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Magic Graphs

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Preface

Magic labelings

Magic squares are among the more popular mathematical recreations. Their origins are lost in antiquity; over the years, a number of generalizations have been proposed. In the early 1960s, Sedláček asked whether "magic" ideas could be applied to graphs.

Shortly afterward, Kotzig and Rosa formulated the study of graph labelings, or *valuations* as they were first called. A labeling is a mapping whose domain is some set of graph elements — the set of vertices, for example, or the set of all vertices and edges — whose range was a set of positive integers. Various restrictions can be placed on the mapping. The case that we shall find most interesting is where the domain is the set of all vertices and edges of the graph, and the range consists of positive integers from 1 up to the number of vertices and edges. No repetitions are allowed.

In particular, one can ask whether the set of labels associated with any edge – the label on the edge itself, and those on its endpoints — always add up to the same sum. Kotzig and Rosa called such a labeling, and the graph possessing it, *magic*. To avoid confusion with the ideas of Sedláček and the many possible variations, we would call it an *edge-magic total labeling*. A related concept, a *vertex-magic total labeling*, is one in which the label on any vertex and the labels on the edges containing it are always constant. Any labeling that has both these properties (usually with two different constants) is called *totally magic*.

Magic labelings were studied briefly, but they were overshadowed by other graph valuations, in particular by *graceful labelings*, which have a number of applications and are related to the problems of decomposing graphs into trees. There has been a resurgence of interest in the last decade.

One reason for this resurgence is the deceptively simple question, "does every tree have an edge-magic total labeling?" Although it is so easy to ask, no progress has been made toward answering this question for three decades. A number of interesting results about other families of graphs have been discovered, but trees remain elusive.

Several mathematicians have become intrigued by the problem of discovering which graphs are magic. In studying this, a number of small theorems of combinatorics and graph theory have been used. For example, in the text, we need to discuss coloring problems and Vizing's Theorem. The construction of magic arrays other than squares is needed. Small structural graph-theoretic theorems need to be invented. And some of the problems seem to be deeper, and more difficult, than one would at first expect.

Some applications have been studied, mainly in network-related areas. Suppose it is required to assign addresses to the possible links in a communications network. It is required that the addresses all be different, and that the address of a link be deduced from the identities of the two nodes linked, without having the need to use a lookup table. This has been modeled using edge-magic labelings. Another application is in the construction of *ruler models*, which have been applied to the study of *radar pulse codes*.

However, the main reasons for a monograph studying magic labelings are threefold:

1. Magic labelings provide an introduction to the more general topic of graph labelings. (I thought of writing on generalizations such as anti-magic labelings, and on the subject of graph labelings more generally, but it is a very big subject and I don't think the same reader would be interested in all aspects of it. Moreover, a comprehensive book would be massive.)

2. A focussed book, on one particular problem such as this, is a good guide for graduate students beginning research, so they can see how new mathematics comes into existence. In fact, I used the draft version of this book as notes for a graduate "special topics" course. Students see some small graph-theoretic proofs and get some idea of how different areas of graph theory interact (as, for example, when Vising's Theorem on the edge-chromatic number is used).

3. In recent years a number of researchers have found the topic fascinating; unfortunately, they have not all communicated very well with each other, and I hope this volume will obviate unnecessary repetition of intellectual effort and help unify notation, which is currently diverse and self-contradictory.

About this book

The book begins with a survey of the main ingredients. Magic properties are introduced by a discussion of magic squares, also touching on the related Latin squares and on Latin rectangles, and the basics of graph theory are covered briefly. We then define graph labelings, and magic labelings in particular. The first chapter also includes a brief sketch of applications. Subsequent chapters explore the three main types of magic labelings — edge-magic, vertex-magic and totally magic — in turn.

Throughout the text there are exercises and research problems. The exercises are designed to aid understanding. Some are quite easy; some ask the reader to do a complete search for labelings of a particular graph or labelings of a particular type; a few are quite difficult. Some of the research problems require very little work, but a few are substantial. A brief commentary on the research problems is included in the volume.

There is an extensive bibliography, and solutions to the majority of the exercises. The book closes with an index, in which the convention has been followed of italicizing the entries where a definition occurs.

Some knowledge of groups and fields is assumed in the preliminary chapter, in the discussion of magic squares and Latin squares, but these details can be skipped if desired. Most readers will have a background in graph theory, but a summary has been provided. So there are not many mathematical prerequisites. However, the reader is assumed to have some mathematical maturity, to understand proofs, and to use matrices and modular arithmetic with reasonable facility.

Acknowledgments

The study of magic labelings owes a great deal to Alex Rosa and to the late Anto Kotzig, and I am happy to acknowledge that I have learned a great deal of combinatorics from both gentlemen.

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Preliminaries

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1.1.1 Magic squares

Magic squares are among the best known mathematical recreations. Their origins are lost in antiquity. A classical reference is [1], while one of the better recent books is [40].

A *magic square* of side n is an $n \times n$ array whose entries are an arrangement of the integers $\{1, 2, \ldots, n^2\}$, in which all elements in any row, any column, or either the main diagonal or main back-diagonal, add to the same sum. Examples include

1	15	8	10
12	6	13	3
14	4	11	5
7	9	2	16

1	7	13	19	25
18	24	5	6	12
10	11	17	23	4
22	3	9	15	16
14	20	21	2	8

34	6	5	32	1	33
9	25	11	8	30	28
15	18	20	23	19	16
22	24	14	17	13	21
27	7	26	29	12	10
4	31	35	2	36	3

Variations in the set of entries have frequently been studied — for example, one might ask that the entries all be primes, or all be perfect squares —

but we shall only need to discuss cases in which the entries are the first n^2 positive integers. In fact, we usually do not need the constancy of the diagonal and back-diagonal. We shall say a square is *RCmagic* if all row-sums and column-sums equal the same constant.

1.1.2 Latin squares

Latin squares are useful in constructing magic squares, and will arise in some other contexts, so we shall briefly survey the main definitions. For more details, see any design theory text, such as [53] or [9]. A *Latin square* of *order* (or *side*) n is an $n \times n$ array based on some set S of n symbols, with the property that every row and every column contains every symbol exactly once. In other words, every row and every column is a permutation of S. Since the arithmetical properties of the symbols are not used, the nature of the elements of S is immaterial, but it is usually most convenient to use $\{1, 2, \ldots, n\}$.

As an example, let G be a finite group with n elements $g_1, g_2, \ldots, g_n$. Define an array $A = (a_{ij})$ by $a_{ij} = k$ where $g_k = g_i g_j$. The elements of column j of A are the $g_i g_j$, where g_j is fixed. Now g_j has an inverse element g_j^{-1} in G. If i and k are any two integers from 1 to n, then $g_i g_j = g_k g_j$ would imply $g_i g_j g_j^{-1} = g_k g_j g_j^{-1}$, whence $g_i = g_k$, so $i = k$. The elements $g_1 g_j, g_2 g_j, \ldots, g_n g_j$ are different. So column j contains a permutation of $\{1, 2, \ldots, n\}$. A similar proof applies to rows. A is a Latin square. This example gives rise to infinitely many Latin squares, including one of every positive integer order. For example, at orders 1, 2, and 3 we have

$$\boxed{1} \qquad \boxed{\begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array}} \qquad \boxed{\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{array}}$$

However, the majority of Latin squares do not come from groups.

Two Latin squares A and B of the same side n are called *orthogonal* if the n^2 ordered pairs (a_{ij}, b_{ij}) — the pairs formed by superimposing one square on the other — are all different. We say "A is orthogonal to B" or "B is orthogonal to A" — clearly, the relation of orthogonality is symmetric. More generally, one can speak of a set of *k-mutually orthogonal* Latin squares: squares $A_1, A_2, \ldots, A_k$ such that A_i is orthogonal to A_j whenever $i \neq j$.

A set of n cells in a Latin square is called a *transversal* if it contains one cell from each row and one cell from each column and if the n entries include every symbol precisely once. Using this idea we get an alternative definition of orthogonality: A and B are orthogonal if and only if for every symbol x in A, the n cells where A has entry x form a transversal in B. The symmetry of the concept is not so obvious when this definition is used.

If n is a prime power, there exists a set of $n-1$ mutually orthogonal Latin squares of side n. The standard construction can be described as follows. Suppose the elements of the n-element field are $x_1, x_2, \ldots, x_{n-1} = -1$, $x_n = 0$. The squares are $A_1, A_2, \ldots, A_{n-1}$, and A_k has (i, j) entry m where $x_m = x_i + x_k x_j$.

There can never be more than $n - 1$ squares of side n in a mutually orthogonal set (except in the trivial case $n = 1$). It is unknown whether this upper bound can ever be achieved when n is not a prime power, but we know there exists a pair of orthogonal squares of every order other than 1, 2 and 6, and a set of three for all orders other than 1, 2, 3, 6 and possibly 10.

Orthogonal Latin squares have been used in the construction of magic squares. If A and B are orthogonal Latin squares of side n based on $\{1, 2, \ldots, n\}$, the array with (i, j) entry $(a_{ij} + n(b_{ij} - 1))$ is RCmagic. If A, B and C are mutually orthogonal Latin squares and C has all diagonal entries equal and all back-diagonal entries equal, the array formed from A and B is magic. The magic square of order 4, given in Section 1.1.1, was constructed in this way from the following A, B, C:

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

1	4	2	3
3	2	4	1
4	1	3	2
2	3	1	4

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

For odd orders, the square C cannot have both its main and back-diagonals constant, because they have a common member. But if $n = 2t - 1$, one can find orthogonal squares A and B, where A has constant main diagonal $(t, t, \ldots, t)$ and B has constant back-diagonal $(t, t, \ldots, t)$, and $(a_{ij} + n(b_{ij} - 1))$ is magic.

The magic square of order 5 in Section 1.1.1 was constructed from orthogonal Latin squares, but the order 6 example obviously was not; it was taken from [40].

Exercise 1.1 *Construct a pair of orthogonal Latin squares of order 3, and a magic square of order 3.*

Exercise 1.2 *An $n \times n$ array A is called* circulant *if $a_{i+1,j+1} \equiv a_{ij} + 1 \pmod n$ in every case. Use a circulant construction to find a pair of orthogonal Latin squares, and a magic square, of every odd order.*

1.1.3 Magic rectangles

Magic rectangles are a generalization of magic squares. A *magic rectangle* $A = (a_{ij})$ of size $m \times n$ is an $m \times n$ array whose entries are $\{1, 2, \ldots, mn\}$, each appearing once, with all its row sums, and all its column sums, equal. The sum of all entries in the array is $\frac{1}{2}mn(mn + 1)$; it follows that

$$\begin{aligned} \sum_{i=1}^{m} a_{ij} &= \tfrac{1}{2}n(mn + 1), \text{ all } j, \\ \sum_{j=1}^{n} a_{ij} &= \tfrac{1}{2}m(mn + 1), \text{ all } j, \end{aligned}$$

so m and n must either both be even or both be odd. It was shown in [23, 24] that such an array exists whenever m and n have the same parity, except for the impossible cases where exactly one of m and n is 1, and for $m = n = 2$. Simpler constructions appear in [22].

Magic rectangles of size $3 \times m$

For our purposes we need the existence of a $3 \times m$ magic rectangle for every odd m, so for completeness we present constructions (from [22]) for this case. The construction uses smaller arrays as building blocks. For positive integers n and i, define

$$C(i) = \begin{array}{|c|c|} \hline i+1 & m+1-i \\ \hline \frac{1}{2}(3m+1)+i & \frac{1}{2}(3m+1)-i \\ \hline 3n-2i & 2n+2i \\ \hline \end{array},$$

$$D(i) = \begin{array}{|c|c|} \hline 3n-2i & 2n+2i \\ \hline \frac{1}{2}(3m+1)+i & \frac{1}{2}(3m+1)-i \\ \hline i+1 & m+1-i \\ \hline \end{array}.$$

Theorem 1.1 *There exists a $3 \times m$ magic rectangle whenever m is odd.*

Proof. First suppose $m \equiv 1 (\text{mod}\, 4)$. Define

$$A = \begin{array}{|c|c|c|} \hline 1 & 2n & \frac{1}{2}(n+3) \\ \hline 3n & \frac{1}{2}(n+1) & n+1 \\ \hline \frac{1}{2}(3n+1) & 2n+1 & 3n-1 \\ \hline \end{array}.$$

Then a $3 \times m$ magic rectangle is formed by adjoining to A the $\frac{1}{4}(n-5)$ arrays $C(i) : 1 \leq i \leq \frac{1}{4}(n-5)$ and the $\frac{1}{4}(n-1)$ arrays $D(j) : \frac{1}{4}(n-1) \leq j \leq \frac{1}{2}(n-3)$.

When $m \equiv 3 (\text{mod}\, 4)$, the case $m = 3$ is easy to construct. So assume $m > 3$, and define t to be $\lfloor \frac{1}{3}(n+1) \rfloor$. Then $n - 3m = 1, 0$ or -1. Define

$$A = \begin{array}{|c|c|} \hline 1 & \frac{1}{2}(n+3) \\ \hline 3n & n+1 \\ \hline \frac{1}{2}(3n+1) & 3n-1 \\ \hline \end{array},$$

and define an array B to equal

$m+1$	$2n+1$	$2n+2m$	
$\frac{1}{2}(3n+1)+m$	$\frac{1}{2}(n+1)$	$\frac{1}{2}(3n+1)-m$	when $n-3m=1$,
$3n-2m$	$2n$	$n+1-m$	

$m+1$	$2n$	$2n+2m$	
$\frac{1}{2}(3n+1)+m$	$\frac{1}{2}(n+1)$	$\frac{1}{2}(3n+1)-m$	when $n-3m=0$,
$3n-2m$	$2n+1$	$n+1-m$	

$3n-2m$	$2n+1$	$n+1-m$	
$\frac{1}{2}(3n+1)+m$	$\frac{1}{2}(n+1)$	$\frac{1}{2}(3n+1)-m$	when $n-3m=-1$.
$m+1$	$2n$	$2n+2m$	

The required magic rectangle is formed by adjoining to A, B and $D(1)$ the $\frac{1}{4}(n-7)$ arrays $C(i), 2 \le i \le \frac{1}{4}(n-3)$ and the $\frac{1}{4}(n-7)$ arrays $D(j), \frac{1}{4}(n+1) \le j \le \frac{1}{2}(n-3), j \ne m$. □

Exercise 1.3 *Show that the rectangles constructed in Theorem 1.1 are magic.*

Kotzig arrays

We define a *Kotzig array of size* $m \times n$ to be an $m \times n$ matrix, each row of which is a permutation of $\{0, 1, 2, \ldots, n-1\}$, in which every column has the same sum. The common sum must be $\frac{m}{n}\binom{n}{2} = \frac{1}{2}m(n-1)$, which is an integer only if m is even or n is odd.

An example of a $2 \times n$ Kotzig array is

$$\begin{bmatrix} 0 & 1 & \ldots & n-2 & n-1 \\ n-1 & n-2 & \ldots & 2 & 1 \end{bmatrix}. \tag{1.1}$$

A $3 \times n$ example can exist only if n is odd. If $n = 2r + 1$, the following example is due to Kotzig [27] (which is why we chose the name):

$$\begin{bmatrix} 0 & 1 & \dots & r & r+1 & r+2 & \dots & 2r \\ 2r & 2r-2 & \dots & 0 & 2r-1 & 2r-3 & \dots & 1 \\ r & r+1 & \dots & 2r & 0 & 1 & \dots & r-1 \end{bmatrix}. \tag{1.2}$$

Arrays with more rows can be constructed by appending copies (1.1) and (1.2).

Lemma 1.2 *There is a Kotzig array of size $m \times n$ whenever $m > 1$ and $m(n-1)$ is even.*

One application is another construction for some magic rectangles. For example, we have the following

Theorem 1.3 *There exists a $3 \times m$ magic rectangle whenever m is odd and $m \equiv 0 \bmod 3$.*

Proof. Write $m = 3n = 6r + 3$, and denote by A the $3 \times n$ array (1.2). For $j = 1, 2, \dots, s$, columns $3j - 2$, $3j - 1$ and $3j$ of the magic rectangle are

$9j + 1$	$9j + 6$	$9j + 8$
$9(2r - 2j) + 9$	$9(2r - 2j) + 2$	$9(2r - 2j) + 4$
$9(r + j) + 5$	$9(r + j) + 7$	$9(r + j) + 3$

(reduced modulo $3m$ when necessary). In other words, the entries are derived from a 3×3 RCmagic square by adding 9 times the element from the relevant row in column j of A. □

Exercise 1.4 *What orders of magic rectangles can be derived from Kotzig arrays using the ideas of Theorem 1.3?*

1.2 Graphs

The basic ideas of graph theory will be surveyed here, primarily to ensure that writer and readers use the terminology in the same way. For further details, proofs, etc., consult a book on the subject (recent examples include [54] and [58]).

A *graph* G consists of a finite set $V(G)$ of objects called *vertices* together with a set $E(G)$ of unordered pairs of vertices; the elements of $E(G)$ are called *edges*. We write $v = v(G)$ and $e = e(G)$ for the orders of $V(G)$ and $E(G)$, respectively; these are called the *order* and *size* of G. In terms of the more general definitions sometimes used, we can say that "our graphs are finite and contain neither loops nor multiple edges." A *multigraph* is defined in the same way as a graph except that there may be more than one edge corresponding to the same unordered pair of vertices. Unless otherwise mentioned, all definitions pertaining to graphs will be applied to multigraphs in the obvious way.

The edge containing x and y is written xy or (x, y); x and y are called its *endpoints*. We say this edge *joins* x to y. $G - xy$ denotes the result of deleting edge xy from G; if x and y were not adjacent, then $G + xy$ is the graph constructed from G by adjoining an edge xy. Similarly, $G - x$ is the graph derived from G by deleting one vertex x (and all the edges on which x lies). Similarly, $G - S$ denotes the result of deleting some set S of vertices.

If vertices x and y are endpoints of one edge in a graph or multigraph, then x and y are said to be *adjacent* to each other, and it is often convenient to write $x \sim y$. The set of all vertices adjacent to x is called the *neighborhood* of x, and denoted $N(x)$. We define the *degree* or *valency* $d(x)$ of the vertex x to be the number of edges that have x as an endpoint. If $d(x) = 0$, x is an *isolated* vertex, and a vertex x with $d(x) = 1$ is a *leaf* or *pendant vertex*. A graph is called *regular* if all its vertices have the same degree; in particular, if the common degree is 3, the graph is called *cubic*. We write $\delta(G)$ for the smallest of all degrees of vertices of G, and $\Delta(G)$ for the largest. (One also writes $\Delta(G)$ for the common degree of a regular graph G.) If G has v vertices, so that its vertex-set is, say,

$$V(G) = \{x_1, x_2, \dots, x_v\},$$

then its *adjacency matrix* M_G is the $v \times v$ matrix with entries m_{ij}, such that

$$m_{ij} = \begin{cases} 1 & \text{if } x_i \sim x_j, \\ 0 & \text{otherwise.} \end{cases}$$

A vertex and an edge are called *incident* if the vertex is an endpoint of the edge, and two edges are called incident if they have a common endpoint. A set of edges is called *independent* if no two of its members are incident, while a set of vertices is independent if no two of its members are adjacent.

Theorem 1.4 *In any graph or multigraph, the number of edges equals half the sum of the degrees of the vertices.*

Corollary 1.4.1 *In any graph or multigraph, the number of vertices of odd degree is even. In particular, a regular graph of odd degree has an even number of vertices.*

Given a set S of v vertices, the graph formed by joining all pairs of members of S is called the *complete* graph on S, and denoted K_S. We also write K_v to mean any complete graph with v vertices. The set of all edges of $K_{V(G)}$ that are *not* in a graph G will form a graph with $V(G)$ as a vertex-set; this new graph is called the *complement* of G, and written $\overline{G}$. More generally, if G is a subgraph of H, then the graph formed by deleting all edges of G from H is called the *complement of* G *in* H, denoted $H-G$. The complement $\overline{K}_S$ of the complete graph K_S on vertex-set S is called a null graph; we also write $\overline{K}_v$ for a null graph with v vertices.

An *isomorphism* of a graph G onto a graph H is a one-to-one map ϕ from $V(G)$ onto $V(H)$ with the property that a and b are adjacent vertices in G if and only if $a\phi$ and $b\phi$ are adjacent vertices in H; G is isomorphic to H if and only if there is an isomorphism of G onto H. From this definition it follows that all complete graphs on n vertices are isomorphic. The notation K_n can be interpreted as being a generic name for the typical representative of the isomorphism class of all n-vertex complete graphs.

If G is a graph, it is possible to choose some of the vertices and some of the edges of G in such a way that these vertices and edges again form a graph, say H. H is then called a *subgraph* of G; one writes $H \leq G$. Clearly every graph G has itself and the 1-vertex graph (which we shall denote K_1) as subgraphs; we say H is a *proper* subgraph of G if it equals neither G nor K_1. If U is any set of vertices of G, then the subgraph consisting of U and all the edges of G that joined two vertices of U is called an *induced* subgraph, the *subgraph induced by* U, and is denoted $\langle U \rangle$. A subgraph G of a graph H is called a *spanning* subgraph if $V(G) = V(H)$.

A graph is called *disconnected* if its vertex-set can be partitioned into two subsets, V_1 and V_2, which have no common element, in such a way that there is no edge with one endpoint in V_1 and the other in V_2; if a graph is not disconnected, then it is *connected.* A disconnected graph consists of a number of disjoint subgraphs; a maximal connected subgraph is called a *component.*

The *complete bipartite graph* on V_1 and V_2 has two disjoint sets of vertices, V_1 and V_2; two vertices are adjacent if and only if they lie in different sets. We write $K_{m,n}$ to mean a complete bipartite graph with m vertices in one set and n in the other. $K_{1,n}$ in particular is called an n*-star*; the vertex of degree n is called the *center*. Any subgraph of a complete bipartite graph is called "bipartite."

More generally, the *complete* r*-partite graph* $K_{n_1,n_2,\ldots,n_r}$ has vertex-set $V_1 \cup V_2 \cup \ldots \cup V_r$, where the V_i are disjoint sets and V_i has order n_i, and xy is an edge if and only if x and y are in different sets. Any subgraph of this graph is called an r-partite graph. If $n_1 = n_2 = \ldots = n_r = n$, we use the abbreviation $K_n^{(r)}$.

A *walk* in a graph G is a finite sequence of vertices $x_0, x_1, \ldots, x_n$ and edges $a_1, a_2, \ldots, a_n$ of G:

$$x_0, a_1, x_1, a_2, \ldots, a_n, x_n,$$

where the endpoints of a_i are x_{i-1} and x_i for each i. A *simple walk* is a walk in which no edge is repeated. A *path* is a walk in which no vertex is repeated; the *length* of a path is its number of edges. A walk is *closed* when the first and last vertices, x_0 and x_n, are equal. A *cycle* of length n is a closed simple walk of length n, $n \geq 3$, in which the vertices $x_0, x_1, \ldots, x_{n-1}$ are all different.

A graph that contains no cycles at all is called *acyclic*; a connected acyclic graph is called a *tree*. Clearly all trees are bipartite graphs. Vertices of a tree other than leaves are called *internal.* A union of disjoint trees is called a *forest.*

It is clear that the set of vertices and edges that constitute a path in a graph is itself a graph. We define a path P_n to be a graph with n vertices $x_1, x_2, \ldots, x_n$ and $n-1$ edges $x_1x_2, x_2x_3, \ldots, x_{n-1}x_n$. A cycle C_n is defined similarly, except that the edge $x_n x_1$ is also included, and (to avoid the triviality of allowing K_2 to be defined as a cycle) n must be at least 3. The latter convention ensures that every C_n has n edges.

A cycle that passes through every vertex in a graph is called a *Hamilton cycle* and a graph with such a cycle is called *Hamiltonian*.

If G is any graph, then a *factor* or *spanning subgraph* of G is a subgraph with vertex-set $V(G)$. A *factorization* of G is a set of factors of G that are pairwise *edge-disjoint* — no two have a common edge — and whose union is all of G.

Every graph has a factorization, quite trivially: since G is a factor of itself, $\{G\}$ is a factorization of G. However, it is more interesting to consider factorizations in which the factors satisfy certain conditions. In particular a *one-factor* is a factor that is a regular graph of degree 1. In other words, a one-factor is a set of pairwise disjoint edges of G that between them contain every vertex. A *one-factorization* of G is a decomposition of the edge-set of G into edge-disjoint one-factors. similarly a *two-factor* is a factor that is a regular graph of degree 2 — a union of disjoint cycles — and a *two-factorization* of G is a decomposition of the edge-set of G into edge-disjoint two-factors.

1.3 Labelings

A *labeling* (or *valuation*) of a graph is a map that carries graph elements to numbers (usually to the positive or non-negative integers). The most common choices of domain are the set of all vertices and edges (such labelings are called *total* labelings), the vertex-set alone (*vertex*-labelings), or the edge-set alone (*edge*-labelings). Other domains are possible.

In many cases, it is interesting to consider the sum of all labels associated with a graph element. This will be called the *weight* of the element. For example, the weight of vertex x under labeling λ is

$$wt(x) = \lambda(x) + \sum_{y \sim x} \lambda(xy),$$

and the weight of the edge xy is

$$wt(xy) = \lambda(x) + \lambda(xy) + \lambda(y).$$

If necessary, the labeling can be specified by a subscript, as in $wt_\lambda(x)$.

We shall define two labelings of the same graph to be *equivalent* if one can be transformed into the other by an automorphism of the graph.

The most complete recent survey of graph labelings is [16].

1.4 Magic labeling

Various authors have introduced labelings that generalize the idea of a magic square. Sedláček [45] defined a graph to be magic if it had an edge-labeling, with range the real numbers, such that the sum of the labels around any vertex equals some constant, independent of the choice of vertex. These labelings have been studied by Stewart (see, for example, [48]), who called a labeling *supermagic* if the labels are consecutive integers, starting from 1. Several others have studied these labelings; a recent reference is [17]. Some writers simply use the name "magic" instead of "supermagic" (see, for example, [25]).

Kotzig and Rosa [29] defined a magic labeling to be a total labeling in which the labels are the integers from 1 to $|V(G)|+|E(G)|$. The sum of labels on an edge and its two endpoints is constant. In 1996 Ringel and Llado [44] redefined this type of labeling (and called the labelings *edge-magic*, causing some confusion with papers that have followed the terminology of [31], mentioned below); see also [18]. Recently Enomoto et al. [11] introduced the name *super edge-magic* for magic labelings in the sense of Kotzig and Rosa, with the added property that the v vertices receive the smaller labels $\{1, 2, \ldots, v\}$. (To avoid confusion with the earlier use of "super," we shall call such labelings *strong*.)

In 1983, Lih [32] introduced magic labelings of planar graphs where labels extended to faces as well as edges and vertices, an idea which he traced back to 13th century Chinese roots. Bača (see, for example, [2, 3]) has written extensively on these labelings. A somewhat related sort of magic labeling was defined by Dickson and Rogers in [10].

Lee, Seah and Tan [31] introduced a weaker concept, which they called *edge-magic*, in 1992. The edges are labeled and the sums at the vertices are required to be congruent modulo the number of vertices.

Total labelings have also been studied in which the sum of the labels of all edges adjacent to the vertex x, plus the label of x itself, is constant. The first paper on these labelings was [34].

To clarify the terminological confusion described above, we define a labeling to be *edge-magic* if the sum of all labels associated with an edge equals a constant independent of the choice of edge, and *vertex-magic* if the same property holds for vertices. (This terminology could be extended to other substructures: face-magic, for example.) The domain of the labeling is specified by a modifier on the word "labeling." We shall always require that the labeling be an injection (one-to-one map) *onto* the appropriate set of consecutive integers starting from 1. For example, Stewart studies vertex-magic edge-labelings, and Kotzig and Rosa define edge-magic total labelings. Our main interest is in edge-magic total labelings, which we sometimes abbreviate to EMTLs, and vertex-magic total labelings (VMTLs). We shall use the term *magic injection* (edge-magic injection, and so on) to denote a labeling with the magic property in which the labels are required to be positive integers but no upper bound is required (and therefore the map is not necessarily onto). If m is the largest label used in a magic injection, and the graph has v vertices and e edges, the difference $m - v - e$ is called the *deficiency* of the injection.

It is conceivable that the same labeling could be both vertex-magic and edge-magic for a given graph (not necessarily with the same constant). In that case the labeling and the graph are called *totally magic*. Totally magic graphs appear to be very rare.

1.5 Some applications of magic labelings

1.5.1 Efficient addressing systems

The problem of efficient addressing systems is introduced in [5]. Suppose it is necessary to assign addresses to the possible links in a communications network. It is required that the addresses all be different, and that the address of a link be deduced from the identities of the two nodes linked, without having to use a lookup table.

The solution proposed in [5] is as follows. First, a graph (the *underlying graph* of the network) is constructed with the nodes as vertices and edges between all pairs of nodes where a link is provided. The graph vertices are labeled in such a way that the differences between endpoints of edges are all distinct – this is called a *semi-graceful* labeling. Then the address of a link is the difference between the labels on its endpoints.

The following solution, using an edge-magic total labeling, provides some additional information. Suppose the underlying graph has an edge-magic total labeling λ with edge-weight k. The nodes and links are assigned the labels of the corresponding vertices and edges. Then the address of the link from x to y is readily calculated as $k - \lambda(x) - \lambda(y)$. Moreover, a unique address is available for messages from the system operator to the nodes: the link from the system operator to node x receives address $\lambda(x)$.

If no edge-magic total labeling of the underlying graph is available, one can use an edge-magic injection. An injection with small excess will lead to a smaller range of possible addresses.

1.5.2 Ruler models

Suppose λ is an edge-magic injection of K_n. It is easy to see that, for any vertices x, y, z, t of K_n, $\lambda(xy) = k - \lambda(x) - \lambda(y)$ and $\lambda(zt) = k - \lambda(z) - \lambda(t)$, so $\lambda(x)+\lambda(y)$ cannot equal $\lambda(z)+\lambda(t)$. So $\lambda(x)-\lambda(z) \neq \lambda(t)-\lambda(y)$. Therefore the $\binom{n}{2}$ differences between the labels of K_n are all different.

If λ is any labeling of K_n, a *ruler model* of λ is constructed as follows. For each vertex of K_n, place a mark distance $\lambda(x)$ from the start of the ruler. The ruler can be used to measure all distances corresponding to the distance between two marks. Ruler models are discussed, for example, in [4] and [5].

The ruler models derived from edge-magic injections have the following special property. No two of the measurable differences are equal, except possibly for differences of the form $\lambda(x) - \lambda(y)$ and $\lambda(y) - \lambda(z)$. This is very nearly the semi-graceful property discussed in the preceding section, so magic sublabelings can be used in many of the situations where semi-graceful labelings have been applied.

One important application of semi-graceful labelings is to *radar pulse codes*. Radar distance ranging is accomplished by transmitting a pulse or train of pulses and waiting for its return after reflection. Only a small fraction of the transmitted energy ever returns to the detector. Because accuracy in measuring target distance is determined by accuracy in measuring the time until the reflected signal is received, it is desirable to have a very narrow transmitted radar pulse whose moment of return can be measured precisely. High energy pulses are broader than low energy pulses. On the other hand, a low energy pulse may be too weak to detect after reflection. The solution is to send a set of low amplitude, narrowly defined pulses, and to increase the total energy when necessary by increasing the number of pulses.

The most efficient way to proceed is to time the pulses in accordance with the marks on a ruler derived from a semi-graceful labeling. An array of detectors is distributed like a template of the transmitted pulse-train. The signal will match up precisely with the detectors when it returns. At no other time will more than one signal excite a detector. If the ruler from a magic sublabeling is used, there may be two detectors excited at once (if $\lambda(x) - \lambda(y) = \lambda(y) - \lambda(z)$ occurs) but it is still very unlikely that the reflected pulse will be misidentified even if part of the signal is lost through dispersion.

In this application, the duration of the train will be minimized if the size of the largest label is minimized, so again sublabelings with small excess are better.

2
Edge-Magic Total Labelings

2.1 Basic ideas

2.1.1 *Definitions*

Definition. An *edge-magic total labeling* on a graph G is a one-to-one map λ from $V(G) \cup E(G)$ onto the integers $1, 2, \ldots, v + e$, where $v = |V(G)|$ and $e = |E(G)|$, with the property that, given any edge (xy),

$$\lambda(x) + \lambda(xy) + \lambda(y) = k$$

for some constant k. In other words, $wt(xy) = k$ for any choice of edge xy. k is called the *magic sum* of G. Any graph with an edge-magic total labeling will be called *edge-magic*.

As an example of edge-magic total labelings, Figure 2.1 shows an edge-magic total labeling of $K_4 - e$.

An edge-magic total labeling will be called *strong* if it has the property that the vertex labels are the integers $1, 2, \ldots, v$, the *smallest* possible labels. A graph with a strong edge-magic total labeling will be called *strongly edge-magic*.

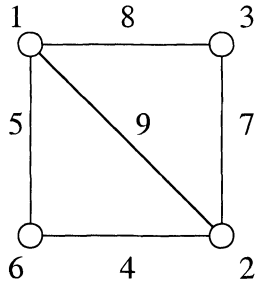

Figure 2.1. An edge-magic total labeling of $K_4 - e$ with $k = 12$.

2.1.2 *Some elementary counting*

As a standard notation, assume the graph G has v vertices $\{x_1, x_2, \ldots, x_v\}$ and e edges. For convenience, we always say vertex x_i has degree d_i and receives label a_i. As we shall frequently refer to the sum of consecutive integers, we define

$$\sigma_i^j = (i+1) + (i+2) + \cdots j = i(j-i) + \binom{j-i+1}{2}. \qquad (2.1)$$

The necessary conditions in order that $\{a_1, a_2, \ldots, a_v\} = \lambda(V(G))$, where λ is an edge-magic total labeling of a graph G with magic sum k, are

(i) $a_h + a_i + a_j = k$ cannot occur if any two of x_i, x_j, x_k are adjacent;

(ii) the sums $a_i + a_j$, where $x_i x_j$ is an edge, are all distinct;

(iii) $0 < k - (a_i + a_j) \leq v + e$ when x_i is adjacent to x_j.

Suppose λ is a magic labeling of a given graph. If x and y are adjacent vertices, then edge xy has label $k - \lambda(x) - \lambda(y)$. Since the sum of all these labels plus the sum of all the vertex labels must equal the sum of the first $v + e$ positive integers, k is determined. So the vertex labels specify the complete labeling.

