OXFORD LOGIC GUIDES • 45

Reductive Logic and Proof-search

Proof Theory, Semantics, and Control

DAVID J. PYM
and
EIKE RITTER

OXFORD SCIENCE PUBLICATIONS

OXFORD LOGIC GUIDES: 45

General Editors

ANGUS MACINTYRE
DOV M. GABBAY
DANA SCOTT

OXFORD LOGIC GUIDES

1. Jane Bridge: *Beginning model theory: the completeness theorem and some consequences*
2. Michael Dummett: *Elements of intuitionism* (*1st edition*)
3. A.S. Troelstra: *Choice sequences: a chapter of intuitionistic mathematics*
4. J.L. Bell: *Boolean-valued models and independence proofs in set theory* (*1st edition*)
5. Krister Seberberg: *Classical propositional operators: an exercise in the foundation of logic*
6. G.C. Smith: *The Boole–De Morgan correspondence 1842–1864*
7. Alec Fisher: *Formal number theory and computability: a work book*
8. Anand Pillay: *An introduction to stability theory*
9. H.E. Rose: *Subrecursion: functions and hierarchies*
10. Michael Hallett: *Cantorian set theory and limitation of size*
11. R. Mansfield and G. Weitkamp: *Recursive aspects of descriptive set theory*
12. J.L. Bell: *Boolean-valued models and independence proofs in set theory* (*2nd edition*)
13. Melvin Fitting: *Computability theory: semantics and logic programming*
14. J.L. Bell: *Toposes and local set theories: an introduction*
15. R. Kaye: *Models of Peano arithmetic*
16. J. Chapman and F. Rowbottom: *Relative category theory and geometric morphisms: a logical approach*
17. Stewart Shapiro: *Foundations without foundationalism*
18. John P. Cleave: *A study of logics*
19. R.M. Smullyan: *Gödel's incompleteness theorems*
20. T.E. Forster: *Set theory with a universal set: exploring an untyped universe* (*1st edition*)
21. C. McLarty: *Elementary categories, elementary toposes*
22. R.M. Smullyan: *Recursion theory for metamathematics*
23. Peter Clote and Jan Krajíček: *Arithmetic, proof theory, and computational complexity*
24. A. Tarski: *Introduction to logic and to the methodology of deductive sciences*
25. G. Malinowski: *Many valued logics*
26. Alexandre Borovik and Ali Nesin: *Groups of finite Morley rank*
27. R.M. Smullyan: *Diagonalization and self-reference*
28. Dov M. Gabbay, Ian Hodkinson, and Mark Reynolds: *Temporal logic: mathematical foundations and computational aspects: Volume 1*
29. Saharon Shelah: *Cardinal arithmetic*
30. Erik Sandewall: *Features of fluents: Volume I: a systematic approach to the representation of knowledge about dynamical systems*
31. T.E. Forster: *Set theory with a universal set: exploring an untyped universe* (*2nd edition*)
32. Anand Pillay: *Geometric stability theory*
33. Dov M. Gabbay: *Labelled deductive systems*
34. Raymond M. Smullyan and Melvin Fitting: *Set theory and the continuum problem*
35. Alexander Chagrov and Michael Zakharyaschev: *Modal logic*
36. G. Sambin and J. Smith: *Twenty-five years of Martin-Löf constructive type theory*
37. María Manzano: *Model theory*
38. Dov M. Gabbay: *Fibring logics*
39. Michael Dummet: *Elements of intuitionism* (*2nd edition*)
40. Dov M. Gabbay, Mark A. Reynolds, and Marcelo Finger: *Temporal logic: mathematical foundations and computational aspects volume 2*
41. J.M. Dunn and G. Hardegree: *Algebraic methods in philosophical logic*
42. H. Rott: *Change, choice and inference: a study of belief revision and nonmontonic reasoning*
43. P.T. Johnstone: *Sketches of an elephant: a topos theory compendium: Volume 1*
44. P.T. Johnstone: *Sketches of an elephant: a topos theory compendium: Volume 2*
45. David J. Pym and Eike Ritter: *Reductive logic and proof-search: proof theory, semantics, and control*

Visit our website: www.oup.com

Reductive Logic and Proof-search

Proof Theory, Semantics, and Control

DAVID J. PYM

Hewlett-Packard Labs., Bristol & University of Bath
England, UK

and

EIKE RITTER

University of Birmingham
England, UK

CLARENDON PRESS • OXFORD

2004

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford OX2 6DP

Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship,
and education by publishing worldwide in

Oxford New York

Auckland Bangkok Buenos Aires Cape Town Chennai
Dar es Salaam Delhi Hong Kong Istanbul Karachi Kolkata
Kuala Lumpur Madrid Melbourne Mexico City Mumbai Nairobi
São Paulo Shanghai Taipei Tokyo Toronto

Oxford is a registered trade mark of Oxford University Press
in the UK and in certain other countries

Published in the United States
by Oxford University Press Inc., New York

© Oxford University Press 2004

The moral rights of the author have been asserted

Database right Oxford University Press (maker)

First published 2004

All rights reserved. No part of this publication may be reproduced,
stored in a retrieval system, or transmitted, in any form or by any means,
without the prior permission in writing of Oxford University Press,
or as expressly permitted by law, or under terms agreed with the appropriate
reprographics rights organization. Enquiries concerning reproduction
outside the scope of the above should be sent to the Rights Department,
Oxford University Press, at the address above

You must not circulate this book in any other binding or cover
and you must impose this same condition on any acquirer

A catalogue record for this title is available from the British Library

Library of Congress Cataloging in Publication Data

(Data available)

ISBN 0 19 852633 4 (Hbk)

10 9 8 7 6 5 4 3 2 1

Typeset by Newgen Imaging Systems (P) Ltd., Chennai, India
Printed in Great Britain
on acid-free paper by
Biddles Ltd., King's Lynn

PREFACE

This a research monograph about logic.

Logic is both part of and has rôles in many disciplines, including, inter alia, mathematics, computing, and philosophy. Our topic in this monograph, the mathematical theory of reductive logic and proof-search, draws upon the techniques and culture of all three disciplines but is mainly about mathematics and computation.

Since its earliest presentations, mathematical logic has been formulated as a formalization of deductive reasoning: given a collection of hypotheses, a conclusion is derived. However, the advent of computational logic has emphasized the significance of reductive reasoning: given a putative conclusion, what are sufficient premisses? Whilst deductive systems typically have a well-developed semantics of proofs, reductive systems are typically well-understood only operationally. Typically, a deductive system can be read as a corresponding reductive system. The process of calculating a proof of a given putative conclusion, for which non-deterministic choices between premisses must be resolved, is called proof-search and is an essential enabling technology throughout the computational sciences. We suggest that the reductive view of logic is (at least) as fundamental as the deductive view and discuss some of the problems which must be addressed in order to provide a semantics of proof-searches of comparable value to the corresponding semantics of proofs. Just as the semantics of proofs is intimately related to the model theory of the underlying logic, so too should be the semantics of reductions and of proof-search. We discuss how to solve the problem of providing a semantics for proof-searches in intuitionistic logic which adequately models not only the logical but also, via an embedding of intuitionistic reductive logic into classical reductive logic, the operational aspects, that is, control of proof-search, of the reductive system. We conclude with a naturally motivated example of our semantics of proof-search: a class of games.

In summary, then, the principal contributions of this monograph are the ones listed below.

- In Chapter 1, we introduce our perspective on reductive logic and proof-search, starting from a motiviation and explanation of the basic concepts. We also provide a discussion of the mathematical prerequisites for readers of this monograph. Specifically, we cover key basic topics in logic and algebra. In logic, we cover basics of classical logic, intuitionism, and axiomatic proof systems; in algebra, we begin with basic ordered structures but focus mostly on category theory.

- In Chapter 2, we provide an explanation of natural deduction proof systems and their corresponding λ-calculi. Beginning with the necessary basics of natural deduction systems for intuitionistic logic and the simply-typed λ-calculus, we then present the $\lambda\mu$-calculus, giving both the basic definitions and essential metatheory, before proceeding with a discussion of the (delicate matter of the) addition of disjunction to $\lambda\mu$, based on recent papers of the authors [97, 108, 111].
- We begin Chapter 3 with an account of the semantics of intuitionisitic natural deduction proof, based on models of the simply-typed λ-calculus. Here we introduce a new form of games semantics which forms the basis for a running example throughout the monograph. Our games combine features of the games of Lorenzen [28, 72], used to model intuitionistic provability, and of the games used by Hyland and Ong [64] to interpret fragments of linear logic, and Ong [89] to interpret terms of the $\lambda\mu$-calculus. We then present the semantics of classical natural deduction proofs via recently-developed models of the $\lambda\mu$-calculus [89, 97], and its disjunctive extenstions [97], based on fibrations of models of simply-typed calculi. We consider also continutations in this context, adumbrating some of our concerns in later chapters.
- In Chapter 4, we provide a systematic account of reductive proof theory. Beginning with a somewhat historical account of (automated and interactive) theorem proving, we provide a systematic account of sequent calculi, including a summary of their essential meta-theory and their representation in the classical λ-calculus [111]. We then provide a systematic account of reductive proof theory, based on the sequent calculus and classical λ-calculi, including a rational reconstruction of Mints' intuitionitic resolution [82].
- In Chapter 5, we provide a systematic model-theoretic account of reductive logic. Here the challenge is to provide semantic structures that are rich enough to account not only for the space of proofs but also for the (larger) space of reductions—all proofs may be seen as successful reductions whereas many reductions fail to determine proofs. Our techniques are those of categorical model theory and categorical proof theory, and we make essential use of the interplay between the semantics of proofs given by algebraic realizers and the meaning of propositions given by Kripke's account of truth-functional semantics.
- Finally, in Chapter 6, we provide a semantics for proof-search in reductive logic which properly incorporates the semantics of the principal control mechanism for proof-search, namely backtracking, within the model theory of the logic. Here our focus is on proof-search in intuitionistic reductive logic, and we exploit an embedding of intuitionistic proofs within classical proofs as a framework within which control structures may be represented.

A file of 'Errata and Remarks', giving corrections to any known errors in this monograph, and providing clarifications and other remarks for this monograph, will be maintained at the following:

`http://www.cs.bath.ac.uk/~pym/reductive-logic-errata.html`
`http://www.cs.bham.ac.uk/~exr/reductive-logic-errata.html`

or via `http://www.oup.co.uk/isbn/0-19-852633-4`

This page intentionally left blank

ACKNOWLEDGEMENTS

The content of this monograph arises from a project initiated by Pym and Lincoln Wallen in the late 1980s, joined by Ritter in the early 1990s, supported by the UK EPSRC-funded projects GR/J46616/01, GR/J42380/01, and GR/K41687/01.

All of the chapters of this monograph, beyond the basic introductory material and with the exception of Chapters 5 and Chapter 6, together with some minor aspects of Chapter 3, are based on papers by the authors which have been published in major, rigorously refereed journals. Chapters 5 and Chapter 6 are, however, based on an article which has been privately reviewed for us by Edmund Robinson, at Queen Mary, University of London, and by Didier Galmiche and Daniel Méry, at Université Henri Poincaré, Nancy. Whilst we are most grateful for all their careful work, any scientific errors which remain are wholly our responsibility.

We also thank Carsten Führmann (at Bath) and Brian Monahan (at HP Labs, Bristol) for their comments and encouragement.

The authors accept no responsibility either for any grammatical errors inherent in the publisher's house style or for any typographical or grammatical errors introduced by the publisher's copy editors. Any other such errors are wholly our responsibility. The monograph has been typeset in LaTeX, using Paul Taylor's 'diagrams' and 'prooftree' packages.

This page intentionally left blank

FOREWORD

The relationship between logic and computation is historically complex and varied, ranging from the traditional use of mechanical procedures in the basic definition of a formal system, to the classification of computable functions via first-order and type-theoretic logical systems.

In recent years the link between principles of assertion and comprehension on the one hand, and of operation and application on the other, has brought logical techniques to centre stage in the mathematical analysis of programming languages and their features. Central to this study has been links forged between systems of logic and systems of typed λ-calculi. These links, sustained at both syntactic and semantic levels, provide an element of revisionist insight into both formalist and intuitionist perspectives on logic, and at the same time provide a framework for semantic study of programming language features sensitive to some degree to operational concerns.

But logic has not only been a significant tool for analysis but also a mechanism for problem representation and direct solution. This latter application rests on semi-decision procedures for natural systems of logic such as first-order predicate calculus and its non-classical variants, and on computer assisted proof construction environments. The former is typically organized around 'reductive' formalisms such as sequent calculi, and systems of tableaux has been a standard part of artificial intelligence and computer problem solving since the birth of modern computers; the latter has been mainly used to support human reasoning about programs and program schemes.

In the early 1980s in Edinburgh—building on earlier work of Gordon, Milner, and Plotkin in LCF, de Bruijn, Constable, and Martin-Löf in systems of intuitionistic logic and language—the Logical Frameworks (LF) initiative provided a focus for the development of systems of logical language suitable for the direct representation of formal proofs in natural form. LF aimed to provide a systematic linguistic and practical basis to working with a multiplicty of programming logics—a formalized meta-logic—around which syntactic, semantic, and programmatic approaches to working with formal systems could be organized.

It is fair to say that this project and its international counterparts around the world, have had a major influence on subsequent investigations in semantics and syntax of programming logic(s) on the one hand, and computer-assisted formal proof environments on the other. Research on the representational elements of this programme, and then subsequently on the semantic elements that grew out of the increasing understanding of typed λ-calculi and their categorical semantics today forms a major branch of theoretical computer science.

Much less developed and explored was one of the original motivations for LF: the practical calculation of proofs. This book goes some way to summarizing the progress that has been made using the LF approach in the last decade or so.

Whereas it is traditional in automated deduction to motivate techniques such as sequent calculi and unification in semantic terms—Löwenheim-Skolem, model properties, etc.—this type of semantics gives little insight into the role these mechanisms play in organizing systematic search and structuring the search space. This has hindered, and continues to hinder, the improvement of techniques of automated deduction and their application across the spectrum of formal systems used for practical representation and reasoning.

By adopting a type-theoretic framework inspired by LF and successor systems, Pym and Ritter are in a position to represent the meta-language of reductive formulations of logic such as sequent calculi, and reduction-oriented operations such as unification. When rendered in this form, these meta-linguistic techniques are laid open to proof-theoretical and ultimately a semantic analysis every bit as refined as that conducted for programming languages, and in many cases using similar techniques.

Thus one finds a representation of sequent calculi as type systems classifying terms in a typed λ-calculus ($\lambda\mu\nu$) for the representation of natural deductions. The substitutional treatment of indeterminates in this setting provides a uniform view of the notion of partiality or partial proof, the natural space of values that proof-search algorithms work with. The search for (individual) terms and the search for subproofs, traditionally separated in a classical treatment, in this setting can be elegantly unified.

While these syntactical advances are presented in detail, the most significant parallel development concerns the semantic approach to the interpretation of the type-theoretic languages that extend the treatment beyond proofs to the larger space of reductions, and hence to an analysis of reduction systems as methods for calculating proofs. Though this is by far the most complex part of this text, requiring of the reader significant mathematical maturity, it contains the main motivation for the earlier syntactic studies, and ultimately will come to be seen as the main contribution of this line of research.

This text thus provides the reader with an introduction, albeit at a sophisticated level, to new approaches to the semantic investigation of syntactic re-formulations of logical systems that designed to support efficient search for proofs in formal logics.

Lincoln Wallen
Witney, Oxfordshire
October, 2003

CONTENTS

FIGURES

TABLES

This page intentionally left blank

1

DEDUCTIVE LOGIC, REDUCTIVE LOGIC, AND PROOF-SEARCH

1.1 Introduction

Axiomatizations of logics as formal systems are usually formulated as calculi for deductive inference. Deductive inference proceeds from established, or supposed, premisses, or hypotheses, to a conclusion, regulated by the application of *inference rules*, R,

$$\Downarrow \qquad \frac{\textbf{Premiss}_1 \ldots \textbf{Premiss}_m}{\textbf{Conclusion}} \quad R$$

A proof is constructed, inductively, by applying instances of rules of this form to proofs of established premisses, thereby constructing a proof of the given conclusion.

A conceptually valuable semantics of proofs is provided by a correspondence between the propositions and proofs of a logic, the types and terms of a λ-calculus [54], and the objects and arrows of a category [70], *q.v.* Fig. 1.1, in which (e.g., natural deduction) proofs correspond to (e.g., typed λ-terms) which correspond to classes of arrows in categories with specified structure.

The leading examples of this form of semantics arise in intuitionistic logic [129], in which natural deduction proofs correspond to simply-typed λ-terms and to arrows in Cartesian closed categories [70], in intuitionistic linear logic—in which natural deduction proofs correspond to linear λ-terms and to the arrows of symmetric monoidal closed categories [15]—and bunched logic [88, 102, 105]—in which natural deduction proofs correspond to $\alpha\lambda$-terms and to arrows of doubly closed categories. Another example, which we exploit extensively in this monograph, is provided by classical natural deduction and the $\lambda\mu\nu$-calculus [90, 97, 111]. Although of great value, particularly from the point of view of our development here, the propositions-as-types relationship between classical proofs and $\lambda\mu\nu$-terms requires a commitment to a particular choice (call-by-value) of reduction strategy in normalization. In this propositions-as-types setting, normalization provides a notion of computation corresponding to function evaluation.

Theorem proving, or algorithmic proof-search, is both an essential enabling technology within the computational sciences and of independent philosophical interest. More specifically, in computing, many problems are formulated as judgements about formal texts, typically representable in logical formalisms.

$$
\begin{tikzcd}
\Phi \Rightarrow \Gamma \vdash \phi \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & {[\Gamma] \vdash [\Phi] : [\phi]} \arrow[dl, leftrightarrow] \\
& {[\![\Gamma]\!] \xrightarrow{[\![\Phi]\!]} [\![\phi]\!]} &
\end{tikzcd}
$$

FIG. 1.1. Propositions-as-types-as-objects

For example, well-formedness (parsing), well-typedness (type-checking), as well as logical consequence (e.g., for specification and correctness) itself.

There are, indeed, many useful formal languages and, for each language, typically many useful procedures for judging properties of sentences. As the complexity of the languages and their properties increases, the possibility of obtaining efficient, total procedures recedes, but partial procedures which fail quickly are of great value and interactive theorem provers, such as the Boyer–Moore system [18], the logic for computable functions (LCF) system [43], and its derivatives, such as Paulson's Isabelle system [85], as well as more complex systems, such as Coq [23], based on dependent type theory [77] are used in a wide range of system-critical applications (e.g., [115]).[1]

With widely varying complexity and efficiency characteristics, these systems have, however, a common underlying logical basis: *reductive inference*.

Reductive inference proceeds from a putative (i.e., supposed [1]) conclusion to sufficient premisses, regulated by *reduction operators*, O_R,

$$\Uparrow \qquad \frac{\textbf{Sufficient Premiss}_1 \ldots \textbf{Sufficient Premiss}_m}{\textbf{Putative Conclusion}} O_R,$$

corresponding to (admissible) inference rules, R, read from conclusion to premisses.[2] Here the idea is the following:

1. The putative conclusion is an assertion, or a *goal*, such as a sequent $\Gamma \vdash \Delta$, the *endsequent*, in our chosen logic. We should like to know whether or not the sequent is provable in our chosen logic. Often, we write

$$\Gamma \ \textbf{?-} \ \Delta,$$

[1]The literature on mechanical and interactive theorem proving is large and rich. The citations we give here are intended only to be representative. More comprehensive discussions are provided in, for example, [134], which provides a more technical perspective on a range of technical issues in first-order theorem proving for both classical and non-classical systems, [36], which provides a more up-to-date view, and broadens the perspective to include type-theoretic languages, and [75], which provides a quite comprehensive and fairly well-balanced history, as well as discussing a range of current issues and challenges.

[2]Henceforth we refer to just R rather than O_R.

borrowing a notation from Prolog [22], to indicate $\Gamma \vdash \Delta$ as a putative conclusion;

2. Here we are assuming that our given logic comes along with a proof system.[3] Each inference rule in the system, including any admissible rules, gives rise to a reduction operator. To apply a reduction operator to particular assertion we must find an instance of a reduction operator such that instance of the putative conclusion matches the assertion;
3. The assertions which must be proved in order to have a proof of the initial assertion, or *subgoals*, are then given by the corresponding instances of the sufficient premisses of the operator.

We believe that this idea of reduction was first explained *in these terms* by Kleene [65].

As well as being a natural form of logical reasoning for humans and the basis of wide variety of reasoning tools, reductive proof can also be seen as a basis for logic programming [7, 67, 80]. Although formulated as *deductive systems* for *refutation*, the resolution calculi upon which Prolog and many of its derivations are based can be systematically reformulated as reductive systems (see Chapter 4). Indeed, the notion of computation provided by reduction is not the evaluation of functions but the calculation of (evidence for the) membership of the provability relation $\vdash$.

So, in reductive logic, an attempt to construct a proof, that is, a *reduction*, proceeds, inductively, by applying instances of reduction operators of this form to putative conclusions of which a proof is desired, thereby yielding a collection of sufficient premisses, proofs of which would be sufficient to imply the existence of a proof, obtainable by deduction, of the putative conclusion.

Note, however, that a reduction may fail to yield a proof: having removed all of the logical structure, that is, the connectives, by reduction, we may be left with p **?-** q, for distinct atoms p and q.

The inherent partiality of reductions presents a clear semantic difficulty: we must be able to interpret those reductions which cannot be completed to be proofs. In particular, we aim to recover a semantics for proofs of utility comparable to that of the propositions-as-types-as-objects triangle for proofs.

The desired set-up is summarized in Fig. 1.2, in which Γ **?-** ϕ denotes a sequent which is a putative conclusion and $\Phi \Rightarrow \Gamma$ **?-** ϕ denotes that Φ is a search with root Γ **?-** ϕ. The judgement $[\Gamma] \mid\!\sim [\Phi] : [\phi]$ indicates that $[\Phi]$ is a *realizer* of $[\phi]$ with respect to assumptions $[\Gamma]$.

The provision of a model-theoretically adequate such framework is non-trivial. The main difficulty is that the objects constructed during a reduction are—in contrast to the objects, that is, proofs, constructed during deduction—inherently *partial*. Whilst any deduction proceeds from axioms to a guaranteed

[3] We could, however, formulate much of our subsequent analysis purely semantically.

$$\begin{tikzcd}
\Phi \Rightarrow \Gamma \ \textbf{?-}\ \phi \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & {[\Gamma] \mid\!\sim [\Phi] : [\phi]} \arrow[dl, leftrightarrow] \\
& {[\![\Gamma]\!] \overset{[\![\Phi]\!]}{\rightsquigarrow} [\![\phi]\!]} &
\end{tikzcd}$$

FIG. 1.2. Reductions-as-realizers-as-arrows

conclusion and so constructs a proof, reductions proceed from a putative conclusion to sufficient premisses. At any intermediate stage, it may be that it is impossible to complete the reduction so as to obtain a proof, that is, all possible reductions lead to trees in which there are leaves of the form ϕ **?-** ψ in which the formulæ ϕ and ψ are both distinct and irreducible.[4]

Suppose, then, that we have a deductive system D which is interpreted in a category $\mathcal{C}$. Consider the interpretation of an axiom sequent, $\phi \vdash \phi$, given by

$$[\![\phi]\!] \xrightarrow{\mathsf{Id}} [\![\phi]\!],$$

the identity arrow from $[\![\phi]\!]$ to itself. Proof trees over D have the property that all leaves have this form (or something very like it).

Now consider the reductive system $R(D)$, obtained by reading each of D's inference rules as reduction operators. Reduction trees over $R(D)$ can have leaves of the form $\phi \vdash \psi$, where ϕ and ψ are distinct, irreducible formulæ, so that there is no way to reduce the leaf to an axiom of the deductive system. A semantics of reductions in $R(D)$ must interpret leaves of this form. One solution is to interpret searches not in the category $\mathcal{C}$ but in the polynomial category $\mathcal{C}[\alpha]$ over an indeterminate α.[5]

[**Aside**: If A and B are objects of a category $\mathcal{C}$, we can adjoin an indeterminate $A \xrightarrow{\alpha} B$ by forming the *polynomial category* $\mathcal{C}[\alpha]$. The objects of $\mathcal{C}[\alpha]$ are the objects of $\mathcal{C}$ and the arrows of $\mathcal{C}[\alpha]$ are formed freely from the arrows of $\mathcal{C}$ together with the new arrow α. The basic ideas may be found in [70]. Although much of the detailed theory which we develop in this chapter will rely on the Cartesian and Cartesian closed structure of our underlying categories, the essential idea of using the polynomial structure should, we conjecture, work for monoidal and monoidal closed underlying categories and so be candidate for a basis for a semantics for proof-search in substructural logics.]

[4]We say that an occurrence of a formula ϕ, in a search tree over a system S, is *irreducible* if it is not the principal formula of an instance of any reduction operator of S.

[5]In general, the polynomial over a set of indeterminates.

Then the interpretation of a leaf of the form p **?-** q, where p and q denote propositional letters,[6] can be defined as follows:

$$[\![p]\!] \xrightarrow{\alpha} [\![q]\!],$$

The corresponding language of realizers (*cf.* Fig. 1.2) is the internal language of $\mathcal{C}[\alpha]$.

Whilst polynomials over categories of proofs provide a place within which reductions can be interpreted, they are deficient in several ways. Firstly, whilst it is an adequate framework for interpreting natural deduction reductions, it is not adequate for interpreting the computationally far more desirable sequent calculus proofs (because of a loss of critical information in the interpretation; see Chapter 5 for a detailed explanation). Secondly, the interpretation available provides no scope for modelling the key computational feature which takes us from reductive logic to proof-search, namely *control*: there is much more to consider in the semantics of proof-search. In computing the existence of a proof, one must make choices: for example, choose which component of a disjunction to work on, which implication on the left to reduce next, etc.. Some of these choices will lead to a proof, some not. Upon failure, one must *backtrack* and make a different choice. How are we to account for these control operations within our semantics?

The first problem is solved by moving to a framework based not merely on indeterminates but also on a notion of Kripke world which maintains the otherwise lost information: thus we take seriously the view that the search process is a constructive one in which the agent performing the computation increases its knowledge as computation proceeds.

For example, as illustrated in Fig. 1.3, suppose we have, in intuitionistic logic, the endsequent

$$\phi, \phi \supset \psi, \phi \supset \chi \vdash (\psi \vee \psi') \wedge (\chi \vee \chi'),$$

in which ϕ, ψ, ψ', χ, and χ' are atomic. Informally, we can provide a semantics in which worlds represent the state of the computation which is attempting to construct a proof of the endsequent. To see this, we can borrow an idea from computational logic, namely the *Herbrand base*, and consider the 'established facts' in the computation to be the atomic formulæ in the hypotheses, that is, on the left of the turnstile, $\vdash$.

At the root world, w_1, the only atomic proposition established on the left, and so potentially capable, in the presence of a matching ϕ on the right, of forming an axiom sequent $\phi \vdash \phi$, is ϕ.[7]

The next two reductions, $\wedge R$ and $\vee R$, take us to worlds w_2 and then w_3 and w_4 without adding to the atomic propositions established on the left. Next

[6]We call such a leaf *atomic*.

[7]Our use of just atoms to form axioms should be considered analogous to the use of atoms in a least Herbrand model [7, 130].

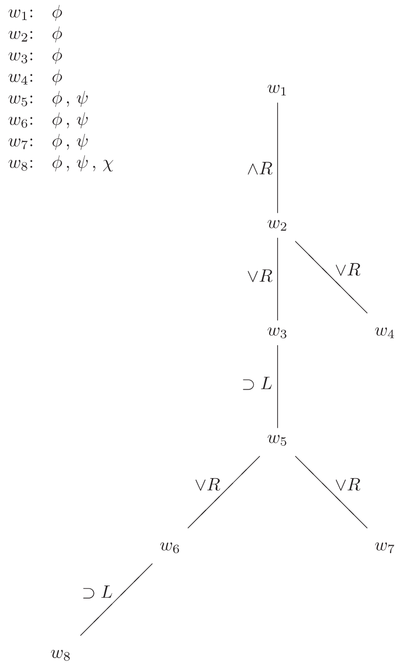


FIG. 1.3. Reduction semantics as Kripke semantics

comes a $\supset L$, with principal formula $\phi \supset \psi$. This step adds ψ to the atomic formulæ established on the left, and so capable of contributing to axioms. As before, the accession to worlds w_6 and w_7, via an $\vee R$, adds no atoms to the left. Finally, the $\supset L$ leading to w_8 adds χ to the collection of formulæ established on the left.

We can use this declarative point of view as way of including, via worlds, a notion of state within our semantics (*cf.* O'Hearn and Tennant's possible worlds semantics of state [86]), and our use, in Chapters 5 and 6, of possible worlds takes a rather more general form than that suggested by this example.

The second problem is solved, for intuitionistic logic, by embedding intuitionistic reductions inside classical reductions and exploiting the classical structure to model the control of the intuitionistic search. Here, the control structure on which we concentrate is *backtracking*. We show how to represent a failed intuitionistic reduction and the one obtained by backtracking within the same classical reduction. We give a semantic account of when backtracking can potentially occur by incorporating a suitable notion of world into the semantic structures used in our mathematical framework.

Our chosen mathematical framework is a representation of classical logic consequences as families of intuitionisitic consequences as provided by the $\lambda\mu\nu$-calculus, which we introduce in Chapter 2, and its categorical models, which we introduce in Chapter 3.

Our semantic account of backtracking is given by adding to the categorical semantics for the $\lambda\mu\nu$-calculus structures which capture the Kripke worlds discussed above. We also provide a games model for intuitionistic proof-search, using the intuition that Opponent (Proponent)-questions are challenges to provide evidence for conclusion and premiss, respectively. Again, intuitionistic searches can be seen as special cases of classical searches, and backtracking can be identified as additional disjunctive choices by Proponent which are not available in intuitionistic searches. These disjunctive choices correspond to backtracking points identified in the semantic account established in an earlier chapter.

1.2 Logical prerequisites

This research monograph is intended for readers who are either established researchers and teachers or graduate students, working in logic and related subjects. Administratively, they might, for example, be working in the mathematical sciences, computing sciences, or philosophy departments of universities, or in the more mathematical parts of industrial research laboratories.

A certain background in basic mathematical, logical and compuational topics is assumed but beyond that we intend the monograph to be essentially self-contained. The assumed background is summarized in this and the next section, although some key basic concepts are nevertheless explained in some detail as and when they are required.

Readers who are familiar with natural deduction, the $\lambda\mu$-calculus, and its categorical models may wish to proceed directly to Chapter 4, referring to specific parts of the next two sections on logical and algebraic prerequisites, and Chapters 2 and 3, such as those parts describing recently-published research on the $\lambda\mu\nu$-calculus and its semantics, and the description of our class of games, as required.

1.2.1 *Basics of classical logic*

Logic may be seen as the study of *consequences*, that is, assertions that the truth of a given proposition follows from the truth of a given *collection* of *propositions*. Propositions are *declarative statements*. We can give a simple definition, as described in Hodges [51], as follows:

A proposition is that situation which is described by an English phrase which may be substituted for X in

It is the case that X.

so as to give a grammatically correct English sentence.

Examples are phrases like 'the sky is blue', 'the sea is green', or 'the program terminates'. Mathematically, propositions are denoted by the *formulæ* of a formal language.

Logic is about more than propositions, however; it is also about reasoning. In classical logic, reasoning is captured by the idea of a *consequence relation* [10, 116, 125]

$$\phi_1, \ldots, \phi_m \vdash \psi_1, \ldots, \psi_n$$

between finite sequences of propositions. It should be read as follows: If all of the ϕs hold, then at least one of the ψs holds.

Formally, a consequence relation on set of formulæ is a binary relation $\vdash$ between finite sequences of formulæ such that:

1. Reflexivity: for every formula $\phi, \phi \vdash \phi$;
2. Transitivity (or Cut): if $\Gamma \vdash \Delta, \phi$ and $\phi, \Gamma' \vdash \Delta'$, then $\Gamma, \Gamma' \vdash \Delta, \Delta'$;
3. Exchange: if $\Gamma \vdash \Delta$, then $\rho(\Gamma) \vdash \sigma(\Delta)$, for permutations ρ and σ;
4. Weakening: if $\Gamma \vdash \Delta$, then $\Gamma, \Gamma' \vdash \Delta, \Delta'$;
5. Contraction: if $\Gamma, \Theta, \Theta, \Gamma' \vdash \Delta$, then $\Gamma, \Theta, \Gamma' \vdash \Delta$ and if $\Gamma \vdash \Delta, \Theta, \Theta, \Delta'$, then $\Gamma \vdash \Delta, \Theta, \Delta'$.

Consequence relations may be realized both model-theoretically and proof-theoretically. In classical logic, the key semantic notion is *truth*. Explanations of truth in mathematical logic usually begin with the idea of a *truth table* in which a proposition is assigned a *truth value*, 0 (false) or 1 (true). The assignment of truth values is performed by induction on the structure of propositions, connective by connective. For example, the truth table for classical implication is the following:

ϕ	ψ	$\phi \supset \psi$
0	0	1
0	1	1
1	0	0
1	1	1

Here the idea is that we assume, inductively, that we have assignments of truth values for ϕ and ψ—there are four possible combinations—and proceed to assign a value to $\phi \supset \psi$ in each case. We can write similar tables for conjunction, $\wedge$, disjunction, $\vee$, and negation, $\neg$, as follows:

ϕ	ψ	$\phi \wedge \psi$
0	0	0
0	1	0
1	0	0
1	1	1

ϕ	ψ	$\phi \vee \psi$
0	0	0
0	1	1
1	0	1
1	1	1

ϕ	$\neg\phi$
0	1
1	0

From these tables, the *de Morgan Laws*,

$$\begin{aligned}\neg(\phi \vee \psi) &= (\neg\phi) \wedge (\neg\psi)\\ \neg(\phi \wedge \psi) &= (\neg\phi) \vee (\neg\psi)\\ \phi \supset \psi &= (\neg\phi) \vee \psi\end{aligned}$$

may be obtained.

Mathematically, we think of such an assignment of truth values, or *model*, sometimes called an *interpretation*, as a function

$$\mathcal{I} : \mathbf{Prop} \to \{0, 1\}$$

from the set of atomic propositions, that is, propositional letters, p, q, etc., to the two-element set. Then we can define $\mathcal{I} \models \phi$, read as '$\mathcal{I}$ satisfies ϕ', by

$$\mathcal{I} \models p \quad \text{iff } \mathcal{I}(p) = 1.$$

Using the relation $\models$, we can now express the meanings of the classical connectives as follows:

$$\begin{array}{rcl}\mathcal{I} \models p & \text{iff} & \mathcal{I}(p) = 1\\ \mathcal{I} \models \phi \wedge \psi & \text{iff} & \mathcal{I} \models \phi \text{ and } \mathcal{I} \models \psi\\ \mathcal{I} \models \phi \vee \psi & \text{iff} & \mathcal{I} \models \phi \text{ or } \mathcal{I} \models \psi\\ \mathcal{I} \models \phi \supset \psi & \text{iff} & \mathcal{I} \models \phi \text{ implies } \mathcal{I} \models \psi.\end{array}$$

Starting from this point, we can define the notion of *semantic consequence for truth* in a given model $\mathcal{I}$:

$$\phi_1, \ldots, \phi_m \models_{\mathcal{I}} \phi \quad \text{iff } \mathcal{I} \models \phi_i, \quad \text{for each } 1 \leq i \leq m, \\ \text{implies } \mathcal{I} \models \phi.$$

A stronger notion is *semantic consequence for validity*, defined as follows:

$$\phi_1, \ldots, \phi_m \models \phi \qquad \text{iff for all } \mathcal{I}, \mathcal{I} \models \phi_i, \quad \text{for each } 1 \leq i \leq m, \\ \text{implies } \mathcal{I} \models \phi.$$

These ideas form the starting point for *classical model theory*, the area of logic which has been developed in the deepest integration with mainstream pure mathematics.

Instead of building all of our logical structure out of atomic propositional letters, we can work with a richer notion of atomic propostion, built from *predicate letters*, p, q, etc., and *terms*, t, u, etc. A predicate letter p comes along with an arity, or number of arguments. A term t is built up from *function symbols*, f, g, etc, which have arities, and variables, x, y, etc. A function symbol of arity 0 is a *constant* and a predicate letter of arity 0 is a propositional letter.

By adding *quantifiers*, such as $\forall$, or 'for all', and $\exists$, or 'there exists', and *theories*, or collections of special symbols and axioms, to the analysis described above, model theory is able to provide a logical study of important mathematical structures.

For example, the *model theory of fields* is a major area in its own right. Its axioms include propositions such as

$$\forall x.(x + 0 = x), \qquad \forall x.\forall y.\forall z.(x \times (y + z) = x \times y + x \times z)$$

and

$$\forall x.((x \neq 0) \supset \exists y.(x \times y = 1))$$

where $+, 0, \times$, and 1 are *function* symbols used to build the *terms* of the logic, and $=$ is a special *predicate* symbol, taken in addition to the logical connectives and quantifiers. The equality symbol, $=$, is used to build the atomic propositions by *predicating* terms: if s and t are terms we can form the proposition that they are equal by writing $=(s, t)$ or, more simply, $s = t$. Similarly, we write $s \neq t$ as a shorthand for $\neg(s = t)$. From this point of view, a field is a model, which satisfies these (and some other) axioms.

The semantics of classical propositions can be extended to classical predicate logic. Roughly, each predicate p letter of arity m is interpreted as an m-ary relation $[\![p]\!]$ on a set D, called the domain of the model I. Similarly, each function symbol f of arity n is interpreted as a function $[\![f]\!]$ from D^n to D. Finally, we take an *environment*, also denoted, $[\![-]\!]$, which assigns to each variable an element $[\![x]\!] \in D$. Then we have the following semantics for predicates and first-order quantifiers:

$$\begin{array}{rl} \mathcal{I} \models p(t_1, \ldots, t_m) & \text{iff } \langle [\![t_1]\!], \ldots, [\![t_m]\!] \rangle \in [\![p]\!] \\ \mathcal{I} \models \forall x.\phi & \text{iff for all } t,\, \mathcal{I} \models \phi[t/x] \\ \mathcal{I} \models \exists x.\phi & \text{iff for some } t,\, \mathcal{I} \models \phi[t/x]. \end{array}$$

The formulæ of classial logic may be written in certain *normal forms* [20, 34, 79]. The following sequence is a useful way to think of the possibilities:

- Conjunctive Normal Form (CNF) and Disjunctive Normal Form (DNF): formulæ are, respectively, conjunctions of disjunctions of literals, that is, atoms or negated atoms, and disjunctions of conjunctions of literals;

- Prenex Normal Form (PNF): formulæ have quantifiers outermost, that is, not dominated by any propositional connective;
- Skolem Normal Form (SNF): formulæ have all quantifiers removed, and replaced by variables and function symbols;
- Clausal Form (CF): conjunctions are removed from formulæ, leaving sets of disjunctions of literals.

With a little care about interpretations [20, 34], we can obtain that any formula has a classically equivalent CNF, DNF, PNF, SNF, and CF.

1.2.2 *Basics of intuitionistic logic*

Consider the following familiar theorem (see, for example, [26]):

Theorem 1.1 *There exist irrational numbers a and b such that a^b is rational.*

Proof Set $a = b = \sqrt{2}$ and proceed by cases. Either $(\sqrt{2})^{\sqrt{2}}$ is rational or it is not.

1. Suppose not. Then $(\sqrt{2})^{\sqrt{2}}$ is irrational. Then set $a = (\sqrt{2})^{\sqrt{2}}$, $b = \sqrt{2}$ and consider $\left((\sqrt{2})^{\sqrt{2}}\right)^{\sqrt{2}}$, we have

$$\begin{aligned}\left((\sqrt{2})^{\sqrt{2}}\right)^{\sqrt{2}} &= (\sqrt{2})^{(\sqrt{2})^2}\\ &= (\sqrt{2})^2\\ &= 2,\end{aligned}$$

   which is rational.
2. Otherwise, $(\sqrt{2})^{\sqrt{2}}$ is rational.

In either case, we are done. □

This proof is not acceptable to intuitionists because it is not *constructive*, that is, it fails to construct specific irrationals a and b such that a^b is rational.

The proof fails to be constructive because it makes essential use of a classical principle, the Law of the Excluded Middle: for any proposition ϕ, either ϕ or $\neg\phi$ is true.

In the absence of the Law of the Excluded Middle, or any equivalent principle, the relationship between classical implication and disjunction

$$\mathcal{I} \models \phi \supset \psi \quad \text{iff } \mathcal{I} \models (\neg\phi) \vee \psi$$

for any model, $\mathcal{I}$, breaks down. But now we have a problem. The definition of the truth of $\phi \supset \psi$ relative to a model I is given by

$$\mathcal{I} \models \phi \supset \psi \quad \text{iff } \mathcal{I} \models \phi \textbf{ implies } \mathcal{I} \models \psi.$$

Here, the bold face **implies** is implication in the *classical* metatheory, expressed in English. We can therefore rewrite the definiens as

$$(\mathbf{not}(\mathcal{I} \models \phi)) \ \mathbf{or}\ \mathcal{I} \models \psi.$$

But, by our earlier definitions, this implies

$$\mathcal{I} \models \phi \supset \psi \quad \text{iff } \mathcal{I} \models (\neg\phi) \vee \psi.$$

Intuitionistic logic avoids this contradiction by changing the semantics of implication. The idea, attributed to Saul Kripke [69], is simple and beautiful and has deep consequences. Kripke's solution abandons the idea that propositions have absolute truth values. Rather, truth is defined relative to a state of knowledge, *possible world.* Possible worlds, denoted v, w, etc., are related by a pre-order, denoted $\sqsubseteq$, with $w \sqsubseteq v$ being interpreted as 'v is a state of knowledge which is greater than or equal to v'.[8] The intuition here is that an agent, the *creative subject* [129] explores the set of possible worlds by travelling up the ordering, increases his knowledge at each step. We require the *monotonicity condition* that travelling up the ordering does not decrease knowledge.

The meaning of implication is then that $\phi \supset \psi$ holds at world w iff, for every world v which is greater than or equal to w, that is, at every increased state of knowledge, if ϕ holds at v implies ψ holds at v. Formally,

$$w \models \phi \supset \psi \quad \text{iff for all } w \sqsubseteq v, v \models \phi \text{ implies } v \models \psi,$$

where $w \models \phi$ is read as 'w forces ϕ'. Note that the metatheoretic 'implies' here is, as before, the classical one. But the definition of implication has been relativized to a world. If there is just one world, or if the order is discrete, then classical and intuitionistic implication coincide.

The Kripke semantics of intuitionistic logic can be extended to predicate logic by generalizing the extension of models of classical propositional logic to classical predicate logic to account for the ordered worlds. There is a domain $D(w)$ at each world w and the key point is to extend the monotonicity conditions to ensure that the collection of true predicates increases with the ordering on worlds. The semantics of the first-order universal quantifier is then given by

$$w \models \forall x.\phi \quad \text{iff for all } w \sqsubseteq v \text{ and all } t \text{ such that } [\![t]\!] \in D(v),\ v \models \phi[t/x],$$

with that of the existential being the same as the classical one at given world.

Finally, since the definition of classical negation ensures that it is dualizing, that is, $\neg\neg\phi$ is equivalent to ϕ, we must also provide an intuitionistic version of negation. We define $\neg\phi$ to be $\phi \supset \bot$, where is $\bot$ the absurd proposition, intuitionistically.

[8]A pre-order is a relation relexive and transitive. Some treatments of Kripke's semantics use a partial order (which is also anti-symmetric).

1.2.3 *Basics of proof systems*

In our summary of the prerquisites for this monograph so far, we have described the elementary semantics of classical and intuitionistic logic. We now turn to formal systems for these logics. We begin with *Hilbert-type* systems.

A Hilbert-type system consists of collections of axioms, rules, and definitions.

Beginning with propositional classical logic, we have a collection of axioms involving just implication an negation, and the single rule of modus ponens. We then recover, following Dummett's style of presentation [26] the other connectives via definitions.

1. Classical axioms
   (i) $\phi \supset (\psi \supset \phi)$
   (ii) $\phi \supset (\psi \supset \chi) \supset ((\phi \supset \psi) \supset (\phi \supset \chi))$
   (iii) $((\neg\phi) \supset (\neg\psi)) \supset (((\neg\psi) \supset \phi) \supset \psi)$
2. Classical rules
   (i) $\dfrac{\phi \quad \phi \supset \psi}{\psi} \quad MP$
3. Classical definitions
   (i) $(\phi \wedge \psi) ::= \neg(\phi \supset \neg\psi)$
   (ii) $(\phi \vee \psi) ::= (\neg\phi) \supset \psi$

Turning to predicate logic and the first-order quantifiers, we need one axiom, one rule, and one definition.

1. Classical predicate axioms
   (i) $\forall x.\phi(x) \supset \phi(t)$
2. Classical predicate rules
   (i) $\dfrac{\chi \supset \phi(y)}{\chi \supset \forall x.\phi(x)}$ (y not free in χ) $\quad G$
3. Classical predicate definitions
   (i) $\exists x.\phi ::= \neg\forall x.(\neg\phi)$
   We call the Hilbert-type system for classical logic HK.

Turning to intuitionistic logic, in which the connectives are not interdefinable, we need many more axioms and rules but, of course, no definitions.

1. Intuitionistic axioms
   (i) $\phi \supset (\psi \supset \phi)$
   (ii) $\phi \supset (\psi \supset (\phi \wedge \psi))$
   (iii) $(\phi \wedge \psi) \supset \phi$
   (iv) $(\phi \wedge \psi) \supset \psi$
   (v) $\phi \supset (\phi \vee \psi)$
   (vi) $\psi \supset (\phi \vee \psi)$
   (vii) $(\phi \vee \psi) \supset ((\phi \supset \chi) \supset ((\psi \supset \chi) \supset \chi))$

(viii) $(\phi \supset \psi) \supset ((\phi \supset (\psi \supset \chi)) \supset (\phi \supset \chi))$
(ix) $(\phi \supset \psi) \supset ((\phi \supset (\psi \supset \bot)) \supset (\phi \supset \bot))$
(x) $\phi \supset ((\phi \supset \bot) \supset \psi)$

2. Intuitionistic rules
   (i) $\dfrac{\phi \quad \phi \supset \psi}{\psi} \quad MP$

As for classical logic, the Hilbert-type system for intuitionistic logic can be extended to intuitionistic predicate logic:

1. Intuitionistic predicate axioms
   (i) $\forall x.\phi(x) \supset \phi(t)$
   (ii) $\phi(t) \supset \exists x.\phi(x)$
2. Intuitionistic predicate rules
   (i) $\dfrac{\chi \supset \phi(y)}{\chi \supset \forall x.\phi(x)}$ (y not free in χ) $\quad G$
   (ii) $\dfrac{\phi(y) \supset \chi}{\exists x.\phi(x) \supset \chi}$ (y not free in χ) $\quad G$

We call the Hilbert-type system for intuitionistic logic HJ.

In both HK and HJ, we have the following key theorem, originally due, in the classical case, to Herbrand:

Theorem 1.2 (deduction theorem) *If* $\Gamma, \phi \vdash \psi$, *then* $\Gamma \vdash \phi \supset \psi$.

It is important to understand the nature of rules in Hilbert-type systems [10]. They are *rules of proof*, that is, the premisses must be axioms or consequences of axioms, so that a judgement $\Gamma \vdash \phi$ in Hilbert-type systems is a not a judgement about hypothetical consequence. Rather, it asserts that ϕ is provable and gives a record, in Γ, of the axioms used. Of course, given a collection of axioms Γ, one can add a collection Δ of hypotheses as temporary axioms and use a Hilbert-type system to prove $\Delta \vdash \phi$ by proving $\Gamma, \Delta \vdash \phi$ [10].

In contrast, the rules of *natural deduction* systems are *rules of inference*, or *rules of deduction*, that is, the premisses may be *hypotheses*, or *assumptions*, so that a judgement $\Gamma \vdash \phi$ in a natural deduction system asserts that ϕ is provable from assumptions Γ. For example, in the disjunction elimination rule,

$$\dfrac{\begin{matrix} \\ \vdots \\ \phi_1 \vee \phi_2 \end{matrix} \quad \begin{matrix} [\phi_1] \\ \vdots \\ \psi \end{matrix} \quad \begin{matrix} [\phi_2] \\ \vdots \\ \psi \end{matrix}}{\psi} \vee E,$$

which may be seen as reasoning by cases, we assume proofs of $\phi_1 \vee \phi_2$, of ψ from ϕ_1, and of ψ from ϕ_2, all relative, implicitly, to a set of proofs of propositions $\chi_1, \ldots, \chi_m$. The $\vee E$ rule may then be written as

$$\dfrac{\Gamma \vdash \phi_1 \vee \phi_2 \quad \Gamma, \phi_1 \vdash \psi \quad \Gamma, \phi_2 \vdash \psi}{\Gamma \vdash \psi},$$

where $\Gamma = \{ \chi_1, \ldots, \chi_m \}$. Corresponding to the $\vee E$ rule is the rule for introducing disjunction, $\vee I$:

$$\frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2} \quad (i = 1, 2) \qquad \vee I.$$

This pattern of introduction and elimination on the right-hand side of consequences is the defining characteristic of natural deduction systems [95, 121].

Although we expect that most of our readers will have already encountered natural deduction, we provide a brief introduction in Chapter 2. Those familiar with natural deduction will immediately see the relationship between the HJ and the natural deduction rules for intuitionistic logic. For example, the $\vee E$ rule corresponds directly to Intuitionistic Axiom (vii).

Much of the proof-theoretic analysis of this monograph is concerned with the *sequent calculus*, introduced by Gentzen [37]. The basic difference between the sequent calculus and natural deduction is that the former replaces the latter's elimination rules with introduction rules on the *left-hand side*. For example, $\vee E$ is replaced by

$$\frac{\Gamma, \phi \vdash \chi \quad \Gamma, \psi \vdash \chi}{\Gamma, \phi \vee \psi \vdash \chi} \quad \vee L.$$

We make no assumption of any detailed knowledge of sequent calculi, which we introduce Chapter 4.

Both natural deduction systems and sequent calculi can be extended to predicate logic.

The relationship between formal systems of the kind we have described in this section, and others, and the semantics we described in the previous section is summarized by *soundness* and *completeness* theorems; see, for example, [26, 79, 128].

1.3 Algebraic prerequisites

As well as some basic logical background, as sketched above, we also need to assume some basic ideas about categories. We assume that the reader is familiar with the basics of sets with algebraic structure. For example, we assume that the reader has encountered structures such as the following:

1. A *lattice*, that is, a set with two *associative*, *commutative*, *idempotent*, *absorptive*[9] binary operations, $\vee$ and $\wedge$;
2. A *distributive* lattice, that is, a lattice such that $\vee$ distributes over $\wedge$ and $\wedge$ distributes over $\vee$;
3. We can define a *partial order*, $\sqsubseteq$, on a lattice by $x \sqsubseteq y$, for elements x and y, by $x \sqsubseteq y$ iff $x \vee y = y$ or, equivalently, $x \wedge y = x$;

[9]For all x and y, $x \wedge (x \vee y) = x = x \vee (x \wedge y)$.

4. A *Boolean* lattice is a distributive lattice with distinguished elements, 0 and 1, which are neutral elements for $\vee$ and $\wedge$, respectively, and a unary *complementation* operation, $\neg$, which is involutive and satisfies the de Morgan Laws for $\vee$ and $\wedge$.

For consistency with categorical ideas (see below), it is common to include the neutral elements 0 and 1 in the definition of a distributive lattice (see, for example, Chapter 3).

Let $\mathcal{A} = (A, \vee, \wedge)$ and $\mathcal{B} = (B, \oplus, \otimes)$ be lattices. A *lattice homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ is map $f : A \rightarrow B$ such that

$$f(a_1 \vee a_2) = f(a_1) \oplus f(a_2) \quad \text{and} \quad f(a_1 \wedge a_2) = f(a_1) \otimes f(a_2).$$

1.3.1 *Basics of categories*

Turning to categories, we provide a very brief summary of what we need, adding a few ideas later on in the monograph as and when they are required.

This section is *not* intended to be a tutorial in category theory. Rather, it is a summary of the basic ideas from category theory required in order to read this monograph; it also serves to fix some notation. A few good references which cover all that is required and much more besides are the following:

- S. Mac Lane and G. Birkhoff, *Algebra* [76];
- S. Vickers, *Topology via Logic* [132];
- S. Mac Lane, *Categories for the Working Mathematician* [74];
- M. Barr and C. Wells, *Categories for Computing Science*, First and Second Editions [12, 13];
- J. Lambek and P. Scott. *Introduction to Higher-Order Categorical Logic* [70];
- A. Asperti and G. Longo. *Categories, Types and Structure: An Introduction to Category Theory for the Working Computer Scientist*, MIT Press, 1991 [9].

Nevertheless, we introduce additional ideas in the text as they are required.

Definition 1.3 A category, $\mathcal{C}$, consists of the following data:

1. A collection, $Obj(\mathcal{C})$, of objects, denoted $A, B, C, \ldots$;
2. A collection, $Arr(\mathcal{C})$, of arrows or morphisms, denoted $f, g, h, \ldots$;
3. Two operations, *dom* and *cod*, which assign to each arrow, f, two objects, respectively called the domain and co-domain of f;
4. An operation, Id, which assigns to each object, A, an arrow called the identity on A, Id_A, such that $cod(\mathsf{Id}_A) = dom(\mathsf{Id}_A) = A$;
5. An operation, $\circ$, which assigns to each pair of arrows, f and g, with $dom(f) = cod(g)$ an arrow, $f \circ g$, called the composition of f and g such

that $dom(f \circ g) = dom(g)$ and $cod(f \circ g) = cod(f)$; such that identity and composition satisfy the following conditions:

(i) Identity: for all arrows, f and g, such that $cod(f) = A = dom(g)$, $\mathsf{Id}_A \circ f = f$ and $g \circ \mathsf{Id}_A = g$;

(ii) Associativity: for all arrows, f, g, and h, such that $dom(f) = cod(g)$ and $dom(g) = cod(h)$, $(f \circ g) \circ h = f \circ (g \circ h)$.

We have the evident notion of *subcategory*. For any category, $\mathcal{C}$, we have the *dual category*, $\mathcal{C}^{op}$, which has the same objects as $\mathcal{C}$ ($A^{op} = A$) but in which the arrows (f^{op}) are reversed. It follows that $(\mathcal{C}^{op})^{op} = \mathcal{C}$.

We denote by $\hom_{\mathcal{C}}(A, B)$ the set of arrows from A to B in $\mathcal{C}$. Such a set is called a hom set.[10]

We shall often express categorical properties in diagrammatic notation. For example, to require that the diagram

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "g_1"'] & B \arrow[d, "g_2"] \\ C \arrow[r, "f_2"'] & D \end{tikzcd}$$

commutes is to require that $g_2 \circ f_1 = f_2 \circ g_1$.

Examples of categories which are of relevance to this monograph include:

- **Pos**: Objects are partially ordered sets and arrows are order-preserving maps;
- **Lat**: Objects are lattices and arrows are lattice homomorphisms;
- **Set**: Objects are sets and arrows are functions.

Definition 1.4 Let A and B be objects of the category $\mathcal{C}$.

1. An arrow $f : A \to B$ is an epimorphism iff $g \circ f = h \circ f$ implies $g = h$.
2. An arrow $f : A \to B$ is a monomorphism iff $f \circ g = f \circ h$ implies $g = h$.
3. An arrow $f : A \to B$ is an isomorphism iff there is some $g : B \to A$ such that $f \circ g = \mathsf{Id}_A = g \circ f$. The f and g are said to be inverses in the evident way.

[10] We suppress any discussion of issues about the *size* of categories. See [74] for a discussion.

Two objects, A and B, of a category $\mathcal{C}$ are said to be isomorphic (written as $A \cong B$) iff there is an isomorphism $f \in \hom_C(A, B)$.

Homomorphisms between categories are called *functors*.

Definition 1.5 Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A covariant functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a pair of maps $F_{Obj} : Obj(\mathcal{C}) \rightarrow Obj(\mathcal{D})$ and $F_{Arr} : Arr(\mathcal{C}) \rightarrow Arr(\mathcal{D})$ such that, for every $f : A \rightarrow B$ and $g : B \rightarrow C$ in $\mathcal{C}$,

- $F_{Arr}(f) : F_{Obj}(A) \rightarrow F_{Obj}(B)$,
- $F_{Arr}(g \circ f) = F_{Arr}(g) \circ F_{Arr}(f)$, and
- $F_{Arr}(\mathsf{Id}_A) = \mathsf{Id}_{F_{Arr}(A)}$.

Similarly, a contravariant functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a pair of maps $F_{Obj} : Obj(\mathcal{C}) \rightarrow Obj(\mathcal{D})$ and $F_{Arr} : Arr(\mathcal{C}) \rightarrow Arr(\mathcal{D})$ such that, for every $f : A \rightarrow B$ and $g : B \rightarrow C$ in $\mathcal{C}$,

- $F_{Arr}(f) : F_{Obj}(B) \rightarrow F_{Obj}(A)$,
- $F_{Arr}(g \circ f) = F_{Arr}(f) \circ F_{Arr}(g)$, and
- $F_{Arr}(\mathsf{Id}_A) = \mathsf{Id}_{F_{Arr}(A)}$.

The subscripts Obj and Arr are usually suppressed.

For example, the covariant powerset functor, $\wp : \mathbf{Set} \rightarrow \mathbf{Set}$, takes each set A to its powerset $\wp(A)$ and each function $f : A \rightarrow B$ to the function $\wp(f) : \wp(A) \rightarrow \wp(B)$ given by, for all $A' \subseteq A$,

$$\wp(f)(A') = \{\, y \in B \mid \text{there is } x \in A' \text{ s.t. } y = f(x) \,\}.$$

A contravariant powerset functor may also be defined.

For another example, the duality functor $(-)^{op} : \mathcal{C} \rightarrow \mathcal{C}^{op}$, such that $(A)^{op} = A$ and $(f)^{op} = f^{op}$, is contravariant.

Hom sets give rise to hom functors:

$$\hom_{\mathcal{C}}(-, B) : \mathcal{C} \rightarrow \mathcal{C} \text{ and } \hom_{\mathcal{C}}(A, -) : \mathcal{C} \rightarrow \mathcal{C},$$

in the evident notation.

Definition 1.6 Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be functors with the same domain and co-domain. A natural transformation $\tau : F \rightarrow G$ is given by a family of maps

$\tau_A : F(A) \to G(A)$, for each $A \in Obj(\mathcal{C})$, such that

$$\begin{tikzcd} A \arrow[d, "f"'] & F(A) \arrow[r, "\tau_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ B & F(B) \arrow[r, "\tau_B"'] & G(B) \end{tikzcd}$$

commutes. The set of natural transformations from F to G is denoted $\mathsf{Nat}(F, G)$.

A natural transformation $\tau : F \to G$ is a *natural isomorphism*, written $\tau : F \cong \mathcal{G}$, if every *component*, τ_A, of τ has an inverse in $\mathcal{D}$.

Further examples of categories which are of relevance to this monograph include:[11]

- **Cat**: Objects are categories and arrows are functors;
- The *functor category* $[\mathcal{C}, \mathcal{D}]$: objects are functors $F : \mathcal{C} \to \mathcal{D}$ and arrows are natural transformations $\tau : F \to G$.

Categories may carry additional structure. Let $\mathcal{C}$ be a category.

1. An object 0 of $\mathcal{C}$ is *initial* iff, for all $A \in Obj(\mathcal{C})$, there is a unique $f \in \hom_{\mathcal{C}}(0, A)$. For example, the empty set is the initial object in **Set**. Initial objects are unique up to isomorphism.
2. An object 1 of $\mathcal{C}$ is *terminal* iff, for all $A \in Obj(\mathcal{C})$, there is a unique $f \in \hom_{\mathcal{C}}(A, 1)$. For example, the one element set $\{*\}$ is terminal in **Set**. Terminal objects are unique up to isomorphism, and are dual to initial objects: $0^{op} = 1$ and $1^{op} = 0$.
3. The *categorical product* of A and B is an object $A \times B$ together with two arrows $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$, and, for every object C, and all $f : A \to B$, $g : B \to C$, there is a unique $\langle f, g \rangle : C \to A \times B$ such that the diagram

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, "\langle f, g \rangle"'] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

[11] Again, we suppress any discussion of issues about the *size* of these categories. See [74] for a discussion.

commutes, that is, $\pi_1 \circ \langle f, g \rangle = f$, $\pi_2 \circ \langle f, g \rangle = g$, and, for all $h : C \to A \times B$, $\langle \pi_1 \circ h, \pi_2 \circ h \rangle$. In **Set**, the product is given by

$$A \times B = \{ \langle x, y \rangle \mid x \in A \text{ and } y \in B \}.$$

Categories with all products are said to be *Cartesian*. Terminal objects are nullary products. A functor is *Cartesian* if it preserves products.[12]

4. *Co-products*, denoted $A + B$ in the evident notation, are dual to products. Initial objects are nullary co-products. Categories with both products and co-products are said to be *bi-Cartesian*.

For example, functor categories $[\mathcal{C}, \mathbf{Set}]$ have terminal objects and products.

Just as the categorical product generalizes the set-theoretic product, so there is a generalization of set-theoretic function spaces.

Definition 1.7 Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. Then F is a left adjoint to G (written $F \dashv G$) and G is a right adjoint to F (written $G \vdash F$) if there is a natural transformation $\eta : \mathsf{Id} \to G \circ F$ between the identity functor on $\mathcal{C}$ and the functor $G \circ F : \mathcal{C} \to \mathcal{C}$ such that, for every object $B \in Obj(\mathcal{D})$ and every arrow $f : A \to G(B)$, there is a unique arrow $\hat{f}$ such that the diagram

$$\begin{tikzcd} A \arrow[d, "\eta_A"'] \arrow[dr, "f"] & \\ G(F(A)) \arrow[r, "G(\hat{f})"'] & G(B) \end{tikzcd}$$

$$F(A) \xrightarrow[\hat{f}]{} B$$

commutes.

The natural transformation η is called the *unit* of the adjunction (and we have evident notion of *co-unit*, $\epsilon : F \circ G \to \mathsf{Id}$).

Examples of adjunctions are found in *closed categories*. For us, the construction in Cartesian categories will suffice.

[12] Generally, we say a functor is X if it preserves Xs. Often this notion is taken up to isomorphism, necessitating coherence conditions [74].

Definition 1.8 Let A and B be objects of a Cartesian category, $\mathcal{C}$. The *exponent* of A and B is an object B^A of $\mathcal{C}$, together with a map $eval_{A,B} : B^A \times A \to B$, such that there is a unique $\Lambda(f) : C \to B^A$ such that the diagram

$$\begin{tikzcd} B^A \times A \arrow[r, "eval_{A,B}"] & B \\ C \times A \arrow[u, "\Lambda(f) \times \mathsf{Id}_A"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes.

Exponents provide and example of an adjunction. Let $\mathcal{C}$ be Cartesian. Then the functor $(-)^A : \mathcal{C} \to \mathcal{C}$ is a right adjoint to the functor $(-) \times A : \mathcal{C} \to \mathcal{C}$. So there is an isomorphism,

$$\hom_{\mathcal{C}}(A \times B, C) \xrightarrow{\sim} \hom_{\mathcal{C}}(A, C^B),$$

which is natural in A, B, and C.[13] In Chapter 3, we show that functor categories $[\mathcal{C}, \mathbf{Set}]$ have exponents.

Categories with products and exponents are said to be *Cartesian closed*, categories with products, exponents, and co-products are said to be *bi-Cartesian closed*.

Functors and natural transformations, which compose in the evident way, may be combined to describe structure systematically [70, 74].

Definition 1.9 Let $\mathcal{C}$ be a category. A monad on $\mathcal{C}$ is a functor $T : \mathcal{C} \to \mathcal{C}$ together with natural transformations $\eta : 1_{\mathcal{C}} \to T$ and $\mu : T^2 \to T$ such that the diagrams

$$\begin{tikzcd} T \arrow[r, "T\eta"] \arrow[d, "\eta T"'] \arrow[dr, "1_T"] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

commute.

[13]That is, for each of A, B, and C, there is a natural isomorphism between each evident pair of hom functors, such as $\hom_{\mathcal{C}}(- \times B, C)$ and $\hom_{\mathcal{C}}(-, C^B)$.

If $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to $G : \mathcal{D} \to \mathcal{C}$, with unit and co-unit η and ϵ, respectively, then

$$(G \circ F, \eta, G \,\epsilon\, F)$$

is monad on $\mathcal{C}$ [70].

Definition 1.10 Let (T, η, μ) be a monad on $\mathcal{C}$. Then the Kleisli category $\mathcal{C}_T$ is defined as follows:

- $Obj(\mathcal{C}_T) = Obj(\mathcal{C})$;
- The arrows $A \to B$ in $\mathcal{C}_T$ are arrows $A \to T(B)$ in $\mathcal{C}$. The composition in $\mathcal{C}_T$ of two arrows $f : A \to T(B)$ and $g : B \to T(C)$, $g * f : A \to T(C)$, is given by

$$g * f : \mu(C)(T(g)) \circ f\ .$$

The identity is given by $\eta(A) : A \to T(A)$.

We shall also use the dualized version of a Kleisli-category, the so-called *co-Kleisli* category. Co-Kleisli categories are based on the notion of co-monad, which is again the dualization of the notion of a monad. The details are as follows:

Definition 1.11 Let $\mathcal{C}$ be a category. A co-monad on $\mathcal{C}$ is a functor $U : \mathcal{C} \to \mathcal{C}$ together with natural transformations $\epsilon : U \to 1_{\mathcal{C}}$ and $\delta : T \to T^2$ such that the diagrams

$$\begin{tikzcd} T \arrow[r, "\delta"] \arrow[d, "\delta"'] \arrow[dr, "1_T"] & T^2 \arrow[d, "\epsilon T"] \\ T^2 \arrow[r, "T\epsilon"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\delta"] \arrow[d, "\delta"'] & T^2 \arrow[d, "T\delta"] \\ T^2 \arrow[r, "\delta T"'] & T^3 \end{tikzcd}$$

commute.

If $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to $G : \mathcal{D} \to \mathcal{C}$, with unit and co-unit η and ϵ, respectively, then

$$(F \circ G, \epsilon, F\eta G)$$

is co-monad on $\mathcal{C}$ [70].

Definition 1.12 Let (U, ϵ, δ) be a co-monad on $\mathcal{C}$. Then the co-Kleisli category $\mathcal{C}_U$ is defined as follows:

- $Obj(\mathcal{C}_T) = Obj(\mathcal{C})$;
- The arrows $A \to B$ in $\mathcal{C}_T$ are arrows $U(A) \to B$ in $\mathcal{C}$. The composition in $\mathcal{C}_T$ of two arrows $f : U(A) \to B$ and $g : U(B) \to C$, $g * f : U(A) \to C$, is given by
$$g * f = g \circ U(f) \circ \delta_A \,.$$
The identity is given by $\epsilon(A) : UA \to A$.

1.4 Outline of the monograph

In Chapter 2, we first introduce the natural deduction proof theory of intuitionisitic logic, giving both graphical and sequential systems, and giving a brief account of the simply-typed λ-calculus, with products and sums, which provides a language of realizers for intuitionistic consequences. Then we move on to classical logic, discussing how a graphical system and a sequential system, based on the idea of free deduction, may be obtained. Finally, we give a language, the $\lambda\mu\nu$-calculus, of realizers for sequential classical natural deduction and develop its proof-theoretic properties.

In Chapter 3, we provide semantics for the systems introduced in Chapter 2. Specifically, we set up categorical models of both the simply-typed λ-calculus, by way of background, and the $\lambda\mu\nu$-calculus, using our recent research, providing a range of metatheory and examples which is appropriate for our subsequent development. In particular, we begin our running example, that of *games models*, by providing games models of both intuitionistic and, as represented by the $\lambda\mu\nu$-calculus, classical proofs.

In Chapter 4, we provide our main explanation of reductive proof. Specifically:

1. We introduce Gentzen's sequent calculus, explaining its relationship with natural deduction and its significance as a basis for reductive proof;
2. We provide a summary of the conceptual and technical background to our proof-theoretic perspective;
3. We provide a representation of sequent calculus proofs in the $\lambda\mu\nu$-calculus with explicit substitutions, $\lambda\mu\nu\epsilon$;
4. We then provide a systematic analysis of the analytic view of resolution provided by the notion of *uniform proof* by Miller et al. [80].
5. In particular, we use our characterization intuitionistic proof within the $\lambda\mu\nu$-calculus to provide a rational reconstruction of Mints' intuitionistic resolution [82].

6. We conclude with a brief discussion of the computational complexity of our representation methods, and some remarks upon the rôle of the control régimes to be considered in Chapter 6.

In Chapter 5, we define a class of (categorical) models for reductions. Specifically:

1. We consider appropriate structures for modelling the reductions which properly generalize the structures used to provide models of proofs. These include polynomial constuctions over categories of proofs and fibrations;
2. We define models for both intuitionistic and classical reduction;
3. We ensure that the chosen structures are able to interpret sequent calculi without suppressing the structure of left rules;
4. We establish that our models are non-trivial, with significant examples.

We prove an appropriate soundness theorem for reductions and, via a term model construction, obtain completeness.

Finally, in Chapter 6, we bring together all of the ideas of this monograph by considering the addition of algorithmic control régimes to reductive proof, thereby obtaining *proof-search*. We consider proof-search in intuitionistic logic by embedding intuitionistic reduction inside classical reduction. Models of classical reduction require extra structure which we use to interpret the control regimes required for proof-search in intuitionistic logic. Thus, for intuitionistic logic, we provide a semantics which properly captures the slogan,

$$\textit{Proof-search} \;=\; \textit{Reduction} \;+\; \textit{Control.}$$

Following our running example, we conclude with a detailed description of a games model of proof-search in intuitionistic logic.

1.5 Discussion

This work is a research monograph. Nevertheless, its presentation requires the inclusion of a good deal of quite familiar, established material, as well as some minor developments of established ideas. We therefore provide a brief summary, with locations, of the research contributions made herein.

1. In this chapter, we have introduced our perspective on reductive logic and proof-search, starting from a motiviation and explanation of the basic concepts. We have also provided a discussion of the mathematical prerequisites for readers of this monograph.
2. In Chapter 2, we provide an explanation of natural deduction proof systems and their corresponding λ-calculi. Beginning with the necessary basics of natural deduction systems for intuitionistic logic and the simply-typed λ-calculus, we then present the $\lambda\mu$-calculus, giving both the basic definitions and essential metatheory, before proceeding with a discussion of the (delicate matter of the) addition of disjunction to $\lambda\mu$, based on recent papers of the authors [97, 108, 111].

