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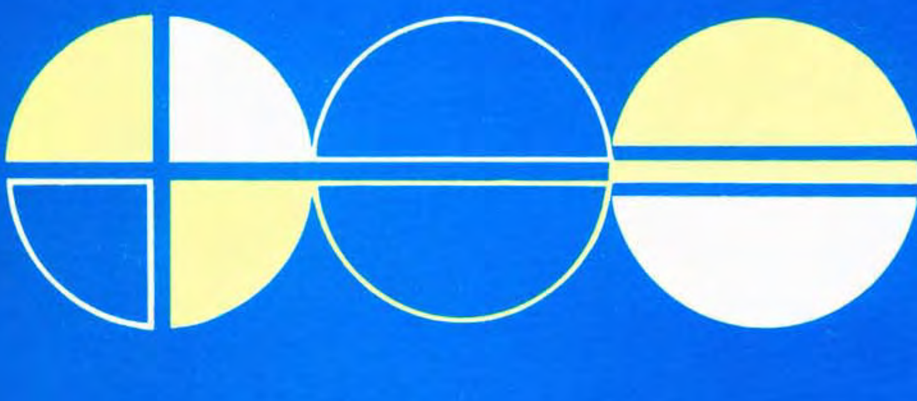
MATHEMATICS STUDIES

8

Graphs, Groups and Surfaces

ARTHUR T. WHITE

Completely revised and enlarged edition



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NORTH-HOLLAND – AMSTERDAM • NEW YORK • OXFORD

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Arthur T. WHITE
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Completely revised and enlarged edition

1984

NORTH-HOLLAND – AMSTERDAM • NEW YORK • OXFORD

First edition: First printing 1973
Revised edition: First printing 1984

ISBN: 0 444 87643 x

Publishers:
ELSEVIER SCIENCE PUBLISHERS B.V.
P.O. BOX 1991
1000 BZ AMSTERDAM
THE NETHERLANDS

Sole distributors for the U.S.A. and Canada:
ELSEVIER SCIENCE PUBLISHING COMPANY, INC.
52 VANDERBILT AVENUE
NEW YORK, N.Y. 10017
U.S.A.

Library of Congress Cataloging in Publication Data

White, Arthur T.
Graphs, groups, and surfaces.

(North-Holland mathematics studies ; 8)
Bibliography: p.
Includes indexes.
1. Graph theory. 2. Groups, Theory of. 3. Combinatorial topology. I. Title. II. Series.
QA166.W45 1984 511'.5 84-18853
ISBN 0-444-87643-X

PRINTED IN THE NETHERLANDS

FOREWORD TO THE FIRST EDITION

This text is based on a course that I taught at Western Michigan University in 1971. The material is suitable for presentation at the graduate or advanced undergraduate level, and assumes only an introductory knowledge of group theory and of point-set topology. One of the features of the subject matter is that many results of the general theory can be readily visualized, or modeled, so that the student is constantly reassured that what he is doing has meaning, is more than a formal manipulation of symbols.

The focus of attention is an interaction amoung graphs, groups, and surfaces (see Figure 0-1). After a brief introduction to the theory of graphs, the relationship between graphs and groups will be explored. For every graph there is an associated group, called the automorphism group of the graph. Conversely, for every group presentation, there is an associated graph, called a Cayley color graph of the group. Both associations will be studied, with emphasis on the latter. An excursion into combinatorial topology will follow; Euler's generalized polyhedral formula and the classification theorem for closed 2-manifolds will be discussed. Imbedding problems in graph theory will be examined at some length (here a graph and a surface come together, often with the aid of a group); it will be seen that the famous four-color conjecture can be stated in this context. The five-color theorem will be established. The Heawood map-coloring theorem will then be studied in detail. (This theorem determines a chromatic number for every closed 2-manifold except the sphere -- a truly astounding result!) The theory of quotient graphs and quotient manifolds (and quotient groups) is at the heart of the proof of this theorem (as well as many others) and will be presented; this theory relies upon each of the concepts mentioned in this paragraph and serves as the unifying feature of this text.

Several peripheral (but significant) results are stated without proof. An effort has been made to provide those proofs of theorems which are most indicative of the charm and beauty of the subject and which illustrate the techniques employed. Proofs missing in the text can be supplied by the reader, as part of the problem sets (problems marked "*" are difficult; those marked "**" are as yet unsolved), or can be found in the references. A bibliography is provided, for further reading; items (c) or (j), (g), and (h) respectively are particularly suitable for more extensive treatments of the theories of graphs, groups, and surfaces, which are seen interacting in this text.

A.T.W.

FOREWORD TO THE SECOND EDITION

The field of topological graph theory has greatly expanded in the ten years since the first edition of this book appeared, with many able mathematicians contributing to this expansion. As with the first edition, I have tried to focus on those aspects of the subject germane to the interaction of the structures listed in the title. I apologize in advance, for any omissions of relevant material.

The nine chapters of the first edition have been revised and updated. Thirty-two new problems have been distributed among these chapters.

Six new chapters have been added, dealing with: voltage graphs (extending the work of Jacques even further), non-orientable imbeddings (emphasizing parallels to the orientable theory and results), block designs associated with graph imbeddings (tying topological graph theory securely to more traditional combinatorics), hypergraph imbeddings (as generalizing graph imbeddings and as depicting block designs), map automorphism groups (groups acting (primarily) on graphs of groups on surfaces), and change ringing (finding how to ring church bells by picturing the right graph of the right group on the right surface.)

Several new problems have been given for each new chapter, so that there are now 181 problems in all; 22 of these have been designated as "difficult" (*) and 9 as "unsolved" (**). Three of the four unsolved problems from the first edition have been solved during the ten years between editions; they are now marked as "difficult."

I wish to thank everyone who read the first edition, particularly those who sent me comments and corrections, and especially Jonathan Gross. I also owe thanks (but delegate no responsibility) to many, including the following, for their contributions to my

understanding of the material in the new chapters as indicated: Thomas Tucker (10), Saul Stahl (11) and (13), Seth Alpert (12), Derek Waller (13) and (14), Norman Biggs (14), and Sabra Anderson, Joan Hutchinson, and T. Jefferson Smith (15). Finally, I thank Margo Johnson for typing the manuscript, and Paul Himelwright for drawing the figures.

A.T.W.

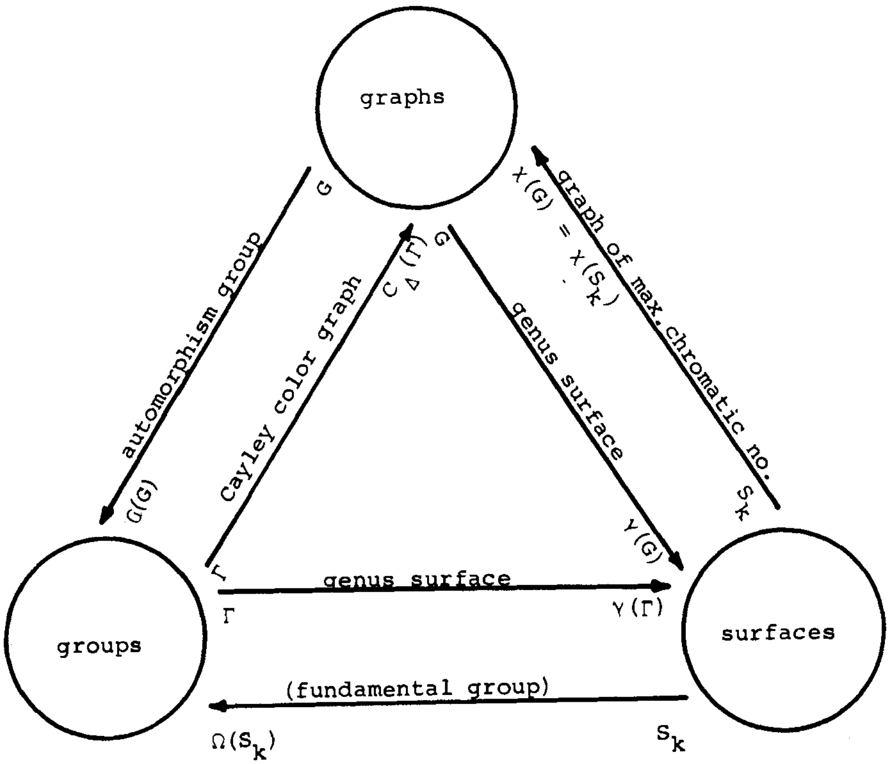
graphs
groups
surfaces
automorphism group
G
G(G)
Cayley color graph
Γ
$C_\Delta(\Gamma)$
G
genus surface
$\gamma(G)$
graph of max.chromatic no.
S_k
$\chi(G) = \chi(S_k)$
genus surface
Γ
$\gamma(\Gamma)$
(fundamental group)
S_k
$\Omega(S_k)$

Figure 0-1

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CHAPTER I

HISTORICAL SETTING

In coloring the regions of a map, one must take care to color differently any two countries sharing a common boundary line, so that the two countries can be distinguished. One would think that an economy-minded map-maker would wish to minimize the number of colors to be used for a given map, although there appears to be no historical evidence of any such effort. Nevertheless a conjecture was made, over thirteen decades ago, to the effect that four colors would always suffice for a map drawn on the sphere, the regions of which were all connected. The first reported mention of this problem (see [BCL1] and [O1]) was by Francis Guthrie, through his brother Frederick and Augustus de Morgan, in 1852. The first written references were by Cayley, in 1878 and 1879. Incorrect "proofs" of the Four Color Conjecture were published soon after by Kempe and Tait. The error in Kempe's "proof" was found by Heawood [H3] in 1890; this error has reappeared in various guises in subsequent years. Ore and Stemple [OS1] showed that any counterexample to the conjecture must involve a map of at least 72 regions. The conjecture continued to provide one of the most famous unsolved problems in mathematics, until Appel and Haken affirmed it in 1976 [AH1].

It is an astonishing fact that several related, seemingly much more difficult, map-coloring problems have been completely solved. Chief among these is the Heawood Map-coloring Conjecture, which gives the chromatic number for <u>every</u> closed 2-manifold <u>other than the spere</u>; we state the orientable case:

$$\chi(S_k) = f(k) = \left\lfloor \frac{7 + \sqrt{1+48k}}{2} \right\rfloor, \quad \text{for} \quad k > 0 ,$$

where k is the genus of the closed orientable 2-manifold S_k. Heawood showed in 1980 [H3] that $\chi(S_k) \leq f(k)$, and in 1891 Heffter [H4] showed the reverse inequality for a possibly infinite set of natural numbers k; almost eight decades passed before it was shown that $\chi(S_k) \geq f(k)$, for all $k > 0$. In 1965 this problem was given the place of honor on the dust jacket for Tietze's <u>Famous Problems</u>

of Mathematics [T2]. An outline of the major portion of the solution now follows.

The dual of a map drawn on S_k is a pseudograph imbedded in S_k, and it can be shown (see Section 8-4) that $\chi(S_k) \geq f(k)$, for $k > 0$, provided the complete graph K_n has genus given by

$$\gamma(K_n) = \left\{\frac{(n-3)(n-4)}{12}\right\}, \quad n \geq 7. \qquad (*)$$

Heawood established (*) for $n = 7$ in 1890, and Heffter for $8 \leq n \leq 12$ in 1891; Ringel handled $n = 13$ in 1952. The first major breakthrough occurred in 1954, when Ringel showed (*) for $n \equiv 5 \pmod{12}$. During 1961-1965, Ringel treated the residue cases $7, 10$, and $3 \pmod{12}$, while independently Gustin settled the cases $3, 4$, and 7. Gustin's method involved the powerful and beautiful idea of quotient graph and quotient manifold, and relies upon the fact that K_n can be regarded as a Cayley color graph for a group presentation; thus graph theory, group theory, and surface topology are combining to solve this famous problem of mathematics.

In 1965, Terry, Welch, and Youngs announced their solution to case 0. Gustin, Ringel, and Youngs finished the remaining residue cases (mod 12), except for the isolated values $n = 18, 20$, and 23; their work was announced in 1968 [RY1]. In 1969, Jean Mayer (a Professor of French Literature) [M4] eliminated the last three obstinate graphs by ad hoc techniques.

Much of the work of Ringel, Terry, Welch, and Youngs was made possible by Gustin's theory of quotient graphs and quotient manifolds; this theory was developed and modified by Youngs, who also introduced the theory of vortices [Y3]. The theory is considerably more general than was needed to prove the Heawood Map-coloring Theorem, and has been unified and developed in more generality by Jacques [J2], in 1969. Jacques' results will be presented in Chapter 9, together with many applications to other imbedding problems in graph theory. This will be a focal point of the text, and it illustrates vividly the fruitful interaction among graphs, groups, and surfaces.

We continue this development through the theory of voltage graphs and by extending to nonorientable imbeddings. We consider the related structures of block designs and hypergraph imbeddings. In map automorphism groups we study groups acting on graphs of groups on surfaces. Finally, in studying change ringing, we use graphs of groups on surfaces to compose pieces of music.

The conjunctions of graph theory, group theory, and surface topology described above are foreshadowed, in this text, by several pairwise interactions among these three disciplines. The Heawood Map-coloring Theorem is proved by finding, for each suface, a graph of largest chromatic number that can be drawn on that surface. Equivalently (as it turns out) we find, for each complete graph, the surface of smallest genus in which it can be drawn. The extension of this latter problem to arbitrary graphs is natural; the solution is particularly elegant for graphs which are the Cayley color graphs of a group. We are led in turn to the problem of finding, for a given group, a surface of minimum genus which represents the group in some way.

Dyck [D5] (see also Burnside [B18], Chapters 18 and 19) considered maps, on surfaces, that are transformed into themselves in accordance with the fixed group Γ, acting transitively on the regions of the map. Any such map gives an upper bound for the parameter $\gamma(\Gamma)$ discussed in Chapter 7 of this text, as a "dual" formed in terms of Burnside's white regions gives a Cayley color graph for Γ. (Cayley [C4] defined his color graphs as complete symmetric digraphs, corresponding to the choice Γ less the identity element as a generating set for Γ; it is sensible to extend his definition to any generating set for the group in question.) Brahana [B15] studied groups represented by regular maps on surfaces; these maps correspond to presentations on two generators, one of which is of order two. In this context the group acts transitively on the edges of the map, and again an upper bound for $\gamma(\Gamma)$ is obtained. In Chapter 7, we regard Γ as acting transitively on the vertices of the map induced by imbedding a Cayley color graph $C_\Delta(\Gamma)$ for Γ in a surface; in Chapter 4, we show that the automorphism group of $C_\Delta(\Gamma)$ is isomorphic to Γ, independent of the generating set Δ selected for Γ, so that in this sense $C_\Delta(\Gamma)$ provides a "picture" of Γ. But more: many properties of Γ, such as commutivity, normality of certain subgroups, the entire multiplication table, can be "seen" from the picture provided by $C_\Delta(\Gamma)$. Thus it is natural to seek the simplest surface on which to draw this picture; this is given by the parameter $\gamma(\Gamma)$.

This point of view may give a surface of lower genus for a given group than the other two approaches listed above; for example, the group $\Gamma = Z_2 \times Z_4$ is toroidal for Dyck (or Burnside) and for

Brahana, yet $\gamma(Z_2 \times Z_4) = 0$.

There is one correspondence depicted in Figure 0-1 which we do not discuss in this text: to every surface S_k there corresponds a unique group, $\Omega(S_k)$, called the fundamental group of the surface; the groups $\Omega(S_k)$ have been completely determined -- they are given by $2k$ generators $a_1, b_1, \ldots, a_k, b_k$ and the single defining relation $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_k b_k a_k^{-1} b_k^{-1} = e$ (see, for example, [S6] .) Each of the other five correspondences illustrated in Figure 0-1 (where the inner triangle commutes, for proper choice of Λ) is germane, as outlined above, to the conjunction of graph theory, group theory, and surface topology described in this introduction and which we now begin to develop.

CHAPTER 2

A BRIEF INTRODUCTION TO GRAPH THEORY

In this chapter we introduce basic terminology from the theory of graphs that will be used in this text. We will give several binary operations on graphs; these will enable us to construct more complicated graphs, and hence to build up our store of examples of frequently encountered graphs.

We emphasize that the material introduced here is primarily for the purpose of later use in this text; for a considerably more thorough introduction to graph theory, see [BCL1] or [H2].

2-1. Definition of a Graph

Def. 2-1. A graph G consists of a finite non-empty set $V(G)$ of vertices together with a set $E(G)$ of unordered pairs of distinct vertices, called edges. If $x = [u, v] \in E(G)$, for $u, v \in V(G)$, we say that u and v are adjacent vertices, and that vertex u and edge x are incident with each other, as are v and x. We also say that the edges $[u, v]$ and $[u, w]$, $w \neq v$, are adjacent. The degree, $d(v)$, of a vertex v is the number of edges with which v is incident. (Equivalently, $d(v)$ is the number of vertices to which v is adjacent; i.e.,

$$d(v) = |\{u \in V(G) \mid [u, v] \in E(G)\}| \, .)$$

If the vertices of G are labeled, G is said to be a labeled graph.

As a matter of notation, we usually write uv for $[u, v]$; $p = |V(G)|$; $q = |E(G)|$. The order of G is given by p. The size of G is given by q.

<u>Example</u>: Let G be defined by:

$$V(G) = \{v_1, v_2, v_3, v_4\}$$
$$E(G) = \{v_1v_2, v_2v_3, v_3v_1, v_1v_4\} ;$$

then G may be represented by either Figure 2-1a or 2-1b, where the latter representation is more accurate, in a sense we will describe in Chapter 6.

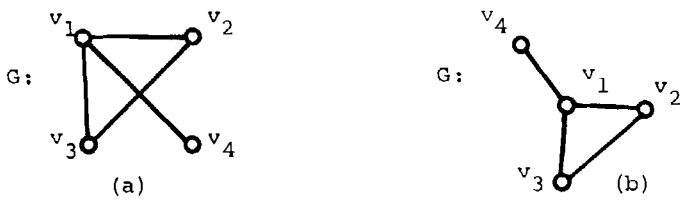

Figure 2-1.

<u>Note</u>: A <u>graph</u> may be more briefly defined as a finite one-dimensional simplicial complex.

<u>Thm. 2-2.</u> For any graph G, $\sum_{i=1}^{p} d(v_i) = 2q$.

<u>Proof</u>: In summing the degrees, each edge is counted exactly twice.

<u>Cor. 2-3.</u> In any graph G, the number of vertices of odd degree is even.

2-2. Variations of Graphs

<u>Def. 2-4.</u> A <u>loop</u> is an edge of the form vv . A <u>multiple edge</u> is an edge that appears more than once in $E(G)$. A <u>directed edge</u> is an ordered pair of distinct vertices. A <u>multigraph</u> allows multiple edges. A <u>pseudograph</u> allows loops and multiple edges. A <u>directed graph</u> (<u>digraph</u>) has every edge directed. The corresponding graph (with all edge directions deleted) is called the <u>underlying graph</u>. An <u>infinite graph</u> has infinite vertex set.

For example, see Figure 2.2.

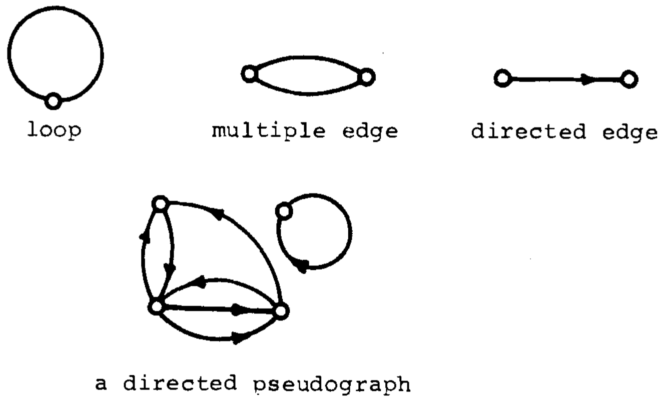

Figure 2-2.

The term "graph," unless qualified appropriately, disallows any and all of the above variations.

2-3. Additional Definitions

<u>Def. 2-5.</u> A graph H is said to be a <u>subgraph</u> of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, H is called a <u>spanning subgraph</u>. For any $\phi \neq S \subseteq V(G)$, the <u>induced subgraph</u> $\langle S \rangle$ is the maximal subgraph of G with vertex set S .

<u>Notation</u>: For $v \in V(G)$, $G - v$ denotes $\langle V(G) - v \rangle$. For $x \in E(G)$, $V(G - x) = V(G)$, and $E(G - x) = E(G) - x$.

Certain subgraphs are given special names. We indicate these by a series of definitions.

<u>Def. 2-6.</u> A <u>walk</u> of a graph G is an alternating sequence of vertices and edges $v_0, x_1, v_1, \ldots, v_{n-1}, x_n, v_n$ (or, briefly: $v_0, v_1, \ldots, v_{n-1}, v_n$) beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it; n is the <u>length</u> of the walk. If $v_0 = v_n$, the walk is said to be <u>closed</u>; it is said to be <u>open</u> otherwise. The

walk is called a trail if all its edges are distinct, and a path if all the vertices are distinct. A cycle is a closed walk with $n \geq 3$ distinct vertices (i.e., $v_0 = v_n$, but otherwise the v_i are distinct).

Two famous problems in graph theory may be described in terms of the above definitions. A graph is said to be eulerian if the graph itself can be expressed as a closed trail. (This corresponds to the "highway inspector" problem; eulerian graphs have been completely and simply characterized: see Harary [H2], p. 64-65.) A graph is said to be hamiltonian if it has a spanning cycle. (This corresponds to the "traveling salesman" problem; hamiltonian graphs have not been completely characterized. See Harary, p. 65-69, for some partial results.)

Def. 2-7. A graph G is connected if $u, v \in V(G)$ implies there exists a path in G joining u to v. A component of G is a maximal connected subgraph of G.

Def. 2-8. The distance, $d(u,v)$, between two vertices u and v of G is the length of a shortest path joining them if such exists; if not, $d(u,v) = \infty$.

Thm. 2-9. A connected graph may be regarded as a (finite) metric space.

Proof: See Problem 2-7. #

For a partial converse to the above theorem, see Chartrand and Kay [CK1]. By Theorem 2-9, every connected graph may be regarded as a topological space. (Actually, since the metric induces the discrete topology, we knew this already.) In Chapter 6 we will see that this is true in another sense also; that is every graph may be regarded as a subspace of R^3, with all edges represented as straight lines. If we consider G as a topological space in this latter sense, then G is connected as a graph if and only if it is connected as a topological space (see Problem 2-8). The term "component" is easily seen to mean the same in both contexts. Furthermore, a graph (as a subspace of R^3) is connected if and only if it is path connected; (see Problem 2.9.)

Def. 2-10. Two graphs G_1 and G_2 are said to be isomorphic ($G_1 \cong G_2$, or $G_1 = G_2$) if there exists a one-to-one, onto map $\theta : V(G_1) \to V(G_2)$ preserving adjacency; that is, $uv \in E(G_1)$ if and only if $\theta(u)\theta(v) \in E(G_2)$.

Note: Isomorphism is an equivalence relation on the set of all graphs.

Notation: $\delta(G) = \min\{d(v) | v \in V(G)\}$.
$\Delta(G) = \max\{d(v) | v \in V(G)\}$.

Def. 2-11. If $\delta(G) = \Delta(G) = r$, we say that G is regular of degree r. (If $r = 3$, G is said to be cubic.)

Thm. 2-12. Let $\theta : V(G_1) \to V(G_2)$ give $G_1 \cong G_2$; then $d(\theta(v)) = d(v)$, for all $v \in V(G_1)$.

Proof: See Problem 2-10. #

Cor. 2-13. If G_1 is regular of degree r and $G_1 \cong G_2$, then G_2 is regular of degree r.

Cor. 2-14. Let the vertices of a graph G_1 have degrees $d_1 \leq d_2 \leq \cdots \leq d_n$, and the vertices of a graph G_2 have degrees $c_1 \leq c_2 \leq \cdots \leq c_n$. If $d_i \neq c_i$, for some $1 \leq i \leq n$, then G_1 and G_2 are not isomorphic.

The inverse of the above corollary need not be true; see Problem 2-3.

Def. 2-15. The complement $\overline{G}$ of a graph G has $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv | u \neq v \text{ and } uv \notin E(G)\}$.

2-4. Operations on Graphs

We now define several binary operations on graphs. In what follows, we assume that $V(G_1) \cap V(G_2) = \phi$.

Def. 2-16. 1.) The union $G = G_1 \cup G_2$ has:
$V(G) = V(G_1) \cup V(G_2)$
$E(G) = E(G_1) \cup E(G_2)$.

Notation: $2G = G \cup G$
$nG = (n-1)G \cup G,\ n \geq 3$.

2.) The join $G = G_1 + G_2$ has:
$V(G) = V(G_1) \cup V(G_2)$
$E(G) = E(G_1) \cup E(G_2) \cup \{v_1v_2 | v_i \in V(G_i),\ i = 1,2\}$.

3.) The cartesian product $G = G_1 \times G_2$ has:
$V(G) = V(G_1) \times V(G_2)$
$E(G) = \{[(u_1,u_2),(v_1,v_2)] | u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)\}$.

4.) The composition (or lexicographic product) $G = G_1[G_2]$ has:
$V(G) = V(G_1) \times V(G_2)$
$E(G) = \{[(u_1,u_2),(v_1,v_2)] | u_1v_1 \in E(G_1)$ or $(u_1 = v_1$ and $u_2v_2 \in E(G_2))\}$.

For additional products, see Harary and Wilcox [HW1].

We are now in a position to conveniently define several infinite families of graphs.

Def. 2-17. a.) P_n denotes the path of length $n-1$ (i.e. of order n.)

b.) C_n denotes the cycle of length n.

c.) K_n denotes the complete graph on n vertices; that is all $\binom{n}{2}$ possible edges are present.

d.) $\overline{K}_n$ denotes the totally disconnected graph on n vertices; that is, $E(\overline{K}_n) = \phi$.

e.) $K_{m,n}$ denotes a complete bipartite graph:

$$K_{m,n} = \overline{K}_m + \overline{K}_n.$$

(Equivalently, $K_{m,n}$ is defined by:

$$\overline{K_{m,n}} = K_m \cup K_n.)$$

f.) $K_{p_1,p_2,\ldots,p_n}$ denotes a <u>complete</u> <u>n-partite</u> graph:

$$K_{p_1,p_2,\ldots,p_n} = \overline{K}_{p_1} + \overline{K}_{p_2} + \cdots + \overline{K}_{p_n},$$

an iterated join. In the special case where $p_1 = p_2 = \cdots = p_n$ (= m, say), we get a <u>regular</u> <u>complete</u> <u>n-partite</u> graph:

$$K_{m,m,\ldots m} = K_n[\overline{K}_m].$$

We introduce $K_{n(m)}$ as a shorter notation for this graph.

g.) Q_n denotes the <u>n-cube</u> and is defined recursively:

$$Q_1 = K_2$$
$$Q_n = K_2 \times Q_{n-1}, \quad n \geq 2.$$

The complete bipartite graphs are a subclass of an extremely important class of graphs -- the bipartite graphs.

<u>Def. 2-18</u>. A <u>bipartite</u> <u>graph</u> G is a graph whose vertex set $V(G)$ can be partitioned into two non-empty subsets V' and V'' so that every edge of G has one vertex in V' and the other in V''.

<u>Thm. 2-19</u>. A nontrivial graph G is bipartite if and only if all its cycles are even.

<u>Proof</u>: (i). Let $v_1v_2\cdots v_nv_1$ be a cycle in a bipartite graph G, and assume, without loss of generality, that $v_1 \in V'$; then $v_n \in V''$, and n must be even.

(ii). We may assume that G is connected, with only even cycles, since the argument in general follows directly from this special case. Consider a fixed $v_0 \in V(G)$. Let $V_i = \{u \in V(G) | d(u,v_0) = i\}$, $i = 0,1,\ldots,n$. Then n is finite, since G is connected, and

$V_0, V_1, \ldots, V_n$ provides a partition of $V(G)$. Now, no two vertices in V_1 are adjacent, since G contains no 3-cycles. Also, no two vertices in V_2 are adjacent, or G would contain either a 3-cycle or a 5-cycle. In fact, every edge in G is of the form uv, where $u \in V_i, v \in V_{i+1}$, for some $i = 0,1,\ldots,n-1$. Letting V' be the union of the V_i for i odd, and V'' be the union of the V_i for i even, we see that G is bipartite. #

This completes our brief introduction; other terms will be defined, and theorems developed, as needed.

2.5. Problems

2-1.) Prove that if G is not connected, then $\overline{G}$ is connected. Given an example to show that the converse need not hold.

2-2.) A graph is said to be perfect if no two vertices have the same degree. Prove that no graph is perfect, except $G = K_1$.

2-3.) Show that, even though $K_{3,3}$ and $K_2 \times K_3$ are both regular of order 6 and degree 3, they are not isomorphic.

2-4.) Prove that $G_1 \times G_2$ is bipartite if and only if both G_1 and G_2 are bipartite. Give an example to show that a similar result need not hold for the lexicographic product.

2-5.) Show that $\overline{G_1[G_2]} = \overline{G}_1[\overline{G}_2]$.

2-6.) Show that $\overline{G_1 + G_2} = \overline{G}_1 \cup \overline{G}_2$.

2-7.) Prove Theorem 2-9.

2-8.) Consider the graph G as a subspace of R^3. Show that G is connected as a topological space if and only if it is connected as a graph.

2-9.) Show that the first occurence of "connected" in Problem 2-8 may be replaced with "path-connected." (Recall that a path-connected topological space must be connected, but that the converse does not always hold. However, a connected space for which every point has a path connected neighborhood must be path-connected.)

2-10.) Prove Theorem 2-12.

2-11.) Show that the set of all graphs, under the operation of cartesian product, forms a commutative semigroup with unity (i.e. a commutative monoid).

2-12.) Let G and H both be hamiltonian. Show that $G[H]$ and $G \times H$ are hamiltonian also. Show that Q_n is hamiltonian, for $n \geq 2$.

2-13.) The <u>line graph</u> $H = L(G)$ of a graph G is defined by $V(H) = E(G)$, $E(H) = \{ef \mid e$ and f are adjacent, $e, f \in E(G)\}$. Show that $L(K_{m,n}) = K_m \times K_n$. (This result appears in Palmer [P2].)

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CHAPTER 3

THE AUTOMORPHISM GROUP OF A GRAPH

In this chapter we show that there is associated, with each graph, a group, known as the automorphism group of the graph. We introduce various binary operations on permutation groups to aid in computing automorphism groups of graphs. Several powerful results relating graph and group products are stated, sometimes without proof (see [H2] for a further discussion); these results will not be used in the sequel. Indeed, the concept of automorphism group of a graph is, in the main, peripheral to the present course; it is introduced here primarily as one example of an interaction between graphs and groups. (In Section 2 of Chapter 4 we find a more direct bearing on subsequent material.)

3-1. Definitions

Def. 3-1. A one-to-one mapping from a finite set onto itself is called a permutation. A permutation group is a group whose elements are all permutations acting on the same finite set, called the object set. (The group operation is composition of mappings.) If X is the object set and A the permutation group, then $|A|$ is the order of the group, and $|X|$ is the degree.

A permutation P partitions its object set by the equivalence relation $x \equiv y$ if and only if $P^k(x) = y$ for some integer k. The equivalence classes are called the orbits of X, under the action of P. If there is just one orbit in the action of A on X, then A is said to be transitive on X. If $|A| = |X|$, and if A is transitive on X, then A is said to be a regular permutation group.

Def. 3-2. Two permutation groups A and B are said to be isomorphic $(A \cong B)$ if there exists a one-to-one onto map

$\theta: A \to B$ such that $\theta(a_1 a_2) = \theta(a_1)\theta(a_2)$, for all $a_1, a_2 \in A$.

Def. 3-3. Two permutation groups A and B (acting on object sets X and Y respectively) are said to be identical (or equivalent) $(A \equiv B)$ if:

(i) $A \cong B$ (given by $\theta: A \to B$)

(ii) there exists a one-to-one, onto map $f: X \to Y$ such that $f(ax) = \theta(a)f(x)$, for all $x \in X$ and $a \in A$.

For a general treatment of permutation groups acting on combinatorial structures, see Biggs and White [BW1]. Here, we consider only the graph automorphism case.

Def. 3-4. An automorphism of a graph G is an isomorphism of G with itself. (The set of all automorphisms of G forms a permutation group, $\mathcal{G}(G)$, acting on the object set $V(G)$.) $\mathcal{G}(G)$ is called the automorphism group of G.

Remark. An automorphism of G, which is a permutation of $V(G)$, also induces a permutation of $E(G)$, in the obvious manner.

Def. 3-5. An identity graph is a graph G having trivial automorphism group; that is, the identity permutation on $V(G)$ is the only automorphism of G.

It is easy to see that the graph pictured in Figure 3-1 is an identity graph. That there is no identity graph of smaller order (other than K_1) is established in Problem 3-1.

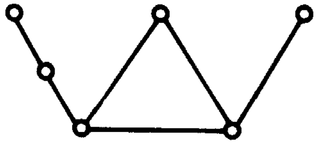
G:  $(\mathcal{G}(G) \cong 1)$

Figure 3-1

Thm. 3-6. $\mathfrak{G}(\overline{G}) \equiv \mathfrak{G}(G)$.

Proof: Let θ: $\mathfrak{G}(G) \to \mathfrak{G}(\overline{G})$ and f: $V(G) \to V(\overline{G})$ both be identity maps, and observe that adjacency is preserved in a graph if and only if non-adjacency is preserved. #

3-2. Operations on Permutation Groups

From a theorem due to Cayley, we recall that any finite group is abstractly isomorphic (as opposed to necessarily being identical) with a permutation group; in fact, if the group G has order n, then G is isomorphic to a subgroup of S_n. In this light, the operations soon to be defined could be regarded as applying to groups in general; however, the definitions will be given in terms of action upon a specified object set.

Let A and B be permutation groups acting on object sets X and Y respectively. We define three binary operations on these permutation groups as follows:

Def. 3-7. 1.) The sum, $A + B$, (or direct product) acts on the disjoint union $X \cup Y$; $A + B = \{a + b | a \in A, b \in B\}$, and

$$(a+b)(z) = \begin{cases} az, & \text{if } z \in X \\ bz, & \text{if } z \in Y \end{cases}.$$

2.) The product, $A \times B$, (or cartesian product) acts on $X \times Y$; $A \times B = \{a \times b | a \in A, b \in B\}$, and $(a \times b)(x, y) = (ax, by)$.

3.) The composition, $A[B]$, (or wreath product) acts on $X \times Y$ as follows: for each $a \in A$ and any sequence $b_1, b_2, \ldots, b_d$ (where $d = |X|$) in B, there is a unique permutation in $A[B]$, written $(a; b_1, b_2, \ldots, b_d)$, and $(a; b_1, b_2, \ldots, b_d)(x_i, y_j) = (ax_i, b_i y_j)$.

Note: The order of $A[B]$ is $|A||B|^d$.

Thm. 3-8. $A + B \cong A \times B$.

Proof: Let $\theta : A + B \to A \times B$ be given by $\theta(a+b) = (a \times b)$. #

Note that, in general, we cannot claim $A + B \equiv A \times B$.

3-3. Computing Automorphism Groups of Graphs

The following theorems indicate some connections between the graphical operations defined in Section 2-4 and the group operations defined above. The groups S_n, A_n, Z_n, D_n are respectively the symmetric and alternating groups of degree n, the cyclic group of order n, and the dihedral group of order $2n$. The first theorem is due to Frucht [F5].

Thm. 3-9. If G is a connected graph, then $\mathcal{A}(nG) \equiv S_n[\mathcal{A}(G)]$.

Thm. 3-10. If no component of G_1 is isomorphic with a component of G_2, then $\mathcal{A}(G_1 \cup G_2) \equiv \mathcal{A}(G_1) + \mathcal{A}(G_2)$.

Proof: See Problem 3-2. #

Thm. 3-11. Let $G = n_1G_1 \cup n_2G_2 \cup \cdots \cup n_rG_r$, where n_i is the number of components of G isomorphic to G_i. Then

$$\mathcal{A}(G) \equiv S_{n_1}[\mathcal{A}(G_1)] + S_{n_2}[\mathcal{A}(G_2)] + \cdots + S_{n_r}[\mathcal{A}(G_r)].$$

Proof: Apply Theorems 3-9 and 3-10, using induction. #

Note: Any graph G may be written as in Theorem 3-11, but if $r = n_r = 1$, the theorem gives no information.

Thm. 3-12. If no component of $\overline{G}_1$ is isomorphic with a component of $\overline{G}_2$, then $\mathcal{A}(G_1 + G_2) \equiv \mathcal{A}(G_1) + \mathcal{A}(G_2)$.

Proof: Apply Theorems 3-6 and 3-10, together with problem 2-6. #

The following two theorems are due to Sabidussi ([S2] and [S2] respectively).

Def. 3-13. A non-trivial graph G is said to be prime if $G = G_1 \times G_2$ implies that either G_1 or G_2 must be trivial (i.e. $= K_1$). If G is not prime, G is composite. Two graphs G_1 and G_2 are relatively prime if $G_1 = G_3 \times G_4$ and $G_2 = G_3 \times G_5$ imply $G_3 = K_1$.

Thm. 3-14. $\mathcal{G}(G_1 \times G_2) \equiv \mathcal{G}(G_1) \times \mathcal{G}(G_2)$ if and only if G_1 and G_2 are relatively prime.

Def. 3-15. The neighborhood of a vertex u is given by: $N(u) = \{v \in V(G) | uv \in E(G)\}$. The closed neighborhood is $N[u] = N(u) \cup \{u\}$.

Thm. 3-16. If G_1 is not totally disconnected, then $\mathcal{G}(G_1[G_2]) \equiv \mathcal{G}(G_1)[\mathcal{G}(G_2)]$, if and only if:

(i) if there are two vertices in G_1 with the same neighborhood, then G_2 is connected.

and

(ii) if there are two vertices in G_1 with the same closed neighborhood, then $\overline{G}_2$ is connected.

We are now able to list the automorphism groups for several common families of graphs.

Thm. 3-17. 1.) $\mathcal{G}(K_n) \equiv S_n$

2.) $\mathcal{G}(C_n) \equiv D_n$

3.) $$\mathcal{G}(K_{m,n}) \equiv \begin{cases} S_2[S_n], & \text{if } m = n \\ S_m + S_n, & \text{if } m \neq n \end{cases}$$

4.) $\mathcal{G}(K_n[\overline{K}_m]) \equiv S_n[S_m]$.

3-4. Graphs with a Given Automorphism Group

Thm. 3-18. Every finite group is the automorphism group of some graph.

For a proof of this theorem, due to Frucht [F4], see Section 4-2.

3-5. Other Groups of a Graph

Def. 3-19. Two graphs G and H (with non-empty edge sets) are said to be edge-isomorphic if there exists a one-to-one, onto map $\phi: E(G) \to E(H)$ preserving adjacency; that is x, y share a vertex in G if and only if $\phi(x), \phi(y)$ share a vertex in H. ϕ is called an edge-isomorphism.

Thm. 3-20. If G and H are isomorphic (with non-empty edge sets), then they are edge-isomorphic.

Proof: See Problem 3-6.

Def. 3-21. An induced edge-isomorphism is an edge isomorphism $\phi: E(G) \to E(H)$ determined by $\phi(uv) = \theta u \theta v$, where $\phi: V(G) \to V(H)$ is an isomorphism. An edge-automorphism of a non-empty graph G is an edge-automorphism of G with itself. (The set of all edge-automorphisms of G forms a permutation group, $\mathfrak{G}_1(G)$, acting on the object set $E(G)$.) $\mathfrak{G}_1(G)$ is called the edge-automorphism group of G. Similarly, the induced edge-automorphism group of G, $\mathfrak{G}^*(G)$, is the set of all induced edge-automorphisms of G under composition.

We note that $\mathfrak{G}^*(G)$ is a (possibly proper; see Problem 3-8) subgroup of $\mathfrak{G}_1(G)$. However, in almost every case, the three groups introduced in this chapter are isomorphic (see, for example, [BCL1]); let $K_{1,3} + x$ denote the graph obtained by adding one edge to $K_{1,3}$:

Thm. 3-22. For $G \neq K_1$, $\mathcal{G}(G) \cong \mathcal{G}^*(G)$ if and only if G contains neither K_2 as a component nor two or more isolated vertices.

Thm. 3-23. Let $E(G) \neq \phi$; then $\mathcal{G}_1(G) \cong \mathcal{G}^*(G)$ if and only if:
(1) not both C_3 and $K_{1,3}$ are components of G , and
(2) none of $K_{1,3} + x$, $K_4 - x$, K_4 is a component of G.

Thm. 3-24. Let G be a connected graph of order $p \geq 3$; then $\mathcal{G}(G) \cong \mathcal{G}_1(G) \cong \mathcal{G}^*(G)$ if and only if $G \neq K_{1,3} + x$, $K_4 - x$, K_4.

Thus it is customary to focus attention on the automorphism group, $\mathcal{G}(G)$.

3-6. Problems

3-1.) Does there exist an identity graph (other than K_1) of order five or less? (Hint: Check to see that every graph in Appendix 1, Harary [H2], with $p \leq 5$, has at least one non-trivial automorphism.)

3-2.) Prove Theorem 3-10.

3-3.) Prove Theorem 3-17.

3-4.) Let $G = K_{p,q,r}$; find $\mathcal{G}(G)$.

3-5.) $\mathcal{G}(K_4) \cong S_4$, yet K_4 is the 1-skeleton of the tetrahedron, and the symmetry group of the tetrahedron is A_4. Explain!

3-6.) Prove Theorem 3-20.

3-7.) Show that the converse of Theorem 3-20 is false.

3-8.) Show that not all edge-automorphisms are induced.

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CHAPTER 4

THE CAYLEY COLOR GRAPH OF A GROUP PRESENTATION

In this chapter we see that any group may be defined in terms of generators and relations and that corresponding to such a presentation there is a unique graph, called the Cayley color graph of the presentation. A "drawing" of this graph gives a "picture" of the group, from which may be determined certain properties of the group. We will establish some basic results about Cayley color graphs, including a rather natural correspondence between direct products of groups and cartesian products of associated Cayley color graphs. In Chapter 7 we will ask which groups have Cayley color graphs that can be represented properly in the plane, and associated questions. In Chapter 9 appropriate answers will solve the Heawood map-coloring problem, as well as many others. The voltage graph theory of Chapter 10, the block design connection of Chapter 12, and the map automorphisms of Chpater 14 will be especially natural in the context of Cayley graphs. And, each change-ringing graph of Chapter 15 will be a Cayley color graph.

4-1. Definitions

Def. 4-1. Let Γ be a group, with $\{g_1, g_2, g_3, \ldots\}$ a subset of the element set of G. A word W in $g_1, g_2, g_3, \ldots$ is a finite product $f_1 f_2 \cdots f_n$, where each f_i is in the set $\{g_1, g_2, g_3, \ldots, g_1^{-1}, g_2^{-1}, g_3^{-1}, \ldots\}$. If every element of Γ can be expressed as a word in $g_1, g_2, g_3, \ldots$, then $g_1, g_2, g_3, \ldots$ are said to be generators for Γ. A relation is an equality between two words in $g_1, g_2, g_3, \ldots$.

Thm. 4-2. Given an arbitrary set of symbols and an arbitrarily prescribed set (possibly empty) of relations in these symbols, there is a unique (up to isomorphism) group with the symbols as generators and with structure determined by the prescribed relations.

(For a proof, see [MKS1].) #

Def. 4.3. If Γ is generated by $g_1, g_2, g_3, \ldots$ and if every relation in G can be deduced from the relations $P = P'$, $Q = Q'$, $R = R'$, ..., then we write $\Gamma = \langle g_1, g_2, g_3, \ldots \mid P = P', Q = Q', R = R', \ldots \rangle$, and the right hand side of the equation is said to be a presentation of Γ. A presentation is said to be a finitely generated (finitely related) if the number of generators (defining relations) is finite. A finite presentation is both finitely generated and finitely related.

Thm. 4-4. Every finite group has a finite presentation.

Proof: Take Γ itself as the set of generators, with all relations of the form $g_i g_j = g_k$, as determined by the group operation. (i.e. the multiplication table serves as a finite presentation.) #

Def. 4-5. For every group presentation there is associated a Cayley color graph: the vertices correspond to the elements of the group; next, imagine the generators of the group to be associated with distinct colors. If vertices v_1 and v_2 correspond to group elements g_1 and g_2 respectively, then there is a directed edge (of the color (or label) of generator h) from v_1 to v_2 if and only if $g_1 h = g_2$; see Figure 4-1.

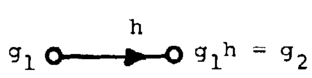


Figure 4-1.

Let P be a presentation for the group Γ ; we denote the Cayley color graph of P for Γ by $C_P(\Gamma)$, or (when convenient) by $C_\Delta(\Gamma)$, where Δ denotes the generating set. (Since a group may have more than one generating set, the Cayley color graph depends on Δ , as well as Γ). Then $C_\Delta(\Gamma)$ is a labeled, directed graph, with a color (or label) assigned to each edge. We observe that the following correspondences occur:

Group	Cayley Color Graph
element	vertex
generator	a set of directed edges of the same color
inverse of a generator	the same set of edges (now directed against the arrow)
word	walk
multiplication of elements	succession of walks
identity word	closed walk
solvability of $rx = s$	(weakly) connected di-graph

Note: A characterization is given in [MKS1] of those graphs G which can be oriented and colored so as to form Cayley color graphs.

Historical note: Max Dehn (in 1911) formulated three fundamental decision problems concerning group presentations. One of these is: "determine in a finite number of steps, for two arbitrary words W and W' in the generators, whether $W = W'$ or not." Equivalently: "construct the Cayley color graph for a given group presentation."

The term "connected" may have a "stronger" meaning for directed graphs than for graphs in general, since we may be allowed to travel only in the direction of the arrow along a given directed edge.

Def. 4-6. A directed graph D is said to be strongly connected if, for every pair u, v of distinct vertices, there is a directed path from u to v. D is said to be unilaterally connected if, for every pair of distinct vertices, one is joined to the other by a directed path. D is called <u>weakly connected</u> if the (undirected) pseudograph underlying D is connected.

For example, see Figure 4-2, where D is strongly connected, D' is unilaterally connected (but not strongly connected) and D'' is weakly connected (but not unilaterally connected).

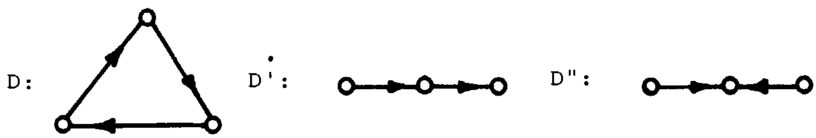


Figure 4-2.

4-2. Automorphisms

We have previously defined an automorphism of a graph G (as a permutation of $V(G)$ preserving adjacency). An automorphism of a directed graph must preserve <u>directed</u> adjacency; and an automorphism of a Cayley color graph must also preserve the <u>color</u> corresponding to each adjacency. We summarize in:

<u>Def. 4-7</u>. An <u>automorphism of a Cayley color graph</u> $C_\Delta(\Gamma)$ is a permutation θ of $V(C_\Delta(\Gamma))$ such that, for each g_1, g_2 in Γ and h in Δ, $g_1h = g_2$ if and only if $\theta(g_1)h = \theta(G_2)$.

Equivalently (see Problem 4-1), θ is an automorphism of $C_\Delta(\Gamma)$ if and only if: for each g in Γ and generator h in Δ, $\theta(gh) = \theta(g)h$; i.e. the diagram in Figure 4-3 commutes.

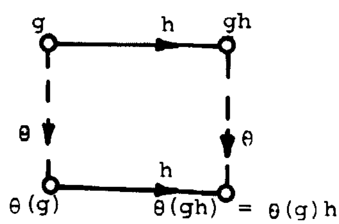


Figure 4-3

As expected, the collection of all automorphisms of $C_\Delta(\Gamma)$ forms a group, called the <u>automorphism group</u> of $C_\Delta(\Gamma)$, and denoted by $\mathcal{G}(C_\Delta(\Gamma))$. The next result is perhaps not expected.

<u>Thm. 4-8</u>. Let $C_\Delta(\Gamma)$ be any Cayley color graph for the finite group Γ; then $\mathcal{G}(C_\Delta(\Gamma)) \cong \Gamma$ (independent of the presentation selected for G.)

<u>Proof</u>: Define $\alpha: \Gamma \to \mathcal{G}(C_\Delta(\Gamma))$ by $\alpha(g) = \theta_g$, where $\theta_g: V(C_\Delta(\Gamma)) \to V(C_\Delta(\Gamma))$ is given by $\theta_g(g_i) = gg_i$. First we show that $\theta_g \in \mathcal{G}(C_\Delta(\Gamma))$. Clearly θ_g is one-to-one and onto (and hence permutes $V(C_\Delta(\Gamma))$). Also, $\theta_g(g_i h) = g(g_i h) = (gg_i)h = \theta_g(g_i)h$, so that α is well-defined.

Now, α preserves products: $\alpha(gg^*) = \theta_{gg^*}$, defined by $\theta_{gg^*}(g_i) = gg^*g_i = \theta_g(g^*g_i) = \theta_g(\theta_{g^*}(g_i)) = (\theta_g\theta_{g^*})(g_i)$; that is $\alpha(gg^*) = \alpha(g)\alpha(g^*)$.

It is clear that α is one-to-one, since $\ker \alpha = \{e\}$.

It remains to show that α is onto. Let $\theta \in \mathcal{G}(C_\Delta(\Gamma))$. Let $\theta(e) = g$, where e is the identity of Γ. Now any g^* in Γ can be written as a word in the generators for Γ; i.e. $g^* = h_1^{a_1} h_2^{a_2} \cdots h_m^{a_m}$, where h_i is a generator for Γ and $a_i = \pm 1$. Then $\theta(g^*) = \theta(eg^*) = \theta(e)h_1^{a_1}h_2^{a_2} \cdots h_m^{a_m} = gg^*$; that is, $\theta = \theta_g$, so that α is onto. This completes the proof. #

From the above theorem (and its proof) it is evident that any vertex of $C_\Delta(\Gamma)$ can be labeled with the identity e of Γ, and that once this has been done (for fixed assignment of colors to the generators) all other vertex labelings are determined; that is, $C_\Delta(\Gamma)$ is vertex transitive. Moreover, $\mathcal{G}(C_\Delta(\Gamma))$ is a regular permutation group, on $V(C_\Delta(\Gamma))$.

We are now able to provide a proof of Frucht's Theorem, Theorem 3-18: Every finite group is the automorphism group of some graph.

Proof: Let Γ be a finite group, and let Δ be a generating set for Γ. Form the Cayley color graph $C_\Delta(\Gamma)$; by Theorem 4-8, we know that $\mathcal{G}(C_\Delta(\Gamma)) \cong \Gamma$. It only remains to convert $C_\Delta(\Gamma)$ to a graph G having the same automorphism group, Γ. This is done as follows: let $\Delta = \{\delta_1, \delta_2, \ldots, \delta_n\}$. Replace each edge (g_i, g_j), where $g_j = g_i\delta_k$, by a path: $v_i, u_{ij}, u_{ij}', v_j$. At vertex $u_{ij}(u_{ij}')$ we attach a new path $P_{ij}(P_{ij}')$ of length $2k - 1$ $(2k)$ $(1 \le k \le n)$; see Figure 4-4, for the case $k = 2$. In this way the "non-graphical" features of direction and label, present in $C_\Delta(\Gamma)$, are incorporated into the graph G. It is clear that $\mathcal{G}(G) \cong \mathcal{G}(C_\Delta(\Gamma)) \cong \Gamma$. #

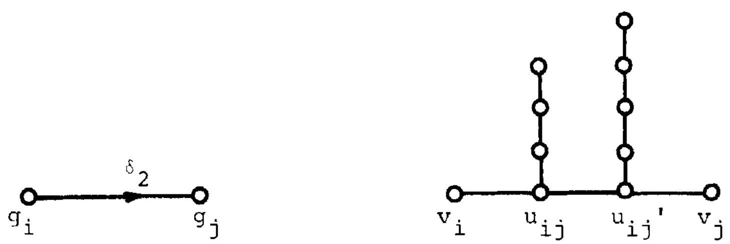


Figure 4-4

4-3. Properties

It is clear that every Cayley color graph is both regular and connected (as a graph); the converse is not true (see Problem 4-10). For two characterizations of graphs which may be regarded as Cayley color graphs, see [J2] and [MKS1].

We may study additional properties for Γ (apart from the multiplication table so conveniently summarized in $C_\Delta(\Gamma)$) from $C_\Delta(\Gamma)$ as follows:

Thm. 4-9. Γ is commutative if and only if, for every pair of generators h_i and h_j, the walk $h_i h_j h_i^{-1} h_j^{-1}$ is closed.

Proof: See Problem 4-2. #

Def. 4-10. An element of a generating set for a group Γ is said to be redundant if it can be written as a word in the remaining generators. A generating set is said to be minimal if it contains no redundant generators.

Example: $\{x^2, x^3\}$ is a minimal generating set for $Z_6 = \langle x \mid x^6 = e \rangle$, even though $\{x\}$ is a generating set with fewer elements.

Thm. 4-11. Let Γ be a finite (infinite) group. A generator h is redundant if and only if the deletion of all edges colored h in $C_\Delta(\Gamma)$ leaves a strongly (weakly) connected directed graph.

Proof: See Problem 4-3. #

Thm. 4-12. If H is not redundant, the removal of all edges colored h leaves a collection of isomorphic disjoint subgraphs, each representing the subgroup of Γ generated by the generating set of Γ minus h.

Proof: See Problem 4-4. #

Thm. 4-13. Let Γ be a finite group with minimal generating set $\{h_1, h_2, \ldots, h_r\}$, and Ω a (necessarily proper) subgroup with generating set $\{h_2, h_3, \ldots, h_r\}$. Let $C_1, C_2, \ldots, C_k$ be the weak components of the directed graph $C_{h_1}(\Gamma)$, obtained from $C_\Delta(\Gamma)$ by deleting the edges colored h_1. Then Ω is normal in Γ if and only if the deleted directed edges from any given component C_i all lead to a single other component C_j.

Proof: (i) Assume the condition holds. Let $C_1 = \Omega$ be the component containing e, let $g \in C_1$ and $r \in \Gamma$. We must show that $rgr^{-1} \in C_1$. We write $r = a_1^{b_1} a_2^{b_2} \cdots a_m^{b_m}$, where a_i is a generator of Γ and $b_i = \pm 1$. If h_1 occurs in r exactly w times with $b_i = +1$ and v times with $b_i = -1$, then the walk corresponding to r leads from e (in C_1) through $w - v$ components, ending in C_{1+w-v}. The walk corresponding to g in C_{1+w-v} now leads to another vertex in C_{1+w-v}, and the walk $a_m^{-b_m} \cdots a_2^{-b_2} a_1^{-b_1}$, corresponding to r^{-1}, returns us to C_1.

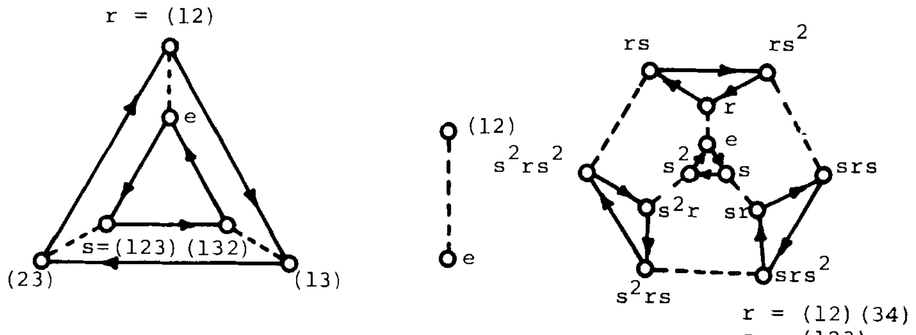
(ii) Suppose that edges colored h_1 lead from C_i to C_1 and C_j, $1 \neq j$. (Again assume $e \in C_1$.) Then there exists $g \in C_1$ such that $h_1^{-1}gh_1 \in C_j$, so that Ω is not normal in Γ. #

It now follows that, for Ω (as above) normal in Γ, the elements of the factor group Γ/Ω (i.e. the right cosets) are the components of $C_{h_1}(G)$. By shrinking these components, each to a single vertex, and restoring the edges colored h_1 (this can be done unambiguously, by Theorem 4-13), a Cayley color graph of Γ/Ω is obtained. (This "shrinking" may be described by adjoining, to the defining relations for G, the additional relations $h_2 = h_3 = \ldots = h_r = e$.)

In general, given a Cayley color graph $C_\Delta(\Gamma)$, whether a subgroup Ω of Γ is normal or not, we obtain a Schreier (right) coset graph as follows: the vertices are the right cosets of Ω in Γ, and there is an edge directed from Ωg to $\Omega g'$, labeled with $\delta \in \Delta$, if and only if $\Omega g\delta = \Omega g'$ (i.e. if and only if $\delta \in g^{-1}\Omega g'$.) That Ωgd is a right coset follows from the fact that the right cosets of Ω in Γ partition Γ. Note that a Schreier coset graph may actually be a pseudograph, as loops and/or multiple edges may result from this process. For the special case $\Omega = \{e\}$, the Schreier coset graph is just the Cayley color graph $C_\Delta(\Gamma)$.

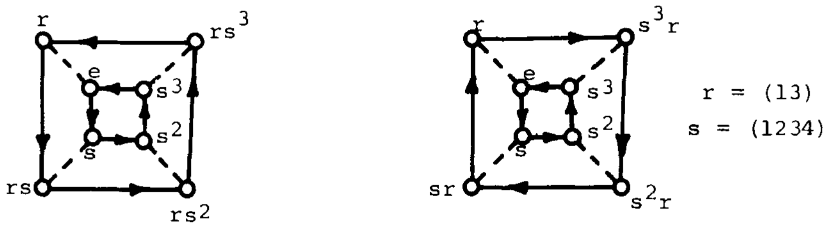
Several of the ideas discussed above are illustrated in Figure 4-5. Note that $\Omega = Z_3$ is normal in S_3, but not in A_4.

As a further example, contrast the groups $Z_2 \times Z_4$ and D_4, as in Figure 4-6. Note that the subgroup of order 2 generated by r is normal in $Z_2 \times Z_4$, but not in D_4. This comment extends in an obvious way to the groups $Z_2 \times Z_n$ and D_n, $n \geq 3$. For example, see $S_3 = D_3$ (in Figure 4-5); the subgroup generated by r is not normal here, either. For a generator δ of order 2, we adopt the standard convention of representing the two directed edges $(g, g\delta)$ and $(g\delta, g)$ in $C_\Delta(\Gamma)$ by a single undirected edge $[g, g\delta]$.



S_3: $r^2 = s^3 = (rs)^2 = e$ $\quad S_3/Z_3 \quad$ A_4: $r^2 = s^3 = (rs)^3 = e$

Figure 4-5.



$Z_2 \times Z_4$: $r^2 = s^4 = rsrs^{-1} = e$ $\quad$ D_4: $r^2 = s^4 = (rs)^2 = e$

Figure 4-6

In Figure 4-7 we give the Schreier coset graph for $\Omega = \{e,r\}$, in $\Gamma = D_4$ (see Figure 4-6).

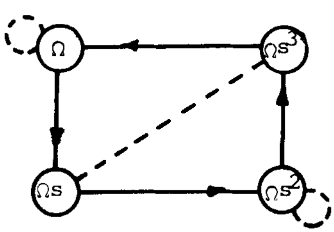


Figure 4-7.

We close this section with a theorem due to Gross [G4].

<u>Thm. 4-14</u>. Every connected regular graph of even degree underlies a Schreier coset graph.

4-4. Products.

We now develop a relationship between the direct product for groups and the cartesian product for graphs. Recall the following from group theory:

<u>Def. 4-15</u>. Let Γ_1 and Γ_2 both be subgroups of the same group Γ, with $\Gamma_1 \cap \Gamma_2 = \{e\}$ and $gh = hg$ for all $g \in \Gamma_1$, $h \in \Gamma_2$. Then $\Gamma_1 \times \Gamma_2 = \{gh \mid g \in \Gamma_1 , h \in \Gamma\}$ is also a subgroup of Γ, called the <u>direct product</u> of Γ_1 and Γ_2.

If $\Gamma_1 = \langle k_1 , \dots , k_m | w_1 = \cdots = w_r = e \rangle$ and

$$\Gamma_2 = \langle k_{m+1} , \dots , k_n | w_{r+1} = \cdots = w_{r+s} = e \rangle ,$$

then $\Gamma_1 \times \Gamma_2 = \langle k_1 , \dots , k_n | w_1 = \cdots = w_{r+s} = k_i k_j k_i^{-1} k_j^{-1} = e ,$

$$\text{for all } 1 \leq i \leq m < j \leq n\rangle$$

is a presentation for $\Gamma_1 \times \Gamma_2$, called the <u>standard presentation</u> for $\Gamma_1 \times \Gamma_2$.

This binary operation may be extended to the class of all groups, by noting that $\Gamma_1 \cong \Gamma_1' = \{(g , e_2) | g \in \Gamma_1 , e_2$ is the identity of $\Gamma_2\}$, $\Gamma_2 \cong \Gamma_2' = \{(e_1 , h) | h \in \Gamma_2 , e_1$ is the identity of $\Gamma_1\}$, and defining $\Gamma_1 \times \Gamma_2 = \{(g , h) | g \in \Gamma_1 , h \in \Gamma_2\}$, with $(g_1 , h_1)(g_2 , h_2) = (g_1 g_2 , h_1 h_2)$ giving the group operation.

Also recall the following (see, for example [BM1, p. 348]):

<u>Thm. 4-16</u>. (The Fundamental Theorem of Finite Abelian Groups): Let Γ be a finite abelian group of order n; then $\Gamma = Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r}$, where m_i divides m_{i-1} $i = 2 , \dots , r$ and $\prod_{i=1}^{r} m_i = n$; furthermore, this decomposition is unique.

(We assume $m_r > 1$, unless $n = 1$, in which case $m_r = r = 1$.)

<u>Def. 4-17</u>. The number r of Theorem 4-15 is called the <u>rank</u> of the abelian group Γ.

Theorem 4-16 completely specifies the structure of finite abelian groups. The next theorem specifies, as a corollary, a Cayley color graph for every finite abelian group. We first extend the definition of cartesian product for graphs to Cayley color graphs, in the natural way.

<u>Def. 4-18</u>. The <u>cartesian product</u>, $C_{\Delta_1}(\Gamma_1) \times C_{\Delta_2}(\Gamma_2)$, <u>of two Cayley color graphs</u> is given by: $V(C_{\Delta_1}(\Gamma_1) \times C_{\Delta_2}(\Gamma_2)) = V(C_{\Delta_1}(\Gamma_1)) \times V(C_{\Delta_2}(\Gamma_2))$; and (g_1, g_2) is joined to (g_1', g_2') by an edge colored h if and only if either:

(i) $g_1 = g_1'$ and $g_2 h = g_2'$, for h a generator in Δ_2

or

(ii) $g_2 = g_2'$ and $g_1 h = g_1'$, for h a generator in Δ_1.

Figure 4-8 shows $C_{\Delta_1}(Z_3) \times C_{\Delta_2}(Z_2)$, where $Z_3 = \langle x | x^3 = e \rangle$ and $Z_2 = \langle y | y^2 = e \rangle$.

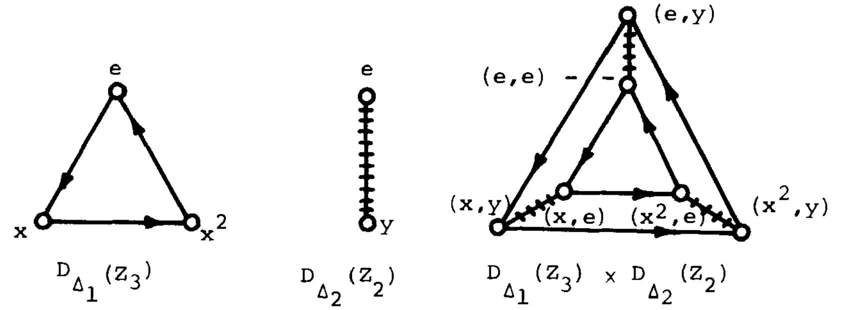


Figure 4-8

<u>Thm. 4-19.</u> Let $C_{P_i}(\Gamma_i)$ be the Cayley color graph associated with presentation P_i for group Γ_i, $i = 1, 2$,. Let P be the standard presenation for $\Gamma_1 \times \Gamma_2$. Then

$$C_P(\Gamma_1 \times \Gamma_2) = C_{P_1}(\Gamma_1) \times C_{P_2}(\Gamma_2).$$

<u>Proof</u>: First we note that $V(C_{P_1}(\Gamma_1) \times C_{P_2}(\Gamma_2)) = V(C_{P_1}(\Gamma_1)) \times V(C_{P_2}(\Gamma_2)) = V(C_P(\Gamma_1 \times \Gamma_2))$. We now show that the edge sets of the two Cayley color graphs coincide (in <u>colored</u> <u>directed</u> <u>adjacency</u>.)