Of course, not every possible assignment will result in an edge-magic labeling: the above process may give a non-integral value for k, or give repeated labels.

Among the labels, write S for the set $\{a_i : 1 \leq i \leq v\}$ of vertex labels, and s for the sum of elements of S. Then S can consist of the v smallest labels, the v largest labels, or somewhere in between, so

$$\sigma_0^v \leq s \leq \sigma_e^{v+e},$$

$$\binom{v+1}{2} \leq s \leq ve + \binom{v+1}{2}. \tag{2.2}$$

Clearly, $\sum_{xy \in E}(\lambda(xy) + \lambda(x) + \lambda(y)) = ek$. This sum contains each label once, and each vertex label a_i an additional $d_i - 1$ times. So

$$ke = \sigma_0^{v+e} + \sum (d_i - 1)a_i. \tag{2.3}$$

If e is even, every d_i is odd and $v+e \equiv 2(\bmod\ 4)$, then (2.3) is impossible. We have

Theorem 2.1 [44] *If G has e even and $v + e \equiv 2(\mathrm{mod} 4)$, and every vertex of G has odd degree, then G has no edge-magic total labeling.*

(A generalization of this theorem will be proven in Section 2.2).

Corollary 2.1.1 *The complete graph K_v is not magic when $v \equiv 4(\mathrm{mod}\ 8)$. The n-spoke wheel W_n, formed from C_n by adding a new vertex and joining it to every existing vertex, is not magic when $n \equiv 3(\mathrm{mod}\ 4)$.*

(We shall see in Section 2.3.2 that K_n is never magic for $n > 6$, so the first part of the corollary really only eliminates K_4.)

If G is strongly edge-magic, then S must consist of the first v integers. In many cases equation (2.3) can be used to show that this is impossible, because there is no assignment of these integers to the vertices such that $\sigma_0^{v+e} + \sum (d_i - 1)a_i$ is divisible by e. For example, for the cycle C_v where v is even, (2.3) is

$$kv = \sigma_0^{2v} + \sigma_0^{v} = v(2v + 1) + \tfrac{1}{2}v(v + 1),$$

so $k = 2v + 1 + \frac{1}{2}(v + 1)$, which is not integral. Thus no even cycle is strongly edge-magic.

Exercise 2.1 *Prove that the graph tK_4, consisting of t disjoint copies of K_4, has no edge-magic total labeling when t is odd. Is the same true of tK_v when $v \equiv 4(\mathrm{mod}\ 8)$?*

Exercise 2.2 *Prove that the union of t n-wheels, tW_n, has no edge-magic total labeling when t is odd and $n \equiv 3(\mathrm{mod}\ 4)$.*

Research Problem 2.1 *Investigate graphs G for which equation* (2.3) *implies the non-existence of an edge-magic total labeling of $2G$.*

Equation (2.3) may be used to provide bounds on k. Suppose G has v_j vertices of degree j, for each i up to Δ, the largest degree represented in G. Then the ke cannot be smaller than the sum obtained by applying the v_Δ smallest labels to the vertices of degree Δ, the next smallest values to the vertices of degree $\Delta - 1$, and so on; in other words,

$$\begin{aligned} ke \quad \geq \quad & (d_\Delta - 1)\sigma_0^{v_\Delta} + (d_{\Delta-1} - 1)\sigma_{v_\Delta}^{v_\Delta+(v_{\Delta-1})} + \cdots \\ & + \sigma_{v_\Delta+(v_{\Delta-1})+\cdots v_3}^{v_\Delta+(v_{\Delta-1})+\cdots v_2} + \binom{v+e+1}{2}. \end{aligned}$$

An upper bound is achieved by giving the *largest* labels to the vertices of highest degree, and so on.

Exercise 2.3 *Suppose a regular graph G of degree d is edge-magic. Prove*

$$ke = (d-1)s + \sigma_0^{v+e} = (d-1)s + \tfrac{1}{2}(v+e)(v+e+1), \qquad (2.4)$$

$$kdv = 2(d-1)s + (v+e)(v+e+1). \qquad (2.5)$$

2.1.3 Duality

Given a labeling λ, its *dual* labeling λ' is defined by

$$\lambda'(x_i) = (v+e+1) - \lambda(x_i),$$

and for any edge xy,

$$\lambda'(xy) = (v+e+1) - \lambda(xy).$$

It is easy to see that if λ is a magic labeling with magic sum k, then λ' is a magic labeling with magic sum $k' = 3(v+e+1) - k$. The sum of vertex labels is $s' = v(v+e+1) - s$.

Either s or s' will be less than or equal to $\frac{1}{2}v(v+e+1)$. This means that, in order to see whether a given graph has an edge-magic total labeling, it suffices to check either all cases with $s \leq \frac{1}{2}v(v+e+1)$ or all cases with $s \geq \frac{1}{2}v(v+e+1)$ (equivalently, either check all cases with $k \leq \frac{3}{2}(v+e+1)$ or all with $k \geq \frac{3}{2}(v+e+1)$).

2.2 Graphs with no edge-magic total labeling

All the results of this section are from the unpublished technical report [26].

2.2.1 Main theorem

Theorem 2.2 *Suppose G is a graph with v vertices and e edges, where e is even, and suppose every vertex of G has odd degree. Select a positive integer δ such that for each vertex x_i*

$$d(x_i) = 2^\delta d_i + 1$$

for some non-negative integer d_i. If $T = \sum d_i$, define τ and Q by

$$T = 2^\tau \cdot Q, \quad \tau \text{ integral, } Q \text{ odd.}$$

If G has an edge-magic total labeling λ, then

$$\begin{aligned}
\tau = 0 \quad &\Rightarrow \quad v = 2^\delta V \text{ for some } V \equiv 1 \bmod 2 \\
&\qquad\; e = 2^\delta E \text{ for some } E \equiv 1 \bmod 2 \\
\tau = 1 \quad &\Rightarrow \quad v = 2^{\delta+1} V' \text{ for some } V' \equiv 1 \bmod 2 \\
&\qquad\; e = 2^{\delta+1} E' \text{ for some } E' \equiv 1 \bmod 2 \\
\tau \geq 2 \quad &\Rightarrow \quad 2^{\delta+2} \text{ divides } v \text{ and } 2^{\delta+2} \text{ divides } e
\end{aligned}$$

Proof. Since e is even, let us write $e = 2^\nu E$ for some odd integer E. The familiar equation $2e = \sum d(x_i)$ yields

$$2e = v + 2^\delta T,$$

so

$$v + e = 3e - 2^{\delta+\tau} \cdot Q = 2^\nu \cdot 3E - 2^{\delta+\tau} \cdot Q.$$

So (2.3) is

$$(2^\nu \cdot 3E - 2^{\delta+\tau} \cdot Q)(2^\nu \cdot 3E - 2^{\delta+\tau} \cdot Q + 1)/2 = 2^\nu \cdot Ek - 2^\delta \cdot \sum d_i \lambda(x_i),$$

which is of the form

$$2^\delta \cdot (2R - 2^\tau \cdot X) = 2^\nu \cdot Y$$

where $R = \sum d_i \lambda(x_i)$, $X = (2^{\delta+\tau} Q - 3 \cdot 2^{\nu+1} - 1) Q$ and $Y = 2Ek - 2^\nu \cdot 9E^2 - 3E$. The actual values of X and Y are unimportant; what matters is that both are odd, and the result follows. □

2.2.2 Forests

Theorem 2.3 *Suppose G is a forest with c component trees and suppose G satisfies the conditions of Theorem 2.2. Then*

$$\begin{aligned}
\tau = 0 &\;\Rightarrow\; c \equiv 0 \bmod 2^{\delta+1} \text{ and } e \equiv 2^{\delta} \bmod 2^{\delta+1}\\
\tau = 1 &\;\Rightarrow\; c \equiv 0 \bmod 2^{\delta+2} \text{ and } e \equiv 2^{\delta+1} \bmod 2^{\delta+2}\\
&\quad\text{or}\;\; c \equiv 2^{\delta+1} \bmod 2^{\delta+2} \text{ and } e \equiv 0 \bmod 2^{\delta+2}\\
\tau \geq 2 &\;\Rightarrow\; c \equiv e \equiv x \bmod 2^{\delta+2} \text{ where } x = 0 \text{ or } 2^{\delta+1}.
\end{aligned}$$

Proof. For a forest, $v = c + e$; the results follow from Theorem 2.2. □

A family $\mathcal{F}$ of forests is defined as follows. If δ is any positive integer, a δ-tree is a tree in which each vertex x has degree $d(x_i) = 1 + 2^{\delta} d_i$ for some non-negative integer d_i, and a δ-forest is a forest *with an even number of edges* in which every component tree is a δ-tree. Figure 2.2 shows a 2-forest.

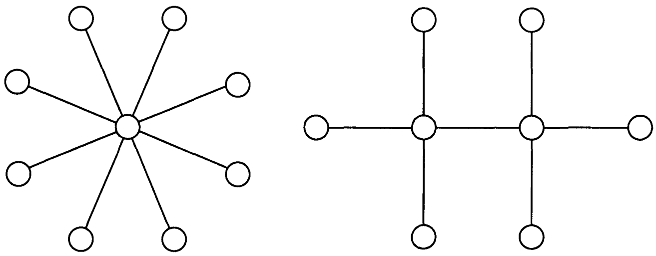

Figure 2.2. A 4-forest.

If F is a δ-forest, define $\tau(F) = \tau$ to be the non-negative integer such that $\sum_i d_I = 2^{\tau} \cdot T$ where T is odd. Then $\mathcal{F}_{\delta}$ is the set of all δ-forests F that satisfy one of the following conditions, where e is the total number of edges and c is the number of component trees of F:

(i) $\tau(F) = 0$ and $c \not\equiv 0 \bmod 2^{\delta+1}$ or $e \not\equiv 0 \bmod 2^{\delta}$;

(ii) $\tau(F) = 1$ and $c \not\equiv 0 \bmod 2^{\delta+1}$ or $e \not\equiv 0 \bmod 2^{\delta+1}$ or $c + e \not\equiv 2^{\delta+1} \bmod 2^{\delta+2}$;

(iii) $\tau(F) \geq 2$ and $c \not\equiv e \bmod 2^{\delta+2}$ or $c \not\equiv 0 \bmod 2^{\delta+1}$.

Finally, $\mathcal{F}$ is the union of all the sets $\mathcal{F}_{\delta}$.

Theorem 2.4 *For every non-negative integer τ there are infinitely many forests F belonging to $\mathcal{F}$ that have $\tau(F) = \tau$.*

Proof. Select natural numbers p, q and t such that $p \geq q$. F_{pqt} is the disjoint union of $(2t+1)\cdot 2^q$ copies of the star with 2^p+1 edges. This forest has $c = (2t+1)\cdot 2^q$ components. Vertex x_i of F_{pqt} has degree $2^p d_i + 1$, where $d_i = 0$ or 1 and $\sum d_i = (2t+1)\cdot 2^q$. In terms of the definition of $\mathcal{F}$, $\delta = p$ and $\tau = q$. Clearly c is not divisible by 2^{p+1}, since the largest power of 2 dividing c is 2^q, so $F_{pqt} \in \mathcal{F}$. Thus, for any natural number τ we have constructed infinitely many members of $\mathcal{F}$ with that τ-value.

For $\tau = 0$, consider the graph F_n consisting of the disjoint union of a 5-star with $(4n-3)$ 3-stars. One vertex has degree $2^1\cdot 2+1$ ($d_i = 2$), $4n-3$ vertices have degree $2^1\cdot 1+1$ ($d_i = 1$), and the other vertices have degree $2^1\cdot 0+1$ ($d_i = 0$). So $\delta = 1$ and $\tau = 0$. Moreover $c = 4x - 2$ is not divisible by $4 = 2^{\delta+1}$, so $F_x \in \mathcal{F}$, and we have constructed infinitely many members of $\mathcal{F}$ that have $\tau = 0$. □

It is clear from Theorem 2.3 that a member of $\mathcal{F}$ cannot have an edge-magic total labeling. So we have:

Corollary 2.4.1 *There are infinitely many forests with no edge-magic total labeling.*

2.2.3 Regular graphs

Theorem 2.5 *Suppose G is a regular graph of odd degree d with v vertices and e edges, where e is even. Write*

$$d = 2^r s + 1$$

where s is an odd positive integer and r is a positive integer. If G is edge-magic, then 2^{r+2} divides v.

Proof. As usual, $2e = dv$. Since e is even and d is odd, 4 divides d; write $v = 4V$, so that $e = 2dV$. Then (2.3) becomes

$$(4V + 2dV)(4V + 2dV + 1)/2 = 2dVk - (d-1)\sum_x \lambda(x),$$

and

$$V[(d+2)(4V+2dV+1)-2dk] = 2^r s) \sum_x \lambda(x).$$

The coefficient of V on the left-hand side is odd, so 2^r divides V, giving the result. □

Remark. This theorem could also be expressed as follows. If a regular graph with v vertices is edge-magic, then

$$\begin{array}{lcl} d \equiv 3(\bmod 4) & \Rightarrow & v \equiv 2(\bmod 4) \text{ or } v \equiv 0(\bmod 8) \\ d \equiv 5(\bmod 8) & \Rightarrow & v \equiv 2(\bmod 4) \text{ or } v \equiv 0(\bmod 16) \\ & \cdots & \\ d \equiv 2^{p-1}+1(\bmod 2^p) & \Rightarrow & v \equiv 2(\bmod 4) \text{ or } v \equiv 0(\bmod 2^{p+1}). \end{array}$$

A family $\mathcal{R}$ of regular graphs is now defined as follows. If G is regular of degree d and has v vertices, then $G \in \mathcal{R}$ if and only if there exist natural numbers α, β, γ and δ satisfying

$$\begin{array}{rcl} \delta & < & 2^\alpha, \\ d & = & 2^{\alpha+1}\beta - 2^\alpha + 1, \\ v & = & 4(2^\alpha \cdot \gamma - \delta). \end{array}$$

Theorem 2.6 *No member of $\mathcal{R}$ is edge-magic.*

Proof. Suppose α, β, γ and δ are natural numbers and $\delta < 2^\alpha$. Consider a regular graph G of degree $d = 2^{\alpha+1}$ with $v = 2^\alpha \cdot \beta - 2^\alpha + 1$ vertices (a typical element of $\mathcal{R}$). Writing $r = \alpha$ and $s = 2\beta - 1$ we have $d = 2^r s + 1$ so by Theorem 2.5 G can have an edge-magic total labeling only if v is divisible by $2^{r+2} = 2^{\alpha+2}$. However $v \equiv -4\delta \not\equiv 0(\bmod 2^{\alpha+2})$ as $0 < \delta < 2^\alpha$. So G has no edge-magic total labeling. □

Lemma 2.7 *If d is any odd natural number greater than 1, there are infinitely many graphs of degree d that belong to $\mathcal{R}$.*

Remark. Lemma 3 of [26] allows the case $d = 1$. The first line of the proof there is "For the case $d = 1$, lemma 4 (*sic*) follows immediately from theorem 3." (His Theorem 3 is our Theorem 2.5.) However, if $d = 1$, the definition of $\mathcal{R}$ would require $2^{\alpha+1}\beta = 2^\alpha$, which is impossible for natural β.

Proof of Lemma 2.7. We construct solutions in the special case $\delta = 1$ (so that $\delta < 2^\alpha$ is obviously true).

Assume d is an odd positive integer greater than 1. There exist unique integers α and b such that b is odd and $d = 2^\alpha b$. Define $\beta = (b-1)/2$. Then $d = 2^{\alpha+1}\beta - 2^\alpha + 1$.

If d is any natural number and v is any even integer greater than d, there exists a regular graph of degree d on v vertices (select any one-factorization of K_v and delete any $v - d - 1$ factors). If d is odd, then among the even integers greater than d, there will be infinitely many of the form $4(2^\alpha \cdot \gamma - 1)$ where α was defined above, and each gives rise to a member of $\mathcal{R}$. □

Corollary 2.7.1 *If d is any odd natural number, there are infinitely many regular graphs of degree d that have no edge-magic total labeling.*

Proof. For $d = 1$, the graphs eK_2, e even, suffice. For larger values of d, use Lemma 2.7. □

2.3 Cliques and complete graphs

2.3.1 Sidon sequences and labelings

Suppose the graph G has an edge-magic total labeling λ, and suppose G contains a complete subgraph (or *clique*) H on n vertices. Let us write $x_1, x_2, \dots, x_n$ for the vertices of H, and denote $\lambda(x_i)$ by a_i. Without loss of generality we can assume the names x_i to have been chosen so that $a_1 < a_2 < \cdots < a_n$.

If k is the magic sum, then $\lambda(x_i x_j) = k - a_i - a_j$, so the sums $a_i + a_j$ must all be distinct. This property is called being *well spread*; in particular, a *Sidon sequence* or *well spread sequence* $A = (a_1, a_2, \dots, a_n)$ of length n is a sequence with the properties:

1. $0 < a_1 < a_2 < \cdots < a_n$;
2. If a_i, a_j, a_k, a_ℓ are all different then $a_i + a_j \neq a_k + a_\ell$.

Such sequences are related to the work of S. Sidon [46]. Their study was initiated by Erdös and Turan [13]. A survey of work on them appears in [21].

In discussing Sidon sequences (or, equivalently, cliques in edge-magic graphs), the difference $d_{ij} = |a_j - a_i|$ (the absolute difference between the labels on the endpoints of the edge $x_i x_j$) will be important.

Lemma 2.8 *Suppose A is a Sidon sequence of length n. If $d_{ij} = d_{pq}$, then $\{a_i, a_j\}$ and $\{a_p, a_q\}$ have a common member. No three of the differences d_{ij} are equal.*

Proof. Suppose $d_{ij} = d_{pq}$. We can assume that $i > j$ and $p > q$. Without loss of generality we can also assume $p \geq i$. Then $a_i - a_j = a_p - a_q$, so $a_i + a_q = a_j + a_p$. Therefore $a_i = a_q$ and $i = q$ ($a_j = a_p$ is impossible), and $p > i > j$ — the common element is the middle one in order of magnitude.

Now suppose three pairs have the same difference. By the above reasoning there are two possibilities: the pairs must have a common element, or form a triangle. In the former case, suppose the differences are d_{ij}, d_{ik} and $d_{i\ell}$. From $d_{ij} = d_{ik}$ we must have either $k > i > j$ or $j > i > k$; let's assume the former. Then $d_{ij} = d_{i\ell}$ implies $i > j > \ell$. So j is greater than both k and ℓ. But $d_{ik} = d_{i\ell}$ must mean that either $k > j > \ell$ or $\ell > j > k$, both of which are impossible. On the other hand, form a triangle, say $d_{ij} = d_{ik} = d_{jk}$. We can assume $i > j$. Then $d_{ij} = d_{jk}$ implies $i > j > k$, and $j > k$ and $d_{ik} = d_{jk}$ imply $k > i$, again a contradiction. □

Lemma 2.9 *Suppose A is a Sidon sequence of length n. If $d_{ij} = d_{ik}$, then $d_{ij} \leq \frac{1}{2} d_{1n}$.*

Proof. Suppose $d_{ij} = d_{ik}$, and assume $j < k$. Then $d_{jk} = a_k - a_j = a_i - a_j + a_k - a_i = d_{ij} = d_{ik} = 2d_{ij}$. But $a_1 \leq a - j$ and $a_k \leq a_n$, so $d_{jk} \leq d_{1n}$, giving the result. □

Theorem 2.10 *In any Sidon sequence of length n, $\binom{n}{2} \leq \lfloor \frac{3}{2} d_{1n} \rfloor$, or equivalently $d_{1n} \geq \lceil \frac{1}{3} n(n-1) \rceil$.*

Proof. There are $\binom{n}{2}$ unordered pairs of elements in the sequence, so there are $\binom{n}{2}$ differences. From Lemmas 2.8 and 2.9, the collection of values of these differences can contain the integers $1, 2, \dots, \lfloor \frac{1}{2}r \rfloor$ at most twice each, and $\lfloor \frac{1}{2}r \rfloor + 1, \dots, d_{1n}$ at most once each. The result follows. □

2.3.2 *Complete graphs*

Theorem 2.11 [51] *Suppose K_v has an edge-magic total labeling with magic sum k. The number p of vertices that receive even labels satisfies the following conditions:*

(i) *If $v \equiv 0$ or $3 \pmod 4$ and k is even then $p = \frac{1}{2}(v - 1 \pm \sqrt{v+1})$.*

(ii) *If $v \equiv 1$ or $2 \pmod 4$ and k is even then $p = \frac{1}{2}(v - 1 \pm \sqrt{v-1})$.*

(iii) *If $v \equiv 0$ or $3 \pmod 4$ and k is odd then $p = \frac{1}{2}(v + 1 \pm \sqrt{v+1})$.*

(iv) *If $v \equiv 1$ or $2 \pmod 4$ and k is odd then $p = \frac{1}{2}(v + 1 \pm \sqrt{v+3})$.*

Proof. Suppose λ is an edge-magic total labeling of K_v with magic sum k. Let V_e denote the set of all vertices x such that $\lambda(x)$ is even, and V_o the set of vertices x with $\lambda(x)$ odd; define p to be the number of elements of V_e. Write E_1 for the set of edges with both endpoints in the same set, either V_o or V_e, and E_2 for the set of edges joining the two vertex-sets, so that $|E_1| = \binom{p}{2} + \binom{v-p}{2}$ and $|E_2| = p(v-p)$.

If k is even, then $\lambda(yz)$ is even whenever yz is an edge in E_1 and odd when yz is in E_2, so there are precisely $p + \binom{p}{2} + \binom{v-p}{2}$ even labels. But these labels must be the even integers from 1 to $\binom{v+1}{2}$, taken once each; so

$$p + \binom{p}{2} + \binom{v-p}{2} = \left\lfloor \frac{1}{2}\binom{v+1}{2} \right\rfloor. \tag{2.6}$$

If $\binom{v+1}{2}$ is even, this equation has solutions $p = \frac{1}{2}(v - 1 \pm \sqrt{v+1})$, while $\binom{v+1}{2}$ odd gives solutions $p = \frac{1}{2}(v - 1 \pm \sqrt{v-1})$.

If k is odd, the edges in E_1 are those that receive the odd labels, and instead of 2.6 we have

$$p + p(v-p) = \left\lfloor \frac{1}{2}\binom{v+1}{2} \right\rfloor \tag{2.7}$$

with solutions $p = \frac{1}{2}(v+1 \pm \sqrt{v+1})$ when $\binom{v+1}{2}$ is even and $p = \frac{1}{2}(v+1 \pm \sqrt{v+3})$ when $\binom{v+1}{2}$ is odd.

Using the fact that $\binom{v+1}{2}$ is even when $v \equiv 0$ or $3 \pmod 4$ and odd otherwise, we have the result. □

Now p must be an integer, so the functions whose roots are taken must always be perfect squares. Therefore:

Corollary 2.11.1 *Suppose K_v has an edge-magic total labeling. If $v \equiv 0$ or $3 \pmod 4$, then $v+1$ is a perfect square. If $v \equiv 1$ or $2 \pmod 4$, then either $v-1$ is a perfect square and the magic sum of the labeling is even, or $v+3$ is a perfect square and the magic sum of the labeling is odd.*

This corollary rules out edge-magic total labelings of K_4 (again! — see Corollary 2.1.1) and K_7, as well as infinitely many larger values. The larger values, however, will all be excluded by the next theorem.

Suppose there is an edge-magic total labeling of K_v, where $v \geq 8$. The vertex labels will form a Sidon sequence of length v, say A. Let us denote the edge labels $b_1, b_2, \ldots, b_e$, where $b_1 < b_2 < \cdots < b_e$; of course, e $= \binom{v}{2}$. If the magic sum is k, then

$$\begin{aligned} k &= a_1 + a_2 + b_e && (2.8)\\ &= a_1 + a_3 + b_{e-1} && (2.9)\\ &= a_v + a_{v-1} + b_1 && (2.10)\\ &= a_v + a_{v-2} + b_2. && (2.11)\end{aligned}$$

Subtracting (2.8) from (2.9),

$$a_3 - a_2 = b_e - b_{e-1}, \tag{2.12}$$

while (2.10) and (2.11) yield

$$a_{v-1} - a_{v-2} = b_2 - b_1. \tag{2.13}$$

Suppose labels 1, 2, $v+e-1$ and $v+e$ were all edge labels. Then $b_1 = 1, b_2 = 2, b_{e-1} = v+e-1$ and $b_e = v+e$. So, from (2.12) and (2.13), $a_3 - a_2 = a_{v-1} - a_{v-2} = 1$. But 2, 3, $v-2$ and $v-1$ are all distinct, so this contradicts Lemma 2.8. So one of $1, 2, v+e-1, v+e$ is a vertex label. Without loss of generality we can assume either 1 or 2 is a vertex label (otherwise, the dual labeling will have this property). So $a_1 = 1$ or 2.

Equations (2.8) and (2.10) give

$$a_v = b_e - (a_{v-1} - a_2) - (b_1 - a_1). \tag{2.14}$$

Since $(a_2, a_3, \ldots, a_{v-1})$ is a Sidon sequence of length $v - 2$ (any subsequence of a Sidon sequence is also well spread), Lemma 2.10 applies to it, and $(a_{v-1} - a_2) \geq \lceil \frac{1}{3}(v - 2)(v - 3) \rceil$, which is at least 10 because $v \geq 8$. Also $(b_1 - a_1) \geq -1$ ($b - 1$ is at least 1 and a_1 is at most 2), and $b_e \leq v + e$. So, from (2.14),

$$a_v \leq v + e - 9.$$

So the six largest labels are all edge labels:

$$b_{e-5} = v + e - 5, b_{e-4} = v + e - 4, \ldots, b_e = v + e.$$

From (2.8) and (2.9) we get

$$k = a_1 + a_2 + v + e = a_1 + a_3 + v + e - 1,$$

so $a_3 = a_2 + 1$. The next smallest sum of two vertex labels, after $a_1 + a_2$ and $a_1 + a_3$, may be either $a_2 + a_3$ or $a_1 + a_4$.

If it is $a_2 + a_3$, then

$$k = a_2 + a_3 + v + e - 2$$

and by comparison with (2.9), $a_2 = a_1 + 1$. The next smallest sum is $a_1 + a_4$, so

$$k = a_1 + a_4 + v + e - 3$$

and $a_4 = a_3 + 2$. Two cases arise. If $a_1 = 1$, then $a_2 = 2, a_3 = 3, a_4 = 5$. Also, a_5 cannot equal 6, because that would imply $a_1 + a_5 = 7 = a_2 + a_4$, contradicting the well spread property. Every integer up to $v+e$ must occur as a label, so $b_1 = 4$ and $b_2 = 6$. So (2.13) is $a_{v-1} - a_{v-2} = b_2 - b_1 = 2$. But $a_4 - a_3 = 2$, so $d_{v-1,v-2} = d_{34}$, in contradiction of Lemma 2.8. In the other case, $a_1 = 2$, we obtain $a_2 = 3, a_3 = 4, a_4 = 6$, so $b_1 = 1, b_2 = 5$, and $a_{v-1} - a_{v-2} = 4 = a_4 - a_1$, again a contradiction.

If $a_1 + a_4$ is the next smallest difference, we have

$$k = a_1 + a_4 + v + e - 2,$$

so $a_4 = a_3 + 1$. If $a_1 = 1$ and $a_2 = 3$, it is easy to see that $b_1 = 2, b_2 = 6$, and we get the contradiction $a_{v-1} - a_{v-2} = a_4 - a_1 = 4$. Otherwise $a_2 \geq 4$, so 3 is an edge-label. If $a_1 = 1$, then $b_1 = 2, b_2 = 3$, and $a_{v-1} - a_{v-2} = 1 = a_3 - a_2$. If $a_1 = 2$, then $b_1 = 1, b_2 = 3$, and $a_{v-1} - a_{v-2} = 2 = a_4 - a_2$. In every case, a contradiction is obtained. So we have

Theorem 2.12 *The complete graph K_v does not have an edge-magic total labeling if $v > 6$.*

This theorem was first proven in [30] (see also [29]) but the above proof follows that in [8].

2.3.3 All edge-magic total labelings of complete graphs

The proof in the preceding section used the fact that $v > 7$. There are edge-magic total labelings for all smaller complete graphs except K_4 and K_7 (which were excluded by Corollary 2.1.1 and Corollary 2.11.1). A complete search has been made for smaller orders, and we now list all edge-magic total labelings for complete graphs.

In each case we list the possible values for the magic sum k, the corresponding sum of vertex labels s, and the set S of vertex labels that realize that value s and give an edge-magic total labeling.

The values to be considered are determined by equations (2.2) and (2.4). For K_v, $e = \binom{v}{2}$. So (2.2) becomes $\frac{1}{2}v(v+1) \leq s \leq v(2v+3)$, while (2.4) is

$$k = \frac{v(v+1)(v^2+v+2) + 8(v-2)s}{4v(v-1)}.$$

For example, when $v = 6$, we have $15 \leq s \leq 90$ and $k = \frac{1}{15}(231 + 4s)$. As k is an integer, $s \equiv 6 (\mathrm{mod}\, 15)$, and the possibilities are $s = 21, 36, 51, 66, 81$, $k = 21, 25, 29, 33, 37$.

- K_2 is trivially possible. Label 1, 2 or 3 can be given to the edge; in each case $k = 6$.

- K_3 The magic sums to be considered are $k = 9, 10, 11, 12$.

$$\begin{array}{llll}
k = 9, & s = 6, & S = \{1, 2, 3\}. \\
k = 10, & s = 9, & S = \{1, 3, 5\}. \\
k = 11, & s = 12, & S = \{2, 4, 6\}. \\
k = 12, & s = 15, & S = \{4, 5, 6\}.
\end{array}$$

- K_4 No solutions, by Corollary 2.1.1.

- K_5 The magic sums to be considered are $k = 18, 21, 24, 27, 30$. Theorem 2.11 tells us that no solutions exist when k is odd, so only 18, 24 and 30 are listed.

$$\begin{array}{llll}
k = 18, & s = 20, & S = \{1, 2, 3, 5, 9\}. \\
k = 24, & s = 40, & S = \{1, 8, 9, 10, 12\}. \\
k = 24, & s = 40, & S = \{4, 6, 7, 8, 15\}. \\
k = 30, & s = 60, & S = \{7, 11, 13, 14, 15\}.
\end{array}$$

- K_6 The magic sums to be considered are $k = 21, 25, 29, 33, 37, 41, 45$.

$$\begin{array}{llll}
k = 21, & s = 21, & \text{no solutions.} \\
k = 25, & s = 36, & S = \{1, 3, 4, 5, 9, 14\}. \\
k = 29, & s = 51, & S = \{2, 6, 7, 8, 10, 18\}. \\
k = 33, & s = 66, & \text{no solutions.} \\
k = 37, & s = 81, & S = \{4, 12, 14, 15, 16, 20\}. \\
k = 41, & s = 96, & S = \{8, 11, 17, 18, 19, 21\}. \\
k = 45, & s = 111, & \text{no solutions.}
\end{array}$$

2.3.4 *Complete subgraphs*

If $A = (a_1, a_2, \ldots, a_n)$ is any Sidon sequence of length n, we define

$$\begin{aligned}
\sigma(A) &= a_n - a_1 + 1 \\
\rho(A) &= a_n + a_{n-1} - a_2 - a_1 + 1 \\
&= \sigma(A) + a_{n-1} - a_2 \\
\sigma^*(n) &= \min \sigma(A) \\
\rho^*(n) &= \min \rho(A)
\end{aligned}$$

where the minima are taken over all Sidon sequences A of length n. σ is called the *size* of the sequence. Without loss of generality one can assume

$a_1 = 1$ when constructing a sequence, and then the size equals the largest element.

We now revert to the more general case, where the graph G has an edge-magic total labeling λ and G contains a complete subgraph H on n vertices. $x_1, x_2, \ldots, x_n$ are the vertices of H, and $a_i = \lambda(x_i)$. We assume $a_1 < a_2 < \cdots < a_n$, so $A = (a_1, a_2, \ldots, a_n)$ is a Sidon sequence of length n. Then

$$\lambda(x_n x_{n-1}) = k - a_n - a_{n-1},$$

and since $\lambda(x_n x_{n-1})$ is a label,

$$k - a_n - a_{n-1} \geq 1. \tag{2.15}$$

Similarly

$$\lambda(x_2 x_1) = k - a_2 - a_1,$$

and since $\lambda(x_2 x_1)$ is a label,

$$k - a_2 - a_1 \leq v + e. \tag{2.16}$$

Combining (2.15) and (2.16) we have

$$v + e \geq a_n + a_{n-1} - a_2 - a_1 + 1 = \rho(A) \geq \rho^*(n).$$

Theorem 2.13 [30] *If the edge-magic graph G contains a complete subgraph with n vertices, then the number of vertices and edges in G is at least $\rho^*(n)$.*

Exercise 2.4 *Suppose $G = K_n + tK_1$. In other words, G consists of a K_n together with t isolated vertices. Prove that if G is edge-magic, then*

$$t \geq \rho^*(n) - n - \binom{n}{2}.$$

From Theorem 2.13, $v(K_n + tK_1) + e(K_n + tK_1) \geq \rho^*(n)$. But $v(K_n + tK_1) = t + n$ and $e(K_n + tK_1) = \binom{n}{2}$.

The **magic number** $M(n)$ of K_n is defined to be the smallest t such that $K_n + tK_1$ is edge-magic. So the preceding exercise shows that

$$M(n) \geq \rho^*(n) - n - \binom{n}{2}.$$

Evaluation, bounds

In view of the above theorem, it is interesting to know more about Sidon sequences. Some bounds on $\sigma^*(n)$ and $\rho^*(n)$ are known:

Theorem 2.14 [28] $\sigma^*(n) \geq 4 + \binom{n-1}{2}$ *when* $n \geq 7$.

The proof appears in [28].

Theorem 2.15 [28] $\rho^*(n) \geq 2\sigma^*(n-1)$ *when* $n \geq 4$.

