Progress in Mathematical Physics
70

Bertrand Eynard

Counting Surfaces

CRM Aisenstadt Chair lectures

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To my wife Sandra
To my children, Maève, Salomé, Thibault.

Preface

This book is based on several lectures that I gave at various places over the time, including IPHT CEA Saclay, CRM Montréal, Les Houches summer school, Barcelona (2007 school organized by the European Network on Random Geometry), Geneva University, and Chebyshev Laboratory in St Petersbourg, Russia.

The largest part of the research presented in this book was pursued at the Institut de Physique Théorique CEA Saclay that hired me in 1995. A large part was also conducted at the Centre de Recherche Mathématiques de Montréal, QC, Canada, of which I became full member in 2013. In fact the idea of topological recursion first occurred as I was CRM visitor in 2002.

The CRM and Aisenstadt Chair

The final stage of writing of this book corresponds to the CRM Aisenstadt chair that I occupied in the fall of 2015, during the thematic semester "AdS-CFT, Holography, Integrability," and corresponds to a series of lectures given there in October 2015. This is why this book is part of the CRM series of lectures and part of the Aisenstadt chair lectures.

The Centre de Recherches Mathématiques (CRM) was created in 1968 to promote research in pure and applied mathematics and related disciplines. Among its activities are special theme years, summer schools, workshops, postdoctoral programs, and publishing. The CRM receives funding from the Natural Sciences and Engineering Research Council (Canada), the FRQNT (Quebec), the NSF (USA), and its partner universities (Université de Montréal, McGill, UQAM, Concordia, Université Laval, Université de Sherbrooke, and University of Ottawa).

Two or three Aisenstadt chairs are awarded by CRM each year to prominent mathematicians chosen because of their relevance and impact, within the thematic program. The recipients of the chair give a series of conferences, the first of which, in compliance to the donor André Aisenstadt's wish, must be accessible to a large public. They are also invited to write a monograph.

Extract from the CRM's presentation.

Acknowledgments

I wish to thank a number of people and institutions without whom it would have been hard to finish this book. First, I was initiated to random surfaces and random matrices by my advisor Jean Zinn-Justin and Francois David. I also learned a lot from colleagues who were the inventors of many notions presented in this book V. Kazakov, I. Kostov, L. Chekhov, J. Ambjorn, C. Kristjansen, J. B. Zuber, E. Brézin, M.L. Mehta, M. Mulase, and G. Akemann. I also thank colleagues who have read early versions and made comments and helped pointing out misprints or improve presentation, in particular G. Borot, G. Schaeffer, and G. Chapuis.

I thank the IPHT of which I was a member all those years, and in particular the stimulating coffee room atmosphere in which many discussions have started. I thank the CRM and also the places at which I have given those lectures, in particular Les Houches school, Geneva University, and St Petersbourg's Chebyshev Laboratory, and I thank S. Smirnov for his invitation to give lectures on those topics in Geneva and St Petersbourg.

I thank my colleagues and research collaborators on topics contained in this book M. Bergère, M. Bertola, G. Bonnet, L. Chekhov, F. David, P. Di Francesco, E. Guitter, J. Harnad, C. Kristjansen, A. Pratts–Ferrer, and J. B. Zuber and my former students N. Orantin, O. Marchal, G. Borot, and R. Belliard.

I thank my physics professor Marc Serrero in undergraduate class, who was always pushing to look beyond the school programs towards the horizon of research and communicated his passion.

I thank my parents, my sister, and my grandfather who encouraged developing my taste to mathematics and physics.

And before everyone, I thank my wife and my children for their moral help, their love, and accompanying and encouraging, and their continued support all this time.

Topic of the Book

A map is a discrete surface, built by gluing polygons—like countries on a world.

This book aims at presenting to mathematicians, the so-called "random matrix" approach to 2D quantum gravity developed by physicists mostly in the 1980s and 1990s of the twentieth century.

The "random matrix method" started with Nobel Prize awardee Gerard t'Hooft's discovery in 1974, from the study of strong nuclear interactions, that matrix integrals are naturally related to graphs drawn on surfaces, weighted by their topology. t'Hooft's first example was then turned into a general paradigm for enumerating maps, by physicists E. Brezin, C. Itzykson, G. Parisi, and J.B. Zuber in 1978: "enumerating maps with random matrices." By their method, they recovered some results due to the Canadian mathematician William Tutte in the early 1960s, about counting the numbers of triangulations or quadrangulations of the sphere. And beyond that, their work triggered a major new trend in quantum gravity, in string theory, in random matrices and in enumerative geometry and had an enormous influence on modern physics and mathematics.

For instance, this approach of quantum gravity is what motivated Edward Witten to formulate his famous "Witten's conjectures," about the geometry of moduli spaces of Riemann surfaces, later proved by Maxim Kontsevich, and which gave rise to an incredible amount of beautiful geometry, still going on nowadays.

Also, over the years, it was understood how to go from planar surfaces and planar graphs to higher topologies. The "topological recursion" method was discovered in 2004 and was promoted to a mathematical theory of geometric invariants in 2007. It gives a formula for counting maps of a certain topology, only in terms of planar maps enumeration; somehow it consists in gluing planar pieces to make surfaces of higher topology.

Here in this book, we explain the relationship between matrix models and enumeration of maps, and we formulate in a precise mathematical language what is a "formal matrix integral." After Brezin, Itzykson, Parisi, and Zuber, a precise definition of a formal matrix integral was never written explicitly in the physics literature, but it was always implicitly assumed; indeed, all integrals in physics after Feynman's works were always treated as formal integrals, and it was so deeply impregnated in physicists' culture that most physics article didn't care of mentioning it. This was of course confusing and sometimes led to misunderstandings and wrong statements.

At first, in the 1980s and 1990s, it was believed that using matrix integrals to count maps was a huge progress:

– Replacing the rather difficult problem of counting graphs on surfaces, by a problem of finding the large N expansion of a $N \times N$ matrix integral. It replaced combinatorics by analysis and algebra. This may look simpler. However, it was then realized that computing the large N behavior of an integral with N^2

variables, is in fact a very difficult problem. Making heuristic guesses about the asymptotic expansions was often easy, but proving them was often very difficult.

– During many years, in the 1980s and 1990s, there was a confusion between genuine *convergent* matrix integrals and *formal* matrix integrals. Formal matrix integrals are a mere rewriting of the combinatorial sum over graphs they are just a nice mnemotechnic way of writing generating series for graphs. These have a large N expansion by their very definition. On the other side, convergent matrix integrals belong to the realm of analysis and probabilities. The issue of their large N expansion is considered a very difficult and challenging problem in asymptotic analysis.
– In the late 1990s, some people started to realize that convergent matrix integrals and formal matrix integrals are in fact not the same thing. For instance, Brézin and Deo in 1996 [19] raised a puzzling paradox: they computed the (apparently) same expectation value by two different methods: one combinatorial (loop equations) and one based on asymptotic analysis (orthogonal polynomials method, the method of Bleher, Its, Deift, and coworkers [16, 25]), and they didn't find the same result! This puzzled the community for a few years before it was clearly understood that the two kinds of matrix integrals were different.
– From the 2000s, the point of view changed. The new point of view is that formal matrix integrals are just a nice way to write the combinatorics of maps; they are identical to generating functions of maps. Manipulating them is just combinatorics. Tutte's equations (recursively deleting or contracting edges) are identical to loop equations (integration by parts). However, writing a formal matrix integral remains very useful because: *it is much easier and faster to integrate by parts than finding bijections among sets of maps.*

The large N expansion of convergent matrix integrals is a very difficult problem, much more difficult than the combinatorics of maps. So we have reversed the BIPZ point of view: nowadays, it is considered that:

counting maps is the easy side, it helps computing large N asymptotics of $N \times N$ matrix integrals (the difficult side).

William Tutte was a combinatorist, whose goal was to enumerate discrete objects: maps. For physicists, maps were supposed to be only an intermediate step; it was supposed to be a discretization of the set of surfaces, more precisely of Riemann surfaces. The ultimate goal was to be able to do quantum gravity, i.e., counting the "numbers" of Riemann surfaces, i.e., measure the volumes of the set of Riemann surfaces weighted by various kinds of weights. This is "string theory."

An important issue was thus to understand the continuum limit: maps, i.e., discrete surfaces made of polygons, are an approximation of continuous smooth Riemann surfaces. Going to the limit means sending the number of polygons to infinity while sending the size of polygons (the mesh size) to zero. Physicists called it the "double scaling limit" in the 1990s.

Most of the physicist's derivations in the 1990s about the double scaling limit were based on a heuristic link between formal matrix integrals and genuine convergent matrix integrals and were not mathematically proved. However, this led to a lot of understanding about the geometry of the moduli space of Riemann surfaces. For example, it led Witten in 1991 to formulate his conjecture:

> the generating function enumerating asymptotically large maps, is equal to the Korteweg de Vries (KdV) Tau function

This was later proved by Kontsevich. Kontsevich also used a set of graphs on surfaces, but rather different from those of BIPZ. Instead of approximating the space of Riemann surfaces by a discrete subset, he cut the full space of Riemann surfaces into cells, each cell labeled by a graph. In this way, no information is lost, and there is no need to take any limit.

Kontsevich graphs are also studied in this book, with a proof of Witten's conjecture based on topological recursion.

Another approach to 2D quantum gravity had also been introduced by Polyakov in 1981 [76]; it is called "Liouville" quantum gravity or Liouville CFT (conformal field theory). Instead of summing over surfaces, Polyakov was summing over metrics on a surface. Modulo changing of coordinates, any metric tensor in 2 dimensions can be brought to a "conformal" metric, characterized by a scalar, called the "Liouville field" on the surface. The Jacobian of the change of variable from an arbitrary metric to a conformal one, i.e., from metric tensor to Liouville field, is the Liouville action. It is called the Liouville action, because its extremum is reached at the constant curvature metric, called the Liouville metric. The Liouville action has the property of being conformally invariant, i.e., invariant under all conformal transformations.

Conformal invariance implies huge constraints and relationships among expectation values. It implies that all expectation values are algebraic combinations of building blocks labeled by representations of the conformal group. Those representations have been deeply studied in conformal field theory. In particular finite representations have been classified in Kac's table [41]. Also, the exponents appearing in asymptotic formulae are dictated by the theory of representations of the conformal group they are classified.

In particular, the famous KPZ (Knizhnik, Polyakov, Zamolodchikov) formula gives, for instance, the Hausdorf dimensions and various other exponents on surfaces with random metrics [55].

Using heuristic asymptotics of matrix integrals, it was found in the 1990s, that continuum limits of large maps should indeed coincide with the results of Polyakov's approach, in particular with the so-called minimal models, the finite representations of the conformal group, and indeed satisfy KPZ relations.

The physicist's derivations of the 1990s were based on heuristic asymptotics, which have been mathematically proved in the 2000s. A complete derivation is presented in this book.

What Is Not Done in This Book

The goal of this book is just to perform the enumeration of maps, not to analyze geometric properties of most probable maps; for instance, we don't consider the statistical properties of geodesic distance on maps.

We focus only on the link between formal matrix integrals and enumeration of maps. We don't look at other approaches, like the relationships between maps and trees, the so-called Schaeffer's bijections, or Bouttier, DiFrancesco, Guitter bijections, and all the subsequent continuum limit notions of Brownian maps. We also don't look at the approach from SLE (Schramm–Loewner evolution) and the Gaussian free field approaches. Books and review articles exist about these notions and we refer the reader to them.

Gif-sur-Yvette, France B. Eynard

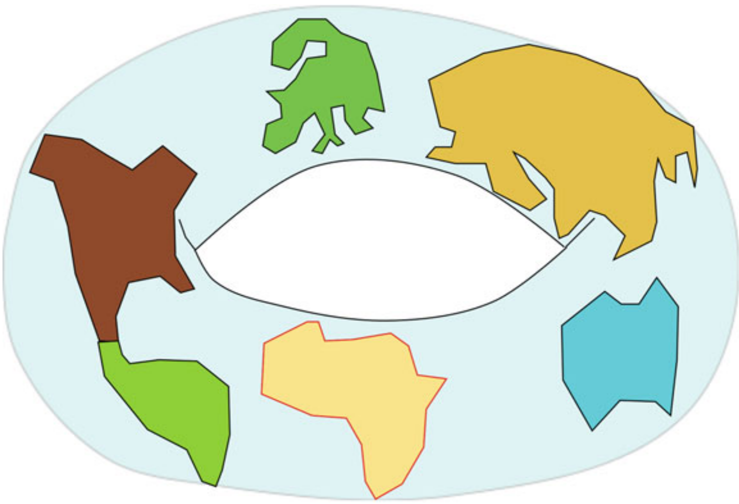
Contents

Chapter 1
Maps and Discrete Surfaces

In this chapter we introduce definitions of maps, which are discrete surfaces obtained by gluing polygons along their sides, and we define generating functions to count them. We also derive Tutte's equations, which are recursive equations satisfied by the generating functions.

We will also rederive Tutte's equations in the matrix model language in Chap. 2, and we will give their solution for any topology in Chap. 3.

Several classical books exist about maps, for instance: Berge [9], Tutte [84, 85], Gross and Tucker [43], as well as [46, 50, 61, 66].



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1.1 Gluing Polygons

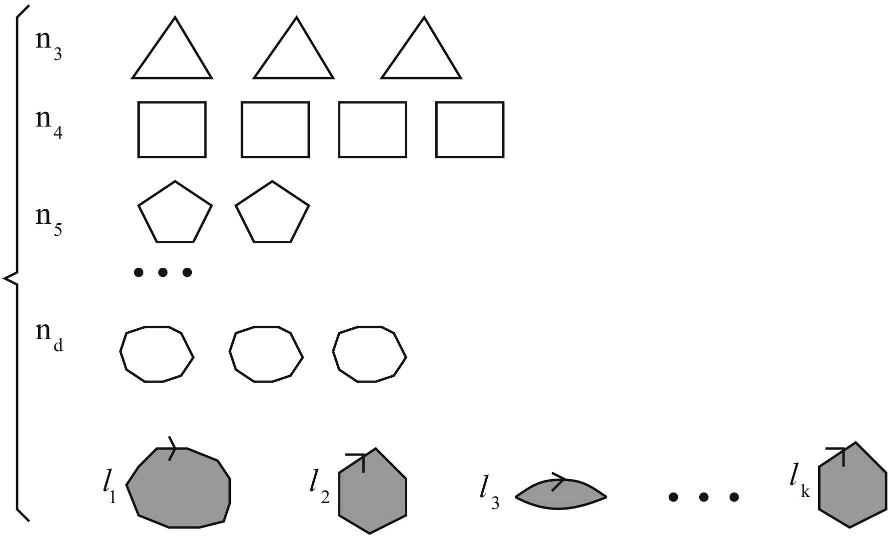
The idea of a map, is a collection of countries and seas on a world. However, in our case, the world is not a sphere or a plane, but a surface with an almost arbitrary topology, and the countries are polygons. Maps are also called **"discrete surfaces"**.

From now on, we shall consider only **orientable surfaces**, the case of maps on non-orientable surfaces can be treated in a similar approach, and satisfies the same topological recursion, but this subject is under development at the time this book is being written.

1.1.1 Intuitive Definition

Here, we give an intuitive and informal definition of a map. A formal definition is given in the next subsection, or can be found in many books [9, 43, 46, 61, 66, 85], and examples can be found in [50]. The idea is a collection of polygons glued together side by side. We consider two kinds of polygons: some unmarked ones, and some marked ones.

Definition 1.1.1 (Intuitive Definition) A map, with n_3 unmarked triangles, n_4 unmarked quadrangles, n_5 unmarked pentagons, ..., n_d unmarked d-gons, and with k labeled "**boundaries**" of lengths $l_1, \ldots, l_k$ (a boundary of length l is a marked polygon with l sides, and with one marked clockwise oriented side, boundaries are labeled from 1 to k), is an oriented connected gluing of those polygonal pieces along their edges:



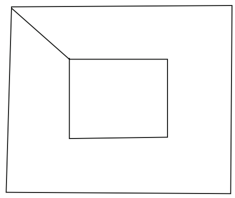
or more precisely, it is the equivalence class of such gluings under graph isomorphisms (i.e. composition of permutations and rotations of the unmarked i-gons, which preserve the oriented marked edges).

The polygons are called faces, and the number of edges of a face, is sometimes called its degree, its size, its length, its perimeter.

Moreover, we require that unmarked polygons have at least three sides, and marked polygons have length at least $l_i \geq 1$.

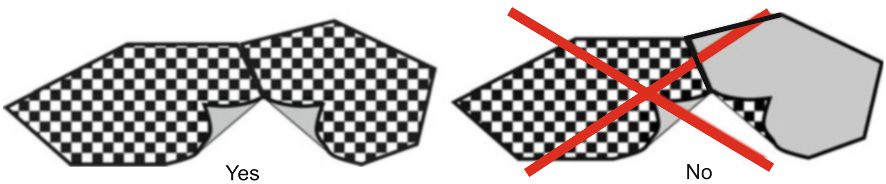
Remark 1.1.1 As mentioned above, this is only an intuitive definition. The actual definition is given in Definition 1.1.2 below, or in many books or works [9, 43, 46, 50, 61, 66, 85].

Remark 1.1.2 Notice that nothing in the definition prevents from gluing a side of a polygon to another side of the same polygon, in particular to an adjacent side. This means that the corresponding surfaces can be rather singular. In the following example we glue a 10-gon to a quadrangle (a 4-gon), the 10-gon has two of its sides glued together.

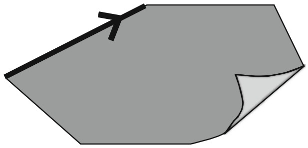


There exists standard combinatorial methods to relate the counting of singular maps to non-singular ones.

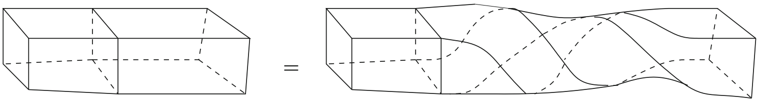
Remark 1.1.3 We recall that we consider **oriented polygons**, which means that each polygon has a recto and a verso, that can be imagined with two different colors, and polygons must be glued by their sides together, matching the colors, as in the following figure:



The boundaries have a marked edge on their side, and by convention, we represent the marked edge with an arrow, in such a way that the boundary's face sits on the right side of the marked edge.



Remark 1.1.4 A “gluing” of polygons means a set of incidence relations, i.e. which edge is glued to which edge. For example, twists will be irrelevant in this book.



Remark 1.1.5 Let us emphasize again that we assume that each unmarked polygonal face (which is not a boundary) has a degree ≥ 3, and $\leq d$:

$$3 \leq \text{degree of unmarked faces} \leq d$$

whereas for the boundaries we only require that the length of the ith boundary is:

$$l_i \geq 1$$

The union of all faces of the map, is a surface, and the map can be seen as an embedding of a graph into a surface. We may consider a marked point at the center of each boundary face, and thus a map with k boundaries is naturally embedded into a surface with k-marked points.

The intuitive way of thinking about a boundary would be to exclude the boundaries from the surface, and thus a map with k boundaries would be naturally embedded into a surface with k “holes”, however one should be careful with that too simple picture. Indeed, one should notice that those surfaces can be rather singular, because nothing in our definition prevents from gluing a polygon, and in particular a boundary, to itself or to another boundary. This means that, although the interior of the boundary is an open disk, the boundary with its border might not be a disk, it might be not simply connected. Therefore, if we remove the interior of boundaries, the remaining surface may be singular, and if we remove the boundaries together with their borders, the removed parts are not necessarily disks removed from the surface.

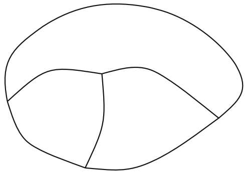
Examples Maps with no boundary ($k = 0$) are called “closed maps”, and maps with boundaries are called “open maps” or “bordered maps”.

Maps with $k = 1$ boundary, i.e. with only one marked oriented edge, are also called **rooted maps**. The knowledge of the marked edge is sufficient to recover the marked face, it is the face sitting to the right of the marked edge. We shall see below that they play an important role.

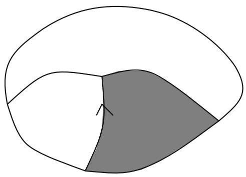
A map made of only triangles is called a **triangulation**, a map made with quadrangles is a **quadrangulation**.

Even Maps A map with only even polygons is called **even**. Planar even maps are also called "bipartite" (indeed a planar map whose faces have even side, can have its vertices bicolored so that adjacent vertices have different colors). This is not true for non-planar maps, even non-planar maps are in general not bipartite.

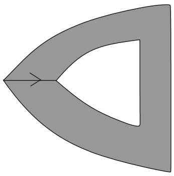
Example of a triangulation with $n_3 = 4$ triangles (the exterior face is a triangle), drawn on the plane, i.e. on the Riemann sphere:



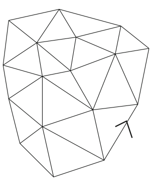
Example of a triangulation with $n_3 = 3$ triangles (one is the exterior one), and one boundary $k = 1$ of length $l_1 = 3$, drawn on the plane, i.e. on the Riemann sphere:



Example of a map with $n_3 = 2$ triangles, and with $k = 1$ boundary of length $l_1 = 8$. Notice that the octogon is glued with itself along the marked edge.



Example of a planar map with $n_3 = 22$ triangles, and one boundary (the exterior) of length $l_1 = 10$:

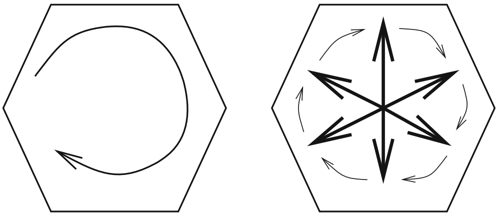


1.1.2 *Formal Definition*

There exists several equivalent definitions of maps. Let us give the following ones, and refer the reader to the literature [9, 50, 61, 66, 84, 85] for other ones (for instance in terms of fatgraphs, or trees,...).

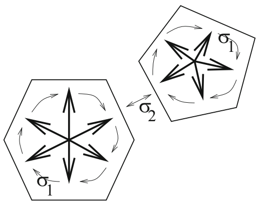
1.1.2.1 Definition with Permutations

A polygon can be seen as a set of edges, together with a cyclic permutation which encodes which edge is next to which along the oriented polygon's side. In a dual manner, an edge of the polygon can be seen as an half edge from the center of the polygon to the edge. We call this half–edge a "dart".



Then, gluing edges together means gluing darts by pairs. The gluing can be encoded into an involutive application from the set of darts to itself, and with no fixed point (a dart can't be glued to itself).

Definition 1.1.2 A labeled map $G = (B, \sigma_1, \sigma_2)$ is the data of a finite ensemble B (whose elements are called "darts" or "half-edges") of even cardinal, and two permutations σ_1 and σ_2, such that σ_2 is an involution without fixed points. The cycles of σ_1 are called faces (or polygons), the cycles of σ_2 are called edges, and the cycles of $\sigma_1 \circ \sigma_2$ are called vertices.



Two labeled maps (B, σ_1, σ_2) and $(B', \sigma_1', \sigma_2')$ are isomorphic iff there exists a bijection $\phi : B \to B'$, such that $\sigma_1' = \phi \circ \sigma_1 \circ \phi^{-1}$ and $\sigma_2' = \phi \circ \sigma_2 \circ \phi^{-1}$.

A map is an equivalence class of labeled maps modulo isomorphisms.

The map is said connected if σ_1 and σ_2 act transitively, i.e. if any two elements of B can be related by a sequence of applications of σ_1 and σ_2.

The Euler characteristics of a map is $\chi = \#\text{faces} - \#\text{edges} + \#\text{vertices}$.

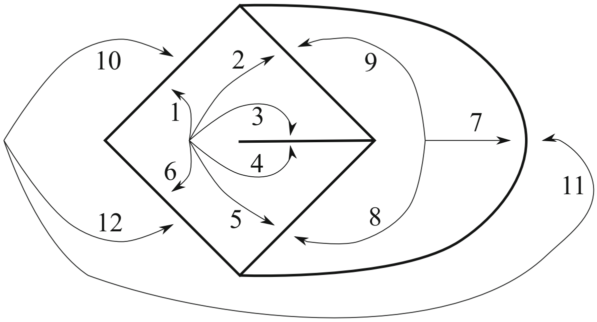
The Automorphism group of a labeled map, is the set of bijections $\phi : B \to B$, such that $\sigma_1 = \phi \circ \sigma_1 \circ \phi^{-1}$ and $\sigma_2 = \phi \circ \sigma_2 \circ \phi^{-1}$. If two labeled maps are isomorphic, their automorphism groups are isomorphic, and in particular the number of automorphisms of a map is well defined (it depends only on the map, not on a labeled map).

Example: the following two permutations encode a labeled map with one hexagon and two triangles

$$\sigma_1 = (2,3,4,5,6,1,8,9,7,12,11,10) = (2,3,4,5,6,1)\,(8,9,7)\,(10,12,11)$$

$$\sigma_2 = (10,9,4,3,8,12,11,5,2,1,7,6) = (10,1)\,(9,2)\,(4,3)\,(8,5)\,(12,6)\,(11,7)$$

$$\sigma_1 \circ \sigma_2 = (12,7,5,4,9,11,10,6,3,2,8,1) = (1,12)\,(2,7,10)\,(3,5,9)\,(4)\,(6,11,8)$$



This definition can be modified in order to have boundaries, i.e. marked faces.

Definition 1.1.3 A labeled map $G = (B^*, B, \sigma_1, \sigma_2)$ is the data of two finite ensembles B and B^* (whose elements are called "darts" or "half-edges"), such that $B \cup B^*$ has even cardinal, and two permutations σ_1 and σ_2 of elements of $B \cup B^*$, such that σ_2 is an involution without fixed points, and every cycle of σ_1 contains at most one element of B^*. The cycles of σ_1 are called faces, the cycles of σ_2 are called edges, and the cycles of $\sigma_1 \circ \sigma_2$ are called vertices. The cycles of σ_1 which contain one element of B^* are called marked faces, and the elements of B^* are marked edges. Each marked face has exactly one marked edge.

Two labeled maps $(B, B^*, \sigma_1, \sigma_2)$ and $(B', B'^*, \sigma_1', \sigma_2')$ are isomorphic iff there exists a bijection $\phi : B \cup B^* \to B' \cup B'^*$, such that $\phi(B^*) = B'^*$ and $\sigma_1' = \phi \circ \sigma_1 \circ \phi^{-1}$ and $\sigma_2' = \phi \circ \sigma_2 \circ \phi^{-1}$.

A map is an equivalence class of labeled maps modulo isomorphisms.

The map is said connected if σ_1 and σ_2 act transitively, i.e. if any two elements of $B \cup B^*$ can be related by a sequence of applications of σ_1 and σ_2.

The Euler characteristics of a map is $\chi = \#\text{faces} - \#\text{edges} + \#\text{vertices} - \#B^*$ (we don't count marked faces).

The Automorphism group of a labeled map, is the set of bijections $\phi : B \cup B^* \to B \cup B^*$, such that the restriction of ϕ to B^* is the identity, and $\sigma_1 = \phi \circ \sigma_1 \circ \phi^{-1}$ and $\sigma_2 = \phi \circ \sigma_2 \circ \phi^{-1}$. The number of automorphisms depends only on the map.

Here is another equivalent definition of maps, closer to our intuitive definition: maps are graphs embedded on a surface:

1.1.2.2 Definition with Embedded Graphs on Surfaces

Definition 1.1.4 A cellular embedded graph (S, G, f), is the data of a topological connected oriented closed surface S, a graph G, and an injective continuous map $f : G \to S$, such that $f(G)$ is a union of Jordan arcs, and such that $S \setminus f(G)$ is a finite union of simply connected open subsets of S.

Two cellular embedded graphs (S, G, f) and (S', G', f') are said isomorphic, iff there exists a graph homeomorphism $\psi : G \to G'$ and a surface homeomorphism $\phi : S \to S'$, preserving the orientation, and such that $f' \circ \psi = \phi \circ f$.

Maps are equivalence classes of cellular embedded graphs modulo isomorphisms.

This definition can be modified in order to accommodate boundaries, i.e. marked faces (a marked face is a face with a marked point in its interior and a marked oriented edge on its boundary), and we require that isomorphisms map marked faces to themselves and marked edges to themselves.

Also, we require that each boundary face has only one marked edge on its boundary such that the boundary is on the right of the marked edge. This means that there can be several marked edges on the border of a polygon, but at most one of them must be oriented such that the polygon is on the right.

And in addition, we require that unmarked faces are at least triangles and at most d-gons. The boundaries are of length at least $l_i \geq 1$.

Remark 1.1.6 One can check that those two definitions of maps are equivalent. Indeed, start from the Definition 1.1.2. For each face (each c cycle of σ_1) consider a polygon in the Euclidian plane, whose edges are labeled by the elements of B in the cycle c, ordered along the face according to the cycle c. It is easy to define a surface by gluing those polygons together along their edges, and at their vertices, by defining an atlas of charts, whose transition maps are obtained from the two permutations. That leads to an embedded graph on the surface in the sense of Definition 1.1.4.

Conversely, start from an embedded graph on a surface. Each face has the topology of a disk, and can be continuously mapped to a polygon in the Euclidian plane, whose vertices are the vertices incident to the face, and the edges are the edges incident to the face. For each such Euclidian polygon, choose a point in the interior, call it the "center" of the face. Then, for each edge of the Euclidian polygon, define a "dart" as an oriented arc going from the center of the face to a point on the edge. label all darts by distinct elements of a finite set B. Incidence relations define three permutations, corresponding to gluing darts along edges (σ_2), and sequence of darts around a face (σ_1), and sequence of darts around a vertex (σ_3). Smoothness of the surface implies that $\sigma_3 = \sigma_1 \circ \sigma_2$. This defines a labeled map in the sense of Definition 1.1.2.

1.1.3 Topology

The topology of a map, is the topology of the surface with the interior of marked faces removed, i.e. with k disks removed, it is entirely characterized by its Euler characteristics:

$$\chi = \#\text{vertices} - \#\text{edges} + \#\text{unmarked faces}.$$

It is a classical result, due to Euler, that the Euler characteristics is a topological invariant, related to the genus of the surface. We admit it here:

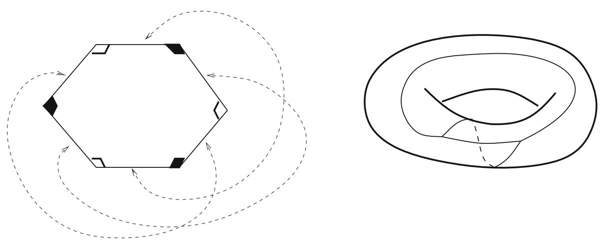
Theorem 1.1.1 (Euler) *For a connected surface of genus g, the Euler characteristic is worth:*

$$\chi = 2 - 2g - k$$

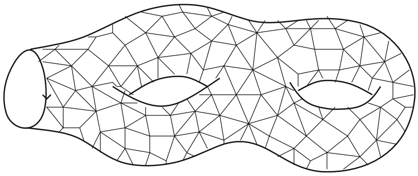
where g is the genus, i.e. the "number of handles", and for non closed surfaces, k is the number of boundaries.

If $g = 0$, i.e. if the surface has the topology of a sphere (with k disks removed), we say that it is a **planar map**.

Example of a map with no boundary ($k = 0$) and only one hexagon ($n_6 = 1$), whose opposite sides are glued together. There is one face, three edges, and two vertices (one black and one white in the picture below), i.e. $\chi = 0$, i.e. $g = 1$. This map cannot be drawn on the plane, it can be drawn on a torus:



Other example: the following map has genus $g = 2$, and $k = 1$ boundary, i.e. $\chi = -3$.



1.1.4 Symmetry Factor

An automorphism of a map defined as in Definition 1.1.2 by a set of darts $B \cup B^*$ and two permutations σ_1, σ_2, is a bijective map $\phi : B \cup B^* \to B \cup B^*$, such that $\phi|_{B^*} = \mathrm{Id}$ and

$$\sigma_1 = \phi \circ \sigma_1 \circ \phi^{-1} \quad , \quad \sigma_2 = \phi \circ \sigma_2 \circ \phi^{-1}.$$

Since $B \cup B^*$ is a finite set, there can be only a finite number of automorphisms for each map. There is always an obvious automorphism which is the identity, and automorphisms always have a group structure, subgroup of the group of permutations of $B \cup B^*$.

Definition 1.1.5 The symmetry factor $\#\,\mathrm{Aut}$ of a map is the number of its automorphisms:

$$\#\,\mathrm{Aut}$$

For generic maps, there is only one automorphism (identity), and $\#\mathrm{Aut} = 1$.

Proposition 1.1.1 *For open graphs with $k \geq 1$ boundaries, the group of automorphisms is always trivial*

$$k \geq 1 \quad \Rightarrow \quad \#\mathrm{Aut} = 1.$$

Proof Since B^* is not empty, and since $\phi|_{B^*} = \mathrm{Id}$, there is at least one element for which $\phi(x) = x$. This implies that $\sigma_1(x)$ and $\sigma_2(x)$ are also fixed by ϕ, and by an easy recursion, since the map is connected, i.e. since σ_1 and σ_2 act transitively, every element of B can be linked to x by σ_1 and σ_2 and thus is a fixed point of ϕ. This implies that $\phi = \mathrm{Id}$, and thus the only possible automorphism is the identity. □

There is another way of computing the symmetry factor.

Definition 1.1.6 For a given map $m = (B, B^*, \sigma_1, \sigma_2)$, let G_m be the group of relabelings of unmarked darts leaving faces invariants, i.e. the group of bijections $\phi : B \cup B^* \to B \cup B^*$ such that $\phi|_{B^*} = \mathrm{Id}$ and $\sigma_1 = \phi \circ \sigma_1 \circ \phi^{-1}$

$$G_m = \{\phi \mid \sigma_1 = \phi \circ \sigma_1 \circ \phi^{-1}\}.$$

The group G_m acts on the set of permutation of darts by conjugation, and in particular, $\mathrm{Aut}(m)$ is the stabilizer of σ_2 under the G_m action, i.e. the subgroup of G_m which fixes σ_2:

$$\mathrm{Aut}(m) = \{\phi \in G_m \mid \sigma_2 = \phi \circ \sigma_2 \circ \phi^{-1}\}.$$

We define the set of "gluings" as the orbit of σ_2 under the G_m action, i.e.

$$\mathrm{Gluings}(m) = \{\phi \circ \sigma_2 \circ \phi^{-1} \mid \phi \in G_m\}.$$

The following is a classical result of group theory

Lemma 1.1.1 (Orbit-Stabilizer Theorem) *Let G be a group of permutations, acting on set X, and let $x \in X$.*

The orbit of x is $G.x = \{g.x \mid g \in G\}$. The stabilizer of x is $G_x = \{g \in G \mid g.x = x\}$. The quotient G/G_x is the set of equivalence classes of G modulo G_x, i.e. an element of G/G_x can be written $g.G_x = \{g.h \mid h \in G_x\}$.