3. We begin Chapter 3 with an account of the semantics of intuitionisitic natural deduction proof, based on models of the simply-typed λ-calculus. Here we introduce a new form of games semantics which forms the basis for a running example throughout the monograph. Our games combine features of the games of Lorenzen [28, 72], used to model intuitionistic provability, and of the games used by Hyland and Ong [64] to interpret fragments of linear logic, and Ong [89] to interpret terms of the $\lambda\mu$-calculus. We then present the semantics of classical natural deduction proofs via recently-developed models of the $\lambda\mu$-calculus [89, 97], and its disjunctive extenstions [97], based on fibrations of models of simply-typed calculi. We consider also continutations in this context, adumbrating some of our concerns in later chapters.
4. In Chapter 4, we provide a systematic account of reductive proof theory. Beginning with a somewhat historical account of (automated and interactive) theorem proving, we provide a systematic account of sequent calculi, including a summary of their essential metatheory and their representation in the classical λ-calculus [111]. We then provide a systematic account of reductive proof theory, based on the sequent calculus and classical λ-calculi, including a rational reconstruction of Mints' intuitionitic resolution [82].
5. In Chapter 5, we provide a systematic model-theoretic account of reductive logic. Here the challenge is to provide semantic structures that are rich enough to account not only for the space of proofs but also for the (larger) space of reductions—all proofs may be seen as successful reductions whereas many reductions fail to determine proofs. Our techniques are those of categorical model theory and categorical proof theory, and we make essential use of the interplay between the semantics of proofs given by algebraic realizers and the meaning of propositions given by Kripke's account of truth-functional semantics.
6. Finally, in Chapter 6, we provide a semantics for proof-search in reductive logic which properly incorporates the semantics of the principal control mechanism for proof-search, namely backtracking, within the model theory of the logic. Here, our focus is on proof-search in intuitionistic reductive logic, and we exploit an embedding of intuitionistic proofs within classical proofs as a framework within which control structures may be represented.

Turning to further developments of the work we have presented in this monograph, we can readily identify a few possibilities, in order of likely increasing conceptual significance:

1. A complete development of the theory presented here in the presence of predicates with first- and higher-order quantifiers;[14]
2. A complete treatment of the other aspects of the control of proof-search, as discussed briefly in the conclusion to Chapter 6;

[14] We provide brief sketches of how to handle the first-order case throughout our present development.

3. A re-development of our existing analysis for a range of substructural logics. Of particular interest would be the following:
   (i) Various fragments of linear logic [42];
   (ii) The bunched logic, **BI** [8, 35, 88, 101, 102, 105], and its applied variants, including the Reynolds' Separation Logic [109] and Ishtiaq and O'Hearn's Pointer Logic [58]. A semantic analysis of Harland and Pym's 'Resource-distribution via Boolean constraints', which systematically and tractably characterizes the relationship between the different branches of reductions in substructural systems with multiplicative rules [45], would be a worthwhile challenge;
4. A reconstruction of our analysis using the categorical semantics of classical proofs provided by Führmann and Pym [30];
5. The development of a clear understanding of the relationship between our semantics of reductive logic and proof-search, and Girard's 'Ludics' [25, 41]. Informally, Girard claims that Ludics is closely related to proof-search.[15] More formally, Hyland and Faggian [27, 56] have given an explicit explanation of Ludics in terms of games semantics. A first step would be to relate our games models of reductive logic and proof-search to their account.

It would seem that the last two points are in fact conceptually rather closely related.

1.6 Errata and remarks

A file of 'Errata and Remarks', giving corrections to any known errors and providing clarifications and other remarks for this monograph, will be maintained at the following:

`http://www.cs.bath.ac.uk/~pym/reductive-logic-errata.html`
`http://www.cs.bham.ac.uk/~exr/reductive-logic-errata.html`

or via `http://www.oup.co.uk/isbn/0-19-852633-4`

[15] It is unclear whether Girard means reductive logic or proof-search, or whether he intends no distinction.

2

LAMBDA-CALCULI FOR INTUITIONISTIC AND CLASSICAL PROOFS

2.1 Introduction

We have explained, in Chapter 1, our overall purpose in this monograph: To provide a proof theory and semantics for reductive proof of value comparable to the well-established semantics for deductive proof. In this chapter, we provide our view of the deductive proof theory of intuitionistic and classical proof theory, concentrating on the relationship between natural deduction proof and the λ-calculus.

In Section 2.2, we introduce the usual natural deduction (ND) systems for intuitionistic logic. We start with the more familiar, graphical treatment and explain its sequentialization. Then, in Section 2.3, we introduce the simply-typed λ-calculus, with product and sum types, as representation of the proofs of propositional intuitionistic logic. We provide a summary of its basic metatheory. In Section 2.4, we recall how the graphical presentation of intuitionistic natural deduction may be strengthened to classical logic by adding appropriate rules for classical negation. We then proceed to discuss the issues that arise in providing sequentialized version of classical natural deduction. This discussion leads us naturally, in Section 2.5, to a presentation of a family of classical λ-calculi, starting from Parigot's $\lambda\mu$-calculus and adding disjunction, in two alternative forms, and explicit substitutions. We provide a detailed account of the metatheory of these calculi.

2.2 Intuitionistic natural deduction

Natural deduction systems were introduced in Gerhard Gentzen's paper from 1934, 'Untersuchungen über das logische Schliessen' ('Investigations into logical deduction') [37]. Natural deduction systems for classical logic and intuitionistic logic are described by pairs of rules which manipulate proofs by either *introducing* a connective into a proof or *eliminating* it from a proof [95]. Proofs are constructed by starting with assumptions and deriving conclusions. As such, that process may be represented as a tree.

A good example is provided by the formulation in natural deduction of *reasoning by cases*, which may be summarized as follows:

- Let ϕ_1, ϕ_2, and ψ be propositions;
- Suppose (i) that we have a proof of ψ assuming ϕ_1, and (ii) that we have a proof of ψ assuming ϕ_2;

- Suppose (iii) we have a proof that $\phi_1 \vee \phi_2$ holds;
- From (i), (ii), and (iii), we can construct a proof of ψ.

In a natural deduction presentation of classical logic, this argument is described by the rule of $\vee$-*elimination*, $\vee$E for short, in contrast to its corresponding $\vee$-*introduction* rules:

$$\frac{\phi_1 \vee \phi_2 \quad \begin{array}{c}[\phi_1]\\ \vdots \\ \psi\end{array} \quad \begin{array}{c}[\phi_2]\\ \vdots \\ \psi\end{array}}{\psi}\,\vee E \qquad \frac{\phi_1}{\phi_1 \vee \phi_2}\,\vee I_1 \qquad \frac{\phi_2}{\phi_1 \vee \phi_2}\,\vee I_2.$$

Notice that we have *discharged* our assumptions ϕ_1 and ϕ_2: given that we have a proof of $\phi_1 \vee \phi_2$, we need not retain the assumptions in order to get a proof of the conclusion.

We can also give similar rules for conjunction:

$$\frac{\phi_1 \qquad \phi_2}{\phi_1 \wedge \phi_2}\,\wedge I \qquad \frac{\phi_1 \wedge \phi_2 \quad \begin{array}{c}[\phi_1, \phi_2]\\ \vdots \\ \psi\end{array}}{\psi}\,\wedge E.$$

Note, however, that the $\wedge E$ may be written in the less systematic but rather simpler 'projective' form

$$\frac{\phi_1 \wedge \phi_2}{\phi_1} \qquad \frac{\phi_1 \wedge \phi_2}{\phi_2}.$$

The rules for implication provide another example:

$$\frac{\phi \;\; \phi \supset \psi}{\psi} \supset \mathrm{E} \qquad \frac{\begin{array}{c}[\phi]\\ \vdots \\ \psi\end{array}}{\phi \supset \psi} \supset I.$$

So suppose that we have proofs of ψ from either ϕ_1 or ϕ_2 and that we have a proof of χ assuming ψ. Then the following is an example of a proof of χ assuming $\phi_1 \vee \phi_2$:

$$\frac{\dfrac{\phi_1 \vee \phi_2 \quad \begin{array}{c}[\phi_1]\\ \vdots \\ \psi\end{array} \quad \begin{array}{c}[\phi_2]\\ \vdots \\ \psi\end{array}}{\psi}\,\vee E \qquad \dfrac{\begin{array}{c}[\psi]\\ \vdots \\ \chi\end{array}}{\psi \supset \chi} \supset I}{\chi} \supset E. \tag{2.1}$$

So we can see that the assumptions made in a proof are represented as the *undischarged* leaves of the tree—in this case, just $\phi_1 \vee \phi_2$.

The pairing of rules for introducing and eliminating connectives is the key characteristic of natural deduction. The natural deduction system for intuitionistic logic that we have sketched here was introduced by Gentzen [37] and is called NJ. The modus ponens (MP) rule of a Hilbert-type system corresponds to $\supset$-elimination.

The key property which a natural deduction system may have is *normalization*. Normalization comes in several parts.

1. Firstly, in any natural deduction proof, all occurrences of an introduction rule immediately[16] followed by the elimination rule for the same occurrence of the same connective may be eliminated from the proof, so as to yield a proof in NJ of the same conclusion from the same assumptions. For example,

$$\dfrac{\phi \qquad \dfrac{\begin{matrix}[\phi] \\ \vdots \\ \psi\end{matrix}}{\phi \supset \psi} \supset I}{\psi} \supset E, \tag{2.2}$$

   is a proof of ψ assuming ϕ. However, such a proof is what we started out with in the right-hand branch of the proof tree. The $\supset I$ rule is immediately followed by the $\supset E$ rule, which gets us back to a proof of ψ assuming ϕ. The introduction followed by the corresponding elimination is a 'pointless detour': we could have just used our original proof. Eliminating all such pointless detours leads us to the *β-normal form* of a proof.

2. Secondly, we may have elimination rules immediately preceding introduction rules. For example, if Φ is a proof in NJ of $\phi \wedge \psi$, then the proof

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \phi \wedge \psi\end{matrix}}{\phi} \wedge E \qquad \dfrac{\begin{matrix}\vdots \\ \phi \wedge \psi\end{matrix}}{\psi} \wedge E}{\phi \wedge \psi} \wedge I,$$

   which first eliminates $\wedge$ from two copies of $\phi \wedge \psi$ only to eliminate it again, can be considered to reduce to Φ. Performing all such reductions leads us to the *η-normal form* of a proof.

3. Finally, the form of the elimination rule for $\vee$, known as Prawitz's generalized form [95], introduces the possibility of a third form of reduction.

[16] Up to some permutabilities of rules [65].

For example, consider the proof figure

$$\dfrac{\dfrac{\begin{matrix} & [\phi_1] & [\phi_2] \\ \vdots & \vdots & \vdots \\ \phi_1 \vee \phi_2 & \psi & \psi \end{matrix}}{\psi} \vee E}{\chi} R,$$

where R is any applicable rule. This figure can be reorganized by permuting R above $\vee E$ to give

$$\dfrac{\begin{matrix} & [\phi_1] & [\phi_2] \\ & \vdots & \vdots \\ \vdots & \dfrac{\psi}{\chi} R & \dfrac{\psi}{\chi} R \\ \phi_1 \vee \phi_2 & & \end{matrix}}{\chi} \vee E.$$

Performing all such reductions, which are also known as *commuting conversions*, leads us to the ζ*-normal form* of a proof.

The third class of reductions considered above requires a little more consideration. For, whereas the β- and η- reductions may readily be seen to reduce the conceptual complexity of proofs,[17] by removing pointless steps, the same is not true of ζ-reductions: They are driven by the syntax of propositions (proofs) and so seem somewhat arbitrary.

We call two natural deduction proofs Φ and Ψ $\beta\eta\zeta$-equal if they have the same $\beta\eta\zeta$-normal form. We refer to this equality also as the *extensional equality* between proofs.

2.2.1 *Sequential natural deduction*

The consequences established using natural deduction rules may be represented as *sequents* (from the German 'Sequenzen') and natural deduction rules may be represented in *sequential* or *linearized* form, in which the assumptions made globally, at the leaves of the proof tree, are represented *locally* within the rules. For example, the $\vee$-elimination and $\vee$-introduction rules go as follows:

$$\dfrac{\Gamma \vdash \phi_1 \vee \phi_2 \quad \Gamma, \phi_1 \vdash \psi \quad \Gamma, \phi_2 \vdash \psi}{\Gamma \vdash \psi} \vee E$$

and

$$\dfrac{\Gamma \vdash \phi_1}{\Gamma \vdash \phi_1 \vee \phi_2} \vee I_1 \qquad \dfrac{\Gamma \vdash \phi_2}{\Gamma \vdash \phi_1 \vee \phi_2} \vee I_2.$$

[17] It is quite possible—fortunately—to devise measures which are diminished by ζ-reductions.

For example, the proof in (2.1) is represented as

$$\dfrac{\dfrac{\Gamma \vdash \phi_1 \vee \phi_2 \quad \Gamma, \phi_1 \vdash \psi \quad \Gamma, \phi_2 \vdash \psi}{\Gamma \vdash \psi} \vee E \qquad \dfrac{\Gamma, \psi \vdash \chi}{\Gamma \vdash \psi \supset \chi} \supset I}{\Gamma \vdash \chi} \supset E. \tag{2.3}$$

Two important things may be seen from this example.

1. Firstly, that discharge corresponds to removing formulæ from the left-hand side of $\vdash$. Note that we can see two versions of this, one in which the discharged formula simply moves to form part of the right-hand side ($\supset I$) and one in which the discharged formulæ are witnessed by a formula on the right-hand side $\vee E$.
2. Secondly, that the rôle of the Γ is somewhat arbitrary. In particular, we could replace Γ with Γ, Γ', that is, do a Weakening operation,

$$\frac{\Gamma \vdash \phi}{\Gamma, \Gamma' \vdash \phi} \quad \text{Weakening},$$

and still have a perfectly good proof of ϕ, with more (unused) assumptions. A related structural is that of Contraction,

$$\frac{\Gamma, \Delta, \Delta, \Gamma' \vdash \phi}{\Gamma, \Delta, \Gamma' \vdash \phi} \quad \text{Contraction},$$

in which duplications of assumptions are removed.

2.2.2 *Natural deduction for intuitionistic predicate logic*

Natural deduction rules may also be written for intuitionistic predicate logic with the existential and universal quantifiers. The propositional rules are unchanged. Each quantifier comes along with an introduction rule and an elimination rule:

$$\frac{\phi(x)}{\forall x.\phi(x)} \; \forall I \qquad \frac{\forall x.\phi(x)}{\phi(t)} \; \forall E,$$

$$\frac{\phi(t)}{\exists x.\phi(x)} \; \exists I \qquad \frac{\exists x.\phi(x) \quad \begin{matrix}[\phi(x)] \\ \psi\end{matrix}}{\psi} \; \exists E,$$

where, in $\forall I$ and $\exists E$, x does not occur freely in any assumption upon which the conclusion depends.

The generalization, G, rule of a Hilbert-type system corresponds to $\forall$-introduction.

The analysis of reduction rules, equations, and normal forms, as well as sequentialization, for propositional natural deduction extends straightforwardly to the quantifiers [42].

TABLE 2.1. Sequential intuitionistic propositional natural deduction: SNJ

Identity and structure

$$\frac{}{\phi \vdash \phi}\ Ax \qquad \frac{\Gamma, \phi, \psi, \Gamma' \vdash \chi}{\Gamma, \psi, \phi, \Gamma' \vdash \chi}\ \text{Exchange}$$

$$\frac{\Gamma \vdash \phi}{\Gamma, \Gamma' \vdash \phi}\ W \qquad \frac{\Gamma, \Delta, \Delta, \Gamma' \vdash \phi}{\Gamma, \Delta, \Gamma' \vdash \phi}\ C$$

Operational rules

$$\frac{}{\langle\rangle \vdash \top}\ \top I$$

$$\frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}\ \wedge I \qquad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi}\ \wedge E \quad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi}\ \wedge E$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \supset \psi}\ \supset I \qquad \frac{\Gamma \vdash \phi \supset \psi \quad \Gamma \vdash \phi}{\Gamma \vdash \psi}\ \supset E$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}\ \bot E$$

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi}\ \vee I \quad \frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi}\ \vee I \qquad \frac{\Gamma \vdash \phi \vee \psi \quad \Gamma, \phi \vdash \chi \quad \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi}\ \vee E$$

2.3 The simply-typed λ-calculus

The proof trees generated by sequential intuitionistic natural deduction proofs may be represented as terms of the simply-typed λ-calculus, in addition to the basic function types, also product and sum types may be added. Formally, the simply-typed λ-calculus with product and sum types stands in propositions-as-types correspondence [11, 54] with natural deduction proofs.

We call the system described below the *simply-typed λ-calculus with products and sums*. Rather than give an explicit presentation of the propositions-as-types correspondence, we take the types of the simply-typed λ-calculus to be given directly by the formulæ of intuitionistic propositional logic. The general set-up is, however, quite easy to see. If Φ is an intuitionistic natural deduction proof of ϕ from assumptions $\phi_1, \ldots, \phi_m$,

$$\Phi : \phi_1, \ldots, \phi_m \vdash \phi,$$

then there is a λ-term t_Φ such that

$$x_1 : \phi_1, \ldots, x_m : \phi_m \vdash t_\Phi : \phi,$$

where each x_i is variable of the λ-calculus, is provable in the natural deduction calculus for the λ-calculus. The x_is are understood to stand for proofs of the

ϕ_is and t_Φ is a realizer of ϕ relative to these assumptions. Moreover, this correspondence is bijective in the evident sense.

The raw types, raw terms, and raw contexts of the simply-typed λ-calculus with products and sums are then given by the following grammars:

1. Types
$$\phi ::= p \mid \top \mid \phi \wedge \phi \mid \bot \mid \phi \vee \phi \mid \phi \supset \phi,$$
where p ranges over atoms;
2. Terms
$$\begin{aligned} t ::= {} & x \mid \top \mid \langle t, u\rangle \mid \pi_1 t \mid \pi_2 t \\ & \mid \bot_\phi(t) \mid \mathtt{in}_1 t \mid \mathtt{in}_2 t \mid \mathtt{case}\, t\, \mathtt{of}\, \mathtt{in}_1\,(x) \Rightarrow t\, \mathtt{or}\, \mathtt{in}_2\,(y) \Rightarrow t \\ & \mid \lambda x : \phi . t \mid t\, t, \end{aligned}$$
where x ranges over variables, the first line gives the evident terms for products, the second line the terms for sums, and the third line the terms for function types;
3. Contexts
$$\begin{array}{rlr} \Gamma ::= & x : \phi & \text{variables} \\ \mid & \Gamma, x : \phi & \text{extension} \\ \mid & \langle\rangle & \text{unit (empty context).} \end{array}$$
We associate distinct variables with each proposition that occurs in a context.

   The natural deduction rules for the simply-typed λ-calculus with products and sums are given in Table 2.2, in which $\bot_\phi(t)$ is the canonical term of type ϕ constructed from any term t of type $\bot$. Substitution of the term u for the variable x in the term t is denoted by $t[u/x]$. A term t is said to be *well-typed*, with type ϕ, if $\Gamma \vdash t : \phi$ is provable in this system.

   We usually omit the notation $\langle\rangle$ for the empty context provided no confusion is possible;
4. Reduction
Next, we turn to reduction of λ-terms. As λ-terms correspond to SNJ-proofs there are also three classes of normalizations, called *reductions*, corresponding to the three classes of normalization of SNJ-proofs. We will write $t \rightsquigarrow s$ if t reduces to s in one or more steps, and $t \rightsquigarrow^* s$ if t reduces to s in 0 or more steps. The β- and η-reductions are listed in Table 2.3.

To state the ζ-reductions, we adopt the notion of a *term context*, or a term with holes. Such a term $\mathcal{C}$ with holes of type ϕ is a λ-term which may have also the additional term constructor $\_$ with the rule $\Gamma \vdash \_ : \phi, \Delta$. The term $\mathcal{C}(u)$ denotes the term $\mathcal{C}$ with the holes textually (with possible variable capture) replaced by u. The ζ-reductions may now be stated as in Table 2.4.[18]

[18]Recall that the necessity of the ζ-reductions for, say, disjunction may be understood as a consequence of the failure of $\vee E$ to be suitably 'syntax-directed'.

TABLE 2.2. Natural deduction for the simply-typed λ-calculus with products and sums

Identity and structure

$$\frac{}{\Gamma, x : \phi, \Gamma' \vdash x : \phi} \ \text{Axiom}$$

Units

$$\frac{\Gamma \vdash t : \bot}{\Gamma \vdash \bot_{\phi}(t) : \phi} \ \bot E$$

$$\frac{}{\langle\rangle \vdash \top : \top} \ \top I$$

Operational rules

$$\frac{\Gamma, x : \phi \vdash t : \psi}{\Gamma \vdash \lambda x : \phi \,.\, t : \phi \supset \psi} \ \supset I \qquad \frac{\Gamma \vdash t : \phi \supset \psi \quad \Gamma \vdash u : \phi}{\Gamma \vdash tu : \psi} \ \supset E$$

$$\frac{\Gamma \vdash t : \phi \quad \Gamma \vdash u : \psi}{\Gamma \vdash \langle t, u\rangle : \phi \wedge \psi} \ \wedge I \qquad \frac{\Gamma \vdash t : \phi \wedge \psi}{\Gamma \vdash \pi_1 t : \phi} \quad \frac{\Gamma \vdash t : \phi \wedge \psi}{\Gamma \vdash \pi_2 t : \psi} \ \wedge E$$

$$\frac{\Gamma \vdash t : \phi_i}{\Gamma \vdash \mathtt{in}_i(t) : \phi_1 \vee \phi_2} \quad (i = 1, 2) \quad \vee I$$

$$\frac{\Gamma \vdash t : \phi \vee \psi \quad \Gamma, x : \phi \vdash u : \chi \quad \Gamma, y : \psi \vdash v : \chi}{\Gamma \vdash \mathtt{case}\, t \,\mathtt{of}\, \mathtt{in}_1\, (x) \Rightarrow u \,\mathtt{or}\, \mathtt{in}_2\, (y) \Rightarrow v : \chi} \ \vee E$$

TABLE 2.3. $\beta\eta$-reductions

β-reductions	η-reductions
$(\lambda x : \phi \,.\, t)u \rightsquigarrow t[u/x]$	$\lambda x : \phi . tx \rightsquigarrow t \quad (x \notin \mathrm{FV}(t))$
$\pi_1\langle t_1, t_2\rangle \rightsquigarrow t_1$	$\langle \pi_1 t, \pi_2 t\rangle \rightsquigarrow M$
$\pi_2\langle t_1, t_2\rangle \rightsquigarrow t_2$	
$(\mathtt{case}\ \mathtt{in}_1(t)\ \mathtt{of}\ \mathtt{in}_1\ (x) \Rightarrow u_1\ \mathtt{or}\ \mathtt{in}_2\ (y) \Rightarrow u_2)$ $\rightsquigarrow u_1[t/x]$	$t \rightsquigarrow \mathtt{case}\, t\, \mathtt{of}$ $\mathtt{in}_1(x) \Rightarrow \mathtt{in}_1(x)\ \mathtt{or}\ \mathtt{in}_2(y) \Rightarrow \mathtt{in}_2(y)$
$(\mathtt{case}\ \mathtt{in}_2(t)\ \mathtt{of}\ \mathtt{in}_1\ (x) \Rightarrow u_1\ \mathtt{or}\ \mathtt{in}_2\ (y) \Rightarrow u_2)$ $\rightsquigarrow u_2[t/y]$	

Equality of λ-terms is defined in terms of reductions. Firstly, we take the obvious α-reductions. Secondly, we write $t = t'$ as the smallest congruence relation containing $\rightsquigarrow$ and satisfying $\Gamma \vdash t = t' : \phi$ for $\Gamma \vdash t : \phi$. We shall be concerned throughout with $\beta\eta\zeta$-equality and $\beta\eta\zeta$-normal forms. We say that an λ-term is *well-typed* in Γ if there is a ϕ such that $\Gamma \vdash t : \phi$ is provable.

TABLE 2.4. ζ-reductions

ζ-reductions for $\vee$
$\mathcal{C}(\texttt{case}\,t\,\texttt{of}\ \texttt{in}_1\,(x) \Rightarrow u\ \texttt{or}\ \texttt{in}_2\,(y) \Rightarrow v) \rightsquigarrow$ $\texttt{case}\,t\,\texttt{of}\ \texttt{in}_1\,(x) \Rightarrow \mathcal{C}(u)\ \texttt{or}\ \texttt{in}_2\,(y) \Rightarrow \mathcal{C}(v)$

There are no term constructors in the simply-typed λ-calculus for the structural rules of Weakening, Exchange, and Contraction nor for Cut. All structural rules are admissible, and Cut is arises as substitution.

Lemma 2.1 (admissibility of structural rules) *The following rules are admissible in the simply-typed λ-calculus:*

Exchange:

$$\frac{\Gamma, x_1 : \phi_1, x_2 : \phi_2 \vdash t : \phi}{\Gamma, x_2 : \phi_2, x_1 : \phi_1 \vdash t : \phi}$$

Contraction:

$$\frac{\Gamma, x : \phi, y : \phi \vdash t : \psi}{\Gamma, x : \phi \vdash t[x/y] : \psi}$$

Weakening:

$$\frac{\Gamma \vdash t : \psi}{\Gamma, x : \phi \vdash t : \psi}$$

Cut:

$$\frac{\Gamma, x : \phi \vdash t : \psi \quad \Gamma \vdash u : \phi}{\Gamma \vdash t[u/x] : \psi}.$$

Moreover, the context Γ in the judgement $\Gamma \vdash t : \phi$ can be restricted to the variables occurring freely in t. This property is called *strengthening*:

Lemma 2.2 (strengthening) *If $\Gamma, x : \phi \vdash t : \psi$ is provable and if $x \notin \mathrm{FV}(t)$, then $\Gamma \vdash t : \psi$ is provable.*

Another important theorem states that reduction preserves typing. This should hold, as reduction only simplifies proofs but does not construct new proofs.

Theorem 2.3 (subject reduction) *If $\Gamma \vdash t : \phi$ is provable and $t \rightsquigarrow t'$, then $\Gamma \vdash t' : \phi$ is provable.*

Proof By induction on the structure of the derivation of the reduction $t \rightsquigarrow t'$, using Lemma 2.1.

For example, the base case for β-reduction (e.g., for $\supset$) has $\supset E$,

$$\frac{\Gamma \vdash \lambda x : \phi.t : \phi \supset \psi \quad \Gamma \vdash u : \phi}{\Gamma \vdash (\lambda x : \phi.t)u : \psi}$$

as the last inference in the derivation (possibly followed by instances of structural rules). Hence we have $\Gamma, x : \phi \vdash t : \psi$ and so, by Lemma 2.1, $\Gamma \vdash t[u/x] : \psi$.

The other cases are similar. □

2.3.1 *Normalization and subject reduction*

We give a sketch of the proof of the strong normalization (SN) theorem for β-reduction for the simply-typed λ-calculus, with the usual functional ($\supset$) and conjunctive ($\wedge$) types. This basic result, which states that all reduction sequences $t \rightsquigarrow t_1 \rightsquigarrow t_2 \rightsquigarrow \cdots$ for every term t are finite, including the extension to disjunctive ($\vee$) types is well-documented, with [95, 96] and [42] being the most accessible.[19] The extension to η-reduction is more complex, though quite straightforward. Note, however, that in order to give a categorical semantics to the system with disjunctive types, which are then interpreted as co-products, care must be taken to handle the ζ-reductions, or commuting conversions, properly [38, 61]. All of these issues arise, and are handled in detail, in our treatment of the $\lambda\mu\nu$-calculus in subsequent sections and chapters. For now, by way of an introduction to these techniques, we content ourselves with sketch in a simple, restricted setting.

The basic idea of the proof of strong normalization is Tait's [122] notion of *reducibility*, taken together with Girard's [40] notion of *neutrality*, which facilitates a technical improvement of Tait's proof.

For the remainder of this section, we confine our attention to λ-terms without $\vee$ (or $\perp$). However, the methods discussed in [61, 95, 97, 111] can be used to extend strong normalization and subject reduction to $\vee$ (and $\perp$). We follow the excellent, and very concise, presentation in [42].

The basic idea of reducibility is to structure the reductions of terms according to the structure of their types. Without $\vee$, we define the set $Red(\phi)$, of *reducible terms of type* ϕ, relative to a given context, by induction over the structure of ϕ as follows:[20]

1. If t has atomic type, p, then $t \in Red(p)$ if it is strongly normalizing;
2. If t has type $\phi_1 \wedge \phi_2$, then $t \in Red(\phi_1 \wedge \phi_2)$ if both $\pi_1 t \in Red(\phi_1)$ and $\pi_2 t \in Red(\phi_2)$;
3. If t has type $\phi \supset \psi$, then $t \in Red(\phi \supset \psi)$ if, for every $u \in Red(\phi)$, $tu \in Red(\psi)$.

The basic idea of neutrality is to pick out those terms which are *not* immediately constructed by introduction rules. In the absence of $\vee$ (and $\perp$), the neutral terms are those of the form x, $\pi_1 t$, $\pi_2 t$, tu.

The key technical lemma is then that the sets $Red(\phi)$ satisfy the following conditions:

CR1 If $t \in Red(\phi)$, then t is SN;
CR2 If $t \in Red(\phi)$ and $t \rightsquigarrow t'$, then $t' \in Red(\phi)$;

[19] The details for $\supset$ and $\wedge$ are provided in [42]; definitions are provided for $\vee$ but for the extension of the proof of SN to $\vee$ one is referred to [96] (see also [126]), which is formulated in terms of natural deduction proof trees rather than a term calculus.

[20] Throughout this section, which is intended as a sketch, we neglect the unit $\top$ (similarly $\perp$, for disjunction). It is a degenerate case of $\wedge$.

CR3 If t is neutral and every redex in t reduces to a term $t' \in Red(\phi)$, then $t \in Red(\phi)$;

CR4 If t is both neutral and normal, then $t \in Red(\phi)$.

Note that **CR4** is a special case of **CR3**. This lemma is proved by induction on the structure of types and on a measure $\nu(t)$, which bounds the length of every reduction sequence beginning with t.

The proof of SN now proceeds, by an induction on the structure of terms, to show that all terms are reducible. The argument uses the following lemma (*q.v.* [42]):

Lemma 2.4 *Let t be any, not necessarily reducible, term with free variables among $x_1 : \phi_1, \ldots, x_m : \phi_m$. If $u_1, \ldots, u_m$ are reducible terms of type $\phi_1, \ldots, \phi_m$, then $t[u_1/x_1, \ldots, u_m/x_m]$ is reducible.*

Theorem 2.5 (strong normalization) *All well-typed λ-terms are strongly normalizing, that is, all reduction sequences terminate.*

2.3.2 *λ-calculi and intuitionistic predicate logic*

The correspondence between the λ-calculus and natural deduction proofs extends quite naturally to predicate quantifiers. Although Howard's paper [54] explains this correspondence in the simply-typed setting, with quantifiers handle pointwise, it is perhaps most naturally seen in the setting of *dependent types* [77, 100, 106].

The basic idea of dependent types is that types are not merely built up using 'propositional' type constructors, such as $\supset$ and $\wedge$, but rather may be dependent upon terms, just as first-order (or, indeed, higher-order) predicates may depend upon terms. Thus a dependently-typed sequent has the form

$$x_1 : A_1, x_2 : A_2(x_1), \ldots, x_m : A_m(x_1, \ldots, x_{m-1}) \vdash t(x_1, \ldots, x_m) : A(x_1, \ldots, x_m).$$

So, given a context $\Gamma = x_1 : A_1, x_2 : A_2(x_1), \ldots, x_m : A_m(x_1, \ldots, x_{m-1})$, we have a rule

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A.t : \Pi x : A.B},$$

which introduces the 'Π-type'. Here both t and B may depend upon x and is described as the 'dependent product' or 'dependent function space'. In the case in which x does not occur freely in B, the dependent type $\Pi x : A.B$ amounts to just the simple type $A \supset B$.

Restricting our attention, for simplicity, to just the $(\supset, \forall)$-fragment, we then get the following propositions-as-types correspondence between constructions:

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \supset \psi} \supset I \quad \text{and} \quad \frac{\Gamma, x : \phi \vdash t : \psi}{\Gamma \vdash \lambda x : \phi.t : \phi \supset \psi} \Pi I, \tag{2.4}$$

where x is not free in ψ, and

$$\frac{\Gamma \vdash \phi(x)}{\Gamma \vdash \forall x.\phi(x)} \ \forall I \quad \text{and} \quad \frac{\Gamma, x : A \vdash t : \psi}{\Gamma \vdash \lambda x : A.t : \Pi x : A.\psi} \ \Pi I \tag{2.5}$$

in general. Note that there is quite a lot going on here. In (2.4), the variable x which is bound by the λ-abstraction stands for a proof of the proposition ϕ. In (2.5), however, the variable x is a first-order variable in the occurring in the predicate ϕ (and must not occur free in other propositions in Γ). With respect to the propositions-as-types correspondence, dependently-typed contexts contain variables of both sorts.[21]

2.4 Classical natural deduction

The system NJ can be made classical by adding stronger rules for negation. Typically, one of the following three rules is added:

- The Law of the Excluded Middle Axiom:

$$\overline{\phi \vee \neg\phi}$$

- The Double-negation Rule:

$$\frac{\neg\neg\phi}{\phi}$$

- The Reductio Ad Absurdum (RAA) Rule:

$$\frac{\begin{matrix}[\neg\phi] \\ \vdots \\ \perp\end{matrix}}{\phi}.$$

We call the resulting system NK .

Classical logic may, in a certain sense, be translated into intuitionistic logic. The following translation, $-^{\circ}$, is called the *Gödel translation* or $\neg\neg$ *translation*:

$$\begin{aligned}
\perp^{\circ} &:= \perp \\
p^{\circ} &:= \neg\neg p \\
(\phi \wedge \psi)^{\circ} &:= \phi^{\circ} \wedge \psi^{\circ} \\
(\phi \vee \psi)^{\circ} &:= \neg(\neg(\phi)^{\circ} \wedge \neg(\psi)^{\circ}) \\
(\phi \supset \psi)^{\circ} &:= \phi^{\circ} \supset \psi^{\circ}.
\end{aligned}$$

[21]See [98, 100, 106] for an account of the proof theory of the $\lambda\Pi$-calculus.

Extending the translation $-^\circ$ to sets of formulæ via $\Gamma^\circ = \{ \phi^\circ \mid \phi \in \Gamma \}$, we then have, with some small abuse of notation, the following:

Theorem 2.6 (Gödel translation) *$\Gamma \vdash \phi$ is provable in NK if and only if $\Gamma^\circ \vdash \phi^\circ$ is provable in NJ.*

The Gödel translation and the associated theorem extend to the quantifiers without difficulty:

$$(\forall x.\phi)^\circ := \forall x.\phi^\circ$$

$$(\exists x.\phi)^\circ := \neg\forall x.\neg(\phi)^\circ.$$

Note that the intuitionistic interpretation of the classical $\vee$ (and $\exists$) by the Gödel translation is quite weak. The asymmetry introduced by the translation corresponds to $\lambda\mu\nu$'s call-by-value reduction strategy and, indeed, to the semantics of classical Cut-reduction provided by continuations. We return to this point briefly in Chapter 3, where we discuss 'Lafont's example'.

Just as for intuitionistic natural deduction, so classical natural deduction can be sequentialized, and the evident single-conclusioned system, adding, respectively, one of the following rules, is equivalent to NK:

- The Law of the Excluded Middle:

$$\frac{}{\Gamma \vdash \phi \vee \neg\phi}\ \text{LEM}$$

- The Double-negation Rule:

$$\frac{\Gamma \vdash \neg\neg\phi}{\Gamma \vdash \phi}\ \neg\neg$$

- The Reductio Ad Absurdum Rule:

$$\frac{\Gamma, \neg\phi \vdash \bot}{\Gamma \vdash \phi}\ \text{RAA.}$$

Not only is the system so obtained equivalent to NK, it retains the major defect of both NJ and NK, namely the necessity of taking the ζ-rules (i.e., the commuting conversions) in order to obtain an extensional equality on proofs.

The $\beta\eta$-equality defined for intuitionistic natural deduction extends to this system.

In the classical setting, however, we can do a bit better. A sequential system of classical natural deduction, with multiple-conclusioned sequents and which requires no ζ-reductions, has been introduced by Parigot [90].

In this setting, we have a choice of forms for $\vee I$. Following the pattern of intuitionistic natural deduction,

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi} \quad \frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi}\ \vee I$$

TABLE 2.5. Sequential natural deduction for classical propositional logic: FNK

$$\frac{}{\Gamma, \phi \vdash \phi, \Delta} \; Ax$$

$$\frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta} \neg I \qquad \frac{\Gamma \vdash \neg\phi, \Delta \quad \Gamma \vdash \phi, \Delta}{\Gamma \vdash \Delta} \neg E$$

$$\frac{}{\Gamma \vdash \top, \Delta} \qquad \text{no elimination rule}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \wedge \psi, \Delta} \wedge I \qquad \frac{\Gamma \vdash \phi_1 \wedge \phi_2, \Delta}{\Gamma \vdash \phi_i, \Delta} \; (i = 1, 2) \wedge E$$

$$\frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \supset \psi, \Delta} \supset I \qquad \frac{\Gamma \vdash \phi \supset \psi, \Delta \quad \Gamma \vdash \phi, \Delta}{\Gamma \vdash \psi, \Delta} \supset E$$

is available. However, given that we have multiple conclusions in our sequents, the following form is available:

$$\frac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \vee \psi, \Delta} \vee I. \tag{2.6}$$

Turning to $\vee E$, having multiple conclusions allows us to use a form which avoids the need for ζ-equalities (commuting conversions) in addition to the $\beta\eta$-equalities:

$$\frac{\Gamma \vdash \phi \vee \psi, \Delta}{\Gamma \vdash \phi, \psi, \Delta} \vee E. \tag{2.7}$$

To see the point here, informally, consider that the ζ-redex

$$\frac{\dfrac{\Gamma \vdash \phi_1 \vee \phi_2 \quad \Gamma, \phi_1 \vdash \psi \quad \Gamma, \phi_2 \vdash \psi}{\Gamma \vdash \psi} \vee E}{\Gamma \vdash \chi} R$$

and its reduct (the rule R may be pushed up the two right-hand branches) in the system which follows the intuitionistic form corresponds to the derivation

$$\frac{\dfrac{\Gamma \vdash \phi_1 \vee \phi_2, \Delta}{\Gamma \vdash \phi_1, \phi_2, \Delta} \vee E}{\Gamma \vdash \phi_1, \phi_2, \Delta'} R',$$

where R' is the rule corresponding to R and Δ results from applying R' with the principal formula in Δ, which is not a redex.

TABLE 2.6. Sequential natural deduction for classical propositional logic: FNK

$$\text{no } \bot I \text{ rule} \qquad \frac{\Gamma \vdash \bot, \Delta}{\Gamma \vdash \phi, \Delta}\bot E$$

$$\frac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \vee \psi, \Delta} \vee I \qquad \frac{\Gamma \vdash \phi \vee \psi, \Delta}{\Gamma \vdash \phi, \psi, \Delta} \vee E$$

The rules for falsity are straightforward:

$$\text{no introduction rule} \qquad \frac{\Gamma \vdash \bot, \Delta}{\Gamma \vdash \phi, \Delta}\bot E.$$

Call the system consisting of the rules above, in Table 2.6, with (2.6) for $\vee I$ and (2.7) for $\vee$-elimination, in Table 2.6, FNK.

Then we have the following:

Proposition 2.7 (classical natural deduction) *If ϕ is provable from assumptions Γ in NJ, then $\Gamma \vdash \phi$ is provable in FNK. Conversely, if $\Gamma \vdash \phi_1, \ldots, \phi_m$ is provable in FNK, then $\phi_1 \vee \cdots \vee \phi_m$ is provable from assumptions Γ in NK.*

Just as the simply-typed λ-calculus, extended with product and sum types as required, provides a representation of intuitionistic natural deduction proofs, so a representation of classical natural deduction proofs is provided by the $\lambda\mu$-calculus [90], extended as required with products and a suitable treatment of disjunction [97, 108, 111].

2.5 The $\lambda\mu$-, $\lambda\mu\oplus$-, $\lambda\mu\nu$-, and $\lambda\mu\nu\epsilon$-calculi

2.5.1 *Proof-objects and realizers*

As we have seen in the previous sections, natural deductions Φ of SNJ can be seen as proof-objects realizing consequences $\Gamma \vdash^{\Phi} \phi$ and can be represented by a λ-term, t, satisfying $\Gamma \vdash t : \phi$. Φ describes how to obtain natural deduction proofs of ϕ from natural deduction proofs of the formulæ in Γ.

This correspondence, between natural deduction proofs and λ-terms on the one hand and propositions and types on the other, does not hold for classical natural deduction. It turns out that, in the presence of any one of the three extensions of intuitionistic natural deduction to give classical natural deduction, $\beta\eta$-equality leads to the identification of all of the proofs of any given consequence. This result is best understood in the context of categorical models of the λ-calculus and is given in detail in Chapter 3.

However, Parigot's $\lambda\mu$-calculus [90] provides an elegant language of proof-objects based on an algorithmic interpretation of FNK. The proof-objects are realizers for multiple-conclusioned sequents $\Gamma \vdash \phi, \Delta$, where, critically, ϕ is a distinguished, or active, formula. $\lambda\mu$-terms provide realizers for the sequents

$\Gamma \vdash \phi, \Delta$ of FNK and the equational theory does not identify all such realizers of a given sequent provided ϕ is treated as a distinguished formula. The trick, then, is to treat $\Gamma \vdash \phi$ as a kind of intuitionistic consequence within the classical context provided by Δ.[22] Consequently, the form of the typing judgement in the $\lambda\mu$-calculus is $\Gamma \vdash t : \phi, \Delta$, where Γ is a context familiar from the typed λ-calculus and Δ is a context containing types indexed by names, $\alpha, \beta, \ldots$, which are distinct from variables. The idea is that each $\lambda\mu$-sequent has exactly one principal formula, ϕ, on the right-hand side, the leftmost one, which is the formula upon which all introduction and elimination rules operate. This formula is the type of the term t.

2.5.2 *The $\lambda\mu$-calculus*

We begin by introducing a minor variation on Parigot's $\lambda\mu$-calculus [90]. In addition to implicational types, we include conjunctive types. We then proceed to add disjunctive types and then *explicit substitutions* $u\{t/x\}$. The last are used in the analysis of search below to give suitable representatives for possibly incompletable sequent derivations. Parigot presents in [90] only a $\lambda\mu$-calculus with implicational types and β-reductions. The addition of conjunctive types is straightforward, but the addition of disjunctive types is more problematic as there are two main alternatives, which we will briefly discuss below. To model the transition from a given proof to a uniform proof we also need η-expansions. We show that strong normalization and confluence still hold for this extended calculus but the reducibility proof needs careful reworking as the η-expansions give rise to additional reduction rules.

We present this calculus in four steps: firstly, we introduce the $\lambda\mu$-calculus with implicational types, conjunctive types, and β-reductions. Secondly, we add disjunctive types and, thirdly, we add η-expansion and prove strong normalization and confluence for this system. Finally, we add explicit substitutions, showing that normalization and confluence are preserved.

2.5.3 *Implication and conjunction*

The raw terms of the $\lambda\mu$-calculus with conjunction are given by the following grammar:

$$t ::= x \mid \lambda x\colon \phi\,.\,t \mid tt \mid \mu\alpha.t \mid [\alpha]t \mid \mu\bot.t \mid [\bot]t \mid \langle t,t\rangle \mid \pi(t) \mid \pi'(t).$$

We assume that the scope of the bracket operator $[\alpha]t$ extends as far to the right as possible, that is, the term $[\alpha]ts$ is implicitly bracketed as $[\alpha](ts)$. The rules for well-formed $\lambda\mu$-terms are given in Table 2.7. The second instances of the rules $[\_]$ and μ model Contraction and Weakening, respectively.

[22]Later, in Chapter 4, we shall see that this formulation is closely related to Dummett's multiple-conclusioned intuitionistic sequent calculus [26].

TABLE 2.7. Well-formed $\lambda\mu$-terms

$$\frac{}{\Gamma, x\colon \phi \vdash x\colon \phi, \Delta}Ax$$

$$\frac{\Gamma, x\colon \phi \vdash t\colon \psi, \Delta}{\Gamma \vdash \lambda x\colon \phi.t\colon \phi \supset \psi, \Delta} \supset I \qquad \frac{\Gamma \vdash t\colon \phi \supset \psi, \Delta \qquad \Gamma \vdash s\colon \phi, \Delta}{\Gamma \vdash ts\colon \psi, \Delta} \supset E$$

$$\frac{\Gamma \vdash t\colon \phi^\alpha, \Delta}{\Gamma \vdash \mu\alpha.t\colon \phi, \Delta}\mu \qquad \frac{\Gamma \vdash t\colon \phi, \Delta}{\Gamma \vdash [\alpha]t\colon \phi^\alpha, \Delta}[\_]$$

$$\frac{\Gamma \vdash t\colon \Delta}{\Gamma \vdash \mu\alpha.t\colon \phi, \Delta}\mu \qquad \frac{\Gamma \vdash t\colon \phi, \phi^\alpha, \Delta}{\Gamma \vdash [\alpha]t\colon \phi^\alpha, \Delta}[\_]$$

$$\frac{\Gamma \vdash t\colon \Delta}{\Gamma \vdash \mu\bot.t\colon \bot, \Delta}\bot I \qquad \frac{\Gamma \vdash t\colon \bot, \Delta}{\Gamma \vdash [\bot]t\colon \Delta}\bot E$$

$$\frac{\Gamma \vdash t\colon \phi, \Delta \qquad \Gamma \vdash s\colon \psi, \Delta}{\Gamma \vdash \langle t, s\rangle\colon \phi \wedge \psi, \Delta} \wedge I \qquad \frac{\Gamma \vdash t\colon \phi \wedge \psi, \Delta}{\Gamma \vdash \pi(t)\colon \phi, \Delta} \wedge E \qquad \frac{\Gamma \vdash t\colon \phi \wedge \psi, \Delta}{\Gamma \vdash \pi'(t)\colon \psi, \Delta} \wedge E$$

The definition of the reduction rules requires not only the standard substitution $t[s/x]$, but also a substitution for names $t[s/[\alpha]u]$, which intuitively indicates the term t with all occurrences of a subterm of the form $[\alpha]u$ replaced by s. Again, we need the notion of a term with holes, adapted for the $\lambda\mu$-calculus. Such a term $\mathcal{C}$ with holes of type ϕ is a $\lambda\mu$-term which may have also the additional term constructor $\_$ with the rule $\Gamma \vdash \_\colon \phi, \Delta$. The term $\mathcal{C}(u)$ denotes the term $\mathcal{C}$ with the holes textually (with possible variable capture) replaced by u. Then we define $t[\mathcal{C}(u)/[\alpha]u]$, where α is a name and u is a metavariable, by

$$\begin{aligned} x[\mathcal{C}(u)/[\alpha]u] &= x \\ ([\alpha]t)[\mathcal{C}(u)/[\alpha]u] &= \mathcal{C}(t[\mathcal{C}(u)/[\alpha]u]) \end{aligned}$$

and defined on all other expressions by pushing the replacement inside.

Again, there are three kinds of reduction rules: β-, η-, and ζ-rules. The β- and η-rules have the same purpose as the β- and η-rules in the simply-typed λ-calculus. The ζ-rules of the $\lambda\mu\nu$-calculus are variants of the β-rules, where the Exchange is applied to the right-hand side before a β-rule is applied. This is different from the simply-typed λ-calculus, where ζ-rules model permuting reductions over $\vee E$-rules. The reduction rules are given in Table 2.8.

The term $[\alpha]t$ realizes the introduction of a name. The term $\mu\alpha.[\beta]t$ realizes the Exchange operation: if ϕ^α was part of Δ before the Exchange, then ϕ is the principal formula of the succedent after the Exchange. Taken together, these terms also provide a notation for the realizers of Contractions and Weakenings on the right of a multiple-conclusioned calculus. It is also easy to detect whether a formula ψ^β in the right-hand side is, in fact, superfluous, that is, that there is a derivation of $\Gamma \vdash t\colon \phi, \Delta'$ in which Δ' does not contain ψ; it is superfluous if β is not a free name in t. This observation is exploited in the sequel.

TABLE 2.8. Reduction rules for the $\lambda\mu$-calculus

β	$(\lambda x\colon \phi.t)s \rightsquigarrow t[s/x]$
$\zeta^{\supset}$	$(\mu\alpha^{\phi\supset\psi}.t)s \rightsquigarrow \mu\beta^{\psi}.t[[\beta]us/[\alpha]u]$
	$(\mu\alpha^{\phi\supset\perp}.t)s \rightsquigarrow \mu\perp.t[[\perp]us/[\alpha]u]$
η^{μ}	$\mu\alpha.[\alpha]s \rightsquigarrow s$ if α not free in s
β^{μ}	$[\gamma](\mu\alpha.s) \rightsquigarrow s[\gamma/\alpha]$
$\eta^{\perp}$	$\mu\perp.[\perp]t \rightsquigarrow t$
$\beta^{\perp}$	$[\perp]\mu\perp.t \rightsquigarrow t$
$\zeta^{\wedge}$	$\pi(\mu\alpha^{\phi\wedge\psi}.s) \rightsquigarrow \mu\beta^{\phi}.s[[\beta]\pi(u)/[\alpha]u]$
	$\pi'(\mu\alpha^{\phi\wedge\psi}.s) \rightsquigarrow \mu\gamma^{\psi}.s[[\gamma]\pi'(u)/[\alpha]u]$
	$\pi(\mu\alpha^{\perp\wedge\psi}.s) \rightsquigarrow \mu\perp.s[[\perp]\pi(u)/[\alpha]u]$
	$\pi'(\mu\alpha^{\phi\wedge\perp}.s) \rightsquigarrow \mu\perp.s[[\perp]\pi'(u)/[\alpha]u]$
$\beta^{\wedge}$	$\pi(\langle t, s\rangle) \rightsquigarrow t$
	$\pi'(\langle t, s\rangle) \rightsquigarrow s$

The $\lambda\mu$-calculus has a special formula $\perp$ and treats the formula $\neg\phi$ as $\phi \supset \perp$. The $\perp I$- and $\perp E$-rule model the fact that the formula $\perp$ can be freely added to the right-hand side of each derivation. As these two rules suggest, we treat $\perp$ as a special name, and when we have a generic term $\mu\alpha.t$ with $\Gamma \vdash t\colon \psi, \phi^{\alpha}, \Delta$, we always include the case $\mu\perp.t$.

2.5.3.1 *Remark* Note that in the version of the $\lambda\mu$-calculus presented above the term $[\alpha]t$ has no principal type. The intuition behind this is that this term occurs only as a subterm in a term like $\mu\beta.[\alpha]t$, which models structural rules on the right-hand side. It is possible to give a different, semantically motivated, presentation of the $\lambda\mu$-calculus in which all terms have a principal type. This is done by choosing the principal type $\perp$ for the term $[\alpha]t$. Hence the rules for $\mu\alpha.t$ and $[\alpha]t$ are changed to

$$\frac{\Gamma \vdash t\colon \phi, \Delta}{\Gamma \vdash [\alpha]t\colon \perp, \phi^{\alpha}, \Delta} \quad \text{and} \quad \frac{\Gamma \vdash t\colon \perp, \phi^{\alpha}, \Delta}{\Gamma \vdash \mu\alpha.t\colon \phi, \Delta}.$$

It follows that there is no need in such a system for the terms $\mu\perp.t$ and $[\perp]t$ or their corresponding rules, $\perp I$ and $\perp E$, because these terms were introduced solely as realizers for $\perp I$ and $\perp E$. Because, in our system, we have $\mu\perp.[\perp]t = t$ and $[\perp]\mu\perp.t = t$, the two systems are equivalent in the sense that there exist translations between them, which preserve equality and which are inverse up to equality. Similar rules express Weakening and Contraction in this way. We use the version outlined above because the algorithms for proof-search rely on the fact that the application of structural rules on the right-hand side is indicated by a term $\mu\alpha.[\beta]t$, where α or β might be $\perp$.

2.5.4 *Disjunctive types: The $\lambda\mu\nu$-calculus*

The key point in the addition of disjunctive types is naturally explained in the setting of the natural deduction calculus for classical logic, FNK. The details of our analysis, using the '$\lambda\mu\nu$-calculus', were originally presented by Pym, Ritter, and Wallen in [97, 108, 111].

One possible formulation, with a single minor formula in the premiss, follows the rules for disjunction in SNJ,

$$\frac{\Gamma \vdash \phi_i, \Delta}{\Gamma \vdash \phi_1 + \phi_2, \Delta} \quad i = 1, 2, \tag{2.8}$$

yielding the usual addition of sums (co-products) to the realizing λ-terms:

$$t ::= \mathtt{in}_1(t) \;\mid\; \mathtt{in}_2(t) \;\mid\; \mathtt{case}\, t\, \mathtt{of}\, \mathtt{in}_1(x) \Rightarrow t\, \mathtt{or}\;\; \mathtt{in}_2(y) \Rightarrow t.$$

An alternative formulation [26] exploits the presence of multiple conclusions, as in FNK:

$$\frac{\Gamma \vdash \phi_1, \phi_2, \Delta}{\Gamma \vdash \phi_1 \vee \phi_2, \Delta}. \tag{2.9}$$

Later, in Chapters 4 and 6, we shall see that this formulation is the more desirable as basis to model reduction operators for proof-search because it maintains a *local representation* of the *global choice* between ϕ_1 and ϕ_2: Given a local representation, we can hope to avoid backtracking to this point in the search space. In particular, this form of disjunction can be exploited to improve the efficiency of certain formulations of logic programming. These points are discussed in more detail in Chapters 4 and 6.

For the $\lambda\mu$-calculus, however, this latter formulation presents a new difficulty. Suppose the $\lambda\mu$-sequent $\Gamma \vdash t\colon \phi, \psi^\beta, \Delta$ is to be the premiss of an application of the $\vee$I rule. In forming the disjunctive active formula $\phi \vee \psi$, we move the named formula ψ^β from the context to the active position. Consequently, $\vee$I is formulated as a binding operation on names and we add the following constructs to $\lambda\mu$, to form the grammar of $\lambda\mu\nu$-terms [108, 111]:

$$t ::= \langle\beta\rangle t \;\mid\; \nu\beta\,.\,t. \tag{2.10}$$

The term $\nu\beta\,.\,t$ introduces a disjunction and the term $\langle\beta\rangle t$ eliminates one. The associated inference rules are as follows:

$$\frac{\Gamma \vdash t\colon \phi, \psi^\beta, \Delta}{\Gamma \vdash \nu\beta.t\colon \phi \vee \psi, \Delta} \vee I \qquad \frac{\Gamma \vdash t\colon \phi \vee \psi, \Delta}{\Gamma \vdash \langle\beta\rangle t\colon \phi, \psi^\beta, \Delta} \vee E$$

$$\frac{\Gamma \vdash t\colon \phi, \Delta}{\Gamma \vdash \nu\bot.t\colon \phi \vee \bot, \Delta} \vee I_\bot \qquad \frac{\Gamma \vdash t\colon \phi \vee \bot, \Delta}{\Gamma \vdash \langle\bot\rangle t\colon \phi, \Delta} \vee E_\bot.$$

To avoid variable capture, we have to add a special clause for the mixed substitution:

$$(\langle\alpha\rangle t)[[\mathcal{C}(u)/[\alpha]u]] = \mu\gamma.\mathcal{C}(\mu\alpha.[\gamma]\langle\alpha\rangle t[\mathcal{C}(u)/[\alpha]u]),$$

where γ is a fresh name. If we had pushed the substitution through, the substitution lemma fails: the term $\mu\beta.[\alpha]\langle\alpha\rangle x$ is well-formed if x is of type $\phi \vee (\psi \supset \chi)$. If the term $(\mu\beta.[\alpha]\langle\alpha\rangle x)[[\alpha']us/[\alpha]u]$ is defined as $\mu\beta.[\alpha'](\langle\alpha\rangle xs)$, we obtain an ill-formed term.

The corresponding reduction rules are

$$\begin{array}{lrl} \beta^{\vee} & \langle\beta\rangle(\nu\alpha.s) \rightsquigarrow & s[\beta/\alpha] \\ \zeta^{\vee} & \langle\beta\rangle\mu\gamma.t \rightsquigarrow & \mu\alpha.t[[\alpha]\langle\beta\rangle s/[\gamma]s] \\ \beta^{\vee}_{\perp} & \langle\beta\rangle\mu\gamma^{\perp\vee\psi}.t \rightsquigarrow & \mu\perp.t[[\perp]\langle\beta\rangle s/[\gamma]s] \\ \zeta^{\vee}_{\perp} & \langle\perp\rangle\mu\gamma^{\phi\vee\perp}.t \rightsquigarrow & \mu\alpha.t[[\alpha]\langle\perp\rangle s/[\gamma]s]. \end{array}$$

The rules $\vee I_{\perp}$, $\vee E_{\perp}$, $\beta^{\vee}_{\perp}$, and $\zeta^{\vee}_{\perp}$ are special cases of $\vee I$, $\vee E$, $\beta^{\vee}$, and $\zeta^{\vee}$, respectively. They are included as convenient abbreviations and need not be analysed separately.

2.5.4.1 *Remark* To avoid loops during reduction, all ζ-rules do not apply if the term t in which the name α is changed is equal to $\langle\alpha\rangle t'$, and α does not occur in t'.

Although the FNK-like $\vee$ can be derived from the SNJ-like $+$ (and vice versa) via Weakening, they are *not* 'isomorphic'; that is, there does not exist a bijection between

$$\{t \mid \Gamma \vdash t : \phi_1 + \phi_2, \Delta \quad \text{is provable}\}$$

and

$$\{t \mid \Gamma \vdash t : \phi_1 \vee \phi_2, \Delta \quad \text{is provable}\},$$

which respects the congruence induced on terms induced by normalization. Indeed, the imposition of a bijection forces for every Γ, Δ, t, and ϕ, the set

$$\{t \mid \Gamma \vdash t : \phi, \Delta \quad \text{is provable}\}$$

to have at most one element. The proper formulation of this result requires *semantic* techniques which make essential use of fibred structure. Such a formulation is provided in Chapter 3.

2.5.5 *The η-rules, strong normalization, and confluence*

Parigot gives only reduction rules for β-reduction. For both proof-theoretic and semantic reasons, we also need *extensionality*, that is, we must have the η-rules. We will work with long η-normal forms in the sequel.

We introduce them here as expansions; that is, each term of functional type is transformed into a λ-abstraction, each term of product type into a product and each term of sum type into a term $\nu\beta.t'$. These rules are

$$\begin{array}{ll} \eta^{\supset} & t \rightsquigarrow \lambda x\colon \phi.tx \\ \eta^{\wedge} & t \rightsquigarrow \langle \pi(t), \pi'(t)\rangle \\ \eta^{\vee} & t \rightsquigarrow \nu\alpha.\langle\alpha\rangle t. \end{array}$$

In these rules, we assume that t is neither a λ-abstraction, nor a product, nor a term $\nu\alpha.t'$, nor that t occurs as the first argument of an application, or as the argument of a projection π or π' or of some term $\langle\beta\rangle\_$. In the $\eta^{\supset}$-, $\eta^{\wedge}$-, and $\eta^{\vee}$-rules, we also assume that t is of function type, product type, and sum type respectively.

These η-rules generate *critical pairs*,[23] which give rise to additional reduction rules. As an example, consider the term $t = [\alpha]\mu\alpha.s$, where α is a name of type $\phi \supset \psi$. This term can reduce via an η-expansion to $[\alpha]\lambda x\colon \phi.(\mu\alpha.s)x$, and via a $\mu\nu$-rule to t. The reduction from $[\alpha]\lambda x\colon \phi.(\mu\alpha.s)x$ to t can be seen as a generalized renaming operation. This operation is denoted by $t\{\beta\}$ and is defined as follows:

Definition 2.8 Define the generalized renaming of a $\lambda\mu\nu$-term t by a name β, written $t\{\beta\}$, by induction over the type of the name β as follows:

Atomic type: $(\mu\alpha.t)\{\beta\} = t[\beta/\alpha]$;
$\phi \supset \psi$: $(\lambda x\colon \phi.t)\{\beta\} = t\{\beta'\}[[\beta]\lambda x\colon \phi.u/[\beta']u]$ for some fresh name β' if x occurs in $t\{\beta'\}$ only within the scope of $[\beta']u$, otherwise $(\lambda x\colon \phi.t)\{\beta\}$ is undefined;
$\phi \wedge \psi$: If $t = \langle t_1, t_2\rangle$ and for some names β_1 and β_2 of type ϕ and ψ, respectively, $t_2\{\beta_2\}$ arises from $t_1\{\beta_1\}$ by replacing each subterm $[\beta_1]s_1$ recursively by some subterm $[\beta_2]s_2$, then $t\{\beta\} = t_1\{\beta_1\}[[\beta]\langle s_1, s_2\rangle/[\beta_1]s_1]$;
$\phi \vee \psi$: $(\nu\alpha.t)\{\beta\} = t\{\beta'\}[[\beta]\nu\alpha.u/[\beta']u]$ for some fresh name β' if α occurs in $t\{\beta'\}$ only within the scope of $[\beta']u$, otherwise $(\nu\alpha.t)\{\beta\}$ is undefined.

The additional reduction rule, which is called ζ^{μ}, can now be stated as:

$$\zeta^{\mu} \quad [\alpha]t \rightsquigarrow t\{\alpha\}. \tag{2.11}$$

Note that this reduction rule specializes to the rule β^{μ}, if α is a name of atomic type. Because the outermost bindings $\mu\alpha.\_$ of names of atomic type disappear

[23] A formal definition of critical pairs may be found in J. W. Klop's comprehensive reference article on term rewriting systems [66]. Informally, the idea is that critical pairs are those pairs of terms upon which the normalization and confluence properties of a rewriting system depend. That is, pairs $\langle t_1, t_2\rangle$ such that there is a term t such that $t_1 \rightsquigarrow t$ and $t_2 \rightsquigarrow t$.

by an application of the ζ^μ-rule, this rule cannot give rise to reduction sequences $t \leadsto^* t$. Logically, the ζ^μ-rule amounts to taking an introduction rule and moving it above a structural rule (i.e., Weakening, Contraction) applied to its principal formula.

Our first lemma gives the local confluence of $\lambda\mu\nu$, extending Parigot's result for $\lambda\mu$ [90].

Lemma 2.9 *The notion of reduction in the $\lambda\mu\nu$-calculus is locally confluent.*

Proof We show that all critical pairs can be completed. For critical pairs arising from the rules β, $\zeta^\supset$, and β^μ this is part of the confluence of Parigot's $\lambda\mu$-calculus.

We show only a few characteristic cases for the rule ζ^μ (2.11). The first case is an overlap with the β^μ-rule. The term

$$u = (\mu\alpha.\cdots[\alpha](\lambda x\colon \phi.\cdots\mu\alpha'.\cdots[\alpha']t\cdots)\cdots)s$$

can reduce via $\zeta^\supset$ to

$$\mu\alpha'.\cdots[\alpha'](\lambda x\colon \phi.\cdots\mu\alpha'.\cdots[\alpha']t\cdots)s\cdots,$$

which in turn reduces via β and β^μ to $\mu\alpha'.\cdots[\alpha']t[s/x]\cdots$. The other reduction sequence via the additional rule is

$$\begin{aligned} u &\overset{\zeta^\mu}{\leadsto} (\mu\alpha.\cdots[\alpha]\lambda x\colon \phi.t\cdots)s \\ &\overset{\beta^\mu}{\leadsto} \mu\alpha'.\cdots[\alpha'](\lambda x\colon \phi.t)s\cdots \\ &\overset{\beta}{\leadsto} \mu\alpha'.\cdots.[\alpha']t[s/x]\cdots. \end{aligned}$$

The second case we consider is the overlap of the β^μ-rule with the η-expansion. This is the case which gives rise to the additional reduction rule ζ^μ. For this, consider the term $w = [\alpha]\mu\alpha.t$, which can be reduced via the β^μ-rule to t. The reduction sequence via the rule ζ^μ is as follows:

$$\begin{aligned} w &\overset{\eta^\supset}{\leadsto} [\alpha](\lambda x\colon \phi.(\mu\alpha.t)x) \\ &\overset{\beta^\mu}{\leadsto} [\alpha]\lambda x\colon \phi.\mu\alpha'.t[[\alpha']ux/[\alpha]u] \\ &\overset{\zeta^\mu}{\leadsto} t[[\alpha]\lambda x\colon \phi.ux/[\alpha]u], \end{aligned}$$

which is t modulo some η-expansions and/or β-reductions. □

The choice of a distinguished formula on the right-hand side of the sequent is required to ensure strong normalization and confluence. We use Parigot's proof [90] and extend it to the conjunctive and disjunctive types and explicit substitution.

We now give the proof of strong normalization for $\lambda\mu\nu$. Our proof is a combination of Parigot's proof of strong normalization for the $\lambda\mu$-calculus and of the SN-proof for the simply-typed λ-calculus with η-expansions, which handles also the ζ-rules by Ghani and Jay [61]. As we consider only a first-order calculus and not a second-order calculus as Parigot, we do not need the notion of reducibility candidates but can define the sets of reducible terms by induction over the type structure. We extend the result to include explicit substitutions in Section 2.5.6. This two-step argument is possible because we do not have composition of substitutions.

Definition 2.10 Suppose $\Gamma \vdash t\colon \phi, \Delta$. By induction over the structure of types in ϕ and Δ we define sets of *reducible* $\lambda\mu\nu$-terms of type ϕ, Δ, written $\mathit{Red}(\phi, \Delta)$, and for each term $\Gamma \vdash t\colon \phi, \Delta$ closure terms of type ϕ, Δ, written $\mathit{cl}_{\phi,\Delta}(t)$ or $\mathit{cl}(t)$ for short, as follows:

- If ϕ and Δ are all atomic types or $\perp$, then

$$\mathit{Red}(\phi, \Delta) = \{t \mid \Gamma \vdash t\colon \phi, \Delta \text{ and } t \text{ is SN}\}$$

  and $\mathit{cl}_{\phi,\Delta}(t) = \emptyset$;
- If one of the types in ϕ or Δ is not an atomic type or $\perp$, define $\mathit{Red}(\phi, \Delta)$ to be the set of all terms $\Gamma \vdash t\colon \phi, \Delta$ such that all terms in $\mathit{cl}(t)$ are reducible;
- The set of closure terms $\mathit{cl}_{\phi,\Delta}(t)$ is defined as the union of the sets

$$\begin{array}{rl}
\{ts \mid s \in \mathit{Red}(\psi, \Delta)\} & \text{if } \phi = \psi \supset \chi \\
\{\pi(t), \pi'(t)\} & \text{if } \phi = \psi \wedge \chi \\
\{\mu\alpha.[\beta]\langle\alpha\rangle t, \langle\alpha\rangle t\} & \text{if } \phi = \psi \vee \chi \\
\{(\mu\alpha.[\beta]t)s \mid s \in \mathit{Red}(\psi, \Delta)\} & \text{if } \alpha^{\psi\supset\chi} \in \Delta \\
\{\pi(\mu\alpha.[\beta]t)), \pi'(\mu\alpha.[\beta]t)\} & \text{if } \alpha^{\psi\wedge\chi} \in \Delta \\
\{\mu\gamma.[\delta]\langle\gamma\rangle\mu\alpha.[\beta]t, \langle\gamma\rangle\mu\alpha.[\beta]t\} & \text{if } \alpha^{\psi\vee\chi} \in \Delta.
\end{array}$$

Next we define the closure properties of the set of reducible terms. We define $\mathit{cl}^n_{\phi,\Delta}(t)$ to be the set of all terms t_n such that there exists a sequence $t_0, t_1, \ldots, t_n$ with $t_i \in \mathit{cl}(t_{i-1})$ and $t = t_0$, for all $1 \leq i \leq n$.

Lemma 2.11 *Every set of reducible $\lambda\mu\nu$-terms has the following properties:*

S1. *If t is reducible, then t is strongly normalizing;*
S2. *For all variables x, each element in $\mathit{cl}^n(x)$ is reducible;*
S3. (i) *If $t[s/x]$ is reducible, so is each element of $\mathit{cl}^n(\lambda x\colon \phi.t)$;*
(ii) *If t and s are reducible, so is each element of $\mathit{cl}^n(\langle t, s\rangle)$;*
(iii) *If $t[\beta/\alpha]$ is reducible, so is each element of $\mathit{cl}^n(\nu\alpha.t)$.*
S4. *If t is reducible, so is $\mathit{cl}^n(\mu\alpha.[\beta]t)$.*

Proof We split each of conditions **S2**, **S3**, and **S4** into two conditions, which we prove by induction, and which together imply the original conditions. We use simultaneous induction over the types of ϕ and Δ to show the following properties:

S1. If t with $\Gamma \vdash t\colon \phi, \Delta$ for some Γ is reducible, then t is strongly normalizing;

S2′. If for any element t of $cl^n(x)$ with $\Gamma \vdash t\colon \phi, \Delta$ for some Γ all elements of $cl^m(t)$ are SN for any $m \geq 0$, then t is reducible;

S2″. If $\Gamma \vdash x\colon \phi, \Delta$ for some Γ, then all elements of $cl^m(x)$ are SN for any $m \geq 0$;

S3′. (i) If all elements of $cl^m(\lambda x\colon \phi.t)$ are SN for all $m \geq 0$, then each element of $cl^n(\lambda x\colon \phi.t)$ is reducible if $\Gamma \vdash cl^n(\lambda x\colon \phi.t)\colon \psi, \Delta$, for some Γ;

(ii) If all elements of $cl^m(t)$ and $cl^m(s)$ are SN for all $m \geq 0$, then each element of $cl^n(\langle t, s\rangle)$ is reducible if $\Gamma \vdash cl^n(\langle t, s\rangle)\colon \phi, \Delta$, for some Γ;

(iii) If all elements of $cl^m(\nu\alpha.t)$ are SN, then each element of $cl^n(\nu\alpha.t)$ is reducible if $\Gamma \vdash cl^n(\nu\alpha.t)\colon \phi, \Delta$, for some Γ;

S3″. (i) If $\Gamma \vdash \lambda x\colon \psi.t\colon \phi, \Delta$ for some Γ and $t[s/x]$ is reducible for all reducible $\Gamma \vdash s\colon \psi, \Delta$, then each element of $cl^m(\lambda x\colon \psi.t)$ is SN;

(ii) If $\Gamma \vdash \langle t, s\rangle\colon \phi, \Delta$ for some Γ and t and s are reducible, then each element of $cl^m(\langle t, s\rangle)$ is SN;

(iii) If $\Gamma \vdash \nu\alpha.t\colon \phi, \Delta$ for some Γ and $t[\beta/\alpha]$ is reducible for each name β, then each element of $cl^m(\nu\alpha.t)$ is SN;

S4′. If t is reducible and $cl^m(\mu\alpha.[\beta]t)$ is SN for all $m \geq 0$, then $cl^n(\mu\alpha.[\beta]t)$ is reducible if $\Gamma \vdash \mu\alpha.[\beta]t\colon \phi, \Delta$ for some Γ;

S4″. If $\Gamma \vdash t\colon \phi, \Delta$ for some Γ and t is reducible, then $cl^m(\mu\alpha.[\beta]t)$ is SN for all $m \geq 0$.

