used in this text. We will give several binary operations on graphs; these will enable us to construct more complicated graphs, and hence to build up our store of examples of frequently encountered graphs.

We emphasize that the material introduced here is primarily for the purpose of later use in this text; for a considerably more thorough introduction to graph theory, see [CL1] or [H3].

2-1. Definition of a Graph

Def. 2-1. A *graph* G consists of a finite non-empty set $V(G)$ of *vertices* together with a set $E(G)$ of unordered pairs of distinct vertices, called *edges*. If $x = \{u, v\} \in E(G)$, for $u, v \in V(G)$, we say that u and v are *adjacent* vertices , and that vertex u and edge x are *incident* with each other, as are v and x. We also say that the edges $\{u, v\}$ and $\{u, w\}, w \neq v$, are *adjacent* . The *degree* , $d(v)$, of a vertex v is the number of edges with which v is incident. (Equivalently, $d(v)$ is the number of vertices to which v is adjacent; i.e.,

$$d(v) = |\{u \in V(G) | \{u, v\} \in E(G)\}|.)$$

If the vertices of G are labeled, G is said to be *labeled graph.*

For brevity, we usually write uv for $\{u, v\}$; $p = |V(G)|$; $q = |E(G)|$. The *order* of G is given by p . The *size* of G is given by q .

Example: Let G be defined by:

$$V(G) = \{v_1, v_2, v_3, v_4\}$$
$$E(G) = \{v_1v_2, v_2v_3, v_3v_1, v_1v_4\}$$

then G may be represented by either Figure 2-1a or 2-1b, where the latter representation is more accurate, in a sense we will describe in Chapter 6.

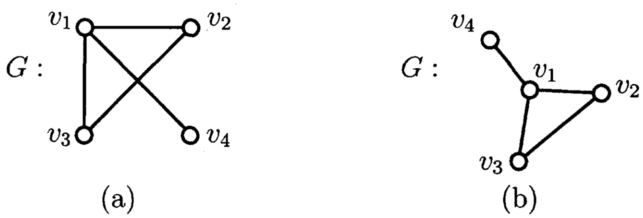


Figure 2-1.

Note: A *graph* may be more briefly defined as a finite one-dimensional simplicial complex.

Thm. 2-2. For any graph G, $\sum_{i=1}^{p} d(v_i) = 2q$.

PROOF. In summing the degrees, each edge is counted exactly twice. □

Cor. 2-3. In any graph G, the number of vertices of odd degree is even.

2-2. Variations of Graphs

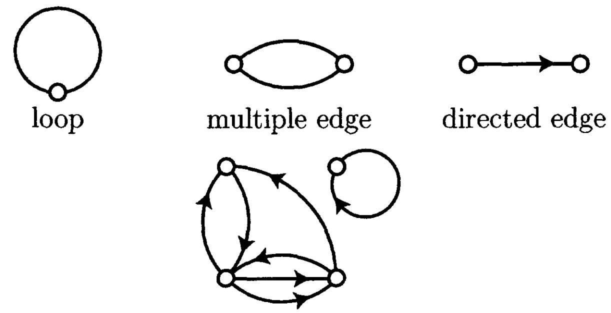
Def. 2-4. A *loop* is an "edge" of the form vv. A *multiple edge* is an edge that appears more than once in $E(G)$. A *directed edge* is an ordered pair of distinct vertices. A *loop graph* allows loops. A *multigraph* allows multiple edges. A *pseudograph* allows loops and multiple edges. A *directed graph (digraph)* has every edge directed. The corresponding graph (with all edge directions deleted) is called the *underlying graph*. (Any multiple edges are coalesced.) An *infinite graph* has infinite vertex set.

For example, see Figure 2-2.

The term "graph," unless qualified appropriately, disallows any and all of the above variations.

2-3. Additional Definitions

Def. 2-5. A graph H is said to be a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, H is called a *spanning subgraph*. For any $\emptyset \neq S \subseteq V(G)$, the *induced subgraph* $\langle S \rangle$ is the maximal subgraph of G with vertex set S.



a directed pseudograph

Figure 2-2.

Notation: For $v \in V(G), G - v$ denotes $\langle V(G) - v \rangle$. For $x \in E(G)$, $V(G - x) = V(G)$, and $E(G - x) = E(G) - x$.

Certain subgraphs are given special names. We indicate these by a series of definitions.

Def. 2-6. A *walk* of a graph G is an alternating sequence of vertices and edges $v_0, x_1, v_1, \ldots v_{n-1}, x_n, v_n$ (or, briefly: $v_0, v_1, \ldots, v_{n-1}, v_n$) beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it; n is the *length* of the walk. If $v_0 = v_n$, the walk is said to be *closed*; it is said to be *open* otherwise. The walk is called a *trail* if all its edges are distinct, and a *path* if all the vertices are distinct. A *cycle* is a closed walk with $n \geq 3$ distinct vertices (i.e., $v_0 = v_n$, but otherwise the v_i are distinct).

Two famous problems in graph theory may be described in terms of the above definitions. A graph is said to be *eulerian* if the graph itself can be expressed as a closed trail. (This corresponds to the "highway inspector" problem; eulerian graphs have been completely and simply characterized: see Problem 2.14 or Harary [H3], p. 64-65.) A graph is said to be *hamiltonian* if it has a spanning cycle. (This corresponds to the "traveling salesman" problem; hamiltonian graphs have *not* been completely characterized. See Harary, p. 65-69, for some partial results.)

Def. 2-7. A graph G is *connected* if $u, v \in V(G)$ implies there exists a path in G joining u to v. A *component* of G is a maximal connected subgraph of G.

Def. 2-8. The *distance*, $d(u, v)$, between two vertices u and v of G is the length of a shortest path joining them if such exists; if not, $d(u, v) = \infty$. The largest distance in a graph is the *diameter* of the graph.

Thm. 2-9. A connected graph may be regarded as a (finite) metric space.

PROOF. See Problem 2-7. $\square$

For a partial converse to the above theorem, see Chartrand and Kay [CK1]. By Theorem 2-9, every connected graph may be regarded as a topological space. (Actually, since the metric induces the discrete topology, we knew this already.) In Chapter 6 we will see that this is true in another sense also; that is every graph may be regarded as a subspace of $\mathbb{R}^3$, with all edges represented as *straight* lines. If we consider G as a topological space in its latter sense, then G is connected as a graph if and only if it is connected as a topological space (see Problem 2-8). The term "component" is easily seen to mean the same in both contexts. Furthermore, a graph (as a subspace of $\mathbb{R}^3$) is connected if and only if it is path connected; (see Problem 2.9.)

Def. 2-10. Two graphs G_1 and G_2 are said to be *isomorphic* ($G_1 \cong G_2$, or $G_1 = G_2$) if there exists a one-to-one, onto map $\theta : V(G_1) \to V(G_2)$ preserving adjacency; that is, $uv \in E(G_1)$ if and only if $\theta(u)\theta(v) \in E(G_2)$.

Note: Isomorphism is an equivalence relation on the set of all graphs.

Notation:

$$\delta(G) = \min\{d(v) | v \in V(G)\}.$$
$$\Delta(G) = \max\{d(v) | v \in V(G)\}.$$

Def. 2-11. If $\delta(G) = \Delta(G) = r$, we say that G is *regular* of degree r. (If $r = 3$, G is said to be *cubic*.)

Thm. 2-12. Let $\theta : V(G_1) \to V(G_2)$ give $G_1 \cong G_2$; then $d(\theta(v)) = d(v)$, for all $v \in V(G_1)$.

PROOF. See Problem 2-10 $\square$

Cor. 2-13. If G_1 is a regular of degree r and $G_1 \cong G_2$, then G_2 is regular of degree r.

Cor. 2-14. Let the vertices of a graph G_1 have degrees $d_1 \leq d_2 \leq \cdots \leq d_n$, and the vertices of a graph G_2 have degrees $c_1 \leq c_2 \leq \cdots \leq c_n$. If $d_i \neq c_i$, for some $1 \leq i \leq n$, then G_1 and G_2 are not isomorphic.

The converse of the above corollary need not be true; see Problem 2-3.

Def. 2-15. The *complement* $\bar{G}$ of a graph G has $V(\bar{G}) = V(G)$ and $E(\bar{G}) = \{uv | u \neq v \text{ and } uv \notin E(G)\}$.

2-4. Operations on Graphs

We now define several binary operations on graphs. In what follows, we assume that $V(G_1) \cap V(G_2) = \emptyset$.

Def. 2-16. 1.) The *union* $G = G_1 \cup G_2$ has:

$$V(G) = V(G_1) \cup V(G_2)$$
$$E(G) = E(G_1) \cup E(G_2).$$

Notation:

$$2G = G \cup G$$
$$nG = (n-1)G \cup G,\ n \geq 3.$$

2.) The *join* $G = G_1 + G_2$ has:

$$V(G) = V(G_1) \cup V(G_2)$$
$$E(G) = E(G_1) \cup E(G_2) \cup \{v_1v_2 | v_i \in V(G_i), i = 1, 2\}.$$

3.) The *cartesian product* $G = G_1 \times G_2$ has:

$$V(G) = V(G_1) \times V(G_2)$$
$$E(G) = \{(u_1, u_2), (v_1, v_2) | u_1 = v_1 \text{ and } u_2v_2 \in E(G_2) \text{ or } u_2 = v_2 \text{ and } u_1v_1 \in E(G_1)\}.$$

4.) The *composition* (or *lexicographic product*) $G = G_1[G_2]$ has:

$$\begin{aligned} V(G) &= V(G_1) \times V(G_2) \\ E(G) &= \{(u_1, u_2), (v_1, v_2) | u_1v_1 \in E(G_1) \text{ or} \\ &\qquad (u_1 = v_1 \text{ and } u_2v_2 \in E(G_2))\}. \end{aligned}$$

For additional products, see Harary and Wilcox [HW1]. We are now in a position to conveniently define several infinite families of graphs.

Def. 2-17.

a). P_n denotes the *path* of length $n-1$ (i.e. of order n.)

b). C_n denotes the *cycle* of length n.

c). K_n denotes the *complete graph* on n vertices; that is all $\binom{n}{2}$ possible edges are present.

d). $\bar{K}_n$ denotes the *totally disconnected* (or *empty*) *graph* on n vertices; that is, $E(\bar{K}_n) = \emptyset$. For $n = 1$, we get the *trivial graph*, K_1.

e). $K_{m,n}$ denotes a *complete bipartite graph*:

$$K_{m,n} = \bar{K}_m + \bar{K}_n.$$

(Equivalently, $K_{m,n}$ is defined by:

$$\overline{K_{m,n}} = K_m \cup K_n.)$$

f). $K_{p_1,p_2,\dots,p_n}$ denotes a *complete n-partite* graph:

$$K_{p_1,p_2,\dots,p_n} = \bar{K}_{p_1} + \bar{K}_{p_2} + \cdots + \bar{K}_{p_n},$$

an iterated join. In the special case where $p_1 = p_2 = \cdots = p_n$ ($= m$, say), we get a *regular complete n-partite* graph:

$$K_{m,m,\dots,m} = K_n[\bar{K}_m].$$

We introduce $K_{n(m)}$ as a shorter notation for this graph.

g). Q_n denotes the *n-cube* and is defined recursively:

$$\begin{aligned} Q_1 &= K_2 \\ Q_n &= K_2 \times Q_{n-1}, n \geq 2. \end{aligned}$$

The complete bipartite graphs are a subclass of an extremely important class of graphs – the bipartite graphs.

Def. 2-18. A *bipartite graph* G is a graph whose vertex set $V(G)$ can be partitioned into two non-empty subsets V' and V'' so that every edge of G has one vertex in V' and the other in V''.

Thm. 2-19. A nontrivial graph G is bipartite if and only if all its cycles are even.

PROOF. (i). Let $v_1v_2\ldots v_nv_1$ be a cycle in a bipartite graph G, and assume, without loss of generality, that $v_1 \in V'$; then $v_n \in V''$, and n must be even.

(ii). We may assume that G is connected, with only even cycles, since the argument in general follows directly from this special case. Consider a fixed $v_0 \in V(G)$. Let $V_i = \{u \in V(G)|d(u,v_0) = i\}$, $i = 0,1,\ldots,n =$ diameter of G. Then n is finite, since G is connected, and $V_0, V_1, \ldots, V_n$ provides a partition of $V(G)$. Now, no two vertices in V_1 are adjacent, since G contains no 3-cycles. Also, no two vertices in V_2 are adjacent, or G would contain either a 3-cycle or a 5-cycle. In fact, every edge in G is of the form uv, where $u \in V_i, v \in V_{i+1}$, for some $i = 0,1,\ldots,n-1$. Letting V' be the union of the V_i for i odd, and V'' be the union of the V_i for i even, we see that G is bipartite. □

This completes our brief introduction; other terms will be defined, and theorems developed, as needed.

2-5. Problems

2-1.) Prove that if G is not connected then $\bar{G}$ is connected. Give an example to show that the converse need not hold.

2-2.) A graph is said to be *perfect* if no two vertices have the same degree. Prove that no graph is perfect, except $G = K_1$.

2-3.) Show that, even though $K_{3,3}$ and $K_2 \times K_3$ are both regular of order 6 and degree 3, they are not isomorphic.

2-4.) For G_1, G_2 nontrivial ($\neq K_1$), prove that $G_1 \times G_2$ is bipartite if and only if both G_1 and G_2 are bipartite. Give an example to show that a similar result need not hold for the lexicographic product.

2-5.) Show that $\overline{G_1[G_2]} = \bar{G}_1[\bar{G}_2]$.

2-6.) Show that $\overline{G_1 + G_2} = \bar{G}_1 \cup \bar{G}_2$.

2-7.) Prove Theorem 2-9.

2-8.) Consider the graph G as a subspace of $\mathbb{R}^3$. Show that G is connected as a topological space if and only if it is connected as a graph.

2-9.) Show that the first occurrence of "connected" in Problem 2-8 may be replaced with "path-connected." (Recall that a path-connected topological space must be connected, but that the converse does not always hold. However, a connected space for which every point has a path connected neighborhood must be path-connected.)

2-10.) Prove Theorem 2-12.

2-11.) Show that the set of all graphs, under the operation of cartesian product, forms a commutative semigroup with unity (i.e. a commutative monoid).

2-12.) Let G and H both be hamiltonian. Show that $G[H]$ and $G \times H$ are hamiltonian also. Show that Q_n is hamiltonian, for $n \geq 2$.

2-13.) The *line graph* $H = L(G)$ of a graph G is defined by $V(H) = E(G)$, $E(H) = \{ef | e \text{ and } f \text{ are adjacent }, e, f \in E(G)\}$. Show that $L(K_{m,n}) = K_m \times K_n$. (This result appears in Palmer [P2]).

2-14.) Find (with proof) a characterization of eulerian graphs.

2-15.) Show that the n-cube Q_n is the lattice graph for the set of all subsets of an n-set. Show that Q_n has diameter n and that each vertex has a unique vertex at distance n from it; two such antipodal vertices represent complementary subsets. Illustrate the case $n = 3$, symmetrically on the sphere.

CHAPTER 3

THE AUTOMORPHISM GROUP OF A GRAPH

In this chapter we show that there is associated, with each graph, a group, known as the automorphism group of the graph. We introduce various binary operations on permutation groups to aid in computing automorphism groups of graphs. Several powerful results relating graph and group products are stated, sometimes without proof (see [H3] for a further discussion); these results will not be used in the sequel. Indeed, the concept of automorphism group of a graph is, in the main, peripheral to the present text; it is introduced here primarily as one example of an interaction between graphs and groups. (In Section 2 of Chapter 4 we find a more direct bearing on subsequent material.)

3-1. Definitions

Def. 3-1. A one-to-one mapping from a finite set onto itself is called a *permutation*. A *permutation group* is a group whose elements are all permutations acting on the same finite set, called the *object set*. (The group operation is composition of mappings.) If X is the object set and A the permutation group, then $|A|$ is the *order* of the group, and $|X|$ is the *degree*.

A permutation P partitions its object set by the equivalence relation $x \equiv y$ if and only if $P^k(x) = y$ for some integer k. The equivalence classes are called the *orbits* of X, under the action of P. If there is just one orbit in the action of A on X, then A is said to be *transitive* on X. If $|A| = |X|$, and if A is transitive on X, then A is said to be a *regular* permutation group.

Def. 3-2. Two permutation groups A and B are said to be *isomorphic* ($A \cong B$) if there exists a one-to-one onto map $\theta : A \to B$ such that $\theta(a_1 a_2) = \theta(a_1)\theta(a_2)$, for all $a_1, a_2 \in A$.

Def. 3-3. Two permutation groups A and B (acting on object sets X and Y respectively) are said to be *identical* (or *equivalent*) ($A \equiv B$) if:

(i) $A \cong B$ (given by $\theta : A \to B$),

(ii) there exists a one-to-one, onto map $f : X \to Y$ such that $f(ax) = \theta(a)f(x)$, for all $x \in X$ and $a \in A$.

For a general treatment of permutation groups acting on combinatorial structures, see Biggs and White [BW1]. Here, we consider only the graph automorphism case.

Def. 3-4. An *automorphism* of a graph G is an isomorphism of G with itself. (The set of all automorphisms of G forms a permutation group, $\mathrm{Aut}(G)$, acting on the object set $V(G)$.) $\mathrm{Aut}(G)$ is called the *automorphism group* of G.

Remark. An automorphism of G, which is a permutation of $V(G)$, also induces a permutation of $E(G)$, in the obvious manner.

Def. 3-5. An *identity graph* is a graph G having trivial automorphism group; that is, the identity permutation on $V(G)$ is the only automorphism of G.

It is easy to see that the graph pictured in Figure 3-1 is an identity graph. That there is no identity graph of smaller order (other than K_1) is established in Problem 3-1.

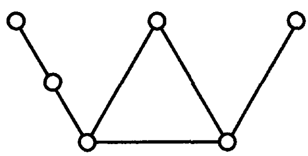
G :  $(\mathrm{Aut}(G) \cong 1)$

Figure 3-1.

Thm. 3-6. $\mathrm{Aut}(\bar{G}) \equiv \mathrm{Aut}(G)$.

PROOF. Let $\theta : \mathrm{Aut}(G) \to \mathrm{Aut}(\bar{G})$ and $f : V(G) \to V(\bar{G})$ both be identity maps, and observe that adjacency is preserved in a graph if and only if non-adjacency is preserved. $\square$

3-2. Operations on Permutations Groups

From a theorem due to Cayley, we recall that any finite group is abstractly isomorphic (as opposed to necessarily being identical) with a permutation group; in fact, if the group G has order n, then G is isomorphic to a subgroup of S_n. In this light, the operations soon to be defined could be regarded as applying to groups in general; however, the definitions will be given in terms of action upon a specified object set.

Let A and B be permutation groups acting on object sets X and Y respectively. We define three binary operations on these permutation groups as follows:

Def. 3-7.

1.) The *sum*, $A+B$, (or *direct product*) acts on the disjoint union $X \cup Y$; $A+B = \{a+b | a \in A, b \in B\}$, and

$$(a+b)(z) = \begin{cases} az, & \text{if } z \in X \\ bz, & \text{if } z \in Y. \end{cases}$$

2.) The *product*, $A \times B$, (or *cartesian product*) acts on $X \times Y$; $A \times B = \{a \times b | a \in A, b \in B\}$, and $(a \times b)(x,y) = (ax, by)$.

3.) The *composition*, $A[B]$, (or *wreath product*) acts on $X \times Y$ as follows: for each $a \in A$ and any sequence $b_1, b_2, \ldots, b_d$ (where $d = |X|$) in B, there is a unique permutation in $A[B]$, written $(a; b_1, b_2, \ldots, b_d)$, and $(a; b_1, b_2, \ldots, b_d)\ (x_i, y_j) = (ax_i, b_i y_j)$. Thus $A[B] = \{(a,f) | a \in A, f : X \to B\}$, $(a,f)(x,y) = (ax, b_x y)$, where $f(x) = b_x$.

Note: The order of $A[B]$ is $|A||B|^d$.

Thm. 3-8. $A + B \cong A \times B$.

PROOF. Let $\theta : A+B \to A \times B$ be given by $\theta(a+b) = (a \times b)$. □

3-3. Computing Automorphism Groups of Graphs

The following theorems indicate some connections between the graphical operations defined in Section 2-4 and the group operations defined above. The groups S_n, A_n, $\mathbb{Z}_n$, D_n are respectively the symmetric and alternating groups of degree n, the cyclic group of order n, and the dihedral group of order $2n$. The first theorem is due to Frucht [F6].

Thm. 3-9. If G is a connected graph, then $\text{Aut}(nG) \equiv S_n[\text{Aut}(G)]$.

Thm. 3-10. If no component of G_1 is isomorphic with a component of G_2, then $\text{Aut}(G_1 \cup G_2) \equiv \text{Aut}(G_1) + \text{Aut}(G_2)$.

PROOF. See Problem 3-3. □

Thm. 3-11. Let $G = n_1G_1 \cup n_2G_2 \cup \cdots \cup n_rG_r$, where n_i is the number of components of G isomorphic to G_i. Then $\mathrm{Aut}(G) \equiv S_{n_1}[\mathrm{Aut}(G_1)] + S_{n_2}[\mathrm{Aut}(G_2)] + \cdots S_{n_r}[\mathrm{Aut}(G_r)]$.

PROOF. Apply Theorems 3-9 and 3-10, using induction. □

Note: Any graph G may be written as in Theorem 3-11, but if $r = n_r = 1$, the theorem gives no information.

Thm. 3-12. If no component of $\bar{G}_1$ is isomorphic with a component of $\bar{G}_2$, then $\mathrm{Aut}(G_1 + G_2) \equiv \mathrm{Aut}(G_1) + \mathrm{Aut}(G_2)$.

PROOF. Apply Theorems 3-6 and 3-10, together with Problem 2-6. □

The following two theorems are due to Sabidussi ([S1] and [S2] respectively).

Def. 3-13. A non-trivial graph G is said to be *prime* if $G = G_1 \times G_2$ implies that either G_1 or G_2 must be trivial (i.e. $= K_1$). If G is not prime, G is *composite.* Two graphs G_1 and G_2 are *relatively prime* if $G_1 = G_3 \times G_4$ and $G_2 = G_3 \times G_5$ imply $G_3 = K_1$.

Thm. 3-14. $\mathrm{Aut}(G_1 \times G_2) \equiv \mathrm{Aut}(G_1) \times \mathrm{Aut}(G_2)$ if and only if G_1 and G_2 are relatively prime.

Def. 3-15. The *neighborhood* of a vertex u is given by: $N(u) = \{v \in V(G) | uv \in E(G)\}$. The *closed neighborhood* is $N[u] = N(u) \cup \{u\}$.

Thm. 3-16. If G_1 is not totally disconnected, then $\mathrm{Aut}(G_1[G_2]) \equiv \mathrm{Aut}(G_1)[\mathrm{Aut}(G_2)]$, if and only if:

(i) If there are two vertices in G_1 with the same neighborhood, then G_2 is connected, and
(ii) if there are two vertices in G_1 with the same closed neighborhood, then $\bar{G}_2$ is connected.

We are now able to list the automorphism groups for several common families of graphs.

Thm. 3-17.

1.) $\text{Aut}(K_n) \equiv S_n$
2.) $\text{Aut}(C_n) \equiv D_n$
3.) $\text{Aut}(K_{m,n}) \equiv \begin{cases} S_2[S_n], & \text{if } m = n \\ S_m + S_n, & \text{if } m \neq n \end{cases}$
4.) $\text{Aut}(K_n[\bar{K}_m]) \equiv S_n[S_m]$.

3-4. Graphs with a Given Automorphism Group

Thm. 3-18. Every finite group is the automorphism group of some graph.

For a proof of this theorem, due to Frucht [F5], see Section 4-2.

3-5. Problems

3-1.) Does there exist an identity graph (other than K_1) of order five or less? (Hint: Check to see that every graph in Appendix 1, Harary [H3], with $p \leq 5$, has at least one non-trivial automorphism.)
3-2.) Supply details for the proof of Theorem 3-8.
3-3.) Prove Theorem 3-10.
3-4.) Prove Theorem 3-17.
3-5.) Let $G = K_{p,q,r}$; find $\text{Aut}(G)$.
3-6.) $\text{Aut}(K_4) \equiv S_4$, yet K_4 is the 1-skeleton of the tetrahedron, and the symmetry group of the tetrahedron is A_4. Explain!
3-7.) For each of the other platonic solids (see Section 5-4), find the symmetry group for both the solid and for its 1-skeleton (the associated graph). Is one a subgroup of the other? If so, what is its index?
3-8.) For $k \geq 2$, the *odd graph* O_k has all $(k-1)$-subsets of a $(2k-1)$-set as vertices, with adjacency corresponding to the property of being disjoint. Then $O_2 = K_3$ and $O_3 = \Pi$, the *Petersen* graph. (Show that this is consistent with Figure 8-8.) It is immediate that $S_{2k-1} \leq \text{Aut}(O_k)$; show that in fact $S_{2k-1} = \text{Aut}(O_k)$. In particular, $\text{Aut}(\Pi) = S_5$.

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CHAPTER 4

THE CAYLEY COLOR GRAPH OF A GROUP PRESENTATION

In this chapter we see that each group can be defined in terms of generators and relations and that corresponding to such a presentation there is a unique graph, called the Cayley color graph of the presentation. A "drawing" of this graph gives a "picture" of the group, from which can be determined certain properties of the group. We will establish some basic results about Cayley color graphs, including a rather natural correspondence between direct products of groups and cartesian products of associated Cayley color graphs. In Chapter 7 we will ask which groups have Cayley color graphs that can be represented properly in the plane, and associated questions. In Chapter 9 appropriate answers will solve the Heawood map-coloring problem, as well as many others. The voltage graph theory of Chapter 10, the block design connection of Chapter 12, and the map automorphisms of Chapter 16 will be especially natural in the context of Cayley graphs. And, each change-ringing graph of Chapter 19 will be a Cayley color graph.

4-1. Definitions

Def. 4-1. Let Γ be a group, with $\{g_1, g_2, g_3, \dots\}$ a subset of the element set of Γ. A *word* W in $g_1, g_2, g_3, \dots$ is a finite product $f_1 f_2 \dots f_n$, where each f_i is in the set $\{g_1, g_2, g_3, \dots, g_1^{-1}, g_2^{-1}, g_3^{-1}, \dots\}$. If every element of Γ can be expressed as a word in $g_1, g_2, g_3, \dots$, then $g_1, g_2, g_3, \dots$ are said to be *generators* for Γ. A *relation* is an equality between two words in $g_1, g_2, g_3, \dots$.

Thm. 4-2. Given an arbitrary set of symbols and an arbitrarily prescribed set (possibly empty) of relations in these symbols, there is a unique (up to isomorphism) group with the symbols as generators and with structure determined by the prescribed relations.

PROOF. (For a proof, see [MKS1].) □

Def. 4-3. If Γ is generated by $g_1, g_2, g_3, \dots$ and if every relation in Γ can be deduced from the relations $P = P'$, $Q = Q'$, $R = R', \dots$, then we write $\Gamma = \langle g_1, g_2, g_3, \dots | P = P',\ Q = Q',\ R = R', \dots \rangle$, and the right hand side of the equation is said to be a *presentation* of Γ. A presentation is said to be a *finitely generated* (*finitely related*) if the number of generators (defining relations) is finite. A *finite presentation* is both finitely generated and finitely related.

For example:

(1) $\mathbb{Z}_n = \langle x | x^n = e \rangle$
(2) $\mathbb{Z} = \langle x | \rangle$
(3) $S_n = \langle x, y | x^n = y^2 = e = (xy)^{n-1} \rangle$
(4) $D_n = \langle x, y | x^n = y^2 = e = (xy)^2 \rangle$
(5) $\mathbb{Z}_2 \times \mathbb{Z}_{2n} = \langle x, y | x^2 = y^{2n} = e = xyxy^{-1} \rangle$

Thm. 4-4. Every finite group has a finite presentation.

PROOF. Take Γ itself as the set of generators, with all relations of the form $g_i g_j = g_k$, as determined by the group operations (i.e. the multiplication table serves as a finite presentation). □

Def. 4-5. For every group presentation there is associated a *Cayley color graph*: the vertices correspond to the elements of the group; next, imagine the generators of the group to be associated with distinct colors. If vertices v_1 and v_2 correspond to group elements g_1 and g_2 respectively, then there is a directed edge (of the color (or label) of generator h) from v_1 to v_2 if and only if $g_1 h = g_2$; see Figure 4-1.

$$g_1 \circ \xrightarrow{h} \circ\ g_1 h = g_2$$

Figure 4-1.

Let P be a presentation for the group Γ; we denote the Cayley color graph of P for Γ by $C_P(\Gamma)$, or (when convenient) by $C_\Delta(\Gamma)$, where Δ denotes the generating set. (Since a group may have more than one generating set, the Cayley color graph depends on Δ, as well as Γ). Then $C_\Delta(\Gamma)$ is a labeled, directed graph, with a color (or label) assigned to each edge. Thus $V(C_\Delta(\Gamma)) = \Gamma$, and $E(C_\Delta(\Gamma)) = \{(g, g\delta)_\delta | g \in \Gamma, \delta \in \Delta\}$. We observe that the following correspondences occur:

Group	Cayley Color Graph
element	vertex
generator	a set of directed edges of the same color
inverse of a generator	the same set of edges (now) directed against the arrow)
word	walk
multiplication of elements	succession of walks
identity word (relator)	closed walk
solvability of $rx = s$	(weakly) connected di-graph

Note: A characterization is given in [MKS1] of those graphs G which can be oriented and colored so as to form Cayley color graphs.

Historical note: Max Dehn (in 1911) formulated three fundamental decision problems concerning group presentations. One of these is: "determine in a finite number of steps, for two arbitrary words W and W' in the generators, whether $W = W'$ or not." Equivalently: "construct the Cayley color graph for a given group presentation."

The term "connected" may have a "stronger" meaning for directed graphs than for graphs in general, since we may be allowed to travel only in the direction of the arrow along a given directed edge.

Def. 4-6. A directed graph D is said to be *strongly connected* if, for every pair u, v of distinct vertices, there is a directed path from u to v. D is said to be *unilaterally connected* if, for every pair of distinct vertices, one is joined to the other by a directed path. D is called *weakly connected* if the (undirected) pseudograph underlying D is connected.

For example, see Figure 4-2, where D is strongly connected, D' is unilaterally connected (but not strongly connected) and D'' is weakly connected (but not unilaterally connected).

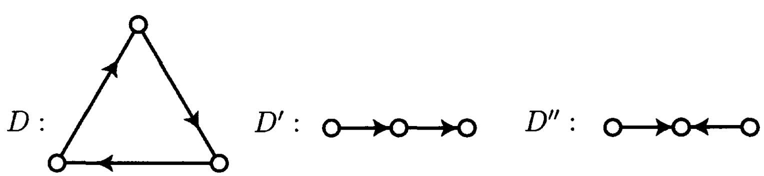


Figure 4-2.

4-2. Automorphisms

We have previously defined an automorphism of a graph G (as a permutation of $V(G)$ preserving adjacency). An automorphism of a directed graph must preserve *directed* adjacency; and an automorphism of a Cayley color graph must also preserve the *color* corresponding to each adjacency. We summarize in:

Def. 4-7. An *automorphism* of a Cayley color graph $C_\Delta(\Gamma)$ is a permutation θ of $V(C_\Delta(\Gamma))$ such that, for each g_1, g_2 in Γ and h in Δ, $g_1 h = g_2$ if and only if $\theta(g_1)h = \theta(g_2)$.

Equivalently (see Problem 4-1), θ is an automorphism of $C_\Delta(\Gamma)$ if and only if: for each g in Γ and generator h in Δ, $\theta(gh) = \theta(g)h$; i.e. the diagram in Figure 4-3 commutes.

$$\begin{tikzcd} g \arrow[r, "h"] \arrow[d, dotted, "\theta"'] & gh \arrow[d, dotted, "\theta"] \\ \theta(g) \arrow[r, "h"] & \theta(gh) = \theta(g)h \end{tikzcd}$$

Figure 4-3.

As expected, the collection of all automorphisms of $C_\Delta(\Gamma)$ forms a group, called the *automorphism group* of $C_\Delta(\Gamma)$, and denoted by $\mathrm{Aut}(C_\Delta(\Gamma))$. The next result is perhaps not expected.

Thm. 4-8. Let $C_\Delta(\Gamma)$ be any Cayley color graph for the finite group Γ; then $\mathrm{Aut}(C_\Delta(\Gamma)) \cong \Gamma$ (independent of the presentation selected for G.)

PROOF. Define $\alpha : \Gamma \to \mathrm{Aut}(C_\Delta(\Gamma))$ by $\alpha(g) = \theta_g$, where $\theta_g : V(C_\Delta(\Gamma)) \to V(C_\Delta(\Gamma))$ is given by $\theta_g(g_i) = gg_i$. First we show that $\theta_g \in \mathrm{Aut}(C_\Delta(\Gamma))$. Clearly θ_g is one-to-one and onto (and hence permutes $V(C_\Delta(\Gamma))$). Also, $\theta_g(g_i h) = g(g_i h) = (gg_i)h = \theta_g(g_i)h$, so that α is well- defined.

Now, α preserves products: $\alpha(gg^*) = \theta_{gg*}$, defined by $\theta_{gg*}(g_i) = gg^* g_i = \theta_g(g^* g_i) = \theta_g(\theta_g^*(g_i)) = (\theta_g \theta_{g^*})(g_i)$; that is $\alpha(gg^*) = \alpha(g)\alpha(g^*)$.

It is clear that α is one-to-one, since $\ker\alpha = \{e\}$.

It remains to show that α is onto. Let $\theta \in \mathrm{Aut}(C_\Delta(\Gamma))$. Let $\theta(e) = g$, where e is the identity of Γ. Now any g^* in Γ can be written as a word in the generators for Γ; i.e. $g^* = h_1^{a_1} h_2^{a_2} \cdots h_m^{a_m}$, where h_i is a generator for Γ and $a_i = \pm 1$. Then $\theta(g^*) = \theta(eg^*) = \theta(e) h_1^{a_1} h_2^{a_2} \cdots h_m^{a_m} = gg^*$; that is, $\theta = \theta_g$, so that α is onto. This completes the proof. □

Cor. 4-9. If $C_{\Delta_1}(\Gamma_1) \cong C_{\Delta_2}(\Gamma_2)$, then $\Gamma_1 \cong \Gamma_2$.

From the above theorem (and its proof) it is evident that any vertex of $C_\Delta(\Gamma)$ can be labeled with the identity e of Γ, and that once this has been done (for fixed assignment of colors to the generators) all other vertex labellings are determined; that is, $C_\Delta(\Gamma)$ is vertex transitive. (More precisely, a graph G is *vertex transitive* if, for each pair $u, v \in V(G)$, there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(u) = v$. Vertex transitive graphs are necessarily regular, but not conversely, as shown by $C_3 \cup C_4$). Moreover, $\mathrm{Aut}(C_\Delta(\Gamma))$ is a regular permutation group, on $V(C_\Delta(\Gamma))$; that is:

Cor. 4-10. Γ acts regularly on $V(C_\Delta(\Gamma))$.

We are now able to provide a proof of Frucht's Theorem, Theorem 3-18: Every finite group is the automorphism group of some graph.

PROOF. Let Γ be a finite group, and let Δ be a generating set for Γ. Form the Cayley color graph $C_\Delta(\Gamma)$; by Theorem 4-8, we know that $\mathrm{Aut}(C_\Delta(\Gamma)) \cong \Gamma$. It only remains to convert $C_\Delta(\Gamma)$ to a *graph* G having the same automorphism group, Γ. This is done as follows: let $\Delta = \{\delta_1, \delta_2, \cdots, \delta_n\}$. Replace each edge (g_i, g_j), where $g_j = g_i \delta_k$, by a path: $v_i, u_{ij}, u'_{ij}, v_j$. At vertex $u_{ij}(u'_{ij})$ we attach a new path $P_{ij}(P'_{ij})$ of length $2k - 1(2k)(1 \le k \le n)$; see Figure 4-4, for the case $k = 2$. In this way the "non-graphical" features of direction and label, present in $C_\Delta(\Gamma)$, are incorporated into the graph G. It is clear that $\mathrm{Aut}(G) \cong \mathrm{Aut}(C_\Delta(\Gamma)) \cong \Gamma$. □

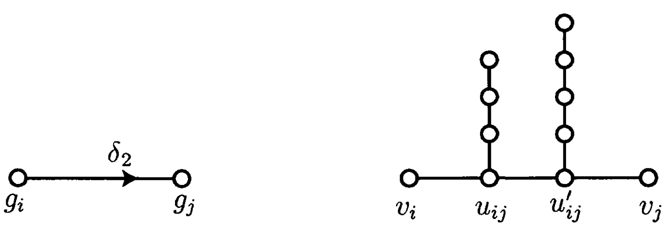


Figure 4-4.

4-3. Properties

It is clear that every Cayley color graph is both regular (and in fact vertex transitive) and connected (as a graph); the converse is not true (see Problem 4-10). For two characterizations of graphs which can be regarded as Cayley color graphs, see [J3] and [MKS1].

We may study additional properties for Γ (apart from the multiplication table so conveniently summarized in $C_\Delta(\Gamma)$) from $C_\Delta(\Gamma)$ as follows:

Thm. 4-11. Γ is commutative if and only if, for every pair of generators h_i and h_j, the walk $h_i h_j h_i^{-1} h_j^{-1}$ is closed.

PROOF. See Problem 4-2. □

Def. 4-12. An element of a generating set for a group Γ is said to be *redundant* if it can be written as a word in the remaining generators. A generating set is said to be *minimal* if it contains no redundant generators.

Example: $\{x^2, x^3\}$ is a minimal generating set for $\mathbb{Z}_6 = \langle x | x^6 = e\rangle$, even though $\{x\}$ is a generating set with fewer elements.

Thm. 4-13. Let Γ be a finite (infinite) group. A generator h is redundant if and only if the deletion of all edges colored h in $C_\Delta(\Gamma)$ leaves a strongly (weakly) connected directed graph.

PROOF. See Problem 4-3. □

Thm. 4-14. If h is not redundant, the removal of all edges colored h leaves a collection of isomorphic disjoint subgraphs, each representing the subgroup of Γ generated by the generating set of Γ minus h.

PROOF. See Problem 4-4. □

Thm. 4-15. Let Γ be a finite group with minimal generating set $\{h_1, h_2, \cdots, h_n\}$, and Ω a (necessarily proper) subgroup with generating set $\{h_2, h_3, \cdots, h_n\}$. Let $C_1, C_2, \cdots, C_k$ be the weak components of the directed graph $C_{h_1}(\Gamma)$, obtained from $C_\Delta(\Gamma)$, by deleting the edges colored h_1. Then Ω is normal in Γ if and only if the deleted directed edges from any given component C_i all lead to a single other component C_j.

PROOF. (i) Assume the condition holds. Let $C_1 = \Omega$ be the component containing e, let $g \in C_1$ and $r \in \Gamma$. We must show that $rgr^{-1} \in C_1$. We write $r = a_1^{b_1} a_2^{b_2} \cdots a_m^{b_m}$, where a_i is a generator of Γ and $b_i = \pm 1$. If h_1 occurs in r exactly w times with $b_i = +1$ and v times with $b_i = -1$, then the walk corresponding to r leads from e (in C_1) through $w - v$ components, ending in C_{1+w-v}. The walk corresponding to g in C_{1+w-v} now leads to another vertex in C_{1+w-v}, and walk $a_m^{-b_m} \cdots a_2^{-b_2} a_1^{-b_1}$, corresponding to r^{-1}, returns us to C_1.

(ii) Suppose that edges colored h_1 lead from C_i to C_1 and $C_j, 1 \neq j$. (Again assume $e \in C_1$.) Then there exists $g \in C_1$ such that $h_1^{-1} g h_1 \in C_j$, so that Ω is not normal in Γ. □

It now follows that, for Ω (as above) normal in Γ, the elements of the factor group Γ/Ω (i.e. the right cosets) are the components of $C_{h_1}(\Gamma)$. By shrinking these components, each to a single vertex, and restoring the edges colored h_1 (this can be done unambiguously, by Theorem 4-15), *a Cayley color graph of* Γ/Ω *is obtained.* (This "shrinking" may be described by adjoining, to the defining relations for Γ, the additional relations $h_2 = h_3 = \cdots = h_r = e$.)

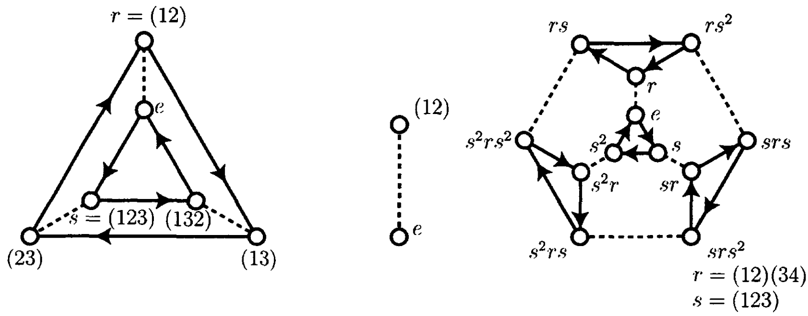
In general, given a Cayley color graph $C_\Delta(\Gamma)$, whether a subgroup Ω of Γ is normal or not, we obtain a *Schreier* (right) *coset graph* $S_\Delta(\Gamma/\Omega)$ as follows: the vertices are the right cosets of Ω in Γ, and there is an edge directed from Ωg to $\Omega g'$, labeled with $\delta \in \Delta$, if and only if $\Omega g \delta = \Omega g'$, (i.e. if and only if $\delta \in g^{-1}\Omega g'$.) That $\Omega g d$ is a right coset follows from the fact that the right cosets of Ω in Γ partition Γ. Note that a Schreier coset graph may actually be a pseudograph, as loops and/or multiple edges may result from this process. For the special case $\Omega = \{e\}$, the Schreier coset graph is just the Cayley color graph $C_\Delta(\Gamma)$.

Several of the ideas discussed above are illustrated in Figure 4-5. Note that $\Omega = \mathbb{Z}_3$ is normal in S_3, but not in A_4.

As a further example, contrast the groups $\mathbb{Z}_2 \times \mathbb{Z}_4$ and D_4, as in Figure 4-6. Note that the subgroup of order 2 generated by r is normal in $\mathbb{Z}_2 \times \mathbb{Z}_4$, but not in D_4. This comment extends in an obvious way to the groups $\mathbb{Z}_2 \times \mathbb{Z}_n$ and $D_n, n \geq 3$. For example, see $S_3 = D_3$ (in Figure 4-5); the subgroup generated by r is not normal here, either. For a generator δ of order 2, we adopt the standard convention of representing the two directed edges $(g, g\delta)$ and $(g\delta, g)$ in $C_\Delta(\Gamma)$ by a single undirected edge $\{g, g\delta\}$.

In Figure 4-7 we give the Schreier coset graph for $\Omega = \{e, r\}$, in $\Gamma = D_4$ (see Figure 4-6).

We close this section with a theorem due to Gross [G6].



$S_3 : r^2 = s^3 = (rs)^2 = e$ $\quad$ $S_3/\mathbb{Z}_3$ $\quad$ $A_4 : r^2 = s^3 = (rs)^3 = e$

Figure 4-5.

$\mathbb{Z}_2 \times \mathbb{Z}_4 : r^2 = s^4 = rsrs^{-1} = e$ $\quad$ $D_4 : r^2 = s^4 = (rs)^2 = e$

Figure 4-6.

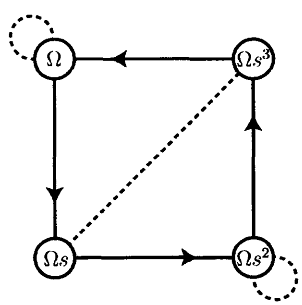


Figure 4-7.

Thm. 4-16. Every connected regular graph of even degree underlies a Schreier coset graph.

4-4. Products

We now develop a relationship between the direct product for groups and the cartesian product for graphs. Recall the following from group theory:

Def. 4-17. Let Γ_1 and Γ_2 both be subgroups of the same group Γ, with $\Gamma_1 \cap \Gamma_2 = \{e\}$ and $gh = hg$ for all $g \in \Gamma_1$, $h \in \Gamma_2$. Then $\Gamma_1 \times \Gamma_2 = \{gh | g \in \Gamma_1, h \in \Gamma_2\}$ is also a subgroup of Γ, called the *direct product* of Γ_1 and Γ_2.

$$\begin{aligned}
\text{If } \Gamma_1 &= \langle k_1, \cdots, k_m | w_1 = \cdots = w_r = e\rangle \text{ and} \\
\Gamma_2 &= \langle k_{m+1}, \cdots, k_n | w_{r+1} = \cdots w_{r+s} = e\rangle, \\
\text{then } \Gamma_1 \times \Gamma_2 &= \langle k_1, \cdots, k_n | w_1 = \cdots = w_{r+s} = k_i k_j k_i^{-1} k_j^{-1} = e, \\
&\qquad \text{for all } 1 \leq i \leq m < j \leq n\rangle
\end{aligned}$$

is a presentation for $\Gamma_1 \times \Gamma_2$, called the *standard presentation* for $\Gamma_1 \times \Gamma_2$.

This binary operation may be extended to the class of all groups, by noting that $\Gamma_1 \cong \Gamma_1' = \{(g, e_2) | g \in \Gamma_1, e_2 \text{ is the identity of } \Gamma_2\}$, $\Gamma_2 \cong \Gamma_2' = \{(e_1, h) | h \in \Gamma_2, e_1 \text{ is the identity of } \Gamma_1\}$, and defining $\Gamma_1 \times \Gamma_2 = \{(g, h) | g \in \Gamma_1, h \in \Gamma_2\}$, with $(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$ giving the group operation.

Also recall the following (see, for example [BM1, p. 348]):

Thm. 4-18. (The Fundamental Theorem of Finite Abelian Groups): Let Γ be a finite abelian group of order n; then $\Gamma = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_r}$, where m_i divides m_{i-1}, $i = 2, \cdots, r$ and $\prod_{i=1}^{r} m_i = n$; furthermore, this decomposition is unique.

(We assume $m_r > 1$, unless $n = 1$, in which case $m_r = r = 1$.)

Def. 4-19. The number r of Theorem 4-18 is called the *rank* of the abelian group Γ.

Theorem 4-18 completely specifies the structure of finite abelian groups. The next theorem specifies, as a corollary, a Cayley color graph for every finite abelian group. We first extend the definition of cartesian product for graphs to Cayley color graphs, in the natural way.

Def. 4-20. The *cartesian product*, $C_{\Delta_1}(\Gamma_1) \times C_{\Delta_2}(\Gamma_2)$, of two Cayley color graphs is given by: $V(C_{\Delta_1}(\Gamma_1) \times C_{\Delta_2}(\Gamma_2)) = V(C_{\Delta_1}(\Gamma_1)) \times V(C_{\Delta_2}(\Gamma_2))$; and (g_1, g_2) is joined to $(g_1\prime, g_2\prime)$ by an edge colored h if and only if either:

(i) $g_1 = g_1\prime$ and $g_2 h = g_2\prime$, for h a generator in Δ_2

or

(ii) $g_2 = g_2\prime$ and $g_1 h = g_1\prime$, for h a generator in Δ_1.

Figure 4-8 shows $C_{\Delta_1}(\mathbb{Z}_3) \times C_{\Delta_2}(\mathbb{Z}_2)$, where $\mathbb{Z}_3 = \langle x | x^3 = e\rangle$ and $\mathbb{Z}_2 = \langle y | y^2 = e\rangle$.

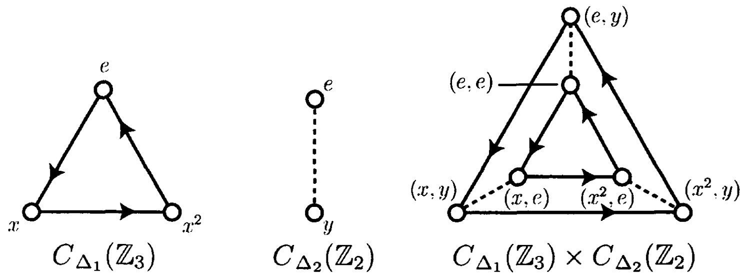


Figure 4-8.

Thm. 4-21. Let $C_{P_i}(\Gamma_i)$ be the Cayley color graph associated with presentation P_i from group $\Gamma_i, i = 1, 2$. Let P be the standard presentation for $\Gamma_1 \times \Gamma_2$. Then

$$C_P(\Gamma_1 \times \Gamma_2) = C_{P_1}(\Gamma_1) \times C_{P_2}(\Gamma_2).$$

PROOF. First we note that $V(C_{P_1}(\Gamma_1) \times C_{P_2}(\Gamma_2)) = V(C_{P_1}(\Gamma_1)) \times V(C_{P_2}(\Gamma_2)) = V(C_P(\Gamma_1 \times \Gamma_2))$. We now show that the edge sets of the two Cayley color graphs coincide (in colored directed adjacency.)

(i) Let (g_1, g_2) be joined to $(g_1\prime, g_2\prime)$ by an edge colored h in $C_P(\Gamma_1 \times \Gamma_2)$. Then $h = k_i$, for some $1 \le i \le n$. If $1 \le i \le m$, then h is a generator of Γ_1, and

$$(g_1\prime, g_2\prime) = (g_1, g_2)(h, e_1) = (g_1 h, g_2),$$

so that $g_1\prime = g_1 h$ and $g_2\prime = g_2$; i.e. this directed, colored edge in $C_P(\Gamma_1 \times \Gamma_2)$ is also in $C_{P_1}(\Gamma_1) \times C_{P_2}(\Gamma_2)$. A similar argument applies for $m < i \le n$, so that

$$E(C_P(\Gamma_1 \times \Gamma_2)) \subseteq E(C_{P_1}(\Gamma_1) \times C_{P_2}(\Gamma_2)).$$

(ii) The argument is reversible, to show that

$$E(C_{P_1}(\Gamma_1) \times C_{P_2}(\Gamma_2)) \subseteq E(C_P(\Gamma_1 \times \Gamma_2)).$$

This completes the proof. $\square$

Since the cyclic group $\mathbb{Z}_n$ with presentation $P : \mathbb{Z}_n = \langle x | x^n = e\rangle$ has the readily constructed Cayley color graph $C_P(\mathbb{Z}_n) = C_n'$ (where C_n' denotes the directed cycle of length n), it is a simple matter to construct, using Theorems 4-18 and 4-21, a Cayley color graph for *any* finite abelian group.

Thm. 4-22. Let Γ be a finite abelian group; then $C'_{m_1} \times C'_{m_2} \times \cdots \times C'_{m_r}$ is a Cayley color graph for Γ, where $\Gamma = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_r}$.

The class of groups for which we can construct Cayley color graphs using Theorem 4-21 can be enlarged as follows:

Def. 4-23. A non-abelian group Γ is said to be *hamiltonian* if every subgroup of Γ is normal in Γ.

Clearly all abelian groups have this normality property for subgroups. That non-abelian groups may also have all subgroups normal is illustrated by Q, the quaternions (one of the two non- abelian groups of order eight). But more: the finite hamiltonian groups are characterized (see Coxeter and Moser [CM1], p. 8):

Thm. 4-24. Γ is a finite hamiltonian group if and only if $\Gamma = Q \times A_1 \times A_2$, where A_1 is a finite abelian group of odd order, and A_2 is a group for which $a^2 = e$, for every $a \in A_2$.

Since elementary group theory shows that A_2 must be abelian, we can apply Theorem 4-21 to find a Cayley color graph for Γ, providing we know a Cayley color graph for Q. This latter Cayley color graph will be produced in Chapter 7; it turns out to be the Cayley color graph of minimum order which cannot be drawn properly in the plane (among those on minimal generating sets.)

4-5. Cayley Graphs

Let Δ be a generating set for the group Γ subject to the following conditions:

(i) $e \notin \Delta$
(ii) If $\delta \in \Delta, \delta^{-1} \notin \Delta$ (unless $\delta^2 = e$).
Also, we adopt the following convention:
(iii) If $\delta \in \Delta, \delta^2 = e$, each pair $(g, g\delta)$ and $(g\delta, g)$ of directed edges are coalesced into a single undirected edge $\{g, g\delta\}$.

Then the pseudograph obtained from the Cayley color graph $C_\Delta(\Gamma)$ by suppressing all edge directions and all edge labels (colors) has no loops (by (i)) and no multiple edges (by (ii) and (iii)); it is in fact a graph.

Def. 4-25. If Δ satisfies (i), (ii), and (iii) above, then the graph underlying the Cayley color graph $C_\Delta(\Gamma)$ is called a *Cayley graph* and is denoted by $G_\Delta(\Gamma)$.

It is clear that, in passing from $C_\Delta(\Gamma)$ to $G_\Delta(\Gamma)$, only structural properties are lost. Thus in the topological considerations to follow in the rest of the book, it will be without loss that we restrict our attention to $G_\Delta(\Gamma)$. Of course, $V(G_\Delta(\Gamma)) = \Gamma$, whereas $E(G_\Delta(\Gamma)) = \{\{g, g\delta\} | g \in \Gamma, \delta \in \Delta \cup \Delta^{-1}\}$, where $\Delta^{-1} = \{\delta^{-1} | \delta \in \Delta\}$.

4-6. Problems

4-1.) Show that θ (a permutation of Γ) is an automorphism of $C_\Delta(\Gamma)$ if and only if: for each g in Γ and h in Δ, $\theta(gh) = \theta(g)h$.

4-2.) Prove Theorem 4-11.

4-3.) Prove Theorem 4-13. Give an example of a infinite group with a redundant generator whose deletion does not leave a strongly connected digraph.

4-4.) Prove Theorem 4-14.

4-5.) How many isomorphic disjoint subgraphs are there, as in the statement of Theorem 4-14?

4-6.) Give a graph -theoretic proof of the fact that a finite subgroup of index 2 must be normal.

4-7.) Let Γ be an abelian group of order pq, where p and q are distinct primes. By one of the Sylow theorems, Γ has $\mathbb{Z}_p$ as a subgroup. Give a graph-theoretic proof that $\mathbb{Z}_p$ is normal, first finding $C_P(\mathbb{Z}_p \times \mathbb{Z}_q)$. Then modify this Cayley color graph, to obtain a Cayley color graph of $\Gamma/\mathbb{Z}_p$.

4-8.) Compile an appendix of Cayley color graphs for all groups of order ≤ 12. (This appendix should be useful for reference, both in this course and in later life. You might want to save some work by doing $\mathbb{Z}_n, \mathbb{Z}_2 \times \mathbb{Z}_n$, and D_n in general, rather than in each case where appropriate. Also, you will find that some of your graphs cannot be properly represented in the plane; these should be re-drawn, following Chapter 7.)

4-9.) Show that, if Γ is finite, then $C_\Delta(\Gamma)$ is always strongly connected. Give an example to show that this need not be true, if Γ is infinite.

4-10.) Show that the Petersen graph Π (see Figure 8-8 and Problem 3-8) *cannot* be colored and labeled so as to be a Cayley color graph. (Note that Π is vertex transitive, by Problem 3-8.)

4-11.) Show that the Heawood graph (see Figure 8-5) *can* be colored and labeled so as to be a Cayley color graph. (This is curious, as the Petersen graph is the unique 5-cage while the Heawood graph is the unique 6-cage).

4-12.) Find a connected regular (p, q) graph which is *not* a Cayley graph, of (i) minimum p; (ii) minimum q (by Problem 4-10, $p \leq 10$ and $q \leq 15$).

4-13.) Find Δ so that $C_3 \times C_3 = G_\Delta(\mathbb{Z}_3 \times \mathbb{Z}_3)$. Show that there is *no* Δ such that $C_3 \times C_3 = G_\Delta(\mathbb{Z}_9)$.

4-14.) A *cut-vertex (bridge)* for a connected graph G is a vertex $v \in V(G)$ (edge $e \in E(G)$) such that $G - v(G - e)$ is disconnected. Show that the Cayley graph $G_\Delta(\Gamma)$ has no cut-vertices, and hence no bridges, for $|\Gamma| \geq 3$

4-15.) An *n-factor* of a graph G is a spanning n-regular subgraph of G. Petersen's Theorem is: a bridgeless cubic graph is the edge disjoint union of a 1-factor and a 2-factor. Illustrate this theorem, for Cayley graphs $G_\Delta(\Gamma)$, where Δ consists of two generators, exactly one of which has order two.

4-16.) Show that every Cayley graph $G_\Delta(\Gamma)$ can be expressed as an edge disjoint union of m 1-factors and n 2-factors, for some m and n such that $m + n = |\Delta|$.

4-17.) A graph G is *n-factorable* if it can be expressed as an edge disjoint union of n-factors (cf Problem 4-15). Find a non-trivial sufficient condition for a Cayley graph $G_\Delta(\Gamma)$ to be:

(i) 1-factorable
(ii) 2-factorable
(iii) 3-factorable
(iv) eulerian

4-18.) Show that the n-cube Q_n is m-factorable if and only if m divides n. (In particular, Q_n is 1-factorable, for all n.)

4-19.) **Babai [B2] conjectured that, for $|\Gamma| \geq 3$, $G_\Delta(\Gamma)$ is always hamiltonian. Prove or disprove!

4-20.) *Show that if Γ is a finite abelian group of order at least three and if Δ is a minimal generating set for Γ, then $G_\Delta(\Gamma)$ is hamiltonian. (Klerlein [K2] showed that, in fact, $C_\Delta(\Gamma)$ is hamiltonian.)

4-21.) Let P denote a property (in adjectival form) that a graph might possess (such as "eulerian" or "hamiltonian",) and let Γ denote a finite group. We say that Γ is P (universally P) if there exists a Δ for Γ such that (for all Δ for Γ) $G_\Delta(\Gamma)$ is P. Discuss:

(i) If Γ is abelian, $|\Gamma| \geq 3$, then Γ is hamiltonian and universally hamiltonian.
(ii) If Γ is hamiltonian (in the sense of Definition 4-23), then Γ is hamiltonian (cf. Problem 2-12 and Theorem 4-24) and (**) universally hamiltonian. (Klerlein and Starling

[KS1] showed that, in fact, $C_\Delta(\Gamma)$ is hamiltonian, for Γ hamiltonian and Δ minimal.)

(iii) If $|\Gamma| \geq 3$, then Γ is hamiltonian.

(iv) If $|\Gamma|$ is odd, then Γ is universally eulerian.

(v) If Γ is abelian, $|\Gamma| \geq 3$, then Γ is eulerian.

(vi) The abelian group Γ is universally eulerian if and only if $|\Gamma|$ is odd.

(vii) What other properties P might be studied?

4-22.) *Is Theorem 4-15 true for Γ infinite?

4-23.) Let Γ be a finite group, with $\Gamma = \Gamma_1 \cup \Gamma_2$. Prove that if Γ_1 does not generate Γ, then Γ_2 does. (Hint: use Problem 2-1.)

4-24.) Show that if Δ is a minimal generating set for Γ, then $|\Delta| \leq \log_2 |\Gamma|$; equality holds if and only if $\Gamma = \mathbb{Z}_2^m$.

CHAPTER 5

AN INTRODUCTION TO SURFACE TOPOLOGY

In this chapter we present an introduction to surface topology, including the statement and a brief discussion of the classification theorem for closed 2-manifolds and a complete development of the euler identity for the orientable case. One motivation for this material is that it gives us alternatives to the plane for drawing graphs in (for example, no Cayley color graph for the quaternions can be drawn properly in the plane); these alternatives are completely classified, and the euler identity gives us important information about them. We give a topological proof that there are exactly five regular polyhedra, and conclude the chapter with a brief discussion of pseudosurfaces.

5-1. Definitions

In this chapter, a *surface* will be a closed, orientable 2-manifold. Any such figure may be considered as a topological subspace of euclidean 3-space, $\mathbb{R}^3$. We consider the subspace topology to be that induced by the standard distance-measuring metric in $\mathbb{R}^3$. To pin down this idea of "surface", we must define the terms used in the first sentence of this paragraph. First, we specify that by the *open unit disk* we mean $\overset{\circ}{D} = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$.

Def. 5-1. A *2-manifold* is a connected topological space in which every point has a neighborhood homeomorphic to the open unit disk. Such a neighborhood is called a *2-cell.*

Note: In Definition 5-1, $\overset{\circ}{D}$ may be replaced by $\mathbb{R}^2$, since these two spaces are themselves homeomorphic.

Example: Only one of the conical spaces (the third) in Figure 5-1 is a 2-manifold.

Definition 5-1 may be extended as follows: an *n-manifold* is a connected topological space in which every point has a neighborhood

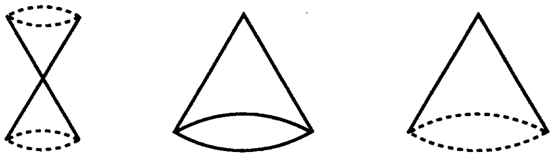


Figure 5-1.

homeomorphic to

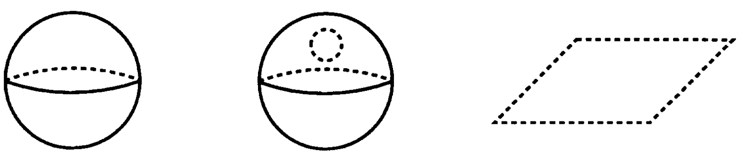
$$B_n = \left\{ (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} x_i^2 < 1 \right\}.$$

However, we are only concerned here with the case $n = 2$.

Def. 5-2. A subspace M of $\mathbb{R}^3$ is *bounded* if there exists a natural number n such that $M \subseteq B(O; n) = \{(x, y, z) | x^2 + y^2 + z^2 < n^2\}$.

Def. 5-3. Let $M \subseteq \mathbb{R}^3$ be a 2-manifold. M is said to be *closed* if it is bounded and the boundary of M coincides with M.

For example, M of Figure 5-2 is closed, while M' and M'' are not.



$M = S^2$ $\qquad$ $M' = S^2 - \overline{D}$ $\qquad$ $M'' = \mathbb{R}^2$

Figure 5-2.

Note that the term "closed" does not mean quite the same thing to a surface topologist as it does to a point-set topologist. What a surface topologist calls a "closed 2-manifold", a point-set topologist calls a "compact 2-manifold." (Recall that $M \subseteq \mathbb{R}^3$ is compact if and only if M is closed (point-set sense) and bounded.)

Def. 5-4. Let M be a 2-manifold: M is said to be *orientable* if, for every simple closed curve C on M, a clockwise sense of rotation is preserved by traveling once around C. Otherwise, M is *nonorientable.*

It can be shown that a 2-manifold M is orientable if and only if it is two-sided. For example, a cylinder open at both ends is orientable, where a Möbius strip (imbedded in $\mathbb{R}^3$ in the usual way) is not.

We offer an equivalent definition of orientability.

Def. 5-4′. M is *orientable* if it admits a 2-cell decomposition with *coherent* orientation (i.e. the boundary of each 2-cell is given an orientation so that a 1-cell portion of the boundary incident with two adjacent 2-cells is oppositely oriented within those two 2-cells.)

5-2. Surfaces and Other 2-manifolds

We finally know what a surface is (abstractly); now, exactly which subspaces of $\mathbb{R}^3$ are surfaces?

Let us begin to answer this question by representing certain familiar 2-manifolds as polygons with appropriate edges identified. See Figure 5-3 for the sphere, open cylinder, torus, projective plane, möbius strip, and klein bottle, respectively. The top three 2-manifolds are orientable, the bottom three nonorientable. Only the cylinder and möbius strip are not closed.

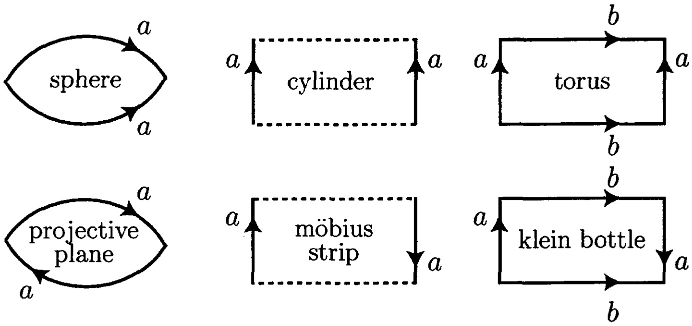


Figure 5-3.

It turns out that every closed 2-manifold (whether orientable or not) can be represented in this manner. In fact (see Fréchet and Fan, [FF1] p. 63) we have the following theorem:

Thm. 5-5. Every closed 2-manifold is elementarily associated with a polygon whose symbolic representation is of one of the following forms:

(i) aa^{-1}
(ii) $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\cdots a_pb_pa_p^{-1}b_p^{-1}$, $p = 1, 2, 3, \cdots$
(iii) $a_1a_1a_2a_2\cdots a_qa_q$, $q = 1, 2, 3, \cdots$

The form (i) corresponds to the sphere; (ii) to the sphere with p handles (a torus is a sphere with one handle) and (iii) to the sphere with q cross-caps (a projective plane is a sphere with one cross-cap; a klein bottle is a sphere with two cross-caps.) Only the forms(iii) correspond to nonorientable closed 2-manifolds. (None of these can be realized in $\mathbb{R}^3$.)

As a byproduct of the development in Fréchet and Fan, an invariant called the *characteristic* is determined for each closed 2-manifold. Then it is shown that:

Thm. 5-6. (The Classification Theorem) Two closed 2-manifolds are homeomorphic if and only if they have the same characteristic and are both orientable or both nonorientable.

It follows that a closed orientable 2-manifold (i.e. a surface) M is a sphere with k handles, where k is a non-negative integer; k is said to be the *genus* of M, and we write $\gamma(M) = k$ and $M = S_k$. A closed nonorientable 2-manifold M is a sphere with k crosscaps, where k is a positive integer; k is said to be the (nonorientable) genus of M, and we write $\tilde{\gamma}(M) = k$ and $M = N_k$.

5-3. The Characteristic of a Surface

We now give an independent determination of the characteristic of a surface, using the notion of a pseudograph. The proof will be by induction on k, the genus of the surface. We first need a few definitions, and one preliminary theorem. The first definition is intuitive; it will be made more precise in Chapter 6.

Def. 5-7. A pseudograph is said to be *imbedded* in a surface M if it is "drawn" in M so that edges intersect only at their common vertices.

For example, Figure 5-4 shows two drawings of K_4 in the plane, but only the second is an imbedding.

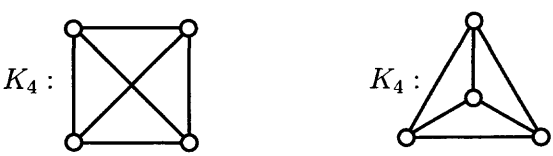


Figure 5-4.

Def. 5-8. A *tree* is a connected graph having no cycles.

For example, Figure 5-5 shows three graphs, but only G_3 is a tree.

Thm. 5-9. Let G be a tree, with p vertices and q edges; then $p = q+1$.

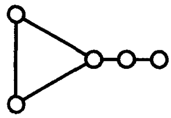
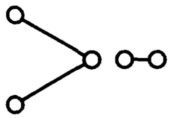
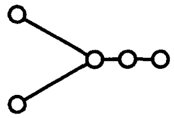
G_1 :  G_2 :  G_3 : 

Figure 5-5.

PROOF. (by induction on p): The result is clearly true for $p = 1$, for then $q = 0$. Now assume the result holds for all trees with fewer than p vertices, and let G be a tree with p vertices and q edges. Since G has no cycles and is finite, we can find $v \in V(G)$ such that $d(v) = 1$. Then $G - v$ is a tree with $p - 1$ vertices and $q - 1$ edges, so that $(p - 1) = (q - 1) + 1$; i.e. $p = q + 1$. □

Def. 5-10. Let a pseudograph G be imbedded in a surface M; the components of $M - G$ are called *regions* (or *faces*) of the imbedding.

For example, the imbedding of K_4 in Figure 5-4 has four regions.

The following theorem is attributed to both Descartes and Euler, independently; we perhaps indicate our preference by calling it the *euler polyhedral identity*:

Thm. 5-11. Let G be a connected graph imbedded in the sphere, S_0. Let G have p vertices and q edges, with r the number of regions of the imbedding. Then $p - q + r = 2$.

PROOF. (by induction on $q \geq 0$): The result is clearly true for $q = 0$, for then $p = 1$ and $r = 1$. Now assume the result holds for all connected graphs with fewer than q edges, and let G be a connected graph with q edges, p vertices, and r regions for an imbedding in S_0. We have two cases to consider:

(i) If G is a tree, then $p = q + 1$ by Theorem 5-9, and $r = 1$ (since there are no cycles), so that $p - q + r = 2$.
(ii) If G is not a tree, then (since G is connected) G contains a cycle; let x be any edge of this cycle. Then $G - x$ has p vertices, $q - 1$ edges, is still connected, and is imbedded in S_0 with $r-1$ regions. Hence $p - (q - 1) + (r - 1) = 2$; i.e. $p - q + r = 2$.

□

Cor. 5-12. Let G be a connected pseudograph imbedded in S_0, with p vertices, q edges, and r regions; then $p - q + r = 2$.

PROOF. See Problem 5-3. □

We observe here that imbedding a graph in the sphere is equivalent to imbedding it in the plane. To see this, perform a *stereographic projection* (see Figure 5-6) with the north pole of the sphere any point in the interior of some region of the imbedding. For each point of the sphere, there corresponds a unique point of the plane: the intersection of the line L through $(0,0,2)$ and (x,y,z) with the plane. The mapping is given explicitly by $f : S^2 - P \to \mathbb{R}^2$, where

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + (z-1)^2 = 1\},$$
$$P = (0,0,2),$$
$$\mathbb{R}^2 = \{(x,y,z) \in \mathbb{R}^3 | z = 0\},$$

and

$$f(x,y,z) = (x', y', 0),$$

with

$$x' = \frac{2x}{2-z},$$
$$y' = \frac{2y}{2-z} \text{ (see Problem 5-4).}$$

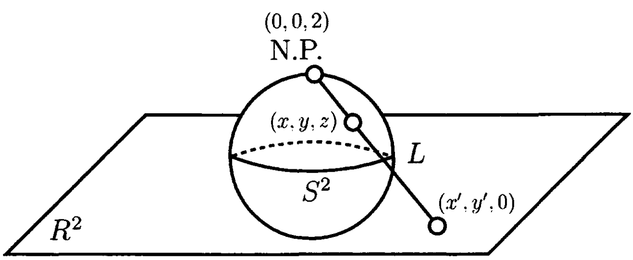


Figure 5-6.

The image of the graph G from S^2 is an imbedding of G in $\mathbb{R}^2$, with the unbounded region corresponding to the region in S^2 from the interior of which the north pole was selected. Clearly, this process is reversible. In fact, the map f gives a homeomorphism between $S^2 - P$ and $\mathbb{R}^2$, where P is any point of S^2. (Rotate S^2 so that P is at the north pole.) Note that neither space is closed from the point of view of surface topology, yet $\mathbb{R}^2$ (and *not* $S^2 - P$) is closed – in the point-set sense – as a subspace of $\mathbb{R}^3$.

Def. 5-13. A region of an imbedding of a graph G in a surface M is said to be a 2-*cell* if it is homeomorphic to the open unit disk. If every region for an imbedding is a 2-cell, the imbedding is said to be a *2-cell imbedding.*

The next theorem, giving the *euler identity*, is perhaps the most important in all of topological graph theory.

Thm. 5-14. Let G be a connected pseudograph, with a 2-cell imbedding in S_k, with the usual parameters p, q, and r. Then

$$p - q + r = 2 - 2k.$$

PROOF. (by induction on k); the case $k = 0$ has been settled by Corollary 5-12. Now assume the theorem is true for fewer than k handles ($k \geq 1$), and let G be as in the statement of the theorem. Without loss of generality, we assume all the vertices of G to be on the "sphere" portion of S_k; and since the imbedding is 2-cell, each handle has at least one edge of G running over it. Select one handle, and draw two disjoint simple closed curves C_1 and C_2 around this handle. Suppose edges $x_1, x_2, \cdots, x_n$ run over the handle, where $n \geq 1$. Then C_i meets x_j in a point of S_k which we designate by $u_{ij}, i = 1, 2;\ j = 1, 2, \cdots, n$. Consider the points u_{ij} to be vertices of a new pseudograph, with edges determined in the natural manner. Now remove the portion of the handle between C_1 and C_2 and "fill in" the two resulting holes (bounded by C_1 and C_2 respectively) with two disks (this is called a *capping* operation). The result is a 2- cell imbedding of a connected pseudograph in S_{k-1}, with parameters p', q', and r' (say). But

$$\begin{aligned} p' &= p + 2n \\ q' &= q + 3n \\ r' &= r + n + 2. \end{aligned}$$

Thus, by the inductive assumption,

$$\begin{aligned} 2 - 2(k-1) &= p' - q' + r' \\ &= (p + 2n) - (q + 3n) + (r + n + 2); \\ &= p - q + r + 2; \end{aligned}$$

that is, $p - q + r = 2 - 2k$. □

Cor. 5-15. Let G be a connected graph, with a 2-cell imbedding in S_k, with the parameters p, q, and r; then $p - q + r = 2 - 2k$.

PROOF. The result is immediate, since any graph is also a pseudograph. □

We have shown that the number $p - q + r$ is invariant for S_k, for any 2-cell imbedding of any connected pseudograph; $p-q+r = 2-2k$, depending only on k. This invariant number, $2-2k$, is called the *euler characteristic* for the surface S_k. It then follows that S_n and S_m are homeomorphic if and only if $m = n$. In the nonorientable case, the characteristic is given by $p - q + r = 2 - k$, where k is the number of cross-caps (see Fréchet and Fan.) In notation, $\chi(S_k) = 2 - 2k$; $\chi(N_k) = 2 - k$.

5-4. Three Applications

The ramifications of Theorem 5-14 are enormous. In this section we give only three of these, each pertaining to the case $k = 0$.

Def. 5-16. A graph is said to be *planar* if it can be imbedded in the plane (or, equivalently, in S_0). A graph imbedded in S_0 is called a *plane graph.*

Note: Suppose a graph G is 2-cell imbedded in a surface S_k. Let v_i be the number of vertices of degree i, and let r_i designate the number of regions having i sides (i.e. the number of regions having as boundary a closed walk of length i; i is also called the *length* of the region). We assume that $v_0 = v_1 = v_2 = 0$, as we focus on polyhedral graphs in this section; moreover, for G a nontrivial graph, $r_0 = r_1 = r_2 = 0$.

Lemma 5-17.

(i) $p = \sum_{i\geq 3} v_i$
(ii) $r = \sum_{i\geq 3} r_i$
(iii) $2q = \sum_{i\geq 3} iv_i$
(iv) $2q = \sum_{i\geq 3} ir_i$

PROOF. (i) and (ii) are obvious; (iii) is Theorem 2-2, and (iv) follows in like manner to (iii); in summing the number of sides in the regions, each edge is counted exactly twice. □

Thm. 5-18. The graph K_5 is not planar.

PROOF. Suppose that the connected graph K_5 were imbedded in the plane; then $2q = 20 = \sum_{i\geq 3} ir_i \geq 3\sum_{i\geq 3} r_i = 3r$, and, by Theorem 5-11,

$$
\begin{aligned}
2 &= p - q + r \\
&\leq 5 - 10 + \frac{20}{3} = \frac{5}{3},
\end{aligned}
$$

a contradiction! Hence, K_5 is not planar. □

Lemma 5-19. Let the *planar* connected graph G ($\delta(G) \geq 3$) be imbedded in the plane; then

(i) G has a vertex of degree 5 or less; *and*
(ii) G has a region with 5 or less sides.

PROOF.

(i) Suppose, to the contrary, that $v_i = 0$, $i = 0, 1, 2, 3, 4, 5$; then $2q = \sum_{i\geq 3} iv_i \geq 6\sum_{i\geq 3} v_i = 6p$. As before, $2q = \sum_{i\geq 3} ir_i \geq 3\sum_{i\geq 3} r_i = 3r$. Then, by Theorem 5-11, $p - q + r = 2$; i.e.

$$
\begin{aligned}
q &= p + r - 2 \\
&\leq \frac{q}{3} + \frac{2q}{3} - 2 = q - 2,
\end{aligned}
$$

a contradiction.

(ii) This follows by duality (soon to be explained); it also follows from Problem 5-6.

□

This explains why a Petoskey stone (the state stone of Michigan), although nearly a hexagonal tessellation, can never be perfectly so. The infinite tessellation of the plane by congruent regular hexagons (or triangles) is *not* precluded by Lemma 5-19, however.

We are now prepared to give a *topological* proof of what the Greeks knew, geometrically, over two thousand years ago: there are exactly five regular polyhedra. A *polyhedron* is a finite, connected collection of at least four polygons, fit together in $\mathbb{R}^3$ so that: (i) each side of each polygon coincides exactly with one side of one other polygon, and (ii) around each vertex there is one circuit of polygons; together with the region of $\mathbb{R}^3$ bounded by these polygons. These two conditions rule out the anomalies depicted in Figure 5-7.

A *regular polyhedron* is a convex polyhedron for which: (i) the polygons are congruent regular polygons, and (ii) the same number of polygons surround each vertex.

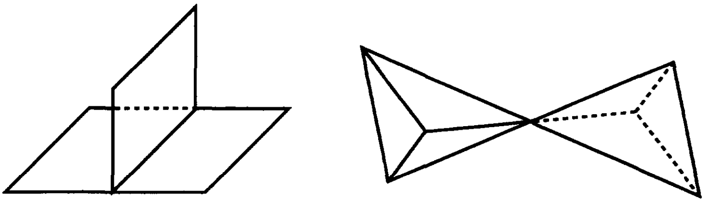


Figure 5-7.

Thm. 5-20. There are exactly five regular polyhedra.

PROOF. Let P be a regular polyhedron. Associated with P is a regular planar graph G (to picture this, first bound P with a sphere, then place a light source inside the polyhedron–the shadow of the vertices and edges of P gives a graph imbedded in the sphere; finally, perform a stereographic projection.) This planar graph G has $v_0 = v_1 = v_2 = 0$, and in fact: $p = v_k, r = r_h$, for $k, h \in \{3, 4, 5\}$, by Lemma 5-19. Next by Theorem 5-11, $p - q + r = 2$; we re-write this as follows:

$$\begin{aligned} 8 &= 4p + 4r - 2q - 2q \\ &= \sum_{i \geq 3} (4 - i)(r_i + v_i) \\ &= (4 - h)r_h + (4 - k)v_k. \end{aligned}$$

But also, $hr_h = kv_k$, since both $= 2q$, by Lemma 5-17. Of the nine possibilities for (h, k) in positive integers, only the following satisfy both of the above equations in r_h and v_k : $(h, k) =$

(i) $(3, 3); r_3 = v_3 = 4$ (the *tetrahedron*)
(ii) $(3, 4); r_3 = 8, v_4 = 6$ (the *octahedron*)
(iii) $(3, 5); r_3 = 20, v_5 = 12$ (the *icosahedron*)
(iv) $(4, 3); r_4 = 6, v_3 = 8$ (the *hexahedron*; i.e. the cube)
(v) $(5, 3); r_5 = 12, v_3 = 20$ (the *dodecahedron*)

This completes the proof. □

The reader may have noticed a certain interchangeability between the roles of vertices and regions (compare (ii) and (iv) above, (iii) and (v) above; see Lemmas 5-17 and 5-19, and Theorem 5-14). This is no accident.

Def. 5-21. Let a connected pseudograph G be 2-cell imbedded in S_k. The *dual* pseudograph of G, $D_I(G)$ (relative to this imbedding I), is given by: the vertices of $D_I(G)$ are the regions of G in S_k, and two such vertices are adjacent if and only if their corresponding regions share a

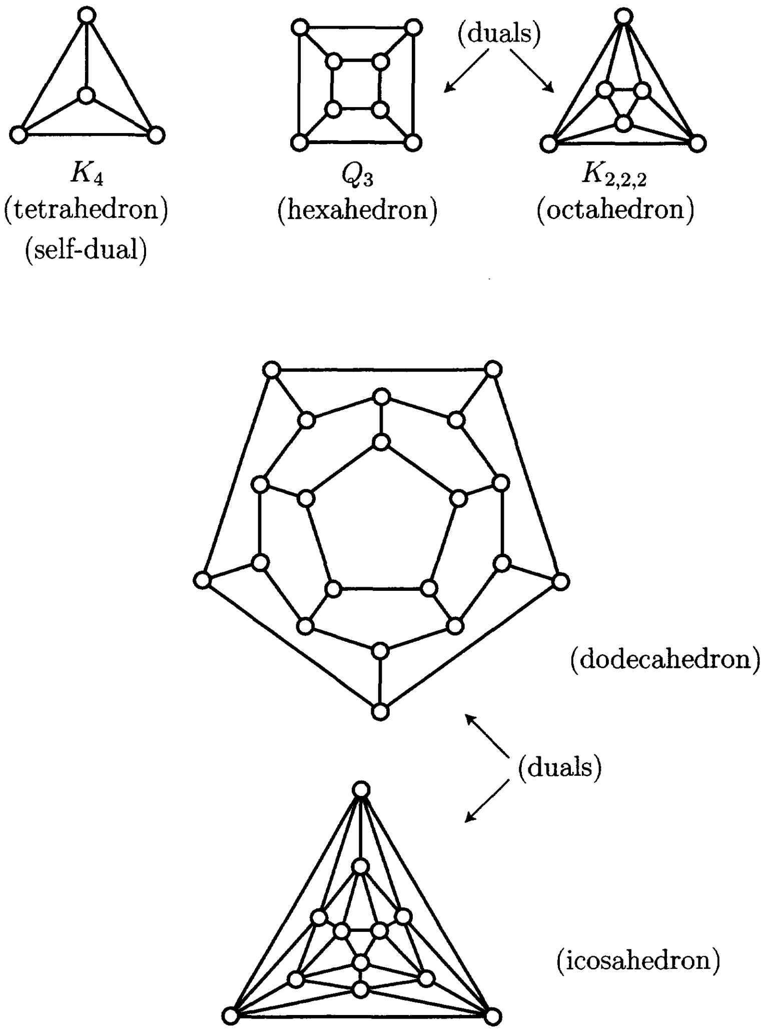


Figure 5-8.

common edge in their boundaries. (Each edge of G is associated with exactly one edge of $D_I(G)$, which therefore may have loops and multiple edges.)

For example, Figure 5-8 not only gives the regular polyhedral graphs, but also indicates duality relationships. Figure 5-9 shows, for instance, that the tetrahedron is self-dual. Thus, having established Lemma 5-19 (i), we establish part (ii) by applying (i) to the dual. Similarity, having found the hexahedron, we discover the octahedron as its dual; and so forth.

Although there are only five regular polyhedra (also called the *Platonic Solids*), there are infinitely many convex polyhedra, as the classes

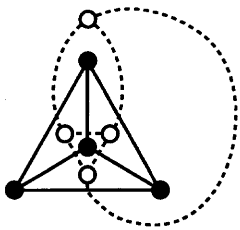


Figure 5-9.

of all prisms and antiprisms show. The thirteen Archimedean Solids are also all convex polyhedra. There are many non-convex polyhedra, some of which are uniform with regard to face structure and vertices, which have planar graphs as 1-skeletons; see, for example, [W4]. For a splendid study of orientable polyhedra (of possibly positive genus) with regular faces, consult B. M. Stewart's *Adventures Among the Toroids* [S23].

Def. 5-22. A graph G is said to be *3-polytopal* if it is the 1-skeleton (the graph induced by the vertices and edges) of a convex polyhedron.

Def. 5-23. A graph G is said to be *n-connected* ($n \geq 1$) if the removal of fewer than n vertices from G neither disconnects G nor reduces G to the trivial graph K_1.

Graphs which are 3-polytopal have been characterized by Steinitz [S22].

Thm. 5-24. A graph G is 3-polytopal if and only if it is planar and 3-connected.

The next theorem, due to Whitney [W27], applies precisely to 3-polytopal graphs, by Theorem 5-24.

Thm. 5-25. A 3-connected planar graph is uniquely imbeddable on the sphere.

One readily verifies that the five planar graphs of Figure 5-8 are also 3-connected. The following theorem of Weinberg [W3] gives information about the automorphism groups of 3-polytopal graphs:

Thm. 5-26. Let G be 3-polytopal, with q edges. Then $|\operatorname{Aut}(G)| \leq 4q$, with equality holding if and only if G is the 1-skeleton of a Platonic Solid.

The Greeks identified four of the Platonic Solids with the four basic elements: earth with hexahedron, air with octahedron, fire with tetrahedron (the sharpest of the solids), and water with icosahedron (the roundest); the dodecahedron became the all-encompassing universe. Indeed, it can be shown geometrically that the Platonic solids nest, sequentially, snugly within one another with the dodecahedron outermost. The Platonic solids appear in nature, in art, and in games of chance. The tetrahedron appears as chrome alum, the hexahedron as sodium chloride (common table salt), and the others as skeleta of micro-organisms called radiolaria. All five are prominent in the works of M. C. Escher (see particularly "Reptiles,", where an alligator-like creature evolves from 2-zto 3-space, giving a triumphant snort upon surmounting a dodecahedron) and of Salvador Dalí (in "The Last Supper", the scene is set within a dodecahedron; in both "Corpus Hypercubus" and "Galacidalacideoxyribonucleic acid", the hexahedron is featured). Finally, whereas the common die is hexahedral, the other Platonic solids could be used for games of chance featuring 4, 8, 12, or 20 equally likely outcomes. The dodecahedron is also ideally suited for desk-calendar paperweights.

For our third application of the euler identity for the sphere (or plane), we consider Pick's Theorem, which surprisingly calculates certain areas by a combinatorial-topology approach. By a *lattice* point in the plane, we mean a point with both coordinates integers. A *simple* polygon is bounded by a simple closed curve.

Thm. 5-27. Let P be a simple polygon placed in the plane with all vertices at lattice points. Then the area of P is given by

$$A = I + \frac{1}{2}B - 1,$$

where I is the number of lattice points in the interior of P, and B is the number of lattice points on the boundary of P.

PROOF. The simple polygon P, regarded as a plane graph, is the cycle C_B. Augment this graph with the I vertices inside P, and then add edges (joining lattice points) inside P so as to subdivide the interior into triangles, each having area $\frac{1}{2}$. (It is always possible to do this; see Problem 5-16.) This yields a plane graph with $p = I + B$, r regions, and, since $2q = 3(r-1) + B$ by Lemma 5-17 (iv), $q = \frac{3}{2}(r-1) + \frac{B}{2}$. Now, we seek area $A = \frac{1}{2}(r-1)$, since we exclude the exterior region.

But $p - q + r = 2$, so

$$I + B - \frac{3}{2}(r-1) - \frac{B}{2} + r - 1 = 1,$$

and

$$\begin{aligned} A &= \frac{1}{2}(r-1) \\ &= I + \frac{B}{2} - 1. \end{aligned}$$

□

5-5. Pseudosurfaces

We now consider topological spaces akin to surfaces, but which fail to be 2-manifolds at a finite number of points; these spaces form additional candidates for the imbedding of graphs, and were studied extensively by Petroelje [P5].

Def. 5-28. Let A denote a set of $\sum_{i=1}^{t} n_i m_i \geq 0$ distinct points of S_k, with $1 < m_1 < m_2 < \cdots < m_t$. Partition A into n_i sets of m_i points each, $i = 1, 2, \cdots, t$. For each set of the partition, identify all the points of that set. The resulting topological space is called a *pseudosurface*, and is designated by $S(k; n_1(m_1), n_2(m_2), \cdots, n_t(m_t))$. Each point resulting from an identification of m_i points of S_k is called a *singular point*. If a graph G is imbedded in a pseudosurface, we assume that each singular point is occupied by a vertex of G; such a vertex is called a *singular vertex.* A *generalized pseudosurface* results when finitely many identifications, of finitely many points each, are made on a topological space of finitely many components, each of which is a pseudosurface, with a connected topological space resulting.

Thm. 5-29. Let G be a graph having a 2-cell imbedding in

$$S(k; n_1(m_1), n_2(m_2), \cdots, n_t(m_t));$$

then $p - q + r = 2 - 2k - \sum_{i=1}^{t} n_k(m_i - 1)$.

The number $2 - 2k - \sum_{i=1}^{t} n_i(m_i - 1)$ is said to be the *characteristic for the pseudosurface*, and is a topological invariant, just as $2 - 2k$ is for the surface S_K.

5-6. Problems

5-1.) A *forest* is a graph for which every component is a tree. Show that, if G is a forest with p vertices, q edges, and k components, then $p = q + f(k)$, where $f(k)$ must be determined.

5-2.) Let G, a graph with p vertices, q edges, and k components, be imbedded in the sphere, with r regions. Show that $p - q + r = g(k)$, where $g(k)$ must be determined. Illustrate, for $G = 2K_4$. For what value of k will the imbedding be 2-cell?

5-3.) Prove Corollary 5-12.

5-4.) Verify that $f(x, y, z) = \left(\frac{2x}{2-z}, \frac{2y}{2-z}, 0\right)$ gives the stereographic projection.

5-5.) Show that $K_{3,3}$ is not planar.

5-6.) Prove Lemma 5-19 (ii), without using duality.

5-7.) Where might the proof of Theorem 5-14 break down, for graphs (instead of pseudographs)?

5-8.) Consider the 2-manifolds in Figure 5-3. Determine in which of these K_5 can be imbedded. For each 2-manifold, compute the characteristic. Note that the characteristics agree for the torus and the klein bottle; are these two homeomorphic? Why? The characteristics also agree for the möbius strip and the projective plane; are they homeomorphic? How about the sphere and the cylinder?

5-9.) Show that the two symbolic representations $ab^{-1}ab$ (as in Figure 5.3) and $a_1a_1a_2a_2$ (as in Theorem 5-5 (iii)) both give the klein bottle. (Hint: cut along an appropriate diagonal of the rectangle $a_1a_1a_2a_2$ and then make an appropriate identification to obtain the rectangle $ab^{-1}ab$.)

5-10.) For G the 1-skeleton of a Platonic Solid, show that $|\operatorname{Aut}(G)| = 4q$.

5-11.) The *wheel* graph W_m is defined as the join (see Definition 2-16) $K_1 + C_{m-1}, m \geq 4$. Show that $|\operatorname{Aut}(W_m)| \leq 4q$, with equality holding if and only if $m = 4$. Is this consistent with Theorem 5-26?