The orbit-stabilizer theorem says that

$$G.x \sim G/G_x$$

and thus

$$|G| = |G.x|.|G_x|.$$

Proof The bijection $G.x \to G/G_x$ is $g.x \mapsto g.G_x$. This map is well defined, because if there exists g and g' such that $g.x = g'.x$, this means that $g^{-1} \circ g'$ belongs to G_x and thus $g.G_x = g'.G_x$. The inverse map is also well defined for the same reason. □

The "orbit-stabilizer theorem" implies

Proposition 1.1.2 (Symmetry Factor and Gluing Number) *Let m be a map with n_3 triangles, n_4 quadrangles,... n_d d-gons,... We have:*

$$\#\mathrm{Aut} \times \#\mathrm{gluings} = \prod_{j=3}^{d} j^{n_j}\, n_j! \qquad (1.1.1)$$

where #gluings is the number of ways of obtaining the map m by gluing together the n_3 triangles, n_4 quadrangles,... n_d d-gons, and k marked faces of length $l_1, \ldots, l_k$ with marked edges.

Proof $\mathrm{Gluings}(m)$ is the orbit of σ_2 under G_m, while $\mathrm{Aut}(m)$ is the stabilizer of σ_2, therefore the orbit-stabilizer theorem implies that

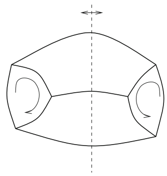
$$\#\mathrm{Aut} \times \#\mathrm{gluings} = \#G_m.$$

G_m is the conjugacy class of σ_1, it depends only on its cycles, i.e. the faces of the map. There are n_j faces of size j. Each cycle of length j can be conjugated by j possible cyclic permutations, and cycles of same length can be permuted together, therefore

$$\#G_m = \prod_{j=3}^{d} j^{n_j}\, n_j!$$

□

Example of a closed planar map with no marked edge ($k = 0$, $g = 0$, $\chi = 2$) with two triangles and three quadrangles (including the exterior one), drawn on the sphere. It has six vertices, nine edges, and five faces. Its symmetry factor is 6.



Indeed, Aut $= \mathbb{Z}_2 \times \mathbb{Z}_3$, where $\mathbb{Z}_2$ is generated by the automorphism which exchanges the two triangles (central symmetry on the figure), and $\mathbb{Z}_3$ is generated by the simultaneous rotation of the triangles (which permutes cyclically the three quadrangles).

How many gluings of two triangles and three quadrangles correspond to that map ? Chose one of the triangles, and label its three sides 1, 2, 3. There is $6 = 3!$ ways of gluing the three quadrangles to its three sides, and each quadrangle can be glued to the triangle along any of its four edges. Then, there is only three possibilities to glue the last triangle. Therefore:

$$\#\text{gluings} = 3! \times 4^3 \times 3 = 2^7 \times 3^2.$$

And thus we verify Eq. (1.1.1) on that example:

$$6 = \frac{3^2\, 2! \times 4^3\, 3!}{2^7 \times 3^2}.$$

1.2 Generating Functions for Counting Maps

Our goal is to count the number of maps having a fixed topology, and fixed numbers of polygons of given sizes, fixed number of boundaries,... Those numbers of maps can be collectively encoded in some generating functions. Let us define them.

1.2.1 Maps with Fixed Number of Vertices

Definition 1.2.1 Let $\mathbb{M}_k^{(g)}(v)$, be the set of connected maps of genus g, (with unmarked faces of degree ≥ 3 and $\leq d$), and k boundaries (marked faces of degree ≥ 1 with one marked edge), and such that the total number of vertices is v. In addition we define $\mathbb{M}_1^{(0)}(1) = \{.\}$ i.e. we have defined a virtual planar rooted map with 1-vertex to be a point, it has no faces, and its unique boundary has length $l_1 = 0$.

Theorem 1.2.1 *$\mathbb{M}_k^{(g)}(v)$ is a finite set.*

Proof Indeed, for any map $m \in \mathbb{M}_k^{(g)}(v)$, write its Euler characteristics:

$$2 - 2g = k + \overbrace{\sum_{i=3}^{d} n_i(m)}^{\#\text{ of faces}} - e(m) + v$$

where $n_i(m)$ is the number of unmarked faces of degree i, and where $e(m)$ is the number of edges of m, that is, half the number of half-edges:

$$2\, e(m) = \sum_{i=1}^{k} l_i(m) + \sum_{i=3}^{d} i\, n_i(m)$$

thus we have:

$$v + 2g - 2 + k = \frac{1}{2} \sum_{i=1}^{k} l_i(m) + \frac{1}{2} \sum_{i=3}^{d} (i-2) n_i(m) \tag{1.2.1}$$

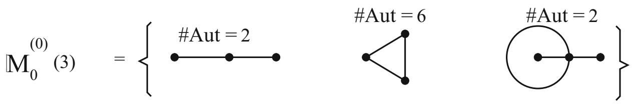
and since $i \geq 3$ in the last sum we have $i-2 \geq 1$ and therefore we find the inequality:

$$v + 2g - 2 + k \geq \frac{1}{2} \sum_{i=1}^{k} l_i(m) + \frac{1}{2} \sum_{i=3}^{d} n_i(m)$$

in particular, this inequality implies that n_i and l_i are bounded, and therefore there is a finite number of such maps. □

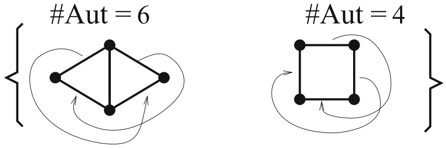
Examples

- planar maps with no marked faces and three vertices:



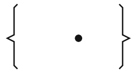
- genus 1 maps with no marked faces and one vertex

$\mathbb{M}_0^{(1)}(1) =$



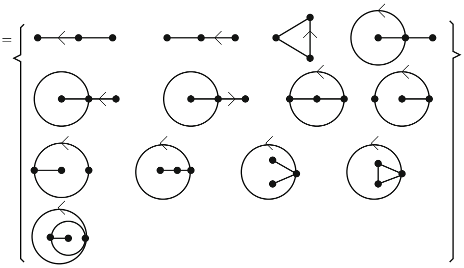
- planar maps with one marked face (the marked face is on the right of the marked edge, i.e. on the exterior):

$\mathbb{M}_1^{(0)}(1) =$



$\mathbb{M}_1^{(0)}(2) =$

$\mathbb{M}_1^{(0)}(3) =$



Definition 1.2.2 We define the generating function of maps of genus g with k boundaries as the formal power series in t:

$$W_k^{(g)}(x_1, \ldots, x_k; t_3, \ldots, t_d; t)$$
$$= \sum_{v=1}^{\infty} t^v \sum_{m \in \mathbb{M}_k^{(g)}(v)} \frac{t_3^{n_3(m)}\, t_4^{n_4(m)} \ldots\, t_d^{n_d(m)}}{x_1^{1+l_1(m)}\, x_2^{1+l_2(m)} \ldots\, x_k^{1+l_k(m)}} \frac{1}{\#\mathrm{Aut}(m)}.$$

$W_k^{(g)}$ is a formal series of t whose coefficients are rational polynomials of the t_j's and $1/x_j$'s:

$$W_k^{(g)} \in \mathbb{Q}[1/x_1, \ldots, 1/x_k, t_3, t_4, \ldots]\,[[t]].$$

Notational remark: In the rest of this book, we shall write only the dependence in the x_i's explicitly, whereas the dependence in $t, t_3, \ldots, t_d$, will be implicitly assumed, by convention we write:

$$W_k^{(g)}(x_1, \ldots, x_k; t_3, \ldots, t_d; t) \equiv W_k^{(g)}(x_1, \ldots, x_k).$$

By convention we also denote:

$$F_g \equiv W_0^{(g)}.$$

Examples (See the Pictures Just Before Definition 1.2.2)

$$W_1^{(0)}(x) = \frac{t}{x} + t^2\left(\frac{1}{x^3} + \frac{t_3}{x^2}\right) + t^3\left(\frac{2}{x^5} + \frac{4t_3}{x^4} + \frac{t_3^2}{x^3} + \frac{2t_4}{x^3} + \frac{2t_5}{x^2} + \frac{2t_3t_4}{x^2}\right) + O(t^4)$$

$$F_0 = W_0^{(0)} = t^3\left(\frac{1}{2}t_4 + \left(\frac{1}{6} + \frac{1}{2}\right)t_3^2\right) + O(t^4)$$

$$F_1 = W_0^{(1)} = t\left(\frac{1}{6}t_3^2 + \frac{1}{4}t_4\right) + O(t^2).$$

Remark 1.2.1 As usual in combinatorics, $W_k^{(g)}$ is a generating function, that is a **formal power series** in t, or also called an "asymptotic series". It is meaningful even when it is not convergent. It is nothing but a convenient short hand notation for the collection of all coefficients. However, it turns out (proved in Chap. 3) that each $W_k^{(g)}$ happens to be an algebraic function of t, and therefore it has a finite radius of convergency. The behaviour in the vicinity of non-analytical points (i.e. at the boundary of the convergency disk) can be used to find the asymptotic numbers of large maps (see Chap. 5).

Remark 1.2.2 Notice that for each v, the sum over $m \in \mathbb{M}_k^{(g)}(v)$ is finite, and therefore the coefficient of t^v in $W_k^{(g)}(x_1, \ldots, x_k)$ is a polynomial in the $1/x_i$'s.

1.2.2 Fixed Boundary Lengths

If we wish to compute generating functions for maps with fixed boundary lengths l_i, we simply pick the coefficient of $1/x_i^{1+l_i}$ by taking a residue. We define:

Definition 1.2.3 The following is the generating function counting maps of genus g, and with k boundaries of respective lengths $l_1, \ldots, l_k$:

$$\begin{aligned}\mathcal{T}_{l_1,\ldots,l_k}^{(g)} &= (-1)^k \operatorname*{Res}_{x_1\to\infty} \cdots \operatorname*{Res}_{x_k\to\infty} x_1^{l_1} \cdots x_k^{l_k}\, W_k^{(g)}(x_1, \ldots, x_k)\, dx_1 \ldots dx_k \\ &= \sum_{v=1}^{\infty} t^v \sum_{m\in\mathbb{M}_k^{(g)}(v)} \frac{t_3^{n_3(m)}\, t_4^{n_4(m)} \ldots\, t_d^{n_d(m)}}{\#\mathrm{Aut}(m)} \prod_{i=1}^{k} \delta_{l_i,l_i(m)}. \end{aligned} \tag{1.2.2}$$

They belong to

$$\mathcal{T}^{(g)}_{l_1,\dots,l_k} \in \mathbb{Q}[t_3, t_4, \dots, t_d][[t]].$$

Remark 1.2.3 (Residues) The residue at $x = \alpha$ of a function f, picks the coefficient of the simple pole at $x = \alpha$, that is the coefficient of $(x - \alpha)^{-1}$ in the Laurent expansion of f in the vicinity of α. For instance the residue at $x = 0$ of a Laurent series $f(x)$ is

$$f(x) = \sum_k f_k\, x^k \quad \Rightarrow \quad \operatorname*{Res}_{x\to 0} f = f_{-1}.$$

The Residue can also be computed by Cauchy's theorem: a contour integral along a small circle $\mathcal{C}_\alpha$ encircling the pole α counterclockwise (small circle means small enough so that it doesn't encircle any other singularity of the integrand):

$$\operatorname*{Res}_{x\to\alpha} f = \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}_\alpha} f(x)\, dx\,.$$

This shows that a **Residue is an integral**. The proper notation for residues should include the integration measure dx:

$$\operatorname*{Res}_{x\to\alpha} f(x)\, dx.$$

The notion of Residue applies to differential forms, not functions. In the literature, one often writes $\operatorname{Res} f(x)$, omitting the dx. This abuse of notation can be done only when there is no ambiguity on the integration variable, and the dx is implicitly assumed. It is particularly important to write the dx, when one wants to use changes of variables $x \to z$, and thus $dx = \frac{dx}{dz}\, dz$. Since changes of variables will play an important role in this book, we shall always write residues of differential forms.

Remark 1.2.4 (Residue at ∞) When changing variable $x \to 1/x$, we have $d(1/x) = -\, dx/x^2$, and thus residues at ∞ come with a $-$ sign:

$$\operatorname*{Res}_{x\to\infty} \frac{1}{x}\, dx = -1.$$

This is why we have the coefficients $(-1)^k$ in Eq. (1.2.2).

Remark 1.2.5 (Differential Forms) One sees, for example from Eq. (1.2.2), that $W^{(g)}_k(x_1, \dots, x_k)$ will always be used to compute residues, i.e. integrals, and in fact, it will always appear together with $dx_1 \dots dx_k$ as in $W^{(g)}_k(x_1, \dots, x_k)\, dx_1 \dots dx_k$. The true nature of $W^{(g)}_k(x_1, \dots, x_k)$, is to be a differential form, and—anticipating on Chap. 7—we define the fundamental intrinsic object :

$$\omega^{(g)}_k = W^{(g)}_k(x_1, \dots, x_k)\, dx_1 \dots dx_k.$$

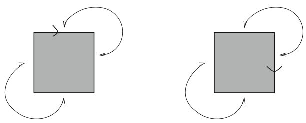
In this notation, $dx_1 \dots dx_k$ is the tensor product of 1-forms $dx_1 \otimes \cdots \otimes dx_k$, it must not be confused with an exterior product $dx_1 \wedge \cdots \wedge dx_k$. It is symmetric in all dx_i's, not antisymmetric. In other words, $\omega_k^{(g)}$ is a linear combination of 1-form of x_1, whose coefficients are linear combinations of 1-form of x_2, whose coefficient is a 1-form of $x_3, \dots$.

Examples

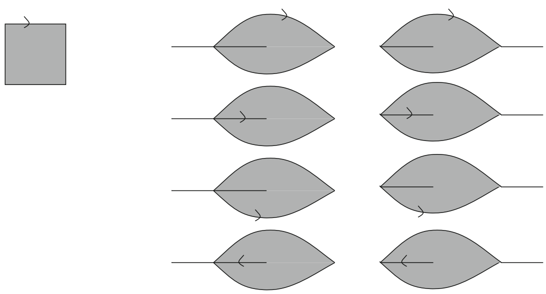
- If we choose all $t_j = 0$ except $t_4 \neq 0$, we count only **quadrangulations**, and $\mathcal{T}_4^{(g)}$ is the number of rooted quadrangulations of genus g, where all faces (including the one on the right of the marked edge) are quadrangles. The total number of faces is $n = n_4 + 1$, and [thanks to Eq. (1.2.1)] the number of vertices is $v = n+2-2g$. In the 60's, Tutte (this is the famous Tutte's formula [84, 85]) computed that (and we shall prove it in Chap. 3):

$$\mathcal{T}_4^{(0)} = t^3 \sum_{n=1}^{\infty} (tt_4)^{n-1} \frac{2\ (2n)!\, 3^n}{n!\,(n+2)!} = 2t^3 + 9t^4 t_4 + 54 t^5 t_4^2 + \dots$$

  In this formula, the coefficient of $t_4^0 t^3$ is 2. The two maps of genus 0 with one marked quadrangle, three vertices and no unmarked quadrangles contributing to the term $2t^3$, are



  The nine maps of genus 0 with one marked quadrangle, four vertices and one unmarked quadrangle, contributing to the term $9t^4 t_4$, are (where one face is the exterior face, and the marked face is the one on the right of the oriented marked edge):



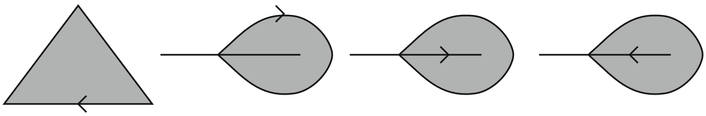
- Similarly, if we choose all $t_j = 0$ except $t_3 \neq 0$, we count only **triangulations**, and $\mathcal{T}_3^{(g)}$ is the number of rooted triangulations of genus g, where all faces (including the one on the right of the marked edge) are triangles. According to Eq. (1.2.1) with $k = 1$ and $l_1 = 3$, the total number of faces is $n_3 + 1 = 2(v + 2g - 2)$ which is always even, and which we denote

$$n = \frac{n_3 + 1}{2} = v + 2g - 2.$$

In Chap. 3 for genus $g = 0$, we shall find:

$$\mathcal{T}_3^{(0)} = t^{5/2} \sum_{n=1}^{\infty} (t_3 \sqrt{t})^{2n-1} \, 2^{3n+1} \, \frac{\Gamma(\frac{3n}{2} + 1)}{(n+2)! \, \Gamma(\frac{n}{2} + 1)} = 4t^3 t_3 + 32 t^4 t_3^2 + \dots.$$

which was also computed by Tutte [84, 85]. The four maps of genus 0 with one marked triangle, three vertices and one unmarked triangle, contributing to the term $4t^3 t_3$, are (where one face is the exterior face, and the marked face is the one on the right of the oriented marked edge):



1.2.3 Redundancy of the Parameters

One can remark that the number of vertices is redundant, because at fixed genus and boundary lengths, the number of vertices can be deduced from the numbers of polygonal faces. In other words the parameter t is redundant with the t_j's and x_j's.

Indeed, using Eq. 1.2.1: $v - (2 - 2g - k) = \frac{1}{2} \sum_{i=1}^{k} l_i + \frac{1}{2} \sum_{i=3}^{d} (i-2) n_i$, we may rewrite:

$$\begin{aligned} & t^{2g-2+k} \, W_k^{(g)}(x_1, \dots, x_k) dx_1 \dots dx_k \\ &= \sum_{v=1}^{\infty} \sum_{m \in \mathbb{M}_k^{(g)}(v)} \frac{\prod_{i=3}^{d} (t_i \, t^{\frac{i}{2}-1})^{n_i(m)}}{\prod_{i=1}^{k} (x_i / \sqrt{t})^{l_i(m)}} \, \frac{1}{\#\mathrm{Aut}(m)} \, \prod_{i=1}^{k} \frac{dx_i}{x_i}. \end{aligned}$$

This means that we can redefine:

$$t_i \to t_i \, t^{\frac{i}{2}-1} \quad , \quad x_i \to x_i / \sqrt{t} \quad , \quad t \to 1$$

and work with $t = 1$.

In other words, $W_k^{(g)}$ was defined as a formal power series in a single variable t (coupled to the number of vertices v), but we may also view it as a formal multiple power series in each t_i and x_i, coupled to the number of polygons n_i and degrees of boundaries l_i. We could chose to define:

$$W_k^{(g)} \in \mathbb{Q}[[1/x_1, \dots, 1/x_k, t_3, t_4, \dots]].$$

However, although t is a redundant parameter, we find more convenient to work with formal series in only one formal variable t than with multiple formal variables, and we shall keep t as the sole formal variable throughout this book.

1.2.4 All Genus

It is convenient to define the generating function of maps regardless of their genus. Thanks to Eq. (1.2.1), the number of vertices of a map is always such that $v + 2g - 2 + k \geq 0$, and this allows to define the following formal power series of t:

$$W_k = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g-k} W_k^{(g)}. \tag{1.2.3}$$

We emphasize that the meaning of this definition, is an equality between formal power series of t, i.e. an equality between the coefficients of terms with equal powers of t. To any order in t, the sum over g is finite. Therefore the sum over g is **NOT a large** N **expansion**, it is a small t expansion.

Moreover, since any $\mathbb{M}_k^{(g)}(v)$ is a finite set, there is a maximal genus $g \leq g_{\max(v)}$ for each v, and the coefficients of W_k in powers of t are polynomials in $1/N$:

$$W_k \in \mathbb{Q}[1/x_1, \dots, 1/x_k, t_3, t_4, \dots, 1/N]\,[[t]].$$

Similarly, for closed maps $k = 0$, we note $W_0^{(g)} = F_g$, and we can define the all genus generating function of closed maps:

$$F = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g} F_g. \tag{1.2.4}$$

Remark 1.2.6 We will see later in this book, that each series $W_k^{(g)}$ has a finite radius of convergency, and is in fact an algebraic function of t. But the all genus generating function W_k is not algebraic, and may have a vanishing radius of convergency.

1.2.5 Non Connected Maps

When we have a formal generating series counting disconnected objects multiplicatively, it is well known that the log is the generating function which counts only the connected objects, see [46]. If we denote $Z_N(t_3,\dots,t_d;t)$ the generating function for closed maps not necessarily connected

$$Z_N(t_3,t_4,\dots,t_d;t) = 1+\sum_{v=1}^{\infty} t^v \sum_{m=\text{map with } v \text{ vertices}} (N/t)^{\chi(m)}\, \frac{t_3^{n_3(m)}\, t_4^{n_4(m)}\dots t_d^{n_d(m)}}{\#\text{Aut}(m)}$$

where $\chi(m) = \#\text{vertices} - \#\text{edges} + \#\text{faces}$, we have:

$$\ln\left(Z_N(t_3,t_4,\dots,t_d;t)\right) = F(t_3,\dots,t_d;t;N) \tag{1.2.5}$$

where again this equality is to be taken as an equality between formal series in $\mathbb{Q}[t_3,\dots,t_d,N,1/N][[t]]$, i.e. equality between the coefficients in the small t expansion.

For open maps with $k \geq 1$ boundaries, there are several ways of obtaining disconnected surfaces, because each disconnected piece may carry either no boundary, or subsets of the set of boundaries. The generating functions of connected objects are **cumulants** of the non-connected ones.

Let $W_k^*(x_1,\dots,x_k)$ be the generating function of not-necessarily connected maps of all genus. We have:

$$W_1^*(x_1) = Z_N\ W_1(x_1)$$
$$W_2^*(x_1,x_2) = Z_N\ \left(W_2(x_1,x_2) + W_1(x_1)\,W_1(x_2)\right)$$
$$W_3^*(x_1,x_2,x_3) = Z_N\ \left(W_3(x_1,x_2) + W_1(x_1)\,W_2(x_2,x_3) + W_1(x_2)\,W_2(x_1,x_3) \right.$$
$$\left. +W_1(x_3)\,W_2(x_1,x_2) + W_1(x_1)\,W_1(x_2)\,W_1(x_3)\right)$$

and so on, if we note $K = \{x_1,\dots,x_k\}$:

$$W_k^*(K) = Z_N \sum_{\mu\vdash K,\ \mu=(J_1,J_2,\dots,J_n)} \prod_{i=1}^{n} W_{\#J_i}(J_i)$$

where we sum over all possible partitions μ of K.

The converse is called **cumulants**, or sometimes **connected parts**:

$$W_1(x_1) = \frac{1}{Z_N}\ W_1^*(x_1)$$
$$W_2(x_1,x_2) = \frac{W_2^*(x_1,x_2)}{Z_N} - \frac{W_1^*(x_1)}{Z_N}\,\frac{W_1^*(x_2)}{Z_N}$$

$$W_3(x_1,x_2,x_3) = \frac{W_3^*(x_1,x_2,x_3)}{Z_N} - \frac{W_1^*(x_1)}{Z_N}\,\frac{W_2^*(x_2,x_3)}{Z_N} - \frac{W_1^*(x_2)}{Z_N}\,\frac{W_2^*(x_1,x_3)}{Z_N}$$
$$-\frac{W_1^*(x_3)}{Z_N}\,\frac{W_2^*(x_1,x_2)}{Z_N} + 2\,\frac{W_1^*(x_1)}{Z_N}\,\frac{W_1^*(x_2)}{Z_N}\,\frac{W_1^*(x_3)}{Z_N}$$

and so on…

1.2.6 Rooted Maps: One Boundary

The case $k = 1$, i.e. one boundary plays a special role.

A map with one boundary, is also, by definition, a map with one oriented marked edge, it is also called a **rooted map**. The marked face is the face on the right of the oriented marked edge.

A reason of the special role of maps with one boundary, is that there is only one automorphism (the identity) which can conserve the map and the oriented marked edge. Therefore if $k = 1$ we have

$$k = 1 \quad \Rightarrow \quad \#\mathrm{Aut} = 1$$

A planar map with one boundary is called a **"disk"**.

1.3 Tutte's Equations

1.3.1 Planar Case: The Disk

The Canadian mathematician Tutte discovered a combinatoric recursive equation in 1963 [84], for counting planar maps with one boundary, i.e. rooted planar maps, or equivalently disks.

The idea is the following. A planar map with one boundary of length $l + 1$, is in fact a planar map with one marked face of degree $l + 1$, with one marked oriented edge on it. Let us draw the map on the plane, such that ∞ is in the marked face, i.e. the marked face is the exterior. The marked edge, separates two faces (not necessarily distinct).

If one removes the marked edge, two situations may occur:

- either the face on the other side of the marked edge was the same, thus if we remove the edge, we get two planar maps, each having one boundary (we mark the edges adjacent to the removed edge), one has a boundary of length j, the other $l-1-j$, for some j such that $0 \le j \le l-1$.
- either the face on the other side is not the same, and thus it is an unmarked face of some degree j such that $3 \le j \le d$. When we remove the marked edge, the map

remains connected, and we get a new map, with a boundary of length $l + j - 1$ (the new marked edge is the one adjacent to the removed edge). Removing the edge was thus equivalent to removing a j-gone, which has weight t_j.

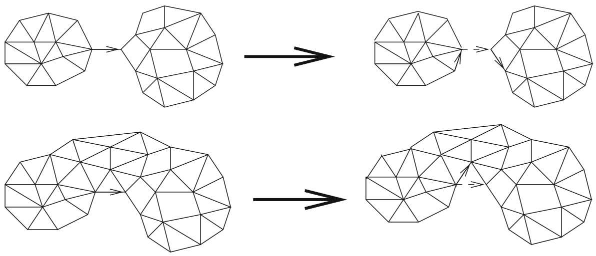
It is easy to see that this procedure of removing the marked edge is a bijection

$$\mathbb{M}^{(0)}_{1;l+1} \to \cup_{j=0}^{l-1} \mathbb{M}^{(0)}_{1;j} \times \mathbb{M}^{(0)}_{1;l-1-j} + \cup_{j=3}^{d} \mathbb{M}^{(0)}_{1;l+j-1}$$

and thus Tutte's equation follows:

$$\boxed{\mathcal{T}^{(0)}_{l+1} = \sum_{j=0}^{l-1} \mathcal{T}^{(0)}_j \mathcal{T}^{(0)}_{l-1-j} + \sum_{j=3}^{d} t_j \, \mathcal{T}^{(0)}_{l+j-1}} \tag{1.3.1}$$

Tutte's proof is illustrated as follows:



1.3.2 Higher Genus Tutte Equations

A similar recursive equation can be found for higher genus or higher number of boundaries.

Consider a map of genus g, with k boundaries of respective lengths $l_1 + 1, l_2, \ldots, l_k$, and let us denote collectively

$$K = \{l_2, \ldots, l_k\}$$

Then we erase the marked edge of the first boundary. Several mutually exclusive possibilities can occur:

- the marked edge separates the marked face with some unmarked face (let us say a j-gone with $3 \leq j \leq d$), and removing that edge is equivalent to removing a j-gone (with weight t_j), and we thus get a map of genus g with the same number of boundaries, and the length of the first boundary is now $l_1 + j - 1$.
- the marked edge separates two distinct marked faces (face 1 and face m with $2 \leq m \leq k$,), thus the marked edge of the first boundary is one of the l_m edges of

the mth boundary. We thus get a map of genus g with $k-1$ boundaries. the other $k-2$ boundaries remain unchanged, and there is now one boundary of length l_1+l_m-1.

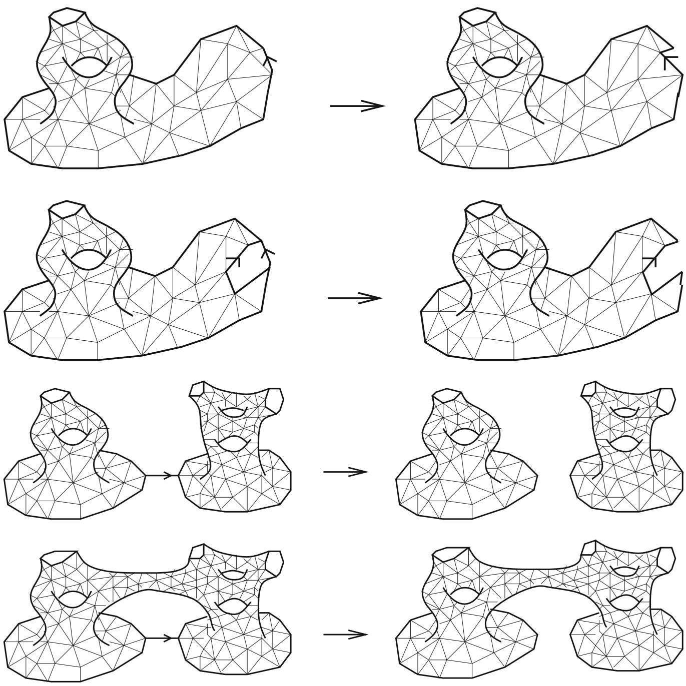
- the same marked face lies on both sides of the marked edge, therefore by removing it, we disconnect the boundary. Two cases can occur: either the map itself gets disconnected into two maps of genus h and $g-h$, one having $1+\#J$ boundaries of lengths (j,J), where J is a subset of K, and the other map having $k-\#J$ boundaries of lengths $(l_1-1-j, K\setminus J)$, or the map remains connected because there was a handle connecting the two sides, and thus by removing the marked edge, we get a map of genus $g-1$, with $k+1$ boundaries of lengths (j, l_1-j-1, K).

Again, this procedure is (up to the symmetry factors) bijective, and all those possibilities correspond to the following recursive equation:

$$\sum_{j=0}^{l_1-1}\Big[\sum_{h=0}^{g}\sum_{J\subset K}\mathcal{T}^{(h)}_{j,J}\mathcal{T}^{(g-h)}_{l_1-1-j,K\setminus J}+\mathcal{T}^{(g-1)}_{j,l_1-1-j,K}\Big]+\sum_{m=2}^{k} l_m\,\mathcal{T}^{(g)}_{l_m+l_1-1,K\setminus\{l_m\}}$$
$$=\mathcal{T}^{(g)}_{l_1+1,K}-\sum_{j=3}^{d} t_j\mathcal{T}^{(g)}_{l_1+j-1,K} \tag{1.3.2}$$

which we call "Loop equation" or "higher genus Tutte's equation".

This equation is illustrated as follows:



Here, we have presented only an intuitive derivation, and we present a more rigorous derivation in Chap. 2, with a very different technique, called loop equations for formal matrix integrals.

1.4 Exercises

Exercise 1 Count all connected quadrangulations with $n_4 = 1$ and $n_4 = 2$ quadrangles, count them with their symmetry factors and according to their topology.

Answer: There is one planar quadrangulation with $n_4 = 1$ quadrangle, and it has symmetry factor 2, and one quadrangulation of genus $g = 1$ with $n_4 = 1$ quadrangle, and it has symmetry factor 4.

There are two planar quadrangulations with $n_4 = 2$ quadrangles, one has symmetry factor 8, one has symmetry factor 1. And there are four quadrangulations of genus $g = 1$ with $n_4 = 2$ quadrangles, one has symmetry factor 8, one has symmetry factor 4, one has symmetry factor 2, one has symmetry factor 1.

This can be summarized as:

$$F = t\,t_4 \left(\frac{N^2}{2} + \frac{1}{4}\right) + t^2\,t_4^2 \left(N^2 \left(1 + \frac{1}{8}\right) + \frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1\right) + O(t_4^3).$$

Exercise 2 Find all planar maps with one marked face of arbitrary length l, and whose unmarked faces are only pentagons, and with up to five vertices.

Check that

$$W_1^{(0)} = \frac{t}{x} + \frac{t^2}{x^3} + t^3 \left(\frac{2}{x^5} + \frac{2t_5}{x^2}\right) + t^4 \left(\frac{5}{x^7} + \frac{9t_5}{x^4}\right) + t^5 \left(\frac{12}{x^9} + \frac{17t_5}{x^6} + \frac{3t_5^2}{x^3}\right) + O(t^6)$$

Chapter 2
Formal Matrix Integrals

In this chapter we introduce the notion of a formal matrix integral, which is very useful for combinatorics, as it turns out to be identical to the generating function of maps of Chap. 1.

A formal integral is a formal series (an asymptotic series) whose coefficients are Gaussian integrals, it is not necessarily a convergent series (in fact in our case it is always not convergent, it has a vanishing radius of convergency).

Wick's theorem [86] gives a method to compute Gaussian matrix integrals in a combinatorial way, it relates formal matrix integrals to generating functions for maps.

The relationship between formal matrix integrals and maps, was first noticed in 1974 by 't Hooft (1999 physics Nobel prize) in the context of the study of strong nuclear interactions [48], and then really introduced as a tool for studying maps by physicists Brezin-Itzykson-Parisi-Zuber in 1978 [74].

2.1 Definition of a Formal Matrix Integral

2.1.1 Introductory Example: 1-Matrix Model and Quartic Potential

Consider the following polynomial moment of a gaussian integral over the set H_N of hermitian $N \times N$ matrices:

$$A_k(N) = \frac{N^k}{k!\,4^k} \int_{H_N} dM \ \ (\operatorname{Tr} M^4)^k \ \ \mathrm{e}^{-N \operatorname{Tr} \frac{M^2}{2}}$$

B. Eynard, *Counting Surfaces*, Progress in Mathematical Physics 70,
DOI 10.1007/978-3-7643-8797-6_2

where M is a $N \times N$ hermitian matrix, and dM the $U(N)$ invariant Lebesgue measure on H_N

$$dM = \frac{1}{2^{N/2}\,(\pi/N)^{N^2/2}} \prod_{i=1}^{N} dM_{ii} \,\prod_{i<j} d\mathrm{Re}M_{ij}\; d\mathrm{Im}M_{ij}$$

normalized so that $\int dM\ \mathrm{e}^{-N\,\mathrm{Tr}\,\frac{M^2}{2}} = 1$, i.e. $A_0(N) = 1$.

We shall see below that $A_k(N)$ is a polynomial in N and $1/N$, so it can be analytically continued to any $N \in \mathbb{C}^*$.

With the sequence $A_k(N)$, $k = 0, 1, 2, \ldots, \infty$, we define a **formal power series** in powers of a variable which we choose to call t_4 because it is associated to $\mathrm{Tr}\, M^4$ (later we shall associate t_j to $\mathrm{Tr}\, M^j$):

Definition 2.1.1 We define the "formal matrix integral"

$$Z_N(t_4) = \sum_{k=0}^{\infty} t_4^k\, A_k(N) = \sum_{k=0}^{\infty} t_4^k\, \frac{N^k}{k!\,4^k} \int_{H_N} dM\ \ (\mathrm{Tr}\, M^4)^k\ \ \mathrm{e}^{-N\,\mathrm{Tr}\,\frac{M^2}{2}}$$

We shall denote it:

$$Z_N(t_4) = \int_{\text{formal}} dM\ \mathrm{e}^{-N\,\mathrm{Tr}\,(\frac{M^2}{2} - t_4 \frac{M^4}{4})}.$$

(as if we could exchange the order of sum and integral).

$Z_N(t_4)$ is well defined as a **formal power series** in t_4

$$Z_N(t_4) \in \mathbb{Q}[N, 1/N][[t_4]],$$

in other words, $Z_N(t_4)$ is **nothing but a notation** which summarizes all the coefficients $A_k(N)$ in only one symbol $Z_N(t_4)$. This means that every time we are going to write properties or equations for $Z_N(t_4)$, we actually mean properties of the coefficients A_k of the t_4 expansion. Writing equations for $Z_N(t_4)$ is merely a shorter way of writing equations for $A_k(N)\ \ \forall k$.

We are never going to consider $Z_N(t_4)$ as a usual function of t_4, and in fact, for $t_4 > 0$ the series $Z_N(t_4)$ is never convergent (in the Borel sense for instance).