The induction proceeds now as follows:

S1. If ϕ and Δ are all atomic or $\perp$, then t is SN by definition. If not, one does a case analysis of ϕ and Δ. We consider here only the cases of $\phi = \psi \vee \chi$ and $\phi = \psi \supset \chi$. In the first case, if $t \rightsquigarrow^* t'$, then either $\langle\alpha\rangle t \rightsquigarrow^* \langle\alpha\rangle t'$, or $t' = \nu\alpha.\langle\alpha\rangle t''$, and $t \rightsquigarrow t''$ via all reduction rules except top-level η-expansions. Hence any infinite reduction sequence starting with t can be extended to an infinite reduction sequence of $\langle\alpha\rangle t$. This is a contradiction because by the induction hypothesis, $\langle\alpha\rangle t$ is SN. Now suppose $\phi = \psi \supset \chi$. Choose a variable x of type ψ that does not occur freely in t. By **S2″** and **S2′**, x is reducible. Hence, by definition, tx is reducible, and SN by **S1** by the induction hypothesis. Hence all reduction sequences of t which do not involve outermost η-expansions terminate. The case of an outermost η-expansion is treated in the same way as in the case of $\phi = \psi \vee \chi$;

S2′. If ϕ and Δ are all atomic or $\perp$, the claim is trivial. If not, we have to show that all elements of $cl^{n+1}(x)$ are reducible. This follows directly from the induction hypothesis;

S2″. Here we do an induction over m and use the fact that, by the induction hypothesis, for all reducible terms s which occur in $cl^m(x)$, $cl^k(s)$ is SN for all k. In particular, the restriction of the ζ-rules mentioned in Section 2.5.4.1 prevents an infinite loop in the term $(\mu\alpha.[\beta]\langle\alpha\rangle x)s$;

S3′. Same argument as for **S2′**;

S3″. Here we again use induction over m. We consider only one case; all other cases are similar. Consider a reduction sequence

$$\begin{aligned}(\mu\alpha.[\beta](\lambda x\colon \psi.t)s)u &\rightsquigarrow \mu\alpha'[\beta](\lambda x\colon \psi.t[[\alpha']wu/[\alpha]w])s[[\alpha']wu/[\alpha]w]\\ &\rightsquigarrow \mu\alpha'.[\beta]t[[\alpha']wu/[\alpha]w][s[[\alpha']wu/[\alpha]w]/x]\\ &= \mu\alpha'.[\beta]t[s/x][[\alpha']wu/[\alpha]w],\end{aligned}$$

for an element of $cl^2(\lambda x\colon \phi.t)$. By the induction hypothesis (**S1**),

$$(\mu\alpha.[\gamma]s)u,\ (\mu\alpha.[\beta](\lambda x\colon \phi.t)x)u \quad \text{and} \quad \mu\alpha'.[\beta]t[s/x][[\alpha']wu/[\alpha]w]$$

are SN. Hence the term $(\mu\alpha.[\beta](\lambda x\colon \psi.t)s)u$ is SN;

S4′. Same argument as for [**S2′**];

S4″. One shows that any infinite reduction sequence for $s \in cl^m(\mu\alpha.[\beta]t)$ yields an infinite reduction sequence for an element of $cl^m(t)$, which is SN by the induction hypothesis (**S1**). □

The key theorem states that every term is reducible. For this we need a generalized induction hypothesis which includes all possible substitutions of reducible terms for free variables and all mixed substitutions for free names. Mixed substitutions arise as contracta of the ζ-rules in the same way as the ordinary substitution arises as a contractum of the β-rule.

Theorem 2.12 *For each $\lambda\mu\nu$-term t such that $\Gamma \vdash t\colon \phi, \Delta$ and reducible terms s_i and u_i, all terms*

$$\begin{aligned}t[s_i/x_i,\ \ &[\alpha'_j]wu_j/[\alpha_j]w,\quad [\alpha'_k]\pi(u)/[\alpha_k]u,\\ &[\alpha'_m]\pi'(u)/[\alpha_m]u,\ [\alpha'_n]\langle\beta_n\rangle u/[\alpha_n]u,\ \beta_r/\alpha_r],\end{aligned}$$

are reducible, where the names α_j, α_k, α_m, and α_n range over all subsets of names in Δ of implication type, conjunction type, and disjunction type, respectively, and each of the α_m is different from each of the α_k. The names α_r form some subset of the names in Δ.

Proof We write f for the substitution

$$[s_i/x_i,\ [\alpha'_j]wu_j/[\alpha_j]w,\ [\alpha'_k]\pi(u)/[\alpha_k]u,$$
$$[\alpha'_m]\pi'(u)/[\alpha_m]u,\ [\alpha'_n]\langle\beta_n\rangle u/[\alpha_n]u,\ \beta_r/\alpha_r]$$

and write $t[f]$ for the application of the substitution f to t. The proof proceeds by induction over the derivation of t.

x_i: Obvious, as $x_i[f] = s_i$, which is reducible by assumption.

$\lambda x\colon \phi.t$: By the induction hypothesis $t[f, s/x]$ is reducible for every reducible term s, hence $(\lambda x\colon \phi.t)[f]$ is reducible by **S3**.

ts: By the induction hypothesis, $t[f]$ and $s[f]$ are reducible, hence by definition of reducibility $t[f]s[f] = (ts)[f]$ is reducible.

$\mu\alpha^\phi.t$: By the induction hypothesis, $t[f]$ is reducible. Hence by **S4**, $\mu\alpha.t[f]$ is reducible as well.

$[\alpha^\phi]t$: If α occurs in f only as part of a substitution $[\beta/\alpha]$ or not at all, then $([\alpha]t)[f] = [\alpha](t[f])$ or $[\beta](t[f])$, depending whether α occurs in f or not. By the induction hypothesis all elements of $cl(t[f])$ are reducible. Because $cl([\alpha](t[f])) \subseteq cl(t[f])$ and $cl([\beta](t[f])) \subseteq cl(t[f])$, respectively, $[\alpha]t[f]$ is reducible. So now assume that α does occur in f in a different position. In this case $\mu\gamma.(([\alpha]t)[f])$ is an element of $cl(t[f])$, hence it is reducible by induction hypothesis.

$\langle t, s\rangle$: By the induction hypothesis, $t[f]$ and $s[f]$ are reducible, hence by **S3**, $\langle t[f], s[f]\rangle$ is reducible.

$\pi(t)$, $\pi'(t)$: By induction hypothesis, $t[f]$ is reducible, hence $\pi(t[f])$ and $\pi'(t[f])$ are reducible by definition.

$\nu\alpha.t$: By induction hypothesis, $t[f, \beta/\alpha]$ is reducible. Hence property **S3** now implies the claim.

$\langle\beta\rangle t$: By induction hypothesis, $t[f]$ is reducible, and hence by definition $\langle\beta\rangle t[f]$ is reducible, too. □

Finally, we obtain the desired result as a corollary.

Corollary 2.13 *All well-typed $\lambda\mu\nu$-terms are SN.*

Now we are in a position to deduce confluence from local confluence and termination via Newman's Lemma [66].

Theorem 2.14 *The $\lambda\mu\nu$-calculus is confluent.*

Proof The proof is a straightforward application of Newman's Lemma [66], which states that a locally confluent and terminating notion of reduction is confluent. □

2.5.6 *Explicit substitutions: The $\lambda\mu\nu\epsilon$-calculus*

Our final extension of the $\lambda\mu$-calculus involves adding a form of *explicit substitution.*

The presentation of the $\lambda\mu$-calculus in [90] and of $\lambda\mu\nu$ herein is as a system of linearized natural deduction with multiple conclusions, with implicational types both introduced and eliminated on the right-hand side. An alternative formulation of Parigot's system, not affecting the structure of the derivable terms, would be as a sequent calculus, with the elimination of implicational types on the right replaced by the introduction of implicational types on the left, as follows:[24]

$$\lambda\mu \supset \mathrm{L} \quad \frac{\Gamma, w\colon \psi \vdash t\colon \chi, \Delta \qquad \Gamma \vdash s\colon \phi, \Delta}{\Gamma, x\colon \phi \supset \psi \vdash t[xs/w]\colon \chi, \Delta}.$$

Such a rule is admissible in Parigot's system since the Cut rule,

$$\lambda\mu\mathrm{Cut} \quad \frac{\Gamma \vdash s\colon \phi, \Delta \qquad \Gamma, w\colon \phi \vdash t\colon \psi, \Delta}{\Gamma \vdash t[s/w]\colon \psi, \Delta},$$

is also admissible. In these rules, the substitution $[t/x]$ is the usual implicit, metatheoretic one. An analysis such as this for a system of first-order dependent function types is presented in [100] and exploited as a basis for a theory of proof-search in [106].

The rule (ϵL),

$$\frac{\Gamma, w\colon \psi \vdash t\colon \chi, \Delta \qquad \Gamma \vdash s\colon \phi, \Delta}{\Gamma, x\colon \phi \supset \psi \vdash t\,\{xs/w\}\,\colon \chi, \Delta}\epsilon L, \tag{2.12}$$

which introduces the explicit substitution $u\{xs/w\}$, corresponds exactly to the usual left rule for implication, but with explicit substitution replacing implicit substitution. The $\lambda\mu\epsilon$-calculus contains this left rule for explicit substitution together with the usual introduction and elimination rules for the implication. Similarly, we use explicit substitutions for the $\vee L$-rule. We call a term u a *substitution term* if it occurs as a subterm of u' in the term $s\,\{u'/x\}$.

$\lambda\mu\nu\epsilon$-terms are thus $\lambda\mu\nu$-terms enriched by the presence of explicit substitutions. More precisely, the grammar of $\lambda\mu\nu\epsilon$-terms is the grammar for the $\lambda\mu\nu$-calculus with the added clause

$$t ::= t\,\{t/x\}\,. \tag{2.13}$$

If the substitution were implicit, and so evaluated when introduced, some parts of a derivation would not be represented by the corresponding term. This happens if the variable being replaced does not occur in the term. The rule for explicit substitution ϵL can thus be used to model the $\supset L$-rule of the classical sequent calculus directly. In [107], a similar analysis is provided for a proof system for SLD-resolution over propositional implicational Horn clauses. Herbelin [44] also uses explicit substitutions, for a similar reason, in his version of a translation of intuitionistic sequent calculus (LJ) into a modified λ-calculus. His concern, however, is to restrict LJ so as obtain a bijective correspondence between λ-terms and LJ-derivations.

[24] For details of this system, see Chapter 4.

Now we extend strong normalization and confluence to the $\lambda\mu\nu\epsilon$-calculus. The reduction rules corresponding to ϵL are as follows, where we assume standard variable capture rules [11]:

$$\begin{aligned}
(\lambda x\colon \phi.t)\{s/z\} &\rightsquigarrow \lambda x\colon \phi.t\{s/z\}\\
(ts)\{u/z\} &\rightsquigarrow t\{u/z\}s\{u/z\}\\
(\mu\alpha.t)\{s/z\} &\rightsquigarrow \mu\alpha.t\{s/z\}\\
([\alpha]t)\{s/z\} &\rightsquigarrow [\alpha]t\{s/z\}\\
\langle t,s\rangle\{u/z\} &\rightsquigarrow \langle t,\{u/z\},s\{u/z\}\rangle\\
\pi(t)\{u/z\} &\rightsquigarrow \pi(t\{u/z\})\\
\pi'(t)\{u/z\} &\rightsquigarrow \pi'(t\{u/z\})\\
(\nu\alpha.t)\{u/z\} &\rightsquigarrow \nu\alpha.t\{u/z\}\\
(\langle\alpha\rangle t)\{u/z\} &\rightsquigarrow \langle\alpha\rangle.t\{u/z\}.
\end{aligned} \tag{2.14}$$

Intuitively, these rules push substitutions under all term constructors but do not include the rule $x\,\{s/x\} \rightsquigarrow s$, which actually carries out the substitution. Note also that the rules for explicit substitution distribute substitutions only to variables and allow no interactions between the substitutions themselves. Hence termination is ensured as the execution of substitution rules cannot create any redexes that were not present in the term with all substitutions eliminated. The precise formulation of this idea uses tree-orderings to ensure that non-substitution redexes do not create a possibility of an infinite reduction sequences by copying redexes of substitution rules. The following metatheorems are extended to $\lambda\mu\nu\epsilon$, as defined by adding the rule (2.12), the syntax (2.13), and the reductions (2.14) to $\lambda\mu\nu$:

Theorem 2.15 *The $\lambda\mu\nu\epsilon$-calculus is strongly normalizing.*

Proof Define the height of a term t inductively by

$$\begin{aligned}
h(x) &= 0\\
h(\alpha) &= 0\\
h(C(t)) &= h(t)+1 \text{ for any unary term constructor}\\
h(C(t_1,t_2)) &= max(h(t_1),h(t_2))+1 \text{ for any binary term constructor.}
\end{aligned}$$

Now we assign a complexity tree $\mathcal{T}(t)$ to each term t. The nodes of the tree are labelled by a pair of natural numbers. The first number is $\nu(t^e)$, where t^e is the term t with all explicit substitutions deleted, and the second number indicates the height of the term t in a subexpression $t\,\{u/x\}$. This tree is inductively defined as follows:

x: $\mathcal{T}(x)$ is the tree which consists only of the root with the label $(0,0)$;

α: $\mathcal{T}(\alpha)$ is the tree, which consists only of the root with the label $(0,0)$;

$C(t)$: If the root of $\mathcal{T}(t)$ is labelled (n, h), then $\mathcal{T}(C(t))$ is the tree $\mathcal{T}(t)$ with the label of the root changed to $(\nu(C(t)^e, h)$;

$C(t_1, t_2)$: $\mathcal{T}(C(t_1, t_2))$ is the tree with the root labelled $(\nu(C(t_1, t_2))^e, 0)$ and where the children of the root are the trees $\mathcal{T}(t_1)$ and $\mathcal{T}(t_2)$;

$(t\,\{u/x\})$: The tree $\mathcal{T}(t\,\{u/x\})$ is the tree with the root labelled $(\nu(t^e), h(t))$ with children $\mathcal{T}(t)$ and $\mathcal{T}(u)$.

We order these trees by the tree-ordering: The tree t_1 is smaller than the tree t_2 if any node of t_2 can be mapped to a node of t_1 such that the root of t_2 is greater than the node of t_1 to which it is mapped and that all other nodes of t_2 are mapped to nodes which are not bigger and that a child of a node in t_2 is mapped to some grandchild (including itself) of the image of the parent. The ordering of the nodes is the lexicographic ordering on natural numbers.

It is well-known that this tree-ordering is a well-ordering, so for the termination proof it suffices to check that whenever $t \rightsquigarrow t'$, the term t is bigger than t' in the tree-ordering. Firstly, consider any non-substitution reduction $t \rightsquigarrow t'$. If the redex that is contracted occurs in a substitution term s, say $u\,\{s/x\}$, then only the subtree $\mathcal{T}(s)$ is affected by the reduction $t \rightsquigarrow t'$, so we can assume without loss of generality that the redex that is contracted in t is not a substitution term in t. Hence the first component of the root $\mathcal{T}(t)$ decreases when the redex is contracted. Any substitution term u is unaffected by the reduction $t \rightsquigarrow t'$. Hence we can use the simple mapping which maps any nodes that correspond to non-substitution terms to the root and maps the subtree corresponding to substitution terms u to the same subtree in $\mathcal{T}(t)$. Secondly, consider any substitution reduction $t \rightsquigarrow t'$. In this case the decrease in the tree-ordering results from the decrease of the second component of the node as the height of the subtree where the substitution occurs decreases. □

Confluence follows from strong normalization and local confluence by Newman's Lemma [66], as usual.

Theorem 2.16 *The $\lambda\mu\nu\epsilon$-calculus is confluent.*

Proof Again, it suffices to check local confluence. Because the $\lambda\mu\nu$-calculus is confluent it suffices to consider only overlaps between explicit substitution rules or an explicit substitution rule and a rule which does not involve explicit substitution. There are no critical pairs in the first case. For the second case, one has to show a substitution lemma. This lemma states that

$$t[s/x]\,\{u/z\} \rightsquigarrow^* t\,\{u/z\}\,[s\,\{u/z\}\,/x].$$

As an example for the completion of the critical pairs, consider the redex

$$((\lambda x\colon \phi.t)s)\,\{u/z\}\,.$$

It can reduce to $t[s/x]\{u/z\}$, and to $((\lambda x\colon \phi.t\{u/z\})s\{u/z\})$. The substitution lemma now implies that $t[s/x]\{u/z\} \rightsquigarrow^{*} t\{u/z\}[s\{u/z\}/x]$. This term is also the contractum of $((\lambda x\colon \phi.t\{u/z\})s\{u/z\})$ via β-reduction. □

2.6 Discussion

We remark that Parigot [90] shows that $\lambda\mu$'s analysis of classical proofs extends to predicate logic with the universal quantifier, $\forall$. We conjecture that the results we have presented in this chapter also extend to the quantifiers.

3

THE SEMANTICS OF INTUITIONISTIC AND CLASSICAL PROOFS

3.1 Introduction

In this chapter, we establish the semantics for intuitionistic and classical logic which we shall use in the remainder of this monograph. Much of our presentation is of familiar background material, but two points perhaps deserve a little more attention:

1. In Section 3.6, we give the semantics of the $\lambda\mu\nu$-calculus. This summarizes quite recently published work of the authors [97], building on the disjunction-free work in [89];
2. In Section 3.4, we provide a games semantics for intuitionistic *proofs* which is *not* constructed via a games semantics for linear logic. Then, in the concluding part of Section 3.6, we generalize this semantics to the representation of classical proofs provided by the $\lambda\mu\nu$-calculus.

We begin, in Section 3.2, with the semantics for intuitionistic proofs. We first consider Heyting algebras, which model provability but not proofs, and then move on to bi-Cartesian closed categories, which do model proofs. Next we present a categorical version of Kripke models—we have summarized the basic ideas in Chapter 1—for intuitionistic logic, showing that Kripke models can capture both truth and proof, and finish this section with a games model for intuitionistic logic.

Section 3.5, reviews categorical structures used for modelling structures with parameters, so-called fibred (or indexed) categories. We use them in this monograph to model the embedding of intuitionistic logic into classical logic.

Afterwards, in Section 3.6, we describe the semantics for classical logic and relate it to the corresponding semantics for intuitionistic logic. We start by defining Boolean algebras and then show that bi-Cartesian closed categories cannot be extended to model classical proofs. Next we present categorical models for the $\lambda\mu\nu$-calculus which are based on an embedding of intuitionistic into classical logic. We finish with an extension of the games semantics of Section 3.4 to classical logic.

3.2 The semantics of intuitionistic proofs

Intuitionism, as proposed by Brouwer, is built on a very different conception from that of classical logic of what constitutes valid reasoning and, as we have seen, intuitionistic natural deduction does not include the law of the excluded middle. Semantically, this rejection of reasoning by contradiction amounts to a requirement that an intuitionistic proof of a theorem must construct sufficient evidence for that theorem.

Heyting [50] and Kolmogorov [2] provided a meaning for the syntactic formalism of intuitionistic proof which captures this evidential character of reasoning. The idea both simple and beautiful, and is usually termed Brouwer–Heyting–Kolmogorov, or BHK, semantics. We start from a primitive judgement that a primitive object P is proof of an atomic proposition p, given by a construction:[25]

$$P \text{ proves } p$$

The meaning of complex proofs is then given by the following inductive clauses:

1. There is a unique construction of $\top$;
2. A proof of a conjunction $\phi_0 \wedge \phi_1$ is a pair $\langle \Phi_0, \Phi_1 \rangle$, where Φ_0 is a proof of ϕ_o and Φ_1 is a proof of ϕ_1;
3. There is no construction of $\perp$;
4. A proof of a disjunction $\phi_0 \vee \phi_1$ is a pair $\langle b, \Phi_i \rangle$, where b is a Boolean, Φ_i is a proof of ϕ_i, and $i = 0$ if $b = 0$ and $i = 1$ if $b = 1$;
5. A proof of $\phi \supset \psi$ is a construction f which converts a proof Φ of ϕ into a proof $f(\Phi)$ of ψ.

The BHK semantics of intuitionistic propositional proofs extends to intuitionistic predicate proofs, over a given language of terms, quite straightforwardly:

1. For atomic predicates $p(x)$, there is for every term, t, which is well-formed in the language and for which $p(t)$ is provable, a construction $P(t)$ which proves $p(t)$;
2. A proof of $\forall x.\phi(x)$ is a function, f, which maps every term, t, which is well-formed in the language, to a proof, $f(t)$, of $\phi[t/x]$;
3. A proof of $\exists x.\phi(x)$ is a pair, $\langle t, \Phi \rangle$, in which t is a term which is well-formed in the language and Φ is a proof of $\phi[t/x]$.

An algebraic account of BHK semantics (for intuitionistic logic) may be given in terms of bi-Cartesian closed categories [54]. Before providing a summary of this set-up, in a form which is suitable for the purposes of this monograph, we begin, for developmental completeness, with a brief account of intuitionistic semantics in Heyting algebras, the intuitionistic counterpart to Boolean algebras.

[25]We do not discuss here what constitutes an acceptable notion of construction. The usual choice is based on the theory of recursive functions but this definition is amenable to relativization.

3.2.1 *Heyting algebras*

We start by recalling the definition of Heyting algebras, which may be used to interpret intuitionistic propositional logic. Heyting algebras are special cases of distributive lattices, which we begin by recalling.

Definition 3.1 A distributive lattice $(A, \vee, \wedge, 0, 1)$ is given by a set A, two binary operations, $\vee$ and $\wedge$, and two distinguished elements 0 and 1 of A, such that

(i) $\vee$ and $\wedge$ are associative and commutative;
(ii) $\vee$ and $\wedge$ are idempotent: $x \vee x = x$ and $x \wedge x = x$ for all $x \in A$;
(iii) 0 and 1 are neutral elements for $\vee$ and $\wedge$, respectively: $x \vee 0 = 0 \vee x = x$ and $x \wedge 1 = 1 \wedge x = x$ for all $x \in A$;
(iv) $\vee$ and $\wedge$ are absorptive: $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ for all $x, y \in A$;
(v) $\wedge$ distributes over $\vee$ and $\vee$ distributes over $\wedge$: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in A$.

On every lattice we define a partial order by $x \leq y$ iff $x \vee y = y$ or equivalently iff $x \wedge y = x$.

Definition 3.2 A Heyting algebra is a distributive lattice $(A, 0, 1, \wedge, \vee)$ together with a binary operation $\supset$ on A such that, for all $x, y, z \in A$, $x \leq (y \supset z)$ iff $x \wedge y \leq z$.[26]

In any Heyting algebra one can define a negation by $\neg x = x \supset 0$. This negation does satisfy $\neg x \wedge x = 0$, but not necessarily other familiar laws like $\neg\neg x = x$ or $x \vee \neg x = 1$.

We give several examples of Heyting algebras. The first one consists of the powerset of a given set A. The operation $\wedge$ and $\vee$ are intersection and union of sets respectively, the partial order $\leq$ is set inclusion, $X \supset Y$ is defined as the union of Y and the complement of X, and 0 and 1 are the empty set and A, respectively. The set $\neg X$ is the complement of X relative to A.

The second example is given by the set of open sets of a topological space. $\vee$ and $\wedge$ are union and intersection, respectively. $\neg X$ is the interior of the complement of X, and $X \supset Y$ is defined as $\neg X \vee Y$. Note that in this example $X \vee \neg X$ is not necessarily equal to 1, which is the whole space.

[26]Such a lattice is said to be residuated.

Formulæ of intuitionistic logic modulo logical equivalence[27] provide another example of a Heyting algebra. The operations $\vee$, $\wedge$, and $\supset$ are the logical operations $\vee$, $\wedge$, and $\supset$, respectively, and $\phi \leq \psi$ holds iff $\phi \supset \phi$ is valid in intuitionistic logic. This example shows that Heyting algebras provide models which do not distinguish between proofs: any proof of $\phi \supset \psi$ gives rise to an inequality $\phi \leq \psi$ in this algebra.

3.2.2 *Bi-Cartesian closed categories*

The next class of models interprets not only formulæ and provability but also distinguishes different proofs. We describe here suitable categorical structures, which provide very rich models for intuitionistic logic and its proofs. We present here bi-Cartesian closed categories, which are widely used to model intuitionistic logic and its proofs.

Definition 3.3 A bi-Cartesian closed category $\mathcal{C}$ is a category with finite products, finite sums, and a right adjoint G_A to the functor $- \times A$.

We write 1 for the terminal object, 0 for the initial object, $A \times B$ for the product, $A+B$ for the co-product, $A \Rightarrow B$ for $G_A(B)$, $\mathsf{Cur}_A(f)$ for the morphism obtained from $f : C \times A \to B$ by applying the bijection between hom-sets defining G and $\mathsf{App}_{A,B} : (A \Rightarrow B) \times A \to B$ for the co-unit of this adjunction. We call objects $A \Rightarrow B$ function spaces.

An example of a bi-Cartesian closed category is the category of sets and functions, where the objects are sets and the morphisms between A and B are the set-theoretic functions from A to B. The product of two sets A and B is the set of tuples (a, b), where $a \in A$ and $b \in B$. The co-product of two sets A and B is the disjoint union of A and B, and the set $A \Rightarrow B$ is the set of all functions from A to B.

Another example is given by any Heyting algebra. A Heyting algebra

$$(A, \vee, \wedge, \supset, 0, 1),$$

can be regarded as a bi-Cartesian closed category where the objects are the elements of A, and product and co-product of two elements a and b are given by $a \wedge b$ and $a \vee b$, respectively. There is at most one morphism between two objects a and b, and it exists iff $a \leq b$. The defining condition for the operator $\supset$ in a Heyting algebra is exactly the natural isomorphism defining the function spaces in the case of a Heyting algebra.

Natural deduction proofs of intuitionistic formulæ can be interpreted in a bi-Cartesian closed category via a map $[\![-]\!]$ which maps formulæ to objects and

[27] Two formulæ ϕ and ψ are said to be *logically equivalent* iff both $\vdash \phi \supset \psi$ and $\vdash \psi \supset \phi$ are provable.

natural deduction proofs of ψ using assumptions $\phi_1, \ldots, \phi_n$ to morphisms

$$[\![\phi_1]\!] \times \cdots \times [\![\phi_n]\!] \to [\![\psi]\!].$$

This map is defined as follows:

Definition 3.4 Let $\mathcal{C}$ be any bi-Cartesian closed category and consider any map ρ from propositional atoms to objects of $\mathcal{C}$. Then define a map $[\![\_]\!]$ extending ρ from formulæ to objects and from natural deduction proofs of intuitionistic logic to morphisms by induction over the definition of formulæ and proofs as follows:

1. On formulæ:
   (i) Atoms: $[\![p]\!] = \rho(p)$;
   (ii) Conjunction $[\![\phi \wedge \psi]\!] = [\![\phi]\!] \times [\![\psi]\!]$;
   (iii) Disjunction: $[\![\phi \vee \psi]\!] = [\![\phi]\!] + [\![\psi]\!]$;
   (iv) Implication: $[\![\phi \supset \psi]\!] = [\![\phi]\!] \Rightarrow [\![\psi]\!]$.
2. On proofs: (Let $\Gamma = \phi_1, \ldots, \phi_n$ and let $A = [\![\phi_1]\!] \times \cdots \times [\![\phi_n]\!]$)

Axiom: Suppose Φ is the axiom $\Gamma, \phi \vdash \phi$. Then $[\![\Phi]\!]$ is the projection π from $A \times [\![\phi]\!]$ to $[\![\phi]\!]$;

$\supset I$**:** Suppose the proof Φ is given by

$$\frac{\Psi : \Gamma, \phi \vdash \psi}{\Phi : \Gamma \vdash \phi \supset \psi} \supset I$$

and suppose that $f = [\![\Psi]\!]$. Then $[\![\Phi]\!] = \mathsf{Cur}_{[\![\phi]\!]}(f)$;

$\supset E$**:** Suppose the proof Φ is given by

$$\frac{\Phi_1 : \Gamma \vdash \phi \supset \psi \qquad \Phi_2 : \Gamma \vdash \phi}{\Phi : \Gamma \vdash \psi} \supset E.$$

Then $[\![\Phi]\!] = \mathsf{App} \circ \langle [\![\Phi_1]\!], [\![\Phi_2]\!] \rangle$;

$\wedge I$**:** Suppose the proof Φ is given by

$$\frac{\Phi_1 : \Gamma \vdash \phi \qquad \Phi_2 : \Gamma \vdash \psi}{\Phi : \Gamma \vdash \phi \wedge \psi} \wedge I.$$

Then $[\![\Phi]\!] = \langle [\![\Phi_1]\!], [\![\Phi_2]\!] \rangle$;

$\wedge E$**:** Suppose the proof Φ is given by

$$\frac{\Psi : \Gamma \vdash \phi \wedge \psi}{\Phi : \Gamma \vdash \phi} \wedge E$$

and suppose that $f = [\![\Psi]\!]$. Then $[\![\Phi]\!] = \pi$, where π is the projection from $[\![\phi]\!] \times [\![\psi]\!]$ to $[\![\phi]\!]$. The semantic of the other case of $\wedge$-elimination is similar;

$\vee I$**:** Suppose the proof Φ is given by

$$\frac{\Psi : \Gamma \vdash \phi}{\Phi : \Gamma \vdash \phi \vee \psi}$$

and suppose that $f = [\![\Psi]\!]$. Then $[\![\Phi]\!] = \mathsf{inl} \circ \mathsf{f}$. The other case of the $\vee I$-rule is similar;

$\vee E$**:** Suppose the proof Φ is the proof

$$\frac{\Phi_1 : \Gamma \vdash \phi \vee \psi \qquad \Phi_2 : \Gamma, \phi \vdash \chi \qquad \Phi_3 : \Gamma, \psi \vdash \chi}{\Phi : \Gamma \vdash \chi} \vee E.$$

Then $[\![\Phi]\!] = [\![\Phi_3]\!] \circ ([\![\Phi_1]\!] + [\![\Phi_2]\!]) \circ f$, where f is the canonical morphism from $A \times ([\![\phi]\!] + [\![\psi]\!])$ to $(A \times [\![\phi]\!]) + (A \times [\![\psi]\!])$.

The equality in bi-Cartesian closed categories is extensional which, for the interpretation of proofs, means that two extensionally equal proofs Φ and Ψ are also mapped to the same morphism.

Let $\mathcal{C}$ denote a bi-Cartesian closed category. We write $\Gamma \models_{\mathcal{C}} \Phi = \Psi : \phi$ iff

$$[\![\Gamma]\!] \xrightarrow{[\![\Phi]\!]} [\![\phi]\!] = [\![\Gamma]\!] \xrightarrow{[\![\Psi]\!]} [\![\phi]\!]$$

in $\mathcal{C}$. We write $\Gamma \models \Phi = \Psi : \phi$ iff, for all $\mathcal{C}$, $\Gamma \models_{\mathcal{C}} \Phi = \Psi : \phi$.

This semantics is sound; in other words, the mapping $[\![\_]\!]$ maps proofs into the corresponding parts of the categorical structure:

Theorem 3.5 (bi-CCC model soundness) *Consider any bi-Cartesian closed category $\mathcal{C}$ and an interpretation $[\![\_]\!]$ of intuitionistic logic in $\mathcal{C}$. If Φ_1 and Φ_2 are extensionally equal natural deduction proofs with assumptions $\phi_1, \ldots, \phi_n$, then $\Gamma \models_{\mathcal{C}} \Phi = \Psi : \phi$.*

Proof By induction over the structure of proofs. The argument is straightforward. See, for example, [24, 38, 42, 61, 70]. □

We have also completeness: bi-Cartesian closed categories are sufficiently rich to capture those equalities between proofs which hold and those which do not:

Theorem 3.6 (bi-CCC model existence) *Two natural deduction proofs Φ_1 and Φ_2 of ϕ, with assumptions Γ, are extensionally equal iff $\Gamma \models \Phi = \Psi : \phi$.*

We need the following lemma:

Lemma 3.7 (bi-CCC model existence) *There is a bi-Cartesian closed category $\mathcal{T}$ and an interpretation $[\![\_]\!]$ of propositional intuitionistic natural deduction such that if $\Gamma \not\vdash \phi$ in propositional intuitionistic natural deduction, then there is no morphism with domain $[\![\Gamma]\!]$ and codomain $[\![\phi]\!]$.*

Proof The proof works, as usual, by the construction of a term model. In other words, $\mathcal{T}$ is based on formulæ and proofs.

The objects of $\mathcal{C}$ are all propositional formulæ; product, co-product, and function spaces of two objects ϕ and ψ are given by $\phi \wedge \psi$, $\phi + \psi$, and $\phi \supset \psi$, respectively. The terminal and initial object are $\top$ and $\bot$ respectively.

Morphisms from ϕ to ψ are the natural deduction proofs $\Phi : \phi \vdash \psi$ modulo extensional equality. Composition of two morphisms $\Phi : \phi \vdash \psi$ and $\Psi : \psi \vdash \omega$ is given by Cut, and the identity morphism is the axiom $\phi \vdash \phi$. The product of two morphisms Φ and Ψ is given by $(\Phi, \Psi); \wedge R$, the projections are defined by the $\wedge E$-rule. For any morphism $\Phi : \phi \times \psi \to \omega$, the morphism $\mathsf{Cur}(\Phi)$ is the proof $((Ax_\phi, Ax_\psi); \wedge I, \Phi); \mathrm{Cut}; \supset I$ and the application morphism App is defined by the $\supset E$-rule.

We omit the verifications that this indeed defines a bi-Cartesian closed category.

For this term model, we have the following: for any proofs Φ and Ψ, $[\![\Phi]\!]$ and $[\![\Psi]\!]$ are equal morphisms iff Φ and Ψ are extensionally equal proofs. By assumption, $[\![\Phi_1]\!]$ and $[\![\Phi_2]\!]$ are equal in the term model, hence also Φ_1 and Φ_2 are extensionally equal proofs. □

Now we get a proof of completeness:

Proof Suppose we do not have $\Gamma \models \Phi = \Psi : \phi$. Then Lemma 3.7 yields a contradiction. □

3.3 Kripke semantics and functor categories

The semantics for intuitionistic natural deduction proofs given above is quite abstract. In this section we consider (i) a more concrete, truth-functional semantics, (ii) its representation in functor categories, and (iii) that the functor category representation provides a unifying framework for both BHK and Kripke semantics.

The picture we organize in this section, together with the subsequent sections on games and fibred categories, forms the basis for the semantics of reductive logic and proof-search in Chapter 5.

3.3.1 *Kripke semantics*

As we have seen in Chapter 2, intuitionistic natural deduction rejects the law of the excluded middle, and any equivalent classical principle. We have explained that a solution, attributed to Saul Kripke [68, 69], is provided by moving to a semantics based on the idea of *possible worlds*.

This semantics is made precise by the notion of a Kripke model of intuitionistic logic [70, 128].

Definition 3.8 A Kripke structure for propositional intuitionistic logic, over the language $\mathcal{L}$ of proposition letters, is an ordered quadruple

$$\mathcal{S} = \langle W, \sqsubseteq, F, [\![-]\!]^{-} \rangle,$$

where W is a set of worlds, preordered by $\sqsubseteq$; F is a function which assigns to each world $w \in W$ a set $F(w)$ of basic facts at w, such that, for all $w \sqsubseteq v$, $F(w) \subseteq F(v)$; and for each world w and propositional letter p, $[\![p]\!]^{w}$ is the interpretation of p at world w.

A Kripke model for propositional intuitionistic logic, over the language $\mathcal{L}$ of proposition letters, is an ordered pair

$$\mathcal{K} = \langle \mathcal{S}, \models \rangle$$

of a Kripke structure $\mathcal{S}$ and a satisfaction (or forcing) relation, $\models \subseteq W \times \mathcal{P}(\mathcal{L})$, such that the following conditions hold:

$$w \models p \quad \text{iff } [\![p]\!]^{w} \in F(w)$$

$$w \models \top \quad \text{for all } w$$

$$w \models \phi \wedge \psi \quad \text{iff } w \models \phi \text{ and } w \models \psi$$

$$w \models \bot \quad \text{for no } w$$

$$w \models \phi \vee \psi \quad \text{iff } w \models \phi \quad \text{or} \quad w \models \psi$$

$$w \models \phi \supset \psi \quad \text{iff for all } w \sqsubseteq v, w \models \phi \text{ implies } v \models \phi.$$

Further, we require that the Kripke monotonicity (or hereditary) property holds:

$$\text{for all } w \sqsubseteq v, \text{ if } w \models \phi, \text{ then } v \models \phi.$$

We write $w \models \Gamma$ iff $w \models \phi_{\Gamma}$, where the formula ϕ_{Γ} is obtained from Γ by replacing each comma by $\wedge$. We write the following notations:

- $w \models_{\mathcal{K}} \phi$ to emphasize that the satisfaction is within the model $\mathcal{K}$;
- $\Gamma \models_{\mathcal{K}} \phi$ iff, for all worlds w in $\mathcal{K}$, $w \models_{\mathcal{K}} \Gamma$ implies $w \models_{\mathcal{K}} \phi$;
- $\Gamma \models \phi$ iff, for all Kripke models $\mathcal{K}$, $\Gamma \models_{\mathcal{K}} \phi$.

Theorem 3.9 (soundness) *If* $\Gamma \vdash \phi$ *is provable in propositional intuitionistic natural deduction, then* $\Gamma \models \phi$*.*

Proof By induction on the structure of natural deduction proofs of $\Gamma \vdash \phi$. □

Lemma 3.10 (model existence) *There is a Kripke model* $\mathcal{T}$ *such that if* $\Gamma \not\vdash \phi$ *in propositional intuitionistic natural deduction, then there is a world* w *in* $\mathcal{T}$ *such that* $w \models \Gamma$ *and* $w \not\models \phi$*.*

Proof We construct $\mathcal{T}$ as a term model. There are various possible approaches; see, for example [129].

We can construct a preorder of worlds from contexts Γ, ordered by extension. Care must be taken to ensure that all contexts are extended to *prime contexts*, that is, contexts with the property that if $\phi \vee \psi \in \Gamma$, then $\phi \in \Gamma$ or $\psi \in \Gamma$ (and a similar property for existentials in the predicate case; see below). $F(\Gamma)$ is then, essentially, the set of all natural deduction proofs of the form $\Phi : \Gamma \vdash \phi$, for some ϕ. □

Theorem 3.11 (completeness) $\Gamma \models \phi$ *implies* $\Gamma \vdash \phi$*.*

Proof By the contrapositive. Suppose $\Gamma \not\vdash \phi$. Then there is a world w in $\mathcal{T}$ such that $w \models \Gamma$ and $w \not\models \phi$. □

Kripke models can be extended to account for predicate logic too. In addition to the set of basic facts, $F(w)$, at each world, we must take a domain of expressions, $D(w)$, at each world, in which terms are interpreted. We require also the monotonicity condition that, for all $w \sqsubseteq v$, $D(w) \subseteq D(v)$. Predicate symbols of arity m are interpreted at world w as m-ary relations on $D(w)$. We then have the following clauses for atomic predicates, for terms t over the usual language of function symbols and variables:

$$w \models p(t_1, \ldots, t_m) \quad \text{iff } \langle [\![t_1]\!]^w, \ldots, [\![t_m]\!]^w \rangle \in [\![p]\!]^w.$$

Abusing notation slightly, the quantifiers then follow the pattern for disjunction and implication, respectively, as follows:

$$\begin{aligned} w \models \exists x.\phi &\quad \text{iff for some } [\![t]\!]^w \in D(w),\ w \models \phi(t) \\ w \models \forall x.\phi &\quad \text{iff for all } w \sqsubseteq v \text{ and all } [\![t]\!]^v \in D(v),\ v \models \phi(t). \end{aligned}$$

The soundness and completeness theorem extend to intuitionistic predicate logic quite straightforwardly [82, 128].

A set of formulæ, Γ, has the *disjunction property* (DP) if $\Gamma \vdash \phi \vee \psi$ implies $\Gamma \vdash \phi$ or $\Gamma \vdash \psi$. In predicate logic, Γ has the *existence property* (EP) if $\Gamma \vdash \exists x.\phi$ implies $\Gamma \vdash \phi[t/x]$, for some closed term t. Whilst classical logic has neither the DP nor the EP, Kripke models can be used to establish the DP and EP for intuitionistic logic as follows [129]:

1. Intuitionistic propositional logic and predicate logic without predicate symbols have the disjunction property as follows: if $\vdash \phi \vee \psi$, then $\vdash \phi$ or $\vdash \psi$;
2. Suppose the language of intuitionistic predicate logic contains at least one constant and no function symbols, then the existence property holds as follows: if $\vdash \exists x.\phi$, then $\vdash \phi[t/x]$, for some closed term t.

So we have now seen two semantics for intuitionistic logic:

- BHK semantics of proofs; and
- Kripke truth-functional semantics.

In fact, for propositional intuitionistic logic, a categorical treatment of Kripke semantics, based on *functor categories*, provides a setting within which these two semantic perspectives can be handled simultaneously.

The important point is that we interpret formulæ not as objects in a bi-Cartesian closed category but as functors from a category of worlds $\mathcal{W}$ to **Set**. This is possible because the functor category $[\mathcal{W}, \mathbf{Set}]$ inherits lots of structure from **Set**. In the next subsection we give the details of this interpretation.

3.3.2 *Functor categories*

Given two categories, $\mathcal{C}$ and $\mathcal{D}$, we denote the category of functors from $\mathcal{C}$ to $\mathcal{D}$ by $[\mathcal{C}, \mathcal{D}]$.[28]

Theorem 3.12 *For any category $\mathcal{W}$, the functor category $[\mathcal{W}, \mathbf{Set}]$ is bi-Cartesian closed.*

Proof Let F and G be two functors from $\mathcal{W}$ to **Set**. Then the product of these functors is given by $(F \times G)(A) = FA \times GA$, the terminal object is given by the functor assigning the one-element set to every object of $\mathcal{W}$ and the identity morphism to any morphism in $\mathcal{W}$, the co-product is given by $(F + G)(A) = FA + GA$, the initial object is given by the functor assigning the empty set to every object of $\mathcal{W}$ and the identity morphism to any morphism in $\mathcal{W}$. The function space $(F \Rightarrow G)$ is the functor which assigns to each object A of $\mathcal{W}$ the set $\mathsf{Nat}(\hom(-, A) \times F, G)$, and to the morphism $f : C \to B$ the map which maps the natural transformation η into the natural transformation ι with components $\iota_A : (g, a) \mapsto \eta_A(f \circ g, a)$. $\square$

The special case of a Kripke model arises if the category of worlds is a preorder. If we write W, V, etc., for the objects of $\mathcal{W}$ in this case, an intuitionistic formula ϕ is interpreted as a functor from $\mathcal{W}$ to **Set**, which can be thought of as mapping each world W to the set of formulæ ϕ such that $W \models \phi$. The definition of the function space in the functor category captures the Kripke semantics of implication: a formula $\phi \supset \psi$ is forced at a world W iff, for all worlds V which are

[28] We sometimes write $\mathcal{D}^{\mathcal{C}}$.

extensions of W at which ϕ is forced (captured by a morphism from W to V), ψ is also forced.

In this setting, we can interpret a formula ϕ as an object of $[\mathcal{W}, \mathbf{Set}]$,

$$[\![\phi]\!] : \mathcal{W} \to \mathbf{Set}.$$

Then a morphism from $[\![\phi]\!]$ to $[\![\psi]\!]$, which interprets a proof $\Phi\colon \phi \vdash \psi$, is a natural transformation parametrized by worlds. The theory of 'Kripke λ-models' has been described quite elegantly by Mitchell and Moggi [83].

We conclude by remarking that functor categories are not an adequate setting for combining the BHK and truth-functional semantics of intuitionistic *predicate* logic: for that we require fibred (or indexed) categories (*cf.* Section 3.5).

3.4 Games

We conclude our basic discussion of the semantics of intuitionistic proofs with a games model. Games models have been used successfully as models for various computational effects. We present here a version which will turn out to be suitable basis for games models of both reductive intuitionistic logic and proof-search.

We consider games played between two players, Proponent, P, and Opponent, O. In such games for a formula ϕ the aim of Opponent is to falsify the given formula ϕ, and the aim of Proponent is to prove it. A game starts by Opponent challenging the given formula. Proponent wins a game when he can answer Opponent's initial challenge, otherwise he loses. The possible moves of both players in a game for ϕ are determined by the structure of ϕ. A proof of a formula corresponds to a *winning strategy* for Proponent. Such a winning strategy for a formula ϕ is a function, which for every legal O-move in a game for ϕ produces a legal P-move such that if P uses this strategy to determine his moves he wins every game for ϕ. Such games for proofs have been described for a variety of logics, including classical and intuitionistic logic [28, 72]. Usually, in games for classical logic Proponent and Opponent are dual to each other, whereas this is not true for games for intuitionistic logic.

These game models for proofs have been adapted to give models of sequential computations in programming languages [3, 4, 32, 64]. Here, the intuition is that Opponent asks for the value of a computation, and Proponent performs the computation to produce values as answers. In such games there is usually a strict alternation between moves by Proponent and Opponent, corresponding to the absence of concurrent computation. As computations have a clear direction (from inputs to outputs) there is usually no duality between Proponent and Opponent in these games.

The key conceptual difference between the games for proofs and the games for computations is that in logic not all propositions are provable, so that in these games not all propositions have strategies, whereas in the programming languages considered, however, all types are inhabited, so that these games have strategies for every type.

The details of how to present game models differ widely, both within games for proofs and within games for computations. The definition of the games considered in this monograph uses elements of both approaches. We use one important technical notion from the games introduced by Hyland and Ong, namely the notion of an arena: for each formula ϕ the possible moves for a game for ϕ are listed in a forest[29] called an arena, and the rules of the game use this forest extensively. Ong [89] introduces also the notion of a *scratchpad* to model the multiple conclusions in the $\lambda\mu$-calculus. Scratchpads are additional games, which Proponent may start at will. For a detailed explanation of these scratchpads, see Chapter 6.

This idea of games semantics in the context of proof theory was introduced by Lorenzen [28, 72]. For games semantics as a semantics of programming languages see [3, 4, 32, 64]. A comprehensive summary is provided in [63]. The use of game-theoretic methods in model theory, however, has a rather longer history, beginning with Ehrenfeucht–Fraïssé games, in which the *back and forth equivalence* of models is used to analyze completeness properties of (first-order) theories [52].

Hyland [63] provides a useful general comparison, in terms of categorical composition, of the correspondences between λ-calculus, proofs, algorithms, and strategies:

Object	Map	Composition
Type	Proof	Application in context
Proposition	Proof	Composition via the Cut rule
Type	Algorithm	Composition with hiding
Game	Strategy	Scratchpad composition

This organization captures the main themes of this monograph, all of which are expressed within the structures of categorical logic:

- The propositions-as-types (Curry-Howard-de Bruin) correspondence;
- The programs-as-proofs correspondence; and
- Games as a semantics for both proofs and computations.

3.4.1 *Games for intuitionistic proofs*

We introduce a class of games which combines ideas from those for intuitionistic provability and those for programming languages to give a class which models intuitionistic proofs *directly*.[30] Moreover, our games extend cleanly not only to the semantics of classical proofs provided by models of the $\lambda\mu\nu$-calculus, described in Section 3.6 but also to the structures required to interpret reductive logic and proof-search.

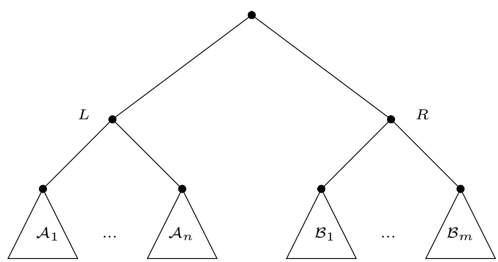
[29] A *forest* is a set of trees.

[30] Games models of intuitionistic proof can be recovered from games models of linear proofs [4] by the exponential ! and, for example, Girard's translation of intuitionistic logic into linear logic.

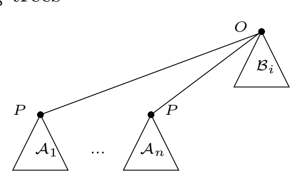
We start the definition of our games semantics by defining arenas. For each formula ϕ, we define an arena, which is a forest, used to characterize legal moves by both players in our games.

Definition 3.13 An arena of type ϕ is a forest with nodes having possibly labels defined inductively by the following:

1. The arena of $\top$ is the empty forest;
2. The arena of $\bot$ is the forest with one node labelled $\bot$;
3. The arena for a propositional atom p is a forest with one node labelled p;
4. The arena for $\phi \wedge \psi$ is the disjoint sum of the arenas for ϕ and ψ;
5. Suppose $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are the trees of the arena for ϕ and $\mathcal{B}_1, \ldots, \mathcal{B}_m$ are the trees of the arena for ψ. Then the arena for $\phi \vee \psi$ is given by



   Note that there are two special nodes called L and R. In the special case that the arena for ϕ or the arena for ψ is empty, the arena for $\phi \vee \psi$ is the empty arena too. The root node of the arena for $\phi \vee \psi$ is labelled $\vee$.
6. Suppose $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are the trees of the arena for ϕ and $\mathcal{B}_1, \ldots, \mathcal{B}_m$ are the trees of the arena for ψ. Then the arena for $\phi \supset \psi$ is the disjoint union of the following trees



   In the special case that the arena for ϕ is empty, the arena for $\phi \supset \psi$ is the arena for ψ. All nodes in the arena for $\phi \supset \psi$ which are root nodes in the arena of ψ are labelled $\supset$ in addition to any other label they might have.

   We call all root nodes in an arena O-nodes, and all children of O-nodes, P-nodes, and all children of P-nodes, O-nodes.

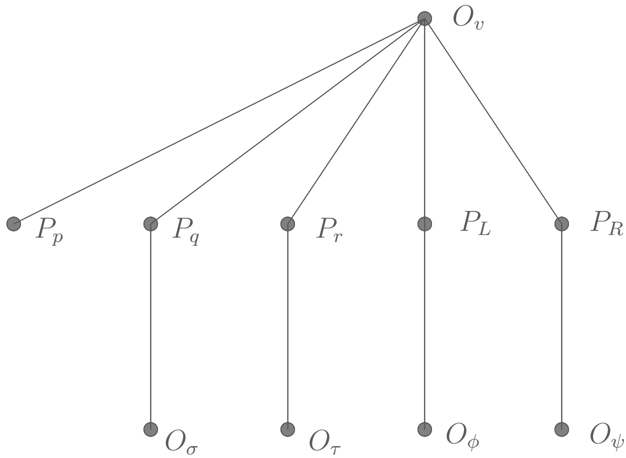


FIG. 3.1. Arena for $p \supset (p \supset q) \supset (q \supset r) \supset (r \vee s)$

Arenas are used to define possible plays. The definition of moves and plays makes this precise.

We illustrate games for intuitionistic proofs using the formula

$$p \supset (p \supset q) \supset (q \supset r) \supset (r \vee s).$$

The arena for this formula is given in Fig. 3.1. Note that we have also labelled all O-nodes with O and all P-nodes with P.

Next, we define possible moves in our games. Each move for a game for ϕ is associated with a node in the arena for ϕ.

There are several types of moves. Firstly, we have moves by Proponent and Opponent, and secondly there are question and answer moves. Questions which correspond to O-(P-)nodes are played by Opponent (Proponent), and answers which correspond to O-(P-)nodes are played by Proponent (Opponent). The definition is as follows:

Definition 3.14 A move m for an arena $\mathcal{A}$ is a node which is classified as either question or answer. Questions which correspond to O-(P-)nodes are moves by Opponent (Proponent), and answers which correspond to O-(P-)nodes are moves by Proponent (Opponent). We call a move by Proponent a P-move and a move by Opponent an O-move.

Next, we define plays, which are instances of the game. Each play consists of a sequence of moves satisfying certain conditions. The intuition is that Opponent starts the play by challenging Proponent to verify the given formula. Proponent responds by asking the Opponent to justify the assumptions which Proponent can make in a natural deduction proof of ϕ. Conjunctive choices are made by Opponent, and disjunctive choices by Proponent. Proponent wins a particular game if he can answer Opponent's initial question.

The moves in a play for ϕ follow the structure of arena of ϕ closely: An O-(P-) question can be played only if there was already a P-(O-) question corresponding to the parent node. An answer can only be given if a question with the same associated node has already been made.

The precise conditions for a play are as follows:

Definition 3.15 A play for an arena $\mathcal{A}$ is a sequence of moves $m_1, \ldots, m_n$ such that:

1. There exists an index $I \geq 1$ such that all moves $m_1, \ldots, m_I$ are O-questions with position $1, \ldots, I$, respectively, and the corresponding nodes are roots in the forest for $\mathcal{A}$. These moves are called initial questions;
2. For each question m_i, with $i > I$, there exists a question m_k, with $k < i$, such that the node corresponding to m_k is the immediate predecessor of the node corresponding to m_i in the arena $\mathcal{A}$. We call m_k the justifying question for m_i;
3. For each answer m_i, with $i > I$, there exists a question m_k, with $k < i$, such that m_k and m_i are the same node in $\mathcal{A}$. If m_j is the justifying question for m_k, we call m_j the justifying question for m_i;
4. Each question can be answered at most once;
5. Any initial questions can only be answered if all non-initial questions have already been answered;
6. For any P-answer m_i there exists a move m_j such that m_j is an O-answer with the same label or $\bot$ and $j < i$ and that the nodes corresponding to m_i and m_j in the arena are on a path which does not contain a P-node n labelled $\supset$ such that the nodes corresponding to m_i and m_j are its children or identical to it;
7. If m is an O-question labelled $\vee$, then at most one P-question is justified by m.

Condition 6 of this definition merits an explanation. During plays we have to ensure that Proponent can answer questions of Opponent only if this answer corresponds to an assumption which Opponent has provided. This matters in the case of Proponent asking a question labelled $\supset$, which corresponds to using an assumption of type $\phi \supset \psi$. The rules of the game work in such a way that in this case two proofs are constructed: one of the original formula using ψ as an additional assumption, and the second one of ϕ. Now we need to ensure that ψ is not available as an assumption during the proof of ϕ. Condition 6 ensures this by making sure that any O-answer for ϕ cannot be used by Proponent.

Conditions 7 and 6 ensure that these games capture intuitionistic proofs: condition 7 enforces the disjunction property of intuitionistic logic, and condition 6 makes sure that only one specific formula can be proved at any one given time.

A possible play for the arena for

$$p \supset (p \supset q) \supset (q \supset r) \supset (r \vee s)$$

starts by Opponent asking the initial question. Here, this means that Opponent is asking for a proof of the formula. Now Proponent has various choices: he can either ask questions labelled L or R, thereby deciding whether to prove r or s respectively, or to ask Opponent for evidence for the assumptions by asking any other question. Let us assume that Proponent asks the question corresponding to the node labelled L. Now Opponent will ask the question labelled r, thereby asking Proponent to prove r. Proponent now needs to use the assumptions. Let us assume that Proponent asks the question labelled r, thereby challenging Opponent to provide evidence for the assumption $q \supset r$. Next, Opponent asks the question labelled q and challenges Proponent to prove the formula r in turn, which is the hypothesis in the implication $q \supset r$. Proponent now asks in a similar way the question labelled q, and Opponent asks the question p. Proponent now asks for the final assumption p. Opponent now has no choice but to answer this question, thereby making it possible for Proponent to answer outstanding questions by Opponent. Now Proponent can use this answer and answer Opponent's question p. Again, Opponent is now forced to answer the question q. This process of answering previously asked questions goes on until finally Opponent is forced to answer the question labelled L, and Proponent can answer the initial question. In this example the condition on paths in clause 6 is not relevant.

The key notion of games semantics is that of a *strategy*. A strategy describes how Proponent responds to arbitrary Opponent moves. Intuitively, a strategy describes how Proponent answers challenges from Opponent to prove the given formula.

Definition 3.16 A strategy is a function from plays $m_1, \ldots, m_k$, where m_k is an O-move, to a sequence of moves $m_{k+1}, \ldots, m_n$ such that $m_1, \ldots, m_k$, $m_{k+1}, \ldots, m_n$ is a play, and the sequence $m_{k+1}, \ldots, m_n$ is non-empty if the sequence $m_1, \ldots, m_k$ contains no unanswered P-move which could be answered by Opponent in the next move according to Definition 3.15.

Note that this definition makes it possible to force Opponent to answer any unanswered questions by Proponent if such a move was allowed by choosing the empty sequence as a result of the function for sequences with unanswered questions by Proponent.

Intuitively, O- and P-questions are challenges for Opponent and Proponent to provide evidence for conclusions and assumptions, respectively. O-answers provide evidence for an assumption, and P-answers provide evidence for a conclusion.

In the example, a strategy for Proponent would be to answer the initial question by asking the question labelled L and then play as indicated above in response to any Opponent move. Note that the choice of asking the question labelled R will not lead to a winning play: Proponent will be unable to answer Opponent's question s.

Next we show that each strategy for the arena corresponding to a formula ϕ gives rise to a natural deduction proof of ϕ. Note that several strategies give rise to the same proof: games make significantly finer distinctions than natural deduction proofs.

Theorem 3.17 *For any formula ϕ and strategy Φ for ϕ there exists a natural deduction proof of ϕ.*

We have to show a stronger version of this theorem, namely:

Lemma 3.18 *Given any set A of O-answers with labels $p_1, \ldots, p_n$ and a strategy for a formula ϕ where Proponent can answer in addition any O-question with label $p_1, \ldots, p_n$ there is a natural deduction proof of*

$$p_1 \wedge \ldots \wedge p_n \supset \phi.$$

We call such a strategy an A-strategy.

Proof By induction over the structure of ϕ. Let Γ be the formula $p_1 \wedge \cdots \wedge p_n$.

Atom p: All possible strategies start with a p-question by Opponent. If p is amongst the labels $p_1, \ldots, p_n$ of A, then the axiom $p \vdash p$ followed by a $\supset$-introduction rule provides the desired derivation. There is no other strategy for such an arena;

$\phi \wedge \psi$: Because every question and answer of a strategy for ϕ and ψ has to be justified eventually by an initial move for ϕ and ψ it is possible to obtain one strategy for ϕ and one strategy for ψ from the given strategy. Hence by the induction hypothesis we obtain natural deduction proofs of $\Gamma \supset \phi$ and $\Gamma \supset \psi$. Hence one obtains also an natural deduction proof of $\Gamma \supset (\phi \wedge \psi)$;

$\phi \supset \psi$: There are several subcases.

Firstly, suppose $\phi = \phi_1 \wedge \phi_2$. Then $(\phi_1 \wedge \phi_2) \supset \psi$ is equivalent to $\phi_1 \supset \phi_2 \supset \psi$, and the arenas for

$$(\phi_1 \wedge \phi_2) \supset \psi \quad \text{and} \quad \phi_1 \supset \phi_2 \supset \psi$$

are identical. Hence we consider the case $\phi_1 \supset \phi_2 \supset \psi$ instead.

Secondly, suppose $\phi = \sigma \vee \tau$. Now define two A-strategies Φ_1 and Φ_2 for $\sigma \supset \psi$ and $\tau \supset \psi$, respectively, where the moves of both players in Φ_1 and Φ_2 are the moves of Φ which are justified by moves not hereditarily justified by τ or σ, respectively. By considering an O-strategy which does not ask the nodes marked L or R corresponding to the disjunction in $\sigma \vee \tau$, one can show that the A-strategies Φ_1 and Φ_2 are well-defined. By the induction hypothesis, we obtain natural deduction proofs of $\Gamma \supset (\sigma \supset \psi)$ and $\Gamma \supset (\tau \supset \psi)$. Hence there is also a natural deduction proof of $\Gamma \wedge (\sigma \vee \tau) \supset \psi$.

Thirdly, suppose $\phi = \sigma \supset \tau$. Again, define A-strategies Φ_1 for $\tau \supset \psi$ and Φ_2 for σ where the moves of both players are the ones not hereditarily justified by σ or τ, respectively. Clause 6 of Definition 3.15 ensure that these A-strategies Φ_1 and Φ_2 are well-defined. By the induction hypothesis we obtain natural deduction proofs of $\Gamma \supset \tau \supset \psi$ and $\Gamma \supset \sigma$. Hence there is also a natural deduction proof of $\Gamma \supset (\sigma \supset \tau) \supset \psi$.

Finally, suppose that ϕ is an atom p. Again, there are two cases. Consider an A-strategy for $p \supset \psi$ without a P-question corresponding to p. In this case, the A-strategy for $p \supset \psi$ is in fact a strategy for ψ, and by the induction hypothesis there is an ND proof of $\Gamma \supset \psi$, hence also a proof of $\Gamma \supset p \supset \psi$. Now suppose there is a P-question corresponding to p. Clause 6 of Definition 3.15 ensures that the strategy which removes the P-question and O-answer for p is an $A \cup \{p\}$-strategy of ψ. By the induction hypothesis, there is a natural deduction proof of $(\Gamma \wedge p) \supset \psi$.

$\phi \vee \psi$: By Clause 7 of Definition 3.15, an A-strategy for $\phi \vee \psi$ gives rise either to an A-strategy for ϕ or an A-strategy for ψ, depending whether Proponent plays the L- or the R-node. By the induction hypothesis there is either a natural deduction proof of $\Gamma \vdash \phi$ or of $\Gamma \vdash \psi$. Hence in both cases there is also a natural deduction proof of $\Gamma \vdash \phi \vee \psi$. $\square$

3.5 Fibred categories

In this section, we introduce fibred categories to the extent needed in this monograph. For a detailed exposition see [12, 60].

Fibred (or indexed) categories model families of mathematical structures which are indexed by elements of some index set. The intuition is that the index set is modelled by a base category $\mathcal{B}$ and for each object of $\mathcal{B}$ there is a category modelling the structure indexed by this object. Morphisms in the base category model functions between indices, and hence for each such a morphism a fibration (defined formally below) provides a so-called reindexing functor between the categories associated to domain and co-domain of this function. We will consider only structures where the index set consists of certain elements of the mathematical structure.

We give here both the definition of indexed categories and fibrations. These concepts are equivalent but the precise formulation of the equivalence is rather complicated. For details see [14, 60]. We start by defining indexed categories, in the particular way we need them in this monograph.

Definition 3.19 A strict indexed category with terminal objects is a functor $\mathcal{E}\colon \mathcal{B}^{op} \to \mathbf{Cat}$ such that the following conditions are satisfied:

1. $\mathcal{B}$ has a terminal object called $\top$;
2. Each fibre $\mathcal{E}(\Gamma)$ has a terminal object 1 which is stable under applying the functor $\mathcal{E}(f)$ for any morphism f in the base category. We often write f^* for the functor $\mathcal{E}(f)$.

As an example, consider functions with parameters. We define the base category $\mathcal{B}$ to be **Set**, and the functor $\mathcal{E}$ by letting $\mathcal{E}(X)$ be the category which has as objects sets again and as morphisms between objects Y and Z functions from $X \times Y$ to Z. The identity morphism is the projection from $X \times Y$ to Y, and the composition $g \circ f$ of two morphisms $f : Y_1 \to Y_2$ and $g : Y_2 \to Y_3$ is the function $g \circ \langle \mathsf{Id}, f\rangle$. For a morphism f between X_1 and X_2, the functor $\mathcal{E}(f)$ is defined as the identity on objects and as mapping the function g to the function $g \circ \langle f, \mathsf{Id}\rangle$, in other words replacing a parametrization over X_2 with a parametrization over X_1 via pre-composing with f.

To capture extensions of parameters in indexed categories, we need additional structure which makes it possible to construct morphisms in the base category out of morphisms in the fibre, thereby modelling the addition of a new, locally defined, parameter.

Definition 3.20 Let $\mathcal{E} : \mathcal{B}^{op} \to \mathbf{Cat}$ be a strict indexed category with terminal objects.

1. Define the Grothendieck completion $Gr(\mathcal{E})$ to be the category whose objects are pairs (Γ, A), where Γ is an object of $\mathcal{B}$ and A an object of $\mathcal{E}(\Gamma)$, and morphisms from (Γ, A) to (Δ, B) are pairs of morphisms (f, g) where f is a morphism from Γ to Δ and g is a morphism from A to $\mathcal{E}(f)(B)$.
2. A strict indexed category with terminal objects is called a strict indexed category with comprehension if the functor $I\colon \mathcal{B} \to Gr(\mathcal{E})$ sending the object Γ to $(\Gamma, 1)$ and the morphism f to $(f, 1)$ has a right adjoint G.

In the above example of functions with parameters, comprehension models extension of parameters: the object $X \cdot Y$ is the product $X \times Y$, modelling the

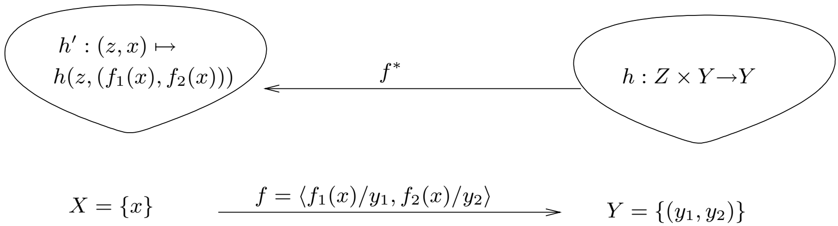


FIG. 3.2. Parametrized sets as an indexed category

additional parameter Y, and the universal property of comprehension is given by the universal property of products, which says a substitution for the parameters in $X \times Y$ is given by a substitution for the parameters in X and a substitution for the parameters in Y. See Fig. 3.2 for an illustration.

Next we present an alternative definition of fibred categories. The definition of a fibration takes the Grothendieck completion of an indexed category as a primitive and axiomatizes its properties. To understand the following definition of a fibration (which is a functor $p : \mathcal{E} \to \mathcal{B}$ with special properties), the reader is encouraged to think of $Gr(\mathcal{E})$ as the category $\mathcal{E}$ in the definition of a fibration:

Definition 3.21 Let $p\colon \mathcal{E} \to \mathcal{B}$ be a functor.

1. A morphism $f : X \to Y$ in $\mathcal{B}$ is called Cartesian over the morphism $u : I \to J$ in $\mathcal{E}$ if $pf = u$ and for every morphism $g : Z \to Y$ in $\mathcal{E}$ such that $pg = u \circ w$ for some $w : pZ \to I$, there exists a unique $h : Z \to X$ in $\mathcal{B}$ such that $pw = h$ and $f \circ h = g$.
2. The functor $p : \mathcal{E} \to \mathcal{B}$ is called a fibration if for every object Y of $\mathcal{E}$ and morphism $u : I \to pY$ in $\mathcal{B}$ there exists a Cartesian morphism $f : X \to Y$ in $\mathcal{E}$ above u.

For a fibration $p : \mathcal{E} \to \mathcal{B}$ and object I of $\mathcal{B}$ define the fibre $\mathcal{E}_I$ as the category whose objects are all objects X such that $pX = I$ and whose morphisms from X to Y are all morphisms h in $\mathcal{E}$ such that $ph = \mathsf{Id}$.

For each indexed category $\mathcal{E} : \mathcal{B}^{op} \to \mathbf{Cat}$ one can obtain a fibration $p : \mathcal{F} \to \mathcal{B}$ by letting $\mathcal{F}$ to be $Gr(\mathcal{E})$, p to be the projection functor mapping (Γ, A) to Γ and a morphism (f, g) to the morphism f. A Cartesian morphism over a morphism f in $\mathcal{B}$ is the morphism (f, Id).

In fact, the fibration arising from a strict indexed category is a special fibration where the mapping from a morphism f in the base category to the Cartesian morphism (f, id) is functorial in a very strong sense. Such a fibration is called a *split* fibration. To state the definition of a split fibration, suppose there exists a choice of Cartesian morphism $\overline{u}(X)$ for each morphism $u : I \to J$ and object X such that $pX = J$. Then define a functor $u^* : \mathcal{E}_J \to \mathcal{E}_I$ by letting $u^*(X)$ to be the domain of the map $\overline{u}(X)$, and for a morphism $f : X \to Y$ by defining $u^*(f)$ as the unique map g such that $\overline{u}(Y) \circ g = f \circ \overline{u}(X)$.

Definition 3.22 A split fibration is a fibration $p : \mathcal{E} \to \mathcal{B}$ such that there exists a choice of Cartesian morphism $\overline{u}(X)$ for which the canonical natural transformations $\mathsf{Id} \Rightarrow \mathsf{Id}^*$ and $u^*v^* \Rightarrow (u \circ v)^*$ are identities.

The importance of split fibrations and strict indexed categories for our purposes is that the functors $\mathcal{E}(f)$, for an indexed category, and u^*, for a fibration, model substitution. The conditions for a strict indexed category and a split fibration state that substitutions compose, and hence iterated substitutions can be modelled by the composition of these substitution functors. Without the additional conditions one requires coherence conditions to be able to model iterated substitutions.

A *fibration with comprehension* is defined in the same way as an indexed category with comprehension:

Definition 3.23 A fibration $p : \mathcal{E} \to \mathcal{B}$ with a terminal object functor $1 : \mathcal{B} \to \mathcal{E}$ has comprehension if the functor 1 has a right adjoint.