(i) Let (g_1, g_2) be joined to (g_1', g_2') by an edge colored h in $C_P(\Gamma_1 \times \Gamma_2)$. Then $h = k_i$, for some $1 \le i \le n$. If $1 \le i \le m$, then h is a generator of Γ_1, and

$$(g_1', g_2') = (g_1, g_2)(h, e_2) = (g_1 h, g_2),$$

so that $g_1' = g_1 h$ and $g_2' = g_2$; i.e. this directed, colored edge in $D_P(\Gamma_1 \times \Gamma_2)$ is also in $C_{P_1}(\Gamma_1) \times C_{P_2}(\Gamma_2)$. A similar argument applies for $m < i \le n$, so that

$$E(C_P(\Gamma_1 \times \Gamma_2)) \subseteq E(C_{P_1}(\Gamma_1) \times C_{P_2}(\Gamma_2)).$$

(ii) The argument is reversible, to show that

$$E(C_{P_1}(\Gamma_1) \times C_{P_2}(\Gamma_2)) \subseteq C_P(\Gamma_1 \times \Gamma_2).$$

This completes the proof. #

Since the cyclic group Z_n with presentation $P: Z_n = \langle x | x^n = e \rangle$, has the readily constructed Cayley color graph $C_P(Z_n) = C_n'$ (where C_n' denotes the directed cycle of length n), it is a simple matter to construct, using Theorems 4-16 and 4-17, a Cayley color graph for <u>any</u> finite abelian group.

Thm. 4-20. Let Γ be a finite abelian group; then $C'_{m_1} \times C'_{m_2} \times \cdots \times C'_{m_r}$ is a Cayley color graph for Γ, where $\Gamma = Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r}$.

The class of groups for which we can construct Cayley color graphs -- using Theorem 4-19 -- can be enlarged as follows:

Def. 4-21. A non-abelian group Γ is said to be hamiltonian if every subgroup of Γ is normal in Γ.

Clearly all abelian groups have this normality property for subgroups. That non-abelian groups may also have all subgroups normal is illustrated by Q, the quaternions (one of the two non-abelian groups of order eight). But more the finite hamiltonian groups are characterized (see Coxeter and Moser [CM1], p. 8):

Thm. 4-22. Γ is a finite hamiltonian group if and only if $\Gamma = Q \times A_1 \times A_2$, where A_1 is a finite abelian group of odd order, and A_2 is a group for which $a^2 = e$, for every $a \in A_2$.

Since elementary group theory shows that A_2 must be abelian, we can apply Theorem 4-19 to find a Cayley color graph for Γ, providing we know a Cayley color graph for Q. This latter Cayley color graph will be produced in Chapter 7; it turns out to be the Cayley color graph of minimum order which cannot be drawn properly in the plane (among those on minimal generating sets.)

4-5. Cayley Graphs

Let Δ be a generating set for the group Γ subject to the following conditions:

(i) $e \notin \Delta$

(ii) If $\delta \in \Delta$, $\delta^{-1} \notin \Delta$ (unless $\delta^2 = e$).

Also, we adopt the following convention:

(iii) If $\delta \in \Delta$, $\delta^2 = e$, each pair $(g, g\delta)$ and $(g\delta, g)$ of directed edges are coalesced into a single undirected edge $[g, g\delta]$.

Then the pseudograph obtained from the Cayley color graph $C_\Delta(\Gamma)$ by suppressing all edge directions and all edge labels (colors) has no loops (by (i)) and no multiple edges (by (ii) and (iii)); it is in fact a graph.

<u>Def. 4-23.</u> If Δ satisfies (i), (ii), and (iii) above, then the graph underlying the Cayley color graph $C_\Delta(\Gamma)$ is called a <u>Cayley graph</u> and is denoted by $G_\Delta(\Gamma)$.

It is clear that, in passing from $C_\Delta(\Gamma)$ to $G_\Delta(\Gamma)$, only structural properties are lost. Thus in the topological considerations to follow in the rest of the book, it will be without loss that we restrict our attention to $G_\Delta(\Gamma)$.

4-6. Problems

4-1.) Show that θ is an automorphism of $C_\Delta(\Gamma)$ if and only if: for each g in Γ and h in Δ, $\theta(gh) = \theta(g)h$.

4-2.) Prove Theorem 4-9.

4-3.) Prove Theorem 4-11. Give an example of an infinite group with a redundant generator whose deletion does not leave a strongly connected digraph.

4-4.) Prove Theorem 4-12.

4-5.) How many isomorphic disjoint subgraphs are there, as in the statement of Theorem 4-12?

4-6.) Give a graph-theoretic proof of the fact that a subgroup of index 2 must be normal.

4-7.) Let Γ be an abelian group of order pq, where p and q are distinct primes. By one of the Sylow theorems, Γ has Z_p as a subgroup. Give a graph-theoretic proof that Z_p is normal, first finding $C_p(Z_p \times Z_q)$. Then modify this Cayley color graph, to obtain a Cayley color graph of Γ/Z_p.

4-8.) Compile an appendix of Cayley color graphs for all groups of order ≤ 12. (This appendix should be useful for reference, both in this course and in later life. You might want to save some work by doing Z_n, $Z_2 \times Z_n$, and D_n in general, rather than in each case where appropriate. Also, you will find that some of your graphs cannot be properly represented in the plane; these should be re-drawn, following Chapter 7.)

4-9.) Show that, if Γ is finite, then $C_\Delta(\Gamma)$ is always strongly connected. Give an example to show that this need not be true, if Γ is infinite.

4-10.) Show that the Petersen graph (see Figure 8-9) cannot be colored and labeled so as to be a Cayley color graph.

4-11.) Find Δ so that $C_3 \times C_3 = G_\Delta(Z_3 \times Z_3)$. Show that there is no Δ such that $C_3 \times C_3 = G_\Delta(Z_9)$.

4-12.) A cut-vertex (bridge) for a connected graph G is a vertex $v \in V(G)$ (edge $e \in E(G)$) such that $G - v(G - e)$ is disconnected. Show that the Cayley graph $G_\Delta(\Gamma)$ has no cut-vertices, and hence no bridges, for $|\Gamma| \geq 3$.

4-13.) An n-factor of a graph G is a spanning n-regular subgraph of G. Petersen's Theorem is: a bridgeless cubic graph is the edge disjoint union of a 1-factor and a 2-factor. Illustrate this theorem, for Cayley graphs $G_\Delta(\Gamma)$, where Δ consists of two generators, exactly one of which has order two.

4-14.) Show that every Cayley graph $G_\Delta(\Gamma)$ can be expressed as an edge disjoint union of m 1-factors and n 2-factors, For some m and n such that $m + n = |\Delta|$.

4-15.) A graph G is n-factorable if it can be expressed as an edge disjoint union of n-factors (cf Problem 4-13). Find a non-trivial sufficient condition for a Cayley graph $G_\Delta(\Gamma)$ to be:

(i) 1-factorable
(ii) 2-factorable
(iii) 3-factorable
(iv) eulerian

4-16.) Show that the n-cube Q_n is m-factorable if and only if m divides n. (In particular, Q_n is 1-factorable, for all n.)

**4-17.) Babai [B2] conjectured that, for $|\Gamma| \geq 3$, $G_\Delta(\Gamma)$ is always hamiltonian. Prove or disprove!

*4-18.) Show that if Γ is a finite abelian group of order at least three and if Δ is a minimal generating set for Γ, then $G_\Delta(\Gamma)$ is hamiltonian. (Klerlein [K2] showed that, in fact, $C_\Delta(\Gamma)$ is hamiltonian.)

4-19.) Let P denote a property (in adjectival form) that a graph might possess (such as "eulerian" or "hamiltonian",) and let Γ denote a finite group. We say that Γ is P (universally P) if there exists a Δ for P (for all Δ for P) $G_\Delta(\Gamma)$ is P. Discuss:

(i) If Γ is abelian, $|\Gamma| \geq 3$, then Γ is hamiltonian and universally hamiltonian.

(ii) If Γ is hamiltonian (in the sense of Definition 4-21), then Γ is hamiltonian (cf Problem 2-12 and Theorem 4-22) and (**) universally hamiltonian. (Klerlein and Starling [KS1] showed that, in fact, $C_\Delta(\Gamma)$ is hamiltonian, for Γ hamiltonian and Δ minimal.)

(iii) If $|\Gamma| \geq 3$, then Γ is hamiltonian.

(iv) If $|\Gamma|$ is odd, then Γ is universally eulerian.

(v) If Γ is abelian, $|\Gamma| \geq 3$, then Γ is eulerian.

(vi) The abelian group Γ is universally eulerian if and only if $|\Gamma|$ is odd.

(vii) What other properties P might be studied?

*4-20.) Is Theorem 4-13 true for Γ infinite?

4-21.) Let Γ be a finite group, with $\Gamma = \Gamma_1 \cup \Gamma_2$. Prove that if Γ_1 does not generate Γ, then Γ_2 does. (Hint: use Problem 2-1.)

CHAPTER 5

AN INTRODUCTION TO SURFACE TOPOLOGY

In this chapter we present an introduction to surface topology, including the statement and a brief discussion of the classification theorem for closed 2-manifolds and a complete development of the euler formula for the orientable case. One motivation for this material is that it gives us alternatives to the plane for drawing graphs in (for example, no Cayley color graph for the quaternions can be drawn properly in the plane); these alternatives are completely classified, and the euler formula gives us important information about them. We give a topological proof that there are exactly five regular polyhedra, and conclude the chapter with a brief discussion of pseudosurfaces.

5-1. Definitions

In this text, a <u>surface</u> will be a closed, orientable 2-manifold. Any such figure may be considered as a topological subspace of euclidean 3-space, R^3. We consider the subspace topology to be that induced by the standard distance-measuring metric in R^3. To pin down this idea of "surface", we must define the terms used in the first sentence of this paragraph. First, we specify that by the <u>open unit disk</u> we mean $\mathring{D} = \{(x, y) \in R^2 | x^2 + y^2 < 1\}$.

<u>Def. 5-1</u>. A <u>2-manifold</u> is a connected topological space in which every point has a neighborhood homeomorphic to the open unit disk.

<u>Note</u>: In Definition 5-1, $\mathring{D}$ may be replaced by R^2, since these two spaces are themselves homeomorphic.

<u>Example</u>: Only one of the conical spaces (the third) in Figure 5-1 is a 2-manifold.

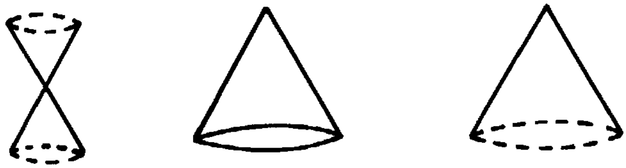


Figure 5-1

Definition 5-1 may be extended as follows: an n-manifold is a connected topological space in which every point has a neighborhood homeomorphic to $B_n = \{(x_1, x_2, \ldots, x_n) \in R^n \mid \sum_{i=1}^{n} x_i^2 < 1\}$. However, we are only concerned here with the case $n = 2$.

Def. 5-2. A subspace M of R^3 is bounded if there exists a natural number n such that $M \subseteq B(O; n) = \{(x, y, z) \mid x^2 + y^2 + z^2 < n\}$.

Def. 5-3. Let $M \subseteq R^3$ be a 2-manifold. M is said to be closed if it is bounded and the boundary of M coincides with M.

For example, M of Figure 5-2 is closed, while M' and M'' are not.

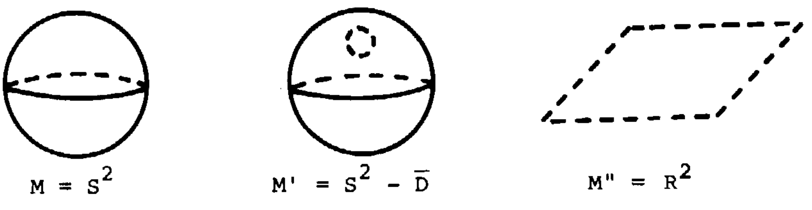


Figure 5-2

Note that the term "closed" does not mean quite the same thing to a surface topologist as it does to a point-set topologist. What a surface topologist calls a "closed 2-manifold", a point-set topologist calls a "compact 2-manifold." (Recall that $M \subseteq R^3$ is compact if and only if M is closed (point-set sense) and bounded.)

Def. 5-4. Let M be a 2-manifold; M is said to be orientable if, for every simple closed curve C on M, a clockwise sense of rotation is preserved by traveling once around C. Otherwise, M is nonorientable.

It can be shown that a 2-manifold M is orientable if and only if it is two-sided. For example, a cylinder open at both ends is orientable, whereas a Möbius strip (imbedded in R^3 in the usual way) is not.

We offer an equivalent defintion of orientability.

Def. 5-4'. M is orientable if it admits a 2-cell decomposition with coherent orientation (i.e. the boundary of each 2-cell is given an orientation so that a 1-cell portion of the boundary incident with two adjacent 2-cells is oppositely oriented within those two 2-cells.)

5-2. Surfaces and Other 2-manifolds

We finally know what a surface is (abstractly); now, exactly which subspaces of R^3 are surfaces?

Let us begin to answer this question by representing certain familiar 2-manifolds as polygons with appropriate edges identified. See Figure 5-3 for the sphere, open cylinder, torus, projective plane, möbius strip, and klein bottle, respectively. The top three 2-manifolds are orientable, the bottom three non-orientable. Only the cylinder and möbius strip are not closed.

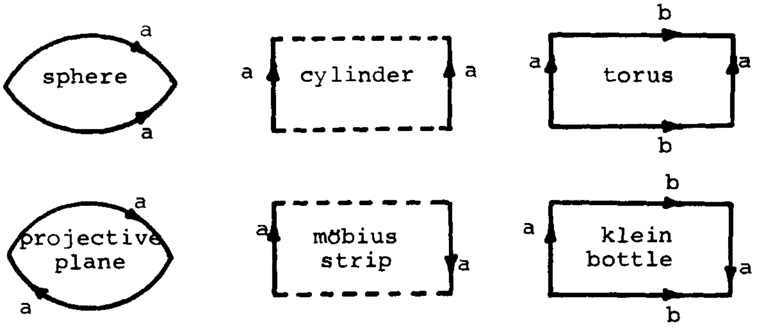


Figure 5-3

It turns out that every closed 2-manifold (whether orientable or not) can be represented in this manner. In fact (see Fréchet and Fan, [FF1] p. 63) we have the following theorem:

Thm. 5-5. Every closed 2-manifold is elementarily associated with a polygon whose symbolic representation is of one of the following forms:

(i) aa^{-1}

(ii) $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\ldots a_pb_pa_p^{-1}b_p^{-1}$, $p = 1,2,3,\ldots$

(iii) $a_1a_1a_2a_2\ldots a_qa_q$, $q = 1,2,3,$

The form (i) corresponds to the sphere; (ii) to the sphere with p handles (a torus is a sphere with one handle) and (iii) to the sphere with q cross-caps (a projective plane is a sphere with one cross-cap; a klein bottle is a sphere with two cross-caps.) Only the forms (iii) correspond to non-orientable closed 2-manifolds. (None of these can be realized in R^3.)

As a byproduct of the development in Fréchet and Fan, an invariant called the characteristic is determined for each closed 2-manifold. Then it is shown that:

Thm. 5-6. (The Classification Theorem) Two closed 2-manifolds are homeomorphic if and only if they have the same characteristic and are both orientable or both non-orientable.

It follows that a closed orientable 2-manifold (i.e. a surface) M is a sphere with k handles, where k is a non-negative integer; k is said to be the genus of M, and we write $\gamma(M) = k$ and $M = S_k$. A closed nonorientable 2-manifold M is a sphere with k crosscaps, where k is a positive integer; k is said to be the (nonorientable) genus of M, and we write $\tilde{\gamma}(M) = k$ and $M = N_k$.

5-3. The Characteristic of a Surface

We now give an independent determination of the characteristic of a surface, using the notion of a pseudograph. The proof will be by induction on k, the genus of the surface. We first need a few definitions, and one preliminary theorem. The first definition is intuitive; it will be made more precise in Chapter 6.

Def. 5-7. A pseudograph is said to be imbedded in a surface M if it is "drawn" in M so that edges intersect only at their common vertices.

For example, Figure 5-4 shows two drawings of K_4 in the plane, but only the second is an imbedding.

Figure 5-4

Def. 5-8. A tree is a connected graph having no cycles.

For example, Figure 5-5 shows three graphs, but only G_3 is a tree.

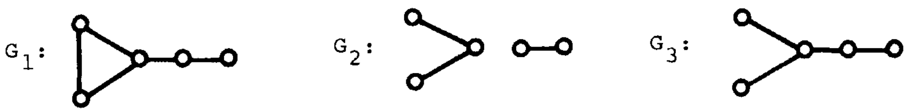


Figure 5-5

Thm. 5-9. Let G be a tree, with p vertices and q edges; then $p = q + 1$.

Proof (by induction on p): The result is clearly true for $p = 1$, for then $q = 0$. Now assume the result holds for all trees with fewer than p vertices, and let G be a tree with p vertices and q edges. Since G has no cycles and is finite, we can find $v \in V(G)$ such that $d(v) = 1$. Then $G - v$ is a tree with $p - 1$ vertices and $q - 1$ edges, so that $(p-1) = (q-1) + 1$; i.e. $p = q + 1$. #

Def. 5-10. Let a pseudograph G be imbedded in a surface M; the components of $M - G$ are called regions (or faces) of the imbedding.

For example, the imbedding of K_4 in Figure 5-4 has four regions.

The following theorem is attributed to both Descartes and Euler, independently; we perhaps indicate our preference by calling it the euler polyhedral identity:

Thm. 5-11. Let G be a connected graph imbedded in the sphere, S_0. Let G have p vertices and q edges, with r the number of regions of the imbedding. Then $p - q + r = 2$.

Proof (by induction on q): The result is clearly true for $q = 1$, for then $p = 2$ and $r = 1$. Now assume the result holds for all connected graphs with fewer than q edges, and let G be a connected graph with q edges, p vertices, and r regions for an imbedding in S_0. We have two cases to consider:

(i) If G is a tree, then $p = q + 1$ by Theorem 5-9, and $r = 1$ (since there are no cycles), so that $p - q + r = 2$.

(ii) If G is not a tree, then (since G is connected) G contains a cycle; let x be any edge of this cycle. Then $G - x$ has p vertices, $q - 1$ edges, is still connected, and is imbedded in S_0 with $r - 1$ regions. Hence $p - (q-1) + (r-1) = 2$; i.e. $p - q + r = 2$.

Cor. 5-12. Let G be a connected pseudograph imbedded in S_0, with p vertices, q edges, and r regions; then $p - q + r = 2$.

Proof See Problem 5-3.

We observe here that imbedding a graph in the sphere is equivalent to imbedding it in the plane. To see this, perform a <u>stereographic projection</u> (see Figure 5-6) with the north pole of the sphere any point in the interior of some region of the imbedding. For each point of the sphere, there corresponds a unique point of the plane: the intersection of the line L through $(0,0,2)$ and (x,y,z) with the plane. The mapping is given explicitly by $f: S^2 - P \to R^2$, where

$$S^2 = \{(x,y,z) \in R^3 | x^2 + y^2 + (z-1)^2 = 1\},$$

$$P = (0,0,2),$$

$$R^2 = \{(x,y,z) \in R^3 | z = 0\},$$

and

$$f(x,y,z) = (x',y',0),$$

with

$$x' = \frac{2x}{2-z},$$

$$y' = \frac{2y}{2-z} \quad \text{(see Problem 5-4)}.$$

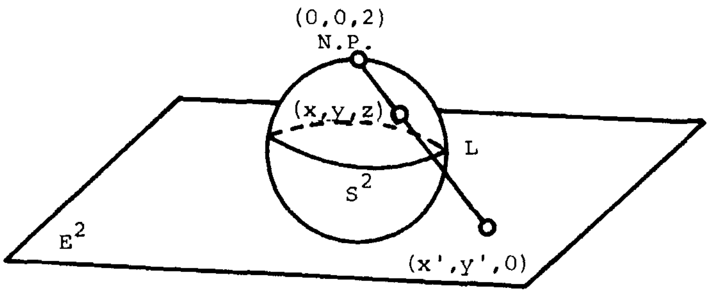


Figure 5-6

The image of the graph G from S^2 is an imbedding of G in R^2, with the unbounded region corresponding to the region in S^2 from the interior of which the north pole was selected. Clearly, this process is reversible. In fact, the map f gives a homeomorphism between $S^2 - P$ and R^2, where P is any point of S^2. Note that neither space is closed from the point of view of surface topology, yet R^2 (and not $S^2 - P$) is closed -- in the point-set sense -- as a subspace of R^3.

Def. 5-13. A region of an imbedding of a graph G in a surface M is said to be a 2-cell if it is homeomorphic to the open unit disk. If every region for an imbedding is a 2-cell, the imbedding is said to be a 2-cell imbedding.

Thm. 5-14. Let G be a connected pseudograph, with a 2-cell imbedding in S_k, with the usual paramters p, q, and r. Then

$$p - q + r = 2 - 2k.$$

Proof (by induction on k); The case $k = 0$ has been settled by Corollary 5-12. Now assume the theorem is true for fewer than k handles ($k \geq 1$), and let G be as in the statement of the theorem. Without loss of generality, we assume all the vertices of G to be on the "sphere" portion of S_k; and since the imbedding is 2-cell, each handle has at least one edge of G running over it. Select one handle, and draw two disjoint simple closed curves C_1 and C_2 around this handle. Suppose edges $x_1, x_2, \ldots, x_n$ run over the handle, where $n \geq 1$. Then C_i meets x_j in a point of S_k which we designate by u_{ij}, $i = 1, 2$; $j = 1, 2, \ldots, n$. Consider the points u_{ij} to be vertices of a new pseudograph, with edges determined in the natural manner. Now remove the portion of the handle between C_1 and C_2 and "fill in" the two resulting holes (bounded by C_1 and C_2 respectively) with two disks (this is called a capping operation). The result is a 2-cell imbedding of a connected pseudograph in S_{k-1}, with parameters p', q', and r' (say). But

$$\begin{aligned} p' &= p + 2n \\ q' &= q + 3n \\ r' &= r + n + 2. \end{aligned}$$

Thus, by the inductive assumption,

$$\begin{aligned} 2 - 2(k-1) &= p' - q' + r' \\ &= (p+2n) - (q+3n) + (r+n+2) \\ &= p - q + r + 2; \end{aligned}$$

that is, $p - q + r = 2 - 2k$. #

Cor. 5-15. Let G be a connected graph, with a 2-cell imbedding in S_k, with the parameters p, q, and r; then $p - q + r = 2 - 2k$.

Proof: The result is immediate, since any graph is also a pseudograph. #

We have shown that the number $p - q + r$ is invariant for S_k, for any 2-cell imbedding of any connected pseudograph; $p - q + r = 2 - 2k$, depending only on k. This invariant number, $2 - 2k$, is called the euler characteristic for the surface S_k. It then follows that S_n and S_m are homeomorphic if and only if $m = n$. In the nonorientable case, the characteristic is given by $p - q + r = 2 - k$, where k is the number of cross-caps (see Fréchet and Fan.) In notation, $\chi(S_k) = 2 - 2k$; $\chi(N_k) = 2 - k$.

5-4. Two Applications

The ramifications of Theorem 5-14 are enormous. In the remainder of this section, we give only two of these, both pertaining to the case $k = 0$.

Def. 5-16. A graph is said to be planar if it can be imbedded in the plane (or, equivalently, in S_0). A graph imbedded in S_0 is called a plane graph.

Notation: Suppose a graph G is 2-cell imbedded in a surface S_k. Let v_i be the number of vertices of degree i, and let r_i designate the number of regions having i sides (i.e. the number of regions having as boundary a closed walk of length i; i is also called the length of the region). We assume that $v_0 = v_1 = v_2 = 0$, as we focus on polyhedral graphs in this section; moreover, since G is a graph, $r_0 = r_1 = r_2 = 0$.

Lemma 5-17.

(i) $p = \sum_{i \geq 3} v_i$

(ii) $r = \sum_{i \geq 3} r_i$

(iii) $2q = \sum_{i \geq 3} iv_i$

(iv) $2q = \sum_{i \geq 3} ir_i$

Proofs: (i) and (ii) are obvious; (iii) is Theorem 2-2, and (iv) follows in like manner to (iii); in summing the number of sides in the regions, each edge is counted exactly twice. #

Thm. 5-18. The graph K_5 is not planar.

Proof: Suppose that the connected graph K_5 were imbedded in the plane; then $2q = 20 = \sum_{i \geq 3} ir_i \geq 3 \sum_{i \geq r} r_i$ $= 3r$, and by Theorem 5-11.

$$2 = p - q + r$$
$$\leq 5 - 10 + \frac{20}{3} = \frac{5}{3},$$

a contradiction! Hence, K_5 is not planar. #

Lemma 5-19. Let the planar connected graph $G(\delta(G) \geq 3)$ be imbedded in the plane; then

(i) G has a vertex of degree 5 or less; and
(ii) G has a region with 5 or less sides.

Proof: (i) Suppose, to the contrary, that $v_i = 0$, $i = 0, 1, 2, 3, 4, 5$; then

$2q = \sum_{i \geq 3} iv_i \geq 6 \sum_{i \geq 3} v_i = 6p$. As before, $2q =$

$\sum_{i \geq 3} ir_i \geq 3 \sum_{i \geq 3} r_i = 3r$. Then, by Theorem 5-11,
$p - q + r = 2$; i.e.

$$\begin{aligned} q &= p + r - 2 \\ &\leq q/3 + 2q/3 - 2 = q - 2, \end{aligned}$$

a contradiction.

(ii) This follows by duality (soon to be explained); it also follows from Problem 5-6. #

We are now prepared to give a <u>topological</u> proof of what the Greeks knew, geometrically, over two thousand years ago: there are exactly five regular polyhedra. A <u>polyhedron</u> is a finite, connected collection of at least four polygons, fit together in R^3 so that: (i) each side of each polygon coincides exactly with one side of one other polygon, and (ii) around each vertex there is one circuit of polygons; together with the region of R^3 bounded by these polygons. These two conditions rule out the anomalies depicted in Figure 5-7.

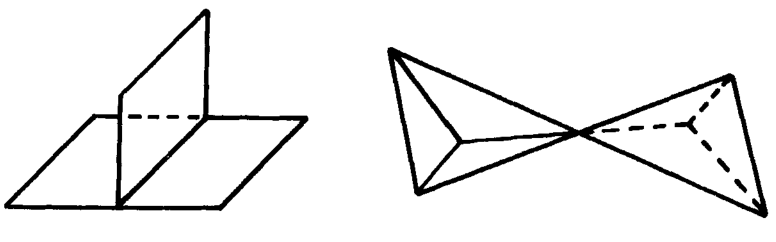


Figure 5-7

A <u>regular</u> <u>polyhedron</u> is a convex polyhedron for which: (i) the polygons are congruent regular polygons, and (ii) the same number of polygons surround each vertex.

<u>Thm. 5-20</u>. There are exactly five regular polyhedra.

<u>Proof</u>: Let P be a regular polyhedron. Associated with p is a regular planar graph G (to picture this, first bound P with a sphere, then place a light source inside the polyhedron -- the shadow of the vertices and edges of P gives a graph imbedded in the sphere; finally, perform a stereographic projection.) This planar graph G has $v_0 = v_1 = v_2 = 0$, and in fact: $p = v_k$, $r = r_h$, for $k, h \quad \{3,4,5\}$, by Lemma 5-19. Next by Theorem 5-11, $p - q + r = 2$; we re-write this as follows:

$$\begin{aligned} 8 &= 4p + 4r - 2q - 2q \\ &= \sum_{i \geq 3} (4-i)(r_i + v_i) \\ &= (4-h)r_h + (4-k)v_k. \end{aligned}$$

But also, $hr_h = kv_k$, since both $= 2q$, by Lemma 5-17. Of the nine possibilities for (h, k) in positive integers, only the following satisfy both of the above equations in r_h and v_k: $(h, k) =$

(i) $(3, 3)$; $r_3 = v_3 = 4$ (the <u>tetrahedron</u>)

(ii) $(3, 4)$; $r_3 = 8$, $v_4 = 6$ (the <u>octahedron</u>)

(iii) $(3, 5)$; $r_3 = 20$, $v_5 = 12$ (the <u>icosahedron</u>)

(iv) $(4, 3)$; $r_4 = 6$, $v_3 = 8$ (the <u>hexahedron</u>; i.e. the cube)

(v) $(5, 3)$; $r_5 = 12$, $v_3 = 20$ (the <u>dodecahedron</u>)

This complete the proof. #

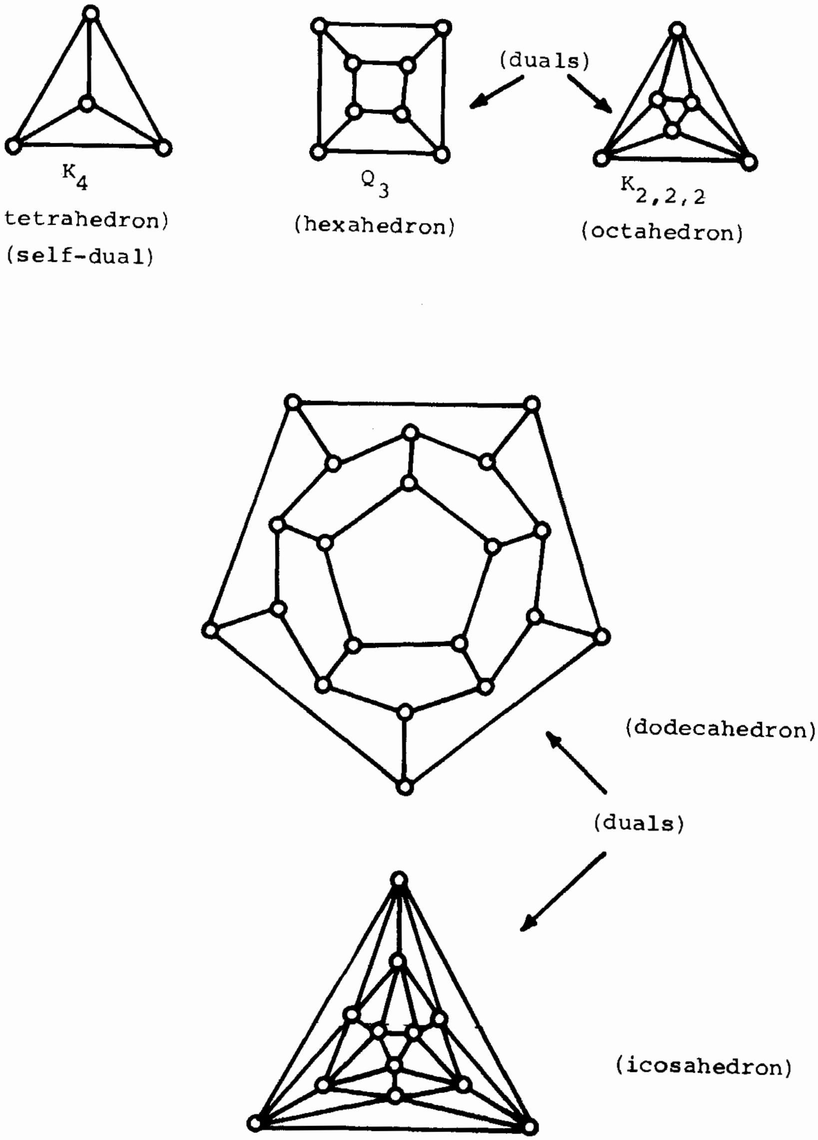

(duals)
K_4
(tetrahedron)
(self-dual)
Q_3
(hexahedron)
$K_{2,2,2}$
(octahedron)
(dodecahedron)
(duals)
(icosahedron)

Figure 5-8

The reader may have noticed a certain interchangability between the roles of vertices and regions (compare (ii) and (iv) above, (iii) and (v) above; see Lemmas 5-17 and 5-19, and Theorem 5-14). This is no accident.

Def. 5-21. Let a connected pseudograph G be 2-cell imbedded in S_k. The dual pseudograph of G, $D_I(G)$ (relative to this imbedding I), is given by: the vertices of $D_I(G)$ are the regions of G in S_k, and two such vertices are adjacent if and only if their corresponding regions share a common edge in their boundaries. (Each edge of G is associated with exactly one edge of $D_I(G)$, which therefore may have loops and multiple edges.)

For example, Figure 5-8 not only gives the regular polyhedral graphs, but also indicates duality relationships. Figure 5-9 shows, for instance, that the tetrahedron is self-dual. Thus, having established Lemma 5-19 (i), we establish part (ii) by applying (i) to the dual. Similarily, having found the hexahedron, we discover the octahedron as its dual; and so forth.

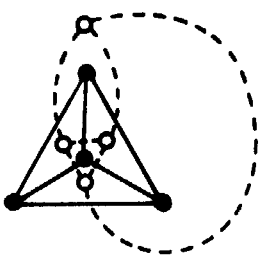


Figure 5-9

Although there are only five regular polyhedra (also called the Platonic Solids), there are infinitely many convex polyhedra, as the classes of all prisms and antiprisms show. The thirteen Archimedean Solids are also all convex polyhedra. There are many non-convex polyhedra, some of which are uniform with regard to face structure and vertices, which have planar graphs as 1-skeletons; see, for example, [W4]. For a splendid study of orientable polyhedra (of possibly positive genus) with regular faces, consult B. M. Stewart's Adventures Among the Toroids [S16].

Def. 5-22. A graph G is said to be 3-polytopal if it is the 1-skeleton (the graph induced by the vertices and edges) of a convex polyhedron.

Def. 5-23. A graph G is said to be n-connected $(n \geq 1)$ if the removal of fewer than n vertices from G neither disconnects G nor reduces G to the trivial graph K_1.

Graphs which are 3-polytopal have been characterized by Steinitz [S15].

Thm 5-24. A graph G is 3-polytopal if and only if it is planar and 3-connected.

One readily verifies that the five planar graphs of Figure 5-8 are also 3-connected. The following theorem of Weinberg [W3] gives information about the automorphism groups of 3-polytopal graphs:

Thm 5-25. Let G be 3-polytopal, with q edges. Then $|\mathcal{G}(G)| \leq 4q$, with equality holding if and only if G is the 1-skeleton of a Platonic Solid.

In the next chapter we consider imbedding graphs (and graphs of groups!) in surfaces of positive genus.

5-5. Pseudosurfaces

We now consider topological spaces akin to surfaces, but which fail to be 2-manifolds at a finite number of points; these spaces form additional candidates for the imbedding of graphs, and were studied extensively by Petroelje [P4].

__Def. 5-26.__ Let A denote a set of $\sum_{i=1}^{t} n_i m_i \geq 0$ distinct points of S_k, with $1 < m_1 < m_2 < \cdots < m_t$. Partition A into n_i sets of m_i points each, $i = 1, 2, \ldots, t$. For each set of the partition, identify all the points of that set. The resulting topological space is called a __pseudosurface__, and is designated by $S(k; n_1(m_1), n_2(m_2), \ldots, n_t(m_t))$. Each point resulting from an identification of m_i points of S_k is called a __singular point__. If a graph G is imbedded in a pseudosurface, we assume that each singular point is occupied by a vertex of G; such a vertex is called a __singular vertex__. A __generalized pseudosurface__ results when finitely many identifications, of finitely many points each, are made on a topological space of finitely many components, each of which is a pseudosurface, with a connected topological space resulting.

__Thm. 5-27.__ Let G be a graph having a 2-cell imbedding in $S(k; n_1(m_1), n_2(m_2), \ldots, n_t(m_t))$; then $p - q + r = 2 - 2k - \sum_{i=1}^{t} n_i(m_i - 1)$.

The number $2 - 2k - \sum_{i=1}^{t} n_i(m_i - 1)$ is said to be the __characteristic for the pseudosurface__, and is a topological invariant, just as $2 - 2k$ is for the surface S_k.

5-6. Problems

5-1.) A __forest__ is a graph for which every component is a tree. Show that, if G is a forest with p vertices, q edges, and k components, then $p = q + f(k)$, where $f(k)$ must be determined.

5-2.) Let G, a graph with p vertices, q edges, and k components, be imbedded in the sphere, with r regions. Show that $p - q + r = g(k)$, where $g(k)$ must be determined. Illustrate, for $G = 2K_4$. For what value of k will the imbedding be 2-cell?

5-3.) Prove Corollary 5-12.

5-4.) Verify that $f(x, y, z) = (2x/(2-z), 2y/(2-z), 0)$ gives the stereographic projection.

5-5.) Show that $K_{3,3}$ is not planar.

5-6.) Prove Lemma 5-19 (ii), without using duality.

5-7.) Where might the proof of Theorem 5-14 break down, for graphs (instead of pseudographs)?

5-8.) Consider the 2-manifolds in Figure 5-3. Determine in which of these K_5 can be imbedded. For each 2-manifold, compute the characteristic. Note that the characteristics agree for the torus and the klein bottle; are these two homeomorphic? Why? The characteristics also agree for the möbius strip and the projective plane; are they homeomorphic? How about the sphere and the cylinder?

5-9.) Show that the two symbolic representations $ab^{-1}ab$ (as in Figure 5.3) and $a_1a_1a_2a_2$ (as in Theorem 5-5 (iii)) both give the klein bottle. (Hint: cut along an appropriate diagonal of the rectangle $a_1a_1a_2a_2$ and then make an appropriate identification to obtain the rectangle $ab^{-1}ab$.)

5-10.) For G the 1-skeleton of a Platonic Solid, show that $|\mathcal{G}(G)| = 4q$.

5-11.) The <u>wheel</u> graph W_m is defined as the join (see Def. 2-10) $K_1 + C_{m-1}$, $m \geq 4$. Show that $|\mathcal{G}(W_m)| \leq 4q$, with equality holding iff $m = 4$. Is this consistent with Thm. 5-25?

*5-12.) Give an example to show that two pseudosurfaces with the same characteristic can be non-homeomorphic (compare the situation for surfaces). Find a formula that gives, for $n \geq -2$, the number of non-homeomorphic pseudosurfaces with characteristic $-n$.

5-13.) Extend Theorem 5-27 to generalized pseudosurfaces.

5-14.) Into how many regions is the plane R^2 divided, by n lines in general position (i.e. no two lines parallel, no three lines concurrent)? The answer can be conjectured inductively and then proved using mathematical induction, but try to obtain it directly by using the euler identity and stereographic projection.

CHAPTER 6

IMBEDDING PROBLEMS IN GRAPH THEORY

Recall from Definition 2-1 that a graph is an abstract mathematical system. It is when we concern ourselves with the geometric realization of a graph as a finite one-dimensional complex that imbedding problems arise. There are practical applications for this view of graphs. For instance, we will see in Chapter 8 that one of the truly famous problems in mathematics can be stated in terms of imbedded graphs. As another example, imagine the task of printing an electronic circuit on a circuit board. Associated with the circuit (in an obvious manner) is a graph, and the circuit can be printed without shorts if and only if the associated graph can be imbedded in the plane. What to do if the graph is not planar will be studied in this chapter.

What do we mean by "the geometric realization" of a graph? In this section, we will normally mean a configuration in R^3, where the vertices of the graph are represented by distinct points, and the edges of the graph by lines; two lines intersect only at a point representing common end vertices of the two corresponding edges. A natural question is: "In what subspaces of R^3 will a given graph imbed in this manner?" We will confine our attention to the following subspaces:

(i) R^3 itself
(ii) R^2
(iii) n-books (see definition below)
(iv) surfaces
(v) pseudosurfaces.
(vi) generalized pseudosurfaces

<u>Def. 6-1</u>. An <u>n-book</u> is the cartesian product of the unit interval with the geometric realization of the graph $K_{1,n}$.

That is, an n-book consists of n rectangles (the pages) joined along a common edge (the spine).

6-1. Answers to Some Imbedding Questions

The imbedding question has been completely answered for (i), (ii), and (iii), as the next three theorems indicate. We will also need some definitions. In the sequel, the term "graph" will be used interchangeably, to represent either the abstract mathematical system, or a realization of this system in R^3. The context should make it clear which use is intended.

Thm. 6-2. If K is a countable and locally finite simplicial complex, with $\dim K \leq n$, then K has a realization (i.e. a linear imbedding) as a closed subset in R^{2n+1}.

(See Spanier [S6] for a discussion of this theorem.)

Cor. 6-3. Any finite one-complex is imbeddable in R^3.

Note that Corollary 6-3 indicates that any graph may be inbedded in R^3, and in such a way that every edge is represented as a straight line.

Another way to see this is as follows. Let C be the curve in R^3 determined by the parametric equations $x = t$, $y = t^2$, $z = t^3$ $(t \geq 0)$. Select p distinct points along C to represent the vertices of G and represent the q edges of G as straight lines joining these points appropriately. Since no four points on C are coplanar (see Problem 6-2) C meets any plane in R^3 at most three times, and no two edges of G intersect extraneously.

Def. 6-4. An elementary subdivision of an edge uv of a graph is the deletion of edge uv, the addition of a new vertex w, and the addition of two new edges, uw and wv.

Def. 6-5. A graph G is said to be homeomorphic from a graph H if G can be obtained from H by a (finite) sequence of elementary subdivisions. (We say that G is a subdivision of H.) G_1 and G_2 are said to be homeomorphic with each other if they are both homeomorphic from a common graph H.

Note that G_1 is homeomorphic with G_2 in the graph-theoretical sense defined above if and only if the realizations of G_1 and G_2 in R^3 are homeomorphic in the topological sense (see Problem 6-1.)

The next theorem is one of the most important in all of graph theory; it is due to Kuratowski [K4].

Thm. 6-6. A graph G is planar if and only if it contains no subgraph homeomorphic with either K_5 or $K_{3,3}$.

It is clear that any graph with q edges can be imbedded in a q-book: place all the vertices along the spine, and use one page for each edge. However, we can do much better; the following theorem, due to Atneosen [A8], is rather surprising.

Thm. 6-7. Any graph G can be imbedded in a 3-book.

Note that by Theorem 6-6, we have a criterion for ascertaining if the third page is needed for a particular graph. Theorem 6-7 is proved as follows: as shown in Massey [M3], any closed 2-manifold with non-void boundary can be represented as a disk with strips attached in a certain way. Clearly any graph G can be imbedded in a closed 2-manifold with non-void boundary (simply remove an open disk from the interior of some region, for any S_k in which G can be imbedded; take $k = q$, for example). Atneosen showed, very neatly, that any disk with strips attached as described by Massey can be imbedded in a 3-book. (An alternate proof has been given by Babai [B1]: Draw G in the plane so that all intersections lie on a straight line and no three lines have a common intersection (except at common end vertices.) Let this line be the spine of the book, and let the plane be the union of two pages. Then the third page can be readily employed to avoid each intersection.

So, it is only for the subspaces (iv), (v), and (vi) of our list above--that is, the surfaces, pseudosurfaces, and generalized pseudosurfaces-- that the imbedding problem is, in general, unsolved, for non-planar graphs. Clearly, any graph will imbed on S_k, for k large enough (for example, take $k = q$ and use one handle for each edge); but this does not characterize which graphs imbed on S_k, for k fixed. The most natural problem here might be: for a given graph, find the surface of minimum genus in which the graph can be imbedded. If the graph is associated with an electronic circuit, the corresponding problem is: find the fewest number of holes that must be punched in the circuit board so that the board can accommodate the circuit. For Cayley color graphs, the problem becomes: find the simplest locally 2-dimensional "drawing board" in which to "paint" a picture of a given group. We will also see, in Chapter 8, that imbedding certain graphs in appropriate surfaces will tell us a good deal about map-coloring problems.

6-2. Definition of "Imbedding"

Let us now give two very careful definitions of "imbedding" (they are easily seen to be equivalent), and then proceed to study this process in some detail.

<u>Def. 6-8</u>. Let G be a graph, with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{x_1, x_2, \ldots, x_m\}$. Let M be a 2-manifold. An <u>imbedding of G in M</u> is a subspace $G(M)$ of M such that

$$G(M) = \bigcup_{i=1}^{n} v_i(M) \cup \bigcup_{j=1}^{m} x_j(M),$$

where

(i) $v_1(M), \ldots, v_n(M)$ are distinct points of M

(ii) $x_1(M), \ldots, x_m(M)$ are m mutually disjoint open arcs in M

(iii) $x_j(M) \cap v_i(M) = \phi$, $i = 1,\ldots,n$; $j = 1,\ldots,m$.

(vi) if $x_j = (v_{j1}, v_{j2})$, then the open arc $x_j(M)$ has $v_{j1}(M)$ and $v_{j2}(M)$ as end points; $j = 1, \ldots, m$.

In the above definition, an <u>arc</u> in M is a homeomorphic image of $[0,1]$; an open arc is an arc less its two end points, the images of 0 and 1.

Equivalently (and much more briefly) we have:

<u>Def. 6-8'</u>. The graph G can be imbedded in the 2-manifold M if the geometric realization of G as a one-dimensional simplicial complex is homeomorphic to a subspace of M.

6-3. The Genus of a Graph

Imbedding question (iv) in this chapter leads directly to:

<u>Def. 6-9</u>. The <u>genus</u>, $\gamma(G)$, <u>of a graph</u> G is the minimum genus among all surfaces in which G can be imbedded.

For example, if G is planar then we write $\gamma(G) = 0$. If $\gamma(G) = k$, $k > 0$, then G has an imbedding in S_k, but not in S_h, for $h < k$. Moreover, G imbeds in S_m, for all $m \geq k$ (merely add $m - k$ handles to an imbedding of G in S_k).

As mentioned above, it is clear that every graph has a genus. Let G have q edges; then place the vertices of G on the sphere, and add one handle for each edge. Thus $\gamma(G) \leq q$.

<u>Def. 6-10</u>. An imbedding of a graph G in a surface S_k is said to be a <u>minimal imbedding</u> if $\gamma(G) = k$.

The next result is extremely useful, as it tells us that the euler formula applies for any minimal imbedding of a connected graph. For a complete proof, see [Y1].

<u>Thm. 6-11</u>. If a connected graph G is minimally imbedded in a suface, then the imbedding is a 2-cell imbedding.

<u>Heuristic Argument</u>: We assume (without loss of generality) that every vertex of G lies on the sphere. Hence only edges can be imbedded on the handles. Suppose that R is a non- 2-cell region. Then there is a simple closed curve C in R which cannot be continuously deformed, in R, to a point. If $\gamma(G) = 0$ C divides S_0 into two parts (by the Jordan curve theorem), each of which must contain a vertex of G. But then G would be disconnected. Hence $\gamma(G) \geq 1$. We consider three cases:

Case (i). If C lies entirely on one handle, we cut the surface along C, cap the two resulting holes, and obtain an imbedding of G in $S_{\gamma(G)-1}$, a contradiction.

Case (ii). If C lies entirely on the sphere, we regard the "sphere" portion of the surface as a handle, and apply case (i).

Case (iii). If C lies partially on some handle H and partially on $S_{\gamma(G)} - H$, we redraw the edges of G formerly carried by H along that portion of C lying in $S_{\gamma(G)} - H$; we obtain an imbedding of G on the surface without using handle H, the final contradiction. #

The corollary below follows directly from Theorems 5-14 and 6-11.

Cor. 6-12. If a connected graph G has a minimal imbedding in S_k, with p vertices, q edges, and r regions, then

$$p - q + r = 2 - 2k.$$

The next two corollaries are often helpful in computing the genus of a graph. We require two new terms.

Def. 6-13. A 2-cell imbedding is said to be a triangular (quadrilateral) imbedding if $r = r_3 (r = r_4)$.

Cor. 6-14. If G is connected, with $p \geq 3$, then, $\gamma(G) \geq \frac{q}{6} - \frac{p}{2} + 1$. Furthermore, equality holds if and only if a triangular imbedding can be found for G.

Proof: Let G be imbedded in $S_{\gamma(G)}$, so that $p - q + r = 2 - 2\gamma(G)$. Since $2q \geq 3r$, with equality if and only if $r = r_3$ (see Lemma 5-17), the result is immediate. #

Cor. 6-15. If G is connected, with $p \geq 3$, and has no triangles, than $\gamma(G) \geq \frac{q}{4} - \frac{p}{2} + 1$. Furthermore, equality holds if and only if a quadrilateral imbedding can be found for G.

(The proof is entirely analogous to that of Cor. 6-14).#

The next corollary will be heavily used in the remainder of this chapter.

Cor. 6-16. If G is a connected bipartite graph having a quadrilateral imbedding, then $\gamma(G) = \frac{q}{4} - \frac{p}{2} + 1$.

Proof: Apply Theorem 2-19 and Corollary 6-15. #

We have shown (among other things) that, for connected graphs, minimal imbeddings are 2-cell imbeddings. Two questions arise: (i) What about minimal imbeddings of disconnected graphs? (ii) Are there 2-cell imbeddings which are not minimal? We discuss these two questions briefly.

Def. 6-17. Given a connected graph G, a cut-vertex is a vertex v such that $G - v$ is disconnected. A block is a maximal connected subgraph of G having no cut-vertices.

For example, the graph in Figure 6-1 has two blocks, both isomorphic to K_4; v is a cut-vertex for this graph. Note that a block is either K_2 or is 2-connected.

G:

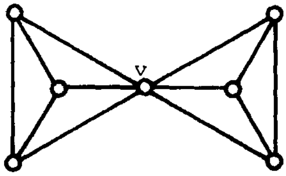


Figure 6-1.

The next theorem and its corollary are due to Battle, Harary, Kodama, and Youngs [BHKY1], and are presented without proof.

Thm. 6-18. The genus of a graph is the sum of the genera of its blocks.

Cor. 6-19. The genus of a graph is the sum of the genera of its components. (i.e., let $G = \bigcup_{i=1}^{n} C_i$; then $\gamma(G) = \sum_{i=1}^{n} \gamma(C_i)$).

6-4. The Maximum Genus of a Graph

That there exist 2-cell imbeddings which are not minimal is evident from Figure 6-2, which shows K_4 in S_1. Note that the euler formula still applies here $(4 - 6 + 2 = 0)$. It is clear that no imbedding of a disconnected graph can be a 2-cell imbedding. To describe all 2-cell imbeddings of a given connected graph, we introduce the following concept:

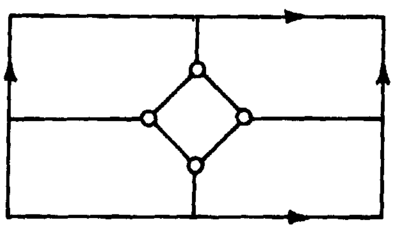


Figure 6-2

Def. 6-20. The maximum genus, $\gamma_M(G)$, of a connected graph G is the maximum genus among the genera of all surfaces in which G has a 2-cell imbedding.

Duke [D4] has shown the following:

Thm. 6-21. If a graph G has 2-cell imbeddings in S_m and S_n, then G has a 2-cell imbedding in S_k, for each k, $m \leq k \leq n$.

Cor. 6-22. A connected graph G has a 2-cell imbedding in S_k if and only if $\gamma(G) \leq k \leq \gamma_M(G)$.

An upper bound for $\gamma_M(G)$ is not difficult to determine.

Def. 6-23. The Betti number $\beta(G)$, of a graph G having p vertices, q edges, and k components, is given by: $\beta(G) = q - p + k$.

$\beta(G)$ is sometimes called the cycle rank of G; it gives the number of independent cycles in a cycle basis for G; see Harary [H2, pp. 37-40].

Recall that $[x]$ denotes the greatest integer less than or equal to x; $\{x\}$ gives the least integer greater than or equal to x. Both symbols will be used frequently in the remainder of this chapter.

Th. 6-24. Let G be connected; then $\gamma_M(G) \leq \left[\frac{\beta(G)}{2}\right]$. Moreover, equality holds if and only if $r = 1$ or 2, according as $\beta(G)$ is even or odd, respectively.

Proof: Let G be connected, with a 2-cell imbedding in S_k; then $r \geq 1$, and $\beta(G) = q - p + 1$; also $p - q + r = 2 - 2k$; thus

$$k = 1 + \frac{q-p-r}{2} \leq \frac{q-p+1}{2} = \frac{\beta(G)}{2},$$

and the result follows.

Nordhaus, Stewart, and White [NSW1] have shown that equality holds in Theorem 6-24 for the complete graph K_n; Ringeisen [R7] has shown that equality holds for the complete bipartite graph $K_{m,n}$; and Zaks [Z1] has shown that equality holds for the n-cube Q_n (if $\gamma_M(G) = [\frac{\beta(G)}{2}]$, G is said to be upper imbeddable.)

Thm. 6-25. $\gamma_M(K_n) = \left[\frac{(n-1)(n-2)}{4}\right]$.

Thm. 6-26. $\gamma_M(K_{m,n}) = \left[\frac{(m-1)(n-1)}{2}\right]$.

Thm. 6-27. $\gamma_M(Q_n) = (n-2)2^{n-2}$, for $n \geq 2$.

Moreover, Kronk, Ringeisen, and White [KRW1] established:

Thm. 6-28. All complete n-partite graphs are upper imbeddable.

Also, Ringeisen [R6] has found $\gamma_M(G)$ for several classes of planar graphs G, including the wheel graphs and the regular polyhedral graphs.

Nordhaus, Ringeisen, Stewart, and White have combined [NRSW1] to establish the following analog to Kuratowski's Theorem (Theorem 6-6): (the graphs H and Q are given in Figure 6-3.)

Thm. 6-29. The connected graph G has maximum genus zero if and only if it has no subgraph homeomorphic with either H or Q. (Furthermore, $\gamma(G) = \gamma_M(G)$ if and only if $\gamma_M(G) = 0$ if and only if G is a cactus with vertex-disjoint cycles.)

Def. 6-30. A cactus is a connected (planar) graph in which every block is a cycle or an edge.

Def. 6-31. A splitting tree of a connected graph G is a spanning tree T for G such that at most one component of $G - E(T)$ has odd size.

The following characterization is due, independently, to Jungerman [J7] and Xuong [X2].

Thm. 6-32. A graph G is upper imbeddable if and only if G has a splitting tree.

Thus, for example, we get an immediate proof of Theorem 6-25 merely by taking $T = K_{1,n-1}$.

Nebesky [N1] has given a sufficient condition for upper imbeddability. First, we need

Def. 6-33. A graph G is said to be locally connected if, for every $v \in V(G)$, the set $N_G(v)$ of vertices adjacent to v is non-empty and the subgraph of G induced by $N_G(v)$ is connected.

Thm. 6-34. If G is connected and locally connected, then G is upper imbeddable.

Although no workable formula is known for the genus of an arbitrary graph, Xuong [X1] developed the following result for maximum genus. Let $\xi_0(H)$ denote the number of components of graph H of odd size, and for G connected set

$$\xi(G) = \min \xi_0 (G - E(T)),$$

the minimum being taken over all spanning trees T of G. Then:

Thm. 6-35. The maximum genus of the connected graph G is given by

$$\gamma_M(G) = \frac{1}{2} (\beta(G) - \xi(G)).$$

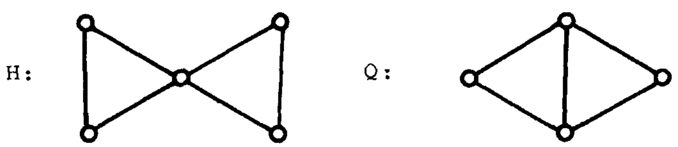


Figure 6-3

6-5. Genus Formulae for Graphs

Prior to the work of Jungerman and Xuong, Theorems 6-25, 6-26, 6-27, and 6-28 and the work of Ringeisen [R6] referred to above gave the only known non-trivial formulas for maximum genus.

Not very many more formulas are known for the genus parameter; we list some of these below. For most of what else was known up to 1978, see Table 1 of Stahl [S9].

<u>Thm. 6-36.</u> (Ringel [R10]; Beineke and Harary [BH1])
$\gamma(Q_n) = 1 + 2^{n-3}(n-4)$, $n \geq 2$.

<u>Thm. 6-37.</u> (Ringel [R11])
$\gamma(K_{m,n}) = \left\{\frac{(m-2)(n-2)}{4}\right\}$; $m,n \geq 2$.

<u>Thm. 6-38.</u> (Ringel and Youngs [RY1])
$\gamma(K_n) = \left\{\frac{(n-3)(n-4)}{12}\right\}$, $n \geq 3$.

<u>Thm. 6-39.</u> (White [W5]; see also [RY5], for the case $m = 1$)
$\gamma(K_{mn,n,n}) = \frac{(mn-2)(n-1)}{2}$.

<u>Th. 6-40.</u> (Stahl and White [SW1])
$\gamma(K_{n,n,n-2}) = \frac{(n-2)^2}{2}$, for n even, $n \geq 2$.

<u>Thm. 6-41.</u> (Stahl and White [SW1])
$\gamma(K_{2n,2n,n}) = \frac{(3n-2)(n-1)}{2}$, $n \geq 1$.

<u>Thm. 6-42.</u> (Jungerman [J5]; see also Garman [G1])
$\gamma(K_{4(m)}) = (m-1)^2$, $m \neq 3$.

<u>Thm. 6-43.</u> (Jungerman and Ringel [JR4]; see also Gross and Alpert [GA1])
$\gamma(K_{n(2)}) = \frac{(n-3)(n-1)}{3}$, for $n \not\equiv 2 \pmod 3$.

<u>Thm. 6-44.</u> (Ringel [R16])
For $2 \leq n \not\equiv 5,9 \pmod{12}$, $\gamma(K_2 \times K_n) = \left\{\frac{(n-2)(n-3)}{6}\right\}$.

<u>Thm. 6-45.</u> (White [W7]) Let G have p vertices of positive degree, q edges, k non-trivial components, and no 3-cycles. Let H have $2n$ ($n \geq 1$) vertices and maximum degree less than two. Then $\gamma(G[H]) = k + n(nq - p)$.

Cor. 6-46. Let G have no 3-cycles. Then $\gamma(G[K_2]) = \gamma(G[\overline{K}_2]) = \beta(G)$.

The special case of Theorem 6-45 when G is bipartite, $p \geq 3$, and $H = \overline{K}_m$ has been generalized to include m odd, by Abu-Sbeih and Parsons [AP1].

For a very readable proof of Theorem 6-36, see Behzad, Chartrand and Lesniak-Foster [BCL1]. Theorems 6-37 and 6-39 are proved using schemes identical or similar to that discussed in the next section. Theorem 6-38 will be discussed at length in Chapter 9. Theorems 6-40 and 6-41 are obtained by means of voltage graph constructions (see Chapter 10). Theorems 6-42 and 6-44 are established using current graph constructions (see Chapter 9), as is Theorem 6-43 - although the latter is in the context of branched covering spaces (see Chapter 10.)

For an outline of the surgical proof of Theorem 6-45, we offer the following: first we note that we may assume G to be connected (the general result will then follow from the Battle, Harary, Kodama, and Youngs Theorem). We then show that $G[H]$ can have at most $2p_Gq_H$ triangular regions in any imbedding (where G has order p_G and H has q_H edges), and proceed to construct an imbedding of $G[H]$ with $r_3 = 2p_Gq_H$ and $r_4 = r - r_3$. The euler formula shows that this imbedding will be minimal, and gives the desired formula for the genus. To construct the imbedding, we begin with q_G copies of $K_{2n,2n}$, each quadrilaterally imbedded in $S_{(n-1)^2}$ as described by Ringel (see Theorem 6-37). By making suitable vertex identifications (and this is the heart of the proof; see [W7]), we obtain the graph $G[\overline{K_{2n}}]$, quadrilaterally imbedded in the desired surface. The construction thus far allows the q_H edges of each copy of H to be added; each such edge converts one quadrilateral region of the imbedding of $G[\overline{K_{2n}}]$ into two triangular regions. This gives the result.

For the above construction, one can compute the genus of the resulting surface directly, without recourse to any euler-type formula. The contributions ot the genus are of three types:

(i) $q_G(n-1)^2$, representing the collective genera of the q_G 2-manifolds with which we began our construction;

(ii) correspondong to every vertex v of G, we make $(\deg_G v-1)$ sets of $2n$ vertex identifications each, each set requiring a "bundle" of n tubes joining two 2-manifolds; this contributes $(2q_G - p_G)(n-1)$ to the genus;

(iii) $\beta(G) = q_G - p_G + 1$, representing the contribution of the bundles of tubes taken collectively.

Adding, we find:

$$(G[H]) = q_G(n-1)^2 + (2q_G - p_G)(n-1) + (q_G - q_G + 1)$$
$$= 1 + n(nq_G - p_G).$$

We mention that Bouchet [B13] has studied $\gamma(K_{n(m)})$, using "generative m-valuations." He considered the residue classes of $n \pmod{12}$ and $m \pmod 6$ and determined $\gamma(K_{n(m)})$ for 32 of these 72 cases, by constructing triangular imbeddings.

Parsons, Pisanski, and Jackson ([PPJ1]) and [JPP1]) employed "wrapped quasi-coverings" to establish:

<u>Thm. 6-46a</u>. Let G have a triangular imbedding in S_0; then there are infinitely many $n \in N$ such that $G(\overline{K}_n)$ has a triangular imbedding.

Finally, we comment that Jackson [J1] has constructed triangular imbeddings, as branched covering spaces (see Section 10), for some complete n-partite graphs of the form $K((n-2)m, m, \ldots, m)$.

6-6. Edmonds' Permutation Technique

Before leaving the theory of graph imbeddings and considering specific imbedding problems, we present a powerful tool for solving such problems: the Edmonds' permutation technique ([E1]; see also Youngs [Y1].) This amounts to an algebraic description, for any 2-cell imbedding of a graph G. It is used, in one form or another, in the proofs of many of the theorems listed above.

Denote the vertex set of a connected graph G by $V(G) = \{1,2,\ldots,n\}$. For each $i \in V(G)$, let $V(i) = \{k \in V(G) \mid [i,k] \in E(G)\}$. Let $p_i\colon V(i) \to V(i)$ be a cyclic permutation on $V(i)$, of length $n_i = |V(i)|$. Then there is a one-to-one correspondence between

2-cell imbeddings of G and choices of the p_i, given by:

<u>Thm. 6-47.</u> Each choice $(p_1, \ldots, p_n)$ determines a 2-cell imbedding $G(M)$ of G in a surface M, such that there is an orientation on M which induces a cyclic ordering of the edges $[i,k]$ at i in which the immediate successor to $[i,k]$ is $[i,p_i(k)]$, $i = 1, \ldots, n$. In fact, given $(p_1, \ldots, p_n)$, there is an algorithm which produces the determined imbedding. Conversely, given a 2-cell imbedding $G(M)$ in a surface M with a given orientation, there is a corresponding $(p_1, \ldots, p_n)$ determining that imbedding.

<u>Proof</u>: Let $D^* = \{(a,b) \mid [a,b] \in E(G)\}$, and define $P^*: D^* \to D^*$ by: $P^*(a,b) = (b,p_b(a))$. Then P^* is a permutation on the set D^* of directed edges of G (where each edge of G is associated with two oppositely-directed directed edges), and the orbits under P^* determine the (2-cell) regions of the corresponding imbedding. These regions may then be "pasted" together -- with (a,b) matched with (b,a) as in Figure 6-4 -- to form a surface M in which G is 2-cell imbedded. (Since every edge (a,b) in the boundary of a given region is matched with an edge -- (b,a) -- in the boundary of another (or possibly the same) region, M is closed. Since (a,b) is matched with (b,a) -- and not with (a,b) -- M is orientable. Since each p_i is a <u>cyclic</u> permutation, M is a 2-manifold.) The genus of M may now be determined by the euler formula, with r given by the number of orbits under P. The converse follows from similar considerations. #

As an example, consider the imbedding of $K_{3,3}$ in S_1 depicted in Figure 6-5. Let $V(K_{3,3}) = \{1,2,3,4,5,6\}$, with $V(1) = V(2) = V(3) = \{4,5,6\}$; $V(4) = V(5) = V(6) = \{1,2,3\}$. Then

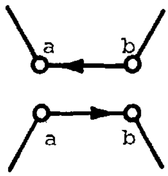


Figure 6-4

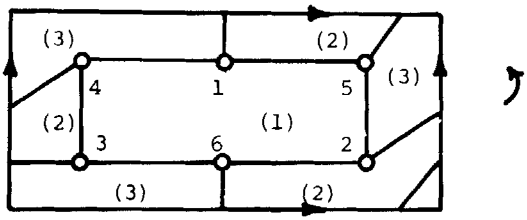


Figure 6-5

p_1: (4,5,6)	p_4: (1,2,3)
p_2: (4,5,6)	p_5: (1,2,3)
p_3: (4,5,6)	p_6: (1,2,3)

describe this imbedding. The orbits under P^* are:

(1) (1,5)(5,2)(2,6)(6,3)(3,4)(4,1)

(2) (5,1)(1,6)(6,2)(2,4)(4,3)(3,5)

(3) (2,5)(5,3)(3,6)(6,1)(1,4)(4,2) .