Proof. Consider the sequences

$$\begin{aligned} A &= (a_1, a_2, \ldots, a_n) \\ B &= (a_1, a_2, \ldots, a_{n-1}) \\ C &= (a_2, a_3, \ldots, a_n) \end{aligned}$$

where $n \geq 4$. Clearly

$$\begin{aligned} \rho^*(n) &\geq \rho(A) \\ &= a_n + a_{n-1} - a_2 - a_1 + 1 \\ &= (a_n - a_2 + 1) + (a_{n-1} - a_1 + 1) - 1 \\ &= \sigma(B) + \sigma(C) - 1 \\ &\geq 2\sigma^*(n-1) - 1. \end{aligned}$$

Moreover, equality can apply only if $\sigma(B) = \sigma(C) = \sigma^*(n-1)$. But

$$\begin{aligned} \sigma(B) = \sigma(C) &\Rightarrow a_n - a_2 = a_{n-1} - a_1 \\ &\Rightarrow a_{n-1} + a_2 = a_n + a_1, \end{aligned}$$

which is impossible for a Sidon sequence A. Since σ^* and ρ^* are integral,

$$\rho^*(n) \geq 2\sigma^*(n-1).$$ □

Exercise 2.5 *Prove that, when* $n \geq 7$,

$$\rho^*(n) \geq n^2 - 5n + 14. \tag{2.17}$$

From Theorems 2.15 and 2.14, $\rho^*(n) \geq 2\sigma^*(n-1) \geq 2(4 + \binom{n-2}{2}) = 8 + (n-2)(n-3) = n^2 - 5n + 14$.

In practice, values of $\sigma^*(n)$ and $\rho^*(n)$ have been calculated using an exhaustive, backtracking approach. The following result proves helpful in restricting the search for $\rho^*(n)$, once some σ^* values are known.

Theorem 2.16 *Suppose the sequence* $A = (1, x, \ldots, y, z)$ *satisfies* $\rho(A) = \rho^*(n)$, *and suppose* B *is any sequence for which* $\rho(B)$ *is known. Then*

$$\sigma^*(n) \leq z \leq \rho(B) - \sigma^*(n-2) + 1$$

and

$$x \leq (\rho(B) - \sigma^*(n-2) + 1) - \sigma^*(n-1) + 1.$$

Proof. Since $(x, \ldots, y)$ is a Sidon sequence,

$$y - x + 1 \geq \sigma^*(n-2).$$

But

$$\rho^*(n) = z + y - x$$

so

$$\begin{aligned} z &= \rho^*(n) - (y - x) \\ &\leq \rho^*(n) - \sigma^*(n-2) + 1 \\ &\leq \rho(B) - \sigma^*(n-2) + 1. \end{aligned}$$

Also $(x, \ldots, y, z)$ is Sidon, so

$$z - x \geq \sigma^*(n-1) - 1$$

and the second part of the theorem follows from the upper bound for z.

We know the following small values of the two functions. The values for $n \leq 8$ are calculated in [28] and listed in [30].

$$\begin{aligned} \sigma^*(3) &= 3 & \qquad \rho^*(3) &= 3 \\ \sigma^*(4) &= 5 & \rho^*(4) &= 6 \\ \sigma^*(5) &= 8 & \rho^*(5) &= 11 \\ \sigma^*(6) &= 13 & \rho^*(6) &= 19 \\ \sigma^*(7) &= 19 & \rho^*(7) &= 30 \\ \sigma^*(8) &= 25 & \rho^*(8) &= 43 \\ \sigma^*(9) &= 35 & \rho^*(9) &= 62 \\ \sigma^*(10) &= 46 & \rho^*(10) &= 80 \\ \sigma^*(11) &= 58 & \rho^*(11) &= 110 \\ \sigma^*(12) &= 72 & \rho^*(12) &= 137 \end{aligned}$$

Sample sequences attaining the σ^* values are:

$\sigma^*(1)$ through $\sigma^*(6)$: 1 2 3 5 8 13 (or part thereof);

$\sigma^*(7)$: 1 2 3 5 9 14 19;

$\sigma^*(8)$: 1 2 3 5 9 15 20 25;

$\sigma^*(9)$: 1 2 3 5 9 16 25 30 35;

$\sigma^*(10)$: 1 2 8 11 14 22 27 42 44 46;

$\sigma^*(11)$: 1 2 6 10 18 32 34 45 52 55 58;

$\sigma^*(12)$: 1 2 3 8 13 23 38 41 55 64 68 72.

The same sequences attain $\rho^*(n)$ for $n, = 1, 2, 3, 4, 5, 6, 8$. For the other values, examples are

$\rho^*(7)$: 1 6 8 10 11 14 22;

$\rho^*(9)$: 1 5 7 9 12 17 26 27 40;

$\rho^*(10)$: 1 2 3 5 9 16 25 30 35 47;

$\rho^*(11)$: 1 2 3 5 9 16 25 30 35 47 65.

$\rho^*(12)$: 1 3 5 8 11 21 30 39 51 62 63 77.

Note: The only other sequence of length 7 with $\rho = 30$ is 1, 9, 12, 13, 15, 17, 22.

From Theorem 2.16, we have

For $n = 7$	$x \leq 12$	$19 \leq z \leq 24$
For $n = 8$	$x \leq 13$	$25 \leq z \leq 31$
For $n = 9$	$x \leq 21$	$35 \leq z \leq 45$
For $n = 10$	$x \leq 30$	$46 \leq z \leq 64$
For $n = 11$	$x \leq 32$	$58 \leq z \leq 77$
For $n = 12$	$x \leq 36$	$72 \leq z \leq 93$

and these bounds were used in calculating the example sequences for $\rho^*(n)$ when $n \geq 7$.

A greedy approach

Here is a simple observation. If $(a_1, a_2, \ldots, a_{n-1})$ is well spread, then none of its sums can exceed $a_{n-2} + a_{n-1}$. Put $a_n = a_{n-1} + a_{n-2}$. Then all the sums $a_i + a_n$ are new, and (since the sequence is strictly monotonic) they are all different. So we have a new well spread sequence. This will be useful in constructing the smallest well spread sequences for small orders: for example, after observing that $(1, 2, 3, 5, 8, 13)$ is a minimal example for $n = 6$, one need not test any sequence in the case $n = 7$ which has size greater than 21. (Unfortunately, 21 is not small enough.)

Suppose $(a_1, a_2, \ldots, a_{n-1})$ had minimal size, and put $a_1 = 1$. Then $\sigma^*(n-1) = a_{n-1}$. Clearly $a_{n-2} < a_{n-1}$, so we have the (bad) bound

$$\sigma^*(n) \le 2\sigma^*(n-1) - 1.$$

Another application of this idea comes from noticing that the recursive construction $a_1 = 1, a_2 = 2, a_n = a_{n-1} + a_{n-2}$ gives a well spread sequence. This is the Fibonacci sequence, except that the standard notation for the Fibonacci numbers has $f_1 = f_2 = 1$, $f_3 = 2$, etc. So we have a well spread sequence with its size equal to the $(n+1)$-th term of the Fibonacci sequence: $a_n = f_{n+1}$. Therefore

$$\sigma^*(n) \le \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

The same reasoning shows that

$$\rho^*(n) \le f_{n+1} + f_n - 2 = f_{n+2} - 2.$$

Note. For further information on the Fibonacci numbers, see for example Section 7.1 of [7].

Exercise 2.6 (Continuation of Exercise 2.4.) *The **magic number** $M(n)$ of K_n was defined to be the smallest t such that $K_n + tK_1$ is edge-magic. We know that $M(n) \ge \rho^*(n) - n - \binom{n}{2}$. Find an upper bound for $M(n)$. (It need not be a good upper bound. The point is to show that* some *upper bound exists.)*

Research Problem 2.2 (Continuation of Exercise 2.6.) *Find $M(7)$. Find $M(8)$.*

Research Problem 2.3 *Find good lower and upper bounds on $M(n)$, as a function of n.*

Exercise 2.7 *Use the results on the evaluation of Sidon sequences to prove Theorem* 2.12. *(Note: this was the original proof of Theorem* 2.12*, as given in* [29, 30].*)*

2.3.5 Split graphs

A *split graph* $K_{m+n\backslash m}$ consists of a complete graph K_n, a null graph $\overline{K}_m$, and the $K_{m,n}$ joining the two vertex-sets.

Suppose $K_{m+n\backslash m}$ is edge-magic. This graph has $v = m + n$, $e = mn + \binom{n}{2}$, and $v + e = m + n + mn + \binom{n}{2} = \frac{1}{2}(2m + n)(n + 1)$, so by Theorem 2.13

$$\frac{1}{2}(2m + n)(n + 1) \geq \rho^*(n).$$

This gives no useful information for small n. However, provided $n > 6$, (2.17) applies and from

$$\frac{1}{2}(2m + n)(n + 1) \geq n^2 - 5n + 14$$

we find that there is no edge-magic total labeling of $K_{m+n\backslash m}$ unless

$$m \geq \frac{n - 12}{2} + \frac{20}{n + 1}. \tag{2.18}$$

Research Problem 2.4 *Observe that $K_{2n\backslash n}$ always satisfies* (2.18). *Does it always have an edge-magic total labeling?*

2.4 Cycles

The cycle C_v is regular of degree 2 and has v edges. So (2.2) becomes

$$v(v + 1) \leq 2s \leq 2v^2 + v(v + 1) = v(3v + 1),$$

and (2.4) is

$$kv = s + v(2v + 1),$$

whence v divides s; in fact $s = (k - 2v - 1)v$. When v is odd, s has $v + 1$ possible values $\frac{1}{2}v(v+1), \frac{1}{2}v(v+3), \ldots, \frac{1}{2}v(v+2i-1), \ldots, \frac{1}{2}v(3v+1)$, with corresponding magic sums $\frac{1}{2}(5v + 3), \frac{1}{2}(5v + 5), \ldots, \frac{1}{2}(5v + 2i + 1), \ldots, \frac{1}{2}(7v + 3)$. For even v, there are v values $s = \frac{1}{2}v^2 + v, \frac{1}{2}v^2 + 2v, \ldots, \frac{1}{2}v^2 + iv, \ldots, \frac{3}{2}v^2$, with corresponding magic sums $\frac{5}{2}v + 2, \frac{5}{2}v + 3, \ldots, \frac{5}{2}v + i + 1, \ldots, \frac{7}{2}v + 1$.

Kotzig and Rosa [29] proved that all cycles are magic, producing examples with $k = 3v + 1$ for v odd, $k = \frac{5}{2}v + 2$ for $v \equiv 2 (\bmod 4)$ and $k = 3v$ for $v \equiv 0(\bmod 4)$. In [18], labelings are exhibited for the minimum values of k in all cases. For convenience we give proofs for all cases, not exactly the same as the proofs in the papers cited. In each case the proof consists of exhibiting a labeling. If vertex names have to be cited, we assume the cycle to be $(u_1, u_2, \ldots, u_v)$.

Theorem 2.17 *If v is odd, then C_v has an edge-magic total labeling with $k = \frac{1}{2}(5v + 3)$.*

Proof. Say $v = 2n + 1$. Consider the cyclic vertex labeling $(1, n + 1, 2n + 1, n, \ldots, n + 2)$, where each label is derived from the preceding one by adding $n (\bmod 2n + 1)$. The successive pairs of vertices have sums $n + 2, 3n + 2, 3n + 1, \ldots, n + 3$, which are all different. If $k = 5n + 4$, the edge labels are $4n + 2, 2n + 2, 2n + 3, \ldots, 4n + 1$, as required. We have an edge-magic total labeling with $k = 5n + 4 = \frac{1}{2}(5v + 3)$ and $s = \frac{1}{2}v(v + 1)$ (the smallest possible values). □

By duality, we have:

Corollary 2.17.1 *Every odd cycle has an edge-magic total labeling with $k = \frac{1}{2}(7v + 3)$.*

Theorem 2.18 *Every odd cycle has an edge-magic total labeling with $k = 3v + 1$.*

Proof. Again write $v = 2n+1$. Consider the cyclic vertex labeling $(1, 2n+1, 4n + 1, 2n - 1, \ldots, 2n + 3)$; in this case each label is derived from the preceding one by adding $2n (\bmod 4n + 2)$. The construction is such that the second, fourth, $\ldots$, $2n$th vertices receive labels between 2 and $2n+1$ inclusive, while the third, fifth, $\ldots$, $(2n+1)$th receive labels between $2n + 2$ and $4n + 1$. The successive pairs of vertices have sums $2n + 2, 6n +$

$2, 6n, 6n-2, \ldots, 2n+4$; if $k = 3v+1 = 6n+4$, the edge labels are $4n+2, 2, 4, \ldots, 4n$. We have an edge-magic total labeling with $k = 3v+1$ and $s = v^2$ (the case $i = \frac{1}{2}(v+1)$ in the list). □

Corollary 2.18.1 *Every odd cycle has an edge-magic total labeling with $k = 3v+2$.*

Figure 2.3 shows examples with $v = 7$ of the constructions in Theorems 2.17 and 2.18; they have $k = 19$ and 22, respectively. (Only the vertex labels are shown in the figure; the edge labels can be found by subtraction.)

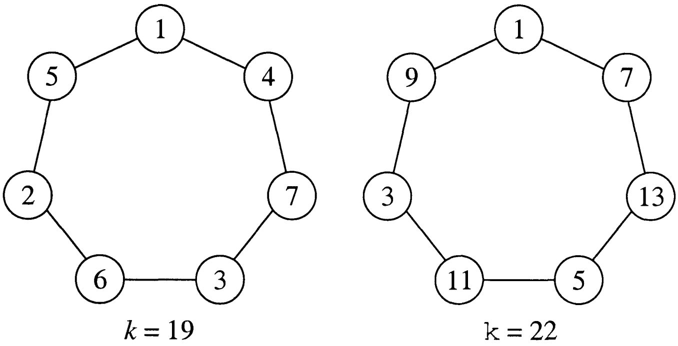

Figure 2.3. Two edge-magic total labelings of C_7.

Theorem 2.19 *If v is even, then C_v has an edge-magic total labeling with $k = \frac{1}{2}(5v+4)$.*

Proof. Write $v = 2n$. If n is even,

$$\lambda(u_i) = \begin{cases} (i+1)/2 & \text{for } i = 1, 3, \ldots, n+1 \\ 3n & \text{for } i = 2 \\ (2n+i)/2 & \text{for } i = 4, 6, \ldots, n \\ (i+2)/2 & \text{for } i = n+2, n+4, \ldots, 2n \\ (2n+i-1)/2 & \text{for } i = n+3, n+5, \ldots, 2n-1, \end{cases}$$

while if n is odd,

$$\lambda(u_i) = \begin{cases} (i+1)/2 & \text{for } i = 1, 3, \dots, n \\ 3n & \text{for } i = 2 \\ (2n+i+2)/2 & \text{for } i = 4, 6, \dots, n-1 \\ (n+3)/2 & \text{for } i = n+1 \\ (i+3)/2 & \text{for } i = n+2, n+4, \dots, 2n-1 \\ (2n+i)/2 & \text{for } i = n+3, n+5, \dots, 2n-2 \\ n+2 & \text{for } i = 2n. \end{cases}$$

□

Corollary 2.19.1 *Every cycle of length divisible by 4 has an edge-magic total labeling with* $k = \frac{1}{2}(7v + 2)$.

In Figure 2.4 we present two examples, for $v = 8$ and 10. The constructions of Theorem 2.19 yield $k = 22$ and 27.

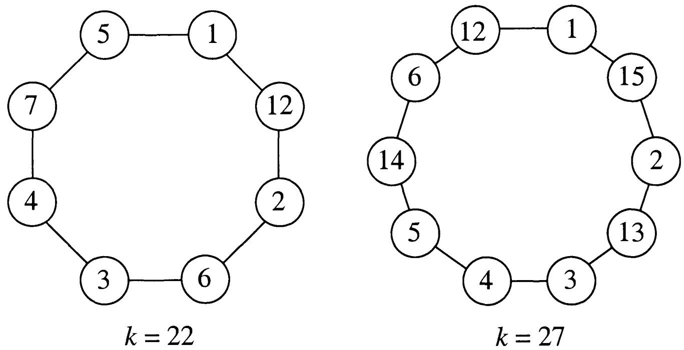

Figure 2.4. Edge-magic total labelings of C_8 and C_{10}.

Theorem 2.20 *Every cycle of length divisible by 4 has an edge-magic total labeling with* $k = 3v$.

Proof. For $v = 4$ the result is given by Theorem 2.19. So assume $v \geq 8$, write $v = 4n$, $n > 1$. The required labeling is

$$\lambda(u_i) = \begin{cases} i & \text{for } i = 1, 3, \dots, 2n-1 \\ 4n+i+1 & \text{for } i = 2, 4, \dots, 2n-2 \\ i+1 & \text{for } i = 2n, 2n+2, \dots, 4n-2 \\ 4n+i & \text{for } i = 2n+1, 2n+3, \dots, 4n-3 \\ 2 & \text{for } i = 4n-1 \\ 2v-2 & \text{for } i = 4n. \end{cases}$$

□

Corollary 2.20.1 *Every cycle of length divisible by 4 has an edge-magic total labeling with $k = 3v + 3$.*

Research Problem 2.5 [18] *If v is odd, does C_v have an edge-magic total labeling for every magic sum k satisfying $\frac{1}{2}(5v + 3) \leq k \leq \frac{1}{2}(7v + 3)$? If v is even, does C_v have an edge-magic total labeling for every magic sum k satisfying $\frac{5}{2}v + 2 \leq k \leq \frac{7}{2}v + 1$?*

2.4.1 Small cycles

We list all edge-magic total labelings of cycles up to C_6.

There are four labelings of C_3: see under K_3, in Section 2.3.3.

For C_4, the possibilities are $k = 12, 13, 14, 15$, with $s = 12, 16, 20, 24$, respectively. The unique solution for $k = 12$ is the cyclic vertex-labeling $(1, 3, 2, 6)$. For $k = 13$ there are two solutions: $(1, 5, 2, 8)$ and $(1, 4, 6, 5)$. The other cases are duals of these two.

For C_5, one must consider $k = 14, 15, 16$ ($s = 15, 20, 25$) and their duals. The unique solution for $k = 14$ is $(1, 4, 2, 5, 3)$ (the solution from Theorem 2.17). There are no solutions for $k = 15$. For $k = 16$, one obtains $(1, 5, 9, 3, 7)$ (the solution from Theorem 2.18) and also $(1, 7, 3, 4, 10)$. Many other possible sets S must be considered when $k = 15$ or 16, but all can be eliminated using the following observation. The set S cannot contain three labels that add to k: for, in C_5, some pair of the corresponding vertices must be adjacent (given any three vertices of C_5, at least two must be adjacent), and the edge joining them would require the third label.

C_6 has possible sums $k = 17, 18, 19$ ($s = 24, 30, 36$) and duals. For $k = 17$ there are three solutions: $(1, 5, 2, 3, 6, 7)$, $(1, 6, 7, 2, 3, 5)$ and $(1, 5, 4, 3, 2, 9)$. Notice that two non-isomorphic solutions have the same set of vertex labels. There is one solution for $k = 18$, $(1, 8, 4, 2, 5, 10)$, and six for $k = 19$, i.e., $(1, 6, 11, 3, 7, 8)$, $(1, 7, 3, 12, 5, 8)$, $(1, 8, 7, 3, 5, 12)$, $(1, 8, 9, 4, 3, 11)$, $(2, 7, 11, 3, 4, 9)$ and $(3, 4, 5, 6, 11, 7)$.

In the case of C_7, the possible magic sums run from 19 to 26, and Godbold and Slater [18] found that all can be realized; there are 118 labelings up to isomorphism. The corresponding numbers for C_8, C_9 and C_{10} are 282, 1540 and 7092 [18].

2.4.2 Generalizations of cycles

Paths

The path P_n can be viewed as a cycle C_n with an edge deleted.

Say λ is an edge-magic total labeling of C_n with the property that label $2n$ appears on an edge. If that edge is deleted, the result is a P_n with an edge-magic total labeling.

For every n, there is an edge-magic total labeling of C_n in which $2n$ appears on an edge — the labelings in Theorem 2.17 and 2.19 have this property. Deleting this edge yields a path, on which the labeling is edge-magic.

Theorem 2.21 *All paths have edge-magic total labelings.*

Suns

An n-sun is a cycle C_n with an edge terminating in a vertex of degree 1 attached to each vertex.

Theorem 2.22 *All suns are edge-magic.*

Proof. First we treat the odd case. Denote by λ the edge-magic total labeling of C_n given in Theorem 2.17. We construct a labeling μ which has $\mu(u) = \lambda(u) + n$ whenever u is a vertex or edge of the cycle. If a vertex has label x, then the new vertex attached to it has label a_x, where $a_x \equiv x - \frac{1}{2}(n-1) (\operatorname{mod} n)$ and $1 \leq a_x \leq n$, and the edge joining them has label b_x, where $b_x \equiv n + 1 - 2x (\operatorname{mod} n)$ and $3n + 1 \leq b_x \leq 4n$. Then μ is an edge-magic total labeling with $k = \frac{1}{2}(11n + 3)$.

In the even case, λ is the edge-magic total labeling of C_n given in Theorem 2.19. The labeling μ again has $\mu(u) = \lambda(u) + n$ whenever u is an element of the cycle. The vertex with label x is adjacent to a new vertex

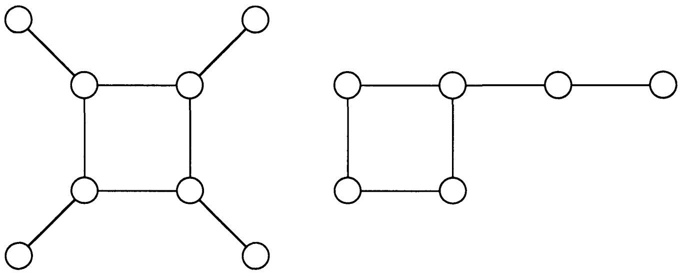

Figure 2.5. A 4-sun and a (4, 2)-kite.

with label a_x, and the edge joining them has label b_x, where:

- if $1 \leq x \leq \frac{1}{2}n$, then
  - $a_x \equiv x + \frac{1}{2}n \pmod n$ and $1 \leq a_x \leq n$,
  - $b_x \equiv 2 - 2x \pmod n$ and $3n + 1 \leq b_x \leq 4n$;
- if $1 + \frac{1}{2}n \leq x < n$, then
  - $a_x \equiv x + \frac{1}{2}n + 1 \pmod n$ and $1 \leq a_x \leq n$,
  - $b_x \equiv 1 - 2x \pmod n$ and $3n + 1 \leq b_x \leq 4n$;
- $a_{\frac{3n}{2}} = b_{\frac{3n}{2}} = 1$.

Then μ is an edge-magic total labeling with $k = \frac{1}{2}(11n + 4)$. □

Kites

An (n, t)-kite consists of a cycle of length n with a t-edge path (the *tail*) attached to one vertex. We write its labeling as the list of labels for the cycle (ending on the attachment point), separated by a semicolon from the list of labels for the path (starting at the vertex nearest the cycle).

Theorem 2.23 *An $(n, 1)$-kite (a kite with tail length 1) is edge-magic.*

Proof. For convenience, suppose the tail vertex is y and its point of attachment is z.

First, suppose n is odd. Denote by λ the edge-magic total labeling of C_n given in Theorem 2.17, with the vertices arranged so that $\lambda(z) = \frac{1}{2}(n + 1)$. Define a labeling μ by $\mu(x) = \lambda(x) + 1$ whenever x is an element of the cycle, $\mu(y) = 2v + 2)$ and $\mu(y, z) = 1$. Then μ is an edge-magic total labeling with $k = \frac{1}{2}(5n + 9)$.

If v is even, λ is the edge-magic total labeling of Theorem 2.19, with $\lambda(z) = \frac{1}{2}(v + 2)$. Define a labeling μ by $\mu(x) = \lambda(x) + 1$ whenever x is an element of the cycle, $\mu(y) = 2v + 2$ and $\mu(y, z) = 1$. Then μ is an edge-magic total labeling with $k = \frac{1}{2}(5v + 10)$. □

Research Problem 2.6 *Investigate the edge-magic properties of (n, t)-kites for general t.*

Exercise 2.8 A triangular book $B_{3,n}$ *consists of n triangles with a common edge. Prove that all triangular books are edge-magic.*

Research Problem 2.7 *The k-cycle book $B_{k,n}$ consists of n copies of C_k with a common edge. Are all k-cycle books edge-magic?*

Research Problem 2.8 *The books described in the preceding exercise and problem can be generalized by replacing the common edge by a path. Investigate the edge-magic properties of these graphs.*

2.5 Complete bipartite graphs

An edge-magic total labeling of a complete bipartite graph can be specified by giving two sets S_1 and S_2 of vertex labels.

Theorem 2.24 [29] *The complete bipartite graph $K_{m,n}$ is magic for any m and n.*

Proof. The sets $S_1 = \{n+1, 2n+2, \ldots, m(n+1)\}$, $S_2 = \{1, 2, \ldots, n\}$, define an edge-magic total labeling with $k = (m+2)(n+1)$. □

Research Problem 2.9 *The complete tripartite graph $K_{m,n,p}$ has three sets of vertices, of sizes m, n, p. Does $K_{m,n,p}$ always have an edge-magic total labeling?*

Research Problem 2.10 *Generalize Research Problem 2.9 to complete t-partite graphs (t parts).*

2.5.1 Small cases

A computer search has been carried out for edge-magic total labelings of $K_{2,3}$. The usual considerations show that $14 \le k \le 22$, with cases $k = 19, 20, 21, 22$ being the duals of cases $k = 17, 16, 15, 14$. The solutions up to $k = 18$ are

$$\begin{array}{lll}
k = 14, & \text{no solutions} & \\
k = 15, & S_1 = \{1, 2\}, & S_2 = \{3, 6, 9\} \\
k = 16, & S_1 = \{1, 2\}, & S_2 = \{5, 8, 11\} \\
 & S_1 = \{1, 3\}, & S_2 = \{5, 6, 11\} \\
 & S_1 = \{4, 6\}, & S_2 = \{1, 2, 7\} \\
 & S_1 = \{4, 8\}, & S_2 = \{1, 2, 3\} \\
k = 17, & S_1 = \{1, 8\}, & S_2 = \{5, 6, 7\} \\
 & S_1 = \{5, 6\}, & S_2 = \{1, 4, 9\} \\
k = 18, & S_1 = \{1, 5\}, & S_2 = \{9, 10, 11\} \\
 & S_1 = \{7, 11\}, & S_2 = \{1, 2, 3\}.
\end{array}$$

(The last two are of course duals.)

For $K_{3,3}$ one has $18 \leq k \leq 30$, and k must be even. Cases $k = 26, 28, 30$ are dual to cases $k = 22, 20, 18$. The solutions are

$$\begin{array}{lll}
k = 18, & \text{no solutions} & \\
k = 20, & S_1 = \{1, 2, 3\}, & S_2 = \{4, 8, 12\} \\
 & S_1 = \{1, 2, 9\}, & S_2 = \{4, 6, 8\} \\
k = 22, & S_1 = \{1, 2, 3\}, & S_2 = \{7, 11, 15\} \\
 & S_1 = \{1, 3, 5\}, & S_2 = \{7, 8, 15\} \\
 & S_1 = \{1, 5, 12\}, & S_2 = \{6, 7, 8\} \\
k = 24, & \text{no solutions.} &
\end{array}$$

2.5.2 Stars

Lemma 2.25 *In any edge-magic total labeling of a star, the center receives label* 1, $n + 1$ *or* $2n + 1$.

Proof. Suppose the center receives label x. Then

$$kn = \binom{2n+2}{2} + (n-1)x. \tag{2.19}$$

Reducing (2.19) modulo n we find

$$x \equiv (n+1)(2n+1) \equiv 1$$

and the result follows. □

Theorem 2.26 *There are $3 \cdot 2^n$ edge-magic total labelings of $K_{1,n}$, up to equivalence.*

Proof. Denote the center of a $K_{1,n}$ by c, the leaves by $v_1, v_2, \dots, v_n$ and edge (c, v_i) by e_i. From Lemma 2.25 and (2.19), the possible cases for an edge-magic total labeling are $\lambda(c) = 1, k = 2n + 4$, $\lambda(c) = n + 1, k = 3n + 3$ and $\lambda(c) = 2n + 1, k = 4n + 2$. As the labeling is magic, the sums $\lambda(v_i) + \lambda(e_i)$ must all be equal to $M = k - \lambda(c)$ (so $M = 2n+3$, $2n+2$ or $2n + 1$). Then in each case there is exactly one way to partition the $2n + 1$ integers $1, 2, \dots, 2n + 1$ into $n + 1$ sets

$$\{\lambda(c)\}, \{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_n, b_n\}$$

where every $a_i + b_i = M$. For convenience, choose the labels so that $a_i < b_i$ for every i and $a_1 < a_2 < \cdots < a_n$. Then up to isomorphism, one can assume that $\{\lambda(v_i), \lambda(e_i)\} = \{a_i, b_i\}$. Each of these n equations provides two choices, according to whether $\lambda(v_i) = a_i$ or b_i, so each of the three values of $\lambda(c)$ gives 2^n edge-magic total labelings of $K_{1,n}$. □

Exercise 2.9 *A graph is derived from a star by adding a pendant edge to one of the vertices of degree* 1. *Prove every such graph is edge-magic.*

Double stars

The *double star* $S_{m,n}$ has two adjacent central vertices x and y. There are m leaves $x_1, x_2, \dots, x_m$ adjacent to x and n leaves $y_1, y_2, \dots, y_n$ adjacent to y. An edge-magic total labeling of this graph can be specified by the list

$$(\{\lambda(x_1), \lambda(x_2), \dots, \lambda(x_m)\}, \lambda(x), \lambda(y), \{\lambda(y_1), \lambda(y_2), \dots, \lambda(y_n), \}).$$

One solution for $S_{2,2}$ is $(\{8, 11\}, 2, 5, \{4, 10\})$ with $k = 16$.

2.6 Wheels

Enomoto et al. [11] have checked all wheels up to $n = 29$ and found that the graph is magic if $n \not\equiv 3 \pmod 4$. We shall prove that W_n is magic whenever $n \equiv 0, 1, 4, 5$ or $6 \pmod 8$.

To describe a wheel, we refer to the central vertex as the *center*, the edges adjacent to the center as *spokes*, and the other vertex on a spoke as its *terminal*. The remaining edges are *arcs*. We write c for the label on the center; the spokes receive labels $a_1, a_2, \ldots, a_n$, the terminal of the spoke a_i gets label b_i, and the arc from b_i to b_{i+1} is labeled s_i. (This scheme is illustrated in Figure 2.6.) It then follows that all the sums $a_i + b_i$ in an edge-magic total labeling are the same; they equal $k - c$, where k is the magic sum.

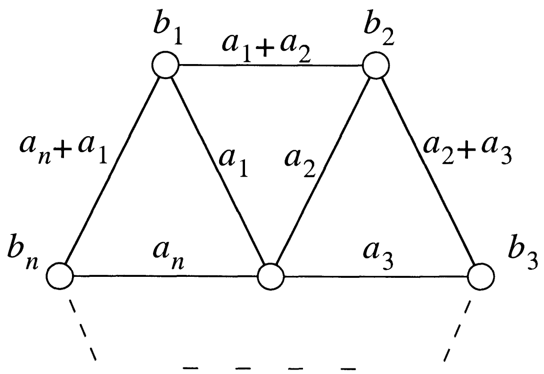

Figure 2.6. Labeling a wheel.

For cases other than $n \equiv 6 \pmod 8$, our constructions (taken from [42]) use $k = 2c$. Then $a_i + b_i = c$ for every i. It follows that c is greater than each a_i and b_i; since labels are distinct and positive, $c \geq 2n + 1$. Also $s_i = a_i + a_{i+1}$. We then need to do the following:

(i) partition the integers from 1 to $2n$ into two classes A and B so that exactly one member of $\{i, 2n + 1 - i\}$ is in each set, for every i;

(ii) order the elements of A in a cyclic sequence so that the n sums of consecutive pairs comprise the integers from $2n+2$ to $3n+1$ in some order (or, what is equivalent, the elements of B must be ordered so that the consecutive sums are a permutation of $\{n + 1, \ldots, 2n\}$).

To visualize this process, it is perhaps easiest to consider a graph with vertices $1, 2, \ldots, 2n$, drawn in two sets so that i and $2n + 1 - i$ are vertical opposite pairs. Two vertices are connected if their sum lies between $n + 1$ and $2n$. The graph for $n = 5$ is shown in Figure 2.7. We need to find a cycle that contains exactly one member from each opposite pair, all of whose edges have different sums. From this representation it is clear, for example, that both 1 and 2 must be terminal labels, because neither $2n$ nor $2n-1$ have degree 2 or greater.

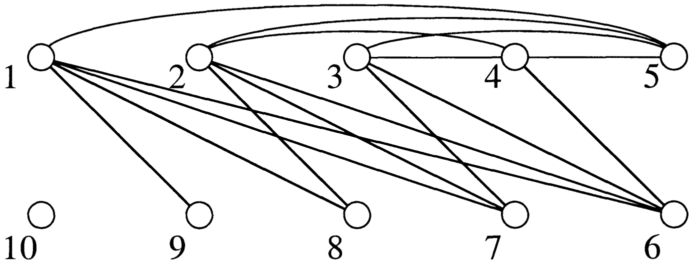

Figure 2.7. The spoke-terminal graph.

2.6.1 *The constructions for $n \not\equiv 6$*

We construct, for each $n \equiv 0, 1, 4$ or $5 \pmod 8$, a labeling of the terminal vertices of the wheel W_n. That is, we construct a sequence of n members of $\{1, 2, \ldots, 2n\}$ such that

(i) for each $i = 1, 2, \ldots, n$, exactly *one* of i and $2n+1-i$ is a member of the sequence, and

(ii) the set of sums of pairs of successive elements of the sequence (including last and first elements) is precisely $n+1, n+2, \ldots, 2n$.

In every case the sequence is defined in terms of several subsequences which are then joined.

It will be observed that the constructions do not provide solutions for $n = 5$ or 13. There is no sequence for $n = 5$; for $n = 13$ one example is

$$(12\ 4\ 20\ 1\ 24\ 2\ 16\ 6\ 13\ 10\ 5\ 9\ 8).$$

Case $n \equiv 0 \pmod 8$, $n \geq 8$.