As another example of indexed categories, consider the so-called hyperdoctrines, which are models for intuitionistic predicate logic [71]. This indexed category is based on the idea that models for propositional intuitionistic logic are parametrized by the free variables of a formulæ. The precise definition is as follows:

Definition 3.24 A hyperdoctrine is an indexed category $\mathcal{E} : \mathcal{B}^{op} \to \mathbf{Cat}$ such that the following conditions are satisfied:

1. $\mathcal{B}$ is a Cartesian category;
2. Each fibre $\mathcal{E}(D)$ is a bi-Cartesian closed category and each functor f^* preserves the bi-Cartesian closed structure on the nose;
3. Each Weakening functor $\mathsf{Fst}^* : \mathcal{E}(D \times E) \to \mathcal{E}(D)$ has a right adjoint Π;

4. The following Beck-Chevalley condition is satisfied: For every morphism $f\colon D \to E$, and objects A in $\mathcal{E}(E)$ and B in $\mathcal{E}(E \times A)$, the canonical morphism between $f^*(\Pi(A.B))$ and $\Pi(f^*A, (f \cdot \mathsf{Id})^*B)$ is an isomorphism.
5. Each Weakening functor $\mathsf{Fst}^* : \mathcal{E}(D \times E) \to \mathcal{E}(D)$ has a left adjoint Σ.
6. The following Beck-Chevalley condition is satisfied: For every morphism $f\colon D \to E$, and objects A in $\mathcal{E}(E)$ and B in $\mathcal{E}(E \times A)$, the canonical morphism between $\Sigma(f^*A, (f \cdot \mathsf{Id})^*B)$ and $f^*(\Sigma(A.B))$ is an isomorphism.

First-order multi-sorted intuitionistic logic is interpreted in a hyperdoctrine. The base category $\mathcal{B}$ interprets the sorts and functions between sorts; in particular there is an object D of $\mathcal{B}$ for each sort. Each formula ϕ with free variables $x_1, \dots, x_n$ of sort $A_1, \dots, A_n$ respectively is interpreted as an object in $\mathcal{E}(D_1 \times \cdots \times D_n)$, where D_i is the object in $\mathcal{B}$ interpreting the sort A_i. The logical operators except quantifiers are interpreted in the fibres using their bi-Cartesian closed structure. The adjunction defining Π models universal quantifiers: the formula $\forall x : A.\phi$ is interpreted as $\Pi(A, B)$, where B is the interpretation of ϕ. The universal property of the adjunction says that a proof of $\forall x : A.\phi$ is equivalent to a proof of ϕ with an additional parameter x which does not occur in any of the hypotheses. The formula $\exists x : A.\phi$ is similarly interpreted by $\Sigma(A, B)$, where B is the interpretation of ϕ. The universal property of the adjunction captures the fact that using an assumption $\exists x : \phi$ to prove a formula ψ which does not contain x free is equivalent to proving ψ using the assumption ϕ. The Beck-Chevalley conditions for Π and Σ ensure that substitution in formulæ with quantifiers is modelled correctly, that is, that applying a substitution for the free variables of a quantified formula is the same as quantifying the formula to which that substitution has been applied.

3.6 The semantics of classical proofs

In this section, we present the semantics of classical proofs. As classical logic can be seen as intuitionistic logic with either the double-negation elimination rule or the law of the excluded middle added, a semantics for intuitionistic logic with additional clauses for double-negation or excluded middle should give rise to a semantics for classical logic. We will follow this idea as much as possible in this section, with one important exception: The naïve addition of a clause for double-negation to bi-Cartesian closed categories yields a category which has at most one morphism between any two objects; in other words, the semantics becomes proof-irrelevant.

Instead, we present here a semantics which is closely linked to the $\lambda\mu\nu$-calculus: the embedding of the λ-calculus (which provides realizers for intuitionistic proofs) into the $\lambda\mu\nu$-calculus is modelled by a fibration where each fibre is a Cartesian closed category modelling proofs with a given right-hand side and the base category models the change of the right-hand side.

3.6.1 *Boolean algebras*

Boolean algebras arose, historically, as algebras describing the manipulation of truth of logical statements. In the context of this monograph, they are perhaps best seen as special cases of Heyting algebras:

Definition 3.25 A Boolean algebra is a Heyting algebra such that, for all elements x, $\neg\neg x = x$.

The usual first example of a Boolean algebra is the algebra of truth values. The elements 0 and 1 are $\bot$ and $\top$, respectively, and the operations $\wedge$, $\vee$, and $\supset$ are logical conjunction, disjunction, and implication, respectively. $\neg$ is logical negation, and the extra condition $\neg\neg 0 = 1$ says that $\bot$ and $\top$ are logical complements.

Not all examples of Heyting algebras mentioned in Section 3.2.1 are also examples of Boolean algebras. The powerset of a set A is a Boolean algebra, as the complement of the empty set is the whole set and vice versa. The algebra of open sets of a topological space $\mathcal{X}$ is not necessarily a Boolean algebra as $\neg\neg A$, for an open set A, is always included in A but not necessarily equal to A. The formulæ of intuitionistic logic modulo provable equivalence do not provide an example of a Boolean algebra, as $\neg\neg\phi$ is not necessarily equivalent to ϕ. This is true in classical logic, and indeed formulæ of classical logic modulo provable equivalence provide an example of a Boolean algebra.

Boolean algebras also satisfy some other well-known identities from classical logic:

Lemma 3.26 *In any Boolean Algebra* $(A, 0, 1, \vee, \wedge, \leq)$ *the following equations hold:*

$$\begin{aligned} x \vee \neg x &= 1 \\ \neg(x \vee y) &= \neg x \wedge \neg y \\ \neg(x \wedge y) &= \neg x \vee \neg y \\ x \supset y &= \neg x \vee y. \end{aligned}$$

A consequence of this lemma is that in a Boolean algebra once negation and either disjunction or conjunction are defined, all other connectives can be uniquely defined. The second and third equalities are called the De Morgan laws.

3.6.2 *Models of classical proofs*

As we have seen, Boolean algebras are proof-irrelevant. Here we describe the rather more complex structures which are models for classical proofs. The naïve first attempt is to generalize the bi-Cartesian closed categories described in the previous section. However, this attempt fails even if we simply add the expected condition that $\neg\neg A$ is isomorphic to A:

Proposition 3.27 *Any bi-Cartesian closed category such that $\neg\neg A$ is isomorphic to A for any object A is a pre-order* [*70*].

Proof We start by showing that $\hom(A, 0)$ is either a singleton set or empty. Let ι_A be the unique morphism from 0 to any object A. In any bi-Cartesian closed category we have

$$\hom(A \times 0, C) \cong \hom(0, A \Rightarrow C)$$

and hence $\hom(A \times 0, C)$ is a one-element set. Hence the morphism $\iota_{A\times 0} \circ \pi'$, where π' is the projection from $A \times 0$ to 0, is the identity.

Now assume there is a morphism $f : A \to 0$. Then we have

$$\pi \circ \iota_{A\times 0} \circ \pi' \circ \langle \mathsf{Id}, f \rangle = \pi \circ \langle \mathsf{Id}, f \rangle = \mathsf{Id}$$

but also

$$\pi \circ \iota_{A\times 0} \circ \pi' \circ \langle \mathsf{Id}, f \rangle = \iota_A \circ f.$$

As also $f \circ \iota_A = \mathsf{Id}$, $A \cong 0$, and hence $\hom(A, 0)$ is a one-element set.

Now we can show the claim

$$\hom(A, B) \cong \hom(A, (B \Rightarrow 0) \Rightarrow 0) \cong \hom(A \times (B \Rightarrow 0), 0),$$

and as we have just shown, the last set is either empty or a singleton. Hence the category $\mathcal{C}$ is a pre-order. □

In [42], it is stated that there are just two possible solutions: one is to decompose the formulæ and arrive, for example, at linear logic, the other one is to weaken the equational theory and consider fewer equalities between classical proofs. The semantics we will present later follows the latter approach: We consider classical logic as an extension of intuitionistic logic—as a family of intuitionistic consequence relations—in which the equality between classical proofs is an extension of the equality between intuitionistic proofs and does not satisfy certain equalities arising from simplifying classical proofs.

Note, however, that recent work has shown that the view expressed in [42] is mistaken. Recent work of Führmann and Pym [30] has solved the problem of giving non-trivial categorical models of classical proofs without recourse to non-symmetric systems with restricted equality. In our work in this monograph, however, we exploit the asymmetry of the view of classical logic as an extension of intuitionistic logic and so their solution, which we describe very briefly in Section 3.8, is not appropriate for our purposes.

With all this background in mind, we recall the Ong-Ritter models described in [89], beginning with a sketch of the basic idea. We must interpret $\lambda\mu\nu$-sequents, of the form

$$\Gamma \vdash t : \phi, \Delta .$$

Such a sequent represents, as the term t via the propositions-as-types correspondence [90], a proof of the classical sequent $\Gamma \vdash \phi, \Delta$, in which we forget variables and names. Now, sequents $\Gamma \vdash t : \phi$, which represent, via the propositions-as-types correspondence [42], proofs in intuitionistic propositional logic, can be interpreted in a bi-Cartesian closed category [70]. However, it is well-known that any attempt to extend this interpretation to classical sequents by adding an involutive negation must fail because bi-CCCs with involutions collapse to Boolean algebras, thereby causing the interpretation to identify all proofs of a given sequent. The solution adopted in Ong-Ritter models [89] is to use a fibration, as follows:

1. The base $\mathcal{B}$, which is a category with finite products, interprets the named part of the sequent, Δ. Its arrows $f : [\![\Delta]\!] \longrightarrow [\![\Delta']\!]$ interpret compositions of Weakenings, Contractions, and Permutations;
2. The fibre $\mathcal{E}_{[\![\Delta]\!]}$ over each object $[\![\Delta]\!]$ of the base is Cartesian closed. It interprets sequents of the form $\Gamma \vdash \phi$, with side-formulæ Δ;
3. Finally, we must add sufficient structure to interpret the structural operations, including negation. In particular, we must be able to interpret the Exchange rule
$$\frac{\Gamma \vdash t : \psi, \phi^{\alpha}, \Delta}{\Gamma \vdash \mu\alpha.[\beta]t : \phi, \psi^{\beta}, \Delta},$$
described in Section 2.5. The key point here is that we move from the fibre over ϕ^{α}, Δ to the fibre over ψ^{β}, Δ and must have sufficient structure in the fibration, corresponding to the interpretation of μ and $[-]$, to interpret this swap.

It follows that the appropriate categorical definitions of models of $\lambda\mu$, $\lambda\mu\oplus$, and $\lambda\mu\nu$ are as fibrations with universally-defined extra structure corresponding, respectively, to each additional logical connective, $\oplus$ or $\vee$.

Such models, because they are fibrations, require Beck-Chevalley conditions [59, 117] for each connective which is to be interpreted. These conditions interpret the ζ-rules for the corresponding type-constructors, ensuring the interpretation of the connectives is stable with respect to change of base (*cf.* the use of Beck-Chevalley conditions to ensure that substitution is modelled correctly in hyperdoctrines, Definition 3.24). The requisite definitions follow.

Definition 3.28 A $\lambda\mu$-structure is a split fibration $p\colon \mathcal{E} \to \mathcal{B}$ satisfying the following conditions:

1. $p\colon \mathcal{E} \to \mathcal{B}$ is a fibred Cartesian closed category, that is, each fibre is Cartesian closed and re-indexing, that is, applications of functors f^*, preserves products and function spaces on the nose;
2. The fibre $\mathcal{E}_1$ over the terminal object 1 in $\mathcal{B}$ is canonical: that is, for any object D of $\mathcal{B}$, there is a bijection between the objects of $\mathcal{E}_D$ and $\mathcal{E}_1$, with one direction given by re-indexing along the terminal arrow $!_D : D \to 1$, that is, applications of the functor $!_D{}^*$;
3. The base category $\mathcal{B}$ is the free category with finite products generated from the set of objects of the canonical fibre $\mathcal{E}_1$ less a distinguished object $\perp$ and all objects isomorphic to it (note that all arrows in $\mathcal{B}$, a free category with finite products, are compositions of Weakening, Contractions, and Permutations);
4. For each projection
$$w_A : D \times A \to D,$$
in the base, there is an isomorphism
$$\mathcal{E}_D(C, A) \cong \mathcal{E}_{D\times A^\alpha}({w_A}^*(C), \perp),$$
written as $s \overset{[-]}{\mapsto} [\alpha^A]s$ and $\mu\alpha^A \,.\, t \overset{\mu}{\leftarrow\!\shortmid} t$, natural in C and D;
5. For any object A of a category $\mathcal{C}$ with finite products, the flat fibre $\mathcal{C}^A$ is the category whose objects are objects of $\mathcal{C}$ and the morphisms from B to C are morphisms from $B \times A$ to C. The previous conditions imply the existence of a bijection $\zeta\colon \mathcal{E}^{\Gamma}_{\Delta\times A\to B}(C, D) \overset{\sim}{\to} \mathcal{E}^{\Gamma\times A}_{\Delta\times B}(C, D)$. We require the action ζ to be functorial, natural in Γ and Δ, and to make the following diagram commute:
$$\begin{tikzcd}
\mathcal{E}^{\Gamma}_{\Delta\times A\to B}(C, D) \arrow[r, "\zeta_{\Delta;A,B}"] \arrow[d, "{[\delta^D]}"'] & \mathcal{E}^{\Gamma\times A}_{\Delta\times B}(C, D) \arrow[d, "{[\delta^D]}"] \\
\mathcal{E}^{\Gamma}_{\Delta\times A\to B\times D}(C, \perp) \arrow[r, "\zeta_{\Delta\times D;A,B}"'] & \mathcal{E}^{\Gamma\times A}_{\Delta\times B\times D}(C, \perp)
\end{tikzcd}$$
6. A Beck-Chevalley condition holds for $\Rightarrow$: for each contraction map
$$c\colon \Delta \times (A \Rightarrow B) \to \Delta \times (A \Rightarrow B) \times (A \Rightarrow B)$$

in $\mathcal{B}$ we require the following diagram to commute:

$$\begin{tikzcd}
\mathcal{E}^{\Gamma}_{\Delta\times A\Rightarrow B\times A\Rightarrow B} \arrow[r, "\zeta\circ\zeta"] \arrow[d, "c^*_{A\Rightarrow B}"'] & \mathcal{E}^{\Gamma\times A\times A}_{\Delta\times B\times B} \arrow[d, "c^*_B\circ c^*_A"] \\
\mathcal{E}^{\Gamma}_{\Delta\times A\Rightarrow B} \arrow[r, "\zeta"'] & \mathcal{E}^{\Gamma\times A}_{\Delta\times B}
\end{tikzcd}$$

Note that in the composite arrow $c^*_B \cdot c^*_A$, and subsequent similar situations, we overload our notation (as in [89]) by writing c^*_A for re-indexing along the relevant 'contraction map' in the flat fibration over $\mathcal{E}_{\Delta\times B\times B}$;

7. A Beck-Chevalley condition holds for products: for the canonical isomorphism and the contraction functor, namely

$$\phi\colon \mathcal{E}_{\Delta\times(A\times B)} \to \mathcal{E}_{\Delta\times A}\times\mathcal{E}_{\Delta\times B} \quad \text{and} \quad c_A\colon \mathcal{E}_{\Delta\times A\times A} \to \mathcal{E}_{\Delta\times A},$$

the two functors

$$(c^*_A\times\langle\rangle\times\langle\rangle\times c^*_B)\circ(\phi\times\phi)\circ\phi : \mathcal{E}_{\Delta\times A\times B\times A\times B} \to \mathcal{E}_{\Delta\times A}\times\mathcal{E}_{\Delta\times B}$$

and

$$\phi\circ c^*_{A\times B} : \mathcal{E}_{\Delta\times A\times B\times A\times B} \to \mathcal{E}_{\Delta\times A}\times\mathcal{E}_{\Delta\times B}$$

are equal.

Definition 3.29 A $\lambda\mu$-model is a pair $\mathcal{P} = \langle p, [\![-]\!]\rangle$, where $p\colon \mathcal{E}\to\mathcal{B}$ is a $\lambda\mu$-structure and the interpretation $[\![-]\!] : L_{\lambda\mu}\to p$ is a function from the syntax of $\lambda\mu$ (denoted $L_{\lambda\mu}$) to (the components of) p such that $[\![\Delta]\!]$ is an object of $\mathcal{B}$ and $\Gamma\vdash t\colon A,\Delta$ is interpreted as morphism $[\![t]\!]\colon[\![\Gamma]\!]\to[\![A]\!]$ in the fibre over $[\![\Delta]\!]$. The interpretations of variables, pairs, and λ-abstractions are given in the usual way via projections, products, and the exponentials in the fibres, respectively. The terms $\mu\alpha.t$ and $[\alpha]t$ are interpreted by the isomorphism given in Definition 3.28 (4).

We will sometimes write $\mathcal{E}_{\mathcal{P}}(D)$ for the fibre over D in the model $\mathcal{P}$ and $\mathcal{B}_{\mathcal{P}}$ for the base in the model $\mathcal{P}$. Also, we will sometimes write $[\![-]\!]_{\mathcal{P}}$ to denote interpretation in the model $\mathcal{P}$. We extend structures to account for each of the two forms of disjunction in the next two definitions. In each case, the corresponding definition of model requires an interpretation $[\![-]\!]$, extended to $L_{\lambda\mu\oplus}$ and $L_{\lambda\mu\nu}$, respectively, as in Definition 3.29.

Definition 3.30 A $\lambda\mu$-structure is called a $\lambda\mu\oplus$-structure if each fibre has a co-product which is stable under re-indexing, that is, applications of the functor f^*, where f is any morphism of $\mathcal{B}$. Additionally, we require the following Beck-Chevalley condition: the diagram

$$\begin{tikzcd}
\mathcal{E}^{\Gamma\times(A+B)}_{\Delta\times C}(w^*_C E,\bot) \arrow[r, "\mu"] \arrow[d, "\iota_{(A+B)}"'] & \mathcal{E}^{\Gamma\times(A+B)}_{\Delta}(E,C) \arrow[d, "\iota_{(A+B)}"] \\
\mathcal{E}^{\Gamma\times A}_{\Delta\times C}(w^*_C E,\bot)\times\mathcal{E}^{\Gamma\times B}_{\Delta\times C}(w^*_C E,\bot) \arrow[r, "\mu\times\mu"'] & \mathcal{E}^{\Gamma\times A}_{\Delta}(E,C)\times\mathcal{E}^{\Gamma\times B}_{\Delta}(E,C)
\end{tikzcd}$$

commutes, where $\iota_{(A+B)}$ is the defining isomorphism for the co-product in the fibres. The definition of interpretation $[\![-]\!]$ can be adapted to $\lambda\mu\oplus$-structures in order to give $\lambda\mu\oplus$-models as follows: the term constructors `case`, $\mathtt{in}_1$, and $\mathtt{in}_2$ are interpreted by the corresponding co-product constructions.

Given this definition of $\lambda\mu\oplus$-models, we can establish soundness and completeness for $\lambda\mu\oplus$ quite straightforwardly.

Definition 3.31 A $\lambda\mu$-structure is a called a $\lambda\mu\nu$-structure if each Weakening functor $w^*_{\Delta,A}\colon \mathcal{E}_\Delta \vdash \mathcal{E}_{\Delta\times A}$ has a right adjoint. We denote by ν the defining isomorphism

$$\nu : \hom_{\mathcal{E}(\Delta\times B)}(\Gamma,A) \xrightarrow{\sim} \hom_{\mathcal{E}(\Delta)}(\Gamma, A\vee B).$$

We also ask for this adjunction to satisfy a Beck-Chevalley condition, that is, that the diagram

$$\begin{tikzcd}
\mathcal{E}_{\Delta\times A\vee B\times A\vee B} \arrow[r, "\zeta_\vee\circ\zeta_\vee"] \arrow[d, "c^*"'] & \mathcal{E}_{\Delta\times A\times B\times A\times B} \arrow[d, "c^*\circ c^*"] \\
\mathcal{E}_{\Delta\times A\vee B} \arrow[r, "\zeta_\vee"'] & \mathcal{E}_{\Delta\times A\times B}
\end{tikzcd}$$

commutes, where $\zeta_\vee$ is the functor given by assigning each morphism $f\colon C \vdash D$ in $\mathcal{E}_{\Delta\times A\vee B}$ the morphism $\mu\gamma.[\alpha](\nu^{-1}(\mu\beta.[\gamma]f))$. The definition of interpretation $[\![-]\!]$ can be adapted to $\lambda\mu\nu$-structures in order to give $\lambda\mu\nu$-models as follows: the interpretation of terms $\nu\alpha.t$ and $\langle\alpha\rangle t$ uses the defining isomorphism for $\vee$.

3.6.3 *Soundness and completeness for $\lambda\mu\nu$*

We take an explicit definition of satisfaction.

Definition 3.32 (satisfaction) Let $\mathcal{P} = \langle p, [\![-]\!]\rangle$, where $p\colon \mathcal{E} \to \mathcal{B}$, be a $\lambda\mu\nu$-model. Define

$$\mathcal{P}, \Delta \models (t\colon \phi)[\Gamma]$$

if and only if there is an arrow $[\![t]\!] : [\![\Gamma]\!] \vdash [\![\phi]\!]$ in $\mathcal{E}_{[\![\Delta]\!]}$ and satisfaction respects the structure of $t\colon \phi$ (i.e. $\models$ must be consistent with the reduction relation *Red* of Definition 2.10. For example, if $t = uv$, then we must have $\mathcal{P}, \Delta \models (u\colon \psi \supset \phi)[\Gamma]$ and $\mathcal{P}, \Delta \models (v\colon \psi)[\Gamma]$, etc.). If, for every $\lambda\mu\nu$-model $\mathcal{P}$, $\mathcal{P}, \Delta \models (t : \phi)[\Gamma]$, then we write $\Gamma \models t : \phi, \Delta$.

Proposition 3.33 (soundness) *Let $\mathcal{P} = \langle p, [\![-]\!]\rangle$, where $p\colon \mathcal{E} \vdash \mathcal{B}$, be a $\lambda\mu\nu$-model. If $\Gamma \vdash t\colon \phi, \Delta$ is provable in the $\lambda\mu\nu$-calculus and if each of $[\![\Gamma]\!]$, $[\![\Delta]\!]$, $[\![\phi]\!]$, and $[\![t]\!]$ is defined in $\mathcal{P}$, then $\mathcal{P}, \Delta \models (t : \phi)[\Gamma]$. Moreover, if $t \leftrightarrow^* s$ and t and s are well-formed, then $[\![t]\!] = [\![s]\!]$.*

Proof By induction on the structure of proofs in the $\lambda\mu\nu$-calculus. As usual, we need substitution lemmas for each kind of substitution. The standard one states that substitution for variables is given by categorical composition. For the mixed substitution, we show that if $\Gamma \vdash t\colon \chi, (\phi \supset \psi)^\alpha, \Delta$, and $\Gamma \vdash s\colon \phi, \Delta$, then $[\![t[[\beta]us/[\alpha]u]]\!]$ is given by $\zeta([\![t]\!]) \circ \langle \mathsf{Id}, [\![s]\!]\rangle$. Similarly, if $\Gamma \vdash t\colon \chi, (\phi \wedge \psi)^\alpha, \Delta$, then $[\![t[[\beta]\pi(u)/[\alpha]u]]\!] = \pi(\phi([\![t]\!]))$, and if $\Gamma \vdash t\colon \chi, (\phi \vee \psi)^\alpha$, then $[\![t[[\alpha']\langle\beta\rangle u/[\alpha]u]]\!] = \zeta_\vee([\![t]\!])$. Note the rôle of the Beck-Chevalley conditions here.

For example, we give the constructions for $\nu\alpha.t$. Suppose we are given a term $\Gamma \vdash t\colon A, B^\beta, \Delta$ and let $[\![t]\!]\colon [\![\Gamma]\!] \to [\![A]\!]$ be the corresponding morphism in $\mathcal{E}_{[\![B\times\Delta]\!]}$ which, by the induction hypothesis, exists. By using the isomorphism ν, we obtain a morphism $t'\colon \Gamma \to [\![\phi \vee \psi]\!]$ which is equal to $[\![\nu\alpha.t]\!]$, and hence we have $\mathcal{P}, \Delta \models (t\colon \phi \vee \psi)[\Gamma]$. The other cases are similar. □

Lemma 3.34 (model existence) *If $\Gamma \not\vdash t\colon \phi, \Delta$, then there is a $\lambda\mu\nu$-model $\mathcal{T} = \langle \tau, [\![-]\!]_{\mathcal{T}}\rangle$ such that $\mathcal{T}, \Delta \not\models (t\colon \phi)[\Gamma]$.*

Proof As usual, $\mathcal{T}$ is the term model. Hence, define the split fibration $\tau : \mathcal{E}_{\mathcal{T}} \to \mathcal{B}_{\mathcal{T}}$ as follows.

The objects of $\mathcal{B}_{\mathcal{T}}$ are succedents Δ. The arrows of $\mathcal{B}_{\mathcal{T}}$ are terms s that are compositions of the basic terms for Permutation, $p : \Delta \to \Theta$, 'Weakening', $w_{\phi} : \Delta \times \phi \to \Delta$, and 'Contraction', $c_{\phi} : \Delta \times \phi \to \Delta \times \phi \times \phi$. The mapping $[\![-]\!]_{\mathcal{T}}$ is then simply $[\![\Delta]\!]_{\mathcal{T}} = \Delta$, *etc.*.

The fibre over each Δ has as objects lists of types and as morphisms from Γ to $\Gamma' = (\phi_1, \ldots, \phi_n)$ tuples $(t_1, \ldots, t_n)$ of normal forms such that $\Gamma \vdash t_i \colon \phi_i, \Delta$. The λ-abstraction provides exponentiation in the fibres.

The isomorphism between $\mathcal{E}_{\Delta}(C, A)$ and $\mathcal{E}_{\Delta \times A}(w_A^*(C), \perp)$ is given by the term constructors μ and $[-]$. The isomorphism defining disjunctions is given by the term constructors for disjunction. $\square$

Proposition 3.35 (completeness)

$$\Gamma \vdash t \colon \phi, \Delta \quad \textit{iff}\ \Gamma \models t \colon \phi, \Delta$$

Proof The (only if) is just soundness (Proposition 3.33). For the (if), we suppose that $\Gamma \not\vdash t : \phi, \Delta$. Then Lemma 3.34 yields a contradiction. $\square$

It follows that we can regard $\lambda\mu\nu$ as the *internal language* of a $\lambda\mu\nu$-structure.

We remark upon the similarity of the soundness and completeness arguments for $\lambda\mu\nu$ and $\lambda\mu\oplus$ (see the paragraph after Definition 3.30). Because both the $\lambda\mu\nu$-calculus and the $\lambda\mu\oplus$-calculus are confluent, they provide, respectively, instances of $\lambda\mu\nu$- and $\lambda\mu\oplus$-structures which are non-degenerate, that is, in which not all hom-sets have only one element. However, the existence of non-trivial instances of models of both disjunctions reveals much about the semantic structure of classical proofs. This is the subject of Section 3.7.

3.6.4 *Continuations: Concrete, computational models*

In the denotational semantics of programming languages, for example, [93, 114], in which programs are given a functional interpretation over structures such as the category of *complete partial orders*, an important technique is to interpret not only the *linguistic constructs* of the programming language but also its *control régime.* The semantic structures commonly used for this purpose are called *continuations*.

The idea is that a continuation models a *change of control* during the evaluation of a program with respect to given data: we temporarily suspend the current computation, carry out another, subsidiary, one and after a while resume the original one. Thus a continuation describes how to complete the subsidiary computation and return to the original computation. Continuations are commonly used to describe, inter alia, backtracking [48, 114], co-routines [114] and evaluation strategies [92]. A survey of the various origins of the idea can be found in [110].

Rather than attempt a general definition, we describe a category of continuations, introduced by Hofmann and Streicher [53], which can be extended so as to correspond to a semantics of classical proofs as represented by the terms of the $\lambda\mu\nu$-calculus.

The $\lambda\mu$-calculus can be used to describe continuations as follows: a continuation of type ϕ is described as the type $\neg\phi$. The intuition is that a continuation expects a term of type ϕ and produces some value which is never used because the control context changes. One could take any type R (for *responses*) for the type of these values, but as it is never used, the $\lambda\mu$-calculus uses $\perp$ for the type of these values. The creation of a continuation is then described by a term of type $\phi \supset \neg\neg\phi$ because it transforms a value of type ϕ into a continuation $\neg\phi$. The other direction, namely the evaluation of a continuation, gives a term of type $\neg\neg\phi \supset \phi$. With these two control operators it is possible to define an operational semantics which treats each term as a continuation rather than having a value.

This syntactic view has a semantic counterpart: Hofmann and Streicher define a category of continuations as a category C with a distinguished class T of objects of C called type objects and a distinguished type object R of responses. In addition, there is a chosen Cartesian product $\Gamma \cdot \phi$ for every object Γ and type ϕ, and chosen terminal objects $[\ ]$ and $1 \in T$. Moreover, for each type object ϕ there is a chosen exponential $R^\phi \in T$, and for any two type objects ϕ and ψ a chosen Cartesian product $R^\phi \cdot \psi \in T$ of R^ϕ and ψ. A $\lambda\mu$-term $\Gamma \vdash t\colon \psi, \Delta$ is interpreted in such a category as a map $R^{[\![\Gamma]\!]} \cdot [\![\Delta]\!] \to R^{[\![\phi]\!]}$.

To interpret conjunctions, we ask in addition for sums of types in the category, and can then define $[\![\phi \wedge \psi]\!] = [\![\phi]\!] + [\![\psi]\!]$, and use standard isomorphisms involving sums, products, and exponentials to define the interpretation of $\lambda\mu$-terms involving products or projections.

The classical disjunction requires the closure of T under products $\phi \cdot \psi$ for every $\phi, \psi \in T$: we can define

$$[\![\phi \vee \psi]\!] = [\![\phi]\!] \cdot [\![\psi]\!]$$

and use the natural isomorphism between

$$\hom(R^{[\![\Gamma]\!]} \cdot [\![\Delta]\!], R^{[\![\phi]\!] \cdot [\![\psi]\!]}) \quad \text{and} \quad \hom(R^{[\![\Gamma]\!]} \cdot [\![\Delta]\!] \cdot [\![B]\!], R^{[\![\phi]\!]})$$

as the categorical counterpart of the introduction and elimination rules for disjunction.

A similar construction for the intuitionistic disjunction $\oplus$ seems to be more difficult to obtain. For the soundness theorem we require

$$\hom(R^{[\![\phi\oplus\psi]\!]} \cdot [\![\Delta]\!], R^{[\![\chi]\!]}) \cong \hom(R^{[\![\phi]\!]} \times [\![\Delta]\!], R^{[\![\chi]\!]}) \cdot \hom(R^{[\![\psi]\!]} \cdot [\![\Delta]\!], R^{[\![\chi]\!]})$$

but there is no obvious way of defining $[\![\phi \oplus \psi]\!]$ in a Cartesian closed category such that $R^{[\![\phi\oplus\psi]\!]} \cong R^{[\![\phi]\!]} + R^{[\![\psi]\!]}$. As we will see in the next section intuitionistic and classical disjunction do not coincide proof-theoretically: we show that a $\lambda\mu\nu$-calculus in which classical and intuitionistic disjunction coincide is trivial in the sense that all terms of the same type are equal.

Hofmann and Streicher prove completeness for $\lambda\mu$-categories by defining a continuation category $\mathcal{C}$ from the syntax of the $\lambda\mu$-calculus. Objects are (continuation) contexts $\Delta = \phi_1^{\alpha_1}, \ldots, \phi_m^{\alpha_m}$; a morphism from Δ to ϕ is a certain $\lambda\mu$-term t such that $\vdash t \colon \phi \supset \bot, \Delta$. The intuition is that t transforms the name α^i of type ϕ_i to a continuation of type ϕ, which is the type $\phi \supset \bot$. The condition on the term is that for any observer o (any $\lambda\mu$-term of type $\neg\neg\phi$) the two possible terms for execution of the continuations t by the observer, namely ot and $t(\mu\alpha^{\phi}.o(\lambda x \colon \phi\,.\,[\alpha]x))$, are equal. The type of responses is fixed as $\bot \supset \bot$. It follows from the naturality of their definitions, that is, they respect substitution, that the completeness result can be extended to cover conjunction and classical disjunction.

Hofmann and Streicher also prove that the continuation categories are universal for the $\lambda\mu$-calculus in the sense that for each $\lambda\mu$-theory (i.e. a $\lambda\mu$-calculus with some additional judgemental equalities between terms) there is a continuation category (namely the term model) such that there is a map from this model to any other $\lambda\mu$-model which respects the interpretation of $\lambda\mu$-terms in both models. Again, it follows from the naturality of their definitions, that is, they respect substitution, that the universality result can be extended to cover conjunction and classical disjunction.

The completeness of our categorical model implies that we must be able to transform each continuation category into a $\lambda\mu\nu$-structure. For this construction, we view this category as a category of display maps [57]; then we exploit a standard construction which transforms categories of display maps into fibrations [59]. We sketch this construction, but omit the detailed verification that the structure we define is indeed a $\lambda\mu\nu$-structure, as follows:

1. The base category $\mathcal{B}$ has as objects the objects of $\mathcal{C}$ and all morphisms necessary to make $\mathcal{C}$ a Cartesian category;
2. Objects of the fibre $\mathcal{E}_\Delta$ are projection morphisms $\Delta \cdot \phi \to \Delta$;
3. Morphisms from $\Delta \cdot \phi \to \Delta$ to $\Delta \cdot \psi \to \Delta$ are morphisms f in $\mathcal{C}$ such that $\pi_\psi \circ f = \pi_\phi$, where π_ψ and π_ϕ are the projections corresponding to $\Delta \cdot \psi$ and $\Delta \cdot \phi$, respectively;
4. Given a morphism $f \colon \Gamma \to \Delta$ the functor $\mathcal{E}(f)$ transforms an object $\Delta \cdot A \to \Delta$ to $\Gamma \cdot \phi \to \Gamma$ and a morphism h into $\pi' \circ (\mathsf{Id} \times h) \circ (\mathsf{Id} \times f)$, where π' is the projection from $\Gamma \cdot \Delta \cdot \psi$ to $\Gamma \cdot \psi$;
5. The object $\bot$ is R;
6. The isomorphism between $\mathcal{E}_\Delta(\chi, \phi)$ and $\mathcal{E}_{\Delta\,\cdot\,\phi}(\chi, \bot)$ is captured by the bijection between $\hom(\Delta \cdot \phi, R)$ and $\hom(\Delta, R^{\phi})$ in $\mathcal{C}$;
7. The naturality and Beck-Chevalley condition of the bijection ζ follow from the fact that $\mathcal{E}(f)$ is defined by composition.

The verification that interpretations of $\lambda\mu\nu$ are indeed well-defined in this structure, so yielding our definition of a $\lambda\mu\nu$-model, is routine.

Finally, we remark that Hofmann and Streicher also show that the interpretation of a $\lambda\mu$-term t in the syntactic continuation category is obtained by replacing each object variable x by a term which describes the execution of a continuation given by a new name α. This interpretation transforms each term into a continuation. This property too extends to $\lambda\mu\nu$.

3.6.5 *Games: Another concrete model*

We extend the games considered in Section 3.4 to games for classical logic. The main difference between the games for intuitionistic logic and those for classical logic is a consequence of the fact that for classical logic we are working with sequents with multiple conclusions, $\Gamma \vdash \Delta$, with the intuitive meaning that (at least) one of the formulæ in Δ must to be proved, whereas in intuitionistic logic we work with only one conclusion. This means that, in classical games, when Opponent challenges a formula ϕ in Δ, Proponent might choose to defend a different formula ψ in Δ, which has to be accepted also as a valid defence of ϕ.

The definitions of arenas, moves, and justification for classical games are the same as those for intuitionistic games. We call a strategy (play) classical if it is the one for classical games. Otherwise we call the strategy (play) intuitionistic.

The conditions for classical plays are not as strong as the conditions for intuitionistic plays. In particular, the rules for disjunction have been changed to allow Proponent to select both disjuncts, thereby possibly violating the disjunction property of intuitionistic logic. More precisely, we have relaxed Clause 6 and Clause 7. We drop the latter clause, and replace the former as follows:

Definition 3.36 A play for an arena $\mathcal{A}$ is a sequence of moves $m_1, \ldots, m_n$ such that conditions 1–5 for intuitionistic plays, and the following additional condition are satisfied:

6. For any P-answer m_i there exists an O-question m_k and an O-answer m_j such that m_i is hereditarily justified by m_k, m_j is an O-answer with the same label as m_k or $\perp$ and $k < j < i$, and that the nodes corresponding to m_k and m_j in the arena are on a path which does not contain a P-node n labelled $\supset$ such that the nodes corresponding to m_i and m_j are its children or identical to it.

This relaxation captures the possibility of pending O-questions (arising from the multiple conclusions on the right-hand side) being answered as well as the immediate justifying question.

This games semantics is sound for classical logic:

Theorem 3.37 *For any formula ϕ and classical strategy Φ for ϕ there exists a classical natural deduction proof of ϕ.*

The proof follows the same line as the proof for the corresponding theorem for intuitionistic games (Theorem 3.17). Again, we have to show a stronger version of the theorem with a stronger notion of strategy.

> **Definition 3.38** For a set A, of propositional atoms or $\perp$, and a sequence $\phi_1, \ldots, \phi_k$ of formulæ, define an $A, \phi_1, \ldots, \phi_k$-strategy for the formula ϕ to be any strategy for ϕ where both players may make additional moves according to the arenas for $\phi_1, \ldots, \phi_k$.

The key lemma is now the following:

Lemma 3.39 *Given formulæ $\phi_1, \ldots, \phi_k$ and any set A of O-answers with labels $p_1, \ldots, p_n$ and an $A, \phi_1, \ldots, \phi_k$-strategy for a formula ϕ there is a classical proof of $p_1, p_n \vdash \phi, \phi_1, \ldots, \phi_k$.*

Proof By induction over the structure of $\phi, \phi_1, \ldots, \phi_k$. As the definition of $A, \phi_1, \ldots, \phi_k$-strategy is invariant under permutation of any of the ϕ_i's and ϕ and natural deduction admits the Exchange rule, it suffices to do a case analysis regarding the structure of ϕ. We will write Δ for the sequence of formulæ $\phi_1, \ldots \phi_k$ and Γ for the sequence $p_1, \ldots, p_n$.

Atoms. Firstly, assume $\phi = p$ for some propositional atom p, and $\phi_1, \ldots, \phi_k = q_1, \ldots, q_k$, where all q_i's are atoms or $\perp$. Any possible strategy starts by Opponent asking at least one question labelled p or $q_1, \ldots, q_k$. Proponent only has an answer if either $p_i = p$, for some i, or $p_i = q_j$, for some i and j. In both cases, the classical axiom $p_1, \ldots, p_n \vdash p, q_1, \ldots, q_k$ is the desired natural deduction proof;

$\psi_1 \vee \psi_2$. Any possible strategy starts with Opponent asking question corresponding to the root of the arena for $\psi_1 \vee \psi_2$. There are now several cases. If Opponent never asks any initial question for the arenas ψ_1 and ψ_2, then the given strategy is also a strategy for $\phi_1, \ldots, \phi_k$. Hence, there is a natural deduction proof of $\Gamma \vdash \Delta$ and hence also of $\Gamma \vdash \psi_1 \vee \psi_2, \Delta$. If Proponent never asks the question corresponding to the node labelled $R(L)$ of this disjunction or Opponent never asks any of the initial questions of the arena for ψ_2 (ψ_1) then the given strategy is also a strategy for ψ_1 (ψ_2). By induction hypothesis there is a natural deduction proof of $\Gamma \vdash \psi_1, \Delta$ ($\Gamma \vdash \psi_2, \Delta$) and hence also a natural deduction proof of $\Gamma \vdash \psi_1 \vee \psi_2, \Delta$. If Opponent asks any initial questions for both arenas ψ_1 and ψ_2, then the strategy has to consider all initial moves for ψ_1 and ψ_2. Hence by induction hypothesis for ψ_1, ψ_2, Δ there exists a natural deduction proof $\Gamma \vdash \psi_1, \psi_2, \Delta$ and hence also a natural deduction proof of $\Gamma \vdash \psi_1 \vee \psi_2, \Delta$;

$\psi_1 \wedge \psi_2$. Because every question and answer of a strategy for ψ_1 and ψ_2 has to be justified eventually by an initial move for ψ_1 and ψ_2 it is possible to obtain one strategy for ψ_1 and one strategy for ψ_2 from the given strategy. Hence by induction hypothesis we obtain natural deduction proofs for $\Gamma \vdash \psi_1, \Delta$ and $\Gamma \vdash \psi_2, \Delta$. Hence one obtains also a natural deduction proof for $\Gamma \vdash \psi_1 \wedge \psi_2, \Delta$;

$\phi' \supset \psi$. There are several subcases. Firstly, assume $\phi' = \psi_1 \wedge \psi_2$. Then $(\psi_1 \wedge \psi_2) \supset \psi$ is equivalent to $\phi_1 \supset \psi_2 \supset \psi$, and the arenas for $(\psi_1 \wedge \psi_2) \supset \psi$ and $\psi_1 \supset \psi_2 \supset \psi$ are identical. Hence we consider the case $\psi_1 \supset \psi_2 \supset \psi$ instead.

Secondly, assume $\phi' = \sigma \vee \tau$. Now define two A, Δ-strategies Φ_1 and Φ_2 for $\sigma \supset \psi$ and $\tau \supset \psi$, respectively, where the moves of both players in Φ_1 and Φ_2 are the moves of Φ which are justified by moves not hereditarily justified by τ or σ, respectively. By the induction hypothesis, we obtain natural deduction proofs for

$$\Gamma \vdash (\sigma \supset \psi), \Delta \quad \text{and} \quad \Gamma \vdash \tau \supset \psi, \Delta\,.$$

Hence there is also a natural deduction proof for $\Gamma \vdash (\sigma \vee \tau) \supset \psi, \Delta$.

Thirdly, suppose $\phi' = \sigma \supset \tau$. Again, define A, Δ-strategies Φ_1 for $\tau \supset \psi$ and Φ_2 for σ where the moves of both players are the ones not hereditarily justified by σ or τ, respectively. By induction hypothesis we obtain natural deduction proofs for $\Gamma \vdash \tau \supset \psi, \Delta$ and $\Gamma \vdash \sigma, \Delta$. Hence there is also a natural deduction proof for $\Gamma \vdash (\sigma \supset \tau) \supset \psi, \Delta$.

Fourthly, suppose ϕ' is an atom p. Again, there are two cases. Consider an A, Δ-strategy for $p \supset \psi$ without a P-question corresponding to p. In this case, the A, Δ-strategy for $p \supset \psi$ is in fact a strategy for ψ, and by induction hypothesis there is a natural deduction proof of $\Gamma \vdash \psi, \Delta$, hence also a proof of $\Gamma \vdash p \supset \psi, \Delta$. Now suppose there is a P-question corresponding to p. Then the strategy that removes the P-question and O-answer for p is an $A \cup \{p\}$, Δ-strategy for ψ. By induction hypothesis there is a natural deduction proof for $\Gamma, p \vdash \psi, \Delta$ and hence also for $\Gamma \vdash p \supset \psi, \Delta$.

Finally, suppose $\phi' = \bot$. In this case there is always a natural deduction proof of $\Gamma, \bot \vdash \psi, \Delta$, and hence also a proof of $\Gamma \vdash \bot \supset \psi, \Delta$. $\square$

3.7 Comparing the disjunctions; De Morgan Laws

The 'intuitionistic' (*i.e.* single-conclusioned) and 'classical' (*i.e.* multiple-conclusioned) versions of classical disjunction have the same proof-theoretic strength. However, when we consider the semantics of proof-terms, the two disjunctions are fundamentally different in the sense that their identification leads to a model in which any provable sequent has at most one proof, that is, a model in which (the interpretations of) $\oplus$ and $\vee$ are isomorphic must be such that each fibre is a Boolean algebra. This can be seen very easily using the notion of model we have given.

Theorem 3.40 *Let p be a $\lambda\mu$-structure. If p is also both a $\lambda\mu\oplus$-structure and a $\lambda\mu\nu$-structure and if the objects $1+1$ and $1\vee 1$ are isomorphic in each fibre, then each fibre of p is a Boolean algebra.*

Proof There is exactly one map $1 \to 1 \vee 1$ in any fibre $\mathcal{E}_\Delta$, namely the map corresponding to the unique map $! : 1 \to 1$ in $\mathcal{E}_{\Delta\times 1}$ under the adjunction defining $\vee$. Because $1 \vee 1$ and $1+1$ are isomorphic by assumption, there is exactly one map $1 \to 1+1$, and hence

$$\mathtt{in}_1 = \mathtt{in}_2 : 1 \to 1+1.$$

We have to show that there is at most one morphism between any two objects in any given fibre. So suppose $f\colon A \to B$ and $g\colon A \to B$ are two morphisms in any fibre. Because each fibre is Cartesian closed, it suffices to show that for the curried morphisms we have

$$\hat{f} = \hat{g} : 1 \to A \Rightarrow B.$$

As $\mathtt{in}_1 = \mathtt{in}_2$, we obtain the following sequence of equations:

$$\begin{aligned} \hat{f} &= (\hat{f} + \hat{g}) \circ \mathtt{in}_1 \\ &= (\hat{f} + \hat{g}) \circ \mathtt{in}_2 \\ &= \hat{g}. \end{aligned}$$

□

Note that this argument relies critically on extensionality: it does not apply to non-extensional systems. Because it is difficult to define the intuitionistic disjunction in the continuations model, this theorem indicates that the classical disjunction is the more appropriate one for the $\lambda\mu$-calculus: it has a natural interpretation in both the fibred and the continuations model, whereas it is difficult to reconcile the intuitionistic disjunction with the continuations interpretation.

Proof-theoretically, this result asserts the non-triviality of the structural rules. At the level of consequence, the equivalence of $\oplus$ and $\vee$ relies on the structural rules of LK. Forcing the interpretations of $\oplus$ and $\vee$ to be *isomorphic*, forces the interpretation of the structural rules to be too trivial and collapse follows.

Some classical identities also work at the level of provability. One example is the classical equivalence between $\neg\phi \vee \psi$ and $\phi \supset \psi$ and one of the De Morgan laws, namely $\neg(\phi \wedge \psi) \cong \neg\phi \vee \neg\psi$. For brevity, we will abuse notation and use logical expressions to denote the corresponding categorical structures, thereby facilitating proofs via the internal language of $\lambda\mu\nu$-structures.

Theorem 3.41 *In any $\lambda\mu\nu$-structure, we have $\neg\phi \vee \psi \cong \phi \supset \psi$.*

Proof We use the internal language for the proof. Consider the $\lambda\mu\nu$-terms

$$f\colon \phi \supset \psi \vdash \nu\beta^{\psi}.\lambda a\colon \phi.[\beta]fa\colon \neg\phi \vee \psi,$$

which we will abbreviate by t, and

$$v\colon \neg\phi \vee \psi \vdash \lambda a\colon \phi.\mu\beta.(\langle\beta\rangle v)a\colon \phi \supset \psi,$$

which we will abbreviate by u. These two terms show that $\neg\phi \vee \psi$ and $\phi \supset \psi$ are isomorphic.

Firstly, we calculate $t[u/f]$, with

$$v\colon \neg\phi \vee \psi \vdash t[v/f]\colon \neg A \vee B,$$

then:

$$\begin{aligned} t[u/f] &= (\nu\beta.\lambda a\colon A.[\beta]fa)[\lambda a\colon A.\mu\beta.(\langle\beta\rangle v)a/f] \\ &= \nu\beta.\lambda a\colon A.(\langle\beta\rangle v)a \\ &= \nu\beta.\langle\beta\rangle v \\ &= v. \end{aligned}$$

Secondly, we calculate the other direction, $u[t/v]$, with

$$f : \phi \supset \psi \vdash u[t/v] : \phi \supset \psi,$$

then:

$$\begin{aligned} u[t/v] &= (\lambda a\colon \phi.\mu\beta.(\langle\beta\rangle v)a)[\nu\beta.\lambda a\colon \phi.[\beta]fa/v] \\ &= \lambda a\colon \phi.\mu\beta.(\langle\beta\rangle\nu\beta.\lambda a\colon \phi.[\beta]fa)a \\ &= \lambda a\colon \phi.\mu\beta.[\beta].fa \\ &= f. \end{aligned}$$ □

The equivalence of this theorem can also be shown semantically using the continuation category: We have $[\![\neg\phi \vee \psi]\!] = R^{[\![\phi]\!]} \cdot [\![\psi]\!] = [\![\phi \supset \psi]\!]$. Note that this result does not imply that we can use $\vee$ to define $\supset$ or vice versa. Because we do not have $\neg\neg\phi \cong \phi$, we do not have $\phi \vee \psi \cong \neg\phi \supset \psi$. As we have defined negation $\neg\phi$ as $\phi \supset \bot$, we cannot eliminate $\supset$ either.

The absence of $\neg\neg\phi \cong \phi$ also makes it impossible to infer statements about the De Morgan dualities from the previous two theorems. In fact, one of the dualities holds proof-theoretically, the other one does not. Again, we give both an argument in the internal language and using the continuations category.

Theorem 3.42 *In any $\lambda\mu\nu$-structure, we have $\neg(\phi \wedge \psi) \cong \neg\phi \vee \neg\psi$ but not in general $\neg(\phi \vee \psi) \cong \neg\phi \wedge \neg\psi$.*

Proof First, we give the arguments using the internal language. For the isomorphism between $\neg(\phi \wedge \psi)$ and $\neg\phi \vee \neg\psi$, consider the terms

$$t = f\colon \neg(\phi \wedge \psi) \vdash \nu\beta.\lambda a.[\beta]\lambda b.f\langle a, b\rangle$$

and

$$u = \lambda c\colon \phi \wedge \psi.(\mu\beta.(\langle\beta\rangle h)\pi c)\pi' c.$$

We must show that $u[t/h]$ and $t[u/f]$ are f and h, respectively:

$$\begin{aligned}
u[t/h] &= \lambda c\colon \phi \wedge \psi.(\mu\beta.(\langle\beta\rangle\nu\beta.\lambda a.[\beta]\lambda b.f\langle a, b\rangle)\pi c)\pi' c \\
&= \lambda c.(\mu\beta[\beta]\lambda b.f\langle \pi c, b\rangle)\pi' c \\
&= \lambda c.f\langle \pi c, \pi' c\rangle \\
&= f. \\
t[u/f] &= \nu\beta.\lambda a.[\beta]\lambda b.(\lambda c\colon \phi \wedge \psi.(\mu\beta.(\langle\beta\rangle h)\pi c)\pi' c)\langle a, b\rangle \\
&= \nu\beta.\lambda a.[\beta]\lambda b.(\mu\beta.(\langle\beta\rangle h)a)b \\
&= \nu\beta.\lambda a.(\langle\beta\rangle h)a \\
&= h.
\end{aligned}$$

Now consider the terms $t = h\colon \neg(\phi \vee \psi) \vdash \langle \lambda a.h(\nu\beta.a), \lambda b.h(\nu\beta.\mu\alpha.[\beta]b\rangle$ and $u = d\colon \neg\phi \wedge \neg\psi \vdash \lambda c.\pi' d(\mu\beta.\pi d(\langle\beta\rangle c))$. We have

$$\begin{aligned}
t[u/h] &= \langle \lambda a.(\lambda c.\pi' d(\mu\beta.\pi d(\langle\beta\rangle c)))(\nu\beta.a), \lambda b.(\lambda c.\pi' d(\mu\beta.\pi d(\langle\beta\rangle c)))(\nu\beta.\mu\alpha.[\beta]b)\rangle \\
&= \langle \lambda a.\pi' d(\mu\beta.(\pi d)a), \lambda b.\pi' d(\mu\beta.(\pi d)\mu\alpha.[\beta]b)\rangle. \\
u[t/d] &= \lambda c.(\lambda b.h(\nu\beta.\mu\alpha.[\beta]b))(\mu\beta.(\lambda a.h(\nu\beta.a))(\langle\beta\rangle c)) \\
&= \lambda c.(\lambda b.h(\nu\beta.\mu\alpha.[\beta]b))(\mu\beta.hc) \\
&= \lambda c.h(\nu\beta.\mu\alpha.hc).
\end{aligned}$$

Both terms are irreducible and so not the identity.

In continuation categories we can reason as follows:

$$[\![\neg(\phi \wedge \psi)]\!] = R^{[\![\phi]\!]+[\![\psi]\!]} \cong R^{[\![\phi]\!]} \cdot R^{[\![\psi]\!]} = [\![\neg\phi \vee \neg\psi]\!]$$

and

$$[\![\neg(\phi \vee \psi)]\!] = R^{[\![\phi \cdot \psi]\!]} \not\cong R^{[\![\phi]\!]} + R^{[\![\psi]\!]} = [\![\neg\phi \wedge \neg\psi]\!]. \qquad \square$$

3.8 Discussion

Girard, Lafont, and Taylor's claim [42] that classical logic is algorithmically inconsistent arises from the following example of a classical Cut-reduction, due to Lafont [42, 127], in which the Cut redex has two possible reducts:[31]

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\Phi_1\\ \vdash \phi\end{array}}{\vdash \phi, \psi}\,\text{Weakening} \qquad \dfrac{\begin{array}{c}\vdots\,\Phi_2\\ \vdash \phi\end{array}}{\psi \vdash \phi}\,\text{Weakening}}{\vdash \phi, \phi}\,\text{Cut}}{\vdash \phi}\,\text{Contraction}
\quad \rightsquigarrow \quad
\begin{array}{c}\vdots\,\Phi_1\\ \vdash \phi\end{array} \quad \text{or} \quad \begin{array}{c}\vdots\,\Phi_2\\ \vdash \phi\end{array}
$$

If we try to enforce equality of the two choices of reduct, Φ_1 and Φ_2, then we find that, as described in [42], all proofs of a classical sequent $\Gamma \vdash \Delta$ are identified. This corresponds to the collapse to the naïve categorical models, as discussed in Section 3.6.2. The loss of the symmetry of the (sequent) calculus forced by $\lambda\mu\nu$'s choice of $\neg\neg$-translation, and the corresponding choice of fibred model, admits only the reduction to Φ_2. In functional programming jargon, $\neg\neg$-transforms are called continuation-passing-style (CPS) transforms [92], and the transform chosen above validates equalities (between $\lambda\mu\nu$-terms) typical for call-by-name. A call-by-value CPS transform would admit only the reduction to Φ_1.

As we remarked, in Section 3.6.2, recent work of Führmann and Pym [30] has solved the problem of giving non-trivial categorical models of classical proofs without recourse to non-symmetric systems with restricted equality. Starting from a convenient formulation of the well-known categorical semantics of linear classical sequent proofs, and from Robinson's classical proof nets [112], Führmann and Pym give models of Weakening and Contraction that do not collapse. Cut-reduction is interpreted by a partial order between morphisms. Their models make no commitment to any translation of classical logic into intuitionistic logic and distinguish non-deterministic choices of Cut-elimination. They establish soundness and completeness via initial models built from proof nets, and describe models built from sets and relations.

However, is hard to reconcile Führmann and Pym's semantics with the view of a classical system as a family of intuitionistic systems. Since we exploit this structural perspective, in full recognition of its shortcomings, in order to give

[31] This proof figure is actually defined in the classical sequent calculus, LK, which we do not introduce formally until Chapter 4. However, readers who are unfamiliar with the sequent calculus may understand this figure in terms of the sequentialized classical natural deduction system, FNK, introduced in Chapter 2.

a semantics for intuitionistic *proof-search*, we do not consider the models given in [30] any further.

Finally, we conjecture that the results of this chapter may be extended to predicate logic with first-order quantifiers. The basic idea is to take our $\lambda\mu\nu$-models and fibre them over a category which is suitable for interpreting first-order terms (*cf.* the hyperdoctrines discussed in Section 3.5).

4

PROOF THEORY FOR REDUCTIVE LOGIC

4.1 Introduction

As we have seen in the introduction, reductive logic is based not on deductive rules,

$$\frac{\text{Premiss}_1 \ldots \text{Premiss}_m}{\text{Conclusion}} \quad R,$$

read from premisses to conclusion, but rather on reduction operators,

$$\frac{\text{Sufficient Premiss}_1 \ldots \text{Sufficient Premiss}_m}{\text{Putative Conclusion}} \quad O_R,$$

read from putative conclusion to sufficient premisses. Clearly, an inference rule may be read as a reduction operator, and reduction operators correspond to admissible rules.[32] We believe that this idea of reduction was first explained *in these terms* by Kleene [65].

As we have seen, an attempt to construct a proof, that is, a *reduction*, proceeds, inductively, by applying instances of reduction operators of this form to putative conclusions of which a proof is desired, thereby yielding a collection of sufficient premisses, proofs of which would be sufficient to imply the existence of a proof, obtainable by deduction, of the putative conclusion. We emphasize again, however, that a reduction may fail to yield a proof: having removed all of the logical structure, that is, the connectives, by reduction, we may be left with $p \vdash q$, for distinct atoms p and q.

In the previous two chapters, we have introduced systems of natural deduction (ND) for intuitionistic and classical logics. We have also provided sequential presentations, and given semantics for these systems via classes of categorical models for which soundness and completeness theorems are available.

Semantically, we have introduced both truth-functional semantics, in the tradition of Boole and Tarski, and proof-functional semantics, in the tradition of Brouwer, Heyting, and Kolmogorov, and have discussed the appropriate mathematical structures for their formulation, including a range of examples. In particular, we have presented a novel formulation of games semantics which provides a unifying theme throughout this monograph and which will provide our most convincing semantics for proof-search.

Thus we have given definitions of intuitionistic and classical proofs as deductive systems. In this chapter, we consider how Gentzen's sequent calculus [37]

[32] Henceforth we refer to just R rather than O_R.

provides an appropriate basis for reductive proof and consider its use as basis for uniform proof [80] and resolution proof, and hence for logic programming, in both classical [108, 111, 112] and intuitionistic logic [82, 108, 111].

Whilst the natural deduction systems we have presented are a very convenient basis for deductive proof, they are seriously defective as a basis for reductive proof. The problem is that natural deduction rules fail, in general, to have the subformula property [26, 84]:

An inference rule

$$\frac{\Gamma_1 \vdash \Delta_1 \ldots \Gamma_m \vdash \Delta_m}{\Gamma \vdash \Delta}$$

has the *subformula property* if every subformula of each $\Gamma_i \vdash \Delta_i$, for $1 \leq i \leq m$, is also a subformula of $\Gamma \vdash \Delta$. We apply the same definition to reduction operators.[33]

For example, the intuitionistic ND rule of $\supset E$,

$$\frac{\Gamma \vdash \phi \quad \Gamma \vdash \phi \supset \psi}{\Gamma \vdash \psi},$$

fails to have the subformula property because ϕ need not, in general, be a subformula of $\Gamma \vdash \psi$.

Systems for proof-search based on rules which fail to have the subformula property are defective as bases for reductive proof because the transition from the putative conclusion to the sufficient premisses requires the executing agent to generate, or discover, any subformula of the sufficient premisses which is not already a subformula of the putative conclusion. For example, faced with constructing a proof of $\Gamma \vdash \psi$ using the $\supset E$ operator, one must generate, or discover, the subformula ϕ in order to generate the sufficient premisses. In general, this is computationally very expensive.

An alternative characterization of intuitionistic and classical proofs is provided by Gentzen's sequent calculus [37]. The intuitionistic sequent calculus, LJ, is presented in Table 4.1 and the classical sequent calculus, LK, is presented in Table 4.2.

The sequent calculus has two introduction rules, right and left, for each connective. The right-rules introduce the connective on the right-hand side of the sequent; for example,

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma' \vdash \psi, \Delta}{\Gamma, \Gamma' \vdash \phi \wedge \psi} \wedge R,$$

and reproduce exactly the introduction rules of the corresponding natural deduction systems.

[33]It is quite common (e.g. see, [84]) to define the subformula property for complete proofs. It should be evident, however, that the property may be defined for rules, from which the property for proofs may be derived.

TABLE 4.1. Intuitionistic sequent calculus: LJ

$$\frac{}{\phi \vdash \phi}\ Ax \qquad \frac{\Gamma, \psi \vdash \phi \quad \Gamma' \vdash \psi}{\Gamma, \Gamma' \vdash \phi}\ \mathrm{Cut}$$

$$\frac{\Gamma, \psi, \chi, \Gamma' \vdash \phi}{\Gamma, \chi, \psi, \Gamma' \vdash \phi}\ E \qquad \frac{\Gamma, \Gamma' \vdash \phi}{\Gamma, \psi, \Gamma' \vdash \phi}\ W \qquad \frac{\Gamma, \psi, \psi, \Gamma' \vdash \phi}{\Gamma, \psi, \Gamma' \vdash \phi}\ C$$

$$\frac{}{\bot \vdash}\ \bot L \qquad \frac{}{\vdash \top}\ \top R$$

$$\frac{\Gamma, \psi, \psi', \Gamma' \vdash \phi}{\Gamma, \psi \wedge \psi', \Gamma' \vdash \phi}\ \wedge L \qquad \frac{\Gamma \vdash \phi \quad \Gamma' \vdash \phi'}{\Gamma, \Gamma' \vdash \phi \wedge \phi'}\ \wedge R$$

$$\frac{\Gamma \vdash \phi \quad \Gamma', \psi \vdash \chi}{\Gamma, \Gamma', \phi \supset \psi \vdash \chi}\ \supset L \qquad \frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \supset \psi}\ \supset R$$

$$\frac{\Gamma, \phi \vdash \psi \quad \Gamma', \phi' \vdash \psi}{\Gamma, \Gamma', \phi \vee \phi' \vdash \psi}\ \vee L \qquad \frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \phi'} \qquad \frac{\Gamma \vdash \phi'}{\Gamma \vdash \phi \vee \phi'}\ \vee R$$

TABLE 4.2. Classical sequent calculus: LK

$$\frac{}{\phi \vdash \phi}\ Ax \qquad \frac{\Gamma, \phi \vdash \Delta \quad \Gamma' \vdash \phi, \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \mathrm{Cut}$$

$$\frac{\Gamma, \phi, \psi, \Gamma' \vdash \Delta}{\Gamma, \psi, \phi, \Gamma' \vdash \Delta}\ EL \qquad \frac{\Gamma \vdash \Delta, \phi, \psi, \Delta'}{\Gamma \vdash \Delta, \psi, \phi, \Delta'}\ ER$$

$$\frac{\Gamma, \Gamma' \vdash \Delta}{\Gamma, \phi, \Gamma' \vdash \Delta}\ WL \qquad \frac{\Gamma \vdash \Delta, \Delta'}{\Gamma \vdash \Delta, \phi, \Delta'}\ WR$$

$$\frac{\Gamma, \phi, \phi, \Gamma' \vdash \Delta}{\Gamma, \phi, \Gamma' \vdash \Delta}\ CL \qquad \frac{\Gamma \vdash \Delta, \phi, \phi, \Delta'}{\Gamma \vdash \Delta, \phi, \Delta'}\ CR$$

$$\frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}\ \neg L \qquad \frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}\ \neg R$$

$$\frac{\Gamma, \phi, \phi', \Gamma' \vdash \Delta}{\Gamma, \phi \wedge \phi', \Gamma' \vdash \Delta}\ \wedge L \qquad \frac{\Gamma \vdash \phi, \Delta \quad \Gamma' \vdash \phi', \Delta'}{\Gamma, \Gamma' \vdash \phi \wedge \phi', \Delta, \Delta'}\ \wedge R$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma', \psi \vdash \Delta'}{\Gamma, \Gamma', \phi \supset \psi \vdash \Delta, \Delta'}\ \supset L \qquad \frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \supset \psi, \Delta}\ \supset R$$

$$\frac{\Gamma, \phi \vdash \Delta \quad \Gamma', \phi' \vdash \Delta'}{\Gamma, \Gamma', \phi \vee \phi' \vdash \Delta, \Delta'}\ \vee L \qquad \frac{\Gamma \vdash \phi, \Delta}{\Gamma \vdash \phi \vee \phi', \Delta} \qquad \frac{\Gamma \vdash \phi', \Delta}{\Gamma \vdash \phi \vee \phi', \Delta}\ \vee R$$

The left-rules introduce the connective on the left-hand side of the sequent; for example,

$$\frac{\Gamma, \phi, \psi, \Gamma' \vdash \Delta}{\Gamma, \phi \wedge \psi, \Gamma' \vdash \Delta} \quad \wedge L.$$

The left-rules replace the elimination rules of the corresponding natural deduction systems.

In both LJ and LK, the set of side-formulæ of the left-hand side of the putative conclusion of binary rules is the union of the sets of side-formulæ of the sufficient premisses. This is unsatisfactory from the point of proof-search as it requires to split the side-formulæ during proof-search, thereby introducing another possible point of failure by choosing a splitting which cannot lead to a proof. This way of combining side-formulæ is called *multplicative*. Because both intuitionistic and classical logic have Weakening and Contraction, it is possible to present a variation of both LJ and LK in which the side-formulæ of the left-hand side of putative conclusions of binary rules are also the side-formulæ of the left-hand side of the sufficient premisses. This way of combining side-formulæ is called *additive*. With an additional change of the axiom rule to allow arbitrary formulæ on the left-hand side and also on the right-hand side for classical logic, the rules of Weakening and Contraction become admissible and may be omitted from the system. Hence no splitting of side-formulæ is required during proof-search. We call these systems LJ′ and LK′, respectively. They are presented in Tables 4.3 and 4.4, respectively. From now on we will use only LJ′ and LK′.

The relationship between natural deduction and sequent calculus may be understood in terms of the Cut rule,

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma, \phi \vdash \Delta}{\Gamma \vdash \Delta, \Delta'}\,\mathrm{Cut}.$$

We explain the translations between natural deduction and sequent calculus by giving examples, from which the general pattern should be apparent. Firstly, we describe the mapping L from natural deduction proofs to sequent calculus proofs. Consider, for example, that a proof which ends with the intuitionistic natural deduction rule of $\vee E$,

$$\frac{\dfrac{\vdots\,\Phi}{\Gamma \vdash \phi \vee \psi} \quad \dfrac{\vdots\,\Phi_1}{\Gamma, \phi \vdash \chi} \quad \dfrac{\vdots\,\Phi_2}{\Gamma, \psi \vdash \chi}}{\Gamma \vdash \chi}\,\vee E$$

maps under L to the LJ proof

$$\frac{\dfrac{\vdots\,L(\Phi)}{\Gamma \vdash \phi \vee \psi} \quad \dfrac{\dfrac{\vdots\,L(\Phi_1)}{\Gamma, \phi \vdash \chi} \quad \dfrac{\vdots\,L(\Phi_2)}{\Gamma, \psi \vdash \chi}}{\Gamma, \phi \vee \psi \vdash \chi}\,\vee L}{\Gamma \vdash \chi}\,\mathrm{Cut}.$$

TABLE 4.3. Modified intuitionistic sequent calculus: LJ$'$

$$\frac{}{\Gamma, \phi \vdash \phi}\ Ax \qquad \frac{\Gamma, \psi \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi}\ \mathrm{Cut}$$

$$\frac{\Gamma, \psi, \chi, \Gamma' \vdash \phi}{\Gamma, \chi, \psi, \Gamma' \vdash \phi}\ E$$

$$\frac{}{\Gamma, \bot \vdash}\ \bot L \qquad \frac{}{\Gamma \vdash \top}\ \top R$$

$$\frac{\Gamma, \psi, \psi' \vdash \phi}{\Gamma, \psi \wedge \psi' \vdash \phi}\ \wedge L \qquad \frac{\Gamma \vdash \phi \quad \Gamma \vdash \phi'}{\Gamma \vdash \phi \wedge \phi'}\ \wedge R$$

$$\frac{\Gamma \vdash \phi \quad \Gamma, \psi \vdash \chi}{\Gamma, \phi \supset \psi \vdash \chi} \supset L \qquad \frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \supset \psi} \supset R$$

$$\frac{\Gamma, \phi \vdash \psi \quad \Gamma, \phi' \vdash \psi}{\Gamma, \phi \vee \phi' \vdash \psi}\ \vee L \qquad \frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \phi'} \qquad \frac{\Gamma \vdash \phi'}{\Gamma \vdash \phi \vee \phi'}\ \vee R$$

TABLE 4.4. Modified classical sequent calculus: LK$'$

$$\frac{}{\Gamma, \phi, \Gamma' \vdash \Delta, \phi, \Delta'}\ Ax \qquad \frac{\Gamma, \phi \vdash \Delta \quad \Gamma \vdash \phi, \Delta}{\Gamma \vdash \Delta}\ \mathrm{Cut}$$

$$\frac{\Gamma, \phi, \psi, \Gamma' \vdash \Delta}{\Gamma, \psi, \phi, \Gamma' \vdash \Delta}\ EL \qquad \frac{\Gamma \vdash \Delta, \phi, \psi, \Delta'}{\Gamma \vdash \Delta, \psi, \phi, \Delta'}\ ER$$

$$\frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}\ \neg L \qquad \frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}\ \neg R$$

$$\frac{\Gamma, \phi, \phi', \Gamma' \vdash \Delta}{\Gamma, \phi \wedge \phi', \Gamma' \vdash \Delta}\ \wedge L \qquad \frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \phi', \Delta}{\Gamma \vdash \phi \wedge \phi', \Delta}\ \wedge R$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \supset \psi \vdash \Delta} \supset L \qquad \frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \supset \psi, \Delta} \supset R$$

$$\frac{\Gamma, \phi \vdash \Delta \quad \Gamma, \phi' \vdash \Delta}{\Gamma, \phi \vee \phi' \vdash \Delta}\ \vee L \qquad \frac{\Gamma \vdash \phi, \phi', \Delta}{\Gamma \vdash \phi \vee \phi', \Delta}\ \vee R$$

Secondly, we describe the mapping N from sequent calculus proofs to natural deduction. Consider, for example, a proof which ends with the intuitionistic sequent calculus rule of $\supset L$,

$$\frac{\dfrac{\vdots\, \Phi_1}{\Gamma \vdash \phi} \quad \dfrac{\vdots\, \Phi_2}{\Gamma\,,\ \psi \vdash \chi}}{\Gamma\,,\ \phi \supset \psi \vdash \chi} \supset L.$$

Under N, this proof maps to the natural deduction proof

$$\dfrac{\dfrac{}{\Gamma, \phi \supset \psi \vdash \phi \supset \psi}\,Ax \quad \dfrac{\dfrac{\vdots\, N(\Phi_1)}{\Gamma \vdash \phi}}{\Gamma, \phi \supset \psi \vdash \phi}\,W}{\Gamma, \phi \supset \psi \vdash \psi}\supset E \quad \dfrac{\dfrac{\vdots\, N(\Phi_2)}{\Gamma, \psi \vdash \chi}}{\Gamma, \phi \supset \psi, \psi \vdash \chi}\,W$$
$$\dfrac{\Gamma, \phi \supset \psi \vdash \psi \qquad \Gamma, \phi \supset \psi, \psi \vdash \chi}{\Gamma, \phi \supset \psi \vdash \chi}\,\text{Cut}$$

Thus we obtain the following:

Theorem 4.1 (equivalence of ND and SC) $\Gamma \vdash \phi$ *has a natural deduction (sequentialized NJ) proof if and only if it has a proof in LJ.*

A similar result obtains for our natural deduction (sometimes described as *free deduction* [90]) formulation of classical logic using the $\lambda\mu\nu$-calculus and its relationship with LK. Note, however, that these correspondences can be obtained only for *provability*. The relationship between proofs—see Section 4.2—is much more complex and represents a substantial part of our subsequent analysis of reductive proof methods.