5-12.) *Give an example to show that two pseudosurfaces with the same characteristic can be non-homeomorphic (compare the situation for surfaces). Find a formula that gives, for $n \geq -2$, the number of non-homeomorphic pseudosurfaces with characteristic $-n$.

5-13.) Prove Theorem 5-29 and then extend it to generalized pseudosurfaces.

5-14.) Into how many regions is the plane $\mathbb{R}^2$ divided, by n lines in general position (i.e. no two lines parallel, no three lines concurrent)? The answer can be conjectured inductively and then proved using mathematical induction, but try to obtain it directly by using the euler identity and stereographic projection.

5-15.) Verify Pick's Theorem, for (i) $m \times n$ rectangles; (ii) isosceles right triangles of side length n.

5-16.) *Complete the proof of Pick's Theorem, by showing that the interior of p can always be subdivided, using line segments joining lattice points, into triangles of area $\frac{1}{2}$.

5-17.) *Now show that the euler identity for the plane can be derived from Pick's Theorem. Thus the two identities are equivalent.

5-18.) Show that a plane graph of order p and size q satisfies $q \leq 3p - 6$, with equality if and only if $r = r_3$.

CHAPTER 6

IMBEDDING PROBLEMS IN GRAPH THEORY

Recall from Definition 2-1 that a graph is an abstract mathematical system. It is when we concern ourselves with the geometric realization of a graph as a finite one-dimensional complex that imbedding problems arise. There are practical applications for this view of graphs. For instance, we will see in Chapter 8 that one of the truly famous problems in mathematics can be stated in terms of imbedded graphs. As another example, imagine the task of printing an electronic circuit on a circuit board. Associated with the circuit (in an obvious manner) is a graph, and the circuit can be printed without shorts if and only if the associated graph can be imbedded in the plane. What to do if the graph is not planar will be considered in this chapter.

What do we mean by "the geometric realization" of a graph? In this section, we will normally mean a configuration in $\mathbb{R}^3$, where the vertices of the graph are represented by distinct points, and the edges of the graph by arcs; two arcs intersect only at a point representing common end vertices of the corresponding edges. A natural question is: "In what subspaces of $\mathbb{R}^3$ will a given graph imbed in this manner?" We will confine our attention to the following subspaces:

(i) $\mathbb{R}^3$ itself
(ii) $\mathbb{R}^2$
(iii) n-books (see definition below)
(iv) surfaces
(v) pseudosurfaces
(vi) generalized pseudosurfaces.

Def. 6-1. An *n-book* is the cartesian product of the unit interval with a geometric realization of the graph $K_{1,n}$.

That is, an n-book consists of n rectangles (the pages) joined along a common edge (the spine).

6-1. Answers to Some Imbedding Questions

The imbedding question has been completely answered for (i), (ii), and (iii), as the next three theorems indicate. We will also need some definitions. In the sequel, the term "graph" will be used interchangeably, to represent either the abstract mathematical system, or a realization of this system in $\mathbb{R}^3$. The context should make it clear which use is intended.

Thm. 6-2. If K is a countable and locally finite simplicial complex, with $\dim K \leq n$, then K has a realization (i.e. a linear imbedding) as a closed subset in $\mathbb{R}^{2n+1}$.

(See Spanier [S11] for a discussion of this theorem.)

Cor. 6-3. Any finite one-complex is imbeddable in $\mathbb{R}^3$.

Note that Corollary 6-3 indicates that any graph may be imbedded in $\mathbb{R}^3$, and in such a way that every edge is represented as a *straight line segment.*

Another way to see this is as follows. Let C be the curve in $\mathbb{R}^3$ determined by the parametric equations $x = t$, $y = t^2$, $z = t^3$ $(t \geq 0)$. Select p distinct points along C to represent the vertices of G and represent the q edges of G as straight line segments joining these points appropriately. Since no four points on C are coplanar (see Problem 6-2) C meets any plane in $\mathbb{R}^3$ at most three times, and no two edges of G intersect extraneously.

Def. 6-4. An *elementary subdivision* of an edge uv of a graph is the deletion of edge uv, the addition of a new vertex w, and the addition of two new edges, uw and wv.

Def. 6-5. A graph G is said to be *homeomorphic from* a graph H if G can be obtained from H by a (finite) sequence of elementary subdivisions. (We say that G is a subdivision of H.) G_1 and G_2 are said to be *homeomorphic with* each other if they are both homeomorphic from a common graph H.

Note that G_1 is homeomorphic with G_2 in the graph-theoretical sense defined above if and only if realizations of G_1 and G_2 in $\mathbb{R}^3$ are homeomorphic in the topological sense (see Problem 6-1.)

The next theorem is one of the most important in all of graph theory; it is due to Kuratowski [K4]. Note that the necessity of the condition follows from Theorem 5-18 and Problem 5-5.

Thm. 6-6. A graph G is planar if and only if it contains no subgraph homeomorphic with either K_5 or $K_{3,3}$.

It is clear that any graph with q edges can be imbedded in a q-book: place all the vertices along the spine, and use one page for each edge. However, we can do much better; the following theorem, due to Atneosen [A13], is rather surprising.

Thm. 6-7. Any graph G can be imbedded in a 3-book.

Note that by Theorem 6-6, we have a criterion for ascertaining if the third page is needed for a particular graph. Theorem 6-7 can be proved as follows: as shown in Massey [M3], any closed 2-manifold with non-void boundary can be represented as a disk with strips attached in a certain way. Clearly any graph G can be imbedded in a closed 2-manifold with non-void boundary (simply remove an open disk from the interior of some region, for any S_k in which G can be imbedded; take $k = q$, for example). Atneosen showed, very neatly, that any disk with strips attached as described by Massey can be imbedded in a 3-book. (An alternate proof has been given by Babai [B1]: Draw G in the plane so that all intersections lie on a straight line and no three edges have a common intersection (except at common end vertices). Let this line be the spine of the book, and let the plane be the union of two pages. Then the third page can be readily employed to avoid each intersection.)

So, it is only for the subspaces (iv), (v), and (vi) of our list above—that is, the surfaces, pseudosurfaces, and generalized pseudosurfaces—that the imbedding problem is, in general, unsolved, for non-planar graphs. Clearly, any graph will imbed on S_k, for k large enough (for example, take $k = q$ and use one handle for each edge); but this does not characterize which graphs imbed on S_k, for k fixed. The most natural problem here might be: for a given graph, find the surface of minimum genus in which the graph can be imbedded. If the graph is associated with an electronic circuit, the corresponding problem is: find the fewest number of holes that must be punched in the circuit board so that the board can accommodate the circuit. For Cayley color graphs, the problem becomes: find the simplest locally 2-dimensional "drawing board" in which to "paint" a picture of a given group. We will also see, in Chapter 8, that imbedding certain graphs in appropriate surfaces will tell us a good deal about map-coloring problems.

6-2. Definition of "Imbedding"

Let us now give two very careful definitions of "imbedding" (they are easily seen to be equivalent), and then proceed to study this process in some detail.

Def. 6-8. Let G be a graph, with $V(G) = \{v_1, v_2, \cdots, v_p\}$ and $E(G) = \{x_1, x_2, \cdots, x_q\}$. Let M be a 2-manifold. An *imbedding* of G in M is a subspace $G(M)$ of M such that

$$G(M) = \bigcup_{i=1}^{p} v_i(M) \cup \bigcup_{j=1}^{q} x_j(M),$$

where

(i) $v_1(M), \cdots, v_p(M)$ are distinct points of M
(ii) $x_1(M), \cdots, x_q(M)$ are M mutually disjoint open arcs in M
(iii) $x_j(M) \cap v_i(M) = \emptyset, i = 1, \cdots, p;\ j = 1, \cdots, q$
(iv) if $x_j = \{v_{j1}, v_{j2}\}$, then the open arc $x_j(M)$ has $v_{j1}(M)$ and $v_{j2}(M)$ as end points; $j = 1, \cdots, q$.

In the above definition, an *arc* in M is a homeomorphic image of $[0, 1]$; an open arc is an arc less its two end points, the images of 0 and 1.

Equivalently (and much more briefly) we have:

Def. 6-8′. The graph G can be *imbedded* in the 2-manifold M if the geometric realization of G as a one-dimensional simplicial complex is homeomorphic to a subspace of M.

6-3. The Genus of a Graph

Imbedding questions (iv) in this chapter leads directly to:

Def. 6-9. The *genus*, $\gamma(G)$, of a graph G is the minimum genus among all surfaces in which G can be imbedded.

For example, if G is planar then we write $\gamma(G) = 0$. If $\gamma(G) = k$, $k > 0$, then G has an imbedding in S_k, but not in S_h, for $h < k$. Moreover, G imbeds in S_m, for all $m \geq k$ (merely add $m - k$ handles to an imbedding of G in S_k).

As mentioned above, it is clear that every graph has a genus. Let G have q edges; then place the vertices of G on the sphere, and add one handle for each edge. Thus $\gamma(G) \leq q$.

Def. 6-10. An imbedding of a graph G in a surface S_k is said to be a *minimal imbedding* if $\gamma(G) = k$.

The next result is extremely useful, as it tells us that the euler identity applies for any minimal imbedding of a connected graph. For a complete proof, see [Y1].

Thm. 6-11. If a connected graph G is minimally imbedded in a surface, then the imbedding is a 2-cell imbedding.

Heuristic Argument: We assume (without loss of generality) that every vertex of G lies on the sphere. Hence only edges can be imbedded on the handles. Suppose that R is a non-2-cell region. Then there is a simple closed curve C in R which cannot be continuously deformed, in R, to a point. If $\gamma(G) = 0$, C divides S_0 into two parts (by the Jordan curve theorem), each of which must contain a vertex of G. But then G would be disconnected. Hence $\gamma(G) \geq 1$. We consider three cases:

Case (i):: If C lies entirely on one handle, we cut the surface along C, cap the two resulting holes, and obtain an imbedding of G in $S_{\gamma(G)-1}$, a contradiction.

Case (ii):: If C lies entirely on the sphere, we regard the "sphere" portion of the surface as a handle, and apply case (i).

Case (iii):: If C lies partially on some handle H and partially on $S_{\gamma(G)} - H$, we redraw the edges of G formerly carried by H along that portion of C lying in $S_{\gamma(G)} - H$; we obtain an imbedding of G on the surface without using handle H, the final contradiction.

The corollary below follows directly from Theorems 5-14 and 6-11.

Cor. 6-12. If a connected graph G has a minimal imbedding in S_k, with p vertices, q edges, and r regions, then

$$p - q + r = 2 - 2k.$$

The next two corollaries are often helpful in computing the genus of a graph. We require two new terms.

Def. 6-13. A 2-cell imbedding is said to be a *triangular* (*quadrilateral*) imbedding if $r = r_3 (r = r_4)$.

Cor. 6-14. If G is connected, with $p \geq 3$, then, $\gamma(G) \geq \frac{q}{6} - \frac{p}{2} + 1$. Furthermore, equality holds if and only if a triangular imbedding can be found for G.

PROOF. Let G be imbedded in $S_{\gamma(G)}$, so that $p - q + r = 2 - 2\gamma(G)$. Since $2q \geq 3r$, with equality if and only if $r = r_3$ (see Lemma 5-17), the result is immediate. □

Cor. 6-15. If G is connected, with $p \geq 3$, and has no triangles, then $\gamma(G) \geq \frac{q}{4} - \frac{p}{2} + 1$. Furthermore, equality holds if and only if a quadrilateral imbedding can be found for G.

PROOF. (The proof is entirely analogous to that of Corollary 6-14). □

The next corollary will be heavily used in the remainder of this chapter.

Cor. 6-16. If G is a connected bipartite graph having a quadrilateral imbedding, then $\gamma(G) = \frac{q}{4} - \frac{p}{2} + 1$.

PROOF. Apply Theorem 2-19 and Corollary 6-15. □

We have shown (among other things) that, for connected graphs, minimal imbeddings are 2-cell imbeddings. Two questions arise: (i) What about minimal imbeddings of disconnected graphs? (ii) Are there 2-cell imbeddings which are not minimal? We discuss these two questions briefly.

Def. 6-17. Given a connected graph G, a *cut-vertex* is a vertex v such that $G - v$ is disconnected. A *block* is a maximal connected subgraph of G having no cut-vertices.

For example, the graph in Figure 6-1 has two blocks, both isomorphic to K_4; v is a cut- vertex for this graph. Note that a block is either K_2 or is 2-connected.

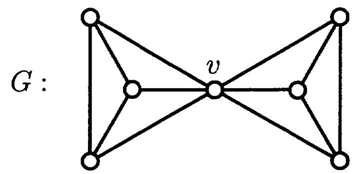


Figure 6-1.

The next theorem and its corollary are due to Battle, Harary, Kodama, and Youngs [BHKY1], and are presented without proof.

Thm. 6-18. The genus of a connected graph is the sum of the genera of its blocks.

Cor. 6-19. The genus of a graph is the sum of the genera of its components. (i.e., let $G = \bigcup_{i=1}^{n} G_i$; then $\gamma(G) = \sum_{i=1}^{n} \gamma(G_i)$).

6-4. The Maximum Genus of a Graph

That there exist 2-cell imbeddings which are not minimal is evident from Figure 6-2, which shows K_4 in S_1. Note that the euler identity still applies here $(4 - 6 + 2 = 0)$. It is clear that no imbedding of a disconnected graph can be a 2-cell imbedding. To describe all 2-cell imbeddings of a given connected graph, we introduce the following concept:

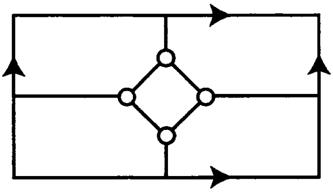


Figure 6-2.

Def. 6-20. *The maximum genus*, $\gamma_M(G)$, of a connected graph G is the maximum genus among the genera of all surfaces in which G has a 2-cell imbedding.

Duke [D6] has shown the following:

Thm. 6-21. If a graph G has 2-cell imbeddings in S_m and S_n, then G has a 2-cell imbedding in S_k, for each k, $m \leq k \leq n$.

Cor. 6-22. A connected graph G has a 2-cell imbedding in S_k if and only if $\gamma(G) \leq k \leq \gamma_M(G)$.

An upper bound for $\gamma_M(G)$ is not difficult to determine.

Def. 6-23. The *Betti number* $\beta(G)$, of a graph G having p vertices, q edges, and k components, is given by : $\beta(G) = q - p + k$.

$\beta(G)$ is sometimes called the *cycle rank* of G; it gives the number of independent cycles in a cycle basis for G; see Harary [H3, pp. 37-40].

Recall that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x; $\lceil x \rceil$ gives the least integer greater than or equal to x. Both symbols will be used frequently in the remainder of this chapter.

Thm. 6-24. Let G be connected; then $\gamma_M(G) \leq \left\lfloor \frac{\beta(G)}{2} \right\rfloor$. Moreover, equality holds if and only if $r = 1$ or 2, according as $\beta(G)$ is even or odd, respectively.

PROOF. Let G be connected, with a 2-cell imbedding in S_k; then $r \geq 1$, and $\beta(G) = q - p + 1$; also $p - q + r = 2 - 2k$; thus

$$k = 1 + \frac{q - p - r}{2} \leq \frac{q - p + 1}{2} = \frac{\beta(G)}{2},$$

and the results follows. □

Nordhaus, Stewart, and White [NSW1] showed that equality holds in Theorem 6-24 for the complete graph K_n; Ringeisen [R9] showed that equality holds for the complete bipartite graph $K_{m;n}$; and Zaks [Z1] showed that equality holds for the n-cube Q_n (if $\gamma_M(G) = \left\lfloor \frac{\beta(G)}{2} \right\rfloor$, G is said to be *upper imbeddable*).

Thm. 6-25. $\gamma_M(K_n) = \left\lfloor \frac{(n-1)(n-2)}{4} \right\rfloor$.

Thm. 6-26. $\gamma_M(K_{m,n}) = \left\lfloor \frac{(m-1)(n-1)}{2} \right\rfloor$.

Thm. 6-27. $\gamma_M(Q_n) = (n-2)2^{n-2}$, for $n \geq 2$.

Moreover, Kronk, Ringeisen, and White [KRW1] established:

Thm. 6-28. All complete n-partite graphs are upper imbeddable.

Also, Ringeisen [R8] found $\gamma_M(G)$ for several classes of planar graphs G, including the wheel graphs and the regular polyhedral graphs.

Nordhaus, Ringeisen, Stewart, and White combined [NRSW1] to establish the following analog to Kuratowski's Theorem (Theorem 6-6): (The graphs H and Q are given in Figure 6-3.)

Thm. 6-29. The connected graph G has maximum genus zero if and only if it has no subgraph homeomorphic with either H or Q. (Furthermore, $\gamma(G) = \gamma_M(G)$ if and only if $\gamma_M(G) = 0$ if and only if G is a cactus with vertex-disjoint cycles.)

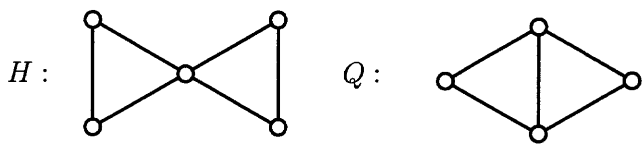


Figure 6-3.

Def. 6-30. A *cactus* is a connected (planar) graph in which every block is a cycle or an edge.

Def. 6-31. A *splitting tree* of a connected graph G is a spanning tree T for G such that at most one component of $G - E(T)$ has odd size.

The following characterization is due, independently, to Jungerman [J9] and Xuong [X2].

Thm. 6-32. A graph G is upper imbeddable if and only if G has a splitting tree.

Thus, for example, we get an immediate proof of Theorem 6-25 merely by taking $T = K_{1,n-1}$.

Nebesky [N1] has given a sufficient condition for upper imbeddability. First, we need

Def. 6-33. A graph G is said to be *locally connected* if, for every $v \in V(G)$, the set $N_G(v)$ of vertices adjacent to v is non-empty and the subgraph of G induced by $N_G(v)$ is connected.

Thm. 6-34. If G is connected and locally connected, then G is upper imbeddable.

Although no workable formula is known for the genus of an arbitrary graph, Xuong [X1] developed the following result for maximum genus. Let $\xi_0(H)$ denote the number of components of graph H of odd size, and for G connected set

$$\xi(G) = \min \xi_0(G - E(T)),$$

the minimum being taken over all spanning trees T of G. Then:

Thm. 6-35. The maximum genus of the connected graph G is given by

$$\gamma_M(G) = \frac{1}{2}(\beta(G) - \xi(G)).$$

6-5. Genus Formulae for Graphs

Prior to the work of Jungerman and Xuong, Theorems 6-25, 6-26, 6-27, and 6-28 and the work of Ringeisen [R8] referred to above gave the only known non-trivial formulas for maximum genus. Not very many more formulas were known for the genus parameter prior to 1978; we list some of these below. For most of what else was known up to that time, see Table 1 of Stahl [S14].

Thm. 6-36. (Ringel [R12]; Beineke and Harary [BH1])
$\gamma(Q_n) = 1 + 2^{n-3}(n-4), n \geq 2.$

Thm. 6-37. (Ringel [R13])
$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil; m, n \geq 2.$

Thm. 6-38. (Ringel and Youngs [RY1])
$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil, n \geq 3.$

Thm. 6-39. (White [W5]; see also [RY5], for the case $m = 1$)
$\gamma(K_{mn,n,n}) = \frac{(mn-2)(n-1)}{2}.$

Thm. 6-40. (Stahl and White [SW2])
$\gamma(K_{n,n,n-2}) = \frac{(n-2)^2}{2}$, for n even, $n \geq 2$.

Thm. 6-41. (Stahl and White [SW2])
$\gamma(K_{2n,2n,n}) = \frac{(3n-2)(n-1)}{2}, n \geq 1.$

Thm. 6-42. (Jungerman [J7]; see also Garman [G1])
$\gamma(K_{4(m)}) = (m-1)^2, m \neq 3.$

Thm. 6-43. (Jungerman and Ringel [JR4]; see also Gross and Alpert [GA1]) $\gamma(K_{n(2)}) = \frac{(n-3)(n-1)}{3}$, for $n \not\equiv 2 \pmod 3$.

Thm. 6-44. (Ringel [R18])
For $2 \leq n \not\equiv 5, 9 \pmod{12}$, $\gamma(K_2 \times K_n) = \left\lceil \frac{(n-2)(n-3)}{6} \right\rceil$.

Thm. 6-45. (White [W7]) Let G have p vertices of positive degree, q edges, k non-trivial components, and no 3-cycles. Let H have $2n (n \geq 1)$ vertices and maximum degree less than two. Then $\gamma(G[H]) = k + n(nq - p)$.

Cor. 6-46. Let G have no 3-cycles. Then $\gamma(G[K_2]) = \gamma(G[\bar{K}_2]) = \beta(G)$.

The special case of Theorem 6-45 when G is bipartite, $p \geq 3$, and $H = \bar{K}_m$ has been generalized to include m odd, by Abu-Sbeih and Parsons [AP1].

The genus of a graph is typically determined as follows: first a lower bound is calculated, using Corollary 6-12 or one of its two refinements (Corollaries 6-14 and 6-15). Then a specific imbedding is constructed to attain that lower bound. If a class of graphs is being studied, a general construction (perhaps using mathematical induction) is sought. The construction employed is usually of one of the three forms:

(1) surgery;
(2) lifting (of a current or voltage graph imbedding);
  or, as a last resort
(3) generating a rotation scheme by *ad hoc* methods.

Surgery is often useful for graphs which can be factored as a graphical product; see for example Theorems 6-36 and 6-45 above. (We will give a surgical proof for the first of these shortly.) Lifting is often appropriate for graphs with a high degree of symmetry, especially those which are Cayley graphs for a suitable group; see for example Theorems 6-38 and 6-40 through 6-44 above. (This method will be discussed in detail in Chapters 9 and 10.) If all else fails, *ad hoc* methods can be tried, such as for Theorems 6-37 and 6-39 above. (Rotation schemes will be introduced in Section 6-6.)

Now we present the Beineke/Harary proof, using surgery, of Theorem 6-36: For $n > 1$, $\gamma(Q_n) = 1 + 2^{n-3}(n-4)$. The idea of the proof extends readily to many other cartesian product graphs; see Chapter 7, [W6], [P6], [P8], [C9], [A1], for example.

PROOF. (i) For $Q_n, p = 2^n$ and $q = n2^{n-1}$ (since $2q = n2^n$). By Problem 2-4 and induction, Q_n is bipartite; thus by Corollary 6-15,

$$\begin{aligned}\gamma(Q_n) &\geq 1 - \frac{p}{2} + \frac{q}{4} \\ &= 1 - \frac{2^n}{2} + \frac{n2^{n-1}}{4} \\ &= 1 + 2^{n-3}(n-4).\end{aligned}$$

(ii) By Corollary 6-16, it will suffice to construct a quadrilateral imbedding for Q_n, $n > 1$. To facilitate the inductive proof, we claim more than might seem necessary. Let $P(n)$ be the proposition that Q_n has a quadrilateral imbedding, including 2^{n-2} regions that partition $V(Q_n)$. We establish $P(n)$, $n > 1$, by mathematical induction.

The anchor for $n = 2$ is immediate, as Q_2 imbeds nicely in S_0 as a 4-cycle.

Now assume $P(n)$, $n > 1$, and consider $Q_{n+1} = K_2 \times Q_n$. Start with two disjoint copies of Q_n, identically vertex-labelled, each imbedded in accordance with $P(n)$ (with the 2^{n-2} special regions from the two copies agreeing as to vertex labels), but with one imbedding being the mirror image of the other. Now excise an open disk from the interior of each special region, and attach topological cylinders between disk boundaries for each corresponding pair of special regions. Imbed four new edges over each cylinder, in the manner depicted in Figure 6-4. In aggregate, this provides precisely the $4 \cdot 2^{n-2} = 2^n$ additional edges needed to complete $Q_{n+1} = K_2 \times Q_n$. We have destroyed $2 \cdot 2^{n-2} = 2^{n-1}$ old quadrilateral regions, but created $4 \cdot 2^{n-2} = 2^n$ new ones. Thus we have a quadrilateral imbedding of Q_{n+1}. Now, from each added cylinder, select either pair of opposite new regions; in Figure 6-4 we have selected (by shading) the top and bottom regions. This yields the $2 \cdot 2^{n-2} = 2^{n-1}$ special regions needed to complete the verification of $P(n+1)$.

□

It follows from Corollary 6-16 that the genus of the surgically-constructed surface for Q_{n+1} is $1 + 2^{n-2}(n-3)$. But we can verify this independently. We commenced with two disjoint surfaces, having

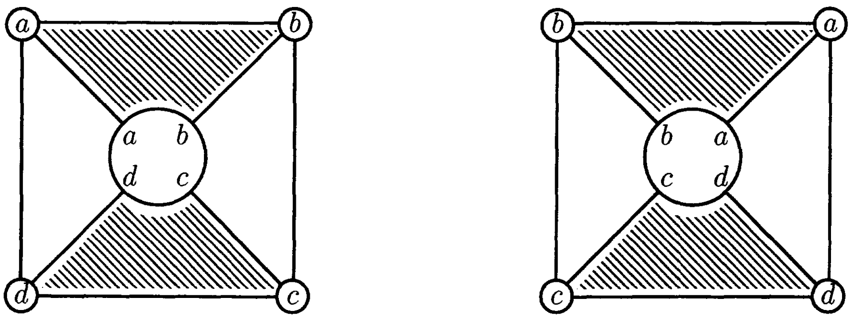


Figure 6-4.

$2(1+2^{n-3}(n-4))$ handles in all. The first cylinder added produces one (connected) surface of that genus; we then added $2^{n-2}-1$ additional handles to that surface, for a total genus of:

$$2(1+2^{n-3}(n-4))+2^{n-2}-1=1+2^{n-2}(n-3).$$

This illustrates the charm of surgical procedures: one can readily visualize the handles (equivalently the holes) of the surface constructed.

We note that the genus is known for all complete graphs (Theorem 6-38) and all complete bipartite graphs (Theorem 6-37), but results are only partial for complete tripartite graphs–even after 30 years of study of the genus parameter. Perhaps this is because, for $K_{a,b,c}$, with $a \geq b \geq c$, $r \neq r_3$ unless $a=b=c$; in fact, $r_3 \leq 2bc$, as every 3-cycle contains a vertex from each partite set. The remaining regions should be as nearly quadrilateral as possible. This leads to the lower bound of the following lemma:

Lemma 6-47. For $a \geq b \geq c$, $\gamma(K_{a,b,c}) \geq \left\lceil \frac{(a-2)(b+c-2)}{4} \right\rceil$.

We conjecture that equality holds. This conjecture is largely affirmed, by Theorems 6-39, 6-40, and 6-41, and especially by work of Craft [C9], who developed new surgical procedures for imbedding nonregular complete tripartite graphs (even though they are *not* graphical products; compare $K_{n,n,n}=K_3[\bar{K}_n]$). Here is one of Craft's results:

Thm. 6-48. Let a, b and c be positive integers with $a \geq b \geq c$. If either:

(i) $b+c$ is even and $a \geq 2b$
   or
(ii) $b+c$ is odd and $a \geq 4\max\{b', c\}+2$, where b' is the smallest integer such that $b' \geq \frac{b}{2}$ and $b'+c \equiv 2 \pmod 4$,

then

$$\gamma(K_{a,b,c}) = \left\lceil \frac{(a-2)(b+c-2)}{4} \right\rceil .$$

We mention that Bouchet [B16] has studied $\gamma(K_{n(m)})$, using "generative m- valuations." He considered the residue classes of n (mod 12) and m (mod 6) and determined $\gamma(K_{n(m)})$ for 32 of the 72 cases, by constructing triangular imbeddings.

Parsons, Pisanski, and Jackson ([PPJ1]) and [JPP1]) employed "wrapped quasi-coverings" to establish:

Thm. 6-49. Let G have a triangular imbedding in S_0; then there are infinitely many $n \in \mathbb{N}$ such that $G(\bar{K}_n)$ has a triangular imbedding.

Finally, we comment that Jackson [J1] has constructed triangular imbeddings, as branched covering spaces (see Section 10-1), for some complete n-partite graphs of the form $K((n-2)m, m, \cdots, m)$.

6-6. Rotation Schemes

Before leaving the theory of graph imbeddings and considering specific imbedding problems, we present a powerful tool for solving such problems: the Edmonds' permutation technique ([E1]; see also Youngs [Y1].) This amounts to an algebraic description, for every 2-cell imbedding of a graph G. It is used, in one form or another, in the proofs of many of the theorems listed above.

Denote the vertex set of a connected graph G by $V(G) = \{1, 2, \cdots, n\}$. For each $i \in V(G)$, let $V(i) = \{k \in V(G) | \{i, k\} \in E(G)\}$. Let $p_i : V(i) \to V(i)$ be a cyclic permutation on $V(i)$, of length $n_i = |V(i)|$; p_i is called the *rotation* at i. The set $\{p_1, p_2, \cdots, p_n\}$ of rotations is called a *rotation scheme*, or *rotation system*. Then there is a one-to-one correspondence between 2-cell imbeddings of G and rotation schemes for G, given by:

Thm. 6-50. Each choice $\{p_1, \cdots, p_n\}$ determines a 2-cell imbedding $G(M)$ of G in a surface M, such that there is an orientation on M which induces a cyclic ordering of the edges $\{i, k\}$ at i in which the immediate successor to $\{i, k\}$ is $\{i, p_i(k)\}$, $i = 1, \cdots, n$. In fact, given $\{p_1, \cdots, p_n\}$, there is an algorithm which produces the determined imbedding. Conversely, given a 2-cell imbedding $G(M)$ in a surface M with a given orientation, there is a corresponding $\{p_1, \cdots, p_n\}$ determining that imbedding.

PROOF. Let $D^* = \{(a,b) | \{a,b\} \in E(G)\}$, and define $P^* : D^* \to D^*$ by: $P^*(a,b) = (b, p_b(a))$. Then P^* is a permutation on the set D^* of directed edges of G (where each edge of G is associated with two oppositely-directed directed edges), and the orbits under $P*$ determine the (2-cell)regions of the corresponding imbedding. These regions may then be "pasted" together with (a,b) matched with (b,a) as in Figure 6-5 – to form a surface M in which G is 2-cell imbedded. (Since every edge (a,b) in the boundary of a given region is matched with an edge – (b,a) – in the boundary of another (or possibly the same) region, M is closed. Since (a,b) is matched with (b,a) – and not with (a,b) – M is orientable. Since each p_i is a *cyclic* permutation, M is a 2-manifold.) The genus of M may now be determined by the euler formula, with r given by the number of orbits under P^*. The converse follows from similar considerations. □

As an example, consider the imbedding of $K_{3,3}$ in S_1 depicted in Figure 6-6. Let $V(K_{3,3}) = \{1,2,3,4,5,6\}$, with $V(1) = V(2) = V(3) = \{4,5,6\}$; $V(4) = V(5) = V(6) = \{1,2,3\}$. Then

$$p_1 : (4,5,6) \qquad p_4 : (1,2,3)$$
$$p_2 : (4,5,6) \qquad p_5 : (1,2,3)$$
$$p_3 : (4,5,6) \qquad p_6 : (1,2,3)$$

describe this imbedding. The orbits under P^* are:

(1) (1,5)(5,2)(2,6)(6,3)(3,4)(4,1)
(2) (5,1)(1,6)(6,2)(2,4)(4,3)(3,5)
(3) (2,5)(5,3)(3,6)(6,1)(1,4)(4,2).

(Note that $P^*(4,1) = (1,5)$; $P^*(3,5) = (5,1)$; $P^*(4,2) = (2,5)$.)

As a matter of notation, from this point on, we will abbreviate an orbit such as (1) above by: 1-5-2-6-3-4; it is implicit that $p_4(3) = 1$, and $p_1(4) = 5$.

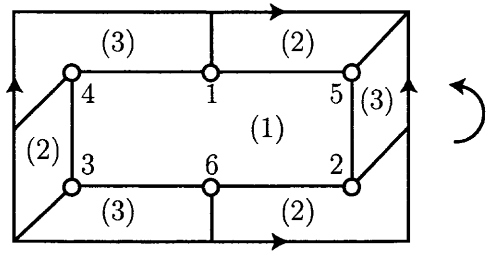


Figure 6-6.

It now follows that the genus of any connected graph (and hence, by Corollary 6-19, of any graph) can be computed, by selecting, from among the $\prod_{i=1}^{n}(n_i - 1)!$ possible permutations P^* (i.e. rotation schemes), one which gives the maximum number of orbits, and hence determines the genus of the graph (component). (Since, by Theorem 6-11, a minimal imbedding must be a 2-cell imbedding, it corresponds to some P^*; by Corollary 5-15, r will be maximal for this imbedding.) The obvious difficulty in applying this procedure is that of selecting a suitable P^* from the (usually) vast number of possible ordered n-tuples of rotations p_i.

Stahl has studied "permutation pairs" as a purely combinatorial generalization of graph imbeddings, and his powerful approach suffices to establish many of the classical results (Theorems 6-18 and 6-21, for example) as well as to obtain new information about the genus of the amalgamation of graphs; see [S15], [S17], [S18], and [S19].

6-7. Imbedding Graphs on Pseudosurfaces

In Section 5-5 we introduced the pseudosurfaces $S(k; n_1(m_1), \cdots, n_t(m_t))$. Recall that for any imbedding of a graph G in a pseudosurface S', we assume that each singular point of S' is occupied by a (singular) vertex of G. The number $2-2k-\sum_{i=1}^{t} n_i(m_i-1)$ gives the characteristic of S', denoted by $\chi'(S')$.

Def. 6-51. The *pseudocharacteristic*, $\chi'(G)$, of a graph G is the largest integer $\chi(S')$ for all pseudosurfaces S' in which G can be imbedded. The *generalized pseudocharacteristic*, $\chi''(G)$, is the largest integer $\chi(S'')$ such that G imbeds in the generalized pseudosurface S''.

Of course, since G has no loops, $r_1 = 0$. But we also require $r_0 = 0$ (else we could identify arbitrarily many K_1 in S_0 at any given vertex, and $\chi''(G)$ would be unbounded) and $r_2 = 0$ (else, by imbedding each edge in its own sphere and then identifying the various images of each vertex, we would have $r = q$ and $\chi''(G) = p$).

A surface can be considered as a (degenerate) pseudosurface, and a pseudosurface as a (degenerate) generalized pseudosurface. Hence we have

$$\chi''(G) \geq \chi'(G) \geq 2 - 2\gamma(G).$$

The second inequality may be strict (i.e. pseudosurfaces may be more efficient, from the point of view of maximizing characteristic, for imbedding graphs into); for example $\chi'(K_5) = 1$, as Figure 6-7 shows. (Now, see Problem 6-13.)

Petroelje [P5] found that many of the basic theorems for imbedding graphs in surfaces carry over for pseudosurfaces. For example:

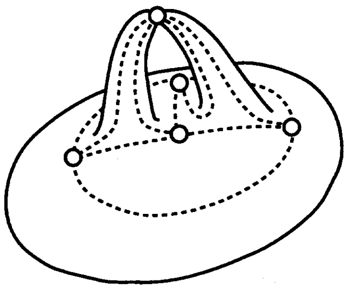


Figure 6-7.

Thm. 6-52. Let G be a connected graph minimally imbedded in the pseudosurface S'; then the imbedding must be 2-cell.

Thm. 6-53. If $G(p,q)$ is connected, then $\chi'(G) \leq p - \frac{q}{3}$; equality holds if and only if G has a triangular imbedding in some pseudosurface.

Petroelje also developed an analogue of Edmonds' permutation technique for pseudosurfaces (the permutations p_i are no longer required to be cyclic; see Problems 6-9 and 6-10.)

Thm. 6-54. $\chi'(K_{n,n,n,n}) = 2n(2-n)$.

(This is consistent with Theorem 6-42 that, for $n \neq 3$, $\gamma(K_{n,n,n,n}) = (n-1)^2$.)

Thm. 6-55. $\chi'(K_{2m,2n,r}) = 2(m+n-mn) - r(m-1)$, where $2m \geq 2n \geq r \geq 1$.

Ringeisen and White [RW1] showed:

Thm. 6-56. $\chi'(K_{m,n}) = 2 - \left\lceil \frac{(m-1)(n-2)}{2} \right\rceil$.

In the cases where m and n are both even or either m or $n \equiv 2 \pmod 4$, the pseudocharacteristic agrees with the characteristic of Theorem 6-37; in all other cases, $\chi'(K_{m,n}) = \chi(K_{m,n}) + 1$. That is, in terms of pseudocharacteristic these imbeddings are more efficient than those for the genus case.

Thm. 6-57. $\chi'(Q_n + \bar{K}_r) = r - (n+r-4)2^{n-2}$, for $0 \leq r \leq n$, $n \geq 2$.

Generalized pseudosurface imbeddings will have relevance in Chapter 12.

6-8. Other Topological Parameters for Graphs

We have seen that, if a graph is not planar, we can still make a "proper" drawing of the graph in some surface and/or pseudosurface. Two common topological parameters (other than genus) which arise if modified drawings are allowed are the thickness and crossing number.

Def. 6-58. The *thickness* $\theta(G)$ of a graph G is the minimum number of planar subgraphs whose union is G. (The union is usually taken over spanning subgraphs).

For sample formulae we have:

Thm. 6-59. (Beineke and Harary [BH2]; Alekseev and Gonchakov [AG1]; Vasak [V1])

$$\theta(K_n) = \left\lfloor \frac{n+7}{6} \right\rfloor, \text{ except that } \theta(K_9) = \theta(K_{10}) = 3.$$

Thm. 6-60. (Beineke [B5])

$$\theta(K_{n,n}) = \left\lfloor \frac{n+5}{4} \right\rfloor.$$

Thm. 6-61. (Kleinert [K1])

$$\theta(Q_n) = \left\lceil \frac{n+1}{4} \right\rceil.$$

Some thickness results have been obtained for surfaces of positive genus.

Def. 6-62. The *thickness* $\theta_n(G)$ of a graph G is the minimum number of subgraphs, each imbeddable on S_n, whose union is G.

Thus $\theta(G) = \theta_0(G)$. The *nonorientable thickness* $\tilde{\theta}_n(G)(n > 0)$ is defined similarly, for N_n.

Thm. 6-63. (Beineke [B6] and [B7])

For $n \geq 3$:

(i) $\tilde{\theta}_1(K_n) = \left\lfloor \frac{n+5}{6} \right\rfloor$;
(ii) $\theta_1(K_n) = \left\lfloor \frac{n+4}{6} \right\rfloor$;
(iii) $\theta_2(K_n) = \left\lfloor \frac{n+3}{6} \right\rfloor$.

Thm. 6-64. (Beineke [B5]) For $n \geq 2$;

(i) $\tilde{\theta}_1(K_{n,n}) = \left\lfloor \frac{n+4}{4} \right\rfloor$;
(ii) $\tilde{\theta}_2(K_{n,n}) = \theta_2(K_{n,n}) = \theta_1(K_{n,n}) = \left\lfloor \frac{n+3}{4} \right\rfloor$;
(iii) $\theta_3(K_{n,n}) = \left\lfloor \frac{n+2}{4} \right\rfloor$.

Thm. 6-65. (Anderson [A4], [A5], [A6])

(i) $\theta_1(K_n - C_n) = \left\lfloor \frac{n+2}{6} \right\rfloor$
(ii) $\theta_1(K_{3(n)}) = \frac{n-1}{2}$, for n an odd prime.
(iii) $\theta_1(K_{4(n)}) = \left\lfloor \frac{n+1}{2} \right\rfloor$
(iv) $\theta_{8rs(2s+1)+1}(K_{4r(2s+1),4r(2s+1)}) = r$.

Def. 6-66. The *crossing number* $v(G)$ of a graph G is the minimum number of pairwise intersections of its (open) edges, among all drawings of G in the plane.

One might say that the crossing number tells us, if we insist upon drawing G on S_0, just how bad this drawing must be. For this parameter, exact values are scarce; we mention the following bounds (see [G12]):

Thm. 6-67. $v(K_n) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$; equality holds for $n \leq 10$.

Thm. 6-68. $v(K_{m,n}) \leq \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$; equality holds for $m \leq 6$, where $m \leq n$.

As an indication of the kind of techniques that might be employed, we prove the following:

Thm. 6-69. $v(K_{3,2,2}) = 2$.

PROOF. Suppose $v(K_{3,2,2}) = x$. Since $K_{3,3}$ is a subgraph of $K_{3,2,2}, \gamma(K_{3,2,2}) \geq 1$ (by Problem 5-5 and the hint for Problem 6-4). Thus $x \geq 1$. Consider such an optimal drawing of $K_{3,2,2}$ in S_0; this gives rise to a plane graph G, with $p = 7 + x$, $q = 16 + 2x$, and $r \neq r_3$. (If $r = r_3$ the configuration in Figure 6-8a, which must exist since

$x \geq 1$, must correspond to the configuration in Figure 6-8b, which cannot occur in any complete tripartite graph.) Hence $2q = 32 + 4x \geq 4+3(r-1) = 3r+1$, and $9+x = q-p = r-2 \leq \frac{31}{3}+\frac{4}{3}x-2 = \frac{25}{3}+\frac{4}{3}x$. Thus $\frac{2}{3} \leq \frac{1}{3}x$, so that $x \geq 2$.

□

(a) (b)

Figure 6-8.

Figure 6-9 shows that $x \leq 2$, to complete the proof. Figure 6-9 depicts an immersion, not an imbedding (since $f : K_{3,2,2} \to S_0$ is not $1-1$).

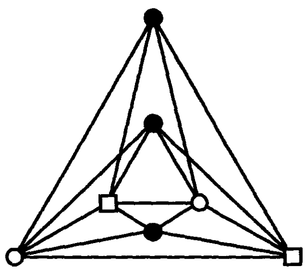


Figure 6-9.

A natural extension of the construction in Figure 6-9 gives the following:

Thm. 6-70. $v(K_{m,n,r}) \leq f(m,n) + f(m,r) + f(n,r)$, where $f(x,y) = (x-1)\binom{y-1}{2} + (y-1)\binom{x-1}{2} + \binom{x-2}{2}\binom{y-1}{2} = \frac{(x-1)(y-1)(xy-4)}{4}$.

Thm. 6-71. $v(K_{n,n,n}) \leq \frac{3}{4}(n-2)(n-1)^2(n+2)$; equality holds for $n = 1, 2$.

The exact results below are due to Beineke and Ringeisen ([BR1] and [RB1]), and to Klešč, Richter, and Stobert [KRS1] for (iii):

Thm. 6-72. The following cartesian products have crossing numbers as indicated:

(i) $v(C_3 \times C_n) = n$, for $n \geq 3$;
(ii) $v(C_4 \times C_n) = 2n$, for $n \geq 4$;

(iii) $v(C_5 \times C_n) = 3n$, for $n \geq 5$;
(iv) $v(K_4 \times C_n) = 3n$, for $n \geq 3$.

Other exact results have been found for crossing numbers on surfaces of positive genus.

Def. 6-73. The *crossing number* $v_n(G)$ of a graph G is the minimum number of pairwise intersections of its (open) edges, among all drawings of G in S_n.

Thus $v(G) = v_0(G)$. The *nonorientable crossing number* $\tilde{v}_n(G)(n > 0)$ is defined similarly, for N_n.

Thm. 6-74. (Guy and Jenkins [GJ1])

$$v_1(K_{3,s}) = \left\lfloor \frac{(s-3)^2}{12} \right\rfloor .$$

Thm. 6-75. (Gross [G7])

Let $h = \frac{(n-1)(n-4)}{4}$, where $n \equiv 1 \pmod 4$ is a prime power; then $v_h(K_{n(2)}) = \frac{n(n-1)}{2}$.

We deduce from Theorem 4-24 that the graphs $Q_n \times K_{4,4}$ are Cayley graphs for all finite Hamiltonian p-groups (in fact, as we will see in Chapter 7, these graphs are of minimum genus for these groups.) From Problem 6-14 we see that $\gamma(Q_n \times K_{4,4}) = 1 + n2^n$, and from Problem 11-8 we will learn that the corresponding nonorientable genus is $\tilde{\gamma}(Q_n \times K_{4,4}) = 2 + n2^{n+1}$.

Thm. 6-76. (Kainen and White [KW1])

Let $h = \gamma(Q_n \times K_{4,4}) - m$, and $k = \tilde{\gamma}(Q_n \times K_{4,4}) - 2m$, with $n \geq 0$; then

(i) $v_h(Q_n \times K_{4,4}) = 4m$, if $0 \leq m \leq 2^n$.
(ii) $\tilde{v}_k(Q_n \times K_{4,4}) = 4m$, if $0 \leq m \leq 2^n$.

6-9. Applications

For applications of the four basic topological parameters discussed in this chapter, consider the problem of printing an electronic circuit on a circuit board. If the associated graph G is planar, one board will suffice, without modification. If G is not planar, at least four alternatives are available to avoid short circuits (the choice depending upon relevant considerations of an engineering and/or economic nature): 1.) the circuit can be accommodated by drilling holes through the board; $\gamma(G)$ gives the minimum number of holes; 2.) some of the vertices can be printed on both sides of the board, with connections made through the board between corresponding images of the same vertex; here we seek the minimum n such that G can be imbedded in the pseudosurface $S(0; n(2))$; however, it may occur that no such n exists (see Problem 6-12.) If a given vertex can appear arbitrarily often, with connections made through the board among corresponding images of the same vertex, and if in addition holes can be drilled as in 1.), then we seek the value of the parameter $\chi'(G)$, for maximum efficiency; 3.) If several circuit boards are used, each containing a planar portion of the circuit, and jumpers are run between successive boards to connect corresponding images of the same junction, then we are studying the parameter $\theta(G)$; 4.) If the circuit is stamped on one side of one circuit board (with no holes yet drilled) and if wherever two connections cross extraneously two holes are now drilled to allow one connection to temporarily pass to the other side of the board, enabling it to "cross" the second connection while avoiding a short circuit, it is the parameter $v(G)$ that dictates economy of effort here.

As an example, consider the modified wheatstone bridge circuit of Figure 6-10(a); the associated graph is $G = K_{3,3}$. Figures 6-9(b) - (e) correspond respectively to: $\gamma(K_{3,3}) = 1$, $\chi'(K_{3,3}) = 1$, $\theta(K_{3,3}) = 2$, and $v(K_{3,3}) = 1$.

6-10. Problems

6-1.) Show that two graphs are homeomorphic in the graph-theoretical sense if and only if their realizations in $\mathbb{R}^3$ are homeomorphic in the topological sense.

6-2.) Show that no four points on $C = \{(x, y, z) \in \mathbb{R}^3 | x = t,\ y = t^2,\ z = t^3; t \geq 0\}$ are coplanar.

6-3.) Prove Corollary 6-15.

6-4.) Prove the easy half of Kuratowski's Theorem: If G contains a Kuratowski subgraph, then G is non-planar. (Hint: show that if H is a subgraph of G, then $\gamma(H) \leq \gamma(G)$.)

6-5.) Show that the Petersen graph (see Figure 8-8 or Figure 11-4) is non-planar. What is its genus?

6-6.) Show that Corollary 6-19 follows from Theorem 6-18.

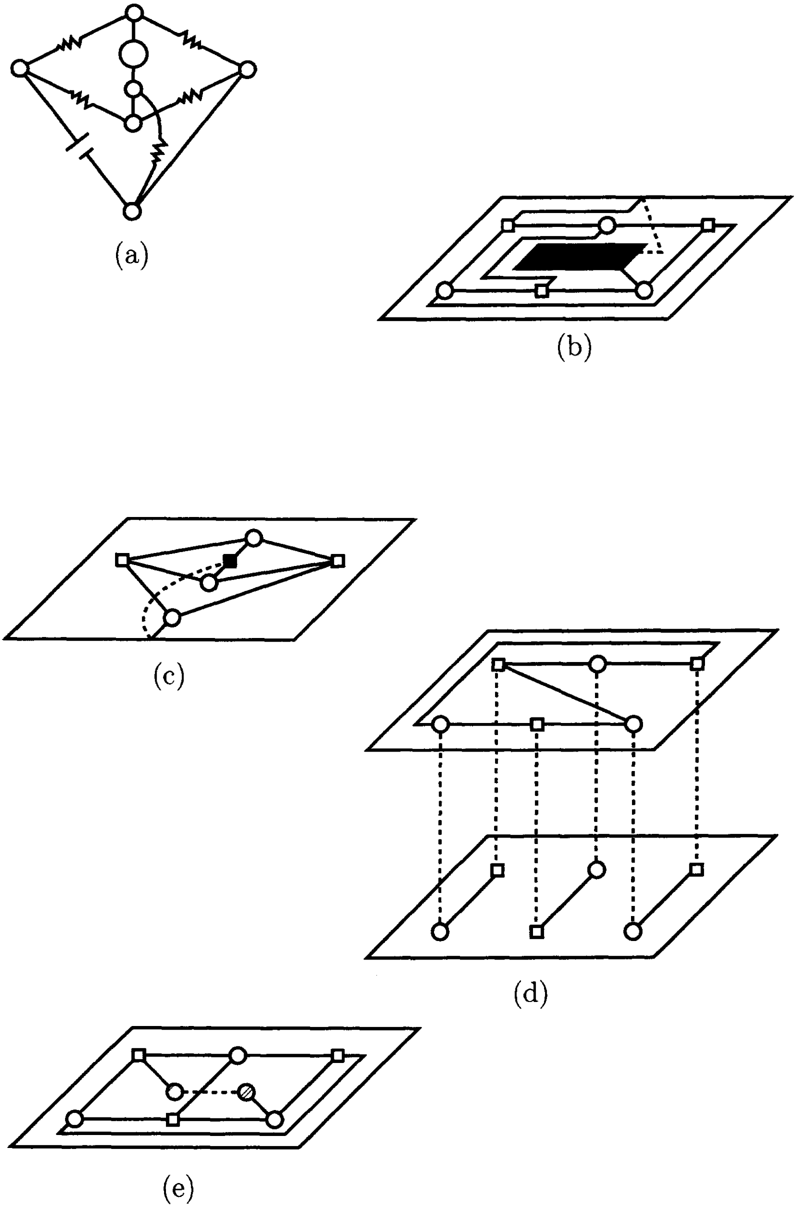


Figure 6-10.

6-7.) Show that

$$
\begin{aligned}
p_1, p_3 &: (5, 6, 7, 8) \\
p_2, p_4 &: (8, 7, 6, 5,) \\
p_5, p_7 &: (1, 2, 3, 4) \\
p_6, p_8 &: (4, 3, 2, 1)
\end{aligned}
$$

describe a 2-cell imbedding of $K_{4,4}$ in S_1, with $r = r_4$. Sketch the imbedding.

6-8.) Give a rotation scheme for the plane imbedding of $K_{3(2)}$ given in Figure 5-8. Then, check that your rotation scheme produces the imbedding you started with.

6-9.) Label the vertices of Figure 6-7 and then give the corresponding (non-cyclic) rotation scheme.

6-10.) Sketch an imbedding of K_6 in the pseudosurface $S(0; 2(2))$. Find the corresponding (non-cyclic) rotation scheme. How does this imbedding compare with that of K_6 on the torus, in terms of maximizing characteristic?

6-11.) Why is $\chi'(G)$ defined as a maximum characteristic, instead of a minimum genus?

6-12.) Show that $G = K_n$, $n \geq 13$, imbeds on no pseudosurface $S(0; k(2))$.

6-13.) Can the first inequality following Definition 6-51 be strict also?

6-14.) *Show that $\gamma(Q_n \times K_{4,4}) = 1 + n2^n, n \geq 0$.

6-15.) *Find $\gamma(K_{3,3} \times K_{3,3})$. Give both a careful proof and a "visualization" of your genus surface.

6-16.) Let G be a connected graph of order $p \geq 2$; show that G^2 $(V(G^2) = V(G),\ E(G^2) = \{uv | u, v \in V(G), 1 \leq d(u,v) \leq 2\})$ is upper imbeddable, $\gamma_M(G^2) \geq \frac{p-2}{2}$, and the lower bound is sharp.

CHAPTER 7

THE GENUS OF A GROUP

To get an accurate and efficient "picture" of a group, we seek a surface of minimum genus on which we can imbed a Cayley color graph of some presentation of the group. This suggests the following definition. Let $\gamma(C_\Delta(\Gamma))$ denote the genus of the *underling graph* $G_\Delta(\Gamma)$ (called the *Cayley graph*) determined from $C_\Delta(\Gamma)$ by removing all arrows and colors from the edges (recall that, by convention $C_\Delta(\Gamma)$ has no loops or multiple edges). Then:

Def. 7-1. The *genus* of a group Γ is given by:

$$\gamma(\Gamma) = \min\{\gamma(C_\Delta(\Gamma))\} = \min\{\gamma(G_\Delta(\Gamma))\},$$

where the minimum is taken over all generating sets Δ for Γ.

Def. 7-2. A group Γ is said to be *planar* if $\gamma(\Gamma) = 0$.

7-1. Imbeddings of Cayley Color graphs

Finite planar groups have been catalogued by Maschke [M2] (see also Anderson [A8]). The finite planar groups on one generator are exactly the cyclic groups $\mathbb{Z}_n$; on two generators, they include the dihedral groups D_n, groups of the form $\mathbb{Z}_2 \times \mathbb{Z}_n$, S_4, A_4, and A_5 (the last three groups are the symmetry groups of the regular polyhedra), and $\mathbb{Z}_2 \times A_4$; on three generators (each must be of order 2) finite planar groups include $\mathbb{Z}_2 \times D_n$, $\mathbb{Z}_2 \times S_4$, and $\mathbb{Z}_2 \times A_5$. In summary:

Thm. 7-3. The finite group Γ is planar if and only if $\Gamma = \Gamma_1 \times \Gamma_2$, where $\Gamma_1 = \mathbb{Z}_1$ or $\mathbb{Z}_2$ and $\Gamma_2 = \mathbb{Z}_n$, D_n, S_4, A_4, or A_5.

We consider infinite groups temporarily, preparatory to establishing a startling result, due to Levinson [L2]. In this chapter, an infinite graph is given by:

Def. 7-4. An *infinite graph* is a graph with denumerable vertex set.

There are two natural (but non-equivalent!) definitions of planarity, for infinite graphs.

Def. 7-5. An infinite graph is said to be *planar* if it can be imbedded in the plane.

Def. 7-5′. An infinite graph is said to be *planar* if it can be imbedded in the plane so that the vertex set has no limit points.

We adopt Definition 7-5, for reasons soon to be obvious.

Thm. 7-6. An infinite graph is planar if and only if it contains no subgraph homeomorphic with K_5 or $K_{3,3}$.

For a proof of this extension of Kuratowski's theorem, see Dirac and Schuster [DS1]. To see that this extension does *not* hold for Definition 7-5′, consider the graph of Figure 7-1, where an infinite path is attached at each vertex of K_4. In fact, if we prohibit limit points to the vertex set, then by the Bolzano-Weierstrass Theorem, no infinite graph imbeds on any closed 2-manifold.

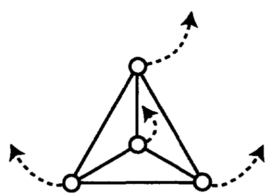


Figure 7-1.

Def. 7-7. An infinite graph G has *infinite genus* ($\gamma(G) = \infty$), if, for every natural number n, there exists a finite subgraph G_n of G such that $\gamma(G_n) \geq n$.

Lemma 7-8. Let G be the graph of a presentation of an infinite group Γ. Let H be an induced finite subgraph of G. Then there exist two disjoint, isomorphic copies of H in G.

PROOF. The vertex set of H corresponds to a finite set $\{g_1, \cdots, g_n\}$ of elements of Γ. Form the (finite) set:

$$S = \{g_i g_j^{-1} | 1 \leq i, j \leq n\}.$$

Pick $x \in \Gamma - S$. Form H^*, the subgraph of G induced by $\{xg_i | i = 1, \cdots, n\}$; then H^* is isomorphic to H since $g_i h = g_j$ if and only if

$xg_ih = xg_j$. Now, suppose that $v \in V(H) \cap V(H^*)$; then there exist i and j such that $xg_j = g_i$, so that $x = g_ig_j^{-1} \in S$, a contradiction.

□

We now present Levinson's result.

Thm. 7-9. Let Γ be an infinite group, with G the graph of a presentation for Γ. Then either $\gamma(G) = 0$, or $\gamma(G) = \infty$.

PROOF. Suppose G is not planar; then, by Theorem 7-6, G contains K, a Kuratowski (and hence finite) subgraph. Thus $\gamma(G) \geq 1$. Let n be an arbitrary natural number. But by Lemma 7-8, we can find a second, disjoint copy of K in G, so that $\gamma(G) \geq \gamma(2K) = 2$ by Corollary 6-19. Now apply the lemma again, with $H = 2K$, to obtain two disjoint copies of $2K$ in G, so that $\gamma(G) \geq 4$. Continuing in this fashion, we eventually find two disjoint copies of $2^{n-1}K$ in G, so that $\gamma(G) \geq 2^n > n$; then $\gamma(G) = \infty$, since n was arbitrary. □

For example, $\gamma(G) = \infty$, for the standard presentation for $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$; see Problem 7-9.

Cor. 7-10. Let Γ be an infinite group; then either $\gamma(\Gamma) = 0$, or $\gamma(\Gamma) = \infty$.

In Figure 7-2, portions of planar Cayley color graphs for presentations of three infinite groups are given. The second group (b) is called the *infinite dihedral* group; the third group (c) is the *free group* on two generators. For an infinite group having infinite genus, see Problem 7-10.

Returning our attention to finite groups, we produce examples of groups of positive genus. The following lemma will be useful. Note that if P gives $\gamma(\Gamma)$, then P may be assumed to have no redundant generators; i.e. P is minimal. We also note that, in any imbedding of a Cayley Color graph, every region boundary corresponds to an identity word.

Lemma 7-11. Let Γ be a finite group, with $3 \not| |\Gamma|$; let Δ be a minimal generating set for Γ. Then $C_\Delta(\Gamma)$ contains no triangles.

PROOF. Suppose $C_\Delta(\Gamma)$ contains a triangle; then we find a closed walk $h_1^{a_1}h_2^{a_2}h_3^{a_3} = e$ in $C_\Delta(\Gamma)$, where h_i is a generator in P, and $a_i = \pm 1$. If any two of the h_i are distinct, then one of these two is redundant.

If, on the other hand, $h_1 = h_2 = h_3$, then the a_i all have the same sign (or else all three $= e$). But then $h_1^3 = e$, and $3||\Gamma|$, a contradiction. $\square$

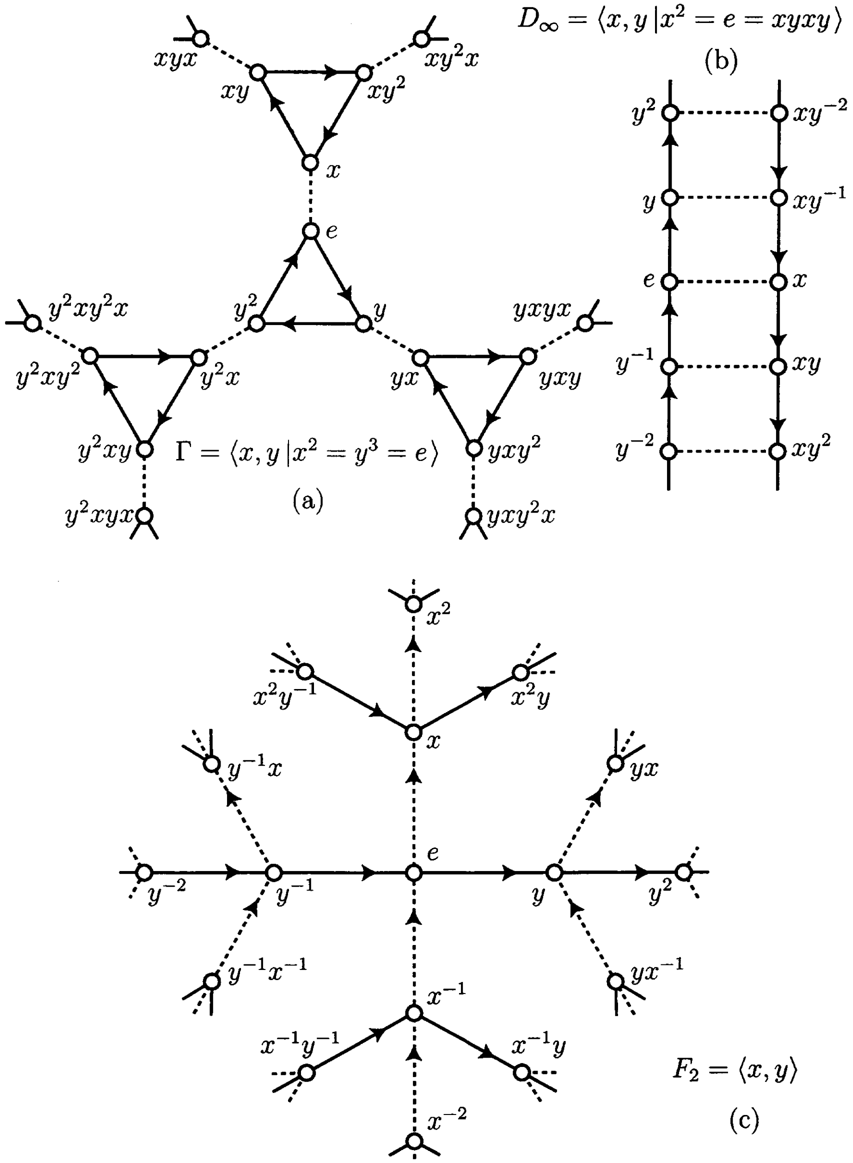


Figure 7-2.

Now consider Q, the group of the quaternions. Let P be a presentation for Q, such that $\gamma(Q) = \gamma(C_\Delta(Q))$; then P is minimal. By Lemma 7-11, $C_\Delta(Q)$ has no triangles, since $|Q| = 8$. It is not difficult to see that Δ has at least two generators and that if Δ has exactly two generators, neither can be of order 2; furthermore, Δ cannot have three generators of order 2 (see Problem 7-1). Thus $C_\Delta(Q)$ is regular of degree at least four. Then $2q \geq 4p = 32$; i.e. $q \geq 16$. Now, by Corollary 6-15,

$$\gamma(Q) \geq \frac{q}{4} - \frac{p}{2} + 1 \geq 1.$$

Then $\gamma(Q) = 1$ is shown by Figure 7-3, where

$$Q = \langle x, y | x^2 = y^2 = (xy)^2 \rangle .$$

The underlying Cayley graph is $K_{4,4}$. Thus Q is the smallest nonplanar group.

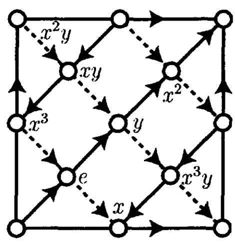


Figure 7-3.

7-2. Genus Formulae for Groups

We now find a non-trivial genus formula for an infinite class of groups: those groups (necessarily abelian) in which every element is of order 2. Let Γ_n denote this group; then $\Gamma_n = (\mathbb{Z}_2)^n$, and $|\Gamma| = 2^n$.

Thm. 7-12. $\gamma(\Gamma_n) = 1 + 2^{n-3}(n-4), n \geq 2$.

PROOF. Γ_n may be expressed as follows: $\Gamma_1 = \mathbb{Z}_2$; $\Gamma_n = \mathbb{Z}_2 \times \Gamma_{n-1}$, for $n \geq 2$. Writing Γ_n as an iterated direct product in this way, we see that any P for Γ_n must have at least n generators; hence $2q \geq np = n2^n$; thus by Lemma 7-11 and Corollary 6-15,

$$\begin{aligned}\gamma(\Gamma_n) &\geq \frac{q}{4} - \frac{p}{2} + 1 \\ &\geq n2^{n-3} - 2^{n-1} + 1 \\ &= 1 + 2^{n-3}(n-4).\end{aligned}$$

But now let P be determined by repeated application of Theorem 4-21; then $G_\Delta(\Gamma_n) = Q_n$, the n-cube, and

$$\begin{aligned}\gamma(\Gamma_n) &\leq \gamma(C_\Delta(\Gamma_n)) \\ &= \gamma(Q_n) \\ &= 1 + 2^{n-3}(n-4),\end{aligned}$$

by Theorem 6-36. This completes the proof. □

Let us extend this result somewhat. We will need the following genus formula, involving $(n+1)$ parameters. Define the graph H_n as follows: let $H_1 = C_{2m_1}$, the cycle on $2m_1$ vertices, and recursively define $H_n = H_{n-1} \times C_{2m_n}$, for $n \geq 2$, where each $m_i \geq 2$. Let $M^{(n)} = \prod_{i=1}^{n} m_i$.

Thm. 7-13. $\gamma(H_n) = 1 + 2^{n-2}(n-2)M^{(n)}$, $n \geq 2$.

PROOF. By Theorem 2-19, Problem 2-4, and a trivial induction argument, H_n is a bipartite graph. We produce a quadrilateral imbedding for H_n, and compute $\gamma(G_n)$ using Corollary 6-16. For H_n, let $p^{(n)}$ and $q^{(n)}$ denote the number of vertices and edges respectively. Then $p^{(n)} = 2^n M^{(n)}$; and since H_n is regular of degree $2n$, it is a simple matter to compute $q^{(n)} = 2^n n M^{(n)}$.

Let the statement $S(n)$ be: there is an imbedding of H_n for which $r = r_4 = n2^{n-1}M^{(n)}$, including two disjoint sets of $2^{n-2}M^{(n)}$ mutually vertex-disjoint quadrilateral regions each, both sets containing all $2^n M^{(n)}$ vertices of H_n. We claim that $S(n)$ is true for all $n \geq 2$, and we verify this claim by mathematical induction.

That $S(2)$ is true is apparent from Figure 7-4 (which shows an imbedding of $C_4 \times C_6$ in S_1), with the regions designated by (1) making up one set, and those designated by (2) making up the other. We now assume $S(n)$ to be true and establish $S(n+1)$, for $n \geq 2$.

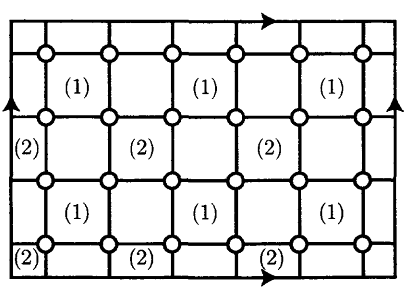


Figure 7-4.

For the graph H_{n+1}, we start with $2m_{n+1}$ copies of H_n, minimally imbedded as described by $S(n)$. We partition the corresponding surfaces into m_{n+1} copies of one orientation, and m_{n+1} copies of the reverse orientation, corresponding to the vertex set partition of the bipartite graph $C_{2m_{n+1}}$. From each copy, two joins of $p^{(n)}$ edges each must be made, both to copies of opposite orientation, in order to construct H_{n+1}. From the statement $S(n)$, it is clear that these two joins can be made, each one over $2^{n-2}M^{(n)}$ tubes carrying four edges each. (Attach one end of a tube in the interior of each region designated by (1) for one join; use the regions designated by (2) for the second join.) Each new region formed by this process is a quadrilateral. In this fashion the required $2m_{n+1}$ joins can be made to imbed H_{n+1}, with $r = r_4$. Now form one set of regions by selecting opposite quadrilaterals from each tube added in alternate joins in this construction. Form the second set by selecting the remaining quadrilaterals on the same tubes. It is clear that the two sets of regions thus selected are disjoint, and that each contains $(2)(m_{n+1})(2^{n-2}M^{(n)}) = 2^{n-1}M^{(n+1)}$ mutually vertex-disjoint quadrilaterals; both sets contain all $2^{n+1}M^{(n+1)}$ vertices of H_{n+1}. Furthermore, $r^{(n+1)} = 2m_{n+1}r^{(n)}+\Delta r$, where $\Delta r = (2m_{n+1})(2^{n-2}M^{(n)})(2)$, where $2m_{n+1}$ joins have been made with $2^{n-2}M^{(n)}$ tubes per join, and a net increase in r of 2 per tube. Hence,

$$\begin{aligned} r^{(n+1)} &= 2m_{n+1}(n2^{n-1}M^{(n)}) + 2^nM^{(n+1)} \\ &= (n+1)2^nM^{(n+1)}, \end{aligned}$$

and we have established that $S(n+1)$ follows from $S(n)$. Therefore, $S(n)$ holds, for all $n \geq 2$.

We can now compute:

$$\begin{aligned} \gamma(H_n) &= 1 + \frac{2^n nM^{(n)}}{4} - \frac{2^nM^{(n)}}{2} \\ &= 1 + 2^{n-2}(n-2)M^{(n)}. \end{aligned}$$

□

For the special case where $m_i = m$, $i = 1, \cdots, n$, we have $M^{(n)} = m^n$, and:

Cor. 7-14. The genus of $H_n^{(m)}$ is given by:

$$\gamma(H_n^{(m)}) = 1 + 2^{n-2}(n-2)m^n.$$

Furthermore, if $m = 2$ in the above formula, since $C_4 = K_2 \times K_2, H_n^{(2)}$ is the $2n$-cube, and we obtain the result (compare with Theorem 6-36):

Cor. 7-15. $\gamma(Q_{2n}) = 1 + 2^{2n-2}(n-2)$.

For further results concerning the genus of repeated cartesian products of bipartite graphs, see [W6], and also Pisanski, [P6].

Now, let $\Gamma_n^{(m)}$ be the abelian group with minimal Cayley color graph $H_n^{(m)}, m \geq 2$; i.e. $\Gamma_1^{(m)} = \mathbb{Z}_{2m}$, and $\Gamma_n^{(m)} = \mathbb{Z}_{2m} \times \Gamma_{n-1}^{(m)}$, for $n \geq 2$. Then we have:

Cor. 7-16. $\gamma(\Gamma_n^{(m)}) = 1 + 2^{n-2}(n-2)m^n$.

The reader may wish to combine Theorems 4-18, 4-21, and 7-13 to obtain genus formulae for additional abelian groups. For example, using the notation of Theorem 4-18, consider Γ where m_r is even. Using current graph constructions, rather than the surgery techniques of Theorem 7-13, Jungerman and White [JW1] found the genus of "most" of the remaining finite abelian groups; the next theorem summarizes to include previous results as well.

Thm. 7-17. Let $\Gamma = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_r}$, where for $2 \leq i \leq r$, m_i divides m_{i-1} (and $m_r > 1$, unless $|\Gamma| = 1$.) Let $N(\Gamma) = 1 + \frac{(r-2)|\Gamma|}{4}$; then

(i) If $r = 1$, then $\gamma(\Gamma) = 0$.
(ii) If $r = 2$ and $m_r = 2$, then $\gamma(\Gamma) = 0$.
(iii) If $N(\Gamma)$ is an integer, $m_r > 3$, $r > 1$, and either m_r is even or $r \neq 3$, then $\gamma(\Gamma) = N(\Gamma)$.
(iv) If $r \geq 3$ and for some $k, 1 \leq k \leq r$, k is minimal so that $m_k = 2$, then $\gamma(\Gamma) = N(\Gamma) - (r-k+1)\frac{|\Gamma|}{8}$.
(v) If $N(\Gamma)$ is an integer, $m_r = 3$, and $1 < r \neq 3$, then $\gamma(\Gamma) \leq N(\Gamma)$.

The argument of Theorem 7-12 can be modified to assist in the computation of the genus for certain hamiltonian groups; the following results are due to Himelwright [H8]:

Thm. 7-18. $\gamma(Q \times (\mathbb{Z}_2)^n) = n2^n + 1$.

Thm. 7-19. $\gamma(Q \times \mathbb{Z}_m \times (\mathbb{Z}_2)^n) = mn2^n + 1$, for m odd.

Cor. 7-20. The groups $Q \times \mathbb{Z}_m \times (\mathbb{Z}_2)^8$, for m odd, have genus asymptotic to the order.

By Theorems 4-18 and 4-24, if G is a hamiltonian group, then $G = Q \times \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_r} \times (\mathbb{Z}_2)^n$, where the m_i are odd ($i = 1, \cdots, r$) and $m_i | m_{i-1}$ ($i = 2, \cdots, r$). Himelwright has also shown:

Thm. 7-21. The genus of the hamiltonian group $Q \times \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_r} \times (\mathbb{Z}_2)^n$ is asymptotic to $2_n(r+n-1)\prod_{i=1}^{r} m_i$, if $1 \leq r \leq n+1$.

There are many open questions in this area. If a generalization of Theorem 7-13 for products of arbitrary (not necessarily even) cycles could be found, then the genus of any abelian (and also of any hamiltonian) group could be easily computed. What is $\gamma(S_n)$? $\gamma(A_n)$? The following theorem produces upper bounds for these (and other) group genera.

Thm. 7-22. If Γ is finite and is minimally generated by $\{g_1, , \cdots, g_n\}$ and satisfies at least the relations $g_i^{m_i} = e = (\prod_{j=1}^{n} g_j)^k, (1 \leq i \leq n)$ then

$$\gamma(\Gamma) \leq 1 + \frac{|\Gamma|}{2}\Big(n - 1 - \frac{1}{k} - \sum_{j=1}^{n} \frac{1}{m_j}\Big).$$

PROOF. Select $p_g = (gg_1, gg_1^{-1}, gg_2, gg_2^{-1}, \cdots, gg_n, gg_n^{-1})$, for all $g \in G$. Then, using Edmonds' algorithm (see Theorem 6-50), we compute orbits as follows:

(i) An orbit containing the directed edge (a, ag_i^{-1}) continues with $p_{ag_i^{-1}}(a) = ag_i^{-2}$; hence this orbit corresponds to the relation $g_i^{m_i} = e$ and has length m_i. (If $m_i = 2$, we draw edges for both $gg_i = g'$ and $g'g_i = g$, obtaining $\frac{|\Gamma|}{2}$ 2-sided regions; for each such region, the two sides can be identified and the arrows removed, so that the region is destroyed but the genus is unaffected.)
(ii) An orbit containing the directed edge (a, ag_i) continues with $p_{ag_i}(a) = ag_ig_{i+1}$; hence this orbit corresponds to the relation $(\prod_{j=1}^{n} g_j)^k = e$ and has length nk. As there are no other orbits, we find

$$\begin{aligned} r &= \sum_{i=1}^{n} r_{m_i} + r_{nk} \\ &= \sum_{i=1}^{n} \frac{|\Gamma|}{m_i} + \frac{|\Gamma|}{k}; \end{aligned}$$

the euler formula now gives the genus γ of the theorem for this imbedding of $C_\Delta(\Gamma)$, for this presentation P for Γ. Hence $\gamma(\Gamma) \leq \gamma(C_\Delta(\Gamma)) \leq \gamma$.

□

We note that an equivalent formula was obtained by Burnside [B21, p, 398] in a different context. Theorem 7-22 gives $\gamma(G)$ exactly, for $\Gamma = \mathbb{Z}_m$, D_m, A_4, S_4, A_5, $\mathbb{Z}_3 \times \mathbb{Z}_3$, or S_5, (for example).

We also obtain the following two corollaries:

Cor. 7-23. $\gamma(S_n) \le 1 + \frac{(n-2)!}{4}(n^2 - 5n + 2), n \ge 2$.

PROOF. Take $s = (123\cdots n)$ and $t = (12)$ as generators for S_n; then $s^n = t^2 = (st)^{n-1} = e$. □

Cor. 7-24.

$$\gamma(A_n) \le \begin{cases} 1 + \frac{(n-1)(n-3)!(n^2-6n+4)}{8}, & n \text{ odd} \\ 1 + \frac{n(n-2)!(n-5)}{8}, & n \text{ even.} \end{cases}$$

PROOF. For n odd, $s = (12\cdots n-2)$ and $t = (1n-1)(2n)$ generate A_n, and $s^{n-2} = t^2 = (st)^n = e$. For n even, $s = (12\cdots n-1)$ and $t = (12)(3n)$ generate A_n, with $s^{n-1} = t^2 = (st)^{n-1} = e$ (see [B19]). □

The two formulas given above for S_n and A_n respectively were also found by Brahana [B18], using a different method and in a slightly different context. For related results, see [W8].

There has been much activity in the study of the genus parameter for groups, in recent years, perhaps at least in part motivated by Chapter 7 of the first edition of *Graphs, Groups and Surfaces.* We have already presented Theorem 7-17. Here is a continued sample of this research.

Thm. 7-25. (Proulx [P11])

$$\gamma(S_n) \le 1 + \frac{n!}{8}, \text{ for } n \text{ even } n \ge 4.$$

This improves on Corollary 7-23, for n even. For n odd, a sharper bound than that of Corollary 7-23 was developed in [W8]; but the next results detract from the interest that might accrue to the improvement of bounds for $\gamma(S_n)$:

Thm. 7-26. (Proulx [P12])

$$\gamma(S_5) = 4.$$

(The proof is not easy!)

Thm. 7-27. (Tucker [T9])

$$\gamma(S_n) = 1 + \frac{n!}{168}, \text{ for } n \ge 168.$$

We also find

Thm. 7-28. (Tucker [T9])

$$\gamma(A_n) < 1 + \frac{n!}{336}, \text{ for } n \geq 168.$$

During the period 1972-77, Gross [G8], Gross and Lomonaco [GL1], and White [W8] showed that certain metacyclic and dicyclic groups are toroidal. Then in 1977 Proulx [P11] completed work begun by Baker in 1931 [B4], to classify *all* toroidal groups in a major effort. The classification consists of nineteen presentations on two generators, ten presentations on three generators, and one presentation on four generators. In the process, Proulx claimed to have found the genus of each group having order less than 32, except for the group $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. This group thus became a focus of attention.

Let $\gamma(\Gamma) = \gamma(G_\Delta(\Gamma))$, $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. A minimal generating set Δ will have three generators, each of order three, so that $G_\Delta(\Gamma)$ will have at most 27 3-cycles and any imbedding of $G_\Delta(\Gamma)$ will have $r_3 \leq 27$ (with $p = 27$ and $q = 81$). Corollary 6-14 gives $\gamma(G_\Delta(\Gamma)) \geq 1$. But if we incorporate $r_3 \leq 27$, so that $r \leq 46$ (using $2q = 162$ and r even), we find $\gamma(G_\Delta(\Gamma)) \geq 5$. From Theorem 7-22, we find that $\gamma(\Gamma) \leq 10$; thus $5 \leq \gamma(\Gamma) \leq 10$; If we take $\Delta = \{(1,0,0),(0,1,0),(0,0,1)\}$ as the standard basis, then by Theorem 4-22 $G_\Delta(\Gamma) = C_3 \times C_3 \times C_3$. Our first attempt to imbed this graph is by surgery. But efficient surgical procedures for cartesian products require all factors to be bipartite. This graph misses badly! (Note that part (iii) of Theorem 7-17 prohibits both $m_3 = 3$ and $r = 3$.) If we try lifting techniques, we find no improvement on the upper bound of 10 already obtained. (See Section 10-6.) (The construction of Theorem 7-22 is actually one of lifting, as we will see in Chapter 10.)

Thus we turn to the *ad hoc* generation of a rotation scheme. The guiding principle is to enforce $r_3 = 27$ and then try to maximize r_4. Set $A = (1,0,0)$, $B = (0,1,0)$, and $C = (0,0,1)$, so that $\Delta = \{A, B, C\}$. Forcing $r_3 = 27$ requires that the rotation at each vertex X of $C_3 \times C_3 \times C_3$ have $X + E$ and $X - E$ adjacent, for each $E \in \Delta$, in one of the two possible orders. In early 1985, Mohar, Pisanski, Skoviera, and White [MPSW1] encoded a rotation scheme for $C_3 \times C_3 \times C_3$ as follows:

(+A, +B, +C)	000, 111, 222
(-A, -B, -C)	012, 120, 201
(+A, -C, +B)	010, 122
(-A, +C, +B)	020, 112
(+A, +C, -B)	022, 110
(-A, +C, -B)	001, 002, 021, 220, 221
(+A, -C, -B)	011, 100, 200, 210, 211
(-A, -C, +B)	101, 102, 121, 202, 212

In this code, $+E$ in the rotation at X is replaced by $X-E, X+E$; whereas $-E$ becomes $X+E$, $X-E$. For example, if $X=010$, then the code (+A, -C, +B) determines $p_x = (210, 110, 011, 012, 000, 020)$. Then the algorithm of Theorem 6-50 yields an imbedding of $C_3 \times C_3 \times C_3$ on S_7, with $r_3 = 27$, $r_4 = 9$, $r_6 = 5$, and $r_{15} = 1$. Thus $5 \leq \gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) \leq 7$.

Coffee-house discussions in Dubrovnik in April of 1985, led by Tucker, raised the lower bound to 6, and then Brin and Squier [BS1] applied the *coup de grace* by raising the lower bound again; thus (after much effort):

Thm. 7-29. $\gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) = 7$.

It then transpired that another group of order 27 had apparently escaped Proulx's attention altogether, perhaps due to a misreading of Table 1 in Coxeter and Moser[CM1] . This group is the semidirect product $\mathbb{Z}_9 \ltimes \mathbb{Z}_3$. In 1989, David Rauschenberg (then an undergraduate at Towson University) combined with Brin and Squier [BSR1] to show that $\gamma(\mathbb{Z}_9 \ltimes \mathbb{Z}_3) = 4$. Thus the genus is now known for all "small" order groups.

We have remarked that odd-order cyclic factors limit the extent of Theorem 7-17. However, Mohar, Pisanski, and White [MPW1] have shown that $\gamma(\Gamma)$ is asymptotic to $N(\Gamma)$, if $r \geq 3$ is fixed and m_r tends to infinity.

7-3. Related Results

In 1977 Babai [B2] solved a problem posed in the first edition of *Graphs, Groups and Surfaces*, when he established:

Thm. 7-30. If Γ_1 is a subgroup of Γ_2, then $\gamma(\Gamma_1) \leq \gamma(\Gamma_2)$.

Since S_n is isomorphic to a subgroup of A_{n+2}, we obtain

Cor. 7-31. $\gamma(S_n) \leq \gamma(A_{n+2})$.

For example, we learn from Theorem 7-26 that $\gamma(A_7) \geq 4$.

Despite the fact that there are infinitely many planar groups (Maschke's Theorem) and infinitely many toroidal groups (Proulx's classification) Tucker [T8] showed, in 1978:

Thm. 7-32. For each $g \geq 2$, there are at most finitely many groups Γ such that $\gamma(\Gamma) = g$.

The following 1981 result, also due to Tucker [T10], is even more surprising:

Thm. 7-33. There is exactly one group of genus two. It has order 96 and presentation

$$\langle x, y, z | x^2 = y^2 = z^2 = (xy)^2 = (yz)^3 = (xz)^8 = y(xz)^4 y(xz)^4 = e \rangle .$$

The unique group of genus two is the automorphism group of the generalized Petersen graph $G(8,3)$; see Frucht, Graver, and Watkins [FGW1].

Theorem 7-32 suggests studying the function $f : A \to B$, where $A = \mathbb{N} \cap [2, \infty)$; $B = \mathbb{N} \cup \{0\}$; and $f(g) = |\{\Gamma | \gamma(\Gamma) = g\}|$. (We recall that $f(0)$ and $f(1)$ are both infinite, *via*, for example, $\Gamma = \mathbb{Z}_n$ and $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_n$, $n \geq 3$, respectively.) By Theorem 7-33, $f(2) = 1$. We also note that $f(3) \geq 1 (Q \times \mathbb{Z}_2)$, $f(4) \geq 2$ (S_5 and $\mathbb{Z}_9 \ltimes \mathbb{Z}_3$)), and $f(5) \geq 3(\mathbb{Z}_2^5, \mathbb{Z}_4 \times \mathbb{Z}_2^3, \mathbb{Z}_4^2 \times \mathbb{Z}_2)$. However, no group of genus 6 (or 8, 12, 14, 16, 18, ...) is yet known. In [W20], the following are established:

Thm. 7-34. For the function f defined above:

(i) $f(g) \geq 2$, for g odd and ≥ 5;
(ii) $f(g) \geq 3$, for $g \equiv 1 \pmod{18}$;
(iii) $f(g) \geq 1$, for $g \equiv 10 \pmod{18}$;
(iv) f is unbounded.

Now, see Problem 7-12.

We mention that other definitions of the "genus of a group" appear in the literature, due to Levinson [L3], Machlachlan [M1], and Burnside [B21]. For Γ a finite group, we denote these parameters by $\gamma_L(\Gamma)$, $\gamma_M(\Gamma)$, and $\gamma_B(\Gamma)$ respectively. The Levinson parameter, like the parameter $\gamma(\Gamma)$ advocated in this chapter, regards Γ as being depicted by a Cayley graph $G_\Delta(\Gamma)$ minimally imbedded on a surface S_k; but S_k is always an $|\Gamma|$-fold (possibly branched) cover of some S_n (i.e. the imbedding is index one.) The Machlachlan and Burnside parameters also regard S_k as an $|\Gamma|$-fold (possibly branched) cover of some S_n, but represent the group Γ via its action on the Riemann surface S_k; for the Burnside parameter, it is always the case that $n = 0$. The four parameters are related as follows , where equality holds except for certain small Γ.

Thm. 7-35. For Γ a finite group, $\gamma(\Gamma) \leq \gamma_L(\Gamma) = \gamma_M(\Gamma) \leq \gamma_B(\Gamma)$; all are bounded above by the bound of Theorem 7-22.

Thus the most efficient genus, among all these, is that given by $\gamma(\Gamma)$. We remark that both the inequalities of Theorem 7-35 can be strict, as the groups $\Gamma = (\mathbb{Z}_{2n})^4$, for $n \geq 2$ indicate:

Thm. 7-36. Let $\Gamma = (\mathbb{Z}_{2n})^4 n \geq 2$; then

(i) $\gamma(\Gamma) = 1 + 8n^4$
(ii) $\gamma_L(\Gamma) = \gamma_M(\Gamma) = 1 + 16n^4$
(iii) $\gamma_B(\Gamma) = 1 + 4n^3(6n - 5)$

In 1991, Babai [B3] reinforced portions of Theorem 7-35;

Thm. 7-37. If a finite group Γ acts on surface S, then Γ has a Cayley map on S.

7-4. The Characteristic of a Group

If we allow nonorientable surfaces also, as our "drawing boards" for "picturing" groups, then we are led naturally to the parameter of this section. Recall from Section 5-3 that the characteristic of a surface S is $\chi(S) = 2 - 2k$, if $S = S_k$; $\chi(S) = 2 - k$, if $S = N_k$.

Def. 7-38. The *characteristic of a graph* G, denoted by $\chi(G)$, is the maximum surface characteristic $\chi(S)$ such that G imbeds in S.

For example, a graph has characteristic two if and only if it is planar; K_5 and K_6 have characteristic one; K_7 has characteristic zero; and so forth.

Def. 7-39. The *characteristic of a group* Γ is given by:

$$\chi(\Gamma) = \max\{\chi(G_\Delta(\Gamma))\},$$

where the maximum is taken over all generating sets Δ for Γ.

Thus a finite group has characteristic two if and only if it appears in Maschke's list (Theorem 7-3.)

Here are some less obvious sample results.

Thm. 7-40. (White [W10])

Let Γ be finite and abelian; then

$$\chi(\Gamma) = \begin{cases} 2, & \text{if } \Gamma = \mathbb{Z}_n, \mathbb{Z}_{2n} \times \mathbb{Z}_2, \text{ or } (\mathbb{Z}_2)^3 \\ 1, & \text{if } \Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3 \\ 0, & \text{if } \Gamma = \mathbb{Z}_{2n} \times \mathbb{Z}_2 \times \mathbb{Z}_2 (n > 1), (\mathbb{Z}_2)^4, \\ & \text{or } \mathbb{Z}_{mn} \times \mathbb{Z}_m (m > 2, mn \neq 3). \end{cases}$$

$\chi(\Gamma) < 0$, otherwise.

Thm. 7-41. (White [W10])

Let $(|\Gamma|, 6) = 1$, with Γ not cyclic. Then $\chi(\Gamma) = 0$ if and only if Γ has a presentation of the form

$$\Gamma = \left\langle a, b | a^{2k+1} = b^{2n+1} = w = \cdots = e \right\rangle,$$

where w is either $aba^{-1}b^{-1}$ or $abab^{-1}$.

Thm. 7-42. (Tucker [T9]) There is no group Γ having $\chi(\Gamma) = -1$.

Thm. 7-43. (Tucker [T9]) For $n \geq 168$, $\chi(A_n) = \frac{-n!}{168}$.

Thm. 7-44. (Tucker [T9]) For $n \geq 168$, $\chi(S_n) = \frac{-n!}{84}$.

We observe that, for $n \geq 168$, $\frac{\chi(S_n)}{\chi(A_n)} = [S_n : A_n]$.

7-5. Problems

7-1.) Show: that any presentation P for Q, the quaternions, has at least two generators; that if P has exactly two generators, neither can be of order 2; and that P can not have exactly three generators, each of order 2. (Hence $\delta(C_\Delta(Q)) \geq 4$).

7-2.) Find an example of a group Γ and a presentation P for Γ such that $\gamma(C_\Delta(\Gamma)) = \infty$.

7-3.) Find $\gamma(\mathbb{Z}_m \times \mathbb{Z}_n)$, for all m and n.

7-4.) Show that the only finite planar abelian groups are $\mathbb{Z}_n$, $\mathbb{Z}_2 \times \mathbb{Z}_{2n}$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where $n \geq 1$.

7-5.) *Use the imbedding of Problem 6-7 to find $\gamma(Q \times Q)$.

7-6.) Find a non-normal subgroup in $Q \times Q$. (Thus the product of hamiltonian groups need not be hamiltonian.)

7-7.) *Show that the *dicyclic group*

$$G_n = \left\langle x, y x^{2n} = x^n y^{-2} = y^{-1} x y x = e \right\rangle$$

has genus 1, for all $n > 1$. ($G_2 = Q; G_3$ is the "least familiar group of order 12".)

7-8.) Find an infinite group with a presentation of the form given in Theorem 7-22.

7-9.) Let $\Delta = \{(1,0,0),(0,1,0),(0,0,1)\}$ for $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Show that $\gamma(G_\Delta(\Gamma)) = \infty$.

7-10.) Show that $\gamma(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}) = \infty$. Now let $\Gamma_1 = \mathbb{Z}$, and $\Gamma_n = \mathbb{Z} \times \Gamma_{n-1}, n \geq 2$; find $\gamma(\Gamma_n)$, for all natural numbers n.

7-11.) **The smallest order groups whose genera are unknown are non-abelian of order 32. There are 44 such; D_{16} and $\mathbb{Z}_2 \times D_8$ are both planar. Pick any one of the other 42, and find its genus.