For each $i \equiv 1 \pmod 4$ with $1 \leq i \leq \frac{n}{2} - 3$ define

$$S_i = (i, 2n+1-3i, i+1, 2n+1-3i-4, i+3, 2n+1-3i-8),$$

and define

$$S = (\frac{n}{2} + j : j \equiv 0, 1 \pmod 4 \text{ and } 1 \leq j \leq \frac{n}{2}).$$

Then the desired sequence is

$$S_1 S_5 \ldots S_{\frac{n}{2}-3} S.$$

Case $n \equiv 4 \pmod 8$, $n \geq 4$.

For each $i \equiv 1 \pmod 4$ with $1 \leq i \leq \frac{n}{2} - 5$ define

$$S_i = (i+1, 2n+1-3i, i, 2n+1-3i-4, i+3, 2n+1-3i-8),$$

and define

$$S = (\frac{n}{2} - 2 + j : j \equiv 0, 1 \pmod 4 \text{ and } 1 \leq j \leq \frac{n}{2} + 2).$$

Then the desired sequence is

$$S_1 S_5 \ldots S_{\frac{n}{2}-5} (\frac{n}{2}, \frac{n}{2} + 4) S.$$

(Note that when $n = 4$, the sequence is simply $(\frac{n}{2}, \frac{n}{2} + 4)S = (2, 6, 1, 4)$.)

Case $n \equiv 1 \pmod 8$, $n \geq 9$.

For each $i \equiv 1 \pmod 4$ with $1 \leq i \leq \frac{n-1}{2} - 3$ define

$$S_i = \{i, 2n+1-3i, i+1, 2n+1-3i-4, i+3, 2n+1-3i-8\},$$

and define

$$S = \{\frac{n-1}{2} + j : j \equiv 1, 2 \pmod 4 \text{ and } 1 \leq j \leq \frac{n+1}{2}\}.$$

Then the desired sequence is

$$S_1 S_5 \ldots S_{\frac{n-1}{2}-3} S.$$

Case $n \equiv 5 \pmod{24}$, $n \geq 29$.

Define

$$S = (\frac{n+10}{3}, n-5, \frac{n+1}{3}, n+3, \frac{n+7}{3}, n-1, 4, 2n+1-7, 1,$$
$$2n+1-3, 2, 2n+1-11)$$

and

$$T = (\frac{n-2}{3}, n, n-3, n-4, n-7, \ldots, \frac{n+5}{2}, \frac{n-1}{2}, \frac{n+3}{2}, \frac{n+11}{2}).$$

For each $i \equiv 6 \pmod 8$, $i \geq 6$, define

$$S_i = (i, 2n+1-(3i+1), i+2, 2n+1-(3i-3), i-1, 2n+1-(3i+5)),$$

and for each $i \equiv 1 \pmod 8$, $i \geq 9$, define

$$S_i = (i, 2n+1-(3i+4), i+3, 2n+1-3i, i+1, 2n+1-(3i+8)).$$

For each $j = 1, 2, \ldots, \frac{n-29}{24}$ write

$$C_j = (\frac{n+10}{3} + 4j, n-5-12j, \frac{n+10}{3} + 4j - 3,$$
$$n-5-12j+4, \frac{n+10}{3} + 4j - 1, n-5-12j+8).$$

Then the desired sequence is

$$SS_6S_9 \ldots S_{\frac{n-35}{3}} S_{\frac{n-26}{3}} S_{\frac{n-11}{3}} T C_{\frac{n-29}{24}} \ldots C_2C_1.$$

(Note that when $n = 29$ this sequence is SS_6T.)

Case $n \equiv 21 \pmod{24}$, $n \geq 21$.

Define

$$S = (\frac{n+6}{3}, n-1, 4, 2n+1-7, 1, 2n+1-3, 2, 2n+1-11)$$

and

$$\begin{aligned} T \;=\; & (\frac{n-3}{3}, 2n+1-(n-2), \frac{n-3}{3}+2, 2n+1-(n-6), \\ & \frac{n-3}{3} - 1, n, n-3, n-4, n-7, \ldots, \\ & \frac{n+5}{2}, \frac{n-1}{2}, \frac{n+3}{2}, \frac{n+11}{2}). \end{aligned}$$

For each $i \equiv 6 \pmod 8$, $i \geq 6$, and for each $i \equiv 1 \pmod 8$, $i \geq 9$, define S_i as in the $n \equiv 5 \pmod{24}$ case. Then, for each $j = 1, 2, \ldots, \frac{n-21}{24}$ write

$$C_j = (\frac{n+6}{3} + 4j, n-1-12j, \frac{n+6}{3} + 4j - 3, n-1-12j+4,$$
$$\frac{n+6}{3} + 4j - 1, n-1-12j+8).$$

Then the desired sequence is

$$SS_6S_9 \dots S_{\frac{n-27}{3}} S_{\frac{n-18}{3}} T C_{\frac{n-21}{24}} \dots C_2C_1.$$

(Note that when $n = 21$ this sequence is ST.)

Case $n \equiv 13 \pmod{24}$.

Define

$$S = (n-1-8, \frac{n+5}{3}+2, n-1-4, \frac{n+5}{3}+6, \frac{2n+1}{3}, \frac{n+2}{3}, \\ n-1, 4, 2n+1-7, 1, 2n+1-3, 2, 2n+1-11)$$

and

$$T = (\frac{n+5}{3}, n, n-3, n-4, n-7, \dots, \frac{2n+1}{3}+4, \frac{2n+1}{3}+1, \\ \frac{2n+1}{3}-4, \frac{2n+1}{3}-3, \dots, \frac{n+5}{2}, \frac{n+7}{2}, \frac{n-3}{2}, \frac{n+11}{2}).$$

For each $i \equiv 6 \pmod 8$, $i \geq 6$, and for each $i \equiv 1 \pmod 8$, $i \geq 9$, define S_i as in the $n \equiv 5 \pmod{24}$ case. Now we must break into two subcases:

(i) $n \equiv 13 \pmod{48}$, $n \geq 61$.

Define

$$D = (\frac{n+5}{3}+10, n-1-20, \frac{n+5}{3}+4, n-1-16, \frac{n+5}{3}+8, \\ n-1-12, \frac{n+5}{3}+3),$$

and for each $j = 1, 2, \dots, \frac{n-61}{48}$ write

$$C_j = (\frac{n+5}{3}+10+8j, n-1-24j-20, \frac{n+5}{3}+4+8j, \\ n-1-24j-16, \frac{n+5}{3}+8+8j, n-1-24j-12, \\ \frac{n+5}{3}+6+8j, n-1-24j-8, \frac{n+5}{3}-1+8j, \\ n-1-24j-4, \frac{n+5}{3}+3+8j, n-1-24j).$$

Then the desired sequence is

$$SS_6S_9\ldots S_{\frac{n-19}{3}}S_{\frac{n-10}{3}}TC_{\frac{n-61}{48}}\ldots C_2C_1D.$$

(Note that when $n = 61$ this sequence is $SS_6S_9S_14S_1TD$.)

(ii) $n \equiv 37 \pmod{48}, n \geq 37$.

Write $D = (\frac{n+5}{3} + 4)$, and for each $j = 1, 2, \ldots, \frac{n-37}{48}$ define

$$\begin{aligned} C_j &= (\frac{n+5}{3} + 6 + 8j, n - 1 - 24j - 8, \frac{n+5}{3} + 8j, \\ &\quad n - 1 - 24j - 4, \frac{n+5}{3} + 4 + 8j, n - 1 - 24j, \\ &\quad \frac{n+5}{3} + 2 + 8j, n - 1 - 24j + 4, \frac{n+5}{3} - 5 + 8j, \\ &\quad n - 1 - 24j + 8, \frac{n+5}{3} - 1 + 8j, n - 1 - 24j + 12). \end{aligned}$$

Then the desired sequence is

$$SS_6S_9\ldots S_{\frac{n-19}{3}}S_{\frac{n-10}{3}}TC_{\frac{n-37}{48}}\ldots C_2C_1D.$$

(Note that when $n = 37$ this sequence is $SS_6S_9S_14S_17TD$.)

Theorem 2.27 *Every wheel W_n with $n \equiv 0$ or $1 \pmod 4$ has an edge-magic total labeling.*

Proof. It will suffice to prove that the above constructions have the required properties (i) and (ii). We prove this for the hardest cases, the case $n \equiv 1 \pmod 8$ and $n \equiv 13 \pmod{48}, n \geq 61$; the proofs for $n \equiv 0$ and $4 \pmod 8$ are similar to the former, while the proofs in the other cases with $n \equiv 5 \pmod 8$ are similar to the latter.

$n \equiv 1 \pmod 8$

Property (i): Suppose $m \equiv 1$ or $2 \pmod 4$, $1 \leq m \leq n$.If $1 \leq m \leq \frac{n-1}{2} - 2$, then m appears in one of the S_i (as an 'i' or '$i+1$' term). If $\frac{n-1}{2} + 1 \leq m \leq n$, then m appears in S. Now suppose $m \equiv 0 \pmod 4$, $4 \leq m \leq 2n - 2$. If $4 \leq m \leq \frac{n-1}{2}$ then m appears in one of the S_i (as an '$i+3$' term) while if $\frac{n-1}{2}+4 \le m \le 2n-2$, then again m appears in one of the S_i, as follows. If $m \equiv \frac{n-1}{2}+4 \mod 12$, then m appears as a term of the form '$2n+1-3i-8$'; if $m \equiv \frac{n-1}{2}+8 \mod 12$, then m appears as a '$2n+1-3i-4$' term; if $m \equiv \frac{n-1}{2} \mod 12$ (which means $m \equiv 2n-2 \mod 12$ since $n \equiv 1 \pmod 8$), then m appears as a '$2n+1-3i$' term. Now since we have all $m \equiv 0 (\mod 4), 4 \le m \le 2n-2$, we also have represented all of their inverses mod $2n+1$, which yields all of the 3 (mod 4) cases between 3 and $2n-3$. Hence we have property (i).

Property (ii): First we note that those sums from $n+2$ to $2n-3$ inclusive that are congruent to 3 (mod 4) appear as successive sums of pairs in S. Then those sums from $n+4$ to $2n-1$ inclusive which are congruent to 1 mod 4 appear as the union of the sums of the first and second terms of each S_i, the last term in S_i with the first term in S_{i+4} and the last term in $S_{\frac{n-1}{2}-3}$ with the first term in S. The sums from $n+1$ to $2n$ inclusive which are congruent to 2 mod 4 appear as the union of the sums of the second and third, and third and fourth terms in each S_i, and the sum of the last term in S with the first term in S_1. Finally, those sums from $n+3$ to $2n-2$ inclusive that are congruent to 0 mod 4 appear as the union of the sums of the fourth and fifth, and fifth and sixth, terms in each S_i. Hence we have property (ii).

$n \equiv 13 \pmod{48}$

The sequence for the case $n = 13$ can be verified directly, so we assume $n \ge 61$.

Property (i): Suppose $m \equiv 1$ or $2 (\mod 4)$, $1 \le m \le n$. If $m = 1, 2, \frac{n+2}{3}$ or $\frac{2n+1}{3}$, then m appears in S. If $m = \frac{n+5}{3}+3, \frac{n+5}{3}+4$ or $\frac{n+5}{3}+8$, then m appears in D. If $m = \frac{n+5}{3}$ or $\frac{n-3}{2}$, or if $\frac{n+5}{2} \le m \le n, m \ne \frac{2n+1}{3}$, then m appears in T. If $m \equiv 5$ or $6 \pmod 8$, $5 \le m \le \frac{n+5}{3}-8$, then m appears in an S_i with $i \equiv 6 \pmod 8$, as the $i-1$ or i term, while if $m \equiv 1$ or $2 \pmod 8$, $9 \le m \le \frac{n+5}{3-4}$, then m appears in an S_i with $i \equiv 1 \pmod 8$, as the i or $i+1$ term. There remain those $m \equiv 1 \pmod 4$, $\frac{n+5}{3+7} \le m \le \frac{n-11}{2}$, and the $m \equiv 2 \pmod 4$, $\frac{n+5}{3+12} \le m \le \frac{n-1}{2}$. These terms appear in the C_js, the former as the $\frac{n+5}{3-1+8j}$ and $\frac{n+5}{3+3+8j}$ terms, $1 \le j \le \frac{n-61}{48}$, and the latter as the $\frac{n+5}{3+4+8j}$ and $\frac{n+5}{3+8+8j}$ terms, $1 \le j \le \frac{n-61}{48}$.

Now suppose $m \equiv 0 \pmod 4$, $4 \le m \le 2n-2$. If $4 \le m \le \frac{n+5}{3+6}$ or if $n-9 \le m \le 2n-2$, then m appears in S or in one of the S_is. If $m = \frac{n+5}{3} + 10, n-21, n-17$ or $n-13$, then m appears in D. There remain the cases $m \equiv 0 \pmod 4$, $\frac{n+5}{3} + 14 \le m \le n-25$. Now the term $\frac{n+11}{2}$ appears in T. The terms $\frac{n+5}{3} + 14 \le m \le \frac{n+3}{2}$ appear in the C_js as the $\frac{n+5}{3} + 10 + 8j$ and $\frac{n+5}{3} + 6 + 8j$ terms, $1 \le j \le \frac{n-61}{48}$, while the terms $\frac{n+19}{2} \le m \le n-25$ appear in the C_js as the $n-1-24j-4k$ terms, $0 \le k \le 5$, $1 \le j \le \frac{n-61}{48}$. Since we have all $0 \pmod 4$ terms between 4 and $2n-2$ we have also represented all $3 \pmod 4$ terms between 3 and $2n-3$ (i.e., the inverses modulo $(2n+1)$). This verifies property (i).

Property (ii): Sums from $n+2$ to $2n-3$ inclusive that are congruent to $3 \pmod 4$ appear as succesive sums of pairs of elements in T. Sums from $\frac{4n+11}{3}$ to $2n-1$ inclusive that are congruent to $1 \pmod 4$ appear as sums in S involving 1 (i.e., $2n-1$ and $42n-5$) and as sums of the fourth and fifth, and fifth and sixth terms in S_i, the sixth term in S_i with the first term in S_{i+3}, and the first and second terms in S_{i+3}, where $i \equiv 6 \pmod 8$, $6 \le i \le \frac{n-19}{3}$. The sum $\frac{4n-1}{3}$ appears as the sum of the sixth and seventh terms in S, while the sum $n+8$ appears as the sum of the fourth and fifth terms in S; $n+4$ appears as the sum of the last two terms in T. Now $\frac{4n-25}{3}$ and $\frac{4n-13}{3}$ appear as the sum of the last two terms in D and the sum of the last term in D with the first term in S, respectively.

There remain the sums congruent to $1 \pmod 4$ between $n+12$ and $\frac{4n-37}{3}$ inclusive; these appear as successive sums over the eighth through twelfth terms in the C_js, $1 \le j \le \frac{n-61}{48}$. Sums from $\frac{4n+2}{3}$ to $2n$ inclusive that are congruent to $2 \pmod 4$ appear as sums of the eleventh and twelfth, and twelfth and last terms in S, the last term in S with the first term in S_6, and the first two terms in S_6; then as sums of the fourth and fifth, and fifth and sixth terms in S_i, the sixth term in S_i with the first term in S_i+5, and the first and second terms in S_i+5, where $i \equiv 1 \pmod 8$, $9 \le i \le \frac{n-10}{3-8}$; then finally as sums of the fourth and fifth, and fifth and sixth terms in $S_{\frac{n-10}{3}}$, and the sixth term in $S_{\frac{n-10}{3}}$ with the first term in T. The sum $n+1$ appears as the sum of the fifth and sixth terms in S, while the sums $\frac{4n-10}{3-4k}$, $3 \ge k \ge 0$, appear as successive sums over the second through sixth terms in D.

There remain the sums congruent to $2 \pmod 4$ between $n+5$ and $\frac{4n-58}{3}$ inclusive; these appear as successive sums over the second through sixth terms in the C_js, $1 \le j \le \frac{n-61}{48}$. Sums from $\frac{4n+20}{3}$ to $2n-2$ inclusive which are congruent to 0 (mod 4) appear as sums of the eighth and ninth terms in S, and as sums of the second and third, and third and fourth terms in S_i and $S_i + 3$, where $i \equiv 6 \pmod 8$ and $6 \leq i \leq \frac{n-19}{3}$. The sum $n + 3$ appears as the sum of the seventh and eighth terms in S, while the sums $\frac{4n+20}{3-4k}$, $4 \geq k \geq 1$, appear as the sum of the first two terms in D and as succesive sums over the first through fourth terms in S. There remain the sums congruent to 0 (mod 4) between $n + 7$ and $\frac{4n-40}{3}$ inclusive. When $n = 61$, this sum (namely 68) appears as the sum of the last term in T (namely $\frac{n+11}{2} = 36$ and the first term in D ($\frac{n+5}{3} + 10 = 32$); when $n > 61$ these appear as the sum of the last term in T with the first term in $C_{\frac{n-61}{48}}$, the sums of the first and second, sixth and seventh, and seventh and eighth terms in each C_j, $\frac{n-61}{48} \geq j \geq 1$, the sum of the last term in C_j with the first term in $C_j - 1$, $\frac{n-61}{48} \geq j \geq 2$, and the sum of the last term in C_1 with the first term in D. This completes the verification of property (ii). □

2.6.2 *The construction for* $n \equiv 6$

If $n \equiv 6 \pmod 8$, the following construction (taken from [47]) provides an edge-magic total labeling with $k = 5n + 2$.

$$c = 2n.$$

$$\text{For } i = 1, 3, \ldots, \tfrac{n}{2}, \quad \begin{cases} a_i = \frac{1}{2}(i+1) \\ b_i = 3n + 2 - \frac{1}{2}(i+1) \\ s_i = n + i. \end{cases}$$

$$\text{For } i = 2, 4, \ldots, \tfrac{n-2}{2}, \quad \begin{cases} a_i = 2n + 1 + \frac{1}{2}i \\ b_i = n + 1 - \frac{1}{2}i \\ s_i = n + i. \end{cases}$$

$$\text{For } i = \tfrac{n+4}{2}, \tfrac{n+8}{2}, \ldots, \tfrac{3n-6}{4}, \quad \begin{cases} a_i = \frac{1}{2}(i+3) \\ b_i = 3n + 1 - \frac{1}{2}(i+1) \\ s_i = n + i + 1. \end{cases}$$

$$\text{For } i = \tfrac{n+2}{2}, \tfrac{n+8}{2}, \ldots, \tfrac{3n-2}{4}, \quad \begin{cases} a_i = 2n + 1 + \frac{1}{2}i \\ b_i = n + 1 - \frac{1}{2}i \\ s_i = n + i + 1. \end{cases}$$

$$\text{For } i = \tfrac{3n+2}{4}, \quad \begin{cases} a_i = \frac{1}{8}(5n+2) \\ b_i = \frac{1}{8}(19n+14) \\ s_i = 2n + 1. \end{cases}$$

For $i = \frac{3n+6}{4}, \frac{3n+14}{4}, \ldots, n,$ $\begin{cases} a_i &= 2n+2+\frac{1}{2}i \\ b_i &= n-\frac{1}{2}i \\ s_i &= n+i+1. \end{cases}$

For $i = \frac{3n+10}{4}, \frac{3n+18}{4}, \ldots, n-3,$ $\begin{cases} a_i &= \frac{1}{2}(i+5) \\ b_i &= 3n-\frac{1}{2}(i+1) \\ s_i &= n+i+1. \end{cases}$

For $i = n-1,$ $\begin{cases} a_i &= \frac{1}{4}(11n+2) \\ b_i &= \frac{1}{4}(n+6) \\ s_i &= \frac{1}{4}(7n+6). \end{cases}$

with the exceptions $\begin{cases} s_{\frac{1}{4}(3n-2)} &= 2n-1 \\ s_{n-2} &= \frac{1}{4}(7n+2) \\ s_n &= \frac{1}{2}(3n+2). \end{cases}$

Research Problem 2.11 *Show that the wheel W_n with $n \equiv 2 (\mathrm{mod}\, 8)$ is edge-magic.*

Exercise 2.10 *The* Petersen graph *P consists of two 5-cycles $x_1x_2x_3x_4x_5$ and $y_1y_3y_5y_2y_4$, together with the five edges $x_1y_1, x_2y_2, x_3y_3, x_4y_4, x_5y_5$. What is the range of possible magic sums for an edge-magic total labeling of P? Prove that P is edge-magic.*

Exercise 2.11 *A* fan *F_n is constructed from a wheel W_n by deleting one arc. Prove that all fans are edge-magic.*

Research Problem 2.12 *A* helm *H_n is constructed from a wheel W_n by adding n vertices of degree 1, one adjacent to each terminal vertex. Which helms are edge-magic?*

Research Problem 2.13 *A* flower *F_n is constructed from a helm H_n by joining each vertex of degree 1 to the center. Which flowers are edge-magic?*

2.7 Trees

It has been conjectured ([29], also [44]) that all trees are edge-magic. However, this seems to be a difficult problem.

We have already seen that stars and and paths are edge-magic. A *caterpillar* is a graph derived from a path by hanging any number of leaves from

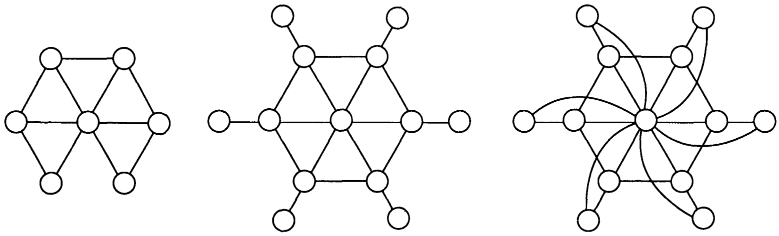

Figure 2.8. A fan, a helm and a flower.

the vertices of the path, so it can be seen as a sequence of stars where each star shares one edge with the next one. We shall show that all caterpillars are edge-magic.

The typical caterpillar is a graph $G = S^1 \cup S^2 \cup \ldots \cup S^n$, where S_i is a star with center c_i and e_i edges, and in every case S_i shares an edge with S_{i+1}. Then G has $e = \sum_1^n e_i - n + 1$ edges ($n - 1$ edges are shared by two stars), and $v = \sum_1^n e_i - n + 2$ vertices. c_i will be a leaf in S^{i-1} (unless $i = 1$) and in S^{i+1} (unless $i = n$), or equivalently the leaves of S^i will include c_{i-1} and c_{i+1}.

Theorem 2.28 [29] *All caterpillars are edge-magic.*

Proof. We describe an edge-magic total labeling λ of the caterpillar G described above. First the stars are ordered $S^1, S^3, S^5, \ldots, S^2, S^4, S^6, \ldots$ and then the leaves of the stars are labeled with the smallest positive integers, starting from 1 as a label on S^1 and ascending. When the leaves of S^i are labeled, c_{i-1} receives the smallest label (except when $i = 1$) and c_{i+1} the largest one. So the leaves of S^1 receive $1, 2, \ldots, e_1,$, with $\lambda(c_2) = e_1$, then the vertices of S_3 receive $v_1, v_1 + 1, \ldots, e_1 + e_3 - 1$, with $\lambda(c_2) = e_1$ and $\lambda(c_2) = e_1 + e_3 - 1$, and so on. This uses labels $1, 2, \ldots, \sum_1^n e_i - n + 2 = v$. Then the edges are labeled. The smallest available labels, namely $v + 1, v + 2, \ldots, v + e_n$, are applied to the edges of S^n, then the next e_{n-1} to the edges of S^{n-1}, and so on until the edges of S^1 are labeled. In each star, the smallest label is given to the edge whose perimeter vertex has the largest label, and so on. The labeling is illustrated in Figure 2.9.

Verification that the labeling has the edge-magic property is an easy exercise. □

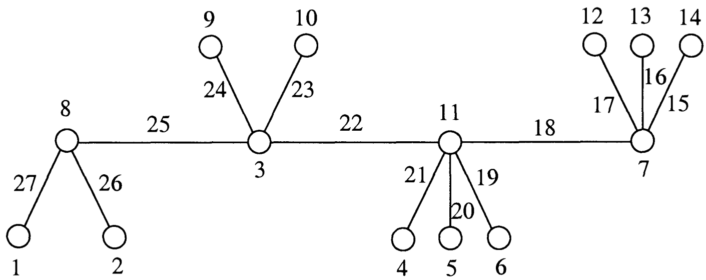

Figure 2.9. Labeling a caterpillar.

Exercise 2.12 *Verify that the labeling described in the proof of Theorem 2.9 is edge-magic.*

This exhausts our systematic knowledge of edge-magic total labelings of trees. However, Enomoto et al. [11] carried out a computer check and showed that all trees with fewer than 16 vertices are edge-magic. In fact, labelings were easy to find: every tree on v vertices had a strong edge-magic labeling.

Research Problem 2.14 *Are all trees edge-magic?*

2.8 Disconnected graphs

2.8.1 Some easy cases

Kotzig and Rosa [29] showed that the one-factor F_{2n}, consisting of n independent edges, is edge-magic if and only if n is odd (see Exercise 2.13), and nK_4 is not edge-magic for n odd (see Exercise 2.1).

Exercise 2.13 *Prove that the one-factor F_{2n}, consisting of n independent edges, is edge-magic if and only if n is odd.* [29]

Exercise 2.14 *Prove that $K_2 \cup C_3$ is not edge-magic, but $K_2 \cup C_4$ is edge-magic.*

Research Problem 2.15 *For which values of n is $K_2 \cup C_n$ edge-magic?*

Research Problem 2.16 *Prove or disprove that nK_4 is edge-magic when n is even.*

2.8.2 *Trichromatic graphs*

Partitionings of sets of graph vertices are usually called *colorings*. Instead of talking about a partition of the vertex-set V into three parts V_1, V_2, V_3, we take a set C of three *colors* and define a map $\xi : V \to \{1, 2, 3\}$. The two approaches are equivalent if V_i is defined as $\{x \mid \xi(x) = i\}$.

Formally, Suppose $C = \{c_1, c_2, \ldots\}$ is a set of undefined objects called *colors*. A *C-coloring* (or C-vertex coloring) ξ of a graph G is a map

$$\xi : V(G) \to C.$$

The sets $V_i = \{x : \xi(x) = c_i\}$ are called *color classes*. A *proper coloring* of G is a coloring in which no two adjacent vertices belong to the same color class. In other words,

$$x \sim y \Rightarrow \xi(x) \neq \xi(y).$$

A proper coloring is called an *n-coloring* if C has n elements. If G has an n-coloring, then G is called *n-colorable*. 2-colorable and 3-colorable graphs are called *bipartite* and *trichromatic*, respectively.

A *total coloring* is an assignment ξ of colors to the vertices and edges of a graph G. A total coloring is called *proper* if the colors on a vertex and all edges touch it contain no repeats.

Suppose G is a 3-colorable graph. Select a proper coloring $\xi : V(G) \to \{1, 2, 3\}$. Then one can define a 3-total coloring $\eta : V \cup E \to \{1, 2, 3\}$ by

$$\text{if } x \in V,\, \eta(x) = \xi(x)$$

$$\text{if } x \sim y,\, \{\eta(x)\eta(y), \eta(xy)\} = \{1, 2, 3\}.$$

Theorem 2.29 [27] *Say G is a 3-colorable edge-magic graph and H is the union of t disjoint copies of G, t odd. Then H is edge-magic.*

Proof. Suppose G has been totally 3-colored with a coloring η, as described above, and suppose λ is an edge-magic total labeling of G with magic sum k. Denote the copies of G by $G_0, G_1, \ldots, G_{2r}$, where $t = 2r + 1$, and write s for $v + e$. Write $A = a_{ij}$ for the matrix (1.2) that was constructed in Section 1.1.3. Then vertex x of G_i receives label

$$\lambda(x) + sa_{\eta(x),i}$$

and edge xy of G_i receives label

$$\lambda(xy) + sa_{\eta(xy),i}.$$

This is an edge-magic total labeling with magic sum $3sr + k$. □

The trichromatic graphs include all cycles and paths, so tC_v and tP_v are edge-magic for all odd t. (This was later independently proven by Wijaya and Baskoro [59], who gave a direct construction.)

The wheel W_n is trichromatic for n even. The union of an odd number of W_n's is not edge-magic when $n \equiv 3 \pmod 4$ (see Exercise 2.2), but the case $n \equiv 1 \pmod 4$ remains in doubt.

Research Problem 2.17 *Is tW_n edge-magic when $n \equiv 1 \pmod 4$ and t is odd?*

The single edge K_2 is of course trichromatic, so half of Exercise 2.13 follows from Theorem 2.29.

2.9 Strong edge-magic total labelings

Recall that a strong edge-magic total labeling is one in which the vertex labels are the integers $1, 2, \ldots, v$. As we noticed in Section 2.1.2, equation (2.3) can sometimes be used to show that a graph is not strongly edge-magic: if vertex x_i has degree d_i and is to receive label a_i, it is necessary to find an arrangement $\{a_i\}$ of the first v integers that makes $\sigma_0^{v+e} + \sum (d_i - 1)a_i$ divisible by e. We used this to show that the even cycles are not strongly edge-magic. However, Theorem 2.17 provides a strongly edge-magic labeling of every odd cycle.

Exercise 2.15 *Prove that all edge-magic one-factors are strongly edge-magic.*

Suppose G has a strong edge-magic total labeling λ. Excluding the trivial case where G has no edges, some edge receives label $v + e$, so the weight of that edge is at least $1 + 2 + (v + e)$. On the other hand, the edge with label $v + 1$ will have weight at most $(v - 1) + v + (v + 1)$. So

$$1 + 2 + (v + e) \leq k \leq (v - 1) + v + (v + 1). \qquad (2.20)$$

So we have

Lemma 2.30 [11] *Any strongly edge-magic graph other than K_1 satisfies $e \leq 2v - 3$.*

This simple lemma has a number of consequences. For example, no wheel is strongly edge-magic, nor is any complete graph with more than three vertices.

Lemma 2.30 also implies that no regular graph of degree greater than 3 can be strongly edge-magic. Suppose G is a regular graph of degree 3 with an edge-magic labeling λ. Necessarily v is even, say $v = 2n$. Then $e = 3n$. From equation (2.5), $6kn = 4s + 5n(5n + 1)$ where s is the sum of the vertex labels. If the labeling is strong, $s = n(2n + 1)$, so $6k = 4(2n + 1) + 5(5n + 1) = 33n + 9$, so n must be odd. The first case is $n = 3, v = 6$.

There are two cubic graphs on six vertices. One is the triangular prism, which is strongly edge-magic (one triangle receives labels 1, 2, 3, and the other receives labels 5, 6, 4 in the corresponding places). The other, $K_{3,3}$, has no strong edge-magic total labeling, as was seen in the complete search in Section 2.5.1. In fact, this is an instance of a stronger result:

Theorem 2.31 [11] *The complete bipartite graph $K_{m,n}$ is strongly edge-magic if and only if $m = 1$ or $n = 1$.*

Proof. One of the labelings in Theorem 2.26 is strong, so the "if" part is easy. Now assume $m \geq n > 1$. From Lemma 2.30, a strongly edge-magic $K_{m,n}$ must satisfy $mn \leq 2(m+n) - 3$, or $(m-2)(n-2) \leq 1$. So we only need to investigate cases $m = n = 3$ and $m = 2$.

We know from the complete enumerations that $K_{2,2}$ and $K_{3,3}$ are not strongly edge-magic. So assume $m = 2, n \geq 3$. (2.20) gives $3n + 5 \leq k \leq 3n + 6$, and these two cases are duals of each other. So let us assume that $K_{2,n}$ has a strong edge-magic total labeling with $k = 3n + 6$. Denote the two vertex sets as U and W.

The largest (edge) label is $3n + 2$, so x_1x_2 cannot be an edge. Say x_1 and x_2 belong to U. To accommodate the edge with label $3n + 1$, x_1x_3 must be an edge, so $x_3 \in W$. Then x_2x_3 is the edge with label $3n + 1$, so x_1x_4 is not an edge, and $x_4 \in U$. At this stage we see that the only possible edge with label $3n$ is x_1x_5. But then $x_5 \in W$, and the $K_{2,n}$ contains two edges with label $3n - 1$, namely x_2x_5 and x_3x_4. □

Exercise 2.16 *Prove that a graph G is strongly edge-magic if and only if there is a map λ from $V(G)$ onto $\{1, 2, \ldots, v\}$ such that*

$$\{\lambda(x) + \lambda(y) \mid xy \in E(G)\}$$

is a set of consecutive integers. [15]

Research Problem 2.18 *Which unions of disjoint cycles are strongly edge-magic?*

Exercise 2.17 *Suppose G is a bipartite graph with vertex-sets V_1 and V_2, of sizes v_1 and v_2, respectively. An edge-magic total labeling of G is* super-strong *if the elements of V_1 receive labels $\{1, 2, \ldots, v_1\}$ and the elements of V_2 receive $\{v_1 + 1, v_1 + 2, \ldots, v\}$. Prove that every super-strongly edge-magic bipartite graph satisfies $e \leq v - 1$.* [41]

2.10 Edge-magic injections

Recall that an edge-magic injection is like an edge-magic total labeling, except that the labels can be any positive integers. We define an $[m]$-edge-magic injection of G to be an edge-magic injection of G in which the largest label is m, and call m the *size* of the injection. The *edge deficiency* $def_e(G)$ of G is to be the minimum value of $m - v(G) - e(G)$, such that an $[m]$-edge-magic injection of G exists.

Theorem 2.32 *Every graph has an edge-magic injection.*

Proof. Suppose G is a graph with v vertices and e edges. The empty graph is trivially edge-magic, so we assume that G has at least one edge. Let$(a_1, a_2, \ldots, a_v)$ be any Sidon sequence of length v with first element $a_1 = 1$. Define $k = a_{v-1} + 2a_v + 1$.

We now construct a labeling λ as follows. Select any edge of G and label its endpoints with a_{v-1} and a_v, and label the remaining vertices with the other members of the Sidon sequence in any order. If xy is any edge, define $\lambda(xy) = k - \lambda(x) - \lambda(y)$. Every edge weight will be equal to k. The smallest edge label will be $k - a_{v-1} - a_v = a_v + 1$, which is greater than any vertex label. If two edge labels were equal, say $\lambda(xy) = \lambda(zt)$, then $\lambda(x) + \lambda(y) = \lambda(z) + \lambda(t)$, and as the labels of vertices are members of a Sidon sequence this implies that $xy = zt$. The vertex labels are distinct by definition. So λ is an edge-magic injection. □

The proof of Theorem 2.32 gives us an upper bound on the deficiency:

Corollary 2.32.1 *If G is a graph with v vertices and $(a_1, a_2, \ldots, a_v)$ is any Sidon sequence of length v with $a_1 = 1$, then*

$$def_e(G) \leq a_{v-1} + 2a_v - a_2 - v - e(G).$$

Proof. In the above construction, no label can be greater than $k - 1 - a_2$. □

This upper bound will not usually be very good. For example, consider the graph constructed from C_5 by joining two non-adjacent vertices. Using the Sidon sequence (1, 2, 3, 5, 8), a labeling with $k = 22$ is obtained, and the best assignment of the sequence to the vertices gives largest label 17, and deficiency 6. However, the graph is actually edge-magic. See Figure 2.10.