(Note that $P^*(4,1) = (1,5)$; $P^*(3,5) = (5,1)$; $P^*(4,2) = (2,5)$.)

As a matter of notation, from this point on, we will abbreviate an orbit such as (1) above by: 1-5-2-6-3-4; it is implicit that $p_4(3) = 1$, and $p_1(4) = 5$.

It now follows that the genus of any connected graph (and hence, by Corollary 6-19, of any graph) can be computed, by selecting, from among the $\prod_{i=1}^{n} (n_i-1)!$ possible permutations P^*, one which gives the maximum number of orbits, and hence determines the genus of the graph (component). (Since, by Theorem 6-11, a minimal imbedding must be a 2-cell imbedding, it corresponds to some P^*; by Corollary 5-15, r will be minimal for this imbedding.) The obvious difficulty in applying this procedure is that of selecting a suitable P^* from the (usually) vast number of possible ordered n-tuples of local vertex permutations p_i.

Each p_i is called a <u>rotation</u>, and $P = (p_1, p_2, \ldots, p_n)$ is called a <u>rotation scheme</u> for G.

Stahl has studied "permutation pairs" as a purely combinatorial generalization of graph imbeddings, and his powerful approach suffices to establish many of the classical results (Theorems 6-18 and 6-21, for example) as well as to obtain new information about the genus of the amalgamation of graphs; see [S10], [S12], [S13], and [S14].

6-7. Imbedding Graphs on Pseudosurfaces

In Section 5-5 we introduced the pseudosurfaces $S(k; n_1(m_1), \ldots, n_t(m_t))$. Recall that for any imbedding of a graph G in a pseudosurface S', we assume that each singular point of S' is occupied by a (singular) vertex of G. The number $2 - 2k - \sum_{i=1}^{t} n_i(m_i - 1)$ gives the characteristic of S', denoted by $\chi'(S')$.

<u>Def. 6-48</u>. The <u>pseudocharacteristic</u>, $\chi'(G)$, <u>of a graph</u> G is the largest integer $\chi'(S')$ for all pseudosurfaces S' in which G can be imbedded. The <u>generalized pseudocharacteristic</u>, $\chi''(G)$, is the largest integer $\chi(S'')$ such that G imbeds in the generalized pseudosurface S''.

A surface can be considered as a (degenerate) pseudosurface, and a pseudosurface as a (degenerate) generalized pseudosurface. Hence we have

$$\chi''(G) \geq \chi'(G) \geq 2 - 2\gamma(G).$$

The second inequality may be strict (i.e. pseudosurfaces may be more efficient, from the point of view of maximizing characteristic, for imbedding graphs into); for example $\chi'(K_5) = 1$, as Figure 6-6 shows. (Now, see Problem 6-10.)

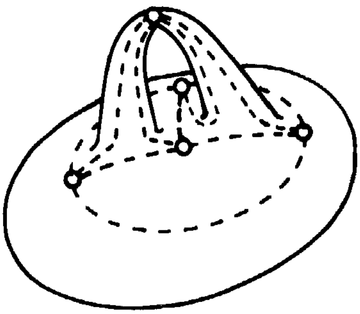


Figure 6-6

Petroelje [P4] has found that many of the basic theorems for imbedding graphs in surfaces carry over for pseudosurfaces. For example:

Thm. 6-49. Let G be a connected graph minimally imbedded in the pseudosurfaces S'; then the imbedding must be 2-cell.

Thm. 6-50. If $G(p,q)$ is connected, then $\chi'(G) \leq p-q/3$; equality holds if and only if G has a triangular imbedding in some pseudosurface.

Petroelje also developed an analogue of Edmonds' permutation technique for pseudosurfaces, and found the following formulae (among others):

Thm. 6-51. $\chi'(K_{n,n,n,n}) = 2n(2-n)$.

(This is consistent with Theorem 6-42 that, for $n \neq 3$, $\gamma(K_{n,n,n,n}) = (n-1)^2$, with $r = r_3$; see also Problem 9-10.)

Thm. 6-52. $\chi'(K_{2m,2n,r}) = 2(m+n-mn) - r(m-1)$, where $2m \geq 2n \geq r \geq 1$.

Ringeisen and White [RW1] showed:

Thm. 6-53. $\chi'(K_{m,n}) = 2 - \left\{ \frac{(m-2)(n-2)}{2} \right\}$.

In the cases where m and n are both even or either m or $n \equiv 2 \pmod 4$, the pseudocharacteristic agrees with the characteristic of Theorem 6-37; in all other cases, $\chi'(K_{m,n}) = \chi(K_{m,n}) = 1$. That is, in terms of pseudocharacteristic these imbeddings are more efficient than those for the genus case.

<u>Thm. 6-54</u>. $\chi'(Q_n + \overline{K}_r) = r - (n+r-4)2^{n-2}$, for $0 \leq r \leq n$, $n \geq 2$.

Generalized pseudosurface imbeddings will have relevance in Chapter 12.

6-8. Other Topological Parameters for Graphs

We have seen that, if a graph is not planar, we can still make a "proper" drawing of the graph in some surface and/or pseudosurface. Two common topological parameters (other than genus) which arise if modified drawings are allowed are the thickness and crossing number.

<u>Def. 6-55</u>. The <u>thickness</u> $\theta(G)$ of a graph G is the minimum number of planar subgraphs whose union is G. (The union is usually taken over spanning subgraphs).

For sample formulae we have:

<u>Thm. 6-56</u>. (Beineke and Harary [BH2]; Alekseev and Gonchakov [AG1]; Vasak [V1])

$$\theta(K_n) = \left[\frac{n+7}{6} \right], \quad \text{except that}$$

$$\theta(K_9) = \theta(K_{10}) = 3$$

<u>Thm. 6-57</u>. (Beineke [B4])

$$\theta(K_{n,n}) = \left[\frac{n+5}{4} \right].$$

<u>Thm. 6-58</u>. (Kleinert [K1])

$$\theta(Q_n) = \left\{ \frac{n+1}{4} \right\}.$$

Some thickness results have been obtained for surfaces of positive genus.

Def. 6-59. The thickness $\theta_n(G)$ of a graph G is the minimum number of subgraphs, each imbeddable on S_n, whose union is G.

Thus $\theta(G) = \theta_0(G)$. The nonorientable thickness $\tilde{\theta}_n(G)\,(n > 0)$ is defined similarly, for N_n.

Thm. 6-60. (Beineke [B5] and [B6])

For $n \geq 3$:

(i) $\tilde{\theta}_1(K_n) = \left\lfloor \frac{n+5}{6} \right\rfloor$;

(ii) $\theta_1(K_n) = \left\lfloor \frac{n+4}{6} \right\rfloor$;

(iii) $\theta_2(K_n) = \left\lfloor \frac{n+3}{6} \right\rfloor$.

Thm. 6-61. (Beineke [B4])

For $n \geq 2$:

(i) $\tilde{\theta}_1(K_{n,n}) = \left\lfloor \frac{n+4}{4} \right\rfloor$;

(ii) $\tilde{\theta}_2(K_{n,n}) = \theta_2(K_{n,n}) = \theta_1(K_{n,n}) = \left\lfloor \frac{n+3}{4} \right\rfloor$;

(iii) $\theta_3(K_{n,n}) = \left\lfloor \frac{n+2}{4} \right\rfloor$.

Thm. 6-62. (Anderson [A3], [A4], [A5])

(i) $\theta_1(K_n - C_n) = \left\lfloor \frac{n+2}{6} \right\rfloor$

(ii) $\theta_1(K_{3(n)}) = \frac{n-1}{2}$, for n an odd prime.

(iii) $\theta_1(K_{4(n)}) = \left\lfloor \frac{n+1}{2} \right\rfloor$

(iv) $\theta_{8rs(2s+1)+1}(K_{4r(2s+1),4r(2s+1)}) = r$.

Thm. 6-63. The crossing number $\nu(G)$ of a graph G is the minimum number of pairwise intersections of its (open) edges, among all drawings of G in the plane.

One might say that the crossing number tells us, if we insist upon drawing G on S_0, just how bad this drawing must be. For this parameter, exact values are scarce; we mention the following bounds (see [G9]):

<u>Thm. 6-64.</u> $v(K_n) \leq \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$; equality holds for $n \leq 10$.

<u>Thm. 6-65.</u> $v(K_{m,n}) \leq \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$; equality holds for $m \leq 6$.

As an indication of the kind of techniques that might be employed, we prove the following:

<u>Thm. 6-66.</u> $v(K_{3,2,2}) = 2$.

<u>Proof</u>: Suppose $v(K_{3,2,2}) = x$. Since $K_{3,3}$ is a subgraph of $K_{3,2,2}$, $\gamma(K_{3,2,2}) \geq 1$ (by the hint for Problem 6-3). Thus $x \geq 1$. Consider such an optimal drawing of $K_{3,2,2}$ in S_0; this gives rise to a plane graph G, with $p = 7 + x$, $q = 16 + 2x$, and $r \neq r_3$. (If $r = r_3$ the configuration in Figure 6-7a, which must exist since $x \geq 1$, must correspond to the configuration in Figure 6-7b, which cannot occur in any complete tripartite graph.) Hence $2q = 32 + 4x \geq$

(a) (b)

Figure 6-7

$4 + 3(r-1) = 3r + 1$, and $9 + x = q - p = r - 2 \leq \frac{31}{3} + \frac{4}{3}x - 2 = \frac{25}{3} + \frac{4}{3}x$. Thus $\frac{2}{3} \leq \frac{1}{3}x$, so that $x \geq 2$.

Figure 6-8 shows that $x \leq 2$, to complete the proof. #

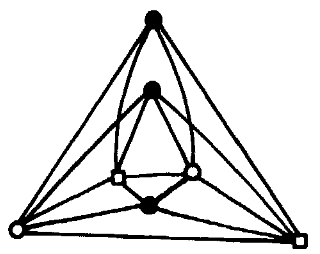


Figure 6-8

A natural extension of the construction in Figure 6-8 gives the following:

Thm. 6-67. $\nu(K_{m,n,r}) \leq f(m,n) + f(m,r) + f(n,r)$, where $f(x,y) = (x-1)\binom{y-1}{2} + (y-1)\binom{x-1}{2} + \binom{x-1}{2}\binom{y-1}{2} = \frac{(x-1)(y-1)(xy-4)}{4}$.

Thm. 6-68. $\nu(K_{n,n,n}) \leq \frac{3}{4}(n-2)(n-1)^2(n+2)$; equality holds for $n = 1,2$.

The exact results below are due to Beineke and Ringeisen ([BR1] and [RB1]):

Thm. 6-68a. The following cartesian products have crossing numbers as indicated:

(i) $\nu(C_3 \times C_n) = n$, for $n \geq 3$;

(ii) $\nu(C_4 \times C_n) = 2n$, for $n \geq 4$;

(iii) $\nu(K_4 \times C_n) = 3n$, for $n \geq 3$.

Other exact results have been found for crossing numbers on surfaces of positive genus.

Def. 6-69. The crossing number $\nu_n(G)$ of a graph G is the minimum number of pairwise intersections of its (open) edges, among all drawings of G in S_n.

Thus $\nu(G) = \nu_0(G)$. The nonorientable crossing number $\tilde{\nu}_n(G)\ (n > 0)$ is defined similarly, for N_n.

Thm. 6-70. (Guy and Jenkins [GJ1])

$$\nu_1(K_{3,s}) = \left\lfloor \frac{(s-3)^2}{12} \right\rfloor .$$

Thm. 6-71. (Gross [G5])
Let $h = \frac{(n-1)(n-4)}{4}$, where $n \equiv 1 \pmod 4$ is a prime power; then

$$\nu_h(K_{n(2)}) = \frac{n(n-1)}{2} .$$

We deduce from Theorem 4-22 that the graphs $Q_n \times K_{4,4}$ are Cayley graphs for all finite Hamiltonian p-groups (in fact, as we will see in Chapter 7, these graphs are of minimum genus for these groups.) From Problem 6-11 we see that $\gamma(Q_n \times K_{4,4}) = 1 + n2^n$, and from Problem 11-8 we will learn that the corresponding nonorientable genus is $\tilde{\gamma}(Q_n \times K_{4,4}) = 2 + n2^{n+1}$.

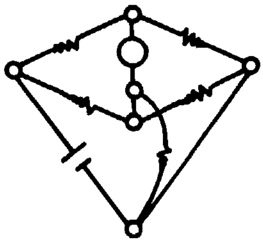
Thm. 6-72. (Kainen and White [KW1])
Let $h = \gamma(Q_n \times K_{4,4}) - m$, and $k = \tilde{\gamma}(Q_n \times K_{4,4}) - 2m$, with $n \geq 0$; then

(i) $\nu_h(Q_n \times K_{4,4}) = 4m$, if $0 \leq m \leq 2^n$;

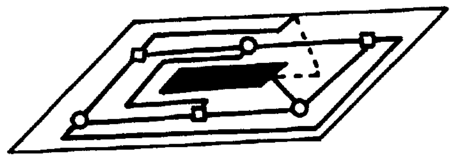
(ii) $\tilde{\nu}_k(Q_n \times K_{4,4}) = 4m$, if $0 \leq m < 2^n$.

6-9. Applications

For applications of the four basic topological parameters discussed in this chapter, consider the problem of printing an electronic circuit on a circuit board. If the associated graph G is planar, one board will suffice, without modification. If G is not planar, at least four alternatives are available to avoid short circuits (the choice depending upon relevant considerations of an engineering and/or economic nature): 1.) the circuit can be accommodated by drilling holes through the board; $\gamma(G)$ gives the minimun number of holes; 2.) some of the vertices can be printed on both sides of the board, with connections made through the board



(a)



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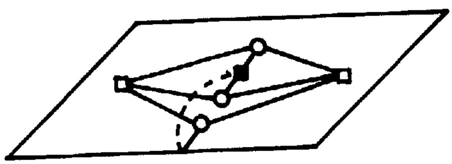

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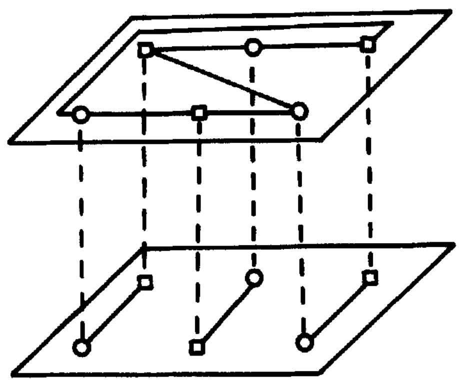

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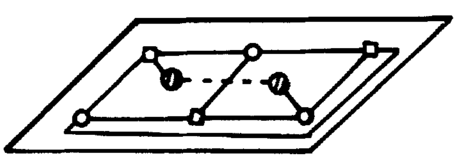


(e)

Figure 6-9

between corresponding images of the same vertex; here we seek the minimum n such that G can be imbedded in the pseudosurface $S(0; n(2))$; however, it may occur that no such n exists (see Problem 6-9.) If a given vertex can appear arbitrarily often, with connections made through the board among corresponding images of the same vertex, and if in addition holes can be drilled as in 1.), then we seek the value of the parameter $\gamma'(G)$, for maximum efficiency; 3.) If several circuit boards are used, each containing a planar portion of the circuit, and jumpers are run between successive boards to connect corresponding images of the same junction, then we are studying the parameter $\theta(G)$; 4.) If the circuit is stamped on one side of one circuit board (with no holes yet drilled) and if wherever two connections cross extraneously two holes are now drilled to allow one connection to temporarily pass to the other side of the board, enabling it to "cross" the second connection while avoiding a short circuit, it is the parameter $\nu(G)$ that dictates economy of effort here.

As an example, consider the modified wheatstone bridge circuit of Figure 6-9(a); the associated graph is $G = K_{3,3}$. Figures 6-9(b)-(e) correspond respectively to: $\gamma(K_{3,3}) = 1$, $\gamma'(K_{3,3}) = 1$, $\theta(K_{3,3}) = 2$, and $\nu(K_{3,3}) = 1$.

6-10. Problems

6-1.) Show that two graphs are homeomorphic in the graph-theoretical sense if and only if their realizations in R^3 are homeomorphic in the topological sense.

6-2.) Show that no four points on $C = \{(x,y,z) \in R^3 \mid x = t, y = t^2, z = t^3;\ t \geq 0\}$ are coplanar.

6-3.) Prove Corollary 6-15.

6-4.) Prove the easy half of Kuratowski's Theorem: if G contains a Kuratowski subgraph, then G is non-planar. (Hint: show that if H is a subgraph of G, then $\gamma(H) \leq \gamma(G)$.)

6-5.) Show that the Petersen graph (see Figure 8-9) is non-planar. What is its genus?

6-6.) Show that Corollary 6-19 follows from Theorem 6-18.

6-7.) Show that p_1, p_3: $(5,6,7,8)$

p_2, p_4: $(8,7,6,5)$

p_5, p_7: $(1,2,3,4)$

p_6, p_8: $(4,3,2,1)$

describe a 2-cell imbedding of $K_{4,4}$ in S_1, with $r = r_4$.

6-8.) Why is $\chi'(G)$ defined as a maximum characteristic, instead of a minimum genus?

6-9.) Show that $G = K_n$, $n \geq 13$, imbeds on no pseudosurface $S(0;\ k(2))$.

6-10.) Can the first inequality following Definition 6-48 be strict also?

6-11.) Show that $\gamma(Q_n \times K_{4,4}) = 1 + n2^n$, $n \geq 1$.

6-12.) Let G be a connected graph of order $p \geq 2$; show that $G^2(V(G^2) = V(G),\ E(G^2) = \{uv | u,v \in V(G),\ 1 \leq d(u,v) \leq 2\})$ is upper imbeddable, $\gamma_M(G^2) \geq \frac{p-2}{2}$, and the lower bound is sharp.

CHAPTER 7

THE GENUS OF A GROUP

To get an accurate and efficient "picture" of a group, we seek a surface of minimum genus on which we can imbed a Cayley color graph of some presentation of the group. This suggests the following definition; Let $\gamma(C_\Delta(\Gamma))$ denote the genus of the underlying graph $G_\Delta(\Gamma)$ (called the Cayley graph) determined from $C_\Delta(\Gamma)$ by removing all arrows and colors from the edges (recall that, by convention $C_\Delta(\Gamma)$ has no loops or multiple edges). Then:

Def. 7-1. The genus of a group Γ is given by:

$$\gamma(\Gamma) = \min\{\gamma(C_\Delta(\Gamma))\},$$

where the minimum is taken over all generating sets Δ for Γ.

Def. 7-2. A group Γ is said to be planar if $\gamma(\Gamma) = 0$.

7-1. Imbeddings of Cayley Color Graphs

Finite planar groups have been cataloged by Maschke [M2] (see also Anderson [A6]). The finite planar groups on one generator are exactly the cyclic groups Z_n; on two generators, they include the dihedral groups D_n, groups of the form $Z_2 \times Z_n$, S_4, A_4, and A_5 (the last three groups are the symmetry groups of the regular polyhedra), and $Z_2 \times A_4$; on three generators (each must be of order 2) finite planar groups include $Z_2 \times D_n$, $Z_2 \times S_4$, and $Z_2 \times A_5$. In summary:

Thm. 7-3. The finite group G is planar if and only if $G \cong G_1 \times G_2$, where $G_1 = Z_1$ or Z_2 and $G_2 = Z_n, D_n, S_4, A_4$, or A_5.

We consider infinite groups temporarily, preparatory to establishing a startling result, due to Levinson [L1]. In this chapter, an infinite graph is given by:

Def. 7-4. An infinite graph is a graph with denumerable vertex set.

There are two natural (but non-equivalent!) definitions of planarity, for infinite graphs.

Def. 7-5. An infinite graph is said to be planar if it can be imbedded in the plane.

Def. 7-5'. An infinite graph is said to be planar if it can be imbedded in the plane so that the vertex set has no limit points.

We adopt Definition 7-5, for reasons soon to be obvious.

Thm. 7-6. An infinite graph is planar if and only if it contains no subgraph homeomorphic with K_5 or $K_{3,3}$.

For a proof of this extension of Kuratowski's theorem, see Dirac and Schuster [DS1]. To see that this extension does not hold for Definition 7.5', consider the graph of Figure 7-1, where an infinite path is attached at each vertex of K_4.

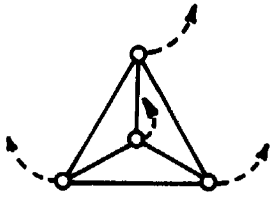


Figure 7-1

Def. 7-7. An infinite graph G has infinite genus $(\gamma(G) = \infty)$, if, for every natural number n, there exists a finite subgraph G_n of G such that $\gamma(G_n) \geq n$.

Lemma 7-8. Let G be the graph of a presentation of an infinite group Γ. Let H be an induced finite subgraph of G. Then there exist two disjoint, isomorphic copies of H in G.

Proof: The vertex set of H corresponds to a finite set $\{g_1,\ldots,g_n\}$ of elements of Γ. Form the (finite) set:

$$S = \{g_i g_j^{-1} \mid 1 \leq i,\ \ j \leq n\}.$$

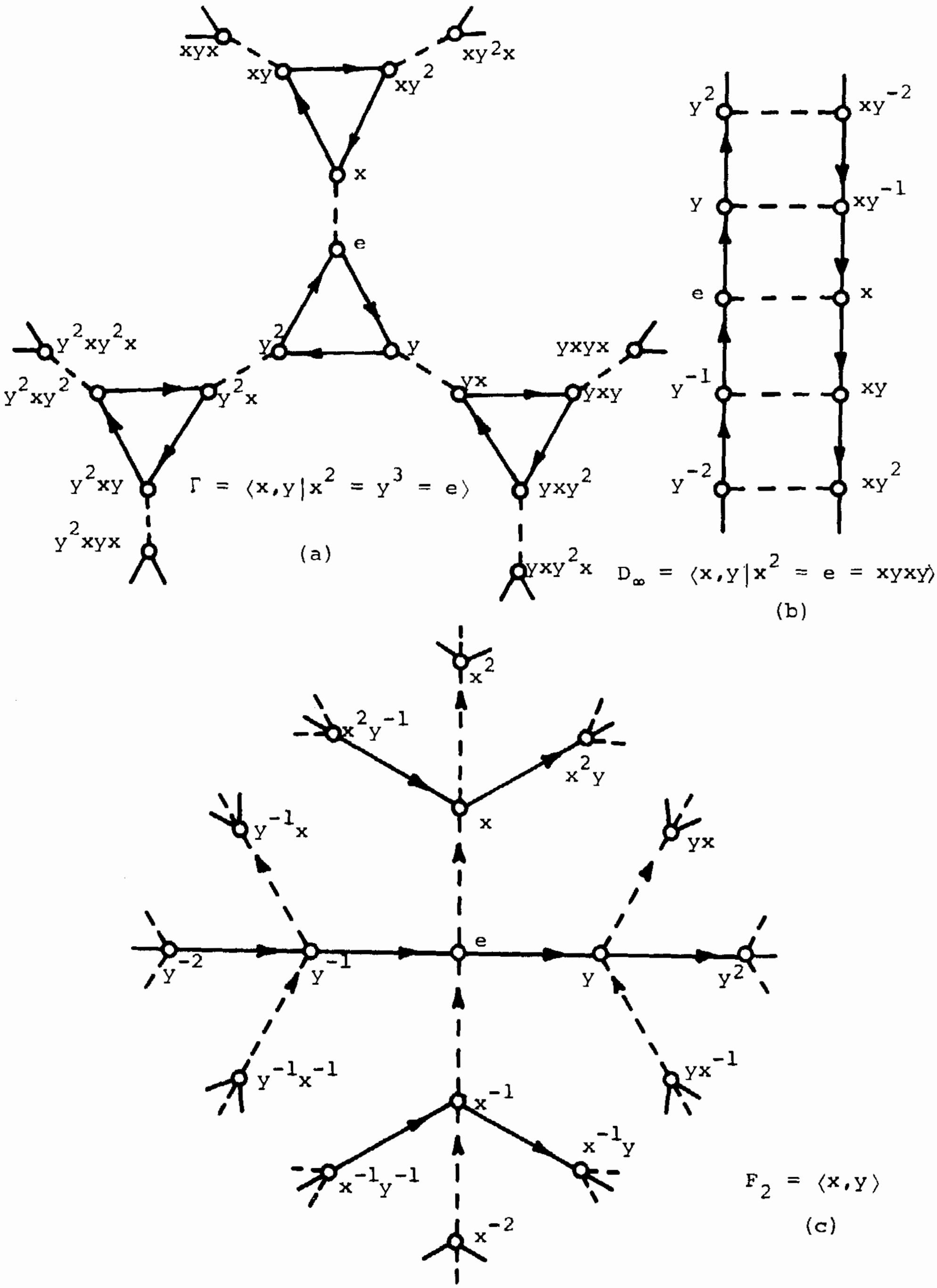
Pick $x \in \Gamma - S$. Form H^*, the subgraph of G induced by $\{xg_i \mid i = 1,\ldots,n\}$; then H^* is isomorphic to H, since $g_i h = g_j$ if and only if $xg_i h = xg_j$. Now, suppose that $v \in V(G) \cap V(H^*)$; then there exist i and j such that $xg_j = g_i$, so that $x = g_i g_j^{-1} \in S$, a contradiction. #

We now present Levinson's result.

Thm. 7-9. Let Γ be an infinite group, with G the graph of a presentation for Γ. Then either $\gamma(G) = 0$, or $\gamma(G) = \infty$.

Proof: Suppose G is not planar; then, by Theorem 7-6, G contains K, a Kuratowski (and hence finite) subgraph. Thus $\gamma(G) \geq 1$. Let n be an arbitrary natural number. But by Lemma 7-8, we can find a second, disjoint copy of K in G, so that $\gamma(G) \geq \gamma(2K) = 2$ by Corollary 6-19. Now apply the lemma again, with $H = 2K$, to obtain two disjoint copies of $2K$ in G, so that $\gamma(G) \geq 4$. Continuing in this fashion, we eventually find two disjoint copies of $2^{n-1}K$ in G, so that $\gamma(G) \geq 2n > n$; then $\gamma(G) = \infty$, since n was arbitrary. #

For example, $\gamma(G) = \infty$. For the standard presentation for $\Gamma = Z \times Z \times Z$; see Problem 7-9.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Γ = ⟨x,y|x2 = y3 = e⟩
(a)
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x-1
x-1y
x-1y-1
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F2 = ⟨x,y⟩
(c)

Figure 7-2

Cor. 7-10. Let Γ be an infinite group; then either $\gamma(\Gamma) = 0$, or $\gamma(\Gamma) = \infty$.

In Figure 7-2, portions of planar Cayley color graphs for presentations of three infinite groups are given. The second group (b) is called the infinite dihedral group; the third group (c) is the free group on two generators. For an infinite group having infinite genus, see Problem 7-10.

Returning our attention to finite groups, we produce examples of groups of positive genus. The following lemma will be useful. Note that if P gives $\gamma(\Gamma)$, then P may be assumed to have no redundant generators; i.e. P is minimal. We also note that, in any imbedding of a Cayley Color graph, every region boundary corresponds to an identity word.

Lemma 7-11. Let Γ be a finite group, with $3 \nmid |\Gamma|$; let Δ be a minimal generating set for Γ. Then $C_\Delta(\Gamma)$ contains no triangles.

Proof: Suppose $C_\Delta(\Gamma)$ contains a triangle; then we find a closed walk $h_1^{a_1}h_2^{a_2}h_3^{a_3} = e$ in $C_\Delta(\Gamma)$, where h_1 is a generator in P, and $a_i = \pm 1$. If any two of the h_i are distinct, then one of these two is redundant. If, on the other hand, $h_1 = h_2 = h_3$, then the a_i all have the same sign (or else all three = e). But then $h_1^3 = e$, and $3 \mid |\Gamma|$, a contradiction. #

Now consider Q, the group of the quaternions. Let P be a presentation for Q, such that $\gamma(Q) = \gamma(C_\Delta(Q))$; then P is minimal. By Lemma 7-11, $C_\Delta(Q)$ has no triangles, since $|Q| = 8$. It is not difficult to see that Δ has at least two generators and that if Δ has exactly two generators, neither can be of order 2; furthermore, Δ cannot have three generators of order 2 (see Problem 7-1). Thus $C_\Delta(Q)$ is regular of degree at least four. Then $2q \geq 4p = 32$; i.e. $q \geq 16$. Now, by Corollary 6-15,

$$\gamma(Q) \geq \frac{q}{4} - \frac{p}{2} + 1 \geq 1.$$

Then $\gamma(Q) = 1$ is shown by Figure 7-3, where $Q = \langle x,y \quad x^2 = y^2 = (xy)^2 \rangle$.

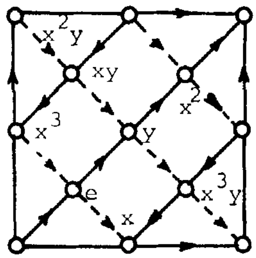

Figure 7-3

7-2. Genus Formulae for Groups

We now find a non-trivial genus formula for an infinite class of groups: those groups (necessarily abelian) in which every element is of order 2. Let Γ_n denote this group; then $\Gamma_n = (Z_2)^n$, and $|\Gamma| = 2^n$.

<u>Thm. 7-12.</u> $\gamma(\Gamma_n) = 1 + 2^{n-3}(n-4)$, $n \geq 2$.

<u>Proof</u>: Γ_n may be expressed as follows: $\Gamma_1 = Z_2$; $\Gamma_n = Z_2 \times \Gamma_{n-1}$, for $n \geq 2$. Writing Γ_n as an iterated direct product in this way, we see that any P for Γ_n must have at least n generators; hence $2q \geq np = n2^n$; thus by Lemma 7-11 and Corollary 6-15,

$$\begin{aligned}\gamma(\Gamma_n) &\geq \frac{q}{4} - \frac{p}{2} + 1 \\ &\geq n2^{n-3} - 2^{n-1} + 1 \\ &= 1 + 2^{n-3}(n-4).\end{aligned}$$

But now let P be determined by repeated application of Theorem 4-18; then $G_\Delta(\Gamma_n) = Q_n$, the n-cube, and

$$\begin{aligned}\gamma(\Gamma_n) &\leq \gamma(C_\Delta(\Gamma_n)) \\ &= \gamma(Q_n) \\ &= 1 + 2^{n-3}(n-4),\end{aligned}$$

by Theorem 6-36. This completes the proof.

Let us extend this result somewhat. We will need the following genus formula, involving $(n+1)$ parameters. Define the graph H_n as follows: let $H_1 = C_{2m_1}$, the cycle on $2m_1$ vertices, and recursively define $H_n = H_{n-1} \times C_{2m_n}$, for $n \geq 2$, where each $m_i \geq 2$. Let $M^{(n)} = \prod_{i=1}^{n} m_i$.

Thm. 7-13. $\gamma(H_n) = 1 + 2^{n-2}(n-2)M^{(n)}$, $n \geq 2$.

Proof: By Theorem 2-19, Problem 2-4, and a trivial induction argument, H_n is a bipartite graph. We produce a quadrilateral imbedding for H_n, and compute $\gamma(G_n)$ using Corollary 6-15. For H_n, let $p^{(n)}$ and $q^{(n)}$ denote the number of vertices and edges respectively. Then $p^{(n)} = 2^n M^{(n)}$; and since H_n is regular of degree $2n$, it is a simple matter to compute $q^{(n)} = 2^n n M^{(n)}$.

Let the statement $S(n)$ be: there is an imbedding of H_n for which $r = r_4 = n2^{n-1}M^{(n)}$, including two disjoint sets of $2^{n-2}M^{(n)}$ mutually vertex-disjoint quadrilateral regions each, both sets containing all $2^n M^{(N)}$ vertices of H_n. We claim that $S(n)$ is true for all $n \geq 2$, and we verify this claim by mathematical induction.

That $S(2)$ is true is apparent from Figure 7-4 (which shows an imbedding of $C_4 \times C_6$ in S_1), with the regions designated by (1) making up one set, and those designated by (2) making up the other. We now assume $S(n)$ to be true and establish $S(n+1)$, for $n \geq 2$.

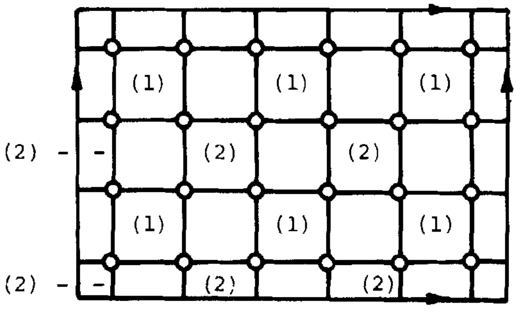

Figure 7-4

For the graph H_{n+1}, we start with $2m_{n+1}$ copies of H_n, minimally imbedded as described by $S(n)$. We partition the corresponding surfaces into m_{n+1} copies of one orientation, and m_{n+1} copies of the reverse orientation, corresponding to the vertex set partition of the bipartite graph $C_{2m_{n+1}}$. From each copy, two joins of $p^{(n)}$ edges each must be made, both to copies of opposite orientation, in order to construct H_{n+1}. From the statement $S(n)$, it is clear that these two joins can be made, each one over $2^{n-2}M^{(n)}$ tubes carrying four edges each. (Attach one end of a tube in the interior of each region designated by (1) for one join; use the regions designated by (2) for the second join.) Each new region formed by this process is a quadrilateral. In this fashion the required $2m_{n+1}$ joins can be made to imbed H_{n+1}, with $r = r_4$. Now form one set of regions by selecting opposite quadrilaterals from each tube added in alternate joins in this construction. Form the second set by selecting the remaining quadrilaterals on the same tubes. It is clear that the two sets of regions thus selected are disjoint, and that each contains (2). $(m_{n+1})(2^{n-2}M^{(n)}) = 2^{n-1}M^{(n+1)}$ mutually vertex-disjoint quadrilaterals; both sets contain all $2^{n+1}M^{(n+1)}$ vertices of H_{n+1}. Furthermore, $r^{(n+1)} = 2m_{n+1}r^{(n)} + \Delta r$, where $\Delta r = (2m_{n+1})(2^{n-2}M^{(n)})(2)$, where $2m_{n+1}$ joins have been made, with $2^{n-2}M^{(n)}$ tubes per join, and a net increase in r of 2 per tube. Hence,

$$r^{(n+1)} = 2m_{n+1}(n2^{n-1}M^{(n)}) + 2^n M^{(n+1)}$$
$$= (n+1)2^n M^{(n+1)},$$

and we have established that $S(n+1)$ follows from $S(n)$. Therefore, $S(n)$ holds, for all $n \geq 2$.

We can now compute:

$$\gamma(H_n) = 1 + \frac{2^n n M^{(n)}}{4} - \frac{2^n M^{(n)}}{2}$$
$$= 1 + 2^{n-2}(n-2)M^{(n)}. \qquad \#$$

For the special case where $m_i = m$, $i = 1, \ldots, n$, we have $M^{(n)} = m^n$, and:

Cor. 7-14. The genus of $H_n^{(m)}$ is given by:

$$\gamma(H_n^{(m)}) = 1 + 2^{n-2}(n-2)m^n.$$

Furthermore, if $m = 2$ in the above formula, since $C_4 = K_2 \times K_2$, $H_n^{(2)}$ is the $2n$-cube, and we obtain the result (compare with Theorem 6-36):

Cor. 7-15. $\gamma(Q_{2n}) = 1 + 2^{2n-2}(n-2)$.

For further results concerning the genus of repeated cartesian products of bipartite graphs, see [W6], and also Pisanski, [P5].

Now, let $\Gamma_n^{(m)}$ be the abelian group with minimal Cayley color graph $H_n^{(m)}$, $m \geq 2$; i.e. $\Gamma_1^{(m)} = Z_{2m}$, and $\Gamma_n^{(m)} = Z_{2m} \times \Gamma_{n-1}^{(m)}$, for $n \geq 2$. Then we have:

Cor. 7-16. $\gamma(\Gamma_n^{(m)}) = 1 + 2^{n-2}(n-2)m^n$.

The reader may wish to combine Theorems 4-16, 4-19, and 7-13 to obtain genus formulae for additional abelian groups. For example, using the notation of Theorem 4-16, consider Γ where m_r is even. Using current graph constructions, rather than the surgery techniques of Theorem 7-13 , Jungerman and White [JW1] found the genus of "most" of the remaining finite abelian groups; the next theorem summarizes to include previous results as well.

<u>Thm. 7-17.</u> Let $\Gamma = Z_{m_1} \times Z_{m_2} \times \ldots \times Z_{m_r}$, where for $2 \leq i \leq r$ m_i divides m_{i-1} (and $m_r > 1$, unless $|\Gamma| = 1$.) Let $N(\Gamma) = 1 - (r-2)\,|\Gamma|/4$; then

(i) If $r = 1$, then $\gamma(\Gamma) = 0$.

(ii) If $r = 2$ and $m_r = 2$, then $\gamma(\Gamma) = 0$.

(iii) If $N(\Gamma)$ is an integer, $m_r > 3$, $r > 1$, and either m_r even or $r \neq 3$, then $\gamma(\Gamma) = N(\Gamma)$.

(iv) If $N(\Gamma)$ is an integer, $m_r = 3$, and $1 < r \neq 3$, then $\gamma(\Gamma) \leq N(\Gamma)$.

The argument of Theorem 7-12 can be modified to assist in the computation of the genus for certain hamiltonian groups ; the following results are due to Himelwright [H7] :

<u>Thm. 7-18.</u> $\gamma(Q \times (Z_2)^n) = n\,2^n + 1$.

<u>Thm. 7-19.</u> $\gamma(Q \times Z_m \times (Z_2)^n) = mn2^n + 1$, for m odd.

<u>Cor. 7-20.</u> The groups $Q \times Z_m \times (Z_2)^8$, for m odd, have genus asymptotic to the order.

By Theorems 4-16 and 4-22, if G is a hamiltonian group, then $G = Q \times Z_{m_1} \times \ldots \times Z_{m_r} \times (Z_2)^n$, where the m_i are odd $(i = 1, \ldots, r)$ and $m_i | m_{i-1}$ $(i = 2, \ldots, r)$. Himelwright has also shown:

<u>Thm. 7-21.</u> The genus of the hamiltonian group $Q \times Z_{m_1} \times \ldots \times Z_{m_r} \times (Z_2)^n$ is asymptotic to $2^n(r + n - 1) \prod_{i=1}^{r} m_i$, if $1 \leq r \leq n + 1$.

There are many open questions in this area. If a generalization of Theorem 7-13 for products of arbitrary (not necessarily even) cycles could be found, then the genus of any abelian (and also of any hamiltonian) group could be easily computed. What is $\gamma(S_n)$? $\gamma(A_n)$? The following theorem produces upper bounds for these group genera.

<u>Thm. 7-22.</u> If Γ is finite and is minimally generated by $\{g_1, \ldots, g_n\}$ and satisfies at least the relations $g_i^{m_i} = e = (\prod_{j=1}^{n} g_j)^k$, $(1 \leq i \leq n)$ then

$$\gamma(\Gamma) \leq 1 + \frac{|\Gamma|}{2}\left\{n-1 - \frac{1}{k} - \sum_{j=1}^{n} \frac{1}{m_j}\right\}.$$

<u>Proof</u>: Select $P_g = (gg_1, gg_1^{-1}, gg_2, gg_2^{-1}, \ldots, gg_n gg_n^{-1})$, for all $g \in G$. Then, using Edmond's algorithm (see Theorem 6-42), we compute orbits as follows:

(i) An orbit containing the directed edge (a, ag_i^{-1}) continues with $p_{ag_i^{-1}}(a) = ag_i^{-2}$; hence this orbit corresponds to the relation $g_i^{m_i} = e$ and has length m_i. (If $m_i = 2$, we draw edges for both $gg_i = g'$ and $g'g_i = g$, obtaining $\frac{|\Gamma|}{2}$ 2-sided regions; for each such region, the two sides may be identified and the arrows removed, so that the region is destroyed but the genus is unaffected.)

(ii) An orbit containing the directed edge (a, ag_i) continues with $P_{ag_i}(a) = ag_i g_{i+1}$; hence this orbit corresponds to the relation $(\prod_{j=1}^{n} g_j)^k = e$ and has length nk. As there are no other orbits, we find:

$$r = \sum_{i=1}^{n} r_{m_i} + r_{nk}$$

$$= \sum_{i=1}^{n} \frac{|\Gamma|}{m_i} + \frac{|\Gamma|}{k};$$

the euler formula now gives the genus γ of the theorem for this imbedding of $C_\Delta(\Gamma)$, for this presentation P for Γ. Hence $\gamma(\Gamma) \leq \gamma(C_\Delta(\Gamma)) \leq \gamma$. #

We note that an equivalent formula was obtained by Burnside [B18, p. 398] in a different context. Theorem 7-22 gives $\gamma(G)$ exactly, for $\Gamma = Z_m, A_4, S_4, A_5$, or $Z_3 \times Z_3$. We also obtain the following two corollaries:

<u>Cor. 7-23.</u> $\gamma(S_n) \leq 1 + \frac{(n-2)!}{4}(n^2-5n+2)$, $n \geq 2$.

<u>Proof:</u> Take $s = (1\ 2\ 3\ \ldots\ n)$ and $t = (1\ 2)$ as generators for S_n; then $s^n = t^2 = (st)^{n-1} = e$. #

<u>Cor. 7-24.</u>

$$\gamma(A_n) \leq \begin{cases} 1 + \frac{(n-1)(n-3)!}{8}(n^2-6n+4), & n \text{ odd} \\ 1 + \frac{n(n-2)!(n-5)}{8}, & n \text{ even.} \end{cases}$$

<u>Proof:</u> For n odd, $s = (1\ 2\ \ldots\ n-2)$ and $t = (1\ n-1)(2\ n)$ generate A_n, and $s^{n-2} = t^2 = (st)^n = e$. For n even, $s = (1\ 2\ \ldots\ n-1)(2\ n)$ and $t = (1\ 2)(3\ n)$ generate A_n, with $s^{n-1} = t^2 = (st)^{n-1} = e$ (see [9]). #

The two formulas given above for S_n and A_n respectively were also found by Brahana [B15], using a different method and in a slightly different context. For related results, see [W8].

There has been much activity in the study of the genus parameter for groups, in recent years, perhaps at least in part motivated by Chapter 7 of the first edition of this book. We have already presented Theorem 7-16. Here is a continued sample of this research.

<u>Thm. 7-25.</u> (Proulx [P9])

$$\gamma(S_n) \leq 1 + \frac{n!}{8}, \quad \text{for } n \text{ even}, \quad n \geq 4.$$

This improves on Corollary 7-23, for n even. For n odd, a sharper bound than that of Corollary 7-23 was developed in [W8]; but the next results detract from the interest that might accrue to the improvement of bounds for $\gamma(S_n)$:

Thm. 7-26. (Proulx [P9])

$$\gamma(S_5) = 4.$$

(The proof is not easy!)

Thm. 7-27. (Tucker [T4])

$$\gamma(S_n) = 1 + \frac{n!}{168}, \quad \text{for} \quad n \geq 168.$$

We also find

Thm. 7-28. (Tucker [T4])

$$\gamma(A_n) < 1 + \frac{n!}{336}, \quad \text{for} \quad n \geq 168$$

During the period 1972-77, Gross [G6], Gross and Lomonaco [GL1], and White [W8] showed that certain metacyclic and dicyclic groups are toroidal. Then in 1977 Proulx [P8] completed work begun by Baker in 1931 [B3], to classify all toroidal groups in a major effort. The classification consists of nineteen presentations on two generators, ten presentations on three generators, and one presentation on four generators.

7-3. Related Results

In 1977 Babai [B2] solved a problem posed in the first edition of this book, when he established:

Thm. 7-29. If Γ_1 is a subgroup of Γ_2, then $\gamma(\Gamma_1) \leq \gamma(\Gamma_2)$.

Since S_n is isomorphic to a subgroup of A_{n+2}, we obtain

Cor. 7-30. $\gamma(S_n) \leq \gamma(A_{n+2})$.

For example, we learn from Theorem 7-25 that $\gamma(A_7) \geq 4$.

Despite the fact that there are infinitely many planar groups (Maschke's Theorem) and infinitely many toroidal groups (Proulx's classification) Tucker [T3] showed, in 1978:

Thm. 7-31. For each $g \geq 2$, there are at most finitely many groups Γ such that $\gamma(\Gamma) = g$.

The following 1981 result, also due to Tucker [T5], is even more surprising:

Thm. 7-32. There is exactly one group of genus two. It has order 96 and presentation $\langle x,y,z \mid x^2 = y^2 = z^2 = (xy)^2 = (yz)^3 = (xz)^8 = y(xz)^4 y(xz)^4 = e \rangle$.

The (unique) group of genus two is the automorphism group of the generalized Petersen graph $G(8, 3)$; see Frucht, Graver, and Watkins [FGW1].

We mention that other definitions of the "genus of a group" appear in the literature, due to Levinson [L2], Machlachlan [M1], and Burnside [B18]. For Γ a finite group, we denote these parameters by $\gamma_L(\Gamma)$, $\gamma_M(\Gamma)$, and $\gamma_B(\Gamma)$ respectively. The Levinson parameter, like the parameter $\gamma(\Gamma)$ advocated in this chapter, regards Γ as being depicted by a Cayley graph $G_\Delta(\Gamma)$ minimally imbedded on a surface S_k; but S_k is always an $|\Gamma|$-fold (possibly branched) cover of some S_n (i.e. the imbedding is index one.) The Machlachlan and Burnside parameters also regard S_k as an $|\Gamma|$-fold (possibly branched) cover of some S_n, but represent the group Γ via its action on the Riemann surface S_k; for the Burnside parameter, it is always the case that $n = 0$. The four parameters are related as follows, where the equality holds except for certain small Γ:

Thm. 7-33. For Γ a finite group, $\gamma(\Gamma) \leq \gamma_L(\Gamma) = \gamma_M(\Gamma) \leq \gamma_B(\Gamma)$; all are bounded above by the bound of Theorem 7-22.

Thus the most efficient genus, among all these, is that given by $\gamma(\Gamma)$. We remark that both the inequalities of Theorem 7-33 can be strict, as the groups $\Gamma = (Z_{2n})^4$, for $n \geq 2$ indicate:

Thm. 7-34. Let $\Gamma = (Z_{2n})^4$, $n \geq 2$; then

(i) $\gamma(\Gamma) = 1 + 8n^4$

(ii) $\gamma_L(\Gamma) = \gamma_M(\Gamma) = 1 + 16n^4$

(iii) $\gamma_B(\Gamma) = 1 + 4n^3(6n-5)$

7-4. The Characteristic of a Group

If we allow nonorientable surfaces also, as our "drawing boards" for "picturing" groups, then we are led naturally to the parameter of this section. Recall from Section 5-3 that the characteristic of a surface S is $\chi(S) = 2 - 2k$, if $S = S_k$; $\chi(S) = 2 - k$, if $S = N_k$.

Def. 7-35. The characteristic of a graph G, denoted by $\chi(G)$, is the maximum surface characteristic $\chi(S)$ such that G imbeds in S.

For example, a graph has characteristic two if and only if it is planar; K_5 and K_6 have characteristic one; K_7 has characteristic zero; and so forth.

Def. 7-36. The characteristic of a group Γ is given by:

$$\chi(\Gamma) = \max\{\chi(G_\Delta(\Gamma))\},$$

where the maximum is taken over all generating sets Δ for Γ.

Thus a group has characteristic two if and only if it appears in Maschke's list (Theorem 7-3.)

Here are some less obvious sample results.

Thm. 7-37. (White [W10])

Let Γ be finite and abelian; then

$$\chi(\Gamma) = \begin{cases} 2, \text{ if } \Gamma = Z_n, Z_{2n} \times Z_2, \text{ or } (Z_2)^3 \\ 1, \text{ if } \Gamma = Z_3 \times Z_3 \\ 0, \text{ if } \Gamma = Z_{2n} \times Z_2 \times Z_2 \ (n > 1), (Z_2)^4, \\ \quad \text{or } Z_{mn} \times Z_m \ (m > 2, mn \neq 3). \end{cases}$$

$\chi(\Gamma) < 0$, otherwise.

Thm. 7-38. (White [W10])

Let $(|\Gamma|, G) = 1$, with Γ not cyclic. Then $\chi(\Gamma) = 0$ if and only if Γ has a presentation of the form

$$\Gamma = \langle a,b \mid a^{2k+1} = b^{2n+1} = w = \ldots = e \rangle ,$$

where w is either $aba^{-1}b^{-1}$ or $abab^{-1}$.

Thm. 7-39. (Tucker [T4])

There is no group Γ having $\chi(\Gamma) = -1$.

Thm. 7-40. (Tucker [T4])

For $n \geq 168$, $\chi(A_n) = \frac{-n!}{168}$.

Thm. 7-41. (Tucker [T4])

For $n \geq 168$, $\chi(S_n) = \frac{-n!}{84}$.

We observe that, for $n \geq 168$, $\chi(S_n)/\chi(A_n) = [S_n : A_n]$.

7-5. Problems

7-1.) Show: that any presentation P for Q, the quaternions, has at least two generators; that if P has exactly two generators, neither can be of order 2; and that P can not have exactly three generators, each of order 2. (Hence $\delta(C_\Delta(Q)) \geq 4$).

7-2.) Find an example of a group Γ and a presentation P for Γ such that $\gamma(C_\Delta(\Gamma)) = \infty$.

7-3.) Find $\gamma(Z_m \times Z_n)$, for all m and n.

7-4.) Show that the only finite planar abelian groups are A_n, $Z_2 \times Z_{2n}$, and $Z_2 \times Z_2 \times Z_2$, where $n \geq 1$.

*7-5.) Use the imbedding of Problem 6-7 to find $\gamma(Q \times Q)$.

7-6.) Find a non-normal subgroup in $Q \times Q$. (Thus the product of hamiltonian groups need not be hamiltonian.)

*7-7.) Show that the <u>dicyclic</u> <u>group</u> $G_n = \langle x, y \; x^{2n} = x^n y^{-2} = y^{-1}xyx = e \rangle$ has genus 1, for all $n > 1$. ($G_2 = Q$; G_3 is the "least familiar group of order 12.")

7-8.) Find an infinite group with a presentation of the form given in Theorem 7-22.

7-9.) Let $\Delta = \{(1,0,0), (0,1,0), (0,0,1)\}$ for $\Gamma = Z \times Z \times Z$. Show that $\gamma(G_\Delta(\Gamma)) = \infty$.

7-10.) Show that $\gamma(Z \times Z \times Z) = \infty$. Now let $\Gamma_1 = Z$, and $\Gamma_n = Z \times \Gamma_{n-1}$, $n \geq 2$; find $\gamma(\Gamma_n)$, for all natural numbers n.

**7-11.) The smallest order group whose genus is unknown is the abelian $Z_3 \times Z_3 \times Z_3$. It is known that $5 \leq \gamma(Z_3 \times Z_3 \times Z_3) \leq 10$; resolve the issue!

7-12.) Verify Theorem 7-34.

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CHAPTER 8

MAP-COLORING PROBLEMS

In this chapter we will see that the famous four-color conjecture (now the four-color theorem) can be formulated -- and studied-- in graph-theoretical terms. Graph theory will be used to establish the five-color theorem. The Heawood Map-coloring theorem will be introduced; this powerful theorem, whose proof was completed in 1968, answers the coloring question -- which was still unanswered for the sphere -- for every other closed 2-manifold. The easy half of the proof -- found by Heawood in 1890 -- is presented in this section. The difficult half of the proof -- developed primarily by Ringel and Youngs -- will be discussed in Chapter 9.

Consider any map of the world. Suppose we desire to color the countries of the world (or the states of a particular country, or the counties of a particular state, etc.) so that the distinct countries are distinguishable. This means that if two countries share a border at other than isolated points, then they must be colored differently. We make only one assumption as to the countries themselves: each country must be connected (this rules out Pakistan of the last decade, and the United States, for example.) Note that a country need not be a 2-cell; that is, it may entirely surround some collection of other countries (such is the case for a certain region in France; see Frechet and Fan [FF1], p. 3).

We mention in passing that several generalizations of this map-coloring problem are possible. One of the most appealing is the following: allow disconnected countries, with each country having at most k components. (It is not hard to see that, without this restriction involving k, arbitrarily many colors may be needed.) Then it can be shown (see Problem 8-6 for the case $k = 2$) that $6k$ colors will always suffice. Ringel ([R8]; p. 26 (see also Heawood [H3])) displays a map, for the case $k = 2$, requiring 12 colors, so that this case is completely solved.

Returning now to the case of classical interest $(k = 1)$, we pose the question thusly: what is the smallest number of colors needed to color any map on the sphere (or, equivalently, on the plane)?

That four colors may be needed is indicated by the map induced by the tetrahedron. That five colors suffice for the sphere will be demonstrated shortly. Whether or not five colors are ever *necessary* has probably stimulated as much work in mathematics as my other single mathematical question; and the answer is finally known. The *four-color theorem* says that five colors are never necessary; *four colors will suffice to color any map on the sphere*. Many "proofs" of the four color theorem have been presented to the mathematical community, but only one has yet survived close scrutiny. The interested reader might wish to read through one of the false "proofs", given by Kempe in 1879 (see [BCL1], for example), and try to spot the error in the "proof."

Graph theory enters the picture in the following way. Form the dual of the map in question. This produces a pseudograph. Attempt to color the vertices of the pseudograph so that no two adajcent vertices have the same color. The pseudograph has no loops, as no self-respecting country ever shares a border with itself. In fact, we may as well drop any multiple edges, since they (the "extra" edges) have no bearing on the coloring question. Then the coloring numbers, or chromatic numbers, of the resulting graph and the map will be identical. This leads to the following definitions.

8-1. Definitions

Def. 8-1. The *chromatic number*, $\chi(G)$, *of a graph* G is the smallest number of colors for $V(G)$ so that adjacent vertices are colored differently.

Def. 8-2. The *chromatic number* $\chi(S_k)$, *of a surface* S_k is the largest $\chi(G)$ such that G can be imbedded in S_k.

8-2. The Four-color Conjecture

In this terminology, the four-color conjecture becomes:

Conjecture 8-3. $\chi(S_0) = 4$.

There are many other equivalent formulations of the four-color conjecture (see, for example, Ore's book: <u>The Four Color Problem</u> [O1], or [BCL1]. We consider two of these.

<u>Thm. 8-4.</u> The four-color conjecture is true if and only if every cubic plane block is 4-region colorable.

<u>Proof</u>: Clearly if every plane graph is 4-region colorable, so is every cubic plane block. For the converse, assume every cubic plane block is 4-region colorable, and let G be a plane block. We obtain a cubic plane block G' from G, by repeated operations of the form (a) -- for vertices of degree 2 -- and (b) -- for vertices of degree 3 or more -- as depicted in Figure 8-1. Thus G' is 4-region colorable (by hypothesis), and any

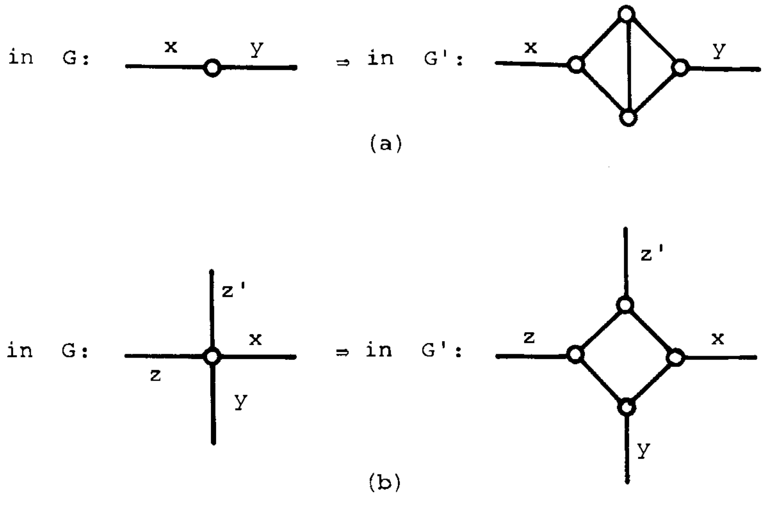


Figure 8-1

4-region coloring of G' induces a 4-region coloring for G. Since it is apparent that the region coloring number of an arbitrary plane graph is the maximum of the corresponding numbers for its blocks, the proof is complete.
#

Def. 8-5. A graph G is said to be n-edge colorable if colors can be assigned to $E(G)$ so that adjacent edges are colored differently.

Thm. 8-6. The Four-Color Conjecture is true if and only if every cubic plane block is 3-edge colorable.

Proof: By Theorem 8-4, it suffices to show that a cubic plane block G is 4-region colorable if and only if it is 3-edge colorable.

Suppose G is 4-region colorable, and let the colors be taken from the group $\Gamma = Z_2 \times Z_2$. Since G is a block, each edge x of G appears in the boundary of two distinct (but adjacent) regions, R_x^1 and R_x^2. Define the color of x by $c(x) = c(R_x^1) + c(R_x^2)$, addition taking place in Γ. Since $c(R_x^1) \neq c(R_x^2)$, $c(x) \neq e$, the identity of Γ (every element is its own inverse, in Γ.) Let x, y, and z be adjacent edges in G; see Figure 8-2. We claim that x, y, and z are colored

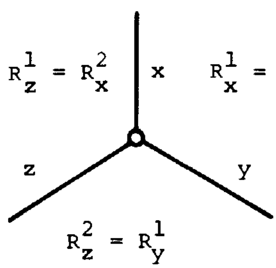


Figure 8-2

distinctly. Suppose to the contrary that, say, $c(x) = c(y)$; that is $c(R_x^1) + c(R_x^2) = c(R_y^1) + c(R_y^2) = c(R_y^1) + c(R_x^1)$. But then $c(R_y^1) = c(R_x^2)$, a contradiction. Thus G is 3-edge colorable (the colors being taken from $\Gamma - \{e\}$.)

Now assume that G is 3-edge colored, with the elements of $\Gamma - \{e\}$. Let R be any region of G, and set $c(R) = e$. Let S be any other region of G; we determine $c(S)$ as follows. Let C' be an arc joining a point in R with a point in S, with $C' \cap V(G) = \phi$. Suppose C' crosses edges $x_1, x_2, \ldots, x_n$ (repetition allowed.) Set $c(S) = \sum_{i=1}^{n} c(x_i)$. To show that $c(S)$ is well-defined, we must show that this element of Γ is independent of the particular arc C' selected. Equivalently, we show that if C is a simple closed curve crossing edges $y_1, y_2, \ldots, y_m$, with $C \cap V(G) = \phi$, then $\sum_{i=1}^{m} c(y_i) = e$. Let C be such a curve. If $V(G) \cap \text{Int } C = \phi$, then each edge crossed by C is crossed an even number of times; hence $\sum_{i=1}^{m} c(y_i) = e$, in this case. If $V(G) \cap \text{Int } C \neq \phi$, we assume (without loss of generality) that each edge crossed by C is crossed exactly once. Let $y_i \subseteq \text{Int } C$, $i = m+1, \ldots, r$. By hypothesis, the sum of the colors on three edges incident with any vertex is e; hence the total of such sums for the set $V(G) \cap \text{Int } C$ is also e; but this sum is also given by

$$\sum_{i=1}^{m} c(y_i) + 2 \sum_{i=m+1}^{r} c(y_i) = \sum_{i=1}^{m} c(y_i).$$

Therefore $\sum_{i=1}^{m} c(y_i) = e$ in this case also, and $c(S)$ is well-defined.

Now consider two adjacent regions, R_x^1 and R_x^2. We have assigned colors to the regions of G so that $c(R_x^2) - c(R_x^1) = c(x) \neq e$; i.e. $c(R_x^2) \neq c(R_x^1)$. Thus G is 4-region colored. #

In 1976, Appel ahd Haken [AH1] announced their resolution of this issue, in the four-color theorem; see Woodall and Wilson [WW1], for one discussion of the proof.

<u>Thm. 8-7.</u> $\chi(S_0) = 4$.

8-3. The Five-Color Theorem

The proof below is found in [BCL1].

Thm. 8-8. Five colors will suffice to color any map on the sphere; i.e. $\chi(S_0) \leq 5$.

Proof: We use induction on p, the order of the graph G, to show that if $\gamma(G) = 0$, then $\chi(G) \leq 5$. The anchor at $p = 1$ is obvious. Now assume that all planar graphs with $p-1$ vertices $(p > 1)$ are 5-colorable. Let G be planar, with p vertices. By Lemma 5-19, G contains a vertex v of degree 5 or less. By the induction hypothesis, $\chi(G-v) \leq 5$; denote the colors in a 5-coloring of $G-v$ by $1,2,3,4,5$. If not all five colors are used for the vertices adjacent to v in G, we can color v with one of the colors not so used, to give $\chi(G) \leq 5$. Otherwise, $d(v) = 5$, and all five colors are used for vertices adjacent to v. We can assume that the situation around v is as in Figure 8-3, and that v_i is colored with color i. Consider now

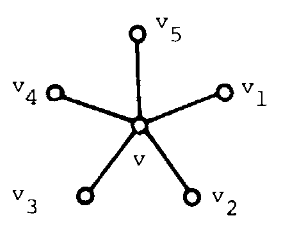


Figure 8-3.

any two colors assigned to non-consecutive vertices v_i, say 1 and 3, and let H be the subgraph of $G - v$ induced by all those vertices colored 1 or 3. If v_1 and v_3 belong to different components of H, then by interchanging the colors in the component of H containing v_1, say, a 5-coloring of $G - v$ is produced in which no vertex adjacent with v is assigned the color 1, and we can use 1 for v. If, on the other hand, v_1 and v_3 are joined by a path in H, the above argument guarantees that we can recolor v_2 with 4, and use 2 for v. This completes the proof. #

8-4. Other Map-coloring Problems; The Heawood Map-coloring Theorem

Now let us consider other subspaces of R^3 in which to pose map-coloring questions such as that above, for the sphere (and plane). Strangely enough, if we allow 3-dimensional countries, arbitrarily many colors may be needed to color the map. This is indicated by Figure 8-4, in which the countries are numbered; it is seen that each country meets each of the other countries. (In general, n rectangular parallelopipeds are laid across n other such solids.)

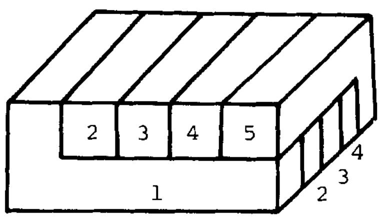


Figure 8-4

Perhaps it seems natural, since the coloring problem is apparently extraordinarily difficult for the sphere, and admits no finite answer in R^3, to consider next the surfaces S_k as candidates for maps and the corresponding map-coloring questions.

The Heawood Map Coloring Theorem (formerly the Heawood Map-Coloring Conjecture) has a particularly colorful background, as outlined in Chapter 1; also see J. W. T. Youngs [Y2]. We state the theorem first for the orientable case:

Thm. 8-9.

$$\chi(S_k) = \left[\frac{7+\sqrt{1+48k}}{2}\right], \quad \text{for} \quad k > 0.$$

Note what happens if we replace k with 0 in this formula. This led many mathematicians to feel that the four color conjecture was probably true, and they were vindicated! Thus we can now take $k \geq 0$ in Theorem 8-9.

The corresponding map-coloring question can also be asked for the closed non-orientable surfaces N_k (spheres with k cross-caps). Ringel [R8] showed the following (the case $k = 2$ was solved by Franklin [F3]):

Thm. 8-10. $\chi(N_k) = \left[\frac{7+\sqrt{1+24k}}{2}\right]$, for $k = 1$ and $k \geq 3$;

$\chi(N_2) = 6.$

For example, the formula gives $\chi(N_1) = 6$ (for the projective plane). Figure 8-5 shows K_6 imbedded in N_1, indicating that $\chi(N_1) \geq \chi(K_6) = 6$.