7-12.) **Does there exist a value of g (g even, $g \not\equiv 10 \pmod{18}$) such that $f(g) = 0$? Is f onto?

7-13.) *Verify Theorem 7-36.

CHAPTER 8

MAP-COLORING PROBLEMS

In this chapter we will see that the famous *four-color theorem* can be formulated – and studied – in graph-theoretical terms. Graph theory will be used to establish the *five-color theorem.* The *Heawood Map-coloring theorem* will be introduced; this powerful theorem, whose proof was completed in 1968, answers the coloring question – which, at that time, was still unanswered for the sphere – for *every other* closed 2-manifold. The easy half of the proof – found by Heawood in 1890 – is presented in this section. The difficult half of the proof – developed primarily by Ringel and Youngs – will be discussed in Chapter 9.

Consider any map of the world. Suppose we desire to color the countries of the world (or the states of a particular country, or the counties of a particular state, etc.) so that the distinct countries are distinguishable. This means that if two countries share a border at other than isolated points, then they must be colored differently. We make only one assumption as to the countries themselves: each country must be connected (this rules out Pakistan of some decades ago, and the United States, for example.) Note that a country need not be 2-cell; that is, it may entirely surround some collection of other countries (such is the case for a certain region in France; see Fréchet and Fan [FF1], p. 3).

We mention in passing that several generalizations of this map-coloring problem are possible. One of the most appealing is the following: allow disconnected countries, with each country having at most k components. (It is not hard to see that, without this restriction involving k, arbitrarily many colors may be needed.) Then it can be shown (see Problem 8-6 for the case $k = 2$) that $6k$ colors will always suffice. Ringel ([R10]; p. 26 (see also Heawood [H4])) displays a map, for the case $k = 2$, requiring 12 colors, so that this case is completely solved.

Returning now to the case of classical interest ($k = 1$), we pose the question thusly: what is the smallest number of colors needed to color any map on the sphere (or, equivalently, on the plane)? That four colors may be needed is indicated by the map induced by the tetrahedron. That five colors suffice for the sphere will be demonstrated shortly. Whether or not five colors are ever *necessary* has probably stimulated as much work in mathematics as any other single mathematical question; and the answer is finally known. The *four-color theorem* says that five colors are never necessary: *four colors will suffice to color any map on the sphere.* Many "proofs" of the four color theorem have been presented to the mathematical community, but most have not yet survived close scrutiny. The interested reader might wish to read through one of the false "proofs", given by Kempe in 1879 (see [BCL1], for example), and try to spot the error in the "proof."

Graph theory enters the picture in the following way. Form the dual of the map in question. This produces a pseudograph. Attempt to color the vertices of the pseudograph so that no two adjacent vertices have the same color. The pseudograph has no loops, as no country ever shares a border with itself. In fact, we may as well drop any multiple edges, since they (the " extra" edges) have no bearing on the coloring question. Then the coloring numbers, or chromatic numbers, of the resulting graph and the map will be identical. This leads to the following definitions.

8-1. Definitions and the Six-Color Theorem

Def. 8-1. The *chromatic number*, $\chi(G)$, *of a graph* G is the smallest number of colors for $V(G)$ so that adjacent vertices are colored differently.

Def. 8-2. The *chromatic number*, $\chi(S_k)$, *of a surface* S_k is the largest $\chi(G)$ such that G can be imbedded in S_k.

We prove that six colors will suffice for every planar graph. Of course, both this result and the five-color theorem of the next section are subsumed by the Four-Color Theorem. But we want to include the proofs, as the techniques they illustrate will have value later.

Thm. 8-3. Six colors suffice to color any map on the sphere; that is, $\chi(S_0) \leq 6$.

PROOF. We use induction on p, the order of the planar graph G, to show that $\chi(G) \leq 6$. The anchor for $p = 1$ is clear. So, assume that all planar graphs with $p - 1$ vertices ($p > 1$) are 6-colorable. Let G be planar, of order p. By Lemma 5-19, G has a vertex v of degree 5 or less. By the induction hypothesis, $\chi(G - v) \leq 6$. Since the neighbors of v use at most 5 colors, there is a sixth color available for v. □

8-2. The Five-Color Theorem

The proof below is found in [BCL1].

Thm. 8-4. Five colors will suffice to color any map on the sphere; i.e. $\chi(S_0) \leq 5$.

PROOF. We use the induction on p, the order of the graph G, to show that if $\gamma(G) = 0$, then $\chi(G) \leq 5$. The anchor at $p = 1$ is obvious. Now assume that all planar graphs with $p - 1$ vertices ($p > 1$) are 5-colorable. Let G be planar, with p vertices. By Lemma 5-19, G contains a vertex v of degree 5 or less. By the induction hypothesis, $\chi(G - v) \leq 5$; denote the colors in a 5-coloring of $G - v$ by 1,2,3,4,5. If not all five colors are used for the vertices adjacent to v in G, we can color v with one of the colors not so used, to give $\chi(G) \leq 5$. Otherwise, $d(v) = 5$, and all five colors are used for vertices adjacent to v. We can assume that the situation around v is as in Figure 8-1, and that v_i is colored with color i. Consider now any two colors assigned to non-consecutive vertices v_i, say 1 and 3, and let H be the subgraph of $G - v$ induced by all those vertices colored 1 or 3. If v_1 and v_3 belong to different components of H, then by interchanging the colors in the component of H containing v_1, say, a 5-coloring of $G - v$ is produced in which no vertex adjacent with v is assigned the color 1, and we can use 1 for v. If, on the other hand, v_1 and v_3 are joined by a path in H, the above argument guarantees that we can recolor v_2 with 4, and use 2 for v. This completes the proof. □

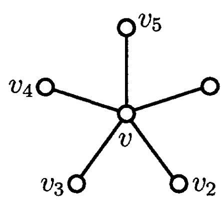


Figure 8-1.

8-3. The Four-Color Theorem

In the notation of Section 8-1, the Four-color Theorem becomes:

Thm. 8-5. $\chi(S_0) = 4$.

The first proof is due to Appel and Haken [AH1] in 1976. See Woodall and Wilson [WW1], for one discussion of this proof; for a second proof, see[RSST1]. See also [SK1], for example.

The proofs so far obtained for Theorem 8-5 are enormously more complicated than those of either Theorem 8-3 or 8-4, in part due to the positive difference upon subtracting the number of colors, four, from the maximum minimum degree of a planar graph, five. Thus no simple induction argument seems to be available.

In fact, the sphere is the only closed orientable 2-manifold for which the maximum minimum degree is not obtained by a complete graph. (See Theorem 8-15.) The sphere is also the only closed orientable 2-manifold of positive characteristic. This is why Theorem 8-13 does not contain a (simple) proof of the Four-Color Theorem.

There are many equivalent formulations of the Four-color Theorem. (See, for example, Ore's book: *The Four Color Problem* [O1], or [BCL1].) Of course, these are all now corollaries of the theorem itself. Following a definition, we present one of these.

Def. 8-6. A graph G is said to be *n-edge colorable* if n colors can be assigned to $E(G)$ so that adjacent edges are colored differently.

Thm. 8-7. Every cubic plane block is 3-edge colorable.

PROOF. Let G be a cubic plane block. By Theorem 8-5, G is 4-region colorable; let the colors be taken from the group $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$. Since G is a block, each edge x of G appears in the boundary of two distinct (but adjacent) regions, R_x^1 and R_x^2. Define the color of x by $c(x) = c(R_x^1) + c(R_x^2)$, addition taking place in Γ. Since $c(R_x^1) \neq c(R_x^2)$, $c(x) \neq e$, the identity of Γ (every element is its own inverse, in Γ.) Let x, y, and z be adjacent edges in G; see Figure 8-2. We claim that x, y, and z are colored distinctly. Suppose to the contrary that, say, $c(x) = c(y)$; that is $c(R_x^1) + c(R_x^2) = c(R_y^1) + c(R_y^2) = c(R_y^1) + c(R_x^1)$. But then $c(R_y^1) = c(R_x^2)$, a contradiction. Thus G is 3-edge colorable (the colors being taken from $\Gamma - \{e\}$). $\square$

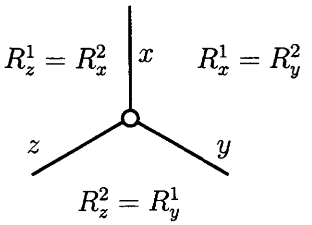


Figure 8-2.

8-4. Other Map-Coloring Problems: The Heawood Map-Coloring Theorem

Now let us consider other subspaces of $\mathbb{R}^3$ in which to pose map-coloring questions such as that above, for the sphere (and plane).

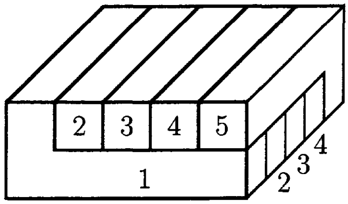


Figure 8-3.

Strangely enough, if we allow 3-dimensional countries, arbitrarily many colors may be needed to color the map. This is indicated by Figure 8-3, in which the countries are numbered; it is seen that each country meets each of the other countries. (In general, n modified rectangular parallelepipeds are laid across n other such solids.)

Perhaps it seems natural, since the coloring problem is apparently extraordinarily difficult for the sphere, and admits no finite answer in $\mathbb{R}^3$, to consider next the surfaces S_k as candidates for maps and the corresponding map-coloring questions.

The Heawood Map Coloring Theorem (formerly the Heawood Map-Coloring Conjecture) has a particularly colorful background, as outlined in Chapter 1; also see J.W.T. Youngs [Y2]. We state the theorem first for the orientable case:

Thm. 8-8. $\chi(S_k) = \left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor$, for $k > 0$.

Note what happens if we replace k with 0 in this formula. This led many mathematicians to feel that the four color conjecture was probably true, and they were vindicated! Thus we can now take $k \geq 0$ in Theorem 8-8.

The corresponding map-coloring question can also be asked for the closed non-orientable surfaces N_k (spheres with k cross-caps). In 1959 Ringel [R10] showed the following (the case $k = 2$ was solved by Franklin [F4]):

Thm. 8-9. $\chi(N_k) = \left\lfloor \frac{7+\sqrt{1+24k}}{2} \right\rfloor$, for $k = 1$ and $k \geq 3$; $\chi(N_2) = 6$.

For example, the formula gives $\chi(N_1) = 6$ (for the projective plane). Figure 8-4 shows K_6 imbedded in N_1, indicating that $\chi(N_1) \geq \chi(K_6) = 6$.

Recalling that the euler characteristics for S_k and N_k are given by $n = 2 - 2k$ and $n = 2 - k$ respectively, we can combine Theorems 8-5, 8-8, and 8-9 as follows:

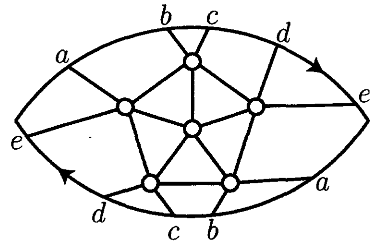


Figure 8-4.

Thm. 8-10. Let M_n be a closed 2-manifold, other than the klein bottle, of characteristic n; then

$$\chi(M_n) = \left\lfloor \frac{7+\sqrt{49-24n}}{2} \right\rfloor .$$

For ease of notation, we let $f(n) = \frac{7+\sqrt{49-24n}}{2}$. We will now establish what Heawood knew in 1890:

$$\chi(M_n) \leq \lfloor f(n) \rfloor .$$

We proceed by a series of steps, following the translator's notes in Fréchet and Fan [FF1].

Lemma 8-11. Let a graph G, with $p \geq 3$, be 2-cell imbedded in M_n, with a denoting the average degree of the vertices of G. Then $a \leq 6\left(1 - \frac{n}{p}\right)$.

PROOF. Note that $a = \frac{2q}{p}$. Now $3r \leq 2q = ap$. Also, $p - q + r = n$. Hence

$$q \leq 3(q - r) = 3(p - n),$$

so that

$$a = \frac{2q}{p} \leq 6\left(1 - \frac{n}{p}\right).$$

□

Thm. 8-12. $\chi(N_1) \leq 6$.

PROOF. We use induction on p, the order of a graph imbedded in N_1. The result is clearly true for $p \leq 6$. Assume $\chi(G) \leq 6$ for all graphs in N_1 with $p-1$ vertices, $p \geq 7$; let G be a graph imbedded in N_1, with p vertices. If the imbedding is not 2-cell, then (see Youngs [Y1]) $\gamma(G) = 0$, and $\chi(G) \leq 5$. Otherwise, by Lemma 8-11, $a < 6$, so that G has a vertex v such that $d(v) \leq 5$. Then $G - v$ is imbedded in N_1, and $\chi(G - v) \leq 6$, by the induction hypothesis. Since there are vertices of at most five colors adjacent to v, the sixth color can be used for v, and $\chi(G) \leq 6$. □

Note that Theorem 8-12, together with Figure 8-4, show that $\chi(N_1) = 6$.

The reader should locate the spot where the following argument fails for the sphere (and the projective plane as well; this is why these two surfaces are treated separately).

Thm. 8-13. $\chi(M_n) \leq \lfloor f(n) \rfloor$, for $n \neq 2$.

PROOF. Thanks to Theorem 8-12, we may assume that $n \leq 0$. We use induction on p, to show that $\chi(G) \leq \lfloor f(n) \rfloor$, if G is imbedded in M_n. (We may assume G to be connected, as the chromatic number of a graph is the largest chromatic number for its components.) It is clear that $\chi(G) \leq \lfloor f(n) \rfloor$ if $p \leq \lfloor f(n) \rfloor$. Now assume that $\chi(G) \leq \lfloor f(n) \rfloor$ for all graphs with fewer than p vertices and imbeddable in M_n. Now, from the definition of $f(n)$, we see that $f^2(n) - 7f(n) + 6n = 0$; i.e. $6\left(1 - \frac{n}{f(n)}\right) = f(n) - 1$. If the imbedding of G (of order $p > \lfloor f(n) \rfloor$) in M_n is 2-cell, then Lemma 8-11 applies, and

$$\begin{aligned} a &\leq 6\left(1 - \frac{n}{p}\right) \\ &\leq 6\left(1 - \frac{n}{f(n)}\right) \\ &= f(n) - 1. \end{aligned}$$

If the imbedding is not 2-cell, then it is not minimal (see again Youngs [Y1]), and we can find a 2-cell imbedding in M_m, where $m > n$. We then apply Lemma 8-11 as above, to get $a \leq f(m) - 1 \leq f(n) - 1$. Thus in either case $a \leq f(n) - 1$, and we can find a vertex v of G having $d(v) \leq \lfloor f(n) \rfloor - 1$, so that (using $\chi(G - v) \leq \lfloor f(n) \rfloor$), $\chi(G) \leq \lfloor f(n) \rfloor$. This completes the proof. □

The task remains to show that $\chi(M_n) \geq \lfloor f(n) \rfloor$, for $M_n \neq N_2$, the klein bottle. This is done by finding a graph G imbeddable in M_n and having $\chi(G) = \lfloor f(n) \rfloor$. For $M_2 = S_0$, K_4 is such a graph; for $M_1 = N_1$, take $G = K_6$ (as in Figure 8-4); for $M_0 = S_1$, pick $G = K_7$ (see Figure 8-5 for the dual of K_7 in S_1); for $M_{-2} = S_2$, let $G = K_8$. In fact, for $M_n \neq N_2$, $\lfloor f(n) \rfloor$ is attained by the largest complete graph imbeddable in M_n. We now confine our attention to the orientable case and explore this claim in some detail.

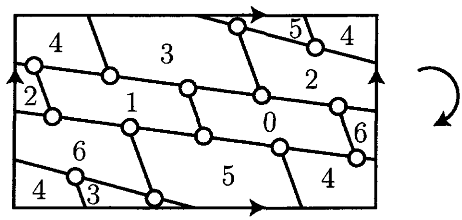


Figure 8-5.

Let us assume the truth of the Complete Graph Theorem (which will be discussed in more detail in Chapter 9):

$$\gamma(K_m) = \left\lceil \frac{(m-3)(m-4)}{12} \right\rceil, \text{ for } m \geq 3.$$

From this it will follow that $\chi(M_n) \geq \lfloor f(n) \rfloor$ (in the orientable case; the non-orientable case is handled similarly).

Thm. 8-14. $\chi(S_k) \geq \lfloor f(2-2k) \rfloor = \left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor$.

PROOF. Consider S_k. Define $m = \lfloor f(2-2k) \rfloor$, and now consider also $S_{\gamma(K_m)}$. Note that $\gamma(K_m) \leq k$, so that $\chi(S_{\gamma(K_m)}) \leq \chi(S_k)$. Now K_m imbeds in $S_{\gamma(K_m)}$. Clearly $\chi(S_{\gamma(K_m)}) \geq m = \lfloor f(2-2k) \rfloor$, so that $\chi(S_k) \geq \lfloor f(2-2k) \rfloor$. □

Theorems 8-13 and 8-14 combine to prove Theorem 8-8, with the understanding that it remains to establish the formula for the genus of K_m. Before indicating how this is done (in the next chapter), we pause for some related results.

8-5. A Related Problem

We have seen that, for every closed 2-manifold , the maximum chromatic number of an imbedded graph is taken on by a complete graph. We now show that the complete graphs play the same role with respect to maximizing the minimum degree of an imbedded graph, with the sphere and the klein bottle as the sole exceptions.

Thm. 8-15. Let M_n be a closed 2-manifold of characteristic n, and G a graph. If G has an imbedding in M_n, then $\delta(G) \leq g(n)$, where

$$g(n) = \begin{cases} \left\lfloor \frac{5+\sqrt{49-24n}}{2} \right\rfloor, & \text{if } n < 2 \\ 5, & \text{if } n = 2. \end{cases}$$

Furthermore, there exists a graph G, imbeddable in M_n, such that $\delta(G) = g(n)$.

PROOF. The theorem is known to be true for $n = 2$, as Lemma 5-19 and the icosahedral graph show. Suppose now that $n < 2$ and that G is a graph having p vertices and q edges, with $\delta(G) > g(n)$, and a 2-cell imbedding in $M_n (\neq S_0)$. By standard arguments, $2q \geq 3r$, and also $2q \geq p(g(n)+1)$. We may assume that G is connected, since if the theorem is true for every component of G, it is also true for G. The euler formula applies, so that

$$\begin{aligned} n &= p - q + r \\ &\leq (\frac{2}{g(n)+1} - \frac{1}{3})q \\ &= (\frac{5 - g(n)}{3(g(n)+1})q. \end{aligned}$$

We may assume that $n \leq 0$, as the above inequality is clearly impossible for $n = 1$. But for $n \leq 0$, $g(n) \geq 6$, so that

$$q \leq \frac{-3n(g(n)+1)}{g(n)-5}.$$

But since $\delta(G) > g(n), p \geq g(n) + 2$, and

$$2q \geq (g(n)+2)(g(n)+1).$$

We note that

$$\begin{aligned} (g(n)+2)(g(n)-5) &= \left\lfloor \frac{9+\sqrt{49-24n}}{2} \right\rfloor \left\lfloor \frac{-5+\sqrt{49-24n}}{2} \right\rfloor \\ &> (\frac{7+\sqrt{49-24n}}{2})(\frac{-7+\sqrt{24-24n}}{2}) \\ &= -6n. \end{aligned}$$

It now follows that

$$\begin{aligned} q &\geq \frac{(g(n)+2)(g(n)+1)}{2} \\ &> \frac{-3n(g(n)+1)}{g(n)-5} \\ &\geq q, \end{aligned}$$

a contradiction. Hence $\delta(G) \leq g(n)$.

Now suppose that G has a non 2-cell imbedding in M_n. By a result of Youngs [Y1], G has a 2-cell imbedding in some $M_{n'}$, where $n < n'$. From what we have shown above, $\delta(G) \leq g(n') \leq g(n)$.

Ringel and Youngs have shown [RY1] (also, see Chapter 9) that the complete graph $K_{[g(2-2k)]+1}$ is imbeddable in S_k, for $k \leq 1$. Ringel [R10] has shown that the complete graph $K_{[g(2-k)]+1}$ is imbeddable in N_k, for all positive k except $k = 2$. It remains to find a graph G

imbeddable in N_2 and having $\delta(G) = 6$. We begin by considering two projective planes, P_1 and P_2, each with a complete graph K_5 imbedded as indicated in Figure 8-6. Cut open disks D_1 and D_2 from the interiors of the five-sided regions of P_1 and P_2, respectively. Let T be a cylinder disjoint from P_1 and P_2, with simple closed boundary curves C_1 and C_2. Identify C_1 with the boundary of D_1 and C_2 with the boundary of D_2. The result, $(P_1 - D_1) \cup T \cup (P_2 - D_2)$, is a klein bottle (see Problem 8-7). The graph G is then constructed by adding the edges $(i, i')(i, (i+1)')$, $i = 1, 2, 3, 4, 5$, (where the vertex $6'$ is the same as the vertex $1'$). This completes the proof. □

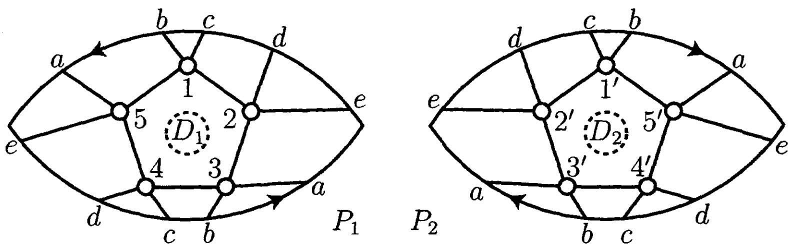


Figure 8-6.

We thus make the following observation. The sphere is the only closed orientable 2-manifold for which the maximum minimum degree is not attained by a complete graph. In contrast, we have seen that for every closed 2-manifold (whether orientable or non-orientable), including the sphere, the maximum chromatic number is attained by a complete graph.

8-6. A Four-Color Theorem for the Torus

Thus far in this chapter we have been discussing, for a given closed 2-manifold M, the chromatic number of arbitrary graphs that can be imbedded in M. In this section we impose a restriction on the girth of the graphs we are considering.

Def. 8-16. The *girth* $g(G)$ of a graph G is the length of a shortest cycle (if any) in G.

Thus a graph G with cycles but no triangles has $g(G) \geq 4$; if G is a forest, we write $g(G) = \infty$. The following theorem was shown by Grötzsch [G10]:

Thm. 8-17. If $\gamma(G) = 0$ and $g(G) \geq 4$, then $\chi(G) \leq 3$.

The graph $G = C_5$ shows that equality can hold in Theorem 8-17. In this section we [KW2] find an upper bound for the chromatic number of toroidal graphs having no triangles, and show that this bound is best possible. We also consider toroidal graphs of arbitrary girth.

Def. 8-18. A connected graph G is said to be *n-edge-critical* $(n \geq 2)$ if $\chi(G) = n$ but, for any edge x of G, $\chi(G - x) = n - 1$.

The next theorem is due to Dirac [D4].

Thm. 8-19. If G is n-edge-critical, $n \geq 4$, and if $G \neq K_n$, then $2q \geq (n-1)p + n - 3$.

We are now able to find the analogue of Grötzsch's Theorem, for the torus.

Thm. 8-20. If $\gamma(G) \leq 1$ and $g(G) \geq 4$, then $\chi(G) \leq 4$.

PROOF. Let $\chi(G) = n \geq 5$. We first assume that G is n-edge-critical, and hence connected. Since $g(G) \geq 4$, $G \neq K_n$. By Theorem 8-19,

$$2q \geq (n-1)p + n - 3.$$

Now if $\gamma(G) = 1$, then by Corollary 6-15,

$$4p \geq 2q \geq (n-1)p + n - 3;$$

thus $n \leq 4$. If $\gamma(G) = 0$, then $n \leq 3$, by Theorem 8-17. In either case we have a contradiction, so that $n \leq 4$.

Now suppose that G is not n-edge-critical. Then G contains an n-edge-critical subgraph H, and the argument above shows that $\chi(G) = \chi(H) = n \leq 4$. □

The graph of Figure 8-7, constructed by Mycielsky [M8] as an example of a graph having no triangles and chromatic number four, also has genus one, so that the bound of Theorem 8-20 cannot be improved.

The situation for the torus is almost completely analyzed in the next theorem.

Thm. 8-21. If $\gamma(G) \leq 1$ and $g(G) = m$, then

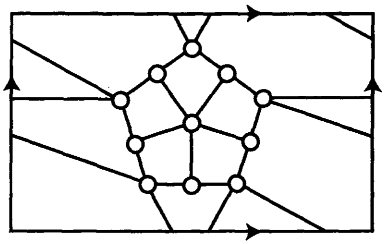


Figure 8-7.

$$\chi(G) \le \begin{cases} 7, & \text{if } m = 3 \\ 4, & \text{if } m = 4 \text{ or } 5 \\ 3, & \text{if } m \ge 6. \end{cases}$$

Moreover, all the bounds are sharp, except possibly for $m = 5$.

PROOF. If $m \ge 6$, then each region in an imbedding for G has at least six edges in its boundary, so that $2q \ge 6r$. As in the proof of Theorem 8-20, we may assume that $\gamma(G) = 1$ and that G is n-edge-critical, where $n = \chi(G)$. If $n \le 4$, then $2q \ge 3p + 1$, by Theorem 8-19. Then, by Corollary 5-14,

$$\begin{aligned} 0 &= p - q + r \\ &\le \frac{2q-1}{3} - q + \frac{q}{3} \\ &= -\frac{1}{3}, \end{aligned}$$

an obvious contradiction. Hence for $\gamma(G) \le 1$ and $g(G) \ge 6$, we must have $\chi(G) \le 3$. This bound is best possible, as an appropriate subdivision G of the Petersen graph (shown imbedded in S_1 in Figure 8-8) can always be found, having $g(G) = m (m \ge 5)$, $\gamma(G) = 1$, and $\chi(G) = 3$.

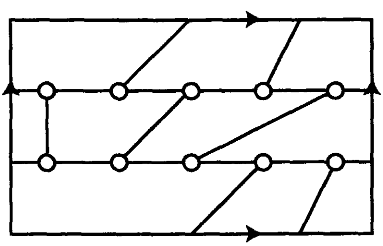


Figure 8-8.

For $m = 4$ or 5, it follows from Theorem 8-20 that $\chi(G) \le 4$. (Now, see Problem 8-9.) Figure 8-7 shows that equality can hold for $m = 4$. For $m = 3$, we refer to the Heawood Map-Coloring Theorem. □

Gimbel and Thomassen [GT1] showed that $\chi(G) \leq 3$, if $\gamma(G) \leq 2$ and $g(G) \geq 6$.

8-7. A Nine-Color Theorem for the Torus and Klein Bottle

The material in this section is due to Ringel [R19].

Def. 8-22. A graph G is said to be 1-*imbeddable* in a surface S if G can be represented on S so that each edge is crossed over by at most one other edge.

Def. 8-23. The 1-*chromatic number* $\chi_1(S)$ is the largest $\chi(G)$ such that G is 1-imbeddable in S.

Thm. 8-24. The 1-chromatic numbers $\chi_1(S_k)$ and $\chi_1(N_h)$ are bounded above as shown:

(i) $\chi_1(S_k) \leq \left\lfloor \frac{9+\sqrt{64k+17}}{2} \right\rfloor$, for $k \geq 1$

(ii) $\chi_1(N_h) \leq \left\lfloor \frac{9+\sqrt{32h+17}}{2} \right\rfloor$, for $h \geq 1$.

Thm. 8-25. The following hold:

(i) $\chi_1(S_1) = 9$,

(ii) $\chi_1(N_2) = 9$,

(iii) $\chi_1(S_{83}) = 41$.

The result (iii) above was obtained using a modified current graph (see Chapter 9, for a discussion of the theory of current graphs.)

8-8. k-degenerate Graphs

Before getting to one focal point of this text, in the next chapter, we digress briefly. The generalization below of Theorem 8-10 might be of interest.

A coloring number for graphs closely related to the chromatic number is the vertex-arboricity (see [CKW1].)

Def. 8-26. The *vertex arboricity*, $a(G)$, *of a graph* G is the minimum number of subsets that $V(G)$ can be partitioned into so that each subset induces an acyclic graph.

Def. 8-27. The *vertex arboricity of a surface* S_k is the maximum vertex-arboricity among all graphs which can be imbedded in S_k.

In 1969, Kronk [K3] showed that the vertex arboricity of S_k, $k > 0$, is $\left\lfloor \frac{9+\sqrt{1+48k}}{4} \right\rfloor$. Chartrand and Kronk [CK2], also in 1969, proved that the vertex-arboricity of the sphere is three. The similarity of Kronk's result to those of Ringel and of Ringel and Youngs for the chromatic number suggested the generalization discussed below.

Def. 8-28. A graph G is said to be *k-degenerate* if every induced subgraph H of G satisfies the inequality $\delta(H) \leq k$.

Def. 8-29. The *vertex partition number*, $\rho_k(G)$, *of a graph* G is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a k-degenerate subgraph of G.

The parameters $\rho_0(G)$ and $\rho_1(G)$ are the chromatic number and vertex arboricity of G, respectively (see Problem 8-4). A general study of k-degenerate graphs has been begun in [LW2], where many of the well-known results for the chromatic number and the vertex-arboricity of a graph have been extended to the parameters $\rho_k(G)$, for all non-negative integers k.

Def. 8-30. The *vertex partition number of the closed* 2*-manifold* M_n, denoted by $\rho_k(M_n)$, is the maximum vertex partition number $\rho_k(G)$ among all graphs G which can be imbedded in M_n.

The following theorem (for a complete proof, see [LW3]) almost completely generalizes the results of Kronk, Ringel, Ringel and Youngs and Haken mentioned above.

Thm. 8-31. The vertex partition numbers for a closed 2-manifold M_n are given by the formula:

$$\rho_k(M_n) = \left\lfloor \frac{(2k+7)+\sqrt{49-24n}}{2k+2} \right\rfloor,$$

where $k = 0, 1, 2, 3, \cdots$; and $n = 2, 1, 0, -1, -2, \cdots$, except for the following cases:

(i) in the orientable case, $\rho_1(S_0) = 3$, $\rho_3(S_0) = \rho_4(S_0) = 2$; and
(ii) in the non-orientable case, $\rho_0(N_2) = 6$, $\rho_1(N_2) = 3$, $\rho_2(N_2) = 2$.

We make the following comments about the proof of Theorem 8-31. Set $f(k,n) = \left\lfloor \frac{(2k+7)+\sqrt{49-24n}}{2k+2} \right\rfloor$. The proof is divided into three parts.

(i) $\rho_k(M_n) \leq f(k,n)$, for $M_n \neq S_0$ (the proof breaks down for the sphere, reminding us how obstinate the four color problem was.)
(ii) $\rho_k(M_n) \geq f(k,n)$, for $M_n \neq N_2$ (the proof fails for the klein bottle).
(iii) the exceptional cases are treated separately:
   (a) for $M_n = S_0$, $\rho_1(S_0) = 3$ appears in [CK2]; for $\rho_k(S_0) = 2$, $k = 2, 3, 4$, see Problem 8-5; finally, $\rho_k(S_0) = 1$ for $k \geq 5$, since any planar graph is 5-degenerate, by Lemma 5-19.
   (b) For $M_n = N_2$, additional *ad hoc* arguments are devised. For example, the graph constructed in Figure 8-6 shows that $\rho_5(N_2) \geq 2$; from (i) we see that $\rho_5(N_2) \leq 2$; thus $\rho_5(N_2) = 2$. The values of $\rho_1(N_2)$ and $\rho_2(N_2)$ were settled by Borodin [B13] in 1976.

8-9. Coloring Graphs on Pseudosurfaces

The pseudosurfaces $S(k; n_1(m_1), \cdots, n_t(m_t))$ have been defined in Section 5-5 and re-encountered in Sections 6-7 and 6-9. Dewdney [D2] has studied a subclass of these pseudosurfaces, namely those of the form $S(0, n(2))$:

Def. 8-32. The *chromatic number*, $\chi(S(0; n(2)))$, *of the pseudosurface* $S(0; n(2))$ is the largest chromatic number $\chi(G)$ of any graph G that can be imbedded in $S(0; n(2))$.

Thm. 8-33. $\chi(S(0; n(2))) \leq n + 4$, for $n > 0$; equality holds for $n = 1, 2, 3, 4$.

For example, Figure 6-7 shows K_5 imbedded in $S(0; 1(2))$, showing that $\chi(S(0; 1(2)) \geq 5$. Similarly, K_6 imbeds in $S(0; 2(2))$, to give equality for the case $n = 2$. (See Problem 6-10.) Note that we state this coloring problem for graphs rather than for maps; the dual of G in $S(0; n(2))$ is *not* a 2-cell imbedding, so that there is not the natural correspondence we find for surfaces. (The cases $n = 3$ and 4 were established by Mark O'Bryan and James Williamson respectively.)

Then in 1974, Borodin and Melnikov [BM2] solved this particular problem completely, except for the case $n = 0$ now covered by the Four-color Theorem; we state the complete solution:

Thm. 8-34.

$$\chi(S(0;n(2))) = \begin{cases} n+4, & 0 \le n \le 4, \\ 8, & n = 5, \\ \left[\frac{7+\sqrt{1+24n}}{2}\right], & 6 \le n \le 12, \\ 12, & n \ge 12. \end{cases}$$

Thus we have the map-coloring numbers for the sphere, where n countries have two components, and that twelve is the largest of all these numbers (see Problem 8-6). Heawood [H4] generalized to ask for the map-coloring number $\chi(S,c)$ for a surface S (orientable or nonorientable) of characteristic n, where each country has at most c components, and showed for every case but the sphere for $c = 1$ that this number is bounded above by:

Thm. 8-35.

$$\chi(S,c) \le \left\lfloor \frac{6c+1+\sqrt{(6c+1)^2 - 24n}}{2} \right\rfloor .$$

Note that the case $c = 1$ is the one of primary interest (the Heawood Map Coloring Theorems), and that the bound *does* hold for $c = 1$ and $n = 2$ as well (the Four-color Theorem.) Moreover, we have seen that $\chi(S_0, 2) = 12$.

Recently it has been shown that equality also holds in Theorem 8-35, for certain other cases:

Thm. 8-36. (Jackson and Ringel [JR2]

$$\chi(S_0, c) = 6c, \text{ for } c \ge 2.$$

Thm. 8-37. (Taylor [T1])

$$\chi(S_1, c) = Gc = 1, \text{ for } c \ge 1.$$

Thm. 8-38. (Jackson and Ringel [JR1])

$$\chi(N_1, c) = 6c, \text{ for } c \ge 1.$$

Thm. 8-39. (Jackson and Ringel [JR3], Borodin [B14])

Let

$$g(c,n) = \left\lfloor \frac{6c+1+\sqrt{(6c+1)^2 - 24n}}{2} \right\rfloor ;$$

then $\chi(S, c) = g(c, n)$, if:

(i) $S = N_k$ and $g(c, n) \equiv 1, 4, 7 \pmod{12}$, unless $k = 2$ and $c = 1$.
(ii) $S = S_k$, c is even, and $g(c, n) \equiv 1 \pmod{12}$.
(iii) $S = S_k$, c is odd, and $g(c, n) \equiv 4, 7 \pmod{12}$.

It remains to construct the "verification figures" (i.e. the appropriate pseudosurface imbeddings) for the cases not covered above. (See Problem 8-17.)

8-10. The Cochromatic Number of Surfaces

The material in this section is taken from Straight ([S25] and [S26].)

Def. 8-40. The *cochromatic number*, $z(G)$, *of a graph* G is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces either an empty or a complete subgraph of G.

Def. 8-41. The *cochromatic number*, $z(S)$, *of a surface* S, is the maximum $z(G)$ such that G imbeds in S.

Thm. 8-42. $z(S_n) \leq \chi(S_n)$, with equality if and only if $n = 0$.

For example, $z(C_5 \cup K_4) = 4$, so that $z(S_0) = 4$.

Thm. 8-43. For $n \geq 4, z(N_n) < \chi(N_n)$.

Thm. 8-44.

(i) $z(S_0) = 4$
(ii) $z(N_1) = 5$
(iii) $z(N_2) = 6$
(iv) $z(N_3) = 6$
(v) $z(N_4) = 7$.

Straight conjectures that, in general, $z(S)$ is the maximum n such that $\cup_{i=1}^{n} K_i$ imbeds in S.

8-11. Problems

8-1.) Let $G \neq \bar{K}_n$; show that $\chi(G) = 2$ if and only if G is bipartite.
8-2.) Find $\chi(C_n)$, for all cycles C_n.

8-3.) Find an imbedding of K_7 on S_1. Form the dual of this imbedding, and explain why this shows that $\chi(S_1) \geq 7$.

8-4.) Show that $\rho_0(G) = \chi(G)$ and that $\rho_1(G) = a(G)$.

8-5.) Show that $\rho_k(S_0) = 2$, for $k = 2, 3, 4$. (Hint: for each k, use induction to show that $\rho_k(S_0) \leq 2$. Then consider graphs of certain regular polyhedra.)

8-6.) *Show (as Ringel and Heawood did) that any map on the surface of the sphere, in which each country has at most two components, can be colored with 12 colors. (Hint: it may be helpful to show that if a graph G is *n-critical*, then $\delta(G) \geq n-1$; i.e. if $\chi(G) = n$, but $\chi(G-v) = n-1$, for all vertices v in G. Then form two "dual" graphs for an arbitrary map, one where the vertices represent countries, the other with vertices representing regions of land. Use also the fact that $q \leq 3p - 6$, for planar connected graphs.) Interpret this result for pseudo-surfaces. Compare Problem 6-12.

8-7.) Show that the connected sum of two projective planes (as in the proof of Theorem 8-15) is a klein bottle. (Hint: find the characteristic of the resulting closed 2-manifold, using the graph G constructed in the same proof.)

8-8.) Show that $\chi(G) = 4$, for the graph G of Figure 8-7.

8-9.) **Does there exist a toroidal graph G having $g(G) = 5$ and $\chi(G) = 4$?

8-10.) *Prove or disprove: K_9 imbeds in $S(0; 5(2))$ (and hence $\chi(S(0; 5(2)) = 9$.) Is $\chi(S(0; n(2))) = n + 4$ for all n? Compare Problems 6-12 and 8-6. Does K_{10} imbed in $S(0; 7(2))$?

8-11.) Define the *chromatic number of a group* to be: $\chi(\Gamma) = \min_\Delta \chi(G_\Delta(\Gamma))$ (cf Babai [B1]). Find $\chi(\Gamma)$, for $\Gamma = \mathbb{Z}_n, \mathbb{Z}_2 \times \mathbb{Z}_n$, $(\mathbb{Z}_2)^n$, $D_n, \mathbb{Z}_2 \times D_n$, S_n. Show $\chi(\Gamma) \leq 3$, for Γ finite abelian or $\Gamma = A_n$.

8-12.) *If Γ has a normal subgroup Γ_1, show that $\chi(\Gamma) \leq \chi(\Gamma/\Gamma_1)$. Thus if Γ is solvable (note that this includes all odd order groups), then $\chi(\Gamma) \leq 3$. If Γ has a subgroup of index 2, then show that $\chi(\Gamma) \leq 2$.

8-13.) Show that $\chi(\Gamma) = 2$ if and only if Γ has a subgroup of index 2. Conclude that $\chi(A_n) = 3$, for $n \geq 3$.

8-14.) **Is $\chi(\Gamma) \leq 3$ for all groups Γ?

8-15.) Is $\chi(\Gamma_1) \leq \chi(\Gamma_2)$, if $\Gamma_1 \leq \Gamma_2$?

8-16.) Extend the definition of $\chi(\Gamma)$ to $\rho_k(\Gamma)$, for arbitrary vertex partition numbers. Study this family of parameters.

8-17.) **Find the imbeddings called for at the conclusion of Section 8-9.

8-18.) **Study the conjecture given at the conclusion of Section 8-10.

8-19.) Show that, for any integer pair (k, n), when $k \geq 0$ and $3 \leq n \leq \chi(S_k)$, these exists a triangulation of S_k by a graph G having $\chi(G) = n$. (Harary, Korzhik, Lawrencenko [HKL1]).

CHAPTER 9

QUOTIENT GRAPHS AND QUOTIENT MANIFOLDS: CURRENT GRAPHS AND THE COMPLETE GRAPH THEOREM

In this chapter we present the beautiful theory of quotient graphs and quotient manifolds, usually called, for short, the theory of current graphs. This theory was introduced by Gustin [G11], developed by Youngs (see, for example, [Y2], [Y3], and [Y6]), and used by Ringel and Youngs to find the genus of K_n, thereby proving the Complete Graph Theorem and, in turn, the Heawood Map-Coloring Theorem. The application of the theory to the graphs K_n falls into 12 cases, depending upon the residue modulo 12 of n. The theory applies directly, for $n \equiv 0, 3, 4, 7 \pmod{12}$, as will be seen shortly. For the remaining eight cases, the theory is augmented (by the *theory of vortices*) to complete the solution. We will treat the case $n \equiv 7 \pmod{12}$ completely, and discuss the case $n \equiv 10 \pmod{12}$; this will give an indication of the power and beauty of the theory. The remaining ten cases are treated similarly, although many complicating details must be handled properly. (Perhaps one should expect a complicated solution, to a complicated problem!)

We will then see how the theory (designed to produce triangular imbeddings for K_n) can be extended to handle first triangular imbeddings for Cayley graphs in general, and then to handle regular imbeddings ($r = r_n$, $n \geq 3$) in general, for Cayley graphs. This is the scope of the theory, as announced by Gustin. But Youngs' theory of vortices [Y3] hints at an even more general theory; we illustrate this general theory, as unified by Jacques [J3]. (For a more theoretical discussion, see [W15].) The even more intuitive dual form will be studied in detail, in Chapter 10.

9-1. The Genus of K_n

Let us now turn our attention to the complete graphs K_n. Recall that if we show that

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil, n \geq 3,$$

the Heawood map-coloring theorem will be established. We see the origin of the number on the right-hand side of the above equality in the following:

Thm. 9-1. Let K_n be minimally imbedded in a surface M. Then

$$\gamma(K_n) = \gamma(M) = \frac{(n-3)(n-4)}{12} + \frac{1}{6}\sum_{i\geq 4}(i-3)r_i.$$

PROOF. From Corollary 6-14, we know that

$$\gamma(K_n) \geq \frac{n(n-1)}{12} - \frac{n}{2} + 1 = \frac{(n-3)(n-4)}{12},$$

with equality if and only if K_n has a triangular imbedding. But we can be more specific than this; we can get some information about the non-triangular regions (if any). If K_n is minimally imbedded in M, then clearly $\gamma(K_n) = \gamma(M)$, and the imbedding is 2-cell, by Theorem 6-11. Thus the euler formula applies, and

$$\begin{aligned}\gamma(M) &= 1 - \frac{p}{2} + \frac{q}{2} - \frac{r}{2}\\ &= 1 - \frac{p}{2} + \frac{q}{6} + \frac{q}{3} - \frac{r}{2}\\ &= 1 - \frac{n}{2} + \frac{n(n-1)}{12} + \frac{1}{6}\sum_{i\geq 3} ir_i - \frac{1}{6}\sum_{i\geq 3} 3r_i\\ &= \frac{(n-3)(n-4)}{12} + \frac{1}{6}\sum_{i\geq 4}(i-3)r_i.\end{aligned}$$

□

We now see that if K_n has a triangular imbedding ($r_i = 0, i \geq 4$), then

$$\gamma(K_n) = \frac{(n-3)(n-4)}{12},$$

and $(n-3)(n-4) \equiv 0 \pmod{12}$; i.e. $n \equiv 0, 3, 4, 7 \pmod{12}$. Moreover, in general, $\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$, if we can show that $\sum_{i\leq 4}(i-3)r_i \leq 5$. It is now perhaps apparent why there are twelve cases for the determination of $\gamma(K_n)$, and why only four of them admit triangular imbeddings. Let us consider these four cases now.

What is needed is a method of constructing triangular imbeddings. The naive trial-and-error method easily handles $n = 3, 4$, and 7; it becomes a bit sticky at $n = 12$. We turn away from the drawing board and employ the algebraic description of 2-cell imbeddings given us by Edmonds' permutation technique. Now we seek a means of selecting judiciously the local vertex permutations, to form the rotation scheme $(p_1, p_2, \cdots, p_n)$; this is what the method of current graphs is all about!

9-2. The Theory of Current Graphs as Applied to K_n

We introduce this theory by means of the example K_7. Let K_7 be imbedded in S_1; by Theorem 9-1, this imbedding must be a triangulation. Select a group Γ for which K_7 is a Cayley color graph; in this case, we can only pick $\Gamma = \mathbb{Z}_7$, and take $1, 2, 3$ as generators for Γ. Label the vertices of K_7 with the elements of Γ (one should also think of the edges as being directed and colored appropriately.) Now take the dual of this imbedding; assume this is as pictured in Figure 8-5. Each region in the dual (formerly a vertex of K_7) is now labeled with a distinct group element: $0, 1, 2, 3, 4, 5$, or 6. We proceed to label the boundary edges of each region of the dual, as indicated in Figure 9-1. (Note that $(g^{-1}h)^{-1} = h^{-1}g$.) We observe that the seven regions of the dual have identical clockwise boundaries: $1, 3, 2, 6, 4, 5$.

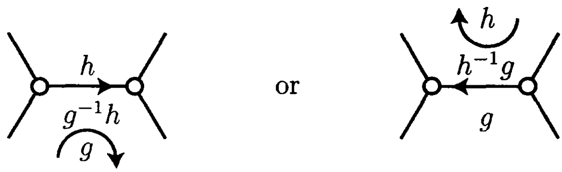


Figure 9-1.

We summarize this information in a map having one region, as shown in Figure 9-2. But $1^{-1} = 6$, $2^{-1} = 5$, and $3^{-1} = 4$; thus the six edges have a natural identification, in three pairs; we make this identification, to form a closed orientable 2-manifold, as in Figure 9-3. The result (in this case, S_1) is the *quotient manifold*; the corresponding graph (actually, in this case, it is a pseudograph) is the *quotient graph*. The subgroup of Γ consisting of all vertices of K_7 whose regions in the dual had the same ordering of directed edges in their boundaries as did $e = 0$ (in this case, $\mathbb{Z}_7$ itself) gives rise to the *quotient group* (in this case, the trivial group). The index of this subgroup in Γ (in this case, 1) is the *index of the imbedding*. The point is this: *all the information needed to describe a triangular imbedding of K_7 in S_1 is contained in the quotient graph, imbedded in its quotient manifold* (which, after all, was obtained by "modding out" the subgroup $\mathbb{Z}_7$).

To see this, let the permutation at vertex 0 be given by the boundary of the single region in the quotient manifold:

$$0 : 1, 3, 2, 6, 4, 5.$$

The remaining local vertex permutations may be obtained by successively adding 1 to every entry in this row (remember, we are in the

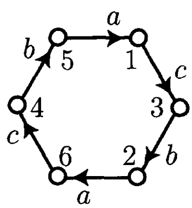


Figure 9-2.

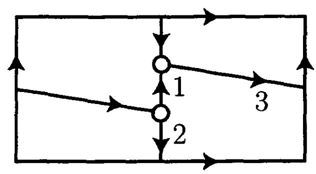


Figure 9-3.

group $\mathbb{Z}_7$!):

$$
\begin{aligned}
0 &: 1,3,2,6,4,5\\
1 &: 2,4,3,0,5,6\\
2 &: 3,5,4,1,6,0\\
3 &: 4,6,5,2,0,1\\
4 &: 5,0,6,3,1,2\\
5 &: 6,1,0,4,2,3\\
6 &: 0,2,1,5,3,4.
\end{aligned}
$$

Now, compute orbits (corresponding to regions in a 2-cell imbedding of K_7):

$$
\begin{array}{lll}
0-1-5 & 1-2-6 & 3-5-6\\
0-2-3 & 1-3-4 & 3-6-4\\
0-3-1 & 1-4-2 & \\
0-4-6 & 1-6-5 & \\
0-5-4 & 2-4-5 & \\
0-6-2 & 2-5-3. &
\end{array}
$$

We see that we have an imbedding of K_7 for which $r = r_3 = 14$; that is – a triangular imbedding. This is no accident as we shall soon see.

We have just encountered a connection between cubic vertices of a quotient graph and triangular regions for an imbedding of interest. As our present concern is triangular imbeddings of complete graphs, the relevant quotient graphs will all be cubic. As the quotient manifold will, in itself, not be crucial to the development, we follow Gustin [G11], Ringel, Youngs, and others (in the body of work proving the Complete

Graph Theory) in suppressing it in the planar diagrams employed. For example, Figure 9-3 is rendered as Figure 9-4. By convention, a solid vertex has its incident edges ordered clockwise; a hollow vertex, counterclockwise. This describes the local vertex permutations for the quotient graph and hence, implicitly, an orientable 2-cell imbedding for it. In the case of Figure 9-4, this is into the torus. We want this imbedding to have only one region, providing the rotation at vertex 0 (and, by translation, at all other vertices) for the complete graph we want to imbed. In Figure 9-4, we use our convention to trace out the region boundary: $1, 3, 2, 6, 4, 5$, as in Figure 9-2.

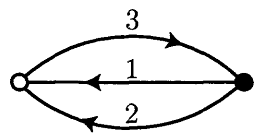


Figure 9-4.

Now let K be a pseudograph, with $K^* = \{(u,v) | \{u,v\} \in E(K)\}$.

Def. 9-2. A *current graph* is a triple (K, Γ, λ), where K is a pseudograph, Γ is a finite group with identity e, and $\lambda : K^* \to \Gamma - e$ is a map satisfying $(\lambda(a))^{-1} = \lambda(a^{-1})$, for all $a \in K^*$. Each value $\lambda(a)$ is called a *current.*

Thus Figure 9-4 is a current graph (with $\Gamma = \mathbb{Z}_7$ understood). Note also that each Cayley graph is a current graph. Moreover, a quotient graph is a current graph. (The former terminology emphasizes the imbedding aspect, whereas the latter stresses the edge labels.)

Def. 9-3. Let a pseudograph K be 2-cell imbedded in a closed orientable 2-manifold M, with $v \in V(K)$. We say that *Kirchoff's Current Law* (KCL) holds at v if the product of the currents directed away from v, taken in the order given by the rotation p_v, is the identity, e. We say the KCL *holds for the imbedding* of K if it holds at v, for each $v \in V(K)$.

The following theorem, and the application we make of it, indicate the power of a much more general theory, which we develop in Chapter 10 – in the dual context of voltage graphs.

Thm. 9-4. If $(K, \mathbb{Z}_n, \lambda)$ is a cubic current graph with:

(i) $\lambda : K^* \to \mathbb{Z}_n - \{0\}$ a bijection, and

(ii) a 2-cell imbedding into a closed orientable 2-manifold with just one region and satisfying the KVL,

then K_n has a triangular imbedding into a closed orientable 2-manifold.

PROOF. The unique region contains each arc $a \in K^*$ exactly once, and since λ is a bijection, the succession of currents on the boundary of the region can be taken as the rotation p_0 at vertex $0 \in V(K) = \{0, 1, 2, \cdots, n-1\}$; we write $p_0 : (a_1, a_2, \cdots, a_{n-1})$. Using $\Gamma = \mathbb{Z}_n$, we cyclically generate $p_i : (a_1+i, a_2+i, \cdots, a_{n-1}+i)$, for each $i \in \mathbb{Z}_n$. Then $(p_0, p_1, \cdots, p_{n-1})$ is a rotation scheme for K_n, and thus determines an orientable 2-cell imbedding for K_n. We need only show that $r = r_3$ for this imbedding.

So, suppose that $p_0(x) = y$, and let $a, b \in K^*$, with $\lambda(a) = x$ and $\lambda(b) = y$. Since K is cubic, in the imbedding for K we have the situation depicted in Figure 9-5. (The arrow gives the orientation at the vertices; by Edmonds' algorithm, the region boundary has the opposite orientation.) From the KCL, we deduce that $-x + y + z = 0$, so that $z = x-y$. Now we also have that $p_0(-y) = z$, so that $p_y(0) = z+y = x$. Now let (u, v) be any directed edge in K. To complete this proof, we will show that (u, v) is in the boundary of a triangular region.

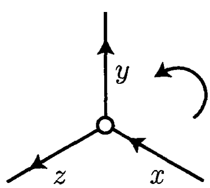


Figure 9-5.

Compute the orbit, beginning

$$u - v - .$$

Let $p_v(u) = w$, then we have

$$u - v - w - .$$

But since $p_v(u) = w$, $p_0(u-v) = w-v$. Letting $u-v = x$ and $w-v = y$ in the above discussion (where $p_0(x) = y$ implies $p_y(0) = x$), we see that $p_{w-v}(0) = u - v$; hence $p_w(v) = u$, and we have

$$u - v - w - u - .$$

Next, let $v - w = x$ and $u - w = y$; then (since $p_w(v) = u$, $p_0(v - w) = u - w$) $p_{u-w}(0) = v - w$, and $p_u(w) = v$. Thus

$$u - v - w - u - v$$

confirms an orbit under P^* (as in the proof of Theorem 6-50) for K_n, of length 3. This completes the proof. □

Next, we prove that $\gamma(K_n)$ is as predicted, for $n \equiv 7 \pmod{12}$. By Theorem 9-4, we need only find a suitable imbedding of a suitable current graph.

Thm. 9-5. K_{12s+7} has a triangular imbedding.

PROOF. Let $\Gamma = \mathbb{Z}_{12s+7}$. We have already treated the case $s = 0$. The cases $s = 1$ and $s = 2$ are shown in Figures 9-6 and 9-7 respectively. The generalization to all s is as in Figure 9-8, with the vertical edges directed alternately and carrying the currents $1, 2, \cdots, 2s$ consecutively. All other currents (not shown) are determined by the KCL. It is straightforward to check that K has $6s + 3$ edges, or $12s + 6$ directed edges, carrying all the currents from

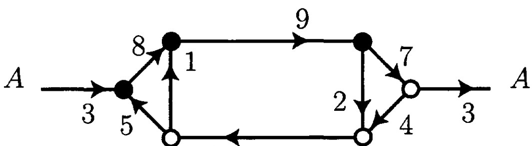


Figure 9-6.

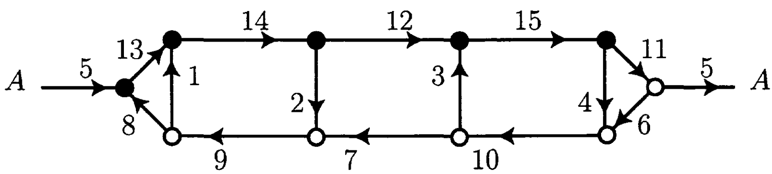


Figure 9-7.

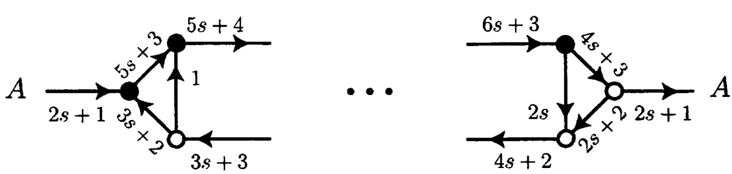


Figure 9-8.

$\mathbb{Z}_{12s+7} - 0$. The single region is:
$(2s+1) - (5s+3) - (5s+4) - \cdots - (6s+3) - (4s+3) - (2s+2) - (-2s) - (-6s-3) - (-2s+1) - (2s+3) - \cdots - (-1) - (3s+2) - (-2s-1) - (-4s-3) - (2s) - (4s+2) - (2s-1) - (-4s-4) - \cdots - (1) - (-5s-3) - (-3s-2) - (-3s-3) - \cdots - (-4s-2) - (-2s-2)$.
Thus K is a current graph (imbedded in S_{s+1}), satisfying the conditions

of Theorem 9-4, and K_{12s+7} has a triangular imbedding, for all non-negative integers s.

□

Cor. 9-6. $\gamma(K_{12s+7}) = (3s+1)(4s+1), s \geq 0$.

The theory of current graphs with *vortices* is employed to find $\gamma(K_n)$, for $n \equiv 10 \pmod{12}$. The group $\mathbb{Z}_{12s+7}$ is used to find a triangular imbedding for $K_{12s+10} - K_3$; the current graph in this case has three vertices of degree one; otherwise it is essentially the K of Figure 9-8. (See [Y3].) Each vertex of degree one has its edge labelled with a generator of $\mathbb{Z}_{12s+7}$ and produces one region, bounded by a hamiltonian cycle in K_{12s+7}. (See Problem 9-2). Then adding a vertex in the interior of these regions, together with $12s+7$ edges from each new vertex to the boundary vertices, gives the triangulation we seek. This is called the *regular* part of the problem. Next, the surface in which $K_{12s+10} - K_3$ is triangularly imbedded is modified – by the addition of one well-chosen handle (See [W12])– so as to accommodate the three edges removed in K_3. This is called the *additional adjacency* part of the problem. The final result is a (non-triangular) imbedding of K_{12s+10} in a surface of the appropriate genus.

The remaining ten cases for $\gamma(K_n)$ are handled similarly, with varying degrees of complexity; see [Y4], [RY2], [Y5], [TWY2], [RY3], [RY4], [TWY1], and [M4]. A constructive proof is given for each case but $n \equiv 0 \pmod{12}$; for this case the theory of finite fields supplements the theory of current graphs in establishing the *existence* of a triangular imbedding (see [TWY1]). (See also [R16], for the complete proof.) We now turn our attention to Cayley graphs in general.

9-3. A Hint of Things to Come

Modifications of the theory of the preceding section allow us to attack other Cayley graphs for which triangular imbeddings are possible. But more: quadrilateral imbeddings can be constructed as well. Even more: imbeddings with regions of varying sizes. We present four examples here; the general theory will be developed, in the more intuitive dual form, in the next chapter.

Example 1. We seek a triangular imbedding for the octahedral graph $K_{2,2,2}$. Choose $\Delta = \{1, 2\}$ for $\Gamma = \mathbb{Z}_6$, so that $G_\Delta(\Gamma) = K_{3(2)}$. We try for an *index one* imbedding of a current graph K having four directed edges (for $\Delta^* = \{1, 5, 2, 4\}$) and one 4-sided region in its quotient manifold. But then K has two edges; no such K will work. Consider, however, the *index two* spherical imbedding of Figure 9-9. This does

work nicely, if we let
$\Omega = \{0,2,4\} : (1,5,4,2)$ and $\bar{\Omega} = \{1,3,5\} : (2,4,5,1)$ produce

$$\begin{array}{ll} 0:(1,5,4,2) & 1:(3,5,0,2) \\ 2:(3,1,0,4) & 3:(5,1,2,4) \\ 4:(5,3,2,0) & 5:(1,3,4,0), \end{array}$$

as a rotation scheme for the $K_{3(2)}$ imbedding of Figure 9-10.

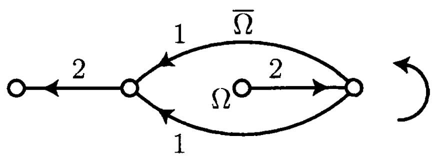


Figure 9-9.

An index one imbedding is possible, if we use $\Gamma = S_3$; see Problem 9-3.

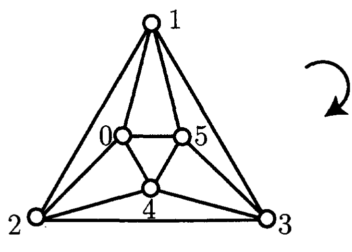


Figure 9-10.

Example 2. The imbedded current graph of Figure 9-11, using $\Gamma = \mathbb{Z}_5$, produces the rotation scheme given for K_5, and in turn a quadrilateral imbedding on the torus.

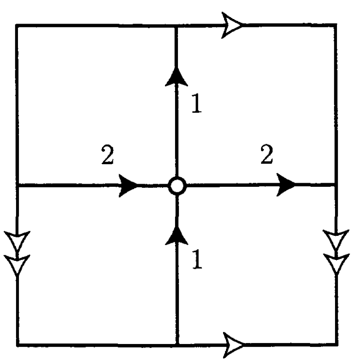


Figure 9-11.

$$\begin{aligned} 0 &: (1,3,4,2) \\ 1 &: (2,4,0,3) \\ 2 &: (3,0,1,4) \\ 3 &: (4,1,2,0) \\ 4 &: (0,2,3,1) \end{aligned}$$

Example 3. The spherical current graph of Figure 9-12 gives $G_\Delta(S_n)$ in S_k, where $k = 1 + \frac{(n-2)!}{4}(n^2 - 5n + 2)$, with $r_n = (n-1)!$, $r_{2n-2} = n(n-2)!$, and $r_2 = \frac{n!}{2}$. (See Problem 9-4.) Here, $\Delta = \{s, t\}$, with $s = (12 \cdots n)$ and $t = (12)$. As is the standard practice, the digons would be collapsed in the Cayley graph imbedding. Note the relevance of this construction to Corollary 7-23 and to Theorem 7-26.

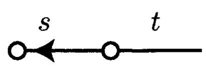


Figure 9-12.

Example 4. The spherical current graph of Figure 9-13 imbeds K_7 in S_3, with $r_3 = 7$ and $r_7 = 3$. Each 7-gon is bounded by a spanning cycle. This index-one imbedding solves the regular part of the problem, for finding $\gamma(K_{10})$; see Section 9-2.

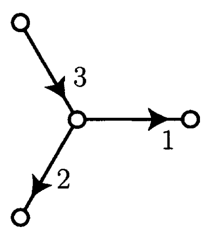


Figure 9-13.

9-4. Problems

9-1.) Show that seven "different" minimal imbeddings of K_5 are compatible with the formula of Theorem 9-1. (For example, $r_3 = 4, r_8 = 1; r_4 = 5$ give two different imbeddings of K_5 on S_1, since the region sizes are distributed differently.) How many of the seven can you actually construct? (**Hint**: not all seven exist!)

9-2.) If a vertex of degree one in a current graph imbedding has its incident edge carrying a current which generates the current group, show that that vertex determines one region bounded by a hamiltonian cycle, in the Cayley graph imbedding being constructed.

9-3.) Use S_3 to find an index-one imbedding of $K_{2,2,2}$.

9-4.) Verify the claims of Example 3, in Section 9-3. (This will be much easier to do, after Chapter 10.)

9-5.) Verify the claims of Example 4, in Section 9-3, by using a theoretical approach. Then, write out the rotation scheme for K_7 determined by Figure 9-13, and use that scheme to verify the claims independently.

9-6.) Use the current graph imbedding of Figure 9-3, but with $\Gamma = \mathbb{Z}_8$. What imbedding, of what graph, does this produce?

9-7.) Repeat Problem 9-6, using $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3$ and

$$\Delta = \{(1,0),(0,1),(1,1)\}.$$

Assign the currents so that the resulting Cayley graph imbedding is as interesting as possible.

9-8.) Use current graph theory to find a hexagonal imbedding ($r = r_6$) for $K_{3,3}$.

9-9.) The current graph of Figure 9-6 is $K_2 \times K_3$. But there is a second cubic graph of order 6 (see Problem 2-3), namely $K_{3,3}$. Label and direct the edges of $K_{3,3}$ with currents from $\mathbb{Z}_{19}$, and specify a rotation at each vertex, so as to obtain a triangular imbedding for K_{19}. How does this imbedding compare with that produced by Figure 9-6?

9-10.) A graph G of size q is said to be *conservative* (see Bange, Barkauskas, and Slater [BBS1]) if the edges can be oriented and distinctly labelled with $1, 2, \cdots, q$ so that at each vertex the sum of the numbers on the inwardly directed edges equals that on the outwardly directed edges. Show that if such a graph has a 2-cell imbedding with $r = 1$, then it serves as a KCL current graph, with currents from $\Gamma = \mathbb{Z}_n$, $n \geq 2q + 1$, determining a 2-cell imbedding of $G_\Delta(\Gamma)$, for $\Delta = \{1, 2, \cdots, q\}$. Study the imbedding. Consider the special cases $n = 2q+1$ and $n = 2q+2$. Use the fact that, for $n \geq 4$, K_n is conservative (see [BBS1]) to find $r = r_{n-1}$ imbeddings for K_{n^2-n+1}, $n \equiv 1, 2 \pmod 4$. If G is cubic, show that, in general, the covering imbedding is minimal for $G_\Delta(\Gamma)$. How does this problem connect with the previous one (9-9)?

CHAPTER 10

VOLTAGE GRAPHS

In the preceding chapter, following the historical development of Gustin, Youngs, Ringel, Jacques, *et al.*, we generated graph imbeddings by means of constructing (simpler) current graph imbeddings. These latter were variously called "quotient graph in quotient manifold" and, in [J3], "reduced constellation." Each approach has the advantage of economy of construction; each approach has the disadvantage of generating regions by vertices and vertices by regions. Moreover, the mode of "generation" is not made explicit, in a way to give maximum aid to intuition.

These defects are corrected by the voltage graph theory initiated by Gross [G4] in 1974 and by its interpretation in the context of branched covering spaces (see papers by Gross and Alpert: [GA1], [GA2], and [AG2].) The key is that the desired imbedding covers its quotient structure directly, rather than in dual form.

In this chapter we present just enough covering space theory for the immediate context (for more details, see [M3]) and then introduce voltage graphs, with examples. We then revisit the Heawood Map-coloring Theorem from this advantageous viewpoint. Next, we describe the strong tensor product construction for graphs; this is an iterative process that often produces an infinite tower of graph imbeddings from one voltage graph imbedding at the base. Finally, we study voltage graphs in conjunction with graphical products.

10-1. Covering Spaces

Def. 10-1. A continuous function $\rho : \tilde{X} \to X$ from one path connected topological space to another is called a *covering projection* if every point $x \in X$ has a neighborhood U_x which is *evenly covered*; i.e. ρ maps each component of $\rho^{-1}(U_x)$ homeomorphically onto U_x. If $Y \subseteq X$ and $\tilde{Y} \subseteq \tilde{X}$ is such that ρ maps $\tilde{Y}$ homeomorphically onto Y, we say that Y *lifts* to $\tilde{Y}$. We call $\tilde{X}$ a *covering space* for X.

A standard result in the theory of covering spaces is that $|\rho^{-1}(x)|$ is independent of the choice of $x \in X$. If $|\rho^{-1}(x)| = n$, then ρ is called an *n-fold* covering projection. Here are some standard examples of covering spaces and covering projections:

Example 1: Let $S^1 = \{(x,y) \in \mathbb{R}^2 | x^2+y^2=1\}$ and consider $\rho : \mathbb{R} \to S^1$, where $\rho(t) = (\cos t, \sin t)$. Then ρ "wraps" the real line $\mathbb{R}$ around the unit circle S^1 infinitely many times, with each half-open interval $[r, r+2\pi)$ "covering" S^1 exactly once. If we pick $(1,0) \in S^1$ and (say) $U_{(1,0)} = \{(x,y) \in S^1 | x > 0, -\frac{1}{2} < y < \frac{1}{2}\}$, then $U_{(1,0)}$ is evenly covered – as $\rho^{-1}(U_{(0,1)}) = \cup_{i \in I} \left(2\pi i - \frac{\pi}{6}, 2\pi i + \frac{\pi}{6}\right)$ and each open interval $\left(2\pi i - \frac{\pi}{6}, 2\pi i + \frac{\pi}{6}\right)$ is clearly homeomorphic to $U_{(1,0)}$.

The covering projection of the preceding example fails to be n-fold, for each $n \in \mathbb{N}$. However, for every $n \in \mathbb{N}$ it is a simple matter to construct an n-fold covering projection.

Example 2: Define $\rho : S^1 \to S^1$ by $\rho(z) = z^n$, where $z \in S^1$ is regarded as a complex number. (Recall that, if $z = \operatorname{cis}\theta = (\cos\theta, \sin\theta)$, then $z^n = \operatorname{cis} n\theta = (\cos n\theta, \sin n\theta)$.) Thus ρ wraps S^1 n times around itself, and – for example – $\rho^{-1}(1,0)$ consists precisely of the nth roots of unity.

The above example is significant for the sequel, as it describes how region boundaries of a desired graph imbedding will project to those of a voltage graph imbedding, both in the simplest of cases (corresponding to the KCL – see Section 9-2 – holding) and, in general.

Example 3: Recall that S_0 is the sphere (S^2 in $\mathbb{R}^3$), and that N_1 is the projective plane (nonorientable surface of genus one, or sphere with one crosscap.) Define $\rho : S_0 \to N_1$ by antipodal identification; i.e. $\rho(x,y,z) = \rho(-x,-y,-z)$; then ρ^2 is a 2-fold covering projection. Intuitively, we could regard ρ as fixing the bottom hemisphere and depressing the top hemisphere in "reverse overlapping" fashion; the antipodal identification along the equator then "sews on" the crosscap. (This takes place in $\mathbb{R}^4$, not $\mathbb{R}^3$.)

The following result is well known (by the Riemann-Hurwitz Theorem.)

Thm. 10-2. If $\rho : \tilde{S} \to S$ is an n-fold covering projection for surfaces, then the surface characteristics are related by: $\chi(\tilde{S}) = n\chi(S)$.

This relationship is quite natural: if G is 2-cell imbedded in S, with $p-q+r = \chi(S)$, then $\rho^{-1}(G)$ is 2-cell imbedded in $\tilde{S}$ with np vertices, nq edges, and nr regions, so that $\chi(\tilde{S}) = np - nq + nr = n(p-q+r) = n\chi(S)$.

Cor. 10-3. If $\tilde{S} = S_k$ and $S = S_h$, then $k = n(h-1)+1$.

Thus, for example, only the torus can cover the torus (since $h = 1$ forces $k = 1$.)

An important generalization of the concept of covering space is required, to cover the case where the KCL does *not* hold.

Def. 10-4. A continuous function $\rho : \tilde{X} \to X$ from one path-connected topological space to another is called a *branched covering projection* (and $\tilde{X}$ is a *branched covering space* of X) if there exists a finite set $B \subseteq X$ such that the restricted function $\rho : \tilde{X} - \rho^{-1}(B) \to X - B$ is a covering projection. The points of B are called *branch points*. For $b \in B$ and U_b a sufficiently small neighborhood of b, the restricted function $\rho : \tilde{U}_b \to U_b - \{b\}$ is n-fold, for some cardinal number n – called the *multiplicity of branching* at b – where $\tilde{U}_b$ is a component of $\rho^{-1}(U_b - \{b\})$ in $\tilde{X}$. (If $n = 1$, then there is no branching.)

Standard examples here are:

Example 4: Let $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$, the unit disk, and give $\rho : D \to D$ by $\rho(z) = z^n$ where, as before, z is regarded as a complex number. Then ρ "wraps" D around itself n times, except that the origin is fixed (and thus is a branch point of multiplicity n.)

Example 5: Define $\rho : S_0 \to S_0$, in spherical coordinates, by $\rho(1, \theta, \phi) = (1, n\theta, \phi)$. Then ρ "wraps" S_0 around itself n times, except that the north and south poles are both fixed; each is thus a branch point of multiplicity n.

Both of the examples above are important for what is to follow, as the former gives the prototype description of the projection of regions in (possibly branched) coverings of one graph imbedding by another – the local picture – while the latter is one instance of the projection between the corresponding ambient surfaces – the global picture.

10-2. Voltage Graphs

Let K be a pseudograph. With each edge $uv \in E(K)$, we associate two oriented edges $e = (u, v)$ and $e^{-1} = (v, u)$, and we set $K^* = \{(u, v) | uv \in E(K)\}$.

Def. 10-5. A *voltage graph* is a triple (K, Γ, ϕ), where K is a connected pseudograph, Γ is a group, and $\phi : K^* \to \Gamma$ satisfies $\phi(e^{-1}) = (\phi(e))^{-1}$ for all $e \in K^*$. Each value $\phi(e)$ is called a *voltage*.

We remark that the pseudograph K is closely related to the quotient graph K of Section 9-2.

Def. 10-6. The *covering graph* $K \times_\phi \Gamma$ for (K, Γ, ϕ) has vertex set $V(K) \times \Gamma$ and each edge $e = (u, v)$ of K determines the edges $(u, g)(v, g\phi(e))$ of $K \times_\phi \Gamma$, for all $g \in \Gamma$.

For pseudographs regarded as topological spaces then, $K \times_\phi \Gamma$ is an $|\Gamma|$ - fold covering space of K; in fact, every regular covering space of K can be obtained in this manner (see [GT3]). (For additional information on covering projections of graphs, see Waller [W1], Farzan and Waller [FW1], Clarke, Thomas, and Waller [CTW1], Sit [S9], and Biggs [B12].)

For any walk $w : e_1, e_2, \cdots, e_m$ beginning at $v \in V(K)$ in (K, Γ, ϕ), we set

$$\phi(w) = \prod_{i=1}^{m} \phi(e_i),$$

and define the *local group* at v by:

$$\Gamma_v = \{\phi(w) | w \text{ is a closed walk at } v\}.$$

Then Γ_v is a subgroup of Γ (see Problem 10-11), and moreover:

Thm. 10-7. If vertices u and v are in the same component of K, then the subgroups Γ_u and Γ_v are conjugate in Γ.

PROOF. (i) Let $\phi(x) \in \Gamma_u$, where x is a closed walk at u. Let y denote a walk from u to v, as shown in Figure 10.1. Then $y^{-1}xy$ is a closed walk at v, so that $\phi(y^{-1}xy) = \phi^{-1}(y)\phi(x)\phi(y) \in \Gamma_v$. Thus $\Gamma_u \subseteq \phi(y)\Gamma_v\phi^{-1}(y)$.

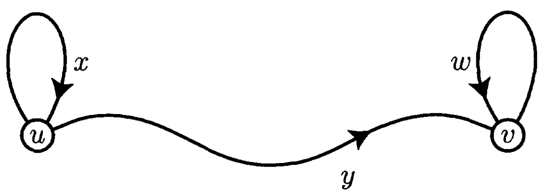


Figure 10-1.

(ii) Now let $\phi(w) \in \Gamma_v$, with w a closed walk at v. But ywy^{-1} is a closed walk at $u, \phi(ywy^{-1}) = \phi(y)\phi(w)\phi^{-1}(y) \in \Gamma_u$, and $\phi(y)\Gamma_v\phi^{-1}(y) \subseteq \Gamma_u$. Hence $\Gamma_u = \phi(y)\Gamma_v\phi^{-1}(y)$, where $\phi(y) \in \Gamma$. $\square$

We see from Theorem 10-7 that, if K is connected, then $[\Gamma : \Gamma_v]$ is independent of $v \in V(K)$.

Thm. 10-8. For (K, Γ, ϕ) a connected voltage graph, the number of components of the covering graph $K \times_\phi \Gamma$ is given by $[\Gamma : \Gamma_v]$, where v is arbitrary in $V(K)$.

PROOF. Fix $v \in V(K)$. Since K is connected, every component of $K \times_\phi \Gamma$ contains a vertex of the form (v, g), for some $g \in \Gamma$. Thus the number of components of $K \times_\phi \Gamma$ is the number of components of $K \times_\phi \Gamma$ having at least one vertex with v as first coordinate. But (v, g) and (v, h) are in the same component of $K \times_\phi \Gamma$ if and only if $g^{-1}h \in \Gamma_v$ if and only if $g\Gamma_v = h\Gamma_v$. Thus the components of $K \times_\phi \Gamma$ are in one-to-one correspondence with the left cosets of Γ_v in Γ. □

In Figure 10.2 we illustrate Theorem 10-8, using $\Gamma = \mathbb{Z}_4$. Since $\Gamma_v = \{0, 2\}$, we have the covering graph $K \times_\phi \Gamma$ consisting of $[\mathbb{Z}_4 : \mathbb{Z}_2] = 2$ components. If we regard K as being imbedded in S_0, then $K \times_\phi \Gamma = 2C_4$ is imbedded in $2S_0$.

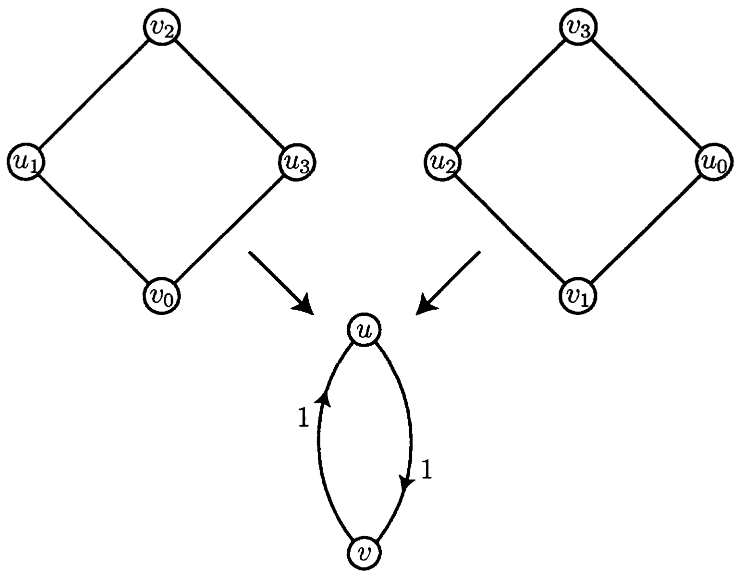


Figure 10-2.

In general let (K, Γ, ϕ) be 2-cell imbedded in an orientable surface S, as described algebraically by the rotation scheme $P = (p_1, p_2, \cdots, p_p)$. We define the *lift* $\tilde{P}$ of P to $K \times_\phi \Gamma$ as follows: if $p_v(v, u) = (v, w)$, then

$$\tilde{p}_{(v,g)}((v, g), (u, g\phi(v, u))) = ((v, g), (w, g\phi(v, w))),$$

for each $g \in \Gamma$. (See Figure 10-3.) Then

$$\tilde{P} = \{\tilde{p}_{(v,g)} | (v, g) \in V(K \times_\phi \Gamma)\}.$$

Thus $\tilde{P}$ determines a 2-cell imbedding of each component of $K \times_\phi \Gamma$. The power of voltage graph theory is that the simpler imbedding of K below gives much information about the more complicated imbedding of $K \times_\phi \Gamma$ above. To see this, let R be a region of the imbedding of K on S induced by P, and let $|R|_\phi$ be the order of $\phi(w)$ in Γ, where $w = e_1, e_2, \cdots, e_m$ is a closed walk in K consisting of the ordered boundary of R. (Since $\phi(w)$ is unique up to inverses and conjugacy, $|R|_\phi$ is independent of the orientation of R and of the initial vertex of w.) We then have the following central result, due to Gross and Alpert [GA1]:

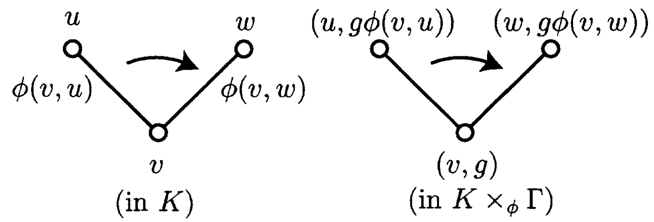


Figure 10-3.

Thm. 10-9. Let (K, Γ, ϕ) be a voltage graph with rotation scheme P and $\tilde{P}$ the lift of P to $K \times_\phi \Gamma$. Let P and $\tilde{P}$ determine 2-cell imbeddings of K and $K \times_\phi \Gamma$ on the orientable surfaces S and $\tilde{S}$ respectively, where $\tilde{S}$ is possibly disconnected. Then there exists a (possibly branched) covering projection $\rho : \tilde{S} \to S$ such that:

(i) $\rho^{-1}(K) = K \times_\phi \Gamma$;
(ii) if R is a region of the imbedding of K which is a k-gon, then $\rho^{-1}(R)$ has $\frac{|\Gamma|}{|R|_\phi}$ components, each of which is a $k|R|_\phi$-gon region of the covering imbedding of $K \times_\phi \Gamma$;
(iii) if $|R|_\phi = n > 1$, then R contains a branch point of multiplicity n. If $n = 1$, then R contains no branch point.

PROOF. For $(v, g) \in V(K \times_\phi \Gamma)$, define $\rho(v, g) = v$. Extend this continuously, first to $E(K \times_\phi \Gamma)$ by sending the image of edge $(u, g)(v, g\phi(u, v))$ in the imbedding of $K \times_\phi \Gamma$ to the image of edge uv in the imbedding of K, and then to the regions of the imbedding of $K \times_\phi \Gamma$.

(i) Then $\rho^{-1}(K) = K \times_\phi \Gamma$, by the definition of ρ.
(ii) Let $v_1, v_2, \cdots, v_k$ be a closed walk w bounding R, with $e_i = (v_i, v_{i+1})$, mod k. Then we can also express w as: $e_1, e_2, \cdots, e_k$. Let $n = |R|_\phi$, the order of $\phi(w)$ in Γ. Then each component of $\rho^{-1}(R)$ will have boundary of the form:

$$\begin{aligned}
&(v_1, g), (v_2, g\phi(e_1)), \cdots, (v_k, g\phi(e_1)\phi(e_2)\cdots\phi(e_{k-1})),\\
&(v_1, g\phi(w)), \cdots, (v_k, g\phi(w)\phi(e_1)\phi(e_2)\cdots\phi(e_{k-1})),\\
&\vdots\\
&(v_1, g\phi^n(w)) = (v_1, g),
\end{aligned}$$

for some $g \in \Gamma$. Hence each component of $\rho^{-1}(R)$ is a kn-gon. The number of such components is $\frac{|\Gamma|}{n}$, as the second coordinates of (v_1, g) range over Γ.

(iii) This follows from the above, after observing that each component of $\rho^{-1}(R)$ is mapped by ρ onto R, essentially by $\rho : D \to D$, $\rho(z) = z^n$, as described in Example 4 of Section 10.1.

□

Figure 10-2 serves to illustrate these ideas also. The covering projection is 4-fold. Each region R below has $|R|_\phi$ = order of $1 + 1$ in $\mathbb{Z}_4 = 2$ and contains a branch point of multiplicity 2. Moreover, each $\rho^{-1}(R)$ has $\frac{4}{2} = 2$ components (one in each S_0 above), each a 4-gon in $2S_0$ (since $k = 2$). Each of these components projects to R, exactly like $\rho : D \to D$, $\rho(z) = z^2$ (wrapping around twice, but fixing a branch point at the origin).

Def. 10-10. The order $|R|_\phi$ of $\phi(w)$ in Γ, where w is a closed walk bounding region R, is called the *period* of R.

Thm. 10-11. Let voltage graph (K, Γ, ϕ) imbedded in S_h have r regions, with periods $\pi_1, \pi_2, \cdots, \pi_r$. Let the covering imbedding of $K \times_\phi \Gamma$ be in S_k. Then

$$k = 1 + |\Gamma|(h-1) + \frac{|\Gamma|}{2}\sum_{i=1}^{r}\left(1 - \frac{1}{\pi_i}\right).$$

PROOF. Let p, q, r and $\tilde{p}, \tilde{q}, \tilde{r}$ apply to the base and covering imbeddings respectively. Since region R_i (a k_i-gon) below determines $\frac{|\Gamma|}{\pi_i}$ regions, each of length $k_i\pi_i$ above, for $1 \le i \le r$, we have

$$2\tilde{q} = \sum_{i=1}^{r}\frac{|\Gamma|}{\pi_i}k_i\pi_i = |\Gamma|\sum_{i=1}^{r}k_i = |\Gamma|(2q),$$

so that $\tilde{q} = |\Gamma|q$. Also, $\tilde{p} = |\Gamma|p$, and $\tilde{r} = \sum_{i=1}^{r} \frac{|\Gamma|}{\pi_i}$. Moreover, $h = 1 + \frac{1}{2}(q - p - r)$. So,

$$\begin{aligned}
k &= 1 + \frac{1}{2}(\tilde{q} - \tilde{p} - \tilde{r}) \\
&= 1 + \frac{1}{2}\left(|\Gamma|q - |\Gamma|p - \sum_{i=1}^{r} \frac{|\Gamma|}{\pi_i}\right) \\
&= 1 + \frac{|\Gamma|}{2}\left(q - p - r + \sum_{i=1}^{r}\left(1 - \frac{1}{\pi_i}\right)\right) \\
&= 1 + |\Gamma|(h-1) + \frac{|\Gamma|}{2}\sum_{i=1}^{r}\left(1 - \frac{1}{\pi_i}\right).
\end{aligned}$$

□

Note that for $|\Gamma| = 1$, each $\pi_i = 1$, so that $k = h$, as expected. Also note that Theorem 7-22 is a special case of Theorem 10-11. For additional corollaries, we need:

Def. 10-12. If $\phi(w) = e$, the identity in Γ, for the closed walk w bounding region R in the voltage graph imbedding of K on S, we say that R satisfies the *Kirchoff Voltage Law* (KVL). If the KVL holds for all regions R of K on S, we say that the imbedding of K satisfies the KVL.

Cor. 10-13. If the KVL holds for the voltage graph imbedding of K on S, then $k = 1 + |\Gamma|(h - 1)$.

The above is also Corollary 3 of Cairns [C1, p. 208], for a covering of S_h by S_k without branching; see also Corollary 10-3 of this book.

Cor. 10-14. If the voltage graph imbedding is on the sphere, then

$$k = 1 - |\Gamma| + \frac{|\Gamma|}{2}\sum_{i=1}^{r}\left(1 - \frac{1}{\pi_i}\right).$$

That is essentially the formula of Fox [F3, p. 255], for branched covers of S_0 by S_k.

If $p = |V(K)| = 1$, the covering imbedding is a Cayley map; see Section 16-3. Then we have:

Cor. 10-15. For Cayley maps,

$$k = 1 + \frac{|\Gamma|}{2}\left(q - 1 - \sum_{i=1}^{r} \frac{1}{\pi_i}\right).$$

That result is Theorem 5.3.8 of [BW1].

Cor. 10-16. For KVL Cayley maps,

$$k = 1 + \frac{|\Gamma|}{2}(q - 1 - r) = 1 + |\Gamma|(h - 1).$$

Cor. 10-17. A KVL Cayley map covering the torus is on the torus.

The last result is independent of the Cayley property (see Corollary 10-13), but it is in the Cayley map context that we will most frequently encounter this situation (see, for example, Section 10-3). The next result follows directly from Theorem 10-9.

Thm. 10-18. The projection $\rho : \tilde{S} \to S$ of Theorem 10-9 is a covering projection (i.e. there is no branching) if and only if the imbedding of the voltage graph (K, Γ, ϕ) satisfies the KVL.

In either case (branching or not), if K has order m, the voltage graph (K, Γ, ϕ) is said to have index m, and the imbedding of $K \times_\phi \Gamma$ is an *index m imbedding*.

Voltage graph theory is even more general than the (dual-form) generalizations of Theorem 9-4 illustrated in Section 9-2, in that the covering graph $K \times_\phi \Gamma$ need not be a Cayley graph. In this book, however, we are concerned primarily with the Cayley graph case. This will arise for Ω a subgroup of Γ and voltage graph (K, Ω, ϕ)–where $\phi : K^* \to \Gamma$– a Schreier coset graph for Ω in Γ. Then $K \times_\phi \Omega = G_\Delta(\Gamma)$, where $\Delta^* = \{\phi(k) | k \in K^*\}$ (see Section 9-5 of [W15], for details of this construction). For example, referring to Figure 10-2 once again, we see that if we lift by $\Omega = \mathbb{Z}_2 = \{0, 2\}$ instead of $\Gamma = \mathbb{Z}_4$, taking vertex v below for the subgroup Ω and vertex u below for the other coset $\{1, 3\}$, then we get just the spherical imbedding of C_4 on the left as the covering space. Or, we could use voltage graph (K, Γ, ϕ)– our preference in this book; then $K \times_\phi \Gamma$ would consist of $\frac{|\Gamma|}{|\Omega|}$ disjoint copies of $G_\Delta(\Gamma)$. This is what the full Figure 10-2 shows.

In the simplest case–index one–K has one vertex ($V(K) = \{v\}$), $\Omega = \Gamma$, the two approaches coincide, and the construction is quite clear: for each $g \in \Gamma$, we identify (v, g) with g, and then $V(K \times_\phi \Gamma) = V(K) \times \Gamma \approx \Gamma$ and $E(K \times_\phi \Gamma) \approx \{\{g, g\phi(e)\} | g \in \Gamma, e = (v, v) \in K^*\}$. Thus $K \times_\phi \Gamma = G_\Delta(\Gamma)$.

We remark that even voltage graph theory has been generalized; see Gross and Tucker [GT3]. Also, see Bouchet [B17] for graph imbeddings as covering spaces with *folds.* (A fold is a 1-dimensional analog of the 0-dimensional branching). And, see Parsons, Pisanski, and Jackson ([PPJ1] and [JPP1]) for graph imbeddings as branched covering spaces, where the restrictions of the branched coverings to the imbedded graphs are "wrapped quasi-coverings." Finally, for a study of voltage-current duality using medial graphs, see Archdeacon [A11].

10-3. Examples

Example 1: Consider the index one voltage graph (a "bouquet" of two circles) of Figure 10-4, shown imbedded on the torus S_1. We take $\Delta = \{a, b\}$ for $\Gamma = \Omega$ abelian ($|\Gamma| \geq 5$), so that the KVL holds; thus by Theorem 10-18, there is no branching. Then, by Corollary 10-17, we see that $G_\Delta(\Gamma) = K \times_\phi \Gamma$ will be imbedded on the torus as well. By Theorem 10-9, this imbedding will have $r = r_4 = |\Gamma|$. Thus Figure 10-4 completely determines an infinite family of quadrilateral imbeddings on the torus. We mention four special cases below.

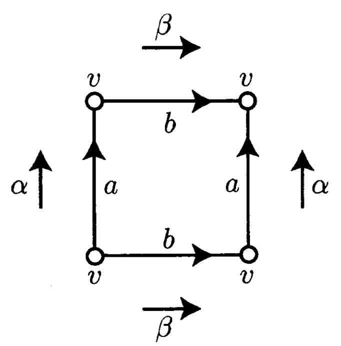


Figure 10-4.

(a) Let $a = 1$ and $b = 2$ in $\Gamma = \mathbb{Z}_5$; then $G_\Delta(\Gamma) = K_5$. This imbedding is self-dual and is the ground case for several infinite families of imbeddings appearing in the literature:

(i) K_{4n+1} has a self-dual imbedding in $\tilde{S} = S_{n(4n-3)}$ (see [W9]); here the voltage graph imbedding is just the normal form representation for $S = S_n$ (see Theorem 5-5 (ii).) The covering projection is $(4n + 1)$-fold, and indeed $\chi(\tilde{S}) = 2 - 2n(4n - 3) = (4n + 1)(2 - 2n) = (4n + 1)\chi(S)$.