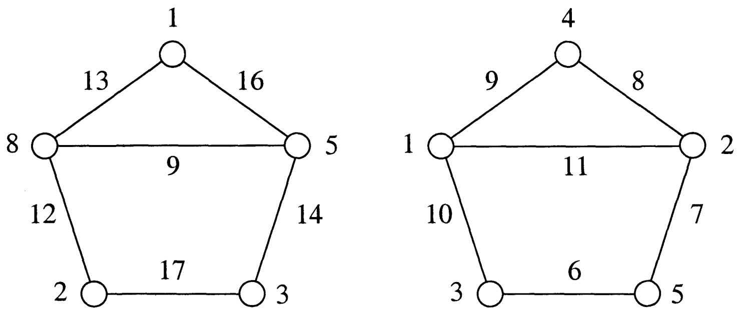

Figure 2.10. Deficiency 6 on left; magic on right.

Exercise 2.18 *If G is an incomplete graph with v vertices and $(a_1, a_2, \ldots, a_v)$ is any Sidon sequence of length v with $a_1 = 1$, prove that*

$$def_e(G) < a_{v-1} + 2a_v - a_2 - v - e(G).$$

In the case of complete graphs, we have essentially encountered the edge-magic deficiency already

Theorem 2.33 *The edge-magic deficiency of K_v equals the magic number $M(v)$.*

Proof. Consider an edge-magic total labeling λ of $K_v + M(v)K_1$. This graph has $v + M(v)$ vertices and $e(K_v)$ edges, so the largest label is $v + M(v) + e(K_v)$, and clearly this label occurs on a vertex or edge of K_v. The labeling constructed by restricting λ to K_v is an $[v + M(v) + e(K_v)]$-edge-magic injection of K_v. Obviously any injection of size $v+m+e(K_v)$ gives rise to an edge-magic total labeling of K_v+mK_1 (apply the m unused labels to the extra vertices), so $v + M(v) + e(K_v)$ is the smallest possible size, and $def_e(K_v) = M(v)$. □

From Theorem 2.33 it is clear that magic number and edge-magic deficiency are essentially equivalent, but the two concepts are rather different when applied to vertex-magic labelings.

Exercise 2.19 *Suppose G is a graph with v vertices. Prove that*

$$def_e(G) \leq M(v) + \binom{v}{2} - e(G).$$

Both Corollary 2.32.1 and Exercise 2.19 give crude upper bounds for the $def_e(G)$. Very little work has been done on finding good bounds for edge-magic deficiencies, and the only known families of graphs for which the exact values are known are the various families of edge-magic graphs (which of course have edge-magic deficiency 0).

3

Vertex-Magic Total Labelings

3.1 Basic ideas

3.1.1 *Definitions*

A one-to-one map λ from $E \cup V$ onto the integers $\{1, 2, \ldots, e + v\}$ is a *vertex-magic total labeling* if there is a constant h so that for every vertex x,

$$\lambda(x) + \sum \lambda(xy) = h \tag{3.1}$$

where the sum is over all vertices y adjacent to x. So the magic requirement is $wt(x) = h$ for all x. The constant h is called the *magic constant* for λ. Again, a graph with a vertex-magic total labeling will be called *vertex-magic*.

It is not hard to find examples of vertex-magic total labelings for some graphs. One labeling for the graph $K_4 - e$ is shown in Figure 3.1. On the other hand, not every graph has a labeling. For the graph K_2, $\lambda(x) \neq \lambda(y)$ implies $\lambda(x) + \lambda(xy) \neq \lambda(y) + \lambda(xy)$, so no labeling is possible. Similarly, any isolated vertex x must have $\lambda(x) = h$, so the prohibition of repeated labels means that there can be at most one isolate. These easy observations will be important, so we state them as a theorem:

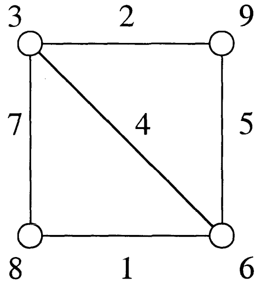

Figure 3.1. A vertex-magic total labeling of $K_4 - e$.

Theorem 3.1 *If G has an isolated edge or two isolated vertices, then G is not vertex-magic.*

Exercise 3.1 *Suppose we wish to define a labeling that is vertex-magic, and uses the labels* $\{0, 1, \ldots, v + e - 1\}$ *once each. Prove that in such a labeling, 0 cannot be the label of any vertex of degree 1. Prove that* P_3 *has no such labeling, but* P_4 *has one.*

3.1.2 Basic counting

Let s_v denote the sum of the vertex labels and s_e the sum of the edge labels in a vertex-magic total labeling λ. Clearly, since the labels are the numbers $1, 2, \ldots, v + e$, the sum of all labels is

$$s_v + s_e = \sigma_0^{v+e} = \binom{v+e+1}{2}.$$

At each vertex x_i we have $\lambda(x_i) + \sum \lambda(x_i y) = h$. Summing this over all v vertices x_i is equivalent to adding each vertex label once and each edge label twice, so

$$s_v + 2s_e = vh. \tag{3.2}$$

Combining these two equations gives

$$s_e + \binom{v+e+1}{2} = vh. \tag{3.3}$$

The edge labels are all distinct (as are all the vertex labels). The edges could conceivably receive the e smallest labels or, at the other extreme, the e largest labels, or anything between. Consequently we have

$$\sigma_0^e \le s_e \le \sigma_v^{v+e}. \tag{3.4}$$

A similar result holds for s_v. Combining (3.3) and (3.4), we get

$$\binom{v+e+1}{2} + \binom{e+1}{2} \le vh \le 2\binom{v+e+1}{2} - \binom{v+1}{2} \tag{3.5}$$

which gives the range of feasible values for h.

It is clear from (3.1) that when h is specified and the edge labels are known, then the vertex labels are determined. So the labeling is completely described by the edge labels. Surprisingly, however, the vertex labels do not completely determine the labeling. Having assigned the vertex labels to a graph, it may be possible to assign the edge labels to the graph in several different ways. Figure 3.2 shows two vertex-magic total labelings of W_4 that have the same vertex labeling but different edge labelings.

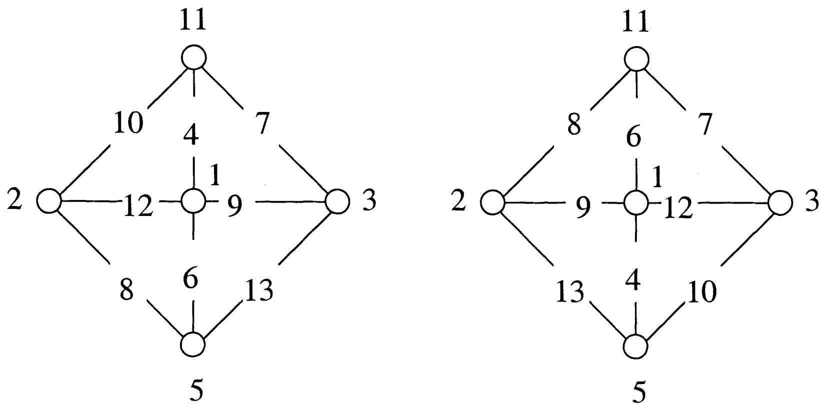

Figure 3.2. Vertex-magic total labelings of W_4 with the same vertex labels.

Research Problem 3.1 *Construct an infinite family of graphs, each having two vertex-magic total labelings with the same vertex labeling but different edge labelings.*

3.2 Regular graphs

If a *regular* graph possesses a vertex-magic total labeling, we can create a new vertex-magic total labeling from it. Given the vertex-magic total labeling λ for graph G, define the map λ' on $E \cup V$ by

$$\lambda'(x) = v + e + 1 - \lambda(x)$$

for any vertex x, and

$$\lambda'(xy) = v + e + 1 - \lambda(xy)$$

for any edge xy. Clearly λ' is also a one-to-one map from the set $E \cup V$ to $\{1, 2, \ldots, e + v\}$. Just as in the case of edge-magic total labelings, we shall call λ' the *dual* of λ. In contrast to the edge-magic case, we have the following theorem:

Theorem 3.2 *The dual of a vertex-magic total labeling for a graph G is vertex-magic if and only if G is regular.*

Proof. Suppose λ is a vertex-magic total labeling for G in which $wt(x) = h$. Then the dual λ' satisfies

$$\begin{aligned} wt_{\lambda'}(x) &= \lambda'(x) + \sum \lambda'(xy) \\ &= v + e + 1 - \lambda(x) + \sum (v + e + 1 - \lambda(xy)) \\ &= (r + 1)(v + e + 1) - h \end{aligned}$$

where r is the number of edges incident at x. Clearly this is constant if and only if r is constant, in other words, if and only if G is regular. Then $(r + 1)(v + e + 1) - h$ is the magic constant for the dual. □

The general problem of whether one can use a vertex-magic total labeling of a graph G to produce such a labeling of some subgraph or supergraph of G appears to be very difficult. The next theorem answers a very special case of this question for regular graphs.

Theorem 3.3 *Let G be a regular graph having a vertex-magic total labeling in which the label 1 is assigned to some edge e'. Then the graph $G - e'$ has a vertex-magic total labeling.*

Proof. Suppose G is an r-regular graph. Define a new mapping Λ by $\Lambda(x_i) = \lambda(x_i) - 1$ and $\Lambda(xy) = \lambda(xy) - 1$. Now Λ is a one-to-one mapping from $E \cup V$ to $\{0, 1, 2, \ldots, e + v - 1\}$. If we now delete the edge e' (which is labeled 0 by Λ) from G, then Λ is a one-to-one map from $(E - e') \cup V$ to $\{0, 1, 2, \ldots, e + v - 1\}$. Also the weight of x under Λ is

$$\begin{aligned} wt_{\Lambda}(x) &= \lambda(x) - 1 + \sum (\lambda(xy) - 1) \\ &= wt_{\lambda}(x) - r - 1 \\ &= h - r - 1. \end{aligned}$$

Thus $wt_\Lambda(x)$ is a constant and Λ is a vertex-magic total labeling for $G-e'$. □

Theorem 3.4 *If G is a regular graph and e an edge such that $G-e$ has a vertex-magic total labeling, then that labeling is derived from a vertex-magic total labeling of G by the process described in Theorem* 3.3.

Proof. Suppose G is an r-regular graph and let λ be any labeling for $G-e$ where e is the edge xy. Adjoin the edge xy to $G-e$ and define $\lambda(xy)=0$. Now adding 1 to all the labels defines a new mapping λ' that is easily seen to be a vertex-magic total labeling of G with label 1 on edge xy. If the magic constant of λ is h, then the magic constant of λ' is $h+r+1$. □

3.3 Cycles and paths

The easiest regular graphs to deal with are the cycles. For cycles (and only for cycles) a vertex-magic total labeling is equivalent to an edge-magic total labeling, and the edge-magic labelings have already received some attention above.

Theorem 3.5 *The n-cycle C_n has a vertex-magic total labeling for any $n \geq 3$.*

Proof. Suppose C_n is the cycle $(x_1, x_2, \ldots, x_n)$. The constructions of Theorems 2.17, 2.18, 2.19 and 2.20 provide an edge-magic total labeling λ' for C_n for every $n \geq 3$. Denote the magic constant of λ' by h. Then define a new mapping λ by $\lambda(x_i)=\lambda'(x_ix_{i+1})$ and $\lambda(x_ix_{i+1})=\lambda'(x_{i+1})$, where the subscripts are integers modulo n. Clearly h is the weight at each vertex, and λ is a vertex-magic total labeling of G. □

Corollary 3.5.1 *P_n, the path with n vertices, is vertex-magic for any $n \geq 3$.*

Proof. For each $n \geq 3$ at least one of the edge-magic labelings λ' referred to in the proof of the theorem assigns the label 1 to a vertex. Then the corresponding vertex-magic labeling λ will assign the label 1 to some edge e. By Theorem 3.3, $C_n - e$ will have a vertex-magic labeling. □

Corollary 3.5.2 *Every vertex-magic total labeling of P_n is derived from a vertex-magic total labeling of C_n.*

Proof. This follows immediately from Theorem 3.4. □

Using equations (3.3) and (3.4), we can readily determine the feasible values of h for the n-cycle. We find

$$\frac{5v+3}{2} \leq h \leq \frac{7v+3}{2}.$$

A systematic search has found exactly 4 labelings for the 3-cycle, one for each feasible value of h. As mentioned above, once h is given, the edge labels completely determine the labeling, so we list only the edge labels.

$$\begin{array}{rclc} h & = & 9 & 1,2,3 \\ h & = & 10 & 1,3,5 \\ h & = & 11 & 2,4,6 \\ h & = & 12 & 4,5,6 \end{array}$$

Since cycles are regular graphs, the duality described in Section 3.2 applies. The labeling with $h = 12$ is dual to the labeling with $h = 9$, and the labeling with $h = 11$ is dual to that with $h = 10$.

There are exactly six vertex-magic total labelings for the 4-cycle and again they come in dual pairs. Once more every feasible value of h admits a labeling. They are listed below, with the edge labels given in cyclic order:

$$\begin{array}{rclc} h & = & 12 & 1,3,2,6 \\ h & = & 13 & 1,4,6,5 \\ h & = & 13 & 1,5,2,8 \\ h & = & 14 & 3,4,8,5 \\ h & = & 14 & 1,7,4,8 \\ h & = & 15 & 3,7,6,8. \end{array}$$

For the 5-cycle, we find $14 \leq h \leq 19$. There are again 6 labelings. None exists corresponding to $h = 15$ or $h = 18$. For $h = 14$ there is a unique solution $(1,4,2,5,3)$, and for $h = 16$ we find $(1,5,9,3,7)$ and also $(1,7,3,4,10)$. Each of these has a dual.

The notion of vertex-magic labeling was at least partially suggested by the following question which appeared on a set of mathematical enrichment problems for high school students. [50] (The problem appeared at the time when corruption charges against International Olympic Committee members were dominating the news):

> The Olympic emblem consists of five overlapping rings containing 9 regions. In order to contribute to a pension fund for a retiring IOC delegate, people are asked to deposit money into each region. The guidelines allow the delegate to take all the money in any one of the rings. Place \$1, \$2, . . . , \$9 in the nine regions so that the amount in each ring is the same.

This problem can be interpreted as asking for a labeling on the path of 5 vertices; one of the solutions the students found is shown in Figure 3.3.

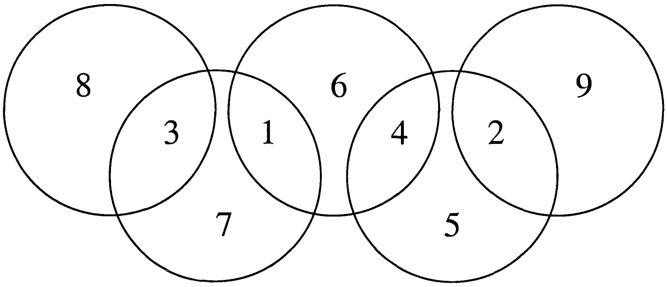

Figure 3.3. Solution of the olympic rings problem.

In the case of a path with v vertices, we have $e = v - 1$ and $v + e = 2v - 1$, and therefore

$$s_e + v(2v - 1) = vh.$$

So $s_e \equiv 0 \pmod v$ and consequently, from (3.2), $s_v \equiv 0 \pmod v$ also. Equations (3.3) and (3.4) then give

$$\frac{v(v-1)}{2} \leq S_e \leq 3\frac{v(v-1)}{2}$$

from which it follows that

$$\frac{5v-3}{2} \leq h \leq \frac{7v-5}{2}. \tag{3.6}$$

For a path with 3 vertices, (3.6) implies that $6 \leq h \leq 8$. According to Theorem 3.4, the vertex-magic total labelings of P_3 will be derived from those of C_3 that assign the label 1 to an edge. There are two such listed previously so there are two vertex-magic total labelings for P_3; they have sums 6 and 7. There is no vertex-magic total labeling of P_3 with magic constant $h = 8$.

For the path with four vertices, equation (3.6) implies $9 \leq h \leq 11$. Four of the six labelings of the 4-cycle listed above provide a derived labeling for the 4-path. This time there is a labeling for all feasible values of h (we list the edge labels only):

$$\begin{array}{rclc} h &=& 9 & 2,1,5 \\ h &=& 10 & 4,5,3 \\ h &=& 10 & 4,1,7 \\ h &=& 11 & 6,3,7. \end{array}$$

For $v = 5$, equation (5) implies $11 \leq h \leq 15$. There are again four labelings; they correspond to $h = 11$, 13 and 14. There is no labeling with $h = 12$ or $h = 15$.

$$\begin{array}{rcll} h &=& 11 & 2,4,1,3 \\ h &=& 12 & \text{none} \\ h &=& 13 & 4,8,2,6 \\ h &=& 13 & 9,3,2,6 \\ h &=& 14 & 6,7,3,9 \\ h &=& 15 & \text{none.} \end{array}$$

Further systematic search discovered ten labelings for $v = 6$; there are 66 for $v = 7$, and for $v = 8$ there are 131 labelings.

Exercise 3.2 *Prove that there is no vertex-magic total labeling of P_3 with magic constant $h = 8$.*

Exercise 3.3 *Find a vertex-magic total labeling of the Petersen graph.*

Exercise 3.4 *The triangular book $B_{3,n}$ was defined in Exercise 2.8. Prove that B_n is vertex-magic if and only if $n \leq 3$.*

3.4 Vertex-magic total labelings of wheels

3.4.1 Large wheels cannot be labeled

Suppose W_n is the wheel whose n rim vertices form the cycle $(x_1, x_2, \ldots, x_n)$. W_n has $v = n + 1$ and $e = 2n$, so the labels are the numbers in the set $\{1, 2, \ldots, 3n + 1\}$. The inequalities (3.4) yield

$$\frac{13n^2 + 7n + 3}{2(n + 1)} \leq h \leq \frac{17n^2 + 15n + 4}{2(n + 1)} \tag{3.7}$$

but these inequalities become largely irrelevant when the structure of W_n is taken into account. In fact, the permissible values of h are determined by the degree of the hub vertex. If the degree of the hub vertex is too high, in other words n is too large, then no labeling is possible:

Theorem 3.6 *The wheel W_n has no vertex-magic total labeling when $n >$ 11.*

Proof. Denote the hub vertex of W_n by u and let $x_1, \ldots, x_n$ be the rim vertices. Then

$$\begin{aligned} h &= wt(u) \\ &\geq \sigma_0^{n+1} \\ &= \tfrac{1}{2}(n + 1)(n + 2). \end{aligned} \tag{3.8}$$

Next we consider the sum of weights of all the rim vertices. An upper bound for this sum is found by assigning the n largest labels to the rim edges (because they are each counted twice), and the $2n$ next largest labels to the rim vertices and the spoke edges. We find that

$$\begin{aligned} wt(x_1) + \cdots + wt(x_n) &\leq \sigma_1^{2n+1} + 2\sigma_{2n+1}^{3n+1} \\ &= \sigma_1^{3n+1} + \sigma_{2n+1}^{3n+1} \\ &= n(7n + 6). \end{aligned}$$

Since there are n rim vertices,

$$h \leq 7n + 6. \tag{3.9}$$

It is easy to see that expression (3.9) is less than expression (3.8) for all $n > 11$. So no labeling is possible. □

Exercise 3.5 *Prove that the* fan F_n *has no vertex-magic total labeling when* $n > 10$. [35]

Exercise 3.6 *The* friendship graph T_n *consists of n triangles with a common vertex. Prove that* T_n *is vertex-magic if and only if* $n \leq 3$. [35]

3.4.2 Small wheels can have many labelings

The labelings for W_3, W_4 and W_5 have been enumerated using a computer search. The feasible values of h are determined by whichever of 3.7 or 3.9 and 3.8 are the more restrictive.

h	N3(h)
19	-
20	2
21	5
22	-
23	5
24	2
25	-

h	N4(h)
26	89
27	149
28	522
29	376
30	573
31	211
32	131
33	29

h	N5(h)
32	239
33	1242
34	2694
35	5180
36	7873
37	7173
38	4124
39	2511
40	776
41	80

3.4.3 Generalizations of wheels

Suppose G is the graph derived from a wheel by duplicating the hub vertex one or more times. We call G a *t-fold wheel* if there are t hub vertices, each adjacent to all rim vertices, and not adjacent to each other. We can do a calculation like that in the proof of Theorem 3.6 and a similar result is obtained:

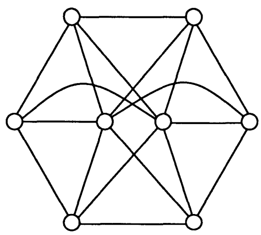

Figure 3.4. A 2-fold wheel.

Theorem 3.7 *Let G be a t-fold wheel with n rim vertices. There is a function $f(t)$ where, for $n > f(t)$, no labeling of G exists.*

Proof. In this case $v = n + t$ and $e = (t + 1)n$, so $v + e = (t + 2)n + t$. Assigning the smallest possible labels to the t hub vertices $u_1, \ldots, u_t$ and the spoke edges, we find that

$$\begin{aligned} wt(u_1) + \cdots + wt(u_t) &\geq \sum_{1}^{tn+t} i \\ &= \frac{(tn+t)(tn+t+1)}{2} \end{aligned}$$

so that

$$h \geq \tfrac{1}{2}(n+1)(t(n+1)+1) \tag{3.10}$$

Assigning the n largest possible labels to the rim edges and the next $(t+1)n$ largest labels to the rim vertices and the spoke edges (that is, all but the hub vertices), we have

$$\begin{aligned} wt(v_1) + \cdots + wt(v_n) &\leq \sum_{t+1}^{M} i + \sum_{M-n+1}^{M} i \\ &= 2\sigma_0^{v+e} - \sigma_0^{t} - \sigma_0^{v+e-n} \\ &= \tfrac{1}{2}n\{n(t+6)(t+1) + (t+3)(2t+1)\}. \end{aligned}$$

Since there are n rim vertices, we have

$$h \leq \frac{(t+6)(t+1)n + (t+3)(2t+1)}{2}. \tag{3.11}$$

No labeling will be possible whenever the expression from 3.11 is less than that from equation 3.10, i.e., when the following inequality in n is satisfied:

$$tn^2 - (t^2 + 2t + 3)n - (2t^2 + 4t + 1) > 0.$$

If we let $f(t)$ denote the larger root of this quadratic, the theorem follows. $\square$

We note that the function $f(t)$ is asymptotically linear in t. When $t = 1$, we get the result from the previous theorem. For double wheels ($t = 2$), there is no labeling for $n > 10$; for wheels with 3 hubs, no labeling exists for $n > 10$; and for wheels with 4 hubs, no labeling exists for $n > 11$.

3.5 Vertex-magic total labelings of complete bipartite graphs

We shall take the complete bipartite graph $K_{m,n}$ to have vertex-set

$$\{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n\}$$

and edge-set

$$\{x_i y_j : 1 \le i \le m, 1 \le j \le n\}.$$

So a vertex-magic total labeling λ of $K_{m,n}$ can be represented by an $m + 1 \times n + 1$ array

$$A = \left[\begin{array}{c|cccc} a_{00} & a_{01} & a_{02} & \cdots & a_{0n} \\ \hline a_{10} & a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{m0} & a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right] \tag{3.12}$$

where

$$\begin{aligned} a_{00} &= 0 & a_{i0} &= \lambda(x_i) \\ a_{0j} &= \lambda(y_j) & a_{ij} &= \lambda(x_i y_j). \end{aligned}$$

The matrix A will be called the *representation matrix* of λ. The magic requirement is that all row-sums and column-sums, except for row 0 and column 0, must be equal (to h say), and that the $(m + 1)(n + 1)$ entries are $\{0, 1, \ldots, mn + m + n\}$ in some order.

We shall call a $K_{m,n}$ *unbalanced* if its parts differ in size by more than 1. We observe that an unbalanced $K_{m,n}$ cannot have a vertex-magic total labeling:

Theorem 3.8 [34] *If $K_{m,n}$ is unbalanced, then it has no vertex-magic total labeling.*

Proof. Without loss of generality, assume $m \le n$. Suppose $K_{m,n}$ has a vertex-magic total labeling with magic constant h. For this graph $v = m+n$ and $e = mn$ so the label set is $\{1, 2, \ldots, mn + m + n\}$. The sum of the weights on $\{x_1, \ldots, x_m\}$ is at least the sum of all but the largest n labels, so

$$\begin{aligned} mh &\geq \sigma_0^{mn+m} \\ &= \frac{(mn+m)(mn+m+1)}{2}; \\ h &\geq \frac{(n+1)(mn+m+1)}{2}. \end{aligned} \tag{3.13}$$

On the other hand, the sum of the weights on $\{y_1, \ldots, y_n\}$ is at most the total of all but the m smallest labels:

$$\begin{aligned} nh &\leq \sigma_m^{mn+m+n} \\ &= \frac{(mn+m+n)(mn+m+n+1) - m(m+1)}{2} \\ &= \frac{(mn^2+2mn+n^2+n)(m+1)}{2}; \\ h &\leq \frac{(mn+2m+n+1)(m+1)}{2}. \end{aligned} \tag{3.14}$$

Combining(3.13) and (3.14),

$$(n+1)(mn+m+1) \leq (mn+2m+n+1)(m+1),$$

and on simplifying one obtains $m \geq n - 2 + \frac{2}{n+2}$, so $m \geq n - 1$. □

In particular, the only star that can have a vertex-magic total labeling is $K_{1,2}$.

3.5.1 *Construction of vertex-magic total labelings of $K_{m,n}$*

We now give constructions for vertex-magic total labelings of complete bipartite graphs in the cases not eliminated by Theorem 3.8.

Labeling $K_{m,m}$

Theorem 3.9 [34] *For every $m > 1$, $K_{m,m}$ has a vertex-magic total labeling with magic constant $\frac{1}{2}[(m+1)^3 - (m+1)]$.*

Proof. Let $S = (s_{ij})$ be any RCmagic square of order $m+1$ on the numbers $\{1, \ldots, (m+1)^2\}$. (For convenience, assume that the rows and columns of

S are numbered $0, 1, \ldots, m$.) Each row and column sums to the magic square constant $\frac{1}{2}(m+1)(m^2+2m+2)$. Form the matrix $A = (a_{ij})$ where $a_{ij} = s_{ij} - 1$. Since S is magic, the rows and columns of A will each sum to the constant

$$h = \tfrac{1}{2}(m+1)(m^2+2m+2) - (m+1) \tag{3.15}$$

and the entries of A will be the numbers in $\{0, \ldots, (m+1)^2 - 1\}$, once each. There are standard constructions ([1], [49]) for magic squares of all orders. We shall assume that the rows and columns of A are permuted so that $a_{00} = 0$. Then A is the representation matrix of a vertex-magic total labeling λ with h given by equation (3.15). The magic constant is easily checked. □

Labeling $K_{m,m+1}$, m odd

A solution for $K_{1,2}$ is easily constructed. So let us write $m = 2n - 1$ where $n > 1$. The construction proceeds for a given n by defining two $2n-1 \times 2n$ matrices, $A = (a_{ij})$ and $B = (b_{ij})$, and then using them to construct a $2n \times 2n + 1$ representation matrix C. For consistency with the earlier notation, C has first row and column indexed with 0.

The value of a_{ij} depends on the parity of i and j, as well as their values. The formula is

$$\begin{array}{rcll} a_{ij} & = & m+1-j & \text{if} \quad i+j \text{ is odd, } j+i \le m+1 \\ & & & \text{or} \quad i+j \text{ is even, } j+i > m+1, \\ a_{ij} & = & j-1 & \text{otherwise.} \end{array}$$

We shall need the row and column sums of this matrix. If i is even, say $i = 2t$, the sum of elements in row i is

$$\begin{aligned} \sum_{j=1}^{2n} a_{ij} &= \sum_{h=1}^{n} (a_{i,2h-1} + a_{i,2h}) \\ &= \sum_{h=1}^{n-t} (a_{i,2h-1} + a_{i,2h}) + \sum_{h=n-t+1}^{n} (a_{i,2h-1} + a_{i,2h}) \\ &= \sum_{h=1}^{n-t} ((2n-2h+1) + (2h-1)) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{h=n-t+1}^{n} ((2h-2) + (2n-2h)) \\
&= \sum_{h=1}^{n-t} 2n + \sum_{h=n-t+1}^{n} (2n-2) \\
&= 2n^2 - i.
\end{aligned}
$$

If i is odd, say $i = 2t + 1$,

$$
\begin{aligned}
&\sum_{j=1}^{2n} a_{ij} \\
&= \sum_{h=1}^{n} (a_{i,2h-1} + a_{i,2h}) \\
&= \sum_{h=1}^{n-t} a_{i,2h-1} + \sum_{h=1}^{n-t-1} a_{i,2h} + \sum_{h=n-t+1}^{n} a_{i,2h-1} + \sum_{h=n-t}^{n} a_{i,2h} \\
&= \sum_{h=1}^{n-t} (2h-2) + \sum_{h=1}^{n-t-1} (2n-2h) \\
&\quad + \sum_{h=n-t+1}^{n} (2n-2h+1) + \sum_{h=n-t}^{n} (2h-1) \\
&= 2n^2 - i,
\end{aligned}
$$

so in either case row i has sum $2n^2 - i$. The column sum is more easily calculated: column j contains n entries equal to $j - 1$ and $n - 1$ equal to $2n - j$. So for each j,

$$
\begin{aligned}
\sum_{i=1}^{2n-1} a_{ij} &= n(j-1) + (n-1)(2n-j) \\
&= 2n^2 - 3n + j.
\end{aligned}
$$

Matrix B has first and last columns $2n-2, 2n-3, \ldots, 1, 0$:

$$b_{1j} = b_{mj} = m - i - 1.$$

The second column begins with the odd integers $2n-5, 2n-7, \ldots, 1$, then has the even integers $2n-2, 2n-4, \ldots, 0$ and ends with $2n-3$. The other columns are formed by back circulating this column. That is,

$$b_{i,j} = b_{i+1,j-1}$$

(with first subscript reduced mod $(2n)$ and second subscript reduced mod $(2n+1)$ where necessary).

In each row of B, columns 2 through $2n-1$ contain all the integers $0, 1, \ldots, m-2$ except

$$\begin{aligned} x_i &= 2n-1-2i \text{ is missing from row } i,\ i = 1, \ldots, n-1, \\ x_i &= 4n-2i-2 \text{ is missing from row } i,\ i = n, n+1, \ldots, 2n-1. \end{aligned}$$

So the row sums are

$$\begin{aligned} \sum_{j=1}^{m} b_{ij} &= 2(2n-i-1) + \sum_{h=0}^{2n-2} h - x_i \\ &= 2n^2 + n - 1 - 2i - x_i \\ \sum_{j=1}^{m} b_{ij} &= 2n^2 - n \text{ if } i \leq n-1, \\ \sum_{j=1}^{m} b_{ij} &= 2n^2 - 3n + 1 \text{ if } i \geq n. \end{aligned}$$

Since each column is a permutation of $\{0, 1, \ldots, 2n-2\}$, each column sum is

$$\sum_{i=1}^{2n-1} b_{ij} = 2n^2 - 3n + 1.$$

We now define a $2n \times (2n+1)$ matrix C by

$$\begin{aligned} c_{00} &= 0, \\ c_{0j} &= 4n^2 + 2n - j,\ 1 \leq j \leq 2n, \\ c_{i0} &= i,\ 1 \leq i \leq n-1, \\ &= 4n^2 - 2n + i,\ n \leq i \leq 2n-1, \\ c_{ij} &= a_{ij} + 2nb_{ij} + n,\ 1 \leq i \leq 2n-1, 1 \leq j \leq 2n. \end{aligned}$$

Theorem 3.10 *The matrix C is the representation matrix of a vertex-magic total labeling of $K_{2n-1,2n}$ with magic constant $4n^3 + 2n^2$.*

Proof. It is necessary to show that the sum of entries in every row and column of C (except possibly row 0 and column 0) equals $4n^3 + 2n^2$ and that every integer from 0 to $v + e = 4n^2 + 2n - 1$ occurs exactly once in C. (We know that 0 appears in the $(0, 0)$ position, as required.)

The row sums of C are

$$\begin{aligned}\sum_{j=0}^{2n} c_{ij} &= c_{i0} + \sum_{j=1}^{2n} a_{ij} + 2n\sum_{j=1}^{2n} b_{ij} + 2n^2 \\ &= c_{i0} + 2n^2 - i + 2n\sum_{j=1}^{2n} b_{ij} + 2n^2 \\ &= 4n^3 + 2n^2,\end{aligned}$$

after inserting the appropriate values of c_{i0} and $\sum_{j=1}^{2n} b_{ij}$, depending on whether or not $i \leq n-1$. Similarly the column sums are

$$\begin{aligned}\sum_{i=0}^{2n-1} c_{ij} &= c_{0j} + \sum_{i=1}^{2n-1} a_{ij} + 2n\sum_{i=1}^{2n-1} b_{ij} + (2n-1)n \\ &= 4n^2 + 2n - j + 2n^2 - 3n + j \\ &\quad + 2n(2n^2 - 3n + 1) + n(2n-1) \\ &= 4n^3 + 2n^2.\end{aligned}$$

Thus all row and column sums (except the first) equal $4n^3 + 2n^2$, as required.