Although the Cut rule does not have the subformula property, we have the Cut-elimination theorem for both classical and intuitionistic sequent calculus [37, 126]:

Theorem 4.2 (Cut-elimination) *If* $\Gamma \vdash \Delta$ *is provable in either LK or LJ with the Cut rule, then it is also provable in either LK or LJ, respectively, without the Cut rule.*

In the classical predicate case, the Cut-elimination theorem has a so-called sharpened form, sometimes known as the *midsequent theorem*[34, 37], in which the resulting Cut-free proof is divided into a purely propositional upper part and a purely quantificational lower part. This result is closely related to an earlier result of Herbrand [49], which may be formulated as follows:

Theorem 4.3 (Herbrand's Theorem) *Let* Δ *be a set of formulæ in prenex normal form. Then* $\vdash \Delta$ *is provable in LK iff there is a quantifier-free formula* $H(\Delta)$*, consisting of a disjunction of substitution instances of the formulæ in* Δ*, such that* $\vdash H(\Delta)$ *is provable in LK.*

The force of Herbrand's Theorem is, essentially, to reduce the problem of finding proofs of consequences in classical predicate logic to that of finding proofs of consequences in classical propositional logic.

The propositional classical sequent calculus not only enjoys Cut-elimination, but also a rather strong *permutation* theorem:

Theorem 4.4 (permutation [65]) *Let* Φ *be a proof in propositional LK of the sequent* $\Gamma \vdash \Delta$*. Let* R *and* R' *be any two inferences (i.e. instances of rules) in*

Φ such that the conclusion of R' is a premiss of R. Then there is a proof Φ' in propositional LK of $\Gamma \vdash \Delta$ in which the conclusion of R is a premiss of R'.

For example, in

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\phi\,,\,\psi \vdash \phi \wedge \psi}}{\phi\,,\,\psi \vdash \phi \wedge \psi\,,\,\phi \wedge \chi}\,WR}{\phi\,,\,\psi \vdash (\phi \wedge \psi) \vee (\phi \wedge \chi)}\,\vee R \quad \dfrac{\dfrac{\dfrac{\vdots}{\phi\,,\,\chi \vdash \phi \wedge \chi}}{\phi\,,\,\chi \vdash \phi \wedge \psi\,,\,\phi \wedge \chi}\,WR}{\phi\,,\,\chi \vdash (\phi \wedge \psi) \vee (\phi \wedge \chi)}\,\vee R}{\dfrac{\phi\,,\,\psi \vee \chi \vdash (\phi \wedge \psi) \vee (\phi \wedge \chi)}{\phi \wedge (\psi \vee \chi) \vdash (\phi \wedge \psi) \vee (\phi \wedge \chi)}\,\wedge L,}\,\vee L$$

the $\vee R$ can be permuted below the $\vee L$, to give

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\phi\,,\,\psi \vdash \phi \wedge \psi} \quad \dfrac{\vdots}{\phi\,,\,\chi \vdash \phi \wedge \chi}}{\phi\,,\,\psi \vee \chi \vdash \phi \wedge \psi\,,\,\phi \wedge \chi}\,\vee L}{\phi\,,\,(\psi \vee \chi) \vdash (\phi \wedge \psi) \vee (\phi \wedge \chi)}\,\vee R}{\phi \wedge (\psi \vee \chi) \vdash (\phi \wedge \psi) \vee (\phi \wedge \chi)}\,\wedge L.$$

However, the permutation theorem fails to hold for the intuitionistic sequent calculus, LJ. To see this, consider the following proof in propositional LJ:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\phi\,,\,\psi \vdash \phi \wedge \psi}}{\phi\,,\,\psi \vdash (\phi \wedge \psi) \vee (\phi \wedge \chi)}\,\vee R \quad \dfrac{\dfrac{\vdots}{\phi\,,\,\chi \vdash \phi \wedge \chi}}{\phi\,,\,\chi \vdash (\phi \wedge \psi) \vee (\phi \wedge \chi)}\,\vee R}{\phi\,,\,\psi \vee \chi \vdash (\phi \wedge \psi) \vee (\phi \wedge \chi)}\,\vee L}{\phi \wedge (\psi \vee \chi) \vdash (\phi \wedge \psi) \vee (\phi \wedge \chi)}\,\wedge L.$$

Here the $\vee L$ must occur below the $\vee R$s. In a classical, LK, proof of the same consequence, the $\vee R$-rule may be permuted below the $\vee L$ because both disjuncts occur in its premiss.

In fact, a permutation theorem for LJ is available only for a rather restricted class of sequents, which we will consider in the sequel; see Section 4.4. The strength of the permutation theorem for the classical sequent calculus, LK, together with Cut-elimination (see above) and the subformula property, render

LK to be more desirable as a basis for reductive logic than LJ. Briefly:

1. The subformula property ensures the analyticity of the construction process: when applying a reduction operator,

$$\frac{\text{Sufficient Premiss}_1 \ldots \text{Sufficient Premiss}_m}{\text{Putative Conclusion}} \; O_R,$$

all of the formulæ required to generate the sufficient premisses are present in the putative conclusion;

2. The full permutation theorem for propositional LK ensures that, at any given application of an operator, the construction makes the least possible commitment to a particular decomposition of the putative conclusion: any operator which was applicable to the putative conclusion remains applicable to (all of) the sufficient premisses;
3. LJ can be embedded in LK so, provided we can find a computationally acceptable characterization of LJ-provability for LK, the computational advantages of LK may be exploited for constructing proofs in LJ. This is the topic of Section 4.4.

The embedding of LJ into LK raises one important issue to which we now turn. The calculus LJ′ is single-conclusioned whereas the calculus LK′ is multiple-conclusioned.[34] In a multiple-conclusioned calculus both subformulæ of the principal formula $\phi \vee \psi$ occur in the sufficient premiss, whereas in the single-conclusioned calculus only one of them occurs. Such multiple-conclusioned systems normally have stronger permutability theorems for inference rules. This is also true for Dummett's multiple-conclusioned intuitionistic sequent calculus [26] which we call LM (see Table 4.5). LM is identical to LK except for the rules $\supset R$ which has only formula on the right-hand side of the sufficient premiss. Hence all other rules except $\supset R$ are freely permutable, which is not true for LJ.

LM is a multiplicative calculus. As with LJ and LK, for search purposes we will use the additive version of LM. In this way we obtain a calculus LM′, which is given in Table 4.6. Both LM and LM′ enjoy Cut-elimination.

Now we consider how to embed derivations in LM′ into LK′. In general, every intuitionistic derivation arises as a subderivation of a classical derivation. Because the $\supset R$-rule allows multiple succedents in its premiss, two different intuitionistic sequent derivations, which are not identical up to a permutation of inference rules, can be subderivations of the same classical sequent derivation up to a choice of axioms. For example, consider the two intuitionistic derivations in LM′

$$\cfrac{\cfrac{}{\psi, \phi \vdash \psi}\, Ax}{\psi \vdash \phi \supset \psi, \chi \supset \psi} \supset R$$

[34] Later in this work, when we consider proof-search, the form of $\vee R$-rule that is available in the multiple-conclusioned calculus will be of computational value.

TABLE 4.5. Multiple-conclusioned sequent calculus for intuitionistic logic: LM

$\dfrac{}{\phi \vdash \phi}$	Ax	$\dfrac{\Gamma_1 \vdash \phi, \Delta_1 \qquad \Gamma_2, \phi \vdash \psi, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \psi, \Delta_1, \Delta_2}$	Cut
$\dfrac{\Gamma, \phi, \psi, \Gamma' \vdash \Delta}{\Gamma, \psi, \phi, \Gamma' \vdash \Delta}$	EL	$\dfrac{\Gamma \vdash \Delta, \phi, \psi, \Delta'}{\Gamma \vdash \Delta, \psi, \phi, \Delta'}$	ER
$\dfrac{\Gamma, \Gamma' \vdash \Delta}{\Gamma, \phi, \Gamma' \vdash \Delta}$	WL	$\dfrac{\Gamma \vdash \Delta, \Delta'}{\Gamma \vdash \Delta, \phi, \Delta'}$	WR
$\dfrac{\Gamma, \phi, \phi, \Gamma' \vdash \Delta}{\Gamma, \phi, \Gamma' \vdash \Delta}$	CL	$\dfrac{\Gamma \vdash \Delta, \phi, \phi, \Delta'}{\Gamma \vdash \Delta, \phi, \Delta'}$	CR
$\dfrac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta}$	$\wedge L$	$\dfrac{\Gamma_1 \vdash \phi, \Delta_1 \qquad \Gamma_2 \vdash \psi, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \phi \wedge \psi, \Delta_1, \Delta_2}$	$\wedge R$
$\dfrac{\Gamma_1, \phi \vdash \Delta_1 \qquad \Gamma_2, \psi \vdash \Delta_2}{\Gamma_1, \Gamma_2, \phi \vee \psi \vdash \Delta_1, \Delta_2}$	$\vee L$	$\dfrac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \vee \psi, \Delta}$	$\vee R$
$\dfrac{\Gamma_1 \vdash \phi, \Delta_1 \qquad \Gamma_2, \psi \vdash \Delta_2}{\Gamma_1, \Gamma_2, \phi \supset \psi \vdash \Delta_1, \Delta_2}$	$\supset L$	$\dfrac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \supset \psi, \Delta}$	$\supset R$
$\dfrac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}$	$\neg L$	$\dfrac{\Gamma, \phi \vdash}{\Gamma \vdash \neg\phi, \Delta}$	$\neg R$

TABLE 4.6. Modified multiple-conclusioned intuitionistic sequent calculus: LM′

$\dfrac{}{\Gamma, \phi \vdash \phi, \Delta}$	Ax	$\dfrac{\Gamma \vdash \phi, \Delta \qquad \Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \psi, \Delta}$	Cut
$\dfrac{\Gamma, \phi, \psi, \Gamma' \vdash \Delta}{\Gamma, \psi, \phi, \Gamma' \vdash \Delta}$	EL	$\dfrac{\Gamma \vdash \Delta, \phi, \psi, \Delta'}{\Gamma \vdash \Delta, \psi, \phi, \Delta'}$	ER
$\dfrac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta}$	$\wedge L$	$\dfrac{\Gamma \vdash \phi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \wedge \psi, \Delta}$	$\wedge R$
$\dfrac{\Gamma, \phi \vdash \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \vee \psi \vdash \Delta}$	$\vee L$	$\dfrac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \vee \psi, \Delta}$	$\vee R$
$\dfrac{\Gamma \vdash \phi, \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \supset \psi \vdash \Delta}$	$\supset L$	$\dfrac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \supset \psi, \Delta}$	$\supset R$
$\dfrac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}$	$\neg L$	$\dfrac{\Gamma, \phi \vdash}{\Gamma \vdash \neg\phi, \Delta}$	$\neg R$

and

$$\dfrac{\dfrac{}{\psi, \chi \vdash \psi}\, Ax}{\psi \vdash \chi \supset \psi, \phi \supset \psi} \supset R.$$

They arise as restrictions to intuitionistic logic from the following classical derivation:

$$\dfrac{\dfrac{\dfrac{}{\psi, \phi, \chi \vdash \psi, \psi}\, Ax}{\psi, \chi \vdash \phi \supset \psi, \psi} \supset R}{\psi \vdash \phi \supset \psi, \chi \supset \psi} \supset R.$$

In this case, both derivations are proofs even in intuitionistic logic, and hence the order in which the $\supset R$-rules are executed does not matter. In general, however, this order matters [134]. As an easy example, consider the sequent

$$\psi \vdash \phi \supset \psi, \sigma \supset \tau.$$

If the formula $\phi \supset \psi$ is reduced first working from root to leaves then the reduction succeeds, otherwise it fails. However, in classical logic the order does not matter. So it becomes apparent already that the reduction in the classical sequent calculus, when viewed as a reduction for intuitionistic proofs, proceeds in parallel: one classical sequent derivation may have many intuitionistic subderivations which are not permutations of each other.

The classical $\vee R$-rule gives rise to another instance of parallelism. The reason is that in the classical rule both disjuncts are side-formulæ, whereas the single-conclusioned intuitionistic $\vee R$-rule keeps only one of the disjuncts. So a classical derivation may contain two completely different intuitionistic subderivations. An example is

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\phi, \chi, \psi, \sigma \vdash \phi, \chi}\, Ax}{\phi, \chi, \psi \vdash \phi, \sigma \supset \chi} \supset R}{\phi, \chi \vdash \psi \supset \phi, \sigma \supset \chi} \supset R}{\phi, \chi \vdash (\psi \supset \phi) \vee (\sigma \supset \chi)} \vee R. \tag{4.1}$$

This derivation contains two intuitionistic subderivations, one for the sequent $\phi, \chi \vdash \psi \supset \phi$ and the other for the sequent $\phi, \chi \vdash \sigma \supset \chi$. These two derivations are obtained by applying the two possible versions of the single-conclusioned intuitionistic $\vee R$-rule to the sequent $\phi, \chi \vdash (\psi \supset \phi) \vee (\sigma \supset \chi)$.

Although inferences in classical logic can be freely permuted, the property of a classical sequent derivation having an intuitionistic subderivation is not always invariant under permutation. Examples of this phenomenon are a bit more complicated. Consider the sequent

$$x\colon \phi \supset \psi, y\colon (\phi \supset \psi) \supset \psi \vdash \psi\,,$$

where we have attached variables to the antecedents to make it easier to refer to a specific formula. If first x is reduced and then y, there is no way of identifying an intuitionistic subderivation. However, if we reduce first y, and then x, then we obtain an intuitionistic derivation. Both derivations are shown in Figs 4.1 and 4.2 respectively (see page 130).

However, our perspective in this monograph is to draw a distinction between the structural proof theory of reductive logic, that is, the formulation of systems of operators and the analysis of their *declarative* properties, and the algorithms used in attempts to construct proofs of specific putative conclusions. In this chapter, our focus is on the structural, proof-theoretic, properties of systems of reduction operators. The addition of algorithmic control to reductive proof, to give proof-search, is considered in Chapter 6.

As usual, before moving on with our main development we pause briefly to consider the issues which arise in extending our analysis to first-order predicate logic. See, for example, [95] or [42] for a suitably systematic treatment of the proof theory of classical and intuitionistic first-order predicate logic.

Most of the analysis we have discussed so far in this chapter extends straightforwardly to first-order predicate logic. The permutation theorem, however, does not. To see this, consider, following Wallen [134], the (LJ- and LK-provable) sequent

$$\vdash \exists w.\forall x.p(w,x) \supset \forall y.\exists z.p(z,y)\,.$$

Starting from this endsequent, the only reduction operator that is applicable is $\supset R$, giving

$$\frac{\exists w.\forall x.p(w,x) \vdash \forall y.\exists z.p(z,y)}{\vdash \exists w.\forall x.p(w,x) \supset \forall y.\exists z.p(z,y)} \supset R. \tag{4.2}$$

At this point, we must choose to proceed either with a $\exists L$ or with a $\forall R$. In either case, we must introduce a parameter. Suppose we proceed with $\exists L$, introducing the parameter a, then we get

$$\frac{\dfrac{\forall x.p(a,x) \vdash \forall y.\exists z.p(z,y)}{\exists w.\forall x.p(w,x) \vdash \forall y.\exists z.p(z,y)} \exists L}{\vdash \exists w.\forall x.p(w,x) \supset \forall y.\exists z.p(z,y)} \supset R. \tag{4.3}$$

Here we have observed a side-condition on the $\exists L$-rule: the parameter must not occur free in the leaf of (4.3).

At this stage in our reduction, we must choose between $\forall L$ and $\forall R$ for the next step. Suppose we proceed with $\forall L$, introducing the parameter b, then we get

$$\dfrac{\dfrac{\dfrac{p(a,b) \vdash \forall y.\exists z.p(z,y)}{\forall x.p(a,x) \vdash \forall y.\exists z.p(z,y)}\,\forall L}{\exists w.\forall x.p(w,x) \vdash \forall y.\exists z.p(z,y)}\,\exists L}{\vdash \exists w.\forall x.p(w,x) \supset \forall y.\exists z.p(z,y)}\,\supset R. \tag{4.4}$$

Note that there is no restriction on the choice of parameter for a universal on the left.

We now have no choice. We must proceed with $\forall R$, for which we must choose a parameter which does not occur free in the leaf of (4.4), that is, neither a nor b. Choosing a parameter c, we get

$$\dfrac{\dfrac{\dfrac{\dfrac{p(a,b) \vdash \exists z.p(z,c)}{p(a,b) \vdash \forall y.\exists z.p(z,y)}\,\forall R}{\forall x.p(a,x) \vdash \forall y.\exists z.p(z,y)}\,\forall L}{\exists w.\forall x.p(w,x) \vdash \forall y.\exists z.p(z,y)}\,\exists L}{\vdash \exists w.\forall x.p(w,x) \supset \forall y.\exists z.p(z,y)}\,\supset R. \tag{4.5}$$

Now, the only remaining possibility is $\exists R$. But any reduction via $\exists R$ will lead to a sequent of the form

$$p(a,b) \vdash p(d,c),$$

for some parameter d. The choice of d is not restricted, so we can choose $d = a$, but even with this choice there is no way to achieve an axiom: c must be different from b, and the reduction we have described fails to construct a proof.

The endsequent is, however, provable. Here is a correct proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(a,b) \vdash p(a,b)}\,Ax}{p(a,b) \vdash \exists z.p(z,b)}\,\exists R}{\forall x.p(a,x) \vdash \exists z.p(z,b)}\,\forall L}{\forall x.p(a,x) \vdash \forall y.\exists z.p(z,y)}\,\forall R}{\exists w.\forall x.p(w,x) \vdash \forall y.\exists z.p(z,y)}\,\exists L}{\vdash \exists w.\forall x.p(w,x) \supset \forall y.\exists z.p(z,y)}\,\supset R. \tag{4.6}$$

So we can see that there is an *order dependence* between the quantifier rules which obstructs the permutation theorem. It is, however, possible to recover a weaker permutation via the idea or a *reduction ordering*. The details of this technique, described in detail in [134], are beyond our present scope. However, the basic idea is quite straightforward. Briefly, for a given reduction, we identify two orderings:

1. $\ll$ is the *formula ordering*: roughly, $\phi \ll \psi$ just in case ϕ is subformula of ψ;
2. $\sqsubset$ is the *substitution ordering*: roughly, $\sqsubset$ represents the ordering constraints on the introduction of parameters.

Clearly, $\ll$ and $\sqsubset$ can be formulated over a common set of labels.

The reduction ordering, $\lhd$, is then given by the transitive closure of the union of the formula ordering and the substitution ordering:

$$\lhd = (\ll \cup \sqsubset)^+.$$

Now, recall that (4.5) could not be completed to a proof because of the side-conditions about the choice of parameters. In the absence of the side-conditions, the reduction could have been completed to give a leaf $p(a,b) \vdash p(a,b)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(a,b) \vdash p(a,b)}\, Ax}{p(a,b) \vdash \exists z.p(z,b)}\, \exists R}{p(a,b) \vdash \forall y.\exists z.p(z,y)}\, \forall R}{\forall x.p(a,x) \vdash \forall y.\exists z.p(z,y)}\, \forall L}{\exists w.\forall x.p(w,x) \vdash \forall y.\exists z.p(z,y)}\, \exists L}{\vdash \exists w.\forall x.p(w,x) \supset \forall y.\exists z.p(z,y)}\, \supset R. \tag{4.7}$$

However, by abandoning the side-conditions, in general, we lose the soundness of the quantifier rules. The reduction ordering provides a solution: we perform a reduction using the quantifier rules without their side-conditions but, for the completed reduction, calculate the reduction ordering, $\lhd$. If $\lhd$ is *acyclic*, then the reduction *determines* a proof. If it is *cyclic*, then it does not [134].

For example, not only is (4.6) acyclic but so is (4.7). The acyclic reduction ordering determines an equivalence class of reductions, which includes the correct proof and those permutation variants of it which fail to be proofs merely because the choice of substitution of parameters is inconsistent with the choice of the order of quantifier reductions, but for which a consistent choice of order of quantifier reductions, that is, a reordering of the given choice, exists.

4.2 Reductive proof theory

We are now ready to begin our proof-theoretic and semantic studies of reductive logic. Recall that our concern is to provide a framework for the semantics of intuitionistic and classical reductive logic and their associated proof-search procedures which is of comparable value to that which is available for the corresponding deductive systems. That is, there should be a ‘Curry-Howard’ correspondence between proofs, functional terms, and their interpretations in a suitable algebraic—here categorical—structure. Moreover, the categorical semantics of reductions should, as far as possible, be formulated within a framework which properly generalizes the existing categorical account of truth-functional semantics [70].

As we have explained, this view may summarized by the diagram

$$\begin{tikzcd}
\Phi \Rightarrow \Gamma \ \mathbf{?\text{-}}\ \phi \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & [\Gamma] \mathrel{|\!\sim} [\Phi] : [\phi] \arrow[dl, leftrightarrow] \\
& [\![\Gamma]\!] \overset{[\![\Phi]\!]}{\leadsto} [\![\phi]\!] &
\end{tikzcd}$$

in which the top left-hand corner denotes reductions in an appropriate calculus, the top right-hand corner denotes the corresponding typing of functional terms, and the bottom corner denotes the categorical semantics.

In this chapter, we are concerned with two parts of this picture, the judgements $\Phi \Rightarrow \Gamma\ \mathbf{?\text{-}}\ \phi$ and $[\Gamma] \mathrel{|\!\sim} [\Phi] : [\phi]$. Recall that the turnstile $\mathbf{?\text{-}}$ is used, in the judgement

$$\Phi : \Gamma\ \mathbf{?\text{-}}\ \phi\,,$$

to denote that Φ is a reduction of ϕ from Γ, that is, Φ denotes a derivation tree, regulated by reduction operators, with root node $\Gamma\ \mathbf{?\text{-}}\ \phi$ and leaf nodes, which need not be axioms. In fact, in this chapter, we will not be concerned directly with the manipulation or semantics of uncompleted reductions. Accordingly, we will work with the more familiar notation $\Phi \Rightarrow \Gamma \vdash \phi$ rather than $\Phi \Rightarrow \Gamma\ \mathbf{?\text{-}}\ \phi$, which will be indispensible in Chapters 5 and 6.

4.2.1 *Background*

The significance of reductive logic derives from a desire among a broad community of logicians, mathematicians, and computing scientists to use mathematical logic, and the vast range of formal languages inspired by its development, as a basis for *automated reasoning* or, more prosaically, *automated theorem proving*.

One could argue for many different starting points for this line of enquiry. However, a prerequisite is the late nineteenth century advancement of logic from its mediæval form to a mathematical theory.[35] So, taking the view that

[35]It was once said by Immanuel Kant that logic was the only science that had made no progress at all since antiquity [131]. We suggest, however, that this view was mistaken: there is a substantial sense of much of modern logic in the work of the *scholastics*; see, for example, [19].

mathematical logic is a essentially prior development, one sensible point of departure is perhaps a pair of papers consisting of Löwenheim's (1915) *On possibilities in the calculus of relatives* [73], and of Skolem's (1920) *Logico-combinatorial investigations in the satisfiability or provability of mathematical propositions: A simplifed proof of a theorem by L. Löwenheim and generalizations of the theorem* [118]. These papers both explore the determination of the validity of formulæ. A second pair of papers then marks a significant advance. Skolem's (1928) *On mathematical logic* [119] and Herbrand's (1930) *Investigations in proof theory: The properties of true propositions* [49]. These two papers develop two important ideas. Firstly, that the arrangement of the connectives in a formula is significant for understanding and calculating with a formula and, secondly, that *proofs* are suitable objects for mathematical investigation. Herbrand's article also marks the starting point for what we now understand as the resolution method (which we consider at length in the sequel).

In particular, 'Herbrand's Theorem' (Theorem 4.3; see, for example, [34] for a detailed explanation) provides a basis for a mechanical proof procedure for first-order logic, implemented by Gilmore in 1960 [39], of manageable complexity. As Bundy [20] remarks, Gilmore's work established the feasibility of automated theorem proving. However, perhaps the most significant development in the 1960s was Julia Robinson's *resolution* procedure [113]. For formulæ in a certain, functionally complete, clausal form the resolution rule is, together with the use of *unification* [34] to calculate terms, both computationally appealing and logically complete. The resolution procedure, together with a control régime for selecting to which clauses from a set the resolution rule should next be applied, forms the basis of the programming language Prolog [21, 22], and Kowalski's famous dictum,

$$Programming = Logic + Control.$$

In recent years, Miller *et al.* [46, 80, 104] have provided more systematic accounts of logic programming via the sequent calculus, and the proof-theoretic basis of that work provides a point of departure for much of our analysis in this mongraph.

Considering once again not logic programming but rather automated theorem proving in general, and from a more systematic perspective, it is standard practice to draw a distinction between *local* methods, inspired historically by technique's such as the resolution calculus [113] and Maslov's inverse method [78], and *global* methods, inspired by the Gentzen's sequent calculi, as discussed above, and Smullyan's tableaux systems [120]. Of course, local methods may have global characterizations, and vice versa. For example, adumbrating much of our forthcoming analysis, the sequent calculus may be used to characterize resolution.

Local methods have the advantage of deriving small independent objects like clauses bearing a simple logical relationship to their parents and allowing wholesale dynamic simplification of the 'search space'[36] using operations such as subsumption.

Global methods, in contrast, are typically *analytic* [29], that is, essentially all formulæ used in the search procedure are subformulæ of the formula to be proved. These global methods benefit from complex implementation methods [6, 16] and produce search spaces which are deep but well-focussed on the shortest proof available; proofs which are, nevertheless, exponentially longer that the shortest proofs available in their local, non-analytic counterparts [17].

The characterization outlined above is broadly coherent for classical propositional and predicate logic. For non-classical logics, in particular for intuitionistic logic, global methods are more easily developed [108] and, as Mints points out in [82], many attempts to formulate local methods for non-classical systems fail to preserve the essential properties of local methods for classical systems. Indeed, he goes on to formulate a list of criteria by which system can qualify as 'resolution'.

A major development since the 1970s and 1980s has been away from fully automated theorem provers into *interactive theorem provers*. These have been developed in response to the so-called *combinatorial explosion*. These systems essentially derive from LCF [43, 91],[37] developed at Stanford and Edinburgh, based on Scott's higher-order logic PPλ. Examples include Isabelle [85], LEGO [94], and Coq [31].

The basic idea of interactive theorem provers is based directly on the idea of reduction. Faced with a putative conclusion, or *goal*, say

$$\phi_1, \ldots, \phi_m \vdash \psi_1, \ldots, \psi_n \,,$$

the user must perform an action which is application to the sequent. Typically, this will be an application of either a single reduction operator, or a *tactic*, that is, a program which systematically applies a combination of reduction operators, or a *tactical*, that is, a program which systematically applies a combination of tactics, to some choice of ϕs and ψs. The result of such an action will be either success, failure, or a new collection of goals (or *subgoals*). In LCF and its derivatives, tactics and tacticals are written in the programming language ML [81].

4.2.2 *Our perspective*

Our perspective in this chapter, perhaps novelly within this area of logic, is not historical, and not merely technical. Rather, it is foundational. That is, we are concerned with the underlying structure of reductive systems, with the choices

[36] This idea is discussed in detail in Chapter 6 but the basic idea is that the search space is the space of possible constructions which may be explored when trying to construct a proof.

[37] LCF stands for 'Logic for Computable Functions'.

that are possible in the design of systems, and with the mathematical analysis of these ideas.

Given that our approach is foundational, it is convenient to restrict our attention to propositional systems, without the added complications of predication and quantification. Whilst such a restriction might, at first sight, seem to eliminate key difficulties, it is in fact essentially harmless. It will be evident that our analyses extend to first-order settings without difficulty.[38] As in the preceding chapters, we shall discuss briefly the extension of the analyses in this chapter to first-order systems.

As we have already explained, in Chapter 1, our guiding theme is the rôle of classical logic as a meta-logic for intuitionistic logic. Thus the focus of our foundational perspective is on the use of classical calculi to construct intuitionistic proofs. Our technical starting point is the sequent calculus, as discussed in Section 4.1, but before we embark on our technical development, we consider, in this section its place within the range of local and global approaches to reductive proof.

We begin our technical development, in Section 4.3, with a representation of proofs in the classical sequent calculus, LK, as terms of the $\lambda\mu\nu\epsilon$-calculus. This representation provides a language of realizers for classical consequences and allows, in Section 4.4, a characterization, via the $\lambda\mu\nu\epsilon$-calculus, of those LK derivations which determine intuitionistic proofs.

We proceed, in Section 4.5, with a development which is inspired by the notion of *uniform proof*, introduced by Miller *et al.* in the context of logic programming [80]. We introduce a larger class of proofs, called *weakly uniform*, which is sufficient to characterize intuitionistic provability within the classical calculus. This section concludes with an application to a restricted class of uniform proofs called *analytic resolution proofs*.

Having set-up our basic analytic theory, we come to our main application, the analysis of *classical and intuitionistic resolution*. Although used as the foundation not only for the logic programming language Prolog but also for many automated theorem provers, resolution is formulated not as a reductive system but as a deductive system for deriving inconsistency, that is, as a *refutation system*.

The basic idea, formulated in classical logic, is as follows:[39]

1. We have a set of formulæ in *clausal form*;[40]
2. Suppose we have two clauses $C_1 \vee p$ and $C_2 \vee \neg p$, where C_1 is a disjunction of m_1 literals and C_2 is a disjunction of m_2 literals, and where p and $\neg p$ are literals. Then the resolution rule for $C_1 \vee p$ and $C_2 \vee \neg p$ is the following

[38] Indeed, even a restriction to first-order is largely unnecessary.

[39] Our technical development, in Section 4.6, uses a more sophisticated formulation.

[40] A set of formulæ is in clausal form if and only each formula is a clause. A clause is a disjunction of literals. A literal is an atomic proposition or a negated atomic proposition.

inference:

$$\frac{C_1 \vee p \quad C_2 \vee \neg p}{C_1 \vee C_2}. \tag{4.8}$$

Here, the conclusion is called the *resolvant* and the two premisses are called its *parents*, within which p and $\neg p$ are called the *complementary literals*. Note that, in each resolution step, an occurrence of a literal is eliminated from the set of clauses;

3. If a clause is a disjunction of 0 literals, then it is logically equivalent to $\perp$ and is said to be *empty*.[41] If $m_1 = m_2 = 0$, then the resolvant is logically equivalent to $\perp$ and we have derived a refutation;
4. Thus, if Γ is a set of clauses and if p is a literal, then to determine whether the sequent $\Gamma \vdash p$ has a proof we seek to deduce a refutation of $\Gamma \cup \{\neg p\}$;
5. In predicate logic, we may have clauses $C_1 \vee p_1$ and $C_2 \vee \neg p_2$, together with a substitution σ for the first-order variables in p_1 and p_2 such that, where $\equiv$ denotes syntactic identity, $p_1\sigma \equiv p_2\sigma$.[42]

The leading application of (predicate) resolution is, perhaps, the logic programming language Prolog [21, 22]. Prolog is based on classical resolution for *Horn clauses*, named after Alfred Horn, who first identified them. Here we consider just the propositional case.

A (propositional) Horn clause is a clause with at most one unnegated literal, that is, it has the following form:[43]

Horn clause: $p_1 \wedge \cdots \wedge p_m \supset \phi$, where ϕ is atomic or $\perp$.

A Horn clause is *definite* if ϕ is not $\perp$. A *program*, $\mathcal{P}$, consists of a finite set of definite clauses. A *goal* or *query*, $\mathcal{G}$, consists of a finite set of atomic formulæ $\{G_1, \ldots, G_m\}$, with each $G_i \neq \perp$ and the corresponding *goal formula* is the formula $\bigwedge \mathcal{G} = G_1 \wedge \cdots \wedge G_m$.

The following use of the resolution rule is called *(propositional) SLD resolution*[44] and derives a goal, $\mathcal{G}, \mathcal{G}'$, from a program, $\mathcal{P}$:

$$\frac{\mathcal{G}, p \qquad \bigwedge \mathcal{G}' \supset p}{\mathcal{G}, \mathcal{G}'}.$$

where $\bigwedge \mathcal{G}' \supset p$ is a clause in $\mathcal{P}$. In general, SLD resolution is defined for predicate Horn clauses and requires a clause $\bigwedge \mathcal{G}' \supset q \in \mathcal{P}$. and a substitution θ (which can be calculated by unification) such that $p\theta = q\theta$.

[41] The empty clause is sometimes written $\Box$.

[42] In first-order predicate logic, such a substitution may be calculated using a unification algorithm (see [113] for a discussion). For higher-order systems, weaker results are available [55, 62, 99].

[43] Note that here we adopt the so-called *Kowalski form* [20, 67] in which a clause $\neg p_1 \vee \cdots \neg p_m \vee q_1 \vee \cdots \vee q_n$ is written in the form $(p_1 \wedge \cdots \wedge p_m) \supset (q_1 \vee \cdots \vee q_n)$.

[44] *SLD* stands for *S*elected *L*iteral for *D*efinite clauses.

The following theorem characterizes classical resolution:

Theorem 4.5 (soundness and completeness of SLD resolution) *SLD resolution is sound and complete with respect to the classical sequent calculus: if Γ is a set of clauses, then Γ has an SLD resolution proof iff $\Gamma \vdash \bot$ is provable in the classical sequent calculus.*

In this chapter, we show that resolution for classical logic may be characterized as a reductive system via our treatment of uniform and weakly uniform proofs. Further, we show that Mints' intuitionistic resolution calculus [82] may be reconstructed from classical resolution together with our characterization of intuitionistic proof within the classical calculus. Thus we provide a systematic, analytic, reductive treatment of classical and intuitionistic resolution via uniform and weakly uniform proofs.

We emphasize once again, however, that at this stage we pay no attention to any of the algorithmic aspects of the construction of reductions. Such issues are treated in Chapter 6.

4.3 Representation of sequent derivations in $\lambda\mu\nu\epsilon$

In this section, we describe the use of the $\lambda\mu\nu\epsilon$-calculus, introduced in Chapter 2, to represent sequent proofs. Below, we show how to use this representation to formulate a condition on classical derivations to determine when they have intuitionistic subderivations. This is formulated as a condition on a $\lambda\mu\nu\epsilon$-term that interprets the classical derivation (see Definition 4.9). Subsequently, we show how transformations on the $\lambda\mu\nu\epsilon$-terms can be used to characterize the search space over a given endsequent (see Theorem 4.16). We prove the completeness of a particular search strategy for classical logic with respect to intuitionistic provability. Again, the formulation of this strategy uses $\lambda\mu\nu\epsilon$-terms (see Theorem 4.23).

Turning to the formal representation, then, we start by giving the translation from classical sequent derivations into the $\lambda\mu\nu\epsilon$-calculus. Note that the classical sequent derivations have to be suitably annotated for the definition. Firstly, each sequent has one principal formula in the succedent together with an arbitrary number of additional formulæ. We introduce a *name* for each additional formula in the succedent and a *variable* for each formula in the antecedent. Secondly, the translation has to take the explicit exchange rule in the $\lambda\mu\nu\epsilon$-calculus into account. For example, the axiom $\Gamma, x\colon \phi \vdash \phi, \psi^{\beta}$ can be translated to the variable x; the axiom $\Gamma, x\colon \phi \vdash \psi, \phi^{\alpha}$, however, must be translated to the $\lambda\mu\nu\epsilon$-term $\mu\alpha.[\beta]x$.

We shall use the following notation: if Φ is a derivation whose last rule is R applied to the derivations $\Phi_1, \ldots, \Phi_n$, we write $(\Phi_1, \ldots, \Phi_n); R$ for Φ.

Definition 4.6 Let $\Phi\colon \Gamma \vdash \phi, \Delta$ be a classical sequent derivation and suppose that each occurrence of a formula in Γ and Δ has a label, that is, the contexts Γ and Δ satisfy $\Gamma = x_1\colon \phi_1, \ldots, x_n\colon \phi_n$ and $\Delta = \psi_1^{\beta_1}, \ldots, \psi_m^{\beta_m}$. (These labels turn into variables and names in the $\lambda\mu\nu\epsilon$-calculus, hence we also use them for the derivations.) We define a $\lambda\mu\nu\epsilon$-term ϕ satisfying $\Gamma \vdash [\![\Phi]\!]\colon \phi, \Delta$ by induction over the structure of ϕ as follows (note the clause for the exchange rule):

Axiom: Suppose $\Phi : \Gamma, x\colon \phi \vdash \phi, \Delta$ is an axiom, then $[\![\Phi]\!] \stackrel{\text{def}}{=} x$;
Exchange: Suppose $\Phi\colon \Gamma \vdash \phi, \psi^{\beta}, \Delta$, and

$$\phi' = \phi;\ E\colon \Gamma \vdash \psi, \phi^{\alpha}, \Delta.$$

We define $[\![\Phi']\!]$ to be the normal form of the term $\mu\beta.[\alpha][\![\Phi]\!]$ with respect to the rules β^{μ} and η^{μ};
$\wedge L$: Suppose we have the derivation

$$\frac{\Phi\colon \Gamma, x\colon \phi, y\colon \psi \vdash \chi, \Delta}{\Phi; \wedge L\colon \Gamma, z\colon \phi \wedge \psi \vdash \chi, \Delta}\ \wedge L,$$

then the corresponding $\lambda\mu\nu\epsilon$-term is

$$[\![\Phi; \wedge L]\!] \stackrel{\text{def}}{=} [\![\Phi]\!][\pi(z)/x, \pi'(z)/y];$$

$\wedge R$: Suppose we have the derivation

$$\frac{\Phi\colon \Gamma \vdash \phi, \Delta \qquad \Psi\colon \Gamma \vdash \psi, \Delta}{(\Phi, \Psi); \wedge R\colon \Gamma \vdash \phi \wedge \psi, \Delta}\ \wedge R,$$

then we define

$$[\![(\Phi, \Psi); \wedge R]\!] \stackrel{\text{def}}{=} \langle [\![\Phi]\!], [\![\Psi]\!] \rangle;$$

$\supset L$: Suppose we have the derivation

$$\frac{\Phi\colon \Gamma \vdash \phi, \chi^{\gamma}, \Delta \qquad \Psi\colon \Gamma, w\colon \psi \vdash \chi, \Delta}{(\Phi, \Psi); \supset L\colon \Gamma, x\colon \phi \supset \psi \vdash \chi, \Delta}\ \supset L$$

then we define $[\![(\Phi, \Psi); \supset L]\!]$ to be the normal form of

$$\mu\gamma.[\gamma][\![\Psi]\!]\,\{x[\![\Phi]\!]/w\}$$

with respect to the reduction rules β^{μ} and η^{μ};
$\supset R$: Suppose we have the derivation

$$\frac{\Phi\colon \Gamma, x\colon \phi \vdash \psi, \Delta}{\Phi; \supset R\colon \Gamma \vdash \phi \supset \psi, \Delta}\ \supset R\ ,$$

then we define $[\![\Phi; \supset R]\!]$ to be $\lambda x\colon \phi.[\![\Phi]\!]$;

$\neg L$: Suppose we have the derivation

$$\frac{\Phi\colon \Gamma \vdash \phi, \chi^{\gamma}, \Delta}{\Phi; \neg L\colon \Gamma, x\colon \neg\phi \vdash \chi, \Delta} \supset L$$

then we define $[\![\Phi; \neg L]\!]$ to be the normal form of $\mu\gamma.[\bot]w\,\{x[\![\Phi]\!]/w\}$ by the reduction rules β^{μ} and $\eta^{\bot}$;

$\neg R$: Suppose we have the derivation

$$\frac{\Phi\colon \Gamma, x\colon \phi \vdash \psi, \Delta}{\Phi; \neg R\colon \Gamma \vdash \neg\phi, \psi^{\beta}, \Delta} \supset R\ ,$$

then we define $[\![\Phi; \neg R]\!]$ to be the normal form of $\lambda x\colon \phi.\mu\bot.[\beta][\![\Phi]\!]$ via the reduction rules β^{μ} and $\eta^{\bot}$;

$\vee L$: Suppose we have the derivation

$$\frac{\Phi\colon \Gamma, x\colon \phi \vdash \Delta \qquad \Psi\colon \Gamma, y\colon \psi \vdash \Delta}{(\Phi, \Psi); \vee L\colon \Gamma, z\colon \phi \vee \psi \vdash \Delta} \vee L$$

we define $[\![(\Phi, \Psi); \vee L]\!]$ to be the normal form of

$$\mu\gamma.[\gamma][\![\Psi]\!]\,\{\mu\alpha.[\gamma][\![\Phi]\!]\,\{\langle\alpha\rangle z/y\}\,/x\}$$

with respect to the reduction rules β^{μ} and η^{μ};

$\vee R$: Suppose we have aderivation

$$\frac{\Phi\colon \Gamma \vdash \phi, \psi^{\beta}, \Delta}{\Phi; \vee R\colon \Gamma \vdash \phi \vee \psi, \Delta}$$

then we define $[\![\Phi; \vee R]\!] = \nu\beta.[\![\Phi]\!]$.

The labelling of the assumptions has one important consequence, namely that there are several possible translations for the same classical sequent derivation. As an example, take the sequent derivation

$$\cfrac{\cfrac{\cfrac{}{\psi, \chi, \phi \vdash \psi, \psi}\ Ax}{\psi, \phi \vdash \chi \supset \psi, \psi} \supset R}{\psi \vdash \phi \supset \psi, \chi \supset \psi} \supset R.$$

There are two possible $\lambda\mu\nu\epsilon$-terms corresponding to this derivation, namely

$$\lambda x\colon \phi.\mu\beta.[\gamma]\lambda y\colon \chi.\mu\delta.[\beta]b$$

and

$$\lambda x\colon \phi.\mu\beta.[\gamma]\lambda y\colon \chi.b\,,$$

where we use the name b to denote the variable corresponding to the formula ψ on the left-hand side. The first proof-term uses the second occurrence of ψ at the leaf for the axiom, whereas the second uses the first occurrence of ψ in the succedent. In this case the difference does not matter—both derivations contain intuitionistic subderivations—but this is not generally true.

4.4 Intuitionistic provability

In this section, we describe how to use the translation of classical LK′-derivations given in the previous section to give a criterion when a classically provable sequent is in fact also intuitionistically provable.

In deciding when a classical derivation indicates that its endsequent is intuitionistically provable, the requirement is to detect superfluous inferences. Consider again the sequent $\psi \vdash \phi \supset \psi, \sigma \supset \tau$. This sequent has an intuitionistic proof in which $\phi \supset \psi$ is reduced first. There is also the following classical proof of this sequent:

$$\dfrac{\dfrac{\dfrac{}{\psi, \phi, \sigma \vdash \psi, \tau}\,Ax}{\psi, \phi \vdash \psi, \sigma \supset \tau}\supset R}{\psi \vdash \phi \supset \psi, \sigma \supset \tau}\supset R\,.$$

We want to be able to detect that the use of the $\supset R$-rule to reduce the formula $\sigma \supset \tau$ is superfluous by using the $\lambda\mu\nu\epsilon$-term corresponding to this proof. We can then conclude that there is an intuitionistic proof of this sequent. The $\lambda\mu\nu\epsilon$-term representing this derivation is

$$\lambda x\colon \phi.\mu\beta.[\gamma]\lambda y\colon \sigma\mu\epsilon.[\beta]b\ ,$$

and the detection amounts to determining when a subterm (here the λ-abstraction over σ) models Weakening on the right. This example motivates the following definition:

Definition 4.7 We define Weakening terms and Weakening occurrences of names by induction over the structure of terms as follows:

1. $\mu\alpha.t$ is a Weakening term if all occurrences of α in t are Weakening occurrences;
2. A term t of type $\perp$ is always a Weakening term;

3. $\langle t, s\rangle$ is a Weakening term if t and s are Weakening terms;
4. $\lambda x\colon \phi.t$ is a Weakening term if t is a Weakening term and if x has only Weakening occurrences in t;
5. The outermost occurrence of α in $[\alpha]t$ and $\langle\alpha\rangle t$ is a Weakening occurrence if t is a Weakening term;
6. $\nu\alpha.t$ is a Weakening term if t is a Weakening term and all occurrences of α are Weakening occurrences;
7. All occurrences of $\perp$ in t are Weakening occurrences;
8. The occurrence of the variable x in tx is a Weakening occurrence if t is a Weakening term and x is not free in t. In this case, the term tx is a Weakening term as well;
9. $t\,\{u/x\}$ is a Weakening term if t is a Weakening term.

As an example, consider the term

$$\lambda x\colon \phi.\mu\beta.[\gamma]\lambda y\colon \sigma.\mu\epsilon.[\beta]b\ ,$$

which we considered before this definition. The term $\lambda y\colon \sigma.\mu\epsilon.[\beta]b$ is a Weakening term, and the only occurrence of γ is a Weakening occurrence. The occurrence of β is not a Weakening occurrence.

Lemma 4.8 *Let t be a term of type $\Gamma \vdash t\colon \phi, \psi^{\beta}, \Delta$. If β has only Weakening occurrences in t, then every term s such that $\mu\beta.[\alpha]t \rightsquigarrow^{*} s$ is a Weakening term.*

Proof It is obvious that $\mu\beta.[\alpha]t$ is a Weakening term. Show by considering each reduction rule that first reductions of t to t' only delete occurrences of α or insert new Weakening occurrences of α but never create non-Weakening instances of a name, second, that any contractum of a Weakening term is a Weakening term, and third that reduction of a term which contains only Weakening occurrences of x yields a term with only Weakening occurrences of x. We just present a few cases here. If the term is $\mu\alpha.t$, then either $\mu\alpha.t \rightsquigarrow \mu\alpha.t'$ with $t \rightsquigarrow t'$, and then α has only Weakening occurrences by the first statement. If the reduction was $\lambda x\colon \phi.(\mu\alpha.t)x \rightsquigarrow \lambda x\colon \phi.\mu\beta.t[[\beta]ux/[\alpha]u]$, then the only occurrences of x and β are Weakening occurrences by the first two statements, and hence t' is a Weakening term. The case $\langle t, s\rangle$ follows from the second statement, the case $\lambda x\colon \phi.t$ from the second and third, and the case $\nu\alpha.t$ follows from the first and second statement. □

Now we can define our first criterion for when a classical sequent derivation determines the existence of an intuitionistic one.

Definition 4.9 Call a $\lambda\mu\nu\epsilon$-term intuitionistic if in any subterm $\lambda x\colon \phi\,.\,t$, which is not a Weakening term, all occurrences of free names are Weakening occurrences.

Let us reconsider the examples at the beginning of this section. There are two $\lambda\mu\nu\epsilon$-terms corresponding to the two derivations of $\psi \vdash \phi \supset \psi, \sigma \supset \tau$. The first one, which corresponds to reducing $\phi \supset \psi$ first, is the term

$$\lambda x\colon \phi.\mu\beta.[\gamma]\lambda y\colon \sigma.\mu\epsilon.[\beta]b\ ,$$

and the second one, which corresponds to reducing $\sigma \supset \tau$ first, is the term

$$\lambda y\colon \sigma.\mu\delta.[\alpha]\lambda x\colon \phi.b\ .$$

In both cases we have an intuitionistic λ-term because the λ-abstraction over σ is a Weakening term. This example shows the parallelism obtained by using a classical sequent calculus: both intuitionistic subderivations of either of the classical proofs are considered simultaneously without any need for backtracking.

In the same way, there are two $\lambda\mu\nu\epsilon$-term for the classical proof of the sequent $\phi, \chi \vdash (\psi \supset \phi) \vee (\sigma \supset \chi)$, given as an example (4.1) in Section 4.1. The first one, namely $\nu\delta.\lambda x\colon \psi.\mu\alpha.[\delta]\lambda y\colon \sigma.\mu\gamma.[\alpha]a$, corresponds to the axiom $\phi \vdash \phi$ and the second one, namely $\nu\delta.\lambda x\colon \psi.\mu\alpha.[\delta]\lambda y\colon \sigma.c$, corresponds to the axiom $\chi \vdash \chi$. Both terms are intuitionistic and incorporate the two single-conclusioned subderivations simultaneously without the need for backtracking.

As an example of a non-intuitionistic term, consider the formula $\neg\phi \vee \phi$. The classical proof is

$$\cfrac{\cfrac{\cfrac{}{\phi \vdash \phi}\ Ax}{\vdash \neg\phi, \phi}\ \neg R}{\vdash \neg\phi \vee \phi}\ \vee R$$

and the corresponding $\lambda\mu\nu\epsilon$-term is $\nu\alpha.\lambda x\colon \phi.\mu\bot.[\alpha]x$. This term is not intuitionistic because the name α and the variable x have a non-Weakening occurrence in the λ-abstraction. For another example of a non-intuitionistic term, consider Peirce's formula, $((\phi \supset \psi) \supset \phi) \supset \phi$. The classical proof of this formula is

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\phi \vdash \psi, \phi}\ Ax}{\vdash \phi \supset \psi, \phi}\supset R \qquad \cfrac{}{\phi \supset \phi}\ Ax}{(\phi \supset \psi) \supset \phi \vdash \phi}\supset L}{\vdash ((\phi \supset \psi) \supset \phi) \supset \phi}\supset R.$$

If this proof is translated into the $\lambda\mu\nu\epsilon$-calculus, the term obtained is

$$\lambda x\colon (\phi \supset \psi) \supset \phi.\mu\alpha.[\alpha]a\,\{x(\lambda y\colon \phi.\mu\beta.[\alpha]y)/a\}\,.$$

The name α has a non-Weakening occurrence in the λ-abstraction over ϕ; hence this term is not intuitionistic.

Next, we show the correctness of the criterion. The crucial point is that a Weakening term corresponds to a superfluous subderivation. The following lemma makes this precise.

Lemma 4.10 *Let Φ be a derivation $\Phi\colon \Gamma, \phi_1, \ldots, \phi_n \vdash \phi, \psi_1, \ldots, \psi_m, \Delta$ such that*

$$\Gamma, a_1 : \phi_1, \ldots, a_n : \phi_n \vdash [\![\Phi]\!] : \phi, \psi_1^{\beta_1}, \ldots, \psi_m^{\beta_m}, \Delta$$

holds. If the variables a_i and names β_j have only Weakening occurrences in $[\![\Phi]\!]$, then there is a procedure to construct a sequent derivation of $\Gamma \vdash \phi, \Delta$. Moreover, if $[\![\Phi]\!]$ is a Weakening term, then there is a procedure to construct a derivation of $\Gamma \vdash \Delta$. These procedures transform sequent derivations which have an intuitionistic subderivation, as described in Table 4.6, into those with the same property.

Proof By induction over the structure of sequent derivations. We give the case of a $\supset L$-rule to illustrate the argument. Suppose we are given a proof ending with

$$\frac{\Gamma \vdash \chi, \phi^{\alpha}, \Delta \qquad \Gamma, \sigma \vdash \phi, \Delta}{\Gamma, x\colon \chi \supset \sigma \vdash \phi, \Delta} \supset L$$

and suppose that its $\lambda\mu\nu\epsilon$-term is $\mu\alpha.[\alpha]t\,\{xs/w\}$. The only interesting case arises if this term is a Weakening term. In this case, the name α has only Weakening occurrences in t and in s, and t is a Weakening term. By the induction hypothesis, we obtain derivations of $\Gamma \vdash \chi, \Delta$ and $\Gamma, \sigma \vdash \Delta$ and hence also a derivation of $\Gamma, \chi \supset \sigma \vdash \Delta$. $\square$

Now we are in a position to show the correctness of the criterion.

Theorem 4.11 *Let $\Phi\colon \Gamma \vdash \phi, \Delta$ be a classical sequent derivation. If $[\![\Phi]\!]$ is an intuitionistic $\lambda\mu\nu\epsilon$-term, then there exists an intuitionistic derivation of the sequent $\Gamma \vdash \phi, \Delta$.*

Proof We proceed by induction over the structure of derivations of sequents. Suppose the last rule is the rule $\supset R$ to obtain a sequent $\Gamma \vdash \phi \supset \psi, \Delta$. By the induction hypothesis, we have an intuitionistic sequent derivation of $\Gamma, \phi \vdash \psi, \Delta$. Let $[\![\Phi]\!] = \lambda a\colon \phi.t$. Either $[\![\Phi]\!]$ is a Weakening term, in which case Lemma 4.10 implies that there is also an intuitionistic derivation of $\Gamma \vdash \Delta$, and hence also of $\Gamma \vdash \phi \supset \psi, \Delta$. If $[\![\Phi]\!]$ is not a Weakening term, then there are no free names in $[\![\Phi]\!]$ that have a non-Weakening occurrence. Hence, by Lemma 4.10 again, there is an intuitionistic derivation $\Gamma, \phi \vdash \psi$. Now the intuitionistic $\supset R$-rule yields the result. □

Finally we show the other direction: each intuitionistic sequent derivation $\Gamma \vdash \Delta$ gives rise to an intuitionistic $\lambda\mu\nu\epsilon$-term $\Gamma \vdash t \colon \phi, \Delta'$, where Δ is a permutation of ϕ, Δ'. To show this, we use the standard inclusion of the multiple-conclusioned LJ-calculus into the LK-calculus: for each $\supset R$-rule

$$\frac{\phi \colon \Gamma, x \colon \phi \vdash \psi}{\Gamma \vdash \phi \supset \psi, \Delta}$$

in a multiple-conclusioned LJ-derivation add Δ to all sequents in the LK-derivation corresponding to Φ and now use the $\supset R$-rule of LK. The same procedure is followed for the $\neg R$-rule.

Theorem 4.12 *Let $\Phi \colon \Gamma \vdash \Delta$ be a multiple-conclusioned LJ-sequent derivation. Then the term $[\![\Psi]\!]$ of the corresponding LK-derivation Ψ is an intuitionistic $\lambda\mu\nu\epsilon$-term.*

Proof Show by induction over the structure of Φ that $[\![\Psi]\!]$ satisfies the typing judgement $\Gamma \vdash [\![\Psi]\!] \colon \phi, \Delta'$, where ϕ, Δ' is a permutation of Δ and that $[\![\Psi]\!]$ is intuitionistic. The only interesting cases are the $\supset R$-rule and $\neg R$-rule. In the first case, the derivation is

$$\frac{\Phi' \colon \Gamma, x \colon \phi \vdash \psi}{\Gamma \vdash \phi \supset \psi, \Delta} \supset R.$$

Let Ψ' be the corresponding LK-derivation. By induction hypothesis, we have $\Gamma, x \colon \phi \vdash [\![\Psi']\!] \colon \psi$, and Ψ' is intuitionistic and contains no free name. We also have also $\Gamma \vdash [\![\Psi']\!] \colon \psi, \Delta$, and hence the term $\Gamma \vdash \lambda x \colon \phi . [\![\Phi']\!] \colon \phi \supset \psi, \Delta$ is an intuitionistic $\lambda\mu\nu\epsilon$-term. The case of the $\neg R$-rule is similar. □

4.5 Uniform proof and analytic resolution

In this section, we specialize the results of the previous section to a special class of sequent proofs and show that a certain classical proof procedure is sound and complete for intuitionistic provability of sequents of propositional hereditary Harrop formulæ. The proof procedure is based on an extension of the notion of *uniform proof* to multiple-conclusioned systems by Miller *et al.* [80] (e.g., see also [47]).

4.5.1 *Uniform proofs*

A uniform proof [80] is a sequent derivation in which, when read from root to leaves, all right rules are applied whenever it is possible so to do, except for axioms with non-atomic principal formulæ.[45] We call a proof *fully uniform* if

[45] An axiom is said to be *atomic* just in case its principal formula is atomic.

right-rules are preferred even over axioms. The notion of a uniform proof leads to a simple, highly deterministic search algorithm: first apply all possible right-rules; then select an appropriate left-rule. Note that Miller *et al.* define uniform proofs for the full, single-conclusioned calculus LJ [37]. In this case, not every LJ-provable propositional sequent has a uniform proof. The reason is that it may be necessary to apply the $\vee L$-rule before the $\vee R$-rule to obtain a proof. As an example in the multiple-conclusioned calculus consider the sequent

$$\phi \vee \psi \vdash \chi \supset \phi, \psi.$$

The only intuitionistic proof of this sequent reduces the formula $\phi \vee \psi$ first. If the formula $\chi \supset \phi$ is reduced first in multiple-conclusioned LJ, we obtain the (intuitionistically unprovable) sequent $\chi, \phi \vee \psi \vdash \phi$.

However, for the fragment of intuitionistic logic consisting only of hereditary Harrop formulæ this is true. Hence in this section we restrict ourselves to hereditary Harrop formulæ.

The definition of propositional hereditary Harrop formulæ (*cf.* [80, 104]) is as follows:

Definition 4.13 Define goal formulæ G and definite formulæ D by

$$\begin{aligned} G &::= p \mid G \wedge G \mid D \supset G \mid G \vee G \\ D &::= p \mid G \supset p \mid D \wedge D, \end{aligned}$$

where p is atomic. Call a sequent $\Gamma \vdash \Delta$ hereditary Harrop if Γ consists of just D-formulæ and Δ consists of just G-formulæ.

It is not the case that each uniform proof in the sense of [80] is a uniform proof in the sense above. The reason is the $\vee R$-rule: in the single-conclusioned calculus, the antecedent throws away one of the disjuncts, whereas both are kept in the multiple-conclusioned calculus. Hence we must expand both formulæ in a uniform proof as defined for the multiple-conclusioned calculus. As an example, consider the derivation of the sequent $\phi, \chi \vdash (\psi \supset \phi) \vee (\sigma \supset \chi)$, given on p. 106. This derivation is uniform in our sense, and the uniform proof in the single-conclusioned calculus, LJ, is

$$\dfrac{\dfrac{\dfrac{}{\phi, \chi, \psi \vdash \phi}\,Ax}{\phi, \chi \vdash \psi \supset \phi}\supset R}{\phi, \chi \vdash (\psi \supset \phi) \vee (\sigma \supset \chi)}\vee R.$$

The uniform derivation in the multiple-conclusioned calculus cannot be obtained by simply adding the formula $\sigma \supset \chi$ to all right-hand sides and then applying the multiple-conclusioned $\vee R$-rule instead.

We make the relation between the two notions of uniform proof precise at the end of the next subsection, after we have studied the effect of permutations on classical uniform proofs.

4.5.2 *Permutations*

The analysis of permutations in proofs is important because there are (well-known) non-permutabilities in intuitionistic logic. We have seen examples of this already, namely with the sequents

$$\psi \vdash \phi \supset \psi, \sigma \supset \tau$$

and

$$(\phi \supset \psi) \supset \psi, \phi \supset \psi \vdash \psi.$$

The first case covers the exchange of two right-rules. There, the order in which the two right-rules were executed did not matter. The second case concerns the exchange of $\supset L$-rules. Whereas in the first case there is a general strategy which renders an exhaustive search of all permutation variants superfluous, in the second case we do have to take into account all possible permutations of $\supset L$-rules for completeness. The invariance under right-rules is covered by the following lemma:

Lemma 4.14 *Let Φ be a classical sequent derivation such that $[\![\Phi]\!]$ is an intuitionistic $\lambda\mu\nu\epsilon$-term.*

1. *If Ψ is the derivation resulting from interchanging any two right-rules apart from an $\supset R$-rule and an $\wedge R$-rule in Φ, then $[\![\Psi]\!]$ is an intuitionistic term.*
2. *If Φ is the derivation*

$$\dfrac{\dfrac{\Gamma, \phi \vdash \psi, \chi, \Delta \qquad \Gamma, \phi \vdash \psi, \sigma, \Delta}{\Gamma, \phi \vdash \psi, \chi \wedge \sigma, \Delta}\ \wedge R}{\Gamma \vdash \phi \supset \psi, \chi \wedge \sigma, \Delta}\ \supset R,$$

then the derivation Ψ obtained by permuting the $\supset R$-rule over the $\wedge R$-rule, towards the leaves, has an intuitionistic $\lambda\mu\nu\epsilon$-term $[\![\Psi]\!]$. Conversely, if we start with a Ψ such that $[\![\Psi]\!]$ is an intuitionistic $\lambda\mu\nu\epsilon$-term, and permute the rules other way around, then at least one of the $\lambda\mu\nu\epsilon$-terms that results from a different choice of axioms in the permuted derivation is intuitionistic.

Proof

1. We show the permutation of a $\supset R$-rule over a $\vee R$-rule as an example. The other cases are similar. So assume we are given a derivation Φ

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \chi \vdash \phi, \psi, \sigma}}{\Gamma, \chi \vdash \phi \vee \psi, \sigma}\vee R}{\Gamma \vdash \phi \vee \psi, \chi \supset \sigma}\supset R,$$

with $[\![\Phi]\!] = \mu\epsilon.[\eta]\lambda z\colon \chi.\mu\delta.[\epsilon]\nu\beta.t$. There are two cases, depending whether $[\![\Phi]\!]$ is a Weakening term or not. We consider only the first case here. In this case the variable z has only Weakening occurrences in t, and the name δ has only Weakening occurrences in t. These conditions are enough to ensure that the $\lambda\mu\nu\epsilon$-term of the permuted derivation Ψ,

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \chi \vdash \phi, \psi, \sigma}}{\Gamma \vdash \sigma, \psi, \chi \supset \sigma}\supset R}{\Gamma \vdash \phi \vee \psi, \chi \supset \sigma}\vee R,$$

which is $[\![\psi]\!] = \nu\beta.\mu\alpha.[\eta]\lambda z\colon \chi.\mu\delta.[\alpha]t$ is an intuitionistic $\lambda\mu\nu\epsilon$-term.

2. The statement about permuting the $\supset R$-rule towards the leaves is shown in the same way as in 1. The additional statement holds because if the term $\lambda x\colon \phi.\mu\beta.[\gamma]t$ is not a Weakening term, then in $[\gamma]t$ the name γ has only Weakening occurrences. Now Lemma 4.10 implies that in this case $\Gamma, \phi \vdash \psi$ has a intuitionistic sequent proof. The derivation is now obvious. □

There are cases in which moving an $\supset R$-rule below a $\wedge R$-rule can lead to a derivation which has no intuitionistic $\lambda\mu\nu\epsilon$-term assigned to it. As an example, consider the (permuted) derivation

$$\dfrac{\dfrac{\dfrac{}{\psi, \sigma, \phi \vdash \psi, \chi}Ax \qquad \dfrac{}{\psi, \sigma, \phi \vdash \psi, \sigma}Ax}{\psi, \sigma, \phi \vdash \psi, \chi \wedge \sigma}\wedge R}{\psi, \sigma \vdash \phi \supset \psi, \chi \wedge \sigma}\supset R$$

the second leaf sequent, the resulting $\lambda\mu\nu\epsilon$-term is not intuitionistic. However, with the other choice, namely the axiom with principal formula ψ, we do obtain an intuitionistic proof.

The key point for the completeness proof of the classical search procedure even for intuitionistic provability below is that the restriction to hereditary Harrop formulæ implies that we can prove a stronger version of the disjunction property. In general, we have only that if a formula $\phi \vee \psi$ is intuitionistically provable, then at least one of ϕ and ψ is intuitionistically provable. Here we can strengthen this property to hold also if there are additional hypotheses under which $\phi \vee \psi$ is provable. When we consider a multiple-conclusioned intuitionistic calculus, we obtain the following lemma:

Lemma 4.15 *Let the sequent $\Gamma \vdash \Delta$ be intuitionistically provable and hereditary Harrop. Then there exists a formula $\phi \in \Delta$ such that $\Gamma \vdash \phi$ is intuitionistically provable too. Moreover, there exists a sequent derivation $\Phi\colon \Gamma \vdash \phi, \Delta$ such that $[\![\Phi]\!]$ has no free names and that in all applications of the $\supset L$-rule in which the principal formula is $G \supset p$, the right branch is the axiom $\Gamma, p \vdash p$.*

Proof By assumption, there exists a normal λ-term t with $\Gamma \vdash t\colon \phi_1 \vee \cdots \vee \phi_n$ if $\Delta = \phi_1, \ldots, \phi_n$. Because the formulæ in Γ and Δ are hereditary Harrop formulæ, the term t is derivable by the grammar

$$t ::= \mathtt{in}_1(t) \mid \mathtt{in}_2(t) \mid \lambda x\colon \phi.t \mid \pi_1(\cdots \pi_m(x) \cdots)t_1 \ldots t_n \mid \langle t, t \rangle,$$

where $\pi_1, \ldots, \pi_m$ is any combination of π and π'. Note that in case $m, n = 0$, the fourth clause reduces to a variable.

Now construct, by induction over the structure of t, a sequent derivation $\Phi\colon \Gamma \vdash \phi, \Delta$ such that $[\![\Phi]\!]$ has no free names and such that in all applications of the $\supset L$-rule the right branch is the axiom $\Gamma, p \vdash p$, in which $G \supset p$ is the principal formula of the $\supset L$-rule. We consider just the case of a term $\Gamma \vdash \mathtt{in}_1(t)\colon \phi \vee \psi$. By induction hypothesis we have a sequent derivation Φ ending in $\Gamma \vdash p$ with all the desired properties. Now consider the derivation Φ' which is Φ with ψ added to the right-hand side of all sequents. We have $[\![\Phi]\!] = [\![\Phi']\!]$, and Φ' has all desired properties as well. It is now easy to see that the derivation $\Phi'; \vee R$ with $[\![\Phi'; \vee R]\!] = \nu\beta.[\![\Phi]\!]$ has all desired properties. □

Now we are in a position to obtain completeness.

Theorem 4.16 *If the hereditary Harrop sequent $\Gamma \vdash \phi, \Delta$ is intuitionistically provable, then, for any possible order of right-rules applied to the succedent, there exists a fully uniform (classical) proof Ψ of the sequent with this order of right rules such that $[\![\Psi]\!]$ is intuitionistic.*

Proof The sequent $\Gamma \vdash \phi, \Delta$ is intuitionistically provable; so, by Lemma 4.15, there exists a formula ψ in ϕ, Δ such that Φ is a fully uniform LJ-proof of $\Gamma \vdash \psi$ in which each leaf of Φ is atomic. Moreover, $[\![\Phi]\!]$ has no free names. Now show by a double induction over the derivation Φ and the structure of formulæ on the right-hand side of Φ that for any such derivation Φ and any antecedent Γ' and succedent Δ', any order of right rules applied to ψ, Δ', there is a fully uniform proof $\Psi : \Gamma, \Gamma' \vdash \psi, \Delta'$, with the order of the right rules such that the following three conditions are met:

1. $[\![\Psi]\!]$ is intuitionistic;
2. Ψ has only Weakening occurrences of free names except possibly a name for the formula ψ, and all subterms corresponding to right rules reducing formulas in Δ' are Weakening terms;
3. The variables occurring in Γ' do not occur in $[\![\Psi]\!]$.

We consider here only the case of the last rule in Φ being a $\supset R$-rule, and the case of the formula $\psi \supset \chi \in \Delta'$. In the first case, suppose that $\Phi = \Phi'; \supset R$ and that $\phi \supset \psi$ is the principal formula of the $\supset R$-rule. Then, by the induction hypothesis, there exists a derivation $\Psi' \colon \Gamma, x \colon \phi, \Gamma', \vdash \psi, \Delta, \Delta'$, where in $[\![\Psi']\!]$ all free names and all free variables in Γ' have only Weakening occurrences. Hence the derivation $\Psi'; \supset R \colon \Gamma, \Gamma' \vdash \phi \supset \psi, \Delta$ satisfies the desired properties. Now we turn to the case of $\psi \supset \chi \in \Delta'$. Here, by the induction hypothesis, there exists a derivation $\Psi' \colon \Gamma, \Gamma', x \colon \psi \vdash \phi, \chi^\gamma, \Delta, \Delta'$ such that x and γ have only Weakening occurrences in $[\![\Psi']\!]$. So the derivation $\Psi'; \supset R$ with $[\![\Psi'; \supset R]\!] = \lambda x \colon \psi.\mu\gamma.[\alpha][\![\Psi']\!]$ has the desired properties. The proof is concluded by setting $\Delta' = \Delta''$, where Δ'' is obtained from Δ by possible exchange of ϕ and ψ. □

Note that the proof of the theorem also shows the way in which the multiple-conclusioned notion of uniform classical proof generalizes the corresponding notion for single-conclusions: each uniform proof in the single-conclusioned sense corresponds to a normal λ-term, and the above proof shows how to construct a multiple-conclusioned uniform proof from this λ-term which contains the original proof as a subproof. As an example, consider the sequent

$$\phi, \chi \vdash (\psi \supset \phi) \vee (\sigma \supset \chi).$$

One uniform derivation in the single-conclusioned calculus LJ is given in Section 4.5.1. The construction in the above proof yields exactly the multiple-conclusioned uniform derivation given in Section 4.10.

Hence to check intuitionistic provability of a sequent with hereditary Harrop formulæ it is enough to construct a uniform proof and then to check, for all possible axiom instances and for all possible exchanges of $\supset L$- and $\neg L$-rules, whether any of the corresponding $\lambda\mu\nu\epsilon$-terms are intuitionistic.

The last two theorems are not true for intuitionistic logic with arbitrary disjunction. In fact, we obtain meaningful results only if we weaken the definition of uniform proof and ask for $\vee L$-rules to occur as close to the root as possible.

Definition 4.17 We define a weakly uniform proof to be a proof in which all possible $\vee L$-rules are as close to the root as possible. In addition, all axioms are only of ground types, and the rightmost branch of an $\supset L$-rule with principal formula $\phi \supset \psi$ is always an axiom if ψ is an atom. Moreover, if the principal formula is $(\phi \supset \chi) \supset \psi$ or $(\phi \wedge \chi) \supset \psi$, then the rule directly preceding the $\supset L$-rule on the left branch is a $\supset R$- or $\wedge R$-rule respectively.

We obtain only the existence of a weakly uniform proof, but not the additional statement that there exists a weakly uniform proof for every order of right-rules.