(ii) The imbedding of K_5 on S_1 can be augmented to an immersion of $K_{5(2)}$ on S_1 attaining the toroidal crossing number $v_1(K_{5(2)}) = 10$. Similarly, $v_k(K_{p(2)}) = \frac{p(p-1)}{2}$, for $k = \frac{(p-1)(p-4)}{4}$ and p a prime power $\equiv 1 \pmod 4$; see Theorem 6-75.

(iii) The imbedding of K_5 on S_1 can also be modified to obtain a genus imbedding for $G_5 = K_{5,5}$ less a 1-factor; this extends to show $\gamma(G_n) = \left\lceil \frac{(n-1)(n-4)}{4} \right\rceil$, which will be useful in Chapter 13.

We observe that the rotation scheme for K_5 in S_1 can be readily obtained from Figure 10-4. First, we find

$$p_v = ((v,v)_a, (v,v)_b, (v,v)_{-a}, (v,v)_{-b}).$$

As is customary for index one imbeddings, we identify (v,g) with g in the covering graph. Next, as K_5 has no loops or multiple edges, we regard each $\tilde{p}_{(v,g)} = \tilde{p}_g$ as permuting the neighbors of g rather than the edges at g. Finally, taking $a = 1$ and $b = 2$, we get:

$$\begin{aligned} p_0 &= (1,2,4,3) \\ p_1 &= (2,3,0,4) \\ p_2 &= (3,4,1,0) \\ p_3 &= (4,0,2,1) \\ p_4 &= (0,1,3,2) \end{aligned}$$

(In practice, the above is written down directly from the figure.) The covering projection is 5-fold: the single vertex, two edges, and one 4-gon for K in S_1 lift respectively to five vertices, ten edges, and five 4-gon regions for $K \times_\phi \Gamma = K_5$ (also in S_1.)

b) Now let $a = (1,0)$ and $b = (0,1)$ for $\Gamma = \mathbb{Z}_m \times \mathbb{Z}_n$; then $G_\Delta(\Gamma) = C_m \times C_n$ and finds a self-dual, quadrilateral, toroidal imbedding. The case $m = n = 3$ is illustrated in Figure 10-5. The covering space nature of this imbedding is readily apparent: the single vertex, two edges, and one 4-gon now lift, respectively, to nine vertices, eighteen edges, and nine 4-gons, under this nine-fold projection ρ.

c) Setting $a = (1,0), b = (1,1)$ for $\Gamma = \mathbb{Z}_4 \times \mathbb{Z}_2$ gives a genus imbedding for $G_\Delta(\Gamma) = K_{4,4}$ in S_1.

d) Now let $\Gamma = \omega(S_1) = \langle a, b | aba^{-1}b^{-1} = e \rangle = \mathbb{Z} \times \mathbb{Z}$, the fundamental group of the torus. In this case $G_\Delta(\Gamma)$ has an imbedding on S_0 (if we allow the vertex set to have limit points), but it is more natural to consider $G_\Delta(\Gamma)$ as being imbedded in $\mathbb{R}^2$, as one of the three regular tessellations of the plane. (Recall that $\mathbb{R}^2$ is the universal covering space for S_1, in the sense that $\mathbb{R}^2$ covers

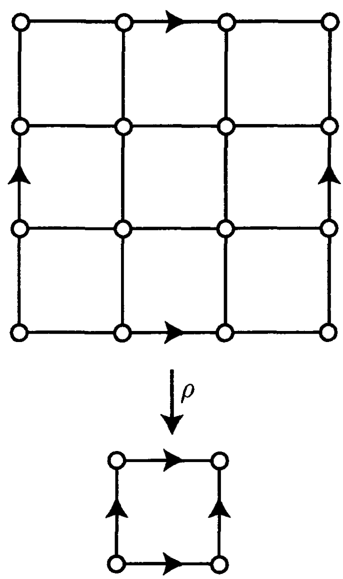


Figure 10-5.

any space which covers S_1; thus $G_\Delta(\Gamma)$ in $\mathbb{R}^2$ also covers each $C_\Delta(\mathbb{Z}_m \times \mathbb{Z}_n)$ imbedding of b) above, in a natural way.

We mention that $\pi(S_1)$ is also the group of one of the seventeen wallpaper designs (i.e. a planar crystallographic group.) The voltage graph theory is applicable to each of the seventeen planar infinite wallpaper groups (as presented, for example, in [B20]); of these imbeddings in $\mathbb{R}^2$, six are index-one branched covers of S_0, six are index-two branched covers of S_0, one is an index-two unbranched cover of S_0, three are index-two unbranched covers of S_1, and the pattern described by $\pi(S_1)$ is an index-one unbranched cover of S_1.

Example 2: Now modify Figure 10-4 slightly, to obtain Figure 10-6.

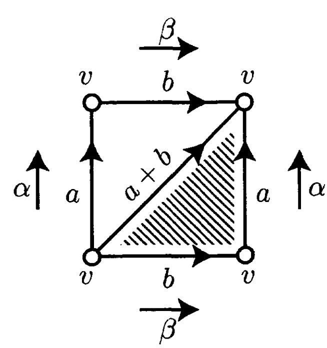


Figure 10-6.

We still require Γ to be abelian, but now take $\Delta = \{a, b, a+b\}$ with $|\Gamma| \geq 7$. Then $G_\Delta(\Gamma)$ is regular of degree six and has an index one $r = r_3$ imbedding in S_1; again infinitely many such imbeddings are determined, one for each choice of Γ. Moreover, each of these has *bichromatic dual* (the geometric dual G^* has chromatic number two, where $G = G_\Delta(\Gamma)$); the indicated 2-coloring of the regions of Figure 10-6 lifts to a 2-coloring of the regions for $G_\Delta(\Gamma)$. (This will be useful in Chapter 12.)

a) For $a = 1$ and $b = 2$ in $\Gamma = \mathbb{Z}_7$, $G_\Delta(\Gamma) = K_7$; this is the same famous toroidal imbedding obtained by the dual of Figure 9-3 (see also Figure 9-4.)
   (i) A fairly natural extension of this voltage graph gives orientable triangular imbeddings for K_{12s+7}, for all $s \in \mathbb{N}$. (The dual is bichromatic only for $s = 0$.)
   (ii) By taking first $a = 1$ and $b = 3$ and then $a = 2$ and $b = 5$ in $\Gamma = \mathbb{Z}_{13}$, toroidal imbeddings are obtained for two complementary graphs in K_{13}; this shows that $N(1,1) = 14$, where $N(\gamma, \gamma')$ is the least integer such that every graph G of order at least $N(\gamma, \gamma')$ is not (γ, γ') *bi-imbeddable* (G imbeddable in S_γ, $\bar{G}$ in $S_{\gamma'}$). (See [AW1]; see also [A3] and [AC1].) It also shows that the *toroidal thickness* of K_{13} is two ($\theta_1(K_{13}) = 2$; see Beineke [B7] and Ringel [R15]; see also Theorem 6-63 (ii).)
b) If $a = 1$ and $b = 2$ for $\Gamma = \mathbb{Z}_8$, we obtain a genus imbedding for $K_{4(2)}$.
c) If $a = (1,0)$ and $b = (0,1)$ for $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3$, a genus imbedding for $K_{3(3)}$ results.

   Voltage graph constructions are by no means unique for a given graph imbedding. For example, the dual of Figure 10-6 serves as an index two voltage graph (using $\Gamma = \mathbb{Z}_3$) for an imbedding of $K_{3,3}$ on S_1 having $r = r_6 = 3$; the dual of this imbedding then serves as an index three voltage graph (again using $\Gamma = \mathbb{Z}_3$) to triangularly imbed $K_{3(3)}$ on S_1 again. This is the ground case of Theorem 4.2 in [KRW1].

   We also mention that taking $a = 1$ and $b = 2$ in Figure 10-6, but replacing $a + b$ with 5 in $\Gamma = \mathbb{Z}_9$, gives an $r = r_{27} = 2$ imbedding for $K_{3(3)}$ (as seen from Theorem 10-9 (ii)); this gives the maximum genus of $K_{3(3)}$ as $\gamma_M(K_{3(3)}) = 9$, and is our first example of a branched covering in this section: each 27-gon above wraps around a triangle R below 9 times, since $|R|_\phi = 9$.
d) We keep $a = (1,0)$ and $b = (0,1)$, but now in $\Gamma = \mathbb{Z}_4 \times \mathbb{Z}_4$; the resulting triangular imbedding of $G_\Delta(\Gamma)$ will be of interest in Section 12-7.
e) In [S28] the following conjecture is attributed to Grünbaum: if a graph G has an orientable triangular imbedding, then the dual

graph G^* has a *Tait coloring* (that is, has edge chromatic number $\chi_e(G^*) = 3$.) We remark that G cannot be allowed loops (consider the G^* of Figure 8.2 in [BCL1]) or multiple edges (consider $G^* = P$, the Petersen graph, in S_1) and that the conjecture is false also for nonorientable triangular imbeddings (consider $G = K_6$ in N_1, where $G^* = P$.) However the conjecture is true, as formulated, for 3-degenerate graphs G, as an easy induction argument shows. It is also true if $\chi(G^*) = 2$: if G^* is cubic and bipartite, then G^* is 1-factorable (see, for example, Theorem 8-7 in [BCL1]), so that $\chi_e(G^*) = 3$. Moreover, any $G_\Delta(\Gamma)$ triangulating S_1 as a covering space of Figure 10-6 has an edge coloring induced by the labels a, b, and $a + b$; now color the dual edges (in G^*) to agree with the colors on the edges of G they cross. Thus the voltage graph of Figure 10-6 also provides an easy algorithm for Grünbaum's conjecture, for infinitely many toroidal triangulations.

Example 3: The graphs $K_{4(n)}$ have genus $\gamma(K_{4(n)}) = (n-1)^2$, for $n \neq 3$, as established by Jungerman [J7] and independently (for n even) by Garman [G1]. (See Theorem 6-42.) Jungerman did not announce $\gamma(K_{4(3)})$, although he did report that $\gamma(K_{4(3)}) > 4$. The index-three (this is *not* a Cayley graph construction) toroidal voltage graph of Figure 10-7, using $\Gamma = \mathbb{Z}_3$, shows that $\gamma(K_{4(3)}) = 5$, by first imbedding $K_{3(3)}$ in S_5 with $r_3 = 6$ and $r_9 = 4$. (Refer to Theorem 10-9 (ii): there are four branch points – indicated by dots inside regions – each of multiplicity three and determining one 9-gon; the two regions having no branch point each lift to three triangles.) It is easily verified that each 9-gon is bounded by a hamiltonian cycle in $K_{3(3)}$. (For example, the clockwise boundary of the lift of the shaded region R is: $(b0, a0, c0, b1, a1, c1, b2, a2, c2)$ where, for ease of notation, we write (v, g) as vg; the fact that $|R|_\phi = 3$ assures that the lift of R wraps around R three times under branched projection by ρ.) Now add a new vertex in the interior of each of three of the four 9-gons, and join each new vertex to all nine boundary vertices, for the region containing that vertex. The result is an imbedding of $K_{4(3)}$ in S_5, with $r_3 = 33$ and $r_9 = 1$. Note that the construction is readily augmented to give a triangular (and hence genus) imbedding for the complete 4-partite graph $K_{4,3,3,3}$.

Example 4: For G a graph and n a natural number, by n-fold G we mean that multigraph which results when each edge uv of G is replaced by n edges uv. The construction of Figure 10-6 suggests the extension in Figure 10-8 (where $S = S_2$). We take $\Delta = \{a, b, a+b\}$–again – for Γ abelian ($n = |\Gamma| \geq 7$) and K a bouquet of nine circles in S_2, with $r = r_3$, KVL, and bichromatic dual. The covering space is an $r = r_3$ imbedding of the 18-regular 3-fold $G_\Delta(\Gamma)$ on S_{n+1}, having bichromatic

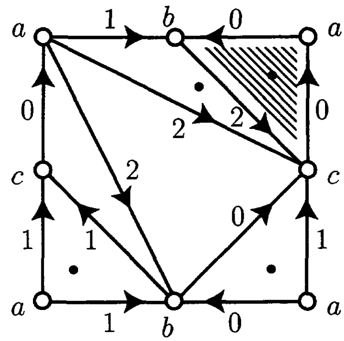


Figure 10-7.

dual. The ground case ($n = 7$) gives an orientable triangular imbedding for 3-fold K_7 (using $a = 1, b = 2$). Jungerman [J11] has found genus imbeddings (no digons allowed) for m-fold K_n, for all m and n (in both the orientable and the nonorientable cases.)

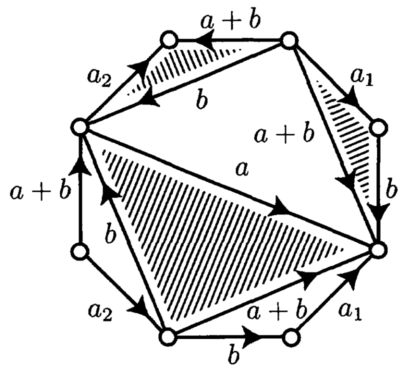


Figure 10-8.

Example 5: The spherical index two voltage graph of Figure 10-9 also appears in [SW2]; it uses $\Gamma = \mathbb{Z}_n$ to imbed $K_{n,n} = K_{2(n)}$ in S_k (as a branched covering space), where $k = \frac{(n-1)(n-2)}{2}$, with $r = r_{2n} = n$ (each $|R|_\phi = n$; now apply Theorem 10-9 (ii).) Moreover, each region boundary is a hamiltonian cycle. (For example, the region covering the shaded digon has clockwise boundary $(a0, b1, a1, b2, a2, \cdots, a(n-1), b0)$.) Thus this imbedding is readily augmented to give an $r = r_3$ imbedding for $K_{3(n)}$ (we discussed the case $n = 3$ in Example 2c) above, so that $\gamma(K_{3(n)}) = \frac{(n-1)(n-2)}{2}$; this proof is far shorter (and more elegant) than those of either [RY5] or [W5]. It is easy (see Problem 10-4) to show that every $r = r_3$ imbedding for $K_{3(n)}$ has bichromatic dual; this will be important in Section 12-7. Moreover, these imbeddings thus all satisfy the Grünbaum conjecture. (In fact, the three types of edges in the complete tripartite graph $G = K_{3(n)}$ determine a natural 3-edge coloring for the dual G^*.)

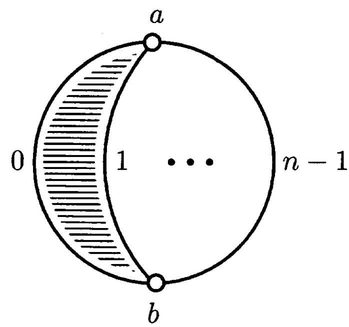


Figure 10-9.

10-4. The Heawood Map-coloring Theorem (again)

We reconsider the crux of the proof of the Heawood map-coloring theorem: the construction of genus imbeddings of complete graphs by the use of current graphs; our context now will be that of branched covering spaces, and we use the voltage graph construction. This viewpoint treats only the regular part of each case; the additional adjacency parts continue to be handled as before (see Section 9-2, [GT2], [W12], and [R16].)

Recall the current graph of Figure 9-3, for imbedding K_7 on the torus. The dual of Figure 10-6 (in particular, see Example 2a) is the corresponding voltage graph. From either the single region of Figure 9-3 or the single vertex of Figure 10-6, we read the rotation $p_0 : (1, 3, 2, 6, 4, 5)$; this generates the entire rotation scheme $P = (p_0, p_1, \cdots, p_6)$ and in turn the triangular imbedding. (Note that the KCL/KVL holds.)

In general, the current graph $K(s)$ for K_{12s+7} (as given in Section 9-3) has as its dual a voltage graph for $\Gamma = \mathbb{Z}_{12s+7}$, with $p = 1$, $q = 6s + 3$, and $r = r_3 = 4s + 2$; the voltage graph, as was the current graph, is imbedded in S_{s+1}. The covering is unbranched (since the KVL holds) and is $(12s + 7)$-fold, giving an imbedding of K_{12s+7} in S_{12s^2+7s+1} as before, with $p = 12s + 7$, $q = (12s + 7)(6s + 3)$, and $r = r_3 = (12s + 7)(4s + 2)$.

The current graph for K_{10} appears in Figure 9-13; it is used, with $\Gamma = \mathbb{Z}_7$ (see, for example, [W12]) to imbed K_7 in S_3 with $r_3 = 7$ and $r_7 = 3$ (determined by the *vortices* $x, y,$ and z.) This is readily augmented to a triangulation of S_3 by $K_7 + \bar{K}_3 = K_{10} - \{xy, xz, yz\}$; then the additional adjacency part of the construction adds the three missing edges over one extra handle.

The corresponding voltage graph is given in Figure 10-10.

The rotation $p_0 = (1, 6, 4, 3, 2, 5)$ is easily read from either figure. The outer region of Figure 10-10 satisfies the KVL ($1 + 4 + 2 = 0$, in $\mathbb{Z}_7$), so by Theorem 10-9 (ii) it lifts, without branching, to seven triangles in the covering imbedding of K_7. Each of the three monogons, however, has voltage sum of order $|R|_\phi = 7$, so that each is covered, with branching, by one seven-sided region (a heptagon.) Thus K_7 is imbedded in S_3 as a branched covering space over S_0. Moreover, since each voltage generates $\mathbb{Z}_7$, each heptagonal region boundary is a hamiltonian cycle for K_7. Thus, if one vertex is added in the interior of each heptagon, a triangular imbedding of $K_7 + \bar{K}_3$ results. The additional adjacency construction is as before.

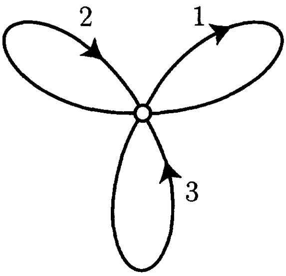


Figure 10-10.

The three vortices for each current graph in case 10 (see [R16]) become monogons in the voltage graph imbedding, with the single boundary edges always carrying a voltage which generates $\mathbb{Z}_{12s+7}$. Thus each monogon is branch-covered by a single $(12s + 7)$-gon, bounded by a hamiltonian cycle for K_{12s+7}, and the construction proceeds as for the case $s = 0$.

For a discussion of all twelve cases in this voltage graph branched covering space setting, see Gross and Tucker [GT2].

10-5. Strong Tensor Products

In [GRW1] the following product operation for graphs was introduced:

Def. 10-19. For two graphs G_1 and G_2, the *strong tensor product* $G_1 \underline{\otimes} G_2$ has vertex set $V(G_1) \times V(G_2)$ and edge set

$$\begin{aligned}\{(u_1, u_2)(v_1, v_2) | (&u_1 = v_1 \text{ and } u_2v_2 \in E(G_2))\\ &\text{or } (u_1v_1 \in E(G_1) \text{ and } u_2v_2 \in E(G_2))\}.\end{aligned}$$

Thus $G_1 \underline{\otimes} G_2$ consists of $|V(G_1)|$ disjoint copies of G_2, together with all the tensor product edges. The following two properties of the strong tensor product follow from the definition:

Thm. 10-20. $K_r \underline{\otimes} K_{n(m)} = K_{n(rm)}$.

Thm. 10-21. $G_1 \underline{\otimes} G_2$ is a subgraph of the lexicographic product $G_2[\bar{G}_1]$, with equality if and only if G_1 is complete.

The theorem below from [GRW1] is especially germane in the present work:

Thm. 10-22. Let G_2 have a triangular imbedding in an orientable surface, with bichromatic dual; let G_1 be connected and bichromatic, with maximum degree at most two. Then $G_1 \underline{\otimes} G_2$ has a triangular imbedding in an orientable surface, with bichromatic dual.

In Chapter 12 we will find that orientable triangulations of strongly regular graphs with bichromatic duals can be particularly useful. We shall see that each $K_{n(m)}$ is strongly regular; thus if we take $G_1 = K_2$, Theorems 10-20 and 10-22 will combine to preserve this usefulness under the strong tensor product operation.

For the present, we observe that each covering imbedding of Figure 10-6 (with $G_\Delta(\Gamma) = G_2$) gives rise to an infinite collection of bichromatic dual triangulations, via repeated application of Theorem 10-22 - for each graph G_1 as in that theorem.

In particular, we can take $G_1 = K_2$ and extend Examples 2a, 2b, and 2c respectively to obtain, for each nonnegative integer k, orientable bichromatic dual triangulations for $K_{7(2^k)}$, $K_{4(2^{k+1})}$, and $K_{3(3 \cdot 2^k)}$.

Finally, we remark that Theorem 10-21 combines with Theorem 6-45 to yield the surprisingly general formula of:

Thm. 10-23. Let G have p vertices of positive degree, q edges, k nontrivial components, and no 3-cycles. Then $\gamma(K_{2n} \underline{\otimes} G) = k + n(nq - p)$.

Cor. 10-24. Let G have no 3-cycles. Then $\gamma(K_2 \underline{\otimes} G) = \beta(G)$.

10-6. Covering Graphs and Graphical Products

This approach to graph imbedding was motivated by the desire to imbed the graph $C_3 \times C_3 \times C_3$ efficiently, so as to find the genus of the group $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ (see Section 7-2). Although it was not successful in that regard, it did bear other fruit ([A1] and [W18]).

A useful way to think of forming a cartesian product $H \times G$ is to replace each vertex of H with a copy of G, and then to join corresponding vertices in G in accordance with the edges of H. We have used this idea in some of our surgical constructions (see the proofs of Theorem 6-36 and Theorem 7-13, for example). Now we combine it with voltage-graph lifting, in a surprisingly general construction.

We are interested in product graphs $H \star G$ for which $V(H \star G) = V(H) \times V(G) = \{(u,v) | u \in V(H), v \in V(G)\}$. We specify four types of possible edges $\{(u_1, v_1), (u_2, v_2)\}$ for $H \star G$:

(i) $u_1 = u_2$ and $\{v_1, v_2\} \in E(G)$;
(ii) $v_1 = v_2$ and $\{u_1, u_2\} \in E(H)$;
(iii) $\{u_1, u_2\} \in E(H)$;
(iv) $\{u_1, u_2\} \in E(H)$ and $\{v_1, v_2\} \in E(G)$.

In Table 10-1 we define six graphical products (the first two were encountered in Section 2-4, the fourth in Section 10-5) $H \star G$, by specifying the types of edges each contains.

name	symbol	types of edge
cartesian	$H \times G$	(i), (ii)
lexicographic	$H[G]$	(i), (iii)
tensor	$H \otimes G$	(iv)
strong tensor	$H \underline{\otimes} G$	(i), (iv)
strong cartesian	$H \underline{\times} G$	(i), (ii), (iv)
augmented tensor	$H \otimes' G$	(ii), (iv)

Table 10-1.

We remark that $H \otimes' G = G \underline{\otimes} H$, so that the augmented tensor product operation is abstractly redundant. We retain it, however, as we want to maintain $G = G_\Delta(\Gamma)$ as a Cayley graph in the second factor. For each of the six products of Table 10-1, we construct $H \star G$, where $G = G_\Delta(\Gamma)$, as an $|\Gamma|$-fold covering graph of the voltage graph $H^* = (K, \Gamma, \phi)$ obtained by modifying H as shown in Table 10-2. Let e represent the identity of Γ. Note that putting, at each vertex of H, a loop for each element in Δ has the effect of replacing that vertex with a copy of G above (in the covering graph), as in our surgical constructions.

The covering graph $K \times_\phi \Gamma$ is $H \star G_\Delta(\Gamma)$, and if we commence with an imbedding of H into some surface, the lift *via* this construction is an imbedding of $H \star G_\Delta(\Gamma)$ into a second surface, uniquely determined by Theorem 10-9.

The motivating example started with $H = C_3 \times C_3$ and $\Delta = \{1\}$ for $\Gamma = \mathbb{Z}_3$. The imbedded voltage graph H^* is shown in Figure 10-11. All loops carry voltage 1, while all other edges carry voltage 0. The covering imbedding is of $(C_3 \times C_3) \times C_3$ on S_{10}. (See Problem 10-17.)

product	for each $u \in V(H)$	for each $\{u, v\} \in E(H)$
cartesian	$u \circlearrowright \quad \forall \delta \in \Delta$	$u \overset{e}{\text{—}} v$
lexicographic	$u \circlearrowright \quad \forall \delta \in \Delta$	$u \xrightarrow{g} v \quad \forall g \in \Gamma$
tensor	u	$u \xrightarrow{\delta} v \quad \forall \delta \in \Delta^*$
strong tensor	$u \circlearrowright \quad \forall \delta \in \Delta$	$u \xrightarrow{\delta} v \quad \forall \delta \in \Delta^*$
strong cartesian	$u \circlearrowright \quad \forall \delta \in \Delta$	$u \xrightarrow{\delta} v \quad \forall \delta \in \Delta^* \cup \{e\}$
augmented tensor	u	$u \xrightarrow{\delta} v \quad \forall \delta \in \Delta^* \cup \{e\}$

Table 10-2.

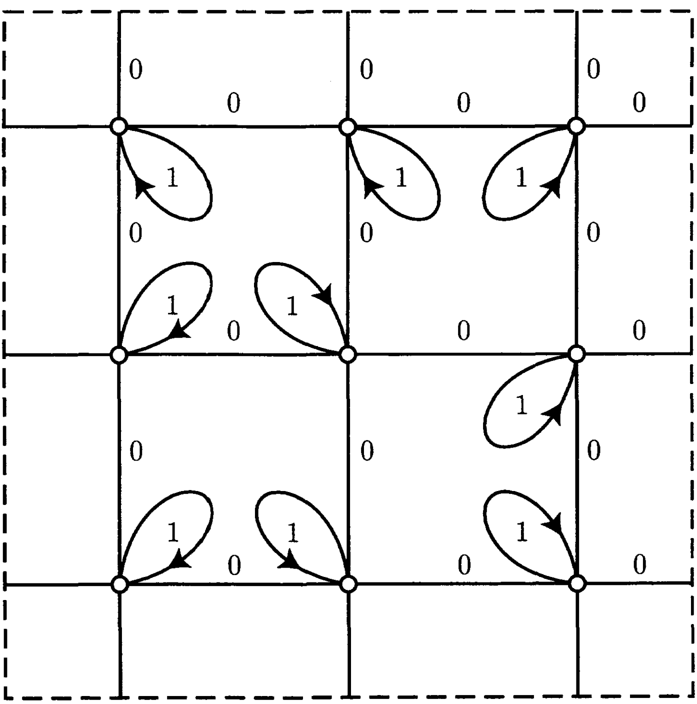


Figure 10-11.

Note that the construction of Figure 10-9 also fits this format, with $K_{n,n} = K_2[\bar{K}_n]$, using $\Delta = \phi$ for $\Gamma = \mathbb{Z}_n$.

We give four other applications here, one for each additional graphical product under study. For a thorough study of these ideas, including many other applications, see Abay-Asmerom [A1].

Thm. 10-25. If a connected graph H of order p and size q has an orientable quadrilateral imbedding, then

$$\gamma(H \otimes K_2) = \begin{cases} 2 + \frac{q}{2} - p, & \text{if } H \text{ is bipartite} \\ 1 + \frac{q}{2} - p, & \text{if } H \text{ is not bipartite} \end{cases}$$

Thm. 10-26. If $H = H_\Delta(\Gamma)$, with $\Delta = \{a, b, c\}$ for Γ satisfying at least the relations $a^4 = b^4 = c^4 = abc = e$, then:

(i) $\gamma(H \underline{\otimes} K_2) = 1 + \frac{3}{4}|\Gamma|$;
(ii) $1 + \frac{7}{4}|\Gamma| \leq \gamma(H \underline{\times} K_2) \leq 1 + \frac{9}{4}|\Gamma|$.

Thm. 10-27. If H is connected and bipartite, with an orientable quadrilateral imbedding, then $\gamma(H \otimes' K_2) = \beta(H)$.

10-7. Problems

10-1.) Show that the identity map $i : X \to X$ is always a covering projection, if X is locally connected (every neighborhood of each point contains a connected neighborhood of that point).
10-2.) Show that if $p : \tilde{X} \to X$ and $q : \tilde{Y} \to Y$ are covering projections, then so is $p \times q : \tilde{X} \times \tilde{Y} \to X \times Y$, where $(p \times q)(x, y) = (px, qy)$.
10-3.) Regard the torus as $S_1 = C \times C$, where C is the unit circle in $\mathbb{R}^2$. Show that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, $\mathbb{R} \times C$, and $S_1 = C \times C$ all cover S_1. Try to visualize ρ, in each case.
10-4.) Show that every orientable triangular imbedding for $K_{3(n)}$ has bichromatic dual.
10-5.) Find the voltage graph corresponding to Figure 9-11.
10-6.) Find the voltage graph corresponding to Figure 9-13.
10-7.) What imbedding (and of what graph) covers the voltage graph imbedding of Figure 10-12 ($\Gamma = \mathbb{Z}_8$)? Is there branching? Where, and of what multiplicity?

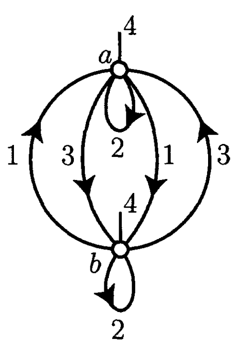


Figure 10-12.

10-8.) *Answer the questions of Problems 10-7, but now for Figure 10-13, with $\Gamma = \mathbb{Z}_{12}$. Then show that $\gamma(K_{6,6,3}) = 7$.

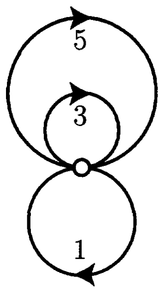


Figure 10-13.

10-9.) *Extend the construction of Problem 10-8, to show that

$$\gamma(K_{2n,2n,2n-2}) = 2(n-1)^2,$$

for $n \equiv 1, 2 \pmod 3$. (See also [SW2].)

10-10.) Show that if we begin with an arbitrary 2-cell imbedding of a Cayley color graph $C_\Delta(\Gamma)$ as a voltage graph imbedding, and use voltage graph Γ, the covering space consists of $|\Gamma|$ disjoint copies of the base configuration.

10-11.) Show that Γ_v is a subgroup of Γ.

10-12.) Construct an interesting voltage graph imbedding, and describe the covering space.

10-13.) Prove Theorem 7-22, using voltage graphs.

10-14.) Does voltage graph theory ever produce a covering surface of lower genus than that of the base space?

10-15.) Prove Theorem 10-20.

10-16.) Prove Theorem 10-21.

10-17.) Show that the covering graph for Figure 10-11 is $C_3 \times C_3 \times C_3$, and that the covering surface is S_{10}. Deduce that $\gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) \leq 10$.

10-18.) What are the covering graph and covering surface for Figure 10-14, using $\Delta = \{1\}$ for $\Gamma = \mathbb{Z}_3$?

10-19.) Prove Theorem 10-23.

10-20.) **Affirm or deny the conjecture of Example 2e.

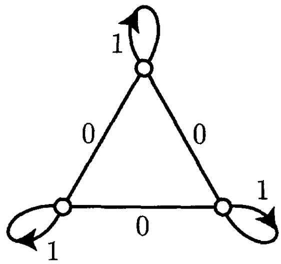


Figure 10-14.

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CHAPTER 11

NONORIENTABLE GRAPH IMBEDDINGS

We reiterate that our primary motivation for this entire work is to depict graphs – and in particular graphs of groups – on surfaces (locally 2-dimensional drawing boards), usually as efficiently as possible. In keeping with our desire to study structures that exist in three-dimensional space, we have concentrated on the orientable surfaces $S_k (k \geq 0)$ – to the almost total exclusion of the non-orientable surfaces $N_h (h \geq 1)$. But much of what we have done has an analog in the nonorientable context, and it is to these analogs that we now turn our attention.

11-1. General Theory

Def. 11-1. The *nonorientable genus*, $\tilde{\gamma}(G)$, of a graph G is the minimum h such that G can be imbedded in the non-orientable surface N_h. Such an imbedding is said to be *nonorientably minimal* for G.

For completeness, we extend Definition 11-1 so that $\tilde{\gamma}(G) = 0$, if G is planar. Since h can be regarded as the number of crosscaps attached to the sphere S_0 to form $N_h, \tilde{\gamma}(G)$ is also called the *crosscap number* of G. Thus a planar graph has crosscap number zero.

The genus and nonorientable genus are related by:

Thm. 11-2. $\tilde{\gamma}(G) \leq 2\gamma(G) + 1$.

PROOF. Imbed G in $S_{\gamma(G)}$ and then attach one crosscap to $S_{\gamma(G)}$, within one region of the imbedding. Then G is imbedded in this new surface, which is homeomorphic to $N_{2\gamma(G)+1}$. □

This upper bound is best possible, as $G = K_7$ shows.

Auslander, Brown, and Youngs [ABY1] have constructed graphs of arbitrarily large genus which all imbed in the projective plane N_1; thus there is no modification of Theorem 11-2 which reverses the inequality.

For a proof of the following analog to Theorem 5-14, see Massey [M3].

Thm. 11-3. Let G be a connected pseudograph, with a 2-cell imbedding in N_h, with the usual parameters p, q, and r. Then $p-q+r = 2-h$.

The number $2 - h$ is the *characteristic* of N_h ($\chi(N_h) = h$), as it is independent of G. Note that in computing the characteristic of a surface, one handle has the same weight as two crosscaps.

Unfortunately, the nonorientable analog of Theorem 6-11 is false. As is implicit in Theorem 8-9, K_7 does not imbed in N_2; but since K_7 *does* imbed in S_1, it has an imbedding in N_3 (add one crosscap in the interior of a region of the K_7 imbedding in S_1) which is *not* 2-cell. This imbedding must be nonorientably minimal for K_7.

Thus we cannot be assured that a nonorientably minimal imbedding of a connected graph satisfies the very useful euler equation of Theorem 11-3. Youngs [Y1] overcame this obstacle by two definitions and one theorem:

Def. 11-4. An imbedding of a connected graph G into a surface S (orientable or nonorientable) is *simplest* if there is no imbedding of G into any S' satisfying $\chi(S') > \chi(S)$.

Def. 11-5. An imbedding of a connected graph G into a surface S (orientable or nonorientable) is *maximal* if there is no imbedding of G into any S' having more regions. (That is, S allows the maximum value of r.)

Thm. 11-6. An imbedding of a connected graph G into a surface S (orientable or nonorientable) is simplest if and only if it is both 2-cell and maximal.

Thus the imbedding of K_7 into N_3 constructed above is neither simplest nor 2-cell (nor maximal.) In general, if a connected graph G has a nonorientable 2-cell imbedding which maximizes r, then that imbedding is simplest (by Theorem 11-6) and hence nonorientably minimal, so that $\tilde{\gamma}(G)$ has been determined. Then we have the following analogs to Corollary 6-14, Corollary 6-15, and Theorem 9-1:

Cor. 11-7. If G is connected, with $p \geq 3$, then $\tilde{\gamma}(G) \geq \frac{q}{3} - p + 2$; equality holds if and only if a nonorientable triangular imbedding can be found for G.

PROOF. Let G be imbedded in $N_{\tilde{\gamma}(G)}$. If the imbedding is 2-cell, then $p - q + r = 2 - \tilde{\gamma}(G)$ by Theorem 11-3. As in the orientable case,

$2q \geq 3r$ – with equality if and only if $r = r_3$. Thus $\tilde{\gamma}(G) = q-p-r+2 \geq \frac{q}{3} - p + 2$. If, on the other hand, the imbedding is not 2-cell, then it is not simplest, by Theorem 11-6. Thus G has a simplest (and hence 2-cell) imbedding on surface S, where $\chi(N_{\tilde{\gamma}(G)}) = 2 - \tilde{\gamma}(G) < \chi(S) = p - q + r \leq p - \frac{q}{3}$ (using $2q \geq 3r$ again); hence $\tilde{\gamma}(G) > \frac{q}{3} - p + 2$, in this case. □

Cor. 11-8. If G is connected, with $p \geq 3$, and has no triangles, then $\tilde{\gamma}(G) \geq \frac{q}{2} - p + 2$; equality holds if and only if a nonorientable quadrilateral imbedding can be found for G.

(The proof is entirely analogous to that of Corollary 11-7.)

Cor. 11-9. Let K_n be nonorientably, minimally, 2-cell imbedded in N_h. Then $\tilde{\gamma}(K_n) = h = \frac{(n-3)(n-4)}{6} + \frac{1}{3}\sum_{i\geq 4}(i-3)r_i$.

(The proof is analogous to that of Theorem 9-1.)

The nonorientable analogs to Theorem 6-18 and Corollary 6-19 are also false, as $G = 2K_7$ shows: since, by Corollary 6-19, $\gamma(2K_7) = 2\gamma(K_7) = 2$, $\tilde{\gamma}(2K_7) \leq 5$, by Theorem 11-2. Similarly, for Theorem 6-18, consider two disjoint copies of K_7 joined by an edge. But, as shown by Franklin [F4], $\tilde{\gamma}(K_7) = 3$, so that $\tilde{\gamma}(2K_7) \neq 2\tilde{\gamma}(K_7)$. What is true follows in two definitions and one theorem, due to Stahl and Beineke [SB1]:

Def. 11-10. The *manifold number* of a graph G is $\mu(G) = \max\{2 - 2\gamma(G), 2 - \tilde{\gamma}(G)\}$.

Def. 11-11. A graph G is *orientably simple* if $\mu(G) \neq 2 - \tilde{\gamma}(G)$; that is, if $\tilde{\gamma}(G) > 2\gamma(G)$.

Thm. 11-12. Let G be a graph with blocks (or components) $G_1, G_2, \cdots, G_k$. If G is orientably simple, then $\tilde{\gamma}(G) = 1 - k + \sum_{i=1}^{k} \tilde{\gamma}(G_i)$; else $\tilde{\gamma}(G) = 2k - \sum_{i=1}^{k} \mu(G_i)$.

11-2. Nonorientable Covering Spaces

Recall from Example 3 of Section 10-1 that the sphere S_0 is a 2-fold covering space of the projective plane. In general (see Stahl [S12], for example), there is a 2-fold covering projection $\rho : S_k \to N_{k+1}$, for every nonnegative integer k. Thus every nonorientable surface has an orientable covering surface. Trivially, each nonorientable surface has at least one nonorientable covering surface, namely itself (see Problem 10-1.) In Section 11-4 we shall see an example of a nonorientable surface (N_3) with infinitely many nonorientable covering surfaces ($N_{n+2}, n \geq 7$).

In contrast, if the base space is an orientable surface, then every covering surface must be orientable also. Thus in Chapter 10 – where each base space is orientable – each covering space is orientable and hence unambiguously determined by its characteristic. In this chapter, however, since the base space will always be nonorientable, if the covering surface has even characteristic, then its orientability character needs to be ascertained to specify the surface uniquely.

In the next section we shall see how to do this, in the context of nonorientable graph imbeddings.

11-3. Nonorientable Voltage Graph Imbeddings

To extend the voltage graph theory to nonorientable imbeddings, we augment the rotation scheme P of section 10-2 to a pair (P, λ) called an *imbedding scheme*: $\lambda : K^* \to \mathbb{Z}_2$ gives a voltage graph $(K, \mathbb{Z}_2, \lambda)$. The region boundaries are computed almost as in the orientable case (see Section 6-6), except that sometimes $p_v^{-1}(v, u)$ is used instead of $p_v(v, u)$; see [S13] and [SW2] for details, and Example 4 of Section 11-4. Now let (K, Γ, ϕ) be a voltage graph with imbedding scheme (P, λ) and let $\tilde{P}$ be the lift of P to $K \times_\phi \Gamma$; define $\tilde{\lambda} : (K \times_\phi \Gamma)^* \to \mathbb{Z}_2$ by $\tilde{\lambda}(\tilde{e}) = \lambda(e)$, for each lift $\tilde{e}$ of $e \in E(K)$. Define $(\tilde{P}, \tilde{\lambda})$ to be the lift of (P, λ) to $K \times_\phi \Gamma$. Then the conclusions of Theorems 10-9 and 10-18 can be shown to follow verbatim.

Thm. 11-13. Let (K, Γ, ϕ) be a voltage graph with imbedding scheme (P, λ) and $(\tilde{P}, \tilde{\lambda})$ the lift of (P, λ) to $K \times_\phi \Gamma$. Let (P, λ) and $(\tilde{P}, \tilde{\lambda})$ determine 2-cell imbeddings of K and $K \times_\phi \Gamma$ on the surfaces S and $\tilde{S}$ respectively, where $\tilde{S}$ is possibly disconnected. Then there exists a (possibly branched) covering projection $\rho : \tilde{S} \to S$ such that:

(i) $\rho^{-1}(K) = K \times_\phi \Gamma$;
(ii) if R is a region of the imbedding of K which is a k-gon, then $\rho^{-1}(R)$ has $\frac{|\Gamma|}{|R|_\phi}$ components, each of which is a $k|R|_\phi$-gon region of the covering imbedding of $K \times_\phi \Gamma$.
(iii) If $|R|_\phi = n > 1$, then R contains a branch point of multiplicity n. If $n = 1$, then R contains no branch point.

Cor. 11-14. The projection $\rho : \tilde{S} \to S$ is a covering projection (i.e. there is no branching,) if and only if the imbedding of $K \times_\phi \Gamma$ satisfies the KVL.

Of course, the imbedding scheme (P, λ) is used in practice only when S is nonorientable; if S is orientable, the rotation scheme P alone suffices, and $\tilde{S}$ is necessarily orientable also, as observed above. If S is nonorientable, we must determine the orientability character of $\tilde{S}$; to this end we give (see [SW2]):

Def. 11-15. For a voltage graph (K, Γ, η), a closed walk c in K is said to be *η-trivial* if $\eta(c) = e$ in Γ.

Thm. 11-16. Under the hypothesis of Theorem 11-13, the derived surface $\tilde{S}$ is orientable if and only if every ϕ-trivial closed walk in K is also λ-trivial.

We mention that Garman [G1] has further extended the theory of voltage graphs, as outlined in the previous and present chapters, to pseudosurface imbeddings. In particular, Theorems 10-8, 10-9, 10-18, 11-13, and 11-16, and Corollary 11-14 also apply for S a pseudosurface; in this case, $\tilde{S}$ is necessarily also a pseudosurface (or a generalized pseudosurface; see Definition 5-28.) Moreover, Rahn [R1] has extended voltage graph theory to hypergraphs.

11-4. Examples

Example 1: Let $m \equiv 2 \pmod 4$. Figure 11-1 presents a projective plane imbedding of a pseudograph K with one vertex and $\frac{m}{2}$ loops: $\lambda(e) = 1$ for each loop e. Let $\Gamma = \mathbb{Z}_n$ (n even), and set $\phi(e) = 1$ for each loop also, with direction as indicated. Then $K \times_\phi \mathbb{Z}_n$ is a graph with vertex set $\{(v, i) | 0 \le i \le n-1\}$ in which (v, i) and $(v, i+1)$ are joined by $\frac{m}{2}$ edges for each i. For each region R of K imbedded below, $|R|_\phi = \frac{n}{2}$, so that the regions of $K \times_\phi \mathbb{Z}_n$ imbedded above are all n-gons; in fact each is a hamiltonian cycle. There are $\frac{m}{2}2 = m$ such regions in all (above.) Thus if we place a new vertex in the interior of each such region, join it by non-intersecting edges to all the vertices on its boundary, and then delete all the original edges of $K \times_\phi \mathbb{Z}_n$, a quadrilateral imbedding of $K_{m,n}$ results. It is clear that every ϕ-trivial closed walk in K is also λ-trivial, so that the covering imbedding is orientable by Theorem 11-16; it is into S_k, where $k = \frac{(m-2)(n-2)}{4}$. This gives a partial proof of Theorem 6-37; the approach appears in [SW2], and originated with Gross [G9].

Example 2. In Figure 11-2 we take $\Delta = \{a, b, a+b\}$ for Γ abelian ($|\Gamma| = n \ge 7$) and K as a bouquet of three circles imbedded in N_3. The single vertex has been given an orientation as indicated. Then those edges which are coherently oriented, as induced by the vertex

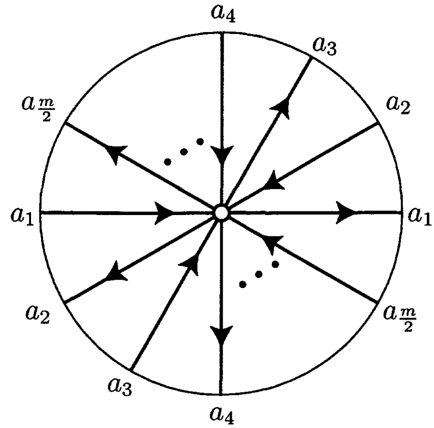

Figure 11-1.

orientation, are assigned $\lambda = 0$ (there are none); all other edges – in this case the three edges bounding the hexagon – are assigned $\lambda = 1$. Then the closed walk $a + b - (a + b)$ is ϕ-trivial but not λ-trivial; thus the covering surface S is nonorientable, by Theorem 11-16. Moreover, $\chi(\tilde{S}) = n\chi(S) = -n$, so that $\tilde{S} = N_{n+2}$. The special case $n = 7$ produces a self-dual imbedding of K_7 on N_9.

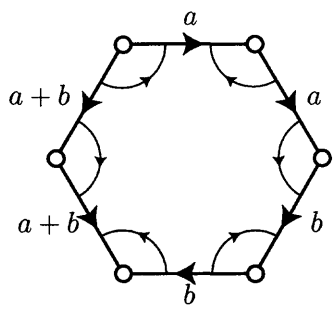

Figure 11-2.

Example 3. Now modify Figure 11-2 slightly, to give Figure 11-3.

The covering imbedding, still on N_{n+2}, is now triangular and of the 12-regular 2-fold $G_\Delta(\Gamma)$. Finally, change ϕ (i.e. relabel the edges of K on N_3; see Problem 11-4) to give 1-fold triangulations of $G_\Delta(\Gamma)$ of characteristic $-2n$; for example, consider $G_\Delta(\mathbb{Z}_{2n})$, where $\Delta = \{1, 2, 3, n-3, n-2, n-1\}$ and still $n \geq 7$. The case $n = 7$ gives $K_{7(2)}$ on N_{16}.

Example 4. The Petersen graph is one of the most famous in graph theory; see [CW2]. We introduced it in Problem 3-8 as the odd graph O_3; encountered it in Problem 4-10 as a connected vertex transitive graph which is not a Cayley graph; and found it imbedded

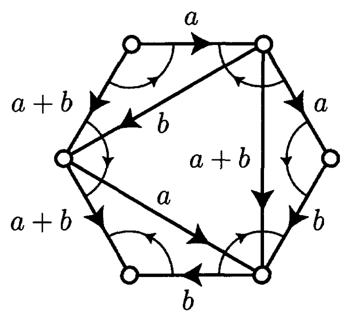

Figure 11-3.

on the torus in Figure 8-8. It then follows from Problem 11-12 and Theorem 6-6 that the orientable genus of the Petersen graph is one. The usual depiction of the Petersen graph is as in Figure 11-4, where we have labelled it as O_3. (For brevity, we set $\{a, b\} = ab$.)

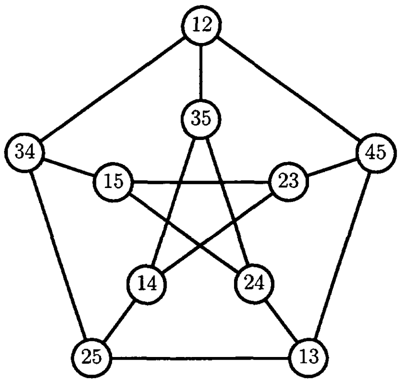

Figure 11-4.

We know that $A(O_3) = S_5$, from Problem 3-8; and 10 of the 120 graph automorphisms, in a pleasing dihedral symmetry, are apparent in Figure 11-4. But the five inappropriate edge intersections are not pleasing, in the context of topological graph theory. In Figure 8-8, however, whereas we have avoided the edge intersections, we have sacrificed the symmetry ($r_5 = 3, r_6 = 1, r_9 = 1$). Consider now Figure 11-5, which depicts the Petersen graph imbedded symmetrically in the projective plane N_1. In fact, it is routine to check that the permutations (123) and (14)(25) both induce map automorphisms, in the sense of Chapter 16 (graph automorphisms preserving oriented region boundaries), except that here we do not require that orientation be preserved. These two permutations generate A_5 (see the proof of Corollary 7-24). Now choose an arbitrary odd permutation in S_5, say (12). This graph automorphism does *not* preserve region boundaries. Thus the map automorphism group is precisely A_5. This is not surprising, as the dodecahedron is a double cover of Figure 11-5, by antipodal identification, and A_5 is the automorphism group of the dodecahedron. Moreover, since we already know that the Petersen graph is not planar, its nonorientable genus is one also. (But the imbedding of Figure 11-5 is most efficient, since it is of highest euler characteristic for the Petersen graph.) We also note that Figure 11-5 is the dual of Figure 8-4 (Figure 8-4 is double covered, by antipodal identification, by the icosahedron, dual to the dodecahedron, as we saw in Figure 5-8); here we have a map of six countries, each bordering on all the others, reaffirming that $\chi(N_1) = 6$.

This is our first instance of obtaining a desired imbedding not as a lift of a simpler imbedding, but as a projection of a more complicated (in the sense of increased p, q, and r) imbedding. If we try to find Figure 11-5 by a voltage graph construction, we encounter two obstacles: (1) since the Petersen graph is not a Cayley graph, no index one voltage graph lifting is possible, and (2) our desired imbedding has $p = 10$, $q = 15$, and $r = r_5 = 6$; but these three numbers have no nontrivial common factor to mod out by. Surgery does not seem relevant to construct our projective planar imbedding either. But here is a rotation scheme that will suffice once we assign $\lambda = 1$ to each edge (a, b, c, d, e) passing through the crosscap and $\lambda = 0$ to the remaining ten edges:

$p_{12} : (35, 45, 34)$	$p_{24} : (15, 35, 13)$
$p_{13} : (45, 24, 25)$	$p_{25} : (34, 13, 14)$
$p_{14} : (23, 25, 35)$	$p_{34} : (15, 12, 25)$
$p_{15} : (23, 24, 34)$	$p_{35} : (24, 14, 12)$
$p_{23} : (14, 15, 45)$	$p_{45} : (12, 23, 13)$.

We modify the algorithm of Theorem 6-50 as follows: if our region boundary commences with walk $w = v_1, v_2, \cdots, v_k$, then $v_{k+1} = p_{v_k}(v_{k-1})$ if $\lambda(w) = 0$, but $v_{k+1} = p_{v_k}^{-1}(v_{k-1})$ if $\lambda(w) = 1$. The central region boundary is calculated as in the orientable case:

$$12 - 34 - 25 - 13 - 45 - 12.$$

The shaded region boundary is calculated as:

$$34 - 12 - 35 - 24 - 15 - 34,$$

using $p_{24}^{-1}(35) = 15$, but $p_{15}(24) = 34$. Intuitively, each time we pass through the crosscap, the orientation is reversed. The other four region boundaries are similar to the preceding one, as is retrospectively evident from the $\mathbb{Z}_5$ rotational symmetry.

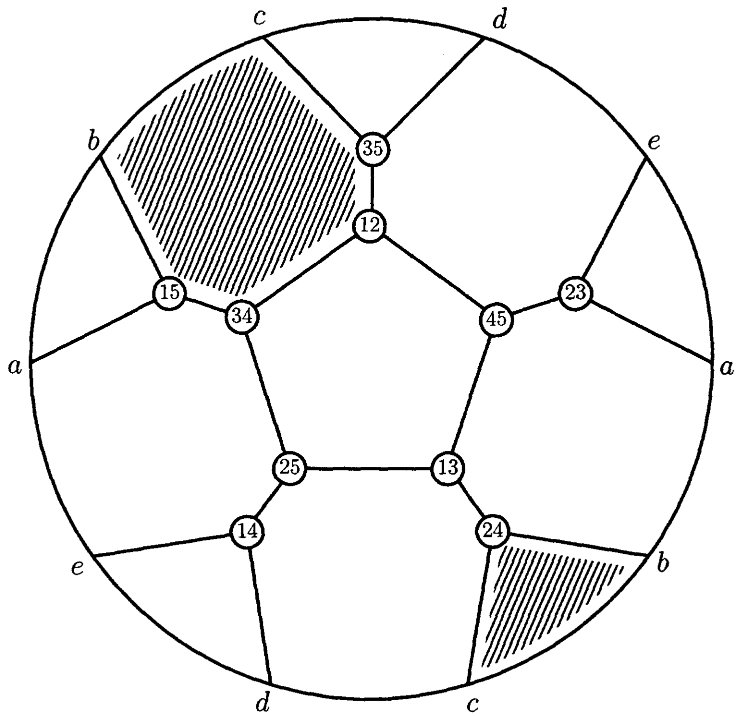

Figure 11-5.

11-5. The Heawood Map-coloring Theorem, Nonorientable Version

If we apply Corollary 11-9 to the graphs $G = K_m$, we obtain:

Lemma 11-17. $\tilde{\gamma}(K_m) \geq \left\lceil \frac{(m-3)(m-4)}{6} \right\rceil$, for $m \geq 3$.

We have noted that equality does not hold for $m = 7$ and that $\tilde{\gamma}(K_7) = 3$. However, equality does hold in every other case:

Thm. 11-18. $\tilde{\gamma}(K_m)] \leq \left\lceil \frac{(m-3)(m-4)}{6} \right\rceil$, for $m \neq 7$.

This is, of course, established by finding an imbedding of K_m on $N_h, h = \left\lceil \frac{(m-3)(m-4)}{6} \right\rceil$. The first proof was by Ringel [R10], without the benefit of current graphs. Later proofs do employ current graph theory (in particular, the theory of cascades), and again split naturally into the residue cases of m modulo 12 (see [R16], [LY1], [J6].) Lemma

11-17 and Theorem 11-18 combine to form the nonorientable version of the complete graph theorem:

Thm. 11-19. $\tilde{\gamma}(K_m) = \left\lceil \frac{(m-3)(m-4)}{6} \right\rceil$, for $m \geq 3$ and $m \neq 7$; $\tilde{\gamma}(K_7) = 3$.

Now we let $M_n = N_h$ in Theorem 8-13, recalling that $f(n) = \frac{7+\sqrt{49-24n}}{2}$; here $n = 2 - h$:

Lemma 11-20. $\chi(N_h) \leq \lfloor f(n) \rfloor = \left\lfloor \frac{7+\sqrt{1+24h}}{2} \right\rfloor$.

To complete the proof of the Heawood Map-coloring Theorem, nonorientable version, we need the following:

Lemma 11-21. $\chi(N_h) \geq \lfloor f(2-h) \rfloor = \left\lfloor \frac{7+\sqrt{1+24h}}{2} \right\rfloor$, $h \neq 2$.

PROOF. Consider $N_h, h \neq 2$. Define $m = \lfloor f(2-h) \rfloor$, and now consider also $N_{\tilde{\gamma}(K_m)}$. Note that $\tilde{\gamma}(K_m) \leq h$, so that $\chi(N_{\tilde{\gamma}(K_m)}) \leq \chi(N_h)$. Now K_m imbeds in $N_{\tilde{\gamma}(K_m)}$. Clearly $\chi(N_{\tilde{\gamma}(K_m)}) \geq m = \lfloor f(2-h) \rfloor$, so that $\chi(N_h) \geq \lfloor f(2-h) \rfloor$. □

Since Franklin [F4] showed that $\chi(N_2) = 6$, we can now combine Lemmas 11-20 and 11-21 to restate Theorem 8-9.

Thm. 11-22. $\chi(N_h) = \left\lfloor \frac{7+\sqrt{1+24h}}{2} \right\rfloor$, for $h \neq 2$; $\chi(N_2) = 6$.

11-6. Other Results

We give a few of the analogs to the orientable results of Chapter 6. For most of the other nonorientable genus results known by 1978, see Table 2 of Stahl [S14].

Thm. 11-23. (Ringel [R14])

$$\tilde{\gamma}(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil ; m, n \geq 2.$$

Thm. 11-24. (Jungerman [J8])
$\tilde{\gamma}(Q_n) = 2 + 2^{n-2}(n-4), n \geq 2$, except that $\tilde{\gamma}(Q_4) = 3$ and $\tilde{\gamma}(Q_5) = 11$.

Thm. 11-25. (Jungerman [J10])
$\tilde{\gamma}(K_{4(n)}) = 2(n-1)^2$, if $n \geq 1$, except that $\tilde{\gamma}(K_{4(2)}) = 3$.

Thm. 11-26. (Stahl and White [SW2])
For $n \geq 3$, $\tilde{\gamma}(K_{n,n,n-2}) = (n-2)^2$.

Thm. 11-27. (Stahl and White [SW2])
For $n \geq 4$ and even, $\tilde{\gamma}(K_{n,n,n-4}) = (n-2)(n-3)$.

Bouchet [B16] has also used his "generative m-valuations" (see Section 6-5) to study $\tilde{\gamma}(K_{n(m)})$. He considered the residue classes of m and n (mod 6) and constructed triangular imbeddings for 18 of these 36 cases, thus determining $\tilde{\gamma}(K_{n(m)})$ for those 18 cases.

We now turn our attention to the maximum nonorientable genus parameter.

Def. 11-28. The *maximum nonorientable genus*, $\tilde{\gamma}_M(G)$, of a connected graph G is the maximum h for which G has a 2-cell imbedding in N_h.

In contrast to the orientable case, the value of this parameter is readily calculated for each connected graph G (see Ringel [R17] and Stahl [S13]); recall that the Betti number $\beta(G) = q - p + 1$.

Thm. 11-29. $\tilde{\gamma}_M(G) = \beta(G)$.

This means that every connected graph has a nonorientable 2-cell imbedding with $r = 1$.

The following theorem, also due to Stahl, gives an analog to Duke's Theorem (6-21) and to Corollary 6-22.

Thm. 11-30. A connected graph G has a 2-cell imbedding in N_h if and only if $\tilde{\gamma}(G) \leq h \leq \beta(G)$.

We give a nonorientable analog to the famous Kuratowski Theorem (6-6); this result is due to Glover, Huneke, and Wang [GHW1] and to Archdeacon [A9]. See also Bodendiek and Wagner [BW2], who present a list of 12 graphs.

Thm. 11-31. A graph G imbeds in the projective plane N_1 if and only if, for each graph H in a prescribed list of 103 graphs, G contains no subgraph homeomorphic with H.

It had long been conjectured that there is a finite number of graphs obstructing imbedding into each surface, whether orientable or nonorientable (by Theorem 6-6 there are two for S_0; by Theorem 11-31 there are 103 for N_1.) Recently Robertson and Seymour [RS1] have affirmed this conjecture:

Thm. 11-32. For each closed 2-manifold M, there is a finite set S_M of graphs such that a graph G imbeds in M if and only if G contains no subgraph homeomorphic from at least one member of S_M.

Finally, we mention that Pisanski [P7] has expanded the surgery techniques he used to generalize work of [W6] (see Theorem 7-13, for example) in the orientable case, to apply to the nonorientable case as well. Moreover, Pisanski and White [PW1] have calculated the nonorientable genus, for many classes of abelian and hamiltonian groups.

11-7. Problems

11-1.) Prove Corollary 11-8.
11-2.) Prove Corollary 11-9.
11-3.) Describe the 2-fold $\rho : S_k \to N_{k+1}$ of Section 11-2, for $k \geq 1$.
11-4.) Relabel the edges of K in Figure 11-3, to yield the $G_\Delta(\mathbb{Z}_{2n})$ imbedding suggested in Example 3.
11-5.) What imbedding, and of what graph, covers the voltage graph imbedding if Figure 11-6 ($\Gamma = \mathbb{Z}_9$)? Is there branching? Where, and of what multiplicity?

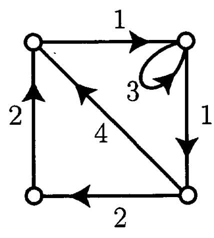

Figure 11-6.

11-6.) Answer the questions of Problem 11-5, but now for $\Gamma = \mathbb{Z}_{10}$.
11-7.) **Try to prove Theorem 11-19 by the use of nonorientable voltage graphs.
11-8.) Show that $\tilde{\gamma}(Q_n \times K_{4,4}) = 2 + n2^{n+1}, n \geq 1$.

11-9.) Show that the Z-metacyclic group

$$\Gamma = \mathbb{Z}_n \times D_m = \langle a, b | a^m = b^{2n} = abab^{-1} = e \rangle,$$

for n odd and $n > 1$, is toroidal. (Hint: use Theorem 7-3 and the voltage graph imbedding that results when the right side arrow is reversed in Figure 10-4.)

11-10.) What other toroidal groups can you find, using the hint of Problem 11-9?

11-11.) Make a small change to Figure 7-3 to imbed $K_{4,4}$ on N_2. Hence verify the case $m = n = 4$ of Theorem 11-23.

11-12.) Find a homeomorph of $K_{3,3}$ in the Petersen graph Π. Is there a homeomorph of K_5 in Π? Find $\tilde{\gamma}(\Pi)$.

11-13.) For G a connected graph having crossing number $v(G)$, show

(i) $\gamma(G) \leq v(G)$

(ii) $\tilde{\gamma}(G) \leq v(G)$.

Are these bounds sharp? Are they useful?

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CHAPTER 12

BLOCK DESIGNS

Block designs are combinatorial structures of interest in their own right, with applications to experimental design and to scheduling problems. Heffter [H6] was the first to observe that certain imbeddings of complete graphs determine BIBDs with $k = 3$ and $\lambda = 2$ (and sometimes $\lambda = 1$.) Alpert [A2] established a one-to-one correspondence between BIBDs with $k = 3$ and $\lambda = 2$ and triangular imbeddings for complete graphs. In [W11] this correspondence is extended to PBIBDs on two association classes with $k = 3$, $\lambda_1 = 0$ and $\lambda_2 = 2$ (and sometimes $\lambda_2 = 1$) and triangular imbeddings for strongly regular graphs. The group divisible designs of Hanani [H2] are used to construct triangular imbeddings (in generalized pseudosurfaces) for the graphs $K_{n(m)}$, in each case permitted by the euler identity. Conversely, triangular imbeddings of $K_{n(m)}$ are constructed (by other means) which lead to new group divisible designs. A process, using the strong tensor product operation for graphs, is developed for "doubling" a given PBIBD of an appropriate form.

In this chapter, the term "surface" includes nonorientable as well as orientable closed 2-manifolds.

12-1. Balanced Incomplete Block Designs

Def. 12-1. A (v, b, r, k, λ)-*balanced incomplete block design* (BIBD) is a set of v *objects* and a collection of b subsets of the object set, each subset being called a *block*, satisfying:

(i) each object appears in exactly r blocks;
(ii) each block contains exactly k $(k < v)$ objects;
(iii) each pair of distinct objects appear together in exactly λ blocks.

If $k = v$, the design would be *complete* ; complete designs are trivial to construct, but have little application. Hence for $k < v$, the design is "incomplete." The "balance" comes from the three uniformity conditions on r, k, and λ respectively.

A trivial example of a BIBD, for each $v > 1$ and $0 < k < v$, is obtained by taking the blocks to be all the k-subsets of the v-set; then $b = \binom{v}{k}$, $r = \binom{v-1}{k-1}$, and $\lambda = \binom{v-2}{k-2}$.

In general, elementary counting arguments (see Problems 12-1 and 12-2) establish the following well-known result:

Thm. 12-2. If a (v, b, r, k, λ)-BIBD exists, then:

(i) $vr = bk$;
(ii) $\lambda(v-1) = r(k-1)$.

These necessary conditions have also been shown to be sufficient, for $k = 3, 4, 5$ (and for some of the cases $k = 6, 7$) by Hanani [H2], and for fixed k and λ, with v large enough, by Wilson [W28]. That they are not sufficient in general is demonstrated by $(43, 43, 7, 7, 1)$; if a BIBD with these parameters existed, so would a projective plane of order 6; but none such exists.

12-2. BIBDs and Graph Imbeddings

What Heffter and Alpert observed is that a triangular imbedding of K_n in an appropriate surface (this is possible exactly when $n \equiv 0, 1 \pmod 3$, $n > 1$; see Sections 9-1 and 11-5), with the regions determining the blocks in the natural fashion, serves as an $(n, \frac{n(n-1)}{3}, n-1, 3, 2)$-BIBD, since: (i) every vertex is adjacent to exactly $n-1$ other vertices and hence is in exactly $n-1$ regions; (ii) each region contains exactly three vertices, by assumption; and (iii) each pair of distinct vertices constitutes an edge (since the graph is complete) and hence belongs to exactly two blocks (the two regions containing that edge in their boundary.)

Conversely, a BIBD on v objects with $k = 3$ and $\lambda = 2$ (a *2-fold triple system*) determines a triangular imbedding of K_v in a generalized pseudosurface, as follows. Each block becomes a 3-sided 2-cell region, with vertices labelled by the objects of the block. Since $\lambda = 2$, each pair of vertices appears exactly twice - so that a 2-manifold (possibly with several components) results from the standard identification process of combinatorial topology. Then identify identically labeled vertices, to form a generalized pseudosurface triangular imbedding for K_v. We summarize, in:

Thm. 12-3. The 2-fold triple systems on v objects are in one-to-one correspondence with triangular imbeddings of K_v in generalized pseudosurfaces.

If the triangular imbedding of K_v has bichromatic dual, then the 2-fold triple system splits naturally into two 1-fold triple systems (*Steiner triple systems*, having $k = 3$ and $\lambda = 1$):

Thm. 12-4. Steiner triple systems on v objects are in two-to-one correspondence with triangular imbeddings of K_v in generalized pseudosurfaces having bichromatic dual.

Letting $k = 3$ and $\lambda = 1$ or 2 in Theorem 12-2, and invoking the work of Hanani (or others) mentioned following that Theorem, we obtain:

Thm. 12-5. (i) Steiner triple systems on v objects exist if and only if $v \equiv 1, 3 \pmod 6$;
(ii) 2-fold triple systems on v objects exist if and only if $v \equiv 0, 1 \pmod 3$.

For independent verification of (ii) above, we follow Alpert in observing that triangular imbeddings of K_v (see Sections 9-3 and 11-5) give 2-fold triple systems on v objects for all $v \equiv 0, 1 \pmod 3$, $v \neq 1$. Moreover, in [GRW1] it is observed that the orientable genus imbeddings for $K_n, n \equiv 3 \pmod{12}$, all have bichromatic dual; thus Steiner triple systems are independently produced for these values of n.

Def. 12-6. A *Mendelsohn triple system* on v objects is a collection of cyclic ordered triples (a, b, c) from the v objects such that each ordered pair of distinct objects appears in exactly one triple.

Thm. 12-7. Mendelsohn triple systems on v objects are in one-to-one correspondence with triangular imbeddings of K_v in orientable generalized pseudosurfaces.

Mendelsohn [M5] showed the following.

Thm. 12-8. A Mendelsohn triple system on v objects exists if and only if $v \equiv 0, 1 \pmod 3$, $v \neq 1$ or 6.

Cor. 12-9. Triangular imbeddings of K_v in orientable generalized pseudosurfaces exist if and only if $v \equiv 0, 1 \pmod 3$, $v \neq 1$ or 6.

Mendelsohn triple systems are 2-fold triple systems, but not conversely. Compare Theorems 12-3, 12-5 (ii), 12-7, and 12-8. The discrepancy for $v = 6$ is because K_6 has a nonorientable triangular imbedding (on N_1), but not an orientable one. Thus K_6 does *not* imbed on the pseudosurface $S(0; 1(2))$, even though such an imbedding would be compatible with the euler identity; see also Theorem 8-33.

12-3. Examples

Example 1: Refer to Example 1a of Section 10-3; the quadrilateral imbedding of K_5 in S_1 obtained there yields a $(5, 5, 4, 4, 3)$-BIBD. This is atypical in that $k > 3$ and $\lambda > 2$, but it does indicate that the scope of the connection between BIBDs and graph imbeddings is even wider than indicated in Section 12.2. This imbedding is a geometric realization of the "all 4-subsets of a 5-set" abstract design.

Example 2: Refer to Example 2a of Section 10-3; the triangular imbedding of K_7 in S_1 obtained there yields a $(7, 14, 6, 3, 2)$-BIBD and, since the dual is bichromatic, two $(7, 7, 3, 3, 1)$-BIBDs. One of these latter is given abstractly in Table 12-1; the blocks may be read off from the seven regions covering the unshaded region in Figure 10-6 (note that they are also half of the "blocks" listed in Section 9-2.) The blocks may also be regarded as the lines of the *Fano Plane* (the finite projective plane of order two; see Chapter 15.) The design could be employed to schedule firemen (say) in a weekly schedule (three men per day, etc.) The extension of the voltage graph of Figure 10-6 to K_{12+7} gives a $(12s + 7, (12s + 7)(4s + 2), 12s + 6, 3, 2)$-BIBD, which is not of bichromatic dual for $s > 0$; hence we obtain no additional Steiner triple systems from this figure.

013
124
235
346
450
561
602

Table 12-1.

Example 3: Refer to Example 3 of Section 11-4; the triangulation of 2-fold K_7 in N_9 obtained there gives a $(7, 28, 12, 3, 4)$-BIBD.

Example 4: Refer to Example 4 of Section 10-3; the triangulation of 3-fold K_7 in S_8 obtained there gives one $(7, 42, 18, 3, 6)$-BIBD and, since the dual is bichromatic, two $(7, 21, 9, 3, 3)$-BIBDs.

12-4. Strongly Regular Graphs

For values of v, b, r, k, λ not meeting the conditions of Theorem 12-2 (and perhaps even for those that do), it is natural to attempt a construction of a related design. For this reason, partially balanced incomplete block designs were introduced by Bose and Nair [BN1]. The partial balance occurs in that there are $\ell > 1$ lambda values, one for each "association class;" we restrict our attention primarily to the case $\ell = 2$.

As the association classes (for the case $\ell = 2$) are determined by adjacencies within a "strongly regular graph" – to ensure an appropriate balance within each class – we first define this latter concept. Let x and y be distinct vertices in a graph G, either non-adjacent $(h = 1)$ or adjacent $(h = 2)$. Let $p_{ij}^h(x, y)$ be the number of vertices which are non-adjacent to both x and $y(i = j = 1)$, adjacent to x but not to $y(i = 2, j = 1)$, adjacent to y but not to $x(i = 1, j = 2)$, or adjacent to both x and $y = (i = j = 2)$.

Def. 12-10. If a graph G is regular of degree n_2 and is of order v, yet $G \neq K_v$ or $\bar{K}_v$, and if $p_{ij}^h(x, y)$ is independent of the choice of x and y, for $h, i, j = 1, 2$, then G is said to be a *strongly regular graph.*

It is well known that the eight conditions involving x and y can be replaced by two of them (see Problem 12-4):

Thm. 12-11. If G is a regular graph which is neither complete nor empty, then G is strongly regular if and only if $p_{22}^h(x, y)$ is independent of x and y, for $h = 1, 2$.

For strongly regular graphs, it is convenient to write p_{ij}^h for $p_{ij}^h(x, y)$, as the choice of x and y is immaterial (except that they must be adjacent in G, for $h = 2$, and adjacent in $\bar{G}$, for $h = 1$.) Two vertices (objects) of a strongly regular graph are said to be *first associates* if they are non-adjacent and *second associates* if they are adjacent. Thus each vertex has exactly n_i ith associates $(i = 1, 2)$, and $n_1 + n_2 = v - 1$.

As a major class of examples of strongly regular graphs, we give:

Thm. 12-12. The regular complete n-partite graphs $G = K_{n(m)}$ are all strongly regular, for $m, n \geq 2$.

PROOF. Clearly G is regular, of degree $n_2 = m(n-1)$. Since $m > 1$, G is not complete; since $n > 1$, G is not empty. Finally, we observe that $p_{22}^1 = m(n-1)$ and $p_{22}^2 = m(n-2)$. □

For the examples of Section 12-8, the following observation will be crucial. If we start with $G_1 = K_r$ and the strongly regular $G_2 = K_{n(m)}$, then the strong tensor product $G_1 \otimes G_2$ is strongly regular also, by Theorems 10-20 and 12-12.

Other standard examples of strongly regular graphs include $G = L(K_n), n \geq 4$, and $G = L(K_{2(m)}), m \geq 2$, where $L(H)$ denotes the line graph of graph H. (We mention in passing that $L(K_{2(m)}) = K_m \times K_m$: see Problem 2-13.) The Petersen graph is strongly regular, with $p_{22}^1 = 1$ and $p_{22}^2 = 0$. (The latter value corresponds to girth greater than three.)

12-5. Partially Balanced Incomplete Block Designs

Def. 12-13. A $(v, b, r, k; \lambda_1, \lambda_2)$-*partially balanced incomplete block design* (PBIBD) is a set of v *objects*, pairwise associated into two association classes (as determined by a strongly regular graph G of order v) and a collection of b subsets of the object set, each subset being called a *block*, satisfying:

(i) each object appears in exactly r blocks;
(ii) each block contains exactly $k (k < v)$ objects;
(iii) each pair of ith associates appear together in exactly λ_i blocks $(i = 1, 2)$.

Again, the requirement $k < v$ corresponds to the incompleteness of the design; the requirement $G \neq K_v$ or $\bar{K}_v$ ensures that $n_1 n_2 > 0$, so that the PBIBD does not collapse to a BIBD – unless $\lambda_1 = \lambda_2$, and the requirement that G be strongly regular is an attempt to restore some of the balance to the experiment that was lost when one λ could not be found for all pairs x, y.

We now give additional terminology, some of which applies to both BIBDs and PBIBDs.

Def. 12-14. A PBIBD is said to be *group divisible* if the strongly regular graph G upon which the design is based has a partition of its vertex set into n groups of m vertices each so that two vertices are first associates if and only if they are in the same group.

Clearly this is possible if and only if $G = K_{n(m)}$.

Def. 12-15. A group divisible PBIBD is a *transversal design* if each block contains exactly one vertex from each group.

Clearly this requires $k = n$.

Def. 12-16. A design (BIBD or PBIBD) is said to be *resolvable* if the set of b blocks can be partitioned into r subsets, of $\frac{v}{k}$ blocks each, each subset containing each object exactly once.

For example, if we take the edges of a strongly regular graph G as blocks ($k = 2$), the resulting PBIBD is resolvable if and only if G is 1-factorable. In particular, $G = K_{n(m)}$ gives an $(mn, \frac{m^2n(n-1)}{2}, m(n-1), 2; 0, 1)$-PBIBD, which is resolvable (see Himelwright and Williamson [HW2]) if and only if mn is even (for $m = 1$, the BIBD on $G = K_{n(1)} = K_n$ is resolvable if and only if n is even) and in fact is a transversal design if and only if $n = 2$. The transversal design on $K_{2(2w+1)}$, for example, is used in duplicate bridge scheduling to ensure that, in $2w+1$ rounds, each north-south couple plays exactly one round against each east-west couple.

Def. 12-17. A design (BIBD or PBIBD) is said to be *z-resolvable* if the block set can be partitioned so that each set in the partition contains each vertex exactly z times.

Thus, a 1-resolvable design is resolvable.

Def. 12-18. Two block designs D_1 and D_2 are said to be *isomorphic* if there exists a one-to-one correspondence $\theta : O_1 \to O_2$ between their object sets such that $\{x_1, x_2, \ldots, x_k\}$ is a block in D_1 if and only if $\{\theta(x_1), \theta(x_2), \ldots, \theta(x_k)\}$ is a block in D_2.

Clearly if D_1 and D_2 are isomorphic, their parameters v, b, r, k, λ (or λ_1 and λ_2) must agree; the converse is not true, as we shall see in Section 12-6.

For an example of all these definitions, consider the designs of Tables 12-2 and 12-3; clearly these are isomorphic, under the map θ sending i to the ith letter of the alphabet, $1 \leq i \leq 6$. The $(6, 8, 4, 3; 0, 2)$-PBIBD is based upon the strongly regular graph $G = K_{3(2)}$ and hence is group divisible; it is also a transversal design and is resolvable (the resolution is given by the horizontal pairing of the blocks.)

This design could be used to compare six wines ($v = 6$) as follows: companies A, B, and C make wines 1 and $4, 2$ and $5, 3$ and 6 respectively. We want to test the wines of the different companies against each other (say twice each: $\lambda_2 = 2$), but do not want to compare two wines made by the same company ($\lambda_1 = 0$.) Each taster tastes exactly 3 wines ($k = 3$), after which his judgement becomes impaired. We have eight tasters ($b = 8$) in all, and each wine is tasted four times ($r = 4$).

$1, 2, 6$	$3, 4, 5$
$2, 4, 6$	$1, 3, 5$
$4, 5, 6$	$1, 2, 3$
$1, 5, 6$	$2, 3, 4$

Table 12-2.

a, b, f	c, d, e
b, d, f	a, c, e
d, e, f	a, b, c
a, e, f	b, c, d

Table 12-3.

Finally we remark that the above design was taken directly from the triangular imbedding of $K_{3(2)}$ in S_0 depicted in Figure 9-10 (writing "6" for "0" and rearranging the orbits to display the resolvability.) This foreshadows the correspondence of the next section.

12-6. PBIBDs and Graph Imbeddings

For triangular imbeddings of strongly regular graphs, we readily obtain analogs to Theorems 12-3 and 12-4. A design is said to be *connected* if its underlying graph is connected; since a complete graph underlies each BIBD, only a PBIBD could fail to be connected.

Thm. 12-19. Connected $(v, b, r, 3; 0, 2)$-PBIBDs are in one-to-one correspondence with triangular imbeddings of strongly regular graphs of order v in generalized pseudosurfaces.

Thm. 12-20. Connected $(v, b, r, 3; 0, 1)$-PBIBDs are in two-to-one correspondence with triangular imbeddings of strongly regular graphs of order v in generalized pseudosurfaces and having bichromatic dual.

When a design is constructed, by any method (here we are advocating graph imbeddings, but other tools of construction include Latin squares, finite projective geometries, finite euclidean (affine) geometries, difference sets), the natural question is: Is it new? Clearly the design *is* new, if no design existed previously on the same parameter set. It is new also if no previously constructed design on the same parameters is isomorphic to the given design. The context of topological graph theory is often very convenient for answering the isomorphism question, as we see in:

Thm. 12-21. Let D_1 and D_2 be two designs (both BIBDs with $k = 3$ and $\lambda = 2$, or both PBIBDs with $k = 3$ and $\lambda_1 = 0$, $\lambda_2 = 2$) on the same parameter set. If the generalized pseudosurfaces they determine are not homeomorphic as topological spaces, then D_1 and D_2 are not isomorphic as designs.

PROOF. The identification procedure of surface topology is well-defined, so that isomorphic designs would yield homeomorphic generalized pseudosurfaces. □

The converse of this theorem is false. For example, K_{19} triangulates S_{20}, both with and without bichromatic dual. (See Problem 9-9.) Thus in one case the 2-fold triple system splits into two Steiner triple systems and in the other case, it does not, so that the two $(19, 114, 18, 3, 2)$-BIBDs are *not* isomorphic.

12-7. Examples

Example 1: Refer to Example lb of Section 10-3; the self-dual quadrilateral imbedding of $C_3 \times C_3$ in S_1 yields a $(9, 9, 4, 4; 1, 2)$-PBIBD. The graph $C_3 \times C_3$ is strongly regular, with $p_{22}^1 = 2$ and $p_{22}^2 = 1$; in fact, $C_3 \times C_3 = L(K_{3,3})$. (see Problem 2-13.) This "topological" design is atypical in that $k > 3$ and $\lambda_1 \neq 0$ and thus indicates that the scope of the connection is even wider than as indicated in Section 12-6. This design will be the ground case of an infinite collection of interesting designs constructed in Section 16-8.

Example 2: Refer to Example 2b of Section 10-3; the bichromatic-dual, triangular imbedding of $K_{4(2)}$ in S_1 obtained there gives one $(8, 16, 6, 3; 0, 2)$-PBIBD and two $(8, 8, 3, 3; 0; 1)$-PBIBDs (with blocks generated by $\{0, 1, 3\}$ and $\{0, 3, 2\}$ – in $\mathbb{Z}_8$ – respectively.) All three of these designs are on parameters for which no BIBD exists. Then Example 2c of Section 10-3 gives a bichromatic-dual triangulation of S_1 also, but this time by $K_{3(3)}$ and giving one $(9, 18, 6, 3, ; 0, 2)$-PBIBD and two $(9, 9, 3, 3; 0, 1)$-PBIBDs (generated by $\{00, 10, 01\}$ and $\{00, 11, 01\}$

– in $\mathbb{Z}_3 \times \mathbb{Z}_3$ – respectively.) Finally Example 2d of Section 10-3 gives a bichromatic-dual triangulation of S_1 once again, now by the strongly regular $G_\Delta(\Gamma)$, $\Gamma = \mathbb{Z}_4 \times \mathbb{Z}_4$ (*not* in the class $K_{n(m)}$) and yielding one $(16, 32, 6, 3; 0, 2)$-PBIBD and two $(16, 16, 3, 3; 0, 1)$-PBIBDS. The latter two designs are on parameters for which no BIBD exists.

Example 3: Refer to Example 5 of Section 10-3; the bichromatic dual (see Problem 10-4) triangulation of $S_{\frac{(n-1)(n-2)}{2}}$ by $K_{3(n)}$ obtained there gives one $(3n, 2n^2, 2n, 3; 0, 2)$-PBIBD and two $(3n, n^2, n, 3; 0, 1)$-PBIBDs. These are all group divisible, transversal designs. In [P5] Petroelje constructed orientable triangulations for $K_{3(n)}$ for which the resulting $\lambda_2 = 2$ PBIBDs are also resolvable. He then used these to obtain pseudosurface triangular imbeddings for $K_{4(n)}$ and the corresponding $(4n, 4n^2, 3n, 3; 0, 2)$-PBIBDs. In [G1] Garman constructed bichromatic-dual orientable surface triangulations for $K_{4(n)}$, n even (no imbedding of $K_{4(n)}$ can have bichromatic dual for n odd, since each vertex then has odd degree $3n$), so that different (by Theorem 12-21) $(4n, 4n^2, 3n, 3; 0, 2)$-PBIBDs, and also pairs of $\left(4n, 2n^2, \frac{3n}{2}, 3; 0, 1\right)$-PBIBDs, are obtained for these values of n. Finally, in [A6] Anderson constructed generalized pseudosurface triangulation for $K_{4(n)}$, for all n, giving yet another realization of these triples of designs.

Example 4: We observe that any $(n, b, n-1, k, \lambda)$-BIBD determined from an $r = r_k$ imbedding of $K_n (k \geq 3)$ determines in turn an $(n^2, 2nb, 2n-2, k; 0, \lambda)$-PBIBD as follows. The line graph $L(K_{2(n)}) = K_n \times K_n$ (see Problem 2-13) is strongly regular, and the cartesian product $K_n \times K_n$ can be obtained by identifying vertices appropriately among $2n$ disjoint copies of K_n. (Let $V^i = \{(i, j) | 1 \leq j \leq n\}$, $1 \leq i \leq n$, and $V_j = \{(i', j') | 1' \leq i' \leq n'\}$, $1' \leq j' \leq n'$, be the $2n$ disjoint vertex sets, each V^i and each V_j inducing a K_n; then identify (i, j) with (i', j'), $1 \leq i, j \leq n$.) Then performing these vertex identification on $2n$ disjoint initial imbeddings of K_n as given yields a generalized pseudosurface imbedding of $K_n \times K_n$ and a PBIBD on the parameter set as claimed.

For instance, take $k = 3$, $\lambda = 2$, $n \equiv 0, 1 \pmod 3$. Or, consider the complete design given by K_4 in N_1 (see Problem 12-8); this $(4, 3, 3, 4, 3)$-BIBD gives a $(16, 24, 6, 4; 0, 3)$-PBIBD by this method. Finally, consider the voltage graph for $\Gamma = \mathbb{Z}_9$ consisting of an octagon with edges labeled, in order, $1, 2, 3, -4, -3, -1, -2, 4$ (a bouquet of four circles in S_2, after identification of the boundary edges of the octagon); the $(9, 9, 8, 8, 7)$-BIBD arising from the covering imbedding of K_9, although a trivial design, gives a non-trivial $(81, 162, 16, 8; 0, 7)$-PBIBD. These designs on $K_n \times K_n$ are called *Latin square* designs; their construction could have been carried out purely combinatorially, but their topological realization is convenient for the application of Theorem 12-21.

Example 5: The imbedding of Figure 11-5 gives a $(10, 6, 3, 5; 1, 2)$-PBIBD based upon the strongly regular Petersen graph.

Example 6: In all of the preceding examples – including those in Section 12-3 – we have used graph imbeddings to obtain block designs. Now we reverse this point of view.

More attention has been paid toward the imbedding of graphs in the class $K_{n(m)}$ then in any other class; see for example [B16], [R10], [RY1], [RY5], [G1], and [J7]; for a survey of results in the orientable case, see [GRW1] and [KRW1]. Except for $n = 2$ (where no triangles are possible), the known results are obtained by constructing triangular imbeddings. For the generalized pseudocharacteristic $\chi''K_{n(m)})$ – see Definition 6-51 – the upper bound is given by the euler characteristic for a triangulation:

Thm. 12-22. $\chi''(K_{n(m)}) \le \frac{mn(6+m-mn)}{6}$.

We observe that the bound is an integer if and only if 3 divides $mn(n-1)$. For $m = 1$, $G = K_n$; it is now well-known that K_n has (in fact surface) triangular imbeddings if and only if $n \equiv 0, 1 \pmod 3$. (See Theorems 6-38 and 11-19; see also Corollary 12-9). For $m > 1$ and $n > 2$ generalized pseudosurface triangular imbeddings of $K_{n(m)}$ are, by Theorem 12-19 and the observation following Definition 12-14, exactly group divisible PBIBDs with $k = 3$, $\lambda_1 = 0$, and $\lambda_2 = 2$. The following is Theorem 6-2 of Hanani [H2], restated in the present context:

Thm. 12-23. A group divisible PBIBD with the object set partitioned into n groups of m objects each $(m > 1, n > 2)$ and $k = 3$, $\lambda_1 = 0$, $\lambda_2 = 2$ exists if and only if 3 divides $mn(n-1)$.

Thus all the triangular imbeddings upon which the estimate of Theorem 12-22 is based actually exist, and we state:

Thm. 12-24. For $n > 2$, $\chi''(K_{n(m)}) = \frac{mn(6+m-mn)}{6}$ if and only if 3 divides $mn(n-1)$.

Hence Hanani's result for block designs serendipitously computes the generalized pseudocharacteristic of $K_{n(m)}$, in $\frac{7}{9}$ of the possible cases.

12-8. Doubling a PBIBD

Given an orientable triangular imbedding of $K_{n(m)}$ with bichromatic dual, we have seen that one $(v, 2b, 2r, 3; 0, 2)$-PBIBD, as well as two $(v, b, r, 3; 0, 1)$-PBIBDs, correspond to this imbedding. Then by Theorem 10-20 (with $r = 2$) and Theorem 10-22 (with $G_1 = K_2$), the strong tensor product $K_2 \underline{\otimes} K_{n(m)} = K_{n(2m)}$ has an orientable triangular imbedding with bichromatic dual, so that one $(2v, 8b, 4r, 3; 0, 2)$-PBIBD and two $(2v, 4b, 2r, 3; 0, 1)$-PBIBDs result; in a sense, each original PBIBD has been doubled. Clearly this process can be iterated indefinitely,

Moreover, the construction given in [GRW1] for Theorem 10-22 provides a prescription for listing the blocks in the doubled designs in terms of those in the initial designs. Thus if the $2b$ initial blocks are

$$\bigcup_{i=1}^{b}\{a_{i1}, a_{i2}, a_{i3}\} \cup \bigcup_{i=1}^{b}\{b_{i1}, b_{i2}, b_{i3}\},$$

grouped by color class, then the $8b$ final blocks are

$$\bigcup_{i=1}^{b}\{a_{i1}, a_{i2}, a_{i3}\} \cup \bigcup_{i=1}^{b}\{b'_{i1}, b'_{i2}, b_{i3}\} \bigcup$$
$$\bigcup_{i=1}^{b}\{b_{i1}, b'_{i2}, b'_{i3}\} \cup \bigcup_{i=1}^{b}\{b'_{i1}, b_{i2}, b'_{i3}\}$$

and

$$\bigcup_{i=1}^{b}\{a'_{i1}, a'_{i2}, a'_{i3}\} \bigcup \bigcup_{i=1}^{b}\{b'_{i1}, b_{i2}, b_{i3}\} \bigcup$$
$$\bigcup_{i=1}^{b}\{b_{i1}, b'_{i2}, b_{i3}\} \bigcup \bigcup_{i=1}^{b}\{b_{i1}, b_{i2}, b'_{i3}\}$$

(again grouped by color class).

Theorem 10-22 always gives a triangular imbedding for $K_2 \underline{\otimes} G$, for G as in the theorem; yet if $K_2 \underline{\otimes} G$ is not strongly regular, then no block designs are provided. It is easily seen that $K_2 \underline{\otimes} G$ *is* strongly regular if and only if G is strongly regular with $p^1_{22} = n_2$ or complete, and that G is strongly regular and connected, with $p^1_{22} = n_2$, or complete, if and only if $G = K_{n(m)}$ for some $m \geq 1$, $n \geq 2$. Thus the doubling process of this section is applicable exactly to triangular imbeddings of $K_{n(m)}$ with bichromatic dual (there is a nonorientable analog to Theorem 10-22; see [G1]). However, every time such an imbedding is found, it determines an infinite tower of triples of PBIBDs, as explained above. Moreover, if the initial imbedding is on a pseudosurface, than all imbeddings derived from it are also pseudosurface imbeddings; thus the associated designs, by Theorem 12-21, differ from those of Hanani (see Example 6 in Section 12-7.)

For example, consider the toroidal imbeddings of K_7, $K_{4(2)}$, and $K_{3(3)}$ given as covering spaces of Figure 10-6 (Example 2 in Section 10-3); each is the base for such an infinite tower of designs. The graphs for K_7, for instance, are the family $K_{7(2^k)}$, $k = 0, 1, 2, \ldots$ – for which the genus is also thereby determined.

As a second example, consider the voltage graph of Figure 12-1, for $\Gamma = \mathbb{Z}_{35}$; this determines a nonorientable pseudosurface triangular imbedding for $K_{7(5)}$ with bichromatic dual. Hence another infinite tower of imbeddings – for $K_{7(5\cdot 2^k)}$ – and of the corresponding block designs is anchored. (In this case, the pseudocharacteristic $\chi'(K_{7(5\cdot 2^k)})$ is also determined.) The $\lambda_2 = 1$ designs, for all k, and the $\lambda_2 = 2$ designs, for k even, have no BIBD counterparts.