2.1.2 Comparison with Convergent Integrals

The definition of a formal matrix integral $Z_N(t_4)$ is not to be confused with the hermitean **convergent matrix integral**:

$$Z_{\text{conv}}(t_4, N) = \int_{H_N} dM \; \mathrm{e}^{-N \operatorname{Tr}\left(\frac{M^2}{2} - t_4 \frac{M^4}{4}\right)}$$
$$= \int_{H_N} \sum_{k=0}^{\infty} t_4^k \; \frac{N^k}{k!\, 4^k} dM \mathrm{e}^{-N \operatorname{Tr} \frac{M^2}{2}} \; (\operatorname{Tr} M^4)^k.$$

One should notice that $Z_{\text{conv}}(t_4, N)$ is well defined for $t_4 < 0$, and not for $t_4 > 0$.

The existence and nature of large N asymptotics of hermitean convergent matrix integrals is a difficult problem which has been solved in a few cases, and which remains an open question in many cases at the time this book is being written (the case of the 2-matrix model with complex potentials for instance is unsolved).

The only difference in the definition of $Z_N(t_4)$ and $Z_{\text{conv}}(t_4, N)$, is that the order of the sum over k and the integral over H_N has been exchanged. In general, the sum and the integral don't commute, and in general:

$$Z_N(t_4) \;\neq\; Z_{\text{conv}}(t_4, N)$$

in other words:

$$\sum_{k=0}^{\infty} \int_{H_N} \; t_4^k \; \frac{N^k}{k!\, 4^k} dM \mathrm{e}^{-N \operatorname{Tr} \frac{M^2}{2}} \; (\operatorname{Tr} M^4)^k \neq \int_{H_N} \sum_{k=0}^{\infty} \; t_4^k \; \frac{N^k}{k!\, 4^k} dM \mathrm{e}^{-N \operatorname{Tr} \frac{M^2}{2}} \; (\operatorname{Tr} M^4)^k$$

the left hand side is a divergent asymptotic series, and it has a meaning either as a formal series (what we shall consider from now on in this book), or for instance it can be Borel resummed for $t_4 < 0$. However, those two definitions of a matrix integral differ even after Borel resummation and analytical continuation from $t_4 > 0$ (which is the interesting regime for combinatorics) to $t_4 < 0$ (where $Z_{\text{conv}}(t_4)$ is well defined) they do not necessarily coincide.

Convergent matrix integrals are not the topic of this book, and readers interested in asymptotic properties of large Matrix integrals, can refer to the many books devoted to it for instance [16, 25, 26, 39, 44, 63, 81].

2.1.3 Formal Integrals, General Case

So far, we have studied the example of a formal matrix integral with quartic potential, now let us give the general definition of a formal integral of the form:

$$\int_{\text{formal}} \mathrm{e}^{-\frac{N}{t} \operatorname{Tr} V(M)} \; dM.$$

The idea is to expand (Taylor series) the exponential of the non-quadratic part of $V(M)$, and write the integral as an infinite sum of polynomial moments of a gaussian integral, and **then invert the integral and the summation**.

Let

$$V(M) = \frac{M^2}{2} - \sum_{j=3}^{d} \frac{t_j}{j} M^j$$

be called the potential, then we define the following polynomial moment of a Gaussian integral:

$$A_k = \frac{1}{k!} \frac{N^k}{t^k} \int_{H_N} dM \; \mathrm{e}^{-\frac{N}{t} \operatorname{Tr} \frac{M^2}{2}} \left(\sum_{j=3}^{d} \frac{t_j}{j} \operatorname{Tr} M^j \right)^k .$$

Lemma 2.1.1 *A_k is a polynomial of t such that:*

$$A_k = \sum_{m=k/2}^{[(d-2)k/2]} A_{k,m} \, t^m .$$

Proof A monomial moment of a Gaussian integral vanishes if the degree of the monomial is odd, and is proportional to t to the power half the degree, if the degree is even. The polynomial $(\sum_{j=3}^{d} \frac{t_j}{j} \operatorname{Tr} M^j)^k$ can be decomposed into a finite sum of monomials in M of the form:

$$\prod_{j=3}^{d} (\operatorname{Tr} M^j)^{n_j} \qquad , \qquad \sum_{j=3}^{d} n_j = k$$

i.e. of degree $\sum_j j n_j$. Therefore such a term contributes to A_k with a power of t^m equal to:

$$m = -k + \frac{1}{2} \sum_{j=3}^{d} j n_j = \frac{1}{2} \sum_{j=3}^{d} (j-2) n_j \geq \frac{1}{2} \sum_{j=3}^{d} n_j = \frac{k}{2}.$$

The upper bound $m \leq (d-2)k/2$ is easily obtained because $j \leq d$ and $n_j \leq k \; \forall j$. □

This Lemma allows to define:

$$\tilde{A}_m = \sum_{k=0}^{2m} A_{k,m}.$$

Definition 2.1.2 The formal integral is the formal power series in t:

$$Z(t) \stackrel{\text{def}}{=} \sum_{m=0}^{\infty} t^m \tilde{A}_m.$$

It is denoted

$$Z(t) \stackrel{\text{notation}}{=} \int_{\text{formal}} e^{-\frac{N}{t} \operatorname{Tr} V(M)} \, dM.$$

Remark 2.1.1 $Z(t)$ can also be viewed as a formal power series in each t_j with $3 \leq j \leq d$. We may choose $t = 1$ and expand in powers of t_3 or $t_4 \ldots$, as we did for the quartic potential. It is clear that t can be absorbed by a redefinition $M \to \sqrt{t}M$ and $t_j \to t^{\frac{j}{2}-1} t_j$, exactly like in Sect. 1.2.3 of Chap. 1.

Remark 2.1.2 The formal integral and the convergent matrix integral differ by the order of integration and sum. In general the two operations do not commute, and the formal integral and the convergent integral are different (see Sect. 2.1.2 above):

$$\int_{\text{formal}} e^{-\frac{N}{t} \operatorname{Tr} V(M)} \, dM \neq \int_{H_N} e^{-\frac{N}{t} \operatorname{Tr} V(M)} \, dM.$$

Remark 2.1.3 In combinatorics, the times t_j's are Boltzman weights for the j-gons, and thus we shall mostly be interested in the $t_j \geq 0$. Convergent integrals converge for instance when d is even and $t_d < 0$. This shows again that formal and convergent matrix integrals are defined in very different domains, and don't have to coincide.

Remark 2.1.4 We shall see below in Sect. 2.2.2, that each $A_{k,m}$ is a Laurent polynomial of N:

$$A_{k,m} = \sum_{g=-g_{\min}(k,m)}^{g_{\max}(k,m)} N^g A_{k,m}^{(g)}$$

so that each $\tilde{A}_m$ is also a Laurent polynomial of N, and thus, to a given order t^m, the formal integral is a Laurent **polynomial of** N, and thus a formal matrix integral **always has a** $1/N$ **expansion**.

In other words, the question of a $1/N$ expansion is trivial for formal integrals, whereas it is a very difficult question for convergent integrals (mostly unsolved for multi-matrix integrals with complex potentials).

Remark 2.1.5 Most of physicist's works in so-called **"2d-quantum-gravity"** are actually using that "formal" definition of a matrix integral (in fact almost all works in quantum field theory after Feynman's works, used formal integrals). Most of the works initiated by Brezin-Itzykson-Parisi-Zuber [74] in 1978 assume the formal definition of matrix integrals, and are correct and rigorous only with that definition, they are often wrong if one uses convergent hermitian matrix integrals instead. See [31, 40, 42, 54]

2.2 Wick's Theorem and Combinatorics

2.2.1 Generalities About Wick's Theorem

Wick's theorem is a very useful theorem for combinatorics. It gives a combinatoric way of computing Gaussian expectation values, or conversely, it gives an algebraic and analytical way of enumerating graphs.

Let A be a positive definite $n \times n$ symmetric matrix, and let $x_1, \dots, x_n$ be n Gaussian random variables, with a probability measure:

$$d\mu(x_1, \dots, x_n) = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \, \mathrm{e}^{-\frac{1}{2}\sum_{i,j} A_{i,j} x_i x_j} \, dx_1 \, dx_2 \, \dots \, dx_n$$

and let

$$B = A^{-1} \tag{2.2.1}$$

which we call the **propagator**.

Let us denote expectation values with brackets (this is the usual notation in physics):

$$< f(x_1, \dots, x_n) > \stackrel{\text{def}}{=} \int f(x_1, \dots, x_n) \, d\mu(x_1, \dots, x_n).$$

Wick's theorem states that:

Theorem 2.2.1 (Wick's Theorem [86]) *The expectation value of a product of two Gaussian random variables, is the inverse of the quadratic form A, and is called the* **propagator***:*

$$< x_i x_j >= B_{i,j} = (A^{-1})_{i,j} = \text{propagator}.$$

The expectation value of any product of an odd number of variables is zero, and the expectation value of an even product of Gaussian random variables, is the sum over all pairings of product of expectation values of pairs:

$$< x_{i_1} x_{i_2} \dots x_{i_{2m}} >= \sum_{\text{pairings}} \prod_{\text{pairs}(k,l)} B_{i_k, i_l}.$$

Example

$$< x_{i_1} x_{i_2} x_{i_3} x_{i_4} >= B_{i_1,i_2} B_{i_3,i_4} + B_{i_1,i_3} B_{i_2,i_4} + B_{i_1,i_4} B_{i_2,i_3}.$$

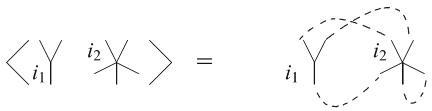
Wick's theorem becomes even more interesting when the indices $i_1, \ldots, i_{2m}$ are not distinct. For instance:

$$< x_{i_1}^2 x_{i_2}^2 >= B_{i_1,i_1} B_{i_2,i_2} + 2 B_{i_1,i_2} B_{i_1,i_2}.$$

2.2.1.1 Graphs

The best way to write Wick's theorem is diagrammatically. To each index i_k associate a vertex, and to each pair (i_k, i_l) associate an edge with weight B_{i_k,i_l}. If an index i_k is repeated, i.e. if it appears as $x_{i_k}^{p_k}$, then we associate to it a vertex with p_k half edges. Wick's theorem says that the expectation value is the sum over all possible ways of linking vertices by edges, of the product of propagators corresponding to edges. In other words, draw all possible graphs with the given vertices, and weight each graph by the product of its edge propagators.

Example

$< x_{i_1}^3 x_{i_2}^5 > =$ 

$+ \ldots$ 104 other pairings (2.2.2)

the represented graph here has weight

$$B_{i_1,i_2}^3 B_{i_2,i_2}.$$

In other words, Wick's theorem allows to count the number of ways of gluing vertices (of given valence) by their edges. Such graphs are called **Feynman graphs**. A Feynman graph is a graph, with given vertices, to which we associate a value, which is the product of the propagators $B_{i,j}$'s of edges:

$$\prod_{e \in \text{edges}} B_{i_e,j_e}.$$

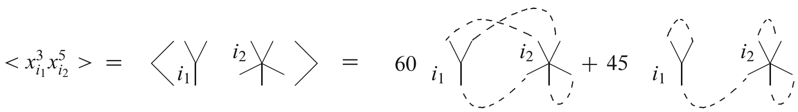
2.2.1.2 Symmetry Factors

The total number of possible graphs with m edges, is the number of pairings of $2m$ half edges, it is:

$$(2m-1)!! = (2m-1)(2m-3)(2m-5)\ldots 1.$$

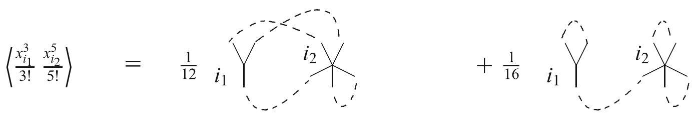
However, many of the graphs obtained, are topologically identical, they have the same weight, and it may be more convenient to write only non equivalent graphs, and associate to them an integer factor (the symmetry factor).

For example, the graph displayed in Eq. (2.2.2), is obtained 60 times, and the only other topological graph is obtained 45 times, which make a total of $60 + 45 = 105 = 7 * 5 * 3 = 7\,!!$:



$$= 60\; B_{i_1,i_2}^3\, B_{i_2,i_2} + 45\; B_{i_1,i_2}\, B_{i_1,i_1}\, B_{i_2,i_2}^2 . \tag{2.2.3}$$

Notice on that example, that both 60 and 45 divide $3! \times 5!$:



(2.2.4)

This is something general: the number of relabelings which leave a graph invariant (i.e. the number of times we obtain the same graph), is equal to the order of the group of relabelings, divided by the number of automorphisms of the graph. This is an application of the orbit-stabilizer theorem, see Proposition 1.1.2.

What we call the symmetry factor, is the number of automorphisms of a graph, it appears in the denominator.

To summarize, one may say that **Gaussian expectation values are generating functions for counting (weighted by the inverse of their integer symmetry factor) the number of graphs with given vertices.**

2.2.2 Matrix Gaussian Integrals

Let us now apply Wick's theorem, to the computation of Gaussian matrix integrals. In that case, the Feynman graphs are going to be fatgraphs also called ribbon-graphs, or maps, or discrete surfaces.

2.2.2.1 Application of Wick's Theorem to Matrix Integrals

Consider a random hermitean matrix M of size N, with Gaussian probability measure:

$$d\mu_0(M) = \frac{1}{Z_0} \mathrm{e}^{-\frac{N}{2t}\,\mathrm{Tr}\,M^2} \prod_{i=1}^{N} dM_{i,i} \prod_{i<j} d\mathrm{Re}M_{i,j}\; d\mathrm{Im}M_{i,j}$$

in other words, the variables $M_{i,i}, \mathrm{Re}M_{i,j}, \mathrm{Im}M_{i,j}$ are independent Gaussian random variables. Z_0 is the normalization constant such that $\int d\mu_0(M) = 1$:

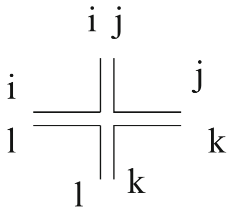
$$Z_0 = 2^N\,(\pi\, t/N)^{\frac{N^2}{2}}. \tag{2.2.5}$$

Since $\mathrm{Tr}\, M^2 = \sum_{i,j} M_{i,j}M_{j,i}$, the Wick's propagator [defined in Eq. (2.2.1)] is easily computed:

$$< M_{i,j}M_{k,l} >_0 = \frac{t}{N}\,\delta_{i,l}\delta_{j,k}$$

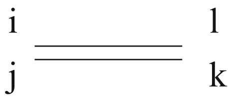
where $\delta_{j,l} = 0$ if $j \neq l$ and 1 if $j = l$ is the Krönecker δ-function, and $<>_0$ means the expectation value with the measure $d\mu_0$.

As a first example, let us compute $<\ \mathrm{Tr}\, M^4 >_0 = \sum_{i,j,k,l} < M_{i,j}M_{j,k}M_{k,l}M_{l,i} >_0$, which we represent as a vertex with four double-line half edges:



We write the half edges as double lines, and associate to each single line its index. Because of the trace, the indices are constant along single lines.

Since the propagator is $< M_{i,j}M_{k,l} >_0 = \frac{t}{N}\,\delta_{i,l}\delta_{j,k}$, it is going to be used to glue together half edges carrying the same oriented pair of indices, we can represent it as a double line edge (a ribbon):



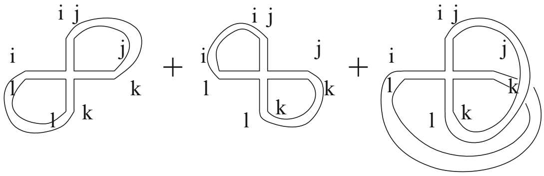
So, let us compute $<\ \mathrm{Tr}\, M^4 >_0$:

$$< \frac{N}{4t}\,\mathrm{Tr}\, M^4 >_0$$

$$= \frac{N}{4t}\sum_{i,j,k,l} < M_{i,j}M_{j,k}M_{k,l}M_{l,i} >_0 \qquad\qquad 3\, possible\, pairings \swarrow$$

$$= \frac{N}{4t}\sum_{i,j,k,l} < M_{i,j}M_{j,k} >_0 < M_{k,l}M_{l,i} >_0$$

$$+ < M_{i,j}M_{l,i} >_0 < M_{j,k}M_{k,l} >_0 + < M_{i,j}M_{k,l} >_0 < M_{j,k}M_{l,i} >_0$$



$$
\begin{aligned}
&= \\
&= \frac{N}{4t}\sum_{i,j,k,l} \frac{t}{N}\delta_{i,k}\delta_{j,j}\ \frac{t}{N}\delta_{k,i}\delta_{l,l} + \frac{t}{N}\delta_{i,i}\delta_{j,l}\ \frac{t}{N}\delta_{j,l}\delta_{k,k} + \frac{t}{N}\delta_{i,l}\delta_{j,k}\ \frac{t}{N}\delta_{j,i}\delta_{k,l} \\
&= \frac{N}{4t}\Big(\frac{t^2}{N^2}N^3 + \frac{t^2}{N^2}N^3 + \frac{t^2}{N^2}N\Big) \\
&= \frac{t}{4}\Big(N^2 + N^2 + N^0\Big) \\
&= \frac{tN^2}{2} + \frac{tN^0}{4}.
\end{aligned}
$$

Notice that there are two steps in that computation:

– the first one consists in applying Wick's theorem, i.e. representing each term as one way of gluing together half edges of the 4-valent vertex with propagators. This gives three pairings, illustrated by three graphs.
– the second step consists in performing the summation over the indices. The special form of the propagator, with Krönecker δ-functions of indices, ensures that there is exactly one independent index per single line. The sum over all indices is thus equal to N to the power the number of single lines, i.e. number of faces of the graph.

Since we also have a factor $1/N$ per propagator i.e. per edge, and a factor N in front of the trace, i.e. a factor N per vertex, in the end the total N dependance for a given graph is:

$$N^{\#\text{vertices}-\#\text{edges}+\#\text{faces}} = N^{\chi}$$

where χ is a topological invariant of the graph, called its **Euler characteristics**, see Sect. 1.1.3.

It should now be clear to the reader that this is something general. The fact that the power of N is a topological invariant, first discovered in 1974 by the physics Nobel prize Gerard 't Hooft [48], is the origin of the name "topological expansion".

Wick's theorem ensures that each term in the expectation value corresponds to one way of gluing vertices by their edges, and the sum over indices coming from the traces ensures that the total power of N for each graph is precisely its Euler characteristics, which we summarize as:

$$< \prod_{k=1}^{m} (N \operatorname{Tr} M^{p_k}) >_0 = \sum_{\text{labeled Fat Graphs } G} N^{\chi(G)} \;\; t^{\#\text{edges}}$$

where the sum is over the set of (labeled) oriented fat graphs having vertices of valence $p_1, \ldots, p_m$ obtained by gluing together half edges.

One should make some remarks:

- the graphs in that sum maybe disconnected.
- several graphs may be topologically equivalent in the sum, i.e. if we remove the labelling of indices. The order of the group of relabellings is (see Sect. 1.1.4):

$$\prod_{k=1}^{m} p_k \; \prod_{p} (\#\{p_i | \; p_i = p\})!$$

indeed, at each vertex of valence p_k one can make p_k rotations of the indices, and if several vertices have the same valence they can be permuted.

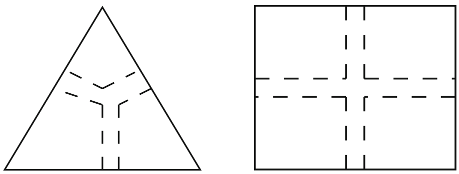
Therefore, it is better to rewrite:

$$< \prod_{k=1}^{m} \frac{1}{n_k!} (\frac{N}{k} \operatorname{Tr} M^k)^{n_k} >_0 = \sum_{\text{Fat Graphs } G} \frac{1}{\#\text{Aut}(G)} N^{\chi(G)} \;\; t^{\#\text{edges}} \tag{2.2.6}$$

where now the sum is over non-topologically equivalent fat graphs made with n_k k-valent vertices, and (using again the orbit-stabilizer Theorem 1.1.2) #Aut(G) is the number of automorphisms of the graph G.

2.2.2.2 From Graphs to Maps

Instead of summing over fatgraphs, let us sum over their duals, using the obvious bijection between a graph and its dual. The dual of a k-valent vertex is a k-gon:



Gluing together vertices by their half-edges is equivalent to gluing (oriented) polygons together by their sides, and thus we obtain a map. Equation (2.2.6) can thus be rewritten:

$$< \prod_{k=1}^{m} \frac{1}{n_k!} (\frac{N}{kt} \operatorname{Tr} M^k)^{n_k} >_0 = \sum_{\text{Maps } \Sigma} \frac{t^{\#\text{edges}-\#\text{faces}}}{\#\text{Aut}(\Sigma)} N^{\chi(\Sigma)}$$

where now the sum is over maps Σ made with n_k k-gons, and $\#\text{Aut}(\Sigma)$ is the number of automorphisms of the map Σ. In the duality graph $G \leftrightarrow$ map Σ, we have:

- vertices of $G \leftrightarrow$ faces of Σ
- edges of $G \leftrightarrow$ edges of Σ
- faces of $G \leftrightarrow$ vertices of Σ

Notice that the Euler characteristics of a graph and its dual is the same. The Euler-Characteristics is

$$\chi = \#\text{vertices} - \#\text{edges} + \#\text{faces}$$

so that the power of t is:

$$t^{\#\text{vertices}-\chi}.$$

All this can be rephrased as the following theorem due to Brezin-Itzykson-Parisi-Zuber in 1978 [74]:

Theorem 2.2.2 (BIPZ 1978) *A Gaussian expectation value of a polynomial moment of a Gaussian random matrix M with measure $d\mu_0$ is a finite weighted sum of maps:*

$$< \prod_{k=1}^{m} \frac{1}{n_k!} (\frac{N}{kt} \operatorname{Tr} M^k)^{n_k} >_0 = \sum_{\text{Maps } \Sigma} \frac{t^{\#\text{vertices}(\Sigma)}}{\#\text{Aut}(\Sigma)} \left(\frac{N}{t}\right)^{\chi(\Sigma)} \tag{2.2.7}$$

where the sum is over all maps (not necessarily connected) having exactly m faces, with given degrees n_k, $k = 1, \ldots, m$. The fact that the power of N is the Euler characteristics $\chi(\Sigma)$ is 't Hooft's discovery in 1974 [48].

As we have seen above, this theorem is a mere consequence of Wick's theorem, applied to matrix Gaussian random variables.

2.3 Generating Functions of Maps and Matrix Integrals

2.3.1 Generating Functions for Closed Maps

Theorem 2.2.2 implies that the generating function Z_N of Eq. (1.2.5), which counts non connected maps, is nothing but the formal integral:

Proposition 2.3.1 (BIPZ [74])

$$Z_N(t; t_3, t_4, \dots, t_d) = \int_{\text{formal}} dM\ \mathrm{e}^{-N \operatorname{Tr} \frac{M^2}{2t}}\ \mathrm{e}^{\frac{N}{t} \operatorname{Tr} \left(\frac{t_3}{3} M^3 + \frac{t_4}{4} M^4 + \dots \frac{t_d}{d} M^d\right)}$$

$$= \sum_{\text{n.c. closed maps } \Sigma} \left(\frac{N}{t}\right)^{\chi(\Sigma)} t_3^{n_3(\Sigma)} t_4^{n_4(\Sigma)} \dots t_d^{n_d(\Sigma)} \frac{t^{\#\text{vertices}(\Sigma)}}{\#\text{Aut}(\Sigma)}$$

where again, formal integral means that we Taylor expand the exponentials of all non quadratic terms, and exchange the Taylor series and the integration. In other words, we perform a formal small t (or equivalently small $t_3, t_4, \dots, t_d$) asymptotic expansion, and order by order we get the number of corresponding maps. The coefficient of t^j is the finite sum over (n.c. = non-connected) closed maps such that $\frac{1}{2} \sum_i (i-2) n_i = j = \#\text{vertices} - \chi$.

2.3.1.1 Connected Maps

When we have a formal generating series counting disconnected objects multiplicatively, it is well known (see [46]) that the logarithm is the generating function which counts only the connected objects, i.e. it is the generating function of Eq. (1.2.5):

$$\begin{aligned} & \ln\left(Z_N(t; t_3, t_4, \dots, n_d)\right) \\ &= F(t; t_3, t_4, \dots, n_d; N) \\ &= \sum_{\text{closed connected maps } \Sigma} \left(\frac{N}{t}\right)^{2-2g(\Sigma)} t_3^{n_3(\Sigma)} t_4^{n_4(\Sigma)} \dots t_d^{n_d(\Sigma)} \frac{t^{\#\text{vertices}}}{\#\text{Aut}(\Sigma)}. \end{aligned}$$

Again, the coefficient of t^j is the finite sum of connected closed maps such that $\frac{1}{2} \sum_i (i-2) n_i = j = \#\text{vertices} - \chi$. And the Euler characteristics of a connected map is $\chi = 2 - 2g$ where g is the genus.

2.3.1.2 Topological Expansion: Maps of Given Genus

We thus see, that order by order in the small t expansion, the coefficients of t^j in $N^{-2}F$ are polynomials of N^{-2}, and thus we can define generating series of coefficients of a given power of N^{-2g}, we define:

$$F = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g} F_g \qquad , \qquad F_g \in \mathbb{Q}[t_3, \ldots, t_d][[t]]$$

where again we emphasize that this is an equality of formal series in powers of t, and order by order, the sum over g is finite, and the coefficients are polynomials in N^{-2}. F_g is obtained by collecting the coefficients of N^{-2g}, and its computation does not involve any large N expansion.

We recognize the generating function of connected closed maps of genus g, of Definition 1.2.4:

$$F_g(t; t_3, t_4, \ldots, t_d) = \sum_v t^v \sum_{\Sigma \in \mathbb{M}_0^{(g)}(v)} t_3^{n_3(\Sigma)} t_4^{n_4(\Sigma)} \ldots t_d^{n_d(\Sigma)} \frac{1}{\#\mathrm{Aut}(\Sigma)}.$$

2.4 Maps with Boundaries or Marked Faces

2.4.1 One Boundary

So far, we have seen how formal matrix integrals, thanks to Wick's theorem, are counting closed maps where all polygons play similar roles. Now let us count maps with some marked faces.

Consider the following formal matrix integral:

$$< \mathrm{Tr}\, M^l > = \frac{\int_{\mathrm{formal}} dM\;\; \mathrm{Tr}\, M^l\;\; \mathrm{e}^{-N \mathrm{Tr}\, \frac{M^2}{2t}}\;\; \mathrm{e}^{\frac{N}{t}\mathrm{Tr}\,\left(\frac{t_3}{3}M^3 + \frac{t_4}{4}M^4 + \ldots \frac{t_d}{d}M^d\right)}}{\int_{\mathrm{formal}} dM\;\; \mathrm{e}^{-N \mathrm{Tr}\, \frac{M^2}{2t}}\;\; \mathrm{e}^{\frac{N}{t}\mathrm{Tr}\,\left(\frac{t_3}{3}M^3 + \frac{t_4}{4}M^4 + \ldots \frac{t_d}{d}M^d\right)}}. \qquad (2.4.1)$$

The bracket $< . >$ now denotes expectation value with respect to the formal measure $\frac{1}{Z}\, \mathrm{e}^{-N \mathrm{Tr}\, \frac{M^2}{2t}}\; \mathrm{e}^{\frac{N}{t}\mathrm{Tr}\,\left(\frac{t_3}{3}M^3 + \frac{t_4}{4}M^4 + \ldots \frac{t_d}{d}M^d\right)} dM$, whereas in the previous section $< . >_0$ meant the expectation value with respect to the gaussian measure $\frac{1}{Z_0}\, \mathrm{e}^{-N \mathrm{Tr}\, \frac{M^2}{2t}}\, dM$.

The numerator in Eq. (2.4.1) is

$$\int_{\text{formal}} dM \;\operatorname{Tr} M^l \; e^{-N \operatorname{Tr} \frac{M^2}{2t}} \; e^{\frac{N}{t} \operatorname{Tr} \left(\frac{t_3}{3} M^3 + \frac{t_4}{4} M^4 + \dots \frac{t_d}{d} M^d\right)}$$
$$= \sum_{n_3,\dots,n_d} \frac{N^{n_3}\, t_3^{n_3}}{3^{n_3}\, n_3!} \frac{N^{n_4}\, t_4^{n_4}}{4^{n_4}\, n_4!} \cdots \frac{N^{n_d}\, t_d^{n_d}}{d^{n_d}\, n_d!} \;\frac{1}{t^{\sum n_j}}$$
$$\int_{H_N} (\operatorname{Tr} M^l) \;\; (\operatorname{Tr} M^3)^{n_3} \,(\operatorname{Tr} M^4)^{n_4} \,\dots\, (\operatorname{Tr} M^d)^{n_d} \;\; d\mu_0(M)$$

i.e. each term is a Gaussian expectation value of a polynomial in M, it can be computed using Wick's theorem, and it gives a sum over all maps (in fact ribbon graphs dual to maps) with n_3 triangles, n_4 squares, ..., n_d d-gons, and one **marked** l**-gon**. The sum may include non connected maps, and the role of the denominator in Eq. (2.4.1) is precisely to kill all non-connected maps (see Sect. 1.2.5 in Chap. 1).

There should be a symmetry factor $1/\#\mathrm{Aut}(\Sigma)$ counting automorphisms which preserve the marked face, and since there is no factor $\frac{1}{l}$ in front of $\operatorname{Tr} M^l$, we get l times the number of maps with no marked edge on the marked face, i.e. we get the number of maps with one marked edge on the marked face. Since there is no N accompanying the $\operatorname{Tr} M^l$, the power of N is $\chi - 1 = 2 - 2g - 1$ which is the Euler characteristic of a surface with one boundary. Therefore we recognize the generating function $\mathcal{T}_l$ of Eq. (1.2.2) in Chap. 1:

$$< \operatorname{Tr} M^l > \; = \mathcal{T}_l$$
$$= \sum_{\text{maps } \Sigma \text{ with 1 boundary of length } l} \left(\frac{N}{t}\right)^{1-2g} \; t_3^{n_3(\Sigma)} t_4^{n_4(\Sigma)} \dots t_d^{n_d(\Sigma)} \; \frac{t^{\#\text{vertices}}}{\#\mathrm{Aut}(\Sigma)}$$
$$= -\operatorname*{Res}_{x\to\infty} x^l \; W_1(x)\, dx.$$

2.4.2 Several Boundaries

The previous paragraph can be immediately generalized to:

$$< \operatorname{Tr} M^{l_1} \operatorname{Tr} M^{l_2} \dots \operatorname{Tr} M^{l_k} >$$
$$= \frac{1}{Z} \int_{\text{formal}} dM \;\operatorname{Tr} M^{l_1} \;\operatorname{Tr} M^{l_2} \,\dots\, \operatorname{Tr} M^{l_k} \;\; e^{-N \operatorname{Tr} \frac{M^2}{2}} \;\; e^{N \operatorname{Tr} \left(\frac{t_3}{3} M^3 + \frac{t_4}{4} M^4 + \dots \frac{t_d}{d} M^d\right)}$$
$$= \frac{1}{Z} \,\mathcal{T}^*_{l_1,\dots,l_k} \tag{2.4.2}$$

where $\mathcal{T}^*_{l_1,\dots,l_k}$ is the generating function of not necessarily connected maps with k boundaries of lengths $l_1,\dots,l_k$ of all genus.

One obtains connected maps by computing cumulants (see Sect. 1.2.5 of Chap. 1), for instance:

$$< \operatorname{Tr} M^{l_1} \operatorname{Tr} M^{l_2} >_c = < \operatorname{Tr} M^{l_1} \operatorname{Tr} M^{l_2} > - < \operatorname{Tr} M^{l_1} >< \operatorname{Tr} M^{l_2} > .$$

And thus the cumulants compute connected maps with k boundaries of lengths $l_1,\dots,l_k$:

$$\begin{aligned} &\mathcal{T}_{l_1,\dots,l_k} \\ &= < \operatorname{Tr} M^{l_1} \operatorname{Tr} M^{l_2} \dots \operatorname{Tr} M^{l_k} >_c \\ &= \sum_{\Sigma \text{ with } k \text{ boundaries of length } l_1,\dots,l_k} \left(\frac{N}{t}\right)^{2-2g-k} t_3^{n_3(\Sigma)} t_4^{n_4(\Sigma)} \dots t_d^{n_d(\Sigma)} \frac{t^{\#\text{vertices}}}{\#\text{Aut}(\Sigma)}. \end{aligned}$$

2.4.3 *Topological Expansion for Bounded Maps of Given Genus*

The Euler characteristics of a connected surface of genus g with k boundaries is:

$$\chi = 2 - 2g - k.$$

Therefore we have:

$$< \operatorname{Tr} M^{l_1} \operatorname{Tr} M^{l_2} \dots \operatorname{Tr} M^{l_k} >_c = \sum_{g=0}^{\infty} \mathcal{T}^{(g)}_{l_1,\dots,l_k} \left(\frac{N}{t}\right)^{2-2g-k} \tag{2.4.3}$$

where $\mathcal{T}^{(g)}_{l_1,\dots,l_k}$ is the generating function defined in Chap. 1, Eq. (1.2.2), which counts connected maps of genus g, with k boundaries of lengths $l_1,\dots,l_k$.

Once more we emphasize that this equality holds term by term in the powers of t, and for each power, the sum over g is finite, i.e. both left hand side and right hand side are Laurent polynomials in N.

In other words, Eq. (2.4.3) is not a large N expansion, it is a small t expansion.

2.4.4 Resolvents

We define the **resolvent**:

$$W_1(x) = \sum_{l=0}^{\infty} \frac{1}{x^{l+1}}\ \mathcal{T}_l = \sum_{l=1}^{\infty} \left\langle \mathrm{Tr}\ \frac{M^l}{x^{l+1}} \right\rangle$$

and conversely:

$$\mathcal{T}_l = -\operatorname*{Res}_{x\to\infty}\ x^l\, W_1(x)\, dx.$$

Very often (in particular in physicist's literature), the resolvent is written:

$$W_1(x)\ \text{"} = \text{"} < \mathrm{Tr}\ \frac{1}{x-M} >$$

which holds in the formal sense, i.e. to each given power of t, the sum over l is finite and each coefficient in the small t or t_j's expansion is a polynomial in $1/x$.

More generally:

$$\begin{aligned} W_k(x_1,\dots,x_k) &= \sum_{l_1,\dots,l_k=0}^{\infty} \frac{1}{x_1^{l_1+1}\dots x_k^{l_k+1}}\ \mathcal{T}_{l_1,\dots,l_k} \\ &= \sum_{l_1,\dots,l_k=0}^{\infty} \left\langle \mathrm{Tr}\ \frac{M^{l_1}}{x_1^{l_1+1}} \dots \mathrm{Tr}\ \frac{M^{l_k}}{x_k^{l_k+1}} \right\rangle_c \\ &= \left\langle \mathrm{Tr}\ \frac{1}{x_1-M} \dots \mathrm{Tr}\ \frac{1}{x_k-M} \right\rangle_c \\ &= \sum_g \left(\frac{N}{t}\right)^{2-2g-k}\ W_k^{(g)}(x_1,\dots,x_k). \end{aligned} \tag{2.4.4}$$

The $W_k^{(g)}$ are the same as those of Definition 1.2.2 in Chap. 1.