Finally, we prove that each integer from 0 to $4n^2 - 2n - 1$ appears exactly once in C. The numbers $0, 1, 2, \ldots, n-1$ and $4n^2 - n, 4n^2 - n + 1, \ldots, 4n^2 + 2n - 1$ appear in the first row and column. The entries in A lie in the (closed) interval $[0, 2n-1]$ and those of B lie in the interval $[0, 2n-2]$. Thus the entries in C outside the first row and column lie in the interval $[n, 3n-1+2n(2n-2)] = [n, 4n^2-n-1]$. There are $4n^2-2n$ such integers so it remains to show that the entries are distinct. To prove this we need to show that the pairs (a_{ij}, b_{ij}) are distinct. The first and last columns of A contain only the integers 0 and $2n-1$, the first and last columns of B contain the integers $0, 1, \ldots, 2n-2$, and it is easy to see that there are no repeated pairs. For the rest of the matrices A and B, note that the entries b_{ij} for a fixed value of $i + j \mod (2n-1)$ are constant, while in matrix A the equivalent entries take all the values $1, 2, \ldots, 2n-2$. Thus all pairs are distinct and so each integer from 1 to $4n^2 - 2n - 1$ appears exactly once in C. □

Labeling $K_{m,m+1}$, m even

In this case we write $m = 2n$. Then $v = 4n + 1$, $e = 4n^2 + 2n$, and a total labeling requires $4n^2 + 6n + 1$ labels.

Theorem 3.11 *There exists a vertex-magic total labeling of $K_{2n,2n+1}$ with magic constant $(n + 1)(2n + 1)^2$.*

Proof. We construct a representation matrix $C = (c_{ij})$ for a vertex-magic total labeling of $K_{2n,2n+1}$ as follows:

(i) Row 0 of C is $0, (2n + 1)^2, (2n + 1)^2 + 1, \dots, (2n + 1)^2 + 2n$, that is $c_{00} = 0$ and $c_{0j} = (2n + 1)^2 + j - 1$ for $1 \le j \le 2n + 1$.

(ii) $c_{i0} = (2n + 2)i$, $1 \le i \le 2n$.

(iii) If $1 \le i < n$ and $1 \le j \le n + 1$, or if $n + 2 \le i \le 2n$ and $n + 2 \le j \le 2n + 1$, then

$$c_{ij} = 2n(2n + 2) - [j + (i - 1)(2n + 2)].$$

(iv) If $1 \le i < n$ and $n + 2 \le j \le 2n + 1$, or if $n + 1 < i \le n$ and $1 \le j \le n + 1$, then

$$c_{ij} = j + (i - 1)(2n + 2).$$

(v) If $1 \le j \le n + 1$, then

$$c_{nj} = n(2n + 2) + 2n - 2j + 3, \quad c_{n+1,j} = (n - 1)(2n + 2) + n + j.$$

If $n + 2 \le j \le 2n + 1$, then

$$c_{nj} = (n - 1)(2n + 2) + 4n - 2j + 4 \;\; c_{n+1,j} = n(2n + 2) + j - n - 1.$$

Part (v) can also be expressed as follows: (except for column 0) rows 0, n and $n + 1$ of C are derived from rows of

$$X = \left[\begin{array}{ccccc|cccc} 1 & 2 & 3 & \dots & n+1 & n+2 & n+3 & \dots & 2n+1 \\ 2n+1 & 2n-1 & 2n-3 & \dots & 1 & 2n & 2n-2 & \dots & 2 \\ n+1 & n+2 & n+3 & \dots & 2n+1 & 1 & 2 & \dots & n \end{array}\right]$$

by adding

$$\begin{bmatrix} (2n+1)^2-1 \\ n(2n+2) \\ (n-1)(2n+2) \end{bmatrix}$$

to each of the first $n+1$ columns and

$$\begin{bmatrix} (2n+1)^2-1 \\ (n-1)(2n+2) \\ n(2n+2) \end{bmatrix}$$

to the remainder. Each row of X is a permutation of $\{1, 2, \ldots, 2n+1\}$, so rows n and $n+1$ between them contain each of $(n-1)(2n+2)+1$, $(n-1)(2n+2)+2, \ldots, (n+1)(2n+2)$ exactly once. When $1 \le i < n$, rows i and $2n+1-i$ contain between them all integers

$$\begin{array}{rcl} t+(i-1)(2n+2) & : & 1 \le t \le 2n+2, \\ t+(2n-i)(2n+2) & : & 1 \le t \le 2n+2 \end{array}$$

precisely once each ($2n+2+(i-1)(2n+2) = i(2n+2) = c_{i0}$, $2n+2+(2n-i)(2n+2) = c_{2n+1-i,0}$, and the others are given by (iii) and (iv)). Row 0 provides 0, $(2n+1)^2$, $(2n+1)^2+1, \ldots, (2n+1)^2+2n = 4n^2+6n+1$. So C contains each of $0, 1, \ldots, 4n^2+6n+1$ exactly once.

From (iii) and (iv) it also follows that

$$\begin{aligned} c_{ij} + c_{2n+1-i,j} &= j+(i-1)(2n+2)+2n(2n+2) \\ &\quad -[j+(i-1)(2n+2)] \\ &= 2n(2n+2) \end{aligned}$$

for $1 \le i < n$. Each column of X has sum $3n+3$, so $c_{nj} + c_{n+1,j} + c_{0,j} = (n+1)(8n+1)$ for $1 \le j \le 2n+1$. Therefore the sum of column j is

$$\begin{aligned} \sum_{i=0}^{2n} c_{ij} &= (n-1)2n(2n+2) + c_{nj} + c_{n+1,j} + c_{0,j} \\ &= (n-1)2n(2n+2) + (n+1)(8n+1) \\ &= (n+1)(2n+1)^2. \end{aligned}$$

If $i \le n$,

$$\begin{aligned}
&\sum_{j=0}^{2n+1} c_{ij} \\
= &\sum_{j=1}^{2n+1} j + c_{i0} + n(i-1)(2n+2) + (n+1)(2n-i)(2n+2) \\
= &\binom{2n+2}{2} + (2n+2)i + (2n+2)[n(i-1) + (n+1)(2n-i)] \\
= &(n+1)(2n+1) + 2(n+1)n(2n+1) \\
= &(n+1)(2n+1)^2,
\end{aligned}$$

and a similar calculation gives the same sum if $i \geq n+1$. □

Remark. The array X is essentially the Kotzig array defined in (1.2). It is remarkable that Kotzig produced the array in solving an *edge*-magic total labeling problem, which did not involve complete bipartite graphs. This surprising connection was not noticed until after Theorem 3.11 had been proven, and it was then noticed that Kotzig's array had the required properties.

3.5.2 The spectrum

In those cases where magic labelings are known to exist, it is interesting to know the set of values h such that there is a magic labeling with magic constant h. This is the *spectrum* of the labeling problem.

Suppose G has a vertex-magic total labeling λ. As usual, s_e is the sum of edge-labels: $s_e = \sum_{x \in E(G)} \lambda(x)$. Then, counting the sum of labels at all vertices, we have from (3.3) and (3.4)

$$s_e + \binom{v+e+1}{2} = vh, \tag{3.16}$$

$$\binom{e+1}{2} \leq s_e \leq \binom{e+1}{2} + ve. \tag{3.17}$$

These equations can be used to put bounds on the spectrum of vertex-magic total labelings. For $K_{m,m}$, (3.16) and (3.17) give

$$\frac{1}{2}[(m+1)^3 - m^2] \leq h \leq \frac{1}{2}[(m+1)^3 + m^2]. \tag{3.18}$$

As $K_{m,m}$ is regular, the duality Theorem, Theorem 3.2, applies, so there will be a vertex-magic total labeling with magic constant $\frac{1}{2}[(m+1)^3+x]$ if and only if there is one with $h=\frac{1}{2}[(m+1)^3-x]$.

For $K_{2,2}$, (3.18) yields $12 \le h \le 15$, and a complete search shows that every value can be realized. In fact, there are exactly six vertex-magic total labelings of $K_{2,2}$ (up to isomorphism). The representation matrices are

$$h=12: \quad \left[\begin{array}{c|cc} 0 & 5 & 7 \\ \hline 8 & 1 & 3 \\ 4 & 6 & 2 \end{array}\right] \qquad h=13: \quad \left[\begin{array}{c|cc} 0 & 7 & 3 \\ \hline 8 & 1 & 4 \\ 2 & 5 & 6 \end{array}\right] \qquad h=13: \quad \left[\begin{array}{c|cc} 0 & 4 & 6 \\ \hline 7 & 1 & 5 \\ 3 & 8 & 2 \end{array}\right]$$

$$h=14: \quad \left[\begin{array}{c|cc} 0 & 6 & 2 \\ \hline 7 & 3 & 4 \\ 1 & 5 & 8 \end{array}\right] \qquad h=14: \quad \left[\begin{array}{c|cc} 0 & 5 & 3 \\ \hline 6 & 1 & 7 \\ 2 & 8 & 4 \end{array}\right] \qquad h=15: \quad \left[\begin{array}{c|cc} 0 & 4 & 2 \\ \hline 5 & 3 & 7 \\ 1 & 8 & 6 \end{array}\right].$$

For $K_{3,3}$, the bounds are $28 \le h \le 36$, and all these values can be realized. There are 35 isomorphism classes with $h=28$ and with $h=36$, 70 with $h=29$ and with $h=35$, 477 with $h=30$ and with $h=34$, 250 with $h=31$ and with $h=33$, and 882 with $h=32$.

In view of the ease with which examples are found for small m, we conjecture that every value of h allowed by (3.18) can be realized, but this is far from established.

Research Problem 3.2 *Prove or disprove that every value of h allowed by (3.18) can be realized as the magic constant of a vertex-magic total labeling of $K_{m,m}$.*

In the case of $K_{m,m+1}$ we can improve on (3.17) by an argument similar to that used in the proof of Theorem 3.8. Write s_1 and s_2 for the sums of the labels on the m-set and $(m+1)$-set of vertices respectively, and s_e again for the sum of the edge-labels. Then every edge is adjacent to exactly one of the vertices in each set. Adding all labels on or adjacent to all vertices in the m-set, we get

$$\begin{aligned} hm &= s_1 + s_e \\ &\ge 1+2+\cdots+(m+m(m+1)), \end{aligned}$$

so

$$h \geq \tfrac{1}{2}(m+1)^2(m+2) \tag{3.19}$$

while the larger set of vertices yields

$$\begin{aligned} h(m+1) &= s_2 + s_e \\ &\leq (m+1) + (m+2) + \cdots + ((2m+1) + m(m+1)), \end{aligned}$$

from which

$$h \leq \tfrac{1}{2}(m+1)(m^2+4m+2). \tag{3.20}$$

For $K_{1,2}$, (3.19) and (3.20) yield $6 \leq h \leq 7$, and both values can be realized. In fact there are exactly two labelings up to isomorphism, with representation matrices

$$\begin{array}{cc} h=6: & h=7: \\ \left[\begin{array}{c|cc} 0 & 4 & 5 \\ \hline 3 & 2 & 1 \end{array}\right] & \left[\begin{array}{c|cc} 0 & 5 & 3 \\ \hline 1 & 2 & 4 \end{array}\right]. \end{array}$$

However, for $K_{2,3}$, the bounds are $18 \leq h \leq 21$, but only 18, 19 and 20 can be realized. There are four labelings up to isomorphism:

$$\begin{array}{cccc} h=18: & h=19: & h=19: & h=20: \\ \left[\begin{array}{c|ccc} 0 & 11 & 10 & 9 \\ \hline 7 & 4 & 6 & 1 \\ 5 & 3 & 2 & 8 \end{array}\right] & \left[\begin{array}{c|ccc} 0 & 11 & 9 & 8 \\ \hline 7 & 5 & 6 & 1 \\ 2 & 3 & 4 & 10 \end{array}\right] & \left[\begin{array}{c|ccc} 0 & 11 & 9 & 8 \\ \hline 5 & 6 & 7 & 1 \\ 4 & 2 & 3 & 10 \end{array}\right] & \left[\begin{array}{c|ccc} 0 & 11 & 9 & 6 \\ \hline 1 & 7 & 8 & 4 \\ 5 & 2 & 3 & 10 \end{array}\right]. \end{array}$$

For $K_{3,4}$, labelings are easily found for each h satisfying the bounds $40 \leq h \leq 46$.

An obvious open question is: for which h satisfying (3.19) and (3.20) do vertex-magic total labelings exist? We lean toward the view that the case $m = 2, h = 21$ is a "small numbers" anomaly, and that all other possible magic constants can be realized.

Research Problem 3.3 *Prove or disprove that every value of h allowed by (3.19) and (3.20) can be realized as the magic constant of a vertex-magic total labeling of $K_{m,m+1}$.*

3.5.3 Joins

Complete bipartite graphs arise in the definition of joins of graphs. Suppose G and H are disjoint graphs. The *join* of G and H, denoted $G \vee H$, is the union of G, H and the complete bipartite graph with vertex-sets $V(G)$ and $V(H)$.

Suppose G and H are graphs that each have v vertices, and suppose the disjoint union $G \cup H$ has a vertex-magic total labeling λ with magic constant h. Suppose there exists a magic square A of size $v \times v$. The magic constant will be $\frac{1}{2}v(v^2+1)$. Define a labeling μ of $G \vee H$ as follows: for the vertices and edges of G and H, $\mu = \lambda$; if x is a vertex of G and y is a vertex of H, then $\mu(xy) = 2v + |E(G)| + |E(H)| + a_{xy}$. Then μ is easily seen to be a vertex-magic total labeling of $G \vee H$ with magic constant $h + v(2v + |E(G)| + |E(H)|) + \frac{1}{2}v(v^2+1)$. Since magic squares of all orders exist, we have:

Theorem 3.12 *If G and H are graphs of the same order such that the disjoint union $G \cup H$ has a vertex-magic total labeling, then $G \vee H$ has a vertex-magic total labeling.*

3.5.4 Unions of stars

We have seen that $K_{1,2}$ is the only vertex-magic star. It is reasonable to ask whether disjoint unions of stars are vertex-magic.

Suppose G is the union $K_{1,n_1} \cup K_{1,n_2} \cup \ldots \cup K_{1,n_t}$ of t stars. Write N for $n_1 + n_2 + \cdots + n_t$. Suppose G has a vertex-magic total labeling with magic constant h. The sum of the weights of the centers of the stars will be th; on the other hand, it will equal at least the sum of the smallest $N+t$ positive integers (the N spokes and the t centers). So

$$th \geq \sigma_0^{N+t} = \tfrac{1}{2}(N+t)(N+t+1). \tag{3.21}$$

On the other hand, the sum of the N weights of the leaves equals the sum of the labels on all the edges and all the vertices except the centers, so

$$Nh \leq \sigma_t^{2N+t} = \tfrac{1}{2}(2N+t)(2N+t+1) - \tfrac{1}{2}t(t+1). \tag{3.22}$$

Combining (3.21) and (3.22),

$$\begin{aligned} N(N+t)(N+t+1) &\leq t(2N+t)(2N+t+1) - t^2(t+1), \\ (N+t)(N+t+1) &\leq N^2(4t) + N(4t^2+2t), \end{aligned}$$

so $N^2 + N(1 - 2t) - (3t^2 + t) \leq 0$. It follows that

$$N \leq \frac{2t - 1 + \sqrt{(16t^2 + 1)}}{2} \leq 3t. \tag{3.23}$$

Theorem 3.13 *If a disjoint union of stars is vertex-magic, then the average size of the component stars is less than* 3.

Exercise 3.7 *Find vertex-magic total labelings of* $2K_{1,2}$ *and* $K_{1,2} \cup K_{1,3}$.

It is clear that Theorem 3.13 is the best possible conclusion from (3.23), because $\frac{2t-1+\sqrt{16t^2+1}}{2} > 3t - 1$. But not every union of stars with average size smaller than 3 is vertex-magic.

Suppose G is a vertex-magic union of t stars that has $3t - 1$ edges. Then (3.21) and (3.22) yield

$$8t - 2 \leq h \leq 8t - 1.$$

Consider any vertex x of degree 1. The label on the edge adjacent to x is no greater than $7t - 2$, so $\lambda(x) \geq t$. Therefore $1, 2, \ldots, t - 1$ must all be labels of centers of stars; and if $h = 8t - 1$, then t is also a center label.

Suppose one of the stars has s edges. Write c for the center of this star. If $h = 8t - 2$, then c has weight at least $1 + t + (t + 1) + \cdots + (t + s - 1) = 1 + st + \frac{1}{2}s(s - 1)$, so

$$8t - 2 \geq 1 + st + \tfrac{1}{2}s(s - 1). \tag{3.24}$$

Clearly $s < 8$, no matter what value t takes. Even for smaller s, not all t are possible. The inequality (3.24) can be written as

$$s^2 + (2t - 1)s + 6 - 16t \leq 0,$$

so

$$s \leq \frac{1 - 2t + \sqrt{4t^2 + 60t - 23}}{2}.$$

In the case $h = 8t - 1$, c has weight at least $1 + (t + 1) + \cdots + (t + s) = 1 + st + \frac{1}{2}s(s + 1)$. Again $s < 8$. For smaller s we obtain the slightly weaker condition

$$s \leq \frac{-1 - 2t + \sqrt{4t^2 + 68t - 15}}{2}.$$

From this we can deduce the following:

Theorem 3.14 *Suppose G is the vertex-magic union of t disjoint stars which between them have $3t - 1$ edges; then no star can contain 8 edges. If the largest star has s edges, then*

$$\begin{aligned} &\textit{if } s = 7 && \textit{then} && t \geq 30 \\ &\textit{if } s = 6 && \textit{then} && t \geq 12 \\ &\textit{if } s = 5 && \textit{then} && t \geq 6 \\ &\textit{if } s = 4 && \textit{then} && t \geq 3 \\ &\textit{if } s = 3 && \textit{then} && t \geq 2. \end{aligned}$$

The two extreme cases are worth considering. If t is any positive integer, then $(t-1)K_{1,3} \cup K_{1,2}$ is always a possibility according to the Theorem, and it is in fact vertex-magic. One labeling is shown in Figure 3.5.

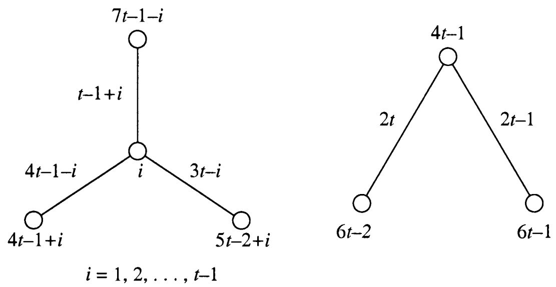

Figure 3.5. Vertex-magic total labeling of $(t-1)K_{1,3} \cup K_{1,2}$.

However, $(t-1)K_{1,2} \cup K_{1,t+1}$ can never have a vertex-magic total labeling when $t > 3$. The cases $t = 2, 3$ are vertex-magic—when $t = 2$, the construction of Figure 3.5 provides an example. The case $t = 3$ is left as an exercise.

Exercise 3.8 *Find a vertex-magic total labeling of $2K_{1,2} \cup K_{1,4}$.*

Research Problem 3.4 *Are the bounds in Theorem 3.14 the best possible?*

3.6 Graphs with vertices of degree one

For a graph to be vertex-magic, the presence of degree one vertices turns out to create a restriction on both the number of edges and on the number of vertices of higher degree in the graph. As an illustration, we first examine a family of graphs in which the number of vertices of degree one is the same as the number of vertices of higher degree.

Let G be any graph of order v and size e. We define a *G-sun* to be a graph G^* of order $2v$ formed from G by adjoining v new vertices of degree 1, one to each vertex of G. We have $v^* = |V(G^*)| = 2v$ and $e^* = |E(G^*)| = e + v$. When G is the cycle C_v and a new vertex is adjoined to each vertex of G, the resulting G^* is a *sun* as defined in Section 2.4.2.

Let us call the edges of G the *inner edges* of the G-sun and the vertices of G the *inner vertices*; the other elements (leaves and their adjacent edges) are the *outer edges* and *outer vertices*. The following theorem shows that for a labeling to exist, the number of edges in G must be bounded above by a function of v that is essentially linear.

Theorem 3.15 *Let G be any graph of order v. If G has e edges, then a G-sun G^* is not vertex-magic total when*

$$e > \frac{-1 + \sqrt{1 + 8v^2}}{2}.$$

Proof. The label set for G^* is $\{1, 2, \ldots, 3v+e\}$. We calculate the minimum possible sum of weights on the inner vertices; this is achieved by putting the e smallest labels on the inner edges and the next $2v$ smallest labels on the inner vertices and outer edges. Remembering that the inner edge labels will each be added twice, this gives us

$$\begin{aligned} \sum wt(x_i) &\geq \sum_1^{2v+e} i + \sum_1^{e} i \\ &= \tfrac{1}{2}\left[(2v + e)(2v + e + 1) + e(e + 1)\right]. \end{aligned}$$

Since there are v inner vertices, we must therefore have

$$h \geq \frac{1}{v}\left[2v^2 + v(2e + 1) + e(e + 1)\right].$$

Calculating the maximum possible sum of weights on the outer vertices, we find (taking the sum of the $2v$ largest labels):

$$\sum wt(x_i) \leq \sum_1^{3v+e} i - \sum_1^{v+e} i$$

$$\begin{aligned} &= \tfrac{1}{2}\left[(3v+e)(3v+e+1)-(v+e)(v+e+1)\right] \\ &= 4v^2+2ve+v. \end{aligned}$$

Since there are v outer vertices, we have

$$\begin{aligned} h &\le \tfrac{1}{v}(4v^2+2ve+v) \\ &= 4v+1+2e. \end{aligned}$$

Consequently, a labeling cannot exist whenever

$$\frac{1}{v}\left[2v^2+v(2e+1)+e(e+1)\right] > 4v+1+2e.$$

This simplifies to

$$e^2+e-2v^2>0$$

and the theorem follows. □

A second illustration of the impact of degree 1 vertices is found by examining trees. This time the existence of a vertex-magic total labeling forces a lower bound on the number of internal vertices in the graph. We have the following :

Theorem 3.16 *Let T be a tree with n internal vertices. If T has more than $2n$ leaves, then T does not admit a vertex-magic total labeling.*

Proof. Suppose T has n internal vertices $\{x_1, x_2, \ldots, x_n\}$and $2n+t$ leaves $\{y_1, y_2, \ldots, y_{2n+t}\}$, so that $v = 3n+t$ and $e = 3n+t-1$. Then the label set is $\{1, 2, \ldots, 6n+2t-1\}$.

The maximum possible sum of weights on the leaves is the sum of the $4n+2$ largest labels:

$$\begin{aligned} \sum wt(y_i) &\le \sum_{1}^{6n+2t-1} i - \sum_{1}^{2n-1} i \\ &= (8n+2t-1)(2n+t) \end{aligned}$$

and since there are $2n+t$ leaves,

$$h \le 8n+2t-1.$$

The minimum possible sum of weights on the internal vertices is the sum of all but the $2n + t$ largest labels:

$$\begin{aligned}\sum wt(x_i) &\geq \sum_{1}^{4n+t-1} i \\ &= \tfrac{1}{2}(4n + t)(4n + t - 1).\end{aligned}$$

Since there are n internal vertices,

$$h \geq \frac{(4n + t)(4n + t - 1)}{2n}$$

and these two inequalities imply that

$$(4n + t)(4n + t - 1) \leq 2n(8n + 2t - 1)$$

and

$$2n(2t - 1) + t(t - 1) \leq 0. \tag{3.25}$$

Since n is at least 1, t cannot be positive. □

The labeling of the tree with 2 internal vertices and 4 leaves displayed in Figure 3.6 shows that the statement of the theorem provides the best possible general result for trees.

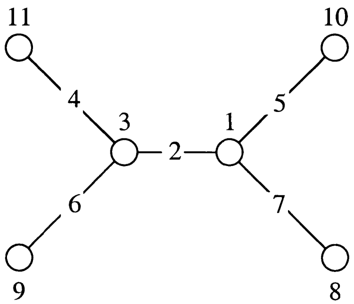

Figure 3.6. Theorem 3.16 is best possible.

Research Problem 3.5 *For each positive integer n, find a tree with n central vertices and 2n leaves, which is vertex-magic.*

3.7 The complete graphs

A systematic search for vertex-magic total labelings of K_4 has revealed interesting results. In this case $\lambda_i \in \{1, 2, \ldots, 10\}$ and equation (3.4) imply that $19 \le h \le 25$. The table shows the number of distinct labelings for each value of h.

h	$=$	19	*none*
h	$=$	20	2
h	$=$	21	5
h	$=$	22	*none*
h	$=$	23	5
h	$=$	24	2
h	$=$	25	*none*

As described in Section 3.2, these come in dual pairs also: those labelings with $h = 24$ are the duals of those with $h = 20$ and those with $h = 23$ are duals of those with $h = 21$. The graph labeling of $K_4 - e$ given in Figure 3.1 was obtained from one of the above labelings with $h = 20$ by deleting an edge labeled with 1, as described in Theorem 3.3.

We can apply equation (3.4) for arbitrary v to find the range of feasible values of h for K_v. Using $e = \frac{v(v-1)}{2}$, we find that

$$\frac{v(v^2+3)}{4} \le h \le \frac{v(v+1)^2}{4}.$$

Theorem 3.17 [20, 33, 34] *There is a vertex-magic total labeling of K_v for all v.*

We shall use two $n \times n$ matrices, where n is odd. A denotes the matrix formed from the row $1, 2, \ldots, v$ by back circulating — $a_{i,j} = a_{i-1,j+1}$, with subscripts reduced modulo v when necessary — so that A is symmetric, and its diagonal entries are all different. If S is the sequence $s_0, s_1, s_2, \ldots, s_{\frac{1}{2}(n-1)}$, then $B(S) = (b_{i,j})$ is the matrix formed by circulating the row

$$s_0, s_1, s_2, \ldots, s_{\frac{1}{2}(n-1)}, s_{\frac{1}{2}(n-1)}, \ldots, s_1$$

— in other words, $b_{1,1} = s_0, b_{1,2} = s_1, \ldots, b_{1,n} = s_1$, and $b_{i,j} = b_{i-1,j-1}$, with subscripts taken modulo v when necessary.

Now define a labeling $\lambda_S(n)$ of K_n by $\lambda(x_i) = a_{i,i} + b_{i,i}$, $\lambda(x_i x_j) = a_{i,j} + b_{i,j}$. It is easy to see that under this labeling, every vertex x has the same weight:

$$wt(x) = s_0 + 2(s_1 + s_2 + \cdots + s_{\frac{1}{2}(n-1)}) + 1 + 2 + \cdots + n.$$

In the case where v is odd we take $v = n$ and use the sequence

$$S = (0, n, 2n, \dots, \tfrac{1}{2}n(n-1)).$$

Every label from 1 to $\frac{1}{2}n(n+1)$ will occur exactly once, so we have a vertex-magic total labeling of K_n. So there is a vertex-magic total labeling of K_n whenever n is odd.

If $v \equiv 2 \pmod 4$, we write $v = 2n$. We find a vertex-magic total labeling of the union of two copies of K_n. Then K_v is the join of these two copies of K_n, so Theorem 3.12 provides a vertex-magic total labeling of K_v.

To label $2K_n$ we distinguish two subcases. If $n = 4m + 1$, consider the two sequences

$$\begin{aligned} S_1 &= 2mn, 0, 2n, \dots, (2m-2)n, (2m+3)n, \\ &\quad (2m+5)n, \dots, (4m+1)n, \\ S_2 &= (2m+2)n, n, 3n, \dots, (2m+1)n, (2m+4)n, \\ &\quad (2m+6)n, \dots, 4mn. \end{aligned}$$

$\lambda_{S_1}(n)$ and $\lambda_{S_2}(n)$ can each be used to label K_n. Each has magic constant $(2m+1)(4m+1)^2$ and between them their sets of labels make up all the integers from 1 to $2\binom{n+1}{2}$. If these labelings are applied to two disjoint copies of K_n, they make up a vertex-magic total labeling of $2K_n$ as required.

In the same way, if $n = 4m + 3$, the sequences

$$\begin{aligned} S_1 &= 2m, 0, 2, \dots, 2m-2, 2m+2, 2m+5, 2m+7, \dots, 4m+3, \\ S_2 &= 2m+4, 1, 3, \dots, 2m+3, 2m+6, 2m+8, \dots, 4m+2 \end{aligned}$$

can be used to label $2K_n$.

If $v \equiv 0 \pmod 4$, say $v = 4m$, we treat K_{4m} as $K_{4m-3} \cup K_3$ with edges joining the two vertex-sets. The copy of K_{4m-3} is labeled using $\lambda_S(4m-3)$, where

$$\begin{aligned}S = {} & 4m, 0, 8m - 3, 12m - 6, 16m - 9, \ldots, \\ & (8m - 3) + (m - 3)(4m - 3), 8m + (m + 1)(4m - 3), \\ & 8m + (m + 2)(4m - 3), \ldots, 8m + (2m - 1)(4m - 3),\end{aligned}$$

yielding constant vertex weight $(2m + 1)(8m^2 - 6m - 3)$. The vertices of K_3 receive labels $4m - 2, 4m - 1, 4m$, and the edges receive $8m - 2 + (m-2)(4m-3), 8m-1+(m-2)(4m-3), 8m+(m-2)(4m-3)$ in such a way as to give each of the three vertices weight $8m^2 - 2m + 9$. Finally, a magic rectangle R of size $3 \times (4m-3)$ is chosen, and the cross-edge joining vertex i of K_3 to vertex j of K_{4m-3} is labeled $8m + (m-2)(4m-3) + r_{ij}$. The magic rectangle has row and column sums $(4m - 3)(6m - 4)$ and $3(6m - 4)$, so the sum on each vertex of K_{4m-3} of the labels on the cross edges is $3[(6m-4) + 8m + (m-2)(4m-3)]$, and for the vertices of K_3 it is $(4m - 3)[(6m - 4) + 8m + (m - 2)(4m - 3)]$. Therefore the combined labeling gives constant vertex weight $16m^3 + 8m^2 - 3m + 3$. Every integer from 1 to $2m(4m+1)$ is used precisely once, so the result is a vertex-magic total labeling. □

Exercise 3.9 *Verify the computations of vertex weight in the case* $v \equiv 0 \pmod 4$, *and verify that every positive integer up to* $2n(4n + 1)$ *appears precisely once among the labels.*

For small values of v, the following result has been established by brute force computation:

Theorem 3.18 *There is at least one vertex-magic total labeling for* K_5 *and for* K_6 *for every feasible value of* h.

The fact that vertex-magic total labelings seem to be plentiful for $n = 5$ and $n = 6$ suggests the following conjecture. The absence of labelings for some values of h in the case of K_4 is probably just the Law of Small Numbers in operation.

Conjecture For each $n > 4$ there is a vertex-magic total labeling for K_n for every feasible value of h.

Research Problem 3.6 *Prove or disprove the above conjecture.*

3.8 Disconnected graphs

Vertex-magic total labelings of disconnected graphs have not been widely studied. The following construction is a modification of the edge-magic total labeling construction in Section 2.8.2.

3.8.1 Multiples of regular graphs

By analogy with vertex colorings, a *C-edge coloring* χ of a graph G is a map

$$\chi : E(G) \to C$$

to some set $C = \{c_1, c_2, \ldots\}$ of colors. A *proper edge coloring* of G is an edge coloring in which no two edges with a common endpoint belong to the same color class. In other words, if $x \sim y$ and $y \sim z$,

$$\chi(xy) \neq \chi(yz).$$

If $|C| = k$, the phrase *k-edge coloring* is used.

Vizing [52] proved the following.

Lemma 3.19 *Suppose the maximum degree among the vertices of G is Δ. Then G has a $(\Delta + 1)$-edge coloring.*

(In fact, Vizing proved that the minimum number of colors needed is either Δ or $\Delta + 1$.)

Suppose G is a regular graph of degree Δ. Then G has maximum degree Δ, and can be properly edge colored in $\Delta + 1$ colors. Choose such a coloring, χ say. At each vertex x, there will be exactly one color not represented on any edge adjacent to x. Define $\eta(x)$ to be that color. Define η to be the same as χ when applied to edges. Then η is a $(\Delta + 1)$-total coloring of g. We have proved:

Lemma 3.20 *Every regular graph of degree Δ has a total coloring in $\Delta +$ 1 colors.*

Theorem 3.21 *Suppose G is a regular graph of degree Δ which has a vertex-magic total labeling.*

(i) *If Δ is even, then nG is vertex-magic whenever n is an odd positive integer.*

(ii) *If Δ is odd, then nG is vertex-magic for every positive integer n.*

Proof. Suppose G has e edges and v vertices. We wish to construct a vertex-magic total labeling of the ne-edge, nv-vertex graph $G_1 \cup G_2 \cup \ldots \cup G_n$, where each G_j is an isomorphic copy of G.

Select a vertex-magic total labeling λ of G, with magic constant h, and a Kotzig array $A = (a_{ij})$ of size $(\Delta + 1) \times n$, as guaranteed by Lemma 1.2. Let η be a total coloring of G in the $\Delta + 1$ colors $1, 2, \ldots, (\Delta + 1)$. We define a labeling Λ as follows. The element (vertex or edge) x_j of G_j corresponding to element x of G receives label $\lambda(x) + (e + v)a_{\eta(x),j}$.

Consider the elements x_j, $1 \le j \le n$. These n elements receive the n labels $\eta(x), \eta(x) + e + v, \ldots, \eta(x) + (n - 1)(e + v)$. These labels are precisely those integers in the range from 1 to $n(e + v)$ that are congruent to $\eta(\text{mod}\,(e + v))$. As x ranges through the $e + v$ elements of G, each congruence class is represented exactly once. So Λ assumes each of the required values exactly once.

Now consider the labels on x_i and its incident edges. They are the labels on the corresponding elements of G, with $0, (e + v), \ldots, (n - 1)(e + v)$ added to them in some order. So the sum is $h+(e+v)\binom{n}{2}$, which is constant, as required. □

Research Problem 3.7 *Can the method of Theorem* 3.21 *be applied to some case $G \cup H$ where G and H are nonisomorphic regular graphs?*

3.9 Vertex-magic injections

A vertex-magic injection is like a vertex-magic total labeling, except that the labels can be any positive integers. As in the edge-magic case, we define an $[m]$-vertex-magic injection of G to be a vertex-magic injection of G in which the largest label is m, and again we call m the *size* of the injection. If G has a vertex-magic injection, the *vertex deficiency* $def_v(G)$ of G is the minimum value of $m - v(G) - e(G)$ such that an $[m]$-vertex-magic injection of G exists.