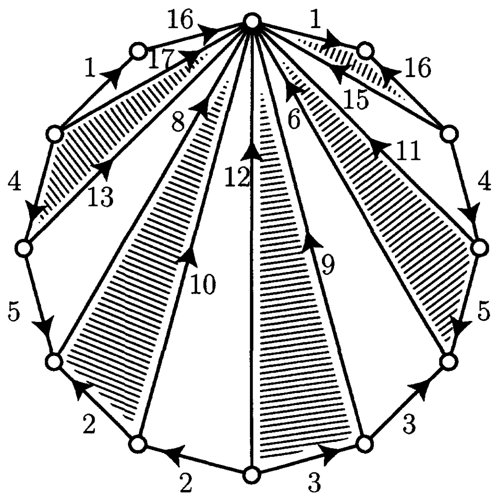

Figure 12-1.

12-9. Problems

12-1.) Prove Theorem 12-2 (i).

12-2.) Prove Theorem 12-2 (ii).

12-3.) Find (and prove) an analog to Theorem 12-2 for PBIBDs.

12-4.) Prove Theorem 12-11.

12-5.) Prove or disprove the following strengthening of Frucht's Theorem (3-18): Every finite group is the automorphism group of some strongly regular graph.

12-6.) Show that the complement of a strongly regular graph is strongly regular.

12-7.) Let a $(v, b, r, k; \lambda_1, \lambda_2)$-PBIBD be based upon the strongly regular graph G. Show that we can use exactly the same objects and blocks, to obtain a corresponding $(v, b, r, k; \lambda_2, \lambda_1)$-PBIBD, based upon $\bar{G}$. (Thus we may assume, without loss of generality, that $\lambda_1 < \lambda_2$.)

12-8.) Construct an imbedding of K_4 in N_1 to give a $(4,3,3,4,3)$-balanced complete block design.

12-9.) Show that $L(K_4) = K_{3(2)}$ and $L(K_5) = \bar{\Pi}$, the complement of the Petersen graph.

12-10.) What design results from the voltage graph imbedding of Figure 12-2, using $\Gamma = \mathbb{Z}_{10}$?

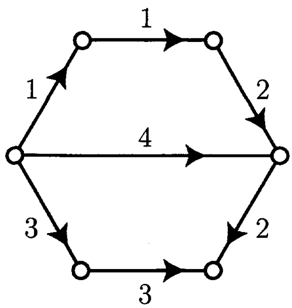

Figure 12-2.

12-11.) What designs result from the voltage graph imbedding of Figure 12-3, using $\Gamma = \mathbb{Z}_{11}$? Find a generalization, for prime powers q congruent to 3 modulo 4. (*Hint*: the quadratic residues mod q give a *perfect difference set*, that is each non-zero difference of two elements in the set occurs exactly $\lambda = \frac{q-3}{4}$ times.) These designs are called *Paley designs*.

12-12.) Show that the design of Figure 12-1 is 3-resolvable, the design of Problem 12-10 is 4-resolvable, and the first design of Problem 12-11 is 5-resolvable.

12-13.) PBIBDs on three or more association classes may also be found from graph imbeddings. For example, take $\Delta = \{(1,0,0),(0,1,0),(0,0,1)\}$ for $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and show that $Q_3 = G_\Delta(\Gamma)$ in S_0 gives an $(8,6,3,4;0,1,2)$-PBIBD$_{(3)}$, where v_1 and v_2 are first associates if $v_2 = v_1 + (1,1,1)$, second associates if non-adjacent but not first associates, and third associates if adjacent.

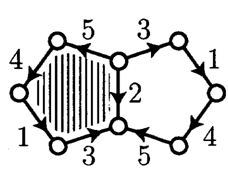

Figure 12-3.

12-14.) Find the parameters v, b, r, k, and λ for the imbeddings of: (1) Example 2a(ii) of Section 10-3, and (2) Theorem 11-19, $m = 13$. Are these designs isomorphic?

12-15.) An isomorphism of a design with itself is called an *automorphism*. If two objects are in a common block, they are said to be *collinear*. Show that if a permutation of the object set of a design preserves blocks (i.e. is an automorphism), then it preserves collinearity. Give an example to show that the converse is false.

12-16.) Modify Figure 10-12, to obtain an imbedding of K_8 on S_2 with all regions triangular except for two disjoint quadrilaterals. (*Hint*: change the placement of the loops and half-edges.) Augment this imbedding to obtain a Mendelsohn triple system of order 9.

12-17.) Given a Steiner triple system of order v, replace each (unordered) triple with two oppositely oriented triples (on the same three elements), to obtain a Mendelsohn triple system of order v. What generalized pseudosurface provides the natural model for this construction?

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CHAPTER 13

HYPERGRAPH IMBEDDINGS

A graph is just a special case of a hypergraph; but although a great deal of attention has been paid, as we have seen, to the geometric realizations of graphs, there has been little effort made to extend these concrete representations to the general setting. In [JSW1] an attempt was made to remedy this situation, as we indicate in the present chapter also.

Our aim is to find a geometric realization for hypergraphs satisfying:

(1) The method should not be unduly cumbersome.
(2) It should include the standard geometric realization of graphs (as points and arcs in appropriate 2-manifolds) as a special case.

Every block design is a hypergraph, so the material of this chapter will connect with that of the previous chapter. The connection will extend to Chapter 15 also, as every finite geometry is a hypergraph too.

13-1. Hypergraphs

Def. 13-1. A *hypergraph* H consists of a finite non-empty set $V(H)$ of *vertices* together with a set $E(H)$, each of whose elements is a subset of $V(H)$ and is called an *edge.* If $e \in E(H)$ $(e \subseteq V(H))$ and if $u, v \in e (u, v \in V(H))$, we say that u and v are *adjacent* vertices, and that the vertex u and edge e are *incident* with each other, as are v and e. Two distinct edges e_1 and e_2 are said to be *adjacent* if $e_1 \cap e_2 \neq \emptyset$. The *degree*, $d(v)$, of $v \in V(H)$ is the number of edges with which v is incident. If $|e| = r (r \geq 0)$ for all $e \in E(H)$, then H is said to be an r-*uniform* hypergraph.

Thus a graph is just a 2-uniform hypergraph.

We write $p_H = |V(H)|$ and $q_H = |E(H)|$. Let $n_i = |e_i|$, for $1 \leq i \leq q_H$ and $e(H) = \{e_1, e_2, \ldots, e_{q_H}\}$. Then as a generalization of Theorem 2-2, we have:

Thm. 13-2. For any hypergraph H,

$$\sum_{i=1}^{p_H} d(v_i) = \sum_{i=1}^{q_H} n_i.$$

PROOF. Merely count total incidences, in the two possible ways. $\square$

Note that for a graph each $n_i = 2$, and we regain Theorem 2-2. In fact:

Cor. 13-3. For an r-uniform hypergraph H,

$$\sum_{i=1}^{p_H} d(v_i) = rq_H.$$

For an example, consider the 3-uniform hypergraph H defined by:

$$V(H) = \{1,2,3,4\},$$
$$E(H) = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\};$$

then Corollary 13-3 observes that $4 \cdot 3 = 3 \cdot 4$. One convention (see Berge [B8]) for representing hypergraphs would depict H as in Figure 13-1; this method does not appear to meet either criterion (1) or (2), as given in the introduction to this chapter. Thus, we seek another method.

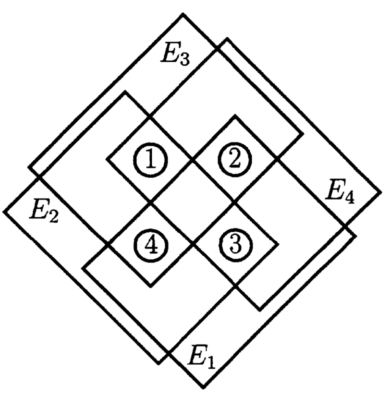


Figure 13-1.

We observe that Definitions 2-6 and 2-7 carry over verbatim, so that a *connected* hypergraph is exactly what one would expect it to be.

13-2. Associated Bipartite Graphs

We use a bijection between connected hypergraphs H and connected bipartite graphs $G(H)$ given by Walsh [W2]. In the present context, we are primarily concerned with the construction of $G(H)$ from H:

$$V(G(H)) = V(H) \cup E(H), \text{ the bipartition;}$$
$$E(G(H)) = \{\{v, e\} | v \in V(H), e \in E(H), v \in e\}.$$

We call $G(H)$ the *Levi graph* of H; see Coxeter[C8].

We next find a 2-cell imbedding of $G(H)$ into some closed orientable 2-manifold $S_k (k \geq 0)$, and denote this 2-cell imbedding by $G(H) \lhd S_k$.

For the example H of Section 13-1, $G(H)$ is $K_{4,4}$ less a 1-factor; that is, $G(H) = Q_3$. It is convenient to take $Q_3 \lhd S_0$, as usual.

In the next section, we shall see how to modify the imbedding of $G(H) \lhd S_k$ so as to obtain an "imbedding" of H into $S_k (H \lhd S_k)$.

13-3. Imbedding Theory for Hypergraphs

Given a 2-cell imbedding of the associated bipartite graph, $G(H) \lhd S_k$, we modify this imbedding to obtain an imbedding of the hypergraph H into S_k, wherein certain of the regions of the modified imbedding $(G^*(H) \lhd S_k)$ represent edges of H; the remaining regions of the modified imbedding $(G^*(H) \lhd S_k)$ become regions for the imbedding of H into S_k. For H connected, $G(H)$ will be connected also, and thus we *can* find a 2-cell imbedding $G(H) \lhd S_k$. The modification we perform preserves the 2-cell aspect of the imbedding, so that the regions for H in S_k are all 2-cell also. Hence the notation $H \lhd S_k$ is justified.

We illustrate the modification process in Figure 13-2. In part (a) of that figure, we see – in the imbedding of $G(H)$ – the vertices representing edge $e = \{v_1, v_2, \ldots, v_k\}$ of H and each vertex in e. In part (b) of the figure we begin modifying the imbedding by adding edge $\{v_i, v_{i+1}\}$, within the region containing v_i, v_{i+1}, and e, for $1 \leq i \leq k$, (mod k). Then the edges $\{e, v_i\}, 1 \leq i \leq k$, and the vertex e, are deleted, so that in part (c) of the figure the edge e appears as a region in the modified imbedding. (More precisely, the set of vertices from the boundary of region (e) is exactly the edge e.)

In Figure 13-3 we illustrate the entire process for the hypergraph H of Section 13-1. In part (a) of this figure we see an imbedding $G(H) = Q_3 \lhd S_0$, and the beginning of the modification. In part (b) of the figure we see the corresponding imbedding $H \lhd S_0$, with the regions of this imbedding being shaded; the unshaded "regions" depict the edges of H.

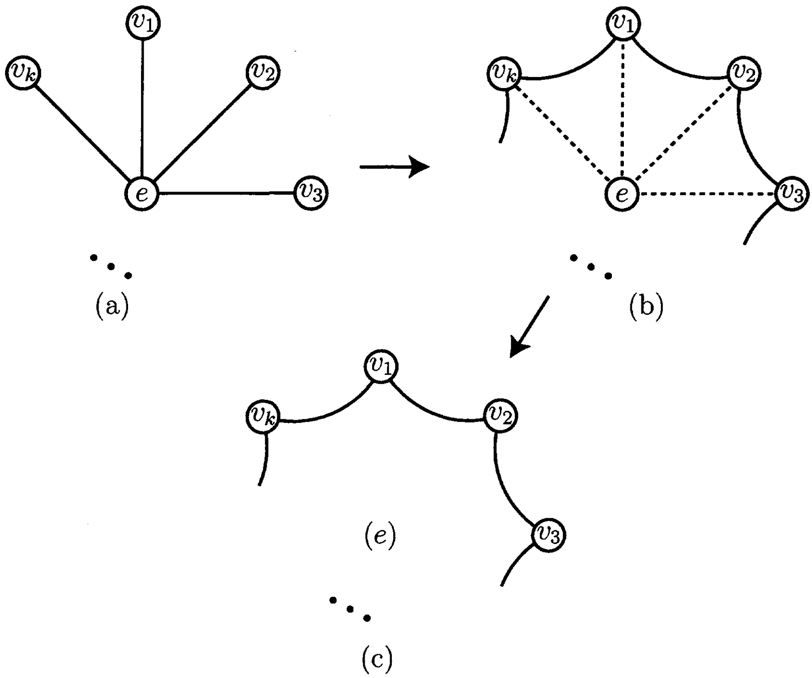
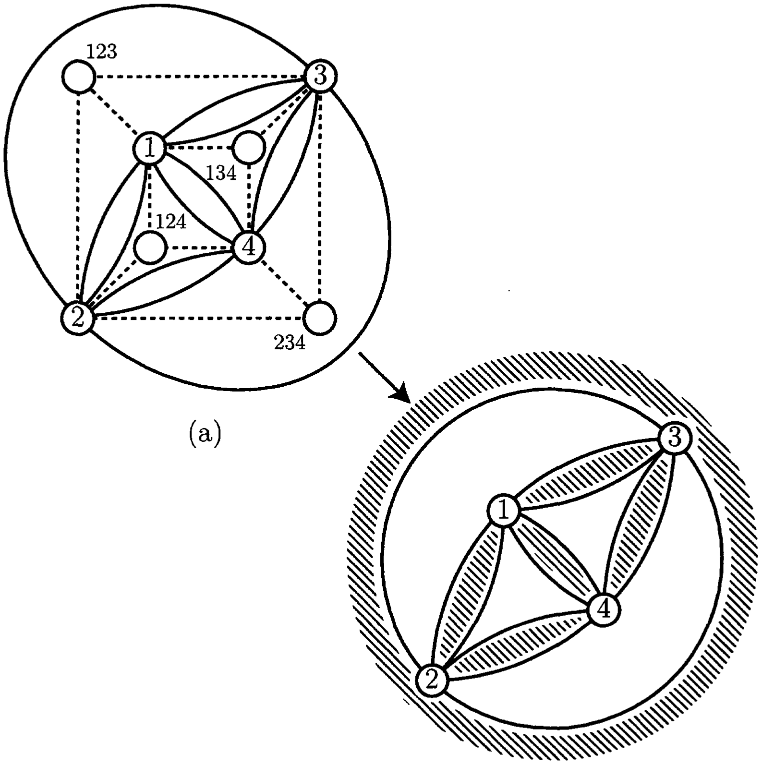


Figure 13-2.

The former (i.e. the *bona fide* regions of the hypergraph imbedding) are all digons here, as each region for the imbedding of $G(H)$ was a quadrilateral. (In general, a k-gonal region results for $H \lhd S_k$ from a $2k$-gonal region of the *bipartite* $G(H) \lhd S_k$.) The process readily reverses. Thus given $H \lhd S_k$, we can obtain $G(H) \lhd S_k$, by inserting an "edge" vertex in the interior of each "edge" region, joining it to the vertices in the boundary, and deleting all edges from $G^*(H) \lhd S_k$.

We remark that: (1) this representation is not cumbersome, in that $V(H)$ and $E(H)$ are readily discernible. (Compare Figure 13-1, for the same hypergraph H.) (2) If H is in fact a graph, then each edge of H becomes a digon for the modified imbedding $G(H) \lhd S_k$, and the collapsing of each digon gives a traditional imbedding $H \lhd S_k$. This is illustrated for $H = K_4$, in Figure 13-4.

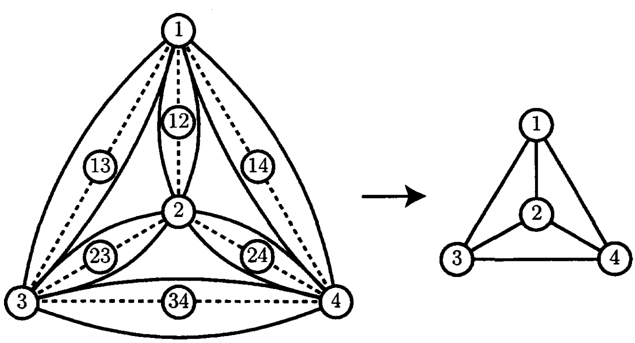
Imbedding problems for hypergraphs now translate directly into the graphical context, and many standard results for graphs have ready generalizations to hypergraphs. (This approach was noted, independently, for the sphere only, by Jones [J5].) For example, here is the generalization of Theorem 5-14. (We let r_H denote the number of *bona fide* regions – i.e. *not* including those regions depicting edges – in a 2-cell imbedding of hypergraph H.)



(a)

(b)

Figure 13-3.



(a) (b)

Figure 13-4.

Thm. 13-4. Let the connected hypergraph H have a 2-cell imbedding in S_k, with the usual parameters p_H, q_H, and r_H. Then

$$p_H + q_H - \sum_{i=1}^{q_H} n_i + r_H = 2 - 2k.$$

PROOF. For the 2-cell imbedding $G(H) \lhd S_k$ which gave rise to $H \lhd S_k$, we have

$$p - q + r = 2 - 2k,$$

by Theorem 5-14, where

$$p = |V(G(H))| = p_H + q_H,$$

$$q = |E(G(H))| = \sum_{i=1}^{q_H} n_i,$$

and $r = r_H$.

The result now follows, by substitution. □

We note that, for H a graph – so that $n_i = 2, 1 \leq i \leq q_H$, then $\sum_{i=1}^{q_H} n_i = 2_{q_H}$ and we have the familiar $p_H - q_H + r_H = 2 - 2k$.

13-4. The Genus of a Hypergraph

Once we accept as worthwhile the task of realizing a hypergraph geometrically, then it is most natural to wish to do this as efficiently as possible. This motivates

Def. 13-5. The *genus*, $\gamma(H)$, of a hypergraph H is the genus of its associated bipartite graph; i.e. $\gamma(H) = \gamma(G(H))$.

Since the genus parameter for graphs is additive over connected components (Corollary 6-19), we obtain immediately:

Thm. 13-6. The genus of a hypergraph is the sum of the genera of its components.

PROOF. Let $H = \bigcup_{i=1}^{n} H_i$ be the decomposition of H into its connected components, with $G(H_i)$ the bipartite graph associated with component H_i. Then

$$\gamma(H) = \gamma(G(H)) = \gamma\left(\bigcup_{i=1}^{n} G(H_i)\right) = \sum_{i=1}^{n} \gamma(G(H_i)) = \sum_{i=1}^{n} \gamma(H_i).$$

□

Thus it is without loss of generality that we continue to restrict our attention to connected hypergraphs.

Def. 13-7. The *maximum genus*, $\gamma_M(H)$, of a connected hypergraph H is given by: $\gamma_M(H) = \gamma_M(G(H))$.

We have the natural generalization of Corollary 6-22:

Thm. 13-8. A connected hypergraph H has a 2-cell imbedding in S_k if and only if $\gamma(H) \leq k \leq \gamma_M(H)$.

We now give a lower bound for the genus parameter, reminiscent of Corollary 6-15 (but not in generalization of that corollary, since if H is a graph, then $G(H)$ will have girth at least six.)

Thm. 13-9. If H is a connected hypergraph, then

$$\gamma(H) \geq 1 + \frac{1}{4}(\sum_{i=1}^{q_H} n_i - 2p_H - 2q_H).$$

PROOF. Let $H \lhd S_k$, where $k = \gamma(H) = \gamma(G(H))$. Since H is connected, so is $G(H)$; thus the minimal imbedding $G(H) \lhd S_k$ is 2-cell, as is $H \lhd S_k$. Since $G(H)$ is bipartite, $4r \leq 2q$. Thus

$$2r_H = 2r \leq q = \sum_{i=1}^{q_H} n_i,$$

and using this in the euler equation for hypergraphs (Theorem 13-4), we get the desired bound. □

We close this section with an upper bound for the maximum genus parameter, in analogy with Theorem 6-24.

Thm. 13-10. Let H be connected; then

$$\gamma_M(H) \leq \frac{1 - p_H - q_H + \sum_{i=1}^{q_H} n_i}{2}.$$

Moreover, equality holds if and only if $r_H = 1$ or 2, according as the numerator is even or odd, respectively.

13-5. The Heawood Map-Coloring Theorem, for Hypergraphs

Def. 13-11. The *chromatic number*, $\chi(H)$, of a hypergraph H is the minimum natural number k for which there is a partition $V(H) = \bigcup_{i=1}^{k} V_i(H)$ such that, for each edge $e \in E(H)$, there is no i with $e \subseteq V_i(H)$.

That is, $\chi(H)$ is the smallest number of colors for $V(H)$ so that no edge of H is uniformly colored. Note that this is the"weak" definition of chromatic number for hypergraphs, as we are not requiring that all vertices in an arbitrary edge be colored differently (just that they not all be colored alike.) Clearly the weak and the "strong" definitions agree, if H is a graph. The latter definition holds less interest, in the following sense: replacing each edge with one complete graph reverts to the chromatic number problem for graphs.

Def. 13-12. The *hypergraph chromatic number of the surface* S_k is defined by: $\chi_H(S_k) =$ the maximum $\chi(H)$ such that $H \triangleleft S_k$.

Thm. 13-13. $\chi_H(S_k) = \left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor, k \geq 0$.

PROOF. Set $f(k) = \left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor$.

(1) Let $H \triangleleft S_k$, and let $G^*(H)$ denote the corresponding modification of $G(H)$. Since $G^*(H) \triangleleft S_k$, $\chi(G^*(H)) \leq f(k)$, by the Heawood Map-Coloring Theorem (or the Four-Color Theorem, if $k = 0$) for graphs. Let $G^*(H)$ be $f(k)$-colored. Now consider an arbitrary edge e of H and any two consecutive vertices in the corresponding region of $G^*(H)$; since they form an edge of $G^*(H)$, these two vertices are colored differently. Thus e is not uniformly colored, and $\chi(H) \leq f(k)$. Since H was arbitrary for $S_k, \chi_H(S_k) \leq f(k)$.

(2) Since $H = K_{f(k)} \triangleleft S_k$, $\chi_H(S_k) \geq \chi(H) = f(k)$.

(3) Thus $\chi_H(S_k) = f(k)$. □

13-6. The Genus of a Block Design

Every block design is a k-uniform hypergraph – with objects as vertices and blocks as edges – so that any realization of a hypergraph associated with a block design is simultaneously a realization of that design. For example, the Steiner triple system H of order 7, arising (for example) from the projective plane of order 2, is depicted in Figure 13-5, where $G(H)$ would be the "Heawood Graph" (see Figure 8-5), the dual of $K_7 \triangleleft S_1$. In this case, $G^*(H)$ *is* K_7, and our realization of H coincides with that of Example 2, Section 12-3. (The blocks of Table 12-1 appear as the unshaded regions in Figure 13-5.)

It will be no surprise that, once we agree to depict block designs realistically, we should desire to do this as efficiently as possible.

Def. 13-14. The *genus of a block design* D, $\gamma(D)$, is the genus of the associated hypergraph H; i.e. $\gamma(D) = \gamma(H) = \gamma(G(H))$, where $G(H)$ is the bipartite graph for H.

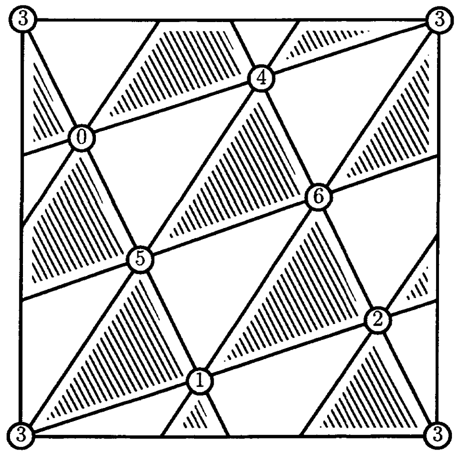


Figure 13-5.

Thus $\gamma(D)$ gives the most efficient orientable surface for the representation of D.

For example, if D is the familiar $(7,7,3,3,1)$-BIBD (Steiner triple system), then Figure 13-5 shows that $\gamma(D) \leq 1$; but one readily finds a homeomorph of $K_{3,3}$ in $G(H)$ for Figure 13-5, so that $\gamma(D) \geq 1$. Thus $\gamma(D) = 1$, and our depiction in Figure 13-5 is optimal.

Rahn [R1, R2] has characterized *planar* (i.e. $\gamma(D) = 0$) BIBDs;

Thm. 13-15. A (v,b,r,k,λ)-BIBD is planar if and only if:

(i) $k = 1$;
(ii) $k = 2$ and $v = 2, 3$, or 4;
(iii) $k = 3$ and

$$(v,b,r,k,\lambda) = (3,1,1,3,1),\ (3,2,2,3,2) \text{ or } (4,4,3,3,2).$$

We note that the case $(4,4,3,3,2)$ is displayed in Figure 13-3.

13-7. An Example

For an additional example of many of the concepts of this chapter, we consider the $(n,n,n-1,n-1,n-2)$-BIBD D_n (for $n \geq 2$) whose object set is $\{1,2,\ldots,n\}$ and whose blocks are the complements of singletons (i.e. all $(n-1)$-subsets of an n-set.) This immediately determines a hypergraph H having the same description: $V(H_n) = \{1,2,\ldots,n\}$, $E(H_n) = \{S \subseteq V(H_n) \mid |S| = n-1\}$. Then $G(H_n)$ is $K_{n,n}$ less a 1-factor (each vertex i is adjacent to every edge except its complement, so that the 1-factor is composed of pairs $\{i, V(H_n) - \{i\}\}$,

$1 \leq i \leq n$.) For $n = 2, 3, 4$ respectively, we have $G(H_n) = 2K_2, C_6, Q_3$; see Figure 13-3 for the case $n = 4$ and the planar imbedding of H_4.

Thm. 13-16. $\gamma(D_n) = \left\lceil \frac{(n-1)(n-4)}{4} \right\rceil$, for $n \geq 2$.

PROOF. From the lower bound of Theorem 13-9, we find that

$$\gamma(D_n) = \gamma(H_n) \geq \left\lceil \frac{(n-1)(n-4)}{4} \right\rceil .$$

To complete the proof, we show the reverse inequality, by construction. This construction splits into four cases, depending upon the residue of n modulo 4; here we provide the details only for the case $n \equiv 1 \pmod 4$. (See Problem 13-2 for the remaining – harder – cases.) The method uses an index two current graph (see Figure 13-6 for the cases $n = 5$ and 9, which have an obvious generalization; the vertex rotations are indicated schematically, to yield a current graph imbedding in S_h, where $h = \frac{n-1}{4}$ and the group $\Gamma = \mathbb{Z}_{2n}$, generated by $\Delta = \{1, 3, 5, \ldots, n-2\}$. Since the KCL holds at each vertex of the current graph, the Cayley graph $(G_\Delta(\Gamma) = G(H_n))$ imbedding which covers the dual of the current graph imbedding is quadrilateral, and hence – since it is bipartite – minimally imbedded in S_k, $k = \frac{(n-1)(n-4)}{4}$. Thus

$$\begin{aligned}\gamma(D_n) &= \gamma(H_n) = \gamma(G(H_n)) \\ &= \frac{(n-1)(n-4)}{4} = \left\lceil \frac{(n-1)(n-4)}{4} \right\rceil\end{aligned}$$

in this case. □

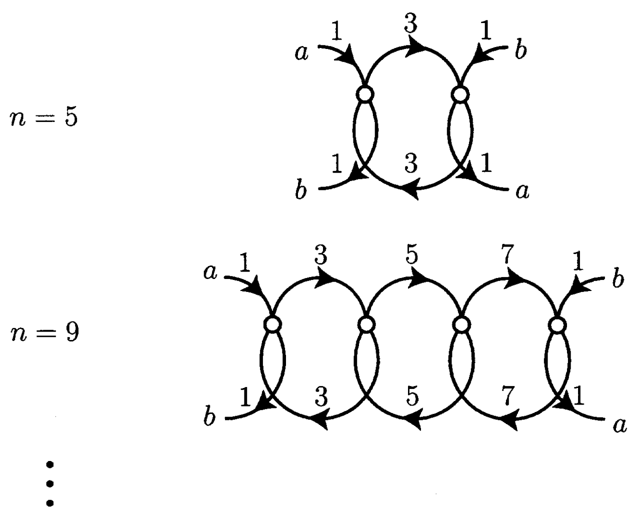


Figure 13-6.

13-8. Nonorientable Analogs

We conclude with a brief discussion of the nonorientable imbedding of hypergraphs. The definition of a 2-cell imbedding of a hypergraph H (via a 2-cell imbedding of the associated bipartite graph $G(H)$) on the nonorientable surface $N_h(H \lhd N_h)$ carries over verbatim from the orientable case, as do all related concepts. We have the following results:

Thm. 13-17. If $H \lhd N_k$, then $p_H + q_H - \sum_{i=1}^{q_H} n_i + r_H = 2 - h$.

Thm. 13-18. If $\tilde{\gamma}(H)$ denotes the nonorientable genus of hypergraph H, then

$$\tilde{\gamma}(H) \geq 2 + \frac{1}{2}\left(\sum_{i=1}^{q_H} n_i - 2p_H - 2q_H\right).$$

Thm. 13-19. The connected hypergraph H has a 2-cell imbedding on N_h if and only if

$$\tilde{\gamma}(H) \leq h \leq 1 + \sum_{i=1}^{q_H} n_i - p_H - q_H.$$

Finally, we give the nonorientable Heawood Map-Coloring Theorem for hypergraphs:

Thm. 13-20. $\chi_H(N_h) = \left\lfloor \frac{7+\sqrt{1+24h}}{2} \right\rfloor$, for $h \neq 2$; $\chi_H(N_2) = 6$.

13-9. Problems

13-1.) Prove Theorem 13-10.

13-2.) *Show that $K_{n,n}$ less a 1-factor imbeds on S_k, $k = \left\lceil \frac{(n-1)(n-4)}{4} \right\rceil$, for $n \equiv 0, 2, 3,\ (\text{mod } 4)$.

13-3.) Prove Theorem 13-17.

13-4.) Prove Theorem 13-18.

13-5.) Prove Theorem 13-19.

13-6.) Prove Theorem 13-20.

13-7.) Find the voltage graph imbeddings corresponding to the two current graph imbeddings of Figure 13-6. Is the generalization to $n \equiv 1 \pmod 4$ as readily apparent in the voltage graph setting?

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CHAPTER 14

FINITE FIELDS ON SURFACES

In Chapter 7 we modelled finite groups on surfaces. First we formed a Cayley graph for a given finite group, depending upon the generating set employed, and we saw that this provided a useful model for the group. Then we strove to depict this model as efficiently as possible, that is, on a surface of minimum genus. Minimizing this genus over all generating sets for the given group then gave the genus of the group, and some interesting mathematics has resulted from the study of this parameter. A group is a set with one binary operation (satisfying certain axioms); in this chapter we impose a second binary operation and seek to model the resulting structure efficiently.

The most general setting is that of a ring. We start modestly with $\langle \mathbb{Z}_n, +, x \rangle$, where n is composite. But even in this simple context, we find no satisfactory graphical model, even before bringing surfaces into the picture. We see that it is the composite nature of n that is thwarting us, so we let n be prime. This gives us a field, and so we study finite fields in general. After some necessary background material, we present results, largely due to Jones [J4], about the genus parameter for finite fields and related questions.

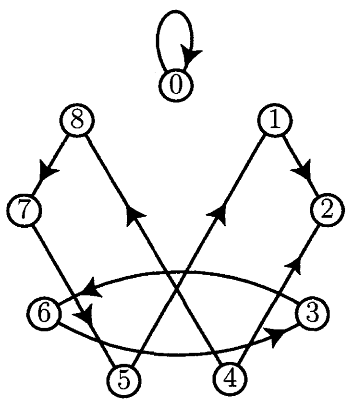
14-1. Graphs Modelling Finite Rings

We begin by considering the ring $\mathbb{Z}_n$, with the usual binary operations of addition and multiplication. Our goal is to model each operation with its own graph (both graphs have vertex set $\mathbb{Z}_n$), and then combine the two graphs. The combined graph would then model the interaction between the two operations. Each separate graph would have its edges determined by generators for the associated operation, in keeping with Chapter 4. Also in keeping with Chapter 4, we require:

(1) With each edge $\{g, h\}$ are associated two arcs: (g, h), with label $\delta = g^{-1}h$ (or $-g + h$ in the additive case); and (h, g), with label $\delta^{-1} = h^{-1}g(-h + g$, respectively) – thus inverse arcs model inverse generators.

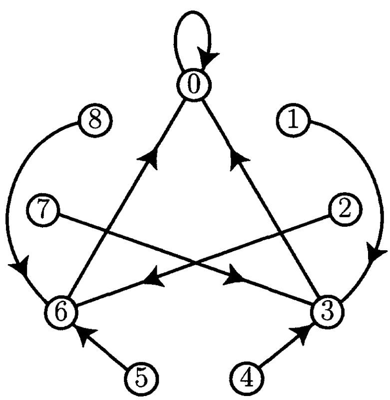
(2) The aggregate of generators for each operation generates the structure for the operation; thus both graphs are connected, except that for multiplication the zero vertex has only loops attached. Thus the combined graph contains all the information of *both* operation tables for the ring.

Addition is readily modelled by taking $\Delta_+ = \{1\}$, so that $G_{\Delta_+}(\mathbb{Z}_n) = C_n$, as in Chapters 4 and 7. Thus the interest lies in the selection of $\Delta_\times$, for the operation of multiplication. The case where n is prime will be treated in Section 14-3, so here we set $n = k \cdot m$, where $1 < k, m < n$. If $\delta \in \Delta_\times$, we want $\delta \neq 1$, for $\delta = 1$ would produce a loop at each vertex. Now condition (1) above requires that δ^{-1} exist in $\mathbb{Z}_n$, that is that $(n, \delta) = 1$, as in Figure 14-1(a), where $n = 9$ $(k = m = 3)$ and $\delta = 2$. The loop at vertex 0 is unavoidable, but we note two additional components, where we want only one more, by condition (2). In general, this unfortunate situation exists and *cannot* be remedied by augmenting $\Delta_\times$: since $n = k \cdot m = 0$, for each δ: $\delta m \in \langle m \rangle$, and we never connect the multiples of m to the other vertices.



$(n, \delta) = (9, 2)$

(a)



$(n, \delta) = (9, 3)$

(b)

Figure 14-1.

If we choose δ so that $(n, \delta) > 1$, a connected graph will sometimes result (see Figure 14-1(b)), but we lose property (1). Moreover (see Problem 14-1) the multiplicative subgraph determined by δ is regular if and only if $(n, \delta) = 1$. (Recall that regularity is a feature of Cayley graphs.)

Thus our attempts to model $\mathbb{Z}_n$ seem doomed to failure, for n composite. On the other hand, for n prime we will find the model we seek. Since our approach extends to finite fields in general, that will be the appropriate setting for this chapter. We start with some background information about finite fields.

14-2. Basic Theorems About Finite Fields

The following theorems can be found in many abstract algebra texts.

Thm. 14-1. There exists a finite field of order n if and only if n is a prime power, $n = p^r$.

Thm. 14-2. All fields of fixed order p^r are isomorphic.

Thm. 14-3. The additive group of the finite field of order p^r is isomorphic to the vector space of dimension r over $\mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p = \mathbb{Z}_p^r$.

Thm. 14-4. The multiplicative group of the nonzero elements of the finite field of order p is cyclic, $\mathbb{Z}_{p^r-1}$.

Thm. 14-5. The elements of the finite field of order p^r are the roots of $\lambda^{p^r} - \lambda = 0$, over $\mathbb{Z}_p$.

We will need to work explicitly with the elements of a given finite field, so we outline a construction of $GF(p^r)$, the Galois field of order p^r, which we take as our representation of the isomorphism class. For $r = 1$, $GF(p) = \mathbb{Z}_p$, with $\Delta_+ = \{1\}$ and $\Delta_\times = \{\delta\}$, where $(p, \delta) = 1$. For $r \geq 2$, start with a monic irreducible polynomial of degree r over $\mathbb{Z}_p$ having a primitive root. (Such a polynomial always exists, by Theorems 14-3, 14-4 and 14-5.) For example, let $p = 3$ and $r = 2$. The polynomial $x^2 + 1$ is monic and irreducible (since neither 0 nor 1 nor 2 is a root). But it is not primitive, since if α is a root, then $\alpha^2 = 2$ and $\alpha^4 = 1$, so that α does *not* generate the multiplicative group. On the other hand, $x^2 + 2x + 2$ is also monic and irreducible, *and* if $\alpha^2 = \alpha + 1$, then we calculate the multiplicative representation of the elements of $GF(9)$ as follows:

$\underline{0}$	$(0,0)$
1	$(0,1)$
α	$(1,0)$
$\alpha^2 = \alpha + 1$	$(1,1)$
$\alpha^3 = \alpha^2 + \alpha = 2\alpha + 1$	$(2,1)$
$\alpha^4 = 2\alpha^2 + \alpha = 2$	$(0,2)$
$\alpha^5 = 2\alpha$	$(2,0)$
$\alpha^6 = 2\alpha^2 = 2\alpha + 2$	$(2,2)$
$\underline{\alpha^7} = 2\alpha^2 + 2\alpha = \alpha + 2$	$(1,2)$
$\alpha^8 = \alpha^2 + 2\alpha = 1$	

The key is that the monic irreducible primitive polynomial provides the connection (which we know exists, by the distributive axiom) between the additive and multiplicative structures of the finite field. We have also listed the corresponding vectors in $\mathbb{Z}_3 \times \mathbb{Z}_3$, isomorphic to $\{a\alpha + b | a, b \in \mathbb{Z}_3\}$. We take the latter to represent the additive group. More formally, this corresponds to $\mathbb{Z}_3[x]/(m(x))$, where $(m(x))$ is the maximal ideal in $\mathbb{Z}_3[x]$ generated by $m(x) = x^2 + 2x + 2$.

For a second example, consider $p = 2$ and $r = 3$, using $x^3 + x + 1$, with root α. Then we have:

$\underline{0}$	$(0,0,0)$
1	$(0,0,1)$
α	$(0,1,0)$
α^2	$(1,0,0)$
$\alpha^3 = \alpha + 1$	$(0,1,1)$
$\alpha^4 = \alpha^2 + \alpha$	$(1,1,0)$
$\alpha^5 = \alpha^3 + \alpha^2 = \alpha^2 + \alpha + 1$	$(1,1,1)$
$\underline{\alpha^6} = \alpha^3 + \alpha^2 + \alpha = \alpha^2 + 1$	$(1,0,1)$
$\alpha^7 = \alpha^3 + \alpha = 1$	

Here we see the eight elements of $GF(8)$ and how they correspond to the seven elements of the cyclic multiplicative group.

14-3. The genus of $\mathbb{F}_p$

The simplest of the Galois fields $GF(p^r)$ to analyze are those of exponent $r = 1$, but we find challenge enough here. We know that 1 generates the additive group, and that the multiplicative group is also cyclic; take g as a generator. Let $\Delta_+ = \{1\}$ and $\Delta_\times = \{g\}$, with $\Delta = \Delta_+ \cup \Delta_\times$. For brevity, we set $\mathbb{F}_p = GF(p)$. Graphically, we represent 1 with a solid arc and g with a dashed arc, and form the analog $C_\Delta(\mathbb{F}_p, g)$ of a Cayley color graph. Denote the underlying pseudograph by $G(\mathbb{F}_p, g)$, where we suppress the Δ, as it is entirely determined by g. We call $G(\mathbb{F}_p, g)$ the *finite field graph* (associated with multiplicative generator g).

Def. 14-6. The *genus of the field* $\mathbb{F}_p$ is given by

$$\gamma(\mathbb{F}_p) = \min \gamma(G(\mathbb{F}_p, g)),$$

the minimum taken over all single generators g for the multiplicative group.

In Figure 14-2 we give plane imbeddings of $C_\Delta(\mathbb{F}_5, 2)$ and $C_\Delta(\mathbb{F}_7, 3)$. We observe that:

(i) For $\mathbb{Z}_5$ (respectively $\mathbb{Z}_7$) there is only one other choice for $g (2^{-1} = 3$ in $\mathbb{Z}_5$, $3^{-1} = 5$ in $\mathbb{Z}_7)$. But in general choosing g^{-1} instead of g merely reverses the multiplicative arcs; thus $G(\mathbb{F}_p, g^{-1}) = G(\mathbb{F}_p, g)$, always. So, both of the finite fields of Figure 14-2 have a unique graphical model. Both fields are planar, as are $\mathbb{Z}_2$ and $\mathbb{Z}_3$. (Problem 14-3.)

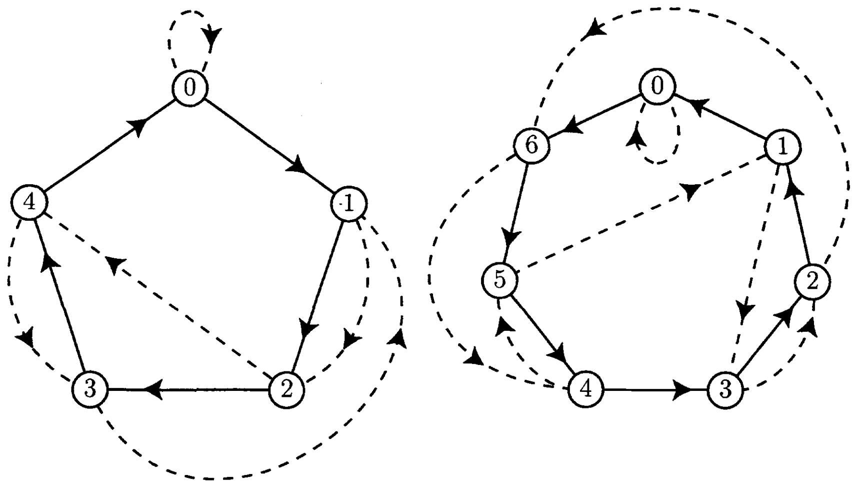
(ii) There is exactly one loop, at 0. This will pertain, in general.

(iii) There are exactly two digons. This will generalize also: a digon (for $p \geq 5$) can result only from the situation of Figure 14-3, where $\delta = g$ or g^{-1}. Suppose $\delta = g$. Then $a + 1 = ag$: $a(g-1) = 1$, and $a = (g-1)^{-1}$ is uniquely determined. (Note the use of the distributive axiom in the above argument.) Similarly, if $\delta = g^{-1}$, a is uniquely determined as $a = (g^{-1} - 1)^{-1}$.

(iv) Both pseudographs $G(\mathbb{F}_p, g)$–not the plane imbeddings – have reflective symmetry fixing vertex 0; this is because $(-a)g = -(ag)$, where we use additional field properties. In general, $G(\mathbb{F}_p, g)$;has this reflective symmetry. This is also consistent with (iii).

The situation is more complex for $p = 11$. The choices for g are: $g = 2$ (or $2^{-1} = 6$) and $g = 7$ (or $7^{-1} = 8$). We find (Problem 14-4) that $\gamma(G(\mathbb{F}_{11}, 2)) = 1$, but $\gamma(G(\mathbb{F}_{11}, 7)) = 0$. Thus $\gamma(\mathbb{F}_{11}) = 0$.

For general p, Jones' analysis proceeds as follows: by algebraic arguments similar to that using Figure 14-3 (see Problem 14-5), we see



$C_\Delta(\mathbb{F}_5, 2)$ $C_\Delta(\mathbb{F}_7, 3)$

(a) (b)

Figure 14-2.

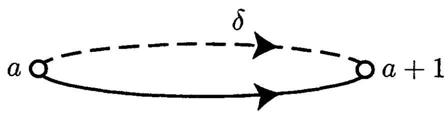


Figure 14-3.

that $G(\mathbb{F}_p, g)$, for $p \geq 7$, always has precisely 4 3-cycles and precisely 7 4-cycles. Thus

$$\begin{aligned} r_1 &\leq 1 \\ r_2 &\leq 2 \\ r_3 &\leq 4 \\ r_4 &\leq 7 \end{aligned}$$

in any imbedding. This leads (*via* Theorem 5-14 and Problem 14-6) to:

Thm. 14-7. For p prime, $\gamma(\mathbb{F}_p) \geq \left\lceil \frac{p-15}{10} \right\rceil$.

If our initial interest is in the planar case, then by Theorem 14-7 $p \leq 13$. But in examining all possible generators for $\mathbb{Z}_{13}$, Jones found a Kuratowski subgraph for each possible pseudograph. Thus the planar values are precisely $p = 2, 3, 5, 7$, and 11, for the fields $\mathbb{F}_p$. Recall that we are insisting (since it seems most natural to do so) that

$\Delta_+ = \{1\}$ and $\Delta_\times = \{g\}$. But note that if we keep $|\Delta_\times| = 1$ but use $|\Delta_+| = 2$ instead of 1, remove the one loop and destroy all digons, we have $q = 3p - 5 > 3p - 6$, so that the resulting pseudograph is not planar, by Problem 5-18. Similarly, if we keep $|\Delta_+| = 1$ but use $|\Delta_\times| = 2$, after removing the two loops, destroying all digons, and reversing the elementary subdivision at vertex 0, we have a nonplanar pseudograph of order $p - 1$ with $q = 3p - 7 > 3p - 9 = 3(p - 1) - 6$. Thus the most natural class of models also produces the most efficient one.

For toroidal fields of the form $\mathbb{Z}_p$, Theorem 14-7 yields $p \leq 23$. By further analyzing possible region distributions, Jones found that $\gamma(\mathbb{Z}_{13}) = \gamma(\mathbb{Z}_{17}) = 1$, $\gamma(\mathbb{Z}_{19}) = 2$, and $\gamma(\mathbb{Z}_{23}) \geq 2$.

14-4. The Genus of $\mathbb{F}_{p^r}$

Again for brevity, set $\mathbb{F}_{p^r} = GF(p^r)$. To find analogs of the theorems of Maschke and Proulx for the genera of finite planar and toroidal groups respectively, Jones combined her analysis of Section 14-3 for the case $r = 1$ with the case $r \geq 2$ of this section. In the general case we set $\Delta_+ = \{\alpha^{r-1}, \dots, \alpha^2, \alpha, 1\}$, a basis for $P_r(\alpha)$ (the vector space of all polynomials of degree less than r over $\mathbb{Z}_p$, isomorphic to $\mathbb{Z}_p^r$), where α is a primitive root of a monic irreducible polynomial of degree r over $\mathbb{Z}_p$; and we set $\Delta_\times = \{\alpha^k\}$, where $(p^r - 1, k) = 1$. Let $\Delta = \Delta_+ \cup \Delta_x$, and form the color graph $C_\Delta(\mathbb{F}_{p^r}, x^k)$. Then let $G(\mathbb{F}_{p^r}, \alpha^k)$ denote the underlying pseudograph; imbedding properties are not affected by this reduction. Again, we call this a *finite field graph.*

Def. 14-8. The *genus of the field* $\mathbb{F}_{p^r}$ is given by

$$\gamma(\mathbb{F}_{p^r}) = \min \gamma(G(\mathbb{F}_{p^r}, \alpha^k)),$$

where the minimum is taken over all k such that $(p^r - 1, k) = 1$.

Several observations are in order:

(1) Definition 14-8 includes Definition 14-6, as the special case $r = 1$.

(2) If $(p^r - 1, k) = 1$, then $(p^r - 1, -k) = 1$ also; but using $\Delta_\times = \{\alpha^{-k}\}$ instead of $\{\alpha^k\}$ merely reverses all multiplicative arrows, so that the underlying pseudographs are isomorphic. Thus the minimum is effectively taken over $\frac{1}{2}\phi(p^r - 1)$ multiplicative generators, where ϕ is the euler phi function.

(3) For $\gamma(\mathbb{F}_{p^r})$ to be well-defined, we need to establish two properties:
   (a) For a fixed monic irreducible polynomial $m(x)$ of degree r over $\mathbb{Z}_p$, $G(\mathbb{F}_{p^r}, \alpha^k)$ depends upon k but not upon the particular primitive root α of $m(x)$ selected. (See Problem 14-12.)

(b) Choosing a different $m(x)$ as above yields an isomorphic collection of pseudographs. (See Problem 14-13. This problem is open in general, yet the claim holds for $p = 4, 8, 9$, and 16–all that we need for the rest of this chapter. For later work, pending resolution of Problem 14-13, we would take the minimum over all monic irreducible polynomials of degree r, also.)

For example, if $p = r = 2$, the only suitable polynomial is x^2+x+1 and we take $\alpha^2 = \alpha + 1$ so that $\langle\alpha\rangle = \{1, \alpha, \alpha+1\}$. The plane color graph $C(\mathbb{F}_4, \alpha)$ is given in Figure 14-4. We use $\Delta_+ = \{\alpha, 1\}$, modelling addition by α with a crossed edge, and $\Delta_\times = \{\alpha\}$.

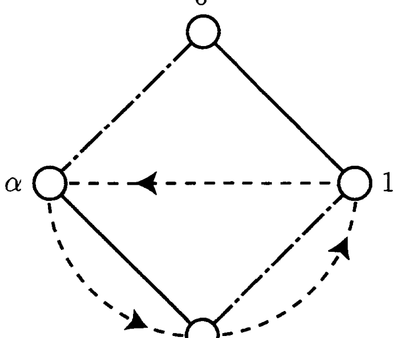


Figure 14-4.

Now consider $p = 2$ still, but with $r = 3$. We use $x^3 + x + 1$, as in the example of Section 14.2. We display $C(\mathbb{F}_8, \alpha)$ in Figure 14-5, drawn in the plane with one crossing. This establishes that $\gamma(\mathbb{F}_8) \leq 1$. To show that $\gamma(\mathbb{F}_8) \geq 1$ we could separately consider color graphs $C(\mathbb{F}_8, \alpha), C(\mathbb{F}_8, \alpha^2)$, and $C(\mathbb{F}_8, \alpha^3)$, finding a Kuratowski subgraph in each case. (These are three mutually non-isomorphic graphs, as examining the configurations of the digons readily shows.) Instead, we note that $G_{\{1,\alpha,\alpha^2\}}(\mathbb{Z}_2^3) = Q_3$ in each case. By Theorem 5-25, Q_3 is uniquely imbeddable in the sphere. If we can show that one multiplicative arc joins antipodal vertices, then $G(\mathbb{F}_8, \alpha^k)$ will not be planar, by the Jordan Curve Theorem, since a 6-cycle divides the plane (obtained under stereographic projection) into two regions, with one antipode in each. So, solve the equation $a + (1 + \alpha + \alpha^2) = a\alpha^k$ for $a = (1 + \alpha + \alpha^2)(\alpha^k - 1)^{-1}$, the unique starting vertex of such an arc. (For Figure 14-5, we calculate that $a = (1 + \alpha + \alpha^2)(\alpha - 1)^{-1} = \alpha^5(\alpha^3)^{-1} = \alpha^2$, as is displayed in the figure.) Thus $\alpha(\mathbb{F}_8) = 1$.

Similar arguments show that $\gamma(\mathbb{F}_{p^r}) \geq 2$ for each $p^r \geq 16$, with $r \geq 2$: for $p = 2$, start with $K_2 \times K_2 \times K_2 \times K_2 = C_4 \times C_4$ on the torus; for $p \geq 5$, find a subgraph of C_{p^r} homeomorphic to $C_5 \times C_5$ on the torus. In either case, we can find an edge that cannot be added on the torus, so that $\gamma(\mathbb{F}_{p^r}) \geq 1 + 1 = 2$. For $p = 3$ and $p^r \geq 16, C_3^3$ is a subgraph, and we know that $\gamma(C_3^3) = 7$ (Theorem 7-29).

Thus to complete the classifications of planar and toroidal finite fields, we have only to study $\mathbb{F}_9$. Consider the construction of $\mathbb{F}_9$ given in Section 14-2, yielding the toroidal imbedding of Figure 14-6. But $\gamma(\mathbb{F}_9) \geq \gamma(C_3 \times C_3) = 1$. Thus $\gamma(\mathbb{F}_9) = 1$.

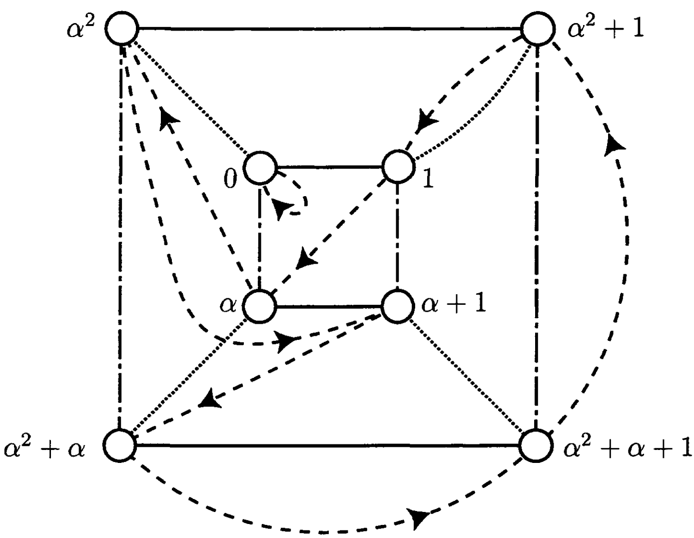


Figure 14-5.

Collecting the results of this and the preceding section, we get the following two theorems of Jones:

Thm. 14-9. The finite field $\mathbb{F}_{p^r}$ is planar if and only if $p^r = 2, 3, 4, 5, 7$, or 11.

Thm. 14-10. The finite field $\mathbb{F}_{p^r}$ is toroidal if and only if $p^r = 8, 9, 13$, or 17.

The next theorem gives two asymptotic results, also developed by Jones in [J4].

Thm. 14-11. The genus of:

(i) $\mathbb{F}_{2^r}$ is asymptotically $1 + 2^{r-3}(r-4)$.
(ii) $\mathbb{F}_{p^p}$ is asymptotically $\frac{p^{p+1}}{4}$.

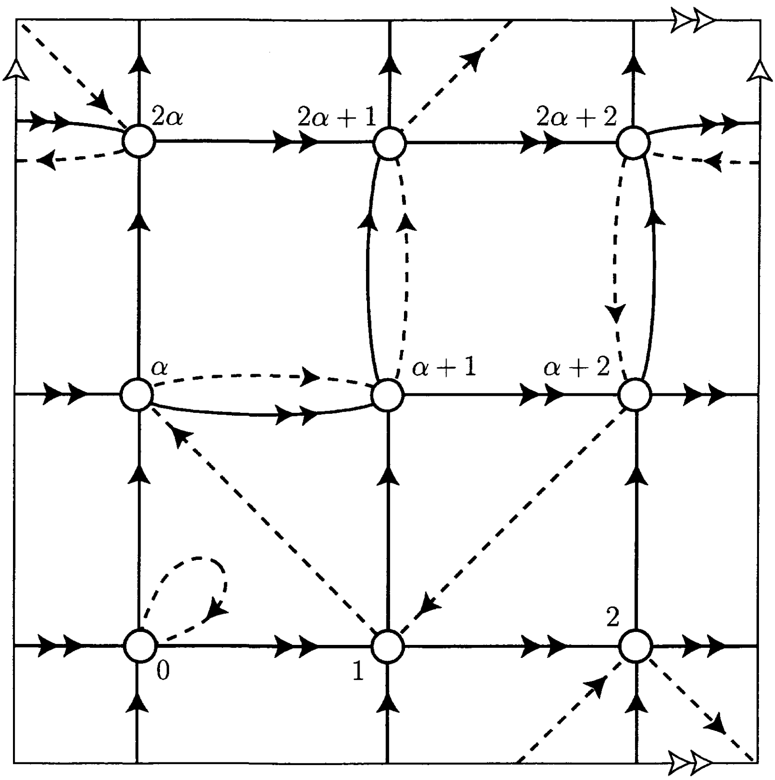


Figure 14-6.

The following bounds [J4] will be useful in the next section:

Thm. 14-12. For p prime:

(i) $\gamma(\mathbb{F}_{p^2}) \geq \frac{2p^2-19}{10}$;
(ii) $\gamma(\mathbb{F}_{p^3}) \geq \frac{p^3-2}{6}$.

14-5. Further Results

Using Theorem 6-32, Jones [J4] found:

Thm. 14-13. Each $G(\mathbb{F}_{p^r}, \alpha^k)$ is upper-imbeddable; thus the maximum genus of $\mathbb{F}_{p^r}$ is given by: $\gamma_M(\mathbb{F}_p r) = \lceil \frac{rp^r+1}{2} \rceil$.

Cor. 14-14. Asymptotically, if $G(\mathbb{F}_{p^p})$ is imbedded on S_k, then $\frac{p^{p+1}}{4} \leq k \leq \frac{p^{p+1}}{2}$.

Now we give a result analogous to that of Theorem 7-30.

Thm. 14-15. If $\mathbb{F}_q$ is a subfield of the finite field $\mathbb{F}_{q'}$, then $\gamma(\mathbb{F}_q) \leq \gamma(\mathbb{F}_{q'})$.

PROOF. Let $q = p^t$ and $q' = (p')^{t'}$; then since p^t divides $(p')^{t'}$ (because of the additive structure), $p' = p$ and $t' \geq t$. Since $p^t - 1$ divides $(p')^{t'} - 1$ (because of the multiplicative structure), $t' = kt$, for some integer k. We can assume $k \geq 2$. We consider two cases, depending upon the parity of p.

(i) $\mathbf{p = 2}$. Note that $\gamma(\mathbb{F}_2) = 0 \leq \gamma(\mathbb{F}_{2^k})$ and $\gamma(\mathbb{F}_4) = 0 \leq \gamma(\mathbb{F}_{2^{2k}})$, so we can assume $t \geq 3$. Then $\gamma(\mathbb{F}_{2^t}) \leq \gamma(Q_t) + 2^t - 1$, as we could use a separate handle for each multiplicative edge other than the loop. So

$$\begin{aligned}\gamma(\mathbb{F}_{2^t}) &\leq 1 + 2^{t-3}(t-4) + 2^t - 1 \\ &\leq 1 + 2^{kt-3}(kt-4) \\ &= \gamma(Q_{kt}) \\ &\leq \gamma(\mathbb{F}_{2^{kt}}).\end{aligned}$$

(ii) $\mathbf{p \geq 3}$. Here we have two subcases, depending upon t.

(a) $\mathbf{t \geq 2}$. We combine lower- and upper-bound arguments, using Corollary 6-14 for the lower bound and Problem 10-13 for the upper bound.

(1) $\gamma(\mathbb{F}_{p^{kt}}) \geq \gamma((C_p)^{kt}) \geq 1 - \frac{p^{kt}}{2} + \frac{ktp^{kt}}{6} = \frac{p^{kt}}{6}(kt-3)+1$.

(2) $\gamma(\mathbb{F}_{p^t}) \leq \gamma((C_p)^t) + p^t - 1 \leq \frac{p^{t-1}(t+1)(p-1)}{2}$. Thus

(3)

$$\begin{aligned}\gamma(\mathbb{F}_{p^t}) &\leq \frac{p^{t-1}(t+1)(p-1)}{2} \\ &\leq \frac{p^{2t}}{6}(2t-3)+1 \\ &\leq \frac{p^{kt}}{6}(kt-3)+1 \\ &\leq \gamma(\mathbb{F}_{p^{kt}}).\end{aligned}$$

(b) $\mathbf{t = 1}$. Now we focus on k.

(1) $\mathbf{k \geq 4}$. $\gamma(\mathbb{F}_p) \leq p-3 < \frac{p^4}{6}+1 \leq \frac{p^k}{6}(k-3)+1 \leq \gamma(\mathbb{F}_{p^k})$, where the last inequality is by (ii)(a)(1) above (whose proof is valid also for $t = 1$).

(2) $\mathbf{k = 3}$. $\gamma(\mathbb{F}_p) \leq p-3 \leq \frac{p^3-2}{6} \leq \gamma(\mathbb{F}_{p^3})$, where the last inequality is that of Theorem 14-12(ii).

(3) $\mathbf{k = 2}$.

(α) $\mathbf{p = 3}$. $\gamma(\mathbb{F}_3) = 0 < 1 = \gamma(\mathbb{F}_9)$.

(β) $\mathbf{p \geq 5}$. $\gamma(\mathbb{F}_p) \leq p-3 \leq \frac{2p^2-19}{10} \leq \gamma(\mathbb{F}_{p^2})$, where we have used Theorem 14-12(i).

□

We give yet another result of Jones [J4], this one analogous to Theorem 7-32.

Thm. 14-16. For each nonnegative integer k, there are at most finitely many finite fields of genus k.

14-6. Problems

14-1.) Let $G(n, \delta)$ be the graph with $\mathbb{Z}_n$ as vertex set and arcs determined by multiplication by $\delta \in \mathbb{Z}_n - \{0\}$.
(a) Show that $G(n, \delta)$ is regular (of degree 2) if and only if $(n, \delta) = 1$.
(b) Show that $G(n, \delta)$ can be either (weakly) connected or disconnected, if $(n, \delta) > 1$.
(c) **Find a characterization of $G(n, \delta)$ being (weakly) connected.

14-2.) Reinterpret $\mathbb{Z}_9$ as consisting of all the roots of $x^9 - x$, by factoring into monic irreducible polynomials over $\mathbb{Z}_3$ and then finding the roots of each polynomial, as numbers in the extension field. Which $\phi(8) = 4$ roots are primitive, and what irreducible factors do they correspond to? (ϕ is the euler phi function.)

14-3.) Find plane graphs for $\mathbb{F}_2$ and $\mathbb{F}_3$.

14-4.) Show that $G(\mathbb{F}_{11}, 2)$ is toroidal, but that $G(\mathbb{F}_{11}, 7)$ is planar.

14-5.) Show that every imbedding of $G(\mathbb{F}_p, g)$, for $p \geq 7$, has:
(i) $r_3 \leq 4$;
(ii) $^*r_4 \leq 7$.

14-6.) Prove Theorem 14-7.

14-7.) **How many finite fields of genus two are there? What are they?

14-8.) **Prove or disprove: Let $\mathbb{F}$ be an infinite field; then either $\mathbb{F}$ is planar or $\mathbb{F}$ has no finite genus (cf. Corollary 7-10).

14-9.) **Find an example of an infinite planar field, or prove that none exists.

14-10.) **We saw in Section 7-2 that the (additive) group $\mathbb{Z}_3^3$ has genus 7 and in Section 14-5 that $\gamma(\mathbb{F}_{27}) \leq 36$ ((ii)(a)(2) of the proof of Theorem 14-15). Find $\gamma(\mathbb{F}_{27})$ precisely.

14-11.) The smallest field whose genus is unknown is $\mathbb{F}_{16}$. Show that $2 \leq \gamma(\mathbb{F}_{16}) \leq 4$.

14-12.) Show that, for a fixed monic irreducible polynomial $m(x)$ of degree r over $\mathbb{Z}_p$, $G(\mathbb{F}_{p^r}, \alpha^k)$ depends on k but not on the particular (primitive) root α of $m(x)$ selected.

14-13.) **We saw in Section 14-4 that fixing a monic irreducible primitive polynomial of degree r for $GF(p^r)$ determines a collection of pseudographs $G(\mathbb{F}_{p^r}, \alpha^k)$, where $(p^r - 1, k) = 1$. Show that selecting a different such polynomial for $GF(p^r)$ determines a collection of pseudographs which are pairwise isomorphic to those of the first collection.

14-14.) Remove requirement (2) of Section 14-1 and define the *genus of the ring* $\mathbb{Z}_n$ (whether n is prime or not) to be the minimum genus $\gamma(G(\mathbb{Z}_n, g))$, taken over all $g \neq \pm 1$ in $\mathbb{Z}_n$ so that $(n, g) = 1$. (If $g = \pm 1$ were allowed, then we would have $\gamma(\mathbb{Z}_n) = 0$ for all n. We also restrict $n \geq 7$, since no suitable g exists for $n = 2, 3, 4$, or 6.) Study this parameter on this common class of rings. Here are some sample results to prove:

(1) $\gamma(G(\mathbb{Z}_9, 2)) = 1$, but $\gamma(G(\mathbb{Z}_9, 4)) = 0$; thus $\gamma(\mathbb{Z}_9) = 0$.

(2) $\gamma(\mathbb{Z}_{13}) = 1$, yet $\gamma(\mathbb{Z}_{16}) = 0$; thus the function is not non-decreasing.

(3) In fact, $\gamma(\mathbb{Z}_{4n}) = 0$, for all $n \geq 2$. (Hint: choose $g = 2n-1$.) Thus there are infinitely many planar rings in this class, under the modified definition.

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CHAPTER 15

FINITE GEOMETRIES ON SURFACES

In previous chapters we have defined, and studied, the genus parameter for a variety of mathematical structures, primarily in the finite case: graphs, groups, block designs, hypergraphs, and fields. For the latter four structures, the genus parameter is defined *via* a relevant graph: the Cayley graph of a group (for a particular generating set), the Levi graph of a design or of a hypergraph, and the finite field graph (for a particular finite field and multiplicative generator). Thus everything is tied back to the genus parameter for graphs.

In this chapter we extend this idea to finite geometries. As each such structure can be regarded as either a design or a hypergraph, it is again the associated Levi graph which is relevant. After illustrating a discussion of axiom systems in general with n-point geometry, we consider the configurations of Fano, Pappus, and Desargues, as well as a common generalization. Then we study the classical protective and affine planes in some detail.

For general references regarding finite geometries, see [CD1], [D1], [H1], [H9], [R20], and [S10].

15-1. Axiom Systems for Geometries

Formally, an axiom system for a geometry starts with a point set P, a line set L, and a symmetric incidence relation $I \subseteq (P \times L) \cup (L \times P)$. We find it convenient to regard each line ℓ as consisting of all points p such that $(p, \ell) \in I$; thus ℓ is a subset of P. The more abstract formulation is useful for discussions involving duality, but our convention facilitates the modelling we seek, as it is in conformity with the usual model for Euclidean geometry. The axiom system continues with statements about points and lines which are assumed to be true, and the challenge is to derive other true statements from these axioms.

The individual axioms specify various properties for the points and lines, such as:

(i) existence
(ii) uniqueness
(iii) uniformity
(iv) finiteness.

The axiom system as a whole might satisfy the desirable properties of:

(a) consistency
(b) independence
(c) completeness.

An axiom system is said to be *consistent* if no contradictions can be derived from the axioms. A *model* for the system consists of a specified set P and a set L of specified subsets of P which together satisfy all the axioms. The existence of a model for a system establishes its consistency, as any contradiction from the axioms would produce a statement both true and false for the model, an impossibility.

An axiom system is said to be *independent* if no axiom in the system can be derived from the others. This property is established by finding, for each axiom, a model failing that axiom but satisfying all the others.

An axiom system is said to be *complete* if every statement in the system can be either proven true or proven false by using the axioms. This property can be verified by showing that any two models for the system are isomorphic: if the system has a unique model, then one needs only check if a given statement is true or false for that model. In the finite case, this can be done finitely.

Thus we see the importance of models for axiom systems in general. We note that Gödel's incompleteness theorem–no axiom system including a formal elementary number theory can be shown to be both consistent and complete–does not apply to geometric axiom systems which make no reference to formal elementary number theory. Thus the systems considered in this chapter might well be shown to be both consistent and complete, and perhaps satisfy independence as well.

Again in conformity with Euclid, all geometries considered in this chapter will satisfy the axiom:

(A) Two distinct points belong to at most one line.

15-2. n-Point Geometry

Our initial example is *n-point geometry*, where $n \in \mathbb{N}$ is fixed. The axioms are:

(1) There are exactly n points.
(2) Each pair of distinct points determine a unique line.
(3) Each line consists of exactly two points.

Observe that (1) provides existence and finiteness; (2) gives existence (if $n \geq 2$) and uniqueness; and (3) imposes uniformity. One model for n-point geometry is the complete graph K_n, so the system is consistent. In fact, with respect to isomorphism K_n is the only model for n-point geometry, so the system is also complete. For $n \geq 4$, the graphs K_{n+1}, C_n, and P_n (one line) show respectively that axiom (1) cannot be derived from the others, and similarly for axioms (2) and (3). Thus, for $n \geq 4$, the system is independent. (This is true for $n = 3$ as well, but now use $K_1 \cup K_2$, in place of C_3.)

Now, either from the axioms or from the unique model of K_n, we can prove theorems, such as:

(4) There are exactly $\frac{n(n-1)}{2}$ lines.
(5) Each point is on exactly $n - 1$ lines.

We also find that, for $n \geq 3$, given a line ℓ and a point p not on ℓ, the number of lines through p and parallel to ℓ (i.e. disjoint from ℓ) is $n-3$. Thus n-point geometry satisfies the Euclidean parallel postulate (Playfair's form) precisely for $n = 4$ (the parabolic case); $n = 3$ and $n \geq 5$ are the elliptic and hyperbolic cases respectively.

15-3. The Geometries of Fano, Pappus, and Desargues

In this section we introduce three classical finite geometries. The *Fano plane* has the following axioms:

(FA1) There is a least one line.
(FA2) Every line contains exactly three points.
(FA3) Not all points lie on one line.
(FA4) Two distinct points belong to exactly one common line.
(FA5) Two distinct lines contain exactly one common point.

Using these axioms, one can construct the unique (up to isomorphism; see Problem 15-1) model of Figure 15-1; the particular labelling employed will be useful in later sections

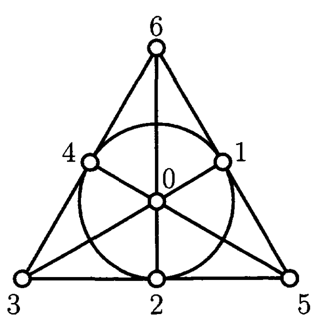


Figure 15-1.

Now, from the axioms (or from the model, since the system is complete), one can deduce theorems, such as:

(FT1) $|P| = 7$.
(FT2) $|L| = 7$.
(FT3) Each point belongs to exactly three lines.

We note that the model also arises from various theorems of concurrency in Euclidean geometry: for any triangle in the Euclidean plane, each of the following triples of lines are concurrent:

(i) the perpendicular bisectors of the sides;
(ii) the altitudes;
(iii) the internal angle bisectors;
(iv) the medians.

For an equilateral triangle, as in Figure 15-1, the four points of concurrency are identical; the three vertices, the three midpoints of the sides, and *the* point of concurrency constitute the point set of the Fano plane. The lines are given by the three sides of the equilateral triangle, the three concurrent lines, and the inscribed circle.

Here is another theorem from Euclidean geometry, due to Pappus of Alexandria (c. 300-350 a.d.).

Thm. 15-1. If A, B, and C are three distinct points on line L and A', B', and C' are three different distinct points on line $L' \neq L$, then the points $AB' \cap A'B$, $AC' \cap A'C$, and $BC' \cap B'C$ are collinear.

One such situation is depicted in Figure 15.2. This gives rise to a geometry of nine lines on nine points, as given in Table 15-1; the final line DEF is the line established by the theorem.

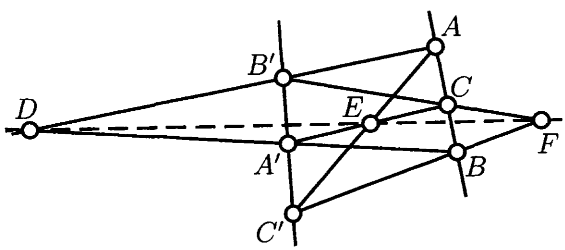


Figure 15-2.

ADB'	$A'DB$	ABC
$A'EC$	AEC'	$A'B'C'$
BFC'	$B'FC$	DEF

Table 15-1.

Here are formal axioms for the *geometry of Pappus*.

(PA1) There is at least one line.

(PA2) Every line contains exactly three points.
(PA3) Not all points belong to the same line.
(PA4) If a point p is not on a line ℓ, then there is a unique line ℓ' containing p such that no point is in both ℓ and ℓ'.
(PA5) If a line ℓ does not contain a point p, then there is a unique point p' in ℓ such that no line contains both p and p'.
(PA6) Except as disallowed by (PA5), two distinct points belong to a unique line.

The system is consistent, as the model of Figure 15-2 attests. It is also complete; see Problem 15-1. Some theorems deducible from either the axioms or the model are:

(PT1) $|P| = 9$.
(PT2) $|L| = 9$.
(PT3) Each point belongs to exactly three lines.
(PT4) For each point p there are exactly two points not sharing a line with p.
(PT5) For each line ℓ there are exactly two lines not sharing a point with ℓ.

Recall that, in a geometry, point p is incident with line ℓ if and only if ℓ is incident with p. Thus each geometry (P, L) has a *dual* geometry (L, P), with the roles of points and lines exchanged. Note that (PA4) and (PA5) are dual statements, as are (PT1) and (PT2), (PT4) and (PT5), and (PT3) and (PT2). In fact, the geometry of Pappus is self-dual, as are the Fano geometry and the geometry we present next (see Problem 15-2). We formalize this idea with two definitions.

Def. 15-2. Two geometries (P, L) and (P', L') are *isomorphic* if there exists a one-to-one, onto function $f : P \longrightarrow P'$ such that $L' = \{f(\ell) | \ell \in L\}$, where $f(\ell) = \{f(p) | p \in \ell\}$.

Def. 15-3. A geometry (P, L) is *self-dual* if there is an isomorphism between (P, L) and the dual geometry (L, P).

As an infinite class of examples of self-dual geometries, we offer the Paley maps of Section 16-8. By Theorem 16-81, these are all self-dual as block designs, and hence as geometries.

Now we consider one more theorem from Euclidean geometry; this one is due to Desargues (1593-1662).

Thm. 15-4. If two triangles are perspective from a point, then they are perspective from a line.

In Figure 15-3, triangles BCD and EFG are perspective from point A. Let $H = BC \cap EF$, $I = CD \cap FG$, and $J = BD \cap EG$. Then the claim of Desargues' Theorem is that H, I, and J are collinear. This yields a geometry of ten lines on ten points, as given in Table 15-2. The final line is the one guaranteed by the theorem. Formal axioms

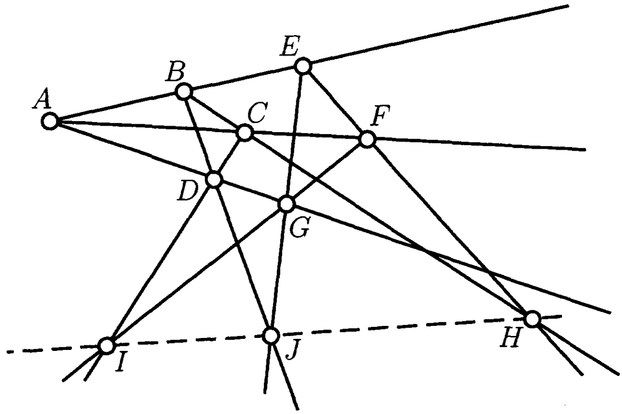


Figure 15-3.

ABE	BCH	EFH	HIJ
ACF	CDI	FGI	
ADG	BDJ	EGJ	

Table 15-2.

for this geometry require two new terms. A line ℓ in the *geometry of Desargues* is a *polar* of point p if no line containing p contains a point of ℓ. Dually, point p is a *pole* of line ℓ if no point of ℓ is in a line containing p. Then we have:

(DA1) There is at least one point.
(DA2) Every point has a unique polar.
(DA3) Every line has a unique pole.
(DA4) Two distinct points belong to at most one common line.
(DA5) Every line contains exactly three points.
(DA6) If point p is not in line ℓ, then there is a point p' in both ℓ and the polar of p.

Some theorems are:

(DT1) $|P| = 10$.
(DT2) $|L| = 10$.
(DT3) Each point belongs to exactly three lines.
(DT4) Two lines parallel to the same line are *not* parallel to each other.
(DT5) Two points are collinear if and only if their polars intersect.
(DT6) If p is on the polar of p', then p' is on the polar of p.

15-4. Block Designs as Models for Geometries

The geometries considered thus far–n-point, Fano, Pappus, Desargues–have all been modelled by graph-like structures. But each has an alternate model as a block design (or as a hypergraph), by specifying the point set and then listing the lines as subsets of the point set. If every point is on at least one line, then just listing the line-subsets suffices, as P is then the union of all the line-subsets.

For n-point geometry, we would list all the 2-subsets of a fixed n-set. This gives an $\left(n, \frac{n(n-1)}{2}, n-1, 2, 1\right)$-BIBD.

For the Fano plane, Table 15-3 gives a $(7,7,3,3,1)$-BIBD, or Steiner Triple System. The lines agree with Figure 15-1, and are cyclically generated from $\{0,1,3\}$– a perfect difference set in $\mathbb{Z}_7$.

0	1	3
1	2	4
2	3	5
3	4	6
4	5	0
5	6	1
6	0	2

Table 15-3.

The Pappus design of Table 15-1 is a $(9,9,3,3;0,1)$-PBIBD, based upon the strongly regular graph $K_{3(3)}$. The design is resolvable (by columns in the table), group divisible, and transversal.

The Desargues design of Table 15-2 is a $(10,10,3,3;0,1)$-PBIBD, based upon the strongly regular graph $\overline{\Pi}$, the complement of the Petersen graph Π. (The strongly-regular parameters for $\overline{\Pi}$ are $p_{22}^2 = 3$ and $p_{22}^1 = 4$.) Figure 15-4 shows Π; the polar of each point p is given by the neighbors of p in Π.

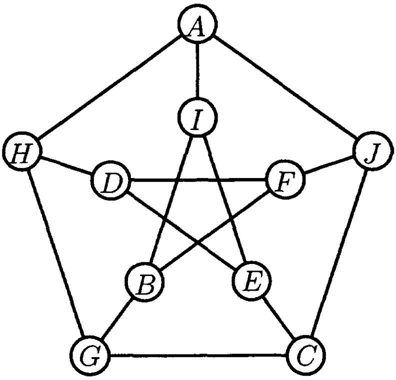


Figure 15-4.

Recall that the Levi graph of a design–and hence of a geometry–has vertex set $P \cup L$ and all edges of the form $\{p, \ell\}$, where $p \in \ell$. Another relevant graph for the study of geometries is the Menger graph.

Def. 15-5. The *Menger graph* of a geometry (P, L) has vertex set P, and two vertices adjacent if and only if they are collinear (i.e. belong to a common line).

Thus the n-point, Fano, Pappus, and Desargues geometries have Menger graphs (respectively) $K_n, K_7, K_{3(3)}$, and $\overline{\Pi}$. Note that non-isomorphic geometries, such as 7-point geometry and the Fano plane, can have the same Menger graph.

We remark that each of the Fano, Pappus, and Desargues geometries corresponds to a K_3-decomposition of the edge set of the associated Menger graph.

15-5. Surface Models for Geometries

The spirit of this book has been to model abstract mathematical structures, using concrete representations. Thus we prefer the graph-like models of Sections 15-2 and 15-3 to the design models of Section 15-4. But we are not satisfied with the former as currently given. For example, Figure 15-1 for the Fano plane has several deficiencies:

(D1) The line $\{1, 2, 4\}$ is differently shaped, yet that line is indistinguishable from the others, *via* the axioms.
(D2) Each other line has two "end" points and one "middle" point, yet there is no axiom for "betweenness."
(D3) There are three "crossings" of lines that have no meaning in the geometry.
(D4) One cannot discern that $r = 3$, by looking at small neighborhoods of points $0, 1, 2$, and 4.

Figures 15-2 and 15-3 suffer all but (D1) above. Of course, it is (D3) that is particularly worrisome to a topological graph theorist.

To overcome these deficiencies, we regard each geometry as a hypergraph (points are hypervertices, lines are hyperedges) and use the construction of Section 13.3. We first imbed the Levi graph of the geometry, and then modify that imbedding to depict the hypergraph: certain regions are the hyperedges; what remain are the *hyperregions*. Of course, we wish to do this as efficiently as possible.

Def. 15-6. The *genus of a geometry* $H = (P, L)$ is the genus of its Levi graph: $\gamma(H) = \gamma(G(H))$.

Note that we are assuming the geometry to be given as $H = (P, L)$, rather than as a perhaps incomplete axiom system, so that $G(H)$ is well-defined.

Recall that the process that modified the $G(H)$ imbedding to an imbedding of $G^*(H)$–depicting the hypergraph H on the same surface–is reversible. If every line of the geometry has exactly three points ($k = 3$ in block-design notation), then $G^*(H)$ is just the Menger graph. If, in addition, two points are on at most one line, then an efficient imbedding of such a geometry corresponds to an imbedding of the Menger graph having bichromatic dual, where the regions of one color class are all triangular, with triangles given by the K_3-decomposition of the Menger graph modelling the lines of the geometry, and the number of regions of the other color class is a maximum. This fact will be exploited in Sections 15-6 and 15-7.

We conclude this section by calculating the genus of n-point geometry, which we denote by H_n. Commence with K_n. Perform an elementary subdivision on each edge; the result is $G(H_n)$. Clearly $G(H_n)$ is homeomorphic to K_n. Thus we have:

Thm. 15-7. The genus of n-point geometry is $\gamma(H_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$, for $n \geq 3$.

We observe that, having imbedded $G(H_n)$ on the genus surface for K_n, we obtain $G^*(H_n)$ by replacing each edge of K_n with a digon (modelling a hyperedge of order 2). The regions for the K_n imbedding become the hyperregions. See Figure 13-4, for the case $n = 4$.

15-6. Fano, Pappus, and Desargues Revisited

We want to find a surface model of minimum genus for each of these geometries. The following lemma will be useful.

Lemma 15-8. If H is a geometry with the property that two points belong to at most one line, then the Levi graph $G(H)$ has girth at least six.

PROOF. Since $G(H)$ is bipartite, it must have even grith. But a 4-cycle in $G(H)$ would describe two points belonging to at least two different lines; see Figure 15-5. Thus the girth is six or more. □

Thm. 15-9. Let H be a geometry satisfying:

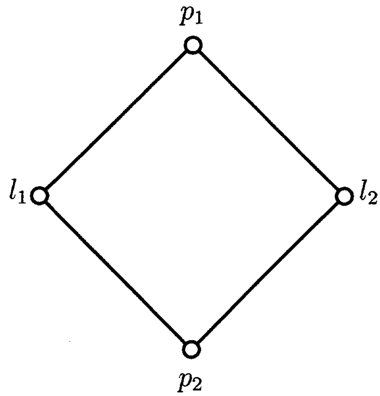


Figure 15-5.

(1) Each point is on at least three lines.
(2) Each line contains at least three points.
(3) Two points belong to at most one line.

Then H is nonplanar.

PROOF. By Lemma 15-8, $G(H)$ has girth at least six. Thus a planar imbedding would contradict Lemma 5-19. □

Hence all three geometries currently under study are nonplanar. By remarks of the preceding section, we seek bichromatic-dual imbeddings of the Menger graphs $K_7, K_{3(3)}$, and $\overline{\overline{\Pi}}$, where the regions of one color give a K_3-decomposition corresponding to the lines of the geometry and the number of regions of the other color class is a maximum. That the latter regions all be triangular is, for girth six, a sufficient condition (but not a necessary one, as we shall see) for the maximization we seek.

Figure 13-5 shows that the Fano plane is toroidal. Figure 10-5, with the nine diagonals of positive slope added within the square regions and the vertices appropriately labelled, shows that the geometry of Pappus is toroidal also. (See [FW2], or Figure 15-17, for an alternative model on the torus.) Both imbeddings cover Figure 10-6 (with $\Gamma = \mathbb{Z}_7$ and $\Delta = \{1,2,3\}$ for Fano, and $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3$ with $\Delta = \{(1,0),(0,1),(1,1)\}$ for Pappus).

The situation for Desargues is more complicated. A toroidal imbedding of $\overline{\overline{\Pi}}$ (with $p = 10, q = 30$, and $r = r_3 = 20$) would seem possible, but unfortunately no such imbedding exists: the neighbors of vertex A, for example, form no wheel graph W_7 with line triangles alternating with hypertriangles (see Problem 15-6). Thus there seems to be insufficient symmetry to allow a useful voltage graph/covering space construction. (The Petersen graph strikes again.)

Figure 15-6, found by *ad hoc* methods [F1], shows that the geometry of Desargues has genus two. Note that the hyperregions include five triangles and three pentagons (instead of the ten triangles a toroidal imbedding required); the latter share point A. This facilitates the discovery of a 3-fold rotational symmetry (about A) which, in fact, generates the map automorphism group $\operatorname{Aut} M$. (Note that some regions–those labelled I, II, and III–appear three times in the figure, to better illustrate this symmetry.

We examine the orbits of the action of $\operatorname{Aut} M = < \alpha >$, where $\alpha = (A)(BCD)(EFG)(HIJ)$, on both P and L, as well as on the hyperregions.

(1) The orbits on P give the pole A, the polar $\{H, I, J\}$, and the two triangles BCD and EFG of perspectivity.
(2) The orbits of the induced action on L give the three lines containing pole A, the three lines bounding triangle BCD, the three lines bounding triangle EFG, and the polar $\{H, I, J\}$.
(3) The orbits of the induced action on the hyperregions give the three pentagons, the three triangles the points of the polar make with the lines at pole A, and each triangle of perspectivity is fixed.

In summary, $\operatorname{Aut} M$ fixes the point of perspectivity, the line of perspectivity, the two triangles of perspectivity, and nothing else.

We summarize this section, in:

Thm. 15-10. The geometries of Fano and of Pappus are toroidal. The geometry of Desargues has genus two.

15-7. 3-Configurations

For this section and the two following, we turn our attention to certain classes of geometries. As with the geometries previously encountered, we impose more structure on a geometry $H = (P, L)$ by specifying various axioms. We say that H is *finite* if both P and L are finite.

Def. 15-11. Let r and k be positive integers. A (k, r)-*configuration* is a geometry $H = (P, L)$ satisfying:

(CA1) P is non-empty and finite.
(CA2) Each point belongs to exactly r lines.
(CA3) Each line consists of exactly k points.
(CA4) Each pair of distinct points belong to at most one line.

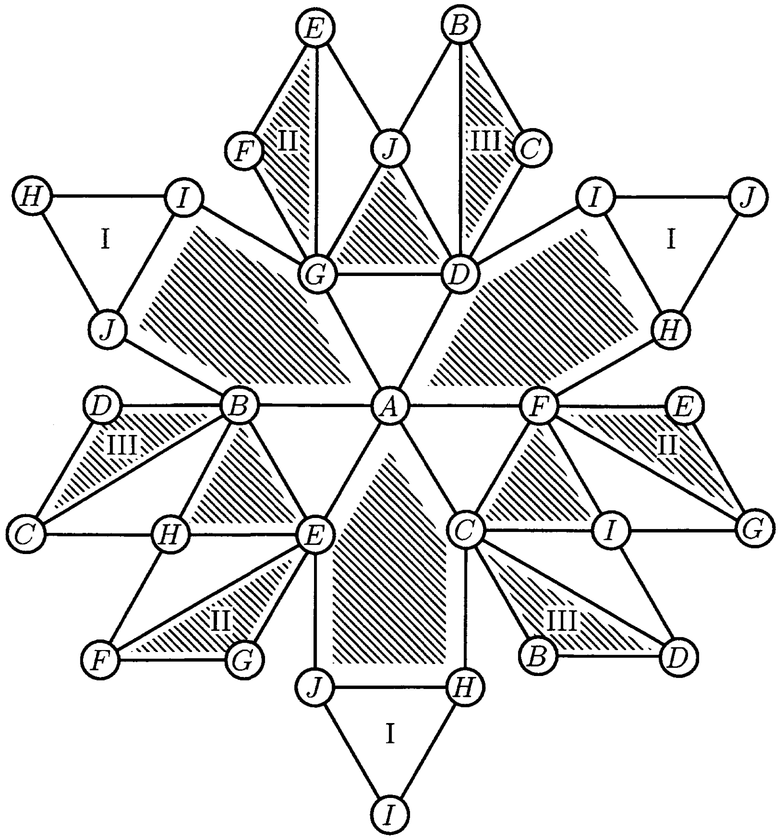


Figure 15-6.

Thus ($CA1$) establishes existence and finiteness (for P), ($CA2$) existence (of lines) and uniformity, ($CA3$) uniformity, and ($CA4$) uniqueness and finiteness (of L and hence of H).

Observe that every $(v, b, r, k, 1)$-BIBD, and every $(v, b, r, k; 0, 1)$-PBIBD, is a (k, r)-configuration, where $v = |P|$ and $b = |L|$.

Def. 15-12. If $r = k$, a (k, r)-configuration is said to be *symmetric.*

Note that the geometries of Fano, Pappus, and Desargues are all symmetric $(3, 3)$-configurations, whereas the $(n-1, 2)$-configuration given by n-point geometry is symmetric only for $n = 3$. The case $k = 3$ is particularly nice in our topological context, for then the modification $G^*(H)$ of the Levi graph $G(H)$ (see Section 13-3) is just the Menger graph, and it is convenient to focus on the latter in constructing surface models of H. Thus we specify $k = 3$ in our study of configurations, and we often suppress the "r" in the notation. There is already much interest in this first non-graphical hypergraph case. (A 2-configuration is just an r-regular graph, and graph imbedding is studied in Chapters 6 and 10.) For instance, every Steiner triple system is a 3-configuration. See Figueroa-Centeno [F1] for a study of topological models of configurations.

The following theorem is due to Gropp [G3].

Thm. 15-13. Let $|P| = v$ and $|L| = b$. There exists a $(3, r)$-configuration $H = (P, L)$ if and only if:

(i) $vr = 3b$; and
(ii) $v \geq 2r + 1$.

As one application of this theorem, we find that the sequence of symmetric 3-configurations commencing with the geometries of Fano, Problem 15-5, Pappus, and Desargues has *no* predecessor. However, the sequence continues indefinitely (see Problem 15-7).

Def. 15-14. A *configuration imbedding* (of *degree* r) of a graph G is a bichromatic-dual imbedding of G for which all the regions of one color class are triangles depicting the lines of a $(3, r)$-configuration $H = (P, L)$ having $P = V(G)$.

We address the following **PROBLEM:** For each pair (v, r) satisfying $vr \equiv 0 \pmod 3$ and $v \geq 2r + 1$, find a "nice" topological model of a $(3, r)$-configuration $H = (P, L)$ having $|P| = v$. By "nice" we mean to take the following into account as much as possible.

(i) We prefer the Menger graph of the configuration to be either complete or strongly regular, so that the corresponding design will be either a BIBD or a PBIBD.
(ii) We prefer surfaces to pseudosurfaces, and pseudosurfaces to generalized pseudosurfaces.
(iii) We prefer orientability to nonorientability.
(iv) We prefer the euler characteristic of the ambient space to be a maximum. (This requires the regions of the second color class to be as nearly triangular as possible.)
(v) We prefer the resulting map to have as many (line-preserving) symmetries as possible. (If we use a Cayley map M, then $|\operatorname{Aut} M| \geq v$, by Theorem 16-24.)