2.5 Loop Equations = Tutte Equations

In this section, we derive a matrix-model proof of Tutte's equation of Chap. 1. In the matrix model framework, those equations were called "**loop equations**" by A. Migdal who introduced them in [64].

Loop equations merely arise from the fact that an integral is invariant under a change of variable (which is called **Schwinger–Dyson equations**), or alternatively from integration by parts.

Although loop equations are equivalent to Tutte's equations, it is often easier to integrate by parts in a matrix integral, than finding bijections between sets of maps, and it is much faster to derive loop equations from matrix models than from combinatorics.

Consider an expectation value of monomial $\tilde{G}(M)$ of total degree $l = l_1 + \cdots + l_k$:

$$< \tilde{G}(M) >= \frac{\int dM\, \tilde{G}(M)\ \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)}}{\int dM\ \ \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)}} \qquad , \qquad \tilde{G}(M) = \prod_{j=1}^{k} \operatorname{Tr} M^{l_j}$$

where $\int$ can mean either the formal matrix integral (i.e., to any order in t, a finite sum of Gaussian integrals) or the convergent matrix integral (both formal and convergent matrix integrals are going to satisfy the same loop equations).

We shall derive a recursion relation on the degrees $l = (l_1, \ldots, l_k)$.

The method is called loop equations, and it is nothing but integration by parts. It is based on the observation that the integral of a total derivative vanishes, and thus, if $G(M)$ is any matrix valued polynomial function of M (for instance $G(M) = M^{l_1}\ \prod_{j=2}^{k} \operatorname{Tr} M^{l_j}$), we have:

$$\begin{aligned} 0 = &\sum_{i<j} \int dM\ \frac{\partial}{\partial \mathrm{Re} M_{i,j}}\ \left((G(M))_{ij}\, \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)} \right) \\ &-i \sum_{i<j} \int dM\ \frac{\partial}{\partial \mathrm{Im} M_{i,j}}\ \left((G(M))_{ij}\, \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)} \right) \\ &+ \sum_{i=1}^{N} \int dM\ \frac{\partial}{\partial M_{i,i}}\ \left((G(M))_{ii}\, \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)} \right). \end{aligned} \qquad (2.5.1)$$

Choosing

$$G(M) = M^{l_1}\ \prod_{j=2}^{k} \operatorname{Tr} M^{l_j}$$

and after computing the derivatives we get:

$$\begin{aligned} &\sum_{j=0}^{l_1-1} <\ \operatorname{Tr} M^j \operatorname{Tr} M^{l_1-1-j} \prod_{i=2}^{k} \operatorname{Tr} M^{l_i} > + \sum_{j=2}^{k} l_j\ <\ \operatorname{Tr} M^{l_j+l_1-1} \prod_{i=2, i\neq j}^{k} \operatorname{Tr} M^{l_i} > \\ &= \frac{N}{t} <\ \operatorname{Tr}\,(M^{l_1} V'(M))\ \prod_{i=2}^{k} \operatorname{Tr} M^{l_i} > . \end{aligned} \qquad (2.5.2)$$

Remark 2.5.1 Again, we emphasize that this equation is valid for both convergent matrix integrals and formal matrix integrals, indeed it is valid for Gaussian integrals, and thus for any finite linear combination of Gaussian integrals, i.e. formal integrals. In case of formal integrals, those equations are valid, of course, only order by order in t. In other words the loop equations are independent of the order of the integral and the Taylor series expansion.

Using the notations of Eq. (2.4.3), we may rewrite the loop equation (2.5.2):

Theorem 2.5.1 *Loop equations* $\forall g$:

$$\begin{aligned}&\sum_{j=0}^{l_1-1}\left[\sum_{h=0}^{g}\sum_{J\subset L}\mathcal{T}^{(h)}_{j,J}\mathcal{T}^{(g-h)}_{l_1-1-j,L/J}+\mathcal{T}^{(g-1)}_{j,l_1-1-j,L}\right]+\sum_{j=2}^{k} l_j\,\mathcal{T}^{(g)}_{l_j+l_1-1,L/\{j\}}\\ &=\mathcal{T}^{(g)}_{l_1+1,L}-\sum_{j=3}^{d} t_j\mathcal{T}^{(g)}_{l_1+j-1,L}\end{aligned}\tag{2.5.3}$$

where we denote collectively $L=\{l_2,\dots,l_k\}$.

Loop equations coincide with Tutte equations (1.3.2) of Chap. 1.

We recall that $\mathcal{T}^{(g)}_{l_1,l_2,\dots,l_k}$ is the generating function that counts the number of connected maps of genus g with k boundaries of perimeters $l_1,\dots,l_k$, and therefore we have re–derived the generalized **Tutte equation** (1.3.2) of Chap. 1.

It is interesting to rewrite the loop equations of Eq. (2.5.3) in terms of resolvents $W_k^{(g)}$'s defined in Eq. (2.4.4). We merely multiply Eq. (2.5.3) by $\prod_{i=1}^{k} 1/x_i^{l_i+1}$ and sum over $l_1,\dots,l_k$ (to any given power of t, the sum is finite).

Theorem 2.5.2 *Loop equations. For any k and g, and $L=\{x_2,\dots,x_k\}$, we have:*

$$\begin{aligned}&\sum_{h=0}^{g}\sum_{J\subset L}W^{(h)}_{1+|J|}(x_1,J)W^{(g-h)}_{k-|J|}(x_1,L\setminus J)+W^{(g-1)}_{k+1}(x_1,x_1,L)\\ &+\sum_{j=2}^{k}\frac{\partial}{\partial x_j}\frac{W^{(g)}_{k-1}(x_1,L\setminus\{x_j\})-W^{(g)}_{k-1}(L)}{x_1-x_j}\\ &=V'(x_1)W^{(g)}_k(x_1,L)-P^{(g)}_k(x_1,L)\end{aligned}\tag{2.5.4}$$

where $P^{(g)}_k(x_1,L)$ is a polynomial in x_1, of degree $d-3$ (except $P^{(0)}_1$ which is of degree $d-2$):

$$P^{(g)}_k(x_1,x_2,\dots,x_k)=-\sum_{j=2}^{d-1}t_{j+1}\sum_{i=0}^{j-1}x_1^i\sum_{l_2,\dots,l_k=1}^{\infty}\frac{\mathcal{T}^{(g)}_{j-1-i,l_2,\dots,l_k}}{x_2^{l_2+1}\dots x_k^{l_k+1}}+t\,\delta_{g,0}\delta_{k,1}.$$

Proof Indeed, if we expand both sides of Eq. (2.5.4) in powers of $x_1 \to \infty$, and identify the coefficients on both side, we find that the negative powers of x_1 give precisely the loop equations (2.5.3), whereas the coefficients of positive powers of x_1 cancel due to the definition of $P_k^{(g)}$, which is exactly the positive part of $V'(x_1)W_k^{(g)}$:

$$P_k^{(g)}(x_1, x_2, \ldots, x_k) = \Big(V'(x_1)\, W_k^{(g)}(x_1, x_2, \ldots, x_k)\Big)_{+x_1}$$

where $(.)_{+x_1}$ means that we keep only the polynomial part, i.e. the positive part of the Laurent series at $x_1 \to \infty$. □

2.6 Loop Equations and "Virasoro Constraints"

We have seen two derivations of the loop equations. One combinatorial proof in Chap. 1, based on Tutte's method, corresponding to recursively removing a marked edge, and one proof based on integration by parts in the formal matrix integral in Chap. 2. However, there exist other possible derivations, of which we shall only give a heuristic idea here.

In particular, in string theory and quantum gravity, it is known that partition functions must satisfy Virasoro constraints. Here, we show how to rewrite the loop equations for generating functions of maps, as Virasoro constraints.

We write the potential:

$$V(x) = -\sum_{j=1}^{\infty} \frac{t_j}{j}\, x^j.$$

In the end, we will be interested in $t_1 = 0, t_2 = -1$ and $t_j = 0$ if $j > d$.

It is easy to see from the definitions of our generating functions, and particularly on the formal matrix integral, that:

$$\mathcal{T}_j^{(g)} = j\,\frac{\partial F_g}{\partial t_j} + t\delta_{j,0}\delta_{g,0} \qquad , \qquad \mathcal{T}_{j_1,j_2}^{(g)} = j_1 j_2\, \frac{\partial^2 F_g}{\partial t_{j_1}\partial t_{j_2}}.$$

If we sum over g we have

$$F = \sum_g (N/t)^{2-2g}\, F_g$$

and thus

$$\mathcal{T}_j = \sum_g (N/t)^{1-2g}\, \mathcal{T}_j^{(g)} = \frac{t}{N}\, j\, \frac{\partial F}{\partial t_j} + N\delta_{j,0}$$

$$\mathcal{T}_{j_1,j_2} = \sum_g (N/t)^{-2g}\ \mathcal{T}^{(g)}_{j_1,j_2} = \frac{t^2}{N^2}\, j_1 j_2\, \frac{\partial^2 F}{\partial t_{j_1}\, \partial t_{j_2}}.$$

The Tutte's equations (2.5.3) can be rewritten for any $k \geq -1$:

$$0 = \frac{N}{t} \sum_{j=1}^{\infty} t_j\, \mathcal{T}_{k+j} + \sum_{k'=0}^{k} (\mathcal{T}_{k'}\, \mathcal{T}_{k-k'} + \mathcal{T}_{k',k-k'})$$

which can be rewritten for $k \geq 2$:

$$0 = \sum_{j=1}^{\infty} (k+j)\, t_j\, \frac{\partial F}{\partial t_{k+j}} + \frac{t^2}{N^2} \sum_{k'=1}^{k-1} k'(k-k') \left(\frac{\partial F}{\partial t_{k'}}\, \frac{\partial F}{\partial t_{k-k'}} + \frac{\partial^2 F}{\partial t_{k'}\, \partial t_{k-k'}} \right) + 2t\,k\, \frac{\partial F}{\partial t_k}.$$

For $k = 1, 0, -1$ it reads:

$$0 = \sum_{j=1}^{\infty} (j+1)\, t_j\, \frac{\partial F}{\partial t_{j+1}} + 2t\, \frac{\partial F}{\partial t_1}$$

$$0 = \sum_{j=1}^{\infty} j\, t_j\, \frac{\partial F}{\partial t_j} + N^2$$

$$0 = t_1\, \frac{N^2}{t} + \sum_{j=2}^{\infty} (j-1)\, t_j\, \frac{\partial F}{\partial t_{j-1}}.$$

Notice that if we write $Z = \mathrm{e}^F$ we have

$$\frac{\partial F}{\partial t_{k'}}\, \frac{\partial F}{\partial t_{k-k'}} + \frac{\partial^2 F}{\partial t_{k'}\, \partial t_{k-k'}} = \frac{1}{Z}\, \frac{\partial^2}{\partial t_{k'}\, \partial t_{k-k'}}\, Z$$

and thus for $k \geq 2$ the quadratic differential equation satisfied by F implies a linear differential equation satisfied by Z:

$$0 = \left(\sum_{j=1}^{\infty} (k+j)\, t_j\, \frac{\partial}{\partial t_{k+j}} + \frac{t^2}{N^2} \sum_{k'=1}^{k-1} k'(k-k')\, \frac{\partial^2}{\partial t_{k'}\, \partial t_{k-k'}} + 2t\,k\, \frac{\partial}{\partial t_k} \right) Z$$

and we have similar expressions for $k = -1, 0, 1$.

2.6.1 Virasoro-Witt Generators

In order to appropriately take into account the special cases $k = -1, 0, 1$, it is convenient to introduce another time t_0 and define:

Definition 2.6.1 We define the Virasoro-Witt generators L_k for $k \geq -1$ as:
For $k \geq 2$:

$$L_k = \frac{t^2}{N^2} \sum_{j=1}^{k-1} j\,(k-j)\, \frac{\partial}{\partial t_j}\, \frac{\partial}{\partial t_{k-j}} + 2\frac{t^2}{N^2}\, k\, \frac{\partial}{\partial t_k}\, \frac{\partial}{\partial t_0} + \sum_{j=1}^{\infty} (k+j)\, t_j\, \frac{\partial}{\partial t_{k+j}},$$

and for $k = -1, 0, 1$:

$$L_1 = 2\frac{t^2}{N^2}\, \frac{\partial}{\partial t_1}\, \frac{\partial}{\partial t_0} + \sum_{j=1}^{\infty} (j+1)\, t_j\, \frac{\partial}{\partial t_{j+1}},$$

$$L_0 = \frac{t^2}{N^2}\, \frac{\partial}{\partial t_0}\, \frac{\partial}{\partial t_0} + \sum_{j=1}^{\infty} j\, t_j\, \frac{\partial}{\partial t_j},$$

$$L_{-1} = t_1\, \frac{\partial}{\partial t_0} + \sum_{j=1}^{\infty} j\, t_{j+1}\, \frac{\partial}{\partial t_j}.$$

The differential operators L_k form a representation of the Witt algebra (the positive part of a Virasoro algebra), indeed one easily verifies that they satisfy the Virasoro-Witt commutation relation:

$$[L_k, L_j] = (k-j) L_{k+j}.$$

In particular, L_{-1}, L_0, L_1 form a representation of the $SU(2)$ algebra:

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1} \qquad , \qquad [L_1, L_{-1}] = 2\, L_0.$$

2.6.2 Generating Series of Virasoro-Witt Generators

It is convenient to introduce generating series:

Definition 2.6.2 The "stress-energy tensor" $T(x)$ is defined as

$$T(x) = \sum_{k=-1}^{\infty} \frac{L_k}{x^{k+2}}.$$

and the "loop insertion operator" is defined as

$$D(x) = \ln x \frac{\partial}{\partial t_0} - \sum_{k=1}^{\infty} \frac{1}{x^k} \frac{\partial}{\partial t_k}.$$

(The names come from the physics literature). The loop insertion operator is called so, because it inserts boundaries, indeed we have:

$$D'(x_1) \dots D'(x_n) \,.\, \ln Z = \sum_{g=0}^{n} (N/t)^{2-2g-n} \; W_n^{(g)}(x_1, \dots, x_n).$$

Those operators satisfy the following algebra:

Proposition 2.6.1 *The Virasoro commutation relations can be rewritten as*

$$[T(x), T(y)] = -2 \frac{T(x) - T(y)}{(x-y)^2} + \frac{T'(x) + T'(y)}{x-y} = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) \frac{T(x) - T(y)}{x-y}.$$

We also have

$$[T(x), D(y)] = -\frac{D'(x) - D'(y)}{x-y}.$$
$$[V'(x), D(y)] = \frac{1}{y-x}.$$

In addition we have that

$$T(x) = \frac{t^2}{N^2} D'(x)^2 + \big(V'(x)\, D'(x)\big)_-$$

where the subscript $()_-$ means that we compute the large x expansion and keep only negative powers of x.

2.6.3 Maps and Virasoro Constraints

Then we have (almost by definition of the L_k's):

Proposition 2.6.2 *The Tutte's equations imply that Z is anihilated by the Virasoro–Witt generators:*

$$\forall k \geq -1 \qquad , \qquad L_k \,.\, \mathrm{e}^{t_0 \frac{N^2}{t}} Z = 0. \tag{2.6.1}$$

In terms of $T(x)$ we have that the Virasoro stress energy tensor $T(x)$ anihilates the partition function:

$$\forall x \qquad , \quad T(x).\left(e^{t_0\frac{N^2}{t}}Z\right) = 0.$$

Remark 2.6.1 As usual, all summations are to be understood as formal power series in powers of t, they are in general not convergent.

The Virasoro algebra method has been extensively used by physicists, but we shall not pursue in that direction in this book, we refer the reader to [40–42, 54].

Let us mention two important properties of the Virasoro equations:

- An important property, is that Eq. (2.6.1) is a linear equation for Z, and thus, linear combinations of solutions are also solutions. In particular, convergent matrix integrals are also solutions of the same Virasoro constraints. In fact the set of all solutions of Virasoro constraints (i.e. a vector space), is in bijection with the homology space of matrix ensembles on which a matrix integral can be absolutely convergent.
- Alexandrov-Mironov-Morozov in 2004 [3], used the fact that the stress energy tensor $T(x)$ should be analytical. This allows to consider contour integrals:

$$0 = \oint_{\mathcal{C}} dx\, D(x_1)\dots D(x_n)\, T(x).\left(e^{t_0\frac{N^2}{t}}Z\right),$$

  and move the integration contour $\mathcal{C}$ to the poles. Using this method, Alexandrov-Mironov-Morozov recovered Theorem 3.3.1 the solution presented in Chap. 3.

2.7 Summary Maps and Matrix Integrals

Let us summarize the concepts introduced in this chapter:

- Formal integral $\int_{\text{formal}}$ means that we exchange the order of the integral and the Taylor expansion of the exponentials of the t_k's.

$$Z_N = \int_{\text{formal}} e^{-\frac{N}{t}\operatorname{Tr} V(M)}\, dM \qquad , \qquad \text{with } V(M) = \frac{M^2}{2} - \sum_{j=3}^{d}\frac{t_j}{j}M^j$$

$$= \sum_{k=0}^{\infty}\frac{N^k}{t^k\, k!}\int_{H_N}\left(\sum_{j=3}^{d}\frac{t_j}{j}M^j\right)^k e^{-\frac{N}{t}\operatorname{Tr}\frac{M^2}{2}}\, dM$$

$$\in \mathbb{Q}[t_3,\dots,t_d,N^2,N^{-2}][[t]].$$

- $\mathbb{M}_0^{(g)}(v)$ = finite set of **connected** maps of **genus** g and no boundary, with v vertices, obtained by gluing n_3 triangles, n_4 squares, n_5 pentagons,..., n_d d-gons.

  Generating function:

$$\ln Z_N = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g} F_g$$

$$= \sum_{j=0}^{\infty} t^j \sum_{v+2g-2=j} N^{2-2g} \sum_{\Sigma \in \mathbb{M}_0^{(g)}(v)} \frac{t_3^{n_3(\Sigma)} t_4^{n_4(\Sigma)} \dots t_d^{n_d(\Sigma)}}{\#\mathrm{Aut}(\Sigma)}$$

  We also denote:

$$F_g = W_0^{(g)}.$$

- $\mathbb{M}_k^{(g)}(v)$ = connected **maps of genus** g with v vertices, obtained by gluing n_3 triangles, n_4 squares, n_5 pentagons, and k boundaries of length $l_1, \dots, l_k$.

  Generating function:

$$< \mathrm{Tr}\, M^{l_1}\ \mathrm{Tr}\, M^{l_2}\ \dots\ \mathrm{Tr}\, M^{l_k}\ >_c$$
$$= \sum_{j=0}^{\infty} t^j \sum_{v+2g+k-2=j} N^{2-2g-k} \sum_{\Sigma \in \mathbb{M}_k^{(g)}(v), \delta\Sigma=\{l_1,\dots,l_k\}} \frac{t_3^{n_3(\Sigma)} t_4^{n_4(\Sigma)} \dots t_d^{n_d(\Sigma)}}{\#\mathrm{Aut}(\Sigma)}$$
$$= \sum_{g} \left(\frac{N}{t}\right)^{2-2g-k} \mathcal{T}^{(g)}_{l_1,\dots,l_k}.$$

- Resolvents for connected **maps of genus** g and with k boundaries.

  Generating function:

$$W_k(x_1, \dots, x_k)$$
$$= \sum_{g} \left(\frac{N}{t}\right)^{2-2g-k} W_k^{(g)}(x_1, \dots, x_k)$$
$$= <\ \mathrm{Tr}\, \frac{1}{x_1 - M}\ \dots\ \mathrm{Tr}\, \frac{1}{x_k - M}\ >_c$$

$$= \sum_{j=0}^{\infty} t^j \sum_{v+2g+k-2=j} N^{2-2g-k} \sum_{\Sigma \in \mathbb{M}_{g,k}(v)} \frac{t_3^{n_3} t_4^{n_4} \dots t_d^{n_d}}{x_1^{l_1+1} \dots x_k^{l_k+1}} \frac{1}{\#\mathrm{Aut}(\Sigma)}\ .$$

- Loop equations (Tutte's equations):

$$\sum_{j=0}^{l_1-1}\Big[\sum_{h=0}^{g}\sum_{J\subset L}\mathcal{T}^{(h)}_{j,J}\,\mathcal{T}^{(g-h)}_{l_1-1-j,L/J}+\mathcal{T}^{(g-1)}_{j,l_1-1-j,L}\Big]+\sum_{j=2}^{k} l_j\,\mathcal{T}^{(g)}_{l_j+l_1-1,L/\{j\}}$$
$$=\mathcal{T}^{(g)}_{l_1+1,L}-\sum_{j=3}^{d} t_j\mathcal{T}^{(g)}_{l_1+j-1,L}$$

where $L=\{l_2,\dots,l_k\}$. Equivalently, the loop equations can be written in terms of $W_k^{(g)}$'s and with $L=\{x_2,\dots,x_k\}$:

$$\sum_{h=0}^{g}\sum_{J\subset L} W^{(h)}_{1+|J|}(x_1,J)W^{(g-h)}_{k-|J|}(x_1,L\setminus J)+W^{(g-1)}_{k+1}(x_1,x_1,L)$$
$$+\sum_{j=2}^{k}\frac{\partial}{\partial x_j}\,\frac{W^{(g)}_{k-1}(x_1,L\setminus\{x_j\})-W^{(g)}_{k-1}(L)}{x_1-x_j}$$
$$=V'(x_1)W^{(g)}_k(x_1,L)-P^{(g)}_k(x_1,L)$$

where $L=\{x_2,\dots,x_k\}$, and $P^{(g)}_k(x_1,L)=\mathrm{Pol}_{x_1}\,V'(x_1)\,W^{(g)}_k(x_1,L)$ is a polynomial in the variable x_1, of degree $d-3$, except $P^{(0)}_1$ which is of degree $d-2$.

2.8 Exercises

Exercise 1 For the quartic formal matrix integral

$$Z=\frac{1}{Z_0}\int_{\text{formal}} dM\,\mathrm{e}^{-\frac{N}{t}\operatorname{Tr}\frac{M^2}{2}-\frac{t_4M^4}{4}}=1+\frac{Nt_4}{4t}\left\langle \operatorname{tr}M^4\right\rangle_0+\frac{1}{2}\left(\frac{Nt_4}{4t}\right)^2\left\langle(\operatorname{tr}M^4)^2\right\rangle_0+O(t_4^3)$$

using Wick's theorem, recover the generating function of quadrangulations to the first orders

$$\ln Z=F=\frac{t_4}{4}\,(2N^2+1)+\frac{t_4^2}{8}\,(9N^2+15)+O(t_4^3).$$

Exercise 2 Prove that with any potential:

$$<\operatorname{Tr}V'(M)>=0\qquad,\qquad \frac{t}{N}<\operatorname{Tr}MV'(M)>=t^2.$$

Hint: this is a loop equation, use integration by parts.

Exercise 3 Prove that for quadrangulations (i.e. with $V(M) = \frac{M^2}{2} - t_4 \frac{M^4}{4}$):

$$\frac{\partial F}{\partial t} = \frac{N t_4}{4t^2} \mathcal{T}_4.$$

Answer: hint: use Exercise 2, and don't forget the t dependance of the normalization factor Z_0 in Eq. (2.2.5).

Chapter 3
Solution of Tutte-Loop Equations

In this chapter, we solve the loop equations (Tutte's equations), we compute explicitly the generating functions counting maps of given genus and boundaries.

We are first going to solve them for planar maps with one boundary (the disk, i.e. planar rooted maps), then two boundaries (the cylinder), and then arbitrary genus and arbitrary number of boundaries. The disk case (planar rooted maps) was already done by Tutte [83–85]. Generating functions for higher topologies have been computed more recently [5, 31].

In this chapter, we shall show that, surprisingly, the first two cases (disk and cylinder) are in fact more irregular than the general case. This is a general feature in enumerative geometry of Riemann surfaces: **unstable surfaces** with Euler characteristics

$$\chi \geq 0$$

are more irregular than **stable surfaces** ($\chi < 0$).

There is a deep algebraic geometry reason to that, because stable Riemann surfaces have a finite group of automorphims, and the volume of their moduli space is well-defined, whereas unstable surfaces have an infinite group of automorphisms (called "zero modes") and need to be "renormalized", see Chap. 5. Physicists would say that sums over unstable surfaces $\chi \geq 0$ involve zero modes, whereas stable surfaces $\chi < 0$ have no zero modes.

The main concepts presented in this chapter are:

- The disk amplitude, i.e. the generating function of planar rooted maps with one marked face, is algebraic. This defines an algebraic curve of genus zero, we call it the spectral curve.
- The cylinder amplitude, i.e. the generating function of planar maps with two boundaries, is universal, it is independent of the t_k's, independent of the type of maps. It is related to a geometric object of the spectral curve: its fundamental second kind differential.

B. Eynard, *Counting Surfaces*, Progress in Mathematical Physics 70,
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- Once we know the disk and cylinder amplitude, then all the higher topology amplitudes can be computed by a universal recursion relation called the "topological recursion". The topological recursion is the same for all types of maps, it works also for Ising model maps of Chap. 8, and it works for Kontsevich graphs of Chap. 6. The Chap. 7 is entirely devoted to the mathematical properties of the topological recursion.

3.1 Disk Amplitude

Disks are planar ($g = 0$) maps with one boundary ($k = 1$), or also "rooted maps" (see Sect. 1.2.6). The generating function of disks i.e. planar rooted maps satisfies the loop equation (2.5.3) of Chap. 2, i.e. Tutte's equation (1.3.1) of Chap. 1:

$$\sum_{j=0}^{l-1} \mathcal{T}_j^{(0)} \mathcal{T}_{l-1-j}^{(0)} = \mathcal{T}_{l+1}^{(0)} - \sum_{j=3}^{d} t_j \mathcal{T}_{l+j-1}^{(0)} \quad , \quad \mathcal{T}_0^{(0)} = t \tag{3.1.1}$$

where $\mathcal{T}_l^{(0)}$ is the generating series of planar maps ($g = 0$) with one boundary ($k = 1$) of perimeter l:

$$\mathcal{T}_l^{(0)} = t\,\delta_{l,0} + \sum_{v=2}^{\infty} t^v \sum_{\Sigma \in \mathbb{M}_1^{(0)}(v),\, l_1(\Sigma)=l} t_3^{n_3(\Sigma)} \dots t_d^{n_d(\Sigma)}.$$

3.1.1 Solving Tutte's Equation

It is more convenient to rewrite Tutte's equation (3.1.1) in terms of the resolvent $W_1^{(0)}$ (see Definition 1.2.2 of Chap. 2):

$$W_1^{(0)}(x) = \frac{t}{x} + \sum_{l=1}^{\infty} \frac{1}{x^{l+1}}\, \mathcal{T}_l^{(0)}$$

and Tutte's equation can be written as Eq. (2.5.4) for $W_1^{(0)}$:

$$\left(W_1^{(0)}(x)\right)^2 = V'(x) W_1^{(0)}(x) - P_1^{(0)}(x) \tag{3.1.2}$$

where we recall that:

$$V'(x) = x - \sum_{j=3}^{d} t_j x^{j-1} \quad , \quad P_1^{(0)}(x) = t - \sum_{j=3}^{d} \sum_{l=0}^{j-2} t_j\, \mathcal{T}_{j-l-2}^{(0)}\, x^l$$

$P_1^{(0)}(x)$ is a polynomial in x of degree $d-2$, we have:

$$P_1^{(0)}(x) = \Big(V'(x)\, W_1^{(0)}(x)\Big)_+$$

where $_+$ means the positive part of the Laurent series at ∞, indeed, the left hand side of Eq. (3.1.2) tends towards zero at ∞.

Equation (3.1.2) is called the **spectral curve**, we will develop the notion of spectral curves in Chap. 7.

Solving the second degree equation (3.1.2), yields:

$$W_1^{(0)}(x) = \frac{1}{2}\left(V'(x) - \sqrt{V'(x)^2 - 4P_1^{(0)}(x)}\ \right).$$

In other words, if we knew how to determine the polynomial $P_1^{(0)}$, i.e. the coefficients $\mathcal{T}_1^{(0)}, \dots, \mathcal{T}_{d-2}^{(0)}$, then we would have determined $W_1^{(0)}$, i.e. $\mathcal{T}_l^{(0)}$ for every l. We do it below.

3.1.2 A Useful Lemma

The following lemma allows to determine the polynomial $P_1^{(0)}$. It is very useful, and it is known under various names in the combinatorics or physics literature. In combinatorics it is more or less equivalent to **Brown's lemma** [20], and in physics it is called the **1-cut assumption** (although it is not an assumption, it is proved) [32, 40].

Lemma 3.1.1 (1-Cut Brown's Lemma) *The polynomial $V'(x)^2 - 4P_1^{(0)}(x)$ has only one pair of simple zeros, all the other zeros are even. More precisely, there exist α, γ^2 and $M(x)$ which are formal power series in t, and $M(x)$ is a polynomial of x*

$$\alpha \in \mathbb{Q}[t_3, \dots, t_d][[t]] \quad , \quad \gamma^2 \in \mathbb{Q}[t_3, \dots, t_d][[t]] \quad , \quad M(x) \in \mathbb{Q}[x, t_3, \dots, t_d][[t]]$$

satisfying:

$$\alpha = O(t) \qquad , \qquad \gamma^2 = t + O(t^2) \qquad , \qquad M(x) = \frac{V'(x)}{x} + O(t)$$

and

$$V'(x)^2 - 4P_1^{(0)}(x) = (M(x))^2\,(x-a)(x-b)$$

with $a = \alpha + 2\gamma$, $b = \alpha - 2\gamma$, i.e. $a, b \in \mathbb{Q}[t_3, \dots, t_d][[\sqrt{t}]]$.

The meaning of this lemma will become clearer in Chap. 4, where we discuss solutions of loop equations which do not satisfy this Lemma. As we shall see below, this lemma determines the polynomial $P_1^{(0)}$ uniquely.

Proof We have $\mathcal{T}_0^{(0)} = t$, and if $l \geq 1$, recall that $\mathcal{T}_l^{(0)}$ counts maps

$$\mathcal{T}_l^{(0)} = t\,\delta_{l,0} + \sum_{v=2}^{\infty} t^v \sum_{\Sigma \in \mathbb{M}_1^{(0)}(v),\, l_1(\Sigma)=l} t_3^{n_3(\Sigma)} \dots t_d^{n_d(\Sigma)}$$

where v is the number of vertices of the maps. Since our maps are disks, i.e. $g = 0$ and $k = 1$ boundary, the Euler characteristics constraint Eq. (1.2.1) implies (we have $l \geq 1$):

$$v = 1 + \frac{l}{2} + \frac{1}{2}\sum_{j=3}^{d}(j-2)\,n_j \quad \geq 2 \tag{3.1.3}$$

i.e. $\mathcal{T}_l^{(0)}$ is a power series that starts as $O(t^2)$ for $l \geq 1$. Therefore $P_1^{(0)}(x)$ vanishes at $t = 0$, it is a power series in t which starts at order 1 in t:

$$P_1^{(0)}(x) = t(1 - \sum_{j=3}^{d} t_j\,x^{j-2}) + O(t^2) = t\,\frac{V'(x)}{x} + O(t^2).$$

This implies that, to leading order in $\sqrt{t}$, $V'(x)^2 - 4P_1^{(0)}(x)$ is a perfect square, its zeros are double zeros close to the zeros of V', they are of the form (here, for simplicity, we assume that $V''(X_i) \neq 0$):

$$\sim X_i \pm 2\,\frac{\sqrt{P_1^{(0)}(X_i)}}{V''(X_i)} + o(\sqrt{t}) \qquad , \qquad V'(X_i) = 0 \quad , \; i = 1, \dots, \deg V'$$

and they are formal power series in $\sqrt{t}$. In other words, the zeros of $V'(x)^2 - 4P_1^{(0)}(x)$ come by pairs $[a_i, b_i]$ centered around the zeros X_i of $V'(x)$, and their distance to X_i is of order at most $O(\sqrt{t})$.

In particular, notice that one of the zeros of $V'(x)$, is $X_1 = 0$, and we have $V''(0) = 1$, and $P_1^{(0)}(0) = tV''(0) + O(t^2)$, thus:

$$a_1 \sim 2\sqrt{t} + o(\sqrt{t}) \qquad , \qquad b_1 \sim -2\sqrt{t} + o(\sqrt{t})$$

And for the other zeros of V', we have $\forall\, i = 2, \dots, \deg V'$, $P_1^{(0)}(X_i) = 0 + O(t^2)$ thus:

$$a_i \sim X_i + O(t) \qquad , \qquad b_i \sim X_i + O(t).$$

Then, recall that for given v, $\mathbb{M}_1^{(0)}(v)$ is a finite set (see Theorem 1.2.1 in Chap. 1), and thus, there is a maximum perimeter $l \leq 2v - 1$ [see Eq. (3.1.3)]. In other words, $W_1^{(0)}(x)$ is, order by order in t, a polynomial in $1/x$ (of degree at most $2v$), and we have:

$$W_1^{(0)}(x) = \frac{t}{x} + \sum_{v=2}^{\infty} t^v \sum_{l=1}^{2v-1} \frac{C_{v,l}}{x^{l+1}}$$

with some coefficients $C_{v,l} \in \mathbb{Q}[t_3, \ldots, t_d]$. This implies that

$$\frac{1}{2i\pi} \oint_{\mathcal{C}} W_1^{(0)}(x) dx = \begin{cases} t & \text{if } \mathcal{C} \text{ encircles } 0 \\ 0 & \text{otherwise} \end{cases} . \tag{3.1.4}$$

This equality holds for any positive oriented closed contour $\mathcal{C}$ in the complex plane, order by order in t.

Assume that $a_i \neq b_i$, and thus there exists $m_i \geq 0$ and $C_i \neq 0$ such that

$$a_i - b_i = C_i\, t^{m_i+1/2}(1 + O(t)).$$

Choose a contour $\mathcal{C}$ (independent of t), which surrounds the zero $X_i \neq 0$ of $V'(x)$, then, order by order in t, the contour $\mathcal{C}$ surrounds the pair $[a_i, b_i]$ of zeros of $V'(x)^2 - 4P_1^{(0)}(x)$. One easily computes that the contour integral $\oint_{\mathcal{C}} \sqrt{V'(x)^2 - 4P_1^{(0)}(x)}dx$ behaves like $i\pi\, \frac{V''(X_i)}{4}\, C_i^2\, t^{2m_i+1}\, (1 + O(t))$ at small t, thus it does not vanish.[1] Another way to see it, is that

$$\sqrt{x - a_i} = \sqrt{x - X_i} \left(1 + \frac{X_i - a_i}{2\,(x - X_i)} - \frac{(X_i - a_i)^2}{8\,(x - X_i)^2} + o((X_i - a_i)^2)\right)$$

and thus

$$\sqrt{(x - a_i)(x - b_i)} = (x - X_i) \left(1 + \frac{2X_i - a_i - b_i}{2\,(x - X_i)} - \frac{(a_i - b_i)^2}{8\,(x - X_i)^2} + o((X_i - a_i))^2\right)$$

and if $(a_i - b_i)^2$ would be non-zero to a certain order in t, $W_1^{(0)}$ would not be, to that order, a polynomial in $1/x$, it would have a pole at $x = X_i$.