Recalling that the euler characteristics for S_k and N_k are given by $n = 2 - 2k$ and $n = 2 - k$ respectively, we can combine Theorems 8-8, 8-9, and 8-10 as follows:

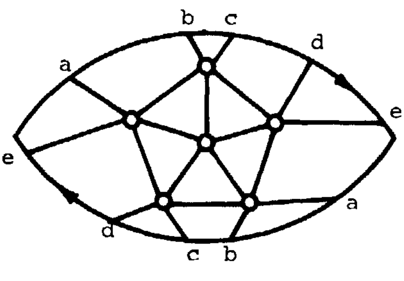


Figure 8-5

Thm. 8-11. Let M_n be a closed 2-manifold, other than the klein bottle, of characteristic n; then

$$\chi(M_n) = \left[\frac{7+\sqrt{49-24n}}{2}\right].$$

For ease of notation, we let $f(n) = \frac{7 + \sqrt{49 - 24n}}{2}$. We will now establish what Heawood knew in 1890:

$$\chi(M_n) \leq [f(n)].$$

We proceed by a series of steps, following the translator's notes in Fréchet and Fan [FF1].

<u>Lemma 8-12</u>. Let a graph G, with $p \geq 3$, be 2-cell imbedded in M_n, with a denoting the average degree of the vertices of G. Then $a \leq 6(1 - n/p)$.

<u>Proof</u>: Note that $a = 2q/p$. Now $3r \leq 2q = ap$. Also, $p - q + r = n$. Hence

$$q \leq 3(q - r) = 3(p - n),$$

so that

$$a = 2q/p \leq 6(1 - n/p). \qquad \#$$

<u>Thm. 8-13</u>. $\chi(N_1) \leq 6$.

<u>Proof</u>: We use induction on p, the order of a graph imbedded in N_1. We need only consider graphs having 2-cell imbeddings in N_1, for otherwise (see Youngs [Y1]), $\gamma(G) = 0$, and $\chi(G) \leq 5$. The result is clearly true for $p \leq 6$. Assume $\chi(G) \leq 6$ for all graphs in N_1 with $p - 1$ vertices, $p \geq 7$; let G be a graph imbedded in N_1, with p vertices. By Lemma 8-12, $a < 6$, so that G has a vertex v such that $d(v) \leq 5$. Then $G - v$ is imbedded in N_1, and $\chi(G - v) \leq 6$, by the induction hypothesis. Since there are vertices of at most five colors adjacent to v, the sixth color can be used for v, and $\chi(G) \leq 6$. #

Note that Theorem 8-13, together with Figure 8-5, show that $\chi(N_1) = 6$.

<u>Thm. 8-14.</u> $\chi(M_n) \leq [f(n)]$, for $n \neq 2$.

<u>Proof</u>: Thanks to Theorem 8-13, we may assume that $n \leq 0$. We use induction on p, to show that $\chi(G) \leq [f(n)]$, if G is imbedded in M_n. (We may assume G to be connected, as the chromatic number of a graph is the largest chromatic number for its components.) It is clear that $\chi(G) \leq [f(n)]$ if $p \leq [f(n)]$. Now assume that $\chi(G) \leq [f(n)]$ for all graphs with fewer than p vertices and imbeddable in M_n. Now, from the definition of $f(n)$, we see that $f^2(n) - 7f(n) + 6n = 0$; i.e. $6(1 - n/f(n)) = f(n) - 1$. If the imbedding of G in M_n is 2-cell, then Lemma 8-12 applies, and

$$\begin{aligned} a &\leq 6(1-n/p) \\ &\leq 6(1-n/f(n)) \\ &= f(n) - 1. \end{aligned}$$

If the imbedding is not 2-cell, then it is not minimal (see again Youngs [Y1]), and we can find a 2-cell imbedding in M_m, where $m > n$. We then apply Lemma 8-12 as above, to get $a \leq f(m) - 1 \leq f(n) - 1$. Thus in either case $a \leq f(n) - 1$, and we can find a vertex v of G having $d(v) \leq [f(n)] - 1$, so that, (using $\chi(G-v) \leq [f(n)]$), $\chi(G) \leq [f(n)]$. This completes the proof. #

The task remains to show that $\chi(M_n) \geq [f(n)]$, for $M_n \neq N_2$, the klein bottle. This is done by finding a graph G imbeddable in M_n and having $\chi(G) = [f(n)]$. For $M_2 = S_0$, K_4 is such a graph; for $M_1 = N_1$, take $G = K_6$ (as in Figure 8-5); for $M_0 = S_1$, pick $G = K_7$ (see Figure 8-6 for the dual of K_7 in S_1); for $M_{-2} = S_2$, let $G = K_8$. In fact, for $M_n \neq N_2$, $[f(n)]$ is attained by the largest complete graph imbeddable in M_n. We now confine our attention to the orientable case and explore this claim in some detail.

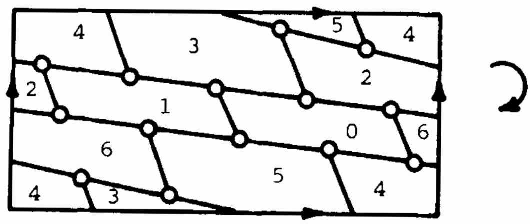


Figure 8-6

Let us assume the truth of the complete graph theorem (which will be discussed in detail in Chapter 9):

$$\gamma(K_m) = \left\{ \frac{(m-3)(m-4)}{12} \right\}, \quad \text{for} \quad m \geq 3.$$

From this it will follow that $\chi(M_n) \geq [f(n)]$ (in the orientable case; the non-orientable case is handled similarly).

Thm. 8-15. $\chi(S_k) \geq [f(2-2k)] = \left\lfloor \frac{7 + \sqrt{1 + 48k}}{2} \right\rfloor$.

Proof: Consider S_k. Define $m = [f(2-2k)]$, and now consider also $S_{\gamma(K_m)}$. Note that $\gamma(K_m) \leq k$, so that $\chi(S_{\gamma(K_m)}) \leq \chi(S_k)$. Now K_m imbeds in $S_{\gamma(K_m)}$. Clearly $\chi(S_{\gamma(K_m)}) \geq m = [f(2-2k)]$, so that $\chi(S_k) \geq [f(2-2k)]$.#

Theorems 8-14 and 8-15 combine to prove Theorem 8-9, with the understanding that it remains to establish the formula for the genus of K_m. Before indicating how this is done (in the next chapter), we pause for two related results.

8-5. A Related Problem

We have seen that, for any closed 2-manifold except the sphere, the maximum chromatic number of an imbedded graph is taken on by a complete graph. We now show that the complete graphs play the same role with respect to maximizing the minimum degree of an imbedded graph.

Thm. 8-16. Let M_n be a closed 2-manifold of characteristic n, and G a graph. If G has an imbedding in M_n, then $\delta(G) \leq g(n)$, where

$$g(n) = \begin{cases} \left\lfloor \dfrac{5 + \sqrt{49 - 24n}}{2} \right\rfloor, & \text{if } n < 2 \\ 5, \text{ if } n = 2. \end{cases}$$

Furthermore, there exists a graph G, imbeddable in M_n, such that $\delta(G) = g(n)$.

Proof: The theorem is known to be true for $n = 2$, as Lemma 5-18 and the icosahedral graph show. Suppose now that G is a graph having p vertices and q edges, with $\delta(G) > g(n)$, and a 2-cell imbedding in $M_n (\neq S_0)$. By standard arguments, $2q \geq 3r$, and also $2q \geq p(g(n) + 1)$. We may assume that G is connected, since if the theorem is true for every component of G, it is also true for G. The euler formula applies, so that

$$\begin{aligned} n &= p - q + r \\ &\leq \left(\frac{2}{g(n)+1} - \frac{1}{3} \right) q \\ &= \left(\frac{5 - g(n)}{3(g(n)+1)} \right) q. \end{aligned}$$

We may assume that $n \leq 0$, as the above inequality is clearly impossible for $n = 1$. But for $n \leq 0$, $g(n) \geq 6$, so that

$$q \leq \frac{-3n(g(n)+1)}{g(n) - 5}$$

But since $\delta(G) > g(n)$, $p \geq g(n) + 2$, and

$$2q \geq (g(n)+2)(g(n)+1).$$

We note that

$$(g(n)+2)(g(n)-5) = \left\lfloor \frac{9 + \sqrt{49-24n}}{2} \right\rfloor \left\lfloor \frac{-5 + \sqrt{49-24n}}{2} \right\rfloor$$

$$> \left(\frac{7 + \sqrt{49-24n}}{2} \right) \left(\frac{-7 + \sqrt{49-24n}}{2} \right)$$

$$= -6n.$$

It now follows that

$$q \geq \frac{(g(n)+2)(g(n)+1)}{2}$$

$$> \frac{-3n(g(n)+1)}{g(n)-5}$$

$$\geq q,$$

a contradiction. Hence $\delta(G) \leq g(n)$.

Now suppose that G has a non 2-cell imbedding in M_n. By a result of Youngs [Y1], G has a 2-cell imbedding in some $M_{n'}$, where $n < n'$. From what we have shown above, $\delta(G) \leq g(n') \leq g(n)$.

Ringel and Youngs have shown [RY1] (also, see Chapter 9) that the complete graph $K_{[g(2-2k)]+1}$ is imbeddable in S_k, for $k \geq 1$. Ringel [R8] has shown that the complete graph $K_{[g(2-k)]+1}$ is imbeddable in N_k, for all positive k except $k = 2$. It remains to find a graph G imbeddable in N_2 and having $\delta(G) = 6$. We begin by considering two projective planes, P_1 and P_2, each with a complete graph K_5 imbedded as indicated in Figures 8-7. Cut open disks D_1 and D_2 from the interiors of the five-sided regions of P_1 and P_2, respectively. Let T be a cylinder disjoint from P_1 and P_2, with simple closed boundary curves C_1 and C_2. Identify C_1 with the boundary of D_1 and C_2 with the boundary of D_2. The result, $(P_1-D_1) \cup T \cup (P_2 - D_2)$, is a klein bottle (see Problem 8-7). The graph G is then constructed by adding the edges $(i,i'),(i,i'+1)$, $i = 1,2,3,4,5$, (where the vertex $6'$ is the same as the vertex $1'$). This completes the proof. #

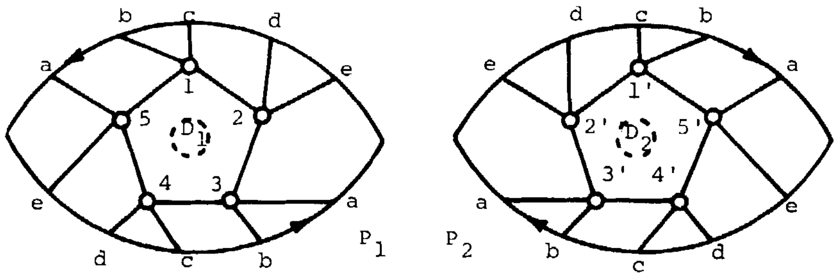


Figure 8-7

We thus make the following observation. The sphere is the only closed orientable 2-manifold for which the maximum minimum degree is not attained by a complete graph. In contrast, we have seen that for every closed 2-manifold (whether orientable or non-orientable), including the sphere, the maximum chromatic number is attained by a complete graph.

8-6. A Four-Color Theorem for the Torus

Thus far in this chapter we have been discussing, for a given closed 2-manifold M, the chromatic number of arbitrary graphs that can be imbedded in M. In this section we impose a restriction on the girth of the graphs we are considering.

Def. 8-17. The girth $g(G)$ of a graph G is the length of a shortest cycle (if any) in G.

Thus a graph G with cycles but no triangles has $g(G) \geq 4$; if G is a forest, we write $g(G) = \infty$. The following theorem was shown by Grötzsch [G7]:

Thm. 8-18. If $\gamma(G) = 0$ and $g(G) \geq 4$, then $\chi(G) \geq 3$.

The graph $G = C_5$ shows that equality can hold in Theorem 8-18. In this section we [KW2] find an upper bound for the chromatic number of toroidal graphs having no triangles, and show that this bound is best possible. We also consider toroidal graphs of arbitrary girth.

Def. 8-19. A connected graph G is said to be n-edge-critical $(n \geq 2)$ if $\chi(G) = n$ but, for any edge x of G, $\chi(G-x) = n - 1$.

The next theorem is due to Dirac [D3].

Thm. 8-20. If G is n-edge-critical, $n \geq 4$, and if $G \neq K_n$, then $2q \geq (n-1)p + n - 3$.

We are now able to find the analogue of Grötzsch's Theorem, for the torus.

Thm. 8-21. If $\gamma(G) \leq 1$ and $g(G) \geq 4$, then $\chi(G) \leq 4$.

Proof: Let $\chi(G) = n \geq 5$. We first assume that G is n-edge-critical, and hence connected. Since $g(G) \geq 4$, $G \neq K_n$. By Theorem 8-20,

$$2q \geq (n-1)p + n-3.$$

Now if $\gamma(G) = 1$, then by Corollary 6-15,

$$4p \geq 2q \geq (n-1)p + n-3;$$

thus $n \leq 4$. If $\gamma(G) = 0$, then $n \leq 3$, by Theorem 8-18. In either case we have a contradiction, so that $n \leq 4$.

Now suppose that G is not n-edge-critical. Then G contains an n-edge-critical subgraph H, and the argument above shows that $\chi(G) = \chi(H) = n \leq 4$. #

The graph of Figure 8-8, constructed by Mycielsky [M5] as an example of a graph having no triangles and chromatic number four, also has genus one, so that the bound of Theorem 8-21 cannot be improved.

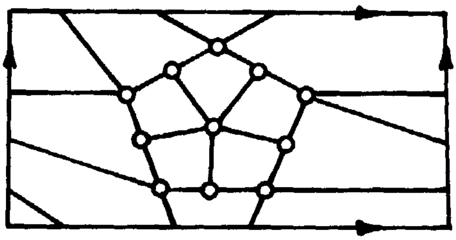


Figure 8-8

The situation for the torus is (almost) completely analyzed in:

<u>Thm. 8-22</u>. If $\gamma(G) \leq 1$ and $g(G) = m$, then

$$\chi(G) \leq \begin{cases} 7, & \text{if } m = 3 \\ 4, & \text{if } m = 4 \text{ or } 5 \\ 3, & \text{if } m \geq 6. \end{cases}$$

Moreover, all the bounds are sharp, except possibly for $m = 5$.

<u>Proof</u>: If $m \geq 6$, then each region in an imbedding for G has at least six edges in its boundary, so that $2q \geq 6r$. As in the proof of Theorem 8-21, we may assume that $\gamma(G) = 1$ and that G is n-edge-critical, where $n = \chi(G)$. If $n \geq 4$, then $2q \geq 3p + 1$, by Theorem 8-20. Then, by Corollary 5-14.

$$\begin{aligned} 0 &= p - q + r \\ &\leq \frac{2q-1}{3} - q + \frac{q}{3} \\ &= -\frac{1}{3}, \end{aligned}$$

an obvious contradiction. Hence for $\gamma(G) \leq 1$ and $g(G) \geq 6$, we must have $\chi(G) \leq 3$. This bound is best possible, as an appropriate subdivision G of the Petersen graph (shown imbedded in S_1 in Figure 8-9) can always be found, having $g(G) = m (m \geq 5)$, $\gamma(G) = 1$, and $\chi(G) = 3$.

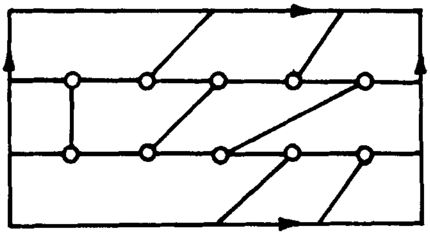


Figure 8-9

For $m = 4$ or 5, it follows from Theorem 8-21 that $\chi(G) \leq 4$. (Now, see Problem 8-9.) Figure 8-8 shows that equality can hold for $m = 4$. For $m = 3$, we refer to the Heawood Map-Coloring Theorem. #

8-7. A Nine-Color Theorem for the Torus and Klein Bottle

The material in this section is due to Ringel [R17].

Def. 8-23. A graph G is said to be 1-imbeddable in a surface S if G can be represented on S so that each edge is crossed over by at most one other edge.

Def. 8-24. The 1-chromatic number $\chi_1(S)$ is the largest $\chi(G)$ such that G is 1-imbeddable in S.

Thm. 8-25. The 1-chromatic numbers $\chi_1(S_k)$ and $\chi_1(N_h)$ are bounded above as shown:

(i) $\chi_1(S_k) \leq \left\lfloor \dfrac{9 + \sqrt{64k+17}}{2} \right\rfloor$, for $k \geq 1$

(ii) $\chi_1(N_h) \leq \left\lfloor \dfrac{9 + \sqrt{32h+17}}{2} \right\rfloor$, for $h \geq 1$.

Thm. 8-26. The following holds:

(i) $\chi_1(S_1) = 9$,

(ii) $\chi_1(N_2) = 9$,

(iii) $\chi_1(S_{83}) = 41$.

The result (iii) above was obtained using a modified current graph (see Chapter 9, for a discussion of the theory of current graphs.)

8-8. k-degenerate Graphs

Before getting to the focal point of this text, in the next chapter, we digress briefly. The generalization below of Theorem 8-11 may be of interest.

A coloring number for graphs closely related to the chromatic number is the vertex-arboricity (see [CKW1].)

Def. 8-27. The vertex arboricity, $a(G)$, of a graph G is the minimum number of subsets that $V(G)$ can be partitioned into so that each subset induces an acyclic graph.

Def. 8-28. The vertex arboricity of a surface S_k is the maximum vertex-arboricty among all graphs which can be imbedded in S_k.

In 1969, Kronk [K3] showed that the vertex arboricity of S_k, $k > 0$, is $\left\lfloor \frac{9 + \sqrt{1 + 48k}}{4} \right\rfloor$. Chartrand and Kronk [CK2], also in 1969, proved that the vertex-arboricity of the sphere is three. The similarity of Kronk's result to those of Ringel and of Ringel and Youngs for the chromatic number suggested the generalization mentioned below.

Def. 8-29. A graph G is said to be k-degenerate if every induced subgraph H of G satisfies the inequality $\delta(H) \leq k$.

<u>Def. 8-30</u>. The <u>vertex partition number</u>, $\rho_k(G)$, <u>of a graph</u> G is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a k-degenerate subgraph of G.

The parameters $\rho_0(G)$ and $\rho_1(G)$ are the chromatic number and vertex arboricity of G, respectively (see Problem 8-4). A general study of k-degenerate graphs has been begun in [LW1], where many of the well-known results for the chromatic number and the vertex-arboricity of a graph have been extended to the parameters $\rho_k(G)$, for all non-negative integers k.

<u>Def. 8-31</u>. The <u>vertex partition number of the closed 2-manifold</u> M_n, denoted by $\rho_k(M_n)$, is the maximum vertex partition number $\rho_k(G)$ of all graphs G which can be imbedded in M_n.

The following theorem (for a complete proof, see [LW2]) almost completely generalizes the results of Kronk, Ringel, Ringel and Youngs, and Appel and Haken mentioned above.

<u>Thm. 8-32</u>. The vertex partition numbers for a closed 2-manifold M_n are given by the formula:

$$\rho_k(M_n) = \left\lfloor \frac{(2k+7) + \sqrt{49-24n}}{2k+2} \right\rfloor ,$$

where $k = 0,1,2,\ldots$; and $n = 2,1,0,-1,-2,\ldots,$ except for the following cases:

(i) in the orientable case, $\rho_1(S_0) = 3$, $\rho_3(S_0) = \rho_4(S_0) = 2$; and

(ii) in the non-orientable case, $\rho_0(N_2) = 6$, $\rho_1(N_2) = 3$ or 4, $\rho_2(N_2) = 2$ or 3.

We make the following comments about the proof of Theorem 8-28. Set $f(k,n) = \left\lfloor \frac{(2k+7) + \sqrt{49-24n}}{2k+2} \right\rfloor$. The proof is divided into three parts.

(i) $\rho_k(M_n) \leq f(k,n)$, for $M_n \neq S_0$ (the proof breaks down for the sphere, reminding us how obstinate the four color problem was.)

(ii) $\rho_k(M_n) \geq f(k,n)$, for $M_n \neq N_2$ (the proof fails for the klein bottle).

(iii) the exceptional cases are treated separately:

(a) for $M_n = S_0$, $\rho_1(S_0) = 3$ appears in the literature; for $\rho_k(S_0) = 2$, $k = 2,3,4$, see Problem 8-5; finally, $\rho_k(S_0) = 1$ for $k \geq 5$, since any planar graph is 5-degenerate, by Lemma 5-18.

(b) For $M_n = N_2$, additional ad hoc arguments are devised. For example, the graph constructed in Figure 8-7 shows that $\rho_5(N_2) \geq 2$; from (i) we see that $\rho_5(N_2) \leq 2$; thus $\rho_5(N_2) = 2$.

8-9. Coloring Graphs on Pseudosurfaces

The pseudosurfaces $S(k;\ n_1(m_1),\ldots,n_t(m_t))$ have been defined in Section 5-5 and re-encountered in Sections 6-7 and 6-8. Dewdney [D1] has studied a subclass of these pseudosurfaces, namely those of the form $S(0,n(2))$:

Def. 8-33. The chromatic number, $\chi(S(0;n(2)))$, of the pseudosurface $S(0;n(2))$ is the largest chromatic number $\chi(G)$ of any graph G that can be imbedded in $S(0;n(2))$.

Thm. 8-34. $\chi(S(0;n(2))) \leq n + 4$, for $n \geq 0$; equality holds for $n = 1, 2, 3, 4$. ǂ

For example, Figure 6-6 shows K_5 imbedded in $S(0;1(2))$, showing that $\chi(S(0;1(2)) \geq 5$. Similarly, K_6 imbeds in $S(0;2(2))$, to give equality for the case $n = 2$. Note that we state this coloring problem for graphs rather than for maps; the dual of G in $S(0;n(2))$ is not a 2-cell imbedding, so that there is not the natural correspondence we find for surfaces. (‡The cases $n = 3$ and 4 were established by Mark O'Bryan and James Williamson respectively.)

Then in 1974, Borodin and Melnikov [BM2] solved this particular problem completely, except for the case $n = 0$ now covered by the four-color theorem; we state the complete solution:

<u>Thm. 8-35</u>.

$$\chi(S(0;n(2))) = \begin{cases} n+4, & 0 \le n \le 4, \\ 8, & n = 5, \\ \left\lfloor \dfrac{7+\sqrt{1+24n}}{2} \right\rfloor, & 6 \le n \le 12, \\ 12, & n \ge 12. \end{cases}$$

Thus we have the map-coloring number for the sphere, where n countries have two components, and that twelve is the largest of all these numbers (see Problem 8-6). Heawood [H3] generalized to ask for the map-coloring number $\chi(S,c)$ for a surface S (orientable or nonorientable) of characteristic n, where each country has at most c components, and showed - for every case but the sphere for $c = 1$ - that this number is bounded above by:

<u>Thm. 8-36</u>.

$$\chi(S,c) \le \left\lfloor \frac{6c+1 \sqrt{(6c+1)^2-24n}}{2} \right\rfloor .$$

Note that the case $c = 1$ is the one of primary interest (the Heawood map coloring theorems), and that the bound <u>does</u> hold for $c = 1$ and $n = 2$ as well (the four-color Theorem.) Moreover, we have seen that $\chi(S_0,2) = 12$.

Recently it has been shown that equality also holds in Theorem 8-36, for certain other cases:

<u>Thm. 8-37</u>. (Jackson and Ringel [JR2]

$$\chi(S_{0,c}) = 6c, \quad \text{for } c \ge 2.$$

Thm. 8-38. (Jackson and Ringel [JR1])

$$\chi(N_{1,c}) = 6c, \quad \text{for} \quad c \geq 1$$

Thm. 8-39. (Taylor [T1])

$$\chi(S_{1,c}) = 6c + 1, \quad \text{for} \quad c \geq 1.$$

Thm. 8-40. (Jackson and Ringel [JR3])

Let $g(c,n) = \left\lfloor \dfrac{6c+1 + \sqrt{(6c+1)^2-24n}}{2} \right\rfloor$;

then $\chi(s,c) = g(c,n)$, if:

(i) $S = N_k$ and $g(c,n) \equiv 1,4,7 \pmod{12}$, unless $k = 2$ and $c = 1$ or 2.

(ii) $S = S_k$, c is even, and $g(c,n) \equiv 1 \pmod{12}$

(iii) $S = S_k$, c is odd, and $g(c,n) \equiv 4,7 \pmod{12}$.

It remains to construct the "verification figures" (i.e. the appropriate pseudosurface imbeddings) for the cases not covered above. (see Problem 8-17.)

8-10. The Cochromatic Number of Surfaces

The material in this section is taken from Straight ([S18] and [S19].)

Def. 8-41. The cochromatic number, $z(G)$, of a graph G is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces either an empty or a complete subgraph of G.

Def. 8-42. The cochromatic number, $z(S)$, of a surface S, is the maximum $z(G)$ such that G imbeds in S.

Thm. 8-43. $z(S_n) \leq \chi(S_n)$, with equality if and only if $n = 0$.

For example, $z(C_5 \cup K_4) = 4$, so that $z(S_0) = 4$.

<u>Thm. 8-44</u>. For $n \geq 4$, $z(N_n) < \chi(N_n)$.

<u>Thm. 8-45</u>. (i) $z(S_0) = 4$

(ii) $z(N_1) = 5$

(iii) $z(N_2) = 6$

(iv) $z(N_3) = 6$

(v) $z(N_4) = 7$.

Straight conjectures that, in general, $z(S)$ is the maximum n such that $\bigcup_{i=1}^{n} K_i$ imbeds in S.

8-11. Problems

8-1.) Let $G \neq \overline{K_n}$; show that $\chi(G) = 2$ if and only if G is bipartite.

8-2.) Find $\chi(C_n)$, for all cycles C_n.

8-3.) Find an imbedding of K_7 on S_1. Form the dual of this imbedding, and explain why this shows that $\chi(S_1) \geq 7$.

8-4.) Show that $\rho_0(G) = \chi(G)$ and that $\rho_1(G) = a(G)$.

8-5.) Show that $\rho_k(S_0) = 2$, for $k = 2,3,4$. (Hint: for each k, use induction to show that $\rho_k(S_0) \leq 2$. Then consider graphs of certain regular polyhedra.)

*8-6.) Show (as Ringel and Heawood did) that any map on the surface of the sphere, in which each country has at most two components, can be colored with 12 colors. (Hint: it may be helpful to show that if a graph G is <u>n-critical</u>, then $\delta(G) \geq n - 1$; i.e. if $\chi(G) = n$, but $\chi(G-v) = n - 1$, for all vertices v in G. Then form two "dual" graphs for an arbitrary map, one where the vertices represent countries, the other with vertices representing regions of land. Use also the fact that $q \leq 3p - 6$, for planar connected graphs.) Interpret this result for pseudo-surfaces. Compare Problem 6-9.

8.7.) Show that the connected sum of two projective planes (as in the proof of Theorem 8-16) is a klein bottle. (Hint: find the characteristic of the resulting closed 2-manifold, using the graph G constructed in the same proof.)

8.8.) Show that $\chi(G) = 4$, for the graph G of Figure 8-8.

**8-9.) Does there exist a toroidal graph G having $g(G) = 5$ and $\chi(G) = 4$?

*8-10.) Prove or disprove: K_9 imbeds in $S(0;5(2))$ (and hence $\chi(S(0;5(2)) = 9.)$ Is $\chi(S(0;n(2))) = n + 4$ for all n? Compare Problems 6-9 and 8-6. Does K_{10} imbed in $S(0;7(2))$?

**8-11.) Note that, in Theorem 8-32, there are exactly six cases not <u>known</u> to comply with the general formula, including four which are known <u>not</u> to comply. The known non-orientable value $(\rho_0(N_2) = 6)$ is one less than that predicted by the formula. The three known orientable values $(\rho_1(S_0), \rho_3(S_0), \rho_4(S_0))$ are each one more than the formula would give. What guess do you make, for $\rho_1(N_2)$ and $\rho_2(N_2)$?

8-12.) Define the <u>chromatic number of a group</u> to be: $\chi(\Gamma) = \min_{\Delta} (G_\Delta(\Gamma))$ (cf Babai [B1]). Find $\chi(\Gamma)$, for

$\Gamma = Z_n,\ Z_2 \times Z_n,\ (Z_2)^n,\ D_n,\ Z_2 \times D_n,\ S_n$. Show $\chi(\Gamma) \leq 3$, for Γ finite abelian or $\Gamma = A_n$.

*8-13.) If Γ has a normal subgroup Γ_1, show that $\chi(\Gamma) \leq \chi(\Gamma/\Gamma_1)$. Thus if Γ is solvable (note that this includes all odd order groups), then $\chi(\Gamma) \leq 3$. If Γ has a subgroup of index 2, then $\chi(\Gamma) \leq 2$.

**8-14.) Is $\chi(\Gamma) \leq 3$ for all groups Γ?

8-15.) Is $\chi(\Gamma_1) \leq \chi(\Gamma_2)$, if $\Gamma_1 \leq \Gamma_2$?

8-16.) Extend the definition of $\chi(\Gamma)$ to $\rho_k(\Gamma)$, for arbitrary vertex partition numbers. Study this family of parameters.

**8-17.) Find the imbeddings called for at the conclusion of Section 8-9.

**8-18.) Study the conjecture given at the conclusion of Section 8-10.

CHAPTER 9

QUOTIENT GRAPHS AND QUOTIENT MANIFOLDS

(and Quotient Groups!)

In this chapter we present the beautiful theory of quotient graphs and quotient manifolds. This theory was introduced by Gustin [G8], developed by Youngs (see, for example, [Y2], [Y3], and [Y6]), and used by Ringel and Youngs to find the genus of K_n and thus solve the Heawood map-coloring problem, as explained in the previous chapter. The application of the theory to the graphs K_n falls into 12 cases, depending upon the residue modulo 12 of n. The theory applies directly, for $n \equiv 0, 3, 4, 7 \pmod{12}$, as will be seen shortly. For the remaining eight cases, the theory is modified (by the theory of vortices) to complete the solution. We will treat the case $n \equiv 7 \pmod{12}$ completely, and discuss the case $n \equiv 10 \pmod{12}$; this will give an indication of the power and beauty of the theory. The remaining ten cases are treated similarly, although many complicating details must be handled properly. (Perhaps one should expect a complicated solution, to a complicated problem!)

We will then see how the theory (designed to produce triangular imbeddings for K_n) may be extended to handle first triangular imbeddings for Cayley graphs in general, and then to handle regular imbeddings $(r = r_n,\ n \geq 3)$ in general, for Cayley graphs. This is the scope of the theory, as announced by Gustin. But Youngs' theory of vortices [Y3] hints at an even more general theory; we present this general theory, as unified by Jacques [J2]. We conclude the chapter with a sampling of applications of this powerful theory.

9-1. The Genus of K_n

Let us now turn our attention to the complete graphs K_n. Recall that if we can show that

$$\gamma(K_n) = \left\{ \frac{(n-3)(n-4)}{12} \right\}, \quad n \geq 3.$$

the Heawood map-coloring theorem will be established. We see the origin of the number on the right-hand side of the above equality in the following:

Thm. 9-1. Let K_n be minimally imbedded in a surface M. Then

$$\gamma(K_n) = \gamma(M) = \frac{(n-3)(n-4)}{12} + \frac{1}{6} \sum_{i \geq 4} (i-3) r_i.$$

Proof: From Corollary 6-14, we know that

$$\gamma(K_n) \geq \frac{n(n-1)}{12} - \frac{n}{2} + 1 = \frac{(n-3)(n-4)}{12},$$

with equality if and only if K_n has a triangular imbedding. But we can be more specific than this; we can get some information about the non-triangular regions (if any!) If K_n is minimally imbedded in M, then clearly $\gamma(K_n) = \gamma(M)$, and the imbedding is 2-cell, by Theorem 6-11. Thus the euler formula applies, and

$$\begin{aligned}
\gamma(M) &= 1 - \frac{p}{2} + \frac{q}{2} - \frac{r}{2} \\
&= 1 - \frac{p}{2} + \frac{q}{6} + \frac{q}{3} - \frac{r}{2} \\
&= 1 - \frac{n}{2} + \frac{n(n-1)}{12} + \frac{1}{6} \sum_{i \geq 3} i r_i - \frac{1}{6} \sum_{i \geq 3} 3 r_i \\
&= \frac{(n-3)(n-4)}{12} + \frac{1}{6} \sum_{i \geq 4} (i-3) r_i. \qquad \#
\end{aligned}$$

We now see that if K_n has a triangular imbedding $(r_i = 0,$ $i \geq 4)$, then

$$\gamma(K_n) = \frac{(n-3)(n-4)}{12},$$

and $(n-3)(n-4) \equiv 0 \pmod{12}$; i.e. $n \equiv 0, 3, 4, 7 \pmod{12}$. Moreover, in general, $\gamma(K_n) = \left\{\frac{(n-3)(n-4)}{12}\right\}$, if we can show that $\sum_{i \geq 4} (i-3)r_i \leq 5$. It is now perhaps apparent why there are twelve cases for the determination of $\gamma(K_n)$, and why only four of them admit triangular imbeddings. Let us consider these four cases now.

What is needed is a method of constructing triangular imbeddings. The naive trial-and-error method easily handles $n = 3, 4$, and 7; it becomes a bit sticky at $n = 12$. We turn away from the drawing board and employ the algebraic description of 2-cell imbeddings given us by Edmonds' permutation technique. Now we seek a means of selecting judiciously the local vertex permutations $(p_1, p_2, \ldots, p_n)$; this is what the method of quotient graphs and quotient manifolds is all about!

9-2. The Theory of Quotient Graphs and Quotient Manifolds, as Applied to K_n

We introduce this theory by means of the example K_7. Let K_7 be imbedded in S_1; by Theorem 9-1, this imbedding must be a triangulation. Select a group Γ for which K_7 is a Cayley color graph; in this case, we can only pick $\Gamma = Z_7 = \langle x \mid x^7 = e \rangle$, but we pick x, x^2, x^3 as generators for Γ. Label the vertices of K_7 with the elements of Γ (one should also think of the edges as being directed and colored appropriately.) Now take the dual of this imbedding; assume this is as pictured in Figure 8-6. Each region in the dual (formerly a vertex of K_7) is now labeled with a distinct group element: 0, 1, 2, 3, 4, 5, or 6. We proceed to label the boundary edges of each region of the dual, as indicated in Figure 9-1. (Note that $(g^{-1}h)^{-1} = h^{-1}g$.) We observe that the seven regions of the dual have identical boundaries: 1, 3, 2, 6, 4, 5.

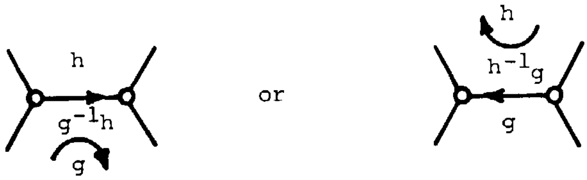


Figure 9-1

We summarize this information in a map having one region, as shown in Figure 9-2. But $1^{-1} = 6$, $2^{-1} = 5$, and $3^{-1} = 4$; thus the six

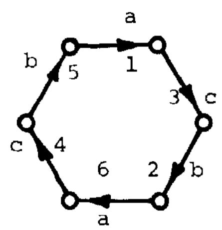


Figure 9-2

edges have a natural identification, in three pairs; we make this identification, to form a closed orientable 2-manifold, as in Figure 9-3. The result (in this case, S_1) is the quotient manifold; the corresponding graph (actually, in this case, it is a pseudograph) is the quotient graph. The subgroup of Γ consisting of all vertices

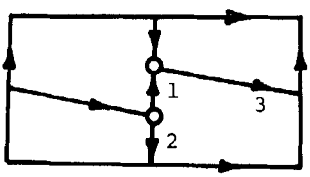


Figure 9-3

of K_7 whose regions in the dual had the same ordering of directed edges in their boundaries as did $e = 0$ (in this case, Z_7 itself) gives rise to the <u>quotient group</u> (in this case, the trivial group). The index of this subgroup in Γ (in this case, 1) is the <u>index of the imbedding</u>. The point is this: <u>all the information needed to describe a triangular imbedding of K_7 in S_1 is contained in the quotient graph</u>, imbedded in its quotient manifold (which, after all, was obtained by (modding out" the subgroup Z_7).

To see this, let the permutation at vertex 0 be given by the boundary of the single region in the quotient manifold:

0: 1, 3, 2, 6, 4, 5.

The remaining local vertex permutations may be obtained by successively adding 1 to every entry in this row (remember, we are in the group Z_7!):

0: 1, 3, 2, 6, 4, 5
1: 2, 4, 3, 0, 5, 6
2: 3, 5, 4, 1, 6, 0
3: 4, 6, 5, 2, 0, 1
4: 5, 0, 6, 3, 1, 2
5: 6, 1, 0, 4, 2, 3
6: 0, 2, 1, 5, 3, 4.

Now, compute orbits (corresponding to regions in a 2-cell imbedding of K_7):

0-1-5	1-2-6	3-5-6
0-2-3	1-3-4	3-6-4
0-3-1	1-4-2	
0-4-6	1-6-5	
0-5-4	2-4-5	
0-6-2	2-5-3.	

We see that we have an imbedding of K_7 for which $r = r_3 = 14$; that is -- a triangular imbedding. This is no accident; it is guaranteed by the theory of quotient graphs and quotient manifolds!

We now examine this theory more closely. (At first we consider only triangular imbeddings for K_n; in sections 9-4 - 9-7, we generalize.) Recall that, for a graph K, $D^* = \{(u,v) \mid [u,v] \in E(K)\}$; in this chapter, we set $D^* = K^*$.

Def. 9-2. A current graph is a triple (K, Γ, λ), where K is a pseudograph, Γ is a finite group with identity e, and $\lambda: K^* \to \Gamma - e$ is a map such that $(\lambda(a))^{-1} = \lambda(a^{-1})$, for all $a \in K^*$.

Note that a Cayley color graph is a current graph.

Def. 9-3. Let K be a 2-cell imbedded in a surface M, with $v \in V(K)$. We say that Kirchoff's Current Law (KCL) holds at v if the product of the currents directed away from v, taken in the order given by p_v, is the identity, e.

Def. 9-4. Given a group Γ of order n and a subgroup Ω of order m, a quotient manifold $M(\Gamma/\Omega)$ is a closed 2-manifold having the following properties:

1.) $M(\Gamma/\Omega)$ is oriented and has a 2-cell decomposition, given by a pseudograph K.
2.) The 2-cells of the decomposition are $N = n/m$ in number, are named $[X]$, $X \in \Gamma/\Omega$, and each is an $(n-1)$-gon. N is called the index of $M(\Gamma/\Omega)$.
3.) The oriented edges in K carry currents from $\Gamma - e$.
4.) K is a current graph.
5.) For $X \in \Gamma/\Omega$, each element of $\Gamma - e$ appears exactly once as a current on some edge of $[X]^*$, the oriented boundary of $[X]$ (with orientation inherited from $M(\Gamma/\Omega)$.)
6. $[X]$ meets $[Y] \neq [X]$ along those edges of $[X]^*$ whose currents are in the set $X^{-1}Y$, and nowhere else.
7. $[X]$ meets itself along those edges of $[X]^*$ whose currents are in the set $(X^{-1}X) - e$, and nowhere else. Moreover, the meeting is along a singular

edge (a loop), if and only if the current is of order 2.

8.) Each vertex of K is of degree 1 or 3.

9.) The current going into each vertex of degree 1 is of order 3.

10.) The KCL holds at each vertex of degree 3.

Thm. 9-5. If K_n is triangularly imbedded in a surface M, then for any group Γ of order n and 1-1, onto map $\lambda: V(K_n) \to \Gamma$, there exists a subgroup Ω and a quotient manifold $M(\Gamma/\Omega)$.

It is clear that $\Omega = \{e\}$ will always satisfy Theorem 9-5; but this gives no progress for the imbedding problem, since $M(\Gamma/\{e\}) = M$. Economy of effort here is inversely proportional to the index of the imbedding.

The graph K of Definition 9-4 is called a quotient graph; it is denoted also by $S(\Gamma/\Omega)$. Note that $S(\Gamma/\Omega)$ is not the graph alone, but the graph together with a collection of local vertex permutations that gives rise to an imbedding of the graph in a quotient manifold.

For additional examples of quotient graphs and quotient manifolds, refer to [Y2] or to subsequent sections of this chapter.

9-3. The Genus of K_n (again)

It is the next theorem which is basic to determining $\gamma(K_n)$.

Thm. 9-6. If $S(\Gamma/\Omega)$ is a quotient graph, where Γ has order n, then there exists a triangular imbedding of K_n.

Thus if we can find an appropriate quotient graph (and this may be a considerably easier task, if Ω is of low enough index), we have found a combinatorial equivalent of a triangular imbedding.

The justification for the above theory is given very nicely by Youngs. We prove here only the special case of an index one imbedding, with $\Gamma = Z_n$.

<u>Thm. 9-7.</u> If $K = S(Z_n/Z_n)$ is a quotient graph such that

(i) K is cubic

(ii) $\lambda: K^* \to Z_n - 0$ is 1-1 and onto,

then K_n has a triangular imbedding.

<u>Proof</u>: Since $\Omega = \Gamma$, K is of index one. Hence there is exactly one region in the cellular decomposition of the quotient manifold $M(Z_n/Z_n)$. This region must contain every edge $\alpha \in K^*$ in its boundary. Since λ is onto, for every $g \in Z_n - 0$, there is an edge $\alpha \in K^*$ such that $\lambda(\alpha) = g$. Since λ is 1-1, there is no edge $\beta \neq \alpha$ such that $\lambda(\beta) = g$. Thus the succession of currents on the boundary of the region is a cyclic permutation of $Z_n - 0$ of length $(n-1)$; we take this permutation as p_0 (where we label the vertices of K_n by: $V(K_n) = \{0,1,2,\ldots,n-1\}$.) Suppose p_0 is given by:

$$p_0: (a_1, a_2, \ldots, a_{n-1});$$

then the remaining p_i are given by:

$$p_i: (a_1 + i,\ a_2 + i,\ \ldots\ ,\ a_{n-1} + i),$$

$i = 1, 2, \ldots, n-1$; all arithmetic is done in $\Gamma = Z_n$. Note that $a_k + i \neq i$, for otherwise, $a_k = 0$. Hence the collection $(p_0, p_1, \ldots, p_{n-1})$ determines a 2-cell imbedding of K_n. It remains to show that $r = r_3$ for this imbedding.

Now suppose that $p_0(a) = b$, and let $\alpha, \beta \in K^*$, with $\lambda(\alpha) = a$, $\lambda(\beta) = b$. The situation, by property (i) of the hypothesis, must be as depicted in Figure 9-4. (The arrow gives the orientation at the vertices; by Edmond's algorithm, the region boundaries have the reverse orientation.)

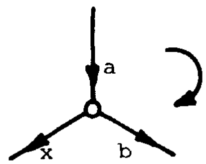


Figure 9-4

But, form the KCL, $-a + b + x = 0$, or $x = a - b$. Now, also, $p_0(-b) = x = a - b$. Therefore, $p_b(0) = x + b = a$. Now let (u,v) be any directed edge in K_n. We will show that (u,v) is in the boundary of a triangular region, to complete the proof. Compute the orbit, beginning

$$u\text{-}v\text{-} \qquad .$$

Let $p_v(u) = w$; then we have

$$u\text{-}v\text{-}w\text{-} \qquad .$$

But since $p_v(u) = w$, $p_0(u-v) = w - v$. Letting $u - v = a$ and $w - v = b$ in the above discussion, we see that

$$p_{w-v}(0) = u - v;$$

hence $p_w(v) = u$, and we have

$$u\text{-}v\text{-}w\text{-}u\text{-} \qquad .$$

Next, let $v - w = a$ and $u - w = b$; then (since $p_w(v) = u$, $p_0(v-w) = u-w$)

$$p_{u-w}(0) = v - w,$$

and $p_u(w) = v$. Thus

u-v-w-u-v

constitutes an orbit under P for K_n, of length 3. This completes the proof. #

We next prove that $\gamma(K_n)$ is as predicted, for $n \equiv 7 \pmod{12}$. By Theorem 9-7, we need only find a cubic quotient graph, $K = S(Z_n/Z_n)$, with a 1-1 onto assignment $\lambda: K^* \to Z_n - 0$. To simplify the presentation of K, instead of showing K imbedded in $M(Z_n/Z_n)$, we draw K in the plane with the following device describing the local vertex permutations: a solid vertex has its incident edges ordered clockwise; a hollow vertex, counterclockwise. For example, Figure 9-3 and 9-5 depict exactly the same $K = S(Z_7/Z_7)$.

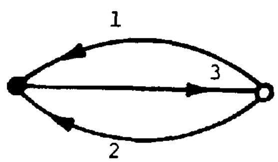


Figure 9-5

Thm. 9-8. K_{12s+7} has a triangular imbedding.

Proof: Let $\Gamma = Z_{12s+7}$. We have already treated the case $s = 0$. The cases $s = 1$ and $s = 2$ are shown in Figures 9-6 and 9-7 respectively. The generalization to all s is as in Figure 9-8, with the vertical edges

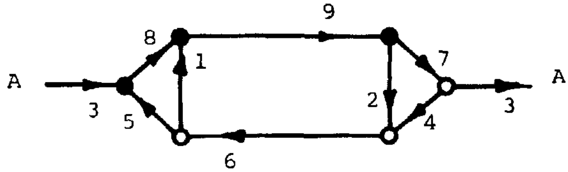


Figure 9-6

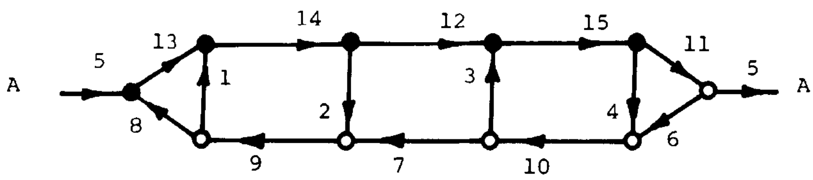


Figure 9-7

directed alternately and carrying the currents 1, 2, ... , 2s consecutively. All other currents are determined by the KCL. It is straightforward to check that K has 6s + 3 edges, or 12s + 6 directed edges,

Figure 9-8

carrying all the currents from $Z_{12s+7} - 0$. The single region in $M(Z_{12s+7}/Z_{12s+7})$ is:

$$
\begin{aligned}
&(2s+1)-(5s+3)-(5s+4)-\ldots-(6s+3)-(4s+3)-(2s+2)-\\
&(-2s)-(-6s-3)-(-2s+1)-(2s+3)-\ldots-(-1)-(3s+2)-\\
&(-2s-1)-(-4s-3)-(2s)-(4s+2)-(2s-1)-(-4s-4)-\ldots-\\
&(1)-(-5s-3)-(-3s-2)-(-3s-3)-\ldots-(-4s-2)-(-2s-2).
\end{aligned}
$$

Thus K is a quotient graph (imbedded in S_{s+1}), satisfying the properties of Theorem 9-7, and K_{12s+7} has a triangular imbedding, for all non-negative integers s.

#

Cor. 9-9. $\gamma(K_{12s+7}) = (3s+1)(4s+1)$, $s \geq 0$.

The theory of current graphs with <u>vortices</u> is employed to find $\gamma(K_n)$, for $n \equiv 10 \pmod{12}$. The group Z_{12s+7} is used to find a triangular imbedding for $K_{12s+10} - K_3$; the current graph in this case has three vertices of degree one; otherwise it is essentially the K of Figure 9-8. (See [Y3].) This is called the <u>regular</u> part of the problem. Next, the surface in which $K_{12s+10} - K_3$ is triangularly imbedded is modified -- by the addition of one well-chosen handle (See [W12])-- so as to accommodate the three edges removed in K_3. This is called the <u>additional adjacency</u> part of the problem. The final result is a (non-triangular) imbedding of K_{12s+10} in a surface of the appropriate genus.

The remaining ten cases for $\gamma(K_n)$ are handled similarly, with varying degrees of complexity; see [Y4], [RY2], [Y5], [TWY2], [RY3], [RY4], [TWY1], and [M4]. A constructive proof is given for each case but $n \equiv 0 \pmod{12}$; for this case the theory of finite fields supplements the theory of quotient graphs in establishing the <u>existence</u> of a triangular imbedding (see [TWY1]). We now turn our attention to graphs other than K_n.

9-4. Extending the Theory

Slight modifications of the above theory allow us to attack other Cayley color graphs for which triangular imbeddings are possible. In fact, there is also a theory for quadrilateral imbeddings of Cayley color graphs. Rather than discuss the modifications of the theory in this section, we will examine three examples; an even more general theory will be given in the next section.

<u>Example 1</u>: Let us try to find a triangular imbedding for $K_{2,2,2}$. Since $K_{2,2,2}$ is of order six, there are two possible groups to work with: Z_6 and S_3. We try them both.

Taking $Z_6 = \langle x,y \mid x^6 = y^3 = e = xyx^{-1}y^{-1}\rangle$, we see that $K_{2,2,2}$ is a Cayley color graph for Z_6. (Here, $x = 1$ and $y = 2$, with $e = 0$; y is clearly redundant, but its presence as a generator enables us to picture Z_6 with $K_{2,2,2}$.) We try for an index one imbedding. This requires a quotient graph K having four directed edges (corresponding to $1,5,2,4$) and one region in its quotient manifold. But then K has two edges; no such K is possible. Next we try for an index two imbedding. This requires a quotient manifold having two quadrilateral regions, and hence a quotient graph with four undirected, and eight directed, edges. The quotient graph is shown in Figure 9-9, imbedded in its quotient manifold, the sphere.

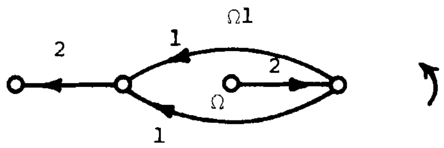


Figure 9-9

Let $\Omega = \{0,2,4\}$: 1, 5, 4, 2

$\Omega_1 = \{2,3,5\}$: 1, 2, 4, 5.

Then:

0: 1, 5, 4, 2	1: 2, 3, 5, 0
2: 3, 1, 0, 4	3: 4, 5, 1, 2
4: 5, 3, 2, 0	5: 0, 1, 3, 4

gives a triangular imbedding of $K_{2,2,2}$.

The orbits are:

0-1-2	1-3-2
0-2-4	1-5-3
0-4-5	2-3-4
0-5-1	3-5-4.

Figure 9-10 shows this imbedding of $K_{2,2,2}$ (the octahedral graph) in S_0.

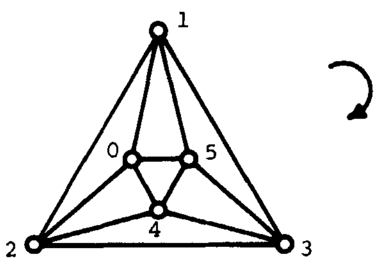


Figure 9-10

Suppose now that we take $\Gamma = S_3 = \langle (23), (12), (132) \rangle$; then $K_{2,2,2}$ is again a Cayley color graph. An index one imbedding <u>is possible</u>, in this case, as shown by Figure 9-11, where the

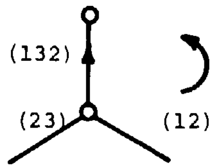


Figure 9-11.

quotient graph is given. Note that the vertex of degree one has a current of order three going into it; the two singular edges carry currents of order two, and the vertex of degree three satisfies the KCL: $(23)(12)(132) = (1)(2)(3) = e$.

There is one coset, $\Omega = S_3$. We have:

$$\begin{aligned}
e&: \ (23),(12),(132),(123)\\
(12)&: \ (132),e,(23),(13)\\
(23)&: \ e\ ,(123),(13),(12)\\
(13)&: \ (123),(132),(12),(23)\\
(123)&: \ (13),(23),e,(132)\\
(132)&: \ (12),(13),(123),e.
\end{aligned}$$

The orbits are:

e-(23)-(123)	(12)-(13)-(23)
e-(12)-(23)	(12)-(132)-(13)
e-(132)-(12)	(23)-(13)-(123)
e-(123)-(132)	(13)-(132)-(123)

Figure 9-12 shows this imbedding, which agrees with Figure 9-10, except for labeling.

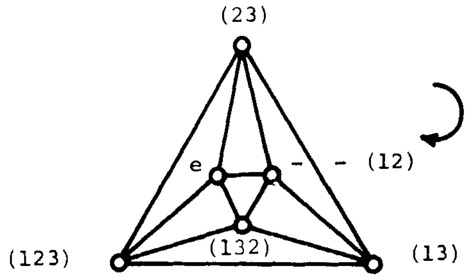


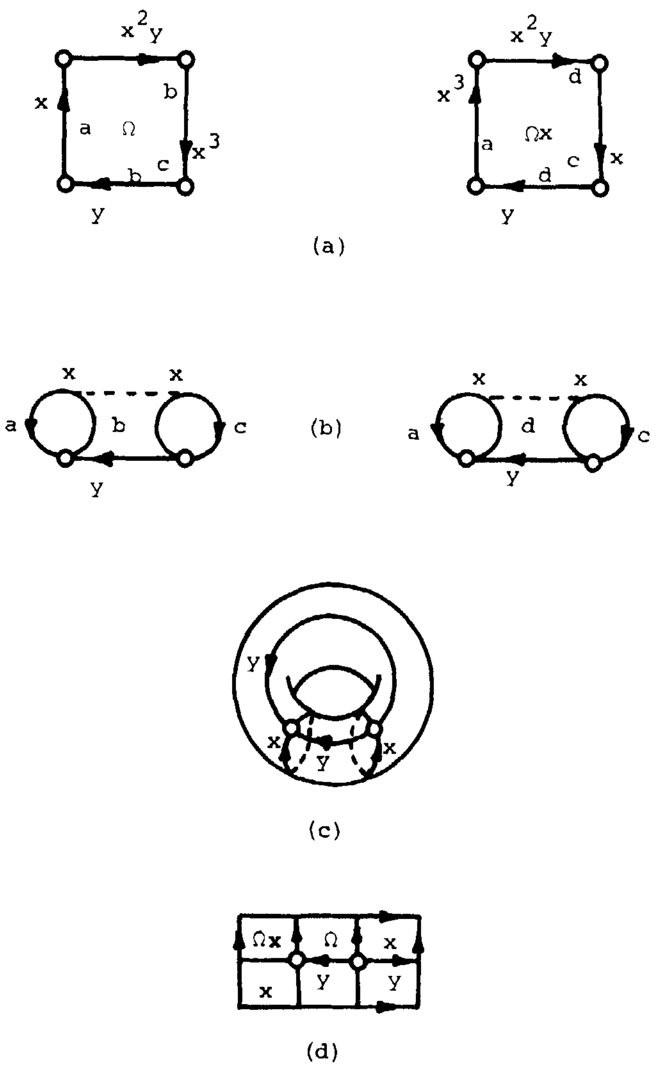
Figure 9-12

Example 2: Consider the quadrilateral imbedding of $K_{4,4}$ (for the quaternions) given in Figure 7-3. We find the quotient graph for this imbedding and choice of groups. Looking at the dual, we find that e, y, x^2, and x^2y have the same boundary: x, x^2y, x^3, y. Furthermore, $\Omega = \{e,y,x^2,x^2y\}$ is a subgroup of Q, since $x^2 = y^2$ and $x^2y = y^3$, with $y^4 = e$. Also, $\Omega x = \{x,x^3y,x^3,xy\}$ is a coset of Q with respect to Ω, and each of these regions (in the dual) has boundary: x^2y,x,y,x^3. Hence this imbedding of $K_{4,4}$ (for Q) is of index two ($[Q:\Omega] = |Q|/|\Omega| = 2.$) We "mod out" regions (in the dual) with identical boundaries, and form the quotient graph, in its quotient manifold S_1, as indicated in Figure 9-13. To make the appropriate identification, we use the general relationship:

$$(X,g)^{-1} = (Xg,g^{-1}),$$

to match the edge g in the boundary of the region for coset X. The computations for this situation are:

$$(\Omega,x)^{-1} = (\Omega x,x^3)$$
$$(\Omega,x^2y)^{-1} = (\Omega,y)$$
$$(\Omega,x^3)^{-1} = (\Omega x,x)$$
$$(\Omega x,y)^{-1} = (\Omega x,x^2y).$$



x²y
x
a
Ω
b
x³
c
b
y
x²y
x³
d
a
Ωx
c
d
x
y
(a)
x
x
a
b
c
y
(b)
x
x
a
d
c
y
y
x
x
y
(c)
Ωx
Ω
x
x
y
y
(d)

Figure 9-13

<u>Example 3</u>: Finally, we find a quadrilateral imbedding of K_{21} in S_{43} (with $r = r_4 = 105$). It is an easy exercise to check that this imbedding is compatible with the euler formula. Let us hope for the best, and try for an index one imbedding with $\Gamma = Z_{21}$. Then we seek a quotient manifold with $r = r_{20} = 1$. This requires a quotient graph K with 20 directed, or 10 undirected, edges. For a quadrilateral imbedding, we would like K to be regular, of degree 4. A likely candidate is $K = K_5$. Now we must make an assignment $\lambda: K^* \to Z_{21} - 0$, satisfying the KCL: then we must make a selection $(p_1, p_2, \ldots, p_5)$ inducing a single orbit, of length 20, for K_5. We find the assignment, using the fact that K_5 is eulerian; the selection is made, using the fact that $\gamma_M(K_5) = 3$ (i.e. S_3 will be the quotient manifold), with $r = r_{20} = 1$; see Figure 9-14.

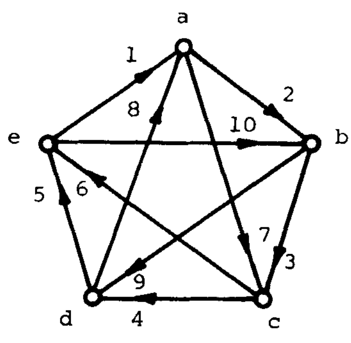


Figure 9-14

The local vertex permutations are:

a: (b,c,d,e)

b: (a,c,d,e)

c: (a,b,d,e)

d: (a,b,c,e)

e: (a,c,d,b)

The single orbit is:

a-b-c-d-e-b-a-c-b-d-c-e-d-a-e-c-a-d-b-e,

giving rise to:

0: 1,2,3,4,5,10,19,7,18,9,17,6,16,8,20,15,14,13,12,11

1: 2,3,4,5,6,11,20,8,19,10,18,7,17,9,0,16,15,14,13,12

.

.

.

20: 0,1,2,3,4,9,18,6,17,8,16,5,15,7,19,14,13,12,11,10,

which in turn gives $r = r_4 = 105$, for K_{21} on S_{43}, using Edmonds' permutation technique.

Although this imbedding may be of little general interest, it does give further indication of the economy of effort possible by seeking quotient graphs in quotient manifolds, rather than graphs in manifolds. Moreover, minimal imbeddings have been found by first finding non-minimal imbeddings of related graphs. For example, Youngs [Y4] imbeds K_{12s+7} on $S_{12s^2+13s+3}$, to find K_{12s+10} on $S_{12s^2+13s+4}$.

9-5. The General Theory

As we have seen, the standard proof-technique for establishing a genus or maximum genus formula is to first use the Euler formula

$$p - q + r = 2 - 2k ,$$

describing a 2-cell imbedding of the graph G in the surface S_k, to find a bound for the parameter under the study. For $\gamma(G)$, we get a lower bound (see, for example, Corollaries 6-14 and 6-15); for $\gamma_M(G)$ we get an upper bound (Theorem 6-24). The second step is to construct an imbedding of G attaining the bound obtained. Inductive constructions are particularly useful for graphs which can be defined as repeated cartesian products (as in the proofs of Theorems 6-36 and 7-13). A local vertex permutation scheme, such as that discussed in Section 6-6, is frequently helpful (as in the proofs of

Theorems 6-25, 6-37, and 6-39). Ad hoc techniques have occasionally been successful (see the proof of Theorem 6-45).

But the most powerful and efficient technique (when it applies, and this is reasonably often!) is the method of quotient graphs and quotient manifolds. In the preceding sections of this chapter, we have discussed this theory, as introduced by Gustin and developed by Youngs, and its application to triangular imbeddings of complete graphs. We have also hinted at a generalization of this theory, and to this we now devote out attention.

The material in this section is based on work by Jacques [J2].

Given a finite group Γ with a set Δ of generators for Γ, recall that the Cayley color graph $C_\Delta(\Gamma)$ has vertex set Γ, with (g,g') a directed edge -- labeled with generator δ_i -- if and only if $g' = g\delta_i$. We assume that, if $\delta_i \in \Delta$, $\delta_i^{-1} \notin \Delta$, unless δ_i has order 2. In this latter case, the two directed edges $(g,g\delta_i)$ and $(g\delta_i,g)$ are represented as a single undirected edge $[g,g\delta_i]$, labeled with δ_i. The graph obtained by deleting all labels and arrows (directions) from the edges of $C_\Delta(\Gamma)$ we called (in Definition 4-23) the Cayley graph, $G_\Delta(\Gamma)$. This graph has all edges of the form $[g,g\delta_i]$, for $g \in \Gamma$ and $\delta_i \in \Delta$.

As examples of graphs which are Cayley graphs, we have Q_n for the elementary 2-group $(Z_2)^n$, $K_{4,4}\times Q_n$ (where "x" denotes the cartesian product of two graphs) for the hamiltonian group $Q\times(Z_2)^n$ (where Q denotes the quaternions and here "x" indicates the direct product of two groups), K_n for the Cyclic group Z_n, $K_{2n,2n}$ for Z_{4n}, $K_{n,n,n}$ for Z_{3n}, and the graph Π_n of Jacques [J2] for the symmetric group S_n. The genus is known for each of the above families of graphs, and in each case the computation is facilitated by a knowledge of the theory we are about to present.

Consider a Cayley graph $G_\Delta(\Gamma)$, 2-cell imbedded in a closed orientable 2-manifold M. We study the imbedding of the Cayley color graph, $C_\Delta(\Gamma)$, thus determined. Let $\Delta^{-1} = \{\delta_i^{-1} \mid \delta_i \in \Delta\}$ and form $\Delta^* = \Delta \cup \Delta^{-1}$. The elements of Δ^* are called currents. The imbedding is characterized (see Section 6-6) by giving, at each $g \in \Gamma$, the cyclic permutation σ_g of $g\Delta^*$ determined by the orientation. For example, see Figure 9-15. Let Ω be a subgroup of Γ such that if $\Omega g = \Omega g'$, then $\sigma_g^* = \sigma_{g'}^*$, where σ_h^* is the cyclic permutation of Δ^* induced by the action of σ_h on $h\Delta^*$; such a subgroup always exists, since we can take $\Omega = \{e\}$,

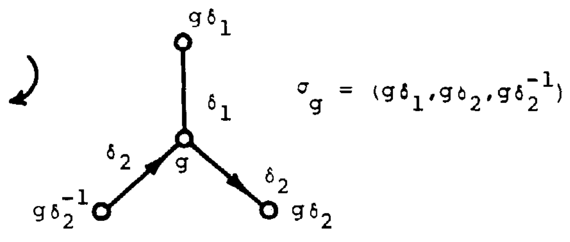


Figure 9-15

where e is the identity of Γ. It is to our advantage, however, to choose Ω as large as possible. In the terminology of Jacques, Γ and Ω determine a quotient constellation C' for the constellation $C = (C_\Delta(\Gamma)$ in $M)$. The quotient constellation is an imbedding of the Schreier coset graph (see Section 4-3) for Ω in Γ, the imbedding being determined by the collection $\{\sigma_h^*\}$, taken over any set $\{h\}$ of right coset representatives of Ω in Γ. The reduced constellation $(C')^*$ is the dual of the quotient constellation. This is a 2-cell imbedding of a pseudograph K (with each edge directed and labeled with the current of its dual edge; see Figure 9-16) in a closed orientable 2-manifold, called by Youngs [Y3] the quotient graph and quotient manifold respectively, for $C_\Delta(\Gamma)$ and Ω. Youngs, as we have seen in Section 9-2, obtains this structure by a different process: he first takes the dual of $C_\Delta(\Gamma)$ in M, and then "mods out" faces with identically labeled boundaries in accordance with the subgroup Ω. Jacques' approach is consistent with that of Gustin [G8] and has the advantage of applying also to irregular imbeddings (i.e. $r \neq r_k$, for each k).

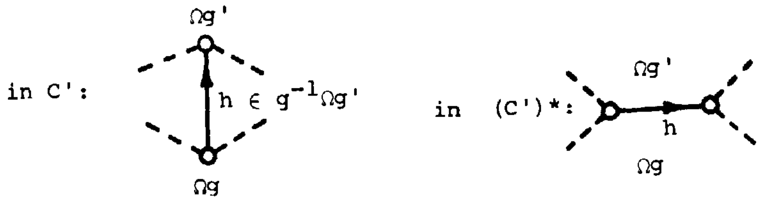


Figure 9-16

Define (after Jacques) a <u>side</u> (or <u>brin</u>) to be an ordered pair (c, δ^*), where c is a vertex in C, C' or $(C')^*$, and $\delta^* \in \Delta^*$. Then we have:

<u>Thm. 9-10.</u> The reduced constellation (quotient graph in its quotinet manifold, or $M(\Gamma/\Omega)$) satisfies the following five properties:

1) each side carries a current from Δ^*.