Thm. 15-15. Let G be a $2r$-regular graph of order v. Then the following statements are equivalent:

(i) G is a Menger graph for a $(3, r)$-configuration.

(ii) G is K_3-decomposable, into $\frac{vr}{3}$ 3-cycles.
(iii) G has a configuration imbedding, of degree r.

PROOF. **(i) implies (ii).** Let G be a Menger graph for a $(3, r)$-configuration. Then the $q = vr = bk = 3b$ edges of G partition into b 3-cycles corresponding to the lines of the geometry.

(ii) implies (iii). If G is K_3-decomposable into $\frac{vr}{3}$ 3-cycles, then let these cycles (each with arbitrary but fixed orientation) bound triangular regions. Any extension of the partial vertex rotations thus determined to a full rotation scheme for G will correspond to a configuration imbedding, with the initial triangular regions comprising one color class. The number of these triangles at each vertex will be r.

(iii) implies (i). Let a configuration imbedding of degree r be given for G. Take the vertices of G as points of a geometry, and the color class of regions having all triangles (or either color class, if the imbedding is triangular) as the line set. Then $k = 3$ uniformly, each pair of points belongs to at most one line, and G is a Menger graph for the $(3, r)$-configuration. □

We illustrate these ideas by restricting $v \leq 10$ and considering all possible values of r for each such v. See [F1] for a study of 3-configurations of low order.

1. **v = 1 or 2.** There are no possible values of r.
2. **v = 3, r = 1.** K_3 on S_0 gives a $(3, 1, 1, 3, 1)$ complete block design and 3 symmetries.
3. **v = 4 or 5.** No possible values of r.
4. **v = 6, r = 1.** $2K_3$ on $2S_0$ gives a $(6, 2, 1, 3; 0, 1)$-PBIBD, and 18 symmetries.
5. **v = 6, r = 2.** $K_{3(2)}$ on S_0 gives a $(6, 4, 2, 3; 0, 1)$-PBIBD, and 12 symmetries.
6. **v = 7, r = 3.** K_7 on S_1 gives the Fano plane, a $(7, 7, 3, 3, 1)$-BIBD, with 21 symmetries.
7. **v = 8, r = 3.** $K_{4(2)}$ on S_1 gives the geometry of Problem 15-5, an $(8, 8, 3, 3; 0, 1)$-PBIBD, with 24 symmetries.
8. **v = 9, r = 1.** $3K_3$ on $3S_0$ gives a $(9, 3, 1, 3; 0, 1)$-PBIBD, and 162 symmetries.
9. **v = 9, r = 2.** $C_3 \times C_3$ on S_1 gives a $(9, 6, 2, 3; 0, 1)$-PBIBD, and 18 symmetries.
10. **v = 9, r = 3.** $K_{3(3)}$ on S_1 gives the geometry of Pappus, a $(9, 9, 3, 3; 0, 1)$-PBIBD, with 27 symmetries.
11. **v = 9, r = 4.** The Menger graph is K_9, the geometry is the affine plane $AG(2, 3)$, and the design will be a $(9, 12, 4, 3, 1)$-BIBD. Competing models will be constructed in Section 15-10.

12. **v = 10, r = 3.** $\overline{\Pi}$ on S_2 gives the geometry of Desargues, a $(10,10,3,3;0,1)$-PBIBD, with 3 symmetries.

It seems that we have covered nearly all the interesting imbeddings of graphs of small order! Each of the constructions above satisfies our preferences (i) through (v), except for Desargues both (iv) and (v) might be improved (but the former at the risk of (ii) and (iii)). In Section 15-10, when modelling $AG(2,3)$, we will see that (iv) is incompatible with (ii) and (iii) taken together.

To illustrate how more complicated models might be constructed, we consider the following situation. Suppose we want a 3-configuration with $v = 21$ and $r = 9$. Start with $K_{7(3)}$, a strongly regular Menger graph. We want a suitable Cayley map for $K_{7(3)}$, so we must choose between $\mathbb{Z}_{21}$ and the semi-direct product $\mathbb{Z}_7 \ltimes \mathbb{Z}_3$. The former is abelian, with one element (and its inverse) of order 3, while the latter is non-abelian and has 7 pairs of elements of order 3. Elements of order 3 are useful in producing triangles, but the abelian property is irresistible (and, as it turns out, one element of order 3 is all we will need here) – so we try $\mathbb{Z}_{21}$. Choosing $\Delta = \{1,2,3,4,5,6,8,9,10\}$, we find $G_\Delta(\Gamma) = K_{7(3)}$. We seek to partition Δ into KVL triples, and as Δ contains four odd numbers (o) and five even numbers (e), we look for a partition of the form $o+o-e=0$, $o+o-e=0$, $e+e-e=0$. (As we see later, it is nice to avoid the form $a+b+c=21$.) This works: $1+9-10=0$, $3+5-8=0$ (we rewrite this as $-3-5+8=0$), $2+4-6=0$. These equations yield three white triangles. (Each will lift to 21 lines of our geometry.) Now we want three black KVL triangles (for preference (iv)), with each generator having the opposite sign to that already used (to meet preference (iii); preferences (i) and (v) have already been attended to). This works: $-2+10-8=0$, $-4-1+5=0$, $6-9+3=0$. We use these six 3-cycles to bound six triangular regions and then identify edges so as to obtain the KVL voltage graph imbedding (in S_2) of Figure 15-7. For preference (ii), we check that our voltage graph has just one vertex, x. (If we had used $-2-8+10=0$ instead of $-2+10-8=0$, for example, this property would seem to fail, leading to a pseudosurface imbedding. But then, since our group is abelian, we could *change* to $-2+10-8=0$ as in Figure 15-7. With our alternative voltage group $\mathbb{Z}_7 \ltimes \mathbb{Z}_3$, this might not be so easy.) The lift will be a bichromatic-dual imbedding of $K_{7(3)}$ in S_{22} ($p = 21, q = 189, r = r_3 = 6 \cdot 21 = 126$), modelling a 3-configuration which is a $(21,63,9,3;0,1)$-PBIBD. (We also get a second such PBIBD from the black covering triangles, and a $(21,126,18,3;0,2)$-PBIBD (not a configuration, since $\lambda_2 = 2$) from all the covering triangles.)

We have two surprises in store. Firstly, since the KVL holds not just in $\mathbb{Z}_{21}$, but–by the way we chose our six equations from Δ –in $\mathbb{Z}_n$, for *all* $n \geq 21$ (voltage-graph theorists say that the KVL holds

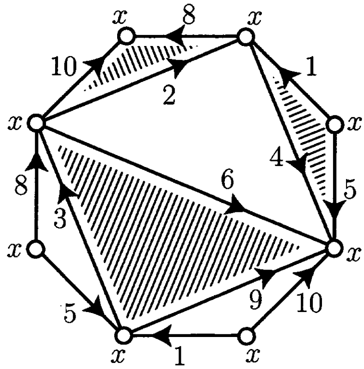


Figure 15-7.

in $\mathbb{Z}_\infty$), we have an *infinite* class of orientable genus surface models, for 3-configurations with $v = n(n \geq 21)$ and $r = 9$, with at least n translational symmetries. (We are not likely to have strong regularity, for $v > 21$.) The covering surface is S_{n+1}, as $p = n$, $q = 9n$, and $r = r_3 = 6n$.

Secondly, if we modify Δ to $\Delta' = \Delta \cup \{7\}$, then $G_{\Delta'}(\Gamma) = K_{21}$. Now add a loop, carrying voltage 7, inside any one of the three black triangles of Figure 15-7. Color the loop region white; it will lift to seven more white triangles. The result is an imbedding of K_{21} on S_{36}, modelling a Steiner triple system of order 21, with at least 21 symmetries. (This probably does not maximize characteristic for this $(21, 70, 10, 3, 1)$-BIBD, as the modified black triangle now lifts to 7 dodecagons. For comparison, the genus of K_{21} is 26, although an imbedding on S_{26} might not model this geometry.)

This gives an idea as to how several of the ideas of this book come together to model 3-configurations in a concrete manner.

Similar considerations have produced topological models of 3-configurations for all possible pairs (v, r) with $v \leq 50$.

Finally we present some counting results.

Thm. 15-16. Let $H = (P, L)$ be a $(3, r)$-configuration, with $v = |P|$ and $b = |L|$ and G and the associated Menger graph. Then:

(i) The total number of orientable topological models (into generalized pseudosurfaces, including pseudosurfaces and surfaces) is $2^b(r!)^v$.
(ii) The number of orientable surface models is $2^b((r-1)!)^v$.

Cor. 15-17. In the probability space of all orientable topological models, with the uniform distribution, the probability of a model being on a surface is $\frac{1}{r^v}$.

15-8. Finite Projective Planes

In the preceding section we studied surface models for 3-configurations, one generalization of our toroidal model of the Fano plane. In this section we generalize the latter in a different way. The Fano plane is not only the smallest symmetrical 3-configuration, but also the smallest projective plane. Here are the axioms for a *projective plane* Π.

(ΠA1) Two distinct points are on a unique common line.
(ΠA2) Two distinct lines contain a unique common point.
(ΠA3) There exist four distinct points, no three on the same line.

We use these axioms to develop some elementary properties for Π.

(ΠT1) There exist four distinct lines, no three containing the same point.

PROOF. Use axioms (ΠA3) and (ΠA1); see Figure 15-8. □

(ΠT2) (Duality) The dual of a valid statement about Π is also valid.

PROOF. Axioms (ΠA1) and (ΠA2) are dual statements, as are axiom (ΠA3) and theorem (ΠT1). Thus the "dual" of the proof of any statement deducible from these four statements establishes the validity of the dual statement. □

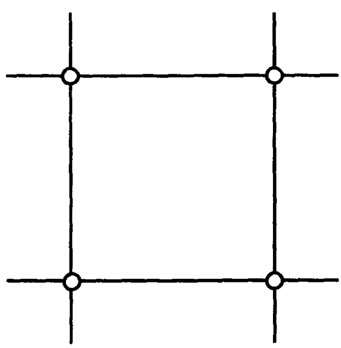


Figure 15-8.

(ΠT3) Any two lines of Π are in one-to-one correspondence with each other (as subsets of the point set).

PROOF. Let $\ell_1 \neq \ell_2$ be given in L. First we find a point $p \notin \ell_1 \cup \ell_2$. If the four points, say A, B, C, and D, guaranteed by (ΠA3) contain such a point p, we are done. Otherwise, the situation must be as in Figure 15-9, and we find $p = AC \cap BD$.

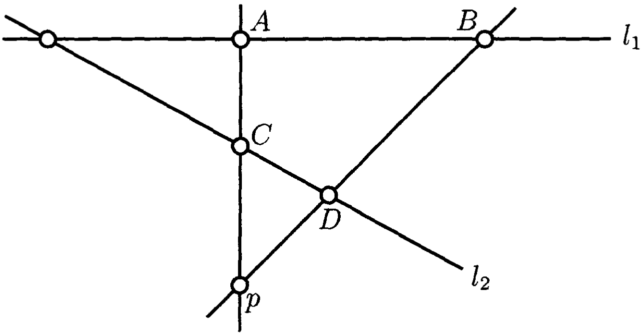


Figure 15-9.

Next, we use p to find a bijection $f : \ell_1 \to \ell_2$, as shown in Figure 15-10, where $f(p_1) = p_2$ is found as the unique intersection of ℓ_2 with the unique line containing p_1 and p.

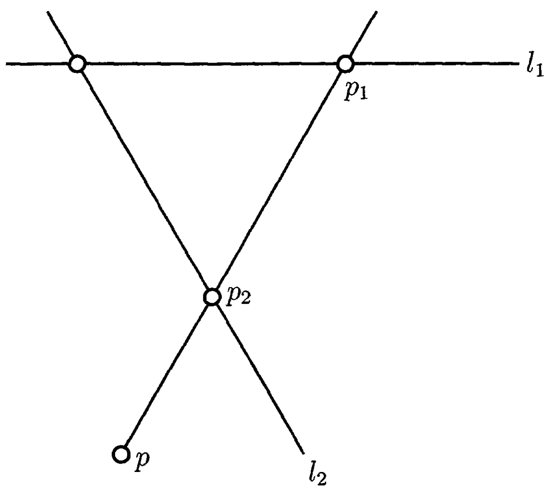


Figure 15-10.

□

We define a projective plane $\Pi = (P, L)$ to be *finite* if P is finite. Then L is finite also, as is each $\ell \in L$. Thus, by (ΠT3), there is an $n \in \mathbb{N}$ so that, for each $\ell \in L$, $|\ell| = n + 1$; n is said to be the *order* of Π. We write $\Pi = \Pi(n)$, and restrict our attention to the finite case for the remainder of this section.

(ΠT4) For each $p \in P$, p belongs to exactly $n + 1$ lines.

PROOF. This is the dual statement to (ΠT3), for $\Pi(n)$. □

(ΠT5) The number of points of $\Pi(n)$ is given by $|P| = n^2 + n + 1$.

PROOF. Let $p \in P$, with $p \in \ell_1, \ell_2, \cdots, \ell_{n+1}$, as in Figure 15-11. As this arrangement, called the *pencil* of lines at p, includes all of P, by (ΠA1), we can use (ΠA2) to count: $|P| = 1 + (n+1)n$. □

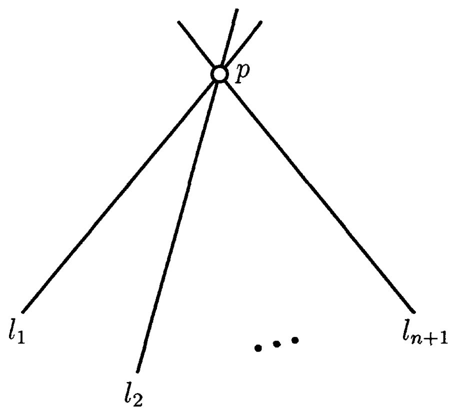


Figure 15-11.

Note that, by (ΠA3) and (ΠT5), the order n of $\Pi(n)$ is at least 2. This confirms that the Fano plane is the smallest projective plane.

(ΠT6) The number of lines of $\Pi(n)$ is given by $|L| = n^2 + n + 1$.

PROOF. By duality. □

We summarize the above development in:

Thm. 15-18. The projective plane $\Pi(n)$ is an $(n^2 + n + 1, n^2 + n + 1, n + 1, n + 1, 1)$-BIBD.

Conversely, every such design is a projective plane $\Pi(n)$. To see this, we note first that (ΠA1) holds, since $\lambda = 1$. For (ΠA2), let B_i and B_j be distinct blocks (lines). Since $\lambda = 1$, $|B_i \cap B_j| \leq 1$. Thus $\sum_{i \neq j} |B_i \cap B_j| \leq \binom{b}{2} = \frac{(n^2+n+1)(n^2+n)}{2}$, with equality if and only if each pair of lines intersect uniquely. But the total number of non-empty pairwise intersections is $v\binom{r}{2} = \frac{(n^2+n+1)(n^2+n)}{2}$. Thus $|B_i \cap B_j| = 1$, for all $i \neq j$. For (ΠA3), consider any two distinct lines. By the above argument, they intersect uniquely. Since $n \geq 2$, we can take, as the four points we seek, any two points from each of the two lines other than their common point. □

Now that we see the power of the axioms for Π, it will be instructive to examine the properties of consistency, independence, and completeness for these axioms. As we indicate below how to construct infinitely many models for the system, it is consistent. As we will construct one model of order n, for each prime power n, the system is not complete. The system as given is independent: 3-point geometry shows that (ΠA3) cannot be deduced from (ΠA1) and (ΠA2); 4-point geometry shows that (ΠA2) does not follows from (ΠA1) and (ΠA3); and the dual of 4-point geometry shows that (ΠA1) cannot be proved using (ΠA2) and (ΠA3) only.

The situation changes if we add:

(ΠA4) For all lines $\ell, |\ell| = n+1$, where $n \geq 2$ is fixed.

The only known models are for n a prime power. It is also known that there are no models possible for $n = 6$ or 10, and that every prime power $n = p^e \geq 9$ of exponent $e \geq 2$ produces at least two non-isomorphic models (there are four, for $n = 9$). Thus the system is still not complete, for such values of n. (It is complete for $n = 2, 3, 4, 5, 7$, and 8, for example.) Consistency seems irrelevant for $n = 6$ and 10, and is an open question for many other non-prime powers. (By the Bruck-Ryser Theorem – see [BR2] – for $n \equiv 1$ or $2 \pmod 4$, if n is not a sum of two squares, then no $\Pi(n)$ exists.)

Now we construct one $\Pi(n)$–denoted $PG(2,n)$–for each prime power n.

A set of m distinct numbers will produce $m(m-1)$ non-zero differences, where we distinguish between $a-b$ and $b-a$. If we take $m = n+1$ and the distinct differences from $\mathbb{Z}_{n^2+n+1}$, then we get n^2+n non-zero differences in $\mathbb{Z}_{n^2+n+1}$. If these differences are distinct, then (together with 0) we have each element of $\mathbb{Z}_{n^2+n+1}$ exactly once. Then using this initial set of $n+1$ elements (called a *perfect difference set*) to generate cyclically n^2+n other perfect difference sets by successively adding 1 to each element in the initial set, we obtain an $(n^2+n+1, n^2+n+1, n+1, n+1, 1)$-BIBD, and hence a $\Pi(n)$. In summary, a $\Pi(n)$ can be constructed by first finding a perfect difference set for $\mathbb{Z}_{n^2+n+1}$. Tables 15-3 (in Section 15-4) and 15-4 show the designs corresponding to planes $\Pi(2)$ and $\Pi(3)$ generated by perfect difference sets $\{0,1,3\}$ and $\{0,1,3,9\}$ for $\mathbb{Z}_7$ and $\mathbb{Z}_{13}$ respectively.

But how do we find an initial perfect difference set? One construction uses finite fields, and this is where the prime powers come into play. (Recall (Theorem 14-1) that a finite field $GF(n)$ of order n exists if and only if n is a prime power.) The construction is general. We illustrate, with the case $n = 5$.

Start with $GF(5) = \{0, 1, 2, 3, 4\}$ and the monic cubic irreducible polynomial $f(x) = x^3 - x - 2$ over $GF(5)$. Set $x^3 = x+2$ and calculate in increasing powers of $x : x^0 = 1, x, x^2, x^3 = x+2, x^4 = x^2+2x, \cdots, x^8 = x+3, \cdots, x^{12} = 2x+2, \cdots, x^{18} = 3x+2, \cdots, x^{31} = 2$.

0	1	3	9
1	2	4	10
2	3	5	11
3	4	6	12
4	5	7	0
5	6	8	1
6	7	9	2
7	8	10	3
8	9	11	4
9	10	12	5
10	11	0	6
11	12	1	7
12	0	2	8

Table 15-4.

Thus $x^{62} = 4, x^{93} = 3$, and $x^{124} = 1$. Hence $< x > \cong \mathbb{Z}_{124} = GF(5^3) - \{0\}$, the multiplicative group of the degree 3 extension, and x is primitive. Now take $\{x^i | 0 \leq i \leq 30\}$ as coset representatives for $(GF(5^3) - \{0\})/(GF(5) - \{0\})$ and $\mathbb{Z}_{31}$ for the points of $\Pi(5) = PG(2, 5)$. (Multiplying powers of x corresponds to adding exponents.) By a theorem of Singer [S7], each line (and all of its translates) in the resultant geometry is a perfect difference set. We take the unique line consisting of all points having coefficient of x^2 equal to 0, called the *standard* perfect difference set (since it contains both 0 and 1). For the present example, this is $\{0, 1, 3, 8, 12, 18\}$. This generates a $(31, 31, 6, 6, 1)$-BIBD, which is $\Pi(5) = PG(2, 5)$.

Similarly, for each prime power n, we obtain an abstract model for $PG(2, n)$, which is a $\Pi(n)$. (There might be other possibilities for $\Pi(n)$, as noted above, for various values of n.) Now we describe how to model the geometries $PG(2, n)$ in a topological manner. Walsh [W2] treated the case $n = 2$ in 1975. We complete the analysis.

For each prime power n, obtain a perfect difference set as described above. This leads (we see how, below) to a generating set Δ for the group $\Gamma = \mathbb{Z}_{n^2+n+1}$ and then an index one voltage graph imbedding which is covered by a Cayley map for $G_\Delta(\Gamma)$ modelling $PG(2, n)$ as follows: the points of the geometry are the vertices of $G_\Delta(\Gamma)$. The lines are modelled by $(n + 1)$-gonal regions. The remaining regions are the hyperregions, which we strive to make as nearly triangular as possible, so as to minimize the complexity of the model. We also want to preserve the action of $\mathbb{Z}_{n^2+n+1}$, already regular on both the points and lines of $PG(2,n)$, as being regular also on each orbit of the set of hyperregions. Full details are given in [W25], where the following results are established.

Thm. 15-19. Let n be a prime power.

(1) If $n \equiv 0 \pmod 3$, then $PG(2,n)$ is modelled on an orientable pseudosurface of characteristic $\frac{(3-2n)(n^2+n+1)}{3}$, with n^2+n+1 hyperregions quadrilateral and all others triangular. These imbeddings are asymptotically efficient, in terms of characteristic.
(2) If $n \equiv 1 \pmod 3$, then $PG(2,n)$ is modelled on the orientable surface of genus $1+\frac{(n-1)(n^2+n+1)}{3}$, with n^2+n+1 hyperregions pentagonal and all others triangular. The genus of these geometries is asymptotic to that of these surfaces.
(3) If $n \equiv 2 \pmod 3$, then $PG(2,n)$ is modelled on the orientable surface of genus $1+\frac{(n-2)(n^2+n+1)}{3}$, with all hyperregions triangular. Thus these are genus imbeddings, for these geometries.

We can consolidate the three parts of Theorem 15-19, but first we need a definition.

Def. 15-20. The *regular pseudocharacteristic* of $PG(2,n)$, $\chi'_r(PG(2,n))$, is the maximum χ' such that the Levi graph $G(PG(2,n))$ imbeds on a pseudosurface S', of characteristic χ', with $\mathbb{Z}_{n^2+n+1}$ acting regularly (as a group of map automorphisms) on each orbit of the region set for the modified imbedding $G^*(PG(2,n))$.

Thm. 15-21. For $i = 0,1,2$ and $n \equiv 2+i \pmod 3$, $\chi'_r(PG(2,n)) = \frac{(4-2n-i)(n^2+n+1)}{3}$.

These constructions give additional information.

Def. 15-22. A (k,g)-*cage* is a graph of minimum order among all k-regular graphs of girth g.

From Proposition 23.1(2) of [B12], it follows that the order of an $(n+1)$-regular graph having girth 6 is at least $2(n^2+n+1)$. By Lemma 15-8, the Levi graph $G(PG(2,n))$ has girth at least 6. But 6-cycles exist in $G(PG(2,n))$; see Problem 15-15. Thus the girth is precisely 6, and by the definition of $\Pi(n)$ and theorems (ΠT4), (ΠT5), and (ΠT6), $G(PG(2,n))$ is $(n+1)$-regular of order $2(n^2+n+1)$. We have shown:

Thm. 15-23. The Levi graph $G(PG(2,n))$ is an $(n+1,6)$-cage, of order $2(n^2+n+1)$.

In fact, Singleton showed [S8]:

Thm. 15-24. A projective plane $\Pi(n)$ exists if and only if there is an $(n+1,6)$-cage of order $2(n^2+n+1)$.

Since the girth of $G(PG(2,n))$ is 6, hexagonal imbeddings will be minimal for these graphs. But hexagonal imbeddings for Levi graphs $G(H)$ correspond with all hyperregions triangular for imbeddings of the modified graphs $G^*(H)$. (Hyperregions double in size by the reversal of the modification process.) Thus we have:

Thm. 15-25. Let G_n denote the $(n+1,6)$-cage associated with $PG(2,n)$. For $n=3m-1$, $\gamma(G_{3m-1}) = m(3m-2)^2$.

The ground case $m=1$ of this theorem gives the Heawood graph on the torus, and the modified imbedding is our toroidal model of the Fano plane. The case $m=2$ we describe below (continuing our earlier example for $n=5$), as it illustrates the constructions of both Theorem 15-19(3) and Theorem 15-25. The other two parts of Theorem 15-19 required only slight modifications to the basic technique of part (3).

In Figure 15-12 we give the voltage graph imbedding, using $\mathbb{Z}_{31}$ and the perfect difference set $\{0,1,3,8,12,18\}$ constructed earlier, that lifts to the model of $PG(2,5)$ – and in turn to the $(6,6)$-cage G_5 – in S_{32}. The generating set Δ for $\mathbb{Z}_{31}$ is found as follows. We need a crucial result (see Theorem 2.5.2 of [A7] or Theorem 11.5.3 of [H1], for example).

Proposition 15-26. Let p be prime, $n=p^m$, $v=n^2+n+1$, and $k=n+1$. Let L_0 be the standard form line for $PG(2,n)$, and let s be the sum of the elements of L_0 in $\mathbb{Z}_v$. Let $j=k^{-1}(-s)$, with $L_j = L_0+j$. Then multiplication by p fixes L_j.

For $n=5$, we calculate that $j=24$ in $\mathbb{Z}_{31}$. We confirm that multiplication by 5 fixes $L_{24} = \{24,25,27,1,5,11\}$. Rewrite L_{24} by orbits under this action:

$$L_{24} = \{1,5,25\} \cup \{11,24,27\} = \{1,11,5,24,25,27\},$$

by interlacing. Use successive differences, to form

$$\Delta = \{10, -6, -12, 1, 2, 5\}$$

for $\Gamma = \mathbb{Z}_{31}$. Observe:

(1) $10 - 12 + 2 = 0$, and
(2) $-6 + 1 + 5 = 0$, forcing
(3) $10 - 6 - 12 + 1 + 2 + 5 = 0$.

(To see why this might work in general, and it does, note that, for instance, $10 - 12 + 2 = (11 - 1)(1 + 5 + 5^2) = 0$ in $\mathbb{Z}_{31}$.) The three equations give the KVL property for the voltage graph imbedding of Figure 15-12.

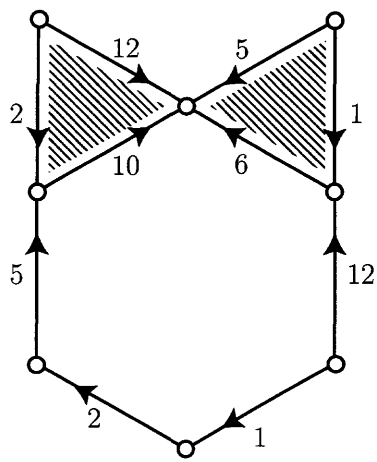


Figure 15-12.

There are six edges in the voltage graph K, one for each generator in Δ, and three regions. There is only one vertex (after identification of the two occurrences of each edge, matching up the arrows), as seen by the (clockwise) ordering of the generators and their inverses: $(-2, 10, -5, -6, -10, -12, 6, 1, 12, 2, -1, 5)$. The surface S in which K is imbedded is orientable, as each edge appears once in each direction, among all the (clockwise, say) region boundaries. The euler identity $\bar{p} - q + r = 2 - 2a$ yields $a = 2$, so we have $S = S_2$. Then $\chi(\tilde{S}) = 31\chi(S_2) = -62 = 2 - 2b$, so $b = 32$ and the covering imbedding of $G_\Delta(\mathbb{Z}_{31})$ is on $\tilde{S} = S_{32}$. The 31 vertices above are the points of $PG(2,5)$. The 31 lifts $\rho^{-1}(R)$ of the hexagon R depict the lines of $PG(2,5)$. Thus if we take $\phi(e_1) = 1$, we find $R_0 = (0, 1, 3, 8, 18, 12)$ producing L_0 in $\rho^{-1}(R)$. Similarly, $R_i = R_0 + i$ depicts L_i, for each i in $\mathbb{Z}_{31}$. In like manner, each triangle below lifts to 31 triangular hyperregions above.

Clearly $\mathbb{Z}_{31}$ acts as a group of automorphisms on the covering imbedding. For example, adding i sends L_j to L_{j+i}. Similarly, $\mathbb{Z}_{31}$

acts regularly on the 31 hyperregions covering each triangle. Using ideas of Chapter 5 of [BW1] (see also Chapter 16 of this book), it can be shown that the full automorphism group $\mathrm{Aut}\, M$ is $\mathbb{Z}_{31} \ltimes (\mathrm{Aut}\, M)_0$, a semidirect product, where $(\mathrm{Aut}\, M)_0$ is the stabilizer of vertex 0. The stabilizer has order dividing 6, as we cannot exchange 6 hexagons with 6 triangles when rotating at a vertex. It can be checked that multiplying by 5 (the generator of the multiplier group for $PG(2,5)$) gives a map automorphism stabilizing 0; as this action has order $3(\sqrt{5} = 6$ in $\mathbb{Z}_{31}$, but 6 is not a multiplier), we conclude that $\mathrm{Aut}\, M = \mathbb{Z}_{31} \ltimes \mathbb{Z}_3$.

We modify this imbedding of $G_\Delta(\mathbb{Z}_{31})$ in S_{32}, to obtain a hexagonal imbedding of the (6,6)-cage $G(PG(2,5))$ in S_{32}, by first inserting a vertex in the interior of each hexagonal region, next adding an edge from each such vertex to every vertex of that hexagonal region, and then deleting all edges of $G_\Delta(\mathbb{Z}_{31})$. (This is the reverse of the procedure of Section 13.3).

Finally, Fink and White [FW3] showed:

Thm. 15-27. The projective plane $PG(2,n)$ has a model on a surface of genus g whose map automorphism group has a regular action on the set $\{(p,\ell) | p \in P, \ell \in L, p \in \ell\}$ of flags if and only if $(n,g) = (2,1), (2,3), (8,147), (8,220)$, or $(8,252)$.

This is consistent with our finding that our model of $PG(2,5)$ has $\mathrm{Aut}\, M = \mathbb{Z}_{31} \ltimes \mathbb{Z}_3$, not $\mathbb{Z}_{31} \ltimes \mathbb{Z}_6$.

In [F1], Figueroa-Centeno greatly generalized the approach of this section, to find topological models for $PG(m,n)$, where $m+1$ is prime and n is a prime power (and $\lambda \geq 2$).

15-9. Finite Affine Planes

Another interesting class of geometries is composed of the affine planes. Here are axioms for an *affine plane* Π'.

(Π'A1) Two distinct points are on a unique common line.
(Π'A2) Through a given point not on a given line, there is a unique parallel line.
(Π'A3) There exist four distinct points, no three on the same line.

Note that the first and third axioms agree with those for a projective plane Π, but that the second axiom (Π') for Π has been replaced with Playfair's axiom for Euclidean geometry (equivalent to the parallel postulate). Thus we have no duality principle for affine planes. Nevertheless, affine and projective planes are intimately related.

Thm. 15-28. To every projective plane $\Pi = (P, L)$ there corresponds an affine plane $\Pi' = (P', L')$, and conversely.

PROOF. Given $\Pi = (P, L)$, choose any $\ell_0 \in L$, and form $\Pi' = (P', L')$ by setting $P' = P - \ell_0$ and $L' = L - \{\ell_0\}$. Thus, for each $\ell \in L$, $\ell \neq \ell_0$, $\ell' = \ell - \{\ell \cap \ell_0\}$. We show that Π' is an affine plane. First, note that (Π'A1) follows directly from axiom (ΠA1) for Π. For (Π'A3), consider any $p \in \ell_0$ and $\ell_1 \neq \ell_2 \in L$ so that $\ell_1 \cap \ell_2 = \{p\}$ in Π. Then take as four points in Π' any pair from each of $\ell_1' = \ell_1 - \{p\}$ and $\ell_2' = \ell_2 - \{p\}$. For (Π'A2), consider $p' \notin \ell'$ in Π', as in Figure 15-13. Let $\ell' \cap \ell_0 = \{p_0\}$. Consider the unique line ℓ_1' containing p' and p_0 in Π. Then ℓ_1' is parallel to ℓ' in Π', since $\ell' \cap \ell_1' = \{p_0\}$ in Π, and $p_0 \notin P'$ for Π'. Moreover, any other line through p' in Π' must intersect ℓ' in Π', for otherwise, by (ΠA2), (ΠA1) is violated in Π. This shows that (Π'A2) holds for Π', so that Π' is an affine plane. □

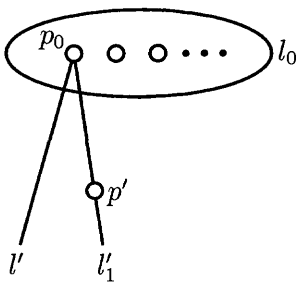


Figure 15-13.

The converse will be established following Theorem 15-30, and we illustrate the converse in Section 15-11. First we need a lemma. Now we regard two lines as being parallel if they are either disjoint or identical.

Lemma 15-29. If ℓ_2' and ℓ_3' are both parallel to ℓ_1' in Π', then they are parallel to each other.

PROOF. If $\ell_2' = \ell_3'$, the conclusion is immediate. If $\ell_2' \neq \ell_3'$, then failure of the conclusion would contradict the uniqueness claim of (Π'A2). □

Thm. 15-30. Let $\Pi' = (P', L')$ be an affine plane. Then L' can be partitioned into classes of parallel lines, with each class partitioning P'. No two lines from distinct classes are parallel.

PROOF. By the definition of "parallel" and by the lemma, "is parallel to" is an equivalence relation on L'. Thus L' is partitioned into classes of parallel lines, and no two lines from separate classes are parallel. Then (Π'A2) and the disjointness of distinct lines in a given class ensure that each class partitions P'. $\square$

We remark that if $\Pi' = \Pi'(n)$ is derived from $\Pi = \Pi(n)$, then (refer to Figure 15-13) each pencil at a point of ℓ_0 forms a parallel class in Π'. Thus L' is partitioned into $n+1$ parallel classes, each consisting of n lines, where each line contains n points.

Now we can prove the converse of Theorem 15-28.

PROOF. Given $\Pi' = (P', L')$, consider the partition of L' given by Theorem 15-30. Form $\Pi = (P, L)$ by adding, for each parallel class, one common point to each line in that class, so that different parallel classes have different added points. This determines P. Finally, complete the formation of L from L' by adding one new line, consisting of all the new points. Then Π is a projective plane.

$\square$

We summarize the above development, in the finite case.

Thm. 15-31. The affine plane $\Pi'(n)$ is a resolvable $(n^2, n^2+n, n+1, n, 1)$-BIBD.

Next we study the properties of consistency, independence, and completeness for the axiom system for the affine plane Π'. As given, the axioms are independent: the graphs P_3, K_5, and $2K_2$ show respectively that neither (Π'A3), nor (Π'A2), nor (Π'A1) can be derived from the other two axioms. The system is consistent, but not complete, as there are models of every prime power order, as we soon see. Now specialize to $\Pi' = \Pi'(n)$, by adding:

(Π'A4) For all lines ℓ, $|\ell| = n$, where $n \geq 2$ is fixed. (We say that n is the order of $\Pi'(n)$.)

Consistency is established, for n a prime power, by deletions from the model for $\Pi(n) = PG(2, n)$ previously constructed, to obtain a model for $\Pi'(n) = AG(2, n)$. (We are obtaining an affine geometry from a projective geometry.) Since there is neither a $\Pi(6)$ nor a $\Pi(10)$ (for example), then by Theorem 15-28 there can be neither a $\Pi'(6)$ nor a $\Pi'(10)$. Since non-isomorphic planes $\Pi(n)$ produce non-isomorphic planes $\Pi'(n)$, then for $n \geq 9$ a prime power with exponent larger than 1, the system for $\Pi'(n)$ is not complete either.

We turn our attention to finding models for $AG(2, n)$. If we delete the first line (and all the points on it) of Table 15-3 (for $n = 2$) and of Table 15-4 (for $n = 3$), we get abstract models for $AG(2, 2)$ and

$AG(2,3)$, as in Table 15-5. (We have reordered the lines of Π' in each case, to display the parallel classes.

2 4	2 4 10
5 6	5 6 8
	7 11 12
2 5	2 5 11
4 6	4 6 12
	7 8 10
2 6	2 6 7
4 5	4 8 11
	5 10 12
	2 8 12
	4 5 7
	6 10 11

Table 15-5.

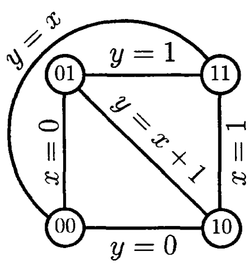
As the reader might suspect, we now seek topological models for $AG(2,n)$. One approach is to commence with the surface (or pseudosurface, when the prime is 3) models constructed for $PG(2,n)$ in Section 15-8, and then *model* $AG(2,n)$ *on the same surface* (or pseudosurface) by making the required deletions, as was done in [W25]. But this is not very satisfying, as the result is neither efficient (since the euler characteristic is too low) nor aesthetically pleasing (since there are too few symmetries). For example, whereas $PG(2,2)$ was modelled on the torus, $AG(2,2)$ is just 4-point geometry, readily modelled on the sphere *via* K_4. More dramatically, whereas $PG(2,3)$ was modelled on a pseudosurface of characteristic -13 (a torus with 13 pairs of points identified), we will find a variety of more satisfactory alternatives to the deletion model of $AG(2,3)$ on that pseudosurface, in Section 15-10.

It is an open problem to find models for $AG(2,n)$, for general prime power orders n, that rival the efficiency and symmetry of the models we found for $PG(2,n)$. (See Problem 15-17.)

We have constructed $AG(2,n)$ by deletions from $PG(2,n)$. But there is also a direct construction, using $\mathbb{F} = GF(n)$, the Galois field of order n. Just as for the plane $\mathbb{R}^2$, form $\mathbb{F}^2 = \{(x,y)|x,y \in \mathbb{F}\}$, giving the n^2 points. Each line consists of all those points satisfying one of the usual equations: $x = a$, for $a \in \mathbb{F}$ (n vertical lines); $y = mx + b$, for $m, b \in \mathbb{F}$ (n^2 lines with finite slope m). This gives $n^2 + n$ lines in all, each line containing n points and each point on $n+1$ lines. Using this coordinatization, we readily see the partition of the n^2+n lines into $n+1$ classes (n classes with finite slope, 1 vertical class) of n parallel lines each, each class partitioning the point set of $\mathbb{F}^2$. Figure 15-14 displays $AG(2,2)$ and $AG(2,3)$ from this perspective. The latter representation appears in many texts. For ease of notation, we represent the ordered pair (x, y) by xy. Also to avoid clutter, for $AG(2,3)$ we omit the labels for the vertical and horizontal lines.

This model for $AG(2,2)$ is already a surface model (K_4 on the sphere, under reverse stereographic projection). However, the $AG(2,3)$ model of Figure 15-14 suffers from all the deficiencies of Figure 15-1 for the Fano Plane.

We can remedy this situation, at least for all odd primes p, and $n = p$, using the voltage graph of Figure 15-15, imbedded in the sphere, and the additive group $\Gamma = \mathbb{Z}_p \times \mathbb{Z}_p$. (The case $p = 2$ gives an alternative model of $AG(2,2)$ on the sphere.) The loop carrying voltage $(0,1)$ lifts to $n = p$ n-gons, modelling the n vertical lines (and partitioning the n^2 points). The other n loops lift to the parallel classes of lines of finite slope: the loop carrying voltage $(1, m)$, where $0 \leq m \leq p-1$, lifts to n n-gons, modelling the n parallel lines of slope m. Thus we see again, quite readily, the $n+1$ parallel classes, of n lines each, and indeed the entire $(n^2, n^2+n, n+1, n, 1)$-BIBD nature of $AG(2,n)$, for $n = p$. But now the surface model avoids the extraneous intersections, and other unfortunate aspects, such as occur in Figure 15-14 for the case $p = 3$.



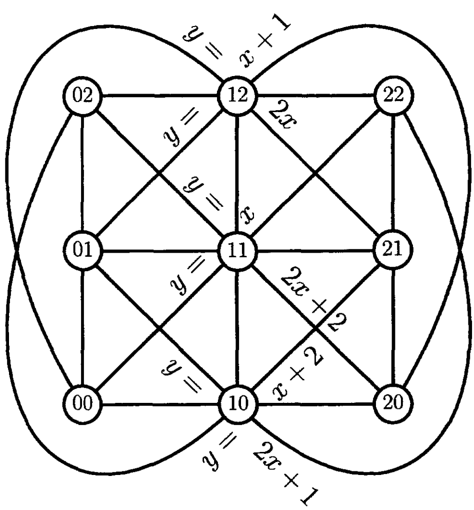


Figure 15-14.

The covering space imbedding over Figure 15-15 has p^2 vertices, $p^2(p+1)$ edges, p^2+p regions modelling the lines of $AG(2,p)$, and p hyperregions (each a (p^2+p)-gon). The covering space is an orientable surface of genus $1 + \frac{p(p-2)(p+1)}{2}$. Although these models have p^2-fold translational symmetry ($\mathbb{Z}_p \times \mathbb{Z}_p$ acts as a group of map automorphisms preserving lines; see Theorem 16-24), the large hyperregions render

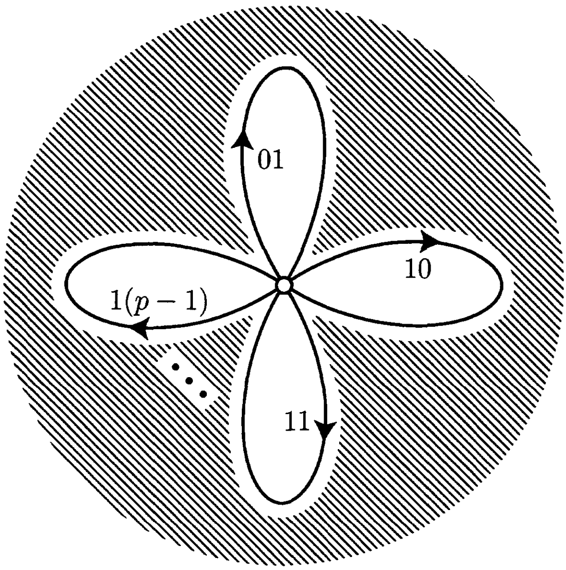


Figure 15-15.

them inefficient–the genus seems too large. Improvements are sought in Problem 15-17.

15-10. Ten Models for $AG(2,3)$

The class of projective planes intersects the class of 3-configurations in the Fano plane $PG(2,2)$, as we have seen. The only affine plane which is also a 3-configuration is $AG(2,3)$. Moreover, as $AG(2,2)$ has been shown to be a planar geometry, $AG(2,3)$ is the first candidate for serious imbedding study, among the affine planes. Finding a suitable topological model for this geometry might indicate how to approach the class $AG(2,n)$ in general. We consider ten models for $AG(2,3)$; six of these are topological.

(1) $AG(2,3)$ is displayed as a $(9,12,4,3,1)$-BIBD, in Table 15-5.
(2) We can specify abstractly that $P = \mathbb{Z}_3 \times \mathbb{Z}_3$, and describe the line set by equations:

$$L = \{x = i | i \in \mathbb{Z}_3\} \cup \{y = mx + b | m, b \in \mathbb{Z}_3\}.$$

(3) Figure 15-14 depicts model (2). As the figure represents an immersion (in the plane) rather than an imbedding, we do not regard it as a truly topological model.
(4) The model of Problem 15-18 improves upon (3), but is still not fully topological.
(5) In Section 15-8 we modelled $PG(2,3)$ on the pseudosurface $(S_1; 13(2))$, so initially the deletion model of $AG(2,3)$ is on $(S_1; 13(2))$ also. But the four deleted points are all points of

identification. If we reverse these four identifications, we obtain a topological model of $AG(2,3)$ on $(S_1; 9(2))$. The characteristic is -9, and the automorphism group is trivial. Each line of the geometry is represented by a path P_3. (In each of the fully topological models below, lines are represented by cycles C_3, and their interiors.)

(6) Specialize the voltage graph imbedding of Figure 15-15 to $p = 3$, to get a model of $AG(2,3)$ on S_7; the characteristic is -12, and the symmetry group is $\mathbb{Z}_3 \times \mathbb{Z}_3$. This model has the nice feature that each loop of the voltage graph lifts to a parallel class of lines.

(7) The rotation scheme below imbeds $AG(2,3)$ on S_3. The characteristic of -4 is optimal, for *orientable surfaces*, and the symmetry group is $\mathbb{Z}_3$. The hyperregions are triangular, with the exception of one hexagon. (Parity does not allow an orientable surface triangulation.) We label the points with elements of $\mathbb{Z}_3 \times \mathbb{Z}_3$, and let xy denote (x, y).

$$\begin{array}{ll}
00: & (02, 01, 10, 20, 21, 12, 11, 22) \\
01: & (21, 11, 20, 12, 22, 10, 00, 02) \\
02: & (01, 00, 22, 12, 20, 11, 10, 21) \\
10: & (20, 00, 01, 22, 21, 02, 11, 12) \\
11: & (01, 21, 22, 00, 12, 10, 02, 20) \\
12: & (10, 11, 00, 21, 02, 22, 01, 20) \\
20: & (12, 01, 11, 02, 22, 21, 00, 10) \\
21: & (12, 00, 20, 22, 11, 01, 02, 10) \\
22: & (12, 02, 00, 11, 21, 20, 10, 01)
\end{array}$$

The final three models will have *all* hyperregions triangular. But the one surface will be nonorientable, and the two orientable spaces include one pseudosurface and one generalized pseudosurface. All have characteristic -3.

(8) The index-one voltage graph of Figure 15-16, shown imbedded in the projective plane, lifts to a model of $AG(2,3)$ in N_5, with symmetry group $\mathbb{Z}_3 \times \mathbb{Z}_3$.

(9) Commence with a toroidal imbedding of $K_{3(3)}$ (see Example 2c in Section 10-3). Recalling that $K_{3(3)} = \overline{3C_3}$, imbed $3C_3$ on $3S_0$ and make nine pairs of vertex identifications, to model $AG(2,3)$ on the generalized pseudosurface $(S_1 \cup 3S_0; 9(2))$. The symmetry group is $(\mathbb{Z}_3 \times \mathbb{Z}_3) \ltimes \mathbb{Z}_3; 27$ symmetries is the best for these topological models. The three added spheres model one parallel class of lines, while the toroidal imbedding gives a Pappus configuration.

(10) The rotation scheme below models $AG(2,3)$ on $S(1; 3(2))$. The symmetry group is $\mathbb{Z}_3$.

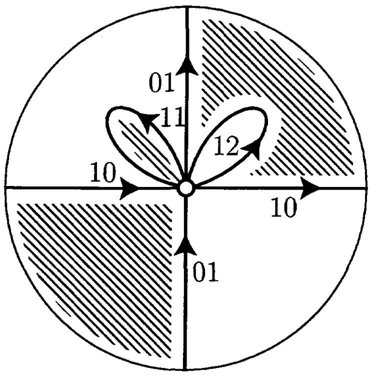


Figure 15-16.

00 :	(12, 21, 02, 01)(11, 22, 20, 10)
01 :	(21, 11, 20, 12, 00, 02, 22, 10)
02 :	(01, 00, 21, 10, 20, 11, 12, 22)
10 :	(12, 11, 00, 20, 02, 21, 01, 22)
11 :	(22, 00, 10, 12, 02, 20, 01, 21)
12 :	(21, 00, 01, 20)(11, 10, 22, 02)
20 :	(02, 10, 00, 22, 21, 12, 01, 11)
21 :	(10, 02, 00, 12, 20, 22, 11, 01)
22 :	(21, 20, 00, 11)(02, 12, 10, 01)

All ten models yield a resolvable $(9, 12, 4, 3, 1)$-BIBD, a Steiner triple system. The last five models imbed K_9, in a variety of ways, but each makes explicit the 3-configuration that is $AG(2, 3)$. Each of the last three models produces a second $(9, 12, 4, 3, 1)$-BIBD (from the hyperregions), as well as one $(9, 24, 8, 3, 2)$-BIBD, a 2-fold triple system. The final two models also give a Mendelsohn triple system, of order nine.

We commented above, in model (9), that $AG(2, 3)$ contains a Pappus configuration, obtained by deleting one parallel class of lines. In fact, any three parallel classes in $AG(2, 3)$ give a Pappus configuration, a group divisible design with groups given by the lines in the fourth class.

In general, the configurations of both Pappus and Desargues occur within each $PG(2, n)$, for $n \geq 3$. (The Fano plane has too few points.) For example, refer to Table 15-4, and see Problem 15-20.

Alternatively, if we model P for $AG(2, 3)$ by $\mathbb{Z}_3 \times \mathbb{Z}_3$ rather than by nine numbers from $\mathbb{Z}_{13}$ (as was done in models (1) and (5)), and refer to the voltage graph of Figure 15-15 (for $p = 3$) again, then by deleting the loop carrying voltage $12 = (1, 2)$, we obtain a toroidal model of the Pappus configuration that combines features of Section 15-6 and models (2), (3), and (4) for $AG(2,3)$. See Figure 15-17, first appearing in [FW2].

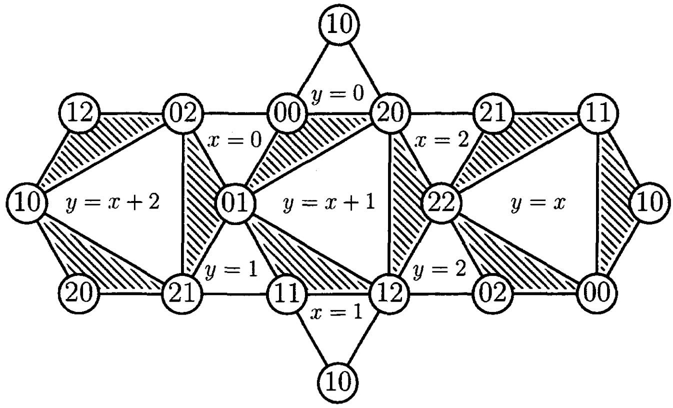


Figure 15-17.

15-11. Completing the Euclidean Plane

The Euclidean plane is an affine plane $\Pi' = (P', L')$, as it satisfies the axioms ($\Pi' A1$), ($\Pi' A2$), and ($\Pi' A3$). Now we complete the Euclidean plane, by applying the process used to prove the converse part of Theorem 15-28. That is, we construct the real projective plane $\Pi = (P, L)$ from Π'. Topologically, this process converts an unbounded orientable 2-manifold into a compact nonorientable one (a surface).

Commence with the equivalence relation "is parallel to", to partition L' into equivalence classes of parallel lines, each class partitioning $P' = \mathbb{R}^2$:

$$L' = \bigcup_{\theta \in [0, \Pi)} L'_\theta,$$

where L'_θ consists of all lines making angle θ with the x-axis. (For $\theta = 0$, these are the horizontal lines.) Then set

$$P = P' \cup [0, \pi),$$
$$L = \cup_{\ell' \in L'} (\ell' \cup \{\theta\}) \cup \{[0, \pi)\},$$

where θ depends on ℓ' *via* $\ell' \in L'_\theta$. Then $\Pi = (P, L)$ is indeed a projective plane, as it satisfies axioms ($\Pi A1$), ($\Pi A2$), and ($\Pi A3$).

Our restriction of θ to the interval $[0, \pi)$ has the effect of identifying the "points at infinity" θ and $\theta + \pi$, for $0 \le \theta < \pi$. This antipodal identification produces the nonorientable surface N_1, a sphere with one crosscap. Thus we see that the two common meanings of the term "projective plane" coincide, for this situation.

In this section, we have modelled an *infinite* geometry on a surface.

15-12. Problems

15-1.) *Show that each geometry–Fano, Pappus, and Desargues–is complete.

15-2.) Show that each geometry–Fano, Pappus, and Desargues–is self-dual. (**Hint**: for Desargues, use the function that sends each point to its unique polar line.)

15-3.) Give a one-line proof of the converse to Desargues' Theorem.

15-4.) Use the Petersen graph (Figure 15-4) and its properties to prove Theorems (DT5) and (DT6).

15-5.) We have seen that the geometries of Fano, Pappus, and Desargues are all self-dual 3-configurations, of order 7, 9, and 10 respectively. What might correspond to the missing number, 8? (**Hint:** use $K_{4(2)}$ to produce an $(8,8,3,3;0,1)$-PBIBD. Is the design resolvable? Group divisible? Transversal? Is the geometry self-dual?) Give an axiom system having the geometry you found as a model. Try to choose the axioms so that the system is complete.

15-6.) Show that the neighbors of vertex A in the Menger graph $\overline{\Pi}$ for the geometry of Desargues induce (with A) no wheel graph W_7 with line triangles alternating with hypertriangles. Explain why this shows that this geometry has genus at least 2.

15-7.) Prove that there exists a toroidal symmetric 3-configuration of order v if and only if $v \geq 7$. Which of the corresponding Menger graphs are either complete or strongly regular?

15-8.) For each $k \geq 4$, find a toroidal Cayley map modelling a 3-configuration with $v = 3k$ and $r = 3$. (**Hint:** Use $\Delta = \{1, 2, 3\}$ for for $\Gamma = \mathbb{Z}_{3k}$.) Then modify the voltage graph you used to model 3-configurations, still with $v = 3k$, but now with $r = 4$.

15.9.) *Find a topological model for a 3-configuration having $v = 36$ and $r = 17$. (The Menger graph is necessarily $K_{18(2)}$.)

15-10.) **A topological model has been found of a 3-configuration having $v = 36$ and $r = 15$, using $\mathbb{Z}_{36}$ and $\overline{\Delta} = \{6, 9, 18\}$. But $G_{\Delta}(\mathbb{Z}_{36})$ is not strongly regular. Try to remedy this defect by replacing the "9" in $\overline{\Delta}$ with "12", so that $G_{\Delta}(\mathbb{Z}_{36}) = K_{6(6)}$.

15-11.) Prove Theorem 15-16.

15-12.) (i) Characterize those generalized pseudosurface rotation schemes which produce pseudosurface imbeddings.
(ii)** Complete Theorem 15-16, by counting the number of orientable pseudosurface imbeddings.

15-13.) **Under what conditions are the two 3-configurations of a configuration imbedding with *all* regions triangular isomorphic?

15-14.) Supply the details of the proofs of (ΠT1) through (ΠT6).

15-15.) For n a prime power, find a 6-cycle in the Levi graph $G(PG(2,n))$.

15-16.) For modelling $AG(2,n)$, why is the voltage graph of Figure 15-15 restricted to *prime* values of n?

15-17.) **Find efficient models for $AG(2,n)$, where n is a prime power.

15-18.) Figure 15-14, for $AG(2,3)$, demands to be redrawn on the torus. Yield to that demand. Try to remedy as many deficiencies–(D1) through (D4) in Section 15-5–as possible.

15-19.) Which of the ten models for $AG(2,3)$ do you prefer? Why? Find a voltage graph, using $\mathbb{Z}_3 \times \mathbb{Z}_3$, consisting of four loops on the sphere, that lifts to a model of $AG(2,3)$ on S_4. How does this model compare with the previous ten?

15-20.) Refer to Table 15-4, listing the lines of $PG(2,3)$.
   (i) Confirm the theorem of Pappus, applied to points $0, 1$, and 3 on line $\{0,1,3,9\}$ and $6, 7$ and 2 on $\{6,7,9,2\}$.
   (ii) Confirm the theorem of Desargues, applied to triangles 056 and 3102, in perspective from point 9.

15-21.) Find an axiom of Euclidean geometry that fails to hold for finite affine planes.

15-22.) Consider the completion of the Euclidean plane to the projective plane (N_1) described in Section 15-11. Sketch familiar graphs, such as $y = \frac{1}{x}$ and $y = \tan x$, on N_1 (with perpendicular coordinate axes as would be depicted by Figure 15-16, with the two loops removed).

15-23.) Define the chromatic number of geometry $H = (P, L)$ to be the chromatic number of H as a hypergraph, as in Section 13-5.
   (a) In the "strong" sense, this would be the chromatic number of the Menger graph for H. Find the strong chromatic number, for:
      (i) $PG(2,n)$
      (ii) $AG(2,n)$.
   (b) In the "weak" sense, show that:
      (i) $PG(2,3)$ and the geometries of Pappus and Desargues all have chromatic number 2.
      (ii) $AG(2,3)$ and the geometry of Fano have chromatic number 3.
      (iii) $AG(2,2)$ has chromatic number 4.

15-24.) In the axiom system given for the geometry of Pappus, replace (PA5) with (P′A5): If a line ℓ does not contain a point p, then there is exactly one point p' in ℓ such that there *is* a line containing both p and p'. (Retain the other five axioms, unchanged). Find a model for this new system, and calculate the genus of the resultant geometry.

15-25.) *Let Π be the Petersen graph. Show that $\overline{\Pi}$ is the line graph of K_5. (**Hint:** think of Π as the odd graph θ_3; see Problem

3-8.) Thus the Menger graph for the Desargues configuration is also the line graph of K_5. To see the connection, take points as edges of K_5, and lines as 3-cycles in K_5. Show that this gives a model for the geometry of Desargues. (**Hint:** verify that axioms (DA1) through (DA6) are satisfied.) Generalize, to find a 3-configuration of order $\binom{n}{2}$, for each $n \geq 3$. Find a suitable topological model, for:

(i) $n = 3$ and 4;

(ii) $^{**}n = 6$

15-26.) Give an axiom system for a geometry of your own devising, and find a suitable model for your geometry.

15-27.) **Project.** Many interesting theorems in Euclidean geometry involve points of concurrency for three lines, for example the median, orthocenter, incenter, and circumcenter for a given triangle. Figure 15-1 illustrates all of these, for equilateral triangles. (Dually, we might have three points collinear, as in the theorems of Pappus and Desargues.) The question is: how likely is it that three lines chosen at random (without replacement) in the Euclidean plane are concurrent? It seems difficult to answer this question directly, for the uncountably infinite sample space of triples of distinct lines in $\mathbb{R}^2$. One heuristic approach would be to analyze the situation for the discrete sample space provided by $AG(2,p)$, for p a prime, and then let p become arbitrarily large. There are $\binom{p^2+p}{3}$ triples of distinct lines in $AG(2,p)$, and four events of interest, with respect to three fixed lines:

(i) There are no intersection points. (The lines are in one parallel class.)

(ii) There is one intersection point. (This is the concurrency case.)

(iii) There are two intersection points. (These arise from two parallel lines and a common transversal.)

(iv) There are three intersection points. (They are the vertices of a triangle. The three lines are said to be *in general position*; see Problem 5-14.)

Find the probability of each event, and draw some conclusions.

15-28.) Does the geometry of Desargues satisfy Playfair's axiom (see (Π'A2))? If not, how "close" does it come?

15-29.) Show that a geometry (P, L) and its dual (L, P) have the same genus. (**Hint:** a one-line proof is available).

CHAPTER 16

MAP AUTOMORPHISM GROUPS

Our focus in this book has been on the various interactions among graphs, groups, and surfaces and - in particular - on the surface imbeddings of graphs depicting groups. In this chapter we go one step further, by considering the automorphism group of the configuration consisting of a graph imbedded in a surface. An important special case will occur when the graph is a Cayley graph for some group. The development here is essentially that of [BW1]; see also Biggs ([B9], [B10], and [B11]), [W9], and [W13].

Recall Corollary 6-22: a connected graph G has a 2-cell imbedding in S_k if and only if $\gamma(G) \leq k \leq \gamma_M(G)$. So far, in this book, we have concentrated on the two extremes of this imbedding range, in calculating various values of the genus and the maximum genus parameters. The connection between block designs and graph imbeddings discussed in Chapter 12 was introduced at the genus end of the spectrum, but extended for larger values of k. In this chapter we consider two additional special types of imbeddings, generally in the interior of the imbedding range: those which are symmetrical, and those which are self-dual. Finally, we combine these two concepts (and others as well) in our study of "Paley maps."

For a theory of maps for orientable surfaces unifying the two standard approaches (that of geometers, studying symmetry properties, and that of combinatorialists, studying graph imbeddings and map colorings), see Jones and Singerman [JS1].

16-1. Map Automorphisms

Recall that a *rotation scheme* for a connected graph G of order n is an ordered n-tuple $P = (p_1, p_2, \cdots, p_n)$, where p_i is the rotation at vertex $i, 1 \leq i \leq n$. Then P determines a 2-cell imbedding in a closed orientable 2-manifold S_k, where k is uniquely specified by the euler equation and the number of orbits of the permutation P^* on the set D^* (see Section 6-6.) Here it will be convenient to let ρ_v denote the rotation at vertex v and $\rho = (\rho_v)_{v \in V(G)}$ denote P.

Def. 16-1. A *map* is a pair (G, ρ), where G is a connected graph and ρ is a rotation scheme for G.

Thus a map may be regarded as a configuration consisting of a representation of a connected graph by its imbedding in a particular closed orientable 2-manifold. This definition can be extended to include nonorientable surfaces as well, as in Section 11-3; but we concentrate on orientable maps here.

We now define an automorphism of a map (G, ρ) to be an automorphism of the graph G which also preserves the rotation ρ. Specifically, we construct an action of the automorphism group $\mathrm{Aut}(G)$ on the set $R(G)$ of all rotations of G. These are the labeled orientable 2-cell imbeddings of G. (We observe that $|R(G)| = \prod_{i=1}^{n}(n_i - 1)!$, where $n_i = d(v_i), 1 \leq i \leq n$; see Problem 16-1.) If $a \in \mathrm{Aut}(G)$ and $\rho \in R(G)$, we define $a(\rho) \in R(G)$ by:

$$(a(\rho))_{a(v)} = a\rho_v a^{-1};$$

that is, if ρ_v takes x to y, then $(a(\rho))_{a(v)}$ takes $a(x)$ to $a(y)$ (see Figure 16-1).

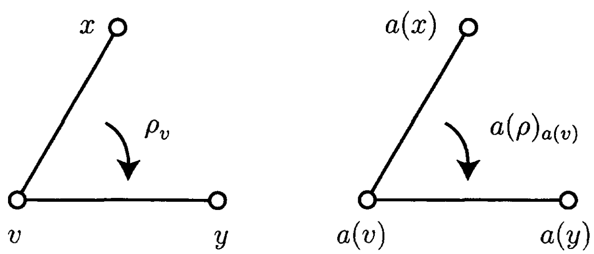


Figure 16-1.

Def. 16-2. Two rotations ρ and σ in $R(G)$ are said to be *equivalent* if there is an $a \in \mathrm{Aut}(G)$ such that $\sigma = a(\rho)$.

This gives an equivalence relation on $R(G)$; see Problem 16-2.

Lemma 16-3. If ρ and σ are equivalent rotations on G, with $\sigma = a(\rho)$ and $(x, y, z, \cdots, w)$ a region of map (G, ρ), then

$$(a(x), a(y), a(z), \cdots, a(w))$$

is a region of map (G, σ).

PROOF. Since $(x, y, z, \cdots, w)$ is a region of (G, ρ), $\rho^*(x, y) = (y, z)$; i.e. $\rho_y(x) = z$. Thus

$$a(\rho)_{a(y)}(a(x)) = a\rho_y a^{-1}(a(x)) = a\rho_y(x) = a(z).$$

So, putting $\sigma = a(\rho)$, we have

$$\sigma_{a(y)}(a(x)) = a(z), \text{ and}$$
$$\sigma^*(a(x), a(y)) = (a(y), a(z)).$$

Hence, $(a(x), a(y), a(z), \cdots, a(w))$ is a region of (G, σ).

$\square$

It follows that, if ρ and σ are equivalent rotations, then there is a one-to-one correspondence between the region sets of the maps (G, ρ) and (G, σ). Hence the two maps are in the same surface; that is, they have the same genus.

Def. 16-4. An *automorphism* of a map $M = (G, \rho)$ is a graph automorphism $a \in \mathrm{Aut}(G)$ such that $a(\rho) = \rho$; i.e. $\rho_{a(v)} = a\rho_v a^{-1}$, for all $v \in V(G)$. (That is, ρ is equivalent with itself, under the action of a).

The following equivalent formulation (see Figure 16-1 and Problem 16-3) readily displays the graph-automorphism nature of each map automorphism. We denote the *automorphism group* of a map $M = (G, \rho)$ by $\mathrm{Aut}(M)$.

Thm. 16-5. For a permutation $a : V(G) \to V(G)$, $a \in \mathrm{Aut}(M)$ if and only if: $(x, y, z, \cdots, w)$ is a region of M implies

$$(a(x), a(y), a(z), \cdots, a(w))$$

is a region of M.

Thus graph automorphisms preserve edges, while map automorphisms preserve oriented region boundaries.

We need two standard results from the theorem of permutation groups (see [BW1], for example). Let (Γ, X) be a permutation group, so that each $\gamma \in \Gamma$ is a permutation of object set X. The *orbit* on $x \in X$ is defined by: $\Gamma x = \{\gamma(x) | \gamma \in \Gamma\}$; then Γx is a subset of X. The *stabilizer* of x is defined as: $\Gamma_x = \{\gamma \in \Gamma | \gamma(x) = x\}$; then Γ_x is a subgroup of Γ. Moreover, we have:

Thm. 16-6. $|\Gamma x| = |\Gamma : \Gamma_x|$.

Finally, for $\gamma \in \Gamma$, the set of fixed points for γ is denoted by: $F(\gamma) = \{x \in X | \gamma(x) = x\}$. Then we have the following theorem of Frobenius, often called "Burnside's Lemma";

Thm. 16-7. The number, t, of orbits of (Γ, X) is given by:

$$t = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} |F(\gamma)|.$$

Now we put the two above results to work.

Thm. 16-8. Let G be a connected graph and ρ a rotation on G; then the number of rotations equivalent to ρ is equal to the index $|\operatorname{Aut}(G) : \operatorname{Aut}(G,\rho)|$.

PROOF. By definition, $\operatorname{Aut}(G,\rho) = \operatorname{Aut}(G)_\rho$, the stabilizer of ρ in the action of $\operatorname{Aut}(G)$ on $R(G)$. The set of rotations equivalent to ρ is just the orbit $\operatorname{Aut}(G)\rho$. Now apply Theorem 16-6. □

Thm. 16-9. The number of equivalence classes of maps with underlying graph G is:

$$\frac{1}{|\operatorname{Aut}(G)|} \sum_{a \in \operatorname{Aut}(G)} |F(a)|,$$

where $F(a) = \{\rho \text{ for } G | a(\rho) = \rho\}$.

PROOF. Apply Theorem 16-6 to the action of $\operatorname{Aut}(G)$ on $R(G)$. □

These are the unlabelled orientable 2-cell imbeddings of G.

We illustrate with $G = K_4$; then $|R(G)| = 2^4 = 16$. Consider $\rho = ((234), (143), (124), (132))$ as shown in Figure 16-2. For this M, $\operatorname{Aut}(M) = A_4$ (see Problem 16-4.) From Theorem 16-8, using $\operatorname{Aut}(K_4) = S_4$ (see Theorem 3-17 (1)), we deduce that there are just two of the sixteen rotations in $R(K_4)$ equivalent to the given one (the other is the "mirror image" $((243), (134), (142), (123))$); both have genus zero. To classify all 16 rotations of K_4, we note that $|F(\gamma)|$ is a *class function*: it is constant on each conjugacy class of elements of S_4; we obtain Table 16.1.

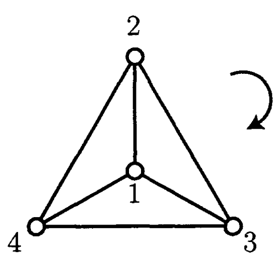


Figure 16-2.

Class representative (a)	e	(12)	(123)	(12)(34)	(1234)
Number in class	1	6	8	3	6
$\lvert F(a)\rvert$	16	0	4	4	2

Table 16-1.

Thus, by Theorem 16-7, the number of equivalence classes is $\frac{16+32+12+12}{24} = 3$. One of these classes is represented by Figure 16-2, and we have just seen that this class contains exactly two rotations. The other two classes are both toroidal (genus one); they are represented in Figure 16-3. The map automorphism groups are $\mathbb{Z}_4$ and $\mathbb{Z}_3$, so that the classes contain (by Theorem 16-8) six and eight rotations respectively. In each case, the rotations pair off by the "mirror image" relationship.

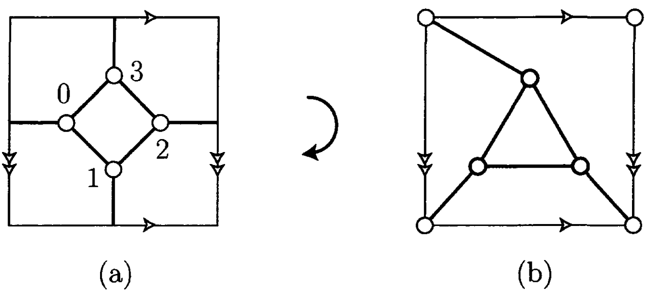


Figure 16-3.

Def. 16-10. The *mirror image* of a map $M = (G, \rho)$ is given by $M^{-1} = (G, \rho^{-1})$, where if $\rho = \{\rho_v\}_{v \in V(G)}$, then $\rho^{-1} = \{\rho_v^{-1}\}_{v \in V(G)}$.

Clearly M^{-1} always exists for a given map M, and $M^{-1} = M$ if and only if G is a cycle (since $\rho_v = \rho_v^{-1}$ if and only if $d(v) = 2$.)

Def. 16-11. A map $M = (G, \rho)$ is said to be *reflexible* if there exists an $\alpha \in \mathrm{Aut}(G)$ such that $(v_1, v_2, \cdots, v_n)$ is a region for M if and only if $(\alpha v_n, \cdots, \alpha v_2, \alpha v_1)$ is a region for M; α is called a *reflection*.

Thus M is reflexible if and only if M and its mirror image M^{-1} are equivalent (Problem 16-5.) For example, every rotation for K_4 gives rise to a reflexible map.

Def. 16-12. The *extended map automorphism group*, $\mathrm{Aut}^*(M)$, of a map M consists of $\mathrm{Aut}(M)$ together with all reflections for M.

Thm. 16-13. Let D be a $(v, b, r, 3, 2)$-BIBD (a 2-fold Steiner triple system), let $\mathrm{Aut}(D)$ be the design automorphism group, and let M be the corresponding map (Theorem 12-3); then $\mathrm{Aut}^*(M) \cong \mathrm{Aut}(D)$.

We have seen that the sixteen rotations for K_4 split into three equivalence classes of maps; we remark that K_5 and K_6 have corresponding numbers as indicated by Table 16-2.

n	2	3	4	5	6
number of rotations	1	1	16	7,776	191,102,976
number of classes	1	1	3	78	265,764

Table 16-2

16-2. Symmetrical Maps

We now study $\mathrm{Aut}(M)$, considered as a permutation group acting on $V(G)$ – where $M = (G, \rho)$.

Lemma 16-14. Let $a \in \mathrm{Aut}(M)$, where $M = (G, \rho)$, and let $\{u, v\} \in E(G)$ with $a(u) = u$ and $a(v) = v$; then $a = e$.

PROOF. Since $a \in \mathrm{Aut}(M), \rho_{a(v)} = a\rho_v a^{-1}$. Thus $a(\rho_v(u)) = a\rho_v a^{-1}(u) = \rho_{a(v)}(u) = \rho_v(u)$; that is, a fixes $\rho_v(u)$ also. But since ρ_v is cyclic on $N(v)$ (the set of vertices adjacent to v), the argument repeats and we see that a fixes each vertex in $N(v)$. Similarly, a fixes each vertex in $N(u)$, each vertex in $N(\rho_v(u))$, and so on; since G is connected, we see that a fixes each $w \in V(G)$, so that $a = e$. □

Thm. 16-15. Let $A = \mathrm{Aut}(M)$, for $M = (G, \rho)$, with $v \in V(G)$. The stabilizer A_v is isomorphic to a subgroup of the cyclic group $\langle \rho_v \rangle$ generated by ρ_v, and hence is a cyclic group whose order divides $d(v)$.

PROOF. We have $a\rho_v a^{-1} = \rho_{a(v)} = \rho_v$, so that $a\rho_v = \rho_v a$. Let $w \in N(v)$; then $a(w) \in N(v)$ also, and since ρ_v is cyclic on $N(v)$ we have $a(w) = \rho_v^i(w)$, for some i. Let $x \in N(v)$, say $x = \rho_v^j(w)$. Then $a(x) = a\rho_v^j(w) = \rho_v^j a(w) = \rho_v^{j+i}(w) = \rho_v^i \rho_v^j(w) = \rho_v^i(x)$. Thus $\bar{a} = \rho_v^i$, where $\bar{a}$ is the restriction of a to $N(v)$, and $\theta : A_v \to \langle \rho_v \rangle$, $\theta(a) = \bar{a}$, is a homomorphism and if $\bar{a}_1 = \bar{a}_2$, then $a_1 a_2^{-1} = e$ by Lemma 16-14; thus θ is a monomorphism. □

If A is transitive on $V(G)$, then all vertex stabilizers are conjugate in A, so that $|A| = p|A_v|$, where $p = |V(G)|$. Thus $|A| = p\delta$, where δ divides d, the common vertex degree of G. For example, the rotation ρ for K_7 given in Section 9-2 gives a vertex-transitive map M (consider $(0, 1, 2, 3, 4, 5, 6) \in \mathrm{Aut}(M)$). Moreover, $p_0 = (1, 3, 2, 6, 4, 5) \in A_0$, so that $|A| = 7 \cdot 6 = 42$. All the K_4 maps are vertex-transitive, except those in the third of the three classes.

Now consider $A = \text{Aut}(M)$ to act on $D^* = \{(u,v) | \{u,v\} \in E(G)\}$, by $a(u,v) = (a(u), a(v))$. Let $x = (u,v)$; then $A_x = e$, by Lemma 16-14. Thus, by Theorem 16-6, $|Ax| = |A : A_x| = |A|$, independent of x. This gives:

Thm. 16-16. $|\text{Aut}(M)|$ divides $2|E(G)|$.

PROOF. By the preceding remarks, each orbit of D^* has length $|\text{Aut}(M)|$. □

Def. 16-17. If $|\text{Aut}(M)| = 2|E(G)|$, then M is said to be a *symmetrical* map.

Thus symmetrical maps display the maximum amount of symmetry. If M is symmetrical, then $\text{Aut}(M)$ is a regular permutation group on the object set D^*; symmetrical maps are also called *regular* maps.

Thm. 16-18. If M is symmetrical, then $\text{Aut}(M)$ is transitive on the vertices, on the edges, and on the regions of M.

PROOF. Let $A = \text{Aut}(M)$ act on D^*, with $|\text{Aut}(M)| = |D^*|$. Let $x = (u,v) \in D^*$ and $a_1, a_2 \in \text{Aut}(M)$; if $a_1 x = a_2 x$, then $a_1 = a_2$ by Lemma 16-14; thus $|A\ x| = |D^*|$, and A is transitive on D^*; hence A is transitive on $E(G)$. Now let $u, w \in V(G)$, with $(u,v), (w,z) \in D^*$. Since A is transitive on D^*, we find $a \in A$ so that $(a(u), a(v)) = a(u,v) = (w,z)$; thus $a(u) = w$, and A is transitive on $V(G)$. Finally, let $r_1 = (v, w, \cdots)$ and $r_2 = (x, y, \cdots)$ be two regions of M, with $a \in \text{Aut}(M)$ such that $a(v) = x$ and $a(w) = y$. Then, since $a\rho_w a^{-1} = \rho_{a(w)} = \rho_y$, $a\rho_w(v) = \rho_{a(w)}a(v) = \rho_y a(v) = \rho_y(x)$, so that the next vertex of r_1 is carried to the next vertex of r_2 by a; this process continues, to show that A is transitive on the region set as well. □

Thus a symmetrical map M has associated with it two important constants: the constant vertex degree d and the constant number k of vertices in each region boundary.

Thm. 16-19. Let M be a symmetrical map of genus γ, with p vertices (all of degree d), q edges, and r regions (all having length k); then:

(i) $dp = kr = 2q$;
(ii) $\gamma = 1 + \frac{(d-2)(k-2)-4}{4k} p$.

PROOF. We use Lemma 5-17 (iii) and (iv) and Theorem 5-14. □

For $d = 2, p = q = k, r = 2$, and $\gamma = 0$; M consists of $G = C_p$ on S_0. Hence we assume $d \geq 3$ (and $k \geq 3$).

If $\gamma = 0$, then $(d-2)(k-2) < 4$ and

$$(d, k) = (3,3), (3,4), (3,5), (4,3),$$

or $(5,3)$. Each pair determines p, q, and r uniquely; see Section 5-4. Thus we have:

Thm. 16-20. A map $M(G, \rho)$ is symmetrical of genus zero if and only if $G = C_p (p \geq 2)$ or the 1-skeleton of a Platonic solid.

If $\gamma = 1$, then $(d-2)(k-2) = 4$ and $(d, k) = (3,6), (4,4)$, or $(6,3)$. For each case, we find infinitely many toroidal maps covered by the corresponding regular tessellation of the plane (see Figures 8-5, 7-3, and 13-5 for one map of each respective type.)

If $\gamma \geq 2$, then again each pair (d, k), for fixed γ, determines p, q, and r uniquely. Moreover, the number of symmetrical maps of genus γ is finite:

Thm. 16-21. If M is a symmetrical map of genus $\gamma \geq 2$, then $|\operatorname{Aut}(M)| \leq 84(\gamma - 1)$; equality holds if and only if $(d, k) = (3,7)$ or $(7,3)$.

PROOF. Using $|\operatorname{Aut}(M)| = 2q$ and both parts of Theorem 16-19, we obtain

$$|\operatorname{Aut}(M)| = 2q = dp = \frac{4kd}{(d-2)(k-2)-4}(\gamma - 1).$$

Since $d \geq 3$ and $k \geq 3$, the coefficient of $\gamma - 1$ has a maximum value of 84, occurring precisely when $(d, k) = (3,7)$ or $(7,3)$. □

For any map M, whether orientable or nonorientable, the order $|\operatorname{Aut}^*(M)|$ of the extended map automorphism group $\operatorname{Aut}^*(M)$ divides $4|E(G)|$, where G is the graph of the map (See Problem 16-9). We get $|\operatorname{Aut}^*(M)| = 4|E(G)|$, in the orientable case, if M is symmetrical and reflexible. In the nonorientable case we lose our sense of order-preservation, so that "map automorphisms" are indistinguishable from "reflections;" a nonorientable map M is thus said to be *symmetrical* if $|\operatorname{Aut}^*(M)| = 4|E(G)|$. We mention just one result about such maps, due to Wilson [W29]; note the connection with the covering projection of Section 11-2.

Thm. 16-22. If N is a nonorientable symmetrical map, then there is a unique orientable map M, both symmetrical and reflexible, which is a 2-fold covering space of N.

For two examples, we give the graph of the dodecahedron on S_0 covering, by antipodal identification, the Petersen graph on N_1 and the dual configurations: the icosahedral graph on S_0 projecting to K_6 on N_1.

Wilson shows that the uniqueness of Theorem 16-22 is not reversible, by giving one M covering two distinct maps N.

16-3. Cayley Maps

Now let $M = (G, \rho)$ be a map, where $G = G_\Delta(\Gamma)$ is a Cayley graph for group Γ, as generated by $\Delta \subseteq \Gamma$ subject to the usual restrictions ($e \notin \Delta$; if $\delta \in \Delta \cap \Delta^{-1}$, then $\delta^2 = e$); recall that $\Delta^* = \Delta \cup \Delta^{-1}$. In this situation, the vertex rotations ρ_v ($v \in V(G) = \Gamma$) can be regarded as permutations not only of $N(v)$, but also of Δ^*; thus two rotations can be more readily compared. Of special interest is the case where the induced permutations of Δ^* are all the same.

Def. 16-23. A *Cayley map* $M(\Gamma, \Delta, \rho)$ is the map $M(G, \rho)$, where $G = G_\Delta(\Gamma)$ and $r : \Delta^* \to \Delta^*$ is a cyclic permutation so that, for $g \in \Gamma$ and $h \in N(g)$,

$$\rho_g(h) = gr(g^{-1}h).$$

Thus the group structure determines the vertex rotations, as in Figure 16-4 – where $r = (\delta_1, \delta_2, \cdots, \delta_k)$: the edge $\{g, g\delta_1\}$ is determined by $\delta_1 \in \Delta^*$, so that the image of $g\delta_1$ under ρ_g is determined by $r(\delta_1) = \delta_2 \in \Delta^*$.

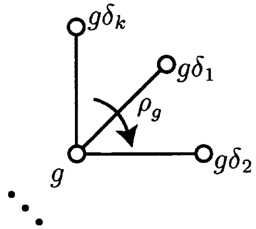


Figure 16-4.

For a specific example, we consider the map of K_7 on S_1 given in Sections 9-2 and 16-2; it is a Cayley map $M(\mathbb{Z}_7, \{1,2,3\}, r)$, with $r = (1,3,2,6,4,5)$. Then $\rho_i(g) = i + r(g - i), 0 \le i \le 6$; that is, $\rho_i = (1+i, 3+i, 2+i, 6+i, 4+i, 5+i)$. In contrast, the graph

Q_3 shown in Figure 5-8 is a Cayley graph $G_\Delta(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$, with $\Delta = \{(1,0,0),(0,1,0),(0,0,1)\}$; but the map shown on S_0 is *not* a Cayley map, since no suitable r exists.

Now we give a strengthening of Theorem 4-8; every automorphism of $C_\Delta(\Gamma)$ is also an automorphism of $M(\Gamma,\Delta,r)$. We remark that the Cayley graph $G_\Delta(\Gamma)$ may well have additional automorphisms.

Thm. 16-24. The Cayley map $M = M(\Gamma,\Delta,r)$ is vertex-transitive; in fact $\mathrm{Aut}(M)$ contains a regular subgroup isomorphic to Γ.

PROOF. The isomorphism $\alpha : \Gamma \to A(C_\Delta(\Gamma))$,

$$\alpha(g) = \theta_g : V(C_\Delta(\Gamma)) \to V(C_\Delta(\Gamma)),$$

$\theta_g(g_i) = gg_i$ given in the proof of Theorem 4-8 works here as well; it only remains to show that $\theta_g \in \mathrm{Aut}(M)$; that is, that $\rho_{\theta_g(h)} = \theta_g \rho_h \theta_g^{-1}$, for each $h \in V(C_\Delta(\Gamma)) = \Gamma$. But let $f \in N(\theta_g(h))$; then

$$\begin{aligned}
\rho_{\theta_g(h)}(f) &= \rho_{gh}(f)\\
&= ghr((gh)^{-1}f)\\
&= \theta_g(hr(h^{-1}g^{-1}f))\\
&= \theta_g\rho_h(g^{-1}f)\\
&= \theta_g\rho_h\theta_{g^{-1}}(f).
\end{aligned}$$

□

We observe that the map of K_4 in S_1 depicted in Figure 16-3(a) is a Cayley map $M(\mathbb{Z}_4, \{1,2\}, r)$, where $r = (1,2,3)$, and has only the automorphisms guaranteed by Theorem 16-24; it is not symmetrical. This situation generalizes:

Thm. 16-25. Let $M = M(\Gamma,\Delta,r)$ be a Cayley map which is not symmetrical, and let $|\Delta^*|$ be prime; then $\mathrm{Aut}(M) \cong \Gamma$.

PROOF. Let $A = \mathrm{Aut}(M)$ and $v \in V(C_\Delta(\Gamma))$. By Theorem 16-15, $|A_v|$ divides $|\Delta^*|$, which is prime; hence $|A_v| = 1$ or $|\Delta^*|$. By Theorem 16-24, M is vertex-transitive, so that by Theorem 16-6, $|A| = |\Gamma||A_v|$. But since M is not symmetrical, $|A_v| \neq |\Delta^*|$; thus $|A_v| = 1$, $|A| = |\Gamma|$, and $\mathrm{Aut}(M) \cong \Gamma$. □

On the other hand, $\mathrm{Aut}(M)$ may be strictly larger than Γ, for M a Cayley map $M(\Gamma,\Delta,r)$. Recall that $\alpha : \Gamma \to \Gamma$ is an *automorphism* of group Γ if α is one-to-one and onto, and $\alpha(\gamma_1\gamma_2) = \alpha(\gamma_1)\alpha(\gamma_2)$, for all $\gamma_1, \gamma_2 \in \Gamma$.

Thm. 16-26. Let $M = M(\Gamma, \Delta, r)$ be a Cayley map and let $\alpha \in \mathrm{Aut}(\Gamma)$ be a group automorphism such that $\alpha|_{\Delta^*} = r^\ell$, for some $\ell, 1 \leq \ell \leq |\Delta^*|$. Then $\alpha \in (\mathrm{Aut}(M))_e$. (That is, α is a map automorphism, fixing e).

PROOF. Since $\alpha \in \mathrm{Aut}(\Gamma)$, $\alpha(e) = e$. Let $\{u, v\} \in E(G_\Delta(\Gamma))$, so that $u^{-1}v \in \Delta^*$; then $\alpha(u^{-1}v) = \alpha(u^{-1})\alpha(v) \in \Delta^*, \{\alpha(u), \alpha(v)\} \in E(G_\Delta(\Gamma))$, and $\alpha \in \mathrm{Aut}(G_\Delta(\Gamma))$. To show that $\alpha \in \mathrm{Aut}(M)$, we verify that for all $g \in \Gamma, \rho_{\alpha(g)} = \alpha\rho_g\alpha^{-1}$: let $h \in N(\alpha(g))$; then $\alpha^{-1}(h) \in N(g)$, $g^{-1}\alpha^{-1}(h) \in \Delta^*$, and

$$\begin{aligned}
\rho_{\alpha(g)}(h) &= \alpha(g)r((\alpha(g))^{-1}h) \\
&= \alpha(g)r(\alpha(g^{-1})h) \\
&= \alpha(g)r(\alpha(g^{-1}\alpha^{-1}(h))) \\
&= \alpha(g)r^{\ell+1}(g^{-1}\alpha^{-1}(h)) \\
&= \alpha(gr(g^{-1}\alpha^{-1}(h))) \\
&= \alpha\rho_g\alpha^{-1}(h).
\end{aligned}$$

□

In the special case $\ell = 1$ of the above theorem, $\mathrm{Aut}(M)$ is as large as possible:

Thm. 16-27. Let $M = M(\Gamma, \Delta, r)$ be a Cayley map, with $a \in \mathrm{Aut}(\Gamma)$ such that $\alpha|_{\Delta^*} = r$. Then M is a symmetrical map.

PROOF. By Theorem 16-26, $\alpha \in (\mathrm{Aut}(M))_e$. By Theorem 16-15, $|(\mathrm{Aut}(M))_e|$ divides $|\Delta^*|$. But $\alpha|_{\Delta^*} = r$, a $|\Delta^*|$-cycle, so that $|(\mathrm{Aut}(M))_e| = |\Delta^*|$. In fact, $(\mathrm{Aut}(M))_e$ is generated by α. Now, by Theorem 16-24, $\mathrm{Aut}(M)$ is transitive on $\Gamma = V(G_\Delta(\Gamma))$, so that Theorem 16-6 gives:

$$\begin{aligned}
|\mathrm{Aut}(M)| &= |\Gamma|\ |(\mathrm{Aut}(M))_e| \\
&= |\Gamma||\Delta^*| \\
&= 2|E(G_\Delta(\Gamma))|,
\end{aligned}$$

and M is symmetrical.

□

We now calculate the genus of an arbitrary Cayley map $M(\Gamma, \Delta, r)$. We first observe (see Problem 16-25) that $M(\Gamma, \Delta, r)$ is just the covering space of the imbedded index one voltage graph K of Figure 16-5 (the ambient orientable surface for K is immaterial here; it is determined uniquely by the rotation at the single vertex.) The region boundaries for K are clearly determined by the (not necessarily cyclic) permutation $\bar{r} : \Delta^* \to \Delta^*$, given by $\bar{r}(\delta) = r(\delta^{-1})$. So, let $\bar{r}$ have t cycles $\Delta_1, \Delta_2, \dots, \Delta_t$ in its action on Δ^*, with m_i the order of $\delta_{i1}\delta_{i2}\dots\delta_{ik_i}$ in Γ, where $\Delta_i = (\delta_{i1}, \delta_{i2}, \dots, \delta_{ik_i})$, $1 \le i \le t$. These t numbers m_i are called the *periods* of M.

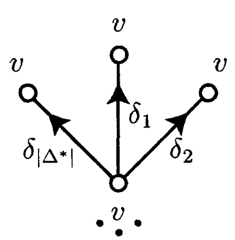


Figure 16-5.

Thm. 16-28. Let $m_1, m_2, \dots, m_t$ be the periods of the Cayley map $M(\Gamma, \Delta, r)$; then the genus γ of M is given by

$$\gamma = 1 + \frac{|\Gamma|}{4}\left(|\Delta^*| - 2 - 2\sum_{i=1}^{t}\frac{1}{m_i}\right).$$

PROOF. By Theorem 10-9 (ii), the cycle $\Delta_i (1 \le i \le t)$ is covered by $\frac{|\Gamma|}{m_i}$ regions, each of length $m_i k_i$. Thus γ is determined by the euler equation (Theorem 5-14), using $p = |\Gamma|$, $q = |\Gamma|\frac{|\Delta^*|}{2}$, and $r = \sum_{i=1}^{t}\frac{|\Gamma|}{m_i}$. □

To illustrate these ideas, consider once again $M = M(\mathbb{Z}_7, \{1,2,3\}, (1,3,2,6,4,5))$. It is immediate that $G_\Delta(\Gamma) = K_7$, and we compute $\bar{r} = (1,4,2)(3,5,6)$; thus $t = 2$, $m_1 = m_2 = 1$. Using Theorem 16-28, we confirm that $\gamma = 1$. Next, we note that $\alpha = (0)(1,3,2,6,4,5) \in \mathrm{Aut}(\mathbb{Z}_7)$, so that Theorem 16-27 applies to show that M is symmetrical.

As a second example, consider now the map $M = M(A_5, \Delta, r)$ as specified by $r = ((1,2,3,4,5), (5,4,3,2,1), (12)(34))$. Then $G_\Delta(\Gamma)$ is the one-skeleton of the familiar soccer ball design. We find $\bar{r} = ((1,2,3,4,5),(12)(34))((5,4,3,2,1))$, so that $t = 2$, $m_1 = 3$, and $m_2 = 5$. We again use Theorem 16-28, to compute $\gamma = 0$ (this is truly fortunate, for the game of soccer!), noting that $r_6 = 20$ (the white panels of the soccer ball) and $r_5 = 12$ (the black panels.) Clearly M is not region-transitive and hence, by Theorem 16-18, M is not symmetrical.

Finally, we observe that r does *not* extend to an automorphism of A_5 (since r does not preserve order), as required by Theorem 16-27.