This shows that the assumption $a_i \neq b_i$ was wrong and therefore this proves that if $i \neq 1$ we must have $a_i = b_i$ to all orders in t. Therefore $V'(x)^2 - 4P_1^{(0)}(x)$ has only one pair $[a_1, b_1]$ of simple zeros, all the others come by pairs:

$$V'(x)^2 - 4P_1^{(0)}(x) = M(x)^2\, (x - a_1)(x - b_1).$$

We have proved the Lemma. □

[1] For simplicity, we assume that $V''(X_i) \neq 0$. The lemma remains true when $V''(X_i) = 0$ but for the proof, one needs to go further in the Taylor expansion...

Notice that $\alpha = \frac{a+b}{2}$ and $\gamma^2 = \frac{(a-b)^2}{16}$ are formal power series of t (whereas a and b are power series of $\sqrt{t}$).

Remark 3.1.1 In Chap. 4, we are going to consider a situation where this lemma does not hold, i.e. we will have more odd zeros of $V'(x)^2 - 4P_1^{(0)}(x)$ centered around the other zeros of V'. That situation is called multi-cut solution of loop equations. In Chap. 4, we are going to see what is the combinatorics meaning of solutions of loop equations for which this lemma is not valid.

3.1.3 1-Cut Solution, Zhukovsky's Variable

Therefore, the solution of loop equation (3.1.2), is:

$$W_1^{(0)}(x) = \frac{1}{2}\left(V'(x) - M(x)\sqrt{(x-a)(x-b)}\ \right) \tag{3.1.5}$$

with

$$V'(x) = x - \sum_{j=3}^{d} t_j x^{j-1} \qquad , \qquad M(x) = \frac{V'(x)}{x} + O(t)$$

$$a = 2\sqrt{t} + o(\sqrt{t}) \qquad , \qquad b = -2\sqrt{t} + o(\sqrt{t}).$$

It remains to compute explicitly $M(x)$ as well as a and b.

3.1.3.1 Zhukovsky's Variable

In that purpose, instead of x, it is more convenient to use another more appropriate variable of expansion z, with the help of Zhukovsky's transformation:

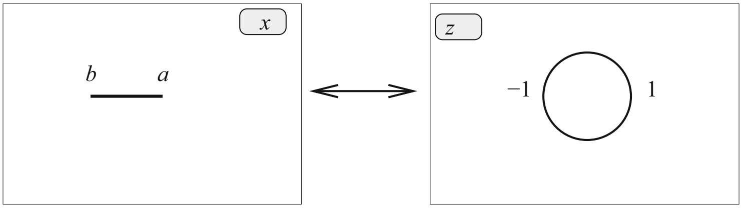
$$x(z) = \frac{a+b}{2} + \frac{a-b}{4}\left(z + \frac{1}{z}\right)$$

which has the property that $\sqrt{(x-a)(x-b)}$ is a rational function of z:

$$\sqrt{(x(z)-a)(x(z)-b)} = \frac{a-b}{4}\left(z - \frac{1}{z}\right)$$

and thus $W_1^{(0)}(x(z))$ is a rational function of z.

Zhukovsky's transformation maps the x–plane cut along the segment $[b, a]$ to the exterior of the unit disk in the z–plane, and the points a, b to $1, -1$. It maps ∞ to ∞, and the other sheet of the x–plane is mapped to the interior of the unit disk:



Zhukovsky was a discoverer of the aerodynamics of wings, and he invented that map in order to transform conformally an infinitely thin wing profile (the segment $[b, a]$), into a circular wing profile (the circle $|z| = 1$), for which equations of aerodynamics are much easier to solve.

The inverse relation between x and z is

$$z = \frac{1}{2\gamma}\left(x - \alpha + \sqrt{(x-\alpha)^2 - 4\gamma^2}\right)$$

where $\alpha = \frac{a+b}{2}$ and $\gamma = \frac{a-b}{4}$. Its large x expansion is

$$z = \frac{x-\alpha}{\gamma} - \sum_{n=0}^{\infty} C_n\,((x-\alpha)/\gamma)^{-2n-1} \qquad , \qquad C_n = \frac{(2n)!}{n!\,(n+1)!}$$

z as a function of x, can be seen as the generating function for Catalan numbers C_n.

3.1.3.2 Solution with Zhukovsky's Variable

In Zhukovsky's variable z, it is clear from Eq. (3.1.5), that $W_1^{(0)}$ is a rational function of z

Lemma 3.1.2 *The disk amplitude $W_1^{(0)}$ is a polynomial of $1/z$, of degree $d-1$.*

$$W_1^{(0)}(x(z)) = \sum_{k=1}^{d-1} u_k\, z^{-k}.$$

In other words

$$W_1^{(0)} \in \mathbb{Q}[1/z, t_3, t_4, \ldots, t_d][[t]].$$

Proof Indeed Eq. (3.1.5) implies that $W_1^{(0)}$ is a Laurent polynomial in z and $1/z$, and there can be no positive power of z because by definition $W_1^{(0)}(x)$ contains no positive power of x at large x, i.e.

$$\lim_{x\to\infty} W_1^{(0)}(x) = 0,$$

and $x = \frac{a+b}{2} + \frac{a-b}{4}(z + 1/z)$, and we take the convention that large x corresponds to large z. □

Then, the coefficients u_k's, as well as a and b can be determined as follows, expand $V'(x)$ into powers of z:

$$V'(x(z)) = \sum_{k=0}^{d-1} u_k(z^k + z^{-k})$$

($V'(x(z))$ is symmetric under $z \to 1/z$ because $x(z)$ is).

Then expand $y = -\frac{1}{2} M(x) \sqrt{(x-a)(x-b)}$, which is antisymmetric under $z \to 1/z$:

$$y(z) = -\frac{1}{2} M(x(z)) \sqrt{(x(z)-a)(x(z)-b)} = -\frac{1}{2} \sum_{k=1}^{d-1} \tilde{u}_k(z^k - z^{-k}). \tag{3.1.6}$$

Since $W_1^{(0)} = \frac{1}{2}V' + y$ must have no positive powers of z, we find:

$$u_k = \tilde{u}_k$$

and thus:

$$W_1^{(0)}(x(z)) = \sum_{k=1}^{d-1} u_k\, z^{-k}$$

In addition we must have

$$u_0 = 0$$

and, since $W_1^{(0)}(x) = \frac{t}{x} + O(1/x^2)$ at large x, and $x \sim \frac{a-b}{4}\, z$, we must have:

$$u_1 = \frac{4t}{a-b}.$$

Let us summarize those results into the following theorem:

Theorem 3.1.1 (Disk Amplitude) *For any α and γ, let $x(z) = \alpha + \gamma(z + 1/z)$, then expand:*

$$V'(x(z)) = \sum_{k=0}^{d-1} u_k\,(z^k + z^{-k}) \tag{3.1.7}$$

where the u_k's are polynomial in α and γ.

α and γ are the unique solutions of:

$$u_0 = 0 \qquad , \qquad u_1 = \frac{t}{\gamma}$$

which behave like

$$\alpha = O(t)\,,\ \gamma = \sqrt{t}(1 + O(t)).$$

Then, the disk amplitude $W_1^{(0)}$, i.e. the generating function of planar rooted maps is:

$$W_1^{(0)}(x(z)) = \sum_{k=1}^{d-1} u_k\, z^{-k}.$$

Examples of applications of this theorem are given in Sects. 3.1.7 and 3.1.8 below, where we compute explicitly the case of quadrangulations and triangulations.

More explicitly, one can expand Eq. (3.1.7), and write the u_k's as explicit polynomials of α and γ:

$$u_k = \alpha\delta_{k,0} + \gamma\delta_{k,1} - \sum_{l=2}^{d-1} \sum_{j=k}^{[(l+k)/2]} t_{l+1}\, \frac{l!}{j!j-k!l+k-2j!}\, \gamma^{2j-k}\, \alpha^{l+k-2j}.$$

In particular with $k = 0$ and $k = 1$, we see that α and γ are determined by two algebraic equations:

$$0 = u_0 = \alpha - \sum_{l=1}^{d-1} \sum_{j=0}^{l} t_{l+j+1}\, \frac{(l+j)!}{j!j!(l-j)!}\, \gamma^{2j}\, \alpha^{l-j}$$

$$\frac{t}{\gamma} = u_1 = \gamma - \sum_{l=2}^{d} \sum_{j=1}^{l} t_{l+j}\, \frac{(l+j-1)!}{j!(j-1)!(l-j)!}\, \gamma^{2j-1}\, \alpha^{l-j}.$$

Those two algebraic equations yield a finite number of solutions for α and γ, and it is easy to see there is a unique solution such that $\alpha \sim O(t)$ and $\gamma^2 \sim t + O(t^2)$ at small t.

To the first few orders we have:

$$\alpha = 2t_3\gamma^2 + (t_3\alpha^2 + 6t_4\alpha\gamma^2 + 6t_5\gamma^4) + \dots$$

$$\gamma^2 = t + (2t_3\gamma^2\alpha + 3t_4\gamma^4) + (3t_4\alpha^2\gamma^2 + 12t_5\alpha\gamma^4 + 20t_6\gamma^6) + \dots$$

i.e.

$$\gamma^2 = t + t^2(4t_3^2 + 3t_4) + O(t^3)\dots$$
$$\alpha = t(2t_3) + t^2(12t_3^3 + 12t_3t_4) + O(t^3)\dots$$

Let us summarize it as follows:

Theorem 3.1.2 (Disk Amplitude, Bis) *Let α and γ be the unique solutions of the algebraic system of equations:*

$$0 = \alpha - \sum_{l=1}^{d-1}\sum_{j=0}^{l} t_{l+j+1}\,\frac{(l+j)!}{j!j!(l-j)!}\,\gamma^{2j}\,\alpha^{l-j} \tag{3.1.8}$$

$$t = \gamma^2 - \sum_{l=2}^{d}\sum_{j=1}^{l} t_{l+j}\,\frac{(l+j-1)!}{j!(j-1)!(l-j)!}\,\gamma^{2j}\,\alpha^{l-j} \tag{3.1.9}$$

which behave like $\alpha = O(t)$ and $\gamma = \sqrt{t} + O(t)$. Then define

$$u_k = \alpha\delta_{k,0} + \gamma\delta_{k,1} - \sum_{l=2}^{d-1}\sum_{j=k}^{(l+k)/2} t_{l+1}\,\frac{l!}{j!j-k!l+k-2j!}\,\gamma^{2j-k}\,\alpha^{l+k-2j}.$$

One then has:

$$W_1^{(0)}(x) = \sum_{k=1}^{d-1} u_k z^{-k} \qquad , \qquad x = \alpha + \gamma(z + 1/z)$$

i.e.

$$W_1^{(0)}(x) = \sum_{k=1}^{d-1} u_k z^{-k} \qquad , \; z = \frac{1}{2\gamma}\left(x - \alpha + \sqrt{(x-\alpha)^2 - 4\gamma^2}\right).$$

It can also be written

$$W_1^{(0)}(x) = \frac{1}{2}\left(V'(x) - M(x)\sqrt{(x-a)(x-b)}\right)$$

where

$$a = \alpha + 2\gamma \qquad , \qquad b = \alpha - 2\gamma, \qquad , \qquad M(x) = \frac{1}{\gamma}\sum_{k=1}^{d-1} u_k\; U_{k-1}\left(\frac{x-\alpha}{2\gamma}\right)$$

*where U_k is the k*th *second kind Chebyshev polynomial.*

The second kind Chebyshev polynomials are defined by:

$$U_k(\cosh\phi) = \frac{\sinh(k+1)\phi}{\sinh\phi}$$

or in other words:

$$U_k\left(\frac{z+1/z}{2}\right) = \frac{z^k - z^{-k}}{z - z^{-1}}.$$

The first few of them are:

$$U_0(x) = 1 \quad , \quad U_1(x) = 2x \quad , \quad U_2(x) = 4x^2 - 1 \quad , \quad U_3(x) = 8x^3 - 4x \quad , \ldots$$

An explicit formula is given by the Taylor expansion at $x \to 1$:

$$U_k(x) = \sum_{n=0}^{k} (x-1)^n\, 2^n\, \frac{(k+n+1)!}{(2n+1)!\,(k-n)!}.$$

or also the Taylor expansion near $x = -1$:

$$U_k(x) = \sum_{n=0}^{k} (-1)^{n+k}\, (x+1)^n\, 2^n\, \frac{(k+n+1)!}{(2n+1)!\,(k-n)!}.$$

Later in Sect. 3.3.3, we shall need to consider the Moments of $M(x)$, defined as:

Definition 3.1.1 ($M(x)$ and Its Moments) We write:

$$W_1^{(0)}(x) = \frac{V'(x)}{2} + y$$

with

$$y = -\frac{1}{2}\, M(x)\, \sqrt{(x-a)(x-b)}$$

and the polynomial $M(x)$ is given by:

$$M(x) = \frac{1}{\gamma} \sum_{k=1}^{d-1} u_k\, U_{k-1}\left(\frac{x-\alpha}{2\gamma}\right).$$

We define the moments $M_{\pm,k}$ of $M(x)$ as the coefficients of its Taylor expansion near $x = a, b$, as:

$$M(x) = M_{+,0}\,(1 - \sum_{k=1}^{d-1} M_{+,k}\ (x-a)^k)$$
$$= M_{-,0}\,(1 - \sum_{k=1}^{d-1} M_{-,k}\ (x-b)^k).$$

We have:

$$M_{+,0} = M(a) = \frac{-y'(1)}{\gamma} \qquad , \qquad M_{-,0} = M(b) = \frac{-y'(-1)}{\gamma}.$$

And for $n \geq 1$, the nth moment at a (reps. at b) of $M(x)$ is:

$$M_{+,n} = \frac{-1}{n!\ M_{+,0}}\ \frac{d^n}{dx^n} M(x)\bigg|_{x=a} \qquad , \qquad M_{-,n} = \frac{-1}{n!\ M_{-,0}}\ \frac{d^n}{dx^n} M(x)\bigg|_{x=b}.$$

We thus have:

$$M_{+,n} = \frac{-1}{\gamma^{n+1}\, M_{+,0}} \sum_{k=n+1}^{d-1} u_k\ \ \frac{(k+n)!}{(2n+1)!\,(k-n-1)!}$$
$$M_{-,n} = \frac{-1}{\gamma^{n+1}\, M_{-,0}} \sum_{k=n+1}^{d-1} (-1)^{k+n+1}\, u_k\ \ \frac{(k+n)!}{(2n+1)!\,(k-n-1)!}.$$

The moments were introduced in a work by Ambjørn–Chekhov–Kristjansen–Makeenko in 1993 [5], and played an important role in solving the Tutte equations for higher topologies, as we shall see in Sect. 3.3.3 below.

3.1.3.3 Variational Principle

There is another way of writing the equations which determine α and γ. Indeed α and γ are critical points of the functional:

$$\mu(\alpha,\gamma) = 2t\ln\gamma + \operatorname*{Res}_{z\to\infty} V(\alpha + \gamma(z+1/z))\ \frac{dz}{z}.$$

Proof We use that $V'(x(z)) = \sum_k u_k(z^k + z^{-k})$, and we compute:

$$\frac{\partial \mu}{\partial \alpha} = \operatorname*{Res}_{z\to\infty} V'(\alpha + \gamma(z+1/z))\ \frac{dz}{z} = -u_0 = 0$$

$$\frac{\partial \mu}{\partial \gamma} = \frac{2t}{\gamma} + \operatorname*{Res}_{z\to\infty} V'(\alpha + \gamma(z+1/z))\ (z+1/z)\,\frac{dz}{z} = \frac{2t}{\gamma} - 2u_1 = 0.$$

□

3.1.4 Even–Bipartite Maps

Even maps are those containing only unmarked faces of even perimeters. In other words we choose all $t_{2k+1} = 0$.

For planar maps, even maps are also Bipartite: vertices can be colored with two colors, in such a way that adjacent vertices are of different color.

For even maps, $V(x)$ is an even polynomial, and thus $V'(x)$ is an odd function of x:

$$V'(x) = x - \sum_{k\geq 2} t_{2k}\ x^{2k-1}.$$

Equation (3.1.8) becomes:

$$0 = u_0 = \alpha \left(1 - \sum_{l=2}^{d/2}\sum_{j=0}^{l-1} t_{2l}\,\frac{(2l-1)!}{j!j!(2l-2j-1)!}\,\gamma^{2j}\,\alpha^{2l-2j-2}\right) = \alpha(1 - O(t)).$$

This implies that

$$\alpha = 0.$$

The Eq. (3.1.9) for γ is now a polynomial equation for γ^2:

$$t = \gamma^2 - \sum_{l=2}^{d/2} t_{2l}\,\frac{(2l-1)!}{l!(l-1)!}\,\gamma^{2l}.$$

whose solution is (see the proof in Lemma 3.1.3 below):

$$\gamma^2 = t + \sum_{k\geq 1} t^{k+1} \sum_{n=1}^{k} \frac{(k+n)!}{(k+1)!\,n!} \sum_{a_1+\cdots+a_n=k,\,a_i>0}\ \prod_{i=1}^{n} \frac{(2a_i+1)!\ t_{2a_i+2}}{a_i!\,(a_i+1)!}.$$

To the first few orders:

$$\gamma^2 = t + t^2(3t_4) + t^3(18\,t_4^2 + 10\,t_6) + t^4(35\,t_8 + 150\,t_4\,t_6 + 40\,t_4^3) + O(t^5)\ldots$$

Moreover, all the u_{2k} vanish, and $W_1^{(0)}(x(z))$ is an odd function of z:

$$W_1^{(0)}(x(z)) = \sum_{k=1}^{d/2} u_{2k-1}\,z^{1-2k}.$$

Let us summarize it as the following theorem

Theorem 3.1.3 (Disk Amplitude Bipartite) *Let γ be the unique solution of*

$$t = \gamma^2 - \sum_{l=2}^{d/2} t_{2l}\,\frac{(2l-1)!}{l!(l-1)!}\,\gamma^{2l}.$$

which behaves like $\gamma = \sqrt{t} + O(t)$, i.e.

$$\begin{aligned}\gamma^2 &= t + \sum_{k\geq 1} t^{k+1} \sum_{n=1}^{k} \frac{(k+n)!}{(k+1)!\,n!} \sum_{a_1+\cdots+a_n=k,\,a_i>0} \prod_{i=1}^{n} \frac{(2a_i+1)!\;t_{2a_i+2}}{a_i!\,(a_i+1)!}\\ &= t + t^2(3t_4) + t^3(18\,t_4^2 + 10\,t_6) + t^4(35\,t_8 + 150\,t_4\,t_6 + 40\,t_4^3) + O(t^5).\end{aligned}$$

Then let

$$x(z) = \gamma(z + 1/z).$$

Let

$$u_{2k+1} = \gamma\left(\delta_{k,0} - \sum_{j\geq 2k+1} t_{2j-2k}\,\frac{(2j-2k-1)!}{j!\,(j-2k-1)!}\,\gamma^{2j-2k-2}\right)$$

Then the disk amplitude is

$$W_1^{(0)}(x(z)) = \sum_{k\geq 0} u_{2k+1}\,z^{-2k-1} \qquad ,\ z = \frac{1}{2\gamma}\left(x + \sqrt{x^2-4\gamma^2}\right).$$

Remark 3.1.2 Notice that γ^2 is a formal series of t, whose coefficients are polynomial of the t_k's with positive integer coefficients:

$$\gamma^2 \in t\,\mathbb{Z}_+[t_4, t_6, \ldots][[t]].$$

We can be even more precisely, let $\tilde{t}_k = \frac{(2k+1)!}{k!\,(k+1)!}\, t_{2k+2}$, we have

$$\gamma^2 \in t\,\mathbb{Z}_+[\tilde{t}_1, \tilde{t}_2, \tilde{t}_3, \dots][[t]],$$

and explicitly

$$\begin{aligned}\gamma^2 &= t + \sum_{k\geq 1} t^{k+1} \sum_{n=1}^{k} \frac{(k+n)!}{(k+1)!\,n!} \sum_{a_1+\dots+a_n=k,\, a_i>0} \prod_{i=1}^{n} \tilde{t}_{a_i} \\ &= t + t^2\,\tilde{t}_1 + t^3(\tilde{t}_2 + 2\,\tilde{t}_1^2) + t^4(\tilde{t}_3 + 5\,\tilde{t}_1\tilde{t}_2 + 5\,\tilde{t}_1^3) \\ &\quad + t^5(\tilde{t}_4 + 6\,\tilde{t}_1\tilde{t}_3 + 3\,\tilde{t}_2^2 + 21\,\tilde{t}_1^2\tilde{t}_2 + 14\,\tilde{t}_1^4) + O(t^6).\end{aligned}$$

Lemma 3.1.3 (Lagrange Inversion) *For bipartite maps, we have for any $m \geq 1$*

$$\gamma^{2m} = t^m \Big(1 + m \sum_{k\geq 1} t^k \sum_{n=1}^{k} \frac{(k+n+m-1)!}{(k+m)!\,n!} \sum_{a_1+\dots+a_n=k,\, a_i>0} \prod_{i=1}^{n} \tilde{t}_{a_i}\Big).$$

where

$$\tilde{t}_k = \frac{(2k+1)!}{k!\,(k+1)!}\, t_{2k+2}.$$

Proof This is the Lagrange inversion formula's method. It works as follows: the equation satisfied by $x = \gamma^2$ is

$$t = \gamma^2\,(1 - P(\gamma^2)) = x(1 - P(x)) \qquad , \qquad P(x) = \sum_{k\geq 1} \tilde{t}_k x^k.$$

The nth coefficient of the asymptotic expansion

$$x^m = \sum_n c_{n,m} t^n$$

is:

$$c_{n,m} = \operatorname*{Res}_{t\to 0} x^m \frac{dt}{t^{n+1}}$$

we integrate by parts:

$$\begin{aligned}c_{n,m} &= \frac{m}{n} \operatorname*{Res}_{x\to 0} t^{-n}\, x^{m-1}\, dx \\ &= \frac{m}{n} \operatorname*{Res}_{x\to 0} x^{-n}\,(1 - P(x))^{-n}\, x^{m-1}\, dx\end{aligned}$$

$$= \frac{m}{n} \operatorname*{Res}_{x \to 0} \frac{dx}{x} x^{m-n} \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!k!} P(x)^k$$

$$= \frac{m}{n} \operatorname*{Res}_{x \to 0} \frac{dx}{x} x^{m-n} \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!k!} \sum_{a_1,\dots,a_k} \tilde{t}_{a_1} \dots \tilde{t}_{a_k} \, x^{a_1+\dots+a_k}$$

$$= \frac{m}{n} \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!k!} \sum_{a_1+\dots+a_k=n-m} \tilde{t}_{a_1} \dots \tilde{t}_{a_k}$$

since $a_i > 0$, we see that the sum over k is in fact bounded by $k \leq n-m$. This gives the lemma. □

3.1.5 Generating Functions of Disks of Fixed Perimeter

The generating function $\mathcal{T}_l^{(0)}$ of disks of perimeter l is the coefficient of $1/x^{l+1}$ in $W_1^{(0)}(x)$, i.e.:

$$\mathcal{T}_l^{(0)} = - \operatorname*{Res}_{x \to \infty} W_1^{(0)}(x) \; x^l \; dx.$$

After changing variable $x \to x(z)$ it can be written

$$\mathcal{T}_l^{(0)} = - \operatorname*{Res}_{z \to \infty} W_1^{(0)}(x(z)) \; (x(z))^l \; x'(z) dz$$

(this is where Remark 1.2.5 is useful, one must not forget the Jacobian $x'(z)$ of the change of variable).

By the binomial formula, we expand:

$$x(z)^l = (\alpha + \gamma(z + 1/z))^l = \sum_{j+k \leq l} \frac{l!}{j!k!(l-j-k)!} \, \alpha^{l-k-j} \, \gamma^{k+j} \, z^{k-j},$$

and we can perform the residue:

$$\mathcal{T}_l^{(0)} = \sum_{j+k \leq l} \frac{l!}{j!k!(l-j-k)!} \, \alpha^{l-k-j} \, \gamma^{k+j+1} \, (u_{k-j+1} - u_{k-j-1})$$

$$= \sum_{j+k \leq l} \frac{l!}{(j+1)!k!(l-1-j-k)!} \, \alpha^{l-1-k-j} \, \gamma^{k+j+2} \, u_{k-j}$$

$$- \sum_{j+k \leq l} \frac{l!}{j!(k+1)!(l-1-j-k)!} \, \alpha^{l-1-k-j} \, \gamma^{k+j+2} \, u_{k-j}.$$

Thus:

Corollary 3.1.1 *The generating function of disks with perimeter l is*

$$\mathcal{T}_l^{(0)} = \sum_{j+k<l,\, j<k<j+d} \frac{(k-j)\ \ l!}{(j+1)!(k+1)!(l-1-j-k)!}\, \alpha^{l-1-k-j}\, \gamma^{k+j+2}\, u_{k-j} \tag{3.1.10}$$

where u_k are such that $V'(x(z)) = \sum_k u_k (z^k + z^{-k})$.

Alternatively this can be written

$$\mathcal{T}_l^{(0)} = \sum_{j+k\leq l,\, j<k<j+d} \frac{l!}{j!k!(l-1-j-k)!}\, \alpha^{l-1-k-j}\, \gamma^{k+j+2}\, v_{k-j}$$

where the coefficients $v_k = u_{k-1} - u_{k+1}$ are such that $V(x(z)) = \sum_k \frac{v_k}{k}(z^k + z^{-k})$.

Corollary 3.1.2 *In particular, for bipartite maps we have $\alpha = 0$ and $u_{2k} = 0$, in which case we obtain:*

$$\mathcal{T}_{2l}^{(0)} = \gamma^{2l+1} \sum_{j=1}^{[d/2]} \frac{(2j-1)\ (2l)!}{(l-j+1)!(l+j)!}\, u_{2j-1}\ , \quad \mathcal{T}_{2l+1}^{(0)} = 0. \tag{3.1.11}$$

or with the v_js:

$$\mathcal{T}_{2l}^{(0)} = \gamma^{2l+1} \sum_{j=1}^{[d/2]} \frac{(2l)!}{(l-j)!(l+j-1)!}\, v_{2j-1}.$$

Remark 3.1.3 (Convergency of Formal Series) So far, all $\mathcal{T}_l^{(0)}$ were defined as formal series in powers of t, and we didn't know whether those series were convergent or not.

Now we see that $\mathcal{T}_l^{(0)}$ is a polynomial of α and γ, and α and γ are solutions of an algebraic equation, therefore all of them are algebraic functions of t. This means that those series are convergent in a certain disk. They may diverge at a finite number of values of t, and they have algebraic singularities.

Those algebraic singularities, and their implication on the behavior of large maps are studied in full details in Chap. 5.

3.1.6 Derivatives of the Disk Amplitude

Let us mention a few useful properties of the disk amplitude, in particular derivatives with respect to the parameters.

Lemma 3.1.4

$$\left.\frac{\partial W_1^{(0)}(x(z))}{\partial t}\right|_{x(z)} x'(z) = \left.\frac{\partial y(z)}{\partial t}\right|_z x'(z) - \left.\frac{\partial x(z)}{\partial t}\right|_z y'(z) = \frac{1}{z}, \tag{3.1.12}$$

$$\begin{aligned}\left.\frac{\partial W_1^{(0)}(x(z))}{\partial t_k}\right|_{x(z)} x'(z) &= -\frac{1}{2k}\frac{d\,x(z)^k}{dz} + \left.\frac{\partial y(z)}{\partial t_k}\right|_z x'(z) - \left.\frac{\partial x(z)}{\partial t_k}\right|_z y'(z) \\ &= -\frac{d}{dz}\frac{1}{k}(x(z)^k)_-, \end{aligned}\tag{3.1.13}$$

where in the leftmost term, the derivative is taken at constant $x(z)$, and in the middle term the derivatives are taken at constant z, and where the function $y(z)$ was defined in Eq. (3.1.6).

Proof Remember that we have $W_1^{(0)}(x(z)) = \frac{V'(x(z))}{2} + y(z)$, and $V'(x(z)) = x(z) - \sum_k t_{k+1}\, x(z)^k$, therefore, from the chain rule, we have:

$$\left.\frac{\partial W_1^{(0)}(x(z))}{\partial t}\right|_{x(z)} x'(z) = \left.\frac{\partial y(z)}{\partial t}\right|_z x'(z) - \left.\frac{\partial x(z)}{\partial t}\right|_z y'(z),$$

$$\left.\frac{\partial W_1^{(0)}(x(z))}{\partial t_k}\right|_{x(z)} x'(z) = -\frac{1}{2k}\frac{d\,x(z)^k}{dz} + \left.\frac{\partial y(z)}{\partial t_k}\right|_z x'(z) - \left.\frac{\partial x(z)}{\partial t_k}\right|_z y'(z).$$

This implies that the right hand side is a polynomial of z and $1/z$. Beside, the antisymmetry $y(1/z) = -y(z)$ and $x(1/z) = x(z)$ implies that

$$\frac{\partial y}{\partial t}\frac{\partial x}{\partial z} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial z} = \frac{1}{z}\sum_j c_j(z^j + z^{-j}).$$

At large $x(z)$, i.e. at large z, we have $W_1^{(0)}(x(z)) \sim t/x(z) + O(z^{-2})$, and therefore $\partial W_1^{(0)}(x)/\partial t \sim 1/x + O(1/x^2) = 1/\gamma z + O(z^{-2})$, and thus

$$\frac{\partial y}{\partial t}\frac{\partial x}{\partial z} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial z} \sim \frac{1}{z} + O(z^{-2}).$$

This implies that we can have only the c_0 term, and thus:

$$\frac{\partial y}{\partial t}\frac{\partial x}{\partial z} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial z} = \frac{1}{z}.$$

Similarly,

$$\frac{\partial y(z)}{\partial t_k}\Big|_z x'(z) - \frac{\partial x(z)}{\partial t_k}\Big|_z y'(z)$$

is clearly a polynomial in z and $1/z$, and from the symmetry $z \leftrightarrow 1/z$, it must be of the form

$$\frac{\partial y}{\partial t_k}\frac{\partial x}{\partial z} - \frac{\partial x}{\partial t_k}\frac{\partial y}{\partial z} = \frac{1}{z}\sum_j c_j(z^j + z^{-j}).$$

Since $W_1^{(0)}(x(z)) \sim t/z + O(1/z^2)$ at large z, we see that

$$-\frac{1}{2k}\frac{d\,x(z)^k}{dz} + \frac{1}{z}\sum_j c_j(z^j + z^{-j}) = O(z^{-2}). \qquad (3.1.14)$$

A derivative can never have a $1/z$ term, and $O(z^{-2})$ also has no $1/z$ term, therefore we must have $c_0 = 0$, and thus

$$\frac{1}{z}\sum_j c_j(z^j + z^{-j}) = \frac{d}{dz}\sum_{j\geq 1}\frac{c_j}{j}(z^j - z^{-j}).$$

Taking only the positive powers of z in Eq. (3.1.14), we see that:

$$\frac{d}{dz}\sum_{j\geq 1} c_j\, z^j = \frac{1}{2k}\frac{d}{dz}(x(z)^k)_+$$

and by symmetry $z \leftrightarrow 1/z$:

$$\frac{d}{dz}\sum_{j\geq 1} c_j\,(z^j - z^{-j}) = \frac{1}{2k}\frac{d}{dz}\left((x(z)^k)_+ - (x(z)^k)_-\right)$$

and the lemma follows. □

Remark 3.1.4 (Hamiltonian Structure)

Those derivatives can be written with Poisson brackets:

$$\{f,g\}_k \stackrel{\text{def}}{=} \left(\frac{\partial f}{\partial t_k}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial t_k}\frac{\partial f}{\partial z}\right). \quad , \quad \{f,g\} \stackrel{\text{def}}{=} z\left(\frac{\partial f}{\partial t}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial t}\frac{\partial f}{\partial z}\right).$$

With those notations, Lemma 3.1.4 can be written

$$\{y,x\} = 1 \quad , \quad \{y,x\}_k = H_k',$$

where $H_k = \frac{1}{2k}\left((x(z)^k)_+ - (x(z)^k)_-\right)$, and $'$ means derivative with respect to z.

The compatibility $\partial_{t_k}\,\partial_{t_l}\; W_1^{(0)}(x) = \partial_{t_l}\,\partial_{t_k}\; W_1^{(0)}(x)$ implies that:

$$\{H'_k, x\}_l = \{H'_l, x\}_k,$$

and

$$\frac{1}{z^2}\{H'_k, x\} = \{1/z, x\}_k = \frac{1}{z^2}\,\frac{\partial x}{\partial t_k},$$

i.e.

$$\frac{\partial x}{\partial t_k} = \{H'_k, x\},$$

i.e. H'_k is a Hamiltonian generating the flow with respect to the time t_k.

Again, the compatibility $\partial_{t_k}\partial_{t_l} = \partial_{t_l}\partial_{t_k}$ implies that those hamiltonians Poisson commute with each other:

$$\{H'_k, H'_l\} = 0.$$

An integrable system can be constructed, and we refer the interested reader to [8, 38, 40] for more details about the integrable structure of matrix models and maps. To summarize we have:

$$\{y, x\} = 1 \qquad , \qquad \frac{\partial x}{\partial t_k} = \{H'_k, x\} \qquad , \qquad \{H'_k, H'_l\} = 0.$$

3.1.7 Example: Planar Rooted Quadrangulations

Let us see how to apply Theorem 3.1.1 to quadrangulations.

We choose all $t_k = 0$ except t_4 (quadrangulations are bipartite maps). We then have:

$$V'(x) = x - t_4 x^3.$$

With that definition, $\mathcal{T}_l^{(0)}$ counts quadrangulations with n_4 quadrangles and a boundary of size l:

$$\mathcal{T}_l^{(0)} = \sum_{n_4} t^v\, t_4^{n_4}\ \#\{\text{planar quadrangulations with one boundary of perimeter } l\,\}$$

where v is the number of vertices. Notice that we have Eq. (3.1.3):

$$v = 1 + n_4 + \frac{l}{2}$$

i.e. l must be even, and v and n_4 are not independent.