Theorem 4.18 *If* $\Gamma \vdash \Delta$ *is intuitionistically provable, then there exists a weakly uniform classical proof* Φ *such that* $[\![\Phi]\!]$ *is intuitionistic.*

Proof $\Gamma \vdash \Delta$ is intuitionistically provable, hence there exists a term t in the simply-typed λ-calculus, with product and sum types,[46] in long $\beta\eta$-normal form with $\Gamma \vdash t \colon \psi$, where $\psi = \phi_1 \vee \cdots \vee \phi_n$ and $\phi_1, \ldots, \phi_n$ are all formulæ of Δ. By induction over the structure of this normal form construct a weakly uniform classical sequent proof ϕ such that $[\![\phi]\!]$ is intuitionistic. This is a special case of the translation of natural deduction into sequent calculus, so we just list the case of an application $xs_1 \cdots s_n$ with x of type $\phi_1 \supset \cdots \supset \phi_n \supset \psi$ with ψ atomic. By the induction hypothesis, we have weakly uniform derivations

$$\phi_i \colon \Gamma \vdash \phi_{i,1}, \ldots, \phi_{i,k_i}, \Delta \quad \text{and} \quad \Gamma \vdash s_i \colon \phi_{i,1} \vee \cdots \vee \phi_{i,k_i}.$$

Then we construct the following derivation

$$\dfrac{\dfrac{\Gamma \supset \phi_{1,1}, \ldots, \phi_{1,k_1}, \Delta}{\Gamma \vdash \phi_1, \Delta}\vee R \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma \supset \phi_{n,1}, \ldots, \phi_{n,k_n}, \Delta}{\Gamma \supset \phi_n, \Delta}\vee R \qquad \Gamma, \psi \vdash \psi, \Delta}{\Gamma, \phi_n \supset \psi \vdash \Delta}\supset L}{\cdots}}{\Gamma, \phi_2 \supset \cdots \supset \phi_n \supset \psi \vdash \psi, \Delta}\supset L}{\Gamma, \phi_1 \supset \cdots \supset \phi_n \supset \psi \vdash \psi, \Delta}\supset L.$$

The additional constraints on the $\supset L$-rule follow from the fact that if ϕ_i is a function type, then s_i is a λ-abstraction and if ϕ_i is a product type, then s_i is a product. Hence the translation of s_i into sequent derivations ends with a $\supset R$-rule and a $\wedge R$-rule, respectively. The fact that we consider a λ-term in long $\beta\eta$-normal form ensures that each subterm $\Gamma, z \colon \phi \vee \psi \vdash s \colon \chi$ is actually a term $\Gamma \vdash \texttt{case}\ z\ \texttt{of}\ \texttt{in}_1(x) \supset t(x)\ \texttt{or}\ \texttt{in}_2(y) \supset s(y) : \chi$. Hence the translation of this subterm ends with an $\vee L$-rule. □

4.5.3 *Application to (hereditary Harrop) analytic resolution*

In this section, we apply the results developed above to an analytic resolution procedure for intuitionistically provable hereditary Harrop formulæ based on the $\supset$L rule. The key point is that in an application of a $\supset L$-rule to the formula $\psi \supset \phi$, the formula ϕ is always atomic, and hence can be matched with a formula

[46] Recall that the typing rules for sum types are given by

$$\vee\mathrm{I}\ \dfrac{\Gamma \vdash t \colon \phi}{\Gamma \vdash \texttt{in}_1(t) \colon \phi \vee \psi} \qquad \vee\,\mathrm{I}\ \dfrac{\Gamma \vdash t \colon \psi}{\Gamma \vdash \texttt{in}_2(t) \colon \phi \vee \psi}$$

and

$$\vee\mathrm{E}\ \dfrac{\Gamma \vdash t \colon \phi \vee \psi \qquad \Gamma, x \colon \phi \vdash s \colon \chi \qquad \Gamma, y \colon \psi \vdash u \colon \chi}{\Gamma \vdash \texttt{case}\ t\ \texttt{of}\ \texttt{in}_1(x) \Rightarrow s\ \texttt{or}\ \texttt{in}_2(y) \Rightarrow u \colon \chi}.$$

in the succedent. We show that there is no loss of generality in this restriction, which greatly simplifies the structure proofs.

Definition 4.19 A sequent derivation is called a resolution derivation if it satisfies the following constraints for rule applications:

1. A $\wedge R$-rule is applied only if no formula on the right-hand side is a disjunction;
2. A $\supset R$-rule is applied only if no formula on the right-hand side is a conjunction or a disjunction;
3. A $\supset L$-rule, with principal formula $G \supset \phi$, is applied only if all formulæ on the right-hand side are atomic and ϕ occurs on the right-hand side;
4. A $\wedge L$-rule is applied only if all formulæ on the right-hand side are atomic;
5. A $\vee L$-rule is applied only if no formula on the left-hand side is a conjunction.

We include condition (4) only for consistency with the usual definition [80, 104]. It is inessential for the analysis presented here.

The primary difference between a fully uniform proof and a resolution proof is the requirement in the latter that the atomic matrix of the principal formula of each $\supset$L rule match with an atom on the succedent of the conclusion of the rule. Note also that the application of both the left and right rules has to be in a specified order—conjunction first—in the case of the latter.

Lemma 4.14 implies that if the restricted order in which the right rules are applied does not succeed in obtaining an intuitionistic proof, then no other ordering will. Moreover, resolution proofs are complete for intuitionistic provability of propositional hereditary Harrop formulæ.

Corollary 4.20 *If $\Gamma \vdash \Delta$ is an intuitionistically provable hereditary Harrop sequent, then there exists a resolution proof ψ of this sequent such that $[\![\psi]\!]$ is intuitionistic.*

Proof The derivation constructed in Theorem 4.16 is in fact not only a uniform proof but also a resolution proof. □

So, in order to search for an intuitionistic proof of the sequent $\Gamma \vdash \Delta$ it is enough to construct a resolution proof and then check, for all possible axiom instances and all possible exchanges of $\supset L$-rules, whether the corresponding $\lambda\mu\nu\epsilon$-terms are intuitionistic. Working on the $\lambda\mu\nu\epsilon$-terms, the first step consists in replacing a variable x by $\mu\alpha.[\beta]y$ or vice versa. The second step is a lot more complicated to capture. The reason is that the $\supset L$-rules introduce arbitrarily complex formulæ in the succedent: these formulæ must be decomposed.

$$
\dfrac{\dfrac{\dfrac{}{\phi \vdash \phi, \psi, \psi}\,Ax}{\vdash \phi, \phi \supset \psi, \psi}\supset R \qquad \dfrac{}{\psi \vdash \phi, \psi}\,Ax}{y\colon (\phi \supset \psi) \supset \psi \vdash \phi, \psi}\supset L \qquad \dfrac{}{y\colon (\phi \supset \psi) \supset \psi, \psi \vdash \psi}\,Ax}{x\colon \phi \supset \psi, y\colon (\phi \supset \psi) \supset \psi \vdash \psi}\supset L
$$

FIG. 4.1. Example derivation before permutation

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\phi \vdash \phi, \psi, \psi}\,Ax \qquad \dfrac{}{\phi, \psi \vdash \psi, \psi}\,Ax}{x\colon \phi \supset \psi, \phi \vdash \psi, \psi}\supset L}{x\colon \phi \supset \psi \vdash \phi \supset \psi, \psi}\supset R \qquad \dfrac{}{\phi \supset \psi, \psi \vdash \psi}\,Ax}{x\colon \phi \supset \psi, y\colon (\phi \supset \psi) \supset \psi \vdash \psi}\supset L
$$

FIG. 4.2. Example derivation after permutation

To see the necessity of exchanging $\supset L$-rules, consider the sequent

$$x : \phi \supset \psi,\ y : (\phi \supset \psi) \supset \psi \vdash \psi\ .$$

One possible derivation is given by Fig. 4.1, in which x is reduced first. The derivation in Fig. 4.2 is obtained from the first one by exchanging the two occurrences of the $\supset L$-rule, that is, exchanging the order of reduction of x and y, and then pushing the right-rules to the root of the derivation, thereby obtaining a uniform derivation. The corresponding $\lambda\mu\nu\epsilon$-terms are

$$\mu\beta.[\beta]b\,\{x(\mu\alpha.[\beta]b\,\{y(\lambda a\colon \phi.\mu\theta.[\alpha]a)/b\})/b\} \quad \text{and} \quad b\,\{y(\lambda a\colon \phi.b\,\{xa/b\})/b\}\,.$$

The first is not an intuitionistic $\lambda\mu\nu\epsilon$-term because the λ-abstraction over ϕ is not a Weakening term, and yet the occurrence of $[\alpha]$ is not a Weakening occurrence. The second one is an intuitionistic $\lambda\mu\nu\epsilon$-term because there are no names (in fact, it is the uniform derivation in the single-conclusioned calculus LJ).

Note that both derivations are not only uniform but are also resolution derivations. This implies that the second premiss in the $\supset L$-rule is always an axiom. However both premisses of the $\supset L$-rule are important for determining when a resolution derivation is intuitionistic. The reason is that the choice of the axiom at the right-hand premiss matters. This is not the case for single-conclusioned intuitionistic resolutions.

A description of the effects of exchanging two $\supset L$-rules requires an operational characterization of the normal form of a $\lambda\mu\nu\epsilon$-term, which is given in the following definition. We use the notion of a term with holes, as introduced in Section 2.5.3.

Definition 4.21 For any type ϕ defined by induction over the structure of ϕ, the uniform term-with-holes $\mathcal{U}^\phi$ to be a $\lambda\mu\nu\epsilon$-term with holes $h_1, \ldots, h_n$ as follows:

1. If ϕ is a base type, then $\mathcal{U}^\phi = h$, where h is a hole;
2. For a function type $\phi \supset \psi$, define $\mathcal{U}^{\phi \supset \psi}$ to be $\lambda x \colon \phi . \mathcal{U}^\psi$;
3. For a product type $\phi \wedge \psi$, define $\mathcal{U}^{\phi \wedge \psi}$ to be $\langle \mathcal{U}^\phi, \mathcal{U}^\psi \rangle$, where all holes in $\mathcal{U}^\phi$ are different from all holes in $\mathcal{U}^\psi$;
4. For a sum type $\phi \vee \psi$, define $\mathcal{U}^{\phi \vee \psi}$ to be $\nu\beta . \mathcal{U}^\phi$.

For each term $\Gamma \vdash t \colon \phi, \Delta$, we define the the list of parameters $\mathcal{P}^\phi(t)$ by induction over the structure of ϕ as follows:

1. If ϕ is a base type, then $\mathcal{P}^\phi(t) = s$, where s is the normal form of t;
2. For a function type $\phi \supset \psi$, define $\mathcal{P}^{\phi \supset \psi}(t)$ to be $\mathcal{P}^\psi(tx)$;
3. For a product type $\phi \wedge \psi$, define $\mathcal{P}^{\phi \wedge \psi}(t)$ to be the list $t_1^1, \ldots, t_n^1, t_1^2, \ldots, t_m^2$, where $\mathcal{P}^\phi(\pi_1(t)) = t_1^1, \ldots, t_n^1$ and $\mathcal{P}^\phi(\pi_2(t)) = t_1^2, \ldots, t_m^2$;
4. For a sum type $\phi \vee \psi$, define $\mathcal{P}^{\phi \vee \psi}(t)$ to be $\mathcal{P}^\phi(\langle\beta\rangle t)$.

We write $\mathcal{U}^\phi(t_1, \ldots, t_n)$ for the term obtained by (textually) substituting t_i for the hole h_i. The intuition is that the term $\mathcal{U}^\phi$ lists all the outermost term constructors in a long $\beta\eta$-normal form of type ϕ. Hence for any term t the long $\beta\eta$-normal form is equal to $\mathcal{U}^\phi(t_1, \ldots, t_n)$, where $\mathcal{P}^\phi(t) = t_1, \ldots, t_n$.

Next we show that the construction of a uniform term corresponds to normalization.

Lemma 4.22 *Let Φ be the sequent derivation*

$$\begin{array}{c} \Phi_i : \Gamma, \psi, \Gamma_i \vdash \phi_i, \psi, \Delta \\ \vdots R^* \\ \Gamma, \psi \vdash \phi, \psi, \Delta, \end{array}$$

where all formulæ in ϕ_i and in Δ are atoms and all the right-rules have ϕ or subformulae of it as prinicipal formulæ. Then $[\![\Phi]\!] = \mathcal{U}^\phi([\![\Phi_1]\!], \ldots, [\![\Phi_n]\!])$, and if all terms $[\![\Phi_i]\!]$ are in normal form, $[\![\Phi]\!]$ is a normal form as well. Moreover, if Φ_i is the proof $\Gamma, b \colon \psi \vdash \phi_i, \psi, \Delta$ using only the axiom rule for ψ, then $[\![\Phi]\!]$ is the long $\beta\eta$-normal form of $\mu\alpha.[\beta]b$.

Proof By induction over the structure of ϕ. □

Now we describe the exchange in detail. Consider Figs 4.3 and 4.4. The former is intended to be a classically valid uniform derivation. The latter is intended to be an intuitionistically valid uniform derivation obtained from the former by permuting $\supset$L-rules with respect to one another and by inserting any right-rules so induced.

$$
\dfrac{\dfrac{\begin{array}{c}\Gamma, \Gamma_i, \Gamma_{i,j} \vdash \chi_j, \phi_i, \psi, \sigma, \Delta \\ \vdots\, R^* \\ \Gamma, \Gamma_i \vdash \chi, \phi_i, \psi, \sigma, \Delta\end{array} \qquad \Gamma, \Gamma_i, v{:}\, \sigma \vdash \phi_i, \psi, \sigma, \Delta}{\begin{array}{c}\Gamma, \Gamma_i, y{:}\, \chi \supset \sigma \vdash \phi_i, \psi, \sigma, \Delta \\ \vdots\, R^* \\ \Gamma, y{:}\, \chi \supset \sigma \vdash \phi, \psi, \sigma, \Delta\end{array}} \supset L \qquad \Gamma, y{:}\, \chi \supset \sigma, w{:}\, \psi \vdash \psi, \sigma, \Delta}{\Gamma, x{:}\, \phi \supset \psi, y{:}\, \chi \supset \sigma \vdash \psi, \sigma, \Delta} \supset L
$$

FIG. 4.3. Derivation before permutation

$$
\dfrac{\dfrac{\begin{array}{c}\Gamma, \Gamma_j, \Gamma_{i,j} \vdash \phi_i, \chi_j, \psi, \sigma, \Delta \\ \vdots\, R^* \\ \Gamma, \Gamma_j \vdash \phi, \chi_j, \psi, \sigma, \Delta\end{array} \qquad \Gamma, \Gamma_j, w{:}\, \psi \vdash \chi_j, \psi, \sigma, \Delta}{\begin{array}{c}\Gamma, \Gamma_j, x{:}\, \phi \supset \psi \vdash \chi_j, \psi, \sigma, \Delta \\ \vdots\, R^* \\ \Gamma, x{:}\, \phi \supset \psi \vdash \chi, \psi, \sigma, \Delta\end{array}} \supset L \qquad \Gamma, x{:}\, \phi \supset \psi, y{:}\, \sigma \vdash \psi, \sigma, \Delta}{\Gamma, x{:}\, \phi \supset \psi, y{:}\, \chi \supset \sigma \supset \psi, \sigma, \Delta} \supset L
$$

FIG. 4.4. Derivation after permutation

Theorem 4.23 *Let ϕ be the uniform derivation given in Figure 4.1, let*

$$s_1, \ldots, s_n = \mathcal{P}^{\phi}(\mu\alpha.[\delta]v)$$

and let

$$w\left\{x(\mathcal{U}^{\phi}(s_i\left\{y\mathcal{U}^{\chi}(u_{i,j})/v\right\}))/w\right\}$$

be the corresponding $\lambda\mu\nu\epsilon$-term. Then the $\lambda\mu\nu\epsilon$-term corresponding to the exchanged derivation, given in Figure 4.2, is the term

$$v\left\{y(\mathcal{U}^{\chi}(t_j\left\{x\mathcal{U}^{\phi}(\mu\alpha_i.[\gamma]u_{i,j})/w\right\}))/v\right\}\ ,$$

where $\mathcal{P}^{\chi}(\mu\gamma.[\beta]w) = (t_1, \ldots, t_m)$. If the first derivation is a resolution derivation, so is the second one.

Proof An easy consequence of Lemma 4.22. □

4.6 Classical resolution

We have introduced resolution in classical logic, informally, as a deductive refutation system based on the rule

$$\frac{C_1 \vee p \quad C_2 \vee \neg p}{C_1 \vee C_2},$$

for clauses C_1 and C_2 and atom p, as given in Equation 4.8. As the reader might suspect, however, the resolution rule may be formulated as a sequential rule. Indeed, Mints' [82] has given explicit translations between resolution systems and the sequent calculus. The basic idea is quite simple and is most readily seen as being based on the multiplicative formulation of the Cut rule,

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma', \phi \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'},$$

in the classical sequent calculus. Given a set Γ of clauses which contains the clauses $C_1 \vee p, C_2 \vee \neg p \in \Gamma$, we have the following derivation of the resolution rule:

$$\dfrac{\dfrac{\Gamma \vdash C_1\,,\, p \qquad \dfrac{\Gamma \vdash C_2\,,\, \neg p}{\Gamma, p \vdash C_2}\,\neg L \text{ (and an Exchange)}}{\Gamma\,,\,\Gamma \vdash C_1\,,\, C_2}\,\text{Cut}}{\Gamma \vdash C_1\,,\, C_2}\,CR.$$

In this section, we show that, under inessential modifications, Mints' translations between resolution systems and the sequent calculus establish tight connections between weakly uniform proofs and resolution derivations in classical logic. The results of Section 4.5 then give realizers for classical resolution derivations.

The key point relies on the following definition of *weakly hereditary Harrop formulæ*, or 'whHfs', in which, as usual, p ranges over atoms:

$$\begin{aligned} D &::= p \mid D \wedge D \mid \neg(p \wedge p) \mid p \vee p \mid G \supset H \\ G &::= p \mid \neg p \mid G \wedge G \mid G \vee G \mid D \supset G \\ H &::= p \mid p \vee p. \end{aligned}$$

Weakly uniform proofs are complete for weakly hereditary Harrop consequences $D_1, \ldots, D_m \vdash G_1, \ldots, G_n$.[47]

Note that the extensions to the usual class of hereditary Harrop formulæ [80] are the inclusion of binary disjunctions of atomic formulæ in implicational goal formulæ, negated atoms in goal formulæ, and negated binary conjunctions of atoms in definite formulæ. These extensions are required if we are to interpret resolution proofs for clauses that include disjunctions as weakly uniform proofs because the translation of Definition 4.27 (at least its modification in Section 4.7.2) make essential use of them. In the absence of disjunction, the simpler notions of uniform proof and hereditary Harrop formula [108, 111] will suffice. We also show that, for classical logic, permutations in the space of resolution proofs correspond to permutations in the space of sequent calculus proofs (Proposition 4.31).

[47]Indeed, weakly uniform proofs are complete for a slightly larger class than this.

We begin by recalling from [82] the construction of a set of clauses of bounded complexity from an arbitrary propositional formula.

Definition 4.24 A formula ϕ is a classical clause if it is either $\bot$ or a disjunction $l_1 \vee \cdots \vee l_m$, with $m > 0$ and each l_i, $1 \leq i \leq n$, a literal. Clauses which differ only in the numbering or order of literals are identified.

We say that a classical clause $l_1 \vee \cdots \vee l_m$, for $m > 0$, has length m and that $\bot$ has length 0.

Lemma 4.25 *For any propositional formula ϕ, a set X_ϕ of clauses of length ≤ 3 can be constructed in linear space and time (in the length of ϕ) such that ϕ is valid if and only if X_ϕ is inconsistent.*

Proof It is enough to show this for formulæ constructed using only of negation and disjunction.

We construct, by induction over the structure of ϕ, a set of clauses for the formulæ $\neg X \vee \phi$ and $\neg\phi \vee X$, where X is a propositional variable. If ϕ is an atom, we simply take these two formulæ.

For the case of a disjunction $\phi \vee \psi$, we introduce new propositional variables ϕ' or ψ' for non-atomic formulæ ϕ or ψ, otherwise let ϕ' or ψ' be ϕ or ψ respectively. We add to the clauses obtained by the induction hypothesis applied to $\neg\phi' \vee \phi$, $\neg\phi \vee \phi'$, $\neg\psi' \vee \psi$, and $\neg\psi \vee \psi'$ the clauses $\neg X \vee \phi' \vee \psi'$, $\neg\phi' \vee X$, and $\neg\psi' \vee X$.

For a negation $\neg\phi$, let ϕ' be a new propositional variable if ϕ is not atomic; otherwise let ϕ' be ϕ. Add to the clauses obtained by the induction hypothesis applied to $\neg\phi' \vee \phi$ and $\neg\phi \vee \phi'$ the clauses $\neg X \vee \neg\phi'$ and $\phi' \vee X$. It is easy to see that the set X_ϕ, defined as the element $\neg X$ together with the clauses for the formulæ $\neg X \vee \phi$ and $\neg\phi \vee X$ as constructed above, satisfies the claims. □

Resolution is defined as a calculus for deriving a judgement $\Gamma \vdash C$, where Γ is a set of clauses and C is a clause. The precise definition follows below.

Definition 4.26 Let Γ be a set of clauses, let C be a clause and let p and q be atoms. A *resolution derivation* of a judgement $\Gamma \vdash C$ is given by:

$$\frac{}{\Gamma, C, \Gamma' \vdash C}\ Ax \qquad \frac{}{\Gamma \vdash p \vee \neg p}\ EM$$

$$\frac{\Gamma \vdash \neg p_1 \vee C_1 \cdots \Gamma \vdash \neg p_n \vee C_n}{\Gamma, p_1 \vee \cdots \vee p_n \vdash C_1 \vee \cdots \vee C_n}\ Res,$$

where $C_1, \ldots, C_n$ are clauses. In the last case, we call the formula $p_1 \vee \cdots \vee p_n$ the input formula of the resolution rule. We identify the clause $C \vee \bot$ with C in the above rules.

Note that Weakening is admissible in this system: whenever $\Gamma \vdash C$ and also $\Gamma \subseteq \Gamma'$, then also $\Gamma' \vdash C$. We call a clause C a *Weakening clause* in a derivation if it is not introduced by one of the rules.

Mints [82] proves the following:

Theorem (Mints) *A formula ϕ is classically provable if and only if there is a resolution derivation $X_\phi \vdash \bot$.*

This is proved by transforming a resolution derivation into a sequent derivation in which the formulæ consist only of disjunction and negation and vice versa.

We start our proof of this theorem, which we will later generalize to intuitionistic logic, with a translation of a resolution proof into a derivation in the classical sequent calculus LK without Cut. This translation is essentially the one given in [82].

Definition 4.27 We define the concatenation of the n sequents $\Gamma_1 \vdash \Delta_1, \dots,$ $\Gamma_n \vdash \Delta_n$ to be the sequent $\Gamma_1, \dots, \Gamma_n \vdash \Delta_1, \dots, \Delta_n$.

1. By induction over the structure of clauses we define a sequent derivation of $\Gamma \vdash \Delta$, for each clause C with a polarity $\{+, -\}$ (defined as in tableaux calculi). A clause has positive (negative) polarity if it is part of Δ (Γ). If C is the clause $C_1^+ \vee C_2^+$, then we define $[\![C_1^+ \vee C_2^+]\!]$ to be the concatenation of the two sequents $[\![C_1^+]\!] = \Gamma_1 \vdash \Delta_1$ and $[\![C_2^+]\!] = \Gamma_2 \vdash \Delta_2$. For the remainder of the clauses the definition is as follows (as usual, p and q are atomic):
$$\begin{aligned}
[\![(\neg p \vee \neg q \vee \neg C)^-]\!] &= \neg(p \wedge q \wedge C) \vdash \\
[\![(\neg p \vee \neg q \vee C)^-]\!] &= (p \wedge q) \supset C \vdash \\
[\![(\neg p \vee q \vee C)^-]\!] &= p \supset (q \vee C) \vdash \\
[\![(p \vee q \vee C)^-]\!] &= (p \vee q \vee C) \vdash \\
[\![(\neg p \vee \neg q)^-]\!] &= \neg(p \wedge q) \vdash \\
[\![(\neg p \vee q)^-]\!] &= p \supset q \vdash \\
[\![(p \vee q)^-]\!] &= (p \vee q) \vdash \\
[\![(\neg p)^-]\!] &= \vdash p \\
[\![(p)^-]\!] &= p \vdash \\
[\![(\neg p)^+]\!] &= p \vdash \\
[\![(p)^+]\!] &= \vdash p.
\end{aligned}$$

2. If X is a set of clauses $C_1, \dots, C_n$ and C is a clause, we denote the sequence resulting from concatenation of $[\![C_1^-]\!], \dots, [\![C_n^-]\!]$ and $[\![C^+]\!]$ by $[\![X^-]\!] \vdash [\![C^+]\!]$. By induction over the derivation of $X \vdash C$, we define a classical sequent derivation of $[\![X^-]\!] \vdash [\![C^+]\!]$ as follows:
   (i) With each axiom $\Gamma, C, \Gamma' \vdash C$, associate the appropriate sequent derivation from the axioms;

(ii) With each axiom $X \vdash p \vee \neg p$, associate the sequent derivation consisting of the axiom $[\![X^-]\!], p \vdash p$;

(iii) If the input formula is $\neg p \vee \neg q$ and if we have resolution derivations of $X \vdash p \vee C_1$ and $X \vdash q \vee C$, then we construct the following sequent derivation:

$$\dfrac{\dfrac{[\![X^-]\!] \vdash p, [\![C_1^+]\!] \qquad [\![X^-]\!] \vdash q, [\![C_2^+]\!]}{[\![X^-]\!] \vdash p \wedge q, [\![C_1^+]\!], [\![C_2^+]\!]}\,\wedge R}{[\![X^-]\!], \neg(p \wedge q) \vdash [\![C_1^+]\!], [\![C_2^+]\!]}\,\neg L;$$

(iv) If the input formula is $\neg p_1 \vee \neg p_2 \vee \neg p_3$, then the construction is similar;

(v) If the input formula is $\neg p \vee \neg q \vee C$ and if we have resolution derivations of $X \vdash p \vee C_1$, $X \vdash q \vee C_2$, and $X \vdash \neg C \vee C_3$, then we construct the following sequent derivation:

$$\dfrac{\dfrac{[\![X^-]\!] \vdash p, [\![C_1^+]\!] \qquad [\![X^-]\!] \vdash q, [\![C_2^+]\!]}{[\![X^-]\!] \vdash p \wedge q, [\![C_1^+]\!], [\![C_2^+]\!]}\,\wedge R \qquad [\![X^-]\!], C \vdash [\![C_3^+]\!]}{[\![X^-]\!], (p \wedge q) \supset C \vdash [\![C_1^+]\!], [\![C_2^+]\!], [\![C_3^+]\!]}\,\supset L;$$

(vi) If the input formula is $\neg p \vee (q \vee C)$ and if we have resolution derivations of $X \vdash p \vee C$, $X, q \vdash C_2$, and $X \vdash \neg C \vee C_3$, then we construct the following sequent derivation:

$$\dfrac{[\![X^-]\!] \vdash p, [\![C_1^+]\!] \qquad \dfrac{[\![X^-]\!], q \vdash [\![C_2^+]\!] \qquad [\![X^-]\!], C \vdash [\![C_3^+]\!]}{[\![X^-]\!], q \vee C \vdash [\![C_2^+]\!], [\![C_3^+]\!]}\,\vee L}{[\![X^-]\!], p \supset (q \vee C) \vdash [\![C_1^+]\!], [\![C_2^+]\!], [\![C_3^+]\!]}\,\supset L;$$

(vii) If the input formula is $\neg p \vee q$ and we have resolution derivations of $X \vdash p \vee C_1$ and $X \vdash \neg q \vee C_2$, then we construct the following sequent derivation:

$$\dfrac{[\![X^-]\!] \vdash p, [\![C_1^+]\!] \qquad [\![X^-]\!], q \vdash [\![C_2^+]\!]}{[\![X^-]\!], p \supset q \vdash [\![C_1^+]\!], [\![C_2^+]\!]}\,\supset L;$$

(viii) If the input formula is $p \vee q$ and if we have resolution derivations of $X \vdash \neg p \vee C_1$ and $X \vdash \neg q \vee C_2$, we obtain the following sequent derivation:

$$\dfrac{[\![X^-]\!], p \vdash [\![C_1^+]\!] \qquad [\![X^-]\!], q \vdash [\![C_2^+]\!]}{[\![X^-]\!], p \vee q \vdash [\![C_1^+]\!], [\![C_2^+]\!]}\,\vee L;$$

(ix) If the input formula is $p_1 \vee p_2 \vee p_3$, then the construction is similar;

(x) If the input formula is p and if we have a resolution derivation of $X \vdash \neg p \vee C$, then we have a sequent derivation of $[\![X^-]\!], p \vdash [\![C^+]\!]$, by assumption, which we simply take;

(xi) If the input formula is $\neg p$ and if we have a resolution derivation of $X \vdash p \vee C$, then we have a sequent derivation of $[\![X^-]\!] \vdash p, [\![C^+]\!]$, by assumption, which we simply take.

By applying the translation of sequent derivations into $\lambda\mu\nu\epsilon$-terms, as given in [108], we obtain a $\lambda\mu\nu\epsilon$-term for each resolution derivation. Moreover, this sequent derivation is weakly uniform.

Theorem 4.28 *The sequent derivation associated with a resolution derivation is weakly uniform.*

Proof Note that the right-hand side of all root sequents of such a sequent derivation is atomic. Furthermore any intermediate non-atomic formula on the right is reduced as soon as it occurs. Hence the sequent derivation is weakly uniform. $\square$

For example, we give the resolution derivation and the corresponding $\lambda\mu\nu\epsilon$-term for the formula $p \supset p$. According to Lemma 4.25 the set $X_{p\supset p}$ is

$$\{\neg X \vee \neg p \vee p, p \vee X, \neg p \vee X, \neg X\}.$$

A resolution derivation of the empty clause from a subset of these clauses can be obtained as follows:

$$\dfrac{\dfrac{\neg X \vdash \neg p \vee p \qquad \neg X \vdash \neg X}{\neg X, p \vee X \vdash p} \qquad \neg X \vdash \neg X}{\neg X, p \vee X, \neg p \vee X \vdash \bot}\,.$$

The corresponding sequent derivation is

$$\dfrac{\dfrac{p \vdash p, X \qquad X \vdash X}{p \vee X \vdash p, X}\vee L \qquad X \supset X}{p \vee X, p \supset X \vdash X}\supset L.$$

The corresponding $\lambda\mu\nu\epsilon$-term, which is obtained by using the translation of Definition 4.6, is $\mu\gamma.[\gamma]y\{\langle\gamma\rangle x/y\}$.

Observe that the sequent derivations obtained by translating resolution derivations do not use Weakening. Moreover, these derivations can be rewritten in such a way that the axioms have the form $p \vdash p$, but at the expense of introducing Weakening at the root of the derivation. These properties are a consequence of the absence of a Weakening rule in the resolution calculus. A translation of classical sequent derivations into resolution derivations can be given only for sequents without Weakening in the middle of the derivation. Mints [82] gives such a translation. Because every sequent derivation where all formulæ are either clauses or elements of $[\![X^-]\!]$ can be transformed into one in which Weakening occurs at the root of the derivation only, for each derivable sequent $\Gamma \vdash \Delta$ with this property there is a subsequent $\Gamma' \vdash \Delta'$ which has a resolution proof. This translation is part of the following theorem:

Theorem 4.29 *Consider a weakly uniform sequent derivation of* $[\![X^-]\!], \Delta_1 \vdash \Delta_2$ *such that (i)* Δ_1 *and* Δ_2 *consist of atoms; (ii) all Weakenings occur only at the root of the derivation; and (iii) all axioms have the form* $p \vdash p$*. Then there is a resolution derivation of* $X \vdash \neg\Delta_1' \vee \Delta_2'$*, in which* Δ_1' *and* Δ_2' *are subsets of* Δ_1 *and* Δ_2*, respectively. Furthermore, all of the formulæ in* Δ_1 *and* Δ_2 *that are not obtained by Weakening are in* Δ_1' *and* Δ_2'*, respectively. As usual,* p *and* q *denote atoms and* C*s denote clauses.*

Proof Let Ψ be the subderivation above the last Weakening rule; proceed by induction over Ψ. It is necessary to strengthen the induction hypothesis and construct a resolution derivation $Y \vdash \neg\Delta_1' \vee \Delta_2'$, where Y is a subset of X, Δ_1' and Δ_2' are as above and Y contains no Weakening formula. Moreover, we show that all formula in $[\![X^-]\!]$ which were not obtained by Weakening are in Y. Because Weakening is an admissible rule in the resolution calculus, this statement implies the claim.

Suppose the last rule in Ψ is an axiom $p \vdash p$. If both atoms are part of Δ_1 and Δ_2 respectively, then the axiom $X \vdash \neg p \vee p$ yields the claim. If both atoms are part of $[\![X^-]\!]$, then the derivation

$$\frac{p \vdash p}{p, \neg p \vdash \bot},$$

yields the claim. If only one atom, say the one on the left-hand side, is part of $[\![X^-]\!]$, then take the resolution axiom $p \vdash p$. If the one atom is the atom on the right-hand side, take the resolution axiom $\neg p \vdash \neg p$.

Suppose the last rule in Ψ is a $\neg L$-rule, with the principal formula $\neg(p \wedge q)$ or $\neg(p_1 \wedge p_2 \wedge p_3)$. We consider the case of a formula $\neg(p \wedge q)$; the other one is similar. The weakly uniform derivation looks like

$$\cfrac{\cfrac{\Gamma_1 \vdash p, \Gamma_1' \qquad \Gamma_2 \vdash q, \Gamma_2'}{\Gamma_1, \Gamma_2 \vdash p \wedge q, \Gamma_1', \Gamma_2'}\, \wedge R}{\Gamma_1, \Gamma_2, \neg(p \wedge q) \vdash \Gamma_1', \Gamma_2'}\, \neg L,$$

where, by hypothesis, neither p nor q is obtained by Weakening. Note that $\Gamma_1, \Gamma_2, \neg(p \wedge q) \vdash \Gamma_1', \Gamma_2'$ is the sequent $[\![X'^-, (\neg p \vee \neg q)^-]\!], \Delta_1 \vdash \Delta_2$, where $X' \cup \{\neg p \vee \neg q\}$ is a subset of X. Furthermore, neither p nor q is an element of $[\![X'^-]\!]$.

The induction hypothesis applied to the sequents $\Gamma_1 \vdash p, \Gamma'_1$ and $\Gamma_2 \vdash q, \Gamma'_2$ yields resolution derivations $Y' \vdash p \vee C_1$ and $Y' \vdash q \vee C_2$ respectively, where Y' is a subset of X'. Hence we can construct the following derivation:

$$\frac{Y' \vdash p \vee C_1 \qquad Y' \vdash q \vee C_2}{Y' \cup \{\neg p \vee \neg q\} \vdash C_1 \vee C_2}.$$

Next we consider the case that the last rule in Ψ is an $\supset L$-rule. We start with the case where the principal formula is $p \supset q$. The weakly uniform derivation looks like

$$\frac{\Gamma \vdash p, \Delta \qquad q \vdash q}{\Gamma, p \supset q \vdash q, \Delta} \supset L,$$

where, by hypothesis, p is not obtained by Weakening. The induction hypothesis applied to the sequent $\Gamma \vdash p, \Delta$ yields a resolution derivation $X' \vdash p \vee C$, where $X' \cup \{\neg p \vee q\}$ is a subset of X. The following resolution derivation now yields the claim:

$$\frac{X' \vee \vdash p \vee C \qquad X' \vdash \neg q \vee q}{X' \cup \{\neg p \vee q\} \vdash C \vee q}.$$

Now consider the case in which the principal formula is $p \supset (q \vee r)$. The weakly uniform derivation looks like

$$\frac{\Gamma \vdash p, \Gamma'_1 \qquad \dfrac{\Gamma_2, q \vdash \Gamma'_2 \qquad \Gamma_3, r \vdash \Gamma'_3}{\Gamma_2, \Gamma_3, q \vee r \vdash \Gamma'_2, \Gamma'_3} \vee L}{\Gamma_1, \Gamma_2, , \Gamma_3, p \supset (q \vee r) \vdash \Gamma'_1, \Gamma'_2, \Gamma'_3},$$

where, by hypothesis, neither p nor $q \vee r$ are obtained by Weakening. The induction hypothesis yields resolution derivations $Y \vdash p \vee D_1$, $Y, q \vdash D_2$ and $Y, C \vdash D_3$, where the Ds denotes clauses. Hence we obtain the following resolution derivation:

$$\frac{Y \vdash p \vee D_1 \qquad Y \vdash \neg q \vee D_2 \qquad Y \vdash \neg r \vee D_3}{Y \cup \{\neg p \vee q \vee r\} \vdash D_1 \vee D_2 \vee D_3}.$$

The remaining case is that we have an $\vee L$-rule as the last rule. The case of a principal formula $p \vee q$ is similar to the case of a principal formula $p \supset q$, and the case of a principal formula $p \vee q \vee r$ is similar to the case of a principal formula $p \supset (q \vee r)$. □

Mints' Theorem can now be obtained as a corollary:

Corollary 4.30 *A formula ϕ is classically provable if and only if there is a resolution derivation $X_\phi \vdash \bot$.*

Proof Suppose ϕ is classically provable. By Lemma 4.25, the set X_ϕ is inconsistent, hence there is a derivation of $[\![X_\phi^-]\!] \vdash$. Theorem 4.29 implies the existence of a resolution derivation $X_\phi \vdash \bot$.

Conversely, given a resolution derivation of $X_\phi \vdash \bot$. The second part of definition 4.27 yields a derivation $[\![X_\phi^-]\!] \vdash$; hence X_ϕ is inconsistent. So ϕ is provable. □

One of our central concerns has been to investigate when permutations transform a weakly uniform sequent derivation which is non-intuitionistic into an intuitionistic derivation. We now show how permutations in the sequent calculus are related to the choice of input formulæ in the resolution calculus. Later on we will transfer this connection to intuitionistic logic. Because the formulæ occurring in sequent derivations arising from resolution derivations have a rather simple structure, it suffices to consider exchanges of $\supset L$-, $\neg L$-, and $\vee L$-rules. These are the only rules whose exchange leads from a weakly uniform derivation to another weakly uniform derivation. The details are contained in the following proposition:

Proposition 4.31 *As usual, let ps and qs denote atoms and let Cs and Ds denote clauses.*

1. *Let*

$$\dfrac{\dfrac{X \vdash \neg p_1 \vee C_1 \qquad X \vdash \neg p_n \vee C_n}{X, p_1 \vee \cdots \vee p_n \vdash C_1 \vee \cdots \vee C_n} \qquad X \vdash \neg q_1 \vee D_1 \cdots X \vdash \neg q_m \vee D_m}{X, p_1 \vee \cdots \vee p_n, \neg C_1 \vee q_1 \vee \cdots \vee q_m \vdash C_2 \vee \cdots \vee C_n \vee D_1 \vee \cdots \vee D_m}$$

   be a resolution derivation and let

$$\dfrac{\dfrac{X \vdash \neg p_1 \vee C_1 \qquad X \vdash \neg q_1 \vee D_1 \cdots X \vdash \neg q_m \vee D_m}{X, \neg C_1 \vee q_1 \cdots \vee q_m \vdash \neg p_1 \vee D_1 \vee \cdots \vee D_m} \qquad X \vdash \neg p_2 \vee C_2 \cdots X \vdash \neg p_n \vee C_n}{X, p_1 \vee \cdots \vee p_n, \neg C_1 \vee q_1 \vee \cdots \vee q_m \vdash C_2 \vee \cdots \vee C_n \vee D_1 \vee \cdots \vee D_m}$$

   be the derivation in which the application of the two instances of the resolution rule are exchanged. The translation of the second resolution derivation into a sequent derivation is obtained by exchanging the two left-rules to which the two applications of the resolution rule are translated.

2. *Conversely, given a weakly uniform sequent derivation of a sequent $\Gamma \vdash \Delta$, where Γ consists only of clauses and Δ only of atoms, the exchange of $\neg L$ and $\supset L$-rules corresponds to the exchange of two resolution rules.*

Proof For first part, check each resolution formula in turn. For the second part, calculate the resolution derivations for all possible exchanges. □

Intuitively, this proposition indicates that the search for a weakly uniform derivation of a sequent with formulæ in clausal form is as complicated as the search for a resolution derivation of the corresponding clauses. In other words, this proposition shows that the essential aspect of the resolution method is the transformation of formulæ into clausal form; the complexity of finding the right input formula in a resolution derivation is the same as finding the right permutation in the sequent derivation.

This analysis carries over to the intuitionistic case (see Section 4.7), including the case of a resolution formula $p \supset q \supset r$. This is important because, in contrast to the classical case, in intuitionistic logic permutations of inferences do matter. Classically, but not intuitionistically, any permutation of a sequent derivation transforms a proof only into a proof and a non-proof only into a non-proof. Moreover, Egly [133] shows that the transformation of sequents into clausal form decreases the complexity of proof-search in intuitionistic logic significantly.

4.7 Intuitionistic resolution

In intuitionistic logic, in the absence of classical negation, the definition of a resolution calculus is less straightforward than in the classical setting, for which the idea was originally developed. A candidate definition of intuitionistic resolution has been proposed by Mints [82]. Mints' calculus is ingenious but lacks the conceptual clarity of the classical system. However, the coincidence of classical and intuitionistic provability for Horn clauses [34] leads us to hope for a notion of intuitionistic resolution of value comparable to that of classical resolution. Maslov [78] and Tammet [123, 124] have also studied these topics.

In this section, we give a systematic development of a resolution calculus for intuitionistic logic based on the ideas in the preceding sections of this chapter. The idea is to retain the resolution calculus for classical logic, because this calculus has no constraints on the order in which input formulæ are taken. The translation of such resolution derivations into $\lambda\mu\nu\epsilon$-terms is used to decide when the derivation provides sufficient evidence that the formula is intuitionistically provable.

We show that our calculus characterizes the provability of exactly the same intuitionistic formulæ via exactly the refutations of exactly the same clauses.

4.7.1 *Mints' intuitionistic resolution*

Mints [82] defines a resolution calculus for intuitionistic logic. It is important to note that Mints' calculus is not a restriction of classical resolution, but has special rules for each connective of the logic.

We begin, for convenience and completeness, with a description of Mints' calculus. We adopt Mints' notation. The rules of Mints' resolution calculus are given in Table 4.7. Clauses are formulæ of the form $p \supset q^* \supset r$, $p \supset q \vee r$ and $p_1, \ldots, p_n \supset q^*$ with $n \leq 3$, where all formulæ are propositional variables and q^* means either a propositional variable or the symbol $\perp$ (falsehood). Similarly, each of s^*, s^{**}, and s^{***} is either a propositional variable or $\perp$.

TABLE 4.7. Mints' resolution calculus

$$\frac{}{X, \Phi \vdash \Phi} \qquad \frac{}{X \vdash \perp \supset p} \qquad \frac{}{X \vdash p \supset p}$$

$$\frac{X \vdash (\Gamma, p^o \supset q^{**})}{X, ((p \supset q^*) \supset r) \vdash \Gamma \supset r}(\supset^-) \qquad \frac{X \vdash \Gamma \supset \perp}{X \vdash \Gamma \supset r}(\perp)$$

$$\frac{X \vdash \Gamma_1 \supset p \qquad X \vdash (\Gamma_2, q \supset s^*) \qquad X \vdash (\Gamma_3, q \supset s^{**})}{X, (p \supset q \vee r) \vdash (\Gamma_1, \Gamma_2, \Gamma_3 \supset s^{***})}(\vee^-)$$

$$\frac{X \vdash \Gamma_1 \supset p \qquad X \vdash \Gamma_2 \supset q}{X, (p, q \supset r^*) \vdash (\Gamma_1, \Gamma_2 \supset r^*)}(c) \qquad \frac{X \vdash \Gamma \supset p}{X, (p \supset q) \vdash \Gamma \supset q}$$

In these rules, Φ is a clause, p, q, r are atoms, Γ_1, Γ_2, and Γ_3 are sets of atoms, the superscript 0 means possible absence of the corresponding formula and in the rule $(\supset^-)$ it is required that either $q^{**} = q^*$ or ($q^{**} = \perp \neq q^*$ and $q^0 = q$); similarly in $(\vee^-)$, $s^{***} = s^*$ if at least one of s^*,s^{**} is s, and $s^{***} = \perp$, if $s^* = s^{**} = \perp$. The rule $(\perp)$ is allowed only as the last rule in the derivation. Mints constructs, for every formula ϕ, a set of clauses X_ϕ, the translation of these clauses into one formula Y_ϕ and an atom F such that ϕ is intuitionistically provable if and only if $Y_\phi \vdash F$ is provable in LJ.

Mints then gives translations between resolution derivations and LJ derivations with Weakening pushed down to the root as much as possible, and obtains as a corollary that a formula ϕ is intuitionistically provable if and only if $X_\phi \vdash \perp$ is derivable in the resolution calculus.[48]

The rules for implication and negation cannot be obtained as special cases of the rules for classical resolution, hence it is not immediately possible to transfer the implementations of classical resolution to the intuitionistic case. The reason is that derivations may contain Weakening at places other than at their roots. For example, consider the LK-derivation

$$\cfrac{\cfrac{\cfrac{\Gamma \vdash q, \Delta}{\Gamma, p \vdash q, \Delta}\, W\, L}{\Gamma \vdash p \supset q, \Delta} \supset R \qquad r \vdash r}{\Gamma, (p \supset q) \supset r \vdash r, \Delta} \supset L,$$

where the Weakening of the formula p cannot be pushed to the root of the derivation. Because the construction of Theorem 4.29 works only for derivations where Weakening is applied only as the last rule of the derivation, there can be

[48]It may be seen that Mints' calculus corresponds to constructing a version of weakly uniform proofs in LJ, with Weakening present and pushed as close to the root as much as possible.

no resolution derivation corresponding to this sequent derivation. Indeed, the method of the previous section, which uses the (classical) equivalence

$$(p \supset q) \supset r \equiv (p \vee r) \wedge (\neg q \vee r),$$

yields only the following resolution derivation:

$$\frac{\Gamma \vdash q \vee \Delta \qquad \bot \vdash \neg r \vee r}{\Gamma, \neg q \vee r \vdash \Delta},$$

where Δ is interpreted as the disjunction of its members, and the input formula $p \vee r$ is added by Weakening at the end and not obtained by a resolution step.

4.7.2 *The intuitionistic force of classical resolution*

In this section, we exploit the results given in the preceding sections to assess the intuitionistic force of classical resolution. In particular, we establish that the association of $\lambda\mu\nu\epsilon$-terms with resolution derivations, as developed in the previous sections, can be used to determine intuitionistic provability for classical resolution proofs. Thus we give a characterization of intuitionistic resolution within our systematic reductive framework.

The translation of formulæ into clauses, referred to in Lemma 4.25, produces clauses given by the grammar

$$C ::= p_1 \vee p_2 \ \mid\ \neg p_1 \vee p_2 \ \mid\ \neg p_1 \vee \neg p_2 \vee p_3 \ \mid\ \neg p_1 \vee \neg p_2 \ \mid\ \neg p_1 \vee p_2 \vee p_3,$$

where p_1, p_2, p_3 are all atomic. In the sequel we restrict attention to such clauses. Note that the transformations leading from formulæ to the clauses arising in the sequent derivations are no longer equivalences: the formula $(p \vee r) \wedge (q \supset r)$ implies $(p \supset q) \supset r$, but not the other way round. In all other cases, the transformations that lead from formulæ to clauses are intuitionistic equivalences.

The translation of resolution derivations into $\lambda\mu\nu\epsilon$-terms leads directly to a criterion for when a resolution derivation gives rise to an intuitionistic proof.

Definition 4.32 A resolution derivation is said to be intuitionistic if it translates into an intuitionistic $\lambda\mu\nu\epsilon$-term.

We want to transfer the soundness theorem for classical resolution to the intuitionistic case. As stated this does not work because the implications

$$(p \vee X) \wedge (q \supset X) \supset ((p \supset q) \supset X)$$

and

$$(p \vee X) \supset \neg p \supset X$$

have the wrong order: for the classical proof to go through we need that the formula with implication implies the clausal form and not vice versa. We address the first case by restricting the class of resolution derivations under consideration and modify the translation of resolution derivations into sequent derivations. We permit for clauses $p \vee r, \neg q \vee r$, arising from the translation of the formula $(p \supset q) \supset r$ into clauses, only derivations of the form

$$\dfrac{\dfrac{X \vdash \neg p \vee q \vee s \qquad X \vdash \neg r \vee r}{X, p \vee r \vdash q \vee s \vee r} \qquad X \vdash \neg r \vee r}{X, p \vee r, \neg q \vee r \vdash s \vee r}\ .$$

Such a resolution derivation is translated into

$$\dfrac{\dfrac{[\![X^-]\!], p \vdash q, [\![s^+]\!]}{[\![X^-]\!] \vdash p \supset q, [\![s^+]\!]} \supset R \qquad r \vdash r}{[\![X^-]\!], (p \supset q) \supset r \vdash [\![s^+]\!], r} \supset L.$$

For the second case, that is, the clause $p \vee q$, we change its translation into $[\![(p \vee q)^-]\!] = \neg p \supset q \vdash$. We allow for this clause only resolution derivations of the form

$$\dfrac{X \vdash \neg p \vee \Delta \qquad X \vdash \neg q \vee q}{X, p \vee q \vdash q \vee \Delta},$$

which we translate into the sequent derivation

$$\dfrac{\dfrac{[\![X^-]\!], p \vdash \Delta}{[\![X^-]\!] \vdash \neg p, \Delta} \neg R \qquad q \vdash q}{[\![X^-]\!], \neg p \supset q \vdash q, \Delta} \supset L.$$

Next we want to show soundness for the translation. The key point is contained in the following Lemma, which is a modification of Lemma 4.25.

Lemma 4.33 *A formula ϕ is intuitionistically provable if there is an intuitionistic sequent derivation of $[\![X_\phi^-]\!] \vdash$.*

Proof Induction over the structure of ϕ. □

The soundness theorem for the translation is as follows:

Theorem 4.34 *A formula ϕ is intuitionistically provable if there is a resolution derivation of $X_\phi \vdash \bot$ such that the $\lambda\mu\nu\epsilon$-term corresponding to the modified translation into the sequent calculus is intuitionistic.*

Proof The translation of the resolution derivation produces a derivation $[\![X_\phi^-]\!] \vdash$. By assumption the $\lambda\mu\nu\epsilon$-term corresponding to this derivation is intuitionistic, hence there is an intuitionistic derivation of this sequent (Theorem 4.11. Now Lemma 4.33 yields the claim. □

Looking at the example of the resolution derivation for the formula $p \supset p$ again, we see that the modified translation yields a derivation

$$\dfrac{\dfrac{p \vdash p}{\vdash p \supset p} \supset R \qquad X \vdash X}{(p \supset p) \supset X \vdash X} \supset L,$$

with the $\lambda\mu\nu\epsilon$-term $w(\lambda a\colon p.a)$, which is in fact a λ-term and hence an intuitionistic $\lambda\mu\nu\epsilon$-term.

We need one extra step for the proof of completeness of the intuitionistic resolution procedure defined prior in this section. Earlier (Theorem 4.18) we have shown that a sequent $\Gamma \vdash \Delta$ is intuitionistically provable if there is a weakly uniform classical sequent derivation such that the corresponding $\lambda\mu\nu\epsilon$-term is intuitionistic. We now strengthen this theorem and show that under the same hypothesis there is also an intuitionistic resolution derivation of the same sequence.

Theorem 4.35 *Suppose we have a weakly uniform classical sequent derivation of a sequent $[\![X^-]\!], \Delta_1 \vdash \Delta_2$ such that the corresponding $\lambda\mu\nu\epsilon$-term is intuitionistic, all formulæ in X are clauses, all formulæ in Δ_1 and Δ_2 are atoms, Weakening is pushed as far as possible to the root of the derivation, and all axioms have the form $p \vdash p$. Then there is an intuitionistic resolution derivation $X \vdash \neg\Delta_1' \vee \Delta_2'$, where Δ_1' and Δ_2' are subsets of Δ_1 and Δ_2, respectively. Moreover, all of the formulæ in Δ_1 and Δ_2 that are not obtained by Weakening are in Δ_1' and Δ_2' respectively.*

Proof We use the proof of Theorem 4.29 to construct a resolution derivation except for the case of the principal formulæ $\neg p \vee r$ and $\neg q \vee r$, if they arise from the translation of $(p \supset q) \vee r$, and for $p \vee q$. So assume we are given a derivation

$$\dfrac{\dfrac{\Gamma, p \vdash q, \Delta}{\Gamma \vdash p \supset q, \Delta} \supset R \qquad r \vdash r}{\Gamma, (p \supset q) \supset r \vdash r, \Delta} \supset L.$$

The Weakening assumption implies that at most one of p and q can be obtained by Weakening. Note also that neither p nor q can be contained in $[\![X^-]\!]$. If neither p nor q is obtained by Weakening, we have the following resolution derivation:

$$\dfrac{\dfrac{X \setminus \{\neg p \vee r, \neg q \vee r\} \vdash p \vee q \vee s \qquad X \setminus \{\neg p \vee r, \neg q \vee r\} \vdash \neg r \vee r}{X \setminus \{\neg q \vee r\} \vdash q \vee s \vee r} \qquad X \setminus \{\neg q \vee r\} \vdash \neg r \vee r}{X \vdash r \vee s}.$$

If p is obtained by Weakening, then the resolution derivation is

$$\frac{X \setminus \{\neg q \vee r\} \vdash q \vee s \qquad X \setminus \{\neg q \vee r\} \vdash \neg r \vee r}{X \vdash r \vee s}$$

and if q is obtained by Weakening, the resolution derivation is

$$\frac{X \setminus \{\neg p \vee r\} \vdash p \vee s \qquad X \setminus \{p \vee r\} \vdash \neg r \vee r}{X \vdash r \vee s}.$$

The modified translation ensures that the translation of the constructed resolution derivation is also an intuitionistic sequent derivation.

Lastly, we have to consider the case of the modification for the clause $p \vee q$. So assume we are given a derivation

$$\frac{\dfrac{\Gamma, p \vdash \Delta}{\Gamma \vdash \neg p, \Delta}\,\neg R \qquad q \vdash q}{\Gamma, \neg p \supset q \vdash q, \Delta}.$$

The Weakening assumption implies that p is not obtained by Weakening. Then we construct the following resolution derivation:

$$\frac{X \setminus p \vee q \vdash \neg p \vee \Delta \qquad X \setminus p \vee q \vdash \neg q \vee q}{X \vdash q \vee \Delta}.$$

The modification ensures that the translation of the constructed resolution derivation is also an intuitionistic sequent derivation. □

Soundness and completeness now follow in exactly the same way as shown for classical logic.

Corollary 4.36 *A formula ϕ is intuitionistically provable if and only if there is an intuitionistic resolution derivation of $X_\phi \vdash \bot$.*

Proof One direction has already been shown; see Theorem 4.34. For the other, the argument as in Corollary 4.30 works for the modified translation. □

Now we turn to the connection between the choice of input formulæ in the resolution calculus and permutations in the sequent calculus.

Consider the translation of a resolution derivation and examine all the permutations of $\supset L$- and $\neg L$-rules. If one permutation yields an intuitionistic $\lambda\mu\nu\epsilon$-term, then permutation of the order of introducing the input formulæ yields the image of an intuitionistic resolution derivation under the translation. Hence, the soundness and completeness properties (Corollary 4.36) imply that the search for an intuitionistic resolution derivation amounts essentially to the search for a permutation of the $\supset L$- and $\neg L$-rules which yields an intuitionistic $\lambda\mu\nu\epsilon$-term.

As an example of this phenomenon, consider the formula

$$(p \supset q \wedge (p \supset q) \supset q) \supset q.$$

This example demonstrates how a permutation can transform a classical sequent derivation with no intuitionistic force into one with such force. The crucial point is that in order to obtain a weakly uniform intuitionistic proof, the $\supset L$-rule with principal formula $(p \supset q) \supset q$ has to be the rule closest to the root of the derivation. This is also true for the resolution derivation of the formula $(p \supset q \wedge (p \supset q) \supset q) \supset q$ in that the resolution step that uses the input formula corresponding to $(p \supset q) \supset q$ must occur as late as possible; this gives rise to a $\lambda\mu\nu\epsilon$-term which is intuitionistic.

4.8 On complexity

Although we have not emphasized this aspect of our work, the reductive proof systems we have described determine *decision procedures* for the logics for which they are formulated (and determine semi-decision prcocedures for the corresponding predicate systems).

The decision procedures presented in this chapter differ significantly in their computational complexity. Egly [133] shows that the general decision procedure of Section 4.4, and also the procedure for weakly uniform proofs of Section 4.5, have a much higher complexity than the analytic resolution presented in Section 4.7. The reason is that the analytic resolution simulates analytic Cuts that is, Cuts in which the Cut-formula is a subformula of the principal formula, and intuitionistic logic with analytic Cuts has significantly lower complexity. The translation of resolution derivations back into sequent derivations preserves these analytic Cuts by turning them into implications.

4.9 Discussion

We have seen that reductive proof theory—in particular, that weakly uniform and uniform proof—provides a systematic basis not only for proof procedures for the sequent calculus but also for resolution systems for both classical and intuitionistic logic. Moreover, we conjecture that our analyses may be extended, *mutatis mutandis*, to other calculi, such as tableaux systems [120], that are closely related to the sequent calculus.

The definition of uniform proof, though expressed structurally, marks our first consideration of the algorithmic aspects of reductive proof: To construct a uniform proof, we must apply the rules in an order which is constrained by the occurrences of connectives on the left- and right-hand sides of the sequent. So the algorithmic content of our development, though suppressed by our analysis so far, is already quite substantial. In our semantic work in subsequent chapters, in Chapter 6 in particular, capturing the algorithmic content of uniform and weakly uniform proof within our analysis will be an objective (which we shall achieve via a games model).

There are, however, many algorithmic choices not specified by the definition of uniform proof yet which have consequences for the structure of the resulting proofs, in addition to any consequences they might have for the complexity of the computation. We identify four main points.

1. Firstly within the context of uniform proof, a sequent

   $$\phi_1, \ldots, \phi_m \vdash \psi_1, \ldots, \psi_n,$$

   may be reducible in a number of ways: there may be several ψ's which are not atomic; even if all the ψ's are atomic, there may be several ϕ's to which left rules are applicable. This situation obtains even if we restrict to hereditary Harrop resolution.
2. Secondly, after the reduction of a sequent $\Gamma \vdash \Delta$ using an operator R,

   $$\frac{\Gamma_1 \vdash \Delta_1 \quad \ldots \quad \Gamma_m \vdash \Delta_m}{\Gamma \vdash \Delta},$$

   a choice of the order in which to reduce the premisses must be made.
3. Thirdly, in predicate settings, a reduction may depend upon a choice of unifier.[49]
4. Finally, we must handle failure. Having made a choice of reduction, in one of the points of above, we may find that even though our sequent is provable, we have made the wrong choice, leading to a failed proof. In these circumstances, we must return to the point at which we made our choice and try a different one. This procedure is known as *backtracking*.

Although all of these aspects of the construction of proofs are important, perhaps the most challenging to understand, because it is the least structural, or logical, is backtracking.

Given its central rôle in the construction of proofs, and the sense in which it is the paradigmatic control régime, we shall seek, in Chapter 6, a semantics for proof-search which gives not only a model-theoretically adequate semantics for reductive proof, which we develop in Chapter 5, but also a model-theoretically adequate semantics for backtracking.

We conclude this chapter with a few remarks on predicate logic. We have already remarked, at the end of Chapter 2, that Parigot's original presentation

[49] First-order terms have most general unifiers but higher-order terms do not [55, 62].

of $\lambda\mu$ includes the universal quantifier. We conjecture that the analysis of this chapter can be extended to the quantifiers but emphasize that the permutability analysis required in the predicate case is somewhat more complex than in the propositional setting, requiring the idea of a reduction ordering, discussed in Section 4.1. In the setting of dependent types, an analysis of permutability via reduction ordering is provided in [106].

5

SEMANTICS FOR REDUCTIVE LOGIC

5.1 Introduction

So far, we have considered the proof-theoretic properties of various calculi as bases for reduction. Moreover, we have shown that reductive proof, based on the ideas of uniform and weakly uniform proof, provides a systematic basis for resolution systems in classical and intuitionistic logic. In this chapter, we turn to the semantics of reductive proof.

As we have seen in the introduction, an attempt to construct a proof, that is, a *reduction*, proceeds, inductively, by applying instances of reduction operators to putative conclusions of which a proof is desired, thereby yielding a collection of sufficient premisses, proofs of which would be sufficient to imply the existence of a proof, obtainable by deduction, of the putative conclusion. We have seen that a reduction may fail to yield a proof: having removed all of the logical structure, that is, the connectives, by reduction, we may be left with p **?-** q, for distinct atoms p and q. This inherent partiality of reductions presents a clear semantic difficulty: we must be able to interpret those reductions that cannot be completed to be proofs. In particular, we aim to recover a semantics for proofs of utility comparable to that of the propositions-as-types-as-objects triangle for proofs.

Recall that the desired set-up is summarized in Fig. 5.1, in which Γ **?-** ϕ denotes a sequent which is a putative conclusion and

$$\Phi \Rightarrow \Gamma \ \textbf{?-}\ \phi$$

denotes that Φ is a search with root Γ **?-** ϕ. The judgement

$$[\Gamma] \mathrel{|\!\sim} [\Phi] : [\phi]$$

$$\begin{tikzcd}
\Phi \Rightarrow \Gamma \ \textbf{?-}\ \phi \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & {[\Gamma] \mathrel{|\!\sim} [\Phi] : [\phi]} \arrow[dl, leftrightarrow] \\
& \llbracket\Gamma\rrbracket \overset{\llbracket\Phi\rrbracket}{\rightsquigarrow} \llbracket\phi\rrbracket &
\end{tikzcd}$$

FIG. 5.1. Reductions-as-realizers-as-arrows

indicates that $[\Phi]$ is a *realizer* of $[\phi]$ with respect to assumptions $[\Gamma]$. The main objective of this chapter is to describe the bottom corner of this triangle, namely the semantics of reductions.

5.2 Semantics for intuitionistic reductive logic

In this section, we describe a semantics for propositional intuitionistic logic viewed as a reductive system. Building on the wealth of proof-theoretic studies of proof-search in intuitionistic logic [36, 98, 106, 108, 111, 134], we take as our point of departure a minor variant of Gentzen's sequent calculus, LJ$'$, introduced in Section 4.1 and given in Table 5.1, in which Contraction and Weakening are built into the other rules. However, for technical reasons, we include, and emphasize, Exchange$L$. For convenience, we shall simply refer to this system as LJ.

The principal virtues of LJ's presentation of intuitionistic proofs as a basis for mechanical proof-search are that it admits Cut-elimination and, in contrast to natural deduction systems, has, in the absence of Cut, the *subformula property*. Note that although Cut forms the basis of the resolution procedure used by Prolog [67, 113, 130], one can simulate the analytic Cuts used in resolution by implication in LJ (see Section 4.7) hence it is possible to use LJ also as a calculus to study analytic resolution. However, for proof-search either wholly or partially by humans, the Cut rule is very useful because it allows the use of lemmas in proofs and leads to shorter proofs [17, 126].

An *LJ-reduction* is a tree regulated by the operators of LJ, that is, the inference rules of LJ read as reduction operators, from conclusion to premisses. As usual, the sequent Γ **?-** Δ at the root of a tree is called an *endsequent*. We use

TABLE 5.1. Intuitionistic sequent calculus

$$\frac{}{\Gamma, \phi \vdash \phi}\ Ax \qquad \frac{\Gamma, \psi \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi}\ \mathrm{Cut}$$

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \psi, \phi \vdash \Delta}\ \mathrm{Exchange}L$$

$$\frac{}{\Gamma, \bot \vdash \phi}\ \bot L \qquad \frac{}{\Gamma \vdash \top}\ \top R$$

$$\frac{\Gamma, \psi, \psi' \vdash \phi}{\Gamma, \psi \wedge \psi' \vdash \phi}\ \wedge L \qquad \frac{\Gamma \vdash \phi \quad \Gamma \vdash \phi'}{\Gamma \vdash \phi \wedge \phi'}\ \wedge R$$

$$\frac{\Gamma \vdash \phi \quad \Gamma, \psi \vdash \chi}{\Gamma, \phi \supset \psi \vdash \chi}\ \supset L \qquad \frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \supset \psi}\ \supset R$$

$$\frac{\Gamma, \phi \vdash \psi \quad \Gamma, \phi' \vdash \psi}{\Gamma, \phi \vee \phi' \vdash \psi}\ \vee L \qquad \frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \phi'} \quad \frac{\Gamma \vdash \phi'}{\Gamma \vdash \phi \vee \phi'}\ \vee R$$

the following notations for reductions: We write $R_1, \ldots, R_n$ for a reduction with operators $R_1, \ldots, R_n$ applied in which the putative conclusion of every R_i, $i \geq 2$, is one of the sufficient premisses of some operator R_j, for $j < i$.

As we have explained in the introduction, a major difference between reductions and proofs is that reductions need not have axiom sequents at their leaves. Whereas all of the leaves of a proof are of the form $\Gamma, \phi, \Gamma' \vdash \phi$, reductions may have leaves of the form p **?-** q, where p and q are distinct propositional letters. Although a branch with such a leaf cannot be extended so as to obtain just axioms at its leaves, a semantics of reductions must nevertheless give meaning to reductions of this form.

In order to give a semantics for reductions, we start by reviewing our first main tool, namely polynomial categories. These polynomical categories are used to model partial reductions.

Definition 5.1 Let $\mathcal{C}$ be a bi-Cartesian closed category, and let A, B be two objects of $\mathcal{C}$. The polynomial category $\mathcal{C}(\xi)$ over an indeterminate $\xi : A \to B$ is the free bi-Cartesian closed category over the graph of $\mathcal{C}$ with an additional edge ξ with source A and target B modulo the equations in $\mathcal{C}$.

Polynomial categories have a universal property similar to polynomials over the natural numbers [70]:

Theorem 5.2 *Let $\mathcal{C}$ be a bi-Cartesian closed category and $\mathcal{C}(\xi)$ be the polynomial category over the indeterminate $\xi\colon A \to B$.*

1. *For every bi-Cartesian closed functor $F\colon \mathcal{C} \to \mathcal{D}$ and any morphism $f\colon FA \to FB$ in $\mathcal{D}$, there is a bi-Cartesian closed functor $\hat{F}\colon \mathcal{C}(\xi) \to \mathcal{D}$.*
2. *Any bi-Cartesian closed functor $G\colon \mathcal{C}(\xi) \to \mathcal{D}$ is equal to $\hat{F}$ for some bi-Cartesian closed functor $F\colon \mathcal{C} \to \mathcal{D}$ and morphism $f\colon FA \to FB$ in $\mathcal{D}$ such that $\hat{F}(\xi) = f$.*

Proof Direct consequence of the freeness of a polynomial category. □

We write $\mathcal{C}(\xi_1, \ldots, \xi_n)$ for $(\cdots((\mathcal{C}(\xi_1))(\xi_2))\cdots)(\xi_n)$. We call a functor

$$\hat{F}\colon \mathcal{C}(\xi) \to \mathcal{C}(\xi_1, \ldots, \xi_n),$$

obtained by the universal property from the inclusion functor $\mathcal{C} \to \mathcal{C}(\xi_1, \ldots, \xi_n)$ and a morphism f in $\mathcal{C}(\xi_1, \ldots, \xi_n)$, a *substitution functor* and write $S_\xi(f)$ for such a functor. This functor is the analogue to substitution of natural numbers for indeterminates in polynomials over natural numbers.

The polynomial category can be defined in more standard categorical terms if the indeterminate ξ is a morphism $\xi\colon 1 \to A$, where the domain is the terminal object. Such a morphism is called a *global section*, and in the case of $\mathcal{C} = \mathit{Set}$

corresponds to an element of the set A. This restriction does not cause a loss of generality: an indeterminate $\xi\colon A \to B$ corresponds via the universal property defining function spaces to an indeterminate $\xi'\colon 1 \to A \Rightarrow B$. The equivalent definition using standard terms is as follows:

Proposition 5.3 *Suppose $\mathcal{C}$ is a bi-Cartesian closed category. Each polynomial category $\mathcal{C}(\xi)$ with an indeterminate $\xi\colon 1 \to A$ is isomorphic to the co-Kleisli category $\mathcal{D}$ for the endofunctor $(- \times A)$ on $\mathcal{C}$.*

Proof Firstly, it is a routine check that the co-Kleisli category is bi-Cartesian closed and that the inclusion $\mathcal{C} \to \mathcal{D}$ is a bi-Cartesian closed functor. Secondly, one checks that the co-Kleisli category $\mathcal{D}$ satisfies the universal property of Theorem 5.2. In particular, given any bi-Cartesian closed functor $F\colon \mathcal{C} \to \mathcal{E}$ and any morphism $g\colon 1 \to FA$, the extension $\hat{F}\colon \mathcal{D} \to \mathcal{E}$ is given by

$$\hat{F}(A) = F(A) \quad \text{and} \quad \hat{F}(f) = F(f) \circ \langle \mathsf{Id}, g \rangle. \qquad \square$$

In the rest of the chapter, we will only consider indeterminates $\xi\colon 1 \to A$. We write 'ξ is an indeterminate of type A' for such an indeterminate.

Next we show how to use polynomial categories to model reductions. The idea is that a reduction with non-atomic leaves $\Gamma_i \vdash \phi_i$ for $1 \leq i \leq n$ is an element of the category $\mathcal{C}(\xi_1, \ldots, \xi_n)$, where $\mathcal{C}(\xi_1, \ldots, \xi_n)$ is the category $\mathcal{C}$ with indeterminates of type $[\![\Gamma_i]\!] \Rightarrow [\![\phi_i]\!]$ adjoined, where $\Rightarrow$ denotes the internal hom.[50] If $\mathcal{C}$ is the free bi-Cartesian closed category over an infinite set of basic objects representing the propositional atoms, then there exists a morphism $1 \to [\![\phi]\!]$ in $\mathcal{C}$ if and only if the formula ϕ is provable in LJ. If $\mathcal{C}$ is not the free category, a morphism $1 \to [\![\phi]\!]$ exists in $\mathcal{C}$ if the formula ϕ is provable in LJ with possible non-logical axioms added.