2) two opposing sides $x = (c, \delta^*)$ and $x^{-1} = (c', \delta^{*-1})$ (see Figure 9-17) carry inverse currents (if $x = x^{-1}$, the current must be of order 2; in this case the side appears as in Figure 9-18).

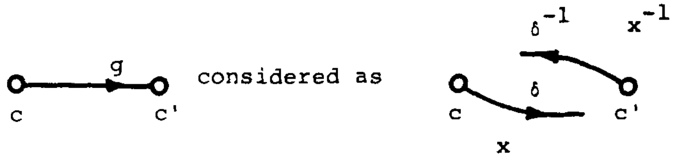


Figure 9-17

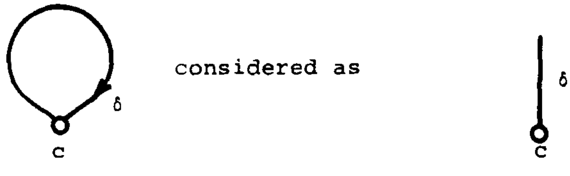


Figure 9-18

3) the regions are in one-to-one correspondence with the right cosets of Ω in Γ. (The index of Ω in Γ is called the <u>index of the imbedding</u>.)

4) the currents appearing in a region boundary are in one-to-one correspondence with Δ^*.

5) if a side x appears in the boundary of a region associated with Ωg and its opposite side x^{-1} in the boundary of a region associated with $\Omega g'$, then the current carried by x is the set $g^{-1}\Omega g'$.

Proof: The properties follow directly from the definitions and the construction of $(C')^*$. #

What is important for imbedding problems is the converse:

Thm. 9-11. A reduced constellation $M(\Gamma/\Omega)$ for $C_\Delta(\Gamma)$ and Ω (i.e. a pseudograph 2-cell imbedded, with edges and regions labeled with elements of Δ^* and right cosets of Ω in Γ respectively) satisfying the five properties above determines a 2-cell imbedding C of $C_\Delta(\Gamma)$ in M such that $(C')^* = M(\Gamma/\Omega)$.

Proof: For each $g \in \Gamma$, define $\sigma_g = (g\delta_1, g\delta_2, \ldots, g\delta_s)$, where $\delta_1, \delta_2, \ldots, \delta_s$ is the oriented boundary of the region $\Omega g'$ in $M(\Gamma/\Omega)$, and $g \in \Omega g'$. By property 3), σ_g is well-defined; by properties 1) and 4), σ_g is a permutation of $g\Delta^*$. By Edmonds' algorithm (Theorem 6-47), the collection $\{\sigma_g\}$ determines a 2-cell imbedding C of $C_\Delta(\Gamma)$ in M. Moreover, by properties 2) and 5), the dual of $M(\Gamma/\Omega)$ is a Schreier coset graph C' for C, so that $(C')^* = M(\Gamma/\Omega)$. #

In short, the region boundaries for $M(\Gamma/\Omega)$ determine the Edmonds' permutation scheme for the Schreier coset graph imbedded in the quotient constellation, and hence for $C_\Delta(\Gamma)$ (and thus $G_\Delta(\Gamma)$) in M.

But we can learn much more from $M(\Gamma/\Omega)$. First we make the following:

Def. 9-12. Let ξ be a vertex of $M(\Gamma/\Omega)$, with π the product of currents (from Δ^*) directed away from ξ, in the order given by the orientation. The order of π in Γ is called the valence of ξ.

$$k = |\Omega|\ (h-1) + 1 + \frac{|\Omega|}{2} \sum_{i=1}^{p'} \left(1 - \frac{1}{v_i}\right)$$

<u>Proof</u>: Let p, q, r and p', q', r' apply to C and $(C')^*$ respectively. Since vertex v_i of $(C')^*$ determines $\frac{|\Omega|}{v_i}$ regions of length $k_i v_i$ in C, $1 \leq i \leq p'$, we have

$$2q = \sum_{i=1}^{p'} \frac{|\Omega|}{v_i}\ k_i v_i = |\Omega| \sum_{i=1}^{p'} k_i = |\Omega|(2q'),$$

so that $q = |\Omega|q'$. Also $p = |\Omega|r'$ and $r = \sum_{i=1}^{p'} \frac{|\Omega|}{v_i}$. Moreover, $h = 1 + \frac{1}{2}(q'-p'-r')$, and

$$\begin{aligned} k &= 1 + \tfrac{1}{2}(q-p-r) \\ &= 1 + \frac{|\Omega|}{v_i}\ \left(q' - r' - \sum_{i=1}^{p'} \frac{1}{v_i}\right) \\ &= 1 + \frac{|\Omega|}{2}\ \left(q'-r'-p' + \sum_{i=1}^{p'} \left(1 - \frac{1}{v_i}\right)\right) \\ &= |\Omega|(h-1) + 1 + \frac{|\Omega|}{2} \sum_{i=1}^{p'} \left(1 - \frac{1}{v_i}\right). \end{aligned}$$

Note that for $|\Omega| = 1$, $C' = C$ and, since all region boundaries in C correspond to identity words, each $v_i = 1$ in $(C')^*$, so that $k = h$, as expected. (Note also the similarity to Theorem 7-22.) The following corollaries are immediate:

<u>Cor. 9-15</u>. If the KCL holds at each vertex of $(C')^*$, then $k = |\Omega|(h-1) + 1$.

(This is also Corollary 3 on page 208 of Cairns [C1], for a covering of S_h by S_k; see Corollary 10-3.)

<u>Remark</u>. If Γ is non-abelian, π may not be uniquely determined; nevertheless, if π and π' are two products at vertex ξ, then π and π' are conjugate elements of Γ and thus have the same order, so that the valence of ξ <u>is</u> well-defined.

We now state:

<u>Thm. 9-13</u>. A vertex of degree k and valence ν in $M(\Gamma/\Omega)$ determines $\frac{|\Omega|}{\nu}$ regions of length $k\nu$ in the imbedding of $C_\Delta(\Gamma)$ in M.

<u>Proof</u>: Let ξ be a vertex in $M(\Gamma/\Omega) = (C')^*$, of degree k and valence ν. Thus, in C', ξ corresponds to a region of length k for which the product $\pi = \delta_1\delta_2\cdots\delta_k$ of currents in the oriented boundary (from, say, coset Ωg to coset Ωg) is of order ν. In C, the walk determined by π is a portion of a region boundary, from g' to g'', where g' and g'' are both in Ωg. This region boundary continues with another walk determined by π (since $\sigma^*_{g'} = \sigma^*_{g''}$) and only concludes (using 4) of Theorem 9-10) at g', for for $\pi^\nu = e$. Thus this region in C is of length $k\nu$. In this region boundary we find exactly ν sides of the form (h, δ_1), with $h \in \Omega g$; hence ξ determines $\frac{|\Omega|}{\nu}$ regions of length $k\nu$ in $C = (C_\Delta(\Gamma)$ in $M)$. #

Thus imbeddings can be studied in terms of possibly much simpler combinatorial structures: the reduced constellations of Jacques or, equivalently, the quotient graphs and quotient manifolds of Youngs. We get the genus of C directly from $(C')^*$, as follows.

<u>Thm. 9-14</u>. Let C in S_k be represented by $(C')^* = M(\Gamma/\Omega)$ in S_h, with $v_1, v_2, \ldots, v_p$, the valences of the vertices of $(C')^*$. Then

Cor. 9-16. If $(C')^*$ is planar, then

$$k = 1 - |\Omega| + \frac{|\Omega|}{2} \sum_{i=1}^{p'} \left(1 - \frac{1}{v_i}\right) .$$

(This is essentially the formula on p. 255 of Fox [F2] for branched coverings of S_0 by S_k; see Section 10-1.)

9-6. Applications to Known Imbeddings

To illustrate the power and beauty of the theory, we present four examples, each giving a new proof of a genus formula appearing in the literature. We then give a fifth example, illustrating what Youngs called the theory of vortices.

Application 1: Take $\Gamma = Z_8$, $\Omega = Z_2 = \{0,4\}$, and $\Delta = \{1,3\}$. Then $M(Z_8/Z_2)$ in Figure 9-19 determines an imbedding of $K_{4,4} = G_\Delta(Z_8)$ in S_1, with $r = r_4 = 4\frac{|\Omega|}{v} = 8$. This is an index four imbedding. The generalization to $K_{2n,2n}$ is fairly easy to obtain, with $\Gamma = Z_{4n}$ and $\Omega = Z_n$; $M(Z_{4n}/Z_n)$ is a multigraph of order $2n$ (with edges appropriately directed and labeled) in S_{n-1}, giving $K_{2n,2n}$ quadrilaterally imbedded in $S_{(n-1)^2}$. (This appears in Ringel [R11].)

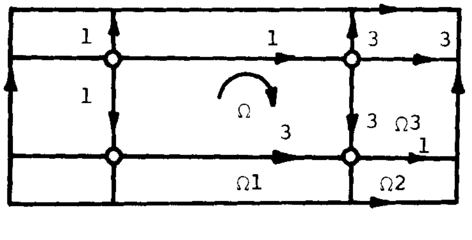


Figure 9-19

Application 2: Take $\Gamma = Z_9$, $\Omega = Z_3 = \{0,3,6\}$, and $\Delta = \{1,2,4\}$. Then $M(Z_9/Z_3)$ in Figure 9-20 gives $K_{3,3,3} = G_\Delta(Z_9)$ in S_1, with $r = r_3 = \frac{6|\Omega|}{\nu} = 18$. This is an index three imbedding. The generalization to $K_{2m+1,2m+1,2m+1}$, with $\Gamma = Z_{6m+3}$ and $\Omega = Z_{2m+1}$, has S_m as quotient manifold, and produces a triangular imbedding of $K_{2m+1,2m+1,2m+1}$ in $S_{m(2m-1)}$; this imbedding appears in [W5].

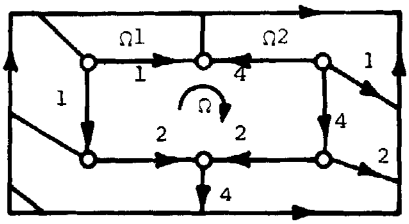


Figure 9-20

Application 3: Take $\Gamma = Q \times Z_m$, where m is odd, $\Omega = Z_4 \times Z_m$, and $\Delta = \{s,ta\}$, where Q is generated by s and t, and Z_m is generated by a. Then $M(Q \times Z_m/Z_4 \times Z_m)$ in Figure 9-21 gives $G_\Delta(Q \times Z_m)$ in S_1, with $r = r_4 = 2\frac{|\Omega|}{\nu} = 8m$. (This index two imbedding was first found by Himelwright [H7]; it shows that the hamiltonian group $Q \times Z_m$ is toroidal.)

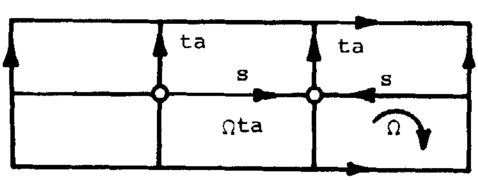


Figure 9-21

<u>Application 4</u>: Consider $\Gamma = \Omega = S_n$ $(n > 3)$, and $\Delta = \{s,t\}$, with $s = (1\ 2\ \ldots\ n)$ and $t = (1\ 2)$. Then $M(S_n/S_n)$ in Figure 9-22 gives $G_\Delta(S_n)$ in S_k, where $k = 1 + \frac{(n-2)!}{4}(n^2 - 5n+2)$, with $r_n = \frac{n!}{n} = (n-1)!$ and $r_{2n-2} = \frac{n!}{n-1}$. The side carrying current t represents a loop. (This index one imbedding appears in [W8], and gives an upper bound for the genus of the symmetric group S_n, for n even.)

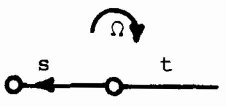


Figure 9-22

<u>Application 5</u>: Take $\Gamma = \Omega = Z_7$, and $\Delta = \{1,2,3\}$. Then $M(Z_7/Z_7)$ in Figure 9-23 gives $G_\Delta(Z_7) = K_7$ in S_3,

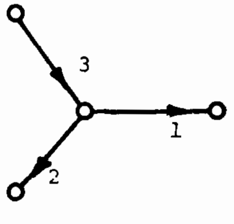


Figure 9-23

with $r_3 = 7$ and $r_7 = 3$. (This index one imbedding solves the regular part of the problem, for finding $\gamma(K_{10})$; see Section 9-3.

9-7. New Applications

Further connections between the theory and imbedding problems can be explored. For example:

Thm. 9-17. If K is a quotient graph for an imbedding of a graph G of index one or two, then $\gamma_M(K) = \left\lfloor \frac{\beta(K)}{2} \right\rfloor$.

Proof: This is a direct consequence of Theorem 6-24. #

Every Cayley graph has an index one imbedding; in fact $G_\Delta(\Gamma)$ has $(|\Delta^*|-1)!$ index one imbeddings. When are these minimal? A partial answer is provided in:

Thm. 9-18. Let the finite group Γ be generated by $\Delta = \{\delta_1, \ldots, \delta_n\}$, where each δ_i is of order $n \geq 3$ and $\prod_{i=1}^{n} \delta_i = e$. Then $G_\Delta(\Gamma)$ has an index one imbedding, with $r = r_n = 2|\Gamma|$, on S_k, where $k = 1 + \frac{|\Gamma|}{2}(n-3)$. If $n = 3$, or if $n = 4$ and no proper subset of Δ contains a redundant generator, then $\gamma(G_\Delta(\Gamma)) = k$.

Cor. 9-19. Let $\Gamma = (Z_n)^{n-1}$, $n \geq 3$, with $\Delta = \{\varepsilon_1, \ldots, \varepsilon_{n-1}, (\prod_{i=1}^{n-1} \varepsilon_i)^{-1}\}$, where ε_i is the $(n-1)$-tuple having 1 in the ith position and 0 elsewhere. Then $\gamma(G_\Delta(\Gamma)) \leq 1 + \frac{n^{n-1}(n-3)}{2}$; equality holds for $n = 3,4$.

Cor. 9-20. Let $\Gamma = (3,3|3,3)$ in the notation of Coxeter and Moser [CM1] (i.e. Γ is the non-abelian group of order 27, generated by s,t and subject to the relations $s^3 = t^3 = (st)^3 = (s^{-1}t)^3 = e$); let $\Delta = \{s,t,(st)^2\}$; then $\gamma(G_\Delta(\Gamma)) = 1$. Moreover, Γ is a toroidal group.

Cor. 9-21. Let $\Gamma = A_4$, the alternating group of degree four, with $\Delta = \{(1\ 2\ 3),(3\ 2\ 4),(1\ 3\ 4)\}$; then $\gamma(G_\Delta(A_4)) = 1$. (However, A_4 is not toroidal, as there exist planar Cayley graphs for A_4.)

In Section 7-2 we saw that we could find the genus of any abelian group $\Gamma = Z_{m_1} \times Z_{m_2} \times \dots \times Z_{m_r}$, where m_r is even (assuming m_i divides m_{i-1}, $i = 2, \dots, r$.) The argument used the fact that $C_{m_1} \times C_{m_2} \times \dots \times C_{m_r} = G_\Delta(\Gamma)$, for the standard presentation for Γ, and Theorem 7-13. The techniques employed fail if even one order m_i is odd. The theory of quotient graphs and quotient manifolds provides a new approach to the genus problem for general cycles, and hence to the genus of an arbitrary finite abelian group. (See also Theorem 7-17).

For example, $M((Z_3 \times Z_{2n} \times Z_{2m})/(Z_3 \times Z_n \times Z_{2m}))$ in Figure 9-24, where Z_3, Z_{2n}, and Z_{2m} are generated by r, s, and t respectively, shows that

$$\gamma(C_3 \times C_{2n} \times C_{2m}) \leq 1 + 4mn;$$

we conjecture equality for this formula. Also, $M((Z_3)^n/(Z_3)^n)$ in Figure 9-25, where ε_i is a generator for the ith copy of Z_3, shows that

$$\gamma((C_3)^n) \leq 1 + (n-2)3^{n-1}.$$

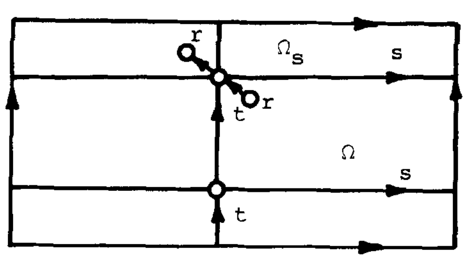


Figure 9-24.

Figure 9-25

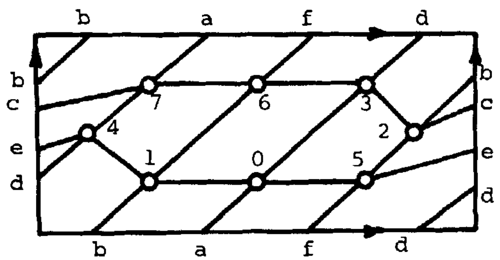


Figure 9-26

9-8. Problems

*9-1.) Show that seven "different" minimal imbeddings of K_5 are compatible with the formula of Theorem 9-1. (For example, $r_3 = 4$, $r_8 = 1$; $r_4 = 5$ give two "different" imbeddings of K_5 on S_1, since the regions are distributed differently). How many of the seven can you actually find? (Hint: not all seven exist!)

9-2.) Find the quotient manifold and the quotient graph for the quadrilateral imbedding of $K_{4,4}$ given (for $\Gamma = Z_8$) in Figure 9-26.

*9-3.) There are twelve different quotient graphs $S(\Gamma/\Omega)$ for $K_{2,2,2,2}$ in S_1 with $\Omega \neq \{e\}$; i.e. twelve different ordered pairs (Γ,Ω) for $K_{2,2,2,2}$. Find these twelve ordered pairs and then obtain an $M(\Gamma/\Omega)$ for each non-trivial index.

9-4.) Use the theory of quotient graphs and quotient manifolds to find a quadrilateral imbedding of K_5.

9-5.) Use the theory of quotient graphs and quotient manifolds to find a hexagonal imbedding $(r = r_6)$ for $K_{3,3}$.

*9-6.) Generalize Example 3 of Section 9-4, to find $r = r_{4n}$ imbeddings of $K_{4n(4n+1)+1}$, where $4n+1$ is prime. Is this an infinite class of graphs? (In fact, only the parity of $4n+1$ is required.)

9-7.) Prove Theorem 9-15.

9-8.) Find $\gamma(Q_n)$ in three ways: (1) by a construction, such as that employed in the proof of Theorem 7-13; (2) by selecting an appropriate $(p_1, p_2, \ldots, p_{2^n})$ and using Edmonds' algorithm (see Theorem 6-47) to show that every orbit has length 4; (3) by finding an appropriate $M(\Gamma/\Omega)$.

9-9.) We see from Theorem 6-25 that K_5 has a 2-cell imbedding in S_3. Show that the theory of this chapter is of no aid in find this imbedding.

*9-10.) Find a triangular imbedding for $K_{n,n,n,n}$ in a suitable surface $(n \neq 3.)$

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CHAPTER 10

VOLTAGE GRAPHS

In the preceeding chapter, following the historical development of Gustin, Youngs, Ringel, Jacques, et al, we generated graph imbeddings by means of constructing (simpler) current graph imbeddings. These latter were variously called "quotient graph in quotient manifold" and "reduced constellation." Each approach has the advantage of economy of construction; each approach has the disadvantage of generating regions by vertices and vertices by regions. Moreover, the mode of "generation" is not made explicit, in a way to give maximum aid to intuition.

These defects are corrected by the voltage graph theory initiated by Gross [G3] in 1974 and by its interpretation in the context of branched covering spaces (see papers by Gross and Alpert: [GA1], [GA2], and [AG2].)

In this chapter we present just enough covering space theory for the immediate context (for more details, see [M3]) and then introduce voltage graphs, with examples. We then revisit the Heawood Map-coloring Theorem from this advantageous viewpoint. Finally, we describe the strong tensor product construction for graphs; this is an iterative process that often produces an infinite tower of graph imbeddings from one voltage graph imbedding at the base.

10-1. Covering Spaces

Def. 10-1. A function $\rho : \tilde{X} \to X$ from one topological space to another is called a covering projection if every point $x \in X$ has a neighborhood U_x which is evenly covered; i.e. ρ maps each component of $\rho^{-1}(U_x)$ homeomorphically onto U_x. If $Y \subseteq X$ and $\tilde{Y} \subseteq \tilde{X}$ is such that ρ maps $\tilde{Y}$ homeomorphically onto Y, we say that Y lifts to $\tilde{Y}$. We call $\tilde{X}$ a covering space for X.

A standard result in the theory of covering spaces is that $|\rho^{-1}(x)|$ is independent of the choice of $x \in X$. If $|\rho^{-1}(x)| = n$, then ρ is called an n-fold covering projection. Here are some standard examples of covering spaces and covering projections:

Example 1: Let $S^1 = \{(x,y) \in R^2 | x^2 + y^2 = 1\}$ and consider $\rho: R \to S^1$, where $\rho(t) = (\cos t, \sin t)$. Then ρ "wraps" the real line R around the unit circle S^1 infinitely many times, with each half-open interval $[r, r + 2\pi)$ "covering" S^1 exactly once. If we pick $(1,0) \in S^1$ and (say) $U_{(1,0)} = \{(x,y) \in S^1 | -\frac{1}{2} < y < \frac{1}{2}\}$, then $U_{(1,0)}$ is evenly covered - as $\rho^{-1}(U_{(1,0)}) = \bigcup_{i \in I} (2\pi i - \pi/6, 2\pi i + \pi/6)$ and each open interval $(2\pi i - \pi/6, 2\pi i + \pi/6)$ is clearly homeomorphic to $U_{(1,0)}$.

The covering projection of the preceeding example fails to be n-fold, for each $n \in N$. However, for every $n \in N$ it is a simple matter to construct an n-fold covering projection.

Example 2: Define $\rho : S^1 \to S^1$ by $\rho(z) = z^n$, where $z \in S^1$ is regarded as a complex number. (Recall that, if $z = \text{cis}\theta = (\cos\theta, \sin\theta)$, then $z^n = \text{cis}\, n\theta = (\cos n\theta, \sin n\theta)$.) Thus ρ wraps S^1 n times around itself, and - for example - $\rho^{-1}(1,0)$ consists precisely of the n^{th} roots of unity.

The above example is significant for the sequel, as it describes how region boundaries of a desired graph imbedding will project to those of a voltage graph imbedding, in the simplest of cases (corresponding to the KCL - see Section 9-2 - holding.)

Example 3: Recall that S_0 is the sphere (in R^3), and that N_1 is the projective plane (nonorientable surface of genus one, or sphere with one crosscap.) Define $\rho : S_0 \to N_1$ by antipodal identification; i.e. $\rho(x,y,z) = (-x,-y,-z)$; then ρ is a 2-fold covering projection. Intuitively, we could regard ρ as fixing the bottom hemisphere and depressing the top hemisphere in "reverse overlapping" fashion; the antipodal identification along the equator then "sews on" the crosscap. (This takes place in R^4, not R^3.)

The following result is well known (by the Riemann-Hurwitz Theorem.)

Thm. 10-2. If $\rho: \tilde{S} \to S$ is an n-fold covering projection for surfaces, then the surface characteristics are related by: $\chi(\tilde{S}) = n\chi(S)$.

This relationship is quite natural: if G is 2-cell imbedded in S, with $p - q + r = \chi(S)$, then $\rho^{-1}(G)$ is 2-cell imbedded in $\tilde{S}$ with np vertices, nq edges, and nr regions, so that $\chi(\tilde{S}) = np - nq + nr = n(p - q + r) = n\chi(S)$.

Cor. 10-3. If $\tilde{S} = S_k$ and $S = S_h$, then $k = n(h-1) + 1$.

Thus, for example, only the torus can cover the torus (since $h = 1$ forces $k = 1$.)

An important generalization of the concept of covering space is required, to cover the case where the KCL does not hold. (See also Theorem 9-14 and its two corollaries.)

Def. 10-4. A function $\rho : \tilde{X} \to X$ from one topological space to another is called a branched covering projection (and $\tilde{X}$ is a branched covering space of X) if there exists a finite set $B \subseteq X$ such that the restricted function $\rho: \tilde{X} - \rho^{-1}(B) \to X - B$ is a covering projection. The points of B are called branch points. For $b \in B$ and U_b a sufficiently small neighborhood of b, the restricted function $\rho: \tilde{U}_b \to U_b - \{b\}$ is n-fold, for some cardinal number n - called the multiplicity of branching at b - where $\tilde{U}_b$ is a component of $\rho^{-1}(U_b - \{b\})$ in $\tilde{X}$. (If $n = 1$, then there is no branching.)

Standard examples here are:

Example 4: Let $D = \{(x,y) \in R^2 \mid x^2 + y^2 \leq 1\}$, the unit disk, and give $\rho: D \to D$ by $\rho(z) = z^n$ where, as before, z is regarded as a complex number. Then ρ "wraps" D around itself n times, except that the origin is fixed (and thus is a branch point of multiplicity n.)

Example 5: Define $\rho: S_0 \to S_0$, in spherical coordinates, by $\rho(1,\theta,\phi) = (1,n\theta,\phi)$. Then ρ "wraps" S_0 around itself n times, except that the north and south poles are both fixed; each is thus a branch point of multiplicity n.

Both of the examples above are important for what is to follow, as the former gives the prototype description of the projection of regions in (possibly branched) coverings of one graph imbedding by another - the local picture - while the latter is one instance of the projection between the corresponding ambient surfaces - the global picture.

10-2. Voltage Graphs

Let K be a pseudograph. With each edge $uv \in E(K)$, we associate two oriented edges $e = (u,v)$ and $e^{-1} = (v,u)$, and we set $K^* = \{(u,v) \mid uv \in E(K)\}$.

Def. 10-5. A voltage graph is a triple (K,Γ,ϕ), where K is a pseudograph, Γ is a group, and $\phi: K^* \to \Gamma$ satisfies $\phi(e^{-1}) = (\phi(e))^{-1}$ for all $e \in K^*$.

We remark that the pseudograph K is closely related to the quotient graph $S(\Gamma/\Omega)$ of Section 9-3.

Def. 10-6. The covering graph $K \times_\phi \Gamma$ for (K,Γ,ϕ) has vertex set $V(K) \times \Gamma$ and each edge $e = uv$ of K determines the edges $(u,g)(v,g\phi(e))$ of $K \times_\phi \Gamma$, for all $g \in \Gamma$.

For pseudographs regarded as topological spaces, then $K \times_\phi \Gamma$ is an $|\Gamma|$ - fold covering space of K; in fact, every regular covering space of K can be obtained in this manner (see [GT2]). (For additional information on covering projections of graphs, see Walker [W1], Farzan and Waller [FW1], Clarke, Thomas, and Waller [CTW1], Sit [S5], and Biggs [B11].)

For any walk $w: e_1, e_2, \ldots, e_m$ beginning at $v \in V(K)$ in (K,Γ,ϕ), we set

$$\phi(w) = \prod_{i=1}^{m} \phi(e_i),$$

and define the <u>local group</u> at v by:

$$\Gamma_v = \{\phi(w) \mid w \text{ is a closed walk at } v\}.$$

Then Γ_v is a subgroup of Γ and moreover Γ_u and Γ_v are conjugate subgroups of Γ if u and v are in the same component of K; thus $[\Gamma: \Gamma_v]$ is independent of v, if K is connected.

<u>Thm. 10-7</u>. For a connected voltage graph (K,Γ,ϕ), the number of components of the covering graph $K\times_\phi\Gamma$ is given by $[\Gamma: \Gamma_v]$, for any $v \in V(K)$.

Now let (K,Γ,ϕ) be 2-cell imbedded in an orientable surface S, as described algebraically by the rotation scheme $P = (p_1, p_2, \ldots, p_p)$. We define the <u>lift</u> $\tilde{P}$ of P to $K\times_\phi\Gamma$ as follows: if $p_v(v,u) = (v,w)$, then

$$\tilde{p}_{(v,g)}((v,g), (u,g\phi(v,u))) = ((v,g), (w,g\phi(v,w))),$$

for each $g \in \Gamma$. (See Figure 10-1.) Then

$$\tilde{P} = \{\tilde{p}_{(v,g)} \mid (v,g) \in V(K\times_\phi\Gamma)\}.$$

Thus $\tilde{P}$ determines a 2-cell imbedding of each component of $K\times_\phi\Gamma$. The power of voltage graph theory is that the simpler imbedding of K below gives much information about the more complicated imbedding of $K\times_\phi\Gamma$ above. To see this, let R be a region of the imbedding of K on S induced by P, and let $|R|_\phi$ be the order of $\phi(w)$ in Γ, where $w = e_1, e_2, \ldots, e_m$ is a closed walk in K consisting of the ordered boundary of R. (Since $\phi(w)$ is unique up to inverses and conjugacy, $|R|_\phi$ is independent of the orientation of R and of the initial vertex of w.) We then have the following central result, due to Gross and Alpert [GA1]:

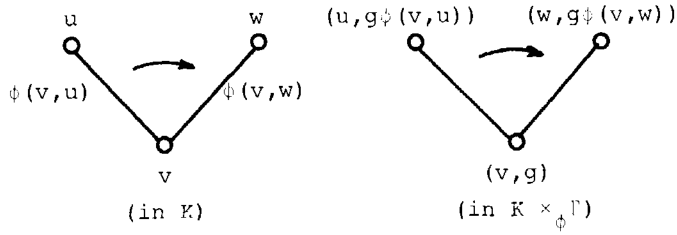

Figure 10-1

<u>Thm. 10-8</u>. Let (K,Γ,ϕ) be a voltage graph with rotation scheme P and $\tilde{P}$ the lift of P to $K\times_\phi\Gamma$. Let P and $\tilde{P}$ determine 2-cell imbeddings of K and $K\times_\phi\Gamma$ on the orientable surfaces S and $\tilde{S}$ respectively. Then there exists a (possibly branched) covering projection $\rho\colon \tilde{S} \to S$ such that:

(i) $\rho^{-1}(K) = K\times_\phi\Gamma$;

(ii) if b is a branch point of multiplicity n, then b is in the interior of a region R such that $|R|_\phi = n$;

(iii) if R is a region of the imbedding of K which is a k-gon, then $\rho^{-1}(R)$ has $|\Gamma|/|R|_\phi$ components, each of which is a $k|R|_\phi$ - gon region of the covering imbedding of $K\times_\phi\Gamma$.

We remark that, for $|R|_\phi = n$, each component of $\rho^{-1}(R)$ is mapped by ρ onto R, essentially by $\rho\colon D \to D$, $\rho(z) = z^n$, as described in Example 4 of Section 10-1.

Note that Theorem 10-8 (iii) is just the voltage graph analog of Theorem 9-13 for current graphs. Now compare Definition 9-3 with the following:

Def. 10-9. If $\phi(w) = e$ in Γ for the closed walk w bounding region R in the voltage graph imbedding, so that $|R|_\phi = 1$, we say that R satisfies the Kirchoff Voltage Law (KVL); if the KVL holds for all regions R of K on S, we say that the imbedding of (K,Γ,Φ) satisfies the KVL.

Cor. 10-10. If the imbedding of (K,Γ,ϕ) satisfies the KVL. then $p: \tilde{S} \to S$ is a covering projection (i.e. there is no branching.)

In either case (branching or not), if K has order m, then the voltage graph (K,Γ,ϕ) is said to have index m, and the imbedding of $K \times_\phi \Gamma$ is an index m imbedding (compare Theorem 9-10 (3).)

Voltage graph theory is even more general than the "general theory" of Section 9-5, in that the covering graph $K \times_\phi \Gamma$ need not be a Cayley graph. In this book, however, we are primarily concerned with the Cayley graph case; this will arise for Ω a subgroup of Γ and voltage graph (K,Ω,ϕ) - where $\phi: K^* \to \Gamma$ - a Schreier coset graph for Ω in Γ. Then $K \times_\phi \Omega = C_\Delta(\Gamma)$, where $\Delta = \{\phi(k) \mid k \in K^*\}$ (reduced as in Section 9-5.) Or, we could use voltage graph (K,Γ,ϕ); then $K \times_\phi \Gamma$ would consist of $|\Gamma|/|\Omega|$ disjoint copies of $C_\Delta(\Gamma)$.

In the simplest case - index 1 - K has one vertex, $\Gamma = \Omega$, the two approaches coincide, and the construction is quite clear: for each $g \in \Gamma$, we identify (v,g) with g (where $V(K) = \{v\}$), and then $V(K \times_\phi \Gamma) = V(K) \times \Gamma \approx \Gamma$ and $E(K \times_\phi \Gamma) \approx \{[g, g\phi(e)] \mid g \in \Gamma, e = (u,v) \in K^*\}$.

Finally, we remark that even voltage graph theory has been generalized; see Gross and Tucker [GT2]. Also, see Bouchet [B14] for graph imbeddings as covering spaces with folds (a fold is a 1-dimensional analog of the 0-dimensional branching.) And, see Parsons, Pisanski, and Jackson ([PPJ1] and [JPP1]) for graph imbeddings as branched covering spaces, where the restrictions of the branched coverings to the imbedded graphs are "wrapped quasi-coverings."

10-3. Examples

Example 1: Consider the index one voltage graph (a "bouquet" of two circles) of Figure 10-2, shown imbedded on the torus S_1. We take

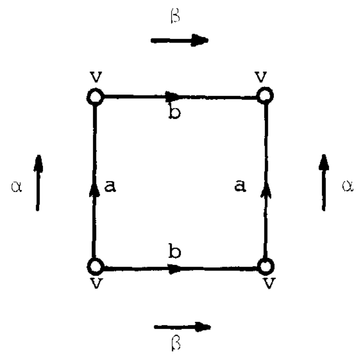

Figure 10-2

$\Delta = \{a,b\}$ for $\Gamma = \Omega$ abelian ($|\Gamma| \geq 5$), so that the KVL holds; thus, by Corollary 10-10, there is no branching. Then, by Corollary 10-3, we see that $C_\Delta(\Gamma) = K \times_\phi \Gamma$ will be imbedded on the torus as well. By Theorem 10-8, this imbedding will have $r = r_4 = |\Gamma|$. Thus Figure 10-2 completely determines an infinite family of quadrilateral imbeddings on the torus. We mention three special cases below.

a) Let $a = 1$ and $b = 2$ in $\Gamma = Z_5$; then $G_\Delta(\Gamma) = K_5$. This imbedding is self-dual and is the ground case for several infinite families of imbeddings appearing in the literature:

(i) K_{4n+1} has a self-dual imbedding in $\tilde{S} = S_{n(4n-3)}$ (see [W9]); here the voltage graph imbedding is just the normal form representation for $S = S_n$ (see Theorem 5-5 (ii).) The covering projection is $(4n+1)$ - fold, and indeed $\chi(\tilde{S}) = 2 - 2n(4n-3) = (4n+1)(2-2n) = (4n+1)\chi(S)$.

(ii) The imbedding of K_5 on S_1 can be augmented to an immersion of $K_{5(2)}$ on S_1 attaining the toroidal crossing number $v_1(K_{5(2)}) = 10$. Similarly, $v_k(K_{p(2)}) = p(p-1)/2$, for $k = (p-1)(p-4)/4$ and p a prime power $\equiv 1 \pmod 4$; see Theorem 6-71.

(iii) The imbedding of K_5 on S_1 can also be modified to obtain a genus imbedding for $G_5 = K_{5,5}$ less a 1-factor; this extends to show $\gamma(G_n) = \left\{(n-1)(n-4)/4\right\}$, which will be useful in Chapter 13.

We observe that the rotation scheme for K_5 in S_1 can be readily obtained from Figure 10-2. First, we find

$$p_v = ((v,v)_a, (v,v)_b, (v,v)_{-a}, (v,v)_{-b}).$$

As is customary for index one imbeddings, we identify (v,g) with g in the covering graph. Next, as K_5 has no loops or multiple edges, we regard each $\tilde{p}_{(v,g)} = \tilde{p}_g$ as permuting the neighbors of g rather than the edges at g. Finally, taking $a = 1$ and $b = 2$, we get:

$$p_0 = (1,2,4,3)$$
$$p_1 = (2,3,0,4)$$
$$p_2 = (3,4,1,0)$$
$$p_3 = (4,0,2,1)$$
$$p_4 = (0,1,3,2) .$$

(In practice, the above is written down directly from the figure.) The covering projection is 5-fold: the single vertex, two edges, and one 4-gon for K in S_1 lift respectively to five vertices, ten edges, and five 4-gon regions for $K\times_\phi\Gamma = K_5$ (also in S_1.)

b) Now let $a = (1,0)$ and $b = (0,1)$ for $\Gamma = Z_m \times Z_n$; then $G_\Delta(\Gamma) = C_m \times C_n$ and finds a self-dual, quadrilateral, toroidal imbedding. The case $m = n = 3$ is illustrated in Figure 10-3. The covering space nature of this imbedding is readily apparent: the single vertex, two edges, and one 4-gon now lift, respectively, to nine vertices, eighteen edges, and nine 4-gons, under this nine-fold projection ρ.

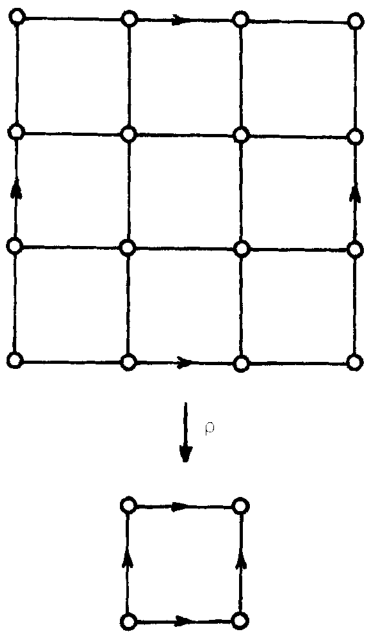

Figure 10-3

c) Now let $\Gamma = \pi(S_1) = \langle a,b \mid aba^{-1}b^{-1} = e \rangle = Z \times Z$, the fundamental group of the torus. In this case $G_\Delta(\Gamma)$ has an imbedding on S_0 (if we allow the vertex set to have limit points), but it is more natural to consider $G_\Delta(\Gamma)$ as being imbedded in R^2, as one of the three regular tessellations of the plane. (Recall that R^2 is the universal covering space for S_1, in the sense that R^2 covers any space which covers S_1; thus $G_\Lambda(\Gamma)$ in R^2 also covers each $C_\Delta(Z_m \times Z_n)$ imbedding of b) above, in a natural way.

We mention that $\pi(S_1)$ is also the group of one of the seventeen wallpaper designs (i.e. a planar crystallographic group.) The voltage graph theory is applicable to each of the seventeen planar infinite wallpaper groups (as presented, for example, in [B17]); of these imbeddings in R^2, six are index one branched covers of S_0, six are index two branched covers of S_0, one is an index two unbranched cover of S_0, three are index two unbranched covers of S_1, and the pattern described by $\pi(S_1)$ is an index one unbranched cover of S_1.

<u>Example 2</u>: Now modify Figure 10-2 slightly, to obtain Figure 10-4.

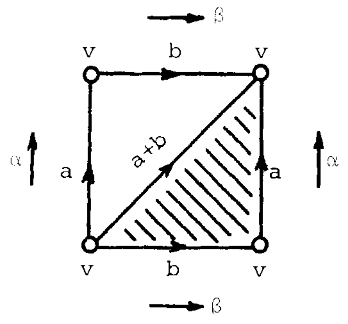

Figure 10-4

We still require Γ to be abelian, but now take $\Delta = \{a,b,a+b\}$ with $|\Gamma| \geq 7$. Then $G_\Delta(\Gamma)$ is regular of degree six and has an index one $r = r_3$ imbedding in S_1; again infinitely many such imbeddings are determined, one for each choice of Γ. Moreover, each of these has <u>bichromatic dual</u> (the geometric dual G^* has chromatic number two, where $G = G_\Delta(\Gamma)$); the indicated 2-coloring of the regions of Figure 10-4 lifts to a 2-coloring of the regions for $G_\Delta(\Gamma)$. (This will be useful in Chapter 12.)

a) For $a = 1$ and $b = 2$ in $\Gamma = Z_7$, $G_\Delta(\Gamma) = K_7$; this is the same famous toroidal imbedding obtained by the dual Figure 9-3 (see also Figure 9-5.)

  (i) A fairly natural extension of this voltage graph gives orientable triangular imbeddings for K_{12s+7}, for all $s \in N$. (The dual is bichromatic only for $s = 0$.)

  (ii) By taking first $a = 1$ and $b = 3$ and then $a = 2$ and $b = 5$ in $\Gamma = Z_{13}$, toroidal imbeddings are obtained for two complementary graphs in K_{13}; this shows that $N(1,1) = 14$, where $N(\gamma,\gamma')$ is the least integer such that every graph G of order at least $N(\gamma,\gamma')$ is not (γ,γ') bi-imbeddable (G imbeddable in S_γ, $\overline{G}$ in $S_{\gamma'}$). (See [AW1]; see also [A2] and [AC1].) It also shows that the toroidal thickness of K_{13} is two ($\theta_1(K_{13}) = 2$; see Beineke [B6] and Ringel [R13]; see also Theorem 6-60 (ii).)

b) If $a = 1$ and $b = 2$ for $\Gamma = Z_8$, we obtain a genus imbedding for $K_{4(2)}$.

c) If $a = (1,0)$ and $b = (0,1)$ for $\Gamma = Z_3 \times Z_3$, a genus imbedding for $K_{3(3)}$ results.

Voltage graph constructions are by no means unique for a given graph imbedding. For example, the dual of Figure 10-4 serves as an index two voltage graph (using $\Gamma = Z_3$) for an imbedding of $K_{3,3}$ on S_1 having $r = r_6 = 3$; the dual of this imbedding then serves as an index three voltage graph (again using $\Gamma = Z_3$) to triangularly imbed $K_{3(3)}$ on S_1 again. This is the ground case of Theorem 4.2 in [KRW1].

We also mention that taking $a = 1$ and $b = 2$ in Figure 10-4, but replacing $a + b$ with 5 in $\Gamma = Z_9$, gives an $r = r_{27} = 2$ imbedding for $K_{3(3)}$ (as seen from Theorem 10-8 (iii)); this gives the maximum genus of $K_{3(3)}$ as $\gamma_M(K_{3(3)}) = 9$, and is our first example of a branched covering in this section: each 27-gon above wraps around a triangle R below 9 times, since $|R|_\phi = 9$.

d) We keep $a = (1,0)$ and $b = (0,1)$, but now in $\Gamma = Z_4 \times Z_4$; the resulting triangular imbedding of $G_\Delta(\Gamma)$ will be of interest in Section 12-7.

e) In [S20] the following conjecture is attributed to Grünbaum: if a graph G has an orientable triangular imbedding, then the dual graph G^* has a Tait coloring (that is, has edge chromatic number $\chi_e(G^*) = 3$.) We remark that G cannot be allowed loops (consider the G^* of Figure 8.2 in [BCL1]) or multiple edges (consider $G^* = P$, the Petersen graph, in S_1) and that the conjecture is false also for nonorientable triangular imbeddings (consider $G = K_6$ in N_1, where $G^* = P$.) However the conjecture is true, as formulated, for 3-degenerate graphs G, as an easy induction argument shows. It is also true if $\chi(G^*) = 2$: if G^* is cubic and bipartite, then G^* is 1-factorable (see, for example, Theorem 8-7 in [BCL1]), so that $\chi_e(G^*) = 3$. Moreover, any $G_\Delta(\Gamma)$ triangulating S_1 as a covering space of Figure 10-4 has an edge coloring induced by the labels a, b, and $a + b$; now color the dual edges (in G^*) to agree with the colors on the edges of G they cross. Thus the voltage graph of Figure 10-4 also provides an easy algorithm for Grünbaum's conjecture, for infinitely many toroidal triangulations.

<u>Example 3</u>: The graphs $K_{4(n)}$ have genus $\gamma(K_{4(n)}) = (n-1)^2$, for $n \neq 3$, as established by Jungerman [J5] and independently (for n even) by Garman [G1]. (See Theorem 6-42.) Jungerman did not announce $\gamma(K_{4(3)})$, although he did report that $\gamma(K_{4(3)}) > 4$. The index three (this is <u>not</u> a Cayley graph construction) toroidal voltage graph of Figure 10-5, using $\Gamma = Z_3$, shows that $\gamma(K_{4(3)}) = 5$, by first imbedding $K_{3(3)}$ in S_5 with $r_3 = 6$ and $r_9 = 4$. (Refer to Theorem 10-8 (iii): there are four branch points -

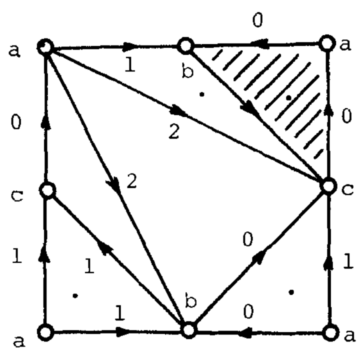

Figure 10-5

indicated by dots inside regions - each of multiplicity three and determining one 9-gon; the two regions having no branch point each lift to three triangles.) It is easily verified that each 9-gon is bounded by a hamiltonian cycle in $K_{3(3)}$. (For example, the clockwise boundary of the lift of the shaded region R is:

(b0, a0, c0, b1, a1, c1, b2, a2, c2) - where, for ease of notation, we write (v,g) as vg; the fact that $|R|_{\phi} = 3$ assures that the lift of R wraps around R three times under branched projection by ρ.) Now add a new vertex in the interior of each of the four 9-gons, and join each new vertex to all nine boundary vertices, for the region containing that vertex. The result is an imbedding of $K_{4(3)}$ in S_5, with $r_3 = 33$ and $r_9 = 1$. Note that the construction is readily augmented to give a triangular (and hence genus) imbedding for the complete 4-partite graph $K_{4,3,3,3}$.

Example 4: For G a graph and n a natural number, by n-fold G we mean that multigraph which results when each edge uv of G is replaced by n edges uv. The construction of Figure 10-4 suggests the extension given below (where $S = S_2$). We take $\Delta = \{a,b,a+b\}$- again - for Γ abelian $(n = |\Gamma| \geq 7)$ and K a bouquet of nine

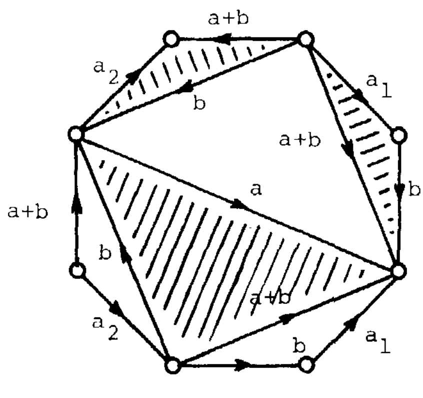

Figure 10-6

circles in S_2, with $r = r_3$, KVL, and bichromatic dual. The covering space is an $r = r_3$ imbedding of the 18-regular 3-fold $G_{\Delta}(\Gamma)$ on S_{n+1}, having bichromatic dual. The ground case $(n = 7)$ gives an orientable triangular imbedding for 3-fold K_7 (using $a = 1$, $b = 2$). Jungerman [J9] has found genus imbeddings (no digons allowed) for m-fold K_n, for all m and n (in both the orientable and the nonorientable cases.)

Example 5: The spherical index two voltage graph of Figure 10-7 also appears in [SW1]; it uses $\Gamma = Z_n$ to imbed $K_{n,n} = K_{2(n)}$ in

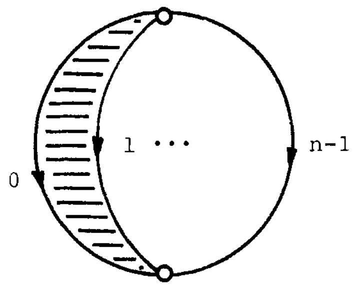

Figure 10-7

S_k (as a branched covering space), where $k = n(n-1)/2$, with $r = r_{2n} = n$ (each $|R|_\phi = n$; now apply Theorem 10-8 (iii).) Moreover, each region boundary is a hamiltonian cycle. (For example, the region covering the shaded digon has clockwise boundary (a0,b1, a1,b2,a2, ... , a(n-1),b0).) Thus this imbedding is readily augmented to give an $r = r_3$ imbedding for $K_{3(n)}$ (we discussed the case $n = 3$ in Example 2c) above), so that $\gamma(K_{3(n)}) = n(n-1)/2$; this proof is far shorter (and more elegant) than those of either [RY5] or [W5]. It is easy (see Problem 10-1) to show that every $r = r_3$ imbedding for $K_{3(n)}$ has bichromatic dual; this will be important in Section 12-7. Moreover, these imbeddings thus all satisfy the Grünbaum conjecture. (In fact, the three types of edges in the complete tripartite graph $G = K_{3(n)}$ determine a natural 3-edge coloring for the dual G^*.)

10-4. The Heawood Map-coloring Theorem (again)

We reconsider the crux of the proof of the Heawood map-coloring theorem: the construction of genus imbeddings of complete graphs by the use of current graphs; our context now will be that of branched covering spaces, and we use the voltage graph construction. This viewpoint treats only the regular part of each case; the additional adjacency parts continue to be handled as before (see Section 9-3, [GT1], [W12], and [R14].)

Recall the current graph of Figure 9-3, for imbedding K_7 on the torus. The dual of Figure 10-4 (in particular, see Example 2a) is the corresponding voltage graph. From either the single region of Figure 9-3 or the single vertex of Figure 10-4, we read the rotation p_0: (1,3,2,6,4,5); this generates the entire rotation scheme $P = (p_0, p_1, \ldots, p_6)$ and in turn the triangular imbedding. (Note that the KCL/KVL holds.)

In general, the current graph $K(s)$ for K_{12s+7} (as given in Section 9-3) has as its dual a voltage graph for $\Gamma = Z_{12s+7}$, with $p = 1$, $q = 6s + 3$, and $r = r_3 = 4s + 2$; the voltage graph, as was the current graph, is imbedded in S_{s+1}. The covering is unbranched (since the KVL holds) and is $(12s + 7)$-fold, giving an imbedding of K_{12s+7} in S_{12s^2+2s+1} as before, with $p = 12s + 7$, $q = (12s+7)(6s+3)$, and $r = r_3 = (12s+7)(4s+2)$.

The current graph for K_{10} appears in Figure 9-23; it is used, with $\Gamma = Z_7$ (see, for example, [W12]) to imbed K_7 in S_3 with $r_3 = 7$ and $r_7 = 3$ (determined by the vortices x, y, and z.) This is readily augmented to a triangulation of S_3 by $K_7 + \overline{K}_3 = K_{10} - \{xy,xz,yz\}$; then the additional adjacency part of the construction adds the three missing edges over one extra handle.

The corresponding voltage graph is given in Figure 10-8.

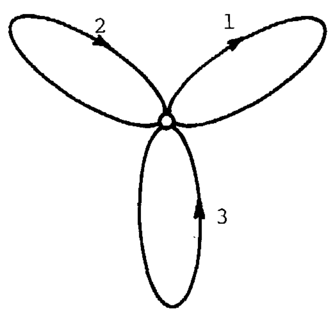

Figure 10-8

The rotation $p_0 = (1,6,4,3,2,5)$ is easily read from either figure. The outer region of Figure 10-8 satisfies the KVL

($1 + 4 + 2 = 0$, in Z_7), so by Theorem 10-8 (iii) it lifts, without branching, to seven triangles in the covering imbedding of K_7. Each of the three monogons, however, has voltage sum of order $|R|_\phi = 7$, so that each is covered, with branching, by one seven-sided region (a heptagon.) Thus K_7 is imbedded in S_3 as a branched covering space over S_0. Moreover, since each voltage generates Z_7, each heptagonal region boundary is a hamiltonian cycle for K_7. Thus, if one vertex is added in the interior of each heptagon, a triangular imbedding of $K_7 + \overline{K}_3$ results. The additional adjacency construction is as before.

The three vortices for each current graph in case 10 (see [R14]) become monogons in the voltage graph imbedding, with the single boundary edges always carrying a voltage which generates Z_{12s+7}. Thus each monogon is branch-covered by a single $(12s+7)$-gon, bounded by a hamiltonian cycle for K_{12s+7}, and the construction proceeds as for the case $s = 0$.

For a discussion of all twelve cases in this voltage graph/ branched covering space setting, see Gross and Tucker [GT1].

10-5. Strong Tensor Products

In [GRW1] the following product operation for graphs was introduced:

<u>Def. 10-11</u>. For two graphs G_1 and G_2, the <u>strong tensor product</u> $G_1 \underline{\otimes} G_2$ has vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u_1, u_2)(v_1, v_2) \mid (u_1 = v_1$ and $u_2v_2 \in E(G_2))$ or $(u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2))\}$.

Thus $G_1 \underline{\otimes} G_2$ consists of $|V(G_1)|$ disjoint copies of G_2, together with all the tensor product edges. The following two properties of the strong tensor product follow from the definition:

<u>Thm. 10-12</u>. $K_r \underline{\otimes} K_{n(m)} = K_{n(rm)}$.

<u>Thm. 10-13</u>. $G_1 \underline{\otimes} G_2$ is a subgraph of the lexicographic product $G_2[G_1]$, with equality if and only if G_1 is complete.

The theorem below from [GRW1] is especially germane in the present work:

Thm. 10-14. Let G_2 have a triangular imbedding in an orientable surface, with bichromatic dual; let G_2 be connected and bichromatic, with maximum degree at most two. Then $G_1 \underline{\otimes} G_2$ has a triangular imbedding in an orientable surface, with bichromatic dual.

In Chapter 12 we will find that orientable triangulations of strongly regular graphs with bichromatic duals can be particularly useful. We shall see that each $K_{n(m)}$ is strongly regular; thus if we take $G_1 = K_2$, Theorems 10-12 and 10-14 will combine to preserve this usefulness under the strong tensor product operation.

For the present, we observe that each covering imbedding of Figure 10-4 (with $G_\Delta(\Gamma) = G_2$) gives rise to an infinite collection of bichromatic dual triangulations, via repeated application of Theorem 10-14 - for each graph G_1 as in that theorem.

In particular, we can take $G_1 = K_2$ and extend Examples 2a, 2b, and 2c respectively to obtain, for each nonnegative integer k, orientable bichromatic dual triangulations for $K_{7(2^k)}$, $K_{4(2^{k+1})}$, and $K_{3(3\cdot 2^k)}$.

Finally, we remark that Theorem 10-13 combines with Theorem 6-45 to yield the surprisingly general formula of:

Thm. 10-15. Let G have p vertices of positive degree, q edges, k nontrivial components, and no 3-cycles. Then $\gamma(K_{2n} \underline{\otimes} G) = k + n(nq-p)$.

10-6. Problems

10-1.) Show that the identity map $i: X \to X$ is always a covering projection.

10-2.) Show that if $p: \tilde{X} \to X$ and $q: \tilde{Y} \to Y$ are covering projections, then so is $p\times q: \tilde{X} \times \tilde{Y} \to X \times Y$, where $(p\times q)(x,y) = (px,qy)$.

10-3.) Regard the torus as $S_1 = C \times C$, where C is the unit circle in R^2. Show that $R^2 = R \times R$, $R \times C$, and

$S_1 = C \times C$ all cover S_1. Try to visualize ρ, in each case.

10-4.) Show that every orientable triangular imbedding for $K_{3(n)}$ has bichromatic dual.

10-5.) Find the voltage graph corresponding to Figure 9-11.

10-6.) Find the voltage graph corresponding to Figure 9-13.

10-7.) What imbedding (and of what graph) covers the voltage graph imbedding of Figure 10-9 ($\Gamma = Z_8$)? Is there branching? Where, and of what multiplicity?

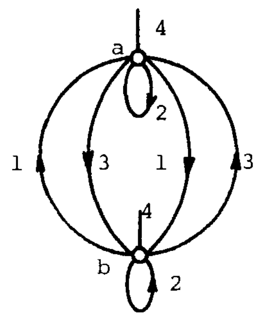

Figure 10-9

$\overset{*}{1}$0-8.) Answer the questions of Problem 10-7, but now for Figure 10-10, with $\Gamma = Z_{12}$. Then show that $\gamma(K_{6,6,3}) = 7$.

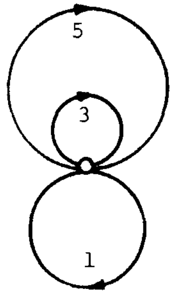

Figure 10-10

★10-9.) Extend the construction of Problem 10-8, to show that $\gamma(K_{2n,2n,2n-2}) = 2(n-1)^2$, for $n \equiv 1,2 \pmod 3$. (See also [SW1].)

10-10.) Show that if we begin with an arbitrary 2-cell imbedding of a Cayley color graph $C_\Delta(\Gamma)$ as a voltage graph imbedding, and use voltage group Γ, the covering space consists of $|\Gamma|$ disjoint copies of the base configuration.

10-11.) A graph G of size q is said to be conservative (see Bange, Barkauskas, and Slater [BBS1]) if the edges can be oriented and distinctly labeled with $1, 2, \ldots, q$ so that at each vertex the sum of the numbers on the inwardly directed edges equals that on the outwardly directed edges. Show that if such a graph has a 2-cell imbedding with $r = 1$, then it serves as an index one current graph, with currents from $\Gamma = Z_n$, $n \geq 2q + 1$. Study the imbedding of $G_\Delta(\Gamma)$ determined, for $\Delta = \{1, 2, \ldots, q\}$. Consider the special cases $n = 2q + 1$, $n = 2q + 2$. Use the fact that, for $n \geq 4$, K_n is conservative (see [BBS1]) to find $r = r_{n-1}$ imbeddings for K_{n^2-n+1}, $n \equiv 1, 2 \pmod 4$. If G is cubic, show that, in general, the covering imbedding is minimal for $G_\Delta(\Gamma)$.

10-12.) Prove Theorem 10-12.

10-13.) Prove Theorem 10-13.

CHAPTER 11

NONORIENTABLE GRAPH IMBEDDINGS

We reiterate that our primary motivation for this entire work is to depict graphs - and in particular graphs of groups - on surfaces (locally 2-dimensional drawing boards), usually as efficiently as possible. In keeping with our desire to study structures that exist in three-dimensional space, we have concentrated on the orientable surfaces S_k $(k \geq 0)$ - to the almost total exclusion of the nonorientable surfaces N_h $(h \geq 1)$. But much of what we have done has an analog in the nonorientable context, and it is to these analogs that we now turn our attention.

11-1. General Theory

Def. 11-1. The nonorientable genus, $\tilde{\gamma}(G)$, of a graph G is the minimum h such that G can be imbedded in the nonorientable surface N_h. Such an imbedding is said to be nonorientably minimal for G.

For completeness, we extend Definition 11-1 so that $\tilde{\gamma}(G) = 0$, if G is planar. Since h can be regarded as the number of crosscaps attached to the sphere S_0 to form N_h, $\tilde{\gamma}(G)$ is also called the crosscap number of G. Thus a planar graph has crosscap number zero.

The genus and nonorientable genus are related by:

Thm. 11-2. $\tilde{\gamma}(G) \leq 2\gamma(G) + 1$.

Proof: Imbed G in $S_{\gamma(G)}$ and then attach one crosscap to $S_{\gamma(G)}$, within one region of the imbedding. Then G is imbedded in this new surface, which is homeomorphic to $N_{2\gamma(G)+1}$. #

Auslander, Brown, and Youngs [ABY1] have constructed graphs of arbitrarily large genus which all imbed in the projective plane N_1; thus there is no modification of Theorem 11-2 which reverses the inequality.

For a proof of the following analog to Theorem 5-14, see Massey [M3].

Thm. 11-3. Let G be a connected pseudograph, with a 2-cell imbedding in N_h, with the usual parameters p, q, and r. Then $p - q + r = 2 - h$.

The number $2 - h$ is the *characteristic* of N_h $(\chi(N_h) = h)$, as it is independent of G. Note that in computing the characteristic of a surface, one handle has the same weight as two crosscaps.

Unfortunately, the nonorientable analog of Theorem 6-11 is false. As is implicit in Theorem 8-10, K_7 does not imbed in N_2; but since K_7 *does* imbed in S_1, it has an imbedding in N_3 (add one crosscap in the interior of a region of the K_7 imbedding in S_1) which is *not* 2-cell. This imbedding must be nonorientably minimal for K_7.

Thus we cannot be assured that a nonorientably minimal imbedding of a connected graph satisfies the very useful euler equation of Theorem 11-3. Youngs [Y1] overcame this obstacle by two definitions and one theorem:

Def. 11-4. An imbedding of a connected graph G into a surface S (orientable or nonorientable) is *simplest* if there is no imbedding of G into any S' satisfying $\chi(S') > \chi(S)$.

Def. 11-5. An imbedding of a connected graph G into a surface S (orientable or nonorientable) is *maximal* if there is no imbedding of G into any S' having more regions. (That is, S allows the maximum value of r.)

Thm. 11-6. An imbedding of a connected graph G into a surface S (orientable or nonorientable) is simplest if and only if it is both 2-cell and maximal.

Thus the imbedding of K_7 into N_3 constructed above is neither simplest nor 2-cell (nor maximal.) In general, if a connected graph G has a nonorientable 2-cell imbedding which maximizes r, then that imbedding is simplest (by Theorem 11-6) and hence nonorientably minimal, so that $\tilde{\gamma}(G)$ has been determined. Then we have the following analogs to Corollary 6-14, Corollary 6-15, and Theorem 9-1:

Cor. 11-7. If G is connected, with $p \geq 3$, then $\tilde{\gamma}(G) \geq q/3 - p + 2$; equality holds if and only if a nonorientable triangular imbedding can be found for G.

Proof: Let G be imbedded in $N_{\tilde{\gamma}(G)}$. If the imbedding is 2-cell, then $p - q + r = 2 - \tilde{\gamma}(G)$ by Theorem 11-3. As in the orientable case, $2q \geq 3r$ - with equality if and only if $r = r_3$. Thus $\tilde{\gamma}(G) = q - p - r + 2 \geq q/3 - p + 2$. If, on the other hand, the imbedding is not 2-cell, then it is not simplest, by Theorem 11-6. Thus G has a simplest (and hence 2-cell) imbedding on surface S, where $\chi(N_{\tilde{\gamma}(G)}) = 2 - \tilde{\gamma}(G) < \chi(S) = p - q + r \leq p - q/3$ (using $2q \geq 3r$ again); hence $\tilde{\gamma}(G) > q/3 - p + 2$, in this case. #

Cor. 11-8. If G is connected, with $p \geq 3$, and has no triangles, then $\tilde{\gamma}(G) \geq q/2 - p + 2$; equality holds if and only if a nonorientable quadrilateral imbedding can be found for G.

(The proof is entirely analogous to that of Corollary 11-7.)

Cor. 11-9. Let K_n be nonorientably, minimally, 2-cell imbedded in N_h. Then $\tilde{\gamma}(K_n) = h = \dfrac{(n-3)(n-4)}{6} + \dfrac{1}{3}\sum_{i\geq 4}(i-3)r_i$.

(The proof is analogous to that of Theorem 9-1.)

The nonorientable analogs to Theorem 6-18 and Corollary 6-19 are also false, as $G = 2K_7$ shows: since, by Corollary 6-19, $\gamma(2K_7) = 2\gamma(K_7) = 2$, $\tilde{\gamma}(2K_7) \leq 5$, by Theorem 11-2. But, as we have seen, $\tilde{\gamma}(K_7) = 3$, so that $\tilde{\gamma}(2K_7) \neq 2\tilde{\gamma}(K_7)$. What is true follows in two definitions and one theorem, due to Stahl and Beineke [SB1]:

Def. 11-10. The manifold number of a graph G is $\mu(G) = \max\{2-2\gamma(G),\ 2-\tilde{\gamma}(G)\}$.

Def. 11-11. A graph G is orientably simple if $\mu(G) \neq 2 - \tilde{\gamma}(G)$; that is, if $\tilde{\gamma}(G) > 2\gamma(G)$.

Thm. 11-12. Let G be a graph with blocks (or components) $G_1, G_2, \ldots, G_k$. If G is orientably simple, then $\tilde{\gamma}(G) = 1 - k + \sum_{i=1}^{k} \tilde{\gamma}(G_i)$; else $\tilde{\gamma}(G) = 2k - \sum_{i=1}^{k} \mu(G_i)$.