16-4. Complete Maps

Def. 16-29. A *complete map* is a map $M(G, \rho)$, where G is a complete graph K_n.

Of course, we have been studying complete maps throughout this book, primarily with the aim of minimizing or maximizing the genus γ of M. Now it is the symmetry of M that we wish to maximize.

Lemma 16-30. If $\alpha \in \mathrm{Aut}(K_n, \rho)$ fixes more than one vertex, then $\alpha = e$.

PROOF. Any two distinct vertices of K_n are adjacent, so Lemma 16-14 applies.

□

Def. 16-31. The transitive permutation group (Γ, X) is said to be a *Frobenius group* if only $e \in \Gamma$ has more than one fixed point in X.

Thus the automorphism group of a vertex-transitive complete map is necessarily Frobenius, by Lemma 16-30. The study of Frobenius groups leads to a classification of vertex-transitive complete maps. We only outline this development here; for full details, see [B10] or [BW1].

Thm. 16-32. Let (Γ, X) be a Frobenius group, with N^* the set of fixed-point-free elements of Γ and $N = N^* \cup \{e\}$. Then:

(i) $|N| = |X|$;
(ii) If Γ_x is abelian, then N is a regular normal subgroup of (Γ, X).

The following result appears in Burnside [B21; p. 172]

Thm. 16-33. If (Γ, X) is a Frobenius group of degree $n = |X| \geq 6$ and order $n(n-1) = |\Gamma|$, then n is a prime power.

Now let Γ be any group of order n, with $\Delta^* = \Gamma - \{e\}$; then $G_\Delta(\Gamma) = K_n$, and if r is any cyclic permutation of $\Gamma - \{e\}$, then the Cayley map (Γ, Δ^*, r) is complete and, by Theorem 16-24, vertex-transitive. Theorem 16-32 is used to provide a converse to this result, so that we have:

Thm. 16-34. Let $M = M(K_n, \rho)$ be a complete map; then $\text{Aut}(M)$ acts transitively on the vertices for M if and only if M is a Cayley map.

PROOF. The sufficiency was established in Theorem 16-24. For the necessity, apply Lemma 16-30 to see that $\text{Aut}(M)$ is a Frobenius group. Now $(\text{Aut}(M))_x$ is cyclic, by Theorem 16-15, and hence Theorem 16-32 (ii) guarantees a regular normal subgroup N of $\text{Aut}(M)$. Now take $\Gamma = \mathbb{Z}_n$ and define a bijection $\beta : V(G_\Delta(\Gamma)) = \Gamma \to N$ by $\beta(i) = \beta_i$, where $\beta_i(0) = i$; then calculations show that ρ is given (homeomorphically) by $r : N - \{e\} \to N - \{e\}$, $r(\beta_i) = \beta_{\rho_e(i)}, i \neq 0$. □

Thus, for example, an imbedding of a complete graph covering a voltage graph of index higher than one (and not projecting to an index one voltage graph) cannot be symmetrical.

Finally, Biggs [B10] established:

Thm. 16-35. There is a rotation ρ for K_n so that (K_n, ρ) is symmetrical if and only if n is a prime power.

PROOF. (i) If $M = (K_n, \rho)$ is symmetrical, then

$$|\text{Aut}(M)| = 2|E(K_n)| = n(n-1);$$

moreover, $\text{Aut}(M)$ is Frobenius, by Theorem 16-18 and Lemma 16-30. Thus Theorem 16-33 applies, to show that n is a prime power.

(ii) Conversely, if $n = p^m$ where p is prime and $m \in N$, then take $\Gamma = (\mathbb{Z}_p)^m$ – the additive group in $GF(n)$ – and let $x \in \Gamma$ generate the multiplicative group. Take $\Delta^* = \Gamma - \{0\}$, and $r : \Delta^* \to \Delta^*$ by $r(\delta) = x\delta$; and r extends (by setting $\alpha(0) = 0$) to $\alpha \in \text{Aut}(\Gamma)$ and, by Theorem 16-27, the Cayley map $M(\Gamma, \Delta, r)$ is symmetrical. □

We close this section by summarizing our knowledge of $A = \text{Aut}(M)$, for $M = M(K_n, \rho)$ a symmetrical complete map:

(i) $|A| = n(n-1)$.
(ii) A is transitive on the vertices, edges, and regions of M, and regular on the directed edges.
(iii) A is a Frobenius group.
(iv) For each $v \in V(K_n)$, $A_v = \mathbb{Z}_{n-1}$.
(v) A has a regular normal subgroup, isomorphic to $(\mathbb{Z}_p)^m$–where $n = p^m$; in fact A is the semi-direct product of $(\mathbb{Z}_p)^m$ and $\mathbb{Z}_{n-1}$.

16-5. Other Symmetrical Maps

In contrast to the situation for K_n, where symmetrical maps exist only for prime powers of n, we have the following three results (see Problems 16-12 and 16-13; for hints, see [BW1]) see also [W16]:

Thm. 16-36. (i) The graph $K_{n,n}$ has a symmetrical map, for all n.

(ii) The graph $K_{n,n,n}$ has a symmetrical map, for all n.

Thm. 16-37. The graph Q_n has a symmetrical map, for all n.

We note that the symmetrical maps for $K_{n,n,n}$ can be taken to be of genus $\gamma(K_{n,n,n})$; those for Q_n can be taken to be of genus $\gamma(Q_n)$ or of genus $\gamma(Q_{n+1})$.

There are several classes of questions that can be asked regarding symmetrical maps; we list three of these, so as to reflect the thrust of this book:

(i) For a given graph G, what are the symmetrical maps $M(G, \rho)$?
(ii) For a given group Γ, what are the symmetrical Cayley maps $M(\Gamma, \Delta, r)$?
(iii) For a given surface S_k, what are the symmetrical maps of genus k?

Theorems 16-35, 16-36, and 16-37 speak to question (i). The discussion following Theorem 16-19 addresses question (iii). And the following two results (see Problems 16-14, and 16-15; for hints, see [BW1]) respond to question (ii):

Thm. 16-38. The group S_n has a symmetrical Cayley map, for all $n \geq 2$.

Thm. 16-39. The group A_n has a symmetrical Cayley map, for n odd, $n \geq 3$.

We could also ask:

(iv) For a given group Γ, what are the symmetrical maps $M = M(G, \rho)$ so that $\mathrm{Aut}(M) \cong \Gamma$?

Thm. 16-40. (Brahana [B19]) The group Γ is the automorphism group of a symmetrical map if and only if Γ is generated by two elements, one of order two.

Cor. 16-41. The symmetric groups S_n $(n \geq 3)$, the alternating groups A_n $(n \geq 4)$, and the dihedral groups D_n each occur as the automorphism group of a symmetrical map.

16-6. Self-Complementary Graphs

Def. 16-42. A graph G is said to be *self-complementary* if it is isomorphic to its complement: $G = \bar{G}$.

Def. 16-43. An *anti-automorphism* of a graph G is a permutation $\beta : V(G) \rightarrow V(G)$ exchanging edges and "non-edges"; that is, $uv \in E(G)$ if and only if $\beta u \beta v \notin E(G)$; $u \neq v$. We denote the set of all anti-automorphisms of G by $\overline{\text{Aut}}(G)$.

It follows that G is self-complementary if and only if $\overline{\text{Aut}}(G) \neq \emptyset$; in this case $\text{Aut}(G) \cup \text{Aut}(G)$ is a group containing $\text{Aut}(G)$ as a (necessarily normal) subgroup of index two.

Self-complementary graphs have been studied in some detail, as the sample results below demonstrate. (See also Sachs [S3], and Gibbs [G2].

Thm. 16-44. (Ringel [R11]) There exists a self-complementary graph G of order p if and only if $p \equiv 0$ or $1 \pmod 4$.

(The necessity is apparent, as the size of G must be $\frac{p(p-1)}{4}$.)

Thm. 16-45. (Ringel [R11]) If $p \equiv 1 \pmod 4$ and β is an anti-automorphism for a self-complementary graph G of order p, then β has exactly one fixed point and every other orbit for the action of β on $V(G)$ has length a multiple of four.

For example, if $\Delta = \{1\}$ for $\Gamma = \mathbb{Z}_5$, then $G_\Delta(\Gamma) = C_5$ is self-complementary, as shown (say) by $\beta = (0)(1,2,4,3)$; $\text{Aut}(C_5) = D_5$, and $\overline{\text{Aut}}(C_5) = \beta D_5$.

Thm. 16-46. (Read [R6]) There are 36 self-complementary graphs of order $p = 9$; 5,600 of order 13; and 11,220,000 of order 17.

Thm. 16-47. (Rao [R4]) If G is a self-complementary graph of order $p \geq 8$ and having minimum degree $\delta \geq \frac{p}{4}$, then G has a 2-factor (a spanning 2-regular subgraph.)

Def. 16-48. A graph G is a *graphical regular representation* of a group Γ if $(\mathrm{Aut}(G), V(G))$ is a regular permutation group and $\mathrm{Aut}(G) \cong \Gamma$.

Thm. 16-49. (Lim [L4]) If G is a graphical regular representation of Γ, then G is not self-complementary.

Thm. 16-50. (Chao and Whitehead [CW1]) If G is self-complementary, then $|V(G)| \leq (\chi(G))^2$.

Recall from Problem 2-1 that at least one of G and $\bar{G}$ is connected; we get immediately:

Thm. 16-51. If G is self-complementary, then G is connected.

Nebeský [N2] has shown that at least one of G and $\bar{G}$ is upper-imbeddable; thus:

Thm. 16-52. If G is self-complementary, then $\gamma_M(G) = \left\lfloor \frac{\beta(G)}{2} \right\rfloor$.

The next result could be useful in seeking a genus imbedding for a self-complementary graph:

Thm. 16-53. (Clapham [C5]) The number of triangles in a self-complementary graph of order p is at least $\frac{p(p-2)(p-4)}{48}$, if $p \equiv 0 \pmod 4$, and at least $\frac{p(p-1)(p-5)}{48}$, if $p \equiv 1 \pmod 4$. These minimum numbers are attained.

Finally, we comment that self-complementary graphs determined by the quadratic residues of a finite field have been used to give lower bounds for the Ramsey numbers $r(k,k)$; see Greenwood and Gleason [GG1] and Burling and Reyner [BR3]. See also Clapham [C6], for a more general construction; he finds a self-complementary graph of order 113 containing no K_7, giving the improved bound $r(7,7) \geq 114$.

16-7. Self-dual Maps

The "self-complementary" property for a graph depends only upon the abstract structure of the graph itself. To obtain an analog in terms of a geometric realization for the graph, we first imbed the graph on a surface, form the dual graph for this imbedding, and then compare the original graph with its dual. (In a sense, duality is a higher-dimensional analog of complementation: in taking a dual, we fix 1-dimensional subsets, while interchanging 2- and 0-dimensional subsets; in taking a complement, we fix 0-dimensional subsets, while interchanging 1-dimensional subsets and "(-1)-dimensional" subsets - the "non-edges.")

Def. 16-54. Let map $M = M(G, \rho)$ have dual map $M^* = M(G^*, \rho^*)$; then M is said to be *self-dual* if $G \cong G^*$.

For example, the maps of Figures 5-4 (b), 7-4, and 10-5 and the map of Example 1a in Section 10-3 are all self-dual.

The first and last of those are special cases of:

Thm. 16-55. (Heffter [H7], Biggs [B10], White [W9], Pengelley [P4], Stahl [S12], and Bouchet [B15]). The complete graph K_n has a self-dual imbedding if and only if $n \equiv 0$ or $1 \pmod 4$.

(Compare with Theorem 16-44.)

We now outline the development of [W9], but in the context of voltage graphs. Let the finite abelian group Γ be generated by $\Delta = \{a_1, b_1, a_2, b_2, \ldots, a_h, b_h\}$, where no generator has order 2. Take as the voltage graph (K, Γ, ϕ) and its imbedding the normal form for $S_h (h \geq 1)$, as given in Theorem 5-5: a regular polygon of $4h$ sides, with clockwise boundary $a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_h, b_h, a_h^{-1}, b_h^{-1}$. Identify the sides, by paired labels, respecting the directions. Then K has one vertex, $2h$ edges, and one region on S_h, and–since Γ is abelian–the index-one imbedding of K satisfies the KVL. (See Figure 16-6, for the case $h = 2$.) Then by Corollary 10-16, the covering imbedding of $G_\Delta(\Gamma)$ is on S_k, where $k = 1 + |\Gamma|(h - 1)$. Moreover, the covering imbedding has $p = p_{4h} = \Gamma$ and $r = r_{4h} = \Gamma$. In fact:

Thm. 16-56. The map $M(G_\Delta(\Gamma), \rho)$ constructed above is self-dual.

Thus for each $h \geq 1$, the normal form for S_h determines a variety of self-dual imbeddings of Cayley graphs (see Problem 16-18.) This has many ramifications.

Thm. 16-57. For $h \geq 1$, there is a self-dual imbedding of some graph G of order p on $S_{p(h-1)+1}$ if and only if $p \geq 4h + 1$.

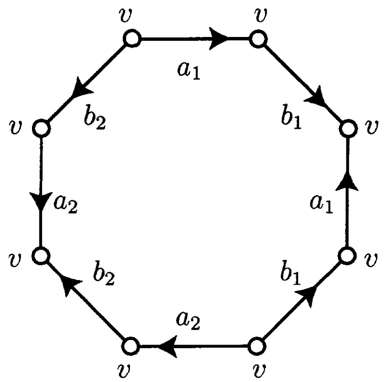


Figure 16-6.

PROOF. (i) For the necessity, we let G be self-dual imbedded on $S_{p(h-1)+1}$; the euler equation (Theorem 5-14) gives $q = 2ph \le \frac{p(p-1)}{2}$ and $p \ge 4h+1$.

(ii) For the sufficiency, choose $\Gamma = \mathbb{Z}_p$ and $\Delta = \{1, 2, \ldots, 2h\}$; that is, $a_i = 2i-1, b_i = 2i$, $1 \le i \le h$. Now apply the construction of Theorem 16-56.

□

Cor. 16-58. There is a self-dual imbedding of some graph G of order p on the torus if and only if $p \ge 5$.

Thm. 16-59. The finite abelian group Γ is self-dual (i.e. has a self-dual imbedding for some $G_\Delta(\Gamma)$) if there exists a generating set Δ for Γ of even order with the property that if $\delta \in \Delta$, then $\delta^{-1} \notin \Delta$.

Cor. 16-60. $\Gamma = \mathbb{Z}_n (n \ge 1)$, is self-dual if and only if $n \ge 4$.

Thm. 16-61. If 4 divides $m(n-1)$, then $K_{n(m)}$ has a self-dual imbedding.

PROOF. Take $\Gamma = \mathbb{Z}_{mn}$, with Δ^* consisting of Γ less all multiples of n; then (since the multiples of n induce the graph nK_m, the complement of $K_{n(m)}$), $G_\Delta(\Gamma) = K_{n(m)}$. Since 4 divides $m(n-1)$, Δ has even order; moreover, if $\frac{mn}{2} \in \Gamma$, $\frac{mn}{2} \notin \Delta$. Thus the construction of Theorem 16-56 applies. □

Now part of Theorem 16-55 follows as an immediate corollary. For additional corollaries, we have:

Cor. 16-62. $K_{m,m}$ has a self-dual imbedding, for $m \equiv 0 \pmod 4$.

Cor. 16-63. For $n \equiv 3 \pmod 4$, $K_{n(m)}$ has a self-dual imbedding if and only if m is even.

Cor. 16-64. $K_{m,m,m}$ has a self-dual imbedding if and only if m is even.

Cor. 16-65. The 1-skeleton of the n-dimensional octahedron, $K_{n(2)}$, has a self-dual imbedding for n odd.

In [S16] Stahl (see also Bouchet [B15]) established the following:

Thm. 16-66. If $m - 1 \equiv n \equiv 0 \pmod 4$, then $K_{n(m)}$ has a self-dual imbedding.

(And, in [Q1] Quitté added:

Thm. 16-67. If $n \equiv 0 \pmod 8$ and $m \equiv 2, 3 \pmod 4$, then $K_{n(m)}$ has a self-dual imbedding.)

Thm. 16-68. For $n \equiv 1 \pmod 4$, the complete maps constructed in the proof of Theorem 16-35 are self-dual.

Thm. 16-69. The fundamental group $\pi(S_k), k \geq 1$, has a self-dual imbedding in the plane.

Thm. 16-70. The finitely generated abelian group Γ has a self-dual imbedding if and only if $\Gamma \neq \mathbb{Z}_2$ or $\mathbb{Z}_3$.

All of the self-dual imbeddings considered thus far have been into orientable surfaces. Stahl [S16] considered the nonorientable case as well.

Thm. 16-71. The graph $K_{n(m)}, n > 1$, has a nonorientable self-dual imbedding if any one of the following holds:

(i) $m \equiv 0 \pmod 4$ and $n > 2$;
(ii) $m \equiv 2 \pmod 4$ and $n \neq 2 \pmod 4$;
(iii) $m = 1$ and $n \equiv 0 \pmod 4$, $n \neq 4$;
(iv) $m \equiv 1 \pmod 2$, $n \not\equiv 2 \pmod 4$, n is not a power of 2, and $(m, n) \neq (1, 3), (1, 5)$, or $(3, 3)$.

Thm. 16-72. The fundamental group $\pi(N_h)$, $h \geq 1$, has a self-dual imbedding in the plane.

Thm. 16-73. The finitely generated abelian group Γ has a nonorientable self-dual imbedding if and only if $|\Gamma| \geq 6$.

16-8. Paley Maps

The properties for graphs and maps we have been discussing are symmetry properties; in the self-complementation or self-duality case, the symmetry is external (G compares with $\bar{G}$ or G^* respectively), while in the context of symmetrical maps, the symmetry is internal ($M = (G, \rho)$ compares with itself.)

In this section we attempt to tie these three properties together.

First we recall, from Definition 3-3, that two permutation groups (Γ, X) and (Γ', X') are equivalent if there exists a bijection $\beta : X \to X'$ and a group isomorphism $\phi : \Gamma \to \Gamma'$ such that, for each $x \in X$ and $\gamma \in \Gamma$, $\phi(\gamma)(\beta(x)) = \beta(\gamma(x))$; that is $\phi(\gamma)\beta = \beta\gamma$, so that $\phi(\gamma) = \beta\gamma\beta^{-1}$ and ϕ is induced by β, which can be regarded as a relabelling. For example, $(\mathrm{Aut}(G), V(G))$ and $(\mathrm{Aut}(\bar{G}), V(\bar{G}))$ are equivalent, with β as the identity function, inducing ϕ as the identity function also. As a less trivial example, if $M = M(G, \rho)$ is a map, with dual map $M^* = M(G^*, \rho^*)$, then $(\mathrm{Aut}(M), D^*)$ and $(\mathrm{Aut}(M^*), (D^*)^*)$ are equivalent, under $\beta : D^* \to (D^*)^*$ assigning, to each $s \in D^*$, the unique $s^* \in (D^*)^*$ 'crossing' s (recall that $D^* = \{(u, v) | uv \in E(G)\}$). However, $(\mathrm{Aut}(M), V(G))$ and $(\mathrm{Aut}(M^*), V(G^*))$ need not be equivalent; in fact, it is quite possible that $|V(G)| \neq |V(G^*)|$.

We combine the symmetry properties of self-complementation, self-duality, and symmetricality, and proceed with the development as in [W13].

Def. 16-74. The map $M = M(G, \rho)$ is said to be *strongly symmetric* if:

(i) $\bar{G} = G$;
(ii) $G^* = G$;
(iii) M is symmetrical
(iv) $\mathrm{Aut}(M)$ and $\mathrm{Aut}(M^*)$ are equivalent, under an anti-isomorphism: $\beta : V(G) \to V(G^*)$.

We need one preliminary result, before characterizing strongly symmetric maps as to order; compare Theorem 16-33.

Thm. 16-75. (Jordan; see Burnside [B21, p. 172]) If (Γ, X) is a Frobenius group of degree $n = |X| \geq 6$ and order $\frac{n(n-1)}{2} = |\Gamma|$, then n is a prime power.

Thm. 16-76. There exists a strongly symmetric map of order n if and only if n is a prime power congruent to 1 (mod 8).

PROOF. (A) Let $M = M(G, \rho)$ be a strongly symmetric map, with $n = |V(G)|$ and $e = |E(G)|$. Since G is self-complementary, $e = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}$; thus $n \equiv 0$ or $1 \pmod 4$ – see also Theorem 16-44. But since M is symmetrical, G is vertex-transitive (by Theorem 16-18) and hence regular of degree $\frac{n-1}{2}$, so that $n \equiv 1 \pmod 4$. Now let M embed G on S_k, with f regions. Since M is self-dual, $f = n$, and the Euler equation (Theorem 5-14) gives $2n - \frac{n(n-1)}{4} = 2 - 2k$, so that $n \equiv 1 \pmod 8$. Since M is symmetrical, $|\operatorname{Aut}(M)| = 2|E(G)| = \frac{n(n-1)}{2}$, and $\operatorname{Aut}(M)$ is a transitive permutation group of order $\frac{n(n-1)}{2}$ and degree $n \geq 9$. We show that $\operatorname{Aut}(M)$ is a Frobenius group and then apply Theorem 16-75, to see that, in fact, n is a prime power.

So, let $\alpha \in \operatorname{Aut}(M)$, with $u \neq v \in V(G)$ such that $\alpha(u) = u$ and $\alpha(v) = v$. If $uv \in E(G)$, then α is the identity permutation, by Lemma 16-14. If $uv \notin E(G)$, then we apply the anti-isomorphism β giving $\operatorname{Aut}(M)$ equivalent to $\operatorname{Aut}(M^*)$. Then β induces an isomorphism ϕ between $\operatorname{Aut}(M)$ and $\operatorname{Aut}(M^*)$, so that $\phi(\alpha) = \beta\alpha\beta^{-1}$.

Thus $(\phi(\alpha))(\beta(u)) = \beta\alpha\beta^{-1}(\beta(u)) = \beta\alpha(u) = \beta(u)$; similarly, $\phi(\alpha)$ also fixes $\beta(v)$. But, since β is an anti-isomorphism and $uv \notin E(G)$, $\beta(u)\beta(v) \in E(G^*)$. Hence, by Lemma 16-14 again, $\phi(\alpha)$ is the identity in $\operatorname{Aut}(M^*)$; but since ϕ is a group isomorphism, α is the identity in $\operatorname{Aut}(M)$ and $\operatorname{Aut}(M)$ *is* a Frobenius group.

(B) For the converse, let $n = p^r \equiv 1 \pmod 8$, p a prime and $r \in N$. We construct a Cayley graph $G_n = G_{\Delta_n}(\Gamma_n)$, where $\Gamma_n = (\mathbb{Z}_p)^r$–the additive group in the Galois field $GF(p^r)$. Take x as a primitive element for $GF(p^r)$, so that x generates the multiplicative group, and let $\Delta_n^* = \{1, x^2, x^4, \ldots, x^{n-3}\}$, the set of all *squares* in $GF(p^r)$. (Equivalently, $uv \in E(G_n)$ if and only if $v - u$ is a square in $GF(P^r)$.) The Cayley graph G_n is called a *Paley graph* (see [P1], where the ideas behind this construction were introduced.) We remark that Paley graphs are defined for all prime powers $p^r \equiv 1 \pmod 4$, since these are precisely the cases for which -1 is a square – so that undirected edges are well-defined; but only in the case $p^r \equiv 1 \pmod 8$ are self-dual imbeddings possible.

Next we define $r_n : \Delta_n^* \to \Delta_n^*$ by $r_n(\delta) = x^2\delta$, so that $M_n = M(\Gamma_n, \Delta_n, r_n)$ is a Cayley map– which we now call a *Paley map*; r_n

induces vertex rotations

$$\rho_v(w) = x^2(w - v) + v,$$

in accordance with Definition 16-23; and the permutation $\bar{r}_n : \Delta_n^* \to \Delta_n^*$, as given by

$$\bar{r}_n(\delta) = -x^2\delta,$$

in accordance with the remarks preceding Theorem 16-28. We observe that Paley maps are in fact defined for all $n = p^r \equiv 1 \pmod 4$, but we continue to specialize to the case $n = p^r \equiv 1 \pmod 8$. We claim that, for $n = p^r \equiv 1 \pmod 8$, the Paley map $M_n = M(\Gamma_n, \Delta_n, r_n)$ is strongly symmetric.

(i) The permutation $\beta : (\mathbb{Z}_p)^r \to (\mathbb{Z}_p)^r$ given by $\beta(v) = xv$ is an anti-automorphism for G_n–since $w = u + x^{2k}$ if and only if $\beta(w) = xw = x(u + x^{2k}) = xu + x^{2k+1} = \beta(u) + x^{2k+1}$; note the use of distributive law in the field $GF(p^r)$– so that G_n is self-complementary.

We remark that Paley graphs, being Cayley graphs of odd order, not only have a 2-factor (as required by Theorem 16-47), but in fact are *2-factorable* (the edge set partitions into 2-factors); see Problem 4-17. We also observe that the anti-automorphism β has exactly one fixed point (the vertex 0) and one orbit of length $n - 1 \equiv 0 \pmod 4$, as required by Theorem 16-45.

(ii) To show that $M(G_n, \rho)$ is a self-dual map, we study the corresponding imbedding of G_n into S_k, where $n = p^r = 8m + 1$ and $k = 8m^2 - 7m$, for $m = 1, 2, 3, 5, 6, 9, \ldots$. This imbedding is an $(8m+1)$-fold covering space (no branching) of a voltage graph imbedding which is the following alternative normal form for $S_m : a_1 a_2 \ldots a_{2m} a_1^{-1} a_2^{-1} \ldots a_{2m}^{-1}$. (Note that the values $n = 8m + 1$, $h = m$ are consistent with Theorem 16-57 and that Theorem 16-70 is also illustrated by this construction.) The $2m$ directed edges bounding this $4m$-gon are labelled with generators from Δ_n by the assignment $a_i \to x^{(i-1)(4m-2)}$, $1 \le i \le 2m$. This assignment describes each region boundary (and each vertex rotation) in the covering space, the map (G_n, ρ), by the voltage graph theory; in particular, the single $4m$-gon below satisfies the KVL and lifts to $8m + 1$ $4m$-gons above.

Figure 10-5 (letting b in Figure 10-4 be 1, $-a = x^2$) depicts the entire situation for $m = 1$, using $x^2 = 2x + 1$ in $GF(9) : a_1 \to 1 = 01, a_2 \to x^2 = 21$; $x = 10$; for just the voltage graph for the case $m = 2$, replace the labels in Figure 16-6 with: $1, x^6, x^{12}, x^2, -1, -x^6, -x^{12}, -x^2$.

We now utilize a method introduced by Bouchet [B15] for showing self-duality. Each region in the covering space imbedding has a unique lift of the directed edge a_1 from the normal-form voltage graph in its boundary; this lift is a side $(g, g + 1)$ in G_n, where g is uniquely determined for that region; label the region with g^*. It is now routine to check that, for each $g \in (\mathbb{Z}_p)^r$, the region g^* $(g^* \in V(G_n^*))$ has neighbors $N(g^*) = \{(g + w(-x^2)^k)^* | 0 \le k \le 4m - 1\}$ in G_n^*, where $w = \frac{1-x^2}{1+x^2}$. In fact, the dual is a Cayley map $M_n^* = M_n(\Gamma_n, w\Delta_n, r_n^*)$, where $r_n^*(w) = -x^2 w\delta(\delta \in \Delta_n)$, r_n^* being taken in the sense opposite to that of r_n. Thus if w is a square in $GF(p^r)$, then $G_n^* = G_n$. On the other hand, if w is a non-square, then $G_n^* = \bar{G}_n = G_n$.

(iii) Using the distributive law in $GF(n)$ again, we readily see that the permutation r_n of Δ_n^* extends to a group automorphism of $\Gamma_n = (\mathbb{Z}_p)^r$; thus Theorem 16-27 applies, to show that $M_n(\Gamma_n, \Delta_n, r_n)$ is a symmetrical map. In fact, $\mathrm{Aut}(M_n)$ is a Frobenius group with Frobenius kernel the regular normal subgroup $\{\gamma_g : \Gamma_n \to \Gamma_n, \gamma_g(h) = g + h | g \in \Gamma_n = V(G_n)\} \cong \Gamma_n$ and Frobenius complement the cyclic group stabilizing the vertex 0, generated by the automorphism ζ_0 extending $r_n, \zeta_0(h) = x^2 h$. (See Theorems 16-32 and 16-15.) The other vertex stabilizers are conjugate, with $(\mathrm{Aut}(M_n))_g$ generated by $\zeta_g = \gamma_g \zeta_0 \gamma_g^{-1}$. Of course, $|\mathrm{Aut}(M_n)| = |\Gamma_n|\,|\Delta_n| = 2|E(G_n)|$. Finally; we emphasize that $\mathrm{Aut}(M_n)$ contains a subgroup isomorphic with Γ_n, as required by Theorem 16-24; this subgroup must be proper, in $\mathrm{Aut}(G_n)$, by Theorem 16-49.

(iv) If $w = \frac{(1-x^2)}{(1+x^2)}$ is a non-square, then the anti-isomorphism $\beta : V(G_n) \to V(G_n^*)$ given by $\beta(g) = g^*$ gives an equivalence between $\mathrm{Aut}(M_n)$ and $\mathrm{Aut}(M_n^*)$, since if $\alpha^* = \beta\alpha\beta^{-1}$, where α is one of the generating automorphisms ζ_0 or γ_g $(g \in \Gamma_n)$ of $\mathrm{Aut}(M_n)$, one readily checks that α^* preserves oriented region boundaries in M_n^*. If, however, $w \in \Delta_n$, then we take $\beta(g) = (xg)^*$, to again see that $\mathrm{Aut}(M_n)$ and $\mathrm{Aut}(M_n^*)$ are equivalent, under an anti-isomorphism β. □

The Paley maps $M_n = (\Gamma_n, \Delta_n, r_n)$ constructed above have additional properties of interest. We use the following facts about Galois fields, taken from Storer [S24].

Thm. 16-77. Let x be a primitive element for $GF(p^r)$, where $p^r = 2f + 1$ and f is even. Let (i, j) denote the number of ordered pairs (s, t) such that $x^{2s+i} + 1 = x^{2t+j}$, $0 \le s, t \le f - 1$; then:

(i) $(0,0) = \frac{f-2}{2}$;
(ii) $(0,1) = (1,0) = (1,1) = \frac{f}{2}$.

Thm. 16-78. For $n = 4m + 1$, the Paley graph G_n is strongly regular, with parameters $p_{22}^1 = m$ and $p_{22}^2 = m - 1$.

PROOF. From $n = 4m + 1 = 2f + 1$, we find $f = 2m$. From Theorem 16-77 (i) we see that $(0,0) = \frac{f-2}{2} = m - 1$. Now clearly

$x^{2s}+1 = x^{2t}$ if and only if $x^{2s+2k}+x^{2k} = x^{2t+2k}$; it follows that $p_{22}^2 = m-1$ for G_n. Similarly, we use Theorem 16-77 (ii) to deduce that $p_{22}^1 = (1,1) = \frac{f}{2} = m$. □

This result overlaps with the next:

Thm. 16-79. If G is self-complementary and has a symmetrical map, then G is strongly regular.

PROOF. In fact, for any graph G having a symmetrical map M, since $\mathrm{Aut}(M)$ is transitive on edges and $\mathrm{Aut}(M) \leq \mathrm{Aut}(G)$, p_{22}^2 is well-defined. But if G is self-complementary, then $\mathrm{Aut}(G)$ is transitive on non-edges also, so that p_{22}^1 is well-defined too. □

Thus the Paley graphs G_n, for $n = p^r \equiv 1 \pmod 8$, are candidates for determining association classes for PBIBDs, and in fact the Paley maps give such designs:

Thm. 16-80. Let x be a primitive element for $GF(n)$, where $n = p^r$; the Paley map M_n, for $n = 8m+1$, yields an $(8m+1, 8m+1, 4m, 4m; \lambda_1, \lambda_2)$-PBIBD, where $\lambda_1 = 2m$ and $\lambda_2 = 2m-1$ if x^2+1 is a square in $GF(n)$, while $\lambda_1 = 2m-1$ and $\lambda_2 = 2m$ if x^2+1 is a non-square.

PROOF. See Problem 16-20.

□

It can be shown that, in a Paley map, either each region boundary consists of neighbors $N(v)$, $v \in (\mathbb{Z}_p)^r$, or each region boundary consists of neighbors in the complementary graph, so that the designs constructed above also exist in a natural, topology-free, context. The topological context, however, does facilitate the next observation: these designs are also self-dual, in a very strong sense. The *dual* of a design has as its objects the blocks of the original design, and a block in the dual design contains all those objects of the dual design which represent blocks of the original design to which the fixed object of the original design (corresponding to the given block) belongs.

Thm. 16-81. Given a Paley map, the dual of the design of the map and the design of the dual of the map coincide, and both are isomorphic to the design of the Paley map itself.

PROOF. See Problem 16-21.

□

We make several additional comments relevant to Theorem 16-76. Firstly, we observe that self-dual imbeddings of self-complementary graphs need not be unique for a given order. Moreover, the order need not be a prime power. (We are temporarily relinquishing the symmetricality condition for strongly symmetric maps.) For the first example, take $\Delta = \{5; 1, 6, 11, 16, 21\}$ in $\Gamma = \mathbb{Z}_{25}$, so that $G_\Delta(\Gamma)$ is the composition $C_5[C_5]$. Since $\overline{C_5} = C_5$ and $\overline{G[H]} = \overline{G}[\overline{H}]$ in general (see Problem 2-5), $C_5[C_5]$ is self-complementary. A self-dual imbedding is constructed by the use of Theorem 16-56. Since $C_5[C_5]$ is not strongly regular, the map M is not symmetrical, by Theorem 16-79; thus M differs from the Paley map M_{25} of the same order. In fact, $\text{Aut}(M) \cong \Gamma = \mathbb{Z}_{25}$ and is equivalent to $\text{Aut}(M^*)$ (both are Frobenius groups) under the anti-isomorphism $\beta(g) = (2g)^*$. (Every map M of a Cayley graph $G_\Delta(\Gamma)$ covering a normal-form voltage graph has $\Gamma \cong \Gamma' \leq \text{Aut}(M)$; in this case the fact that there are no additional automorphisms follows from the observation that each region boundary contains repeated vertices.) We mention that Theorem 16-56 also provides a self-dual imbedding, but not a symmetrical map, for the Paley graph G_{25}.

For the second example, we take

$$\Delta = \{13; 5, 20, 15; 1, 4, 16; 3, 12, -17; 7, 28, -18; 11, -21, -19\}$$

in $\Gamma = \mathbb{Z}_{65}$; $\beta(g) = 2g$ gives an anti-automorphism, so that $G_\Delta(\Gamma)$ is self-complementary. Again a self-dual imbedding is constructed by Theorem 16-56, and again the map fails to be symmetrical, by Theorem 16-79. Again, $\text{Aut}(M)$ and $\text{Aut}(M^*)$ are equivalent under the anti-isomorphism $\beta(G) = (2g)^*$, and both are Frobenius groups isomorphic to $\Gamma = \mathbb{Z}_{65}$. The non-prime-power order is possible, because the automorphism group is not large enough for the Jordan theorem (Theorem 16-75) to apply.

Next we present another sufficient condition to give prime-power order (see Problem 16-22.)

Thm. 16-82. Let G be strongly regular (with $p_{22}^1 = 2m$, $p_{22}^2 = 2m - 1$), and $M = (G, \rho)$ a symmetrical map yielding an $(8m+1, 8m+1, 4m, 4m; 2m-1, 2m)$-PBIBD. Then $|V(G)| = 8m+1$ is a prime power.

We remark that, for the Paley graphs $G_n (n = p^r \equiv 1 \pmod 4)$, $\text{Aut}(G_n)$ consists not only of the map automorphisms $\text{Aut}(M_n)$ as in (iii) of the proof of Theorem 16-76, but also of the field automorphisms, generated by $\theta : (\mathbb{Z}_p)^r \to (\mathbb{Z}_p)^r$, $\theta(g) = g^p$; in fact, $\text{Aut}(G_n) = \langle \text{Aut}(M), \theta \rangle$:

Thm. 16-83. (Carlitz [C3]) The automorphism group of the Paley graph G_n is $\mathrm{Aut}(G_n) = \{g \to x^{2k}g^{p^s} + a | 0 \le k \le 2m-1, 0 \le s \le r-1, a \in (\mathbb{Z}_p)^r\}$, where $n = 4m+1$.

We use this result to study the reflexibility of the Paley maps.

Def. 16-84. If a map M is symmetric but not reflexible, we say that M is *chiral.*

Thm. 16-85. The Paley map M_n is reflexible if and only if $n = 9$; thus M_n is chiral if and only if $n \neq 9$.

PROOF. If M_n is reflexible, with $n = p^r$, then we must have $r = 2s$ and $\theta^s(g) = g^{p^s}$ giving a reflection. Since θ^s fixes both 0 and 1, we must have $\theta^s(x^2) = x^{2p^s} = x^{p^{2s}-3}$ (since $\rho_0 = (1, x^2, \ldots m, x^{p^{2s}-3})$. Thus $p^{2s} - 3 = 2p^s$, so that p divides 3; hence $p = 3$ and $s = 1$ (or else 3 divides 1); that is, $n = 9$.

Conversely, $\theta(g) = g^3$ is readily seen to be a reflection for M_9; refer to Figure 10-5. □

As we have seen, the Paley maps–defined for prime powers congruent to 1 (mod 8) –have several interesting properties: they are regular (symmetrical), self-dual imbeddings of strongly regular, self-complementary graphs which produce self-dual block designs, for example. Now we see how closely we can approximate these properties by similar maps, for all other prime-power orders.

For $p^r = 5 \pmod 8$, the Paley graphs G_n are defined, and are both self-complementary and strongly regular, as before. The euler equation disallows self-dual surface imbeddings, so we utilize pseudosurfaces. Again we use a normal-form voltage graph but, for $n = 8m+5$, it is a $(4m+2)$-gon and thus has two vertices; see Figure 16-7 for the case $m = 1$. The $(8m+5)$-fold covering imbedding has $16m+10$ vertices, each of degree $4m+2$. For each $g \in GF(n)$, we identify the two vertices (a, g) and (b, g); the result is a symmetrical, self-dual pseudosurface imbedding of G_n, for $m > 0$. ($G_5^* = C_5^*$ consists of five disjoint loops. For the pseudosurface theory of voltage graphs, see Garman [G1].) In fact, the map is a Cayley map $M(\Gamma_n, \Delta_n, r_n)$, with $r_n : \delta_n^* \to \Delta_n^*$, $r_n(\delta) = x^4$, having *two* cycles. The dual G^* is G if $w = \frac{1-x^4}{1+x^r}$ is a square, or $\overline{G}$ if w is a non-square. The automorphism group is Frobenius, with Frobenius kernel $\cong (\mathbb{Z}_p)^r$. The stabilizer of vertex 0 is generated by $g \to x^2 g$, which alternates between the two orbits of the vertex rotation at 0. An $(8m+5, 8m+5, 4m+2, 4m+2; 2m, 2m+1)$-PBIBD results if $1 + x^4$ is non-square; otherwise, the two λ values interchange.

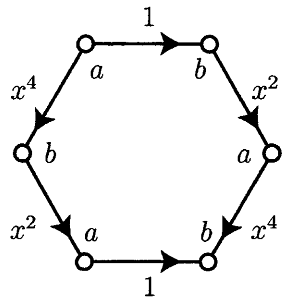


Figure 16-7.

For $p^r \equiv 3 \pmod 4$, the Paley graphs G_n are no longer defined, since $-1 = x^{2m+1}$ ($p^r = 4m+3$) is *not* a square. Instead, we define the *Paley tournament* T_n, by: $(g_1, g_2) \in E(T_n)$ if and only if $g_2 - g_1$ is a square in $GF(n)$; of course, we still take $\Gamma_n = (\mathbb{Z}_p)^r$ as our vertex set. Then $g_1 \neq g_2$ gives $|\{(g_1, g_2), (g_2, g_1)\} \cap e(T_n)| = 1$, so that we do have a tournament. The underlying undirected graph is thus complete, and hence degenerately "strongly regular". The Paley tournaments are self-converse, under the anti-automorphism $\beta : g \to xg$. In Biggs [B10] we find symmetrical imbeddings of the associated K_{4m+3} having $8m+6$ regions, each of length $2m+1$; these are Cayley maps $M(\Gamma_n, \Gamma_n - \{0\}, r_n)$, where $r_n(g) = xg$. Thus the imbedding cannot be self-dual directly, although the dual is bichromatic. Hence we obtain one $(4m+3, 8m+6, 4m+2, 2m+1, 2m)$-BIBD and two $(4m+3, 4m+3, 2m+1, 2m+1, m)$-BIBDs. (The latter are Hadamard designs.) We modify this map to form a self-dual pseudosurface imbedding of K_{4m+3}. Each region contains exactly one edge corresponding to $1 \in (\mathbb{Z}_p)^r$, in one of the two possible senses (clockwise or counterclockwise.) In fact, this distinction determines the 2-coloring of the dual. The region is assigned label $g^*(g \in (\mathbb{Z}_p)^r)$ if either: (i) $(g, g+1)$ bounds the region in the clockwise sense, or (ii) $(g-a, g-a-1)$ bounds the region in the clockwise sense, where $a = \frac{2}{x^{2m}-1}$. Thus each g appears exactly twice as a region label, and if these n pairs of vertices in the dual are identified, we obtain a self-dual pseudosurface imbedding of K_{4m+3}. Moreover, if the edge directions are carried over into the dual, then we have a self-dual imbedding of the self-converse tournament T_n, with each vertex neighborhood $N(g^*)$ partitioned into two sets (corresponding to the g^* identification): one consists of those vertices dominated by g^*, the other consists of those dominating g^*.

Finally, we consider $p = 2$, so that $p^r = 2^r$. Here not even 1 is a square ($1 = x^{2^r-1}$) in $GF(2^r)$, so our previous constructions seem not to apply. However, if we use the planar voltage graph of Figure 16-8, with $n = 2^r$, we obtain an n-fold covering imbedding of a 2-fold K_n, with n $(n-1)$-gons and $\binom{n}{2}$ digons. If each digon is closed (by identifying its two edges), a symmetrical, self-dual imbedding of the "strongly regular" K_n (self-complementary in the 2-fold K_n) results. (In fact, $r_n^* = r_n$ for this case.) The concomitant design is an $(n, n, n-1, n-1, n-2)$-BIBD which, of course, can also be constructed by taking complements of singletons as blocks.

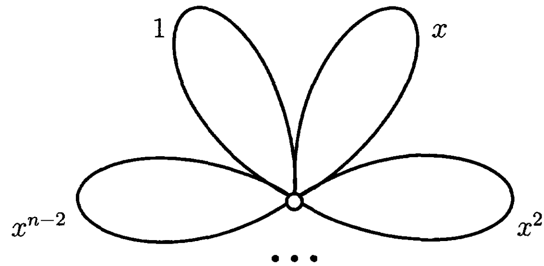


Figure 16-8.

We close this lengthy section by discussing conditions under which the Paley maps are, in some sense, unique. Self-duality, strong regularity, and designs do not enter into the characterizations given, but as it is Paley maps which are being characterized, these additional properties persist – except the self-duality fails for $n \equiv 5 \pmod 8$. For proofs of the following two theorems, refer to [W13] or to Section 16-9.

Thm. 16-86. If $M = (G, \rho)$ is vertex-transitive and if $(\text{Aut}(M), V(G))$ is self-equivalent under an anti-automorphism β of G, then M is isomorphic to a Cayley map $M(\Gamma, \Delta, r)$, where $|\Gamma| = |V(G)|$ and $|\Delta^*| = \frac{(|V(G)|-1)}{2}$.

Thm. 16-87. There exists a symmetrical imbedding of a self-complementary graph G of order n, with $(\text{Aut}(M), V(G))$ self-equivalent under an anti-automorphism β of G, if and only if n is a prime power congruent to $1 \pmod 4$. Moreover, if $\beta^2 \in \text{Aut}(M)$, the maps are essentially unique, with at most six exceptions.

Cor. 16-88. For each prime $p \equiv 1 \pmod 4$, there exists a unique symmetrical imbedding of a self-complementary graph G of order p having $\text{Aut}(M) = \text{Aut}(G)$.

PROOF. The Paley maps give existence. But if $\text{Aut}(M) = \text{Aut}(G)$, then β^2 and $\beta\alpha\beta^{-1}$ (for each $\alpha \in \text{Aut}(M)$ and where β is an anti-automorphism of G) are both in $\text{Aut}(M)$, so that Theorem 16-87 (and its proof) show that M is a Paley map. $\square$

The strength of this uniqueness claim is illustrated by Theorem 16-46. The Paley maps of order 13 and 17, for example, are unique among 5,600 and 11,220,000 self-complementary graphs respectively, with respect to being symmetrical, with $\mathrm{Aut}(M) = \mathrm{Aut}(G)$.

Cor. 16-89. If there exists a symmetrical imbedding of a self-complementary graph G of order n, with $\mathrm{Aut}(M) = \mathrm{Aut}(G)$, then (with at most six exceptions) n is a prime congruent to 1 (mod 4).

PROOF. As in the proof of Corollary 16-88, we find by Theorem 16-87 that n is a prime *power* congruent to 1 (mod 4); moreover, the maps (with at most six exceptions) are unique and thus are Paley maps. But if $n = p^r$ with $r > 1$, then $\mathrm{Aut}(M)$ is a proper subgroup of $\mathrm{Aut}(G)$, by Theorem 16-83; thus $r = 1$, and n is prime. □

16-9. Problems

16-1.) Show that $|R(G)| = \prod_{i=1}^{n}(n_i - 1)!$

16-2.) Show that the relation of Definition 16-2 is an equivalence relation.

16-3.) Prove Theorem 16-5.

16-4.) Show that $\mathrm{Aut}(M) = A_4$, for M as in Figure 16-2.

16-5.) Prove that the following are equivalent, for a map $M = (G, \rho)$:
- (i) M is reflexible;
- (ii) M and its mirror image M^{-1} are equivalent;
- (iii) there exists an $\alpha \in \mathrm{Aut}(G)$ such that, for all $v \in V(G)$, $p_{\alpha(v)} = \alpha \rho_v^{-1} \alpha^{-1}$.

16-6.) Show that the map of Figure 10-5 is reflexible.

16-7.) Prove Theorem 16-13.

16-8.) Verify the entries in Table 16-2.

16-9.) Show that $|\mathrm{Aut}^*(M)|$ divides $4|E(G)|$. (See Theorem 5-26.) Thus a nonorientable map M is defined to be *symmetrical* if $|\mathrm{Aut}^*(M)| = 4|E(G)|$. Is the map of Figure 8-4 symmetrical?

16-10.) Prove or disprove: If G is a strongly regular graph, then there exists a rotation ρ for G so that $M = (G, \rho)$ is a regular map. (Hint: consider the Petersen graph.)

16-11.) Let M be the imbedding of K_5 on S_1 given in Example 1a of Section 10-3. Show that M is symmetrical (i.e. $|\mathrm{Aut}(M)| = 20$), but not reflexible (i.e. M is chiral.) Show that the corresponding design D (Example 1 of Section 12-3) has automorphism group $\mathrm{Aut}(D) \cong S_5$. Does this contradict Theorem 16-13?

16-12.) *Prove Theorem 16-36.

16-13.) Prove Theorem 16-37.

16-14.) Prove Theorem 16-38.

16-15.) Prove Theorem 16-39.

16-16.) Show that $M(\mathbb{Z}_9, \{1,2,4\}, (1,2,4,8,7,5))$ is a symmetrical map for $K_{3(3)}$. Use this map to show that $\gamma(K_{3,3,3,6}) = 7$.

16-17.) Let map $M^* = M(G^*, \rho^*)$ be dual to map $M = M(G, \rho)$. Find a description of ρ^*, in terms of ρ and $\lambda : D^* \to D^*$, where $D^* = \{(u,v) | uv \in E(G)\}$ and $\lambda(u,v) = (v,u)$. (See Biggs [B11].)

16-18.) *Prove Theorem 16-56.

16-19.) Show that $\gamma_M(G_n) = \frac{(n-1)(n-4)}{8}$, where G_n is the Paley graph of order $n = p^r \equiv 1 \pmod 8$.

16-20.) Prove Theorem 16-80.

16-21.) *Prove Theorem 16-81.

16-22.) Prove Theorem 16-82.

16-23.) Prove Theorem 16-86.

16-24.) Prove Theorem 16-87.

16-25.) Show that a Cayley map covers an index-one voltage graph imbedding. Reconcile Theorems 10-11 and 16-28.

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CHAPTER 17

ENUMERATING GRAPH IMBEDDINGS

We have studied various techniques for imbedding a given graph on a given surface, and we have studied the range of possible surfaces for 2-cell imbeddings of a fixed connected graph. In this chapter we will study three counting problems that arise in topological graph theory:

(1) For a fixed connected graph, how many 2-cell imbeddings does it have altogether?
(2) How many of these are on a particular surface?
(3) How many of those on a fixed surface have a particular region distribution?

We will consider these questions both for labelled and for unlabelled graphs. Our surfaces will all be orientable (closed orientable 2-manifolds), but similar studies could be made for nonorientable (and/or pseudosurface) imbeddings as well.

17-1. Counting Labelled Orientable 2-Cell Imbeddings

Let G be a connected graph, with a labelling given by $V(G) = \{1, 2, \ldots, n\}$. As in Section 6-6, let $V(i) = \{k \in V(G) | \{i, k\} \in E(G)\}$, for $1 \leq i \leq n$, and let $p_i : V(i) \to V(i)$ be a cyclic permutation. Set $d_i = |V(i)|$, the degree of vertex i. The set $\{p_1, p_2, \cdots, p_n\}$ is a rotation scheme for G and determines a 2-cell imbedding of G into an orientable surface. In fact the set $R(G)$ of all rotation schemes for G is in one-to-one correspondence with the set of all labelled orientable 2-cell imbeddings of G, by Theorem 6-50 (and its proof). This gives immediately:

Thm. 17-1. The number of labelled orientable 2-cell imbeddings of the connected graph G is:

$$|R(G)| = \prod_{i=1}^{n} (d_i - 1)!$$

Thus question (1) of the introduction to this chapter is entirely answered, for **all** connected graphs G, in the labelled case. Questions (2) and (3) are considerably more difficult. We illustrate, with a series of examples.

Example 1: Let $G = K_4$. From Theorem 17-1, we see that K_4 has $((3-1)!)^4 = 16$ labelled orientable 2-cell imbeddings in all. We saw in Section 16-1 that the 16 imbeddings split into 2 on the sphere and 14 on the torus. Both imbeddings on the sphere have $r = r_3 = 4$. However, on the torus 6 imbeddings have $r_4 = r_8 = 1$ and 8 have $r_3 = r_9 = 1$. Thus we have answered questions (2) and (3) for $G = K_4$. Note that no other region distribution for K_4 is possible for the sphere, but that either $r_5 = r_7 = 1$ or $r_6 = 2$ seems, at first, possible for the torus. (Both distributions satisfy both the euler identity and part (iv) of Lemma 5-17.) But no 5-sided region can occur for K_4; see Figure 17-1. At least one vertex, say v, must be repeated, since only four vertices are available. As K_4 has no loops, the second occurrence of v can be placed, without loss of generality, as shown in the figure. But then $\{u, v\}$ is a multiple edge. Similarly (see Problem 17-1) no 6-sided region can occur for K_4.

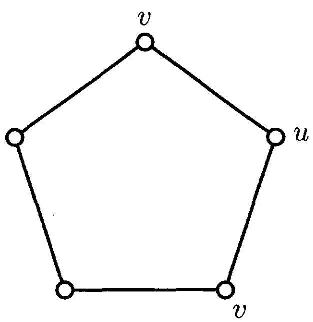

Figure 17-1.

The following definitions will be useful. As usual, r_i denotes the number of regions of length i, $i \geq 3$, and $q = |E(G)|$.

Def. 17-2. The sequence $\{r_3, r_4, r_5, \cdots\}$, called a *region distribution*, is said to be *compatible* for a connected graph G if:

(i) $\sum_{i \geq 3} r_i$ has the same parity as $p - q$; and
(ii) $\sum_{i \geq 3} i r_i = 2q$.

Def. 17-3. A compatible sequence $\{r_3, r_4, r_5, \cdots\}$ is said to be *realizable* if G has an orientable 2-cell imbedding attaining the values of the sequence.

Thus, for K_4, both $r_3 = r_9 = 1$ (that is, the sequence $(1, 0, 0, 0, 0, 0, 1, 0, 0, 0, \cdots))$ and $r_5 = r_7 = 1$ are compatible, but only the former is realizable.

For $G = K_n$ in general, $|R(G)| = ((n-2)!)^n$ grows rapidly. Thus computer programs were written for the next two examples, repeatedly implementing the algorithm of Theorem 6-50 and consolidating the results.

Example 2: Now take $G = K_5$. In Table 17-1 we record the counts for both the labelled and the unlabelled cases (see Section 17-2 for the approach to the latter). We include all compatible region distributions, noting that three of them are not realizable. The final column will be discussed in Chapter 18. The table provides answers to questions (1), (2), and (3) for $G = K_5$.

Example 3: See [LW1], where a similar table appears for K_6. Here, in Table 17-2, we record the answers to questions (1) and (2) for K_6, a column for the number M of non- realizable compatible region distributions, and a column for probabilities.

In general, let $g_k(G)$ denote the number of labelled orientable 2-cell imbeddings of the connected graph G on the surface S_k. Then we have the following:

Thm. 17-4. (i) $g_k(G) > 0$ if and only if $\gamma(G) \le k \le \gamma_M(G)$.
(ii) $\sum_{k=\gamma(G)}^{\gamma_M(G)} g_k(G) = |R(G)|$.

We discuss the function $g_k(G)$ for five families of connected graphs G: cobblestone paths, closed-end ladders, Ringel ladders, bouquets, and dipoles. The *cobblestone path* J_n is obtained by doubling every edge of the n-vertex path P_n and then adding a loop at each end. Thus J_n is a 4-regular pseudograph of order n, and $|R(J_n)| = 6^n$. Furst, Gross, and Statman [FGS1] established:

Thm. 17-5. For the cobblestone path J_n,

$$g_k(J_n) 4^{n-k} 3^k \binom{n-k}{k} = 2 \cdot 4^{n-k} 3^{k-1} \binom{n-k}{k-1}.$$

The *closed-end ladder* L_n is obtained by taking the cartesian product of the n-vertex path P_n with K_2 and then doubling both end edges. Thus L_n is a cubic multigraph of order $2n$, and $|R(L_n)| = 2^{2n}$. The next result is also due to Furst, Gross, and Statman [FGS1] .

Thm. 17-6. For the closed-end ladder L_n,

$$g_k(L_n) = 2^{n-1+k} \binom{n+1-k}{k} \frac{2n+2-3k}{n+1-k}.$$

Region Distribution for K_5	N = number of occurrences		$P = \frac{N}{7776}$
	labeled	unlabeled	
$r_3 = 4,\ r_8 = 1$	150	(3)	.019
$r_3 = 3,\ r_4 = r_7 = 1$	120	(1)	.015
$r_3 = 3,\ r_5 = r_6 = 1$	0		- - -
$r_3 = r_4 = 2,\ r_6 = 1$	120	(2)	.015
$r_3 = r_5 = 2,\ r_4 = 1$	60	(1)	.008
$r_3 = r_5 = 1,\ r_4 = 3$	0		- - -
$r_4 = 5$	12	(2)	.002
Total on S_1	462	(9)	.059
$r_3 = 2,\ r_{14} = 1$	960	(8)	.123
$r_3 = r_4 = r_{13} = 1$	960	(8)	.123
$r_3 = r_5 = r_{12} = 1$	240	(2)	.031
$r_3 = r_6 = r_{11} = 1$	240	(2)	.031
$r_3 = r_7 = r_{10} = 1$	360	(3)	.046
$r_3 = r_8 = r_9 = 1$	720	(6)	.093
$r_4 = 2,\ r_{12} = 1$	240	(2)	.031
$r_4 = r_5 = r_{11} = 1$	120	(1)	.015
$r_4 = r_6 = r_{10} = 1$	420	(4)	.054
$r_4 = r_7 = r_9 = 1$	360	(3)	.046
$r_4 = 1,\ r_8 = 2$	60	(2)	.008
$r_5 = 2,\ r_{10} = 1$	24	(1)	.003
$r_5 = r_6 = r_9 = 1$	120	(1)	.015
$r_5 = r_7 = r_8 = 1$	120	(1)	.015
$r_6 = 2,\ r_8 = 1$	30	(1)	.004
$r_6 = 1,\ r_7 = 2$	0		- - -
Total on S_2	4,974	(45)	.640
$r_{20} = 1$; Total on S_3	2,340	(24)	.301
Grand Total	7,776	(78)	1.000

Table 17-1

Totals for K_6	N = number of occurrences	M	$P = \frac{N}{(24)^6}$
Total on S_1	1,800	0	$9.42 = \times 10^{-6}$
Total on S_2	654,576	3	.00343
Total on S_3	24,613,800	0	.129
Total on S_4	124,250,208	0	.650
Total on S_5	41,582,592	0	.218
Grand Total	$191{,}102{,}976 = (24)^6$	3	1.000

Table 17-2

The *Ringel ladder* graph R_n is obtained from the ladder L_n by performing, at each end of L_n, an elementary subdivision on one of the two multiple edges, and then joining the two new vertices with an edge. Thus Figures 9-6, 9-7, and 9-8 depict, respectively, R_2, R_4, and R_{2s}. Since R_n is a cubic graph of order $2n+2$, $|R(R_n)| = 2^{2n+2}$. Tesar [T3] showed:

Thm. 17-7. For the Ringel ladder R_n,

$$g_k(R_n) = 2^{3k+1}\binom{n-k}{k} + 2^{3k}\binom{n-k}{k-1}$$
$$+ (2^{n+k} - 2^{3k-3})\binom{n-k+1}{k-2} + (2^{n+k-1} - 2^{3k-2})\binom{n-k+1}{k-1}.$$

The even-runged Ringel ladders served as current graphs for Case 7 of the proof of the Complete Graph Theorem (see Section 9-2). Since the solution is of index one for Case 7, the imbedding of $R_{2s}(s \geq 1)$ whose dual is covered by a triangular imbedding of K_{12s+7} has $r = 1$. Thus the following corollary has interest; see Theorem 18-9.

Cor. 17-8. The number of maximum genus imbeddings for R_{2s} is given by :

$$g_{s+1}(R_{2s}) = 2^{3s}(s+2).$$

The *bouquet* B_m consists of one vertex and m loops, so that $|R(B_m)| = (2m-1)!$. Recall that an index-one voltage graph (K, Γ, ϕ) is covered by a Cayley graph $G_\Delta(\Gamma)$, where $\Delta^* = \{\phi(e) | e \in K^*\}$ and $K = B_{|\Delta|}$. Thus bouquets are heavily utilized. For example, in Section 10-3 we find B_2 on S_1 covered by K_5 and $C_s \times C_t (s, t \geq 3)$ on S_1, and B_3 on S_1 is covered by K_7, $K_{4(2)}$, and $K_{3(3)}$, all on S_1. Moreover, B_{2s} on S_s provides the normal form $(a_1 b_2 a_1^{-1} b_2^{-1} \ldots a_s b_s a_s^{-1} b_s^{-1})$ for S_s as a polygon of $4s$ sides, identified in pairs as indicated.

Jackson [J2] used characters of symmetric groups to calculate $e_{m-2k+1}(m)$, the number of **unlabelled** orientable 2-cell imbeddings of B_m on S_k, in terms of generating functions and recurrence relations. Gross, Robbins, and Tucker [GRT1] then showed:

Thm. 17-9. For the bouquet B_m,

$$g_k(B_m) = (m-1)! 2^{m-1} e_{m-2k+1}(m).$$

A closed form expression arises for the special case $m = 2k$, corresponding to $k = \gamma_M(B_{2k})$:

Cor. 17-10. For the even-size bouquet B_{2k},

$$g_k(B_{2k}) = \frac{(4k-1)!}{2k+1}.$$

For example, taking $k = 1$ we find $g_1(B_2) = 2$. Since $|R(B_2)| = 6$ and $\gamma_M(B_2) = 1$, it follows from Theorem 17-4 that $g_0(B_2) = 4$. The six labelled imbeddings are displayed in Figure 17-2.

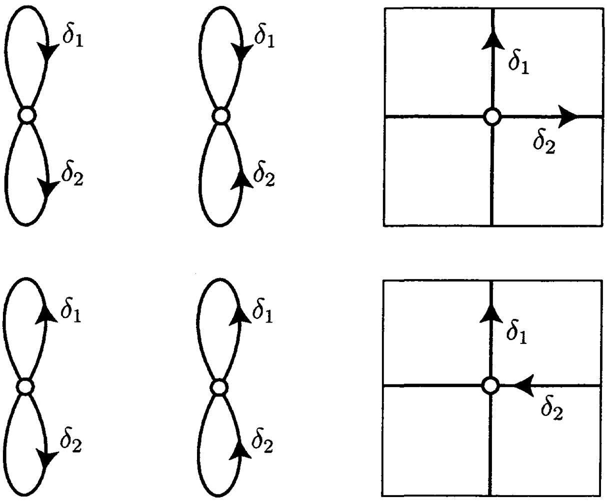

Figure 17-2.

In [R7], Rieper independently determined the genus distribution of bouquets **and** enumerated by region distribution as well. Thus (since $|R(B_m)| = (2m-1)!$ has already been calculated) all three questions of the introductory section for this chapter have been answered, for bouquets. Here is Rieper's solution. Let $S_{m,r}$ denote the number of labelled orientable 2-cell imbeddings of B_m having r regions.

Thm. 17-11. The number $S_{m,r}$ is given by the recurrence relation $(m+1)S_{m,r} = 4(2m-1)(2m-3)(m-1)^2\ (m-2)S_{m-2,r} + 4(2m-1)(m-1)S_{m-1,r-1}$, for $m > 2$, with initial conditions $S_{m,r} = 0$ if $m < 0$ or $r < 1$ or $r > m+1$; $S_{0,1} = S_{1,2} = 1$, $S_{0,r} = 0$ for $r \neq 1$, and $S_{1,r} = 0$ for $r \neq 2$; $S_{2,1} = 2$, $S_{2,2} = 0$, $S_{2,3} = 4$, and $S_{2,r} = 0$ for $r > 3$.

Table 17-3 contains several values for $S_{m,r}$.

$m\backslash r$	1	2	3	4	5	6	Total
1	0	1					1!
2	2	0	4				3!
3	0	80	0	40			5!
4	1008	0	3360	0	672		7!
5	0	185472	0	161280	0	16128	9!

Table 17-3

Note the consistency of the first column with Corollary 17-10. In Table 17-4 we record $g_k(B_m)$, using $k = \frac{(m+1-r)}{2}$.

Rieper also answered questions (2) and (3) for dipoles in [R7]. The *dipole* D_n has 2 vertices and n edges joining them. In Figure 10-9 D_n is used as an index-two voltage graph for $\mathbb{Z}_n$, leading to symmetrical maps for $K_{n,n}$ and $K_{n,n,n}$ (the latter are genus imbeddings) and three infinite families of partially balanced, group divisible, transversal, incomplete block designs–in what must be one of the most elegant constructions in topological graph theory; it is due to Stahl [SW2].

$m\backslash k$	0	1	2	3	Total
1	1				1!
2	4	2			3!
3	40	80			5!
4	672	3360	1008		7!
5	16128	161280	185472		9!
6	506880	8870400	24837120	5702400	11!

Table 17-4

It is easy to see that $|R(D_n)| = (n - 1!)^2$. It is not so easy to compute $g_k(D_n)$. We note that the *Stirling numbers $s(a,b)$ of the first kind* are given by: $s(a,b) = 0$ if either $b = 0$ or $b > a$; $s(1,1) = 1$; and $s(a+1,b) = s(a,b-1) - as(a,b)$. Rieper found:

Thm. 17-12. The genus distribution of D_n is given by:

$$g_k(D_n) = \frac{2(n-1)!}{n(n+1)}|s(n+1,\ n-2k)|,$$

where the $s(n+1, n-2k)$ are Stirling numbers of the first kind.

Cor. 17-13. The genus distribution of $K_{2,n}$ is given by:

$$g_k(K_{2,n}) = \frac{2(n-1)!}{n(n+1)}|s(n+1, n-2k)|.$$

PROOF. We can obtain the graph $K_{2,n}$ from the dipole D_n by performing an elementary subdivision on each edge. But vertices of degree 2 do not affect topological considerations. Thus $g_k(K_{2,n}) = g_k(D_n)$. □

Kim and Lee [KL1] generalized the developments above for B_m and D_n, by studying the bouquet of m n-dipoles, denoted by $B_{m,n}$; it is the multigraph n-fold $K_{1,m}$. Then B_m is homeomorphic to $B_{m,2}$, and $D_n = B_{1,n}$. Kim and Lee calculated all the values $g_k(B_{m,n})$, in terms of Stirling numbers of both the first and second kind. The following special cases give information about B_m and D_n in closed form:

Thm. 17-14. (i) $g_0(B_m) = \frac{2^m(2m-1)!}{(m+1)m!}$

(ii) $g_1(B_m) = \frac{2^{m-2}(2m-1)!}{3(m-2)!}$
(iii) $g_o(D_n) = (n-1)!$
(iv) $g_1(D_n) = \frac{(n+1)!(n-1)(n-2)}{24}$.

17-2. Counting Unlabelled Orientable 2-Cell Imbeddings

Let G be a connected graph, with $R(G)$ the set of labelled orientable 2-cell imbeddings of G, each given by its own rotation scheme. As in Section 16-1, we impose an equivalence relation on $R(G)$; the equivalence classes will be the unlabelled orientable 2-cell imbeddings of G. Recall that the relation is as follows. Rotation schemes ρ and σ in $R(G)$ are equivalent if there is an $a \in \mathrm{Aut}(G)$ such that $a(\rho) = \sigma$; that is, $\sigma_{a(v)} = a\rho_v a^{-1}$. This describes a re-labelling of ρ. We illustrate this by examining the example $G = K_4$ of Section 16-1 in more detail.

Consider $\rho = \{p_1 = (234), p_2 = (143), p_3 = (124), p_4 = (132)\}$, as in Figure 16-2. For any $a \in \mathrm{Aut}\, M = A_4$ (where $M = (K_4, \rho)$), $a(\rho) = \rho$. For example, if $a = (123)$, then $a(\rho) = \{p_2 = (314), p_3 = (241), p_1 = (234), p_4 = (213)\} = \rho$. However, if $a \in \mathrm{Aut}\, G - \mathrm{Aut}\, M = S_4 - A_4$, then $a(\rho) = \rho^{-1} \neq \rho$. For example, if $a = (12)$, then $a(\rho) = \{p_2 = (134), p_1 = (243), p_3 = (214), p_4 = (231)\}$, which we denote by ρ^{-1} since it describes the mirror image of Figure 16-2. The point is that ρ and ρ^{-1} are different, but in the same equivalence class; that is, they arise from the same unlabelled imbedding (Figure 16-2 without the labels).

In general, let $C(G)$ denote the set of equivalence classes–unlabelled orientable 2-cell imbeddings–for G. (See [MRW1], where the term used is "congruence class".) Then we rewrite Theorem 16-9 as:

Thm. 17-15. The number of unlabelled orientable 2-cell imbeddings of the connected graph G is:

$$|C(G)| = \frac{1}{|\mathrm{Aut}\ G|} \sum_{a \in \mathrm{Aut}\, G} |F(a)|, \text{ where}$$
$$F(a) = \{\rho \in R(G) | a(\rho) = \rho\}.$$

We verify the entry $|F((123))| = 4$ in Table 16-1, to reinforce these ideas. We want to count the $\rho \in R(G)$ fixed by $(123) \in S_4 = \mathrm{Aut}(K_4)$. There are two choices for p_1 ((234) and (243)), and each one determines p_2 and p_3 uniquely, under the action of (123). However, neither of the two possibilities for p_4 is affected by (123). Thus $|F((123))| = 2\times 2 = 4$.

The remaining entries of Table 16-1 are calculated similarly, and we easily compute $|C(K_4)| = 3$, using Theorem 17-15.

In [MRW1] Mull, Rieper, and White completely generalize the above procedure. We need some notation. For $a \in \operatorname{Aut} G$ and $v \in V(G)$, let $F_v(a) = \{p_v | a p_v a^{-1} = p_v\}$. Let $\ell(v)$ be the cardinality of the orbit of v under the action of $< a >$ on $V(G)$.

Thm. 17-16. For

$$a \in \operatorname{Aut} G, |F(a)| = \prod_{v \in S} |F_v(a^{\ell(v)})|$$

, where the product is taken over a complete set S of orbit representatives for $< a >$ acting on $V(G)$.

Now for a permutation b of an n-set, write $j(b)$ for the n-tuple whose kth entry is the number of k-cycles in the disjoint cycle representation of b; write j_k for the kth entry. As usual, ϕ is the euler phi function, and $N(v)$ is the set of neighbors of vertex v.

Thm. 17-17. For $a \in \operatorname{Aut} G$ and $v \in V(G)$, with $n = |N(v)|$,

$$|F_v(a^{\ell(v)})| = \begin{cases} \phi(d)\frac{n}{(d-1)!}d^{\frac{n}{d}-1}, \text{ if} \\ \qquad j(a^{\ell(v)}|_{N(v)}) = (0, \cdots, 0, j_d = \frac{n}{d}, 0 \cdots 0) \\ 0, \text{ otherwise.} \end{cases}$$

Revisiting the example $a = (123)(4)$ for $G = K_4$, choose $S = \{1, 4\}$ and set e as the identity of S_4. Then $a^{\ell(1)}|_{N(1)} = e$ has cycle type $(j_1 = \frac{3}{1}, 0, 0)$ and, by Theorem 17-17, $|F_1(a^3)| = \phi(1)(\frac{3}{1} - 1)!1^{\frac{3}{1}-1} = 2$; whereas $a^{\ell(4)}|_{N(4)} = (123)$ has cycle type $(0, 0, j_3 = \frac{3}{3})$ and, by Theorem 17-17 again, $|F_4(a)| = \phi(3)(\frac{3}{3} - 1)!3^{\frac{3}{3}-1} = 2$. Thus by, Theorem 17-16, $|F(a)| = |F_1(a^3)|\ |F_4(a)| = 2 \times 2 = 4$. This formalizes the calculation of $|F(123)|$ given earlier.

Thus in theory, Problem (1) for the unlabelled case is solved in general: Theorem 17-17 enables Theorem 17-16, which in turn enables Theorem 17-15. So far, we have illustrated this process only for $G = K_4$. Two natural generalizations occur: the wheel graphs $W_n = C_{n-1} + K_1$ (the join operation) and the complete graphs K_n. In [MRW1] closed-form expressions for $|C(G)|$ are established for both generalizations. Here we give just the result for complete graphs.

Thm. 17-18. The number of unlabelled orientable 2-cell imbeddings of K_n is:

$$|C(K_n)| = \sum_{d|n} \frac{(n+2)!^{\frac{n}{d}}}{d^{\frac{n}{d}}(\frac{n}{d})!} + \sum_{d|(n-1),\ d\neq 1} \frac{\phi(d)(n-2)!^{\frac{n-1}{d}}}{n-1}.$$

This produces the numbers in the bottom row of Table 16-2. The corresponding numbers in Table 17-1 summing to $78 = |C(K_5)|$ were calculated by Mull, using *ad hoc* methods.

Routine calculations involving Theorem 17-18 produce:

Cor. 17-19. Asymptotically, $|C(K_n)| = \frac{|R(K_n)|}{|\mathrm{Aut}(K_n)|}$; that is

$$\lim_{n\to\infty} \frac{|C(K_n)|}{\frac{((n-2)!)^n}{n!}} = 1.$$

Mull [M7] has applied Theorems 17-16, 17-17, and 17-18 to count $|C(G)|$, first for $G = K_{p_1,p_2}$, and then for complete n-partite graphs in general.

17-3. The Average Number of Symmetries

In this section we find an elementary connection among $\mathrm{Aut}\, G, R(G)$, and $C(G)$, where G is a connected graph. We start with a property of equivalent rotations in $R(G)$.

Thm. 17-20. For G connected and $\rho, \sigma \in R(G)$, if ρ and σ are equivalent, then $\mathrm{Aut}(G,\rho)$ and $\mathrm{Aut}(G,\sigma)$ are conjugate subgroups of $\mathrm{Aut}\, G$.

PROOF. Since ρ and σ are equivalent, we find $a \in \mathrm{Aut}\ G$ so that $a(\rho) = \sigma$; that is

(1) $\sigma_{a(u)} = a\rho_u a^{-1}$, for all $u \in V(G)$.

Now, let $b \in \mathrm{Aut}(G,\rho)$, so that $b(\rho) = \rho$; that is

(2) $\rho_{b(w)} = b\rho_w b^{-1}$, for all $w \in V(G)$.

We **claim** that $aba^{-1} \in \mathrm{Aut}(G,\sigma)$, so that $a\,\mathrm{Aut}(G,\rho)a^{-1} \subseteq \mathrm{Aut}(G,\sigma)$. Similarly, $a^{-1}\,\mathrm{Aut}(G,\sigma)a \subseteq \mathrm{Aut}(G,\rho)$, so that $\mathrm{Aut}(G,\sigma) \subseteq a\,\mathrm{Aut}(G,\rho)a^{-1}$, and $a\,\mathrm{Aut}(G,\rho)a^{-1} = \mathrm{Aut}(G,\sigma)$.

To verify our claim, we must show that $(aba^{-1})(\sigma) = \sigma$, that is: $\sigma_{(aba^{-1})(v)} = aba^{-1}\sigma_v ab^{-1}a^{-1}$, for all $v \in V(G)$. But

$$\begin{aligned}
\sigma_{(aba^{-1})(v)} &= \sigma_{a(ba^{-1}(v))} \\
&= a\rho_{ba^{-1}(v)}a^{-1} \quad ((1), \text{ with } u = ba^{-1}(v)) \\
&= a(b\rho_{a^{-1}(v)}b^{-1})a^{-1} \quad ((2), \text{ with } \ w = a^{-1}(v)) \\
&= ab(a^{-1}\sigma_v a)b^{-1}a^{-1} \quad ((1), \text{ with } \ u = a^{-1}(v)) \\
&= aba^{-1}\sigma_v ab^{-1}a^{-1}.
\end{aligned}$$

□

Since conjugate subgroups are isomorphic, we have the expected result that an unlabelled graph imbedding has a unique symmetry structure. However, note that we cannot claim $\mathrm{Aut}(G, \rho) = \mathrm{Aut}(G, \sigma)$; see Problem 17-2. We can, however, deduce:

Cor. 17-21. For G connected and ρ, σ equivalent in $R(G)$,

$$|\mathrm{Aut}(G, \rho)| = |\mathrm{Aut}(G, \sigma)|.$$

It is instructive to consider the imbedding of K_4 on the torus having $r_3 = r_9 = 1$ once again. The eight labelled imbeddings corresponding to this one unlabelled imbedding arise from the four ways to choose the 3-cycle bounding the small region and then the two ways to orient this 3-cycle. Each of the four pairs of mirror-image imbeddings thus produced has automorphism group generated by a rotation of the 3-cycle. These eight non-identity rotations correspond to the eight elements of order 3 in $S_4 = \mathrm{Aut}(K_4)$, paired by inverse. The four subgroups thus determined are all conjugate in S_4.

Now we present the connection we seek, due to Mull [M6].

Thm. 17-22. For G a connected graph, the average number of map automorphisms, taken over $R(G)$, is $\frac{|\mathrm{Aut}\, G||C(G)|}{|R(G)|}$.

PROOF. Let A denote the average number of map automorphisms for the connected graph G, taken over $R(G)$. Our calculation uses Theorem 16-8 and Corollary 17-21. Let S be a collection of equivalence class representatives, consisting of exactly one rotation scheme for each class in $C(G)$. Then

$$
\begin{aligned}
A &= \frac{\sum_{\rho \in R(G)} |\operatorname{Aut}(G,\rho)|}{|R(G)|} \\
&= \sum_{\rho \in S} \frac{|\operatorname{Aut}(G,\rho)|}{|R(G)|} \frac{|\operatorname{Aut} G|}{|\operatorname{Aut}(G,\rho)|} \\
&= \sum_{\rho \in S} \frac{|\operatorname{Aut} G|}{|R(G)|} \\
&= \frac{|\operatorname{Aut} G|}{|R(G)|}|S| \\
&= \frac{|\operatorname{Aut} G|}{|R(G)|}|C(G)|.
\end{aligned}
$$

□

Cor. 17-23. Let $Q \in C(G)$ be any one equivalence class of rotation schemes for G. Then $\sum_{\rho \in Q} |\operatorname{Aut}(G,\rho)| = |\operatorname{Aut} G|$.

These ideas can be readily checked for our continuing example $G = K_4$. We omit the details.

Cor. 17-24. Almost no labelled orientable 2-cell imbeddings of K_n have non-trivial map automorphisms.

PROOF. Apply Corollary 17-19 in conjunction with Theorem 17-22.

□

17-4. Problems

17-1.) Show that K_4 has no orientable 2-cell imbedding with $r_6 = 2$.

17-2.) Find an example of ρ and σ equivalent in $R(G)$, but $\operatorname{Aut}(G,\rho) \neq \operatorname{Aut}(G,\sigma)$. (Hint: consider K_4 on S_1, with $r_3 = r_9 = 1$). Then find $a \in \operatorname{Aut} G$ so that $a \operatorname{Aut}(G,\rho) a^{-1} = \operatorname{Aut}(G,\sigma)$.

17-3.) Show that if $|\operatorname{Aut} G| = 1$, then $|C(G)| = |R(G)|$. (Thus the numbers of labelled and unlabelled orientable 2-cell imbeddings agree, in this situation.) Give an example to show that the converse is false.

17-4.) Let $G = K_{2,3}$. Find $\operatorname{Aut} G, R(G), C(G)$, and the average number of symmetries. Partition both $R(G)$ and $C(G)$ by region distribution.

17-5.) Repeat Problem 17-4, for $G = K_{3,3}$. How could you cut down your work by a factor of 4? Does $K_{3,3}$ have a compatible region distribution which is not realizable?

17-6.) Repeat Problem 17-5, for $G = K_2 \times K_3$.

17-7.) Use Corollary 17-13 to show that $g_0(K_{2,n}) = (n-1)!$ How many unlabelled imbeddings does this represent?

17-8.) Now verify the special case $g_0(K_{2,n}) = (n-1)!$ of Corollary 17-13 independently, by a direct counting argument.

17-9.) Show that $\gamma_M(K_{2,2k+1}) = k$, and that $g_k(K_{2,2k+1}) = \frac{((2k)!)^2}{k+1}$.

17-10.) Check your work of Problem 17-4 against Corollary 17-13, as far as possible.

17-11.) Find the genus distribution for $K_{2,5}$ and the region distributions for S_0 and S_2. Make a reasonable definition of the "average genus of a labelled graph", and then calculate the average genus for $K_{2,5}$.

17-12.) **Find the average genus for $K_{2,n}$.

17-13.) Use the recurrence relation of Theorem 17-11 and mathematical induction to show that $S_{m,1} = \frac{(2m-1)!}{m+1}$ if m is even, but 0 if m is odd.

17-14.) Use Problem 17-13 to prove Corollary 17-10.

17-15.) Prove that if maps (G, ρ) and (G, σ) are equivalent, then their region distributions (and hence their genera) agree. To show that the converse is false, consider K_5 on S_1, with $r = r_4 = 5$, as described by Example 1a) in Section 10-3. First, demonstrate that this map M is not reflexible. Then, deduce from Problem 14-5 that M and its mirror image M^{-1} (which also has $r = r_4 = 5$) are not equivalent.

17-16.) Following up on Problem 17-15 regarding maps M and M^{-1}, note from Table 17-1 that there are exactly 12 rotation schemes for K_5 producing $r = r_4 = 5$.

(i) Apply Theorem 16-27 to show that M is symmetrical.

(ii) Deduce that $|\operatorname{Aut} M| = 20$.

(iii) Now apply Theorem 16-8 to show that precisely 6 of our 12 rotation schemes produce maps equivalent to M.

(iv) Show that the mirror image M^{-1} of M also covers Figure 10-4, but with both "b" arrows reversed.

(v) Repeat (i), (ii), and (iii) for M^{-1}, thereby showing that the 12 rotation schemes split into two equivalence classes (represented by M and M^{-1}).

17-17.) Gross and Tucker [GT4] show that there are exactly $2 \cdot 5!$ rotation schemes in $R(K_7)$ yielding $r = r_3 = 14$ on S_1. (Thus $g_1(K_7) = 240$.) Show that these split into two equivalence classes of 120 rotation schemes each, with each class containing the mirror images of the members of the other class.

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CHAPTER 18

RANDOM TOPOLOGICAL GRAPH THEORY

In one model for random graph theory, we start with $n \in N$ and $p \in R$, $0 \leq p \leq 1$. The sample space Ω consists of all labelled graphs of order n. If $G \in \Omega$ has m edges, then the probability of G is given by:

$$P(G) = p^m (1-p)^{\binom{n}{2}-m}.$$

Intuitively, we think of p as the constant probability, for each potential edge of G, that that edge actually exists in G, and we assume that the $\binom{n}{2}$ corresponding events are independent. The *uniform* case occurs for $p = \frac{1}{2}$, where each G in Ω has $P(G) = \frac{1}{2}^{\binom{n}{2}} = \frac{1}{|\Omega|}$. If A is the set of all labelled graphs of order n satisfying some property Q, where $\lim_{n\to\infty} P(A) = 1$, we say that *almost all graphs have property Q.* If the limit is 0, we say that *almost no graphs have property Q.* Now fix both p and $k (k \in N)$. The following results appear in [P3], except that (vii) appears in [AG3].

Thm. 18-1. Almost all graphs:

(i) are hamiltonian;
(ii) have diameter 2;
(iii) are k-connected;
(iv) are locally connected;
(v) contain a given subgraph of order k as an induced subgraph;
(vi) are nonplanar.
(vii) have genus in $\left[\frac{(1-\varepsilon)pn^2}{12}, \frac{(1+\varepsilon)pn^2}{12}\right]$, where $\varepsilon > 0$ and $0 < p = p(n) < 1$ with $p^2(1-p) \geq \frac{8(\ln n)^4}{n}$.

We note that (vi) follows from (v) by taking, for instance, $k = 5$ and $G = K_5$ as the given subgraph. Here is a generalization of (vi), and of the proof just given.

Cor. 18-2. For each $g \in N$, almost all graphs have genus at least g.

PROOF. In (v) take $k = 5g$ and $G = gK_5$. $\square$

In this chapter, motivated by Theorem 18-1 (vii) and Corollary 18-2, we introduce a variety of models for random topological graph theory, posing–and attempting to answer–natural questions which arise within these models. (See [W24], [LW1] and [S5].) All of our models will be for labelled graphs, but corresponding models for unlabelled graphs could be constructed in some of the cases. (See, for example, Problem 18-3.)

Model I imposes the uniform probability distribution on $R(G)$, where G is a labelled connected graph. (The corresponding model for unlabelled graphs would be on $C(G)$, with either the uniform distribution or a probability distribution weighting each equivalence class by the number of rotations it contains. But the latter would essentially return us to the uniform distribution for the labelled case.) Model II imposes a variable distribution on $R(G)$, where G is a labelled connected cubic graph. (There is no analogous unlabelled model for Model II.) Models III and IV extend from rotation schemes to imbedding schemes (allowing for the possibility of nonorientable imbeddings), with uniform and variable distributions respectively. Model V combines features of Model II and IV. Finally, we study Model VI, for random Cayley maps.

18-1. Model I

In this model, for each labelled connected graph G of degree sequence $\{d_1, d_2, \ldots, d_n\}$, the sample space Ω consists of all $|R(G)|$ orientable 2-cell imbeddings (G,ρ) of G, with the uniform distribution

$$P(G,\rho) = \left(\prod_{i=1}^{n}(d_i - 1)!\right)^{-1}.$$

The two natural random variables of interest are as follows.

(1) The *genus random variable* g, $g : \Omega \to N \cup \{0\}$, gives the genus of an arbitrary sample point; that is, if (G,ρ) imbeds G on S_k, then $g(G,\rho) = k$.
(2) The *symmetry random variable* M, $M : \Omega \to N$, gives the number of map automorphisms: $M(G,\rho) = |\operatorname{Aut}(G,\rho)|$.

We are especially interested in the expected values (but also in the variances) of these random variables. Since our probability distribution is uniform, an expected value is just an average. Thus $E(g) = \frac{1}{|\Omega|}\sum_{\rho \in R(G)} g(G,\rho)$ has been called the *average genus* (in [A10] and [S19], for example). As $E(M) = \frac{|\operatorname{Aut} G||C(G)|}{|R(G)|}$ has been calculated in Theorem 17-22, we concentrate here on $E(g)$.

Thm. 18-3. For G connected, $E(g) = \frac{1}{|\Omega(G)|}\sum_{k\geq 0} kg_k(G)$, where $g_k(G)$ is the number of $\rho \in R(G)$ having $g(G,\rho) = k$.

Since $|R(G)|$ is known, the evaluation of $E(g)$ depends only on work from enumerative topological graph theory to calculate $g_k(G)$, as given in Chapter 17. In Table 18-1, we summarize information about K_4, K_5 and K_6. The entry for (n, k) is $P(g(K_n, \rho)) = k$. The final column is obtained from Theorem 17-22, using Theorem 17-18.

$n\backslash k$	0	1	2	3	4	5	$E(g)$	$V(g)$	$E(M)$
4	.125	.875					.875	.109	4.500
5		.059	.640	.301			2.242	.301	1.204
6		10^{-5}	.003	.129	.650	.218	4.082	.360	1.001

Table 18-1

These and other data provided empirical evidence leading to the next theorem, due to Lee [L1] and depending upon a result of Stahl [S20].

Thm. 18-4. If G is a connected graph of order n and size asymptotic to $cn^{1+\varepsilon}$, where c and ε are positive constants, then the average genus $E(g(G_n))$ is asymptotic to the maximum genus $\gamma_M(G_n)$.

Thus, for many families of graphs, the genus distribution is skewed considerably to the right, as we begin to see in Table 18-1. (See also Problem 18-1.)

Cor. 18-5. For complete graphs,

$$E(g(K_n)) \approx \gamma_M(K_n) = \left\lfloor \frac{(n-1)(n-2)}{4} \right\rfloor .$$

PROOF. Let $c = \frac{1}{2}$ and $\varepsilon = 1$ in Theorem 18-4, and refer to Theorem 6-25. □

Further calculations, using results of Section 17-1 (see [W24] and [T3]), establish:

Thm. 18-6. For the cobblestone path J_n,

$$E(g(J_n)) = \frac{4n + 1 + \frac{(-1)^{n+1}}{3^n}}{16}.$$

Thm. 18-7. For the closed-end ladder L_n,

$$E(g(L_n)) = \frac{3n + 1 + \frac{(-1)^{n+1}}{2^n}}{9}.$$

Thm. 18-8. For the Ringel ladder R_n,

$$\begin{aligned} E(g(R_n)) = &\ \frac{1}{2^{2n+2}} \left([\frac{2^{n+2}}{9}(2^n(3n+10) + (-1)^{n+1}) \right)] \\ - &\ \frac{1}{2^{n+2}} \left([\frac{1}{2^{n-1}} 8h_n(33) \ + \ h_{n+1}(33)) \right)], \end{aligned}$$

where $h_n(x) = \sum_{k \geq 0} \binom{n}{2k+1} x^k$.

Thm. 18-9. The probability that a random imbedding of R_{2s}, edge-labelled so as to be a KCL current graph K for K_{12s+7}, will be a suitable quotient graph for K_{12s+7} is $P(g(R_{2s}, \rho) = s+1) = \frac{s+2}{2^{s+2}}$.

As this probability goes to 0 with increasing s, we see that such imbeddings of R_{2s} are rare: almost no imbeddings are suitable.

Thm. 18-10. For the bouquet B_m,

$$E(g(B_m)) = \frac{(m-1)!2^{m-1}}{(2m-1)!} \sum_{k=0}^{\lfloor \binom{m}{2} \rfloor} k \ e_{m-2k+1}(m),$$

where $e_{m-2k+1}\ (m)$ is the number of congruence classes of B_m on S_k.

Here is a generalization of Figure 17-2:

Thm. 18-11. For even-size bouquets,

$$P(g(B_{2k}, \rho) = k) = \frac{1}{2k+1}.$$

PROOF. Use Corollary 17-10 and $|\Omega| = (4k-1)!$ ☐

Thm. 18-12. For G_n either the dipole D_n or $K_{2,n}$,

$$E(g(G_n)) = \frac{2}{(n+1)!} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} k|s(n+1, n-2k)|,$$

where the $s(n+1, n-2k)$ are stirling numbers of the first kind.

PROOF. Refer to Theorem 17-12 and Corollary 17-13. ☐

18-2. Model II

We seek a variable-probability distribution, akin to that given for random graph theory at the beginning of this chapter–where a binary choice is made for each pair of distinct vertices (either join or do not join by an edge). Since we are fixing a labelled connected graph G, our edges are determined. What we can vary are the vertex rotations, to produce an element in $R(G)$. Since we want a binary choice, we restrict to *cubic* graphs; then each vertex has exactly two possible rotations. The trick is: how to determine which is "clockwise" (so that the other will be "counterclockwise"). We offer three possibilities:

Model II A. For each labelled connected cubic graph G of order n, with $V(G) = \{1, 2, \dots, n\}$ and $0 \leq p \leq 1$, the sample space Ω consists of all $|R(G)|$ orientable 2-cell imbeddings (G, ρ) of G. For $i \in V(G)$, let $N(i) = \{j, k, l\}$, with $j < k < \ell$. If $p_i = (j, k, l)$, we say that rotation p_i is *clockwise*. On the other hand, if $p_i = (j, l, k)$, we say that p_i is *counterclockwise*. Then $\rho = \{p_1, p_2, \dots, p_n\}$. Let ρ have exactly c clockwise rotations $p_i, 0 \leq c \leq n$. Then we define

$$P(G, \rho) = p^c(1-p)^{n-c}.$$

This does give a probability distribution, as:

(i) $0 \leq p^c(1-p)^{n-c} \leq 1$, for $0 \leq c \leq n$.
(ii) $\sum_{c=0}^{n} \binom{n}{c} p^c(1-p)^{n-c} = (p + (1-p))^n = 1$.