There is a significant difference between vertex-magic and edge-magic injections. An edge-magic injection of G corresponds to an edge-magic total labeling of some $G \cup tK_1$. One cannot add an arbitrary number of disjoint vertices in the vertex-magic case, because (Theorem 3.1) a vertex-magic graph cannot contain two isolates, and there seems to be no equivalent notion.

We saw in Theorem 2.32 that every graph has an edge-magic injection. From Theorem 3.1, it is clear that a graph with isolated edge or two isolated vertices could not have a vertex-magic injection. However, all other graphs contain them:

Theorem 3.22 *If a graph has no component K_2 and no two components K_1, then it has a vertex-magic injection.*

Proof. Let G be a graph with no component K_2 and no isolate. We prove that both G and $G \cup K_1$ have vertex-magic injections.

As usual, say G has v vertices and e edges. Write $h = 2^e + 2^{e-1}$. To define a labeling λ, assign the values $1, 2, 2^2, \ldots, 2^{e-1}$ as labels of the edges of G in any order. Then, for each vertex x, define

$$\lambda(x) = h - \sum_{y \sim x} \lambda(xy).$$

This λ clearly has the constant weight property with magic constant h. All the edge labels are distinct, by construction. If x_1 and x_2 are two vertices, their sets of neighboring edges $\{x_1y \mid y \sim x_1\}$ and $\{x_2y \mid y \sim x_2\}$ must be different (these sets are the same only if x_1 and x_2 are both isolates, so that the sets are empty, or x_1x_2 forms a component K_2), so $\sum_{y \sim x_1} \lambda(x_1y)$ and $\sum_{y \sim x_2} \lambda(x_2y)$ are sums of different sets of powers of 2, and are unequal. Thus $\lambda(x_1) \neq \lambda(x_2)$, and all the vertex labels are distinct. The largest possible value for the sum of edges neighboring a vertex is $\sum_{i=0}^{e-1} 2^i$, so the smallest possible vertex label is

$$h - \sum_{i=0}^{e-1} 2^i = (2^e + 2^{e-1}) - (2^e - 1) = 2^{e-1} + 1,$$

which is greater than the greatest edge label. So all the labels are different and λ is a vertex-magic injection.

No label could be greater than $h - 1$, so h does not occur as a label. So λ can be extended to a vertex-magic injection of $G \cup K_1$ by assigning label h to the isolate. □

The preceding proof used the fact that if S is any set of distinct powers of 2, then different subsets of S have different sums. This is of course true if 2 is replaced by any positive integer n (although it is trivial when $n = 1$, and S has only one member), and is the reason that positional notation works. More generally, we could have used for edge labels any set of positive integers for which all subset sums are distinct (called a *DSS set*). If $\max(S)$ denotes the largest member of the set S, one might hope to find an e-element DSS set S with $\max(S)$ significantly smaller than 2^{e-1}. However, this will not happen. Erdös and Moser [12] proved that if $f(n)$ is the smallest value of $\max(S)$ for any n-element DSS set S, then $f(n) \geq 2^{n-2}n^{-\frac{1}{2}}$. A good survey of the DSS set problem is [6].

4

Totally Magic Labelings

4.1 Basic ideas

4.1.1 Definitions

In this chapter we investigate the question: for a graph G does there exist a total labeling λ that is both edge-magic and vertex-magic? As we said, such a λ is called a *totally magic labeling* and G is a *totally magic* graph. The constants h and k are the magic constant and magic sum respectively. We do not require that $h = k$.

Most of the material in this chapter comes from [14] and [38].

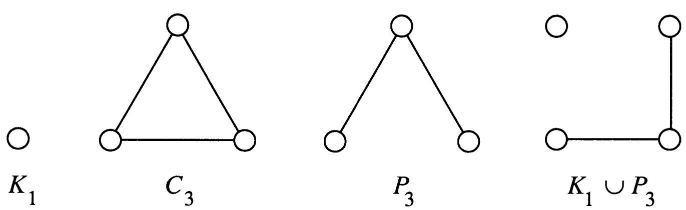

Figure 4.1. The small totally magic graphs.

4.1.2 *Examples*

One quickly constructs three small examples of connected totally magic graphs. An obvious trivial example is the single vertex graph K_1. There are four totally magic labelings of the 3-vertex cycle C_3; if the set of vertex labels is denoted by S_v, then the labelings have

$$\begin{array}{rclrclrcl} h &=& 9, & k &=& 12, & S_v &=& \{4,5,6\} \\ h &=& 10, & k &=& 11, & S_v &=& \{2,4,6\} \\ h &=& 11, & k &=& 10, & S_v &=& \{1,3,5\} \\ h &=& 12, & k &=& 9, & S_v &=& \{1,2,3\}. \end{array}$$

(There is an obvious duality here.) The three-vertex path P_3 has two labelings. Writing the labels in sequence vertex-edge-vertex-edge-vertex, they are

$$\begin{array}{rclrcll} h &=& 6, & k &=& 9, & \text{labels } 4,2,3,1,5, \\ h &=& 7, & k &=& 8, & \text{labels } 3,4,1,2,5. \end{array}$$

Among disconnected graphs, only one small example is known: there is exactly one totally magic labeling of $K_1 \cup P_3$, constructed from the first of the P_3-labelings listed above by mapping the isolated vertex to 6.

4.2 Isolates and stars

We saw in Theorem 3.1 that a graph with two isolated vertices or an isolated edge cannot be vertex-magic. *A fortiori* we have

Lemma 4.1 *No totally magic graph has two isolated vertices or an isolated edge.*

Moreover, if $K_1 \cup G$ is totally magic, the isolated vertex must necessarily receive the largest possible label, so the remaining labels form a totally magic labeling of G. We have:

Lemma 4.2 *If a graph with an isolated vertex is totally magic, then the graph G resulting from the deletion of the isolate has a totally magic labeling with magic constant $|V(G)| + |E(G)| + 1$.*

The labeling of $K_1 \cup P_3$ given above is as described in this lemma.

It was shown in Theorem 3.8 that $K_{m,n}$ is never vertex-magic when $|m - n| > 1$. This means that no star larger than $K_{1,2}$ is vertex-magic. So:

Lemma 4.3 *No star larger than $K_{1,2}$ is totally magic.*

Theorem 4.4 *Suppose the totally magic graph G has a leaf x. Then the component of G containing x is a star.*

Proof. Suppose λ is a totally magic labeling on G, with magic constant h and magic sum k, and suppose x is a leaf with neighbor y. By the vertex-magic property, $\lambda(x)+\lambda(xy) = h$, and by the edge-magic property, $\lambda(x)+\lambda(xy) + \lambda(y) = k$. So $\lambda(y) = k - h$.

By Lemma 4.1, y has a neighbor, z say. Then $k = \lambda(y) + \lambda(yz) + \lambda(z) = \lambda(yz)+\lambda(z)+k-h$, so $\lambda(yz) = h-\lambda(z)$. So $wt(z) \geq \lambda(z)+\lambda(yz) = h$, with equality only if z has degree 1. So every vertex adjacent to y has degree 1, and the component of G containing y is a star with center y. □

From Lemma 4.3 and Theorem 4.4, we conclude the following.

Corollary 4.4.1 *The only connected totally magic graph containing a vertex of degree* 1 *is P_3.*

Every non-trivial tree has at least two vertices of degree 1, so:

Corollary 4.4.2 *The only totally magic trees are K_1 and P_3.*

A totally magic graph cannot have two stars as components, because their centers would each receive label $k - h$. It follows that the components of a totally magic graph can include at most one K_1 and at most one star, and all other components have minimum degree at least 2, and consequently have as many edges as vertices.

Corollary 4.4.3 *The only totally magic proper forest is $K_1 \cup P_3$.*

Theorem 4.5 *The only totally magic graphs with a component K_1 are $K_1 \cup P_3$ and K_1 itself.*

Proof. Suppose $K_1 \cup G$ is a totally magic graph; K_1 has vertex x, and G has v vertices and e edges, as usual. G may have a star as a component, but any other component has minimum degree at least 2, so $e \geq v - 1$. From

Lemma 4.2,

$$h = \lambda(x) = v + e + 1,$$

and G is totally magic with magic constant h. Now from (3.5),

$$v(v + e + 1) = \tfrac{1}{2}(v + e)(v + e + 1) + \sum_{y \in E} \lambda(y).$$

So

$$\sum_{y \in E} \lambda(y) = \tfrac{1}{2}(v - e)(v + e + 1).$$

But $\sum \lambda(y) \geq \frac{1}{2}e(e + 1)$, and thus

$$e(e + 1) \leq (v - e)(v + e + 1),$$

and

$$2e(e + 1) \leq v(v + 1).$$

Clearly $e < v$. So $e = v - 1$, whence

$$2(v^2 - v) \leq v^2 + v,$$

and $v \leq 3$. The only possibility is $G = P_3$. □

Exercise 4.1 *Suppose $K_{1,n} \cup G$ is totally magic, where G has v vertices and e edges. By considering vertex weights in the star, prove that*

$$v + e \geq \tfrac{1}{2}(n^2 - n - 4).$$

4.3 Forbidden configurations

Theorem 4.6 *If a totally magic graph G contains two adjacent vertices of degree* 2*, then the component containing them is a cycle of length* 3.

Proof. Suppose G contains a pair of adjacent vertices b and c, each having degree 2, and suppose λ is a totally magic labeling of G with magic constant h and magic sum k.

First, assume G contains a path $\{a, b, c, d\}$, where a and d are distinct vertices. From $h = wt(b) = wt(c)$ it follows that

$$\lambda(ab) + \lambda(b) + \lambda(bc) = \lambda(bc) + \lambda(c) + \lambda(cd),$$

from which

$$\lambda(cd) = \lambda(ab) + \lambda(b) - \lambda(c) \tag{4.1}$$

while the edge-magic property yields

$$k = \lambda(a) + \lambda(ab) + \lambda(b) = \lambda(b) + \lambda(bc) + \lambda(c) = \lambda(c) + \lambda(cd) + \lambda(d); \tag{4.2}$$

the second equality in (4.2) implies that

$$\lambda(cd) = \lambda(b) + \lambda(bc) - \lambda(d) \tag{4.3}$$

while the first gives

$$\lambda(a) + \lambda(ab) = \lambda(bc) + \lambda(c). \tag{4.4}$$

But (4.1) and (4.3) give

$$\lambda(ab) - \lambda(c) = \lambda(bc) - \lambda(d),$$

so

$$\lambda(d) = \lambda(bc) + \lambda(c) - \lambda(ab) \tag{4.5}$$

and (4.4) and (4.5) together imply $\lambda(a) = \lambda(d)$, a contradiction.

Now assume b and c have a common neighbor, a say, and suppose a has some other neighbor, z. From (4.2) we see that

$$\lambda(ab) = k - \lambda(a) - \lambda(b),$$

and similarly for bc and ca, so

$$\begin{aligned} wt(b) &= \lambda(ab) + \lambda(b) + \lambda(bc) \\ &= 2k - \lambda(a) - \lambda(b) - \lambda(c), \end{aligned}$$

whereas

$$\begin{aligned} wt(a) &= \lambda(ca) + \lambda(a) + \lambda(ab) + \lambda(az) + \cdots \\ &\geq 2k - \lambda(a) - \lambda(b) - \lambda(c) + \lambda(az) \\ &> wt(a), \end{aligned}$$

as $\lambda(az) > 0$. □

The configurations forbidden by Theorem 4.6 are shown in Figure 4.2. In this and subsequent figures, small lines attached to a vertex indicate that there may or may not be further edges incident with it.

Figure 4.2. Configurations forbidden by Figure 4.6.

Corollary 4.6.1 *No totally magic graph contains as a component a path other than P_3 or a cycle other than K_3.*

(Lemma 4.1 must be invoked to rule out P_2.) In particular,

Corollary 4.6.2 *The only totally magic cycle is K_3.*

Theorem 4.7 *Suppose G contains two vertices, x_1 and x_2, that are each adjacent to precisely the same set $\{y_1, y_2, \ldots, y_d\}$ of other vertices. (It is not specified whether x_1 and x_2 are adjacent.) If $d > 1$, then G is not totally magic.*

Proof. Suppose λ is a totally magic labeling of G with magic constant and sum h and k. For convenience, define $\lambda(x_1x_2) = 0$ if x_1 is not adjacent to x_2. Then

$$h = \lambda(x_i) + \sum_{j=1}^{d} \lambda(x_i y_j) + \lambda(x_1 x_2),\ i = 1, 2,$$

and

$$k = \lambda(x_i) + \lambda(x_i y_j) + \lambda(y_j),\ i = 1, 2,\ 1 \le j \le d.$$

So

$$\begin{aligned} dk &= d\lambda(x_i) + \sum_{j=1}^{d} \lambda(x_i y_j) + \sum_{j=1}^{d} \lambda(y_j),\ i = 1, 2, \\ &= (d-1)\lambda(x_i) + (h - \lambda(x_1 x_2)) + \sum_{j=1}^{d} \lambda(y_j),\ i = 1, 2, \end{aligned}$$

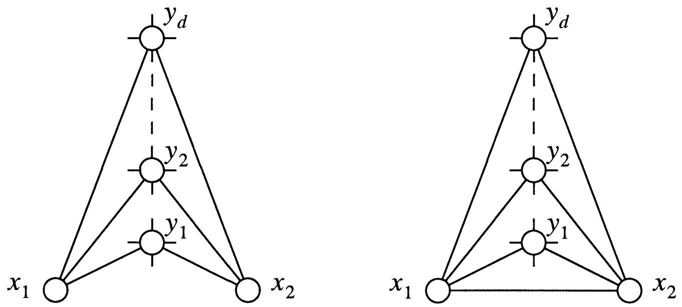

Figure 4.3. Configurations forbidden by Figure 4.7.

and thus

$$(d-1)\lambda(x_1) = (d-1)\lambda(x_2),$$

a contradiction unless $d = 1$. □

Corollary 4.7.1 *The only totally magic complete graphs are K_1 and K_3. The only totally magic complete bipartite graph is $K_{1,2}$.*

Theorem 4.8 *Suppose G contains two vertices, x and y, with a common neighbor. If x and y are inadjacent and each have degree 2, or are adjacent and each have degree 3, then G is not totally magic.*

Proof. Denote the common neighbor by z, the other neighbor of x by x_1, and the other neighbor of y by y_1. (Possibly $x_1 = y_1$.) Suppose λ is a totally magic labeling of G with magic constant h and magic sum k; if x is not adjacent to y in G then define $\lambda(xy) = 0$.

The weight of x is

$$\begin{aligned} wt(x) &= \lambda(x) + \lambda(xx_1) + \lambda(xz) + \lambda(xy) \\ &= \lambda(x) + (k - \lambda(x) - \lambda(x_1)) + (k - \lambda(x) - \lambda(z)) + \lambda(xy) \\ &= 2k - (\lambda(x) + \lambda(x_1) + \lambda(z)) + \lambda(xy) \end{aligned}$$

and similarly

$$wt(y) = 2k - (\lambda(y) + \lambda(y_1) + \lambda(z)) + \lambda(xy).$$

From the vertex-magic property, $wt(x) = wt(y)$, so $\lambda(x) + \lambda(x_1) = \lambda(y) + \lambda(y_1)$. So $\lambda(xx_1) = k - \lambda(x) - \lambda(x_1) = k - \lambda(y) - \lambda(y_1) = \lambda(yy_1)$. But this contradicts the edge-magic property. □

(The case where x and y are adjacent could be rephrased, "a totally magic graph G cannot contain a triangle with two vertices of degree 3.")

Theorem 4.9 *Suppose the totally magic graph G contains a triangle. Then the sum of the labels of all edges outside the triangle and incident with any one vertex of the triangle is the same, whichever vertex is chosen.*

Proof. Suppose the triangle is xyz. Write X for the sum of the labels of all edges other than xy and xz that are adjacent to x; define Y and Z similarly. Then

$$\begin{aligned} h &= wt(x) &&= \lambda(x) + \lambda(xy) + \lambda(xz) + X \\ & &&= 2k - \lambda(x) - \lambda(y) - \lambda(z) + X \\ &= wt(y) &&= \lambda(y) + \lambda(xy) + \lambda(yz) + Y \\ & &&= 2k - \lambda(x) - \lambda(y) - \lambda(z) + Y \\ &= wt(z) &&= \lambda(z) + \lambda(xz) + \lambda(yz) + Z \\ & &&= 2k - \lambda(x) - \lambda(y) - \lambda(z) + Z, \end{aligned}$$

so $X = Y = Z$. □

Corollary 4.9.1 *If the totally magic graph G contains a triangle with one vertex of degree 2, then the triangle is a component of G.*

Exercise 4.2 *Suppose vertices x, y, z, t form a 4-cycle in the totally magic graph G. Denote by $\tau(x)$ the sum of weights of all the edges touching x,* other than *the edges in the cycle. That is, $\tau(x) = wt(x) - \lambda(x) - \lambda(xy) - \lambda(xt)$. $\tau(y)$, $\tau(z)$ and $\tau(t)$ are defined similarly. Prove that $\tau(x)$, $\tau(y)$, $\tau(z)$, $\tau(t)$ are all different.*

Observe that Theorems 4.6, 4.7, 4.8 and 4.9 are essentially *forbidden configuration* theorems. If a graph G is in violation of one of them, then

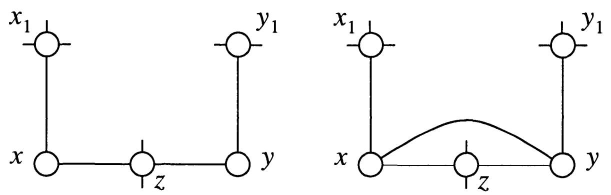

Figure 4.4. Configurations forbidden by Figure 4.8.

not only is G not totally magic, but G cannot be a component or union of components in any totally magic graph.

Exercise 4.3 *Suppose xy is any edge of a totally magic graph whose labeling λ has magic constant and sum h and k. Prove that*

$$\lambda(xy) + \sum_{\substack{z \sim x \\ z \neq y}} \lambda(xz) + \sum_{\substack{t \sim y \\ t \neq x}} \lambda(xz) = 2h - k.$$

Exercise 4.4 *(**The Star Theorem.**) Suppose x is any vertex of a totally magic graph and $x_1, x_2, \ldots, x_n$ are the vertices adjacent to it. Denote by τ_i the sum of weights of all the edges touching x_i,* other than xx_i*. Prove that $\tau_1, \tau_2, \ldots, \tau_n$ are all equal.*

4.4 Unions of triangles

In this section we construct two infinite families of totally magic graphs, both based on triangles. We shall show that the union of an odd number of triangles is always totally magic, as is the graph constructed by deleting one edge from such a union.

Suppose λ is a totally magic labeling of nK_3, with magic constant h and magic sum k. Assume that n is odd. Suppose the vertices of one of the triangles are labeled x, y, z; suppose the edge opposite the vertex labeled x receives the label X, and so on. Then

$$h = x + Y + Z = X + y + Z = X + Y + z, \tag{4.6}$$

$$k = X + y + z = x + Y + z = x + y + Z. \tag{4.7}$$

If we write $s = x + y + z$ and $S = X + Y + Z$, then (4.6) and (4.7) yield

$$\begin{aligned} 3h &= 2S + s, \\ 3k &= S + 2s, \\ h + k &= S + s. \end{aligned}$$

So

$$3h - 3k = S - s,$$

whence

$$\begin{aligned} S &= 2h - k, \\ s &= 2k - h. \end{aligned}$$

Finally, from $X + Y + Z = S$ and $x + Y + Z = h$ we obtain

$$X - x = S - h = h - k \tag{4.8}$$

for every choice of x, that is the difference between an edge label and the vertex label is a constant, $h - k$.

Let us write d for $h - k$. It is clear that h and k determine d. On the other hand, if d is known, then h and k are determined. For each triangle, $S + s = h + k$, so summing over all triangles we get $n(h + k)$. But this is the sum of all vertex and edge labels in nK_3, so it equals $\frac{1}{2}6n(6n + 1)$, and

$$h + k = 3(6n + 1). \tag{4.9}$$

Therefore

$$h = 9n + \tfrac{1}{2}(3 + d) \text{ and } k = 9n + \tfrac{1}{2}(3 - d). \tag{4.10}$$

Without loss of generality, let us assume $k < h$, that is $d = h - k > 0$. ($k = h$ is impossible, as it would imply $x = X$, and if $k > h$, then we can obtain a *dual* labeling with $k < h$ by interchanging the labels on each vertex and its opposite edge.) From (4.8), every edge label is at least $d + 1$, and $1, 2, \ldots, d$ are all vertex labels. The edges opposite those vertices receive labels $d+1, d+2, \ldots, 2d$. Proceeding in this way, we see that the set of vertex labels is

$$T_d(n) = \{1, 2, \ldots, d, 2d + 1, 2d + 2, \ldots, 3d, 4d + 1, \ldots, 6n - d\}$$

and the edge labels are

$$\{d + 1, d + 2, \ldots, 2d, 3d + 1, 3d + 2, \ldots, 4d, 5d + 1, \ldots, 6n\}.$$

So $6n$ is a multiple of $2d$. We have

Theorem 4.10 *If there is a totally magic labeling of nK_3, where n is odd, then the magic constant and sum have the values*

$$h = 9n + \tfrac{1}{2}(3 + d) \textit{ and } k = 9n + \tfrac{1}{2}(3 - d), \tag{4.10}$$

for some divisor d of $3n$.

(The negative divisors d correspond to the labelings that are duals of those with positive d.)

Suppose d is a specified positive divisor of $3n$. Let $3n = ad$. Then the sum of the elements of $T_d(n)$ is

$$\begin{aligned} ms &= \sum_{i=1}^{d}\sum_{j=0}^{a-1}(2jd+i) \\ &= \sum_{i=1}^{d}(a(a-1)d+ai) \\ &= a(a-1)d^2+\tfrac{1}{2}ad(d+1), \\ s &= 3(a-1)d+\tfrac{3}{2}(d+1). \end{aligned} \tag{4.11}$$

So the triples of vertex labels of triangles must form a partition of $T_d(n)$ into n triples, each with sum (4.11). Conversely, any such partition will define a totally magic labeling with the constants (4.10).

Lemma 4.11 *If n is an odd positive integer and d is any divisor of* $3n$, *then there exists a partition of*

$$\begin{aligned} T_d(n) = \{&1,2,\ldots,d,2d+1,2d+2,\ldots,3d,4d+1,\ldots, \\ &(2a-2)d+1,(2a-2)d+2,\ldots,(2a-1)d\} \end{aligned}$$

into n triples, each with sum (4.11), *where $a = 3n/d$.*

Proof. We use two families of $3 \times m$ arrays, m odd, which are denoted A_m and B_m, and are defined as follows. A_m has j-th column $(j, m+2-2j, \frac{1}{2}(m-1)+j)$, where entries are reduced modulo m, if necessary, so that they lie in the range $1, 2, \ldots, m$. This array was used by Kotzig [27]. Each row is a permutation of $\{1, 2, \ldots, m\}$, and each column sums to $\frac{1}{2}(3m+3)$. B_m is constructed from a copy of A_m by adding m to every entry in the second row and $2m$ to every entry in the third row, so each column has sum $\frac{1}{2}(9m+3)$.

First, suppose d is a multiple of 3. Form a master array M by subtracting 1 from each entry of A_a and multiplying by $2d$. From each column of M we construct $d/3$ triples by adding each of the columns of $B_{d/3}$. It is easy to check that the $a(d/3) = n$ triples form a partition of $T_d(n)$. All the triples have the same sum because both component arrays have constant column sum.

If d is not a multiple of 3, then a must be. In this case the master array is formed from $A_{a/3}$, again subtracting 1 and multiplying by $2d$. The columns of B_d are added. $\square$

Table 4.1 provides an illustration of the two constructions.

$$M = \begin{bmatrix} 0 & 18 & 36 & 54 & 72 \\ 72 & 36 & 0 & 54 & 18 \\ 36 & 54 & 72 & 0 & 18 \end{bmatrix} \quad B_{d/3} = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 4 & 5 \\ 8 & 9 & 7 \end{bmatrix}$$

Partition

1	19	37	55	73	2	20	38	56	74	3	21	39	57	75
78	42	6	60	24	76	40	4	58	22	77	41	5	59	23
44	62	80	8	26	45	63	81	9	27	43	61	79	7	25

Example for $a = 5, d = 9, n = 15,$ column sum 123.

$$M = \begin{bmatrix} 0 & 10 & 20 \\ 50 & 30 & 40 \\ 70 & 80 & 60 \end{bmatrix} \quad B_d = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 4 & 2 \\ 3 & 4 & 5 & 1 & 2 \end{bmatrix}$$

Partition

1	2	3	4	5	11	12	13	14	15	21	22	23	24	25
55	53	51	54	52	35	33	31	34	32	45	43	41	44	42
73	74	75	71	72	83	84	85	81	82	63	64	65	61	62

Example for $a = 9, d = 5, n = 15,$ column sum 129.

Table 4.1. Partitions for labeling $15K_3$.

From Lemma 4.11 and the discussion just before it, we have:

Theorem 4.12 *Suppose n is odd. There is a totally magic labeling of nK_3 with magic constant h and magic sum k if and only if $h = 9n + \frac{1}{2}(3 + d)$ and $k = 9n + \frac{1}{2}(3 - d)$, where d is a divisor of $3n$.*

This theorem not only proves the existence of totally magic labelings of odd unions of triangles, it also determines the spectrum of realizable magic constants for such labelings. However, there may be many non-isomorphic labelings that give the same constants. For example, there are exactly eight totally magic labelings of $3K_3$. They arise in dual pairs (the dual is derived by exchanging the label of each vertex with that of its opposite edge). Of the four pairs, two come from the construction of Theorem 4.12 and two do not.

In cases where d is negative, 1 is always an edge label. Delete this edge, and subtract 1 from every label. The resulting labeling is totally magic, with magic constant and sum found by subtracting 3 from the original constants. We have another infinite family:

Corollary 4.12.1 *The graph $P_3 \cup nK_3$ is totally magic when n is even.*

Exercise 4.5 *Prove that every totally magic labeling of $P_3 \cup nK_3$ comes from a labeling of $(n+1)K_3$ with one edge labeled 1, in the manner described above.*

Exercise 4.6 *What are the possible magic constants for totally magic labelings of $P_3 \cup nK_3$?*

In each dual pair of labelings of $3K_3$, one member has 1 as an edge label, so (from Exercise 4.5) there are exactly four totally magic labelings of $P_3 \cup 2K_3$.

Theorem 4.13 *There is no totally magic labeling of nK_3, the disjoint union of n triangles, when n is even.*

Proof. Let us write $n = 2m$. Again we assume that $k < h$, write $d = h-k$, and consider partitioning the set $T_d(2m)$ and its complement into triples.

First, suppose $h-k = 1$. We have to partition edge labels $2, 4, \dots, 12m$ into $2m$ sets of size 3, such that the sum of the labels in each set is constant. If such a partition exists, halving it provides a partition of $1, 2, \dots, 6m$ into $2m$ sets of size 3, such that the sum of the labels in each set is a constant. But if that common sum is S, then $2mS = \sum_{i=1}^{2m} i = m(2m+1)$, and $S = \frac{1}{2}(2m+1)$, which is not an integer. So $h-k = 1$ is impossible.

We assume $h-k \geq 3$. Write $h-k = 2c+1$. From (4.9) we obtain $h = 18m + c + 2$ and $k = 18m - c + 1$. so $2k - h = 18m - 3c$, and

$2k - h \equiv 18m - 3c \equiv -3c \equiv c + 2 (\bmod 4c + 2)$, since $2c + 1$ divides $3m$.

Each vertex label must be congruent to one of $1, 2, \ldots, h - k (\bmod 2(h - k))$, that is $1, 2, \ldots, 2c + 1 (\bmod 4c + 2)$. The sum of the three vertex labels in any triangle will be congruent to $2k - h$. Now, $2k - h \equiv c + 2 (\bmod 4c + 2)$. Since $0 < c < h - k$, $c + 1$ must be a vertex label. But this is impossible, since there do not exist two elements $d, e \in \{1, 2, \ldots, 2c+1\} (\bmod 4c+2)$ such that $c+1+d+e \equiv c+2 (\bmod 4c+2)$.

Hence $2mK_3$ is not totally magic for any positive integer m. □

Corollary 4.13.1 *The graph $P_3 \cup nK_3$ is not totally magic when n is odd.*

Proof. Suppose there is a totally magic labeling of $P_3 \cup nK_3$ with vertex and edge constants h and k. The center vertex of P_3 must receive label $k - h$. Suppose the other vertices of P_3 receive a and b. Then the two edges must be labeled $h - a$ and $h - b$. So the center vertex has weight $(k - h) + (h - a) + (h - b) = k - h - a - b$. Therefore $k - h - a - b = h$, and $a + b = k$. One can construct a totally magic labeling of $(n + 1)K_3$ with vertex and edge constants $h + 3$ and $k + 3$ by adding 1 to every label on $P_3 \cup nK_3$ and joining the endpoints of P_3 with an edge labeled 1. When n is odd, this contradicts Theorem 4.13. □

Although stars other than $K_{1,2}$ cannot be totally magic, no theorem so far known excludes the possibility of a star as a component of a totally magic graph. In view of the results in this section, it seems possible that the union of a larger star with a number of triangles could possibly be totally magic. Now we ask:

Research Problem 4.1 *Is the graph $K_{1,m} \cup nK_3$ ever totally magic?*

Some partial results have been discovered on this topic. Two of them are left as exercises.

Exercise 4.7 *Show that if $m > 2 + \sqrt{4 + 21n}$ then $K_{1,m} \cup nK_3$ is not vertex-magic, and therefore is not totally magic.*

Exercise 4.8 *Suppose $K_{1,m} \cup nK_3$ has a totally magic labeling λ with magic constant and sum h and k. Show that the smallest vertex label is $k - h$, and that $h < k \leq \frac{3}{2}h$.*

4.5 Small graphs

A complete search for small totally magic graphs is described in [14]. The result is that no examples with ten or fewer vertices exist, other than the four graphs given in Section 4.1.2 and the two nine vertex graphs constructed in Theorem 4.12 and Corollary 4.12.1.

The complete search was carried out in two stages. First, using *nauty* [36], lists were prepared of all connected graphs (up to ten vertices) not ruled out by Theorems 4.6, 4.7 and 4.8 and Corollary 4.9.1. Second, the survivors were tested exhaustively. There were four survivors with six or fewer vertices (K_1, K_3 and P_3 and the graph shown in Figure 4.5), 42 with seven, 1,070 with eight, 61,575 with nine and 4,579,637 with 10. The graph of Figure 4.5 was eliminated using Theorem 4.9 (see Exercise 4.9); probably many other survivors could also be eliminated by *ad hoc* applications of that theorem.

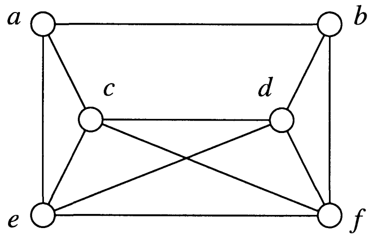

Figure 4.5. A graph to be eliminated using Theorem 4.9.

Exhaustive testing is very time consuming. However, a shortcut is available. If a totally magic labeling exists, it must retain the magic properties after reduction modulo 2. So we tested all $\bmod\ 2$ possibilities. Label 1 or 0 is assigned to each vertex and to the constant k. Then every edge-label can be calculated. Next, one can check whether all the vertex weights are congruent $(\bmod\ 2)$. Moreover, the total number of vertices and edges labeled 1 must either equal the number labeled 0 or exceed that number by 1. This process is quite fast (for example, only 2^9 cases need to be examined in the eight-vertex case), and eliminated over 25% of graphs. Then one can sieve the remaining graphs modulo 3, then modulo 4, and so on. For example, sieving the 1070 eight-vertex graphs mod 2 eliminated 307 graphs, sieving mod 3 eliminated 351 more, and so on: one graph survived after sieving modulo 7, and it was eliminated mod 8.

There were very few disconnected graphs to consider. Except for the graph $K_1 \cup P_3$, the only possibilities are made up of at most one star, copies of K_3, and survivors with more than three vertices. The only cases not already discussed are $K_{1,n} \cup K_3$ for $n = 3, 4, 5, 6$, $K_{1,4} \cup 2K_3$, and the 42 unions of a triangle and a 7-vertex survivor. None of these is totally magic, so there are no further totally magic graphs with ten or fewer vertices.

A further investigation using a variant of simulated annealing has been carried out. This procedure quickly finds the graphs we have described, but has so far found no other examples. This might suggest that we have found all totally magic graphs. However, for larger numbers of vertices (more than 20), it appears that the search gets "nearer" to satisfaction (no, I do not wish to clarify this vague description!), so perhaps there are large totally magic graphs yet to be discovered.

Exercise 4.9 *Show that the graph of Figure 4.5 is not totally magic.*

Research Problem 4.2 *Prove or disprove: no graph of the form* $K_{1,n} \cup G$ *is totally magic, when* $n \geq 3$.

Research Problem 4.3 *Are there any further totally magic graphs?*

4.6 Totally magic injections

Most of the material in this section is taken from [37].

A totally magic injection is an injection that is both edge-magic and vertex-magic. As in the other cases, we define an $[m]$-totally magic injection of G to be a totally magic injection of G in which the largest label is m, and again call m the *size* of the injection. If G has a totally magic injection, the *total deficiency* $def_t(G)$ of G is the minimum value of $m - v(G) - e(G)$, such that there is an $[m]$-totally magic injection of G, and the corresponding minimal value of m is denoted by $m_t(G)$. A labeling achieving this bound is called *minimal*.

Theorems 4.6, 4.7, 4.8 and 4.9 all show that certain classes of graphs have no totally magic injection, so totally magic injections are somewhat rarer than the edge-magic and vertex-magic species. However, some graphs are known to have totally magic injections but are not totally magic.

Theorem 4.14 *The star $K_{1,n}$ has a totally magic injection provided $n > 1$. The total deficiency when $n > 2$ is $def_t(K_{1,n}) = \binom{n+2}{2} - 2n - 3$.*

Proof. Suppose λ is a totally magic injection of a $K_{1,n}$ with center c and leaves $x_1, x_2, \ldots, x_n$; write a_i for $\lambda(cx_i)$, and let a be the smallest a_i. From the proof of Theorem 4.4, $\lambda(c) = k - h$, so $k = wt(cx_i) = (k - h) + a_i + \lambda(x_i)$ implies $\lambda(x_i) = h - a_i$. The weight of c is $(k - h) + \sum a_i$, and this will equal h. So h is the sum of $n + 1$ positive integers, whence $h \geq \binom{n+2}{2}$. The largest vertex weight, $h - a$, is clearly the largest value of λ. Now $h - a$ is the sum of n positive integers, $n - 1$ of which are greater than a, so it must equal at least $1 + 3 + 4 + \cdots + (n + 1)$, or $\binom{n+2}{2} - 2$. So this is the smallest possible size of a totally magic injection.