Each reduction operator is interpreted as a functor between the appropriate polynomial categories, and we show that a reduction is completeable to a proof when there exist morphisms f_i in $\mathcal{C}(\xi_1, \ldots, \xi_{i-1})$ such that there is a functor $S_{\xi_1}(f_1) \circ \cdots \circ S_{\xi_n}(f_n)\colon \mathcal{C}(\xi_1, \ldots, \xi_n) \to \mathcal{C}$.

Before we can state the semantics of LJ-reductions we fix some notation about categorical morphisms. Suppose $f\colon A \times B \times C \to D$ is a morphism in $\mathcal{C}$. Then we denote by $\mathsf{Cur}_B(f)\colon A \times C \to B \Rightarrow D$ the morphism obtained by applying the definition of exponentials to f. We denote by App the morphism $A \times (A \Rightarrow B) \to B$. Furthermore, we denote the projections by $\pi\colon A \times B \to B$ and $\pi'\colon A \times B \to B$, respectively. More generally, projections are denoted by $\pi_{A_i}\colon A_1 \times \cdots \times A_i \times \cdots \times A_m \to A_i$.

Before we give the definition of the translation from LJ-sequent reductions into morphisms in the polynomial category, we present an example. To state the example and the translation, for each indeterminate ξ of type $[\![\Gamma]\!] \Rightarrow [\![\phi]\!]$, ξ' denotes the morphism $\mathsf{App} \circ \langle \xi \circ \langle\rangle, \mathsf{Id} \rangle\colon [\![\Gamma]\!] \to [\![\phi]\!]$.

[50] Note that we use indeterminates to witness reductions for arbitrary leaves rather than just atomic leaves.

The morphism for the sequent reduction

$$\frac{\phi \text{ ?- } \phi \qquad \phi, \psi \text{ ?- } \psi}{\phi, \phi \supset \psi \text{ ?- } \psi} \supset L$$

will be interpreted by a functor $H : \mathcal{C}(\xi) \to \mathcal{C}$, where ξ is the indeterminate of type $[\![(\phi \wedge (\phi \supset \psi))]\!]$. In fact, H is the substitution functor $S_\xi(\mathsf{Cur}_{[\![\phi \wedge (\phi \supset \psi)]\!]}(\mathsf{App}))$. This functor arises in two stages. Firstly, we have the functor F with domain $\mathcal{C}(\xi)$ and co-domain $\mathcal{C}(\xi_1, \xi_2)$, where ξ_1 is an indeterminate of type $[\![\phi \supset \phi]\!]$ and ξ_2 an indeterminate of type $[\![\phi \wedge \psi \supset \psi]\!]$, respectively, such that $F = S_\xi(\xi_2' \circ \langle \pi, \mathsf{App} \circ \langle \pi', \xi_1' \circ \pi \rangle \rangle)$. The functor F is the semantics of the inference rule $\supset L$, which is basically the application morphism and describes how to obtain a reduction for the sequent $\phi, \phi \supset \psi$?- ψ corresponding to the indeterminate ξ from the two reductions for the sequents ϕ ?- ϕ and ϕ, ψ ?- ψ corresponding to the indeterminates ξ_1 and ξ_2. As the reductions for the latter two sequents are axioms, they are represented by the functors $G_1 = S_{\xi_1}(\mathsf{Cur}_{[\![\phi]\!]}(\mathsf{Id}))$ and $G_2 = S_{\xi_2}(\mathsf{Cur}_{[\![\phi \wedge \psi]\!]}(\pi_{[\![\psi]\!]}))$. The functor H is obtained by essentially composing F with G_1 and G_2.

After this example we give the definition of the translation.[51]

Definition 5.4 Let $\mathcal{C}$ be a bi-Cartesian closed category. The interpretation of each unary LJ-reduction operator

$$\frac{\Delta \text{ ?- } \psi}{\Gamma \text{ ?- } \phi}$$

in $\mathcal{C}$ is a functor $\mathcal{C}(\xi) \to \mathcal{C}(\zeta)$, where ξ is an indeterminate of type $[\![\Gamma]\!] \Rightarrow [\![\phi]\!]$ and ζ is an indeterminate of type $[\![\Delta]\!] \Rightarrow [\![\psi]\!]$. The interpretation of a binary reduction operator

$$\frac{\Delta_1 \text{ ?- } \psi_1 \qquad \Delta_2 \text{ ?- } \psi_2}{\Gamma \text{ ?- } \phi}$$

in $\mathcal{C}$ is a functor $\mathcal{C}(\xi) \to \mathcal{C}(\xi_1, \xi_2)$, where ξ_1, ξ_2, and ξ are indeterminates of types $[\![\Delta_1]\!] \Rightarrow [\![\psi_1]\!]$, $[\![\Delta_2]\!] \Rightarrow [\![\psi_2]\!]$, and $[\![\Gamma]\!] \Rightarrow [\![\phi]\!]$, respectively. These functors are defined as follows:

Axiom: If the reduction operator is

$$\overline{\Gamma, \phi \text{ ?- } \phi},$$

then $[\![Ax]\!] = S_\xi(\mathsf{Cur}_{[\![\Gamma]\!] \times [\![\phi]\!]}(\pi_{[\![\phi]\!]}))$;

[51] Note that we include a clause for the Cut-rule. We need it for the completeness of the categorical semantics we are considering later in this chapter.

Cut: If the reduction operator is

$$\frac{\Gamma, \psi \;\textbf{?-}\; \phi \qquad \Gamma \;\textbf{?-}\; \psi}{\Gamma, \;\textbf{?-}\; \phi}$$

and ξ, ξ_1, and ξ_2 are indeterminates of type $[\![\Gamma]\!] \Rightarrow [\![\phi]\!]$, $[\![\Gamma, \psi]\!] \Rightarrow [\![\phi]\!]$, and $[\![\Gamma]\!] \Rightarrow [\![\psi]\!]$, respectively, then we have $[\![\mathrm{Cut}]\!] = \mathrm{S}_\xi(\mathsf{Cur}(\xi_2' \circ (\langle \mathsf{Id}, \xi_1' \rangle)))$;

Exchange L: If the reduction operator is

$$\frac{\Gamma, \phi_2, \phi_1 \;\textbf{?-}\; \phi}{\Gamma, \phi_1, \phi_2 \;\textbf{?-}\; \phi} \;,$$

then $[\![\text{Exchange } L]\!] = S_\xi(\mathsf{Cur}_{[\![\Gamma, \phi_1, \phi_2]\!]}(\zeta' \circ \langle \pi_{[\![\Gamma]\!]}, \pi_{[\![\phi_2]\!]}, \pi_{[\![\phi_1]\!]} \rangle))$;

$\bot L$: If the reduction operator is

$$\frac{}{\Gamma, \bot \;\textbf{?-}\; \phi} \;,$$

then $[\![\bot L]\!] = \mathsf{Cur}_{[\![\Gamma]\!] \times 0}(S_\xi(\iota \circ \pi))$, where ι is the initial morphism $0 \to [\![\phi]\!]$ and π is the projection from $[\![\Gamma]\!] \times 0$ to 0;

$\top R$: If the reduction operator is

$$\frac{}{\Gamma \;\textbf{?-}\; \top} \;,$$

then $[\![\top R]\!] = \mathsf{Cur}_{[\![\Gamma]\!]}(S_\xi(!))$, where ! is the unique morphism with the terminal object 1 as the co-domain;

$\wedge L$: $[\![\wedge L]\!] = S_\xi(\zeta)$;

$\wedge R$: If the reduction operator is

$$\frac{\Gamma \;\textbf{?-}\; \phi \qquad \Gamma \;\textbf{?-}\; \psi}{\Gamma \;\textbf{?-}\; \phi \wedge \psi}$$

and ξ, ξ_1, and ξ_2 are indeterminates of type $[\![\Gamma]\!] \Rightarrow ([\![\phi]\!] \times [\![\psi]\!])$, $[\![\Gamma]\!] \Rightarrow [\![\phi]\!]$ and $[\![\Gamma]\!] \Rightarrow [\![\psi]\!]$, respectively, then we have

$$[\![\wedge R]\!] = S_\xi(f \circ \langle \xi_1, \xi_2 \rangle)$$

where f is the canonical morphism with domain $([\![\Gamma]\!] \Rightarrow [\![\phi]\!]) \times ([\![\Gamma]\!] \Rightarrow [\![\psi]\!])$ and co-domain $[\![\Gamma]\!] \Rightarrow ([\![\phi]\!] \times [\![\psi]\!])$;

$\vee L$: If the reduction operator is

$$\frac{\Gamma, \phi \;\textbf{?-}\; \sigma \qquad \Gamma, \psi \;\textbf{?-}\; \sigma}{\Gamma, \phi \vee \psi \;\textbf{?-}\; \sigma}$$

and ξ_1 and ξ_2 are indeterminates of type $(\llbracket\Gamma\rrbracket \times \llbracket\phi\rrbracket) \Rightarrow \llbracket\sigma\rrbracket$ and $(\llbracket\Gamma\rrbracket \times \llbracket\psi\rrbracket) \Rightarrow \llbracket\sigma\rrbracket$ respectively, then we have

$$\llbracket \vee L \rrbracket = S_\xi(f \circ (\mathsf{Cur}_{\llbracket\phi\rrbracket + \llbracket\psi\rrbracket}(\mathsf{Cur}_{\llbracket\Gamma\rrbracket}(\xi_1') + \mathsf{Cur}_{\llbracket\Gamma\rrbracket}(\xi_2')))),$$

where f is the canonical isomorphism between $(\llbracket\phi\rrbracket + \llbracket\psi\rrbracket) \Rightarrow \llbracket\Gamma\rrbracket \Rightarrow \llbracket\sigma\rrbracket$ and $(\llbracket\Gamma\rrbracket \times (\llbracket\phi\rrbracket + \llbracket\psi\rrbracket)) \Rightarrow \llbracket\sigma\rrbracket$;

$\vee R$: If the reduction operator is

$$\frac{\Gamma \textbf{ ?- } \phi}{\Gamma \textbf{ ?- } \phi \vee \psi}$$

and suppose ζ is an indeterminate of type $\llbracket\Gamma\rrbracket \Rightarrow \llbracket\phi\rrbracket$, then we have

$$\llbracket \vee R \rrbracket = S_\xi(\mathsf{Cur}_{\llbracket\Gamma\rrbracket}(\mathtt{in}_1 \circ \zeta')).$$

The other case is similar;

$\supset L$: If the reduction operator is

$$\frac{\Gamma \textbf{ ?- } \phi \qquad \Gamma, \psi \textbf{ ?- } \sigma}{\Gamma, \phi \supset \psi \textbf{ ?- } \sigma}$$

and ξ_1, ξ_2, and ξ are indeterminates of type $\llbracket\Gamma\rrbracket \Rightarrow \llbracket\phi\rrbracket$, $(\llbracket\Gamma\rrbracket \times \llbracket\psi\rrbracket) \Rightarrow \llbracket\sigma\rrbracket$, and $(\llbracket\Gamma\rrbracket \times (\llbracket\phi\rrbracket \Rightarrow \llbracket\psi\rrbracket)) \Rightarrow \llbracket\sigma\rrbracket$, respectively, then we have

$$\llbracket \supset L \rrbracket = S_\xi(\xi_2' \circ \langle \pi, \mathsf{App} \circ \langle \pi', \xi_1' \circ \pi \rangle\rangle);$$

$\supset R$: If the reduction operator is

$$\frac{\Gamma, \phi \textbf{ ?- } \psi}{\Gamma \textbf{ ?- } \phi \supset \psi}$$

and ζ is an indeterminate of type $(\llbracket\Gamma\rrbracket \times \llbracket\phi\rrbracket) \Rightarrow \llbracket\psi\rrbracket$, then we have

$$\llbracket \supset R \rrbracket = S_\xi(\mathsf{Cur}_{\llbracket\Gamma\rrbracket}(\mathsf{Cur}_{\llbracket\phi\rrbracket}(\zeta'))).$$

The interpretation of an LJ-reduction $R_1 \,;\, \ldots \,;\, R_k$ for Γ **?-** ϕ, where ; denotes the composition of operators and where the non-axiom leaves are Γ_i **?-** ψ_i $(0 \leq i)$, is given by a functor $H\colon \mathcal{C}(\xi) \to \mathcal{C}(\xi_1, \ldots, \xi_n)$, where ξ is an indeterminate of type $\llbracket\Gamma\rrbracket \Rightarrow \llbracket\phi\rrbracket$ and ξ_i are indeterminates of type $\llbracket\Gamma_i\rrbracket \Rightarrow \llbracket\phi_i\rrbracket$ defined inductively as follows:

1. If $k = 1$, then H is the interpretation of the reduction operator R_1;

2. If $k > 1$ and $R_1; \ldots ; R_{k-1}$ is inductively interpreted as a functor

$$H : \mathcal{C}(\xi) \to \mathcal{C}(\xi_1, \ldots , \xi_l, \eta)$$

and the reduction operator R_k is interpreted as the substitution functor $S_\eta(f)$ for some indeterminate η and morphism f in $\mathcal{C}(\xi_1, \ldots , \xi_l)$, then the reduction $R_1; \ldots ; R_k$ is interpreted as the functor $G \circ H$, where G is the functor obtained by the universal property of polynomial categories applied to the maps

$$\xi_i \mapsto \xi_i \quad \text{and} \quad \eta \mapsto f.$$

A way of grouping all these polynomial categories together is as an indexed category, $\mathcal{E} : \mathcal{B}^{op} \to \mathcal{CCC}$, for which in the setting we have been considering $\mathcal{B}$ is a category with finite products, and the functor $\mathcal{E}$ constructs a (semantic) category of reductions parametrized by indeterminates. This indexed category has extra structure. In particular, it has comprehension, which interprets the addition of fresh indeterminates as reduction proceeds.

An example will help to explain how this definition works. Consider the following reduction, which has one non-axiom leaf σ, τ **?-** ϕ :

$$\dfrac{X \qquad \dfrac{\dfrac{}{\sigma, \psi \text{ ?- } \sigma}Ax \qquad \dfrac{}{\sigma, \psi, \tau \text{ ?- } \tau}Ax}{\sigma, \sigma \supset \tau, \psi \text{ ?- } \tau}}{\sigma, \phi \supset \psi, \sigma \supset \tau \text{ ?- } \tau} \supset L$$

where X is the reduction

$$\dfrac{\dfrac{}{\sigma \text{ ?- } \sigma}Ax \qquad \sigma, \tau \text{ ?- } \phi}{\sigma, \sigma \supset \tau \text{ ?- } \phi} \supset L.$$

If π_σ denotes the projection with co-domain $[\![\sigma]\!]$, then the semantics of the reduction X is the morphism

$$S_{\xi_1}(\mathsf{Cur}_{[\![\sigma]\!] \times [\![\sigma \supset \tau]\!]}(\chi' \circ \langle \pi_\sigma, \mathsf{App}_{\sigma,\tau} \rangle))$$

where χ is an indeterminate of type $[\![\sigma \wedge \tau]\!] \Rightarrow [\![\phi]\!]$, and the semantics for the reduction of the sequent $\sigma, \sigma \supset \tau$ **?-** τ is the morphism

$$S_{\xi_2}(\mathsf{Cur}_{[\![\sigma]\!] \times [\![\sigma \supset \tau]\!]}(\mathsf{App} \circ \langle \pi_\sigma, \pi_{\sigma \supset \tau} \rangle)).$$

The semantics for the whole derivation is then

$$S_\xi(\mathsf{Cur}_{[\![\Gamma]\!]}(\mathsf{App}_{\sigma,\tau} \circ \langle \pi_\sigma, \pi_{\sigma\supset\tau} \rangle \circ \langle \pi_\sigma, \pi_{\sigma\supset\tau}, \mathsf{App}_{\phi,\psi}$$
$$\circ \langle \chi' \circ \langle \pi_\sigma, \mathsf{App}_{\sigma,\tau} \circ \langle \pi_\sigma, \pi_{\sigma\supset\tau} \rangle \rangle, \pi_{\phi\supset\psi} \rangle \rangle))$$

which is via projection-equalities equal to

$$S_\xi(\mathsf{Cur}_{[\![\Gamma]\!]}(\mathsf{App}_{\sigma,\tau} \circ \langle \pi_\sigma, \pi_{\sigma \supset \tau}\rangle)),$$

where Γ is the context $\sigma, \phi \supset \psi, \sigma \supset \tau$. This is unsatisfactory: the left-hand side of the reduction is ignored in the semantics; in other words any reduction with the same right-hand side but a different left-hand side has the same semantics.

The problem is that our semantics implicitly uses a translation from sequent calculus into natural deduction, as there is a direct correspondence between introduction and elimination rules of natural deduction and the categorical constructions. As the translation of sequent calculus into natural deduction identifies sequent calculus derivations up to certain permutations and some Cut-eliminations (see [135] for details), some sequent calculus derivations have the same semantics. In this particular case, the translation of $\supset L$ into natural deduction makes essential use of a Cut. Because it is a Cut with a weakened formula, after Cut-elimination the two derivations have identical natural deduction translation, and hence identical semantics.

5.2.1 *Intuitionistic reduction models*

We solve the problem of information loss described above by introducing a Kripke-world structure in which worlds are intended to record the history of application of reduction operators. Hence each application of a reduction operator gives rise to an extension of worlds. In the key case (*cf.* the example above) of $\supset L$, worlds may therefore be seen as recording increasing propositional 'knowledge' in hypotheses (or, in sequents, antecedents).

In Section 4.3, we used the $\lambda\mu\nu\epsilon$-calculus, which is the $\lambda\mu\nu$-calculus with explicit substitutions added, as a calculus of realizers for LK$'$-derivations. We added the explicit substitutions to overcome the same problem of information loss in a syntactic way. It is possible to treat the explicit substitutions semantically via a Kripke-world structure where the worlds do not contain all reduction operators but reductions corresponding to sufficient premisses of the $\supset L$ and $\vee L$-rules which gave rise to explicit substitutions. The setting described below is more uniform, as it regards application of all reduction operators as an increase of knowledge. This is certainly appropriate for models of proof-search.

The categorical model we use to model this Kripke-world structure is a variant of the setting of a categorical semantics for intuitionistic logic described earlier (see Sections 3.3.1 and 3.3.2 for an introduction to this semantics): For deductions, one considers an indexed category with comprehension $F\colon \mathcal{W} \to \mathcal{CCC}$, where $\mathcal{W}$ is a partial order of worlds regarded as a category, and the functor F assigns to each world W a category $F(W)$ which models all derivations which have additional assumptions given by W. In our setting, worlds represent histories of which reductions have been applied.[52] Hence we modify this semantics to require that the co-domain of the functor F is not the category of bi-Cartesian closed categories but rather a category which represents indeterminates. For each set of indeterminates, we require a bi-Cartesian closed category which models all reductions which use that set of indeterminates. An appropriate categorical structure for this modelling of indeterminates is given by an indexed category with comprehension. The base category models the indeterminates and the fibre over an object models the polynomial category over the indeterminates corresponding to this object. The universal properties of comprehension correspond to the universal property of the polynomial categories. The notion of an indexed category with comprehension is as follows (see Section 3.5 for an introduction into fibred categories):

Definition 5.5 An indexed category with comprehension is a functor

$$\mathcal{E}\colon \mathcal{B}^{op} \to \mathbf{Cat}$$

such that, subject to the usual coherence conditions [14], the following conditions are satisfied:

1. $\mathcal{B}$ has a terminal object called $\top$;
2. Each fibre $\mathcal{E}(\Gamma)$ has a terminal object 1, which is stable under re-indexing;
3. If we denote by $Gr(\mathcal{E})$ the category whose objects are pairs (Γ, A), where Γ is an object of $\mathcal{B}$ and A an object of $\mathcal{E}(\Gamma)$, and morphisms from (Γ, A) to (Δ, B) are pairs of morphisms (f, g) where f is a morphism from Γ to Δ and g is a morphism from A to $\mathcal{E}(f)(B)$, then the functor $I\colon \mathcal{B} {\to} Gr(\mathcal{E})$ sending the object Γ to $(\Gamma, 1)$ and the morphism f to $(f, 1)$ has a right adjoint G.

We denote the object $G(\Gamma, A)$ by $\Gamma \cdot A$ and by $\langle f, g\rangle$ the part of the bijection between hom-sets given by the adjunction $I \dashv G$ sending a morphism $f\colon \Gamma \to \Delta$ in $\mathcal{B}$ and a morphism $g\colon 1 {\to} \mathcal{E}(f)A$ in $\mathcal{E}(\Gamma)$ to a morphism from Γ to $\Delta \cdot A$.

Now we explain how to set-up indeterminates in an indexed categorical setting. An indeterminate of type A is modelled by an object $\top \cdot A$, and a morphism in $\mathcal{C}(\xi_1, \ldots, \xi_n)$ is modelled by a morphism in $\mathcal{E}(\top \cdot A_1 \cdot \ldots \cdot A_n)$. The universal property of polynomial categories is captured as follows: if f is a morphism in $\mathcal{C}(\xi_1, \ldots, \xi_n)$ corresponding to a morphism f' in $\mathcal{E}(\top \cdot A_1 \cdot \ldots \cdot A_n)$ and ξ is

[52]We can also think of worlds as representing the propositions which have been added to the hypotheses by the reduction, the key point being that the $\supset L$-operator replaces a hypothesis $\phi \supset \psi$ with ψ, together with a proof obligation (for ϕ) which may be further reduced. This view is discussed briefly in [103].

an indeterminate of type A, the substitution functor $S_\xi(f)$ is modelled by the functor $\langle \mathsf{Id}, f' \rangle$.

With all this technology set up, we can give a definition of a *reduction structure*, that is, a semantic structure within which intuitionistic (LJ) reductions may be interpreted. A few points are noteworthy:

1. As we have seen, the interpretation of LJ-reductions in polynomials over a bi-Cartesian closed category is inadequate. Consequently the interpretation of (Cut-free) LJ-reductions exploits a Kripke-world structure which records the history of the reduction;
2. There is no equality in the semantics: We interpret only Cut-free reductions and do not consider any equality induced by Cut-elimination.

Definition 5.6 (**reduction structure**) Let $\mathcal{W}$ be a small category (of 'worlds') with finite products. A reduction structure $(\mathcal{E}, F)$ is given by

1. a strict indexed category $\mathcal{E} \colon \mathcal{B}^{op} \to \mathbf{Cat}$ with comprehension such that $\mathcal{B}$ has finite products[53] and each fibre $\mathcal{E}(\Gamma)$ is a bi-Cartesian closed category and each functor $\mathcal{E}(f)$ preserves the bi-Cartesian closed structure on the nose;
2. a functor $F \colon \mathcal{W} \to \mathcal{B}$ which preserves finite products.

Next we present a set-theoretic example of a reduction structure.

Example 5.7 (set-theoretic reduction structure) Let $\mathcal{W}$ be the category of sets and functions. Let $\mathcal{E}$ be the indexed category arising from the flat fibration over **Set** (*i.e.*, $\mathcal{B}$ is **Set** again, and $\mathcal{E}(S)$ is the co-Kleisli category of **Set** with respect to the functor $- \times \mathsf{Id}_S$). We define the functor F as the identity functor from **Set** to **Set**.

Note that in this example, indeterminates and the state of knowledge given by worlds coincide, as the functor F is the identity. This is not necessarily true in general.

We now describe the interpretation of reduction operators and reductions in a reduction category. This interpretation depends on the worlds of the reduction category. The details are given in the following definition:

Definition 5.8 (**interpretation**) Let $(\mathcal{E}, F)$ be a reduction structure. A function $[\![-]\!]$, which is parametrized by a list of indeterminates Θ and a world W, mapping reductions and their syntactic constituents to elements of a reduction structure is called an interpretation if it satisfies the following

[53] If the functor $\mathcal{E}(f)$ is constant on objects then comprehension gives rise to finite products in $\mathcal{B}$. This is the case for all the reduction structures we consider in this monograph.

mutually recursive conditions:

1. $[\![\Theta]\!]^W$ is an object of $\mathcal{B}$ and $[\![\Theta]\!]^W = A$ if Θ is the empty list of indeterminates and $F(W) = A$;
2. For any formula ϕ, $[\![\phi]\!]^W_\Theta$ is an object of the category $\mathcal{E}([\![\Theta]\!]^W)$;
3. For any context $\Gamma = \phi_1, \ldots, \phi_n$, $[\![\Gamma]\!]^W_\Theta$ is equal to $(A_1 \times \cdots \times A_n)$, where $[\![\phi_i]\!]^W_\Theta = A_i$;
4. For a reduction $\Phi \colon \Gamma$ **?-** ϕ with indeterminates in Θ, $[\![\Phi]\!]^W_\Theta$ is a pair (W', g), where W' is a world and g a morphism from $[\![\Gamma]\!]^W_\Theta$ to $[\![\phi]\!]^W_\Theta$ such that $g = \langle \mathsf{Id}, F(a)\rangle^* f$ for some morphisms $f \colon [\![\Gamma]\!]^{W'}_\Theta \to [\![\phi]\!]^{W'}_\Theta$ and $a \colon W \to W'$[54];
5. For all reduction operators R, there exists a world W_R and a morphism $a_R \colon 1 \to W_R$;
6. For a reduction $\Phi; R$ with unary reduction operator R and reduction Φ for the putative premiss of R,

$$[\![\Phi; R]\!]^W_\Theta = (W', \langle \mathsf{Id}, F(a)\rangle^*(\mathsf{Cur}^{-1}_{[\![\Gamma]\!]^{W\times W_R}_\Theta}(\mathsf{App} \circ \langle \mathsf{Cur}_{[\![\Gamma_1]\!]^{W\times W_R}_\Theta}(f_1), \mathsf{Snd}\rangle)))$$

   where $W' = W_1 \times W \times W_R$ and furthermore $[\![\Phi]\!]^{W\times W_R}_\Theta = (W_1', f_1)$ and $W_1' = W \times W_R \times W_1$ and $a \colon W \to W'$;
7. For a reduction $(\Phi_1, \Phi_2); R$ with binary reduction operator R and reductions Φ_1 and Φ_2 for the putative premisses of R,

$$\begin{aligned}[\![(\Phi_1, \Phi_2); R]\!]^W_\Theta &= (W', \langle \mathsf{Id}, F(a_{W'})\rangle^*(\mathsf{Cur}^{-1}_{[\![\Gamma]\!]^{W\times W_R}_\Theta} \\ &\qquad (\mathsf{App} \circ \langle\langle \mathsf{Cur}_{[\![\Gamma_1]\!]^{W\times W_R}_\Theta}(f_1), \mathsf{Cur}_{[\![\Gamma_1]\!]^{W\times W_R}_\Theta}(f_2)\rangle\rangle, \mathsf{Snd})))\end{aligned}$$

   where $W' = W_1 \times W_2 \times W \times W_R$ and $[\![\Phi_i]\!]^{W\times W_R}_\Theta = (W_i', f_i)$ and $W_i' = W \times W_R \times W_i$;
8. If $\Theta = \Theta', \xi$, where ξ is an indeterminate for $\phi_1, \ldots, \phi_n$ **?-** ϕ, then $[\![\Theta]\!]^W$ is equal to $[\![\Theta']\!]^W \cdot [\![(\phi_1 \wedge \cdots \wedge \phi_n) \supset \phi]\!]^W_{\Theta'}$.

This definition only specifies which elements of a reduction structure are used to interpret a given syntactic constituent of a reduction: each reduction operator gives rise to a change of worlds (Clause 5), and the functor F describes how extensions of worlds give rise to change of indeterminates corresponding to reduction operators. Clauses 6 and 7 say that the semantics of a reduction Φ is given by a pair (a, f), where a is an extension of worlds induced by the reduction

[54]Here $-^*$ denotes the usual inverse image functor.

operators of Φ, and f is the morphism obtained by applying the changes of indeterminates induced by the reduction operators to the indeterminates representing the premisses of the reduction.

However, this definition does not specify how to interpret the logical connectives and operators in a reduction. As we use bi-Cartesian closed categories for interpreting reductions, this can be done in a canonical way for the logical connectives and operators of intuitionistic logic. In this way, we obtain a canonical interpretation which is a specific function from syntactic constitutents of a reduction to elements of a reduction structure.

Definition 5.9 (**canonical interpretation**) Let $(\mathcal{E}, F)$ be a reduction structure. The following function $[\![-]\!]$ is an interpretation, called the canonical interpretation, where Θ is a list of indeterminates:

1. $[\![\bot]\!]^W_\Theta \stackrel{\text{def}}{=} 0$;
2. $[\![\top]\!]^W_\Theta \stackrel{\text{def}}{=} 1$;
3. $[\![\phi \supset \psi]\!]^W_\Theta \stackrel{\text{def}}{=} [\![\phi]\!]^W_\Theta \Rightarrow [\![\psi]\!]^W_\Theta$;
4. $[\![\phi \wedge \psi]\!]^W_\Theta \stackrel{\text{def}}{=} [\![\phi]\!]^W_\Theta \times [\![\psi]\!]^W_\Theta$;
5. $[\![\phi \vee \psi]\!]^W_\Theta \stackrel{\text{def}}{=} [\![\phi]\!]^W_\Theta + [\![\psi]\!]^W_\Theta$;
6. For all reduction operators R, $F(a_R) = \langle \mathsf{Id}_1, f\rangle$, where $S_\xi(f)$ is the interpretation of R according to Definition 5.4, where the category $\mathcal{C}$ is the category $\mathcal{E}(1)$.

Note that this definition ensures that for each world W_R the object $F(W_R)$ is the object $([\![\Gamma_1]\!] \Rightarrow [\![\phi_1]\!]) \Rightarrow [\![\Gamma]\!] \Rightarrow [\![\phi]\!]$ for a unary reduction operator R with sufficient premiss Γ_1 **?-** ϕ_1 and putative conclusion Γ **?-** ϕ and $F(W_R)$ is the object $(([\![\Gamma_1]\!] \Rightarrow [\![\phi_1]\!]) \times ([\![\Gamma_2]\!] \Rightarrow [\![\phi_2]\!])) \Rightarrow ([\![\Gamma]\!] \Rightarrow [\![\phi]\!])$ for a reduction operator R with sufficient premisses Γ_1 **?-** ϕ_1 and Γ_2 **?-** ϕ_2 and putative conclusion Γ **?-** ϕ. For each reduction operator R, we denote by ξ_R the indeterminate of the above type, and with A_R the corresponding object.

As mentioned before, the interpretation does not enforce any equality between reductions: The reason is that the semantics of a reduction is a pair (f, g), where f is a morphism between worlds, and it is possible that each reduction gives rise to a different morphism f. Two different reductions might give rise to the same morphism g, however.

Now let us reconsider our earlier example. We need to be precise and indicate carefully the changes of the worlds involved in the reduction. We construct the semantics of the whole reduction, $[\![\Phi]\!]^1_\chi$, where χ is an indeterminate of type $[\![\sigma \wedge \tau]\!] \Rightarrow [\![\phi]\!]$ and A the corresponding object in the base category, step by step. We start with the reduction X. Following Clause 7 of Definition 5.8, we have to calculate $[\![X]\!]^{W_{\supset L}}_\chi$. We obtain $[\![X]\!]^{W_{\supset L}}_\chi = (a_X, f_X)$, where a_X is the extension of worlds from the world $W_{\supset L}$ to $W_{\supset L} \times W_{\supset L} \times W_{Ax}$ and f_X is the morphism

$$h \circ \langle \pi_\sigma, \mathsf{App}_{\sigma,\tau} \rangle$$

in the fibre $E(A_{W_{\supset L}} \times A)$, where h is the morphism $\mathsf{App} \circ \langle \mathsf{Id}, \mathsf{Snd} \circ ! \rangle$.

Next, we have to calculate $[\![\Phi]\!]_X^{W_{\supset L}}$, where Φ is the reduction of the sequent $\sigma, \sigma \supset \tau$ **?-** τ. Again, $[\![\Phi]\!]_X^{W_{\supset L}}$ is a pair (a_Φ, f_Φ), where a_Φ is the extension of worlds from the world $W_{\supset L}$ to $W_{\supset L} \times W_{\supset L} \times W_{Ax} \times W_{Ax}$, and f_Φ is the morphism

$$\mathsf{App} \circ \langle \pi_\sigma, \pi_{\sigma \supset \tau} \rangle$$

in the fibre $E(A_{W_{\supset L}} \times A)$. The semantics for the whole reduction Ψ is a pair (a_Ψ, f_Ψ), where a_Ψ is the world extension from the empty world (the terminal object in the category $\mathcal{W}$) to the world $W_{\supset L} \times W_{\supset L} \times W_{Ax} \times W_{\supset L} \times W_{Ax} \times W_{Ax}$, and f_Ψ is the morphism

$$\mathsf{App}_{\sigma,\tau} \circ \langle \pi_\sigma, \pi_{\sigma \supset \tau} \rangle \circ \langle \pi_\sigma, \pi_{\sigma \supset \tau}, \mathsf{App}_{\phi,\psi} \circ \langle h \circ \langle \pi_\sigma, \mathsf{App}_{\sigma,\tau} \circ \langle \pi_\sigma, \pi_{\sigma \supset \tau} \rangle \rangle, \pi_{\phi \supset \psi} \rangle \rangle$$

in the fibre $E(A)$, which is via projection-equalities equal to

$$\mathsf{App}_{\sigma,\tau} \circ \langle \pi_\sigma, \pi_{\sigma \supset \tau} \rangle,$$

where Γ is the context $\sigma, \phi \supset \psi, \sigma \supset \tau$.

The semantics of the reduction Ψ does not ignore the reduction X: the world extension a_Φ explicitly mentions the reduction operators in X, thereby recording the increase of knowledge obtained by the reduction X.

Our objective has been to establish a semantics of reductive logic of comparable value to that which is available for deductive logic. To this end, we now establish soundness and completeness theorems relating reductions and their semantics. We begin with the appropriate semantic judgement,

$$W \models_\Theta (\Phi : \phi)[\Gamma],$$

between worlds, W, indeterminates in Θ, sequents Γ **?-** ϕ and reductions, Φ. This judgement is formulated as a constraint on reduction structures which is required in order to interpret reductions correctly in reduction structures.

Definition 5.10 (reduction model) A reduction model,

$$\mathcal{R} = \langle (\mathcal{E}, F), [\![-]\!], \models \rangle,$$

is given by the following:

1. A reduction structure $(\mathcal{E}, F)$;

2. An interpretation $[\![-]\!]$ of reduction operators and reductions;

3. A forcing relation $W \models_{\Theta} (\Phi\colon \phi)[\Gamma]$, where W is a world, Θ and Γ are contexts, ϕ a formula and Φ a reduction with endsequent Γ **?-** ϕ with indeterminates contained in Θ, such that

$$[\![\Gamma]\!]^W_{\Theta} \xrightarrow{[\![\Phi]\!]^W_{\Theta}} [\![\phi]\!]^W_{\Theta}$$

is a morphism in the reduction structure, and which satisfies the following conditions:
   (i) If $W \models_{\Theta} (\Phi\colon \phi)[\Gamma]$ and $a\colon W \to W'$ is a morphism in $\mathcal{W}$ for some world W', then also $W' \models_{\Theta} (\Phi\colon \phi)[\Gamma]$;
   (ii) $W \models_{\Theta} (Ax\colon \phi)[\Gamma, \phi]$;
   (iii) $W \models_{\Theta,\xi} (\xi\colon \phi)[\Gamma]$ if ξ is an indeterminate of type Γ **?-** ϕ;
   (iv) If R is a reduction operator with premisses Γ_1 **?-** ϕ_1 and Γ_2 **?-** ϕ_2 and conclusion Γ **?-** ϕ, then $W \models_{\Theta} ((\Phi_1, \Phi_2); R)[\Gamma, \phi]$ if $W \times W_R \models_{\Theta} (\Phi_i)[\Gamma_i, \phi_i]$;
   (v) If R is a reduction operator with premiss Γ_1 **?-** ϕ_1 and conclusion Γ **?-** ϕ, then $W \models_{\Theta} (\Phi_1; R)[\Gamma, \phi]$ if $W \times W_R \models_{\Theta} (\Phi_1)[\Gamma_1, \phi_1]$.

Substitutivity for indeterminates is a property of the forcing relation:

Lemma 5.11 *If* $W \models_{\Theta,\xi} (\Phi\colon \phi)[\Gamma]$, $W \models_{\Theta} (\Psi\colon \psi)[\Delta]$ *and* ξ *is an indeterminate of type* Δ **?-** ψ, *then also* $W \models_{\Theta} (\Phi[\Psi/\xi]\colon \phi)[\Gamma]$.

Proof By induction over the structure of Φ. □

Now, we can establish soundness: the existence of a reduction Φ of Γ **?-** ϕ implies that ϕ is forced at every world W in a reduction model and, consequently, that reduction Φ is interpreted as a realizer of the interpretation of ϕ from the interpretation of Γ.

Theorem 5.12 (soundness) *Consider any reduction structure* $(\mathcal{E}, F)$. *Suppose* Φ *is a reduction of* Γ **?-** ϕ *with indeterminates* $\xi_1, \ldots, \xi_n$ *of type* Γ_i **?-** ϕ_i. *Then, for any world* W, $W \models_{\Theta} (\Phi\colon \phi)[\Gamma]$, *where* $\Theta = \{\xi_1, \ldots, \xi_n\}$.

Proof We use induction over the structure of ϕ. The case of an indeterminate and an axiom are trivial. Now, consider the case of a reduction $(\Phi_1, \Phi_2); R$. By induction hypothesis, $W \times W_R \models_{\Theta} (\Phi_i)[\Gamma_i, \phi_i]$. Hence, by Clause (3.iv) of Definition 5.10, $W \models_{\Theta} ((\Phi_1, \Phi_2); R)[\Gamma, \phi]$. The case of a unary reduction rule is similar. □

Turning to completeness, we must first establish a notion of *validity*. We say that the judgement $\Phi : \phi$ is *valid with respect to* Γ *and* Θ, and write

$$\Gamma \models_{\Theta} \Phi : \phi,$$

if and only if, for all worlds, W, in all reduction models, $\mathcal{R}$,

$$W \models^{\mathcal{R}}_{\Theta} (\Phi : \phi)[\Gamma].$$

With respect to this, quite straightforward, notion of validity, we are able to establish completeness. The first step is a model existence lemma based on the construction of a term model.

Lemma 5.13 (model existence) *There exists a reduction model*

$$\mathcal{T} = \langle F, [\![-]\!], \models_{\Theta} \rangle,$$

such that if $1 \models^{\mathcal{T}}_{\Theta} (\Phi : \phi)[\Gamma]$, *then* Γ **?-** ${}_{\Theta}\Phi : \phi$.

Proof We construct a term model from reductions in the calculus LJ. We begin by defining a reduction structure $(\mathcal{E}, F)$.

The category of worlds is the free Cartesian category where

1. the ground objects are reduction operators R with sufficient premisses Γ_1 **?-** ϕ_1 and Γ_2 **?-** ϕ_2, for binary operators, and Γ' **?-** ϕ', for unary operators, and putative conclusion Γ **?-** ϕ, and
2. for each reduction operator, there is a ground morphism $a_R \colon 1 \to R$.

The objects of the category $\mathcal{B}$ are finite sequences of indeterminates $\xi_1, \ldots, \xi_n$ of type $\phi_1, \ldots, \phi_n$, and a morphism from $(\xi_1, \ldots, \xi_n)$ to $(\xi'_1, \ldots, \xi'_m)$ is a list $(f_1, \ldots, f_m)$ of reductions such that f_i is a reduction of $-$ **?-** ϕ'_i, possibly using the indeterminates $\xi_1, \ldots, \xi_n$, where ϕ'_i is the type of the indeterminate ξ'_i. Composition is given by substitution of reductions for indeterminates. For each sequence of indeterminates $(\xi_1, \ldots, \xi_n)$, we define the category $\mathcal{E}((\xi_1, \ldots, \xi_n))$ to be the category where the objects are formulæ and morphisms from ϕ to ψ are reductions with premiss ϕ and conclusion ψ with indeterminates amongst the ones in $(\xi_1, \ldots, \xi_n)$ up to $\beta\eta$-equivalence. Composition in this category is given by Cut. There is no equivalence on propositions, as we do not consider any type dependency.

As $\mathcal{W}$ is the free Cartesian category over the reduction operators R and morphisms a_R, it suffices to define the action of F on reduction operators and morphisms a_R.

For a binary reduction operator, the functor F is given by $F(R) = \xi$, where ξ is an indeterminate of type $((\Gamma_1 \supset \phi_1) \wedge (\Gamma_2 \supset \phi_2)) \supset (\Gamma \supset \phi)$,[55] where

[55]Here we abuse notation slightly and write, where $\Gamma = \psi_1, \ldots, \psi_m$, just $\Gamma \supset \phi$ to denote the formula $(\psi_1 \wedge \ldots \wedge \psi_m) \supset \phi$.

the reduction operator with sufficient premisses Γ_1 **?-** ϕ_1 and Γ_2 **?-** ϕ_2, and putative conclusion Γ **?-** ϕ, and $F(a_R) = h$, where $S_\xi(h)$ is the interpretation of R in the polynomial categories.

For a unary reduction operator, we define $F(R) = \xi$, where ξ is an indeterminate of type $(\Gamma' \supset \phi') \supset (\Gamma \supset \phi)$ for a reduction operator with sufficient premiss Γ' **?-** ϕ' and putative conclusion Γ **?-** ϕ.

Now consider the morphism a_R for the reduction operator with sufficient premisses Γ_1 **?-** ϕ_1 and Γ_2 **?-** ϕ_2 and putative conclusion Γ **?-** ϕ. Let Φ_R be sequent reduction

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma_2 \supset \phi_2, \Gamma_1 \;\textbf{?-}\; \Gamma_1}\,Ax \qquad \dfrac{}{\Gamma_2 \supset \phi_2, \Gamma_1, \phi_1 \;\textbf{?-}\; \phi_1}\,Ax}{\Gamma_1 \supset \phi_1, \Gamma_2 \supset \phi_2, \Gamma_1 \;\textbf{?-}\; \phi_1}\,\supset L \qquad \begin{matrix} \vdots\, \Psi \\ \Gamma_1 \supset \phi_1, \Gamma_2 \supset \phi_2, \Gamma_2 \;\textbf{?-}\; \phi_2 \end{matrix}}{\Gamma_1 \supset \phi_1, \Gamma_2 \supset \phi_2, \Gamma \;\textbf{?-}\; \phi}\,R}{\textbf{?-}\; ((\Gamma_1 \supset \phi_1) \wedge (\Gamma_2 \supset \phi_2)) \supset \Gamma \supset \phi}\,\wedge L; \supset R; \supset R,$$

where Ψ is the sequent reduction of $\Gamma_1 \supset \phi_1, \Gamma_2 \supset \phi_2, \Gamma_2$ **?-** ϕ_2 similar to the reduction of $\Gamma_1 \supset \phi_1, \Gamma_2 \supset \phi_2, \Gamma_1$ **?-** ϕ_1. Now we define $F(a_R) = \Phi_R$. Intuitively, the additional reduction steps in Φ_R are just book-keeping steps to ensure the typing of $F(a_R)$ matches the typing of the corresponding indeterminate.

Next, we show that, for this reduction structure with the obvious interpretation $[\![-]\!]$ of operators and reductions, the relation defined by $W \models_\Theta (\Phi\colon \phi)[\Gamma]$ iff Φ is a reduction of Γ **?-** ϕ with indeterminates in Θ such that $[\![\Phi]\!]^W_\Theta$ is a morphism from $[\![\Gamma]\!]^W_\Theta$ to $[\![\phi]\!]^W_\Theta$ is a forcing relation, and the triple $((\mathcal{E}, F), [\![-]\!], \models)$ is a reduction model. $\square$

In the usual way, we now obtain the following:

Theorem 5.14 (completeness) *If* $\Gamma \models_\Theta \Phi\colon \phi$, *then* Γ **?-** $\Phi : \phi$.

Proof Suppose $\Gamma \models_\Theta \Phi\colon \phi$, then, for all worlds W in all reduction models $\mathcal{R}$,

$$W \models^{\mathcal{R}}_\Theta (\Phi : \phi)$$

This holds also for the term model constructed in Lemma 5.13. By construction of this model, we have Γ **?-** $\Phi : \phi$. $\square$

Under stronger conditions, namely that there exists a canonical interpretation, we can show more, namely for each reduction structure and interpretation there exists a canonical forcing relation:

Lemma 5.15 *Suppose* $(\mathcal{E}, F)$ *is a reduction structure with a canonical interpretation* $[\![-]\!]$. *Then the relation* $\mathcal{R}$ *defined by* $W \models_\Theta (\Phi\colon \phi)[\Gamma]$ *iff* Φ *is a reduction of* Γ **?-** ϕ *with indeterminates in* Θ *such that* $[\![\Phi]\!]^W_\Theta$ *is a morphism from* $[\![\Gamma]\!]^W_\Theta$ *to* $[\![\phi]\!]^W_\Theta$ *is a forcing relation, and the triple* $((\mathcal{E}, F), [\![-]\!], \models)$ *is a reduction model. Moreover, the forcing relation* $\models$ *of Lemma 5.13 is such a relation.*

Proof We have to check that the relation $\mathcal{R}$ is a forcing relation. For this, one shows that Clause 6 of Definition 5.9 implies that $[\![\Phi]\!]^W_\Theta$ is indeed a morphism from $[\![\Gamma]\!]^W_\Theta$ to $[\![\phi]\!]^W_\Theta$. □

Now we consider the translation in the other direction. As the reduction category is not necessarily the free category over some ground objects, we cannot define such a translation inductively but only specify constraints which such a translation should satisfy. If the reduction structure happens to a free structure, the conditions turn out to define a translation uniquely. We define this translation first for polynomial categories and then generalize it to reduction structures.

Definition 5.16 Let $\mathcal{C}$ be any bi-Cartesian category. A translation $(-)^s$ assigning morphisms $f\colon \Gamma{\to}A$ in $\mathcal{C}(\xi_1,\ldots,\xi_n)$ to reductions with non-atomic endsequents contained in $\xi_i\colon \Gamma_i$ **?-** A_i is called sound if:

$$\begin{aligned}
(\pi)^s &= Ax, \text{ where } \pi \text{ is any projection}\\
(\xi_i)^s &= \xi_i\\
(g\circ f)^s &= ((f)^s,(g)^s);\mathrm{Cut}\\
(\langle f,g\rangle)^s &= ((f)^s,(g)^s);\wedge R\\
(\mathsf{Cur}M)^s &= (M)^s;\supset R\\
(\mathsf{App})^s &= \supset L\\
(\mathtt{in}_1)^s &= \vee R\\
(\mathtt{in}_2)^s &= \vee R\\
(f\oplus g)^s &= ((f)^s,(g)^s);\vee L
\end{aligned}$$

Lemma 5.17 *Suppose $\mathcal{C}$ is the free bi-Cartesian closed category over some set of objects $\mathcal{G}$. Then there is a sound translation assigning to each morphism $f\colon \Gamma\to A$ in $\mathcal{C}(\xi_1,\ldots,\xi_n)$ reductions with non-atomic endsequents contained in $\xi_i\colon \Gamma_i$* **?-** *A_i.*

Proof The translation is given in the canonical way by using the Curry–Howard correspondence to derive natural deductions for morphisms, and then translating them into reductions. □

Now we generalize this translation to the translation of morphisms of reduction structures to reductions. Again, we list first conditions which such a translation should satisfy.

Definition 5.18 A translation $(-)^s$ from morphisms $f\colon \Gamma\to A$ in $\mathcal{E}(\Theta)$ of a reduction structure $(\mathcal{E},F)$ to reductions $\Gamma\to A$ where $\Theta=\top\cdot A_1\cdot\cdots\cdot A_n$

and ξ_i is an indeterminate with type $(A_i)^s$ is sound if:

$$\begin{aligned}
(\pi)^s &= Ax, \text{ where } \pi \text{ is any projection} \\
(\mathsf{Fst}^k * \mathsf{Snd})^s &= \xi_k \\
(g \circ f)^s &= ((f)^s, (g)^s); \mathrm{Cut} \\
(\langle f, g\rangle)^s &= ((f)^s, (g)^s); \wedge R \\
(\mathsf{Cur} M)^s &= (M)^s; \supset R \\
(\mathsf{App})^s &= \supset L \\
(\mathtt{in}_1)^s &= \vee R \\
(\mathtt{in}_2)^s &= \vee R \\
(f \oplus g)^s &= ((f)^s, (g)^s); \vee L
\end{aligned}$$

Again, when we have the initial reduction structure these conditions are sufficient to guarantee the existence of such a translation.

Lemma 5.19 *Suppose $(\mathcal{E}, F)$ is a reduction structure such that $Gr(\mathcal{E})$ is the free comprehension category over some set of objects $\mathcal{G}$. Then there is a sound translation assigning to each morphism $f\colon \Gamma \to A$ in $\mathcal{E}(\Theta)$ a reduction with non-atomic endsequents contained in $\xi_i\colon \Gamma_i$* **?-** *A_i, where Θ is the context corresponding to the indeterminates $\xi_1, \ldots, \xi_n$.*

Proof Direct transfer of the previous lemma. □

Now we can show that a reduction can be completed if and only if there exists a functor from the corresponding polynomial category into the ground category.

Theorem 5.20 *Suppose $(\mathcal{E}, F)$ is the free reduction structure over a set of objects $\mathcal{G}$. A reduction Φ of Γ* **?-** *ϕ with leaves Γ_i* **?-** *ϕ_i which are not axioms can be completed to a proof iff there exists a morphism f such that there is a functor $\mathcal{E}(\langle !, f\rangle)\colon \mathcal{E}(\Theta) \to \mathcal{E}(1)$ where Θ is the context corresponding to the indeterminates $\xi_1, \ldots, \xi_n$. Moreover, the completion of a Cut-free reduction is Cut-free.*

Proof If there exists a completion, the soundness theorem guarantees the existence of a morphism f. In the other direction, given such a morphism, the previous lemma provides the sequent derivations, which complete the reduction to a proof. As Cut-elimination holds for LJ without indeterminates, there is also a Cut-free sequent which provides the completion. □

5.2.2 *Games for intuitionistic reductions*

We now describe how to extend our games model of intuitionistic proofs, introduced in Chapter 3, to be a model of intuitionistic reductions. There are two issues which we need to consider. Firstly, we need to consider games for sequent calculus, not for natural deduction, as in Section 3.4, and, secondly, we have to model indeterminates.

We begin with the first issue. It turns out that the games described in Section 3.4 give rise to a very natural interpretation of reductions in the sequent calculus. Intuitively, O- and P-questions are challenges for Opponent and Proponent to provide evidence for conclusions and premisses, respectively. O-answers provide evidence for a premiss, and P-answers provide evidence for a conclusion. Conjunctive choices are made by Opponent and disjunctive choices are made by Proponent.

As usual, left-operators involve operations on the premisses: they are initiated by P-questions. Similarly, right-operators involve operations on the conclusions: they are initiated by O-questions.

We need additional structure to model indeterminates. The key idea is to introduce additional plays which Proponent may start at will.

Definition 5.21 A strategy with oracle of type ϕ is a strategy where, in addition, Proponent is allowed to play using an additional arena for ϕ. The justifying question for the root nodes of ϕ is an initial question.

Substitution of reductions for indeterminates is modelled by substitution of strategies for oracles.

Definition 5.22 Suppose Ψ is a strategy with oracle of type ϕ and Φ is a strategy of type ϕ. We define the substitution of Φ for the oracle in Ψ to be the strategy Ψ except that we replace every answer which is a move given by the arena for ϕ by the move obtained by using Φ to answer Ψ's move in ϕ, then using Ψ to answer this move and so on until Ψ answers with a move outside the arena for ϕ.

Substitution of strategies for oracles is well-defined:

Lemma 5.23 *Let Ψ be a strategy for the arena for σ with oracle of type ψ and Φ is strategy for the arena for ψ with an oracle of type ϕ. Then the substitution of Φ for the oracle in Ψ is a strategy for the arena for σ with oracle of type ϕ.*

Proof By induction over the structure of σ. $\square$

A proof with indeterminate of type ϕ is now modelled as a strategy with oracle of type Φ. More precisely, the proof of **?-** ϕ using only an indeterminate of type ϕ is modelled by the copy-cat strategy, where Proponent simply replays each O-question in the arena for ϕ in the additional arena for ϕ he may use.

Theorem 3.17, presented in Chapter 3, can be extended to games with indeterminates and stated intuitionstic reductions (based not on natural deduction but on the sequent calculus):

Theorem 5.24 *For any formula ϕ and strategy Φ for ϕ with oracles of type $\psi_1, \ldots, \psi_m$ there exists a intuitionistic reduction of ϕ with indeterminates of types $\psi_1, \ldots, \psi_m$.*

Proof By Theorem 3.17 we obtain a reduction of $(\psi_1 \wedge \cdots \wedge \psi_m) \supset \psi$ and hence also a reduction of $\psi_1 \wedge \cdots \wedge \psi_m$ **?-** ψ. Now we use Cuts with the reduction $(\cdots(\Psi_1, \Psi_2); \wedge R \cdots), \Psi_m); \wedge R$ where Ψ_i is the reduction consisting only of an indeterminate of type ψ_i. □

At this point, we have achieved our first objective. We have a class of abstract structures which supports the triangle of Fig. 1.2. The top left-hand corner represents the basic calculus of queries; the top right-hand corner stands formal language of reductions, built using a class of variables corresponding to indeterminates; and the bottom corner is given by the interpretation of reduction models. However, we do not as yet have a declarative, or truth-functional, semantics of *search*. As we have seen, in the setting of reductive logic, such a semantics can be understood in terms of state.

We will now develop a semantics of proof-search for intuitionistic logic by considering the class of intuitionistic reductions to be embedded in the class of classical reductions, using the techniques introduced in [97, 108, 111]. To this end, we extend our semantics of reduction to the formulation of classical logic based on the $\lambda\mu\nu$-calculus [97, 108, 111].

5.3 Semantics for classical reductive logic

In this section, we describe a semantics for propositional classical logic viewed as a reductive system. Building on the wealth of proof-theoretic studies of proof-search in classical logic—see, for example, [36, 98, 106, 108, 111, 134]—we take as our point of departure a minor variant of Gentzen's sequent calculus, LK, given in Table 4.3. Contraction and Weakening are built into the other rules but, for technical reasons, we include Exchange. Note also the absence of the usual rules for negation,

$$\frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}\,\neg L \qquad \frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}\,\neg R.$$

For technical reasons, it is simpler for our semantic purposes to *define* $\neg\phi$ as in the intuitionistic style as $\phi \supset \bot$. In the presence of the classical $\supset R$ rule,

TABLE 5.2. Classical sequent calculus

$$\frac{}{\Gamma, \phi \vdash \phi, \Delta}\ Ax \qquad \frac{\Gamma, \phi \vdash \Delta \quad \Gamma \vdash \phi, \Delta}{\Gamma \vdash \Delta}\ \mathrm{Cut}$$

$$\frac{\Gamma, \phi, \psi, \Gamma' \vdash \Delta}{\Gamma, \psi, \phi, \Gamma' \vdash \Delta}\ \mathrm{Exchange}L \qquad \frac{\Gamma \vdash \Delta, \phi, \psi, \Delta'}{\Gamma \vdash \Delta, \psi, \phi, \Delta'}\ \mathrm{Exchange}R$$

$$\frac{}{\Gamma, \bot \vdash \Delta}\ \bot L \qquad \frac{}{\Gamma \vdash \top, \Delta}\ \top R$$

$$\frac{\Gamma, \psi, \psi' \vdash \Delta}{\Gamma, \psi \wedge \psi' \vdash \Delta}\ \wedge L \qquad \frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \phi', \Delta}{\Gamma \vdash \phi \wedge \phi', \Delta}\ \wedge R$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \supset \psi \vdash \Delta}\ \supset L \qquad \frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \supset \psi, \Delta}\ \supset R$$

$$\frac{\Gamma, \phi \vdash \Delta \qquad \Gamma, \phi' \vdash \psi, \Delta}{\Gamma, \phi \vee \phi' \vdash \psi, \Delta}\ \vee L \qquad \frac{\Gamma \vdash \phi, \phi', \Delta}{\Gamma \vdash \phi \vee \phi', \Delta}\ \vee R$$

$\neg L$ and $\neg R$ are derivable. For convenience, we shall simply refer to this system as LK.

As with the intuitionistic calculus, LJ, the principal virtues of LK's presentation of intuitionistic proofs as a basis for proof-search are that it admits Cut-elimination and, in the absence of Cut, has the subformula property. Note, however, that the advantages of Cut discussed in Section 5.2 apply equally well to classical logic.

Semantically, we aim to extend the definition of a reduction structure to classical logic, that is, to LK proofs. To this end, we require a representation of classical proofs for which a non-trivial semantics is available.[56]

The $\lambda\mu\nu$-calculus [108, 111] is a representation of classical proofs, essentially a multiple-conclusioned form of natural deduction, which has a non-trivial categorical semantics [97]. It is an extension of Parigot's [90] $\lambda\mu$-calculus to account for disjunction.

The relationship between $\lambda\mu\nu$ and LK is delicate. Some of the delicate issues were discussed in detail in Chapter 4. Below we adapt the semantics of LK-proofs in the $\lambda\mu\nu$-calculus to deal with LK-reductions, in the same way as we changed the semantics of LJ-proofs using bi-Cartesian closed categories to deal with LJ-reductions.

5.3.1 *Classical reduction models*

Having established the semantics of $\lambda\mu\nu$ as a deductive system, and given our general prescription for reading inference rules as reduction operators, we can

[56]That is, a semantics that does not identify all proofs of a given sequent.

give the definition of a classical reduction structure. Such a structure arises from a $\lambda\mu\nu$-structure by introducing an additional fibration to model indeterminates and introducing a category of worlds and a functor to the Grothendieck completion of the fibration as for reduction structures. Note that we can merge the two fibrations (one for the formulæ on the right-hand side, and one for indeterminates) into a fibration over a product.

Again, a few points are noteworthy:

1. The addition of indeterminates to models of $\lambda\mu\nu$ follows the same pattern as for (intuitionistic) reduction structures but fibre-wise;
2. The structure of $\lambda\mu\nu$-models reflects the fact that $\lambda\mu\nu$ is essentially a system of natural deduction. Consequently, just as in the intuitionistic case, the interpretation of (Cut-free) LK-reductions exploits a Kripke-world structure which records the history of the reduction;
3. As before, there is no equality between reductions in the semantics: We interpret only Cut-free reductions and do not consider any equality induced by Cut-elimination. A non-trivial, symmetric categorical semantics of LK (essentially in Gentzen's original form [37]), which validates all (in)equalites induced by Cut-elimination, has been introduced by Führmann and Pym [30], but these ideas are beyond our present scope.

Definition 5.25 Let $\mathcal{W}$ be a small category (of 'worlds') with finite products. A classical reduction structure $(\mathcal{E}, F)$ is given by the following:

1. A strict indexed category $\mathcal{E} \colon (\mathcal{B} \times \mathcal{C})^{op} \to \mathbf{Cat}$ with comprehension such that $\mathcal{B}$ has finite products and each fibre $\mathcal{E}(\Gamma, \Delta)$ is a bi-Cartesian closed category and each functor $\mathcal{E}(f, g)$ preserves the bi-Cartesian closed structure on the nose; and
2. A functor $F \colon \mathcal{W} \to \mathcal{B}$ which preserves finite products
   such that the following properties hold:
   (i) There is a natural bijection between $\mathsf{hom}_{\mathcal{B}}(A, B \times C)$ and the pair
   $$(\mathsf{hom}_{\mathcal{B}}(A, B), \mathsf{hom}_{\mathcal{E}(A,1)}(1, C));$$
   (ii) For each object A of $\mathcal{B}$, the functor $\mathcal{E}(\mathsf{Id}_A \times -)$ is $\lambda\mu\nu$-structure, and for each morphism $f \colon A \to B$ in $\mathcal{B}$, the natural transformation $\mathcal{E}(f, -)$ preserves the structure of a $\lambda\mu\nu$-structure on the nose.

We give a set-theoretic example of a classical reduction structure. This example is an adaptation of the set-theoretic example of an intuitionistic reduction structure.

Example 5.26 (*set-theoretic classical reduction structure*) Let $\mathcal{E}$ be the $\lambda\mu\nu$-structure of Chapter 3 defined using continuations, where the category $\mathcal{C}$ is the category of sets and functions. Let $\mathcal{W}$ be the category of sets and functions. Let $\mathcal{B}$ be the category of sets and functions, and let $\mathcal{F}$ be the indexed category defined by $\mathcal{F}(A, B) = \mathcal{E}(A \times B)$. The functor F is the identity.

As each fibre $\mathcal{E}(A)$ in the continuation model is equivalent to the co-Kleisli category for the functor $A \times -$, we do not need to construct co-Kleisli categories to model indeterminates, as we do for the set-theoretic example of intuitionistic reduction structures. Also note that, in this example, indeterminates and the states of knowledge given by worlds coincide, because the functor F is the identity. Again, this is not necessarily true in general.

Next we describe how to interpret LK-reductions in a classical reduction structure. In the same way as for intuitonistic logic, we first spell out the defining conditions for such an interpretation.

Definition 5.27 (interpretation) Let $(\mathcal{E}, F)$ be a classical reduction structure. A function $[\![-]\!]$, which is parametrized by a list of indeterminates Θ and a world W, mapping reductions of LK and their syntactic constituents to elements of a reduction structure is called an interpretation if it satisfies the following mutually recursive conditions:

1. $[\![\Theta]\!]^W$ is an object of $\mathcal{B}$ and $[\![\Theta]\!]^W = A$ if Θ is the empty list of indeterminates and $F(W) = A$;
2. For any formula ϕ, $[\![\phi]\!]^W_\Theta$ is an object of the category $\mathcal{E}(([\![\Theta]\!]^W, 1))$;
3. For any context $\Gamma = \phi_1, \ldots, \phi_n$, $[\![\Gamma]\!]^W_\Theta$ is equal to $(A_1 \times \cdots \times A_n)$, where $[\![\phi_i]\!]^W_\Theta = A_i$;
4. For a reduction $\Phi\colon \Gamma$ **?-** ϕ, Δ with indeterminates in Θ, $[\![\Phi]\!]^W_\Theta$ is a pair (W', g), where W' is a world and g a morphism from $[\![\Gamma]\!]^W_\Theta$ to $[\![\phi]\!]^W_\Theta$ in $\mathcal{E}([\![\Theta]\!]^{W'}, [\![\Delta]\!]^{W'}_\Theta)$ such that $g = (\langle \mathsf{Id}, F(a)\rangle, \mathsf{Id})^* f$, for some morphisms
$$f\colon [\![\Gamma]\!]^{W'}_\Theta \to [\![\phi]\!]^{W'}_\Theta$$
and $a\colon W \to W'$;
5. For all reduction operators R, there exists a world W_R and a morphism $a_R\colon 1 \to W_R$;
6. For a reduction $\Phi; R$, with unary reduction operator R, with sufficient premiss Γ' **?-** ϕ', Δ' and putative conclusion Γ **?-** ϕ, Δ,
$$[\![\Phi; R]\!]^W_\Theta = (W', (\langle \mathsf{Id}, F(a)\rangle, \mathsf{Id})^*(\nu^{-1}_{[\![\Delta]\!]^{W \times W_R}_\Theta}(\mathsf{Cur}^{-1}_{[\![\Gamma]\!]^{W \times W_R}_\Theta} \\ (\mathsf{App} \circ \langle \mathsf{Cur}_{[\![\Gamma_1]\!]^{W \times W_R}_\Theta}(\nu_{[\![\Delta']\!]^{W \times W_R}_\Theta}(f_1)), \mathsf{Snd}\rangle)))),$$
where $W' = W_1 \times W \times W_R$ and $[\![\Phi]\!]^{W \times W_R}_\Theta = (W', f_1)$ and $a\colon W \to W'$;

7. For a reduction $(\Phi_1, \Phi_2); R$, with binary reduction operator R, with sufficient premisses Γ_i **?-** ϕ_i, Δ_i and with putative conclusion Γ **?-** A, Δ,

$$\begin{aligned}[\![(\Phi_1, \Phi_2); R]\!]^W_\Theta = (W', (\langle \mathsf{Id}, F(a)\rangle, \mathsf{Id})^*(\nu^{-1}_{[\![\Delta]\!]^{W\times W_R}_\Theta} \\ (\mathsf{Cur}^{-1}_{[\![\Gamma]\!]^{W\times W_R}_\Theta}(\mathsf{App} \circ \langle\langle \mathsf{Cur}_{[\![\Gamma_1]\!]^{W\times W_R}_\Theta}(\nu_{[\![\Delta_1]\!]^{W\times W_R}\Theta}(f_1)), \\ \mathsf{Cur}_{[\![\Gamma_1]\!]^{W\times W_R}_\Theta}(\nu_{[\![\Delta_2]\!]^{W\times W_R}\Theta}(f_2))\rangle, \mathsf{Snd}\rangle)))),\end{aligned}$$

where $W' = W_1 \times W_2 \times W \times W_R$, $a\colon W \to W'$ and $[\![\Phi_i]\!]^{W\times W_R}_\Theta = (W'_i, f_i)$ and $W'_i = W \times W_R \times W_i$;

8. Suppose $\Theta = \Theta', \xi$, where ξ is an indeterminate $\phi_1, \ldots, \phi_n$ **?-** ϕ. Then $[\![\Theta]\!]^W$ is equal to $[\![\Theta']\!]^W \times [\![(\phi_1 \wedge \cdots \wedge \phi_n) \supset \phi]\!]^W_{\Theta'}$.

We can now give the canonical interpretation of LK-reductions in classical reduction structures.

Definition 5.28 (**canonical interpretation**) Let $(\mathcal{E}, F)$ be a classical reduction structure. The following interpretation, $[\![-]\!]$, where Θ is a list of indeterminates, is called the canonical interpretation (where *ass* is the associativity isomorphism between $(\phi \vee \psi) \vee \Delta$ and $\phi \vee (\psi \vee \Delta)$):

1. $[\![\bot]\!]^W_\Theta \stackrel{\text{def}}{=} 0$;
2. $[\![\top]\!]^W_\Theta \stackrel{\text{def}}{=} 1$;
3. $[\![\phi \supset \psi]\!]^W_\Theta \stackrel{\text{def}}{=} [\![\phi]\!]^W_\Theta \Rightarrow [\![\psi]\!]^W_\Theta$;
4. $[\![\phi \wedge \psi]\!]^W_\Theta \stackrel{\text{def}}{=} [\![\phi]\!]^W_\Theta \times [\![\psi]\!]^W_\Theta$;
5. $[\![\phi \vee \psi]\!]^W_\Theta \stackrel{\text{def}}{=} [\![\phi]\!]^W_\Theta \vee [\![\psi]\!]^W_\Theta$;
6. For all reduction operators R except $\supset L$, $\vee L$, $\vee R$, and Exchange$R$, $F(a_R) = \langle \mathsf{Id}_1, f\rangle$, where $S_\xi(f)$ is the interpretation of R according to Definition 5.4, where the category $\mathcal{C}$ is the category $\mathcal{E}(1,1)$;
7. For the remaining reduction operators, $F(a_R)$ is defined as follows:
   Exchange$R$: Consider the reduction operator

$$\frac{\Gamma \ \textbf{?-}\ \psi, \phi, \Delta}{\Gamma \ \textbf{?-}\ \phi, \psi, \Delta}$$

and let ϕ' be the formula $\Gamma \supset \psi \vee \phi \vee \Delta$. Then

$$F_a(\text{Exchange}R) = \langle \mathsf{Id}, \mathsf{Cur}(\nu(\mu\alpha.[\beta]\mathsf{App} \circ \langle \nu^{-1}(\pi_{[\![\phi']\!]}), \pi_{[\![\Gamma]\!]}\rangle))\rangle;$$

$\supset L$: Consider the reduction operator

$$\frac{\Gamma \ \textbf{?-}\ \phi, \sigma, \Delta \qquad \Gamma, \psi \ \textbf{?-}\ \sigma, \Delta}{\Gamma, \phi \supset \psi \ \textbf{?-}\ \sigma, \Delta}$$

and let ϕ_1 be the formula $(\Gamma \supset \phi \vee \sigma \vee \Delta)$, ϕ_2 be $(\Gamma \wedge \psi) \supset \sigma \vee \Delta$, and let π_1 be the projection

$$[\![\phi_1 \wedge \phi_2 \wedge \Gamma \wedge (\phi \supset \psi)]\!]^1_\emptyset \to [\![\phi_1]\!]^1_\emptyset,$$

and π_2 be the projection

$$[\![\phi_1 \wedge \phi_2 \wedge \Gamma \wedge (\phi \supset \psi)]\!]^1_\emptyset \to [\![\phi_2]\!]^1_\emptyset.$$

Then

$$\begin{aligned} F(a_{\supset L}) = \langle \mathsf{Id}_1, \mathsf{Cur}(\mu\gamma.c^*([\gamma](\nu^{-1}(\mathsf{Cur}^{-1}(\pi_2)) \\ \circ\langle \mathsf{Id}, \pi_{[\![\Gamma]\!]}, \mathsf{App} \circ \langle \pi_{[\![\phi\supset\psi]\!]^1_\emptyset}, \nu^{-1}(\mathsf{Cur}^{-1}(\pi_1)) \circ \langle \mathsf{Id}, \pi_{[\![\Gamma]\!]}\rangle\rangle\rangle)))\rangle; \end{aligned}$$

$\vee L$: Consider the reduction operator

$$\frac{\Gamma, \phi \ \textbf{?-}\ \sigma, \Delta \qquad \Gamma, \psi \ \textbf{?-}\ \sigma, \Delta}{\Gamma, \phi \vee \psi \ \textbf{?-}\ \sigma, \Delta}$$

and let ϕ_1 be the formula $(\Gamma \wedge \phi) \supset \sigma \vee \Delta$, ϕ_2 be $(\Gamma \wedge \psi) \supset \sigma \vee \Delta$, and let π_1 be the projection

$$[\![\phi_1 \wedge \phi_2 \wedge \Gamma \wedge (\phi \vee \psi)]\!]^1_\emptyset \to [\![\phi_1]\!]^1_\emptyset$$

and π_2 be the projection

$$[\![\phi_1 \wedge \phi_2 \wedge \Gamma \wedge (\phi \vee \psi)]\!]^1_\emptyset \to [\![\phi_2]\!]^1_\emptyset.$$

Then

$$\begin{aligned} F(a_{\vee L}) = \langle \mathsf{Id}_1, \mathsf{Cur}(\mu\gamma.c^*[\gamma](w^*\pi_2 \circ \langle \mathsf{Id}, \pi_{[\![\Gamma]\!]^1_\emptyset}, \mu\beta.[\gamma] \\ (w^*\mathsf{Cur}^{-1}\pi_1 \circ \langle \mathsf{Id}, \pi_{[\![\Gamma]\!]^1_\emptyset}, \nu^{-1}(\pi_{[\![\sigma\vee\tau]\!]^1_\emptyset})\rangle\rangle\rangle)))\rangle \end{aligned}$$

$\vee R$: $F(a_{\vee R}) = \langle \mathsf{Id}_1, \mathsf{Cur}(\nu(ass \circ \nu^{-1}(\mathsf{App} \circ \langle \pi_{[\![\Gamma]\!]^1_\emptyset}, \pi_{[\![\Gamma\supset A\vee B\vee\Delta]\!]^1_\emptyset}\rangle))))\rangle$. (Because reduction structures are derived from $\lambda\mu\nu$-structures, in the cases for $\supset L$ and $\vee L$, the formula σ is distinguished in order to define the interpretation.)