Then we compute the resolvent generating series:

$$W_1^{(0)}(x) = \frac{t}{x} + \sum_{l=1}^{\infty} \frac{1}{x^{2l+1}}\ \mathcal{T}_{2l}^{(0)}$$

with the help of Theorem 3.1.1: let $x(z) = \alpha + \gamma(z + 1/z)$, we have:

$$V'(x(z)) = \alpha + \gamma(z + 1/z) - t_4\Big[\alpha^3 + 3\alpha^2\gamma(z + z^{-1}) \\ +3\alpha\gamma^2(z^2 + 2 + z^{-2}) + \gamma^3(z^3 + 3z + 3z^{-1} + z^{-3})\Big]$$

i.e.:

$$u_0 = \alpha - t_4(\alpha^3 + 6\alpha\gamma^2) \quad , \quad u_1 = \gamma - t_4(3\alpha^2\gamma + 3\gamma^3)$$
$$u_2 = -3t_4\alpha\gamma^2 \quad , \quad u_3 = -t_4\gamma^3.$$

The equations $u_0 = 0$ and $u_1 = t/\gamma$ allow to determine α and γ:

$$0 = \alpha - t_4(\alpha^3 + 6\alpha\gamma^2) = \alpha(1 - t_4(\alpha^2 + 6\gamma^2))$$
$$\frac{t}{\gamma} = \gamma - t_4(3\alpha^2\gamma + 3\gamma^3) = \gamma(1 - t_4(3\alpha^2 + 3\gamma^2)).$$

Since α and γ must tend to 0 at small t, we have $1 - t_4(\alpha^2 + 6\gamma^2) \neq 0$ order by order in t, and therefore the first equation implies $\alpha = 0$. Then the second equation gives

$$t = \gamma^2 - 3t_4\gamma^4$$

whose solution is (the sign in front of the square root must be $-$, in order for γ to behave like $\sqrt{t}$ at small t):

$$\begin{cases} \alpha = 0 \\ \gamma^2 = \frac{1}{6t_4}\left(1 - \sqrt{1 - 12\,tt_4}\,\right) = t(1 + 3tt_4 + 18(tt_4)^2 + \dots) = t\sum_n \frac{(3tt_4)^n\ (2n)!}{(n+1)!\,n!}\ . \end{cases} \tag{3.1.15}$$

We shall write

$$r = \sqrt{1 - 12\,t\,t_4}\ ,$$

and thus

$$\gamma^2 = 2t/(1 + r).$$

As an expansion we have (using the Lagrange formula of Lemma 3.1.3)

$$\gamma^2 = t(1 + 3tt_4 + 18(tt_4)^2 + \dots)$$
$$= t\sum_n \frac{(3tt_4)^n\ (2n)!}{(n+1)!\,n!}.$$

From Theorem 3.1.1, we get the disk amplitude:

$$W_1^{(0)}(x(z)) = \frac{t}{\gamma z} - t_4\gamma^3\frac{1}{z^3}. \tag{3.1.16}$$

We can also write it:

$$W_1^{(0)}(x) = \frac{1}{2}\left(x - t_4x^3 - M(x)\sqrt{(x-a)(x-b)}\right)$$

where

$$M(x) = -t_4\,(x^2 - \gamma^2 - \frac{t}{t_4\gamma^2}) = r - \frac{1-r}{6}\left(\frac{x^2}{\gamma^2} - 4\right).$$

The moments are

$$M_{+,0} = M_{-,0} = r,$$
$$M_{+,1} = -M_{-,1} = \frac{1-r}{3\,r}\,\frac{1}{\gamma},$$
$$M_{+,2} = M_{-,2} = \frac{1-r}{3\,r}\,\frac{1}{\gamma^2}.$$

See Fig. 3.1.

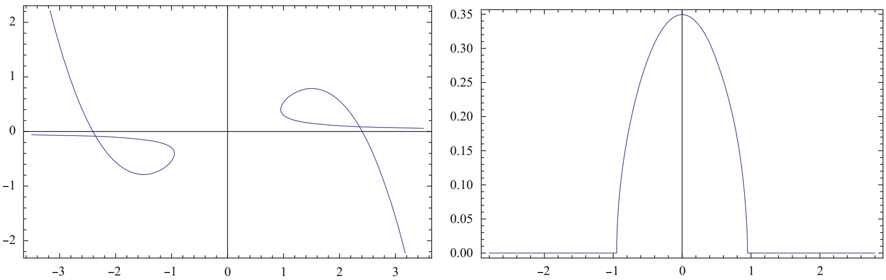


Fig. 3.1 Plot of the spectral curve $W_1^{(0)}(x)$ and $\frac{-1}{2\pi t}\,\mathrm{Im}\,W_1^{(0)}(x)$, for $t = 0.2$ and $t_4 = 1/6$. $\rho(x) = \frac{-1}{2\pi t}\,\mathrm{Im}\,W_1^{(0)}(x)$ would be large N limit of the eigenvalue density of a random matrix with probability measure $e^{-N\,\mathrm{Tr}\,V(M)}dM$

Then using Eq. (3.1.11) for fixed perimeter $2l$:

$$\mathcal{T}_{2l}^{(0)} = \gamma^{2l} \frac{(2l)!}{l!(l+2)!} \left[(2l+2)t - l\gamma^2\right] \quad , \quad \mathcal{T}_{2l+1}^{(0)} = 0$$

which counts the number of planar rooted quadrangulations with a boundary of perimeter $2l$ (resp. $2l+1$).

$$\mathcal{T}_{2l}^{(0)} = \frac{(2l)!}{l!(l+2)!} \left(\frac{2t}{r+1}\right)^{l+1} (r\,l + r + 1).$$

Lemma 3.1.3 gives the expansion into powers of t:

$$\mathcal{T}_{2l}^{(0)} = \frac{(2l)!}{l!\,(l-1)!}\, t^{l+1} \sum_n (3tt_4)^{n-1} \frac{(2n+l-3)!}{(l+n)!(n-1)!}.$$

The number of rooted planar quadrangulations with $n_4 = n-1$ quadrangles, and with a marked face of perimeter $2l$ is thus:

$$3^{n_4} \frac{(2l)!}{l!\,(l-1)!} \frac{(2n_4+l-1)!}{(l+n_4+1)!\,n_4!}.$$

In particular with $l = 2$, we find the number of planar rooted quadrangulations where all n faces, including the marked one, are quadrangles:

$$\begin{aligned}\mathcal{T}_4^{(0)} &= \gamma^4\,(3t-\gamma^2)\\ &= 4t^3\,\frac{1+3r}{(1+r)^3}\\ &= t^3 \sum_n \frac{2\ 3^n\,(2n)!}{n!\,(n+2)!}\,(tt_4)^{n-1}.\end{aligned}$$

Thus we recover the famous result of Tutte [84, 85] that the number of rooted quadrangulations with n faces is:

$$\frac{2\ 3^n\,(2n)!}{n!\,(n+2)!}.$$

3.1.8 Example: Planar Rooted Triangulations

If we want only triangulations, we choose all $t_k = 0$ except t_3. We then have:

$$V'(x) = x - t_3 x^2.$$

The generating series of planar rooted triangulations with n_3 triangles, and a boundary of length l is:

$$\mathcal{T}_l^{(0)} = \sum_{n_3} t^v\, t_3^{n_3}\ \#\{\text{planar triangulations with one boundary of perimeter } l\,\}.$$

Notice that we have Eq. (3.1.3):

$$v = 1 + \frac{l + n_3}{2}$$

i.e. $l + n_3$ must be even, and v and n_3 are not independent.

Then we compute the resolvent generating series

$$W_1^{(0)}(x) = \frac{t}{x} + \sum_{m=1}^{\infty} \frac{1}{x^{m+1}}\ \mathcal{T}_m^{(0)}$$

with Theorem 3.1.1: let $x(z) = \alpha + \gamma(z + 1/z)$, we have:

$$V'(x(z)) = \alpha + \gamma(z + 1/z) - t_3\left[\alpha^2 + 2\alpha\gamma(z + z^{-1}) + \gamma^2(z^2 + 2 + z^{-2})\right]$$

i.e.:

$$u_0 = \alpha - t_3(\alpha^2 + 2\gamma^2) = 0 \qquad , \qquad u_1 = \gamma - 2\alpha\gamma t_3 = \frac{t}{\gamma} \qquad , \qquad u_2 = -t_3\gamma^2.$$

Theorem 3.1.1 requires $u_0 = 0$ and $u_1 = t/\gamma$, which implies (after eliminating α in the equation for γ^2):

$$\alpha = \frac{1}{2t_3}\,(1 - \frac{t}{\gamma^2}) \qquad , \qquad \frac{t^2}{\gamma^6} - \frac{1}{\gamma^2} + 8t_3^2 = 0.$$

The cubic equation for $1/\gamma^2$, can be solved as follows:

$$12\sqrt{3}\ t\,t_3^2 = \sin(3\phi) \qquad , \qquad \frac{\sqrt{3}}{2}\,\frac{t}{\gamma^2} = \sin(\frac{\pi}{3} - \phi)$$

that gives

$$\gamma^2 = \frac{t\ \cos\frac{\pi}{6}}{\cos(\frac{\pi}{6} + \frac{1}{3}\text{Arcsin}(12\sqrt{3}\ t\,t_3^2))} = t(1 + 4tt_3^2 + 40t^2t_3^4 + 2^9\,t^3t_3^6 + \ldots.$$

It is convenient to write $\gamma^2 = t/r$, and thus r obeys the equation:

$$r - r^3 = 8tt_3^2,$$

and we must chose the solution $r = 1 + O(t)$. We then have:

$$\gamma^2 = \frac{t}{r} \qquad , \qquad \alpha\, t_3 = \frac{1-r}{2}.$$

The Lagrange inversion formula allows to expand $t/\gamma^2 = r = \sum_n c_n (8tt_3^2)^n$, in that purpose we write

$$\begin{aligned}
c_n &= \operatorname*{Res}_{t\to 0}\; r\, \frac{d(8tt_3^2)}{(8tt_3^2)^{n+1}} \\
&= \operatorname*{Res}_{r\to 1}\; r\, \frac{(1-3r^2)\, dr}{(r-r^3)^{n+1}} \qquad \text{(change of variable } 8tt_3^2 = r - r^3 \text{)} \\
&= \operatorname*{Res}_{r\to 1}\; r\, \frac{(1-3r^2)\, dr}{(r-r^3)^{n+1}} \\
&= \operatorname*{Res}_{r\to 1}\; \frac{1-3r^2}{r^n}\, \frac{dr}{(1-r^2)^{n+1}} \\
&= \frac{1}{2} \operatorname*{Res}_{u\to 1}\; \frac{1-3u}{u^{(n+1)/2}}\, \frac{du}{(1-u)^{n+1}} \qquad \text{(change of variable } r^2 = u \text{)} \\
&= \frac{-1}{2} \operatorname*{Res}_{v\to 0}\; \frac{3v-2}{(1-v)^{(n+1)/2}}\, \frac{dv}{v^{n+1}} \qquad \text{(change of variable } u = 1-v \text{)} \\
&= \frac{-1}{2} \operatorname*{Res}_{v\to 0}\; (3v-2) \sum_k \frac{\Gamma((n+1)/2+k)}{k!\,\Gamma((n+1)/2)}\, v^k\, \frac{dv}{v^{n+1}} \\
&= \frac{-1}{2} \left[3\frac{\Gamma((n+1)/2+n-1)}{(n-1)!\,\Gamma((n+1)/2)} - 2\frac{\Gamma((n+1)/2+n)}{n!\,\Gamma((n+1)/2)} \right] \\
&= \frac{-1}{2}\, \frac{\Gamma((3n-1)/2)}{n!\,\Gamma((n+1)/2)}.
\end{aligned}$$

and finally

$$\frac{1}{\gamma^2} = \frac{-1}{2t} \sum_n (8tt_3^2)^n\, \frac{\Gamma((3n-1)/2)}{n!\Gamma((n+1)/2)} = \frac{1}{t}(1 - 4tt_3^2 - 24t^2t_3^4 + \dots).$$

By a similar method, we find

$$\begin{aligned}
\gamma^2 &= t \sum_n (8tt_3^2)^n\, \frac{\Gamma((3n+1)/2)}{(n+1)!\,\Gamma((n+1)/2)} \\
\gamma^4 &= 2\,t^2 \sum_n (8tt_3^2)^n\, \frac{(n+1)\,\Gamma((3n+2)/2)}{(n+2)!\,\Gamma((n+2)/2)}.
\end{aligned}$$

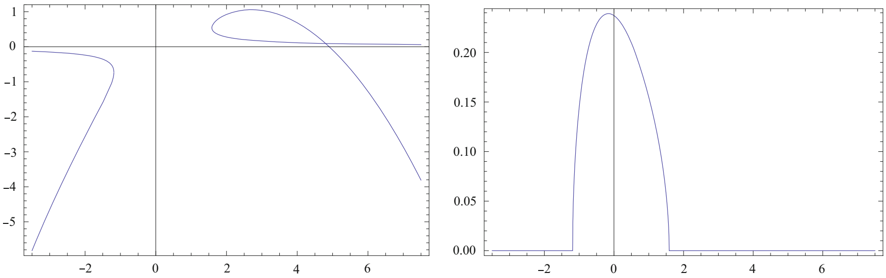


Fig. 3.2 Plot of the spectral curve $W_1^{(0)}(x)$ and $\frac{-1}{2\pi t}\,\mathrm{Im}\,W_1^{(0)}(x)$, for $t_3 = 0.2$ and $t = 0.44$. $\rho(x) = \frac{-1}{2\pi t}\,\mathrm{Im}\,W_1^{(0)}(x)$ would be large N limit of the eigenvalue density of a random matrix with probability measure $e^{-N\,\mathrm{Tr}\,V(M)}dM$

Finally the resolvent $W_1^{(0)}$ is:

$$W_1^{(0)}(x(z)) = \frac{t}{\gamma z} - t_3\gamma^2\frac{1}{z^2}$$

We have

$$W_1^{(0)}(x(z)) = \frac{1}{2}\left(V'(x) - M(x)\sqrt{(x-a)(x-b)}\right)$$

with

$$M(x) = -t_3 x + t_3\alpha + \frac{t}{\gamma^2} = -t_3 x + \frac{1+r}{2}.$$

Its moments are

$$M_{\pm,0} = r \mp \sqrt{\frac{1-r^2}{2}} \qquad , \qquad M_{+,0}M_{-,0} = \frac{3r^2-1}{2}.$$

$$M_{\pm,1} = \frac{\pm 1}{\gamma}\,\frac{1}{3r^2-1}\left(\frac{1-r^2}{2} \pm r\sqrt{\frac{1-r^2}{2}}\right).$$

See Fig. 3.2.

Then we easily compute the number of planar rooted triangulations $\mathcal{T}_3^{(0)}$ where all faces, including the marked one, are triangles, using Eq. (3.1.10):

$$\begin{aligned}\mathcal{T}_3^{(0)} &= \frac{t}{8t_3^3}\left[-\frac{1}{r} + 4 - 4r + r^3\right] \\ &= \frac{t}{2t_3^3} - \frac{t^2}{t_3} - \frac{t}{8t_3^3}\left[\frac{1}{r} + 3r\right]\end{aligned}$$

$$= -\frac{t}{8t_3^3}\sum_{n\geq 2}(8tt_3^2)^n\left[\frac{\Gamma((3n+1)/2)}{(n+1)!\,\Gamma((n+1)/2)} - \frac{3}{2}\,\frac{\Gamma((3n-1)/2)}{n!\,\Gamma((n+1)/2)}\right]$$

$$= -\frac{t}{8t_3^3}\sum_{n\geq 2}(8tt_3^2)^n\,\frac{\Gamma((3n-1)/2)}{(n+1)!\,\Gamma((n+1)/2)}\left[\frac{3n-1}{2} - \frac{3}{2}\,(n+1)\right]$$

$$= \frac{2\,t}{8t_3^3}\sum_{n\geq 2}(8tt_3^2)^n\,\frac{\Gamma((3n-1)/2)}{(n+1)!\,\Gamma((n+1)/2)}$$

$$= \frac{2\,t^2}{t_3}\sum_{n\geq 1}(8tt_3^2)^n\,\frac{\Gamma(3n/2+1)}{(n+2)!\,\Gamma(n/2+1)}$$

i.e. the number of rooted planar triangulations with $2n$ faces is:

$$2^{3n+1}\,\frac{\Gamma(3n/2+1)}{(n+2)!\,\Gamma(n/2+1)},$$

which is again Tutte's result [84, 85].

3.1.9 Example: Gaussian Matrix Integral, Catalan Numbers

For book keeping purpose, as well as for normalization purposes, it is also interesting to consider the quadratic potential $V(x) = \frac{t_2}{2}x^2$, although it is not directly a generating function of maps in the sense of Chap. 2, it has a combinatorial interpretation as Catalan numbers which we won't discuss in this book (it was used for instance in Harrer-Zagier's work [45]).

In that case:

$$Z = \int_{H_N} \mathrm{e}^{-\frac{Nt_2}{2t}\,\mathrm{Tr}\,M^2}\,dM$$

is a Gaussian integral, and thus can be computed exactly. With our definition of dM we find:

$$Z = (t_2)^{-N^2/2} = \exp\left(\sum_{g=0}^{\infty}(N/t)^{2-2g}F_g\right)$$

therefore:

$$F_0 = -\frac{t^2}{2}\ln(t_2) \qquad , \qquad F_g = 0 \quad \forall\, g > 0.$$

We write:

$$x(z) = \alpha + \gamma(z + \frac{1}{z})$$

$$V'(x(z)) = t_2 x(z) = t_2\alpha + t_2\gamma(z + \frac{1}{z})$$

and Theorem 3.1.1 implies:

$$t_2\alpha = 0 \qquad , \qquad \frac{t}{\gamma} = t_2\gamma$$

i.e. $\alpha = 0$ and

$$\gamma = \sqrt{t/t_2}.$$

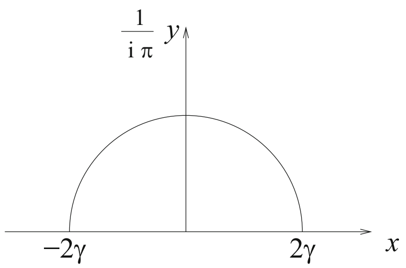
The spectral curve of the Gaussian model is thus [we use def. Eq. (3.1.6)]:

$$\begin{cases} x(z) = \gamma(z + \frac{1}{z}) \\ y(z) = \frac{t}{2\gamma}(\frac{1}{z} - z) = -\frac{t_2}{2}\sqrt{x^2 - 4\gamma^2} \end{cases}$$

in other words

$$W_1^{(0)}(x(z)) = \frac{t_2}{2}x(z) + y(z) = \frac{t}{\gamma\, z}.$$

If we plot $\frac{-1}{i\pi}y$, as a function of x in the range $-2\gamma \leq x \leq 2\gamma$, we obtain a semi–circle of area t, which is the famous Wigner's semi-circle.



We also find, using Eq. (3.1.10):

$$\mathcal{T}_{2l}^{(0)} = t\,\gamma^{2l}\,\frac{(2l)!}{l!\,(l+1)!}$$

which is the lth Catalan number.

3.2 Cylinders/Annulus Amplitude

By "cylinder" or "annulus", we mean genus zero connected maps with two boundaries of perimeters l_1, l_2, i.e. elements of $\mathbb{M}_2^{(0)}$, with generating function:

$$W_2^{(0)}(x_1, x_2) = \sum_v t^v \sum_{\Sigma\in\mathbb{M}_2^{(0)}(v)} \frac{t_3^{n_3}\dots t_d^{n_d}}{x_1^{l_1+1}\,x_2^{l_2+1}}\,\frac{1}{\#\mathrm{Aut}(\Sigma)}$$
$$= \sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty} \frac{\mathcal{T}^{(0)}_{l_1,l_2}}{x_1^{l_1+1}\,x_2^{l_2+1}}.$$

The loop equation (2.5.3) for $g = 0$ and $k = 2$ is:

$$2\sum_{j=0}^{l_1-1} \mathcal{T}_j^{(0)}\mathcal{T}^{(0)}_{l_1-1-j,l_2} + l_2\,\mathcal{T}^{(0)}_{l_2+l_1-1} = \mathcal{T}^{(0)}_{l_1+1,l_2} - \sum_{j=3}^{d} t_j\mathcal{T}^{(0)}_{l_1+j-1,l_2} \tag{3.2.1}$$

which can be rewritten in terms of $W_2^{(0)}$ and $W_1^{(0)}$:

$$2W_1^{(0)}(x_1)\,W_2^{(0)}(x_1,x_2)+\frac{\partial}{\partial x_2}\,\frac{W_1^{(0)}(x_1)-W_1^{(0)}(x_2)}{x_1-x_2} = V'(x_1)W_2^{(0)}(x_1,x_2)-P_2^{(0)}(x_1,x_2) \tag{3.2.2}$$

where

$$P_2^{(0)}(x_1,x_2) = -\sum_{j=3}^{d}\sum_{l=0}^{j-3}\sum_{m_2=1}^{\infty} t_j x_1^l\,\frac{1}{x_2^{m_2+1}}\,\mathcal{T}^{(0)}_{j-2-l,m_2}$$

is a polynomial in x_1 of degree at most $d-3$ (because maps in $\mathbb{M}_2^{(0)}$ must have a boundary of length $j-2-l \geq 1$). Since the left hand side of Eq. (3.2.2) has no positive part, we have:

$$P_2^{(0)}(x_1,x_2) = \Big(V'(x_1)\,W_2^{(0)}(x_1,x_2)\Big)_{+x_1},$$

where the subscript $()_{+x_1}$ means keeping only positive powers of x_1 at large x_1.

Inserting the expression of $W_1^{(0)}$ of Eq. (3.1.5) into Eq. (3.2.2), we obtain:

$$W_2^{(0)}(x_1,x_2) = \frac{P_2^{(0)}(x_1,x_2) + \frac{\partial}{\partial x_2}\frac{W_1^{(0)}(x_1)-W_1^{(0)}(x_2)}{x_1-x_2}}{V'(x_1)-2W_1^{(0)}(x_1)}$$
$$= \frac{P_2^{(0)}(x_1,x_2) + \frac{\partial}{\partial x_2}\frac{W_1^{(0)}(x_1)-W_1^{(0)}(x_2)}{x_1-x_2}}{M(x_1)\,\sqrt{(x_1-a)(x_1-b)}} \tag{3.2.3}$$

i.e.

$$W_2^{(0)}(x_1,x_2) = \frac{P_2^{(0)}(x_1,x_2) + \frac{1}{2}\frac{\partial}{\partial x_2}\frac{V'(x_1)-V'(x_2)}{x_1-x_2}}{M(x_1)\,\sqrt{(x_1-a)(x_1-b)}}$$
$$-\frac{1}{2}\frac{1}{(x_1-x_2)^2} + \frac{1}{2}\frac{\frac{\partial}{\partial x_2}\frac{M(x_2)\,\sqrt{(x_2-a)(x_2-b)}}{x_1-x_2}}{M(x_1)\,\sqrt{(x_1-a)(x_1-b)}}. \qquad (3.2.4)$$

In particular this tells us that $W_2^{(0)}$ is an algebraic function of x_1, with possible $(x_1 - a)^{-1/2}$ and $(x_1-b)^{-1/2}$ square root singularities at the branch points a and b. It may also have simple poles at the zeros of $M(x_1)$ or double poles at $x_1 = x_2$.

This also shows that in terms of Zhukovsky variables, $W_2^{(0)}(x(z_1), x(z_2))$ is a rational function of z_1, with possible poles at $z_1 = +1, -1, z_2, 1/z_2$, and also possible simple poles at the zeros of $M(x(z_1))$.

In order to see that poles at the zeros of $M(x(z_1))$ are not possible, we need a small variation of the 1-cut Brown's Lemma 3.1.1 :

Lemma 3.2.1 (1-Cut Lemma for Cylinders) *$W_2^{(0)}(x(z_1), x(z_2))\, x'(z_1)x'(z_2)$ is a rational function of z_1 and z_2, which behaves as $O(z_1^{-2})$ at large z_1, and it has a pole only at $z_1 = 1/z_2$, and this pole is a double pole with coefficient $-z_2^{-2}$, and with no residue:*

$$W_2^{(0)}(x(z_1), x(z_2))\; x'(z_1)x'(z_2) \underset{z_1\to 1/z_2}{\sim} \frac{-z_2^{-2}}{(z_1 - \frac{1}{z_2})^2} + O(1)$$

Proof From Eq. (3.2.4) we see that $W_2^{(0)}(x_1,x_2)$ may possibly have a simple pole at x_1 a zero of $M(x_1)$. Recall that $M(x_1) = \frac{V'(x_1)}{x_1} + O(t)$, so that the zeros of $M(x_1)$ are zeros of V' plus a power series in t.

Similarly to Lemma 3.1.1, to every order in t, $W_2^{(0)}(x_1,x_2)$ is a polynomial in $1/x_1$, starting with $O(1/x_1^2)$, and thus for every contour $\mathcal{C}$, we have order by order in t:

$$\frac{1}{2i\pi}\oint_{\mathcal{C}} W_2^{(0)}(x,x_2)dx = 0. \qquad (3.2.5)$$

In particular, if we choose a contour which surrounds a zero of $V'(x)$, and thus surrounds a zero of $M(x)$ order by order in t, this implies that the residue of $W_2^{(0)}$ is zero to all orders in t, and therefore $W_2^{(0)}(x,x_2)$ can have no pole at that point.

Also, it is clear that the expressions in the right hand side of Eq. (3.2.3) have no pole at $x_1 = x_2$, provided that both x_1 and x_2 are in the same sheet (same sign of the square-root). In other words, there is no pole at $z_1 = z_2$, but there could be a pole at $z_1 = 1/z_2$.

In the right hand side of expression Eq. (3.2.4), there are terms with a square-root $\sqrt{(x_1-a)(x_1-b)}$ which change sign when $z_1 \to 1/z_1$, and there is a term $-\frac{1}{2(x_1-x_2)^2}$ which does not change sign. Writing that there is no pole at $z_1 = z_2$, gives the coefficient of the double pole of the square-root term, and changing the sign, we get the coefficient of the pole at $z_1 = 1/z_2$.

Moreover, there is no pole at the branchpoints $z_1 = \pm 1$, because the multiplication by $x'(z_1) = \gamma(1 - z_1^{-2})$ cancels the possible $(x_1 - a)^{-1/2}$ and $(x_1 - b)^{-1/2}$.

Also, it is easy to see from the degree of each terms in Eq. (3.2.3), that it behaves as $O(z_1^{-2})$ when $z_1 \to \infty$, and the symmetry $z_1 \to 1/z_1$ implies that there is also no pole at $z_1 = 0$.

This proves the Lemma. □

With this lemma, it is rather easy to find $W_2^{(0)}$:

Theorem 3.2.1 (Cylinder Amplitude) *The cylinder generating function is:*

$$W_2^{(0)}(x(z_1), x(z_2))\ x'(z_1)x'(z_2) = \frac{-1}{(z_1 z_2 - 1)^2} = \frac{1}{(z_1 - z_2)^2} - \frac{x'(z_1)x'(z_2)}{(x(z_1) - x(z_2))^2}. \tag{3.2.6}$$

Proof This is the only rational fraction of z_1 which satisfies Lemma 3.2.1. □

3.2.1 Universality and Fundamental Second Kind Kernel

The cylinder generating function is universal! Written in the Zhukovsky variables, it is independent of the t_j's, it depends on nothing, it is independent of the type of maps we are considering.

In fact, the bi-differential form:

$$B(z_1, z_2) = \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2}$$

is called the **fundamental second kind differential** or Bergman–Schiffer kernel on the Riemann sphere (z lives in the complex plane $\mathbb{C}$, or more precisely, on the Riemann sphere $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$), or "heat kernel of the Riemann sphere", or also derivative of the "Green function", and it plays a very important role in algebraic geometry (see [12–14, 36, 37]), and Chap. 7.

Remark 3.2.1 On an arbitrary compact Riemann surface (here the Riemann sphere), the fundamental form of the second kind, is the unique differential form (in z_1), which has a double pole at $z_1 = z_2$ and no other pole, and which behaves near the pole as

$$B(z_1, z_2) \sim \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2} + O((z_1 - z_2)^0).$$

In the case where z_1 would live on a higher genus Riemann surface, there would be also a cycle integral normalization condition, see Chap. 7. It can be proved that this differential form is unique, and it is always symmetric in z_1 and z_2. For example on the torus, the unique function with a double pole is the Weierstrass function $\wp(z)$, modulo a constant, and thus the Bergman-Schiffer kernel would be $B(z_1, z_2) = (\wp(z_1 - z_2) + C)\, dz_1 \otimes dz_2$, with a constant C chosen to normalize some cycle integral, see Chap. 7 or Chap. 4 for details).

What we have found, is that:

Theorem 3.2.2 (Universal Cylinder Amplitude and Bergman-Schiffer Fundamental Second Kind Kernel) *Up to the addition of a trivial term [rational in $x(z)$], and written as a differential form, the cylinder generating function for any sort of maps, is always the fundamental second kind kernel*

$$W_2^{(0)}(x(z_1), x(z_2))\, dx(z_1)dx(z_2) + \frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2} = B(z_1, z_2). \tag{3.2.7}$$

Remark 3.2.2 We call $dx(z_1)dx(z_2)/(x(z_1) - x(z_2))^2$ a "trivial term" for reasons which will be clear in Sect. 3.2.2 below, basically, it does not contribute to residues, it does not contribute to the computation of $\mathcal{T}^{(0)}_{l_1,l_2}$.

Remark 3.2.3 In our case, the Zhukovsky variable z lives on the Riemann sphere $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$, and the fundamental second kind kernel of the Riemann sphere is simply

$$B(z_1, z_2) = \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2}.$$

However, the fact that the generating function of cylinders $W_2^{(0)}$ is the fundamental second kind kernell, is valid far beyond the case studied in this chapter. It continues to hold for "multicut case" (see Chap. 4), and it holds also for more complicated sorts of maps, like Ising model maps (see Chap. 8).

3.2.2 *Cylinders of Fixed Perimeter Lengths*

We have

$$\mathcal{T}^{(0)}_{l_1,l_2} = \operatorname*{Res}_{x_1\to\infty} \operatorname*{Res}_{x_2\to\infty} x_1^{l_1}\, x_2^{l_2}\, W_2^{(0)}(x_1, x_2)\, dx_1\, dx_2.$$

Changing to Zhukovsky variables, it is easy to see that the last term in Eq. (3.2.6) plays no role in the residue, therefore we may compute the residue with the fundamental second kind kernel only:

$$\mathcal{T}^{(0)}_{l_1,l_2} = \operatorname*{Res}_{z_1\to\infty} \operatorname*{Res}_{z_2\to\infty} x(z_1)^{l_1}\, x(z_2)^{l_2}\, \frac{dz_1 dz_2}{(z_1 - z_2)^2}$$

which is a universal polynomial of α and γ^2.

In particular for bipartite maps, with $\alpha = 0$, and writing

$$x(z)^l = \gamma^l\,(z+1/z)^l = \gamma^l\,z^l \sum_{j=0}^{l} \frac{l!}{j!\,(l-j)!}\,z^{-2j},$$

this simplifies to:

Corollary 3.2.1 *For bipartite maps, assume $l_1 \geq l_2$ and $l_1 + l_2$ even, one has*

$$\mathcal{T}^{(0)}_{l_1,l_2} = \gamma^{l_1+l_2} \sum_{j=0}^{\lfloor l_2/2 \rfloor} (l_2 - 2j)\,\frac{l_1!\,l_2!}{j!\,(\frac{l_1-l_2}{2}+j)!\,(\frac{l_1+l_2}{2}-j)!\,(l_2-j)!}$$

where $\lfloor l_2/2 \rfloor$ is the largest integer $\leq l_2/2$.

The coefficient of $\gamma^{l_1+l_2}$ is an integer number, independent of the t_k's and of t, it is a pure constant.

Example: Quadrangulations

Let us chose $l_1 = l_2 = 4$, and remember that for quadrangulations $\gamma^2 = \frac{1}{6t_4}(1 - \sqrt{1-12tt_4})$, so that we find, using Lemma 3.1.3:

$$\mathcal{T}^{(0)}_{4,4} = 36\,\gamma^8 = 16\,t^4 \sum_{n\geq 2} \frac{(2n-1)!}{(n-2)!\,(n+2)!}\;3^n\;(t\,t_4)^{n-2} = 36\,t^4 + 432\,t^5 t_4 + 4536\,t^6 t_4^2 + \ldots$$

i.e. the number of maps with $n = n_4 + 2$ faces, where all faces including the two marked faces are quadrangles, is $4 \times 3^n \frac{(2n-1)!}{(n-2)!\,(n+2)!}$. One can check that this is consistent with the number of maps with only one boundary:

$$\frac{\partial \mathcal{T}^{(0)}_4}{\partial t_4} = \frac{1}{4}\,\mathcal{T}^{(0)}_{4,4}.$$

Example: Triangulations

Let us chose $l_1 = l_2 = 3$, one gets

$$\mathcal{T}^{(0)}_{3,3} = \operatorname*{Res}_{z_1\to\infty} \operatorname*{Res}_{z_2\to\infty} x(z_1)^3\,x(z_2)^3\,\frac{dz_1 dz_2}{(z_1-z_2)^2} = 9\alpha^4\gamma^2 + 36\alpha^2\gamma^4 + 12\gamma^6.$$

For triangulations $\gamma^2 = t/r$, $\alpha = (1-r)/2t_3$, where $r - r^3 = 8tt_3^2$, so that we find:

$$\mathcal{T}^{(0)}_{3,3} = \frac{9}{4}\,\frac{t}{t_3^4} \sum_{n\geq 1} \frac{(2n-1)\,\Gamma(\frac{3n}{2})}{(n+2)!\,\Gamma(\frac{n}{2})}\;(8\,t\,t_3^2)^{n+1} = 12\,t^3 + 288\,t^4 t_3^2 + 5040\,t^5 t_3^4 + \ldots$$

i.e. the number of planar maps with $2n$ faces, where all faces including the two marked faces are triangles, is $18\,\frac{(2n-1)\Gamma(\frac{3n}{2})}{(n+2)!\,\Gamma(\frac{n}{2})}\,2^{3n}$.

Again, one can check that this is consistent with the number of maps with only one boundary:

$$\frac{\partial \mathcal{T}_3^{(0)}}{\partial t_3} = \frac{1}{3}\,\mathcal{T}_{3,3}^{(0)}.$$

3.3 Higher Topology and Topological Recursion

We are now going to compute $W_k^{(g)}$ for values of $(g,k) \neq (0,1),(0,2)$, i.e. such that $2g-2+k>0$, or in other words, for maps of strictly negative Euler characteristics $\chi = 2-2g-k<0$. In algebraic geometry, Riemann surfaces of strictly negative Euler characteristics are called "stable".

Let us choose (g,k) such that $2g-2+(k+1)>0$. We shall consider maps of genus g with $k+1$ boundaries. Let us call l the length of the first boundary, and let $L=\{l_1,\dots,l_k\}$ denote collectively the boundary lengths of k other boundaries, labeled $1,\dots,k$. We are now going to compute:

$$\mathcal{T}_{l,l_1,\dots,l_k}^{(g)} = \mathcal{T}_{l,L}^{(g)}$$

which is the generating function that counts the number of genus g connected maps with $k+1$ boundaries of respective perimeters $l, l_1, \dots, l_k$.

3.3.1 *Preliminary Results: Analytical Properties*

Lemma 3.3.1 $W_k^{(g)}(x(z_1),\dots,x(z_k))$ *is a rational function of its Zhukovsky variables* $z_1,\dots,z_k$.