11-2. Nonorientable Covering Spaces

Recall from Example 3 of Section 10-1 that the sphere S_0 is a 2-fold covering space of the projective plane. In general (see Stahl [S8], for example), there is a 2-fold covering projection $p\colon S_k \to N_{k+1}$, for every nonnegative integer k. Thus every nonorientable surface has an orientable covering surface. Trivially, each nonorientable surface has at least one nonorientable covering surface, namely itself (see Problem 10-1.) In Section 11-4 we shall see an example of a nonorientable surface (N_3) with infinitely many nonorientable covering surfaces $(N_{n+2},\ n \geq 7)$.

In contrast, if the base space is an orientable surface, then every covering surface must be orientable also. Thus in Chapter 10 - where each base space is orientable - each covering space is orientable and hence unambiguously determined by its characteristic. In this chapter, however, since the base space will always be nonorientable, if the covering surface has even characteristic, then its orientability character needs to be ascertained to specify the surface uniquely.

In the next section we shall see how to do this, in the context of nonorientable graph imbeddings.

11-3. Nonorientable Voltage Graph Imbeddings

To extend the voltage graph theory to nonorientable imbeddings, we augment the rotation scheme P of Section 10-2 to a pair (P,λ)

called an imbedding scheme: $\lambda: K^* \to Z_2$ gives a voltage graph (K, Z_2, λ). The region boundaries are computed almost as in the orientable case (see Section 6-6), except that sometimes $P_v^{-1}(v,u)$ is used instead of $P_v(v,u)$; see [S8] and [SW1] for details. Now let (K, Γ, ϕ) be a voltage graph with imbedding scheme (P, λ) and let $\tilde{P}$ be the lift of P to $K \times_\phi \Gamma$; define $\tilde{\lambda}: (K \times_\phi \Gamma)^* \to Z_2$ by: $\tilde{\lambda}(\tilde{e}) = \lambda(e)$, for each lift $\tilde{e}$ of $e \in E(K)$. Define $(\tilde{P}, \tilde{\lambda})$ to be the lift of (P, λ) to $K \times_\phi \Gamma$. Then the conclusions of Theorem 10-8 and Corollary 10-10 can be shown to follow verbatim.

Thm. 11-13. Let (K, Γ, ϕ) be a voltage graph with imbedding scheme (P, λ) and $(\tilde{P}, \tilde{\lambda})$ the lift of (P, λ) to $K \times_\phi \Gamma$. Let (P, λ) and $(\tilde{P}, \tilde{\lambda})$ determine 2-cell imbeddings of K and $K \times_\phi \Gamma$ on the surfaces S and $\tilde{S}$ respectively. Then there exists a (possibly branched) covering projection $\rho: \tilde{S} \to S$ such that:

(i) $\rho^{-1}(K) = K \times_\phi \Gamma$;

(ii) if b is a branch point of multiplicity n, then b is in the interior of a region R such that $|R|_\phi = n$.

(iii) if R is a region of the imbedding of K which is a k-gon, then $\rho^{-1}(R)$ has $|\Gamma|/|R|_\phi$ components, each of which is a $k|R|_\phi$-gon region of the covering imbedding of $K \times_\phi \Gamma$.

Cor. 11-14. If the imbedding of (K, Γ, ϕ) satisfies the KVL, then $\rho: \tilde{S} \to S$ is a covering projection (i.e. there is no branching.)

Of course, the imbedding scheme (P, λ) is used in practice only when S is nonorientable; if S is orientable, the rotation scheme P alone suffices, and $\tilde{S}$ is necessarily orientable also, as observed above. If S is nonorientable, we must determine the orientability character of $\tilde{S}$; to this end we give:

Def. 11-15. For a voltage graph (K, Γ, η), a closed walk c in K is said to be η-trivial if $\eta(c) = e$ in Γ.

Thm. 11-16. Under the hypothesis of Theorem 11-13, the derived surface $\tilde{S}$ is orientable if and only if every ϕ-trivial closed walk in K is also λ-trivial.

We mention that Garman [Gl] has further extended the theory of voltage graphs, as outlined in the previous and present chapters, to pseudosurface imbeddings. In particular, Theorems 10-7, 10-8, 11-13, and 11-16, and Corollaries 10-10 and 11-14 also apply for S a pseudosurface; in this case, $\tilde{S}$ is necessarily also a pseudosurface (or a generalized pseudosurface; see Definition 5-26.)

11-4. Examples

Example 1: Let $m \equiv 2 \pmod 4$. Figure 11-1 presents a projective plane imbedding of a pseudograph K with one vertex and $m/2$ loops: $\lambda(e) = 1$ for each loop e. Let $\Gamma = Z_n$ (n even), and $\phi(e) = 1$ as indicated. Then $K \times_\phi Z_n$ is a graph with vertex set

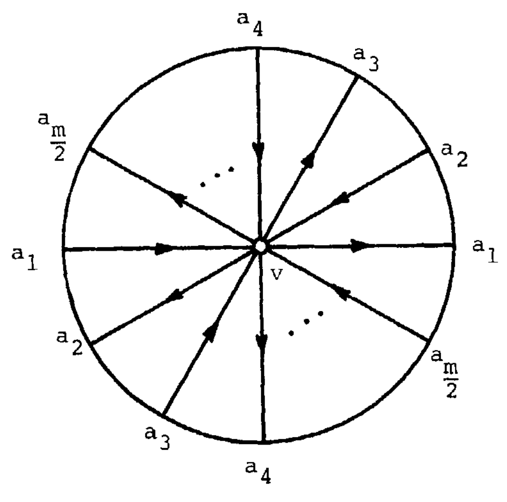

Figure 11-1

$\{(v,i) \mid 0 \le i \le n-1\}$ in which (v,i) and $(v, i + 1)$ are joined by $m/2$ edges for each i. For each region R of K imbedded below, $|R|_\phi = n/2$, so that the regions of $K \times_\phi Z_n$ imbedded above are all n-gons; in fact each is a hamiltonian cycle. There are $(m/2)2 = m$ such regions in all (above.) Thus if we place a new vertex in the interior of each such region, join it by non-intersecting edges to all the vertices on its boundary, and then delete all the original edges of $K \times_\phi Z_n$, a quadrilateral imbedding of $K_{m,n}$ results. It is clear that every ϕ-trivial closed walk in K is also λ-trivial, so that the covering imbedding is orientable, by Theorem 11-16; it is into S_k, where $k = (m-2)(n-2)/4$. This gives a partial proof of Theorem 6-37; the approach appears in [SW1].

Example 2. In Figure 11-2 we take $\Delta = \{a,b,a+b\}$ for Γ abelian ($|\Gamma| = n \geq 7$) and K as a bouquet of three circles imbedded in N_3. The single vertex has been given an orientation as indicated.

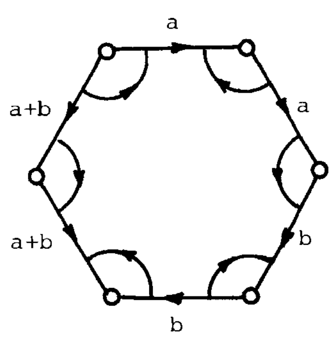

Figure 11-2

Then those edges which are coherently oriented, as induced by the vertex orientation, are assigned $\lambda = 0$ (there are none); all other edges - in this case the three edges bounding the hexagon - are assigned $\lambda = 1$. Then the closed walk $a + b - (a+b)$ is ϕ-trivial but not λ-trivial; thus the covering surface $\tilde{S}$ is nonorientable, by Theorem 11-16. Moreover, $\chi(\tilde{S}) = n\chi(S) = -n$, so that $\tilde{S} = N_{n+2}$. The special case $n = 7$ produces a self-dual imbedding of K_7 on N_9.

Example 3. Now modify Figure 11-2 slightly, to give Figure 11-3.

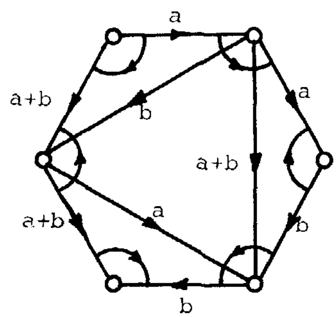

Figure 11-3

The covering imbedding, still on N_{n+2}, is now triangular and of the 12-regular 2-fold $G_\Lambda(\Gamma)$. Finally, change ϕ (i.e. relabel the edges of K on N_3; see Problem 11-4) to give 1-fold triangulations of $G_\Lambda(\Gamma)$ of characteristic $-2n$; for example, consider $G_\Delta(Z_{2n})$, where $\Delta = \{1,2,3,n-3,n-2,n-1\}$ and still $n \geq 7$. The case $n = 7$ gives $K_{7(2)}$ on N_{16}.

11-5. The Heawood Map-coloring Theorem, Nonorientable Version

If we apply Corollary 11-7 to the graphs $G = K_m$, we obtain:

Lemma 11-17. $\tilde{\gamma}(K_m) \geq \left\{\frac{(m-3)(m-4)}{6}\right\}$, for $m \geq 3$.

We have noted that equality does not hold for $m = 7$ and that $\tilde{\gamma}(K_7) = 3$. However, equality does hold in every other case:

Thm. 11-18. $\tilde{\gamma}(K_m) \leq \left\{\frac{(m-3)(m-4)}{6}\right\}$, for $m \neq 7$.

This is, of course, established by finding an imbedding of K_m on N_h, $h = \left\{\frac{(m-3)(m-4)}{6}\right\}$. The first proof was by Ringel [R8], without the benefit of current graphs. Later proofs do employ current graph theory (in particular, the theory of cascades), and again split naturally into the residue cases of m modulo 12 (see [R14], [LY1],

[J4].) Lemma 11-17 and Theorem 11-18 combine to form the nonorientable version of the complete graph theorem:

<u>Thm. 11-19</u>. $\tilde{\gamma}(K_m) = \left\{\frac{(m-3)(m-4)}{6}\right\}$, for $m \geq 3$ and $m \neq 7$;
$\tilde{\gamma}(K_7) = 3$.

Now we let $M_n = N_h$ in Theorem 8-13, recalling that $f(n) = (7 + \sqrt{49-24n})/2$; here $n = 2-h$:

<u>Lemma 11-20</u>. $\chi(N_h) \leq [f(n)] = [(7 + \sqrt{1+24h})/2]$.

To complete the proof of the Heawood Map-coloring Theorem, nonorientable version, we need the following:

<u>Lemma 11-21</u>. $\chi(N_h) \geq [f(2-h)] = [(7 + \sqrt{1+24h})/2]$, $h \neq 2$.

> <u>Proof</u>: Consider N_h, $h \neq 2$. Define $m = [f(2-h)]$, and now consider also $N_{\tilde{\gamma}(K_m)}$. Note that $\tilde{\gamma}(K_m) \leq h$, so that $\chi(N_{\tilde{\gamma}(K_m)}) \leq \chi(N_h)$. Now K_m imbeds in $N_{\tilde{\gamma}(K_m)}$. Clearly $\chi(N_{\tilde{\gamma}(K_m)}) \geq m = [f(2-h)]$, so that $\chi(N_h) \geq [f(2-h)]$.

Since Franklin [F3] showed that $\chi(N_2) = 6$, we can now combine Lemmas 11-20 and 11-21 to restate Theorem 8-10.

<u>Thm. 11-22</u>. $\chi(N_h) = [(7 + \sqrt{1+24h})/2]$, for $h \neq 2$; $\chi(N_2) = 6$.

11-6. Other Results

We give a few of the analogs to the orientable results of Chapter 6. For most of the other nonorientable genus results known by 1978, see Table 2 of Stahl [S9].

<u>Thm. 11-23</u>. (Ringel [R12])

$$\tilde{\gamma}(K_{m,n}) = \left\{\frac{(m-2)(n-2)}{2}\right\} \quad ; \quad m,n \geq 2.$$

Thm. 11-24. (Jungerman [J6])

$\tilde{\gamma}(Q_n) = 2 + 2^{n-2}(n-4)$, $n \geq 2$, except that $\tilde{\gamma}(Q_4) = 3$ and $\tilde{\gamma}(Q_5) = 11$.

Thm. 11-25. (Jungerman [J8])

$\tilde{\gamma}(K_{4(n)}) = 2(n-1)^2$, if $n \geq 1$, except that $\tilde{\gamma}(K_{4(2)}) = 3$.

Thm. 11-26. (Stahl and White [SW1])

For $n \geq 3$, $\tilde{\gamma}(K_{n,n,n-2}) = (n-2)^2$.

Thm. 11-27. (Stahl and White [SW1])

For $n \geq 4$ and even, $\tilde{\gamma}(K_{n,n,n-4}) = (n-2)(n-3)$.

Bouchet [B13] has also used his "generative m-valuations" (see Section 6-5) to study $\tilde{\gamma}(K_{n(m)})$. He considered the residue classes of m and n (mod 6) and constructed triangular imbeddings for 18 of these 36 cases, thus determining $\tilde{\gamma}(K_{n(m)})$ for those 18 cases.

We now turn our attention to the maximum nonorientable genus parameter.

Def. 11-28. The maximum nonorientable genus, $\tilde{\gamma}_M(G)$, of a connected graph G is the maximum h for which G has a 2-cell imbedding in N_h.

In contrast to the orientable case, the value of this parameter is readily calculated for each connected graph G (see Ringel [R15] and Stahl [S8]); recall that the Betti number $\beta(G) = q - p + 1$.

Thm. 11-29. $\tilde{\gamma}_M(G) = \beta(G)$.

This means that every connected graph has a nonorientable 2-cell imbedding with $r = 1$.

The following theorem, also due to Stahl, gives an analog to Duke's Theorem (6-21) and to Corollary 6-22.

Thm. 11-30. A connected graph G has a 2-cell imbedding in N_h if and only if $\tilde{\gamma}(G) \leq h \leq \beta(G)$.

We give a nonorientable analog to the famous Kuratowski Theorem (6-6); this result is due to Glover, Huneke, and Wang [GHW1] and to Archdeacon [A7].

Thm. 11-31. A graph G imbeds in the projective plane N_1 if and only if, for each graph H in a prescribed list of 103 graphs, G contains no subgraph homeomorphic with H.

It has long been conjectured that there is a finite number of graphs obstructing imbedding into each surface, whether orientable or nonorientable (by Theorem 6-6 there are two for S_0; by Theorem 11-31 there are 103 for N_1.) Recently Seymour [S4a] has affirmed this conjecture:

Thm. 11-32. For each closed 2-manifold M, there is a finite set S_M of graphs such that a graph G imbeds in M if and only if G contains no subgraph homeomorphic from at least one member of S_M.

Finally, we mention that Pisanski [P6] has expanded the surgery techniques he used to generalize work of [W6] (see Theorem 7-13, for example) in the orientable case, to apply to the nonorientable case as well.

11-7. Problems

11-1.) Prove Corollary 11-8.

11-2.) Prove Corollary 11-9.

11-3.) Describe the 2-fold $\rho: S_k \to N_{k+1}$ of Section 11-2, for $k \geq 1$.

11-4.) Relabel the edges of K in Figure 11-3, to yield the $G_\Delta(Z_{2n})$ imbedding suggested in Example 3.

11-5.) What imbedding, and of what graph, covers the voltage graph imbedding of Figure 11-4 ($\Gamma = Z_9$)? Is there branching? Where, and of what multiplicity?

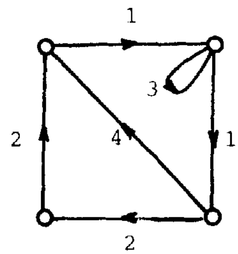

Figure 11-4

11-6.) Answer the questions of Problem 11-5, but now for $\Gamma = Z_{10}$.

**11-7.) Try to prove Theorem 11-19 by the use of nonorientable voltage graphs.

11-8.) Show that $\tilde{\gamma}(Q_n \times K_{4,4}) = 2 + n2^{n+1}$, $n \geq 1$.

11-9.) Show that the Z-metacyclic group $\Gamma = Z_n \times D_m = \langle a,b \mid a^m = b^{2n} = abab^{-1} = e \rangle$, for n odd and > 1, is toroidal. (Hint: use Theorem 7-3 and the voltage graph imbedding that results when the right side arrow is reversed in Figure 10-2.)

11-10.) What other toroidal groups can you find, using the hint of Problem 11-9?

CHAPTER 12

BLOCK DESIGNS

Block designs are combinatorial structures of interest in their own right, with applications to experimental design and to scheduling problems. Heffter [H5] was the first to observe that certain imbeddings of complete graphs determine BIBDs with $k = 3$ and $\lambda = 2$ (and sometimes $\lambda = 1$.) Alpert [A1] established a one-to-one correspondence between BIBDs with $k = 3$ and $\lambda = 2$ and triangulation systems for complete graphs. In [W11] this correspondence is extended to PBIBDs on two association classes with $k = 3$, $\lambda_1 = 0$ and $\lambda_2 = 2$ (and sometimes $\lambda_2 = 1$) and triangulation systems for strongly regular graphs. The group-divisible designs of Hanani [H1] are used to construct triangular imbeddings (in generalized pseudosurfaces) for the groups $K_{n(m)}$, in each case permitted by the euler equation. Conversely, triangular imbeddings of $K_{n(m)}$ are constructed (by other means) which lead to new group-divisible designs. A process, using the strong tensor product operation for graphs, is developed for "doubling" a given PBIBD of an appropriate form.

12-1. Balanced Incomplete Block Designs

Def. 12-1. A (v, b, r, k, λ)-balanced incomplete block design (BIBD) is a set of v objects and a collection of b subsets of the object set, each subset being called a block, satisfying:

(i) each object appears in exactly r blocks;

(ii) each block contains exactly k $(k < v)$ objects;

(iii) each pair of distinct objects appears together in exactly λ blocks.

If $k = v$, the design would be complete; complete designs are trivial to construct, but have little application. Hence for $k < v$, the design is "incomplete." The "balance" comes from the three uniformity conditions on r, k, and λ respectively.

A trivial example of a BIBD, for each $v > 1$ and $0 < k < v$, is obtained by taking the blocks to be all the k-subsets of the v-set; then $b = \binom{v}{k}$, $r = \binom{v-1}{k-1}$, and $\lambda = \binom{v-2}{k-2}$.

In general, elementary counting arguments (see Problems 12-1 and 12-2) establish the following well-known result:

Thm. 12-2. If a (v, b, r, k, λ)-BIBD exists, then:

(i) $vr = bk$;

(ii) $\lambda(v-1) = r(k-1)$.

These necessary conditions have also been shown to be sufficient, for $k = 3,4,5$ (and for some of the cases $k = 6,7$) by Hanani [H1], and for fixed k and λ, with v large enough, by Wilson [W15].

12-2. BIBDs and Graph Imbeddings

What Heffter and Alpert observed is that a triangular imbedding of K_n in an appropriate surface (this is possible exactly when $n \equiv 0,1 \pmod 3$, $n > 1$; see Sections 9-1 and 11-5), with the regions determining the blocks in the natural fashion, serves as an $(n, n(n-1)/3, n-1, 3, 2)$-BIBD, since: (i) every vertex is adjacent to exactly $n - 1$ other vertices and hence is in exactly $n - 1$ regions; (ii) each region contains exactly three vertices, by assumption; and (iii) each pair of distinct vertices constitutes an edge (since the graph is complete) and hence belongs to exactly two blocks (the two regions containing that edge in their boundary.)

Conversely, a BIBD on v objects with $k = 3$ and $\lambda = 2$ (a 2-fold triple system) determines a triangular imbedding of K_v in a generalized pseudosurface, as follows. Each block becomes a 3-sided 2-cell region, with vertices labelled by the objects of the block. Since $\lambda = 2$, each pair of vertices appears exactly twice - so that a 2-manifold (possibly with several components) results from the standard identification process of combinatorial topology.

Then identify identically labelled vertices, to form a generalized pseudosurface triangular imbedding for K_v. We summarize, in:

Thm. 12-3. The 2-fold triple systems on v objects are in one-to-one correspondence with triangular imbeddings of K_v in generalized pseudosurfaces.

If the triangular imbedding of K_v has bichromatic dual, then the 2-fold triple system splits naturally into two 1-fold triple systems (Steiner triple systems, having $k = 3$ and $\lambda = 1$):

Thm. 12-4. Steiner triple systems on v objects are in two-to-one correspondence with triangular imbeddings of K_v in generalized pseudosurfaces having bichromatic dual.

Letting $k = 3$ and $\lambda = 1$ or 2 in Theorem 12-2, and invoking the work of Hanani (or others) mentioned following that Theorem, we obtain:

Thm. 12-5. (i) Steiner triple systems on v objects exist if and only if $v \equiv 2, 3 \pmod 6$;
(ii) 2-fold triple systems on v objects exist if and only if $v \equiv 0, 1 \pmod 3$.

For independent verification of (ii) above, we follow Apert in observing that triangular imbeddings of K_v (see Sections 9-3 and 11-5) give 2-fold triple systems on v objects for all $v \equiv 0, 1 \pmod 3$, $v \neq 1$. Moreover, in [GRW1] it is observed that the orientable genus imbeddings for K_n, $n \equiv 3 \pmod{12}$, all have bichromatic dual; thus Steiner triple systems are independently produced for these values of n.

12-3. Examples

Example 1: Refer to Example 1a of Section 10-3; the quadrilateral imbedding of K_5 in S_1 obtained there yeilds a $(5,5,4,4,3)$-BIBD. This is atypical in that $k > 3$ and $\lambda > 2$, but it does indicate that the scope of the connection between BIBDs and graph imbeddings is even wider than indicated in Section 12-2. This imbedding is a geometric realization of the "all 4-subsets of a 5-set" abstract design.

Example 2: Refer to Example 2a of Section 10-3; the triangular imbedding of K_7 in S_1 obtained there yields a (7, 14, 6, 3, 2)-BIBD and, since the dual is bichromatic, two (7, 7, 3, 3, 1)-BIBDs. One of these latter is given abstractly in Table 12-1;

0	1	3
1	2	4
2	3	5
3	4	6
4	5	0
5	6	1
6	0	2

Table 12-1

the blocks may be read off from the seven regions covering the unshaded region in Figure 10-4 (note that they are also half of the "blocks" listed in Section 9-2.) The blocks may also be regarded as the lines of the Fano Plane (the finite projective plane of order two.) The design could be employed to schedule firemen (say) in a weekly schedule (three men per day, etc.) The extension of the voltage graph of Figure 10-4 to K_{12+7} gives a (12s+7, (12s+7). (4s+2), 12s+6, 3, 2)-BIBD, which is not of bichromatic dual for $s > 0$; hence we obtain no additional Steiner triple systems from this figure.

Example 3: Refer to Example 3 of Section 11-4; the triangulation of 2-fold K_7 in N_9 obtained there gives a (7, 28, 12, 3, 4)-BIBD.

Example 4: Refer to Example 4 of Section 10-3; the triangulation of 3-fold K_7 in S_8 obtained there gives one (7, 42, 18, 3, 6)-BIBD and, since the dual is bichromatic, two (7, 21, 9, 3, 3,)-BIBDs.

12-4. Strongly Regular Graphs

For values of v, b, r, k, λ not meeting the conditions of Theorem 12-2 (and perhaps even for those that do), it is natural to attempt a construction of a related design. For this reason, partially balanced incomplete block designs were introduced by Bose and

Nair [BN1]. The partial balance occurs in that there are $\ell > 1$ lambda values, one for each "association class;" we restrict our attention primarily to the case $\ell = 2$.

As the association classes (for the case $\ell = 2$) are determined by adjacencies within a "strongly regular graph" - to ensure an appropriate balance within each class - we first define this latter concept. Let x and y be distinct vertices in a graph G, either non-adjacent $(h = 1)$ or adjacent $(h = 2)$. Let $p^h_{ij}(x,y)$ be the number of vertices which are non-adjacent to both x and y $(i = j = 1)$, adjacent to x but not to y $(i = 2, j = 1)$, adjacent to y but not to x $(i = 1, j = 2)$, or adjacent to both x and y $(i = j = 2)$.

<u>Def. 12-6</u>. If a graph G is regular of degree n_2 and is of order v, yet $G \neq K_v$ or $\bar{K}_v$, and if $p^h_{ij}(x,y)$ is independent of the choice of x and y, for $h, i, j = 1, 2$, then G is said to be a <u>strongly regular</u> graph.

It is well known that the eight conditions involving x and y can be replaced by two of them (see Problem 12-4):

<u>Thm. 12-7</u>. If G is a regular graph which is neither complete nor empty, then G is strongly regular if and only if $p^h_{22}(x,y)$ is independent of x and y, for $h = 1,2$.

For strongly regular graphs, it is convenient to write p^h_{ij} for $p^h_{ij}(x,y)$, as the choice of x and y is immaterial (except that they must be adjacent in G, for $h = 2$, and adjacent in $\bar{G}$, for $h = 1$.) Two vertices (objects) of a strongly regular graph are said to be <u>first associates</u> if they are non-adjacent and <u>second associates</u> if they are adjacent. Thus each vertex has exactly n_i ith associates $(i = 1, 2)$, and $n_1 + n_2 = v - 1$.

As a major class of examples of strongly regular graphs, we give:

<u>Thm. 12-8</u>. The regular complete n-partite graphs $G = K_{n(m)}$ are all strongly regular, for $m, n \geq 2$.

<u>Proof</u>: Clearly G is regular, of degree $n_2 = m(n-1)$. Since $m > 1$, G is not complete; since $n_1 > 1$, G

is not empty. Finally, we observe that $p^1_{22} = m(n-1)$ and $p^2_{22} = m(n-2)$. #

For the examples of Section 12-8, the following observation will be crucial. If we start with $G_1 = K_r$ and the strongly regular $G_2 = K_{n(m)}$, then the strong tensor product $G_1 \mathbin{\underline{\otimes}} G_2$ is strongly regular also, by Theorems 10-12 and 12-8.

Other standard examples of strongly regular graphs include $G = L(K_n)$ and $G = L(K_{2(m)})$, where $L(H)$ denotes the line graph of graph H. (We mention in passing that $L(K_{2(m)}) = K_m \times K_m$; see Problem 2-13.)

12-5. Partially Balanced Incomplete Block Designs

Def. 12-9. A $(v, b, r, k; \lambda_1, \lambda_2)$-partially balanced incomplete block design (PBIBD) is a set of v objects, pairwise associated into two association classes (as determined by a strongly regular graph G of order v) and a collection of b subsets of the object set, each subset being called a block, satisfying:

(i) each object appears in exactly r blocks;

(ii) each block contains exactly k $(k < v)$ objects;

(iii) each pair of ith associates appear together in exactly λ_i blocks $(k = 1, 2)$.

Again, the requirement $k < v$ corresponds to the incompleteness of the design; the requirement $G \neq K_v$ or $\overline{K}_v$ ensures that $n_1 n_2 > 0$, so that the PBIBD does not collapse to a BIBD - unless $\lambda_1 = \lambda_2$, and the requirement that G be strongly regular is an attempt to restore some of the balance to the experiment that was lost when one λ could not be found for all pairs x,y.

We now give additional terminology, some of which applies to both BIBDs and PBIBDs.

Def. 12-10. A PBIBD is said to be group-divisible if the strongly regular graph G upon which the design is based has a partition of its vertex set into n groups of m

vertices each so that two vertices are first associates if and only if they are in the same group.

Clearly this is possible if and only if $G = K_{n(m)}$.

Def. 12-11. A group-divisible PBIBD is a transversal design if each block contains exactly one vertex from each group.

Clearly this requires $k = n$.

Def. 12-12. A design (BIBD or PBIBD) is said to be resolvable if the set of b blocks can be partitioned into r subsets, of v/k blocks each, each subset containing each object exactly once.

For example, if we take the edges of a strongly regular graph G as blocks $(k = 2)$, the resulting PBIBD is resolvable if and only if G is 1-factorable. In particular, $G = K_{n(m)}$ gives an $(mn, m^2n(n-1)/2, m(n-1), 2; 0, 1)$-PBIBD, which is resolvable (see Himelwright and Williamson [HW2]) if and only if mn is even (for $m = 1$, the BIBD on $G = K_{n(1)} = K_n$ is resolvable if and only if n is even) and in fact is a transversal design if and only if $n = 2$. The transversal design on $K_{2(2w+1)}$, for example, is used in duplicate bridge scheduling to ensure that, in $2w+1$ rounds, each north-south couple plays exactly one round against each east-west couple.

Def. 12-13. A design (BIBD or PBIBD) is said to be z-resolvable if the block set can be partitioned so that each set in the partition contains each vertex exactly z times.

Thus, a 1-resolvable design is resolvable.

Def. 12-14. Two block designs D_1 and D_2 are said to be isomorphic if there exists a one-to-one correspondence $\theta\colon O_1 \to O_2$ between their object sets such that $\{x_1, x_2, \ldots, x_k\}$ is a block in D_1 if and only if $\{\theta(x_1), \theta(x_2), \ldots, \theta(x_k)\}$ is a block in D_2.

Clearly if D_1 and D_2 are isomorphic, their parameters v, b, r, k, λ (or λ_1 and λ_2) must agree; the converse is not true, as we shall see in Section 12-6.

For an example of all these definitions, consider the designs of Tables 12-2 and 12-3; clearly these are isomorphic, under the map θ sending i to the ith letter of the alphabet, $1 \leq i \leq 6$. The (6, 8, 4, 3; 0, 2)-PBIBD is based upon the strongly regular graph $G = K_{3(2)}$ and hence is group-divisible; it is also a transversal design and is resolvable (the resolution is given by the horizontal pairing of the blocks.)

This design could be used to compare six wines $(v = 6)$ as follows: companies A, B, and C make wines 1 and 4, 2 and 5, 3 and 6 respectively. We want to test the wines of the different companies against each other (say twice each: $\lambda_2 = 2$), but do not want to compare two wines made by the same company ($\lambda_1 = 0$.) Each taster tastes exactly 3 wines $(k = 3)$, after which his judgement becomes impaired. We have eight tasters $(b = 8)$ in all, and each wine is tasted four times $(r = 4)$.

1, 2, 6	3, 4, 5
2, 4, 6	1, 3, 5
4, 5, 6	1, 2, 3
1, 5, 6	2, 3, 4

Table 12-2

a, b, f	c, d, e
b, d, f	a, c, e
d, e, f	a, b, c
a, e, f	b, c, d

Table 12-3

Finally we remark that the above design was taken directly from the triangular imbedding of $K_{3(2)}$ in S_0 depicted in Figure 9-10 (writing "6" for "0" and rearranging the orbits to display the resolvability.) This foreshadows the correspondence of the next section.

12-6. PBIBDs and Graph Imbeddings

For triangular imbeddings of strongly regular graphs, we readily obtain analogs to Theorems 12-3 and 12-4. A design is said to be connected if its underlying graph is connected; since a complete graph underlies each BIBD, only a PBIBD could fail to be connected.

Thm. 12-15. Connected $(v, b, r, 3; 0, 2)$-PBIBDs are in one-to-one correspondence with triangular imbeddings of strongly regular graphs of order v in generalized pseudosurfaces.

Thm. 12-16. Connected $(v, b, r, 3; 0, 1)$-PBIBDs are in two-to-one correspondence with triangular imbeddings of strongly regular graphs of order v in generalized pseudosurfaces and having bichromatic dual.

When a design is constructed, by any method (here we are advocating graph imbeddings, but other tools of construction include Latin squares, finite projective geometries, finite euclidean geometries, difference sets), the natural question is: Is it new? Clearly the design is new, if no design existed previously on the same parameter set. It is new also if no previously constructed design on the same parameters is isomorphic to the given design. The context of topological graph theory is often very convenient for answering the isomorphism question, as we see in:

Thm. 12-17. Let D_1 and D_2 be two designs (both BIBDs with $k = 3$ and $\lambda = 2$, or both PBIBDs with $k = 3$ and $\lambda_1 = 0$, $\lambda_2 = 2$) on the same parameter set. If the generalized pseudosurfaces they determine are not homeomorphic as topological spaces, then D_1 and D_2 are not isomorphic as designs.

Proof: The identification procedure of surface topology is well-defined, so that isomorphic designs would yield homeomorphic generalized pseudosurfaces. #

The converse of this theorem is false. For example, K_{19} triangulates S_{20}, both with and without bichromatic dual. Thus in one case the 2-fold triple system splits into two Steiner triple systems and in the other case it does not, so that the two (19, 114, 18, 3, 2)-BIBDs are not isomorphic.

12-7. Examples

Example 1: Refer to Example 1b of Section 10-3; the self-dual quadrilateral imbedding of $C_3 \times C_3$ in S_1 yields a (9,9,4,4;1,2)-PBIBD. The graph $C_3 \times C_3$ is strongly regular, with $p^1_{22} = 2$ and $p^2_{22} = 0$; in fact, $C_3 \times C_3 = L(K_{3,3})$. (See Problem 2-13.) This "topological" design is atypical in that $k > 3$ and $\lambda_1 \neq 0$ and thus indicates that the scope of the connection is even wider than as indicated in Section 12-6. This design will be the ground case of an infinite collection of interesting designs constructed in Section 14-8.

Example 2: Refer to Example 2b of Section 10-3; the bichromatic-dual, triangular imbedding of $K_{4(2)}$ in S_1 obtained there gives one (8,16,6,3;0,2)-PBIBD and two (8,8,3,3;0,1)-PBIBDs (with blocks generated by $\{0,1,3\}$ and $\{0,3,2\}$ - in Z_8 - respectively.) All three of these designs are on parameters for which no BIBD exists. Then Example 2c of Section 10-3 gives a bichromatic-dual triangulation of S_1 also, but this time by $K_{3(3)}$ and giving one (9,18,6,3;0,2)-PBIBD and two (9,9,3,3;0,1)-PBIBDs (generated by $\{00,10,01\}$ and $\{00,11,01\}$ - in $Z_3 \times Z_3$ - respectively.) Finally, Example 2d of Section 10-3 gives a bichromatic-dual triangulation of S_1 once again, now by the strongly regular $G_\Lambda(\Gamma)$, $\Gamma = Z_4 \times Z_4$ (not in the class $K_{n(m)}$) and yielding one (16,32,6,3;0,2)-PBIBD and two (16, 16, 3, 3; 0, 1) -PBIBDS. The latter two designs are on parameters for which no BIBD exists.

Example 3: Refer to Example 5 of Section 10-3; the bichromatic-dual (see Problem 10-4) triangulation of $S_{n(n-1)/2}$ by $K_{3(n)}$ obtained there gives one $(3n,2n^2,2n,3;0,2)$-PBIBD and two $(3n,n^2,n,3;0,1)$-PBIBDs. These are all group-divisible, transversal designs. In [P4] Petroelje constructed orientable triangulations for $K_{3(n)}$ for which the resulting $\lambda_2 = 2$ PBIBDs are also resolvable. He then used these to obtain pseudosurface triangular imbeddings for $K_{4(n)}$ and the corresponding $(4n,4n^2,3n,3;0,2)$-PBIBDs. In [G1] Garman constructed bichromatic-dual orientable surface triangulations for $K_{4(n)}$, n even (no imbedding of $K_{4(n)}$ can have bichromatic dual for n odd, since each vertex then has odd degree $3n$), so that different (by Theorem 12-7) $(4n,4n^2,3n,3;0,2)$-PBIBDs, and also pairs of $(4n,2n^2,3n/2,3;0,1)$-PBIBDs, are obtained for these values of n. Finally, in [A5] Anderson constructed generalized pseudosurface triangulations for $K_{4(n)}$, for all n, giving yet another realization of these triples of designs.

Example 4: We observe that any $(n,b,n-1,k,\lambda)$-BIBD determined from an $r = r_k$ imbedding of K_n $(k \geq 3)$ determines in turn an $(n^2,2nb,2n-2,k;0,\lambda)$-PBIBD as follows. The line graph $L(K_{2(n)}) = K_n \times K_n$ (see Problem 2-13) is strongly regular, and the cartesian product $K_n \times K_n$ can be obtained by identifying vertices appropriately among $2n$ disjoint copies of K_n. (Let $V^i = \{(i,j) \mid 1 \leq j \leq n\}$, $1 \leq i \leq n$, and $V_j = \{(i',j') \mid 1' \leq i' \leq n'\}$, $1' \leq j' \leq n'$, be the $2n$ disjoint vertex sets, each V^i and each V_j *inducing a* K_n; then identify (i,j) with (i',j'), $1 \leq i,\ j \leq n$.) Then performing these vertex identification on $2n$ disjoint initial imbeddings of K_n as given yields a generalized pseudosurface imbedding of $K_n \times K_n$ and a PBIBD on the parameter set as claimed.

For instance, take $k = 3$, $\lambda = 2$, $n \equiv 0,1 \pmod 3$. Or, consider the complete design given by K_4 in N_1 (see Problem 12-6); this $(4,3,3,4,3)$-BIBD gives a $(16,24,6,4;0,3)$-PBIBD by this method. Finally, consider the voltage graph for $\Gamma = Z_9$ consisting of an octagon with edges labelled, in order, 1, 2, 3, -4, -3, -1, -2, 4 (a bouquet of four circles in S_2, after identification of the boundary edges of the octagon); the $(9,9,8,8,7)$-BIBD arising from the covering imbedding of K_7, although a trivial design, gives a non-trivial $(81,162,16,8;0,7)$-PBIBD. These designs on $K_n \times K_n$ are called Latin square designs; their construction could have been carried out purely combinatorially, but their topological realization is convenient for the application of Theorem 12-17.

Example 5: In all of the preceeding examples - including those in Section 12-3 - we have used graph imbeddings to obtain block designs. Now we reverse this point of view.

More attention has been paid toward the imbedding of graphs in the class $K_{n(m)}$ than in any other class; see for example [B13], [R8], [RY1], [RY5], [G1], and [J5]; for a survey of results in the orientable case, see [GRW1] and [KRW1]. Except for $n = 2$ (where no triangles are possible), the known results are obtained by constructing triangular imbeddings. For the generalized pseudocharacteristic $\chi''(K_{n(m)})$ - see Definition 6-48 - the upper bound is given by the euler characteristic for a triangulation:

Thm. 12-18. $\chi''(K_{n(m)}) \leq \frac{mn(6+m-mn)}{6}$.

We observe that the bound is an integer if and only if 3 divides $mn(n-1)$. For $m = 1$, $G = K_n$; it is now well-known that K_n has (in fact surface) triangular imbeddings if and only if $n \equiv 0,1 \pmod 3$. For $m > 1$ and $n > 2$ generalized pseudosurface triangular imbeddings of $K_{n(m)}$ are, by Theorem 12-15 and the observation following Definition 12-10, exactly group-divisible PBIBDs with $k = 3$, $\lambda_1 = 0$, and $\lambda_2 = 2$, The following is Theorem 6.2 of Hanani [H1], restated in the present context:

Thm. 12-19. A group-divisible PBIBD with the object set partitioned into n groups of m objects each $(m > 1, n > 2)$ and $k = 3$, $\lambda_1 = 0$, $\lambda_2 = 2$ exists if and only if 3 divides $mn(n-1)$.

Thus all the triangular imbeddings upon which the estimate of Theorem 12-18 is based actually exist, and we state:

Thm. 12-20. For $n > 2$, $\chi''(K_{n(m)}) = \frac{mn(6+m-mn)}{6}$ if and only if 3 divides $mn(n-1)$.

Hence Hanani's result for block designs serendipitously computes the generalized pseudocharacteristic of $K_{n(m)}$, in 7/9 of the possible cases.

12-8. Doubling a PBIBD

Given an orientable triangular imbedding of $K_{n(m)}$ with bichromatic dual, we have seen that one $(v,2b,2r,3;0,2)$-PBIBD, as well as two $(v,b,r,3;0,1)$-PBIBDs, correspond to this imbedding. Then by Theorem 10-12 (with $r = 2$) and Theorem 10-14 (with $G_1 = K_2$), the strong tensor product $K_2 \underline{\otimes} K_{n(m)} = K_{n(2m)}$ has an orientable triangular imbedding with bichromatic dual, so that one $(2v,8b,4r,3;0,2)$-PBIBD and two $(2v,4b,2r,3;0,1)$-PBIBDs result; in a sense, each original PBIBD has been doubled. Clearly this process can be iterated indefinitely,

Moreover, the construction given in [GRW1] for Theorem 10-14 provides a prescription for listing the blocks in the doubled designs in terms of those in the initial designs. Thus if the 2b initial blocks are $\bigcup_{i=1}^{b} \{a_{i1},a_{i2},a_{i3}\}$ and $\bigcup_{i=1}^{b} \{b_{i1},b_{i2},b_{i3}\}$, then the doubled blocks are

$$\bigcup_{i=1}^{b} \{a_{i1},a_{i2},a_{i3}\} \cup \bigcup_{i=1}^{b} \{b'_{i1},b'_{i2},b_{i3}\} \cup$$

$$\bigcup_{i=1}^{b} \{b_{i1},b'_{i2},b'_{i3}\} \cup \bigcup_{i=1}^{b} \{b'_{i1},b_{i2},b'_{i3}\}$$

and

$$\bigcup_{i=1}^{b} \{a'_{i1},a'_{i2},a'_{i3}\} \cup \bigcup_{i=1}^{b} \{b'_{i1},b_{i2},b_{i3}\} \cup$$

$$\bigcup_{i=1}^{b} \{b_{i1},b'_{i2},b_{i3}\} \cup \bigcup_{i=1}^{b} \{b_{i1},b_{i2},b'_{i3}\}$$

(again grouped by color class.)

Theorem 10-14 always gives a triangular imbedding for $K_2 \underline{\otimes} G$, for G as in the theorem; yet if $K_2 \underline{\otimes} G$ is not strongly regular, then no block designs are provided. It is easily seen that $K_2 \underline{\otimes} G$ <u>is</u> strongly regular if and only if G is strongly regular with $p^1_{22} = n_2$ or complete, and that G is strongly regular and connected, with $p^1_{22} = n_2$, or complete, if and only if $G = K_{n(m)}$ for some $m \geq 1$, $n \geq 2$. Thus the doubling process of this section is applicable exactly to triangular imbeddings of $K_{n(m)}$ with bichromatic dual (there is a nonorientable analog to Theorem 10-14; see [G1].) However, every time such an imbedding is found, it determines an infinite tower of triples of PBIBDs, as explained above. Moreover, if the initial imbedding is on a pseudosurface, than all imbeddings derived from it are also pseudosurface imbeddings; thus the associated designs, by Theorem 12-17, differ from those of Hanani (see Example 5 in Section 12-7.)

For example, consider the toroidal imbeddings of K_7, $K_{4(2)}$, and $K_{3(3)}$ given as covering spaces of Figure 10-4 (Example 2 in Section 10-3); each is the base for such an infinite tower of designs. The graphs for K_7, for instance, are the family $K_{7(2^k)}$, $k = 0,1,2, \ldots$ - for which the genus is also thereby determined.

As a second example, consider the voltage graph of Figure 12-1, for $\Gamma = Z_{35}$; this determines a nonorientable pseudosurface

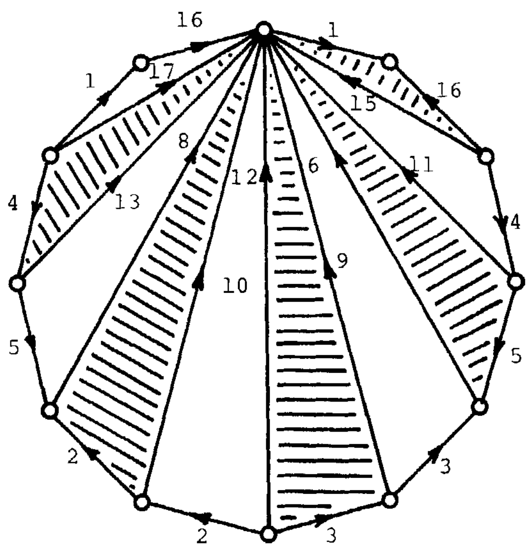

Figure 12-1

triangular imbedding for $K_{7(5)}$ with bichromatic dual. Hence another infinite tower of imbeddings - for $K_{7(5\cdot 2^k)}$ - and of the corresponding block designs is anchored. (In this case, the pseudo-characteristic $\chi'(K_{7(5\cdot 2^k)})$ is also determined.) The $\lambda_2 = 1$

designs, for all k, and the $\lambda_2 = 2$ designs, for k even, have no BIBD counterparts.

12-9. Problems

12- 1.) Prove Theorem 12-2 (i).

12- 2.) Prove Theorem 12-2 (ii).

12- 3.) Find (and prove) an analog to Theorem 12-2 for PBIBDs.

12- 4.) Prove Theorem 12-7.

*12- 5.) Prove or disprove the following strengthening of Frucht's Theorem (3-18): Every finite group is the automorphism group of some strongly regular graph.

12- 6.) Show that the complement of a strongly regular graph is strongly regular.

12- 7.) Let a $(v,b,r,k;\lambda_1,\lambda_2)$-PBIBD be based upon the strongly regular graph G. Show that we may use exactly the same objects and blocks, to obtain a corresponding $(v,b,r,k;\lambda_2,\lambda_1)$-PBIBD, based upon $\overline{G}$. (Thus we may assume, without loss of generality, that $\lambda_1 < \lambda_2$.)

12- 8.) Construct an imbedding of K_4 in N_1 to give a $(4,3,3,4,3)$-PBIBD.

12- 9.) Imbed the Petersen graph in N_1 so as to give a $(10,6,3,5;1,2)$-PBIBD.

12-10.) What design results from the voltage graph imbedding of Figure 12-2, using $\Gamma = Z_{10}$?

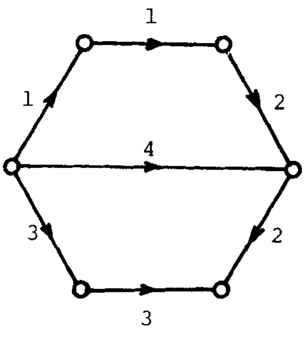

Figure 12-2

12-11.) What designs result from the voltage graph imbedding of Figure 12-3, using $\Gamma = Z_{11}$?

12-12.) Show that the design of Figure 12-1 is 3-resolvable, the design of Problem 12-10 is 4-resolvable, and the design of Problem 12-11 is 5-resolvable.

12-13.) PBIBDs on three or more association classes may also be found from graph imbeddings. For example, take $\Delta = \{(1,0,0), (0,1,0), (0,0,1)\}$ for $\Gamma = Z_2 \times Z_2 \times Z_2$ and show that $Q_3 = G_\Delta(\Gamma)$ in S_0 gives an $(8,6,3,4;0,1,2)$-PBIBD$_{(3)}$, where v_1 and V_2 are first associates if $v_2 = v_1 + (1,1,1)$, second associates if non-adjacent but not first associates, and third associates if adjacent.

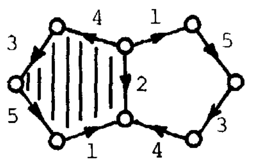

Figure 12-3

CHAPTER 13

HYPERGRAPH IMBEDDINGS

A graph is just a special case of a hypergraph; but although a great deal of attention has been paid, as we have seen, to the geometric realizations of graphs, there has been little effort made to extend these concrete representations to the general setting. In [JSW1] an attempt was made to remedy this situation, as we indicate in the present chapter also.

Our aim is to find a geometric realization for hypergraphs satisfying:

(1) The method should not be unduly cumbersome.

(2) It should include the standard geometric realization of graphs (as points and arcs in appropriate 2-manifolds) as a special case.

13-1. Hypergraphs

Def. 13-1. A *hypergraph* H consists of a finite non-empty set $V(H)$ of *vertices* together with a set $E(H)$, each of whose elements is a subset of $V(H)$ and is called an *edge*. If $e \in E(H)$ $(e \subseteq V(H))$ and if $u,v \in e$ $(u,v \in V(H))$, we say that u and v are *adjacent* vertices, and that the vertex u and edge e are *incident* with each other, as are v and e. Two distinct edges e_1 and e_2 are said to be *adjacent* if $e_1 \cap e_2 \neq \phi$. The *degree*, $d(v)$, of $v \in V(H)$ is the number of edges with which v is incident. If $|e| = r (r \geq 0)$ for all $e \in E(H)$, then H is said to be an r-*uniform* hypergraph.

Thus a graph is just a 2-uniform hypergraph.

We write $p_H = |V(H)|$ and $q_H = |E(H)|$. Let $n_i = |e_i|$, for $1 \leq i \leq q_H$ and $E(H) = \{e_1, e_2, \ldots, e_{q_H}\}$. Then as a generalization of Theorem 2-2, we have:

Thm. 13-2. For any hypergraph H, $\sum_{i=1}^{p_H} d(v_i) = \sum_{i=1}^{q_H} n_i$.

Proof: Merely count total incidences, in the two possible ways. #

Note that for a graph each $n_i = 2$, and we regain Theorem 2-2. In fact:

Cor. 13-3. For an r-uniform hypergraph H, $\sum_{i=1}^{p_H} d(v_i) = rq_H$.

For an example, consider the 3-uniform hypergraph H defined by:

$V(H) = \{1,2,3,4\}$,
$E(H) = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}$;

then Corollary 13-3 observes that $4 \cdot 3 = 3 \cdot 4$. One convention (see Berge [B7]) for representing hypergraphs would depict H as in Figure 13-1; this method does not appear to meet either criterion (1) or (2), as given in the introduction to this chapter. Thus, we seek another method.

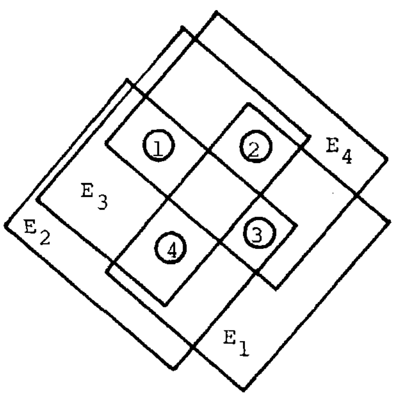

Figure 13-1

We observe that Definitions 2-6 and 2-7 carry over verbatim, so that a connected hypergraph is exactly what one would expect it to be.

13-2. Associated Bipartite Graphs

We use a bijection between connected hypergraphs H and connected bipartite graphs $G(H)$ given by Walsh [W2]. In the present context, we are primarily concerned with the construction of $G(H)$ from H:

$$V(G(H)) = V(H) \cup E(H), \quad \text{the bipartition;}$$

$$E(G(H)) = \{\{v,e\} \mid v \in V(H), \quad e \in E(H), \quad v \in e\}.$$

We next find a 2-cell imbedding of $G(H)$ into some closed orientable 2-manifold S_k $(k \geq 0)$, and denote this 2-cell imbedding by $G(H) \triangleleft S_k$.

For the example H of Section 13-1, $G(H)$ is $K_{4,4}$ less a 1-factor; that is, $G(H) = Q_3$. It is convenient to take $Q_3 \triangleleft S_0$, as usual.

In the next section, we shall see how to modify the imbedding of $G(H) \triangleleft S_k$ so as to obtain an "imbedding" of H into S_k $(H \triangleleft S_k)$.

13-3. Imbedding Theory for Hypergraphs

Given a 2-cell imbedding of the associated bipartite graph $G(H) \triangleleft S_k$, we modify this imbedding to obtain an imbedding of the hypergraph H into S_k, wherein certain of the regions of the modified imbedding $(G(H) \triangleleft S_k)$ represent edges of H; the remaining regions of the modified imbedding $(G(H) \triangleleft S_k)$ become regions for the imbedding of H into S_k. For H connected, $G(H)$ will be connected also, and thus we can find a 2-cell imbedding $G(H) \triangleleft S_k$. The modification we perform preserves the 2-cell aspect of the imbedding, so that the regions for H in S_k are all 2-cell also. Hence the notation $H \triangleleft S_k$ is justified.

We illustrate the modification process in Figure 13-2. In part (a) of that figure, we see - in the imbedding of $G(H)$ - the vertices representing edge $e = \{v_1, v_2, \ldots, v_k\}$ of H and each vertex in e. In part (b) of the figure we begin modifying the imbedding by adding edge $\{v_i, v_{i+1}\}$, within the region containing

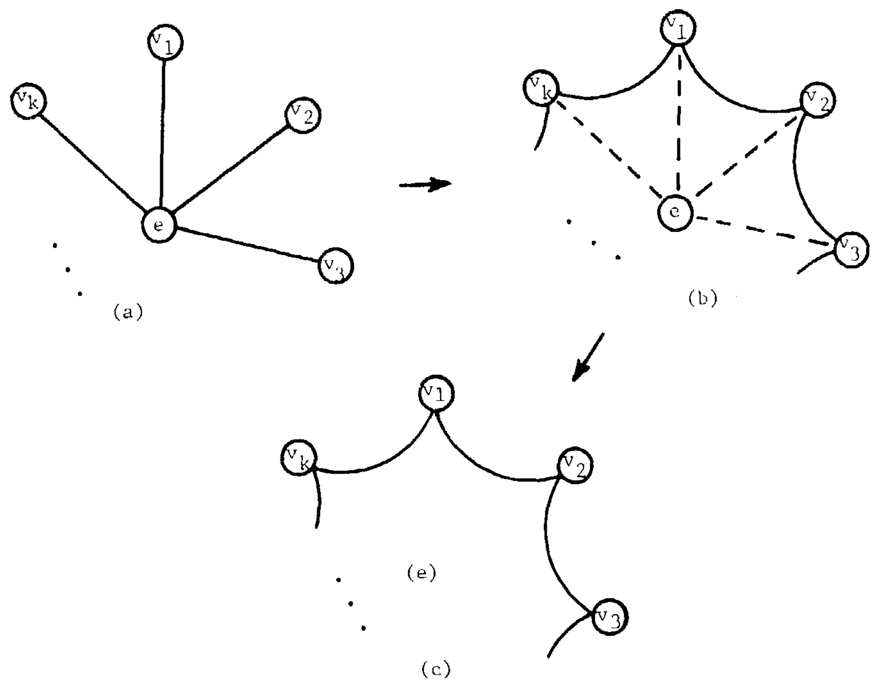

Figure 13-2

v_i, v_{i+1}, and e, for $1 \leq i \leq k$, (mod k). Then the edges $\{e, v_i\}$, $1 \leq i \leq k$, and the vertex e, are deleted, so that in part (c) of the figure the edge e appears as a region in the modified imbedding. (More precisely, the set of vertices from the boundary of region (e) is exactly the edge e.)

In Figure 13-3 we illustrate the entire process for the hypergraph H of Section 13-1. In part (a) of this figure we see an imbedding $G(H) = Q_3 \triangleleft S_0$, and the beginning of the modification. In part (b) of the figure we see the corresponding imbedding $H \triangleleft S_0$, with the regions of this imbedding being shaded; the unshaded "regions" depict the edges of H. The former (i.e. the bonafide regions of the hypergraph imbedding) are all digons here, as each region for the imbedding of G(H) was a quadrilateral. (In general, a k-gonal region results for $H \triangleleft S_k$ from a 2k-gonal region of the

(a)

(b)

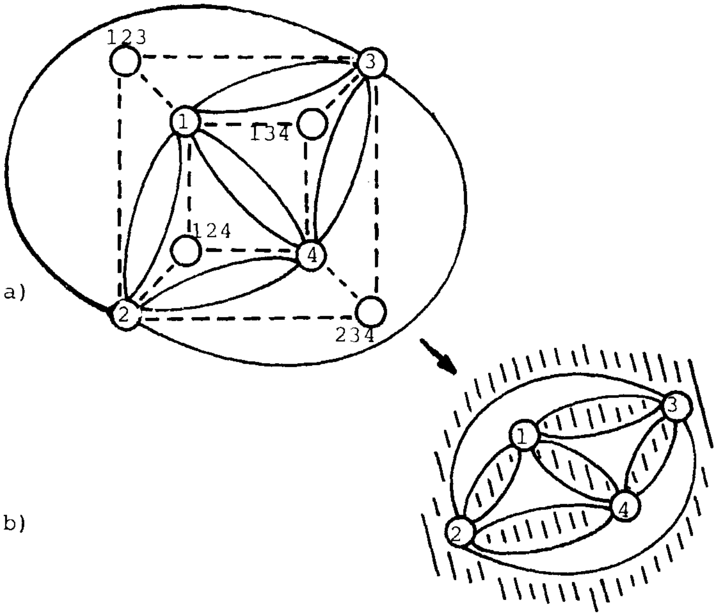

Figure 13-3

<u>bipartite</u> $G(H) \lhd S_k$.)

We remark that: (1) this representation is not cumbersome, in that $V(H)$ and $E(H)$ are readily discernible. (Compare Figure 13-1, for the same hypergraph H.) (2) If H is in fact a graph, then each edge of H becomes a digon for the modified imbedding $G(H) \lhd S_k$, and the collapsing of each digon gives a traditional imbedding $H \lhd S_k$. This is illustrated for $H = K_4$, in Figure 13-4.

Imbedding problems for hypergraphs now translate directly into the graphical context, and many standard results for graphs have ready generalizations to hypergraphs. (This approach was noted, independently, for the sphere only, by Jones [J3].) For example, here is the generalization of Theorem 5-14. (We let r_H denote the number of bonafide regions - i.e. <u>not</u> including those regions depicting edges - in a 2-cell imbedding of hypergraph H.)

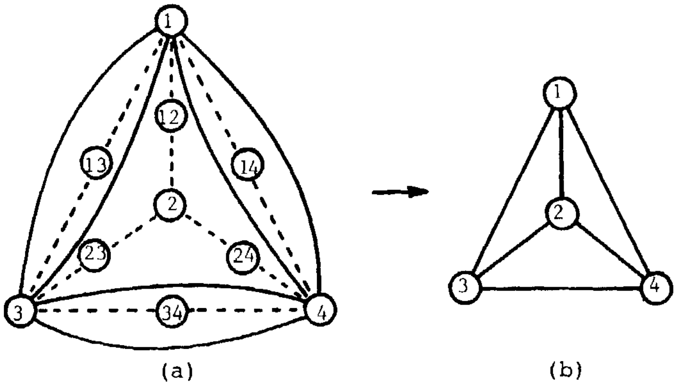

Figure 13-4

Thm. 13-4. Let the connected hypergraph H have a 2-cell imbedding in S_k, with the usual parameters p_H, q_H, and r_H. Then

$$p_H + q_H - \sum_{i=1}^{q_H} n_i + r_H = 2 - 2k.$$

Proof: For the 2-cell imbedding $G(H) \lhd S_k$ which gave rise to $H \lhd S_k$, we have

$$p - q + r = 2 - 2k,$$

by Theorem 5-14, where

$$p = V(G(H)) = p_H + q_H,$$

$$q = E(G(H)) = \sum_{i=1}^{q_H} n_i,$$

and $r = r_H$.

The result now follows, by substitution. #

We note that, for H a graph - so that $n_i = 2$, $1 \leq i \leq q_H$, then $\sum_{i=1}^{q_H} n_i = 2q_H$ and we have the familiar $p_H - q_H + r_H = 2 - 2k$.

13-4. The Genus of a Hypergraph

Once we accept as worthwhile the task of realizing a hypergraph geometrically, then it is most natural to wish to do this as efficiently as possible. This motivates

Def. 13-5. The genus, $\gamma(H)$, of a hypergraph H is the genus of its associated bipartite graph; i.e. $\gamma(H) = \gamma(G(H))$.

Since the genus parameter for graphs is additive over connected components (Corollary 6-19), we obtain immediately

Thm. 13-6. The genus of a hypergraph is the sum of the genera of its components.

Proof: Let $H = \bigcup_{i=1}^{n} H_i$ be the decomposition of H into its connected components, with $G(H_i)$ the bipartite graph associated with component H_i. Then $\gamma(H) = \gamma(G(H)) = \gamma(\bigcup_{i=1}^{n} G(H_i)) = \sum_{i=1}^{n} \gamma(G(H_i)) = \sum_{i=1}^{n} \gamma(H_i)$. #

Thus it is without loss of generality that we continue to restrict our attention to connected hypergraphs.

Def. 13-7. The maximum genus $\gamma_M(H)$, of a connected hypergraph H is given by: $\gamma_M(H) = \gamma_M(G(H))$.

We have the natural generalization of Corollary 6-22:

Thm. 13-8. A connected hypergraph H has a 2-cell imbedding in S_k if and only if $\gamma(H) \leq k \leq \gamma_M(H)$.

We now give a lower bound for the genus parameter, reminiscent of Corollary 6-15 (but not in generalization of that corollary, since if H is a graph, then $G(H)$ will have girth at least six.)

Thm. 13-9. If H is a connected hypergraph, then

$$\gamma(H) \geq 1 + \frac{1}{4}\left(\sum_{i=1}^{q_H} n_i - 2p_H - 2q_H\right).$$

Proof: Let $H \lhd S_k$, where $k = \gamma(H) = \gamma(G(H))$. Since H is connected, so is $G(H)$; thus the minimal imbedding $G(H) \lhd S_k$ is 2-cell, as is $H \lhd S_k$. Since $G(H)$ is bipartite, $4r \leq 2q$. Thus

$$2r_H = 2r \leq q = \sum_{i=1}^{q_H} n_i,$$

and using this in the euler equation for hypergraphs (Theorem 13-4), we get the desired bound. #

We close this section with an upper bound for the maximum genus parameter, in analogy with Theorem 6-24.

Thm. 13-10. Let H be connected; then

$$\gamma_M(H) \leq \frac{1 - p_H - q_H + \sum_{i=1}^{q_H} n_i}{2}.$$

Moreover, equality holds if and only if $r_H = 1$ or 2, according as the numerator is even or odd, respectively.

13-5. The Heawood Map-coloring Theorem, for Hypergraphs

Def. 13-11. The chromatic number, $\chi(H)$, of a hypergraph H is the minimum natural number k for which there is a partition $V(H) = \bigcup_{i=1}^{k} V_i(H)$ such that, for each edge $e \in E(H)$, there is no i with $e \subseteq V_i(H)$.

That is, $\chi(H)$ is the smallest number of colors for $V(H)$ so that no edge of H is uniformly colored. Note that this is the "weak" definition of chromatic number for hypergraphs, as we are not requiring that all vertices in an arbitrary edge be colored differently (just that they not all be colored alike.) Clearly the weak and the "strong" definitions agree, if H is a graph.

Def. 13-12. The hypergraph chromatic number of the surface S_k is defined by: $\chi_H(S_k)$ = the maximum $\chi(H)$ such that $H \lhd S_k$.

Thm. 13-13. $\chi_H(S_k) = \left\lfloor \frac{7 + \sqrt{1 + 48k}}{2} \right\rfloor$, $k \geq 0$.

Proof: Set $f(k) = \left\lfloor \frac{7 + \sqrt{1 + 48k}}{2} \right\rfloor$.

(1) Let $H \lhd S_k$, and let $G^*(H)$ denote the corresponding modification of $G(H)$. Since $G^*(H) \lhd S_k$, $\chi(G^*(H)) \leq f(k)$, by the Heawood Map-Coloring Theorem (or the Four-Color Theorem, if $k = 0$) for graphs. Let $G^*(H)$ be $f(k)$-colored. Now consider an arbitrary edge e of H and any two consecutive vertices in the corresponding region of $G^*(H)$; since they form an edge of $G^*(H)$, these two vertices are colored differently. Thus e is not uniformly colored, and $\chi(H) \leq f(k)$. Since H was arbitrary for S_k, $\chi_H(S_k) \leq f(k)$.

(2) Since $H = K_{f(k)} \lhd S_k$, $\chi_H(S_k) \geq \chi(H) = f(k)$.

(3) Thus $\chi_H(S_k) = f(k)$.

13-6. The Genus of a Block Design

Every block design is a k-uniform hypergraph - with objects as vertices and blocks as edges - so that any realization of a hypergraph associated with a block design is simultaneously a realization of that design. For example, the Steiner triple system H of order 7, arising (for example) from the projective plane of order 2, is depicted in Figure 13-5, where G(H) would be the "Heawood Graph" (see Figure 8-6), the dual of $K_7 \lhd S_1$. In this case, $G^*(H)$ is K_7, and our realization of H coincides with that of Example 2, Section 12-3. (The blocks of Table 12-1 appear as the unshaded regions in Figure 13-5.)

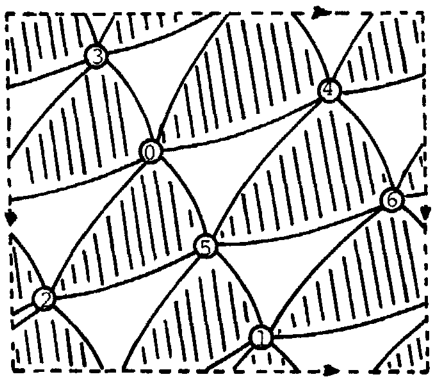

Figure 13-5

It will be no surprise that, once we agree to depict block designs realistically, we should desire to do this as efficiently as possible.

Def. 13-14. The genus of a block design D, $\gamma(D)$, is the genus of the associated hypergraph H; i.e. $\gamma(D) = \gamma(H) = \gamma(G(H))$, where G(H) is the bipartite graph for H.

Thus $\gamma(D)$ gives the most efficient orientable surface for the representation of D.