This size can be realized when $n > 2$. Take $h = \binom{n+2}{2}$, $k = h + 1$, $a_i = i + 1$. Then $\lambda(c) = 1$, $\lambda(cx_i) = \binom{n+2}{2} - i - 1$. This totally magic injection has deficiency $\binom{n+2}{2} - 2 - v(K_{1,n}) - e(K_{1,n}) = \binom{n+2}{2} - 2n - 3$. When $n = 2$, this construction gives two equal labels. However $K_{1,2}$ is known to be totally magic. □

It follows from the discussion in Section 4.2 that a graph with two isolates cannot have a totally magic injection. However, graphs with one isolate are more easily handled.

Theorem 4.15 *Suppose G has a totally magic injection λ with magic constant h. If G has no isolated vertex, then $G \cup K_1$ has a totally magic injection of size h.*

Proof. Extend λ to $G \cup K_1$ by labeling the isolate with h. All that is necessary is to check that h is not already in use as a label on G. But if either $\lambda(x) = h$ or $\lambda(xy) = h$, this would imply that $wt(x) > h$ (there are no isolates), which is impossible. □

Suppose this method is applied to the labeling of $K_{1,n}$ constructed in Theorem 4.14. It is clear that $wt(c)$ cannot be smaller than $\binom{n+2}{2}$ in any totally magic injection, so the isolated vertex in $K_{1,n} \cup K_1$ could not receive a smaller label than $\binom{n+2}{2}$. Therefore the application of Theorem 4.15 to a minimal injection of $K_{1,n}$ produces a minimal injection of $K_{1,n} \cup K_1$. But there is no reason why this should be true in general.

Research Problem 4.4 *Find examples in which, when the method of Theorem 4.15 is applied to a minimal totally magic injection of G, the resulting injection of $G \cup K_1$ is not minimal.*

Exercise 4.10 *Prove that the only forests with totally magic injections are K_1, $K_{1,n}$ for $n \geq 2$, and $K_1 \cup K_{1,n}$ for $n \geq 2$. For each forest F, find $m_t(F)$.*

Research Problem 4.5 *Show that $2nK_3$ has a totally magic injection, and find the total deficiency of $2nK_3$.*

Notes on the Research Problems

Edge-magic labelings

The first problem,

Research Problem 2.1. *Investigate graphs G for which equation* (2.3) *implies the nonexistence of an edge-magic total labeling of* $2G$, is a simple, open-ended, investigative exercise. As a follow-up, one could look at Section 2.2.1 with a view to generalizing it.

The study of magic numbers is almost untouched. Kotzig and Rosa mentioned it in [29], but there are not many results. In **Research Problem 2.2** we ask for $M(7)$ and $M(8)$; these can be found by exhaustive searches. A more general — and much harder — problem is to investigate the function $M(n)$ more generally, and in particular

Research Problem 2.3. *Find good lower and upper bounds on* $M(n)$.

Research Problems 2.4, 2.6, 2.7, 2.8, 2.9, 2.10, 2.12 and **2.13** ask for investigation of the edge-magic properties of various classes of graphs: $K_{2n\backslash n}$, (n, t)-kites, books $B_{k,n}$, generalized books, complete multipartite graphs, helms and flowers. This list could be generalized to taste. I have included classes that seem to be of interest, because they generalize known cases, because someone has asked (in a paper, a talk, or seminar) about their magic properties, or because other labelings of these graphs have been investigated. In general, if a class of graphs is being studied, some interesting graph-theoretic properties may be revealed by investigating the magic properties.

If a graph, or family of graphs, has a magic labeling, one can always ask for the spectrum. The following case was specifically included because the problem is addressed in [18].

Research Problem 2.5. *If v is odd, does C_v have an edge-magic total labeling for every magic sum k satisfying $\frac{1}{2}(5v+3) \le k \le \frac{1}{2}(7v+3)$? If v is even, does C_v have an edge-magic total labeling for every magic sum k satisfying $\frac{5}{2}v+2 \le k \le \frac{7}{2}v+1$?*

The problem of which wheels are edge-magic has proven far harder than originally expected. The different cases have been solved by various methods, until only the case of W_n where $n \equiv 2(\bmod\ 8)$ remains. So it is posed in **Research Problem 2.11**.

Research Problem 2.14 asks whether all trees are edge-magic. This was conjectured in [29], and again in [44]. The problem is interesting partly because it is similar to the conjecture that all trees are graceful, and also because it seems to be difficult (no progress in over 30 years).

We propose three problems about connected graphs, **Research Problems 2.15, 2.16** and **2.17**: are $K_2 \cup C_n$, nK_4 (n is even) or tW_n (t odd, and $n \equiv 1(\bmod\ 4)$) edge-magic? Many other questions could be asked. $K_2 \cup G$ is interesting because $K_2 \cup G$ is never *vertex*-magic, $G = C_n$ is a simple case, probably with enough structure to make a general solution possible. We know that an *odd* number of copies of K_4 is not edge-magic, so the even case is worth considering. The wheel case is also a slight variation on the case $n \equiv 3(\bmod 4)$ that is known not to be edge-magic.

Many questions could be asked about strongly edge-magic graphs.

Research Problem 2.18 discusses unions of disjoint cycles in the hope that they will shed some light on strongly edge-magic disconnected graphs.

Vertex-magic labelings

We know that there are graphs that have two vertex-magic total labelings with the same vertex labeling but different edge labelings. No one has yet constructed an infinite family of graphs with this property, so **Research Problem 3.1** asks for one. As a variant, one could require that the graphs be connected.

The next two problems,

Research Problem 3.2. *Prove or disprove that every value of h allowed by* (3.18) *can be realized as the magic constant of a two vertex-magic total labeling of $K_{m,m}$,*

Research Problem 3.3. *Prove or disprove that every value of h allowed by (3.19) and (3.20) can be realized as the magic constant of a vertex-magic total labeling of $K_{m,m+1}$* concern the spectrum of vertex-magic complete bipartite graphs. It is expected that almost all of the possible constants can be realized, with only a few small exceptions. However, this may not be true — maybe there is another constraint waiting to be discovered — and in any event the problems may be quite difficult. Another spectrum problem, probably equally difficult, is

Research Problem 3.6. *Prove or disprove: for each $n > 4$, there is a vertex-magic total labeling for K_n with magic constant h whenever*

$$\frac{v(v^2+3)}{4} \leq h \leq \frac{v(v+1)^2}{4}.$$

On the other hand, **Research Problem 3.4** and **Research Problem 3.5** ask for the construction of unions of stars, and of trees, that meet the extreme values in terms of numbers of leaves. Their existence should be relatively easy to prove or disprove.

Finally, we asked the following.

Research Problem 3.7. *Can the method of Theorem* 3.21 *be applied to some case $G \cup H$ where G and H are nonisomorphic regular graphs?* Who knows?

Totally magic labelings

The study of totally magic labelings was slow to develop. Researchers felt that totally magic graphs would be relatively rare, but they did not expect them to be particularly interesting. After a little study, reaction changed to: "Hey, where are these graphs?" and later to wondering why they are so rare. It seems likely that there should some stronger theorems waiting to be found that would rule out the existence of totally magic labelings for large classes of graphs.

As an alternative to proving the nonexistence of totally magic graphs, one can seek new families of such graphs. The star $K_{1,m}$ is not totally magic when $m > 2$, but every star has a totally magic injection. so we ask:

Research Problem 4.1. *Is the graph $K_{1,m} \cup nK_3$ ever totally magic?*

The results in Exercises 4.7 and 4.8 were discovered while investigating this problem, and may be useful. A harder question is:

Research Problem 4.2. *Prove or disprove: no graph of the form $K_{1,n} \cup G$ is totally magic, when $n \geq 3$.*

(See Exercise 4.1.)

Of course, the real problem concerning totally magic labelings is

Research Problem 4.3. *Are there any further totally magic graphs?*

Totally magic injections are discussed primarily in the hope that they will give some insight into this problem.

Research Problem 4.4. *Find examples in which, when the method of Theorem 4.15 is applied to a minimal totally magic injection of G, the resulting injection of $G \cup K_1$ is not minimal.*

Research Problem 4.5. *Show that $2nK_3$ has a totally magic injection, and find the total deficiency of $2nK_3$*. This should be quite easy.

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Answers to Selected Exercises

Exercise 1.1 One example is

$$\begin{array}{ccc} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{array} \qquad \begin{array}{ccc} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{array} \qquad \begin{array}{ccc} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2. \end{array}$$

Exercise 1.2 Say the order is $2n - 1$. To get the Latin squares, circulate

$$(n \;\; 1 \;\; 2 \;\; \dots \;\; n-1 \;\; n+1 \;\; n+2 \;\; \dots \;\; 2n-1),$$

and back-circulate

$$(1 \;\; 2 \;\; \dots \;\; n-1 \;\; n+1 \;\; n+2 \;\; \dots \;\; 2n-1 \;\; n).$$

Exercise 1.4 A $t \times t$ magic square and a $t \times m$ Kotzig array will produce a $t \times tm$ magic rectangle. For the first ingredient, t must be at least 3, and for the second, t odd implies m odd. So the construction gives all orders $t \times tm$, $t \geq 3$, $t(m-1)$ not odd.

Exercise 2.1 Every vertex of tK_{8m+4} has odd degree. The number of edges is $t(4m+2)(8m+3)$, which is even, and is congruent to $2(\bmod 4)$ when t is odd, and the number of vertices is $0(\bmod 4)$, so $v + e \equiv 2(\bmod 4)$, and Theorem 2.1 applies.

Exercise 2.3 Each D_i equals d, so (2.3) is

$$ke = \sigma_0^{v+e} + \sum (d-1)a_i = \sigma_0^{v+e} + (d-1)\sum a_i = \sigma_0^{v+e} + (d-1)s,$$

which is (2.4). In a regular graph, $e = \frac{1}{2}dv$, and (2.5) follows.

Exercise 2.4 From Theorem 2.13, $v(K_n + tK_1) + e(K_n + tK_1) \geq \rho^*(n)$. But $v(K_n + tK_1) = t + n$ and $e(K_n + tK_1) = \binom{n}{2}$.

Exercise 2.5 From Theorems 2.15 and 2.14, $\rho^*(n) \geq 2\sigma^*(n-1) \geq 2(4 + \binom{n-2}{2}) = 8 + (n-2)(n-3) = n^2 - 5n + 14$.

Exercise 2.6 We use the Fibonacci numbers

$$f_2 = 1, f_3 = 2, \ldots, f_{n+1}.$$

Write $k = f_{n+3} + 1$ and define a labeling λ of the K_n with vertices $\{x_1, x_2, \ldots, x_n\}$ by

$$\lambda(x_i) = f_{i-1}$$

for $i = 1, 2, \ldots, n$ and

$$\lambda(x_i x_j) = k - \lambda(x_i) - \lambda(x_j).$$

The smallest edge label is the label on $x_{n-1}x_n$, namely $f_{n+3} + 1 - f_n - f_{n+1} = f_{n+1} + f_{n+2} + 1 - f_n - f_{n+1} = f_{n+1} + 1$, and this is greater than any vertex label. The largest label used will be at most $k - f_2 - f_3 = k - 3$. So the labels are all positive, and no label is repeated. For each number between 1 and $k-3$ that has not been used as a label, introduce a new vertex with that number as a label. The result is a magic graph with magic sum k. There will be

$$t = k - 3 - n - \tfrac{1}{2}n(n-1)$$

new vertices, so

$$m(G) \leq f_{n+3} - 2 - n - \tfrac{1}{2}n(n-1).$$

Exercise 2.7 From the computational results on Sidon sequences we know

$$\rho^*(7) = 30, \ \rho^*(8) = 43.$$

When $n \geq 8$, the equation

$$\rho^*(n) \geq n^2 - 5n + 14 \text{ when } n > 8$$

(equation (2.17), proven in Exercise 2.5) applies. Suppose an edge-magic total labeling of K_n existed. From Theorem 2.13,

$$n + \binom{n}{2} \geq \rho^*(n).$$

For $n = 7$, this says $28 \geq 30$, and for $n = 8$, it says $36 \geq 43$. Both are false. And provided $n \geq 8$,

$$\tfrac{1}{2}n(n-1) + n \geq n^2 - 5n + 14,$$

or equivalently

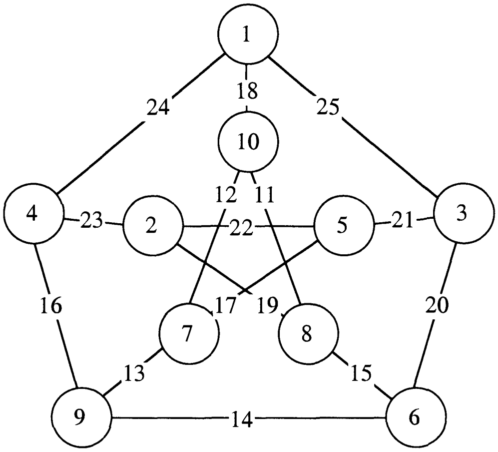
$$n^2 - 11n + 28 \leq 0,$$

which is false for all such n.

Exercise 2.8 Label the endpoints of the common edge with 1 and $n + 2$, and the other vertices $2, 3, \ldots, n + 1$.

Exercise 2.9 Take a $K_{1,n+1}$ with center labeled $n + 2$ and other vertex labels $n + 1, n + 6, n + 7, \ldots, 2n + 5$. Attach an edge to the $n + 1$ and label the other endpoint $n + 3$ (or $n + 4$). This has $k = 3n + 8$. It is by no means unique: a $K_{1,3}$ with center 1 and other labels 5, 8, 9, with 6 attached to the 5, works with $k = 13$.

Exercise 2.10 P is regular of degree 3 with $v = 10$, $e = 15$. So $2\sigma_0^{10} + \sigma_0^{25} \leq 15k \leq 2\sigma_{15}^{25} + \sigma_0^{25}$, or $435 \leq 15k \leq 735$. So $29 \leq k \leq 49$. Below is an example with $k = 29$, which is in fact strongly edge-magic.



Exercise 2.11 Use the same notation as for wheels; assume that the edge deleted would have received label s_n. Then

$$\begin{aligned}
c &= n+1 & \\
a_i &= \tfrac{1}{2}(i+1) & \text{if } i \text{ is odd,}\\
a_i &= 2n + \tfrac{1}{2}i & \text{if } i \text{ is even,}\\
b_i &= 3n+1-\tfrac{1}{2}(i+1) & \text{if } i \text{ is odd,}\\
b_i &= n+1-\tfrac{1}{2}i & \text{if } i \text{ is even,}\\
s_i &= 2n-i+1 &
\end{aligned}$$

is an edge-magic total labeling with $k = 4n+2$ [47].

Exercise 2.13 Say the one-factor consists of vertices x_i and y_i and edges x_iy_i for $1 \le i \le n$. $wt(x_iy_i) = \lambda(x_i) + \lambda(y_i) + \lambda(x_iy_i)$. Adding,

$$\begin{aligned}
nk &= \sum(\lambda(x_i) + \lambda(y_i) + \lambda(x_iy_i))\\
&= \sigma_0^{3n} = \tfrac{1}{2}3n(3n+1),
\end{aligned}$$

so $k = \frac{3(3n+1)}{2}$, which is not an integer unless n is odd. For odd n, select a Kotzig array A of size $3 \times n$ and put

$$\lambda(x_i) = a_{1i}+1, \quad \lambda(y_i) = a_{2i}+n+1, \quad \lambda(x_iy_i) = a_{3i}+2n+1.$$

Exercise 2.14 For a given k, the labeling is essentially determined by the labels on the vertices of the cycle (in order) — there are choices of which label goes on the edge of K_2, but the difference is unimportant. For $K_2 \cup C_3$, (2.2) gives $13 \le k \le 17$. For $k = 13$, the vertex labels sum to

7; the only cycle of labels is $(1, 2, 4)$, which does not work. (A label 11 is required, but 11 is too big.) $k = 14$ gives sum 11, and none of the possible cycles $(1, 2, 8)$, $(1, 3, 7)$, $(1, 4, 6)$, $(2, 3, 6)$, $(2, 4, 5)$ work. If $k = 15$ the sum is 15; the label on a vertex of a triangle would necessarily equal that on the opposite edge. Case $k = 16, 17$ are the duals of $k = 14, 13$. On the other hand, for $K_2 \cup C_4$, one solution with $k = 17$ is given by cycle labels $(1, 4, 3, 10)$.

Exercise 2.15 The solution to Exercise 2.13 gives a strong solution.

Exercise 2.16 Suppose λ is a strong edge-magic total labeling. Say the edge with label $v + i$ is $x_i y_i$ (a vertex may receive more than one name). Then $\lambda(x_i) + \lambda(y_i) + \lambda(x_i y_i) = k$ so $\lambda(x_i) + \lambda(y_i) = k - v - i$ and

$$\begin{aligned} & \{\lambda(x_1) + \lambda(y_1), \lambda(x_2) + \lambda(y_2), \ldots, \lambda(x_e) + \lambda(y_e)\} \\ = \; & \{k - v - 1, k - v - 2, \ldots, k - v - e\}, \end{aligned}$$

a set of consecutive integers. conversely, suppose λ has the given property; say the consecutive integers are $\{t + 1, t + 2, \ldots, t + e\}$. Define $\lambda(xy)$ to be $v - t + \lambda(x) + \lambda(y)$ for each edge xy. This extends λ to a strong edge-magic total labeling.

Exercise 2.17 Suppose λ is a super-strong edge-magic total labeling for G. If xy is any edge, then the smallest possible value for $\lambda(x) + \lambda(y)$ is $v_1 + 2$ and the largest possibility is $v_1 + (v_1 + v_2)$. From Exercise 2.16 the values taken by $\lambda(x) + \lambda(y)$ when $x \sim y$ is a set of consecutive integers, so the number of these sums is at most $v_1 + (v_1 + v_2) - (v_1 + 2) + 1 = v_1 + +v_2 - 1 = v - 1$. The sums are obviously all distinct, so there are e of them. So $e \leq v - 1$.

Exercise 2.18 Apply the labels in such a way that 1 and a_2 are on nonadjacent vertices.

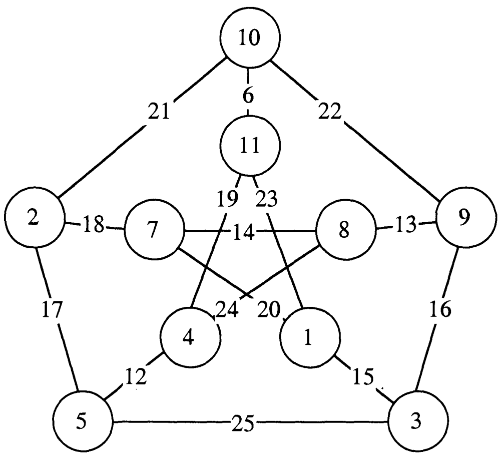
Exercise 2.19 Let λ be an edge-magic injection of K_v with length $m(v)$, as provided by Theorem 2.33. The restriction of λ to G has the desired deficiency.

Exercise 3.1 Call the labeling λ, and denote its magic constant by h. Suppose x is a vertex of degree 1, y is adjacent to it, and $\lambda(x) = 0$. Then $\lambda(xy) = h$. Then $wt(y) \geq \lambda(xy) + \lambda(y) > h$, since $\lambda(y)$ must be greater than 0.

Say P_3 has vertex sequence x, y, z. If 0 is a vertex label, $\lambda(y) = 0$, so $\lambda(xy) + \lambda(yz) = wt(y) = h$. But $\lambda(x) + \lambda(xy) = wt(x) = h$ also, so $\lambda(x) = \lambda(yz)$. If 0 is an edge label, we may assume $\lambda(yz) = 0$, so $\lambda(xy) + \lambda(y) = wt(y) = h$, and $\lambda(x) + \lambda(xy) = wt(x) = h$, so $\lambda(x) = \lambda(y)$.

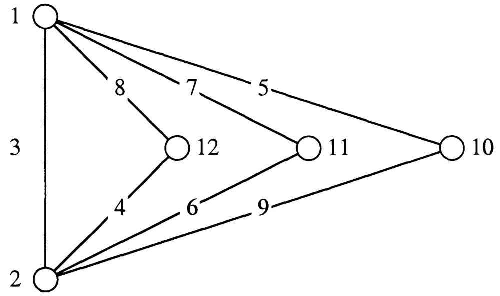
One labeling of P_4 has vertex labels (in sequence) 6, 5, 3, 4, and edges (same order) 0, 1, 2; it has $h = 6$.

Exercise 3.3 The following example has $h = 59$, the maximum possible.



Exercise 3.4 Suppose there is a vertex-magic total labeling with constant h. $3n + 3$ labels are used. The sum of the weights of the n vertices of degree 2 is at most the sum of the $3n$ largest labels, so $nh \leq \sigma_3^{3n+3} = \frac{1}{2}(3n + 4)(3n + 3) - 6$. The sum of the weights of the other 2 vertices includes the label of the edge joining them, twice, and $2n + 2$ other labels, so $2h \geq 2 + \sigma_1^{2n+3} = (n + 2)(2n + 3) + 2$. So $n[(n + 2)(2n + 3) + 2] \leq (3n + 4)(3n + 3) - 12$. It follows that $2n^2 - 2n - 13 \leq 0$. The largest integer solution to this is $n = 3$.

$B_{3,1}$ is C_3 (all cycles are vertex-magic) and $B_{3,2}$ is $K_4 - e$ (a vertex-magic total labeling is given in Figure 3.1). One VMTL of $B_{3,3}$ is



Exercise 3.5 Denote the center vertex of F_n by u and let $x_1, \ldots, x_n$ be the rim vertices. Then

$$\begin{aligned} h &= wt(u) \\ &\geq \sigma_0^{n+1} \\ &= \tfrac{1}{2}(n+1)(n+2). \end{aligned}$$

Now consider the sum of weights of all the rim vertices. An upper bound for this sum is found by assigning the $n-1$ largest labels to the rim edges and the $2n$ next largest labels to the rim vertices and the spoke edges. We find

$$\begin{aligned} nh &= wt(x_1) + \cdots + wt(x_n) \\ &\leq \sigma_1^{2n+1} + 2\sigma_{2n+1}^{3n} \\ &= \sigma_1^{3n} + \sigma_{2n+1}^{3n} \\ &= \tfrac{1}{2}3n(3n+1) - 1 + (2n+1)(n-1) + \tfrac{1}{2}n(n-1) \\ &= \tfrac{1}{2}(9n^2 + 3n - 2 + 4n^2 - 2n - 2 + n^2 - n) \\ &= 7n^2 - 2. \end{aligned}$$

$$h < 7n.$$

It is easy to see that $7n < \frac{1}{2}(n+1)(n+2)$ for all $n > 10$.

Exercise 3.6 Suppose T_n has a vertex-magic total labeling. The weight h of the center vertex is at least $\sigma_0^{2n+1} = (n+1)(2n+1)$. The sum $2nh$ of weights of all the $2n$ rim vertices is at most $2\sigma_{4n+1}^{5n+1} + \sigma_1^{4n+1} = n(17n+9)$.

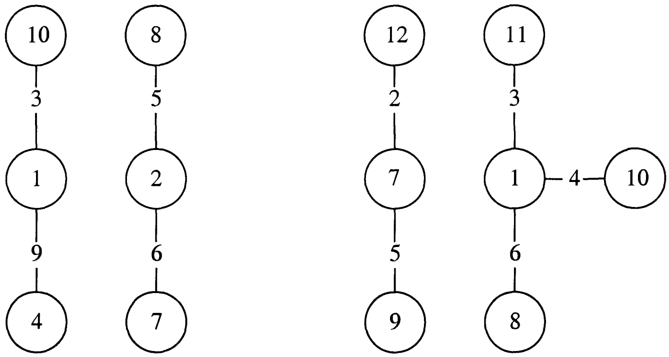
So $2(n+1)(2n+1) \leq 2h \leq 17n+9$, $n \leq \frac{1}{8}(11+\sqrt{233})$. n is integral, so $n \leq 3$. In order to describe the labeling of a triangle in T_n, we write $x\ (a, b, c)\ y$ to mean that one spoke receives label x, its endpoint receives a, the rim edge b, other vertex c and other spoke y. Then the following are examples of labelings for $n = 2, 3$:

T_2 : center 3, triangles 8 (2, 11, 9) 1, 4 (7, 10, 6) 5.

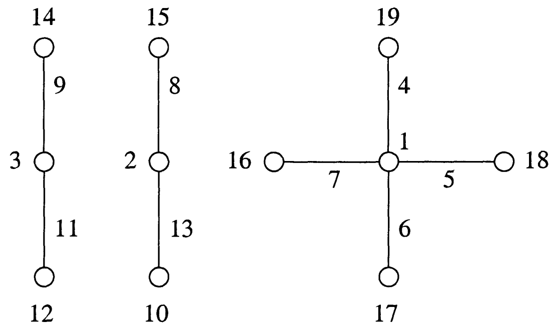
T_3 : center 2, triangles 7 (13, 8, 14) 6, 4 (9, 15, 10) 3, 1 (16, 11, 12) 5.

(T_1 is just K_3.)

Exercise 3.7 Examples, with $h = 13$ and 14 respectively, are



Exercise 3.8 The following is an example with $h = 23$:



Exercise 4.1 Suppose the center of the star receives label c. Suppose the edges of the star receive labels $a_1, a_2, \ldots, a_n$, and the vertex adjacent to label a_i receives b_i. Then $a_i + b_i = h = c + \sum a_i$, so b_i is the sum of n positive integers, and therefore $b_i \geq \frac{1}{2}n(n+1)$ for every

i. In particular, If b_1 is the smallest of the b_i and b_n is the largest, then $b_n \geq b_1 + n - 1 \geq \frac{1}{2}n(n+3) - 1$. As $b_n \leq v + e + 2n + 1$, we have $\frac{1}{2}n(n+3) - 1 \leq v + e + 2n + 1$, so $v + e \geq \frac{1}{2}(n^2 - n - 4)$.

Exercise 4.2 We have

$h = \lambda(xy) + \lambda(x) + \lambda(xt) + \tau(x)$	$k = \lambda(x) + \lambda(xy) + \lambda(y)$
$h = \lambda(yz) + \lambda(y) + \lambda(yx) + \tau(y)$	$k = \lambda(y) + \lambda(yz) + \lambda(z)$
$h = \lambda(zt) + \lambda(z) + \lambda(zy) + \tau(z)$	$k = \lambda(z) + \lambda(zt) + \lambda(t)$
$h = \lambda(tx) + \lambda(t) + \lambda(tz) + \tau(t)$	$k = \lambda(t) + \lambda(tx) + \lambda(x)$

From the first and second "h" equations we get

$$\tau(x) - \tau(y) = (\lambda(y) + \lambda(yz)) - (\lambda(tx) + \lambda(x)).$$

The second and fourth "k" equations give $(\lambda(y) + \lambda(yz)) = k - \lambda(z)$ and $(\lambda(x) + \lambda(tx)) = k - \lambda(t)$ respectively. So

$$\tau(x) - \tau(y) = k - \lambda(z) - (k - \lambda(t)) = \lambda(t) - \lambda(z)$$

which cannot be zero. So $\tau(x) \neq \tau(y)$. $\tau(y) \neq \tau(z)$, $\tau(z) \neq \tau(t)$ and $\tau(t) \neq \tau(x)$ are proven similarly. To show that $\tau(x) \neq \tau(z)$, use the first and third "h" equations to show

$$\tau(x) - \tau(z) = (\lambda(yz) + \lambda(z) + \lambda(zt)) - (\lambda(tx) + \lambda(x) + \lambda(xy)).$$

Now the first and second "k" equations yield $\lambda(yz) + \lambda(z) = \lambda(x) + \lambda(xy)$, so $\tau(x) - \tau(z) = \lambda(zt) - \lambda(tx)$, which again cannot be zero, so $\tau(x) \neq \tau(z)$, and similarly $\tau(y) \neq \tau(t)$.

Exercise 4.3 Consider the weights of x and y.

$$\begin{aligned} h &= \lambda(x) + \lambda(xy) + \sum_{\substack{z \sim x \\ z \neq y}} \lambda(xz) \\ &= \lambda(y) + \lambda(xy) + \sum_{\substack{t \sim y \\ t \neq x}} \lambda(xz). \end{aligned}$$

Adding,

$$\begin{aligned} 2h &= \lambda(x) + \lambda(xy) + \lambda(y) + \lambda(xy) \\ &\quad + \sum_{\substack{z \sim x \\ z \neq y}} \lambda(xz) + \sum_{\substack{t \sim y \\ t \neq x}} \lambda(xz) \\ &= k + \lambda(xy) + \sum_{\substack{z \sim x \\ z \neq y}} \lambda(xz) + \sum_{\substack{t \sim y \\ t \neq x}} \lambda(xz), \end{aligned}$$

giving the result.

Exercise 4.4 From the vertex-magic property,

$$\begin{aligned} h &= \lambda(x_1) + \lambda(xx_1) + \tau_1 \\ &= \lambda(x_2) + \lambda(xx_2) + \tau_2 \\ &\quad \ldots \\ &= \lambda(x_n) + \lambda(xx_n) + \tau_n, \end{aligned}$$

while edge-magic implies that

$$\lambda(x_1) + \lambda(xx_1) = \lambda(x_2) + \lambda(xx_2) = \cdots = \lambda(x_n) + \lambda(xx_n) = k - \lambda(x),$$

so

$$\tau_1 = \tau_2 = \cdots = \tau_n = h - k + \lambda(x).$$

Exercise 4.5 Suppose λ is a totally magic labeling of $P_3 \cup nK_3$ with magic sum and constant k and h. Define a labeling μ by $\mu(x) = 1 + \lambda(x)$ for every element x of $P_3 \cup nK_3$. Then add an edge y joining the endpoints of the P_3, and define $\mu(y) = 1$. We claim that μ is a totally magic labeling of $(n+1)K_3$, with magic sum and constant $k+3$ and $h+3$. It is necessary to verify that $wt_\mu(y) = k + 3$. The center vertex c of P_3 must satisfy $\lambda(c) = k - h$. Suppose the other vertices of P_3 receive labels a and b. Then the two edges must be labeled $h - a$ and $h - b$. So $wt_\lambda(c) = (k-h) + (h-a) + (h-b) = k - h - a - b$. Therefore $k - h - a - b = h$, and $a + b = k$. Thus $wt_\mu(y) = (a + 1) + (b + 1) + 1$.

The result of deleting the y and reducing all labels by 1 is the original labeling λ of $P_3 \cup nK_3$.

Exercise 4.6 It follows from Exercise 4.5 that the possible magic constant and sum values are 3 less than the corresponding values for a totally magic labeling of $(n + 1)K_3$. So there is a totally magic labeling of

$P_3 \cup nK_3$ with magic constant h and magic sum k if and only if n is even and $h = 9n + \frac{1}{2}(d - 15)$ and $k = 9n + \frac{1}{2}(15 - d)$, where d is a divisor of $3n + 3$.

Exercise 4.7 Suppose $G = K_{1,m} \cup nK_3$ has a vertex-magic labeling λ with magic constant h. As G has $m + 3n + 1$ vertices and $m + 3n$ edges, the upper bound in (3.5) is

$$\begin{aligned}(m + 3n + 1)h &\leq 2\binom{2m + 6n + 2}{2} - \binom{m + 3n + 2}{2} \\ &= (m + 3n + 1)(4m + 12n + 2 - \frac{m + 3n + 2}{2})\end{aligned}$$

and $2h \leq 7m + 21n + 2$. On the other hand, h equals the sum of some $m + 1$ labels (consider the weight of the center of the star). So $h \geq \binom{m+2}{2}$. Combining these, $(m+1)(m+2) \leq 7m+21n+2$, and $m^2-4m-21n \leq 0$. From this, $m \leq 2 + \sqrt{4 + 21n}$.

Exercise 4.8 Suppose $K_{1,m} \cup nK_3$ has a totally magic labeling λ with magic constant and sum h and k. From Lemma 4.1 we can assume $m > 1$. Suppose the star has center c. Then $\lambda(c) = k - h$. If x is any vertex of a K_3 and X is the edge opposite x, then $\lambda(x) = k - h + \lambda(X) > k - h$ (see Section 4.4). If a is one of the pendant vertices, then $h = wt(c) > \lambda(c) + \lambda(ac) = k - h + \lambda(ac)$ and also $h = wt(c) = \lambda(a) + \lambda(ac)$, so $\lambda(a) = h - \lambda(ac) > k - h$. So every vertex label is at least $k - h$.

As $k - h$ is a label, $h < k$. If $\frac{3}{2}h < k$, then $3h < 2k$, so $2h - k < k - h$. Now $2h - k$ is positive ($2h \leq k$ implies $h \leq k - h$, but we just saw that $h > k - h$), so $2h - k$ must be an edge label. If it is on an edge of the star, then $wt(c) > (k - h) + (2h - k) = h$, which is imposssible; if it is an edge of a K_3, then its opposite vertex has label h, which is also impossible. So $h < k \leq \frac{3}{2}h$.

Exercise 4.9 Suppose there were a totally magic labeling λ. Consider the triangle ace. By Theorem 4.9,

$$\lambda(ab) = \lambda(ed) + \lambda(ef).$$

Again, considering triangle bdf,

$$\lambda(ab) = \lambda(cf) + \lambda(ef).$$

So $\lambda(ed) = \lambda(cf)$.

Exercise 4.10 Theorem 4.4 implies that a forest with a totally magic injection must consist of stars and isolates. Since the centers of any two stars would receive the same label, the only possibilities are K_1, $K_{1,n}$ and $K_1 \cup K_{1,n}$. Since $K_{1,1}$ is not allowed, we have the required list. The smallest possible greatest labels are

$$m_t(K_1) = 1, \quad m_t(K_{1,2}) = 5, \quad m_t(K_{1,n}) = \binom{n+2}{2} - 2, n > 2,$$

$$m_t(K_1 \cup (K_{1,n}) = \binom{n+2}{2}.$$

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