Proof The general loop equation is Eq. (2.5.3):

$$\sum_{j=0}^{l_1-1}\Big[\sum_{h=0}^{g}\sum_{J\subset L}\mathcal{T}_{j,J}^{(h)}\mathcal{T}_{l_1-1-j,L\setminus J}^{(g-h)} + \mathcal{T}_{j,l_1-1-j,L}^{(g-1)}\Big] + \sum_{j=1}^{|L|} l_j\,\mathcal{T}_{l_j+l_1-1,L\setminus\{j\}}^{(g)}$$
$$= \mathcal{T}_{l_1+1,L}^{(g)} - \sum_{j=3}^{d} t_j \mathcal{T}_{l_1+j-1,L}^{(g)}$$

again, we write it in terms of the resolvent generating function:

$$W_{k+1}^{(g)}(x,x_1,\dots,x_k) = \sum_{l,l_1,\dots,l_k=0}^{\infty}\frac{\mathcal{T}_{l,l_1,\dots,l_k}^{(g)}}{x^{l+1}x_1^{l_1+1}\dots x_k^{l_k+1}}$$

which satisfies (we now write $L = \{x_1, \dots, x_k\}$):

$$\sum_{h=0}^{g} \sum_{J \subset L} \left[W_{|J|+1}^{(h)}(x, J) W_{k-|J|+1}^{(g-h)}(x, L \setminus J) \right] + W_{k+2}^{(g-1)}(x, x, L)$$
$$+ \sum_{j=1}^{|L|} \frac{\partial}{\partial x_j} \frac{W_k^{(g)}(x, L \setminus \{x_j\}) - W_k^{(g)}(L)}{x - x_j}$$
$$= V'(x) W_{k+1}^{(g)}(x, L) - P_{k+1}^{(g)}(x, L) \qquad (3.3.1)$$

where $P_{k+1}^{(g)}(x, L)$ is a polynomial of degree $d-3$ in the variable x.
This equation is sufficient to prove the lemma □

This lemma allows to define

Definition 3.3.1 We define the rational functions:

$$\omega_k^{(g)}(z_1, \dots, z_k) = W_k^{(g)}(x(z_1), \dots, x(z_k))\, x'(z_1) \dots x'(z_k)$$
$$+ \delta_{k,2} \delta_{g,0} \frac{x'(z_1) x'(z_2)}{(x(z_1) - x(z_2))^2}$$
$$- \frac{1}{2} \delta_{k,1} \delta_{g,0}\, V'(x(z_1))\, x'(z_1).$$

For $k = 1$, $g = 0$, we have already found:

$$W_1^{(0)}(x(z)) = \frac{1}{2} V'(x(z)) + y(z) = \sum_{j=1}^{d-1} u_j z^{-j}$$

where $y(z)$ was already computed in Eq. (3.1.6):

$$y(z) = -y(1/z) = -\frac{1}{2} \sum_{j=1}^{\deg V'} u_j (z^j - z^{-j}).$$

In other words

$$\omega_1^{(0)}(z) = y(z)\, x'(z).$$

And for $k = 2$, $g = 0$ we have already computed:

$$\omega_2^{(0)}(z_1, z_2) = \frac{1}{(z_1 - z_2)^2}.$$

Remark 3.3.1 Here we consider $\omega_k^{(g)}$ (rational functions of the z_i's), instead of the $W_k^{(g)}$ (algebraic functions of the x_i's), where we have multiplied by $x'(z_i)$. The fundamental reason to do this change of function, is related to Remark 1.2.5 in Chap. 1, the generating functions $W_k^{(g)}$ should be viewed as differential forms, rather than functions. Indeed, if $f(x)$ is a function of x, then the differential form $f(x)dx$, written in the Zhukovski variable becomes:

$$f(x)dx \quad \to \quad f(x(z))\, x'(z)\, dz$$

this is the reason why we multiplied $W_k^{(g)}$ by $\prod_i x'(z_i)$, which is the Jacobian of the change of variable.

The topological recursion of Chap. 7 is naturally written in terms of differential forms.

Now we are going to explain the method to compute all others $\omega_k^{(g)}$'s.

First, we need the following anti-symmetry lemma:

Lemma 3.3.2 (Anti-Symmetry Lemma) *If* $2g+k-2>0$, $\omega_k^{(g)}(z_1,\dots,z_k)/x'(z_1)$ *is an antisymmetric function under* $z_1 \to 1/z_1$, *i.e.*

$$\frac{1}{x'(1/z_1)}\ \omega_k^{(g)}(1/z_1,\dots,z_k) = -\frac{1}{x'(z_1)}\ \omega_k^{(g)}(z_1,\dots,z_k)$$

or:

$$\omega_k^{(g)}(1/z_1,\dots,z_k) = z_1^2\ \omega_k^{(g)}(z_1,\dots,z_k).$$

Another way of writing this, is with differential forms:

$$\omega_k^{(g)}(1/z_1,\dots,z_k)\, d(1/z_1) = -\ \omega_k^{(g)}(z_1,\dots,z_k)\, dz_1.$$

In other words the differential forms $\omega_k^{(g)}(z_1,\dots,z_k)\, dz_1 \dots dz_k$ are antisymmetric. See Remark 3.3.1 above.

Proof It is easily proved by recursion from the loop equation. Let us temporarily define:

$$\widetilde{W_k}^{(g)}(z_1,\dots,z_k) = W_k{}^{(g)}(x(z_1),\dots,x(z_k)) + \frac{1}{2}\,\delta_{k,2}\delta_{g,0}\,\frac{1}{(x(z_1)-x(z_2))^2}$$

[notice the factor $1/2$, to be compared with Eq. (3.2.7)] we have:

$$\widetilde{W_2}^{(0)}(z_1,z_2) = \frac{1}{2(x(z_1)-x(z_2))^2}\ \frac{(z_1+1/z_1)\,(z_2+1/z_2)-4}{(z_1-1/z_1)\,(z_2-1/z_2)}$$

which has the antisymmetry property.

The loop equation can be written:

$$\begin{aligned}
&-2y(z)\,\widetilde{W}^{(g)}_{k+1}(z,L)\\
&=\sum_{h=0}^{g}\sum_{J\subset L}{}^{\prime}\left[\widetilde{W}^{(h)}_{|J|+1}(z,J)\widetilde{W}^{(g-h)}_{k-|J|+1}(z,L/J)\right]+\widetilde{W}^{(g-1)}_{k+2}(z,z,L)\\
&\quad-\sum_{j=1}^{|L|}\frac{\partial}{\partial x(z_j)}\,\frac{\widetilde{W}^{(g)}_{k}(L)}{x(z)-x(z_j)}+P^{(g)}_{k+1}(x(z),L)
\end{aligned}$$

where the symbol $\sum'_{h,J}$ means that we exclude from the sum the term that we have put in the left hand side, i.e. the terms $(h,J)=(0,\emptyset)$ and $(h,J)=(g,L)$. By recursion hypothesis, the right hand side has the antisymmetry property, so the left hand side is antisymmetric. This proves the lemma. □

Then we have the following lemma:

Lemma 3.3.3 (Analytical Behavior Lemma) *If $2g+k-2>0$, $\omega^{(g)}_k(z_1,\dots,z_k)$ is a rational function of its Zhukovsky variables $z_1,\dots,z_k$, with poles only at the branch points $z_i=\pm 1$, and which behaves as $O(z_i^{-2})$ at large z_i.*

Proof Loop equation seem to imply that $\omega^{(g)}_k(z_1,\dots,z_k)$ could possibly have poles at $z_i=z_j$ or at $z_i=1/z_j$, or at the zeros of $y(z_i)$, or at $z_i=0$ or $z_i=\infty$ or at $z_i=\pm 1$.

What we need to prove is that $\omega^{(g)}_k(z_1,\dots,z_k)$ doesn't have poles at $z_i=0,\infty,z_j,1/z_j,0,\infty$, neither at the zeros of $y(z)$.

This is proved by recursion. Assume that the theorem has been proved for all $\omega^{(g')}_{k'+1}$ with $0<2g'-2+k'\le 2g-2+k$.

Using the antisymmetry, write the loop equation (now $L=\{z_1,\dots,z_k\}$):

$$\begin{aligned}
&2y(z)\,\omega^{(g)}_{k+1}(z,L)\,x'(1/z)\\
&=\sum_{h=0}^{g}\sum_{J\subset L}{}^{\prime}\left[\omega^{(h)}_{|J|+1}(z,J)\,\omega^{(g-h)}_{k-|J|+1}(\frac{1}{z},L\setminus J)\right]+\omega^{(g-1)}_{k+2}(z,1/z,L)\\
&\quad+\sum_{j=1}^{|L|}\frac{\partial}{\partial z_j}\,\frac{\omega^{(g)}_{k}(z,L\setminus\{z_j\})x'(1/z)+\omega^{(g)}_{k}(L)\,x'(z)\,x'(1/z)/x'(z_j)}{x(z)-x(z_j)}\\
&\quad-P^{(g)}_{k+1}(x(z),L)\,x'(z)x'(1/z)\,x'(z_1)\dots x'(z_k)
\end{aligned}$$

where Σ' means that we exclude the terms $(h = 0, J = 0)$, and $(h = g, J = L)$. We thus have

$$\begin{aligned}
&\omega^{(g)}_{k+1}(z, L) \\
&= \sum_{h=0}^{g} \sum_{J\subset L}' \frac{\omega^{(h)}_{|J|+1}(z, L)\omega^{(g-h)}_{k-|J|+1}(\frac{1}{z}, L \setminus J)}{2y(z)x'(1/z)} + \frac{\omega^{(g-1)}_{k+2}(z, 1/z, L)}{2y(z)x'(1/z)} \\
&\quad + \sum_{j=1}^{|L|} \frac{1}{2y(z)} \frac{\partial}{\partial z_j} \frac{\omega^{(g)}_{k}(z, L \setminus \{z_j\}) + \omega^{(g)}_{k}(L)\, x'(z)/x'(z_j)}{x(z) - x(z_j)} \\
&\quad - \frac{x'(z)}{2y(z)} P^{(g)}_{k+1}(x(z), L)\; x'(z_1) \ldots x'(z_k).
\end{aligned} \tag{3.3.2}$$

It is clear that in the Zhukovsky variable, the right hand side is a rational function of z. It may possibly have poles at $z = \pm 1$, at $z = 0$ or $z = \infty$, at the zeros of $y(z)$, or also at $z = z_j$ or $z = 1/z_j$ with $j = 1, \ldots, k$.

First, it is very easy to see that it behaves as $O(z^{-2})$ at $z = \infty$, since $y(z) \sim O(z^{\deg V'})$, and $\deg P^{(g)}_{k+1} \leq \deg V'''$. From the antisymmetry Lemma 3.3.2, or for the same reason $(y(z) \sim O(z^{-\deg V'}))$ there is no pole at $z = 0$.

Also, the recursion relation Eq. (3.3.1) is regular at $x(z) = x(z_j)$, which means that $\omega^{(g)}_{k+1}$ has no pole at $z = z_j$, and from the antisymmetry lemma, it has also no pole at $z = 1/z_j$.

From the recursion hypothesis, $\omega^{(g)}_{k+1}$ could have at most simple poles at the zeros of $y(z)$. To see that there is no pole at those points, it is sufficient to prove that the residues vanish, which is proved by the following "1-cut" lemma:

Lemma 3.3.4 (1-Cut Lemma) *for every contour $\mathcal{C}$ which does not enclose the branch-points ± 1, we have, order by order in t:*

$$\oint_{\mathcal{C}} \omega^{(g)}_{k+1}(z, z_1, \ldots, z_k) dz = 0.$$

Proof This is the generalization of Brown's 1-cut Lemmas 3.1.1 and 3.2.1. This lemma holds because to every order in t, $W^{(g)}_{k+1}(x, x_1, \ldots, x_k)$ is a polynomial in $1/x$ starting at order $1/x^2$. □

Applying this lemma to a contour surrounding a zero of $y(z)$, shows that $\omega^{(g)}_{k+1}$ can have no residue, and thus no pole at that point.

Thus, we have proved the recursion hypothesis, i.e. $\omega^{(g)}_{k+1}$ has poles only at the branch points $z_j = \pm 1$. □

3.3.2 The Topological Recursion

As we shall see now, knowing that the $\omega_k^{(g)}$'s are rational functions with poles only at $z_i = \pm 1$, will allow to considerably simplify the loop equations. It will result in the following theorem, named "topological recursion":

Theorem 3.3.1 (Topological Recursion) *The generating functions $\omega_{k+1}^{(g)}$ counting genus g maps with $k+1$ boundaries can be computed with the following recursion (called "**topological recursion**"):*

$$\omega_{k+1}^{(g)}(z_0, L) = \frac{1}{2} \operatorname{Res}_{z\to\pm 1} \left(\frac{1}{z_0-z} - \frac{1}{z_0-1/z}\right) \frac{dz}{2y(z)x'(1/z)} \Big[\sum_{h=0}^{g} \sum_{J\subset L}' \omega_{1+|J|}^{(h)}(z, J)\omega_{1+k-|J|}^{(g-h)}(\tfrac{1}{z}, L \setminus J) + \omega_{k+2}^{(g-1)}(z, \tfrac{1}{z}, L)\Big]. \tag{3.3.3}$$

Notice that all the terms in the right hand side have $2g' + k' - 2 < 2g + (k + 1) - 2$, and thus this theorem allows an effective recursive computation of $\omega_{k+1}^{(g)}$. It is relatively easy to implement on any symbolic mathematic computer program. See also Sect. 7.4 of Chap. 7, how to represent this recursion in a diagrammatic way, and Sect. 7.4.5 of Chap. 7, how it can be interpreted as cutting surfaces into pairs of pants recursively.

Proof (Cauchy Formula, and Moving the Contour) Since $\omega_{k+1}^{(g)}(z_0, z_1, z_2, \ldots, z_k)$ is a rational function of z_0, we can write Cauchy residue formula:

$$\omega_{k+1}^{(g)}(z_0, z_1, z_2, \ldots, z_k) = \operatorname*{Res}_{z\to z_0} \frac{dz}{z - z_0}\, \omega_{k+1}^{(g)}(z, z_1, z_2, \ldots, z_k).$$

Now, since the only other poles are at the branch-points $z = \pm 1$, and it behaves like $O(1/z^2)$ at ∞, we may deform the integration contour (a residue is a contour integral), as a contour enclosing all the other poles, i.e. $z = \pm 1$:

$$\omega_{k+1}^{(g)}(z_0, z_1, z_2, \ldots, z_k) = \operatorname*{Res}_{z\to\pm 1} \frac{dz}{z_0 - z}\, \omega_{k+1}^{(g)}(z, z_1, z_2, \ldots, z_k).$$

Then we change the variable $z \to 1/z$ and use the antisymmetry Lemma 3.3.2, and thus we also have:

$$\begin{aligned}
\omega_{k+1}^{(g)}(z_0, z_1, z_2, \ldots, z_k) &= \operatorname*{Res}_{z\to\pm 1} \frac{dz}{z_0 - z}\, \omega_{k+1}^{(g)}(z, z_1, z_2, \ldots, z_k) \\
&= \operatorname*{Res}_{z\to\pm 1} -\frac{dz}{z^2}\, \frac{1}{z_0 - \frac{1}{z}}\, \omega_{k+1}^{(g)}(\frac{1}{z}, z_1, z_2, \ldots, z_k) \\
&= -\operatorname*{Res}_{z\to\pm 1} \frac{dz}{z_0 - \frac{1}{z}}\, \omega_{k+1}^{(g)}(z, z_1, z_2, \ldots, z_k).
\end{aligned}$$

Taking the half sum of the two equations, we get

$$\omega_{k+1}^{(g)}(z_0, z_1, z_2, \ldots, z_k) = \frac{1}{2} \operatorname*{Res}_{z\to\pm 1} \left(\frac{dz}{z_0 - z} - \frac{dz}{z_0 - \frac{1}{z}} \right) \omega_{k+1}^{(g)}(z, z_1, z_2, \ldots, z_k).$$

Then substitute $\omega_{k+1}^{(g)}$ in the right hand side, using the loop equation (3.3.2),

$$\begin{aligned}
&\omega_{k+1}^{(g)}(z_0, z_1, z_2, \ldots, z_k) \\
&= \frac{1}{2} \operatorname*{Res}_{z\to\pm 1} \left(\frac{dz}{z_0 - z} - \frac{dz}{z_0 - \frac{1}{z}} \right) \Big[\sum_{h=0}^{g} \sum_{J\subset L}' \frac{\omega_{|J|+1}^{(h)}(z, L)\omega_{k-|J|+1}^{(g-h)}(\frac{1}{z}, L \setminus J)}{2y(z)x'(1/z)} \\
&+ \frac{\omega_{k+2}^{(g-1)}(z, 1/z, L)}{2y(z)x'(1/z)} \\
&+ \sum_{j=1}^{|L|} \frac{1}{2y(z)} \frac{\partial}{\partial z_j} \left(\frac{\omega_k^{(g)}(z, L \setminus \{z_j\}) + \omega_k^{(g)}(L)\, x'(z)/x'(z_j)}{x(z) - x(z_j)} \right) \\
&- \frac{x'(z)}{2y(z)} P_{k+1}^{(g)}(x(z), L)\; x'(z_1) \ldots x'(z_k) \Big]
\end{aligned}$$

where $L = \{z_1, \ldots, z_k\}$. Then, notice that $x'(z)/y(z)$ has no pole at $z = \pm 1$, and $P_{k+1}^{(g)}(x(z), L)$ as well as $\frac{\partial}{\partial z_j} \omega_k^{(g)}(L)/(x'(z_j)(x(z) - x(z_j)))$ have no pole at $z = \pm 1$, so that they don't contribute to the residue. Using the antisymmetry property $y(z) = -y(1/z)$, the term $\frac{\partial}{\partial z_j} \omega_k^{(g)}(z, L \setminus \{z_j\})/(x(z) - x(z_j))$ can be replaced by $\frac{\partial}{\partial z_j} \omega_k^{(g)}(z, L/\{z_j\})/(x(z) - x(z_j)) - z^{-2} \frac{\partial}{\partial z_j} \omega_k^{(g)}(1/z, L \setminus \{z_j\})/(x(z) - x(z_j)) = 0$, and thus it doesn't contribute to the residue either. □

This theorem can be rephrased in the general framework of symplectic invariants, see Chap. 7, it proves that the functions $\omega_{k+1}^{(g)}$ are the symplectic invariant correlators for the spectral curve (x, y). This gives the corollary:

Corollary 3.3.1 *The $\omega_k^{(g)}$'s are the symplectic invariants correlators (defined in Chap. 7) of the spectral curve $\mathcal{E} = (\mathbb{C}P^1, x, y)$ with:*

$$\mathcal{E} \;:\; \begin{cases} z \in \mathbb{C}P^1 \\ x(z) = \alpha + \gamma(z + \frac{1}{z}) \\ y(z) = -\frac{1}{2} \sum_{j=1}^{d-1} u_j (z^j - z^{-j})\,. \end{cases}$$

where $x(z)$ and $y(z)$ are meromorphic functions defined on the Riemann sphere, i.e. the complex projective plane $z \in \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$.

Remark 3.3.2 The recursion of Theorem 3.3.3, is not symmetric in all the variables, z_0 seems to play a special role, but it can be proved by recursion, that this recursion always generates symmetric functions in all variables, this is a general property of symplectic invariants, see Chap. 7. Here this is not surprising, since by definition, generating functions of maps are symmetric.

3.3.3 Topological Recursion for $W_k^{(g)}$'s, and the Method of Moments

The topological recursion can equivalently be rewritten as follows (this is the form under which it was initially found in [30]):

Theorem 3.3.2

$$W_{k+1}^{(g)}(x_0, L) = \frac{1}{\sqrt{(x_0-a)(x_0-b)}} \operatorname*{Res}_{x\to a,b} \frac{dx}{x_0 - x}\, \frac{1}{M(x)}\, \mathcal{W}_{k+1}^{(g)}(x; x_1, \ldots, x_k)$$

where we defined

$$\begin{aligned}\mathcal{W}_{k+1}^{(g)}(x; x_1, \ldots, x_k) &= W_{k+2}^{(g-1)}(x, x, x_1, \ldots, x_k) \\ &\quad + \sum_{h=0}^{g} \sum_{J \uplus J' = \{x_1,\ldots,x_k\}}^{\prime} \tilde{W}_{1+|J|}^{(h)}(x, J)\tilde{W}_{1+|J'|}^{(g-h)}(x, J')\end{aligned}$$

with

$$\tilde{W}_k^{(g)}(x_1, \ldots, x_k) = W_k^{(g)}(x_1, \ldots, x_k) \qquad \text{if } (g, k) \neq (0, 2),$$

$$\begin{aligned}\tilde{W}_2^{(0)}(x_1, x_2) &= W_2^{(0)}(x_1, x_2) + \frac{1}{2\,(x_1-x_2)^2} \\ &= \frac{x_1x_2 - \frac{a+b}{2}\,(x_1+x_2) + ab}{2\,(x_1-x_2)^2\,\sqrt{(x_1-a)(x_1-b)(x_2-a)(x_2-b)}}.\end{aligned}$$

Proof First assume that $(g, k+1) \neq (1, 1)$ and $\neq (0, 3)$.

The topological recursion is:

$$\begin{aligned}\omega_{k+1}^{(g)}(z_0, L) = \tfrac{1}{2}\ \operatorname{Res}_{z\to\pm1}\left(\tfrac{1}{z_0-z} - \tfrac{1}{z_0-1/z}\right) \tfrac{dz}{2y(z)x'(1/z)} \\ \Big[\textstyle\sum_{h=0}^{g}\sum_{J\subset L}^{\prime} \omega_{1+|J|}^{(h)}(z, J)\omega_{1+k-|J|}^{(g-h)}(\tfrac{1}{z}, L\setminus J) + \omega_{k+2}^{(g-1)}(z, \tfrac{1}{z}, L)\Big].\end{aligned}$$

Notice that

$$\left(\frac{1}{z_0 - z} - \frac{1}{z_0 - 1/z}\right) \frac{1}{2y(z)} = \frac{z - 1/z}{z_0\,(z_0 + 1/z_0 - z - 1/z)}\ \frac{1}{2y(z)}$$
$$= \frac{\gamma\,(z - 1/z)}{z_0\,(x(z_0) - x(z))}\ \frac{-1}{M(x(z))\ \sqrt{(x(z)-a)(x(z)-b)}}$$
$$= \frac{\sqrt{(x(z)-a)(x(z)-b)}}{z_0\,(x(z_0) - x(z))}\ \frac{-1}{M(x(z))\ \sqrt{(x(z)-a)(x(z)-b)}}$$
$$= \frac{-1}{z_0}\ \frac{1}{(x(z_0) - x(z))\ M(x(z))}.$$

Then, using the anti–symmetry, and for $(g', k') \neq (0, 2)$, we have $\omega_{k'}^{(g')}(1/z, z_1, \ldots, z_{k'-1}) = z^2 \omega_{k'}^{(g')}(z, z_1, \ldots, z_{k'-1})$, and we recall that

$$W_k^{(g)}(x(z_1), \ldots, x(z_k)) = \omega_k^{(g)}(z_1, \ldots, z_k) \prod_{i=1}^{k} x'(z_i).$$

This gives (writing $L = \{x(z_1), \ldots, x(z_k)\}$:

$$W_{k+1}^{(g)}(x(z_0), L)\, x'(z_0) = \frac{-1}{2\,z_0} \operatorname*{Res}_{z\to\pm 1} \frac{1}{x(z_0) - x(z)}\ \frac{1}{M(x(z))}\ \frac{z^2\,dz}{x'(1/z)}\, x'(z)^2\, \mathcal{W}_{k+1}^{(g)}(x(z); L)$$
$$= \frac{-1}{2\,z_0} \operatorname*{Res}_{z\to\pm 1} \frac{1}{x(z_0) - x(z)}\ \frac{1}{M(x(z))}\ \frac{\gamma^2\, z^2 (1 - 1/z^2)^2\, dz}{\gamma(1 - z^2)}$$
$$\mathcal{W}_{k+1}^{(g)}(x(z); L)$$
$$= \frac{1}{2\,z_0} \operatorname*{Res}_{z\to\pm 1} \frac{1}{x(z_0) - x(z)}\ \frac{1}{M(x(z))}\, x'(z)\, dz\ \mathcal{W}_{k+1}^{(g)}(x(z); L)$$
$$= \frac{1}{2\,z_0} \operatorname*{Res}_{z\to\pm 1} \frac{1}{x(z_0) - x(z)}\ \frac{1}{M(x(z))}\, dx(z)\ \mathcal{W}_{k+1}^{(g)}(x(z); L).$$

Now remark that a circle going around $z = \pm 1$ in the z–plane, goes twice around $x = a$ (reps. $x = b$) in the x-plane, therefore, the residue in the z variable, is twice a residue in the x variable. This gives

$$W_{k+1}^{(g)}(x_0, L) = \frac{1}{\sqrt{(x_0 - a)(x_0 - b)}} \operatorname*{Res}_{x\to a,b} \frac{dx}{x_0 - x}\ \frac{1}{M(x)}\ \mathcal{W}_{k+1}^{(g)}(x; L).$$

The cases $(g, k + 1) = (1, 1)$ or $(0, 3)$, need to be done separately, and we leave them as an exercise for the reader. □

Let us introduce the zeros of $M(x)$:

$$\forall\ i = 1,\dots,d-2, \qquad M(m_i) = 0,$$

i.e.

$$M(x) = -t_{d+1} \prod_{i=1}^{d-2} (x - m_i).$$

In the previous theorem the residues are taken at $x = a$ and $x = b$, which are poles of $\mathcal{W}^{(g)}_{k+1}(x; L)$. One can move the integration contour, and pick the residues at the other poles, i.e. the zeros of $M(x)$, and also at $x = x_0$, or at $x = x_j$. This results into

Theorem 3.3.3 *If* $(g, k+1) \neq (0, 3)$*, we have*

$$\begin{aligned} W^{(g)}_{k+1}(x_0, L) = {} & \frac{1}{\sqrt{(x_0-a)(x_0-b)}} \sum_{i=1}^{d-2} \frac{1}{M'(m_i)} \frac{\mathcal{W}^{(g)}_{k+1}(x_0; L) - \mathcal{W}^{(g)}_{k+1}(m_i; L)}{x_0 - m_i} \\ & - \frac{1}{\sqrt{(x_0-a)(x_0-b)}} \sum_{j=1}^{k} \frac{\partial}{\partial x_j} \frac{W^{(g)}_k(L)}{(x_0 - x_j)\, M(x_j)} \end{aligned}$$

where m_i *are the zeros of* $M(x)$*, and*

$$\mathcal{W}^{(g)}_{k+1}(x; L) = W^{(g-1)}_{k+2}(x, x, L) + \sum_{h=0}^{g} \sum_{J \subset L}' \tilde{W}^{(h)}_{1+\#J}(x, J) \tilde{W}^{(g-h)}_{1+k-\#J}(x, L \setminus J).$$

The case of $W^{(0)}_3$ *will be treated separately, we compute it explicitly in Sect. 3.3.4 below.*

Proof We move the integration contour, and pick the residues at the other poles, i.e. the zeros of $M(x)$, and also at $x = x_0$, or at $x = x_j$:

$$\begin{aligned} \sqrt{(x_0-a)(x_0-b)}\, W^{(g)}_{k+1}(x_0, L) = {} & \operatorname*{Res}_{x \to x_0} \frac{dx}{x - x_0} \frac{1}{M(x)} \mathcal{W}^{(g)}_{k+1}(x; L) \\ & + \sum_{i=1}^{d-2} \operatorname*{Res}_{x \to m_i} \frac{dx}{x - x_0} \frac{1}{M(x)} \mathcal{W}^{(g)}_{k+1}(x; L) \\ & + \sum_{j=1}^{k} \operatorname*{Res}_{x \to x_j} \frac{dx}{x - x_0} \frac{1}{M(x)} \mathcal{W}^{(g)}_{k+1}(x; L). \end{aligned}$$

The residue at $x = x_0$ is simply

$$\operatorname*{Res}_{x \to x_0} \frac{dx}{x - x_0} \frac{1}{M(x)} \mathcal{W}^{(g)}_{k+1}(x; L) = \frac{1}{M(x_0)} \mathcal{W}^{(g)}_{k+1}(x_0; L).$$

We shall write that

$$\frac{1}{M(x_0)} = \sum_{i=1}^{d-1} \frac{1}{(x_0 - m_i)\, M'(m_i)}.$$

Then, the residue at $x = m_i$ is

$$\operatorname*{Res}_{x\to m_i} \frac{dx}{x-x_0}\, \frac{1}{M(x)}\, \mathcal{W}^{(g)}_{k+1}(x;L) = \frac{-1}{(x_0-m_i)\, M'(m_i)}\, \mathcal{W}^{(g)}_{k+1}(m_i;L).$$

And for the residue at $x = x_j$, notice that $\mathcal{W}^{(g)}_{k+1}(x;L)$ has a double pole $1/2(x - x_j)^2$ each time there is a $\tilde{W}_{0,2}(x, x_j)$ in the product. Since any of the two terms in the product can be a $\tilde{W}_{0,2}$, we have a factor 2, i.e.

$$\mathcal{W}^{(g)}_{k+1}(x;L) \sim \frac{1}{(x-x_j)^2}\, W^{(g)}_k(x, L\setminus\{x_j\}) + \text{ analytic at } x_j$$
$$\sim \frac{\partial}{\partial x_j}\, \frac{1}{(x-x_j)}\, W^{(g)}_k(x, L\setminus\{x_j\}) + \text{ analytic at } x_j.$$

Therefore

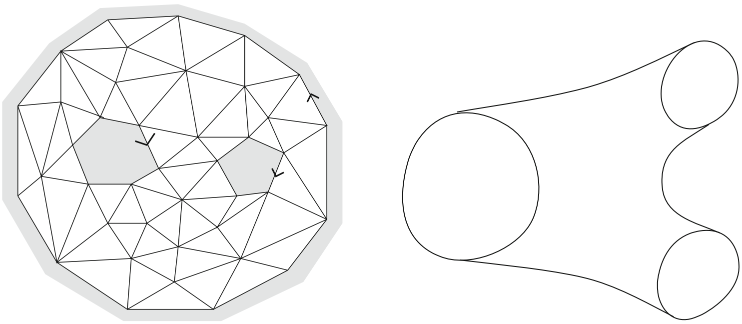
$$\operatorname*{Res}_{x\to x_j} \frac{dx}{x-x_0}\, \frac{1}{M(x)}\, \mathcal{W}^{(g)}_{k+1}(x;L) = \frac{\partial}{\partial x_j}\, \frac{1}{(x_j-x_0)\; M(x_j)}\, W^{(g)}_k(x_j, L\setminus\{x_j\}).$$

This gives the theorem. □

3.3.4 *Examples of Maps of Higher Topology*

Let us illustrate Theorem 3.3.1 on a few examples with low values of g and k.

3.3.4.1 The Pair of Pants ($g = 0, k = 3$)



A pair of pants is a sphere with three disks removed, i.e. a genus zero surface with three boundaries, i.e. an element of $\mathbb{M}_3^{(0)}$. Theorem Eq. (3.3.3) gives:

$$\omega_3^{(0)}(z_0,z_1,z_2) = \frac{1}{2}\operatorname*{Res}_{z\to\pm 1}\left(\frac{1}{(z_0-z)} - \frac{1}{z_0-1/z}\right)\frac{dz}{2y(z)x'(1/z)}$$
$$\left[\omega_2^{(0)}(z,z_1)\,\omega_2^{(0)}(1/z,z_2) + \omega_2^{(0)}(z,z_2)\,\omega_2^{(0)}(1/z,z_1)\right]$$
$$= \frac{-1}{4}\operatorname*{Res}_{z\to\pm 1}\frac{dz}{zz_0(x(z_0)-x(z))\;y(z)}$$
$$\left[\frac{1}{(z-z_1)^2\,(1/z-z_2)^2} + \frac{1}{(z-z_2)^2\,(1/z-z_1)^2}\right].$$

After computing the residues, one finds:

Theorem 3.3.4

$$\omega_3^{(0)}(z_0,z_1,z_2) = \frac{-1}{2\gamma y'(1)}\;\frac{1}{(z_0-1)^2}\;\frac{1}{(z_1-1)^2}\;\frac{1}{(z_2-1)^2}$$
$$+\frac{1}{2\gamma y'(-1)}\;\frac{1}{(z_0+1)^2}\;\frac{1}{(z_1+1)^2}\;\frac{1}{(z_2+1)^2}.$$

One can see that although the recursion relation Eq. (3.3.3) looks non-symmetric in z_0, z_1 and z_2, the result is symmetric. This is a general fact, Eq. (3.3.3) generates only symmetric functions of its $k+1$ variables (see Chap. 7).

In terms of moments we also have

$$\omega_3^{(0)}(z_0,z_1,z_2) = \frac{1}{2\gamma^2 M_{+,0}}\;\frac{1}{(z_0-1)^2}\;\frac{1}{(z_1-1)^2}\;\frac{1}{(z_2-1)^2}$$
$$-\frac{1}{2\gamma^2 M_{-,0}}\;\frac{1}{(z_0+1)^2}\;\frac{1}{(z_1+1)^2}\;\frac{1}{(z_2+1)^2}.$$

and

$$W_3^{(0)}(x_0,x_1,x_2) = \frac{\gamma}{2}\left(\frac{1}{M_{+,0}}\;\frac{1}{(x_0-a)(x_1-a)(x_2-a)} - \frac{1}{M_{-,0}}\;\frac{1}{(x_0-b)(x_1-b)(x_2-b)}\right)$$
$$\prod_{i=0}^{2}\frac{1}{\sqrt{(x_i-a)(x_i-b)}}.$$

Case of Bipartite Maps

If we consider bipartite maps such that $V'(x)$ is odd, we have $\alpha = 0$ and $y'(1) = y'(-1) = -\gamma M_{a,0} = -\gamma M_{b,0}$, and thus:

$$\omega_3^{(0)}(z_0, z_1, z_2) = \frac{-1}{2\gamma y'(1)} \, \frac{1}{(z_0-1)^2} \, \frac{1}{(z_1-1)^2} \, \frac{1}{(z_2-1)^2} + \frac{1}{2\gamma y'(-1)} \, \frac{1}{(z_0+1)^2} \, \frac{1}{(z_1+1)^2} \, \frac{1}{(z_2+1)^2}.$$

Proposition 3.3.1 *The generating function of bipartite maps of genus* 0 *with 3 boundaries of lengths* l_1, l_2, l_3 *such that* $l_1 + l_2 + l_3$ *is even, is:*

$$\mathcal{T}^{(0)}_{l_1,l_2,l_3} = -\tilde{C}_{l_1} \tilde{C}_{l_2} \tilde{C}_{l_3} \, \frac{\gamma^{l_1+l_2+l_3-1}}{y'(1)} \, \frac{1 + (-1)^{l_1+l_2+l_3}}{2}$$

where

$$\tilde{C}_l = \frac{l!}{([l/2])! \;\; ([(l-1)/2])!}$$

and $[k]$ *is the integer part of* k, *i.e. the largest integer* $\leq k$.