We also check that $\sum_{c=0}^{n} \binom{n}{c} = (1+1)^n = 2^n = |\Omega|$.

The distribution of Model IIA is uniform if and only if $p = \frac{1}{2}$, in which case we revert to Model I. (For two methods for calculating the genus distribution and average genus, and some computer results for several cubic graphs of order 96, in the uniform case of Model II A, see Archdeacon [A10].) The number of clockwise rotations has the binomial distribution; but our interest is in $P(G, \rho)$, rather than in $P(c) = \binom{n}{c} P(G, \rho)$.

For example, if $G = K_4$, then in Ω let E_1 be the event that (K_4, ρ) has $r_3 = 4$, let E_2 be the event that (K_4, ρ) has $r_3 = r_9 = 1$, and let E_3 be the event that (K_4, ρ) has $r_4 = r_8 = 1$. Then routine calculations produce the following probability polynomials, as studied by Tesar [T2, T4]:

$$
\begin{aligned}
&P(E_1) = 2p^2(1-p)^2;\\
&P(E_2) = 4p(1-p)^3 \;+\; 4p^3(1-p);\\
&P(E_3) = p^4 \;+\; (1-p)^4 \;+\; 4p^2(1-p)^2;\\
&P(g(K_4,\rho)=0) = P(E_1) = 2p^2(1-p)^2;\\
&P(g(K_4,\rho)=1) = P(E_2 \cup E_3) = 1 - 2p^2(1-p)^2;\\
&E(g(K_4)) = 1 - 2p^2(1-p)^2;\\
&V(g(K_4)) = 2p^2(1-p)^2(1-2p^2(1-p)^2);\\
&E(M(K_4)) = 4(1 - p(p-1)(6p^2-6p+1)).
\end{aligned}
$$

The calculations above are independent of the vertex labelling selected for K_4 from $\{1,2,3,4\}$, as similar calculations are for $K_{3,3}$ from $\{1,2,3,4,5,6\}$. (See Problem 18-5.) But this is not true in general for Model II A, as $G = K_2 \times K_3$ shows. (See Problem 18-6.) This is a shortcoming of the model. Thus we turn to:

Model II B. Let Γ be a group of order $2m$, generated by $\Delta^* = \{\delta_1, \delta_2, \delta_3\}$, where δ_1 is an involution. (It could be that δ_2 and δ_3 are also involutions. If not, then $\delta_3 = \delta_2^{-1}$.) Let $0 \le p \le 1$. The sample space Ω consists of all $|R(G)|$ labelled orientable 2-cell imbeddings (G,ρ) of $G = G_\Delta(\Gamma)$, with vertex labels taken from Γ. If $p_g = (g+\delta_1, g+\delta_2, g+\delta_3)$, then we say that p_g is *clockwise*; otherwise p_g is *counterclockwise*. Let $\rho = \{p_g\}_{g\in\Gamma}$ have exactly c clockwise rotations $p_g, 0 \le c \le 2m$. Then we set

$$P(G,\rho) = p^c(1-p)^{2m-c}.$$

As before, we get a probability distribution. But now there is no ambiguity as to how the vertex labels are assigned. Nor does changing the order in which the elements of Δ^* are given affect the probability calculations we wish to make, as the only change might be a reversal of the roles of "clockwise" and "counterclockwise". (However, changing Γ and/or Δ could have an effect.)

If we take $\Delta^* = \{2,3,1\}$ for $\Gamma = \mathbb{Z}_4$, then $G_\Delta(\Gamma) = K_4$ and we regain the calculations of Model II A for K_4 (writing "4" for "0").

In Model II B, the event $E : \rho$ has $c = 0$ or $2m$ has interest, since then (G,ρ) is a Cayley map. It can be viewed as a $2m$-fold branched covering projection over the bouquet B_2 in the sphere (or B_3 if δ_2 and δ_3 are involutions also.)

For connected cubic graphs which are not Cayley graphs, there is still an alternative to Model II A.

Model II C. In this model we start with a fixed labelled orientable 2-cell imbedding of a connected cubic graph G, and the clockwise vertex rotations are determined by this imbedding. Otherwise, this model agrees with the two previous variants.

For example, if G is the ladder L_n, it is natural to take as the fixed imbedding the planar one derived from Figure 9-8 by deleting edge A and suppressing the two vertices of degree 2 that result. Tesar [T5] found the following:

Thm. 18-13. In Model IIC, $P(g(L_n, g) = k)$ is given by:

$$f_k(L_n, p) = \binom{n+1-k}{k}[2p(1-p)]^k[(1-p)^2+p^2]^{n-2k}[1-\frac{2p(1-p)k}{n-k+1}].$$

Of course, letting $p = \frac{1}{2}$ and multiplying by $|\Omega| = 2^{2n}$ regains Theorem 17-6.

18-3. Model III

Now we expand to allow both orientable and nonorientable labelled 2-cell imbeddings of a connected graph G; we assume that G has minimum degree at least 3. As in Chapter 11, we augment a rotation scheme (G, ρ) to an imbedding scheme (G, ρ, λ), where $\lambda : E(G) \to \mathbb{Z}_2$. From Theorem 11-16, we see that the imbedding is orientable if and only if every cycle of G has an even number of edges e for which $\lambda(e) = 1$. (We say that such a cycle C is *λ-trivial*, since if $C = \sum_{i=1}^{k} e_i$ as an edge sum, then $\lambda(C) = \lambda(\sum_{i=1}^{k} e_i) = \sum_{i=1}^{k} \lambda(e_i) = 0$ in $\mathbb{Z}_2$.)

So, for each labelled connected graph G of degree sequence $\{d_1, d_2, \ldots, d_n\}$, the sample space Ω consists of **all** 2-cell imbeddings (G, ρ, λ), with the uniform distribution

$$P(G, \rho, \lambda) = \frac{1}{2^m \prod_{i=1}^{n}(d_i - 1)!},$$

where $m = \frac{1}{2}\sum_{i=1}^{n} d_i$ is the size of G.

An event of interest here is $E : (G, \rho, \lambda)$ is orientable. Note that E depends only upon λ (for fixed G); it is independent of ρ. Each assignment λ will have $\prod_{i=1}^{n}(d_i - 1)!$ imbeddings of G associated with it, all of the same orientability nature. Thus the uniform probability of (G, λ) is $\frac{1}{2^m}$. The key idea for the following development is due to Schwenk [S6]. As usual, let $\beta(G) = m - n + 1$ denote the Betti number, or cycle independence number, of G. It is the number of edges in a cycle basis for G, and is the size of the complement of every spanning tree for G.

Lemma 18-14. Every cycle in G is λ-trivial if and only if every cycle in a cycle basis for G is λ-trivial.

PROOF. The necessity is immediate. For the sufficiency, let $\{C_i\}_1^k$ be a cycle basis for G, where $k = \beta(G)$. Let $\varepsilon_i \in \mathbb{Z}_2$, $1 \le i \le k$, and let $C = \sum_{i=1}^k \varepsilon_i C_i$ be an arbitrary cycle in G. Then $\lambda(C) = \sum_{i=1}^k \varepsilon_i \lambda(C_i) = \sum_{i=1}^k \varepsilon_i 0 = 0$. □

Thm. 18-15. If λ is uniformly distributed on $E(G)$ and if E is the event that (G, ρ, λ) will be orientable, then $P(E) = \frac{1}{(2)^{\beta(G)}}$.

PROOF. Let T be a spanning tree for G, with $\lambda : E(T) \to \mathbb{Z}_2$ given arbitrarily. Add each of the $\beta(G)$ remaining edges one at a time. Each such edge e completes a unique cycle C in a fixed cycle basis for G ($E(C) - e \subseteq E(T)$) and we assign $\lambda(e)$, with probability $\frac{1}{2}$, so that $\lambda(C) = 0$. Then every cycle in this cycle basis is λ-trivial, with probability $\frac{1}{(2)^{\beta(G)}}$, as these events are independent. Now apply Lemma 18-14. □

Cor. 18-16. If G_n is a family of connected graphs of order n such that $\lim_{n\to\infty} \beta(G_n) = \infty$, then (G_n, ρ, λ) is almost never orientable.

Thus, in Model III, K_n is almost never orientably imbedded. Nor are $K_{n,n}$, $K_{n,n,n}$, or Q_n, for example. In fact:

Cor. 18-17. If G_n is a family of connected graphs of order n and minimum degree at least 3, then (G_n, ρ, λ) is almost never orientable.

This includes all Cayley graphs, except $G_\Delta(\Gamma) = C_n$ for $\Gamma = \mathbb{Z}_n$ and $|\Delta| = 1$, and $G_\Delta(\Gamma) = K_2 \times C_n$ for $\Gamma = D_n$ and $|\Delta^*| = 2$. However, in practice, as graph imbedders use nonorientable imbeddings to construct covering imbeddings of interest, $\lambda(e) = 1$ with probability considerably less than $\frac{1}{2}$. Thus we turn to Model IV.

18-4. Model IV

For each labelled connected graph G and rotation scheme ρ for G, and for $0 \le p \le 1$, the sample space Ω consists of all imbedding schemes (G, ρ, λ). Let $k = |\{e \in E(G) | \lambda(e) = 1\}|$. Then

$$P(G, \rho, \lambda) = p^k (1-p)^{m-k},$$

where $m = |E(G)|$.

We emphasize that both G and ρ are fixed, and that $P(G, \rho, \lambda)$ varies only with λ. We readily check (see Problem 18-11) that we have a probability distribution, with $|\Omega| = 2^m$. The distribution is uniform precisely when $p = \frac{1}{2}$. Then, for questions of orientability, we are effectively back in Model III. We think of p as giving the probability, for each $e \in E(G)$, that $\lambda(e) = 1$. The distribution of k, the number of positive edges, is binomial, but our concern is in the event E that (G, ρ, λ) is orientable. We make three initial observations.

(1) If $p = 0$, then $k = 0$ and $P(E) = 1$.
(2) If $p = 1$, then $k = m$ and $P(E) = 1$ or 0 according to whether G is bipartite or not, respectively.
(3) If $p = \frac{1}{2}$, then $P(E) = \frac{1}{(2)^{\beta(G)}}$ as in Model III. It then follows (see Problem 18-10) that $|E| = 2^{n-1}$. More generally, a slight modification of the proof of Theorem 18-15 gives:

Thm. 18-18. If G is a connected graph, with $0 < p < 1$ in Model IV, then $P((G, \rho, \lambda)$ is orientable$) \leq [\max\{p, 1-p\}]^{\beta(G)}$.

Cor. 18-19. For a family of connected graphs G_n of order n with $\lim_{n\to\infty} \beta(G_n) = \infty$, then (G_n, ρ, λ) is almost never orientable, in Model IV with $0 < p < 1$. Moreover, if G_n is not bipartite, the result holds for $p = 1$ also.

It is also possible to show [W24]:

Thm. 18-20. For $0 < p < 1, q = p - 1$, and $n \geq 4$,

$$P((K_n, \rho, \lambda) \text{ is orientable}) = q^{\binom{n}{2}} \sum_{s=0}^{n-1} \binom{n-1}{s} \left(\frac{p}{q}\right)^{s(n-s)}.$$

18-5. Model V

Here we combine the features of Models II and IV.

Fix a labelled connected cubic graph G of order n and size m. Fix p_1 and p_2, both in $[0, 1]$. The sample space Ω consists of all imbedding schemes (G, ρ, λ). Let ρ have exactly c clockwise rotations, $0 \leq c \leq n$; and let $k = |\{e \in E(G) | \lambda(e) = 1\}|$, $0 \leq k \leq m$. Then $|\Omega| = 2^{n+m}$ and

$$P(G, \rho, \lambda) = p_1^c(1-p_1)^{n-c}p_2^k(1-p_2)^{m-k}.$$

Note that we have not specified the meaning of "clockwise". As in Model II, several options are available.

Here are three natural questions for this model.

(a) What is the probability that (G, ρ, λ) is orientable? The answer is the same as for Model IV, as orientability depends on λ, but not on ρ.
(b) What is the conditional expectation of the genus (or symmetry) random variable, given that (G, ρ, λ) is orientable? The answer could differ from that of Model II, as the use of ρ in evaluating g is affected by λ.
(c) What is the expected value of the euler characteristic, with no assumption of orientability?

18-6. Model VI: Random Cayley Maps

Recall that a Cayley map (Γ, Δ, π) is the map (G, ρ), where $G = G_\Delta(\Gamma)$ and $\pi : \Delta^* \to \Delta^*$, where $\Delta^* = \Delta \cup \Delta^{-1}$, is a cyclic permutation so that, for $g \in \Gamma$ and $h \in N(g)$, $\rho_g(h) = g\pi(g^{-1}h)$; thus $\rho = \{\rho_g | g \in \Gamma\}$.

In Model VI, our sample space Ω consists of all Cayley maps for a fixed finite group Γ and generating set Δ for Γ. The uniform probability distribution is given by

$$P(\Gamma, \Delta, \Omega) = \frac{1}{(|\Delta^*| - 1)!}.$$

Our interest here will be in the expected value of the genus random variable $g : \Omega \to N \cup \{0\}$, $g(\Gamma, \Delta, \pi) = k$ if $g(G, \rho) = k$:

$$E(g) = \frac{1}{(|\Delta^*| - 1)!} \sum_{\pi} g(\Gamma, \Delta, \pi),$$

also called the *average Cayley genus* by Schultz [S5] (see also [SW1]), where the parameters *Cayley genus* and *maximum Cayley genus* are studied also. (See Problems 18-16 and 18-17 respectively.)

We consider the situation where $\Gamma = (\mathbb{Z}_n)^m$, the repeated direct product of m factors, each isomorphic to $\mathbb{Z}_n$, where $n \geq 3$ and $m \geq 1$; we take Δ as the standard basis for Γ. Then $G_\Delta(\Gamma) = (C_n)^m$, the corresponding repeated cartesian product of cycles, by Theorem 4-22. (Note that, for $n = p$ (a prime), this is precisely the model we took for the additive structure of $GF(p^m)$ in Chapter 14.) Now, for any cyclic permutation $\pi : \Delta^* \to \Delta^*$, the Cayley map (Γ, Δ, π) is a covering space for an index-one voltage graph imbedding. To study the former, we first study the latter.

Lemma 18-21. Let R be a region for an index-one voltage graph imbedding covered by the Cayley map (Γ, Δ, π) for $\Gamma = (\mathbb{Z}_n)^m, n \geq 3$, with Δ the standard basis. Then R satisfies the Kirchoff Voltage Law (KVL) if and only if R is the unique region below.

PROOF. (i) If R is the unique region, then each element of Δ^* appears exactly once in the boundary of R. Since Γ is abelian and Δ contains no involutions, the KVL is satisfied.

(ii) Now assume that R satisfies the KVL, but that there is another region R' below; thus R does not contain all of Δ^* in its boundary. But since Γ is abelian, and Δ consists of independent generators, for each δ in the boundary of R, δ^{-1} is in the boundary too. Thus, by identifying each pair δ, δ^{-1} in the boundary of R in the usual way, we obtain a closed orientable 2-manifold that does not exhaust Δ^*; that is, the voltage graph imbedding is into a disconnected space. This is a contradiction, since no one-vertex graph can be disconnected. Thus it must be that all the one-vertex components are identified at that one vertex, giving a voltage graph imbedding into a generalized pseudosurface that is **not** a 2-manifold. This is the final contradiction.

□

Thm. 18-22. Let Δ be the standard basis for $\Gamma = (\mathbb{Z}_n)^m$, $n \geq 3$, and let the Cayley map (Γ, Δ, π) cover an index-one voltage graph imbedding having r regions. Then

$$g(\Gamma, \Delta, \pi) = \begin{cases} 1 + \frac{n^m}{2}(m-2), & \text{if } r = 1 \\ 1 + \frac{n^{m-1}}{2}(mn - n - r), & \text{if } r \geq 2 \end{cases}$$

PROOF. By Theorem 10-9, since every $\delta \in \Delta$ has order m, a region R of size k below lifts to n^m k-gons when R satisfies the KVL (that is, by Lemma 18-21, when $r = 1$) and to n^{m-1} nk-gons otherwise.

Thus the total number $\tilde{r}$ of regions above is:

$$\tilde{r} = \begin{cases} n^m, & \text{if } r = 1 \\ rn^{m-1}, & \text{if } r \geq 2 \end{cases}$$

and order and size of $G_\Delta(\Gamma) = (C_n)^m$ are $p = n^m, q = mn^m$ respectively. Hence, by Corollary 5-15, the result follows by elementary algebra. □

Next, we apply Theorem 18-22 and Rieper's work presented in Theorem 17-11 to calculate $E(g)$ for (Γ, Δ, π) as above, in a striking instance of the usefulness of enumerative topological graph theory to random topological graph theory.

Thm. 18-23. Let Δ be the standard basis for $\Gamma = (\mathbb{Z}_n)^m$, $n \geq 3$; then

$$E(g((\mathbb{Z}_n)^m, \Delta, \pi)) = \frac{1}{(2m-1)!}\left(\left(1 + \frac{n^m}{2}(m-2)\right) S_{m,1} + \sum_{r=2}^{m+1}\left(1 + \frac{n^{m-1}}{2}(mn - n - r)\right) S_{m,r}\right).$$

PROOF. We need only observe that $|\Delta^*| = 2m$, since Δ contains no involutions, and recall that $S_{m,r}$ denotes the number of 2-cell imbeddings of the bouquet B_m having r regions. □

We note that, when $r = 1$, m must be even by the euler identity applied to the voltage graph imbedding (where $p = 1$ also). Thus for m odd, the first term inside the square brackets in Theorem 18-23 vanishes.

Cor. 18-24. The expected value of the genus random variable for the Cayley maps $((\mathbb{Z}_n)^m, \Delta, \pi)$, with $n \geq 3$ and Δ the standard basis, is:

$$E(g) = \begin{cases} 0, & \text{if } m = 1 \\ \frac{1}{3}n^2 - n + 1, & \text{if } m = 2 \\ n^3 - \frac{4}{3}\, n^2 + 1, & \text{if } m = 3 \\ \frac{7}{5}n^4 - \frac{4}{3}n^3 + 1, & \text{if } m = 4 \\ 2n^5 - \frac{23}{15}n^4 + 1, & \text{if } m = 5 \\ \frac{17}{7}n^6 - \frac{23}{15}\, n^5 + 1, & \text{if } m = 6 \\ \quad \cdots \end{cases}$$

PROOF. Use the appropriate values for $S_{m,r}$ from Table 17-3, in Theorem 18-23. □

The polynomials of the corollary lead to the entries in Table 18-2. The entry for (n, m) is $E(g((\mathbb{Z}_n)^m, \Delta, \pi))$. The first column corresponds to the fact that C_n has only a planar imbedding. The case $m = 2$ (of either Corollary 18-24 or Table 18-2) corresponds to Figure 17-2. The four spherical voltage graphs each lift to $C_n \times C_n$ on $S_{\binom{n-1}{2}}$ with $r_n = 2n$ and $r_{2n} = n$, whereas the two toroidal voltage graphs both lift to $C_n \times C_n$ on S_1 with $r = r_4 = n^2$. Thus $E(g) = \frac{4\frac{n^2-3n+2}{2} + 2\cdot 1}{6} = \frac{n^2}{3} - n + 1$.

From Theorem 18-23, one readily sees that $E(g(\Gamma, \Delta, \pi))$ has the form $a_0 n^m + a_1 n^{m-1} + a_2$, where a_0, a_1, and a_2 depend only on m. This is, of course, confirmed by Corollary 18-24. Since $\sum_{r=1}^{m+1} S_{m,r} = (2m-1)!$,

$n\backslash m$	1	2	3	4	5	6
3	0	1	16	78.4	362.8	1398.8
4	0	2.333	43.67	274.1	1656.5	8378.3
5	0	4.333	92.67	709.3	5292.7	33155.8
6	0	7	169	1527.4	13565.8	101385.2

Table 18-2

we easily find that $a_2 = 1$. Since $S_{m,1} = \frac{(2m-1)!}{m+1}$ if m is even, but is 0 if m is odd (see Problem 17-13), it follows (after some work) that $a_0 = \frac{m^2-2}{2m+2}$ if m is even, and $a_0 = \frac{m-1}{2}$ if m is odd. Finally, but less satisfactorily, $a_1 = -\frac{1}{2(2m-1)!}\sum_{r=2}^{m+1} rS_{m,r}$.

18-7. Problems

18-1.) Calculate $|\operatorname{Aut} G|$, $|R(G)|$, $|C(G)|$, and $E(M)$, for $G = K_7$. Compare with $E(M)$ for $G = K_4, K_5$, and K_6 (see Table 18-1). What conclusion do you make?

18-2.) Show that $P(g(K_{2,n}, \rho) = \gamma(K_{2,n})) = \frac{1}{(n-1)!}$, so that

$$P(g(K_{2,2k+1}, \rho) = \gamma(K_{2,2k+1})) = \frac{1}{(2k)!},$$

whereas

$$P(g(K_{2,2k+1}, \rho) = \gamma_M(K_{2,2k+1})) = \frac{1}{(k+1)}.$$

What does this indicate?

18-3.) In Model I, let $\Omega = C(G)$ instead of $R(G)$. Study this revised model.

18-4.) In Model II A, show that $P(g(K_4) = 0)$ is maximized, and $E(g(K_4))$ is minimized, uniquely for $p = \frac{1}{2}$, the uniform case.

18-5.) Analyze $G = K_{3,3}$ in Model II A.

18-6.) Show that, in Model II A, calculations for $G = K_2 \times K_3$ depend on the assignment of labels from $\{1, 2, 3, 4, 5, 6\}$ to $V(G)$.

18-7.) Analyze $K_2 \times K_3$ in Model II B, using $\Delta = \{3, 1, 5\}$ for $\Gamma = \mathbb{Z}_6$.

18-8.) Analyze $K_2 \times K_3$ in Model II C, with reference to the fixed planar imbedding of $K_2 \times K_3$ as a 3-prism.

18-9.) Prove Corollary 18-16.

18-10.) If E is the event that (G, ρ, λ) will be orientable, for fixed G and ρ in Model III, show that $|E| = 2^{n-1}$, where $n = |V(G)|$.

18-11.) Show that Model IV gives a probability distribution.

18-12.) What does Corollary 18-16 have to say about K_n? About $K_{n,n}$? About a family of graphs G_n with minimum degree at least 3?

18-13.) In Model IV, express $P((K_4, \rho, \lambda)$ is orientable$)$ as a function of p. (Hint: use Theorem 18-20.) Verify the uniform case, by a direct count.

18-14.) Consider the random variable $k : \Omega \to N \cup \{0\}, k(G,\rho,\lambda) = |\{e \in E(G) | \lambda(e) = 1\}|$ in Model IV. Find $E(k)$ and $\sigma(k)$, where σ is the standard deviation.

18-15.) **Study Model V.

18-16.) In Model VI, find $P[g((\mathbb{Z}_n)^m, \Delta, \pi) = \min_\pi g((\mathbb{Z}_n)^m, \Delta, \pi)]$, for $m \leq 6$, where Δ is the standard basis.

18-17.) In Model VI, find

$$P[g((\mathbb{Z}_n)^m, \Delta, \pi) = \max_\pi g((\mathbb{Z}_n)^m, \Delta, \pi)],$$

where Δ is the standard basis, for:

(i) $m \leq 6$

(ii) m even.

18-18.) Do Problems 18-16 and 18-17, taken together, reinforce, or undermine, the indication of Problem 18-2?

18-19.) *Use Theorem 18-8 to show that $E(g(R_n)) \leq \frac{n+4}{3}$. Next show that $E(g(R_n)) \geq E(g(L_n)) \geq \frac{n}{3}$ (using Theorem 18-7). Thus $E(g(R_n))$ is asymptotic to $\frac{n}{3}$. Deduce that, for R_n, average genus and maximum genus are not asymptotic. In what sense does this show that Theorem 18-4 is best possible?

CHAPTER 19

CHANGE RINGING

The ancient and continuing art of change ringing, or campanology (how the English ring church bells), is here studied from a mathematical point of view. An "extent" on n bells is regarded as a hamiltonian cycle in a Cayley color graph for the symmetric group S_n, often imbedded in an appropriate surface. Thus–perhaps surprisingly–graphs, groups, and surfaces combine to model something musical. We begin by describing some of the history and lore of change ringing, along with the requisite terminology. Next we give details of the mathematical model we will be using, together with some basic results, illustrated by the only two extents on three bells. Then, in succession, we consider various compositions on four, five, six, seven, and n bells. Along the way we meet Fabian Stedman, a seventeenth-century printer and bell ringer, who was doing coset decomposition in symmetric groups a century before mathematicians happened upon the concept. We also encounter three compositions of the author, all performed to quarter-peal length in Oxford, England. Each was found by imbedding the right graph of the right group on the right surface, in the right way. Most of the material in this chapter is taken from [W14], [W17], [W19], [W21], [W22], [W23], and [W26].

19-1. The Setting

Bells are chimed–swung through an arc, with clapper and bell meeting to produce the sound–using ropes or levers. The purpose is to announce coronations, weddings, funerals, and calls to service, or perhaps just to enjoy the challenge of composing and ringing a piece of music properly. For centuries church towers were almost the only structures substantial enough to accommodate sizeable bells. Prior to the fourteenth century, church bells in Europe were usually hung on a spindle and chimed by pulling a rope attached to the spindle. The succeeding centuries saw the development, in England, of a more sophisticated method of hanging a bell, to improve the control that a ringer had over it. The bell was mounted on first a quarter-wheel, then a half-wheel, and finally on a full wheel, so that it would swing through a full 360-degree arc each time it was rung. The further refinement of the slider and the stay made possible the setting of the bell (in mouth-up position), allowing the ringer to temporarily halt and then restart the bell precisely. That led to the development of change ringing in England. This practice crossed the English Channel into Belgium only, but without the slider and the stay, so purely mechanical methods of ringing eventually led to the carillon there. In Great Britain nonconformist chapels usually had but one bell, and for centuries Roman Catholic churches were allowed no bells at all. Thus change ringing became a peculiarly English art, formalized by Fabian Stedman with the publication of *Tintinnalogia–or the Art of Change Ringing* in 1668 [D5], and *Campanologia: or the Art of Ringing Improved*, in 1677 [S21]. Now there are more than five thousand church towers in England where bells are mounted to be rung in changes, perhaps a score or so in the United States and in Australia, and a half-dozen or so in Canada. The Whitechapel Bell Foundry in London, which manufactured Big Ben and the Liberty Bell among many others, still actively produces bells for the ringing of changes. Let the n bells in a tower be denoted by the natural numbers $1, 2, \ldots, n$–arranged in descending order of pitch, from bell 1 (the *treble*) to bell n (the *tenor*). A *change* is a ringing of the n bells, once each, in some order. The very special change that rings the bells in the natural order $1, 2, \ldots, n$ is called *rounds*. The central problem in change ringing is to ring an *extent* on n bells; this is a sequence of $n! + 1$ changes satisfying:

(i) The first and last change are both rounds.
(ii) No other change is repeated (so that each change other than rounds is rung exactly once).
(iii) From one change to the next, no bell changes its order of ringing by more than one position.

Rule (i) is for musicality, rule (ii) is for thoroughness, and rule (iii) is necessitated by the manner in which the bells are mounted in the tower: a ringer can only advance or retard the motion of his or her bell slightly. Thus a bell cannot be quickly rung twice successively, nor can it not be rung for more than a short period of time, so that English church bells are rung not in melody, but in permutations. Moreover, the way that one permutation follows another is strictly limited. This makes the connection to mathematics, as we see in detail in the next section. Certain additional conditions that an extent might meet are often regarded as desirable (or perhaps even essential), but it is only the three conditions (rules) given above that are **always** required. Among the additional conditions, here regarded as optional, the following three are the most noteworthy:

(iv) No bell occupies the same position in its order of ringing for more than two (sometimes relaxed to four) successive changes.
(v) The "working" bells each do the same work.
(vi) Each lead (or division) of the extent, and thus the plain course and perhaps even the extent itself, is palindromic in the sequence

of transitions employed to pass from one change to the next. That is, the lead (or division) is *symmetric*.

We need additional terminology to understand the last two conditions. But first we remark that (iv) keeps the performance interesting for the ringers, (v) is for balance, and (vi) is to ease the memory burden for the ringers, as no visual aid to memory is allowed within the ringing chamber. One ringer–the conductor–will occasionally make a *call* (an oral instruction, either a *bob* or a *single*, to instruct the ringers to modify their pattern of ringing slightly at an appropriate time. What each ringer must bring into the tower is a clear image of the path (called the *blue line*; see [SS1]) that his or her bell follows through the other bells in the sequence of changes to be rung. This path is normally followed by *ropesight*: pulling your rope about one quarter of a second after the rope of the bell yours is to follow is pulled by your fellow ringer. Waiting to **hear** the sound of that bell will not do.

Here is some additional terminology. There are two basic types of composition in change ringing: methods and principles. A *method* is *treble-dominated*. That is, the treble *plain hunts*–occupying successively positions $1, 2, \ldots, n$; $n, \ldots, 2, 1$; $1, 2, \ldots, n$; $n, \cdots, 2, 1$; and so on. It is not considered to be working (the blue line is as simple as possible). In a *principle*, all the bells are working–that is, they are performing more intricate tasks, such as dodging around other bells, making internal places and so forth. (A plain hunt bell makes only the external places 1 and n.) Finally, each composition is composed of basic blocks called, for methods, *leads* (the changes progressing from one treble lead–such as in rounds–to the next, usually consisting of $2n$ changes), or, for principles, *divisions*. The *plain course* is the sequence of changes starting with rounds and following lead after lead (or division after division) without calls (special generating transitions, either bobs or singles) until rounds comes up again. (If this happens before the extent is completed, then calls are required. If not, then the plain course is the extent, called a *no-call* extent.) Condition (v) can be expressed by saying that the plain course must have the same number of leads (or divisions) as there are working bells. These ideas will be illustrated in the succeeding sections. Typically the number n of bells in a tower is between three and twelve, with eight (tuned to an octave) being common. There is a nomenclature for extents, and the last part of this nomenclature specifies the number of bells; see Table 19-1. The odd-bell names reflect the maximum number of disjoint pairs of bells that could be exchanged, in their order of ringing.

As an extent of Major takes about twenty hours to ring–surely one of the 'major' physical and intellectual feats of mankind–extents on more than eight bells clearly surpass the limits of human endurance. Even on eight bells they are extremely rare: on 27 and 28 July, 1963,

n	name	$n! + 1$
3	Singles	7
4	Minimus	25
5	Doubles	121
6	Minor	721
7	Triples	5041
8	Major	40,321
9	Caters	362,881
10	Royal	3,628,801
11	Cinques	39,916,801
12	Maximus	479,001,601

Table 19-1

Plain Bob Major was rung on tower bells at the Loughborough Bell Foundary; on 27 and 28 December 1977, the same extent was rung on handbells in a private residence in Farnham, Surrey. Each handbell is readily controlled by a flick of the wrist, but then each ringer–having a bell in each hand–has **two** blue lines to memorize. In a tower, each ringer has both hands full–of rope and sally–controlling one bell. In fact, it takes many months of practice to learn this control, let alone learning to strike uniformly as part of rounds, to say nothing of ringing constantly changing changes. An extent of triples, usually requiring just under three hours of concentrated ringing, is much more readily attainable, and many peals are attempted, completed, and duly reported in the weekly publication *The Ringing World* (published in Guildford, Surrey). Technically, a *peal* consists of at least 5,000 and, for $5 \leq n \leq 7$, exactly 5,041 successive changes satisfying the rules (i), (ii), (iii) above–except that (ii) is waived for $n < 7$, where a peal consists of several extents strung together. Thus for $n = 7$, a peal is an extent; and, for $n \geq 7$, a peal is a partial extent, called a *touch* (rules (i) and (iii) still hold). As generally the number of bells in a tower is even, for odd-bell extents (or peals or touches) a *covering bell*–always the tenor–rings last in every change, in forgivable violation of rules (iv) and (v). In fact, some listeners find the stability and regularity this provides to be pleasing musically. For more information about the history and practice of change ringing, the reader could consult Wilson [W30], Camp [C2], Cook [C7], Eisel [E2], and *The Ringing World.* For other mathematicians who have studied this fascinating subject, see, for example, Budden [B20], Dickinson [D3], Fletcher [F2], Price [P9 and P10], and especially Rankin [R3], whom we will encounter in Section 19-7. For a mystery novel in which change ringing figures prominently, read *The Nine Tailors*, by Dorothy Sayers [S4].

19-2. A Mathematical Model

We employ a mathematical model for change ringing extents that uses graphs of groups on surfaces. Our first observation is that a change on n bells is a permutation of degree n and that, by rule (ii), the symmetric group S_n is relevant for extents on n bells. Secondly, by rule (iii), each legal transition (from one change to the next) can be described as a product of disjoint transpositions of adjacent numbers from $\{1, 2, \ldots, n\}$–regarded as a line, not a circle, so that 1 and n are **not** adjacent (for $n \geq 3$). This leads to our surprising first theorem. Let $t(n)$ give the number of legal transitions for n bells.

Thm. 19-1. For $n \geq 2, t(n) = F(n+1) - 1$, where $F(n)$ is the nth Fibonacci number.

PROOF. We check that $t(2) = 1 = F(3) - 1$ and that $t(3) = 2 = F(4) - 1$, and then assume the claim for $n < k$. Consider $t(k), k \geq 4$. Since there are $t(k-1)$ legal transitions fixing the bell in position $k, t(k-2)$ legal transitions exchanging the bells in positions k and $k-1$ and at least one other pair, and the single transposition $(k-1, k)$, we have:

$$\begin{aligned} t(k) &= t(k-1) + t(k-2) + 1 \\ &= (F(k) - 1) + (F(k-1) - 1) + 1 \\ &= F(k+1) - 1. \end{aligned}$$

$\square$

To avoid confusing the number of a given bell with the number of the position it occupies in a particular change, we regard each change as a function f from the set of n positions to the set of n bells; thus $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, where the domain elements represent positions and the range elements represent bells. Then change f, recorded as $f(1), f(2), \ldots, f(n)$, would ring bell $f(1)$ first, bell $f(2)$ second, and so on. Rounds is given by the identity permutation $r(i) = i, 1 \leq i \leq n$. Each transition is regarded as a permutation of the set of n positions. Thus if $d_1, d_2, \ldots, d_k$ represent the first k transitions in an extent, then the $(k+1)$st change is the function $rd_1d_2 \ldots d_k$, where we compose from right to left in S_n. Now we can express conditions (v) and (vi) more precisely. Let word $w = d_1d_2 \ldots d_\ell$ describe the first lead (or division) and the transition to the second. Then if there are m working bells, (v) requires that w be an m-cycle, so that $w^m = I$ gives the plain course–where I is the identity on the set of positions–and there is only one orbit for w in its action on the working bells. The symmetry condition (vi) is that w be a palindrome in the letters $d_1, d_2, \ldots, d_{\ell-1}$. Next let Δ be the set of all transitions employed for a particular extent. (Necessarily, each is a legal transition, by rule (iii).) By rule (ii), Δ must generate S_n. Thus we can form the Cayley color graph $C_\Delta(S_n)$. Since all the generators are involutions, the edges have color, but not direction (by convention (iii) in Section 4-5). By rule (i), in conjunction now with (ii) and (iii), our extent corresponds to a hamiltonian cycle in $C_\Delta(S_n)$. The starting vertex is arbitrary, as $C_\Delta(S_n)$ is vertex-transitive. We have shown:

Thm. 19-2. An extent on n bells, satisfying rules (i), (ii), and (iii) and whose set of transitions is from Δ, can be composed if and only if $C_\Delta(S_n)$ is hamiltonian.

We remark that the Cayley color graph $C_\Delta(S_n)$ can be replaced by the Cayley graph $G_\Delta(S_n)$. But we retain the edge colors, as they are valuable in constructing the extent from the hamiltonian cycle and in visually checking the palindromic condition (vi). Thus we see that graphs of symmetric groups are central to change ringing. Imbedding or immersing such graphs on surfaces is helpful in making a hamiltonian cycle more readily apparent, and it is this idea that we exploit in the remainder of this chapter. We begin with a simple example, for $n = 3$, to clarify many of the points we have discussed. From Theorem 19-1 we see that $t(3) = F(4) - 1 = 2$; and if we take $\Delta = \{(12), (23)\}$, then $G_\Delta(S_3) = C_6$; see $C_\Delta(S_3)$ in Figure 19-1. The hamiltonian cycle and the extent it gives rise to shown in Table 19-2 describe the clockwise interior region boundary of the spherical imbedding of $C_\Delta(S_3)$ depicted. This extent is called *quick six*. The clockwise boundary of the exterior region produces *slow six*, the only other extent on 3 bells. (Not surprisingly, slow six is just quick six in the backwards order.) If we let $a = (12)$–denoted by solid edges–and $b = (23)$-denoted by dashed edges–then quick six is completely described by the identity word $(ab)^3$ in S_3 (and slow six by $(ba)^3$). In the first case the relevant palindrome–for condition (vi) applied to the extent–is *ababa*; in the latter case it is *babab*. Both extents appear to be methods, as the treble is plain hunting. But in both cases, the other two bells are plain hunting also, from one leading position to the next. Thus both extents are actually principles, as reinforced by the identity words $(ab)^3$ and $(ba)^3$ respectively; there are three divisions, one for each bell, and all bells work alike. The word $w = ab = (123)$, for example, gives $w^3 = I$ as the plain course **and** the extent, for quick six. Thus condition (v) is met. Condition (iv) is readily verified by examining the final column of Table 19-2, and also algebraically–by noting the effect of alternating a and b throughout. Of course, rules (i), (ii), and (iii) are guaranteed to hold, by the hamiltonian cycle and the generating set employed.

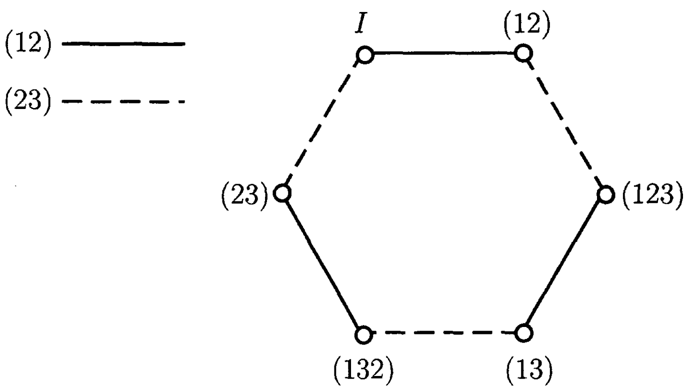

Figure 19-1.

hamiltoniancycle	change	range of change
I	$rI = r$	1,2,3
(12)	$r(12)$	2,1,3
(123)	$r(123)$	2,3,1
(13)	$r(13)$	3,2,1
(132)	$r(132)$	3,1,2
(23)	$r(23)$	1,3,2
I	r	1,2,3

Table 19-2.

Ringers study extents primarily in the form of the final column (without the commas) of Table 19-2, where each change appears as a row in an $(n!+1)\times n$ matrix (see [SS1]). Our contribution is to introduce a graph-of-groups-on-surfaces approach to find that matrix.

19-3. Minimus

Now we turn to 4-bell extents, all of which are called "Minimus" as the last part of their nomenclature. We see from Theorem 19-1 that there are $t(4) = F(5) - 1 = 4$ potential transitions for four bells; they are

$a = (12)(34)$ ————————

$b = (23)$ - - - - - - - - - -

$c = (34)$ ....

$d = (12).$ —·—·—·—·—

It is well known by ringers that the eleven minimus methods given in [SS1] are the only such which satisfy condition (vi). They are algebraically described (by the corresponding identity word in S_4) in Table 19-3. Each word corresponds to a hamiltonian cycle in the Cayley color graph for S_4 as generated by the letters in the word. Note that only the first three satisfy all six of our conditions. The others all fail (iv).

In fact, both Reverse Court and Double Court Minimus fail even the relaxed version of (iv).

Minimus Extent	Algebraic Description
Plain Bob	$((ab)^3\ ac)^3$
Reverse Bob	$(abad(ab)^2)^3$
Double Bob	$(abadabac)^3$
Canterbury	$(abcdcbab)^3$
Reverse Canterbury	$(db(ab)^2dc)^3$
Double Canterbury	$(dbcdcbdc)^3$
Single Court	$(db(ab)^2db)^3$
Reverse Court	$(ab(cb)^2ab)^3$
Double Court	$(db(cb)^2db)^3$
St. Nicholas	$(dbadabdc)^3$
Reverse St. Nicholas	$(abcdcbac)^3$

Table 19-3.

It is particularly instructive to consider Plain Bob Minimus topologically. For $\Delta = \{(12)(34), (23), (34)\}$, $C_\Delta(S_4)$ imbeds on the projective plane N_1, as shown in Figure 19-2. We note from Figure 19-3 that $a = (12)(34)$ and $b = (23)$ generate the dihedral group D_4 (the symmetry group of the square). Thus the first lead of Plain Bob Minimus, described by $abababa = b = (23)$ starting from rounds at, say, the vertex designated in Figure 19-2, covers the subgroup D_4 of S_4 and corresponds to the vertices in the central octagon of the figure. Using transition b a fourth time would return us prematurely to rounds, completing a touch of only nine changes. Using c instead shunts us across to $(ab)^3ac = bc = (234) = w$. Now alternating $abababa$ again traces out the boundary of a second octagon, the left coset wD_4 (the second lead of the extent). Using c a second time takes us to w^2, and then following with $(ab)^3\ ac$ for a third time traces out the third octagon (and the lead corresponding to coset w^2D_4), returning to rounds (so that $w^3 = I$ describes the entire extent). Thus the partitioning of the extent into leads corresponds to a decomposition of S_4 into left cosets of D_4 (called the *hunting group*, as the treble completes one plain hunt for these changes) and a 2-factor of $C_\Delta(S_4)$ consisting of three 8-cycles, each bounding a region on the projective plane. The rows of Plain Bob Minimus are given in Table 19-4; this is the form the ringers use. The listing follows either from the word $((ab)^3\ ac)^3$ or from the hamiltonian cycle in the Cayley color graph (two versions of the same idea). Again, conditions (i), (ii), (iii) are guaranteed by our hamiltonian cycle and choice of Δ. Condition (iv) can be visually checked, but can be even more readily established by noting the alternation of "a" in $((ab)^3\ ac)^3$. Condition (v) follows algebraically from $w = (ab)^3\ ac = (234)$, but we confirm by checking, from the table, that what bell 2 does in column 1 (the first lead), bell 3 does in column 2, and bell 4 does in column 3,

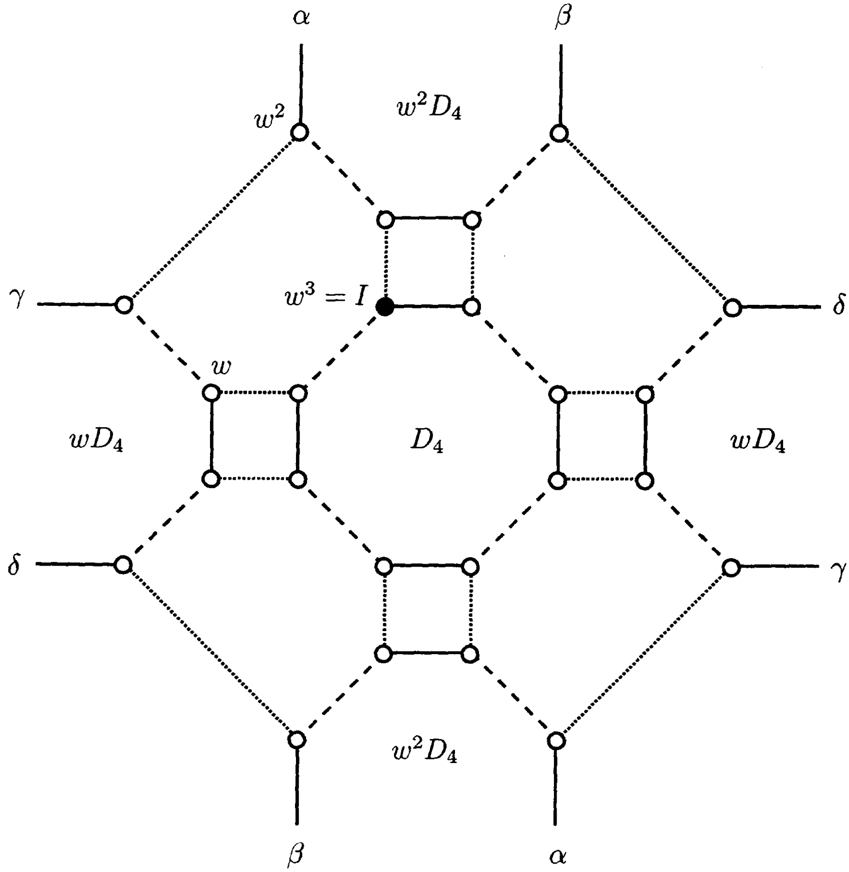

Figure 19-2.

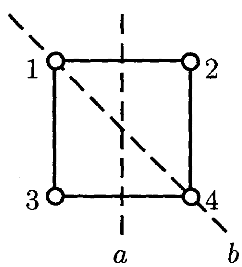

Figure 19-3.

etc., so that all three working bells work alike. Finally, condition (vi) holds, as *abababa* is a palindrome.

The graph imbedding of Figure 19-2 also serves directly to describe Reverse Court Minimus, and indirectly for Reverse Bob and Single Court Minimus (with d replacing c to describe the "reversal", which amounts to a vertical reflection in the rows of the extent). As the graph of Figure 19-2 is non-planar, all four of these extents have no

1234	1342	1423
2143	3124	4132
2413	3214	4312
4231	2341	3421
4321	2431	3241
3412	4213	2314
3142	4123	2134
1324	1432	2143
		1234

Table 19-4.

more efficient graphical imbedding (from the point of view of maximizing euler characteristic). The other seven minimus methods given need two additional topological models. As depicted in [W14], one of these is on the sphere, and the other is a spherical immersion, with six crossings occurring within quadrilaterals of the spherical imbedding, for the added (redundant) generator. Double Court and Double Canterbury Minimus, modelled on the sphere, are the only planar minimus methods. In examining minimus methods, we have been "ringing the changes," that is using the full Cayley color graph. Now we turn our attention to minimum principles, for which Schreier (right) coset graphs introduce a nice efficiency. The result will be "ringing the cosets." We outline the development here; for full details, see [W22]. Take $\Delta = \{(12)(34), (12), (23), (34)\}$; we temporarily use all $t(4) = 4$ potential transitions. Since a minimus principle will be described by an identity word w^4, where w is itself a word in six letters, it makes sense to study the Schreier right coset graph $S_\Delta(S_4/\mathbb{Z}_4)$, as shown in Figure 19-4. Starting, without loss of generality, at the designated vertex

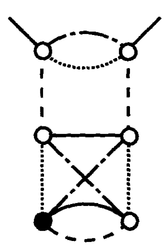

Figure 19-4.

(to exploit the symmetry of the diagram in seeking to satisfy condition (iv)), we find exactly eight hamiltonian cycles in the coset graph, listed below by the associated word w:

(1) $(db)^2da = (1342)$	(5) $cbdbca = (1243)$
(2) $(db)^3 = I$	(6) $cbdbcb = (14)(23)$
(3) $dbcbda = (1243)$	(7) $(cb)^2ca = (1342)$
(4) $dbcbdb = (14)(23)$	(8) $(cb)^3 = I$

Then (2) and (8) give touches of length six, whereas (4) and (6) give touches of length twelve. In [W22] we find the following result. Recall that the "no-call" means that neither a bob nor a single is required; that is, the plain course $w^n = I$ gives the extent.

Thm. 19-3. There is a no-call principle on n bells, using the transitions of Δ, if and only if there is a hamiltonian cycle in $S_\Delta(S_n/\mathbb{Z}_n)$ whose associated word is an n-cycle.

As our first application of this theorem, we see that (1), (3), (5), and (7) all give no-call minimus principles. Since Figure 19-4 contains only the eight hamiltonian cycles already considered, we have proved:

Thm. 19-4. There are exactly four no-call minimus principles.

These were already known to ringers as:

(1) $[(db)^2da]^4$, Erin Minimus
(3) $[dbcbda]^4$, Stanton Minimus
(5) $[cbdbca]^4$, Reverse Stanton Minimus
(7) $[(cb)^2\ ca]^4$, Reverse Erin Minimus.

The contribution of the present study is the exhaustiveness of this list (and the unified approach–one diagram–to obtaining it). We note that all four extents satisfy conditions (v) and (vi), but none satisfies condition (iv). Extents (3) and (5), however, satisfy the relaxation of (iv): no bell rests in one position for more than four successive changes.

19-4. Doubles

The vertices of a regular pentagon can be labelled so that $a = (12)(34)$ and $b = (23)(45)$ are two reflections generating the dihedral group D_5 (see Problem 19-3). In fact, $D_5 = \{I, a, ab, aba, \dots, (ab)^4a\}$. Let $c = (34)$ and form $P = (ab)^4ac = (2354)$; this is the *plain lead* of Plain Bob Doubles, and $P^4 = [(ab)^4\ ac]^4$ gives the plain course. We note the similarity with Plain Bob Minimus. But there the plain course was the extent, while here the plain course gives a touch of 40 (of the requisite 120) different changes, using four (of the requisite twelve) left cosets of D_5 in S_5. To extend the touch to an extent, introduce $d = (23)$ and form $B = (ab)^4ad = (45)$, the *bob lead*. When the conductor calls "bob," the ringers know to replace transition c with transition d. The question is: when should this occur, to get the full extent of Plain Bob Doubles? In Figure 19-5 we give a slight modification of a diagram due to D.W. Struckett [S27]. The graph is the

1-skeleton of a truncated octahedron. The 24 vertices are labelled with the 24 treble leads on five bells. (Bell 1 is first in each of these changes, and is suppressed.) The directed edges represent right multiplication by $P = (2354)$, and the undirected edges represent right multiplication by $B = (45)$. Thus the figure gives a spherical imbedding of the Cayley color graph $C_\Delta(\Gamma)$, where $\Delta = \{P, B\}$ for $\Gamma = (S_5)_1 \cong S_4$, the stabilizer in S_5 of object 1, acting on $\{2,3,4,5\}$-precisely what is needed to study the treble leads on 5 bells. An example calculation might be

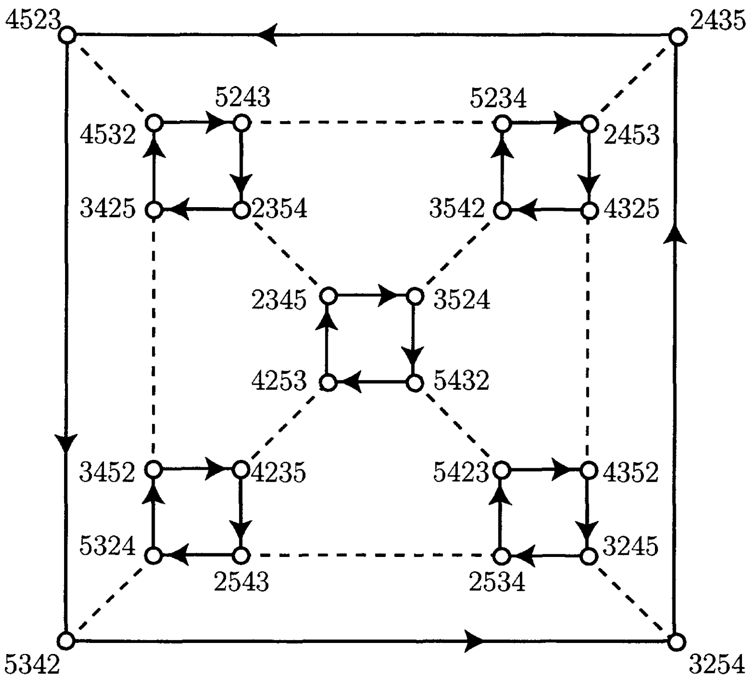

Figure 19-5.

helpful. If we start at vertex 5423 in Figure 19-5, we must remember that the change 15423 is identified with permutation (2534)–in position two, ring bell 5; in position five, ring bell 3; etc.–so that when we multiply by P, the product $(2534)(2354) = (245)$ is identified with change 14352. Thus there is a directed edge from vertex 5423 to vertex 4352, colored with the color of P. But the figure has another interpretation, as given in [W19]. Each vertex represents a block of ten changes, commencing with the treble lead given and continuing by applying the word $w^* = (ab)^4a = b = (23)(45)$ one letter at a time. Note that: (1) $w^*c = P$ and $w^*d = B$: (2) since $w^* \in (S_5)_1$, the tenth change in each block is also a treble lead. Thus each of the 24 treble leads appears in each of two blocks, once as the first change (the one given in the figure) and once as the last. These two blocks are diametrically opposite (that is, *antipodal*) vertices on the spherical imbedding, since antipodes are joined by the path $P^2BP^{-2}B = (23)(34) = w^*$. Moreover, since w^* is a palindrome, in each such pair of blocks one is the backwards version of the other. In fact, each block of ten changes is a left coset of D_5 in S_5, and each coset appears twice–represented by each of the two treble leads it contains. To avoid this redundancy, we perform antipodal identification on the sphere. This gives a 2-fold covering projection to the graph we want, imbedded on the projective plane; see Figure 19-6. Now each hamiltonian cycle properly incorporating the arrows

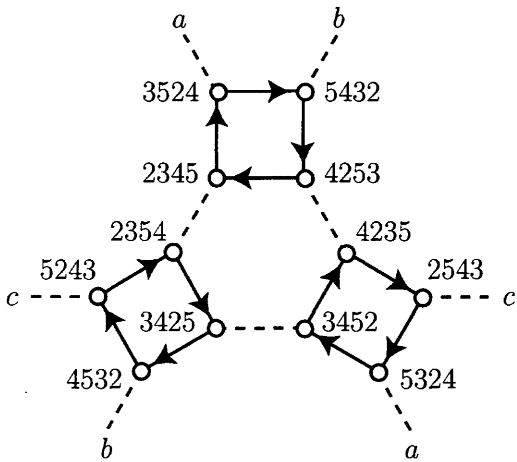

Figure 19-6.

gives an extent of Plain Bob Doubles, and all such extents can be obtained this way. The arrows are respected, except that a bob lead (labelled a, b, c in the figure) requires traversing against the arrows on the plain leads until another bob lead passes through the crosscap. This is required, since passing through the crosscap corresponds to changing hemispheres above, where–for a fixed vantage point–orientation is reversed.

The hamiltonian cycles obtained, all starting at 2345 (rounds), are:

(1) $(P^3B)^3$
(2) $P^2B(P^3B)^2P$
(3) $PB(P^3B)^2P^2$
(4) $B(P^3B)^2P^3$.

Thus topological graph theory has shown very nicely that there are precisely four extents of Plain Bob Doubles using the standard bob b. For example, the extent given by (1) is:

$$[((ab)^4ac)^3(ab)^4ad]^3.$$

Now let us try to compose a doubles principle. Using Theorem 19-1 again, we find $t(5) = F(6) - 1 = 7$; thus we have seven transitions to choose from:

$$\begin{aligned} a &= (12)(34) \\ b &= (23)(45) \\ c &= (34) \\ d &= (23) \\ e &= (12) \\ f &= (12)(45) \\ g &= (45). \end{aligned}$$

No two of these will generate S_5. Choosing more than three seems wasteful, so we fix on $|\Delta| = 3$. We like a, b, and f, as they facilitate satisfying condition (iv). However, $\langle a, b, f\rangle = A_5$, not S_5. We settle on $\Delta = \{a, b, g\}$, which does generate S_5. In [W14] a hamiltonian cycle was found in $C_\Delta(S_5)$, by imbedding that Cayley color graph on N_{10} with 5-fold rotational symmetry, and then finding a path of length 24 which replicated to a spanning cycle by rotation. In [W19] we used this symmetry in a more systematic way. Identify each vertex with its four images under rotation by $72, 144, 216$, and 288 degrees, giving a decomposition of S_5 into right cosets of $\mathbb{Z}_5$. The quotient imbedding is into N_2, as depicted in Figure 19-7. The covering projection is 5-fold, with branching (two of twelve 10-gons above wrap five times each around a respective digon below). In order to satisfy condition (iv), generator a–the only one to affect position 1–must alternate throughout the extent. We find exactly twelve hamiltonian cycles in Figure 19-7 with a alternating; without loss of generality we start at the designated vertex, with the solid edge. The twelve cycles fall into four equivalence classes under rotation (conjugation). These are:

I. $(ag)^3(ab)^3ag(ab)^2(ag)^2ab = (14253)$
II. $(ab)^3(ag)^3ab(ag)^2(ab)^2ag = (12345)$
III. $(ag)^3(ab)^2agab(ag)^4ab = (12345)$
IV. $[(ag)^5ab]^2 = I$.

By Theorem 19-3, I, II, and III above produce no-call doubles extents w^5 where w is the word given in a, b, and g. All three satisfy conditions (i) through (v) by construction, but fail (vi). The extent arising from I was found initially as a hamiltonian cycle in $C_\Delta(S_5)$, as mentioned earlier, but Theorem 19-3 is a more efficient means of composing it. On 9 December 1984, this extent was rung to quarter-peal length (eleven replications, 1,321 changes in all) on the tower bells of the church of St. Thomas the Martyr in Oxford, England. Following the performance, the band (J.D. Alford 1, M.E. Ovenden 2, J.G. Pusey 3, R. L. Wilden 4, I. M. Gardiner 5, R. Pusey (C) 6-rung in cover) named the composition (as is their right) "White's No Call Doubles."

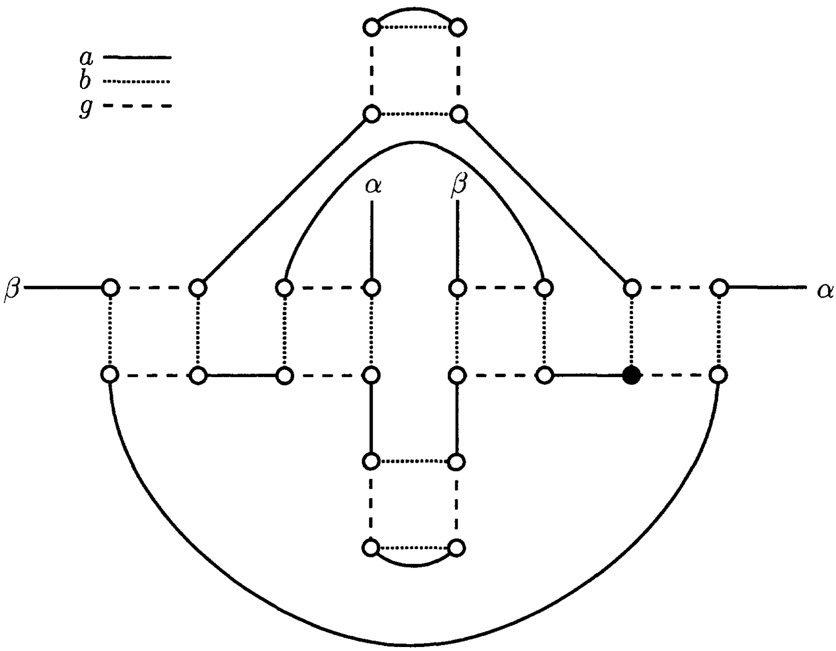

Figure 19-7.

As noted above, this composition does not satisfy the symmetry property (vi). Surprisingly, this turns out to be an asset, as an asymmetric principle automatically generates three companion extents: (i) backwards, (ii) reverse (in this case, interchanging the roles of a and b and replacing g with e), and (iii) backwards reverse (which is the same as reverse backwards). On 27 February 1985, Reverse White's No Call Doubles was rung to quarter-peal length at Carfax Tower, Oxford. It seems that the two backwards versions have yet to be rung.

The extent arising from II above is, in fact, the backwards version of White's No Call Doubles. The two reverse forms do not arise directly from Figure 19-7, as the generating sets differ. The extent arising from III represents another asymmetric doubles principle (with its three variants; all are apparently unrung, and hence unnamed).

In [W22] **all** 102 no-call doubles principles on three generators are constructed, using just five Schreier right coset graphs. All five are nonplanar, including the graph of Figure 19-7; two others have crossing number 1. One of the other two is depicted in Figure 19-8 with one crossing on N_1 and 4-fold rotational symmetry. The hamiltonian cycle $(fc)^5fd(fc)^4(fd)^2 = (15342)$ in that figure lifts to a no-call doubles principle satisfying all but condition (vi). This piece was rung to quarter-peal length at Carfax Tower, Oxford on 19 July 1987, and named "Western Michigan University Doubles." Kalamazoo composer (and Western Michigan University colleague) C. Curtis-Smith then incorporated part of the extent into the third movement ("Moto Perpetuo: Brilliant and Ringing") of his "Concerto for Left hand and Orchestra," which received its world premiere performance on April 17, 1991 with Leon Fleisher and the Kalamazoo Symphony Orchestra; Yoshimi Takeda, conductor. Subsequent performances of the piano concerto have involved the Detroit Symphony and the New York Philharmonic orchestras.

If we rewrite $[(fc)^5fd(fc)^4(fd)^2]^5$ as $[df(cf)^4(df)^2(cf)^5]^5$ and regard the latter as $[PB]^5$, where $P = df(cf)^4df$ is the plain division and $B = df(cf)^4cf$ is the bob division, then $P^* = df(cf)^4d$ is a palindrome, and **this** arrangement of Western Michigan University Doubles **is** symmetric. (But this doubles principle is no longer a no-call extent.) A

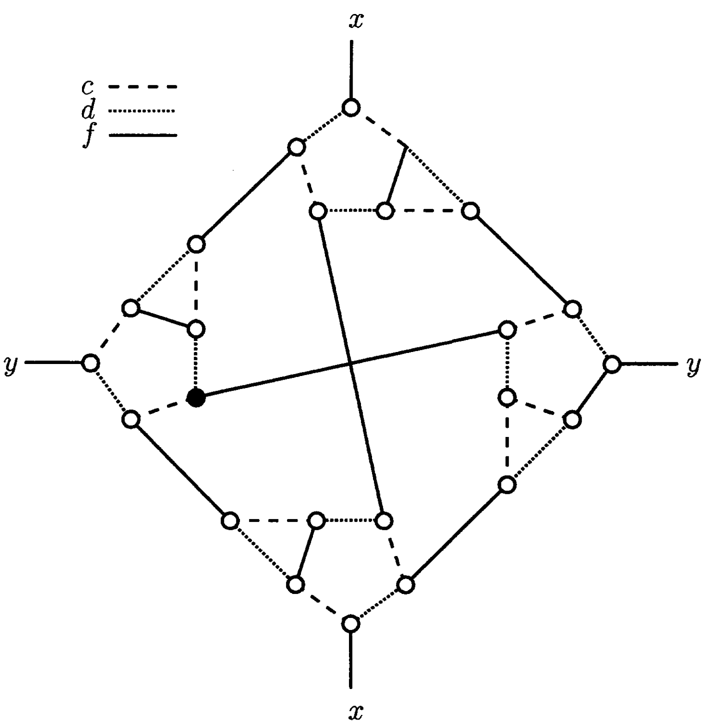

Figure 19-8.

pure doubles extent would be one for which every transition moved the maximum of four bells. But since $\langle a, b, f\rangle = A_5$, not S_5, no such extent exists. (Extents of pure triples–every transition moving six bells–**do** exist.) An approximation to pure doubles occurs when the two cosets of A_5 (each a no-call doubles touch $w^5 = I$ of length 60) are linked by a single (called twice). This is only possible when $S_\Delta(A_5/\mathbb{Z}_5)$ is hamiltonian, for $\Delta = \{a, b, f\}$. Starting at the designated vertex (to display the symmetry) of Figure 19-9, we find exactly four hamiltonian cycles:

A. $fbafbfbfabfa = xa$ (say)
B. $fbafbfbfabfb = xb$
C. $fabfafafbafa = ya$ (say)
D. $fabfafafbafb = yb$.

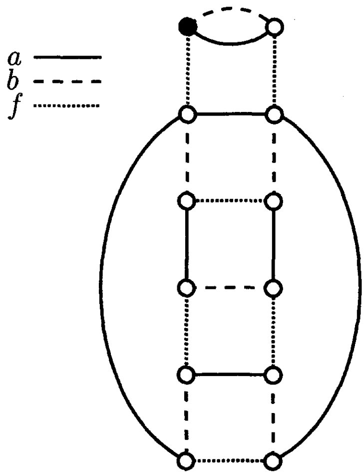

Figure 19-9.

Then for extents of "nearly pure" doubles, we obtain:

A. $[(xa)^4xc]^2$, Reverse Carter Doubles
B. $[(xb)^4xd]^2$, Stedman Doubles
C. $[(ya)^4yc]^2$, Reverse Stedman Doubles
D. $[(yb)^4yd]^2$, Carter Doubles.

These are all well-known. The point is that we have found a unity in their construction, and we have shown that no other extents of this type are possible.

19-5. Minor

Amazingly, a principle like White's No Call Doubles can be used to provide callings for an extent of Plain Bob Minor. In this context it is notationally convenient to rewrite White's No Call Doubles as w^5, where $w = (b's')^3(b'c')^3b's'(b'c')^2(b's')^2\ b'c'; b' = (12)(34)$, $c' = (23)(45)$, and $s' = (45)$. To compose Plain Bob Minor, we start with $a = (12)(34)(56)$ and $b = (23)(45)$ to generate the hunting group D_6 as the first lead; all other leads will be left cosets of this one. Appending $c = (34)(56)$ and $s = (56)$, we get the plain and single leads $P = (ab)^5ac$ and $S = (ab)^5as$ respectively. Noting that, since $(ab)^6 = 1$, $(ab)^5a = b$, we rewrite $P = bc$ and $S = bs$. Now regard

White's No Call Doubles as being rung on the back five of six bells, so that, after renumbering, it becomes $[(bs)^3(bc)^3bs(bc)^2(bs)^2bc]^5$. Then $[S^3P^3SP^2S^2P]^5$ gives a calling, using singles only, for Plain Bob Minor. (Note that, whereas s and c represent one transition each, b represents a sequence of eleven transitions. Thus we account for $720 = 6!$ transitions in all.) That White's No Call Doubles is an extent guarantees that the 120 treble leads in our minor composition are distinct. Then, since each lead (of 12 changes, the first and last being treble leads) is a coset, and two cosets are either disjoint or identical, our minor composition repeats no change, until rounds at the end. That it is indeed **Plain Bob** Minor follows from our choices for a, b and c. It is routine to check that conditions (iv), (v), and (vi) all hold.

Thus the 24-vertex graph of Figure 19-7, when fully interpreted, rings all 721 changes of Plain Bob Minor.

Now consider the Schreier right coset graph $S_\Delta(A_5/\mathbb{Z}_3)$ for $\Delta = \{a, p, b\}$ as given in Figure 19-10. Starting at the designated vertex, the obvious hamiltonian cycle starting with the edge labelled "a" is: $(ap)^4(ab)^2(ap)^3ab = (365)$; thus $[(ap)^4(ab)^2(ap)^3(ab)]^3 = I$. Now, in S_6, set $x = (12)(34)(56)$, $y = (12)(56)$, $z = (34)(56)$, $b = (23)(56)$, and $c = (23)(45)$. Let $a = xyxc(xzxc)^4xyx = (34)(56)$, a palindrome. The word a will determine the leads of the extent we are constructing. The leads will be connected by transitions b and $c = p$. Set $P = ap$ and $B = ab$, the plain and bob leads respectively. Since $a, b, c \in (S_6)_1$, the treble will be leading in the first and last change of each lead; both these treble leads are in A_5. Since $xy = (34)$, the treble will also lead in rows 3 and 22 of each lead (of 24 changes); both these treble leads are in the other coset of A_5 in S_5. Then the identity word above translates to $[P^4\ B^2\ P^3\ B]^3$, a calling for a treble dodging minor method called Oxford Treble Bob Minor. This is a variant of the method-type of composition described in Section 19-1; here the treble is said to be dodge hunting. (It might be helpful to write out the first lead of this composition; see Problem 19-7.) Thus we have improved upon even the efficiency of our construction of Plain Bob Minor, for now we get 721 changes from a graph with only 20 vertices. This analysis is greatly extended in [W23].

19-6. Triples and Fabian Stedman

Fabian Stedman (c. 1640-1713) worked both as a printer (in London and perhaps also in Cambridge) and as a clerk in a London office of Audit of Excise. He was also very active in change ringing: as ringer, composer, and expositor. In 1682 he became Master of the Society of Colledg Youths, a bell-ringing society. *Tintinnalogia* [D5], published in 1668, was written "By a Lover of that ART," and "printed by W.G.

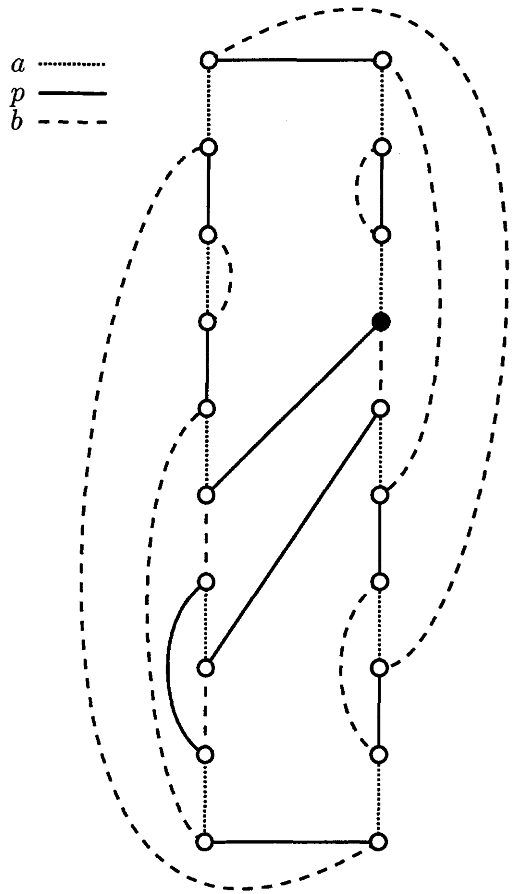

Figure 19-10.

for Fabian Stedman." It is thought [E2] that "W. G." stands for the publisher W. Godbid, that Stedman helped to arrange the printing and to supply material for the book, and that the actual author was Richard Duckworth. Extents on four, five, and six bells were presented, including what we now call Plain Bob Minimus. *Campanalogia* [S21], published in 1677, was printed "by W. Godbid, for F.S.," and historical evidence indicates that "F.S." was indeed Fabian Stedman. *Campanalogia* is a substantial updating of *Tintinnalogia.* Although apparently Stedman had no formal mathematical training, he wrote:

"Although the practick part of *Ringing* is chiefly the subject of this Discourse, yet first I will speak something of the Art of *changes*, its Invention being Mathematical, and produceth incredible effects, as hereafter will appear."

Many group-theoretic ideas (but of course without the terminology and notation that trained mathematicians introduced much later) are implicit in Stedman's exposition and in the many compositions recorded in *Campanalogia.* These concepts include closed systems, axiomatic systems, coset decomposition (including the ideas of coset representative and disjointness), even and odd permutations, factorials, and stabilizers in permutation groups. (See [W26] for details.) Among the extents we have considered in this chapter, Plain Bob Doubles and Stedman Doubles both appear. Let us examine Stedman Doubles more closely. Rewriting this extent as given in Section 19-4 slightly, we obtain

$$[(bfbfbafbfbfa)^4\ (bfbfbafbfbfd)]^2$$

as an identity word ringing Stedman Doubles. As before, $b = (23)(45)$, $f = (12)(45)$, $a = (12)(34)$, and $d = (23)$. The sequence $bfbfb$ gives a slow six on the front three bells, while $fbfbf$ gives a quick six. These two sequences are used in alternation; each yields all the permutations on the front three bells, and thus a subgroup isomorphic to S_3. (See Problem 19-8.) Transition a links successive sixes, bringing one of the back two bells into the front three. The plain course consists of 60 changes, all in $A_5 = \langle a, b, f\rangle$. The single d is used once to change cosets of A_5 and a second time to return to rounds. Stedman Doubles can also be thought of as being composed of the 20 left cosets of S_3 in S_5, and ringers learn the extent in that form; see [SS1]. If we extend Stedman's Principle (as he called it in *Campanalogia*) on five bells to seven bells, we get Stedman Triples. Letting $f = (12)(45)(67)$, $b = (23)(45)(67)$, and $a = (12)(34)(56)$, we obtain the plain course $w^7 = I$, where $w = (fb)^2(fa)(bf)^2ba = (1374562)$. Note that quick and slow sixes alternate, just as for Stedman Doubles; but now there are many more of them! In fact, we need 840 left cosets of S_3 in S_7 for the extent. To expand the plain course touch of 85 changes to a full extent, bob $c = (12)(34)(67)$ and single $d = (12)(34)$ have been used effectively, replacing a in either of its occurrences in certain subwords w to get beyond the plain course, even as far as the full extent. Until recently, the most famous unsolved problem in bell ringing was: Is it possible to ring the full extent of Stedman Triples using only a, b, f, and c? In late 1994, Colin Wyld achieved such a composition, using–out of the 840 positions where the bob c might be called–705 bobs. Then, in early 1995, Andrew Johnson and Philip Saddleton also composed an extent of Stedman Triples using the common bob c only (no singles), and one week later their composition was successfully rung by a Cambridge University Guild band, being called (579 bobs) at the first attempt by Philip Agg. Thus a centuries old (mathematical!) problem derived from the work of Fabian Stedman has finally been settled. The solution corresponds to a hamiltonian cycle in the Cayley graph for the symmetric group S_7, as generated by involutions a, b, c, and f above, incorporating quick and slow sixes (generated by b and f) in alternation, linked by generators a and c. It also corresponds to a hamiltonian cycle in $S_\Delta(S_7/S_3)$, where $\Delta = \{c, d\}$.

19-7. Extents on n Bells

Here we present some results of a more general nature, to complement those of Section 19-2. The first of these is due to Rapaport [R5].

Thm. 19-5. Let $\Delta = \{(12), (12)(34)(56)\dots, (23)(45)(67)\dots\}$ for S_n; then $C_\Delta(S_n)$ is hamiltonian.

If we require only rules (i), (ii), and (iii) for an extent, then we have:

Cor. 19-6. An extent on n bells exists, for all n. Moreover, only three transitions are required.

Examination of Rapaport's constructive proof reveals:

Cor. 19-7. For n odd an extent exists which also satisfies condition (iv) at all but position n, where (at the worst) no bell rests for more than four successive changes.

A standard algorithm for producing all permutations of n objects from all those for $n-1$ objects leads to a sequence of symmetric extents, all with the treble plain hunting. In fact, if we take the $n-1$ objects to be all but the tenor in a ring of n bells, and set $\Delta = \{(12), (23), \dots (n-1, n)\}$ for S_n, it is not hard to visualize how the treble plain hunts (through each row of E_{n-1} replicated n times) to produce E_n, the *permutation extent* on n bells. (See [W17] for details.) Thus we have:

Thm. 19-8. A symmetric extent, with the treble plain hunting, exists for all n.

Define the *genus of the extent* E_n by: $\gamma(E_n) = \gamma(G_\Delta(S_n))$, where $\Delta = \{(12), (23), \dots, (n-1, n)\}$.

Thm. 19-9. The genus of the permutation extent on n bells is given by:

$$\gamma(E_n) = \begin{cases} 0, & \text{if } n \leq 4 \\ 6, & \text{if } n = 5 \\ 1 + \frac{(n-5)n!}{8}, & \text{if } n \geq 6. \end{cases}$$

PROOF. The result is immediate if $n = 1$ or 2. So, let $n \geq 3$. As each generator in Δ is an odd permutation in $S_n, G_\Delta(S_n)$ is a bipartite graph. Thus we obtain a genus imbedding if we first maximize r_4 and then, if r_4 equals T_4, the number of 4-cycles in $G_\Delta(S_n)$, we next maximize r_6. We produce such an imbedding, for each $n \geq 3$, as an $\frac{n!}{2}$-fold branched covering space of the spherical voltage graph imbedding, using voltage group S_n, of Figure 19-11. For either vertex v of the voltage graph, $\Gamma_v = A_5$. Thus by Theorem 10-8 the covering space consists of two homeomorphic components. We take either one of these. For $n = 3, r_4 = T_4 = 0$ and $r_6 = 2$ (see Figure 19-1). For $n = 4, r_4 =$

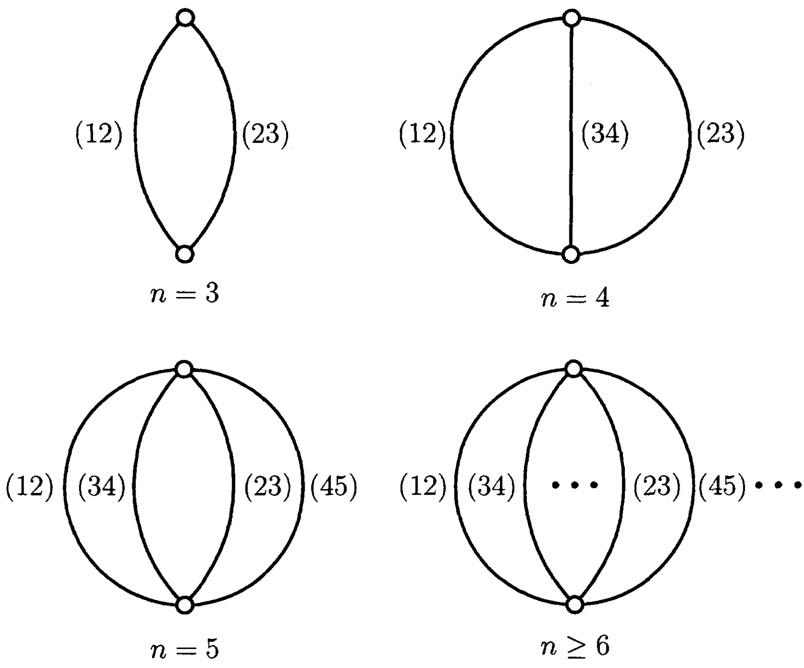

Figure 19-11.

$T_4 = 6$ and $r_6 = 8$ (see Figure 3 of [W14]). For $n = 5, r_4 = T_4 = 90$ and $r_6 = 20$. Finally, for $n \geq 6$, $r = r_4 = \frac{(n-1)n!}{4}$. The result now follows from the euler identity (Corollary 5-15), using $p = n!$ and $q = \frac{(n-1)n!}{2}$. □

These imbeddings were found also by Jacques [J3], by a different method and for a different purpose.

We say that an involution in S_n is of *type A* if it is an allowable transition for ringing changes on n bells: a product of disjoint transpositions of adjacent positions. The following is proved in [W19]: note the potential applicability to Plain Bob on n bells.

Thm. 19-10. Let $\Delta = \{a, b, c, d\}$ be a set of generating involutions for S_n, each of type A. Let $b, c, d, \in \overline{A_{n-1}} \subseteq (S_n)_1$. Let $\langle a, b\rangle = D_n$, with ab of order n. Set $X = (ab)^{n-1}ac$, $Y = (ab)^{n-1}ad$, and assume that $\Delta' = \{X, Y\}$ generates A_{n-1}. Then there is an extent on n bells using transitions from Δ and leads X and Y if and only if $C_{\Delta'}(A_{n-1})$ is hamiltonian.

Thm. 19-11. Let X and Y be as in Theorem 19-10, with the additional condition that $cd = dc$. Then $C_{\Delta'}(A_{n-1})$ is hamiltonian.

The heart of the proof is a construction due to Philip Saddleton (private communication). From $cd = dc$ it is easy to deduce that $YX^{-1} = XY^{-1}$. Then Figure 19-12 shows how to merge cycles in $C_{\Delta'}(A_{n-1})$ recursively, until a hamiltonian cycle is obtained.

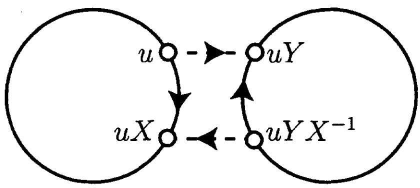

Figure 19-12.

Cor. 19-12. Under the combined hypothesis of Theorems 19-10 and 19-11, there is an extent on n bells, using transitions from Δ and leads X and Y.

We apply Corollary 19-12 to Plain Bob Major. Take $n = 8$, $a = (12)(34)(56)(78)$, $b = (23)(45)(67)$, $c = (34)(56)(78)$, and $d = (78)$. Then $X = (3578642)$ and $y = (23)(45)(678)$. From $ab = (24687531)$ and d, we find that Δ **does** generate S_8. (Use Section 1.5 of [BW1], and repeated conjugation of d by ab.) From $Y^2 = (876)$ and $XY^2 = (23574)$, giving (by conjugation) $(68z)$ in $\langle X, Y\rangle$ for $z = 7, 4, 2, 3, 5$, we see that $\langle X,\ Y\rangle = A_7$. (See 5.8.4 of [BW1].) All the other conditions of Corollary 19-12 are met. As d is a single, we deduce that an extent of Plain Bob Major can be rung on plain and single leads only. The result generalizes.

Thm. 19-13. For $n \equiv 0 \pmod 4$, an extent of Plain Bob on n bells can be rung on plain and single leads only.

Similar considerations, in [W19], establish the following two results.

Thm. 19-14. For n even ($n \geq 6$), there is no extent of Plain Bob on n bells using plain and bob leads only.

Thus a single is both necessary and sufficient for Plain Bob Major.

Thm. 19-15. For n even ($n \geq 6$), there is a Plain Bob $2n!$ on n bells (each change rung exactly twice, followed by rounds for a third time), using plain and single leads only.

The next theorem, restated for the present context, is due to Rankin [R3].

Thm. 19-16. Let group Γ be generated by $\Delta = \{x,\ y\}$, with $k = \frac{|\Gamma|}{|\langle x \rangle|}$, $\ell = \frac{|\Gamma|}{|\langle y \rangle|}$, and $m = |\langle x^{-1}y \rangle|$ odd. If $G_\Delta(\Gamma)$ is hamiltonian, then k and ℓ are both odd.

Rankin used this theorem to give a modern (1948) proof of an 1886 result of Thompson [T6], the case $n = 7$ of the more general theorem below (also based on Rankin's theorem). *Grandsire* is a popular method on $n \geq 5$ (n odd) bells having **two** bells (1 and 2) plain hunting. In [W19] we show that the plain and bob leads on Grandsire (n) are given by $P = (ba)^{n-1}(bf)$ and $B = (ba)^{n-2}(bf)^2$, where $a = (12)(34)\dots\ (n-2, n-1)$, $b = (23)(45)\dots(n-1, n)$, and $f = (12)(45)(67)\dots(n-1, n)$, and that an extent using these leads would require $C_{\{P,B\}}(A_{n-1})$ to be hamiltonian. This yields:

Thm. 19-17. For n odd ($n \geq 5$), there is no extent of Grandsire on n bells using plain and bob lead only.

We close with a result from [W19] generalizing the calling for Plain Bob Minor derived from White's No Call Doubles in Section 19-5. Let $n \geq 6$ be even. Let $a = (12)(34)\dots\ (n-1, n)$, $b = (23)(45)\dots(n-2, n-1)$; then $\langle a, b\rangle = D_n$, the hunting group on n bells. Let $c = (34)(56)\dots(n-1, n)$ form the plain lead $P = (ab)^{n-1}ac$. By Theorem 19-14, single leads are required for an extent of Plain Bob on n bells ($n \geq 6$, n even). Such an extent would be given by a calling $[f(P, S)]^{n-1}$, where $f(P, S)$ is a word in P and S (the single lead) of length $\frac{(n-2)!}{2}$—since P and S each contain $2n$ changes.

Thm. 19-18. For each calling $[f(P,S)]^{n-1}$ of Plain Bob on n bells ($n \geq 6$ and even), there is a principle on $n-1$ bells with

$$b' = (12)(34)\ldots(n-3, n-2)$$

alternating with

$$c' = (23)(45)\ldots(n-2, n-1)$$

and

$$s' = (45)(67)\ldots(n-2, n-1);$$

and conversely.

19-8. Summary

In this chapter we have attempted to illustrate the classical use of a mathematical model: a problem in a context seemingly far removed from mathematics (ringing extents on English church bells) is translated into a mathematical context (finding hamiltonian cycles in graphs); a solution is found to the mathematical problem, in appropriate situations; and that solution, when translated back into the ringing context, has relevance there. But more: there are layers to the model we use. Initially we modelled the music in question by Cayley color graphs for groups, and we found hamiltonian cycles vertex by vertex. Then we introduced the efficiency of decomposing a given group into cosets, and we found hamiltonian cycles in the corresponding Schreier coset graph, coset by coset. This is advantageous for ringers, who know–dating back to Fabian Stedman in the seventeenth century–that two cosets are either disjoint or identical. Thus, verifying that the treble leads are distinct, in an extent of Plain Bob for example, suffices to "prove" the extent; that is, to show that only rounds is repeated. This is a significant simplification for the composing of change ringing music. The final layer of the model comes when we imbed the right graph of the right group on the right surface. This can be helpful in at least four ways: (1) cycles are easier to trace out and verify when edges only intersect where they are supposed to–which does not happen when a non-planar graph is represented in the plane, for instance: (2) significant portions of the cycle can often be taken from some of the region boundaries–as was the case for Plain Bob Minimus in Section 19-3; (3) symmetries of the imbedding can suggest how to extend a partial cycle into a full one, as we did for White's No Call Doubles in Section 19-4; (4) a covering projection can reduce a complicated situation to a simple one, as we saw with Plain Bob Doubles in Section 19-4. We have illustrated these ideas further, with additional extents on $3, 4, 5, 6, 7, 8$, and n bells.

19-9. Problems

19-1.) Write out the changes of Erin Minimus, and ascertain which of the conditions (i) through (vi) are satisfied. Is Erin Minimus a method or a principle?

19-2.) (i) Repeat Problem 19-1, for Double Bob Minimus.

(ii) The word "Double" as the first part of the nomenclature for an extent indicates that the extent is identical to its reverse. Verify this property, for Double Bob Minimus. (**Hint**: in the reverse version, $a = (12)(34)$ and $b = (23)$ are unaffected, but $c = (34)$ and $d = (12)$ are interchanged.)

19-3.) Show that the vertices of a regular pentagon can be labelled so that $a = (12)(34)$ and $b = (23)(45)$ are both reflections, and that $\langle a, b\rangle = D_5$.

19-4.) Write out the plain course of Plain Bob Doubles. Check to see that the working bells all work alike.

19-5.) Let G be the graph underlying Figure 19-8. Show:

(i) $\gamma(G) = 1$ or 2.

(ii) $\tilde{\gamma}(G) = 1$ or 2.

(iii) $v(G) = 1$ or 2.

(iv) $v_1(G) = 0$ or 1.

(v) $\tilde{v}_1(G) = 0$ or 1.

19-6.) *Let $\Delta = \{a, b, d\}$, where $a = (12)(34)$, $b = (23)(45)$, and $d = (23)$.

(i) Show that $S_\Delta(S_5/\mathbb{Z}_5)$ has crossing number one.

(ii) Find **all** no-call doubles principles using Δ for the set of transitions. (**Hint**: each edge **not** used in a hamiltonian cycle for a cubic graph forces all four adjacent edges to **be** used.)

19-7.) Write out the first lead of Oxford Treble Bob Minor. Note the dodge-hunting path of the treble.

19-8.) Show that $b = (23)(45)$ and $f = (12)(45)$ generate a subgroup isomorphic to S_3, and that each path $bfbfb$ (or $fbfbf$) starting from rounds produces all six elements of that subgroup.

19-9.) Let $a = (12)(34)$, $b = (23)(45)$, and $f = (12)(45)$. If we write the plain course of Stedman Doubles as $w^5 = I$, where $w = bfbfbafbfbfa$, then by splitting each division in half we achieve a coset decomposition, of S_5 by S_3. If we use $w = fbafbfbfabfb$ instead, as ringers do in practice, we lose that feature. What do we gain in its place?

19-10.) Show that the permutation extent E_3 of Section 19-7 is quick six, and that E_4 is Double Canterbury Minimus.

19-11.) Let $\Delta = \{a, b, c, d\}$ be as in Theorem 19-10. We know from Problem 4-17 that $C_\Delta(S_n)$ is 1-factorable (by color). What is the significance of the 1-factor formed by generator b for the Plain Bob extents?

19-12.) (i) *Show that $\langle(34675), (165)(347)\rangle = A_6 \leq S_6 \cong (S_7)_2$.

(ii) Now prove the case $n = 7$ of Theorem 19-17.

19-13.) Let an extent E on n bells be given by a hamiltonian cycle in $C_\Delta(S_n)$. Define the *characteristic* of $E, \chi(E)$, to be the maximum euler characteristic among all surfaces (either orientable or nonorientable) in which $G_\Delta(S_n)$ can be imbedded.

(i) Find $\chi(E_n)$, for the permutation extents E_n of Section 19-7.

(ii) *Find $\chi(E)$, for each minimus extent E.

(iii) Set $a = (12)(34)(56)\ldots$, $b = (23)(45)(67)\ldots$, and $c = (34)(56)(78)\ldots$ in S_n. Let $\Delta = \{a, b, c\}$.

(a) Show that Δ generates S_n, and that the plain course of Plain Bob on n bells is depicted by a subgraph of $C_\Delta(S_n)$.

(b) Define $\chi(n)$, the characteristic of the plain course of Plain Bob (n), to be the characteristic of $C_\Delta(S_n)$. How would the latter be defined?

(c) Show that $\chi(4) = 1$.

(d) Find an imbedding of $C_\Delta(S_5)$ on N_5 having $r_4 = 30$, $r_8 = 15$, and $r_{10} = 12$. Then label an arbitrary vertex with rounds, and trace out the plain course of Plain Bob Doubles, using four of the decagons. Finally, use hamiltonian cycle (1) of Section 19-4 to find a hamiltonian cycle in $C_{\Delta'}(S_5)$ corresponding to Plain Bob Doubles where $\Delta' = \Delta \cup \{(23)\}$.

(e) *Show that $\chi(5) = -3$.

(f) **Prove or disprove: for $n \geq 4$,

$$\chi(n) = -(n-2)!\frac{n^2 - 5n + 2}{4}.$$

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INDEX OF SYMBOLS

Note: The number following the symbol gives the first page on which the symbol appears. In the top half of the list, corresponding symbols are displayed along side one another. The six distinguished symbols correspond to the six relationships depicted in Figure 0-1.

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INDEX OF DEFINITIONS