For example, if D is the familiar (7,7,3,3,1)-BIBD (Steiner triple system), then Figure 13-5 shows that $\gamma(D) \leq 1$; but one readily finds a homeomorph of $K_{3,3}$ in G(H) for Figure 13-5, so that $\gamma(D) \geq 1$. Thus $\gamma(D) = 1$, and our depiction of Figure 13-5 is optimal.

Rahn [R1] has characterized <u>planar</u> (i.e. $\gamma(D) = 0$) BIBDs;

<u>Thm. 13-15</u>. A (v,b,r,k,λ)-BIBD is planar if and only if:

(i) $k = 1$;
(ii) $k = 2$ and $v = 2,3,$ or 4;
(iii) $k = 3$ and $(v,b,r,k,\lambda) = (3,1,1,3,1)$, $(3,2,2,3,2)$ or $(4,4,3,3,2)$.

We note that the case $(4,4,3,3,2)$ is displayed in Figure 13-3.

13-7. An Example

As an additional example of many of the concepts of this chapter, we consider the $(n,n,n-1,n-1,n-2)$-BIBD D_n (for $n \geq 2$) whose object set is $\{1,2, \ldots, n\}$ and whose blocks are the complements of singletons (i.e. all $(n-1)$-subsets of an n-set.) This immediately determines a hypergraph H having the same description: $V(H_n) = \{1,2, \ldots, n\}$, $E(H_n) = \{S \subseteq V(H_n) \mid |S| = n-1\}$. Then $G(H_n)$ is $K_{n,n}$ less a 1-factor (each vertex i is adjacent to every edge except its complement, so that the 1-factor is composed of pairs $\{i, V(H_n) - \{i\}\}$, $1 \leq i \leq n$.) For $n = 2, 3, 4$ respectively, we have $G(H_n) = 2K_2, C_6, Q_3$; see Figure 13-3 for the case $n = 4$ and the planar imbedding of H_4.

<u>Thm. 13-16</u>. $\gamma(D_n) = \left\{ \frac{(n-1)(n-4)}{4} \right\}$, for $n \geq 2$.

<u>Proof</u>: From the lower bound of Theorem 13-9, we find that

$$\gamma(D_n) = \gamma(H_n) \geq \left\{ \frac{(n-1)(n-4)}{4} \right\} .$$

To complete the proof, we show the reverse inequality, by construction. This construction splits into four cases, depending upon the residue of n modulo 4; here we provide the details only for the case $n \equiv 1 \pmod 4$. (See Problem 13-2 for the remaining - harder - cases.) The method uses an index two current graph (see Figure 13-6 for the cases $n = 5$ and 9, which have obvious generalization; the vertex rotations are indicated schematically, to yield a current graph imbedding in S_h, where $h = (n-1)/4$ and the group $\Gamma = Z_{2n}$, generated by $\Delta = \{1,3,5, \ldots , n-2\}$. Since the KCL holds at each vertex of the current graph, the Cayley graph $(G_\Delta(\Gamma) = G(H_n))$ imbedding which covers the dual of the current graph imbedding is quadrilateral, and hence - since it is bipartite - minimally imbedded in S_k, $k = (n-1)(n-4)/4$. Thus

$$\begin{aligned}\gamma(D_n) &= \gamma(H_n) = \gamma(G(H_n)) \\ &= \frac{(n-1)(n-4)}{4} = \left\{ \frac{(n-1)(n-4)}{4} \right\}\end{aligned}$$

in this case.

n = 5

n = 9

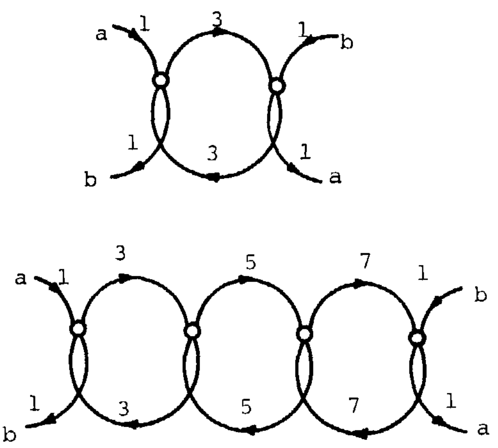

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Figure 13-6.

13-8. Nonorientable Analogs

We conclude with a brief discussion of the nonorientable imbedding of hypergraphs. The definition of a 2-cell imbedding of a hypergraph H (via a 2-cell imbedding of the associated bipartite graph $G(H)$) on the nonorientable surface N_h $(H \triangleleft N_h)$ carries over verbatim from the orientable case, as do all related concepts. We have the following results:

<u>Thm. 13-17</u>. If $H \triangleleft N_h$, then $p_H + q_H - \sum_{i=1}^{q_H} n_i + r_H = 2 - h.$

<u>Thm. 13-18</u>. If $\tilde{\gamma}(H)$ denotes the nonorientable genus of hypergraph H, then

$$\tilde{\gamma}(H) \geq 2 + \frac{1}{2}\left(\sum_{i=1}^{q_H} n_i - 2p_H - 2q_H\right).$$

<u>Thm. 13-19</u>. The connected hypergraph H has a 2-cell imbedding on N_h if and only if

$$\tilde{\gamma}(H) \leq h \leq 1 + \sum_{i=1}^{q_H} n_i - p_H - q_H.$$

Finally, we give the nonorientable Heawood Map-Coloring Theorem for hypergraphs:

<u>Thm. 13-20</u>. $\chi_H(N_h) = \left\lfloor \frac{7 + \sqrt{1 + 24h}}{2} \right\rfloor$, for $h \neq 2$;

$$\chi_H(N_2) = 6.$$

13-9. Problems

13-1.) Prove Theorem 13-10.

*13-2.) Show that $K_{n,n}$ less a 1-factor imbeds on S_k, $k = \{(n-1)(n-4)/4\}$, for $n \equiv 0, 2, 3, \pmod 4$.

13-3.) Prove Theorem 13-17.

13-4.) Prove Theorem 13-18.

13-5.) Prove Theorem 13-19.

13-6.) Prove Theorem 13-20.

CHAPTER 14

MAP AUTOMORPHISM GROUPS

Our focus in this book has been on the various interactions among graphs, groups, and surfaces and - in particular - on the surface imbeddings of graphs depicting groups. In this chapter we go one step further, by considering the automorphism group of the configuration consisting of a graph imbedded in a surface. An important special case will occur when the graph is a Cayley graph for some group. The development here is essentially that of [BW1]; see also Biggs ([B8], [B9], and [B10]), [W9], and [W13].

Recall Corollary 6-22: a connected graph G has a 2-cell imbedding in S_k if and only if $\gamma(G) \leq k \leq \gamma_M(G)$. So far, in this book, we have concentrated on the two extremes of this imbedding range, in calculating various values of the genus and the maximum genus parameters. In this chapter we consider two additional special types of imbeddings, generally in the interior of the imbedding range: those which are symmetrical, and those which are self-dual. Finally, we combine these two concepts (and others as well) in our study of "Paley maps."

For a theory of maps for orientable surfaces unifying the two standard approaches (that of geometers, studying symmetry properties, and that of combinatorialists, studying graph imbeddings and map colorings), see Jones and Singerman [JS1].

14-1. Map Automorphisms

Recall that a rotation scheme for a connected graph G of order n is an ordered n-tuple $P = (p_1, p_2, \ldots, p_n)$, where p_i is the rotation at vertex i, $1 \leq i \leq n$. Then P determines a 2-cell imbedding in a closed orientable 2-manifold S_k, where k is uniquely specified by the euler equation and the number of orbits of the permutation P^* on the set D^* (see Section 6-6.) Here it will be convenient to let ρ_v denote the rotation at vertex v and

$\rho = (\rho_v)_{v \in V(G)}$ denote P.

<u>Def. 14-1</u>. A <u>map</u> is a pair (G, ρ), where G is a connected graph and ρ is a rotation scheme for G.

Thus a map may be regarded as a configuration consisting of a representation of a connected graph by its imbedding in a particular closed orientable 2-manifold. This definition can be extended to include nonorientable surfaces as well, as in Section 11-3; but we concentrate on orientable maps here.

We now define an automorphism of a map (G, ρ) to be an automorphism of the graph G which also preserves the rotation ρ. Specifically, we construct an action of the automorphism group $\mathcal{G}(G)$ on the set $R(G)$ of all rotations of G. (We observe that $|R(G)| = \prod_{i=1}^{n} (n_i - 1)!$, where $n_i = d(v_i)$, $1 \leq i \leq n$; see Problem 14-1.) If $a \in \mathcal{G}(G)$ and $\rho \in R(G)$, we define $a(\rho) \in R(G)$ by:

$$(a(\rho))_{a(v)} = a\rho_v a^{-1};$$

that is, if ρ_v takes x to y, then $(a(\rho))_{a(v)}$ takes $a(x)$ to $a(y)$ (see Figure 14-1).

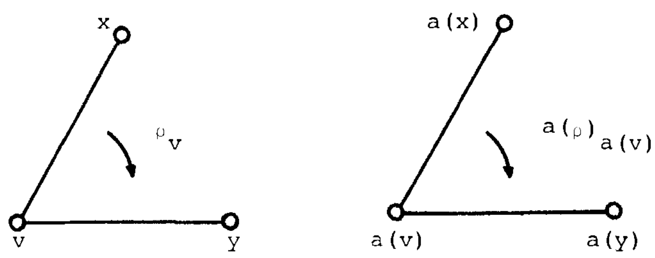


Figure 14-1

Def. 14-2. Two rotations ρ and σ in $R(G)$ are said to be equivalent if there is an $a \in \mathcal{G}(G)$ such that $\sigma = a(\rho)$.

Lemma 14-3. If ρ and σ are equivalent rotations on G, with $\sigma = a(\rho)$ and $(x, y, z, \ldots, w)$ a region of map (G, ρ), then $(a(x), a(y), a(z), \ldots, a(w))$ is a region of map (G, σ).

Proof: Since $(x, y, z, \ldots, w)$ is a region of (G, ρ), $\rho^*(x, y) = (y, z)$; i.e. $\rho_y(x) = z$. Thus

$$a(\rho)_{a(y)}(a(x)) = a\rho_y a^{-1}(a(x)) = a\rho_y(x) = a(z).$$

So, putting $\sigma = a(\rho)$, we have

$$\sigma_{a(y)}(a(x)) = a(z), \text{ and}$$
$$\sigma^*(a(x), a(y)) = (a(y), a(z)).$$

Hence, $(a(x), a(y), a(z), \ldots, a(w))$ is a region of (G, σ). #

It follows that, if ρ and σ are equivalent rotations, then there is a one-to-one correspondence between the region sets of the maps (G, ρ) and (G, σ). Hence the two maps are in the same surface; that is, they have the same genus.

Def. 14-4. An automorphism of a map $M = (G, \rho)$ is a graph automorphism $a \in \mathcal{G}(G)$ such that $a(\rho) = \rho$; i.e. $\rho_{a(v)} = a\rho_v a^{-1}$, for all $v \in V(G)$. (That is, ρ is equivalent with itself, under the action of a.)

The following equivalent formulation (see Figure 14-1 and Problem 14-2) readily displays the graph-automorphism nature of each map automorphism. We denote the automorphism group of a map $M = (G, \rho)$ by $\mathcal{G}(M)$.

Thm. 14-5. For a permutation $a: V(G) \to V(G)$, $a \in \mathcal{G}(M)$ if and only if: $(x, y, z, \ldots, w)$ a region of M implies $(a(x), a(y), a(z), \ldots, a(w))$ is a region of M.

Thus graph automorphisms preserve edges, while map automorphisms preserve oriented region boundaries.

We need two standard results from the theory of permutation groups (see [BW1], for example.) Let (Γ, X) be a permutation group, so that each $\gamma \in \Gamma$ is a permutation of object set X. The orbit of $x \in X$ is defined by: $\Gamma x = \{\gamma(x) \mid \gamma \in \Gamma\}$; then Γx is a subset of X. The stabilizer of x is defined as: $\Gamma_x = \{\gamma \in \Gamma \mid \gamma(x) = x\}$; then Γ_x is a subgroup of Γ. Moreover, we have:

Thm. 14-6. $|\Gamma x| = |\Gamma : \Gamma_x|$.

Finally, for $\gamma \in \Gamma$, the set of fixed points for γ is denoted by: $F(\gamma) = \{x \in X \mid \gamma(x) = x\}$. Then we have the following theorem of Frobenius, often called "Burnside's Lemma":

Thm. 14-7. The number, t, of orbits of (Γ, X) is given by:

$$t = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} |F(\gamma)| \ .$$

Now we put the two above results to work.

Thm. 14-8. Let G be a connected graph and ρ a rotation on G; then the number of rotations equivalent to ρ is equal to the index $|\mathcal{G}(G) : \mathcal{G}(G,\rho)|$.

Proof: By definition, $\mathcal{G}(G,\rho) = \mathcal{G}(G)_\rho$, the stabilizer of ρ in the action of $\mathcal{G}(G)$ on $R(G)$. The number of rotations equivalent to ρ is just the orbit $\mathcal{G}(G)\rho$. Now apply Theorem 14-6. #

Thm. 14-9. The number of equivalence classes of maps with underlying graph G is:

$$\frac{1}{|\mathcal{G}(G)|} \sum_{a \in \mathcal{G}(G)} |F(a)|,$$

Where $F(a) = \{\rho$ for $G \mid a(\rho) = \rho\}$.

Proof: Apply Theorem 14-7 to the action of $\mathcal{G}(G)$ on $R(G)$. #

We illustrate with $G = K_4$; then $|R(G)| = 2^4 = 16$. Consider $\rho = ((234),(143),(124),(132))$ as shown in Figure 14-2. For this M,

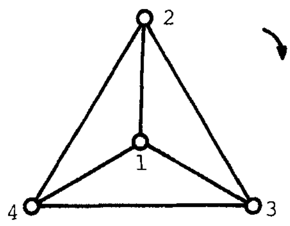


Figure 14-2

$\mathcal{G}(M) = A_4$ (see Problem 14-3.) From Theorem 14-8, using $\mathcal{G}(K_4) = S_4$ (see Theorem 3-17 (1)), we deduce that there are just two of the sixteen rotations in $R(K_4)$ equivalent to the given one (the other is the "mirror image" ((243),(134),(142),(123))); both have genus zero. To classify all 16 rotations of K_4, we note that $|F(\gamma)|$ is a class function: it is constant on each conjugacy class of elements of S_4; we obtain Table 14-1.

Class representative (a)	e	(12)	(123)	(12)(34)	(1234)
Number in class	1	6	8	3	6
$\|F(a)\|$	16	0	4	4	2

Table 14-1

Thus, by Theorem 14-7, the number of equivalence classes is $(16 + 32 + 12 + 12)/24 = 3$. One of these classes is represented by Figure 14-2, and we have just seen that this class contains exactly two rotations. The other two classes are both toroidal (genus one); they are represented in Figure 14-3. The map automorphism groups are Z_4 and Z_3, so that the classes contain (by Theorem 14-8) six and eight rotations respectively. In each case, the rotations pair off by the "mirror image" relationship:

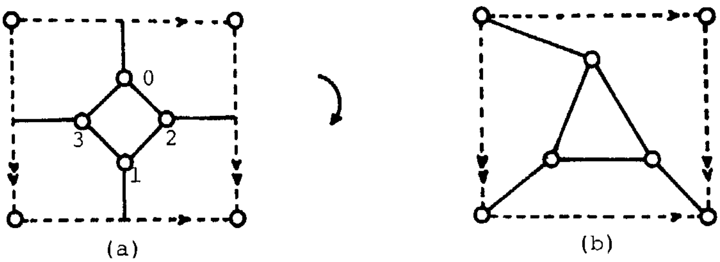


Figure 14-3

Def. 14-10. The mirror image of a map $M = (G,\rho)$ is given by $\tilde{M} = (G,\rho^{-1})$, where if $\rho = \{\rho_v\}_{v \in V(G)}$, then $\rho^{-1} = \{\rho_v^{-1}\}_{v \in V(G)}$.

Clearly $\tilde{M}$ always exists for a given map M, and $M = \tilde{M}$ if and only if G is a cycle (since $\rho = \rho_v^{-1}$ if and only if $d(v) = 2$.)

Def. 14-11. A map $M = (G,\rho)$ is said to be reflexible if and only if there exists an $\alpha \in G(G)$ such that $(v_1,v_2,\ldots v_n)$ is a region for M if and only if $(\alpha v_n,\ldots,\alpha v_2,\alpha v_1)$ is a region for M; α is called a reflection.

Thus M is reflexible if and only if M and the mirror image $\tilde{M}$ are equivalent (Problem 14-4.) For example, every rotation for K_4 gives rise to a reflexible map.

Def. 14-12. The extended map automorphism group, $G^*(M)$, of a map M consists of $G(M)$ together with all reflections for M.

Thm. 14-13. Let D be a $(v,b,r,3,2)$-BIBD (a 2-fold Steiner triple system), let $\mathcal{G}(D)$ be the design automorphism group, and let M be the corresponding map (Theorem 12-3); then $\mathcal{G}^*(M) \cong \mathcal{G}(D)$.

We have seen that the sixteen rotations for K_4 split into three equivalence classes of maps; we remark that K_5 and K_6 have corresponding numbers as indicated by Table 14-2.

n	2	3	4	5	6
number of rotations	1	1	16	7,776	191,102,976
number of classes	1	1	3	78	265,764

Table 14-2

14-2. Symmetrical Maps

We now study $\mathcal{G}(M)$, considered as a permutation group acting on $V(G)$ - where $M = (G,\rho)$.

Lemma 14-14. Let $a \in \mathcal{G}(M)$, where $M = (G,\rho)$, and let $\{u,v\} \in E(G)$ with $a(u) = u$ and $a(v) = v$; then $a = e$.

Proof: Since $a \in \mathcal{G}(M)$, $\rho_{a(v)} = a\rho_v a^{-1}$. Thus $a(\rho_v(u)) = a\rho_v a^{-1}(u) = \rho_{a(v)}(u) = \rho_v(u)$; that is, a fixes $\rho_v(u)$ also. But since ρ_v is cyclic on $N(v)$ (the set of vertices adjacent to v), the argument repeats and we see that a fixes each vertex in $N(v)$. Similarly, a fixes each vertex in $N(u)$, each vertex in $N(\rho_v(u))$, and so on; since G is connected, we see that a fixes each $w \in V(G)$, so that $a = e$.

Thm. 14-15. Let $\mathcal{G} = \mathcal{G}(M)$, for $M = (G,\rho)$, with $v \in V(G)$. The stabilizer $\mathcal{G}_v$ is isomorphic to a subgroup of the cyclic group $\langle\rho_v\rangle$ generated by ρ_v, and hence is a cyclic group whose order divides $d(v)$.

<u>Proof</u>: We have $a\rho_v a^{-1} = \rho_{a(v)} = \rho_v$, so that $a\rho_v = \rho_v a$. Let $w \in N(v)$; then $a(w) \in N(v)$ also, and since ρ_v is cyclic on $N(v)$ we have $a(w) = \rho_v^i(w)$, for some i. Let $x \in N(v)$, say $x = \rho_v^j(w)$. Then $a(x) = a\rho_v^j(w) = \rho_v^j a(w) = \rho_v^{j+i}(w) = \rho_v^i\rho_v^j(w) = \rho_v^i(x)$. Thus $\bar{a} = \rho_v^i$, where $\bar{a}$ is the restriction of a to $N(v)$, and $\theta: \mathcal{G}_v \to \langle\rho_v\rangle$, $\theta(a) = \bar{a}$, is a homomorphism and if $\bar{a}_1 = \bar{a}_2$, then $a_1a_2^{-1} = e$ by Lemma 14-14; thus θ is a monomorphism. #

If $\mathcal{G}$ is transitive on $V(G)$, then all vertex stabilizers are conjugate in $\mathcal{G}$, so that $|\mathcal{G}| = p|\mathcal{G}_v|$, where $p = |V(G)|$. Thus $|\mathcal{G}| = p\delta$, where δ divides d, the common vertex degree of G. For example, the rotation ρ for K_7 given in Section 9-2 gives a vertex-transitive map M (consider $(0,1,2,3,4,5,6) \in \mathcal{G}(M)$). Moreover, $p_0 = (1,3,2,6,4,5) \in \mathcal{G}_0$, so that $|\mathcal{G}| = 7\cdot 6 = 42$. All the K_4 maps are vertex-transitive, except those in the third of the three classes.

Now consider $\mathcal{G} = \mathcal{G}(M)$ to act on $D^* = \{(u,v) \mid \{u,v\} \in E(G)\}$, by $a(u,v) = (a(u),a(v))$. Let $x = (u,v)$; then $\mathcal{G}_x = e$, by Lemma 14-14. Thus, by Theorem 14-6, $|\mathcal{G}_x| = |\mathcal{G}:\mathcal{G}_x| = |\mathcal{G}|$, independent of x. This gives:

<u>Thm. 14-16.</u> $|\mathcal{G}(M)|$ divides $2|E(G)|$.

<u>Proof</u>: By the preceeding remarks, each orbit of D^* has length $|\mathcal{G}(M)|$. #

<u>Def. 14-17.</u> If $|\mathcal{G}(M)| = 2|E(G)|$, then M is said to be a <u>symmetrical</u> map.

Thus symmetrical maps display the maximum amount of symmetry. If M is symmetrical, then $\mathcal{G}(M)$ is a regular permutation group on the object set D^*; symmetrical maps are also called <u>regular</u> maps.

<u>Thm. 14-18.</u> If M is symmetrical, then $\mathcal{G}(M)$ is transitive on the vertices, on the edges, and on the regions of M.

Proof: Let $\mathcal{G} = \mathcal{G}(M)$ act on D^*, with $|\mathcal{G}(M)| = |D^*|$. Let $x = (u,v) \in D^*$ and $a_1, a_2 \in \mathcal{G}(M)$; if $a_1x = a_2x$, then $a_1 = a_2$ by Lemma 14-14; thus $|\mathcal{G}x| = |D^*|$, and $\mathcal{G}$ is transitive on D^*; hence $\mathcal{G}$ is transitive on $E(G)$. Now let $u,w \in V(G)$, with (u,v), $(w,z) \in D^*$. Since $\mathcal{G}$ is transitive on D^*, we find $a \in \mathcal{G}$ so that $(a(u),a(v)) = a(u,v) = (w,z)$; thus $a(u) = w$, and $\mathcal{G}$ is transitive on $V(G)$. Finally, let $r_1 = (v,w, \ldots)$ and $r_2 = (x,y, \ldots)$ be two regions of M, with $a \in \mathcal{G}(M)$ such that $a(v) = x$ and $a(w) = y$. Then, since $a\rho_w a^{-1} = \rho_{a(w)} = \rho_y$, $a\rho_w(v) = \rho_{a(w)}a(v) = \rho_y a(v) = \rho_y(x)$, so that the next vertex of r_1 is carried to the next vertex of r_2 by a; this process continues, to show that $\mathcal{G}$ is transitive on the region set as well. #

Thus a symmetrical map M has associated with it two important constants: the constant vertex degree d and the constant number k of vertices in each region boundary.

Thm. 14-19. Let M be a symmetrical map of genus γ, with p vertices (all of degree d), q edges, and r regions (all having length k); then:

(i) $dp = kr = 2q$;

(ii) $\gamma = 1 + \frac{(d-2)(k-2) - 4}{4k} p$.

Proof: We use Lemma 5-17 (iii) and (iv) and Theorem 5-14. #

For $d = 2$, $p = q = k$, $r = 2$, and $\gamma = 0$; M consists of $G = C_p$ on S_0. Hence we assume $d \geq 3$ (and $k \geq 3$).

If $\gamma = 0$, then $(d-2)(k-2) < 4$ and $(d,k) = (3,3)$, $(3,4)$, $(3,5)$, $(4,3)$, or $(5,3)$. Each pair determines p, q, and r uniquely; see Section 5-4. Thus we have:

Thm. 14-20. A map $M(G,\rho)$ is symmetrical of genus zero if and only if $G = C_p$ $(p \geq 2)$ or the 1-skeleton of a Platonic solid.

If $\gamma = 1$, then $(d-2)(k-2) = 4$ and $(d,k) = (3,6), (4,4)$, or $(6,3)$. For each case, we find infinitely many toroidal maps covered by the corresponding regular tessellation of the plane (see Figures 8-6, 7-3, and 13-5 for one map of each respective type.)

If $\gamma \geq 2$, then again each pair (d,k), for fixed γ, determines p, q, and r uniquely. Moreover, the number of symmetrical maps of genus γ is finite:

<u>Thm. 14-21</u>. If M is a symmetrical map of genus $\gamma \geq 2$, then $|\mathcal{G}(M)| \leq 84(\gamma-1)$; equality holds if and only if $(d,k) = (3,7)$ or $(7,3)$.

<u>Proof</u>: Using $|\mathcal{G}(M)| = 2q$ and both parts of Theorem 14-19, we obtain

$$|\mathcal{G}(M)| = 2q = dp = \frac{4kd}{(d-2)(k-2)-4}(\gamma-1).$$

Since $d \geq 3$ and $k \geq 3$, the coefficient of $\gamma - 1$ has a maximum value of 84, occurring precisely when $(d,k) = (3,7)$ or $(7,3)$. #

For any map M, whether orientable or nonorientable, the order $|\mathcal{G}^*(M)|$ of the extended map automorphism group $\mathcal{G}^*(M)$ divides $4|E(G)|$, where G is the graph of the map. (See Problem 14-8.) We get $|\mathcal{G}^*(M)| = 4|E(G)|$, in the orientable case, if M is symmetrical and reflexible. In the nonorientable case we lose our sense of order-preservation, so that "map automorphisms" are indistinguishable from "reflections;" a nonorientable map M is thus said to be <u>symmetrical</u> if $|\mathcal{G}^*(M)| = 4|E(G)|$. We mention just one result about such maps, due to Wilson [W16]; note the connection with the covering projection of Section 11-2.

<u>Thm. 14-22</u>. If N is a nonorientable symmetrical map, then there is a unique orientable map M, both symmetrical and reflexible, which is a 2-fold covering space of N.

For two examples, we give the graph of the dodecahedron on S_0 covering, by antipodal identification, the Petersen graph on N_1 and the dual configurations: the icosahedral graph on S_0 projecting to K_6 on N_1.

Wilson shows that the uniqueness of Theorem 14-22 is not reversible, by giving one M covering two distinct maps N.

14-3. Cayley Maps

Now let $M = (G, \rho)$ be a map, where $G = G_\Delta(\Gamma)$ is a Cayley graph for group Γ, as generated by $\Delta \subseteq \Gamma$ subject to the usual restrictions ($e \notin \Delta$; if $\delta \in \Delta \cap \Delta^{-1}$, then $\delta^2 = e$); recall that $\Delta^* = \Delta \cup \Delta^{-1}$. In this situation, the vertex rotations ρ_v ($v \in V(G) = \Gamma$) can be regarded as permutations not only of $N(v)$, but also of Δ^*; thus two rotations can be more readily compared. Of special interest is the case where the induced permutations of Δ^* are all the same.

<u>Def. 14-23.</u> A <u>Cayley map</u> $M(\Gamma, \Delta, r)$ is the map $M(G, \rho)$, where $G = G_\Delta(\Gamma)$ and $r\colon \Delta^* \to \Delta^*$ is a cyclic permutation so that, for $g \in \Gamma$ and $h \in N(g)$,

$$\rho_g(h) = gr(g^{-1}h).$$

Thus the group structure determines the vertex rotations, as in Figure 14-4 - where $r = (\delta_1, \delta_2, \ldots, \delta_k)$: the edge $\{g, g\delta_1\}$ is determined by $\delta_1 \in \Delta^*$, so that the image of $g\delta_1$ under ρ_g is determined by $r(\delta_1) = \delta_2 \in \Delta^*$.

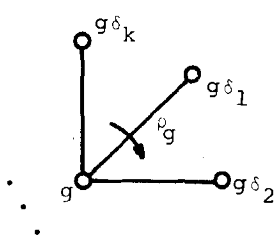


Figure 14-4

For a specific example, we consider the map of K_7 on S_1 given in Sections 9-2 and 14-2; it is a Cayley map $M(Z_7, Z_7-\{0\},r)$, with $r = (1,3,2,6,4,5)$. Then $\rho_i(g) = i + r(g-i)$, $0 \le i \le 6$; that is, $\rho_i = (1+i,3+i,2+i,6+i,4+i,5+i)$. In contrast, the graph Q_3 shown in Figure 5-8 is a Cayley graph $G_\Lambda(Z_2 \times Z_2 \times Z_2)$, with $\Lambda = \{(1,0,0),(0,1,0),(0,0,1)\}$; but the map shown on S_0 is not a Cayley map, since no suitable r exists.

Next we give a strengthening of Theorem 4-8; every automorphism of $C_\Delta(\Gamma)$ is also an automorphism of $M(\Gamma,\Lambda,r)$. We remark that the Cayley graph $G_\Delta(\Gamma)$ may well have additional automorphisms.

Thm. 14-24. The Cayley map $M = M(\Gamma,\Lambda,r)$ is vertex-transitive; in fact $\mathfrak{G}(M)$ contains a regular subgroup isomorphic to Γ.

Proof: The isomorphism $\alpha: \Gamma \to \mathfrak{G}(C_\Delta(\Gamma))$, $\alpha(g) = \theta_g: V(C_\Delta(\Gamma)) \to V(C_\Delta(\Gamma))$, $\theta_g(g_i) = gg_i$ given in the proof of Theorem 4-8 works here as well; it only remains to show that $\theta_g \in \mathfrak{G}(M)$; that is, that $\rho_{\theta_g(h)} = \theta_g\rho_h\theta_g^{-1}$, for each $h \in V(C_\Delta(\Gamma)) = \Gamma$. But let $f \in N(\theta_g(h))$; then

$$\begin{aligned}\rho_{\theta_g(h)}(f) &= \rho_{gh}(f)\\ &= ghr((gh)^{-1}f)\\ &= \theta_g(hr(h^{-1}g^{-1}f))\\ &= \theta_g\rho_h(g^{-1}f)\\ &= \theta_g\rho_h\theta_{g^{-1}}(f).\end{aligned}$$

#

We observe that the map of K_4 in S_1 depicted in Figure 14-3(a) is a Cayley map $M(Z_4,Z_4-\{0\},r)$, where $r = (1,2,3)$, and has only the automorphisms guaranteed by Theorem 14-24; it is not symmetrical. This situation generalizes:

Thm. 14-25. Let $M = M(\Gamma,\Delta,r)$ be a Cayley map which is not symmetrical, and let $|\Delta^*|$ be prime; then $\mathfrak{G}(M) \cong \Gamma$.

Proof: Let $\mathcal{G} = \mathcal{G}(M)$ and $v \in V(C_\Delta(\Gamma))$. By Theorem 14-15, $|\mathcal{G}_v|$ divides $|\Delta^*|$, which is prime; hence $|\mathcal{G}_v| = 1$ or $|\Delta^*|$. By Theorem 14-24, M is vertex-transitive, so that by Theorem 14-6 $|\mathcal{G}| = |\Gamma||\mathcal{G}_v|$. But since M is not symmetrical, $|\mathcal{G}_v| \neq |\Delta^*|$; thus $|\mathcal{G}_v| = 1$, $|\mathcal{G}| = |\Gamma|$, and $\mathcal{G} \cong \Gamma$. #

On the other hand, $\mathcal{G}(M)$ may be strictly larger than Γ, for M a Cayley map $M(\Gamma,\Delta,r)$. Recall that $\alpha\colon \Gamma \to \Gamma$ is an <u>automorphism</u> of group Γ if α is one-to-one and onto, and $\alpha(\gamma_1\gamma_2) = \alpha(\gamma_1)\alpha(\gamma_2)$, for all $\gamma_1, \gamma_2 \in \Gamma$.

<u>Thm. 14-26</u>. Let $M = M(\Gamma,\Delta,r)$ be a Cayley map and let $\alpha \in \mathcal{G}(\Gamma)$ be a group automorphism such that $\alpha|_{\Delta^*} = r^\ell$, for some ℓ, $1 \le \ell \le |\Delta^*|$. Then $\alpha \in (\mathcal{G}(M))_e$. (That is, α is a map automorphism, fixing e.)

Proof: Since $\alpha \in \mathcal{G}(\Gamma)$, $\alpha(e) = e$. Let $\{u,v\} \in E(G_\Delta(\Gamma))$, so that $u^{-1}v \in \Delta^*$; then $\alpha(u^{-1}v) = \alpha(u^{-1})\alpha(v) \in \Delta^*$, $\{\alpha(u), \alpha(v)\} \in E(G_\Delta(\Gamma))$, and $\alpha \in \mathcal{G}(G_\Delta(\Gamma))$. To show that $\alpha \in \mathcal{G}(M)$, we verify that, for all $g \in \Gamma$, $\rho_{\alpha(g)} = \alpha\rho_g\alpha^{-1}$: let $h \in N(\alpha(g))$; then $\alpha^{-1}(h) \in N(g)$, $g^{-1}\alpha^{-1}(h) \in \Delta^*$, and

$$\begin{aligned}
\rho_{\alpha(g)}(h) &= \alpha(g)r((\alpha(g))^{-1}h)\\
&= \alpha(g)r(\alpha(g^{-1})h)\\
&= \alpha(g)r(\alpha(g^{-1}\alpha^{-1}(h)))\\
&= \alpha(g)r^{\ell+1}(g^{-1}\alpha^{-1}(h))\\
&= \alpha(gr(g^{-1}\alpha^{-1}(h)))\\
&= \alpha\rho_g\alpha^{-1}(h).
\end{aligned}$$ #

In the special case $\ell = 1$ of the above theorem, $\mathcal{G}(M)$ is as large as possible:

<u>Thm. 14-27</u>. Let $M = M(\Gamma,\Delta,r)$ be a Cayley map, with $\alpha \in \mathcal{G}(\Gamma)$ such that $\alpha|_{\Delta^*} = r$. Then M is a symmetrical map.

<u>Proof</u>: By Theorem 14-26, $\alpha \in (\mathcal{G}(M))_e$. By Theorem 14-15, $|(\mathcal{G}(M))_e|$ divides $|\Delta^*|$. But $\alpha|_{\Delta^*} = r$, a $|\Delta^*|$-cycle, so that $|(\mathcal{G}(M))_e| = |\Delta^*|$. (In fact, $(\mathcal{G}(M))_e$ is generated by α.) Now, by Theorem 14-24, $\mathcal{G}(M)$ is transitive on $\Gamma = V(G_\Delta(\Gamma))$, so that Theorem 14-6 gives:

$$\begin{aligned} |\mathcal{G}(M)| &= |\Gamma| \; |(\mathcal{G}(M)_e| \\ &= |\Gamma| \; |\Delta^*| \\ &= 2|E(G_\Delta(\Gamma))|, \end{aligned}$$

and M is symmetrical. #

We now calculate the genus of an arbitrary Cayley map $M(\Gamma,\Delta,r)$. We first observe that $M(\Gamma,\Delta,r)$ is just the covering space of the imbedded index one voltage graph K of Figure 14-5 (the ambient orientable surface for K is immaterial here; it is determined uniquely by the rotation at the single vertex.) The region boundaries for K are clearly determined by the (not

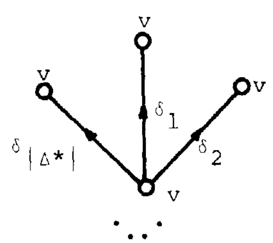


Figure 14-5

necessarily cyclic) permutation $\bar{r}$: $\Delta^* \to \Delta^*$, given by $\bar{r}(\delta) = r(\delta^{-1})$. So, let $\bar{r}$ have t cycles $\Delta_1, \Delta_2, \ldots, \Delta_t$ in its action on Δ^*, with m_i the order of $\delta_{i1}\delta_{i2}\cdots\delta_{ik_i}$ in Γ, where $\Delta_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{ik_i})$, $1 \leq i \leq t$. These t numbers m_i are called the <u>periods</u> of M.

<u>Thm. 14-28</u>. Let $m_1, m_2, \ldots, m_t$ be the periods of the Cayley map $M(\Gamma,\Delta,r)$; then the genus γ of M is given by:

$$\gamma = 1 + \frac{|\Gamma|}{4}\left(|\Delta^*| - 2 - 2\sum_{i=1}^{t} \frac{1}{m_i}\right).$$

<u>Proof</u>: By Theorem 10-8 (iii), the cycle $\Delta_i (1 \leq i \leq t)$ is covered by $|\Gamma|/m_i$ regions, each of length $m_i k_i$. Thus γ is determined by the euler equation (Theorem 5-14), using $p = |\Gamma|$, $q = |\Gamma|\ |\Delta^*|/2$, and $r = \sum_{i=1}^{t} \frac{|\Gamma|}{m_i}$. #

To illustrate these ideas, consider once again $M = M(Z_7, Z_7 - \{0\}, (1,3,2,6,4,5))$. It is immediate that $G_\Delta(\Gamma) = K_7$, and we compute $\overline{r} = (1,4,2)(3,5,6)$; thus $t = 2$, $m_1 = m_2 = 1$. Using Theorem 14-26, we confirm that $\gamma = 1$. Next, we note that $\alpha = (0)(1,3,2,6,4,5) \in G(Z_7)$, so that Theorem 14-27 applies to show that M is symmetrical.

As a second example, consider now the map $M = M(A_5,\Delta,r)$ as specified by $r = ((1,2,3,4,5), (5,4,3,2,1), (12)(34))$. Then $G_\Delta(\Gamma)$ is the one-skeleton of the familiar soccer ball design. We find $\overline{r} = ((1,2,3,4,5), (12)(34)),((5,4,3,2,1))$, so that $t = 2$, $m_1 = 3$, and $m_2 = 5$. We again use Theorem 14-28, to compute $\gamma = 0$ (this is truly fortunate, for the game of soccer!), noting that $r_6 = 20$ (the white panels of the soccer ball) and $r_5 = 12$ (the black panels.) Clearly M is not region-transitive and hence, by Theorem 14-18, M is not symmetrical. Finally, we observe that r does <u>not</u> extend to an automorphism of A_5 (since r does not preserve order), as required by Theorem 14-27.

14-4. Complete Maps

<u>Def. 14-29</u>. A <u>complete map</u> is a map $M(G,\rho)$, where G is a complete graph K_n.

Of course, we have been studying complete maps throughout this book, primarily with the aim of minimizing or maximizing the genus γ of M. Now it is the symmetry of M that we wish to maximize.

<u>Lemma 14-30</u>. If $\alpha \in G(K_n, \rho)$ fixes more than one vertex, then $\alpha = e$.

<u>Proof</u>: Any two distinct vertices of K_n are adjacent, so Lemma 14-14 applies. #

<u>Def. 14-31</u>. The transitive permutation group (Γ, X) is said to be a <u>Frobenius group</u> if only $e \in \Gamma$ has more than one fixed point in X.

Thus the automorphism group of a vertex-transitive complete map is necessarily Frobenius, By Lemma 14-30. The study of Frobenius groups leads to a classification of vertex-transitive complete maps. We only outline this development here; for full details, see [B9] or [BW1].

<u>Thm. 14-32</u>. Let (Γ, X) be a Frobenius group, with N^* the set of fixed-point-free elements of Γ and $N = N^* \cup \{e\}$. Then:

(i) $|N| = |X|$;

(ii) If Γ_x is abelian, then N is a regular normal subgroup of (Γ, X).

The following result appears in Burnside [B18; p. 172]:

<u>Thm. 14-33</u>. If (Γ, X) is a Frobenius group of degree $n = |X| \geq 6$ and order $n(n-1) = |\Gamma|$, then n is a prime power.

Now let Γ be any group of order n, with $\Delta^* = \Gamma - \{e\}$; then $G_\Delta(\Gamma) = K_n$, and if r is any cyclic permutation of $\Gamma - \{e\}$, then the Cayley map (Γ, Δ^*, r) is complete and, by Theorem 14-24, vertex-transitive. Theorem 14-32 is used to provide a converse to this result, so that we have:

<u>Thm. 14-34</u>. Let $M = M(K_n, \rho)$ be a complete map; then $\mathcal{G}(M)$ acts transitively on the vertices of M if and only if M is a Cayley map.

<u>Proof</u>: The sufficiency was established in Theorem 14-24. For the necessity, apply Lemma 14-30 to see that $\mathcal{G}(M)$ is a Frobenius group. Now $(\mathcal{G}(M))_x$ is cyclic, by Theorem 14-15, and hence Theorem 14-32 (ii) guarantees a regular normal subgroup N of $\mathcal{G}(M)$. Now take $\Gamma = Z_n$ and define a bijection $\beta: V(G_\Delta(\Gamma)) = \Gamma \to N$ by $\beta(i) = \beta_i$, where $\beta_i(0) = i$; then calculations show that ρ is given (homeomorphically) by $r: N - \{e\} \to N - \{e\}$, $r(\beta_i) = \beta_{\rho_e(i)}$, $i \neq 0$. #

Thus, for example, an imbedding of a complete graph covering a voltage graph of index higher than one (and not projecting to an index one voltage graph) cannot be symmetrical.

Finally, Biggs [B9] established:

<u>Thm. 14-35</u>. There is a rotation ρ for K_n so that (K_n, ρ) is symmetrical if and only if n is a prime power.

<u>Proof</u>: (i) If $M = (K_n, \rho)$ is symmetrical, then $|\mathcal{G}(M)| = 2|E(K_n)| = n(n-1)$; moreover, $\mathcal{G}(M)$ is Frobenius, by Theorem 14-18 and Lemma 14-30. Thus Theorem 14-33 applies, to show that n is a prime power.

(ii) Conversely, if $n = p^m$ where p is prime and $m \in N$, then take $\Gamma = (Z_p)^m$ - the additive group in $GF(n)$ - and let $x \in \Gamma$ generate the multiplicative group. Take $\Delta^* = \Gamma - \{0\}$, and $r: \Delta^* \to \Delta^*$ by $r(\delta) = x\delta$; then r extends (by setting $\alpha(0) = 0$) to $\alpha \in \mathcal{G}(\Gamma)$ and, by Theorem 14-27, the Cayley map $M(\Gamma, \Delta, r)$ is symmetrical. #

We close this section by summarizing our knowledge of $\mathcal{G} = \mathcal{G}(M)$, for $M = M(K_n, \rho)$ a symmetrical complete map:

(i) $|\mathfrak{G}| = n(n-1)$.

(ii) $\mathfrak{G}$ is transitive on the vertices, edges, and regions of M, and regular on the directed edges.

(iii) $\mathfrak{G}$ is a Frobenius group.

(iv) For each $v \in V(K_n)$, $\mathfrak{G}_v = Z_{n-1}$.

(v) $\mathfrak{G}$ has a regular normal subgroup, isomorphic to $(Z_p)^m$ - where $n = p^m$; in fact $\mathfrak{G}$ is the semi-direct product of $(Z_p)^m$ and Z_{n-1}.

14-5. Other Symmetrical Maps

In contrast to the situation for K_n, where symmetrical maps exist only for prime powers of n, we have the following two results (see Problems 14-11 and 14-12; for hints, see [BW1]):

<u>Thm. 14-36</u>. The graph $K_{n,n}$ has a symmetrical map, for all n.

<u>Thm. 14-37</u>. The graph Q_n has a symmetrical map, for all n.

We note that the symmetrical maps for Q_n can be taken to be of genus $\gamma(Q_n)$ or of genus $\gamma(Q_{n+1})$.

There are several classes of questions that can be asked regarding symmetrical maps; we list three of these, so as to reflect the thrust of this book:

(i) For a given graph G, what are the symmetrical maps $M(G,\rho)$?

(ii) For a given group Γ, what are the symmetrical Cayley maps $M(\Gamma,\Delta,r)$?

(iii) For a given surface S_k, what are the symmetrical maps of genus k?

Theorem 14-35, 14-36, and 14-37 speak to question (i). The discussion following Theorem 14-19 addresses question (iii). And the following two results (see Problems 14-13 and 14-14; for hints, see [BW1]) respond to question (ii):

Thm. 14-38. The group S_n has a symmetrical Cayley map, for all $n \geq 2$.

Thm. 14-39. The group A_n has a symmetrical Cayley map, for n odd, $n \geq 3$.

We could also ask:

(iv) For a given group Γ, what are the symmetrical maps $M = M(G,\rho)$ so that $\mathcal{G}(M) \cong \Gamma$?

Thm. 14-40. (Brahana [B16]) The symmetric groups S_n $(n \geq 3)$ and the alternating groups A_n $(n \geq 4)$ each occur as the automorphism group of a symmetrical map.

14-6. Self-complementary Graphs

Def. 14-41. A graph G is said to be self-complementary if it is isomorphic to its complement: $G = \overline{G}$.

Def. 14-42. An anti-automorphism of a graph G is a permutation $\beta: V(G) \to V(G)$ exchanging edges and "non-edges;" that is, $uv \in E(G)$ if and only if $\beta u \beta v \notin E(G)$; $u \neq v$. We denote the set of all anti-automorphisms of G by $\overline{\mathcal{G}}(G)$.

It follows that G is self-complementary if and only if $\overline{\mathcal{G}}(G) \neq \phi$; in this case $\mathcal{G}(G) \cup \overline{\mathcal{G}}(G)$ is a group containing $\mathcal{G}(G)$ as a (necessarily normal) subgroup of index two.

Self-complementary graphs have been studied in some detail, as the sample results below demonstrate. (See also Sachs [S3], and Gibbs [G2].)

Thm. 14-43. (Ringel [R9]) There exists a self-complementary graph G of order p if and only if $p \equiv 0$ or $1 \pmod 4$.

(The necessity is apparent, as the size of G must be $p(p-1)/4$.)

<u>Thm. 14-44</u>. (Ringel [R9]) If $p \equiv 1 \pmod 4$ and σ is an anti-automorphism for a self-complementary graph G of order p, then σ has exactly one fixed point and every other orbit for the action of σ on $V(G)$ has length a multiple of four.

For example, if $\Delta = \{1\}$ for $\Gamma = Z_5$, then $G_\Delta(\Gamma) = C_5$ is self-complementary, as shown (say) by $\sigma = (0)(1,2,4,3)$; $\mathfrak{G}(C_5) = D_5$, and $\bar{\mathfrak{G}}(C_5) = \sigma D_5$.

<u>Thm. 14-45</u>. (Read [R5]) There are 36 self-complementary graphs of order $p = 9$; 5,600 of order 13; and 11,220,000 of order 17.

<u>Thm. 14-46</u>. (Rao [R3]) If G is a self-complementary graph of order $p \geq 8$ and having minimum degree $\delta \geq n/4$, then G has a 2-factor (a spanning 2-regular subgraph.)

<u>Def. 14-47</u>. A graph G is a <u>graphical regular representation</u> of a group Γ if $(\mathfrak{G}(G), V(G))$ is a regular permutation group and $\mathfrak{G}(G) \cong \Gamma$.

<u>Thm. 14-48</u>. (Lim [L3]) If G is a graphical regular representation of Γ, then G is not self-complementary.

<u>Thm. 14-49</u>. (Chao and Whitehead [CW1]) If G is self-complementary, then $|V(G)| \leq (\chi(G))^2$.

Recall from Problem 2-1 that at least one of G and $\bar{G}$ is connected; we get immediately:

<u>Thm. 14-50</u>. If G is self-complementary, then G is connected.

Nebesky [B2] has shown that at least one of G and $\bar{G}$ is upper-imbeddable; thus:

<u>Thm. 14-51</u>. If G is self-complementary, then $\gamma_M(G) = [\beta(G)/2]$.

The next result could be useful in seeking a genus imbedding for a self-complementary graph:

Thm. 14-52. (Clapham [C5]) The number of triangles in a self-complementary graph of order p is at least $p(p-2)(p-4)/48$, if p 0 (mod 4), and at least $p(p-1)(p-5)/48$, if $p \equiv 1 \pmod 4$. These minimum numbers are attained.

Finally, we comment that self-complementary graphs determined by the quadratic residues of a finite field have been used to give lower bounds for the Ramsey numbers $r(k,k)$; see Greenwood and Gleason [GG1] and Burling and Reyner [BR2]. See also Clapham [C6], for a more general construction; he finds a self-complementary graph of order 113 containing no K_7, giving the improved bound $r(7,7) \geq 114$.

14-7. Self-dual Maps

The "self-complementary" property for a graph depends only upon the abstract structure of the graph itself. To obtain an analog in terms of a geometric realization for the graph, we first imbed the graph on a surface, form the dual graph for this imbedding, and then compare the original graph with its dual. (In a sense, duality is a higher-dimensional analog of complementation: in taking a dual, we fix 1-dimensional subsets, while interchanging 2- and 0-dimensional subsets; in taking a complement, we fix 0-dimensional subsets, while interchanging 1-dimensional subsets and "(-1)-dimensional" subsets - the "non-edges.")

Def. 14-53. Let map $M = M(G,\rho)$ have dual map $M^* = M(G^*,\rho^*)$; then M is said to be self-dual if $G \cong G^*$.

For example, the maps of Figures 5-4 (b), 7-4, and 10-3, and the map of Example 1a in Section 10-3 are all self-dual.

The first and last of those are special cases of:

Thm. 14-54. (Heffter [H6], Biggs, [B9], White [W9], Pengelley [P3], Stahl, [S7], and Bouchet [B12]) The complete graph K_n has a self-dual imbedding if and only if $n \equiv 0$ or 1 (mod 4).

(Compare with Theorem 14-43.)

We now outline the development of [W9]. Let the finite abelian group Γ be generated by $\Lambda = \{a_1, b_1, a_2, b_2, \ldots, a_h, b_h\}$, where no generator has order 2. Take as the reduced constellation $(C')^*$ (see Section 9-5) the normal form for S_h $(h \geq 1)$ (as given in Theorem 5-5): a regular polygon of $4h$ sides, with clockwise boundary $a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_h, b_h, a_h^{-1}, b_h^{-1}$. Then $(C')^*$ has one vertex, $2h$ edges, and one region on S_h and satisfies the KCL - since each current appears exactly once, on a loop, and Γ is abelian. Also, $(C')^*$ is an index one quotient manifold; see Figure 14-6, for the case $h = 2$. (The condition that no generator have order two ensures that property 4 (Theorem 9-10) holds; the other four properties hold trivially here.) Then, by Theorems 9-11 and 9-13, and some routine calculation, $(C')^*$ determines an imbedding of $C_\Lambda(\Gamma)$ on $S_{|\Gamma|(h-1)+1}$, which is a $|\Gamma|$-fold covering space - with no branching - over S_h. Moreover, the covering imbedding has $r = r_{4h} = |\Gamma|$; and, in fact:

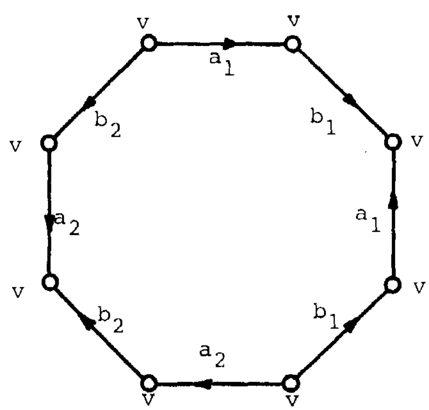


Figure 14-6

Thm. 14-55. The map $M(G_\Lambda(\Gamma), \rho)$ constructed above is self-dual.

Thus for each $h \geq 1$, the normal form for S_h determines a variety of self-dual imbeddings of Cayley graphs (see Problem 14-18.) This has many ramifications.

Thm. 14-56. For $h \geq 1$, there is a self-dual imbedding of some graph G of order p on $S_{p(h-1)+1}$ if and only if $p \geq 4h+1$.

Proof: (i) For the necessity, we let G be self-dual imbedded on $S_{p(h-1)+1}$; the euler equation (Theorem 5-14) gives $q = 2ph \leq p(p-1)/2$ and $p \geq 4h+1$.

(ii) For the sufficiency, choose $\Gamma = Z_p$ and $\Lambda = \{1, 2, \ldots, 2h\}$; that is, $a_i = 2i-1$, $b_i = 2i$, $1 \leq i \leq h$. Now apply the construction of Theorem 14-55.

Cor. 14-57. There is a self-dual imbedding of some graph G of order p on the torus if and only if $p \geq 5$.

Thm. 14-58. The finite abelian group Γ is self-dual (i.e. has a self-dual imbedding for some $G_\Delta(\Gamma)$) if there exists a generating set Λ for Γ of even order with the property that if $\delta \in \Delta$, then $\delta^{-1} \notin \Delta$.

Cor. 14-59. $\Gamma = Z_n$ $(n > 1)$ is self-dual if and only if $n \geq 4$.

Thm. 14-60. If 4 divides $m(n-1)$, then $K_{n(m)}$ has a self-dual imbedding.

Proof: Take $\Gamma = Z_{mn}$, with Δ^* consisting of Γ less all multiples of n; then (since the multiples of n induce the graph nK_m, the complement of $K_{n(m)}$), $G_\Lambda(\Gamma) = K_{n(m)}$. Since 4 divides $m(n-1)$, Λ has even order; moreover, if $mn/2 \in \Gamma$, $mn/2 \notin \Lambda$. Thus the construction of Theorem 14-55 applies. #

Now Theorem 14-54 follows as an immediate corollary. For additional corollaries, we have:

Cor. 14-61. $K_{m,m}$ has a self-dual imbedding, for $m \equiv 0 \pmod 4$.

Cor. 14-62. For $n \equiv 3 \pmod 4$, $K_{n(m)}$ has a self-dual imbedding if and only if m is even.

Cor. 14-63. $K_{m,m,m}$ has a self-dual imbedding if and only if m is even.

Cor. 14-64. The 1-skeleton of the n-dimensional octahedron, $K_{n(2)}$, has a self-dual imbedding for n odd.

In [S11] Stahl (see also Bouchet [B12]) established the following:

Thm. 14-65. If $m-1 \equiv n \equiv 0 \pmod 4$, then $K_{n(m)}$ has a self-dual imbedding.

(And, in [Q1] Quitté added:

Thm. 14-66. If $n \equiv 0 \pmod 8$ and $m \equiv 2,3 \pmod 4$, then $K_{n(m)}$ has a self-dual imbedding.)

Thm. 14-67. For $n \equiv 1 \pmod 4$, the complete maps constructed in the proof of Theorem 14-35 are self-dual.

Thm. 14-68. The fundamental group $\pi(S_k)$, $k \geq 1$, has a self-dual imbedding in the plane.

Thm. 14-69. The finitely generated abelian group Γ has a self-dual imbedding if and only if $\Gamma \neq Z_2$ or Z_3.

All of the self-dual imbeddings considered thus far have been into orientable surfaces. Stahl [S11] considered the non-orientable case as well:

Thm. 14-70. The graph $K_{n(m)}$, $n > 1$, has a nonorientable self-dual imbedding if any one of the following holds:

(i) $m \equiv 0 \pmod 4$ and $n > 2$;

(ii) $m \equiv 2 \pmod 4$ and $n \not\equiv 2 \pmod 4$;

(iii) $m = 1$ and $n \equiv 0 \pmod 4$, $n \neq 4$;

(iv) $m \equiv 1 \pmod 2$, $n \not\equiv 2 \pmod 4$, n is not a power of 2, and $(m,n) \neq (1,3)$, $(1,5)$, or $(3,3)$.

Thm. 14-71. The fundamental group $\pi(N_h)$, $h \geq 1$, has a self-dual imbedding in the plane.

Thm. 14-72. The finitely generated abelian group Γ has a nonorientable self-dual imbedding if and only if $|\Gamma| \geq 6$.

14-8. Paley Maps

The properties for graphs and maps we have been discussing are symmetry properties; in the self-complementation or self-duality case, the symmetry is external (G compares with $\overline{G}$ or G^* respectively), while in the context of symmetrical maps, the symmetry is internal ($M = (G,\rho)$ compares with itself.)

In this section we attempt to tie these three properties together.

First we recall, from Definition 3-3, that two permutation groups (Γ,X) and (Γ',X') are equivalent if there exist a bijection $\beta: X \to X'$ and a group isomorphism $\phi : \Gamma \to \Gamma'$ such that, for each $x \in X$ and $\gamma \in \Gamma$, $\phi(\gamma)(\beta(x)) = \beta(\gamma(x))$; that is $\phi(\gamma)\beta = \beta\gamma$, so that $\phi(\gamma) = \beta\gamma\beta^{-1}$ and ϕ is induced by β, which may be regarded as a relabelling. For example, $(\mathcal{G}(G), V(G))$ and $(\mathcal{G}(\overline{G}), V(\overline{G}))$ are equivalent, with β as the identity function, inducing ϕ as the identity function also. As a less trivial example, if $M = M(G,\rho)$ is a map, with dual map $M^* = M(G^*,\rho^*)$, then $(\mathcal{G}(M), D^*)$ and $(\mathcal{G}(M^*), (D^*)^*)$ are equivalent, under $\beta: D^* \to (D^*)^*$ assigning, to each $s \in D^*$, the unique $s^* \in (D^*)^*$ 'crossing' s (recall that $D^* = \{(u,v) \mid uv \in E(G)\}$). However, $(\mathcal{G}(M), V(G))$ and $(\mathcal{G}(M^*), V(G^*))$ need not be equivalent; in fact, it is quite possible that $|V(G)| \neq |V(G^*)|$.

We combine the symmetry properties of self-complementation, self-duality, and symmetricality, and proceed with the development as in [W13].

<u>Def. 14-73</u>. The map $M = M(G,\rho)$ is said to be <u>strongly symmetric</u> if:

(i) $\overline{G} = G$;

(ii) $G^* = G$;

(iii) M is symmetrical

(iv) $\mathcal{G}(M)$ and $\mathcal{G}(M^*)$ are equivalent, under an anti-isomorphism $\beta: V(G) \to V(G^*)$.

We need one preliminary result, before characterizing strongly symmetric maps as to order; compare Theorem 14-33.

<u>Thm. 14-74</u>. (Jordan; see Burnside [B18, p. 172]) If (Γ,X) is a Frobenius group of degree $n = |X| \geq 6$ and order $n(n-1)/2 = |\Gamma|$, then n is a prime power.

<u>Thm. 14-75</u>. There exists a strongly symmetric map of order n if and only if n is a prime power congruent to $1 \pmod 8$.

<u>Proof</u>: (A) Let $M = M(G,\rho)$ be a strongly symmetric map, with $n = |V(G)|$ and $e = |E(G)|$. Since G is self-complementary, $e = \frac{1}{2}\binom{n}{2} = n(n-1)/4$; thus $n \equiv 0$ or $1 \pmod 4$ - see also Theorem 14-43. But since M is symmetrical, G is vertex-transitive (by Theorem 14-18) and hence regular of degree $(n-1)/2$, so that $n \equiv 1 \pmod 4$. Now let M imbed G on S_k, with f regions. Since M is self-dual, $f = n$, and the euler equation (Theorem 5-14) gives $2n - n(n-1)/4 = 2 - 2k$, so that $n \equiv 1 \pmod 8$. Since M is symmetrical, $|\mathcal{G}(M)| = 2|E(G)| = n(n-1)/2$, and $\mathcal{G}(M)$ is a transitive permutation group of order $n(n-1)/2$ and degree $n \geq 9$. We show that $\mathcal{G}(M)$ is a Frobenius group and then apply Theorem 14-74, to see that, in fact, n is a prime power.

So, let $\alpha \in \mathcal{G}(M)$, with $u \neq v \in V(G)$ such that $\alpha(u) = u$ and $\alpha(v) = v$. If $uv \in E(G)$, then α is the identity permutation, by Lemma 14-14. If $uv \notin E(G)$, then we apply the anti-isomorphism β giving $\mathcal{G}(M)$ equivalent to $\mathcal{G}(M^*)$. Then β induces an isomorphism ϕ between $\mathcal{G}(M)$ and $\mathcal{G}(M^*)$, so that $\phi(\alpha) = \beta\alpha\beta^{-1}$.

Thus $(\theta(\alpha))(\beta(u)) = \beta\alpha\beta^{-1}(\beta(u)) = \beta\alpha(u) = \beta(u)$; similarly, $\theta(\alpha)$ also fixed $\beta(v)$. But, since β is an anti-isomorphism and $uv \notin E(G)$, $\beta(u)\beta(v) \in E(G^*)$. Hence, by Lemma 14-14 again, $\theta(\alpha)$ is the identity in $\mathcal{G}(M^*)$; but since θ is a group isomorphism, α is the identity in $\mathcal{G}(M)$ and $\mathcal{G}(M)$ <u>is</u> a Frobenius group.

(B) For the converse, let $n = p^r \equiv 1 \pmod 8$, p a prime and $r \in N$. We construct a Cayley graph $G_n = G_{\Delta_n}(\Gamma_n)$, where $\Gamma_n = (Z_p)^r$ - the additive group in the Galois field $GF(p^r)$. Take x as a primitive element for $GF(p^r)$, so that x generates the multiplicative group, and let $\Delta_n^* = \{1, x^2, x^4, \ldots, x^{n-3}\}$, the set of all <u>squares</u> in $GF(p^r)$. (Equivalently, $uv \in E(G_n)$ if and only if $v - u$ is a square in $GF(P^r)$.) The Cayley graph G_n is called a <u>Payley graph</u> (see [Pl], where the ideas behind this construction were introduced.) We remark that Paley graphs are defined for all prime powers $p^r \equiv 1 \pmod 4$, since these are precisely the cases for which -1 is a square - so that undirected edges are well-defined; but only in the case $p^r \equiv 1 \pmod 8$ are self-dual imbeddings possible.

Next we define $r_n\colon \Delta_n^* \to \Delta_n^*$ by $r_n(\delta) = x^2\delta$, so that $M_n = M(\Gamma_n, \Delta_n, r_n)$ is a Cayley map- which we now call a <u>Paley map</u>; r_n induces vertex rotations

$$\rho_v(w) = x^2(w-v) + v,$$

in accordance with Definition 14-23; and the permutation $\overline{r}_n\colon \Delta_n^* \to \Delta_n^*$, given by

$$\overline{r}_n(\delta) = -x^2\delta,$$

in accordance with the remarks preceeding Theorem 14-28. We observe that Paley maps are in fact defined for all $n = p^r \equiv 1 \pmod 4$, but we continue to specialize to the case $n = p^r \equiv 1 \pmod 8$. We claim that, for $n = p^r \equiv 1 \pmod 8$, the Paley map $M_n = M(\Gamma_n, \Delta_n, r_n)$ is strongly symmetric.

(i) The permutation $\beta: (Z_p)^r \to (Z_p)^r$ given by $\beta(v) = xv$ is an anti-automorphism for G_n - since $w = u + x^{2k}$ if and only if $\beta(w) = xw = x(u + x^{2k}) = xu + x^{2k+1} = \beta(u) + x^{2k+1}$; note the use of the distributive law in the field $GF(p^r)$ - so that G_n is self-complementary.

We remark that Paley graphs, being Cayley graphs of odd order, not only have a 2-factor (as required by Theorem 14-46), but in fact are <u>2-factorable</u> (the edge set partitons into 2-factors); see Problem 4-14. We also observe that the anti-automorphism β has exactly one fixed point (the vertex 0) and one orbit of length $n - 1 \equiv 0 \pmod 4$, as required by Theorem 14-44.

(ii) To show that $M(G_n,\rho)$ is a self-dual map, we study the corresponding imbedding of G_n into S_k, where $n = p^r = 8m + 1$ and $k = 8m^2-7m$, for $m = 1, 2, 3, 5, 6, 9, \ldots$. This imbedding is an $(8m+1)$-fold covering space (no branching) of a voltage graph imbedding which is the following alternative normal form for S_m: $a_1a_2\cdots a_{2m}a_1^{-1}a_2^{-1}\cdots a_{2m}^{-1}$. (Note that the values $n = 8m+1$, $h = m$ are consistent with Theorem 14-56 and that Theorem 14-69 is also illustrated by this construction.) The $2m$ directed edges bounding this $4m$-gon are labelled with generators from Δ_n by the assignment $a_i \mapsto x^{(i-1)(4m-2)}$, $1 \le i \le 2m$. This assignment describes each region boundary (and each vertex rotation) in the covering space, the map (G_n,ρ), by the voltage graph theory; in particular, the single $4m$-gon below lifts to $8m+1$ $4m$-gons above.

Figure 10-3 (letting b in Figure 10-2 be 1, $-a = x^2$) depicts the entire situation for $m = 1$, using $x^2 = 2x + 1$ in $GF(9)$: $a_1 \mapsto 1 = 01$, $a_2 \mapsto x^2 = 21$; $x = 10$; for just the voltage graph for the case $m = 2$, replace the labels in Figure 14-6 with: 1, x^6, x^{12}, x^2, -1, $-x^6$, $-x^{12}$, $-x^2$.