Note that also in the classical case the definition of interpretation does not force any two reductions to be equal. The reason is the same as for (intuitionistic) reduction structure: No equality between worlds or morphisms between them is forced by the interpretation.

Note that the the semantics of the reduction operators which involve structural rules on the right-hand side or change the side formulæ on the right-hand side involve a change of base $\mathcal{C}$. This is obviously true for Exchange$R$, but also $\supset L$, $\vee L$, and $\vee R$ involve such a change of base: for $\supset L$ and $\vee L$ it is given by a contraction on the right-hand side, and for $\vee L$ by the isomorphism used for modelling $\vee$.

We can now define classical reduction models, which generalize the (intuitionistic) reduction models established in Definition 5.10.

Definition 5.29 (**classical reduction model**) A classical reduction model,

$$\mathcal{R} = \langle (\mathcal{E}, F), [\![-]\!], \models \rangle,$$

is given by the following:

1. A classical reduction structure $(\mathcal{E}, F)$;
2. An interpretation $[\![-]\!]$ of reduction operators and searches for LK;
3. A forcing relation $W \models_\Theta (\Phi\colon \phi)[\Gamma; \Delta]$, where W is a world, Θ and Γ, Δ are contexts, ϕ a formula and Φ a reduction with endsequent Γ **?-** ϕ, Δ with indeterminates contained in Θ, such that

$$[\![\Gamma]\!]^W_\Theta \xrightarrow{[\![\Phi]\!]^W_\Theta} [\![\phi, \Delta]\!]^W_\Theta$$

is a morphism in the reduction structure, and which satisfies the following conditions:

(i) If $W \models_\Theta (\Phi\colon \phi)[\Gamma; \Delta]$ and $a\colon W \to W'$ is a morphism in $\mathcal{W}$ for some world W', then also $W' \models_\Theta (\Phi\colon \phi)[\Gamma; \Delta]$;

(ii) $W \models_\Theta (Ax\colon \phi)[\Gamma, \phi; \Delta]$;

(iii) $W \models_{\Theta,\xi} (\xi\colon \phi)[\Gamma; \Delta]$ if ξ is an indeterminate of type Γ **?-** $\phi; \Delta$;

(iv) If R is a reduction operator with premisses Γ_1 **?-** ϕ_1, Δ_1 and Γ_2 **?-** ϕ_2, Δ_2 and conclusion Γ **?-** ϕ, Δ, then $W \models_\Theta ((\Phi_1, \Phi_2); R)[\Gamma, \phi; \Delta]$ if

$$W \times W_R \models_\Theta (\Phi_i)[\Gamma_i, \phi_i; \Delta_i];$$

(v) If R is a reduction operator with premiss Γ_1 **?-** ϕ_1, Δ_1 and conclusion Γ **?-** ϕ, Δ, then $W \models_\Theta (\Phi_1; R)[\Gamma, \phi; \Delta]$ if $W \times W_R \models_\Theta (\Phi_1)[\Gamma_1, \phi_1; \Delta_1]$;

Soundness and completeness carry over from the intuitionistic case.

Theorem 5.30 (soundness) *Consider any classical reduction structure* $(\mathcal{E}, F)$*. Suppose* Φ *is a LK-reduction of* Γ **?-** ϕ, Δ *with indeterminates* $\xi_1, \ldots, \xi_n$ *of type* Γ_i **?-** ϕ_i, Δ_i*. Then* $W \models_\Theta (\Phi\colon \phi)[\Gamma; \Delta]$ *for any world* W*, where* $\Theta = \{\xi_1, \ldots, \xi_n\}$.

Proof The proof is essentially the same as for Theorem 5.12. □

Again, we write $\Gamma \models_\Theta \Phi\colon \phi; \Delta$ if for all worlds W and all classical reduction models, we have $W \models_\Theta (\Phi\colon \phi)[\Gamma; \Delta]$. Then we have also completeness:

Theorem 5.31 (completeness) *If* $\Gamma \models_\Theta \Phi\colon \phi, \Delta$*, then* Γ **?-** $\Phi : \phi, \Delta$.

Proof The term model construction for the intuitionistic case can be extended easily to give a term model for a classical reduction structure. For the category $\mathcal{C}$ choose the free Cartesian category over the atomic formulæ, and now follow the intuitionistic case in constructing a term model out of reductions. □

We also obtain completeness with respect to searches:

Theorem 5.32 *Suppose* $(\mathcal{E}, F)$ *is the free classical reduction structure over a set of objects* $\mathcal{G}$*. A reduction* Φ *of* Γ **?-** ϕ, Δ *with leaves* Γ_i **?-** ϕ_i, Δ_i *which are not axioms can be completed to a proof iff there exists a morphism* f *such that there is a functor* $\mathcal{E}(\langle !, f\rangle)\colon \mathcal{E}(\Theta) \to \mathcal{E}(1)$*, where* Θ *is the context corresponding to the indeterminates* $\xi_1, \ldots, \xi_n$*. Moreover, the completion of a Cut-free reduction is Cut-free.*

Proof The proof can be transferred directly from the intuitionistic case. □

5.4 Discussion

We have now provided, in the intuitionistic and classical settings, a semantics for reductive proof which satisfies our triangular criterion, summarized by Fig. 5.2.

However, we have not yet provided a semantics for proof-search. Following our slogan,

$$\textit{Proof-search} = \textit{Reductive Proof} + \textit{Control},$$

$$\begin{tikzcd}
\Phi \Rightarrow \Gamma \ \mathbf{?\text{-}}\ \phi \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & {[\Gamma] \mathrel{|\!\sim} [\Phi] : [\phi]} \arrow[dl, leftrightarrow] \\
& \llbracket\Gamma\rrbracket \overset{\llbracket\Phi\rrbracket}{\rightsquigarrow} \llbracket\phi\rrbracket &
\end{tikzcd}$$

FIG. 5.2. Reductions-as-realizers-as-arrows

we must now pay attention to control. Following our discussion at the end of Chapter 4, we shall provide, in Chapter 6, a semantics for backtracking.

We conjecture that the semantics for reductive proof given in this chapter can be easily extended to predicate logic and quantifiers: we have previously described how to use fibrations to obtain models for predicate logic. It should be possible to combine these fibrations in a modular way with the fibrations used to describe reduction structures, so as to produce reduction structures for predicate logic.

6

INTUITIONISTIC AND CLASSICAL PROOF-SEARCH AND THEIR SEMANTICS

6.1 Introduction

So far we have presented reductive logic and its semantics from a wholly extensional perspective. More specifically, when considering reduction operators, such as $\wedge R$, $\vee L$, or $\supset L$, which have multiple premisses, we have not considered any strategy, for the development of the different branches, that is, for the exploration of the *search space* [67]. We now develop a more intensional perspective, which will emphasize the rôle and form of control.

For example, consider again the notions of uniform and weakly uniform proof explained in Chapter 4. Recall that the basic idea is that right rules are preferred over left rules wherever possible (though *weakly* uniform allows $\vee L$s to occur as close to the root of the reduction as possible). So, given a putative conclusion,

$$\phi_1, \ldots, \phi_m \;\textbf{?-}\; \psi_1, \ldots, \psi_n,$$

the attempted construction of, that is, the *search* for, a uniform proof requires that

(1) the structure of the right-hand side be analysed, then,
either a choice of one of the ψ_is is made and reduction operator is applied, or

(2) the structure of the left-hand side is analysed, then
either a choice of one of the ϕ_js is made and a reduction operator is applied, or

(3) the search fails;

(4) and so on.

We should like our semantics of proof-search to capture at least this level of algorithmic detail.

However, as we have already explained in the discussion that concludes Chapter 4, there are many algorithmic choices not specified by the definition of uniform proof yet which have consequences for the structure of the resulting proofs, in addition to any consequences they might have for the complexity of the computation.

We have identified four main points.

1. Firstly, within the context of uniform proof, a sequent

$$\phi_1, \ldots, \phi_m \vdash \psi_1, \ldots, \psi_n$$

may be reducible in a number of ways: There may be several ψs which are not atomic; even if all the ψs are atomic, there may be several ϕs to which left rules are applicable. This situation obtains even if we restrict ourselves to hereditary Harrop resolution.

2. Secondly, after the reduction of a sequent $\Gamma \vdash \Delta$ using an operator R,

$$\frac{\Gamma_1 \vdash \Delta_1 \quad \ldots \quad \Gamma_m \vdash \Delta_m}{\Gamma \vdash \Delta},$$

a choice of the order in which to reduce the premisses must be made.

3. Thirdly, in predicate settings, a reduction may depend upon a choice of unifier.[57]
4. Finally, we must handle failure. Having made a choice of reduction, in one of the above points, we may find that even though our sequent is provable, we have made the wrong choice, leading to a failed proof. In these circumstances, we must return to the point at which we made our choice and try a different one. This procedure is known as *backtracking*.

We would suggest that the last point is conceptually the most significant and, perhaps, technically the most challenging.

In summary, we concentrate, in this chapter, on providing a semantics for proof-search in intuitionistic (propositional) logic which captures, within the framework of models of reductive logic set up in Chapter 5, uniform and weakly uniform proof and backtracking. We achieve this aim firstly, by providing a characterization in classical reduction models of where backtracking can occur in intuitionistic proof-search and, secondly, by constructing a specific games model within which both backtracking and the uniform proof strategies may be understood quite naturally.

Recall that our games models are within the general framework initiated by Hyland and Ong [64] but are generalized to allow the interpretation not only of natural deduction proof but also the sequent calculus. Consequently, we are able to interpret reductive proof in sufficient generality to encompass the whole analysis of this monograph.

In Section 6.2, we give a semantics of backtracking using the semantics of reductions defined in Chapter 5. In Section 6.3, we describe how to model backtracking in the games semantics we introduced earlier. In Section 6.4, we show how to use the games semantics to give a semantics for uniform proofs.

[57] First-order terms have most general unifiers but higher-order terms do not [55, 62].

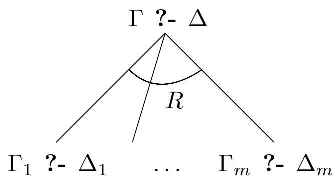
6.2 Towards a semantics of control: Backtracking

Given a system of reduction operators, $\mathcal{R}$, the search space of $\mathcal{R}$, Space($\mathcal{R}$), may be described graphically as an *and–or tree* as follows:[58]

1. Nodes of the tree are labelled by problems, Γ **?-** Δ. The root is labelled by the initial problem;
2. Nodes are connected by arcs labelled by instances of reduction operators,

$$\frac{\Gamma_1 \ \textbf{?-}\ \Delta_1 \quad \ldots \quad \Gamma_m \ \textbf{?-}\ \Delta_m}{\Gamma \ \textbf{?-}\ \Delta} \quad R,$$

which may be denoted



in which arcs are directed (traditionally) down the page. The collection of arcs from a node labelled by some problem Γ **?-** Δ to the nodes labelled by the problems

$$\Gamma_1 \ \textbf{?-}\ \Delta_1 \ \ldots \ \Gamma_m \ \textbf{?-}\ \Delta_m,$$

determined by such an instance of a reduction operator, R, and connected by the curved arc in the figure, is called an *R-bundle*;[59]

3. A problem may be the origin of several different bundles, corresponding to different reduction operators and giving the disjunctive (or) structure of the space. If n different reduction operators R_i,

$$\frac{\Gamma_{i1} \ \textbf{?-}\ \Delta_{i1} \quad \ldots \quad \Gamma_{im} \ \textbf{?-}\ \Delta_{im}}{\Gamma \ \textbf{?-}\ \Delta} \quad R_i,$$

[58] In [98, 106], the search space for an intuitionistic sequent calculus is defined to carry the 'subderivation ordering', $\sqsubseteq$: For reductions R, S, $R \sqsubseteq S$ if R is a labelled subtree of S. In this chapter, we shall make no use of this ordering but remark that orderings of this kind may provide a suitable basis modelling control régimes such as formula-selection strategies. For example, Prolog programs may be seen as antecedents of sequents, ordered from left to right in order to impose the 'leftmost first' strategy.

[59] Whilst this graphical notation is useful for defining search spaces, it is not convenient for performing specific reductions, for which we revert to the use of 'proof trees'.

for $1 \leq i \leq n$, are applicable to a problem Γ **?-** Δ, then the corresponding arcs in the search space may be denoted

$$\begin{tikzcd}[column sep=small] & \Gamma \ \textbf{?-}\ \Delta \arrow[dl, dash, "R_1"'] \arrow[d, dash, "R_i"] \arrow[dr, dash, "R_n"] & \\ (\Gamma_{1j} \ \textbf{?-}\ \Delta_{1j})_{j=1}^{n} & \ldots & (\Gamma_{nj} \ \textbf{?-}\ \Delta_{nj})_{j=1}^{n}, \end{tikzcd}$$

that is, a disjunction of R_i-bundles;

4. Paths through Space($\mathcal{R}$) thus correspond to compositions of instances of reduction operators.

Within a bundle, the search space has conjunctive (and) structure. For example, the problem

$$\phi \wedge \psi \ \textbf{?-}\ \phi \vee \psi,$$

in the search space Space(LK), is the root of bundles arising from $\wedge R$, with two branches, and $\vee R$, with one branch. In the search space Space(LJ), two distinct bundles, each with one branch, arise from $\psi \vee \psi$.

Thus the exploration of a search space requires navigation between disjunctive choices: one might make a choice, such as between the two branches of Space(LJ) generated by the two cases of the $\vee R$ operator, explore that branch of the search space, and perhaps *fail*. One then *backtracks* to the point at which the choice was made, and tries the other branch. Thus backtracking is a key, and we suggest perhaps the prototypical, control mechanism in proof-search. Indeed, the lack of a full permutation theorem for intuitionistic propositional sequent calculus [65, 108, 111], with the consequence that the order of the propositional rules used is criticial in the finding of a proof, renders backtracking an essential component of the control of a search for a proof in LJ. To see this, consider the following example, in which first the use of $\supset L$ on $p \supset q$ leaves the subsequent development of the left-hand branch of the reduction doomed to failure, even though the endsequent is provable:[60]

$$\dfrac{\begin{array}{c}\textbf{fails}^{(1)}\\ \vdots \\ \hline p \supset q \,,\, r \supset s \ \textbf{?-}\ p\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\textbf{succeeds}\\ \vdots \\ \hline r \supset s \,,\, q \,,\, r \ \textbf{?-}\ s\end{array} \qquad \begin{array}{c}\textbf{succeeds}\\ \vdots \\ \hline r \supset s \,,\, q \,,\, r \,,\, t \ \textbf{?-}\ t\end{array}}{r \supset s \,,\, s \supset t \,,\, q \,,\, r \ \textbf{?-}\ t} \supset L_{s \supset t}}{r \supset s \,,\, s \supset t \,,\, q \ \textbf{?-}\ r \supset t} \supset R}{p \supset q \,,\, r \supset s \,,\, s \supset t \ \textbf{?-}\ r \supset t} (1) \ \supset L_{p \supset q}.$$

[60]We adopt the notation R_ϕ to denote the instance of the operator O generated by the formula ϕ, for example, $\supset L_{p \supset q}$.

After the first $\supset L$, we can see that the left-hand branch will fail, and we must backtrack to (1) and make a different choice of reduction. We might try $\supset L_{r \supset s}$ instead. Such a control step lies outside the logical structure we have so far established but we can give a logical account of it by considering the intuitionistic calculus LJ to be embedded in the classical sequent calculus, LK. We quickly review the main points from Chapters 2 and 4 in this context before proceeding to characterize backtracking.

In general, every intuitionistic sequent derivation arises as a subderivation of a classical sequent derivation via (e.g.) Dummett's presentation of intuitionistic logic as a multiple-conclusioned sequent calculus [26]. Because the classical $\supset R$-rule allows multiple succedents in the premiss, two different intuitionistic sequent derivations, which are not identical up to a permutation of inference rules, can be subderivations of the same classical derivation up to a choice of axioms. For example, consider the following two intuitionistic reductions:

$$\dfrac{\dfrac{}{\psi\,,\,\phi \text{ ?- } \psi}\,Ax}{\psi \text{ ?- } \phi \supset \psi\,,\,\chi \supset \psi}\supset R \quad \text{and} \quad \dfrac{\dfrac{}{\psi\,,\,\chi \text{ ?- } \psi}\,Ax}{\psi \text{ ?- } \chi \supset \psi\,,\,\phi \supset \psi}\supset R.$$

They arise as restrictions to intuitionistic logic of the following classical reduction:

$$\dfrac{\dfrac{\dfrac{}{\psi\,,\,\phi\,,\,\chi \text{ ?- } \psi\,,\,\psi}\,Ax}{\psi\,,\,\chi \text{ ?- } \phi \supset \psi\,,\,\psi}}{\psi \text{ ?- } \phi \supset \psi\,,\,\chi \supset \psi}\supset R.$$

Similarly, in LK viewed as reductive system, the $\supset L$-rule has the form

$$\dfrac{\Gamma \text{ ?- } \phi, \Delta \quad \Gamma, \psi \text{ ?- } \Delta}{\Gamma, \phi \supset \psi \text{ ?- } \Delta},$$

in which the Δ is retained in both premisses. Using this operator instead of its intuitionistic counterpart, we are able to restart the computation at (2), and proceed to apply the necessary $\supset R$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{succeeds}}{\vdots}}{r \supset s\,,\,s \supset t\,,\,r \text{ ?- } t\,,\,p}\supset L_{s\supset t}}{r \supset s\,,\,s \supset t \text{ ?- } \boxed{r \supset t}\,,\,p}\supset R}{r \supset s\,,\,s \supset t \text{ ?- } p\,,\,\boxed{r \supset t}}\text{(2) Exchange} \qquad \dfrac{\dfrac{\text{as above}}{\vdots}}{r \supset s\,,\,s \supset t\,,\,q \text{ ?- } r \supset t}\supset R}{p \supset q\,,\,r \supset s\,,\,s \supset t \text{ ?- } r \supset t}\text{(1)} \supset L_{p \supset q}\,.$$

Note, in particular, the use of Exchange at (2). From the point of view of the $\lambda\mu\nu$-calculus, the necessary $\supset R$-rule is applicable only if the implicational formula is leftmost in the succedent.

A successful classical reduction for a problem Γ **?-** ϕ yields a classical proof but not necessarily an intuitionistic proof. So, in order to exploit the structural and combinatorial advantages of classical reduction for intuitionistic logic, we must be able to calculate *syntactically* whether a given classical reduction determines an intuitionistic proof.

To do this we represent the sequent calculus LK in the $\lambda\mu\nu$-calculus (see Chapter 4). More precisely, we represent LK in the $\lambda\mu\nu\epsilon$-calculus, that is, the $\lambda\mu\nu$-calculus with explicit substitutions.

If we represent the classical sequent calculus in the $\lambda\mu\nu\epsilon$-calculus, then we can calculate whether a successful classical reduction determines the existence of an intuitionistic proof by analyzing the structure of the $\lambda\mu\nu\epsilon$-term which realizes the classical proof (see Chapter 4).

We repeat here the basic idea. Consider the difference between the $\supset R$-rule in the classical calculus, LK,

$$\frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \supset \psi, \Delta} \supset R,$$

and the form of its restriction to capture intuitionistic implication, as in Dummett's multiple-conclusioned calculus [26],

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \supset \psi, \Delta} \supset R.$$

Here the key point is that a built-in Weakening,[61] by Δ, is required. To see this, consider the following reduction:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\psi\,,\,\phi\,,\,\theta\ \textbf{?-}\ \tau\,,\,\psi}\,Ax}{\psi\,,\,\phi\ \textbf{?-}\ \theta\supset\tau\,,\,\psi}\supset R}{\psi\,,\,\phi\ \textbf{?-}\ \psi\,,\,\theta\supset\tau}\ \text{Exchange}}{\psi\ \textbf{?-}\ \phi\supset\psi\,,\,\theta\supset\tau}\supset R.$$

We need to be able to detect that the use of the $\supset R$ operator to reduce the formula $\theta \supset \tau$ is superfluous, and so conclude that we could have simply deleted $\theta \supset \tau$ at the first $\supset R$ reduction and so conclude that the initial problem, ψ **?-** $\phi \supset \psi, \theta \supset \tau$ has an intuitionistic proof.

[61] The *Weakening* rules are

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta} WL \quad \text{and} \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \Delta'} WR.$$

Recall, from Chapter 4, the notion of an *intuitionistic term* in the $\lambda\mu\nu\epsilon$-calculus. The definition is complex but the basic idea is that intuitionistic terms identify those implicational realizers, that is, terms of the form $\Gamma \vdash \lambda x\!:\!\phi.t : \phi \supset \psi, \Delta$, in which all of the subterms of $\lambda x\!:\!\phi.t$ corresponding to formulæ in Δ arise from Weakenings. In Chapter 4 we proved the following Theorem (Theorem 4.11):

Theorem 6.1 (intuitionistic provability) *Let Φ be an LK-proof of $\Gamma \vdash \phi, \Delta$ and let t_Φ be the corresponding $\lambda\mu\nu\epsilon$-term. Then t_Φ is an intuitionistic term iff $\Gamma \vdash \phi, \Delta$ is intuitionistically provable.*

If we translate intuitionistic LJ-reductions into classical reductions, backtracking may occur at two points: firstly, at the $\supset L$-rule, and secondly, at the $\vee R$-rule. In both cases we lose potentially useful side formulæ, when we apply the reduction operator. This can be captured semantically as follows:

Theorem 6.2 (backtracking) *An intuitionistic reduction contains a possible backtracking point before the reduction operator R if and only if for the translation of the reduction into a classical reduction, the corresponding reduction operator R with sufficient premisses Γ_i **?-** ϕ_i, Δ_i and putative conclusion Γ **?-** ϕ, Δ, there exists a j such that the fibres $\mathcal{E}(1, [\![\Delta]\!]^1_\emptyset)$ and $\mathcal{E}(1, [\![\Delta_j]\!]^1_\emptyset)$ are not identical.*

Proof All left-operators except $\supset L$ leave the right-hand sides of sequents unchanged, and hence $\mathcal{E}(1, [\![\Delta]\!]^1_\emptyset)$ and $\mathcal{E}(1, [\![\Delta_j]\!]^1_\emptyset)$ are identical for all j. These operators also do not give rise to a possible backtracking point. For the operator $\supset L$, $\mathcal{E}(1, [\![\Delta]\!]^1_\emptyset)$ and $\mathcal{E}(1, [\![\Delta_1]\!]^1_\emptyset)$ are not identical, and indeed $\supset L$ gives rise to a backtracking point. All right-operators except $\vee R$ do not modify the side-formulæ on the right-hand side, and hence $\mathcal{E}(1, [\![\Delta]\!]^1_\emptyset)$ and $\mathcal{E}(1, [\![\Delta_j]\!]^1_\emptyset)$ are identical for all j. These operators also do not give rise to a possible backtracking point. The $\vee R$-rule does change the side formulæ on the right-hand side and models the intuitionistic $\vee R$-rule, which indeed gives rise to a backtracking point. Also, $\mathcal{E}(1, [\![\Delta]\!]^1_\emptyset)$ and $\mathcal{E}(1, [\![\Delta_1]\!]^1_\emptyset)$ are not identical. $\square$

6.3 A games semantics for proof-search

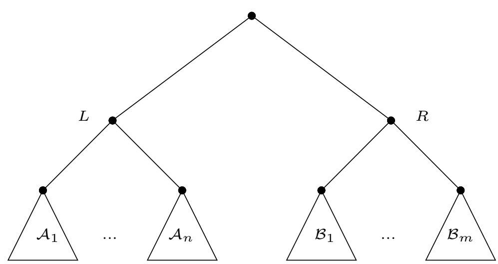
We conclude with an example of our semantics—of intuitionistic reduction with backtracking, embedded in classical reduction—which corresponds closely to our intuitions about the nature of constructing proofs: that is, a games semantics for proof-search. Our semantics is based *mutatis mutandis* on the games semantics for classical logic presented in [97], which in turn is based on the games of Hyland and Ong [64].

The games semantics of Hyland and Ong [64] models natural deduction proofs, whereas we must model the sequent calculus (and so reduction). This implies significant, though we would argue rather natural, changes to the games semantics introduced by Hyland and Ong. The main difference is that we permit both players to make sequences of moves rather than single moves.

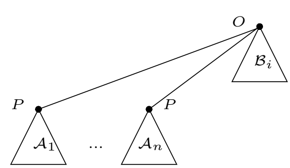
In this section, we use the games semantics described in Section 3.6.5 to model backtracking. We repeat here the definition of earlier chapters.

Definition 6.3 An arena of type ϕ is a forest with nodes having possibly labels defined inductively by the following:

1. The arena of $\top$ is the empty forest;
2. The arena of $\perp$ is the forest with one node labelled $\perp$;
3. The arena for a propositional variable p is a forest with one node labelled p;
4. The arena for $\phi \wedge \psi$ is the disjoint sum of the arenas for ϕ and ψ;
5. Suppose $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are the trees of the arena for ϕ and $\mathcal{B}_1, \ldots, \mathcal{B}_m$ are the trees of the arena for ψ. Then the arena for $\phi \vee \psi$ is given by



   Note that there are two special nodes called L and R. In the special case that the arena for ϕ or the arena for ψ is empty, the arena for $\phi \vee \psi$ is the empty arena too. The root node of the arena for $\phi \vee \psi$ is labelled $\vee$;
6. Suppose $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are the trees of the arena for ϕ and $\mathcal{B}_1, \ldots, \mathcal{B}_m$ are the trees of the arena for ψ. Then the arena for $\phi \supset \psi$ is the disjoint union of the following trees



   In the special case that the arena for ϕ is empty, the arena for $\phi \supset \psi$ is the arena for ψ. All nodes in the arena for $\phi \supset \psi$ which are root nodes in the arena of ψ are labelled $\supset$ in addition to any other label they might have.

We call all root nodes in an arena O-nodes, and all children of O-nodes P-nodes, and all children of P-nodes O-nodes.

Arenas are used to define possible plays. The definition of moves and plays makes this precise. Next, we define possible moves in our games. Each move for a game for ϕ is associated with a node in the arena for ϕ.

There are several types of moves. Firstly, we have moves by Proponent and Opponent, and secondly, there are question and answer moves. Questions which correspond to O-(P-)nodes are played by Opponent (Proponent), and answers which correspond to O-(P-)nodes are played by Proponent (Opponent). The definition is as follows:

Definition 6.4 A move m for an arena $\mathcal{A}$ is a node that is classified as either question or answer. Questions which correspond to O-(P-)nodes are moves by Opponent (Proponent), and answers which correspond to O-(P-)nodes are moves by Proponent (Opponent). We call a move by Proponent a P-move and a move by Opponent an O-move.

Next, we define plays, which are instances of the game. Each play consists of a sequence of moves satisfying certain conditions. The intuition is that Opponent starts the play by challenging Proponent to verify the given formula. Proponent responds by asking the Opponent to justify the assumptions which Proponent can make in a sequent calculus proof of ϕ. Proponent wins a particular game if he can answer Opponent's initial question.

The moves in a play for ϕ follow the structure of arena of ϕ closely: An O-(P-)question can be played only if there was already a P-(O-)question corresponding to the parent node. An answer can only be given if a question with the same associated node has already been made.

The precise conditions for a play are as follows:

Definition 6.5 A play for an arena $\mathcal{A}$ is a sequence of moves $m_1, \ldots, m_n$ such that

(1) there exists an index $I \geq 1$ such that all moves $m_1, \ldots, m_I$ are O-questions with position $1, \ldots, I$, respectively, and the corresponding nodes are roots in the forest for $\mathcal{A}$. These moves are called initial questions;
(2) for each question m_i with $i > I$ there exists a question m_k with $k < i$ such that the node corresponding to m_k is the immediate predecessor of the node corresponding to m_i in the arena $\mathcal{A}$. We call m_k the justifying question for m_i;
(3) for each answer m_i with $i > I$ there exists a question m_k with $k < i$ such that m_k and m_i are the same node in $\mathcal{A}$. If m_j is the justifying question for m_k, we call m_j the justifying question for m_i;

(4) each question can be answered at most once;

(5) any initial questions can only be answered if all non-initial questions have already been answered;

(6) for any P-answer m_i there exists an O-question m_k and an O-answer m_j such that m_i is hereditarily justified by m_k, m_j is an O-answer with the same label as m_k or $\perp$ and $k < j < i$ and that the nodes corresponding to m_k and m_j in the arena are on a path which does not contain a P-node n labelled $\supset$ such that the nodes corresponding to m_k and m_j are its children or identical to it.

As we have seen, the key notion of games semantics is that of a *strategy*. A strategy describes how Proponent responds to arbitrary Opponent moves. Intuitively, a strategy describes how Proponent answers challenges from Opponent to prove the given formula.

Definition 6.6 A strategy is a function from plays $m_1, \ldots, m_k$ where m_k is an O-move, to a sequence of moves $m_{k+1}, \ldots, m_n$ such that $m_1, \ldots, m_k, m_{k+1}, \ldots, m_n$ is a play and the sequence $m_{k+1}, \ldots, m_n$ is non-empty if $m_1, \ldots, m_k$ contains no unanswered P-move which could be answered by Opponent in the next move according to Definition 6.5.

Now, we can explain how this semantics *models* searches in the sequent calculus. Intuitively, O- and P-questions are challenges for Opponent and Proponent to provide evidence for conclusions and premisses, respectively. O-answers provide evidence for a premiss, and P-answers provide evidence for a conclusion. Conjunctive choices are made by Opponent, and disjunctive choices by Proponent.

As usual left-operators involve operations on the premisses: they are initiated by P-questions. Similarly, right-operators involve operations on the conclusions: they are initiated by O-questions. The restriction in Clause 6 that Proponent can answer questions only if Opponent has answered a P-question with the same label before ensures that the axiom rule can be invoked only if there is the same formula on both sides of the sequent.

Contraction is built in implicitly by allowing both players to ask the same question several times. Moreover, Clause 2 of the definition of a play allows parallel reductions in different branches of the search tree: a P-question with position $p \cdot n_1 \cdots n_k \cdots m$ with $k > 0$ and p the position of the justifying O-question represents the application of $\supset L$ in all branches which arise by playing moves with position $p \cdot n_1 \cdot n_i$ for $i < k$.

Note that we allow both players to make several moves at once. This makes it possible to model not only provability but also proofs by having different plays for different sequences of reduction operators applied to a sequent. In particular, the application of several left-operators requires Proponent to be able to make several moves at the same time.

Note also that our games semantics is capable of representing detailed information about how searches are done. The level of detail is sufficient not only to model which reduction operators is applied but also in which order. Some reduction operators are even modelled by several moves, with the possibility of interleaving the moves corresponding to different reduction operators. Hence a mapping from strategies to searches assigns the same search to several strategies.

Two rules are responsible for the fact that we model LK-reduction and not only LJ-reductions. The first rule is the ability of Proponent to play arbitrary moves labelled L and R. For modelling LJ-reductions, one would allow Proponent to play only one switching move which is justified by a given O-question. The second rule is the second part of Clause 6 of the definition of plays. This rule models the possibility of having multiple formulæ on the right-hand side and therefore being able to apply an axiom rule using any formula on the right-hand side. If we omit these two rules, we obtain a representation of LJ-searches.

The games semantics Ong presents in [89] for the $\lambda\mu$-calculus (without disjunction) uses *scratchpads* to model classical logic. Scratchpads are separate plays to be started by Proponent whenever he chooses. As we consider disjunction as well, we have extended the definition of an arena and introduced the concept of switching moves (the moves labelled L and R) to model the $\lambda\mu\nu$-calculus. Proponent choosing a move labelled R corresponds to the switch of fibres in the $\lambda\mu\nu$-structures, which is captured by changing to a scratchpad in Ong's model.

Compared to a games semantics for natural deduction, we allow both Opponent and Proponent more freedom: both players can make several moves at a time, which are subject to fewer restrictions. In this way, we capture the possibility of applying reduction operators to several sequents independently. We also capture the possiblity of sequences of blocks of left and right rules in a play.[62]

In Section 3.4.1, we gave an example of a play for a game for intuitionistic logic. This play is repeated here.

A possible play for the arena for $p \supset (p \supset q) \supset (q \supset r) \supset (r \vee s)$ starts by Opponent asking the initial question. Here, this means that Opponent is asking for a proof of the formula. Now Proponent has various choices: he can either ask questions labelled L or R, thereby deciding whether to prove r or s, respectively, or to ask Opponent for evidence for the assumptions by asking any other question. Let us assume that Proponent asks the question corresponding to the node labelled L. Now Opponent will ask the question labelled r, thereby asking Proponent to prove r. Proponent now needs to use the assumptions. Let us assume that Proponent asks the question labelled r, thereby challenging Opponent to

[62]This latter possibility is critical for modelling proof procedures such as resolution.

provide evidence for the assumption $q \supset r$. Next, Opponent asks the question labelled q and challenges Proponent to prove the formula r in turn, which is the hypothesis in the implication $q \supset r$. Proponent now asks in a similar way the question labelled q, and Opponent asks the question p. Proponent now asks for the final assumption p. Opponent now has no choice but to answer this question, thereby making it possible for Proponent to answer outstanding questions by Opponent. Now Proponent can use this answer and answer Opponent's question p. Again, Opponent is now forced to answer the question q. This process of answering previously asked questions goes on until finally Opponent is forced to answer the question labelled L, and Proponent can answer the initial question.

This play corresponds to the reduction

$$\dfrac{\dfrac{\dfrac{p \;\textbf{?-}\; p, q, r \vee s \qquad p, q \;\textbf{?-}\; q, r \vee s}{p, p \supset q \;\textbf{?-}\; q, r \vee s} \supset L_1 \qquad \dfrac{p, p \supset q, r \;\textbf{?-}\; r, s}{p, p \supset q, r \;\textbf{?-}\; r \vee s} \vee R}{p, p \supset q, q \supset r \;\textbf{?-}\; r \vee s} \supset L_2}{\textbf{?-}\; p \supset (p \supset q) \supset (q \supset r) \supset (r \vee s)} \wedge L_3; (\supset R)s,$$

where the reduction operators are applied in the order $\supset R; \wedge L; \supset L_3; \supset L_2; \supset L_1$ followed by axioms. Note that the Proponent makes in the second move of the play the (disjunctive) choice of which of the two conclusions, r and s, he wants to prove. As this is the important aspect of the $\vee R$-rule, we choose this step to say that a $\vee R$-rule has been applied.

If one substitutes arbitrary formulæ for propositional variables in a proof, one still obtains a valid proof. This substitution lemma has an important analogon for games:

Lemma 6.7 *Suppose we have a strategy for the arena of a type ϕ which contains a propositional variable A. Then there is also a strategy for the arena of type $\phi[\psi/A]$, where ψ is any formula.*

Proof We only sketch the proof here. By definition of plays, in all plays defined by the strategies Opponent asks a question labelled A before Proponent does, and Opponent's answer is then used by Proponent to answer Opponent's original question. Hence Proponent can use a copy-cat strategy whenever the opponent makes a move in the arena for ψ. □

To model reductions, we use *oracles*, that is, additional plays which Proponent may start at will.

Definition 6.8 A strategy with oracle of type ϕ is a strategy where in addition Proponent is allowed to play using an additional arena for ϕ.

The instantiation of non-axiom leaves of a reduction with reductions is modelled by the substitution of strategies for oracles.

Definition 6.9 Suppose Ψ is a strategy with oracle of type ϕ and Φ is a strategy of type ϕ. We define the substitution of Φ for the oracle in Ψ to be the strategy Ψ except that we replace every answer which is a move given by the arena for Φ by the move obtained by using Φ to answer Ψ's move in ϕ, then using Ψ to answer this move and so on until Ψ answers with a move outside the arena for ϕ.

Before we can construct a classical reduction structure from games, we need some preliminary notation.

Definition 6.10 Suppose $\mathcal{C}$ is the free Cartesian category over the set of formulæ and assume π is a morphism from $(\phi_1, \ldots, \phi_n)$ to $(\psi_1, \ldots, \psi_m)$ and assume that Φ is a strategy for $\psi \vee \psi_1 \vee \cdots \vee \psi_m$. Furthermore, let $B_{i,1}, \ldots, B_{i,k_i}$ be the arenas of ψ_i and $A_{j,1}, \ldots, A_{j,l_j}$ be the arenas of ϕ_j, and let $A_1, \ldots, A_n$ be the arenas of ϕ.

We define the strategy $\pi^*(\Phi)$ to be the strategy for $\psi \vee \phi_1 \vee \cdots \vee \phi_n$ answering any question in the arena for ψ by the answer Φ would give to the corresponding question, and by answering any Opponent move in the part of the arena selecting a subarena for ψ_j by the Proponent move selecting the corresponding subarena for ϕ_i, where π maps ϕ_i to ψ_j, and answering any move in any subarena ψ_j by the answer Φ gives to the corresponding subarena in ϕ_i.

We now describe how to construct a classical reduction structure from this notion of game. Intuitively, the base category $\mathcal{B}$ of a reduction structure models the collection of indeterminates. A reduction with indeterminates is modelled as a *game with oracles*. Hence the category $\mathcal{B}$ consists of formulæ as objects (these represent the available oracles) and of games with oracles as morphisms. The indexing functor models substitution of games for oracles. As the category of worlds, we take compositions of reduction operators, as in the construction of the term models in Chapter 5.

The precise definition of the classical reduction structure obtained from games is given in the proof of the following proposition:

Proposition 6.11 *Games form a classical reduction structure.*

Proof We present here only the definition of the categories involved; the natural transformations are straightforward.

The category $\mathcal{C}$ is the free Cartesian category over the set of formulæ.

The category $\mathcal{B}$ has as objects finite lists of formulæ, $(\phi_1, \ldots, \phi_n)$ and as morphisms from $(\phi_1, \ldots, \phi_n)$ to $(\phi'_1, \ldots, \phi'_m)$ finite lists $(\Phi_1, \ldots, \Phi_m)$ of strategies such that Φ_i is a strategy for ϕ'_i possibly with oracles of type $(\phi_1, \ldots, \phi_n)$. We define composition of two morphisms $(\Phi_1, \ldots, \Phi_n)\colon (\sigma_1, \ldots, \sigma_k) \to (\phi_1, \ldots, \phi_n)$ and $(\Psi_1, \ldots, \Psi_m)\colon (\phi_1, \ldots, \phi_n) \to (\psi_1, \ldots, \psi_m)$ in $\mathcal{B}$ as the list of strategies $(\psi'_1, \ldots, \psi'_m)$, where ψ'_i is the strategy ψ_i with every answer that arises from the arena for ϕ_j. The answer for ϕ_j is replaced by the move obtained by first using the strategy Φ_j to answer this move, then Ψ to answer this move, and so on until ψ_i answers with a move outside the arena for ϕ_j.

For each pair of finite lists of formulæ, $(\phi_1, \ldots, \phi_n)$ and $(\psi_1, \ldots, \psi_m)$, we define a category $\mathcal{F}((\phi_1, \ldots, \phi_n), (\psi_1, \ldots, \psi_m))$, where the objects are formulæ and the morphisms from ϕ to ψ strategies for $\phi \supset (\psi \vee \phi_1 \vee \cdots \vee \phi_n)$, with oracles of type $\psi_1, \ldots, \psi_m$. We define composition in the category

$$\mathcal{F}((\phi_1, \ldots, \phi_n), (\psi_1, \ldots, \psi_m)),$$

in the same way as in the category $\mathcal{B}$.

For a morphism $(\Phi_1, \ldots, \Phi_n)$ in $\mathcal{B}$, and π in $\mathcal{C}$, we define a functor

$$\mathcal{E}((\Phi_1, \ldots, \Phi_n), \pi)$$

by leaving the objects unchanged and assigning to each strategy Φ the strategy $\pi^*(\Phi')$, where Φ'_i is the strategy obtained by substituting Φ_i for the indeterminate of type ϕ_i in Φ.

As the category of worlds, we take the free Cartesian category generated from ground objects W_R, where R is an LK-reduction operator, and ground morphism $a_R\colon 1 \to W_R$ for each reduction operator R. The functor F is defined as the functor assigning to W_R the object

$$((\Gamma_1 \supset \phi_1 \vee \Delta_1) \wedge \cdots \wedge (\Gamma_n \supset \phi_n \vee \Delta_n)) \supset (\Gamma \supset \phi \vee \Delta)$$

where R is a reduction operator with sufficient premisses Γ_i **?-** ϕ_i, Δ_i and putative conclusion Γ **?-** ϕ, Δ, and to the morphism a_R the canonical derivation given by R. $\square$

Note that this highly intensional category is non-trivial: equality between morphisms is essentially equality between partial functions. As the arenas for $\perp$ and $\top$, and for ϕ and $\neg\neg\phi$, are different, strategies for them cannot be equal. If we were to try to define an extensional collapse of this category, we must be careful to ensure that the arenas for $\neg\neg\phi$ and ϕ be not identified under the collapse.

Now we explain how backtracking is modelled in our games semantics. Backtracking points are captured by the possibility of Proponent making disjunctive choices which are not available when the moves are restricted to intuitionistic games. This is the case when Proponent plays both switching moves and when Proponent plays a P-question m corresponding to a node arising from a $\supset L$-operator. In the first case, playing the other switching move is not allowed in

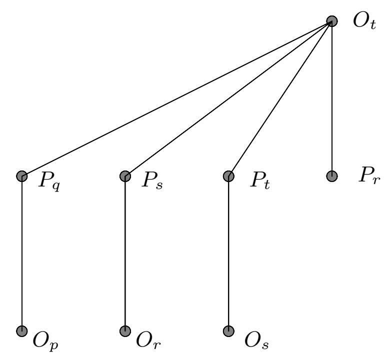


FIG. 6.1. Arena for $(p \supset q) \supset (r \supset s) \supset (s \supset t) \supset r \supset t$

games for LJ, and in the second case no previously pending O-question can be used to justify the P-answer to the O-question which is the immediate successor to the P-question m.

Backtracking actually occurs when Proponent plays a different switching move, or actually answers a question with a different label using Clause 6 of the definition of a play.

To illustrate this point, consider an example of the previous section, namely the reduction for the sequent

$$((p \supset q) \wedge (r \supset s) \wedge (s \supset t) \wedge r) \supset t.$$

The arena is given in Fig. 6.1. Then the following play corresponds to the second reduction in the previous section:

$$O_t^Q P_q^Q O_r^Q P_t^Q O_s^Q P_s^Q O_r^Q P_r^Q O_r^A P_r^A O_s^A P_s^A O_t^A P_r^A O_q^A P_t^A,$$

where moves by Opponent (Proponent) are denoted by the letter $O\,(P)$ with subscripts and superscripts, and the subscript indicates the label of the move and the superscript indicates whether the move is a question or an answer.

Note first the contraction involved in this play: the move P_t^Q models both instances of the $\supset L$-operator reducing $s \supset t$. The backtracking points are the P-questions labelled q, s, and t, and backtracking is reached with the move P_r^A: this move is possible only in games for multiple-conclusioned LK, and models the Exchange which is necessary to make the reduction succeed.

6.4 A concluding example: The semantics of uniform proof

Let us briefly review what we have achieved so far in this monograph.

1. Beginning with a review of the semantics of intuitionistic proofs, we have shown how the mathematical framework used there can be extended to

provide a semantics for classical proofs. We have discussed the failings of this approach and have given a range of examples including, a games semantics which combines ideas from the games semantics of intuitionistic provability and the games semantics of linear proof and programming languages.

2. We have explained the notion of reductive proof. We have shown how, via proof-theoretic analyses based on permutability and uniformity, the key technology of resolution, for both classical and intuitionistic logic, may be understood systematically as reductive systems.
3. We have explained the difficulties which arise in providing models for reductive proof which are of comparable value to those of deductive proof and have explained how our models of intuitionistic and classical proof may be enriched with a notion of indeterminate in order to interpret the inherent partiality and uncompletability of reductions.
4. We have shown how the key control régime of proof-search, that is, backtracking, in the computation of reductive proofs, may be understood semantically for intuitionistic logic by embedding models of intuitionistic reduction in models of classical reduction. In particular, we have given a game-theoretic example of this semantics.

Now, by way of a conclusion which ties together all of the key points in our development, we describe how our games semantics captures uniform proof and weakly uniform proof, that is, the classes of proofs which are the key to our systematic characterization of resolution proof within the reductive framework.

Recall that a uniform proof in (single-conclusioned) LJ is a proof in which right rules are preferred over left rules, so that a left rule is applied only if all formulæ on the right-hand side are atomic. In our games semantics, right-rules correspond to challenges by Opponent and left rules to challenges by Proponent, so uniform proofs correspond to strategies in which Opponent always plays as many rules as possible. The precise definition is as follows:

Definition 6.12 A strategy for ϕ in a game for intuitionistic or classical logic is called a uniform strategy if the following conditions hold: (i) Opponent always makes as many moves as possible; (ii) Proponent makes any move labelled L or R if possible.

If we consider games for intuitionistic logic, then a uniform strategy corresponds to a uniform proof in (single-conclusioned) LJ. If we consider games for classical logic, then a uniform strategy corresponds to a uniform proof in classical LK.

Weakly uniform proofs can be characterized in the same way. Recall that a weakly uniform proof is a uniform proof where, in addition to the conditions for uniform proof, $\vee L$-rules are applied as close to the root as possible. This can be captured in the games semantics by defining a strategy to be a *weakly uniform strategy* if

(i) it is uniform, and
(ii) moves by Proponent corresponding to the root node in the arena for the interpretation of any formula $\phi \vee \psi$ (on the left) are played in preference to any other moves, and
(iii) moves by Opponent labelled L and R are played in preference to any other move.

As we have seen earlier, the embedding of a uniform single-conclusioned LJ-proof Φ in LK is not necessarily uniform, but there exists a uniform multiple-conclusioned uniform LK-proof Φ' which contains the LJ-proof as a subproof. The parts of Φ' that are not contained in Φ correspond to Weakening terms when the translation of Φ' into the $\lambda\mu\nu\epsilon$-calculus is considered.

This has an analogue in games: Any strategy for intuitionistic games is also a strategy for classical games. As Opponent has more possibilities of challenging Proponent, a strategy which is uniform for intuitionistic games is not uniform for classical games. However, any uniform strategy for intuitionistic games gives rise in a canonical way to a uniform strategy for classical games: Proponent ignores the additional questions by Opponent and considers only the questions Opponent asked in the original strategy. Proponent is also able to use the answers he gave in the intuitionistic strategy to answer the additional questions by Opponent.

6.5 Discussion

We have provided a semantics for intuitionistic proof-search, that is, intuitionistic reductive proof with backtracking. In particular, we have provided a semantics for weakly uniform and uniform proof. Thus our semantics accounts for the principal structural aspects of reductive proof theory, which characterizes both classical and intuitionistic resolution, as well as the principal control régime for proof-search in these systems.

We have not, however, addressed all of the aspects of a control régime that are necessary to define a deterministic proof-search procedure. Specifically, we have not considered how to incorporate with our semantics a representation of the selection of formulæ in a sequent and selection between the premisses of a reduction operator. For example, given the sequent

$$\Gamma, \phi_1 \vee \phi_2, \psi_1 \vee \psi_2 \ \textbf{?-}\ \chi$$

we must choose a formula to drive the reduction. Suppose we choose $\phi_1 \vee \phi_2$, then, applying the $\vee L$ operator, we obtain the premisses

$$\Gamma, \phi_1, \psi_1 \vee \psi_2 \ \textbf{?-}\ \chi \quad \text{and} \quad \Gamma, \phi_2, \psi_1 \vee \psi_2 \ \textbf{?-}\ \chi.$$

So we must choose which premiss to attack next.

One approach to these issues might be to incorporate notions of *ordering*, *cf. domain theory* [5], into the basic constructions of the semantics.

We conjecture that our semantics of proof-search (including the treatment of backtracking) can be easily extended to predicate logic. The quantifiers can be added easily to the $\lambda\mu\nu$-calculus, and for the games semantics we should consider games in which a universally quantified formula gives rise to a generic question by Opponent and in which an existentially quantified formula to a generic question by Proponent.

REFERENCES

1. *Concise Oxford Dictionary* (*second edition*). Oxford University Press, 1976.
2. A. Kolmogorov. Zur Deutung der intuitionistischen Logik. *Mathematische Zeitschrift*, 35, 1932.
3. S. Abramsky, R. Jagadeesan, and P. Malacaria. Full abstraction for PCF (extended abstract). LNCS 789, TACS, 1984.
4. S. Abramsky, R. Jagadeesan, and P. Malacaria. Full abstraction for PCF. *Information and Computation*, 163(2):409–470, 2000.
5. R. Amadio and P.-L. Curien. *Domains and Lambda-Calculi*. Cambridge University Press, Cambridge, England, 1998.
6. P. B. Andrews. Theorem-proving with general matings. *Journal of the ACM*, 28:193–214, 1981.
7. K. R. Apt and M. H. van Emden. Contributions to the theory of logic programming. *Journal of the ACM*, 29(3):841–862, 1982.
8. P. Armelín and D. Pym. Bunched logic programming (extended abstract). In *Proceedings of IJCAR 2001*, number 2083 in LNAI, pages 289–304. Springer, Berlin, Heidelberg, New York, 2001.
9. A. Asperti and G. Longo. *Categories, Types and Structure: An Introduction to Category Theory for the Working Computer Scientist*. MIT Press, Cambridge, Mass., 1991.
10. A. Avron. Simple consequence relations. *Information and Computation*, 91(1):105–139, 1991.
11. H. P. Barendregt. Lambda calculi with types. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Background: Computational Structures*, Volume 2 of *Handbook of Logic in Computer Science*, pages 117–309. Oxford University Press, Oxford, England, 1992.
12. M. Barr and C. Wells. *Category Theory for Computing Science*. Prentice-Hall International, London, 1990.
13. M. Barr and C. Wells. *Category Theory for Computing Science* (*second edition*). Prentice-Hall International, London, 1995.
14. J. Bénabou. Fibered categories and the foundations of naive category theory. *Journal of Symbolic Logic*, 50:10–37, 1985.
15. P. N. Benton, G. M. Bierman, V. C. V. de Paiva, and J. M. E. Hyland. Term assignment for intuitionistic linear logic (preliminary report). Technical report, University of Cambridge, Computer Laboratory, August 1992. Report 262.
16. W. Bibel. On matrices with connections. *Journal of the ACM*, 28:633–645, 1981.

17. G. Boolos. Don't eliminate cut. In Richard Jeffrey, editor, *Logic, Logic, and Logic*, pages 365–369. Harvard University Press, 1998.
18. R. S. Boyer and J. S. Moore. *A Computational Logic*. ACM Monograph Series. Academic Press, New York, 1979.
19. Alexander Brodie. *George Lokert: Late-Scholastic Logician*. Edinburgh University Press, Edinburgh, 1983.
20. A. Bundy. *The Computer Modelling of Mathematical Reasoning*. Academic Press: London, 1984.
21. W. Clocksin. *Clause and Effect*. Springer-Verlag 21: Berlin and Heidelberg, 1997.
22. W. Clocksin and C. Mellish. *Programming in Prolog*. Springer-Verlag, 1994.
23. Th. Coquand. *Une théorie des constructions, Thèse de Troisième Cycle*. PhD thesis, Université de Paris VII, 1995.
24. R. L. Crole. *Categories for Types*. Cambridge University Press, Cambridge, 1994.
25. P.-L. Curien. Preface to locus solum. *Mathematical Structures in Computer Science*, 11(3):299–300, 2001.
26. M. Dummett. *Elements of Intuitionism*. Oxford University Press, Oxford, 1977.
27. C. Faggian. Travelling on designs: Ludics dynamics. In *Proceedings of CSL '02*, volume 2471 of *LNCS*. Springer-Verlag, 2002.
28. W. Felscher. Dialogues as a foundation for intuitionistic logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic Volume III: Alternatives to Classical Logic*, pages 341–372. D. Reidel, Dordrecht, 1986.
29. M. D. Fitting. *Proof Methods for Modal and Intuitionistic Logics*. D. Reidel, Dordrecht, 1983.
30. C. Führmann and D. Pym. Order-enriched categorical models of the classical sequence calculus. Submitted. Manuscript available at www.cs.bath.ac.uk/~pym, 2003.
31. G. Huet, G. Kahn, and C. Paulin-Mohring. The Coq Proof Assistant—A Tutorial. 2003.
32. G. McCusker. Games and full abstraction for FPC. In *Proceedings of the 11th Annual Symposium Meeting in New Brunswick on Logic in Computer Science* (*LICS*), pages 174–183. IEEE Computer Society Press, 1996.
33. D. Gabbay and F. Guenthner (editors). *Handbook of Philosophical Logic, Volume III: Alternatives to Classical Logic*. Number 166 in Synthese Library. D. Reidel, Dordrecht, Holland, 1986.
34. J. Gallier. *Logic for Computer Science: Foundations of Automatic Theorem Proving*. Wiley, New York, 1986.
35. D. Galmiche, D. Méry, and D. Pym. Resource Tableaux. In *Proceedings of CSL 2002*, volume 2471 of *LNCS*, pages 183–199, 2002. Preprint available at www.cs.bath.ac.uk/~pym.
36. D. Galmiche and D. J. Pym. Proof-search in type-theoretic languages: An introduction. *Theoretical Computer Science*, 232:5–53, 2000. Special Issue

on Proof-search in Type-theoretic Languages, edited by D. Galmiche and D. Pym.

37. G. Gentzen. Untersuchungen über das logische Schliessen. *Mathematische Zeitschrift*, 39:176–210, 405–431, 1934.
38. N. Ghani. $\beta\eta$-equality for coproducts. In *Proceedings of TLCA 1995*, volume 902 of *Lecture Notes in Computer Science*, pages 171–185, 1995.
39. P. Gilmore. A proof method for quantificational theory. *IBM Journal of Research and Development*, 4:28–35, 1960.
40. J.-Y. Girard. *Interprétation Fonctionnelle et Elimination des Coupures de l'Arithmétique d'Ordre Supérieur*. Thèse de doctorat d'état, Université Paris VII, June 1972.
41. J.-Y. Girard. Locus Solum: From the rules of logic to the logic of rules. *Mathematical Structures in Computer Science*, 11(3):301–506, 2001.
42. J.-Y. Girard, Y. Lafont, and P. Taylor. *Proofs and Types*. Cambridge University Press, Cambridge, 1989.
43. M. Gordon, R. Milner, and C. Wadsworth. *Edinburgh* LCF*, A Mechanized Logic of Computation*, volume 78 of *Lecture Notes in Computer Science*. Springer-Verlag, Berlin, 1979.
44. H. Herbelin. A lambda-calculus structure isomorphic to Gentzen-style sequent calculus structure. In L. Pacholski and J. Tiuryn, editors, *Computer Science Logic, 8th International Workshop, CSL '94, Kazimierz, Poland, September 25–30, 1994*, volume 933 of *Lecture Notes in Computer Science*, pages 61–75. Springer, 1994.
45. J. Harland and D. Pym. Resource-distribution via Boolean constraints. *ACM Transactions on Computational Logic*, 4(1):56–90, 2003.
46. J. A. Harland, D. J. Pym, and M. Winikoff. Programming in Lygon: An overview. In M. Wirsing and M. Nivat, editors, *Proceedings of AMAST '96*, volume 1101 of *LNCS*, pages 391–405. Springer, 1996.
47. James Harland, Tatjana Lutovac, and Michael Winikoff. Goal-directed proof search in multiple-conclusioned intuitionistic logic. In Lloyd et al., editors, *Proceedings of the First International Conference on Computational Logic*, volume 1861 of *Lecture Notes in Artificial Intelligence*, pages 254–268. Springer, 2000.
48. C. Haynes. Logic continuations. *Journal of Logic Programming*, 4:157–176, 1987.
49. Jacques Herbrand. Investigations in proof theory: The properties of true propositions. In van Heijenoort, editor, *From Frege to Gödel*, pages 525–581. Harvard University Press, 1967.
50. A. Heyting. *Intuitionism: An Introduction*. Cambridge University Press, Cambridge, 1989.
51. W. Hodges. *Logic*. Penguin, 1993.
52. W. Hodges. *A Shorter Model Theory*. Cambridge University Press, Cambridge, 1997.

53. M. Hofmann and T. Streicher. Continuation models are universal for $\lambda\mu$-calculus. In *Proceedings of 12th Annual Symposium on Logic in Computer Science (LICS)*, pages 387–397. IEEE Computer Society Press, 1997.
54. W. Howard. The formulæ-as-types notion of construction. In J. P. Seldin and J. R. Hindley, editors, *To H. B. Curry: Essays in Combinatory Logic, Lambda Calculus and Formalism*, pages 479–490. Academic Press, New York, 1980.
55. Gérard Huet. A unification algorithm for typed λ-calculus. *Theoretical Computer Science*, 1:27–57, 1975.
56. J. M. E. Hyland and C. Faggian. Designs, disputes and strategies. In *Proceedings of CSL '02*, volume 2471 of *LNCS*. Springer-Verlag, 2002.
57. J. M. E. Hyland and A. M. Pitts. The theory of constructions: Categorical semantics and topos theoretic models. *Contemporary Mathematics*, 92:137–198, 1989.
58. S. S. Ishtiaq and P. O'Hearn. BI as an assertion language for mutable data structures. In *28th ACM-SIGPLAN Symposium on Principles of Programming Languages, London*, pages 14–26. Association for Computing Machinery, 2001.
59. B. Jacobs. *Categorical Type Theory*. PhD thesis, Catholic University, Nijmegen, September 1991.
60. B. Jacobs. *Categorical Logic and Type Theory*. Elsevier, Amsterdam, 1998.
61. C. B. Jay and N. Ghani. The virtues of eta-expansion. *Journal of Functional Programming*, 5(2):135–154, 1995.
62. D. Jensen and T. Pietrzykowski. Mechanizing ω-order type theory through unification. *Theoretical Computer Science*, 3:123–171, 1976.
63. J. M. E. Hyland. Games Semantics. In A. M. Pitts and P. Dybjer, editors, *Semantics and Logics of Computation*. Cambridge University Press, Cambridge, 1997.
64. J. M. E. Hyland and C.-H. L. Ong. On Full Abstraction for PCF: I, II and III. 2000.
65. S. C. Kleene. *Mathematical Logic*. Wiley and Sons, New York, 1968.
66. J. W. Klop. Term rewriting systems. In S. Abramsky, T. Maibaum, and D. Gabbay, editors, *Handbook of Logic in Computer Science*, volume 1 and 2. Oxford University Press, Oxford, 1992.
67. R. Kowalski. *Logic for Problem-solving*. North-Holland, Elsevier, Amsterdam, 1979.
68. S. A. Kripke. Semantical considerations on modal logic. *Acta Philosophica Fennica*, 16:83–94, 1963.
69. S. A. Kripke. Semantical analysis of intuitionistic logic I. In J. N. Crossley and M. A. E. Dummett, editors, *Formal Systems and Recursive Functions*, pages 92–130. North-Holland, Amsterdam, 1965.
70. J. Lambek and P. Scott. *Introduction to Higher-Order Categorical Logic*. Cambridge University Press, Cambridge, 1986.

71. F. W. Lawvere. Adjointness in foundations. *Dialectica*, 23:281–296, 1969.
72. P. Lorenzen. Ein dialogisches Konstruktivitätskriterium. In *Infinitistic Methods*, pages 193–200. Pergamon Press, 1961.
73. Leopold Löwenheim. On possibilities in the calculus relatives. In van Heijenoort, editor, *From Frege to Gödel*, pages 228–263. Harvard University Press, 1967.
74. S. Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag, New York, 1971.
75. D. MacKenzie. *Mechanizing Proof: Computing, Risk, and Trust* (*Inside Technology*). MIT Press, Cambridge, Mass., 2001.
76. S. MacLane. *Algebra* (fourth edition). American Mathematical Society, New York, 1999.
77. P. Martin-Löf. Constructive logic and computer programming. In L. J. Cohen et al., editors, *Logic, Methodology and Philosophy of Science VI*, pages 153–175. North-Holland, Amsterdam, 1982.
78. S. Yu Maslov. *Theory of Deductive Systems and its Applications*. MIT Press, Cambridge, Mass., 1987.
79. E. Mendelson. *Introduction to Mathematical Logic*. Van Nostrand, Princeton, 1987.
80. D. Miller, G. Nadathur, F. Pfenning, and A. Ščedrov. Uniform proofs as a foundation for logic programming. *Annals of Pure and Applied Logic*, 51:125–157, 1991.
81. R. Milner, M. Tofte, and R. Harper. *The Definition of Standard ML*. MIT Press, Cambridge, MA, 1990.
82. G. Mints. Gentzen-type systems and resolution rules part I: Propositional logic. In P. Martin-Löf and G. Mints, editors, *COLOG-88: Proceedings of International Conference on Computer Logic, Tallinn, USSR*, Volume 417 of *Lecture Notes in Computer Science*, pages 198–231, Berlin, Heidelberg, New York, 1990. Springer-Verlag.
83. J. Mitchell and E. Moggi. Kripke-style models for typed lambda calculus. *Annals of Pure and Applied Logic*, 51:99–124, 1981.
84. S. Negri and J. van Plato. *Structural Proof Theory*. Cambridge University Press, Cambridge, 2001.
85. T. Nipkow, L. C. Paulson, and M. Wenzel. *Isabelle/HOL—A Proof Assistant for Higher-Order Logic*, Volume 2283 of *Lecture Notes in Computer Science*. Springer-Verlag, 2002.
86. P. W. O'Hearn and R. D. Tennent. Parametricity and local variables. *Journal of ACM*, 42(3):658–709, May 1995. Also in *Algol-like Languages* (Volume 2). Birkhäuser, Boston, 1997, pages 109–164.
87. P. W. O'Hearn and R. D. Tennent (editors). *Algol-like Languages* (Volume 2). Birkhäuser, Boston, 1997.
88. P. W. O'Hearn and D. J. Pym. The logic of bunched implications. *Bulletin of Symbolic Logic*, 5(2):215–244, June 1999.

89. C.-H. L. Ong. A semantic view of classical proofs: type-theoretic, categorical, and denotational characterizations (preliminary extended abstract). In *Proceedings, 11th Annual IEEE Symposium on Logic in Computer Science*, pages 230–241, New Jersey, USA, 1996. IEEE Computer Society Press, Los Alamitos, California.
90. M. Parigot. $\lambda\mu$-calculus: An algorithmic interpretation of classical natural deduction. In *Proceedings of the International Conference on Logic Programming and Automated Reasoning, St. Petersburg*, 1992.
91. L. C. Paulson. *Logic and Computation, Interactive Proof with Cambridge* LCF, Volume 2 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge, England, 1987.
92. G. D. Plotkin. Call-by-name, call-by-value, and the λ-calculus. *Theoretical Computer Science*, 1:125–159, 1975.
93. G. D. Plotkin. *The Category of Complete Partial Orders: A Tool for Making Meanings*. Lecture notes for the Summer School on Foundations of Artificial Intelligence and Computer Science, Pisa, June 1978.
94. Randy Pollack. LEGO Proof Development system: User's manual. Technical report, LFCS, University of Edinburgh, May 1992.
95. D. Prawitz. *Natural Deduction: A Proof-Theoretical Study*. Almquist and Wiksell, Stockholm, 1965.
96. D. Prawitz. Ideas and results in proof theory. In *Proceedings of the Second Scandinavian Logic Symposium*. North-Holland, Amsterdam, 1971.
97. D. Pym and E. Ritter. On the semantics of classical disjunction. *Journal of Pure and Applied Algebra*, 159:315–338, 2001.
98. D. J. Pym. *Proofs, Search and Computation in General Logic*. PhD thesis, University of Edinburgh, 1990.
99. D. J. Pym. A unification algorithm for the $\lambda\Pi$-calculus. *International Journal of Foundations of Computer Science*, 3(2):333–378, 1992.
100. D. J. Pym. A note on the proof theory [of] the $\lambda\Pi$-calculus. *Studia Logica*, 54:199–230, 1995.
101. D. J. Pym. On bunched predicate logic. In *Proceedings of LICS'99*, pages 183–192. IEEE Computer Society Press, 1999.
102. D. J. Pym. *The Semantics and Proof Theory of the Logic of Bunched Implications*, Volume 26 of *Applied Logic Series*. Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.
103. D. J. Pym. Editorial. *Journal of Logic and Computation*, 13(5), 2003. The 'Semantics Corner' of the Journal of Logic and Computation.
104. D. J. Pym and J. A. Harland. A uniform proof-theoretic investigation of linear logic programming. *Journal of Logic and Computation*, 4:175–207, 1994.
105. D. J. Pym, P. W. O'Hearn, and H. Yang. Possible worlds and resources: The semantics of BI. To appear, *Theoretical Computer Science*, 2004. Available at www.cs.bath.ac.uk/˜pym, 2000.

106. D. J. Pym and L. A. Wallen. Proof-search in the $\lambda\Pi$-calculus. In G. Huet and G. Plotkin, editors, *Logical Frameworks*, pages 309–340. Cambridge University Press, Cambridge, 1991.
107. D. J. Pym and L. A. Wallen. Logic programming via proof-valued computations. In K. Broda, editor, *ALPUK92, Proceedings of 4th U.K. Conference on Logic Programming*, pages 253–262. Springer-Verlag, 1992.
108. E. Ritter, D. J. Pym, and L. A. Wallen. Proof-terms for classical and intuitionistic resolution. *Journal of Logic and Computation*, 10(2):173–207, 2000.
109. J. C. Reynolds. Separation Logic: A logic for shared mutable data structures. In *Proceedings of LICS '02*, pages 55–74. IEEE Computer Society Press, 2002.
110. John C. Reynolds. The discoveries of continuations. *LISP and Symbolic Computation*, 6(3–4):233–247, 1993.
111. E. Ritter, D. J. Pym, and L. A. Wallen. On the intuitionistic force of classical search. *Theoretical Computer Science*, 232:299–333, 2000.
112. E. Robinson. Proof nets for classical logic. *Journal of Logic and Computation*, 13(5), 2003. Semantic Foundations of Proof-search edited by D. Pym.
113. J. Robinson. A machine-oriented logic based on the resolution principle. *Journal of the ACM*, 12(1):23–49, 1965.
114. D. A. Schmidt. *Denotational Semantics, A Methodology for Language Development*. Allyn and Bacon, Newton, Massachusetts, 1986. Reprinted by W. C. Brown, Dubuque, Iowa (1988).
115. Bruce Schneier. *Secrets and Lies: Digital Security in a Networked World*. Wiley, New York, 2000.
116. D. Scott. Rules and derived rules. In S. Stenlund, editor, *Logical Theory and Semantical Analysis*, pages 147–161. Reidel: Dordrecht, 1974.
117. R. A. G. Seely. Hyperdoctrines, natural deduction and the Beck condition. *Zeitschr. für Math. Logik und Grundlagen der Math.*, 29:505–542, 1983.
118. Thoralf Skolem. Logico-combinatorial investigations in the satisfiability or provability of mathematical propositions: A simplified proof of a theorem by L. Löwenheim and generalizations of the theorem. In van Heijenoort, editor, *From Frege to Gödel*, pages 252–263. Harvard University Press, Cambridge, Mass., 1967.
119. Thoralf Skolem. On mathematical logic. In van Heijenoort, editor, *From Frege to Gödel*, pages 508–524. Harvard University Press, Cambridge, Mass., 1967.
120. R. Smullyan. *First-order Logic*, Volume 43 of *Ergebnisse der Mathematik*. Springer-Verlag, 1968.
121. G. Sundholm. Proof theory and meaning. In Gabbay and Guenthner, editors, *Handbook of Philosophical Logic, Volume III: Alternatives to Classical Logic*, pages 471–506. Number 166 in Synthese Library. D. Reidel, Dordrecht, Holland, 1986.

122. W. W. Tait. The intensional interpretation of functionals of finite type. *Journal of Symbolic Logic*, 32, 1967.
123. T. Tammet. Gandalf. *Journal of Automated Reasoning*, 18(2):199–204, 1997.
124. T. Tammet. Resolution, inverse method and the sequent calculus. In *Kurt Gödel Colloquium 1997*, pages 65–83, 1997.
125. A. Tarski. *Logic, Semantics, Metamathematics.* Oxford University Press, Oxford, 1956.
126. A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory.* Cambridge University Press, Cambridge, 1996.
127. C. Urban. *Classical Logic and Computation.* PhD thesis, Cambridge University, October 2000.
128. D. van Dalen. *Logic and Structure* (*second edition*), Springer, Berlin, 1983.
129. D. van Dalen. Intuitionistic logic. In Gabbay and Guenthner, editors, *Handbook of Philosophical Logic, Volume III: Alternatives to Classical Logic*, pages 225–339. Number 166 in Synthese Library. D. Reidel, Dordrecht, Holland, 1986.
130. M. H. van Emden and R. A. Kowalski. The semantics of predicate logic as a programming language. *Journal of ACM*, 23:733–742, 1976.
131. Jean van Heijenoort (editor). *From Frege to Gödel.* Harvard University Press, Cambridge, Mass., 1967.
132. S. Vickers. *Topology via Logic. Cambridge Tracts in Theoretical Computer Science.* Cambridge University Press, Cambridge, 1996.
133. W. Egly. Some pitfalls of LK-to-LJ-translations and how to avoid them. In W. McCune, editor, *Proceedings of 14th International Conference on Automated Deduction* (*CADE*), Volume 1249 of *Lecture Notes in Computer Science*, pages 116–130. Springer-Verlag, 1997.
134. L. A. Wallen. *Automated Proof Search in Non-Classical Logics: Efficient Matrix Proof Methods for Modal and Intuitionistic Logics.* MIT Press, Cambridge, Mass., 1989.
135. J. Zucker. The correspondence between cut-elimination and normalisation. *Annals of Mathematical Logic*, 7:1–112, 1974.

INDEX