We now utilize a method introduced by Bouchet [B12] for showing self-duality. Each region in the covering space imbedding has a unique lift of the directed edge a_1 from the normal-form voltage graph in its boundary;

this lift is a side $(g,g+1)$ in G_n, where g is uniquely determined for that region; label the region with g^*. It is now routine to check that, for each $g \in (Z_p)^r$, the region g^* $(g^* \in V(G_n^*))$ has neighbors $N(g^*) = \{(g+w(-x^2)^k)^* \mid 0 \leq k \leq 4m-1\}$ in G_n^*, where $w = (1-x^2)/(1+x^2)$. In fact, the dual is a Cayley map $M_n^* = M_n(\Gamma_n, w\Delta_n, r_n^*)$, where $r_n^*(w\ ) = -x^2 w\delta(\delta \in \Delta_n)$, r_n^* being taken in the sense opposite to that of r_n. Thus if w is a square in $GF(p^r)$, then $G_n^* = G_n$. On the other hand, if w is a non-square, then $G_n^* = \overline{G}_n = G_n$.

(iii) Using the distributive law in $GF(n)$ again, we readily see that the permutation r_n of Δ_n^* extends to a group automorphism of $\Gamma_n = (Z_p)^r$; thus Theorem 14-27 applies, to show that $M_n(\Gamma_n, \Delta_n, r_n)$ is a symmetrical map. In fact, $\mathcal{G}(M_n)$ is a Frobenius group with Frobenius kernel the regular normal subgroup $\{\gamma_g: \Gamma_n \to \Gamma_n, \gamma_g(h) = g+h \mid g \in \Gamma_n = V(G_n)\} \cong \Gamma_n$ and Frobenius complement the cyclic group stablizing the vertex 0, generated by the automorphism ζ_0 extending r_n, $\zeta_0(h) = x^2h$. (See Theorems 14-32 and 14-15.) The other vertex stabilizers are conjugate, with $(\mathcal{G}(M_n))_g$ generated by $\zeta_g = \gamma_g\zeta_0\gamma_g^{-1}$. Of course, $|\mathcal{G}(M_n)| = |\Gamma_n||\Delta_n| = 2|E(G_n)|$. Finally; we emphasize that $\mathcal{G}(M_n)$ contains a subgroup isomorphic with Γ_n, as required by Theorem 14-24, this subgroup must be proper, in $\mathcal{G}(G_n)$, by Theorem 14-48.

(iv) If $w = (1-x^2)/(1+x^2)$ is a non-square, then the anti-isomorphism $\beta: V(G_n) \to V(G_n^*)$ given by $\beta(g) = g^*$ gives an equivalence between $\mathcal{G}(M_n)$ and $\mathcal{G}(M_n^*)$, since if $\alpha^* = \beta\alpha\beta^{-1}$, where α is one of the generating automorphisms ζ_0 or γ_g $(G \in \Gamma_n)$ of $\mathcal{G}(M_n)$, one readily checks that α^* preserves oriented region boundaries in M_n^*. If, however, $w \in \Delta_n$, then we take $\beta(g) = (xg)^*$, to again see that $\mathcal{G}(M_n)$ and $\mathcal{G}(M_n^*)$ are equivalent, under an anti-isomorphism β. #

The Paley maps $M_n = (\Gamma_n, \Lambda_n, r_n)$ constructed above have additional properties of interest. We use the following facts about Galois fields, taken from Storer [S17].

Thm. 14-76. Let x be a primitive element for $GF(p^r)$, where $p^r = 2f+1$ and f is even. Let (i,j) denote the number of ordered pairs (s,t) such that $x^{2s+i} + 1 = x^{2t+j}$, $0 \le s, t \le f-1$; then:

(i) $(0,0) = (f-2)/2$;

(ii) $(0,1) = (1,0) = (1,1) = f/2$.

Thm. 14-77. For $n = 4m+1$, the Paley graph G_n is strongly regular, with parameters $p^1_{22} = m$ and $p^2_{22} = m-1$.

Proof: From $n = 4m+1 = 2f+1$, we find $f = 2m$. From Theorem 14-76 (i) we see that $(0,0) = (f-2)/2 = m-1$. Now clearly $x^{2s}+1 = x^{2t}$ if and only if $x^{2s+2k}+x^{2k} = x^{2t+2k}$; it follows that $p^2_{22} = m-1$ for G_n. Similarly, we use Theorem 14-76 (ii) to deduce that $p^1_{22} = (1,1) = f/2 = m$. #

This result overlaps with the next:

Thm. 14-78. If G is self-complementary and has a symmetrical map, then G is strongly regular.

Proof: In fact, for any graph G having a symmetrical map M, since $\mathcal{G}(M)$ is transitive on edges and $\mathcal{G}(M) \le \mathcal{G}(G)$, p^2_{22} is well-defined. But if G is self-complementary, then $\mathcal{G}(G)$ is transitive on non-edges also, so that p^1_{22} is well-defined too. #

Thus the Paley graphs G_n, for $n = p^r \equiv 1 \pmod 8$, are candidates for determining association classes for PBIBDs, and in fact the Paley maps give such designs:

Thm. 14-79. Let x be a primitive element for $GF(n)$, where $n = p^r$; the Paley map M_n, for $n = 8m+1$, yields an $(8m+1, 8m+1, 4m, 4m; \lambda_1, \lambda_2)$-PBIBD, where $\lambda_1 = 2m$ and

${}_2 = 2m-1$ if x^2+1 is a square in $GF(n)$, while
${}_1 = 2m-1$ and ${}_2 = 2m$ if x^2+1 is a non-square.

Proof: See Problem 14-20. #

It can be shown that, in a Paley map, either each region boundary consists of neighbors $N(v)$, $v \in (Z_p)^r$, or each region boundary consists of neighbors in the complementary graph, so that the designs constructed above also exist in a natural, topology-free, context. The topological context, however, does facilitate the next observation: these designs are also self-dual, in a very strong sense. The dual of a design has as its objects the blocks of the original design, and a block in the dual design contains all those objects of the dual design which represent blocks of the original design to which one fixed object of the original design belongs.

Thm. 14-80. Given a Paley map, the dual of the design of the map and the design of the dual of the map coincide, and both are isomorphic to the design of the Paley map itself.

Proof: See Problem 14-21. #

We make several additional comments relevant to Theorem 14-75. Firstly, we observe that self-dual imbeddings of self-complementary graphs need not be unique for a given order. Moreover, the order need not be a prime power. (We are temporarily relinquishing the symmetricality condition for strongly symmetric maps.) For the first example, take $\Delta = \{5;1,6,11,16,21\}$ in $\Gamma = Z_{25}$, so that $G_\Delta(\Gamma)$ is the composition $C_5[C_5]$. Since $\overline{C}_5 = C_5$ and $\overline{G[H]} = \overline{G}[\overline{H}]$ in general (see Problem 2-5), $C_5[C_5]$ is self-complementary. A self-dual imbedding is constructed by the use of Theorem 14-55. Since $C_5[C_5]$ is not strongly regular, the map M is not symmetrical, by Theorem 14-78; thus M differs from the Paley map M_{25} of the same order. In fact, $\mathcal{G}(M) \cong \Gamma = Z_{25}$ and is equivalent to $\mathcal{G}(M^*)$ (both are Frobenius groups) under the anti-isomorphism $\theta(g) = (2g)^*$. (Every map M of a Cayley graph $G_\Delta(\Gamma)$ covering a normal-form voltage graph has $\Gamma \cong \Gamma' \leq \mathcal{G}(M)$; in this case the fact that there are no additional automorphisms follows from the observation that each region boundary contains repeated vertices.) We mention that Theorem 14-55 also provides a self-dual imbedding, but not a symmetrical map, for the Paley graph G_{25}.

For the second example, we take $\Delta = \{13;5,20,15;1,4,16;3,12,-17;$ $7,28,-18;11,-21,-19\}$ in $\Gamma = Z_{65}$; $\theta(g) = 2g$ gives an anti-automorphism, so that $G_\Delta(\Gamma)$ is self-complementary. Again a self-dual imbedding is constructed by Theorem 14-55, and again the map fails to be symmetrical, by Theorem 14-78. Again, $\mathcal{G}(M)$ and $\mathcal{G}(M^*)$ are equivalent under the anti-isomorphism $\theta(g) = (2g)^*$, and both are Frobenius groups isomorphic to $\Gamma = Z_{65}$. The non-prime-power order is possible, because the automorphism group is not large enough for the Jordan theorem (Theorem 14-74) to apply.

Next we present another sufficient condition to give prime-power order (see Problem 14-22.)

<u>Thm. 14-81</u>. Let G be strongly regular (with $p^1_{22} = 2m$, $p^2_{22} = 2m-1$), $M = (G,\rho)$ a symmetrical map yielding an $(8m+1,8m+1,4m,4m;2m-1,2m)$-PBIBD. Then $|V(G)| = 8m+1$ is a prime power.

We remark that, for the Paley graphs $G_n (n = p^r \equiv 1 \pmod 4)$, $\mathcal{G}(G_n)$ consists not only of the map automorphisms $\mathcal{G}(M_n)$ as in (iii) of the proof of Theorem 14-75, but also of the field automorphisms, generated by $\theta : (Z_p)^r \to (Z_p)^r$, $\theta(g) = g^p$; in fact, $\mathcal{G}(G_n) =$ $\langle \mathcal{G}(M), \theta \rangle$:

<u>Thm. 14-82</u>. (Carlitz [C3]) The automorphism group of the Paley graph G_n is $\mathcal{G}(G_n) = \{g \mapsto x^{2k}g^{p^s}+a \mid 0 \leq k \leq 4m-1,$ $0 \leq s \leq r-1,\ a \in (Z_p)^r\}$.

We use this result to study the reflexibility of the Paley maps.

<u>Def. 14-83</u>. If a map M is symmetric but not reflexible, we say that M is <u>chiral</u>.

<u>Thm. 14-84</u>. The Paley map M_n is reflexible if and only if $n = 9$; thus M_n is chiral if and only if $n \neq 9$.

<u>Proof</u>: If M_n is reflexible, with $n = p^r$, then we must have $r = 2s$ and $\theta^s(g) = g^{p^s}$ giving a reflection. Since θ^s fixes both 0 and 1, we must have

$\rho^s(x^2) = x^{2p^s} = x^{p^{2s}-3}$ (since $\rho_0 = (1,x^2,\ldots,x^{p^{2s}-3})$.)
Thus $p^{2s}-3 = 2p^s$, so that p divides 3; hence $p = 3$ and $s = 1$ (or else 3 divides 1); that is, $n = 9$.

Conversely, $\theta(g) = g^3$ is readily seen to be a reflection for M_9; refer to Figure 10-3. #

As we have seen, the Paley maps - defined for prime powers congruent to 1 (mod 8) - have several interesting properties: they are regular (symmetrical), self-dual imbeddings of strongly regular, self-complementary graphs which produce self-dual block designs, for example. Now we see how closely we can approximate these properties by similar maps, for all other prime power orders.

For $p^r \equiv 5 \pmod 8$, the Paley graphs G_n are defined, and are both self-complementary and strongly regular, as before. The euler equation disallows self-dual surface imbeddings, so we utilize pseudosurfaces. Again we use a normal-form voltage graph but, for $n = 8m+5$, it is a $(4m+2)$-gon and thus has two vertices; see Figure 14-7 for the case $m = 1$. The $(8m+5)$-fold covering imbedding has

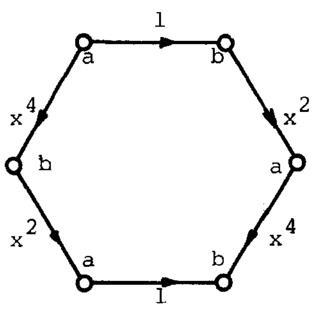


Figure 14-7.

$16m+10$ vertices, each of degree $2m+1$. For each $g \in GF(n)$, we identify the two vertices (a,g) and (b,g); the result is a symmetrical, self-dual pseudosurface imbedding of G_n, for $m > 0$. ($G_5^* = C_5^*$ consists of five disjoint loops. For the pseudosurface theory of voltage graphs, see Garman [G1].) In fact, the map is a Cayley map $M(\Gamma_n,\Delta_n,r_n)$, with $r_n: \Delta_n^* \to \Delta_n^*$, $r_n(\delta) = x^4$, having

<u>two</u> cycles. The dual G^* is G if $w = (1-x^4)/(1+x^4)$ is a square, or $\overline{G}$ if w is a non-square. The automorphism group is Frobenius, with Frobenius kernel $\cong (Z_p)^r$. The stablizier of vertex 0 is generated by $g \mapsto x^2 g$, which alternates between the two orbits of the vertex rotation at 0. An $(8m+5, 8m+5, 4m+2, 4m+2; 2m, 2m+1)$-PBIBD results if $1+x^4$ is non-square; otherwise, the two λ values interchange.

For $p^r \equiv 3 \pmod 4$, the Paley graphs G_n are no longer defined, since $-1 = x^{2m+1}$ $(p^r = 4m+3)$ is <u>not</u> a square. Instead, we define the <u>Paley tournament</u> T_n, by: $(g_1, g_2) \in E(T_n)$ if and only if $g_2 - g_1$ is a square in $GF(n)$; of course, we still take $\Gamma_n = (Z_p)^r$ as our vertex set. Then $g_1 \neq g_2$ gives $|\{(g_1,g_2), (g_2,g_1)\} \cap E(T_n)| = 1$, so that we do have a tournament. The underlying undirected graph is thus complete, and hence degenerately "strongly regular." The Paley tournaments are self-converse, under the anti-automorphism $\rho: g \mapsto xg$. In Biggs [B9] we find symmetrical imbeddings of the associated K_{4m+3} having $8m+6$ regions, each of length $2m+1$; these are Cayley maps $M(\Gamma_n, \Gamma_n - \{0\}, r_n)$, where $r_n(g) = xg$. Thus the imbeddings cannot be self-dual directly, although the dual is bichromatic. Hence we obtain one $(4m+3, 8m+6, 4m+2, 2m+1, 2m)$-BIBD and two $(4m+3, 4m+3, 2m+1, 2m+1, m)$-BIBDs. (The latter are Hadamard designs.) We now modify this map to form a self-dual pseudosurface imbedding of K_{4m+3}. Each region contains exactly one edge corresponding to $1 \in (Z_p)^r$, in one of the two possible senses (clockwise or counterclockwise.) In fact, this distinction determines the 2-coloring of the dual. The region is assigned label g^* $(g \in (Z_p)^r)$ if either: (i) $(g, g+1)$ bounds the region in the clockwise sense, or (ii) $(g-a, g-a-1)$ bounds the region in the clockwise sense, where $a = 2/(x^{2m}-1)$. Thus each g appears exactly twice as a region label, and if these n pairs of vertices in the dual are identified, we obtain a self-dual pseudosurface imbedding of K_{4m+3}. Moreover, if the edge directions are carried over into the dual, then we have a self-dual imbedding of the self-converse tournament T_n, with each vertex neighborhood $N(g^*)$ partitioned into two sets (corresponding to the g^* identification): one consists of those vertices dominated by g^*, the other consists of those dominating g^*.

Finally, we consider $p = 2$, so that $p^r = 2^r$. Here not even 1 is a square $(1 = x^{2^r - 1})$ in $GF(2^r)$, so our previous constructions seem not to apply. However, if we use the planar voltage graph of Figure 14-8, with $n = 2^r$, we obtain an n-fold covering imbedding of a 2-fold K_n, with n $(n-1)$-gons and $\binom{n}{2}$ digons. If each digon is closed (by identifying its two edges), a symmetrical, self-dual imbedding of the "strongly regular" K_n (self-complementary in the 2-fold K_n) results. (In fact, $r_n^* = r_n$ for this case.) The concomitant design is an $(n,n,n-1,n-1,n-2)$-BIBD which, of course, can also be constructed by taking complements of singletons as blocks.

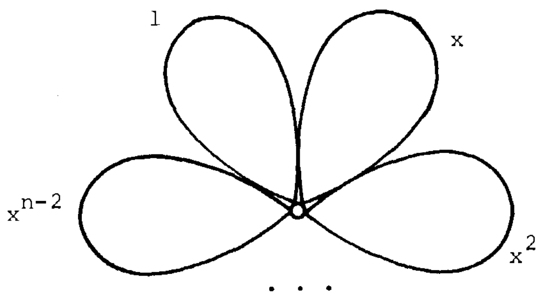


Figure 14-8

We close this lengthy section by discussing conditions under which the Paley maps are, in some sense, unique. Self-duality, strong regularity, and designs do not enter into the characterizations given, but as it is Paley maps being characterized, these additional properties persist - except that self-duality fails for $n \equiv 5 \pmod 8$. For proofs of the following two theorems, refer to [W13] or to Section 14-9.

Thm. 14-85. If $M = (G,\rho)$ is vertex-transitive and if $(G(M), V(G))$ is self-equivalent under an anti-automorphism β of G, then M is isomorphic to a Cayley map $M(\Gamma,\Delta,r)$, where $|\Gamma| = |V(G)|$, $|\Delta^*| = (|V(G)| - 1)/2$.

Thm. 14-86. There exists a symmetrical imbedding of a self-complementary graph G of order n, with $(G(M), V(G))$

self-equivalent under an anti-automorphism β of G, if and only if n is a prime power congruent to 1 (mod 4). Moreover, if $\beta^2 \in (M)$, the maps are essentially unique, with at most six exceptions.

Cor. 14-87. For each prime $p \equiv 1 \pmod 4$, there exists a unique symmetrical imbedding of a self-complementary graph G of order p having $\mathfrak{G}(M) = \mathfrak{G}(G)$.

Proof: The Paley maps give existence. But if $\mathfrak{G}(M) = \mathfrak{G}(G)$, then β^2 and $\beta\alpha\beta^{-1}$ (for each $\alpha \in \mathfrak{G}(M)$ and where β is an anti-automorphism of G) are both in $\mathfrak{G}(M)$, so that Theorem 14-86 (and its proof) show that M is a Paley map. #

The strength of this uniqueness claim is illustrated by Theorem 14-45: the Paley maps of orders 13 and 17, for example, are unique among 5,600 and 11,220,000 self-complementary graphs respectively, with respect to being symmetrical, with $\mathfrak{G}(M) = \mathfrak{G}(G)$.

Cor. 14-88. If there exists a symmetrical imbedding of a self-complementary graph G of order n, with $\mathfrak{G}(M) = \mathfrak{G}(G)$, then (with at most six exceptions) n is a prime congruent to 1 (mod 4).

Proof: As in the proof of Corollary 14-87, we find by Theorem 14-86 that n is a prime power congruent to 1 (mod 4); moreover, the maps (with at most six exceptions) are unique and thus are Paley maps. But if $n = p^r$ with $r > 1$, then $\mathfrak{G}(M)$ is a proper subgroup of $\mathfrak{G}(G)$, by Theorem 14-82; thus $r = 1$, and n is prime. #

14-9. Problems

14-1.) Show that $|R(G)| = \prod_{i=1}^{n} (n_i-1)!$

14-2.) Prove Theorem 14-5.

14-3.) Show that $\mathfrak{G}(M) = A_4$, for M as in Figure 14-2.

14-4.) Prove that the following are equivalent, for a map $M = (G,\rho)$:

(i) M is reflexible;

(ii) M and its mirror image M are equivalent;

(iii) there exists an $\alpha \in \mathcal{G}(G)$ such that, for all $v \in V(G)$, $\rho_{\alpha(v)} = \alpha\rho_v^{-1}\alpha^{-1}$.

14-5.) Show that the map of Figure 10-3 is reflexible.

14-6.) Prove Theorem 14-13.

14-7.) Verify the entries in Table 14-2.

14-8.) Show that $|\mathcal{G}^*(M)|$ divides $4|E(G)|$. (See Theorem 5-25.) Thus a nonorientable map M is defined to be <u>symmetrical</u> if $|\mathcal{G}^*(M)| = 4|E(G)|$. If we augment Figure 8-7(a) in the obvious way to imbed K_6 in N_1, is the resulting map symmetrical?

*14-9.) Prove or disprove: If G is a strongly regular graph, then there exists a rotation ρ for G so that $M = (G,\rho)$ is a regular map. (Hint: consider the Petersen graph.)

14-10.) Let M be the imbedding of K_5 on S_1 given in Example 1a of Section 10-3. Show that M is symmetrical (i.e. $|\mathcal{G}(M)| = 20$), but not reflexible (i.e. M is chiral.) Show that the corresponding design D (Example 1 of Section 12-3) has automorphism group $\mathcal{G}(D) \cong S_5$. Does this contradict Theorem 14-13?

*14-11.) Prove Theorem 14-36.

14-12.) Prove Theorem 14-37.

14-13.) Prove Theorem 14-38.

14-14.) Prove Theorem 14-39.

14-15.) Show that $M(Z_9, \{1,2,4\}, (1,2,4,8,7,5))$ is a symmetrical map for $K_{3(3)}$. Use this map to show that $\gamma(K_{3,3,3,6}) = 7$.

*14-16.) For what other values of n does $K_{3(n)}$ have a symmetrical map?

14-17.) Let map $M^* = M(G^*,\rho^*)$ be dual to map $M = M(G,\rho)$. Find a description of ρ^*, in terms of ρ and : $D^* \rightarrow D^*$, where $D^* = \{(u,v) \mid uv \in E(G)\}$ and $(u,v) = (v,u)$. (See Biggs [B10].

*14-18.) Prove Theorem 14-55.

14-19.) Show that $\gamma_M(G_n) = (n-1)(n-4)/8$, where G_n is the Paley graph of order $n = p^r \equiv 1 \pmod{8}$.)

*14-20.) Prove Theorem 14-79.

*14-21.) Prove Theorem 14-80.

14-22.) Prove Theorem 14-81.

*14-23.) Prove Theorem 14-85.

*14-24.) Prove Theorem 14-86.

CHAPTER 15

CHANGE RINGING

The ancient and continuing art of change ringing, or campanology (how the British ring church bells), is here studied from a mathematical viewpoint. An "extent" on n bells is regarded as a hamiltonian cycle in a Cayley color graph for the symmetric group S_n, imbedded on an appropriate surface. Thus - perhaps surprisingly - graphs, groups, and surfaces combine to model something musical.

After making this model explicit, we discuss two ringing methods for variable n (Plain Bob for all n and Grandsire for $n \equiv 3 \pmod 4$); and a new method for n odd is introduced. All minimus methods $(n = 4)$ and five doubles methods $(n = 5)$ are depicted, one of these being the new "No Call" Doubles.

The material here also appears in [W14].

15-1. Definitions

In the science of change ringing (sometimes called campanology), the central problem is to ring a full extent on n bells. We designate the n bells by the natural numbers $1, 2, \ldots, n$, arranged in order of descending pitch.

Def. 15-1. Bell 1 is called the *treble*, bell n the *tenor*. Each ringing of the n bells, once each in some order, is called a *change*; the special change $1, 2, \ldots, n$ is called *rounds*. An *extent* consists of $n! + 1$ successive changes, satisfying:

(i) the first and last change are both rounds;
(ii) no other change is repeated (so that each possible change other than rounds is rung exactly once);
(iii) from one change to the next, no bell moves more than one position in its order of ringing (positions 1 and n are *not* adjacent, unless $n = 2$);

(iv) no bell rests in the same position for more than two successive changes;

(v) each bell (except perhaps the treble) does the same amount of "work" (hunting, plain hunting, dodging, etc.);

(vi) the method employed is palindromic, or self-reversing (this will be made more precise in Section 15.2.)

Conditions (iv) and (v) are occasionally relaxed in practice, but the other conditions appear to be inviolable.

I venture to guess that rule (i) is for musicality, rule (ii) for thoroughness, rule (iii) for mechanical considerations (dealing with how the bells are hung and rung), rules (iv) and (v) to keep the performance interesting for the ringers, and rule (vi) to ease the memory burden for the ringers.

Typically the number of bells is in the range $3 \leq n \leq 12$, with names assigned to the extents as in Table 15-1. The odd-bell names reflect the maximum number of pairs of bells that could be exchanged, in their order or ringing.

n	name	n! + 1
3	Singles	7
4	Minimus	25
5	Doubles	121
6	Minor	721
7	Triples	5,041
8	Major	40,321
9	Caters	362,881
10	Royal	3,628,801
11	Cinques	39,916,801
12	Maximus	479,001,601

Table 15-1

As an extent of Major takes approximately eighteen hours to ring (the ringers are allowed no visual aids to memory, so this must surely rank as one of the "major" physical and intellectual feats of mankind), extents on more than eight bells are clearly beyond the limits of human endurance; even on eight bells they are extremely rare. Much more tractable, an extent of triples requires under three hours of concentrated ringing. For this reason peals are often attempted, and often succeed; a peal consists of at least 5000 and, for $5 \leq n \leq 7$, exactly 5041 successive changes satisfying rules (i) through (vi) above (but not (ii), if $n < 7$.) For $n < 7$, a peal consists of several extents strung together; for $n = 7$ a peal *is* an extent; and for $n > 7$, a peal is a partial extent. If tower bells are being rung, then each ringer operates one bell (a huge amount of practice is required for the beginner to learn to ring his or her bell even individually, let alone in rounds with the other bells, let alone through all the changes of a peal or an extent); if handbells are employed, then each ringer rings two bells. There is also a bell-ringing machine (designed and constructed by John Carter, a gunsmith and bell-ringer, and currently on loan to the Science Museum in London); it mechanically rings twelve hand bells suspended in a cabinet, after being "programmed" for method.

As generally an even number of bells is provided, for the odd bell methods a "covering" bell - always the tenor - rings last in every change, in forgiven violation of rules (iv) and (v). It might even be argued that the stability and regularity thus produced is musically pleasing.

Change ringing, as described above, began in England in the middle of the seventeenth century and remains almost exclusively a British art. Today there are over five thousand bell towers in Great Britain, about a half dozen in Canada and a dozen in the United States, and perhaps twice that in Australia. On July 27 and 28, 1963, the first extent of Plain Bob Major was successfully rung on tower bells at the Loughborough Bell Foundry in England. On December 27 and 28, 1977, the same extent was accomplished on hand-bells in a private home in Farnham, Surrey, England.

Change ringing has been popularized, to an extent, by Dorothy Sayers in her novel *The Nine Tailors* [S4] and, more recently, as part of the festivities at the Royal Wedding of Prince Charles and Princess Diana. Only recently have mathematicians begun to study change ringing from their vantage point, although curiously the early composers of peals and extents were doing coset decomposition in symmetric groups a full century before Lagrange. The reader could consult works by Rankin [R2], Fletcher [F1], Dickinson [D2], Price [P7], Budden [B17], and White [W10]. For more on the history and practice of change ringing, the reader should seek out Wilson [W17], Camp [C2], and the weekly publication "The Ringing World" (published in Guildford, Surrey, England.)

15-2. Notation

It would be easy to confuse the number of a given bell with the number of the position it occupies in a particular change. To avoid this confusion we regard each change as a function from the set of n positions to the set of n bells, so that domain elements describe positions and range elements describe bells. Thus a change f, recorded as $f(1), f(2), \ldots, f(n)$, would indicate that bell $f(1)$ is to be rung first, bell $f(2)$ second, and so on. Rounds is given by $r(i) = i$, $1 \leq i \leq n$. In order to satisfy condition (iii) of Section 1, we describe a potential transition from one change to the next by a permutation on the set of n positions which either fixes a given position or interchanges it with an adjacent position. Conversely, each transition satisfying (iii) is of this type. Thus each transition is represented by an involution (that is, by a product of disjoint transpositions) in the symmetric group S_n. Now let $t(n)$ give the number of possible transitions for n bells, and let $F(n)$ give the nth Fibonacci number (where $F(0) = F(1) = 1$); then we have the following well-known result:

<u>Thm. 15-2</u>. For $n \geq 2$, $t(n) = F(n) - 1$.

<u>Proof</u>: It is easy to verify that $t(2) = 1$ and that $t(3) = 2$, so we assume the result for $n < k$ and consider $t(k)$, $k \geq 4$. Since there are $t(k-1)$ allowable transitions fixing position k, $t(k-2)$ allowable transitions exchanging positions k and $k-1$ and at least one other pair, and the single transposition $(k-1,k)$, we have:

$$
\begin{aligned}
t(k) &= t(k-1) + t(k-2) + 1 \\
&= F(k-1) - 1 + F(k-2) - 1 + 1 \\
&= F(k) - 1
\end{aligned}
$$

#

Now, if $d_1, d_2, \ldots, d_k$ represent the first k transitions in an extent, then the $(k+1)$st change is the function $rd_1d_2\ldots d_k$, where - in accordance with the standard notation for function composition - we take products in S_n from right to left. Let Λ be any collection of generating involutions for S_n (if Λ does not generate S_n, then condition (ii) fails) and form the Cayley color graph $C_\Delta(S_n)$; then we have the immediate but absolutely central result:

<u>Thm. 15-3</u>. An "extent" on n bells, satisfying conditions (i)-(iii) and using transition rules from Δ, can be rung if and only if $C_\Lambda(S_n)$ is hamiltonian.

Since each $d \in \Delta$ has order two, $C_\Delta(S_n)$ has no directed edges by convention (see, for example, White [W10]): hence hamiltonian cycles in $C_\Lambda(S_n)$ coincide with those in the underlying Cayley graph $G_\Lambda(S_n)$. We retain the edge colors of $C_\Delta(S_n)$, however, as they are valuable in constructing the extent itself from the hamiltonian cycle and in checking the palindromic condition (vi). Each method is based upon a <u>principle</u>, which usually (but not always) consists of $2n$ changes. The word in S_n describing the $2n - 1$ juxtaposed transitions of the principle should be a palindrome: $d_1d_2\ldots d_{n-1}d_nd_{n-1}\ldots d_2d_1$.

These ideas are illustrated in Figure 5-1, with $\Lambda = \{(12),(23)\}$ for the case $n = 3$. The hamiltonian cycle and the extent it gives rise to shown in Table 15-2 describe the interior clockwise region boundary of the spherical imbedding of $C_\Delta(S_3)$ depicted. The extent is called "quick six"; the clockwise boundary of the exterior region produces "slow six", the only other extent possible on 3 bells. (Not surprisingly, "slow six" is just "quick six" in the opposite order.) If we let $a = (12)$ (denoted by solid edges) and $b = (23)$ (denoted by dashed edges), then "quick six" is completely described by the identity word $(ab)^3$ in S_3 and "slow six" by $(ba)^3$. In the former case the relevant palindrome is ababa; in the latter it is babab.

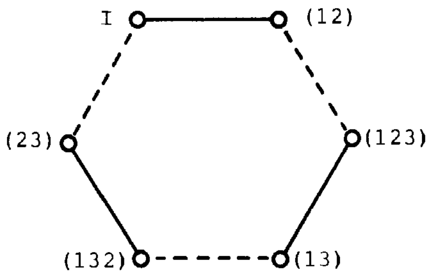


Figure 15-1

hamiltonian cycle	change	range of change
I	rI = I	1 , 2 , 3
(12)	r(12)	2 , 1 , 3
(123)	r(123)	2 , 3 , 1
(13)	r(13)	3 , 2 , 1
(132)	r(132)	3 , 1 , 2
(23)	r(23)	1 , 3 , 2
I	r	1 , 2 , 3

Table 15-2

15-3. General Theory

In [R4], Rapaport provided an inductive construction, to establish the following:

Thm. 15-4. Let $\Lambda = \{(12),(12)(34)(56)\ldots,(23)(45)(67)\ldots\}$ for S_n; then $C_\Lambda(S_n)$ is hamiltonian.

In [W10], we required only conditions (i), (ii), and (iii) for an extent, and gave the following two ramifications of the Rapaport construction:

<u>Thm. 15-5</u>. An extent on n bells exists, for all n. Moreover, only three transition rules are required.

<u>Thm. 15-6</u>. For n odd an extent exists which also satisfies condition (iv) at all but position n, where (at the worst) no bell rests for more than four successive changes.

In [W10] we also displayed an imbedding of $C_\Delta(S_4)$, for Rapaport's Δ, in the projective plane. (This is essentially Figure 15-2 below.) In this imbedding the three left cosets of D_4 in S_4 bound regions which link together nicely to trace out the hamiltonian cycle giving the extent, in extension of the idea behind Figure 15-1. This suggests the efficacy of picturing extents on surfaces, and as we naturally wish to do this as efficiently as possible, we make the following definition:

<u>Def. 15-7</u>. The <u>characteristic of an extent</u> (on n bells, with transition rules Δ) is the maximum characteristic of all surface imbeddings for $C_\Delta(S_n)$.

As certain transition rules (the "calls" - so called because the conductor calls them at the appropriate time, while all other transitions are trusted to the memory of the ringers - known as "bobs" and "singles") are used infrequently in a particular extent (a single is perhaps employed only <u>twice</u> among the $n!$ transitions, for example) we take Δ to include only the other generators for S_n; this has the effect of avoiding unnecessary clutter in $C_\Delta(S_n)$. Thus, in the event that Δ does not generate S_n, a characteristic imbedding of an extent may be on a disconnected surface.

For example, the Rapaport $\Delta = \{a,b,c\}$ (where $a = (12)$, $b = (12)(34)(56)\ldots$, and $c = (23)(45)(67)\ldots$) for S_n $(n \geq 5)$ involves no "call" generators and the relators $(ab)^2$, $(ac)^6$, and $(bc)^n$ bound regions for an imbedding (orientable if and only if $n \equiv 3 \pmod 4$) of $C_\Delta(S_n)$ of characteristic $(n-1)!(3-n)/6$, which is thus a lower bound for the characteristic of the extent. Or, the single relator $(abc)^{n-1}$ can be taken to bound regions for an orientable imbedding of $C_{\Delta}(S_n)$ of characteristic $n(n-2)!(3-n)/2$, but this is less efficient. Note that Δ does generate S_n, as a and bc are known to do so; see Coxeter and Moser [CM1, p. 63]. For $n = 4$, the imbedding in the projective plane cannot be improved (as $C_{\Delta}(S_4)$ contains a homeomorph of $K_{3,3}$), so that the characteristic of this extent is exactly one.

We now consider $\Delta = \{b,c,d\}$, where $b = (12)(34)(56)\ldots$, $c = (23)(45)(67)\ldots$, and $d = (34)(56)(78)\ldots$, for the various <u>Plain Bob</u> extents on n bells. Here the <u>hunting group</u> generated by b and c is always D_n (as can be seen from its action on a regular n-gon with vertices labeled, in order, $1,2,4,6,\ldots,7,5,3$), giving a block of $2n$ changes. The generator d is used to link $n - 1$ of these blocks together, to give a "touch" of $2n(n-1)$ changes. Finally, for $n \geq 5$, the call generators (the bob $e = (23)(56)(78)\ldots$ and, for $n \geq 6$, the single $f = (56)(78)\ldots$) are added to produce the full $n! + 1$ changes of extent. Here the relators $(bc)^n$, $(bd)^2$, and $(cd)^{n-1}$ bound regions for an always nonorientable imbedding of $C_{\Delta}(S_n)$ of characteristic $-(n-2)!(n^2 - 5n + 2)/4$; this is for $n \geq 4$. This appears to be the most useful imbedding for Plain Bob extents, as the relator $(bc)^n$ exactly portrays the principle embodied in the hunting group, and replicated in the left cosets of that group. Again we note that Δ generates S_n, as bc and bd are known to do so.

Next consider $\Delta = \{b,c,g\}$, where $b = (12)(34)(56)\ldots$, $c = (23)(45)(67)\ldots$, and $g = (12)(45)(67)\ldots$ for the various <u>Grandsire</u> extents on n (odd) bells. Here we see that $\langle a,c\rangle = D_n$ and g is used as a linking permutation. The bob for Grandsire is constructed from Δ, but the single is the new $h = (45)(67)(89)\ldots$. To see that $C_{\Delta}(S_n)$ is connected (for $n \equiv 3 \pmod 4$), we observe that $gc = (123)$ and that conjugating alternately by b and g (and then taking inverses) produces $\{(12k) \mid 3 \leq k \leq n\}$; this is known [CM1, p. 66] to generate A_n. But then $\langle A_n,g\rangle = S_n$. We use the relators $(gc)^3, (gb)^{n-2}$, and $(cb)^n$ (the latter is used in the principle for Grandsire) to bound regions for an imbedding of $C_{\Delta}(S_n)$ (orientable for $n \equiv 3 \pmod 4$, nonorientable and in two components for $n \equiv 1 \pmod 4$)) of characteristic $-(n - 1)(n - 3)!(n^2 - 5n + 3)/3$. For $n \equiv 1 \pmod 4$ the single h is odd, so that $\langle A_n,h\rangle = S_n$.

Finally, we endeavor to improve on the Rapaport generating set, so as to satisfy condition (iv) universally. Her construction is based upon the following lemma:

<u>Lemma 15-8</u>: *A connected graph regular of degree three has a hamiltonian cycle if there is a set P of polygons and a set Q of squares, each set partitioning the vertex set of the graph and such that no member of P contains every vertex of a member of Q.*

For the Rapaport $\Delta = \{a,b,c\}$, where $a = (12)$, $b = (12)(34)(56)\ldots$, and $c = (23)(45)(67)\ldots$, the set P is composed of $n!/12$ 12-gons $(ac)^6$ (or, we could use $(n-1)!/2$ $2n$-gons $(bc)^n$) and the set Q consists of $n!/4$ squares $(ab)^2$. The Rapaport construction of a hamiltonian cycle uses b edges to combine 12-gons from P (or a edges to combine $2n$-gons from P); thus the cycle consists of partial words $(ac)^k b(ca)^h$ (or $(bc)^k a(cb)^h$) and we see that condition (iv) is uniformly satisfied, if n is odd. This strengthens Theorem 15-6.

If we modify Δ to $\Delta' = \{d,b,c\}$ as in Plain Bob (so that $d = (34)(56)(78)\ldots$, instead of (12)), then Lemma 15-8 still applies, to show that $C_\Delta(S_n)$ is hamiltonian (it is clear that Δ' generates S_n, since Δ does.) But *now* the hamiltonian cycle found by Rapaport always involves $\ldots cdc \ldots$, which fixes the bell in position one unduly.

To retain the constant movement of Plain Bob (that is, to avoid the "stagnation" of Rapaport's $a = (12)$, which fixes $n - 2$ bells) but to avoid the dawdling in position one, we now try $\Delta'' = \{b,c,h\}$, where $h = (45)(67)(89)\ldots$, and where $n = 2m - 1$ is odd. We first check that Δ'' generates $S_n = \langle ch, (bc)^{2m-3} \rangle$ (this argument fails if n is even.) Lemma 15-8 then applies, with P determined by (say) $(bc)^n$ and Q by $(ch)^2$. Then we obtain a hamiltonian cycle in which b alternates throughout; as b moves every position but the last, which is moved by both c and h, we see immediately that condition (iv) is satisfied. Moreover, there is an efficiency here (over Plain Bob), as only three generators are used; there are no bobs or singles. However, conditions (v) and (vi) have not been checked, and indeed it appears difficult to do so from this inductive construction. The special case $n = 5$ will be considered in detail in Section 5.

The imbedding of the "no call" extent described above is non-orientable, of characteristic $-(n-1)^2(n-4)(n-3)!/4$.

Thus the various extent imbeddings we have constructed in this section have had characteristics asymptotic to $n!/k$, where $k \in \{2,3,4,6\}$.

15-4. Four-bell Extents (Minimus)

The eleven minimus extents [SS1] are algebraically described and graphically depicted in this section. We see from Theorem 15-2 that there are four allowable transition rules for four bells; they are:

$$a = (12)(34)$$

$$b = (23)$$

$$c = (34)$$

$$d = (12)$$

Each extent is described, in Table 15-3, by the identity word in S_4 giving the hamiltonian cycle in the Cayley color graph. Since all Cayley color graphs are vertex-transitive, the initial vertex of the hamiltonian cycle can be arbitrarily selected; the remainder of the cycle is then uniquely determined.

Minimus Extent	Algebraic Description
Plain Bob	$((ab)^3ac)^3$
Reverse Bob	$(abad(ab)^2)^3$
Double Bob	$(abadabac)^3$
Canterbury	$(abcdcbab)^3$
Reverse Canterbury	$(db(ab)^2dc)^3$
Double Canterbury	$(dbcdcbcd)^3$
Single Court	$(db(ab)^2db)^3$
Reverse Court	$(ab(cb)^2ab)^3$
Double Court	$(db(cb)^2db)^3$
St. Nicholas	$(dbadabdc)^3$
Reverse St. Nicholas	$(abcdcbac)^3$

Table 15-3

Obviously, each word uses twenty-four symbols, but it is worth noting that only Plain Bob requires as few as four letters to be written down in its algebraic description. We observe that none of the minimum extents satisfies all six of the conditions given in Section 15-1: all fail (v), and only the first three listed satisfy (iv).

Now follow the graphical pictures of the corresponding Cayley graphs. Figure 15-2 serves directly for Plain Bob Minimums and Reverse Court Minimus; it also applies to Reverse Bob and Single Court, with d replacing c. These four extents all have characteristic one. Figure 15-3 pictures Double Court and Double Canterbury; these two extents have characteristic two. (They are the only <u>planar</u> extents.) Figure 15-4 depicts Reverse St. Nicholas, St. Nicholas, Reverse Canterbury, Canterbury, and Double Bob; these are <u>immersions</u> and not imbeddings, as the redundant generator a is allowed to diagonalize six quadrilaterals. In the language of topological graph theory, this immersion shows that, for Λ = $\{a,b,c,d\}$, the crossing number of $C_\Lambda(S_4)$ is at most six.

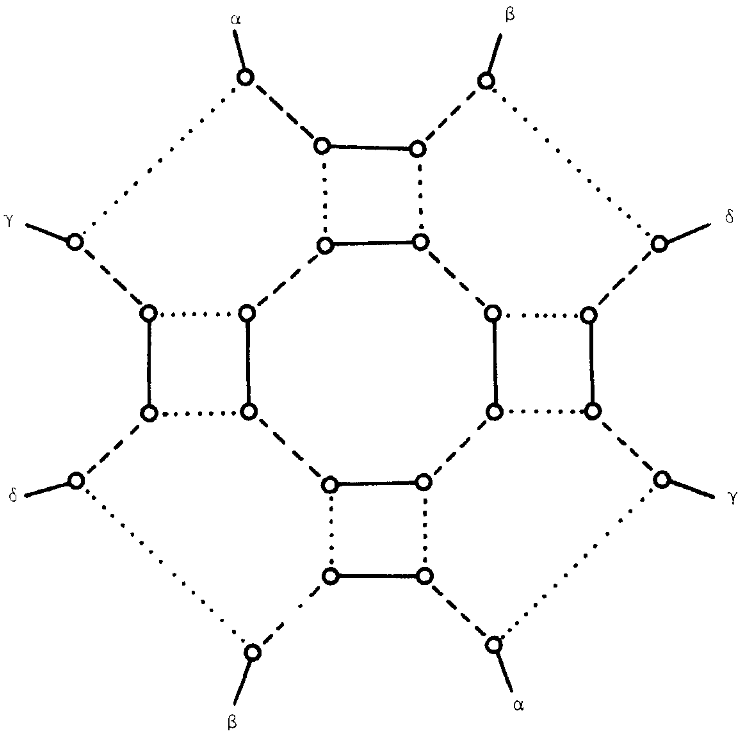


Figure 15-2

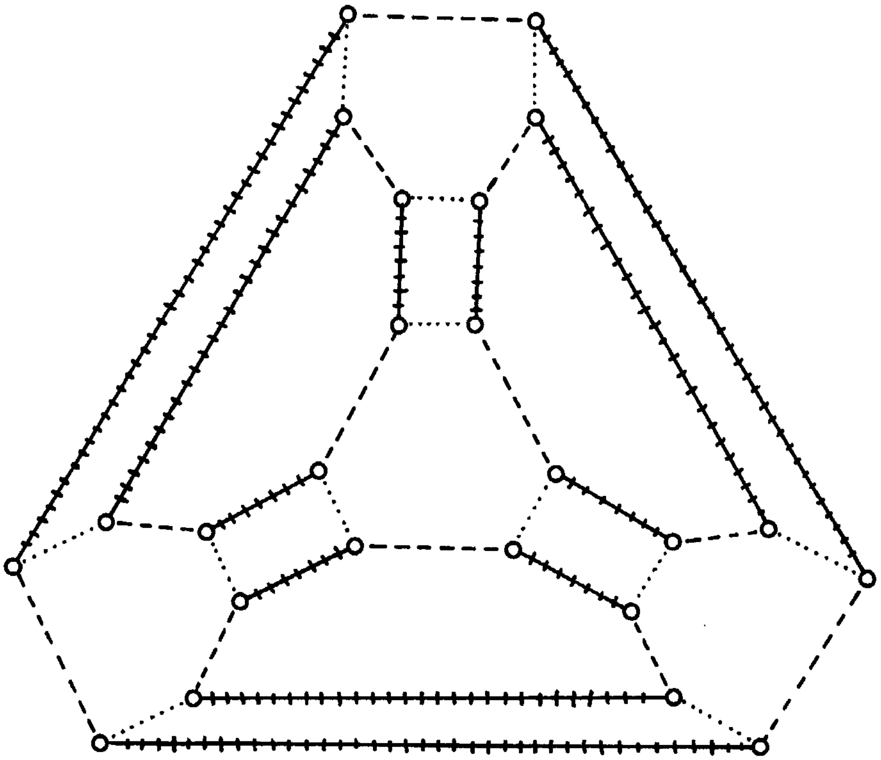


Figure 15-3

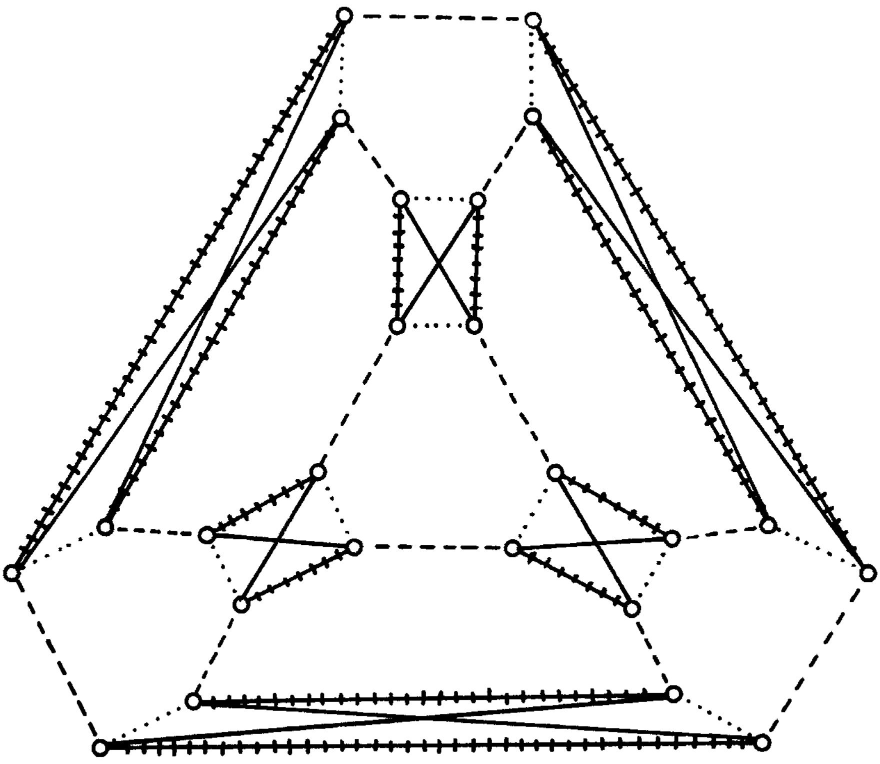


Figure 15-4

15-5. Five-bell Extents (Doubles)

There are sixteen doubles extents given in [SS1]; there are undoubtedly many more. The seven possible transitions (see Theorem 15-2) are:

$a = (12)(34)$	________
$b = (23)(45)$	
$c = (34)$	....
$d = (23)$	++++++++
$e = (12)$	
$f = (12)(45)$	
$g = (45)$	--------

We shall not need to represent e for the extents discussed in this section, which are:

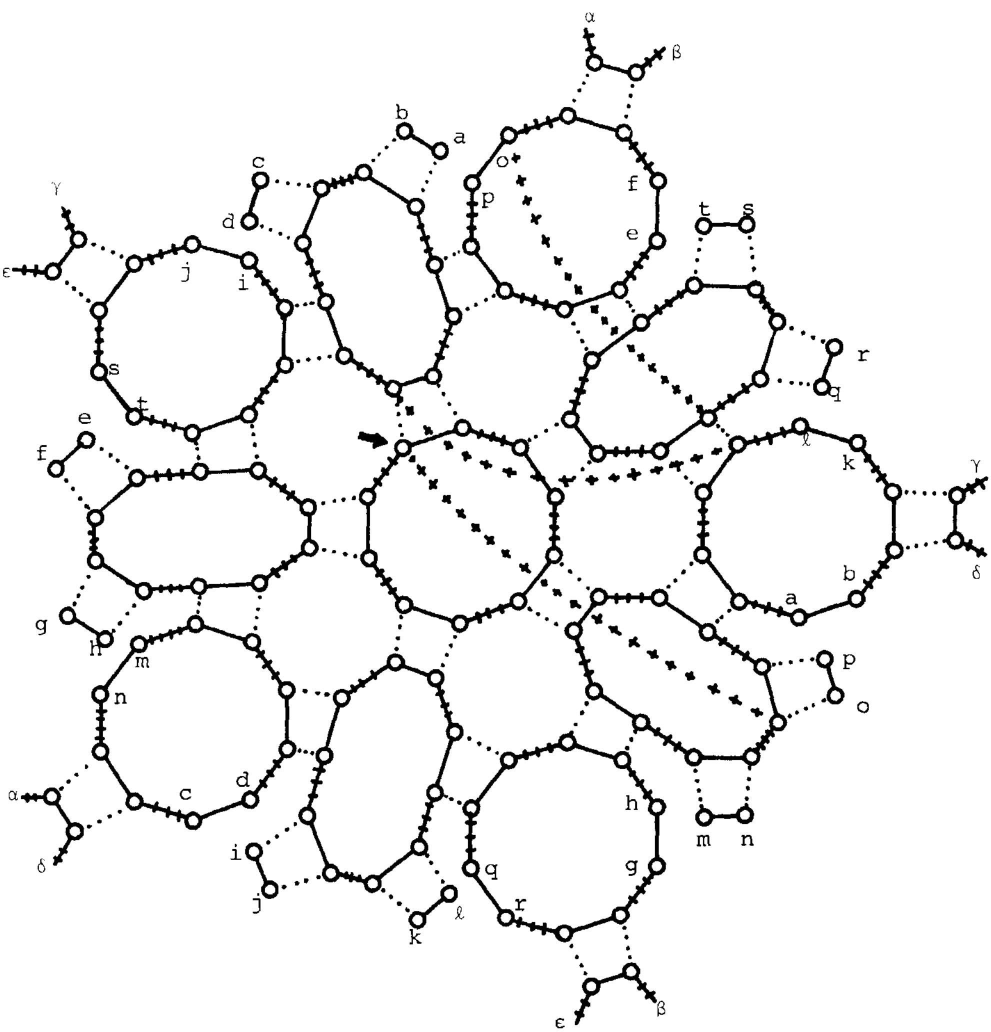
$$\text{Plain Bob} = (((ab)^4ac)^3(ab)^4ad)^3$$

$$\text{Grandsire} = ((fb(ab)^4fb(ab)^3fb)^2fb(ab)^4fb(ab)^3fg)^2$$

$$\text{Stedman} = (fba(f(bf)^2ab(fb)^2a)^4f(bf)^2abfd)^2$$

$$\text{No-Call} = ((ag)^3(ab)^3ag(ab)^2(ag)^2ab)^5.$$

Figure 15-5 shows $C_\Delta(S_5)$ for Plain Bob Doubles ($\Delta = \{a,b,c\}$) imbedded on the nonorientable surface N_5, of characteristic -3. Indeed Plain Bob Doubles <u>has</u> characteristic -3, as the graph which results when each decagon is contracted to a vertex is $K_{6,6}$ less a 1-factor, for which the resulting quadrilateral imbedding is necessarily optimal, and contraction cannot decrease characteristic. If we start the hamiltonian cycle at the distinguished vertex in Figure 15-5 and call the bob d as indicated by the superimposed edges, then the 121 rows of the extent can be readily obtained; see Problem 15-2.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q
g
j
r
ℓ
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Figure 15-5

Figure 15-6 shows $C_\Delta(A_5)$, $\Delta = \{a,b,f\}$ imbedded on the non-orientable surface N_6, of characteristic -4. The first 59 symbols of the identity word for Grandsire Doubles trace all but one edge of a hamiltonian cycle for $C_\Delta(A_5)$ on N_6. The single g then shunts us across to a <u>second copy</u> of the same $C_\Delta(A_5)$ on another, disjoint, surface N_6. (Thus $C_\Delta(S_5)$ has two components, one for each coset of A_5.) The same hamiltonian path is repeated on this second surface, and the single g returns us to the initial vertex on the first surface. The result is a hamiltonian cycle in $C_{\Delta\cup\{g\}}(S_5)$, depicted on a topological space consisting of disconnected surface whose two homeomorphic components are joined by a pair of line segments.

Figure 15-6 also serves for Stedman Doubles, in the same manner, if we use d for the single instead of g. Thus both these extents have characteristic at least -8.

We note that the same surface (two copies of N_6) can be used for St. Simon Doubles - $((ab(af)^2abac)^3ab(af)^2abad)^3$, except that c is used nine times (of 120) as part of Δ, and d is used as a bob three times; the hamiltonian cycle, due to the action of c, does not respect the cosets of A_5 - that is, it transfers frequently from one N_6 to the other.

15-6. A New Composition

Finally, Figure 15-7 shows $C_\Lambda(S_5)$, $\Lambda = \{a,b,g\}$, imbedded on the nonorientable surface N_{10}, also of characteristic -8. The identity word $((ag)^3(ab)^3ag(ab)^2(ag)^2ab)^5$ for "No-call" Doubles (so named by the author, as no bobs or singles are required) traces out a hamiltonian cycle in $C_\Lambda(S_5)$. (In fact, the cycle was found in this figure <u>first</u>, as suggested by the five-fold symmetry, and then the word was written down from the picture; start at the indicated vertex, for example.) The corresponding 121 rows of this extent are given in Table 15-4.

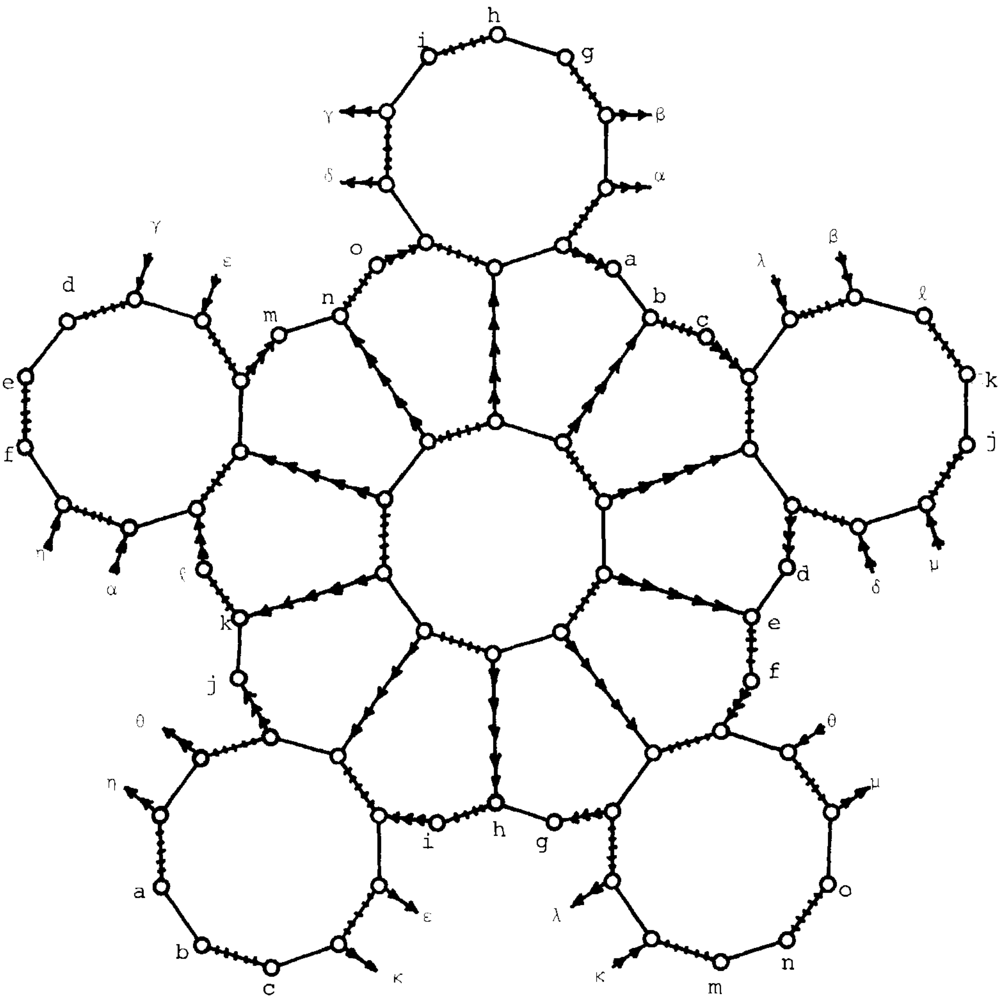
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Figure 15-6

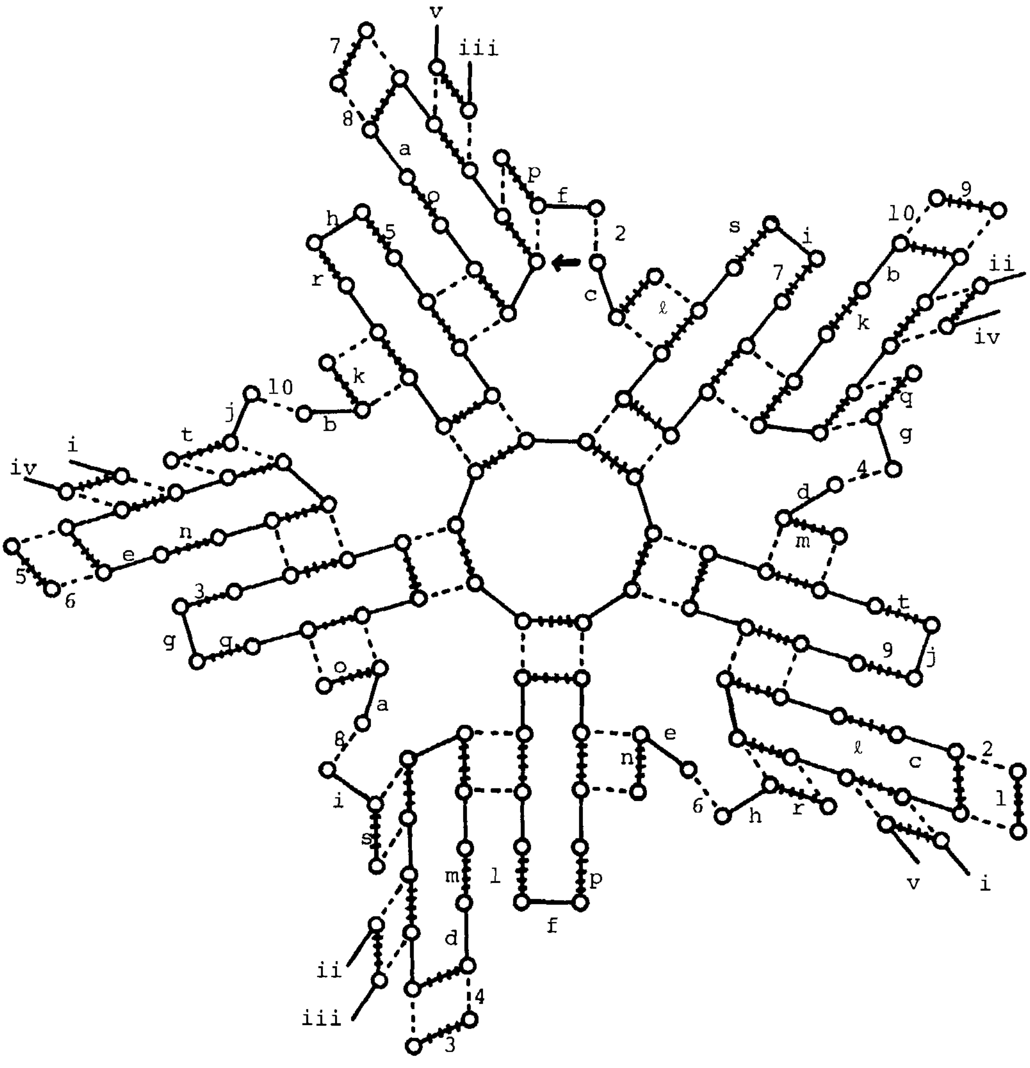

Figure 15-7

"No Call" Doubles

12345	45123	23451	51234	34512
21435	54213	32541	15324	43152
21453	54231	32514	15342	43125
12543	45321	23154	51432	34215
12534	45312	23145	51423	34251
21354	54132	32415	15243	43521
21345	54123	32451	15234	43512
12435	45213	23541	51324	34152
14253	42531	25314	53142	31425
41523	24351	52134	35412	13245
45132	23415	51243	34521	12354
54312	32145	15423	43251	21534
53421	31254	14532	42315	25143
35241	13524	41352	24135	52413
35214	13542	41325	24153	52431
53124	31452	14235	42513	25341
51342	34125	12453	45231	23514
15432	43215	21543	54321	32154
14523	42351	25134	53412	31245
41253	24531	52314	35142	13425
41235	24513	52341	35124	13452
14325	42153	25431	53214	31542
14352	42135	25413	53241	31524
41532	24315	52143	35421	13254
				12345

Table 15-4

From the table it is a simple matter to check not only that conditions (i) through (iv) are satisfied (of course, (i)-(iii) must hold, by Theorem 15-3, if we designate the first (and last) vertex of the hamiltonian cycle as rounds; this is always possible, since Cayley graphs are vertex-transitive. Moreover, we get immediate algebraic verification of (iv) by analyzing the identity word for No-call Doubles: a alternates throughout, moving every position but 5, and both b and g move position 5), but that condition (v) holds as well: each bell does the same amount of work as every other. In particular, the path of bell 1 in column i is exactly mimicked by that of bell 4 in column i + 1, bell 2 in column i + 2, bell 5 in column i + 3, and bell 3 in column i + 4, $1 \le i \le 5$ (arithmetic in Z_5, writing "5" for "0".) Unfortunately, condition (vi) just fails, as agagagabababagababagaga is not palindromic. There are four pairs of symbols (such as the third "g" and the second "b") that could be switched to obtain a palindrome, but sadly each such switch destroys the much more crucial hamiltonian property. Perhaps the efficiency of "No-call" - in that only three transition rules are required (no other doubles extent uses this few) - would argue its merit, in spite of the failure of the final condition. But our final paragraph indicates that such argument is not needed!

John Elliott, in his article "Doubles Principles" [E2], generated by computer eight doubles extents not requiring bobs or singles. Each of these is palindromic, and in fact satisfies all conditions (i) - (iv); but it appears that none employs fewer than five generators. In the June 12, 1981 issue of "The Ringing World," a writer identified only as S. J. P. commented that the Central Council of Church Bell Ringers relaxes rule (iv) by allowing a bell to rest in the same position for up to four successive changes and abolishes rule (vi) - removed "some years ago" - altogether! This latter, the writer continues, allows "any old rubbish to qualify as a method."

15-7. Problems

15-1.) The planar imbedding of Figure 15-3 is a 12-fold branched covering space of an index two voltage graph imbedding; find this latter configuration.

15-2.) Use Figure 15-5 and the corresponding identity word in S_5 to write the rows for Plain Bob Doubles.

**15-3.) Find the characteristic of Plain Bob Minor.

**15-4.) Find the characteristic of Plain Bob Major.

**15-5.) Find $F(n)$, the characteristic of Plain Bob on n bells. (We know that $F(4) = 1$, $F(5) = -3$, and in general $F(n) \geq -(n-2)!(n^2-5n+4)/4$.

15-6.) If the principle for an extent on n bells corresponds to a subgroup of S_n (as is the case for Plain Bob), then contracting each coset of the principle to a single vertex produces a Schreier coset graph. Discuss the ramifications, for the calculation of the characteristic of the extent.

15-7.) Show that the <u>lead-ins</u> (coset leads) for Plain Bob Minor correspond to a 1-factorization of K_6. To what extent (i.e. other minor extents, other even values of n) does this generalize, and why?

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INDEX OF SYMBOLS

Note: The number following the symbol gives the first page on which the symbol appears. In the top half of the list, corresponding symbols are displayed alongside one another. The six distinguished symbols correspond to the six relationships depicted in Figure 0-1.

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INDEX OF DEFINITIONS