Proof Indeed, compute the residue:

$$\mathcal{T}^{(0)}_{l_1,l_2,l_3} = \frac{1}{2\gamma y'(1)} \, \operatorname*{Res}_{z_1\to\infty} \operatorname*{Res}_{z_2\to\infty} \operatorname*{Res}_{z_3\to\infty} \frac{x(z_1)^{l_1}\, x(z_2)^{l_2}\, x(z_3)^{l_3}}{(z_1-1)^2\,(z_2-1)^2\,(z_3-1)^2}\, dz_1 dz_2 dz_3 - \frac{1}{2\gamma y'(-1)} \, \operatorname*{Res}_{z_1\to\infty} \operatorname*{Res}_{z_2\to\infty} \operatorname*{Res}_{z_3\to\infty} \frac{x(z_1)^{l_1}\, x(z_2)^{l_2}\, x(z_3)^{l_3}}{(z_1+1)^2\,(z_2+1)^2\,(z_3+1)^2}\, dz_1 dz_2 dz_3.$$

Therefore we need to compute

$$\operatorname*{Res}_{z\to\infty} \frac{x(z)^l}{(z-a)^2}\, dz$$

with $a = \pm 1$. First integrate by parts

$$\begin{aligned}
\operatorname*{Res}_{z\to\infty} \frac{x(z)^l}{(z-a)^2}\, dz &= l \operatorname*{Res}_{z\to\infty} x(z)^{l-1}\, \frac{x'(z)}{z-a}\, dz \\
&= \gamma\, l \operatorname*{Res}_{z\to\infty} x(z)^{l-1}\, \frac{1-z^{-2}}{z-a}\, dz \\
&= \gamma\, l \operatorname*{Res}_{z\to\infty} x(z)^{l-1}\, (z+a)\, \frac{dz}{z^2}
\end{aligned}$$

then use the binomial formula for $x(z)^{l-1}$:

$$\operatorname*{Res}_{z\to\infty} \frac{x(z)^l}{(z-a)^2}\, dz = l\,\gamma^l \operatorname*{Res}_{z\to\infty} (z+1/z)^{l-1}\,(z+a)\,\frac{dz}{z^2}$$
$$= l\,\gamma^l \sum_{j=0}^{l-1} \frac{(l-1)!}{j!\,(l-1-j)!} \operatorname*{Res}_{z\to\infty} z^{l-1-2j}\,(z+a)\,\frac{dz}{z^2}$$

if $l = 2k+1$ is odd only $j = (l-1)/2 = k$ contributes:

$$-l\,\gamma^l\,\frac{(2k)!}{k!\,k!}$$

and if $l = 2k$ is even, only $j = k-1$ contributes:

$$-a\,l\,\gamma^l\,\frac{(2k-1)!}{(k-1)!\,k!}.$$

In both cases we obtain:

$$-\gamma^l\,a^{l-1}\,\frac{l!}{[l/2]!\,[(l-1)/2]!}$$

This gives:

$$\mathcal{T}^{(0)}_{l_1,l_2,l_3} = -\frac{\gamma^{l_1+l_2+l_3}}{2\gamma y'(1)}\,\tilde{C}_{l_1}\,\tilde{C}_{l_2}\,\tilde{C}_{l_3}\,(1+(-1)^{l_1+l_2+l_3}).$$

□

For instance for quadrangulations we have:

$$T^{(0)}_{4,4,4} = (12)^3\,\frac{\gamma^{12}}{2t}\left(1+\frac{1}{\sqrt{1-12tt_4}}\right) = t^5 \sum_n 2^6\,3^n\,\frac{(2n-1)!}{(n+2)!\,(n-3)!}\,(tt_4)^{n-3}$$

i.e. the number of planar quadrangulations where all n faces, including the three marked faces, are quadrangles, is $2^6\,3^n\,\frac{(2n-1)!}{(n+2)!\,(n-3)!}$.

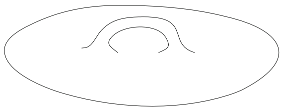
For triangulations we find (with the notations of Sect. 3.1.8, $r - r^3 = 8tt_3^2$)

$$T^{(0)}_{3,3,3} = \frac{27}{32\,t_3^7}\,\frac{(1+r)(1-r)^4(5+7r-3r^2-r^3)}{(3r^2-1)}$$
$$= 27\;t^4\,t_3 \sum_{n\geq 2} \frac{2^{3n}\,(n-1)(2n-1)\,\Gamma(\frac{3n}{2})}{(n+2)!\,\Gamma(\frac{n}{2})}\,(tt_3^2)^{n-2}$$

in other words, the number of planar triangulations where all $2n$ faces, including the three marked faces, are triangles, is $\frac{2^{3n}\,(n-1)(2n-1)\,\Gamma(\frac{3n}{2})}{(n+2)!\,\Gamma(\frac{n}{2})}$.

3.3.4.2 The Genus 1 Disk ($g = 1, k = 1$), the "lid"

Let us compute the generating function $\omega_1^{(1)}$, i.e. count maps of genus 1 with one boundary, elements of $\mathbb{M}_1^{(1)}$, i.e. maps drawn on a disk with one handle (a "lid").



Formula Eq. (3.3.3) gives:

$$\omega_1^{(1)}(z_0) = \frac{1}{2} \operatorname*{Res}_{z\to\pm 1} \left(\frac{1}{(z_0 - z)} - \frac{1}{z_0 - 1/z}\right) \frac{dz}{2y(z)x'(1/z)}\, \frac{1}{(z - 1/z)^2}.$$

After computing the residues this gives

$$\omega_1^{(1)}(z_0) = \frac{-1}{16\gamma y'(1)} \left(\frac{1}{(z_0-1)^4} + \frac{1}{(z_0-1)^3} - \frac{1 + \frac{y''(1)}{y'(1)} + \frac{y'''(1)}{3y'(1)}}{2(z_0-1)^2}\right)$$
$$+ \frac{1}{16\gamma y'(-1)} \left(\frac{1}{(z_0+1)^4} - \frac{1}{(z_0+1)^3} - \frac{1 - \frac{y''(-1)}{y'(-1)} + \frac{y'''(-1)}{3y'(-1)}}{2(z_0+1)^2}\right). \tag{3.3.4}$$

In terms of moments:

$$\sqrt{(x_0-a)(x_0-b)}\, W_1^{(1)}(x_0) = \frac{1}{16M_{+,0}} \left(\frac{1}{(x_0-a)^2} + \frac{M_{+,1}}{(x_0-a)} - \frac{1}{2\gamma\,(x_0-a)}\right)$$
$$+ \frac{1}{16M_{-,0}} \left(\frac{1}{(x_0-b)^2} + \frac{M_{-,1}}{(x_0-b)} + \frac{1}{2\gamma\,(x_0-b)}\right).$$

Example Quadrangulations
Equation (3.3.4) gives (with $r = \sqrt{1 - 12tt_4}$):

$$\omega_1^{(1)}(z) = \frac{z}{2\,t} \left(- \frac{1 + r^{-1}}{(1-z^2)^4} + \frac{1 + r^{-1}}{(1-z^2)^3} + \frac{1 - r^{-2}}{12\,(1-z^2)^2}\right)$$

and:

$$\mathcal{T}_4^{(1)} = \frac{1}{6t_4}\left(\frac{1}{1-12\,tt_4} - \frac{1}{\sqrt{1-12\,tt_4}}\right)$$

$$\mathcal{T}_4^{(1)} = 2t\sum_{n\geq 1}\left(1+\binom{-1/2}{n}(-1)^{n-1}\right)(12tt_4)^{n-1} = 2t\sum_{n\geq 1}\left(1-\frac{(2n-1)!!}{n!\,2^n}\right)(12t_4)^{n-1}$$
$$= t + 15t^2t_4 + 198t^3t_4^2 + \dots$$

i.e. the number of rooted quandrangulations of genus 1 with n faces is:

$$\frac{1}{6}\left(1-\frac{(2n-1)!!}{n!\,2^n}\right)(12)^n = \frac{3^n}{6}\left(2^{2n}-\frac{(2n)!}{n!\,n!}\right). \tag{3.3.5}$$

This is indeed a positive integer.

Example Triangulations
Equation (3.3.4) gives

$$\omega_1^{(1)}(z) = -\frac{t^3z^3+\gamma^3t^2t_3z^2(1+z^2)+\gamma^6tt_3^2z(1-6z^2+z^4)+\gamma^9t_3^3(1-5z^2-5z^4+z^6)}{(t^2-4\gamma^6t_3^2)^2\;(1-z^2)^4}$$

and, after taking the residue $T_3^{(1)} = -\operatorname{Res}_\infty \omega_1^{(1)}(z)\,x(z)^3\,dz$, we get (with the notations of Sect. 3.1.8, $r-r^3 = 8tt_3^2$)

$$T_3^{(1)} = \frac{1}{2t_3}\,\frac{1-r^2}{(1-3r^2)^2}.$$

Let us write it as

$$T_3^{(1)} = \frac{1}{t_3}\sum_n c_n(8tt_3^2)^n$$
$$= tt_3 + 40\,t^2t_3^3 + 1368\,t^3t^5 + O(t^4t_3^7).$$

We have

$$c_n = \frac{1}{2}\operatorname*{Res}_{r\to 1}\frac{1-r^2}{(1-3r^2)^2}\,\frac{(1-3r^2)\,dr}{(r(1-r^2))^{n+1}}$$
$$= \frac{1}{4}\operatorname*{Res}_{u\to 1}\frac{1-u}{(1-3u)}\,\frac{du}{u^{n+3/2}\,(1-u)^{n+1}} \qquad \text{change of variable } r^2 = u$$
$$= \frac{1}{4}\operatorname*{Res}_{u\to 1}\frac{1}{(1-3u)}\,\frac{du}{u^{n+3/2}\,(1-u)^{n}}$$

$$= \frac{1}{4} \operatorname*{Res}_{v \to 0} dv \, \frac{(1+v)^{2n+1/2}}{v^n \, (2-v)} \qquad \text{change of variable } u = 1/(1+v)$$

$$= \frac{1}{8} \sum_{k=0}^{n-1} 2^{-k} \operatorname*{Res}_{v \to 0} dv \, \frac{(1+v)^{2n+1/2}}{v^{n-k}}$$

$$= \frac{1}{8} \sum_{k=0}^{n-1} 2^{-k} \binom{2n+\frac{1}{2}}{n-k-1}$$

$$= \frac{1}{2^{n+2}} \sum_{k=0}^{n-1} \frac{(4n+1)!!}{k! \, (4n+1-2k)!!}.$$

3.3.4.3 Quadrangulations Genus 2

To genus 2, computing $\omega_1^{(2)}$ gives:

$$\omega_1^{(2)}(z) = z \, \frac{(1+r)^3}{r^3 \, t^3} \Big(-\frac{105}{8} (1-z^2)^{-10} + \frac{105}{2} (1-z^2)^{-9}$$
$$- \frac{7\,(29+529\,r)}{48\,r} (1-z^2)^{-8} + \frac{7\,(29+109\,r)}{16\,r} (1-z^2)^{-7}$$
$$- \frac{7+94\,r+67\,r^2}{8\,r^2} (1-z^2)^{-6} + \frac{84+113\,r-71\,r^2}{48\,r^2} (1-z^2)^{-5}$$
$$- \frac{14+57\,r-78\,r^2+7\,r^3}{96\,r^3} (1-z^2)^{-4} + \frac{14-27\,r+12\,r^2+r^3}{96\,r^3} (1-z^2)^{-3}$$
$$- \frac{14-41\,r+39\,r^2-11\,r^3-r^4}{576\,r^4} (1-z^2)^{-2} \Big)$$

from which we have

$$\mathcal{T}_4^{(2)} = \left(\frac{13}{2\,r^6} + \frac{1}{2\,r^5} - \frac{7}{r^7} \right) t_4 \qquad \text{with} \quad r = \sqrt{1-12tt_4}$$
$$= 45\, tt_4^2 + 2007\, t^2 t_4^3 + O(t_4^5).$$

That is, the number of rooted quandrangulations of genus 2 with n faces is:

$$= (3)^{n-2} \, \frac{n(n-1)}{4} \Big(\frac{2n-1!}{(n-1)!\, n!} \, \frac{28n+13}{15} - 13 * 4^{n-2} \Big).$$

3.4 Closed Surfaces

The topological recursion of Theorem 3.3.1 applies only to $k \geq 1$, i.e. maps with at least one boundary.

Closed surfaces are surfaces with no boundary, i.e. elements of $\mathbb{M}_0^{(g)}$. We want to compute their generating function $F_g = W_0^{(g)} = \omega_0^{(g)}$.

$$F_g(t, t_3, \dots, t_d) = \sum_v t^v \sum_{\Sigma \in \mathbb{M}_0^{(g)}(v)} \frac{t_3^{n_3(\Sigma)}\, t_4^{n_4(\Sigma)} \dots t_d^{n_d(\Sigma)}}{\#\mathrm{Aut}(\Sigma)}.$$

As usual in algebraic geometry, unstable maps require a special treatment, i.e. F_0 and F_1 are in fact less regular than F_g with $g \geq 2$. For example, F_g with $g \geq 2$ will always be an algebraic expression, whereas F_0 and F_1 also contain logarithmic terms as we shall see below.

3.4.1 General Considerations

The way to compute F_g is through its derivatives. Indeed, it is clear from the definition of the formal matrix integral that for every $l = 3, \dots, d$:

$$\frac{\partial \ln Z}{\partial t_l} = \frac{N}{l\,t} < \mathrm{Tr}\, M^l >$$

i.e.

$$\frac{\partial F_g}{\partial t_l} = \frac{1}{l}\, \mathcal{T}_l^{(g)} = -\frac{1}{l} \mathop{\mathrm{Res}}_{x\to\infty} x^l\ W_1^{(g)}(x)\, dx$$

and more generally:

$$\frac{\partial < \mathrm{Tr}\, \frac{1}{x_1 - M} \dots Tr \frac{1}{x_k - M} >_c}{\partial t_l} = \frac{N}{l\,t} < \mathrm{Tr}\, \frac{1}{x_1 - M} \dots Tr \frac{1}{x_k - M}\ \mathrm{Tr}\, M^l >_c$$

which can be written (for $2g - 2 + k > 0$):

$$\frac{\partial W_k^{(g)}(x_1, \dots, x_k)}{\partial t_l} = -\frac{1}{l} \mathop{\mathrm{Res}}_{x\to\infty} W_{k+1}^{(g)}(x, x_1, \dots, x_k)\, x^l\ \ dx$$

or:

$$\left.\frac{\partial\, \omega_k^{(g)}(z_1,\dots,z_k)}{\partial t_l}\right|_{x(z_1),\dots,x(z_k)} = -\frac{1}{l}\operatorname*{Res}_{z\to\infty} \omega_{k+1}^{(g)}(z,z_1,\dots,z_k)\, x(z)^l\;\, dz.$$

Let us summarize it as a theorem:

Theorem 3.4.1

$$\frac{\partial F_g}{\partial t_l} = \frac{1}{l}\mathcal{T}_l^{(g)} = -\frac{1}{l}\operatorname*{Res}_{x\to\infty} x^l\;\, W_1^{(g)}(x)\, dx = -\frac{1}{l}\operatorname*{Res}_{z\to\infty} x(z)^l\;\, \omega_1^{(g)}(z)\, dz. \tag{3.4.1}$$

And for $k\geq 1$*:*

$$\frac{\partial W_k^{(g)}(x_1,\dots,x_k)}{\partial t_l} = -\frac{1}{l}\operatorname*{Res}_{x\to\infty} W_{k+1}^{(g)}(x,x_1,\dots,x_k)\, x^l\;\, dx \tag{3.4.2}$$

or for $2g-2+k\geq 0$*:*

$$\left.\frac{\partial \omega_k^{(g)}(z_1,\dots,z_k)}{\partial t_l}\right|_{x(z_1),\dots,x(z_k)} = -\frac{1}{l}\operatorname*{Res}_{z\to\infty} \omega_{k+1}^{(g)}(z,z_1,\dots,z_k)\, x(z)^l\;\, dz. \tag{3.4.3}$$

In other words, taking a derivative is an operation which increases the number of boundaries $k\to k+1$.

For $k\geq 1$ and $2g-2+k>0$, this relation can be inverted, with the following theorem which decreases k by 1:

Theorem 3.4.2 *For* $k\geq 1$ *and* $2g-2+k>0$ *we have*

$$\omega_k^{(g)}(z_1,\dots,z_k) = \frac{1}{2-2g-k}\operatorname*{Res}_{z\to\pm 1} \Phi(z)\, \omega_{k+1}^{(g)}(z_1,\dots,z_k,z)\;\, dz$$

where

$$\Phi'(z) = y(z)x'(z).$$

Proof This theorem is a general property of symplectic invariants (see Chap. 7), and we refer to [34] for the general proof.

Here, let us do the proof for maps.

Assume $k\geq 1$. We shall prove by recursion on $2g+k$ that

$$\operatorname*{Res}_{z\to\pm 1} \Phi(z)\, \omega_{k+1}^{(g)}(z_1,\dots,z_k,z)\;\, dz = (2-2g-k)\, \omega_k^{(g)}(z_1,\dots,z_k).$$

First let us show that it holds for $2g-2+k=0$, i.e. for $(g,k)=(0,2)$. We have

$$\omega_3^{(0)}(z_1,z_2,z) = \frac{-1}{2\gamma y'(1)}\,\frac{1}{(z_1-1)^2\,(z_2-1)^2\,(z-1)^2} + \frac{1}{2\gamma y'(-1)}\,\frac{1}{(z_1+1)^2\,(z_2+1)^2\,(z+1)^2} \tag{3.4.4}$$

which gives

$$\operatorname*{Res}_{z\to\pm1} \Phi(z)\,\omega_3^{(0)}(z_1,z_2,z)\;\;dz = \frac{-\Phi'(1)}{2\gamma y'(1)}\,\frac{1}{(z_1-1)^2\,(z_2-1)^2} + \frac{\Phi'(-1)}{2\gamma y'(-1)}\,\frac{1}{(z_1+1)^2\,(z_2+1)^2} = 0$$

which vanishes because $\Phi'(1)=\Phi'(-1)=0$.

Assume that we have already proved the theorem for all g',k' such that $2g'-2+k' < 2g-2+k$. Now consider $k\geq 1$ and $2g-2+k>0$, write $L=\{z_1,\dots,z_{k-1}\}$. We have, from the topological recursion of Theorem 3.3.1:

$$\begin{aligned}
&\operatorname*{Res}_{z_k\to\pm1}\;\Phi(z_k)\,\omega_{k+1}^{(g)}(z_0,L,z_k)\,dz_k\\
&= \operatorname*{Res}_{z_k\to\pm1}\;\operatorname*{Res}_{z\to\pm1}\left(\frac{\Phi(z_k)\,dz_k}{(z_0-z)} - \frac{\Phi(z_k)\,dz_k}{z_0-1/z}\right)\frac{dz}{4y(z)x'(1/z)}\\
&\Big[\omega_2^{(0)}(z,z_k)\omega_k^{(g)}(\frac{1}{z},L) + \sum_{h=0}^{g}\sum_{J\subset L}{}^{'}\,\omega_{2+|J|}^{(h)}(z,J,z_k)\omega_{k-|J|}^{(g-h)}(\frac{1}{z},L/J)\\
&+\omega_{1+k}^{(g)}(z,L)\omega_2^{(0)}(\frac{1}{z},z_k) + \sum_{h=0}^{g}\sum_{J\subset L}{}^{'}\,\omega_{1+|J|}^{(h)}(z,J)\omega_{1+k-|J|}^{(g-h)}(\frac{1}{z},L/J,z_k)\\
&+\omega_{k+2}^{(g-1)}(z,\frac{1}{z},L,z_k)\Big]
\end{aligned}$$

where as usual $\sum_h \sum_J'$ means that we exclude $(h=0, J=\emptyset)$ and $(h=g, J=L)$ from the sum.

For the moment, we first compute the residue at $z\to\pm1$, and then at $z_k\to\pm1$. We can exchange the order, by pushing the small circle where z_k is integrated through that of z. By doing so, we may pick a pole at $z_k=z$ or $z_k=1/z$:

$$\operatorname*{Res}_{z_k\to\pm1}\operatorname*{Res}_{z\to\pm1} = \operatorname*{Res}_{z\to\pm1}\operatorname*{Res}_{z_k\to\pm1} + \operatorname*{Res}_{z\to\pm1}\operatorname*{Res}_{z_k\to z,1/z}.$$

The only cases where there can be a pole at $z_k=z$ or $z_k=1/z$, is when we have a $\omega_2^{(0)}(z,z_k)$ or $\omega_2^{(0)}(1/z,z_k)$, because all the other $\omega_{k'}^{(g')}$ have no pole at coinciding points. That gives:

$$\begin{aligned}
&\operatorname*{Res}_{z_k\to\pm 1} \Phi(z_k)\,\omega^{(g)}_{k+1}(z_0,L,z_k)\,dz_k \\
&= \operatorname*{Res}_{z\to\pm 1}\operatorname*{Res}_{z_k\to\pm 1}\left(\frac{\Phi(z_k)\,dz_k}{(z_0-z)}-\frac{\Phi(z_k)\,dz_k}{z_0-1/z}\right)\frac{dz}{4y(z)x'(1/z)} \\
&\Big[\sum_{h=0}^{g}\sum_{J\subset L}{}' \omega^{(h)}_{2+|J|}(z,J,z_k)\omega^{(g-h)}_{k-|J|}(\frac{1}{z},L/J) \\
&+\sum_{h=0}^{g}\sum_{J\subset L}{}' \omega^{(h)}_{1+|J|}(z,J)\omega^{(g-h)}_{1+k-|J|}(\frac{1}{z},L/J,z_k)+\omega^{(g-1)}_{k+2}(z,\frac{1}{z},L,z_k)\Big] \\
&+\operatorname*{Res}_{z\to\pm 1}\operatorname*{Res}_{z_k\to\pm 1}\left(\frac{\Phi(z_k)\,dz_k}{(z_0-z)}-\frac{\Phi(z_k)\,dz_k}{z_0-1/z}\right)\frac{dz}{4y(z)x'(1/z)} \\
&\Big[\omega^{(0)}_2(z,z_k)\omega^{(g)}_k(\frac{1}{z},L)+\omega^{(g)}_k(z,L)\omega^{(0)}_2(\frac{1}{z},z_k)\Big] \\
&+\operatorname*{Res}_{z\to\pm 1}\operatorname*{Res}_{z_k\to z,1/z}\left(\frac{\Phi(z_k)\,dz_k}{(z_0-z)}-\frac{\Phi(z_k)\,dz_k}{z_0-1/z}\right)\frac{dz}{4y(z)x'(1/z)} \\
&\Big[\omega^{(0)}_2(z,z_k)\omega^{(g)}_k(\frac{1}{z},L)+\omega^{(g)}_k(z,L)\omega^{(0)}_2(\frac{1}{z},z_k)\Big]
\end{aligned}$$

In the first line, the residues in $z_k \to \pm 1$ are evaluated by the recursion hypothesis, in the second line there is no pole at $z_k = \pm 1$, and in the last line we have

$$\operatorname*{Res}_{z_k\to z} \omega^{(0)}_2(z,z_k)\,\Phi(z_k)\,dz_k = \operatorname*{Res}_{z_k\to z} \frac{\Phi(z_k)\,dz_k}{(z-z_k)^2} = \Phi'(z) = y(z)\,x'(z).$$

That gives

$$\begin{aligned}
&\operatorname*{Res}_{z_k\to\pm 1} \Phi(z_k)\,\omega^{(g)}_{k+1}(z_0,L,z_k)\,dz_k \\
&= \operatorname*{Res}_{z\to\pm 1}\left(\frac{1}{(z_0-z)}-\frac{1}{z_0-1/z}\right)\frac{dz}{4y(z)x'(1/z)} \\
&\Big[\sum_{h=0}^{g}\sum_{J\subset L}{}'(2-2h-1-|J|)\omega^{(h)}_{1+|J|}(z,J)\omega^{(g-h)}_{k-\#J}(\frac{1}{z},L/J) \\
&+\sum_{h=0}^{g}\sum_{J\subset L}{}'(2-2g+2h-k+|J|)\omega^{(h)}_{1+|J|}(z,J)\omega^{(g-h)}_{k-\#J}(\frac{1}{z},L/J) \\
&+(2-2(g-1)-(k+1))\,\omega^{(g-1)}_{k+1}(z,\frac{1}{z},L)\Big]
\end{aligned}$$

$$+\operatorname*{Res}_{z\to\pm 1}\left(\frac{1}{(z_0-z)}-\frac{1}{z_0-1/z}\right)\frac{dz}{4y(z)x'(1/z)}$$
$$\Big[x'(z)\,y(z)\omega_k^{(g)}(\frac{1}{z},L)+x'(1/z)\,y(1/z)\omega_k^{(g)}(z,L)\Big].$$

Notice that $y(1/z)=-y(z)$, $x'(1/z)=-z^2\,x'(z)$, and $\omega_k^{(g)}(1/z,L)=z^2\omega_k^{(g)}(z,L)$, the last line is thus worth:

$$-\operatorname*{Res}_{z\to\pm 1}\left(\frac{1}{(z_0-z)}-\frac{1}{z_0-1/z}\right)\frac{dz}{4}\Big[2\,\omega_k^{(g)}(z,L)\Big].$$

The only poles of the integrand are at $z=\pm 1$ and $z=z_0$ or $z=1/z_0$. By moving the integration contour we find that the last line is thus

$$\begin{aligned}&\operatorname*{Res}_{z\to z_0,1/z_0}\left(\frac{1}{(z_0-z)}-\frac{1}{z_0-1/z}\right)\frac{dz}{4}\Big[2\,\omega_k^{(g)}(z,L)\Big]\\&=-\frac{1}{2}\,\omega_k^{(g)}(z_0,L)-\frac{1}{2}\,z_0^{-2}\omega_k^{(g)}(1/z_0,L)\\&=-\omega_k^{(g)}(z_0,L).\end{aligned}$$

Therefore we find:

$$\begin{aligned}&\operatorname*{Res}_{z_k\to\pm 1}\ \Phi(z_k)\,\omega_{k+1}^{(g)}(z_0,L,z_k)\,dz_k\\&=(2-2g-(k-1))\operatorname*{Res}_{z\to\pm 1}\left(\frac{1}{(z_0-z)}-\frac{1}{z_0-1/z}\right)\frac{dz}{4y(z)x'(1/z)}\\&\Big[\sum_{h=0}^{g}\sum_{J\subset L}{}^{\prime}\,\omega_{1+|J|}^{(h)}(z,J)\omega_{k-|J|}^{(g-h)}(\frac{1}{z},L/J)+\,\omega_{k+1}^{(g-1)}(z,\frac{1}{z},L)\Big]\\&-\omega_k^{(g)}(z_0,L)\\&=(2-2g-k)\,\omega_k^{(g)}(z_0,L)\end{aligned}$$

which proves the theorem. □.

3.4.2 *The Generating Function of Stable Maps of Genus ≥ 2*

F_g is obtained as the $k=0$ case of Theorem 3.4.2. Indeed, what we are going to prove is that

$$\frac{\partial}{\partial t_k}\,\frac{1}{2-2g}\operatorname*{Res}_{z\to\pm 1}\ \Phi(z)\,\omega_1^{(g)}(z)\,dz=\frac{1}{k}\,\mathcal{T}_k^{(g)}=\frac{-1}{k}\operatorname*{Res}_{z\to\infty}x(z)^k\,\omega_1^{(g)}(z)\,dz.$$

This will prove that F_g coincides with $\frac{1}{2-2g}\,\mathrm{Res}_{z\to\pm1}\,\Phi(z)\,\omega_1^{(g)}(z)\,dz$, up to a constant independent of t_k, and which can be computed at $t_k = 0$, i.e. for the Gaussian matrix model. It results:

Theorem 3.4.3

$$\forall g \geq 2\,, \qquad F_g = \frac{B_{2g}\,t^{2-2g}}{2g\,(2-2g)} + \frac{1}{2-2g}\,\mathop{\mathrm{Res}}_{z\to\pm1}\Phi(z)\,\omega_1^{(g)}(z)\,dz$$

where $\Phi'(z) = y(z)x'(z)$, *and* B_{2g} *is the* $2g$th *Bernoulli number* ($B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, $B_{10} = 5/66, \dots$).

Proof We have

$$\begin{aligned}&\frac{\partial}{\partial t_k}\mathop{\mathrm{Res}}_{z\to\pm1}\Phi(z)\,\omega_1^{(g)}(z)\,dz\\ &= \mathop{\mathrm{Res}}_{z\to\pm1}\left.\frac{\partial\Phi(z)}{\partial t_k}\right|_{x(z)}\omega_1^{(g)}(z)\,dz + \mathop{\mathrm{Res}}_{z\to\pm1}\Phi(z)\left.\frac{\partial}{\partial t_k}\omega_1^{(g)}(z)\,dz\right|_{x(z)}.\end{aligned}$$

From Theorem 3.4.1, we have

$$\left.\frac{\partial}{\partial t_k}\omega_1^{(g)}(z)\,dz\right|_{x(z)} = \frac{-1}{k}\mathop{\mathrm{Res}}_{z'\to\infty} x(z')^k\,\omega_2^{(g)}(z,z')\,dz\,dz',$$

and thus

$$\mathop{\mathrm{Res}}_{z\to\pm1}\Phi(z)\left.\frac{\partial}{\partial t_k}\omega_1^{(g)}(z)\,dz\right|_{x(z)} = \frac{-1}{k}\mathop{\mathrm{Res}}_{z\to\pm1}\mathop{\mathrm{Res}}_{z'\to\infty}\Phi(z)\,x(z')^k\,\omega_2^{(g)}(z,z')\,dz\,dz'.$$

The residue at $z \to \pm1$ is computed by Theorem 3.4.2, and gives

$$\mathop{\mathrm{Res}}_{z\to\pm1}\Phi(z)\left.\frac{\partial}{\partial t_k}\omega_1^{(g)}(z)\,dz\right|_{x(z)} = \frac{-1}{k}\,(1-2g)\mathop{\mathrm{Res}}_{z'\to\infty} x(z')^k\,\omega_1^{(g)}(z')\,dz'.$$

And since $d\Phi/dx = y$, we have from Lemma 3.1.4

$$-\left.\frac{\partial\Phi(z)}{\partial t_k}\right|_{x(z)} = \frac{x(z)^k}{2k} - \frac{(x(z)^k)_-}{k} = \frac{(x(z)^k)_+ - (x(z)^k)_-}{2k}.$$

By the symmetry $z \leftrightarrow 1/z$, we see that

$$\mathop{\mathrm{Res}}_{z'\to\pm1}(x(z')^k)_+\,\omega_1^{(g)}(z')\,dz' = -\mathop{\mathrm{Res}}_{z'\to\pm1}(x(z')^k)_-\,\omega_1^{(g)}(z')\,dz'.$$

Then, notice that the only poles of $(x(z')^k)_-\, \omega_1^{(g)}(z')$ are at $z' = \pm 1$ and $z' = 0$, therefore

$$\operatorname*{Res}_{z'\to\pm 1} (x(z')^k)_-\, \omega_1^{(g)}(z')\, dz' = -\operatorname*{Res}_{z'\to 0} (x(z')^k)_-\, \omega_1^{(g)}(z')\, dz' = \operatorname*{Res}_{z'\to\infty} (x(z')^k)_+\, \omega_1^{(g)}(z')\, dz'$$

(where we have used the symmetry $z \leftrightarrow 1/z$ again). Since $(x(z')^k)_-\, \omega_1^{(g)}(z')$ has no pole at ∞, we can add it, and obtain

$$\operatorname*{Res}_{z'\to\pm 1} (x(z')^k)_-\, \omega_1^{(g)}(z')\, dz' = \operatorname*{Res}_{z'\to\infty} x(z')^k\, \omega_1^{(g)}(z')\, dz',$$

which implies

$$\operatorname*{Res}_{z\to\pm 1} \left.\frac{\partial \Phi(z)}{\partial t_k}\right|_{x(z)} \omega_1^{(g)}(z)\, dz = \frac{-1}{k} \operatorname*{Res}_{z'\to\infty} x(z')^k\, \omega_1^{(g)}(z')\, dz',$$

and thus

$$\frac{\partial}{\partial t_k} \operatorname*{Res}_{z\to\pm 1} \Phi(z)\, \omega_1^{(g)}(z)\, dz = \frac{-(2-2g)}{k} \operatorname*{Res}_{z'\to\infty} x(z')^k\, \omega_1^{(g)}(z')\, dz' = (2-2g)\, \frac{\partial F_g}{\partial t_k}.$$

Therefore $F_g - \frac{1}{2-2g} \operatorname{Res}_{z\to\pm 1} \Phi(z)\, \omega_1^{(g)}(z)\, dz$ is independent of t_k. It can be computed at all $t_k = 0$, i.e. using the Gaussian matrix integral, for which, it is known that $F_g(\text{Gaussian}) = \frac{B_{2g}\, t^{2-2g}}{2g\,(2-2g)}$. □

Again this theorem is a general property of symplectic invariants, see Chap. 7.

Example Quadrangulations

A direct computation of the residue in Theorem 3.4.3 gives

$$\begin{aligned} F_2 &= t^{-2} \left(\frac{-89r^5 + 20r^4 + 130r^3 - 100r^2 - 65r + 56}{5 * 9 * 2^8\, r^5} - \frac{B_4}{8} \right) \\ &= \frac{15}{4}\, t\, t_4^3 + \frac{2007}{16}\, t^2\, t_4^4 + \frac{28323}{10}\, t^3\, t_4^5 + \dots \\ &= \frac{1}{5 * 3^3 * 2^8\, t^2} \sum_{n\geq 1} (12tt_4)^{n+2} \left(\frac{(2n+3)!}{2^{2n}\, n!\, (n+2)!}\, (28n+65) - 195(n+1) \right), \end{aligned}$$

where we have written $r = \sqrt{1 - 12\, t\, t_4}$.

3.4.3 Planar Maps

We want to compute F_0. The result has been found by many authors [4, 5, 31], and can be written as follows:

Theorem 3.4.4 *The generating function of planar maps is any of the following three equivalent expressions*

$$F_0 = \frac{1}{2}\left(\operatorname*{Res}_{z\to\infty} V(x)W_1^{(0)}(x)dx + t \operatorname*{Res}_{z\to\infty} V(x)\frac{dz}{z} + \frac{3t^2}{2} + t^2 \ln\left(\frac{\gamma^2}{t}\right)\right)$$
$$= \frac{1}{2}\Big(\sum_{j\geq 1} \frac{\gamma^2}{j}(u_{j+1} - u_{j-1})^2 + \frac{2t\gamma}{j}(-1)^j(u_{2j-1} - u_{2j+1})$$
$$+ \frac{3t^2}{2} + t^2 \ln\left(\frac{\gamma^2}{t}\right)\Big)$$
$$= \frac{1}{2}\Big(\sum_{j\geq 1} j v_j^2 + 4t\,(-1)^j\, v_{2j} + \frac{3t^2}{2} + t^2 \ln\left(\frac{\gamma^2}{t}\right)\Big) \tag{3.4.5}$$

where $V(x(z)) = 2v_0 + \sum_{j\geq 1} v_j(z^j + z^{-j})$.

Proof It is sufficient to prove that its derivative with respect to any t_l, is:

$$\frac{\partial F_0}{\partial t_l} = \frac{1}{l} < \operatorname{Tr} M^l >^{(0)} = -\frac{1}{l} \operatorname*{Res}_{z\to\infty} x^l\ W_1^{(0)}(x)\, dx$$

and that the initial conditions ($t_l = 0$) agree.

- Let us compute $\partial/\partial t_l$ of Eq. (3.4.5).

  First, notice that at fixed $x(z)$ we have:

$$\frac{d\,x(z)}{d\,t_l} = 0 = x'(z)\,\frac{\partial z}{\partial t_l} + \frac{\partial \alpha}{\partial t_l} + \frac{\partial \gamma}{\partial t_l}(z + \frac{1}{z}),$$

and thus

$$\frac{\partial z}{\partial t_l} = -\frac{dz}{dx}\left(\frac{\partial \alpha}{\partial t_l} + \frac{\partial \gamma}{\partial t_l}(z + \frac{1}{z})\right)$$

which implies after dividing by z:

$$\frac{\partial \ln z}{\partial t_l} = -\frac{d\ln z}{dx}\left(\frac{\partial \alpha}{\partial t_l} + \frac{\partial \gamma}{\partial t_l}(z + \frac{1}{z})\right)$$

and applying d/dx:

$$\frac{d}{dx}\frac{\partial}{\partial t_l}\ln z = \frac{\partial}{\partial t_l}\Big(\frac{dz}{zdx}\Big) = -\frac{d}{dx}\Big(\frac{dz}{zdx}\Big(\frac{\partial\alpha}{\partial t_l} + \frac{\partial\gamma}{\partial t_l}(z+\frac{1}{z})\Big)\Big).$$

Since $V(x) = \frac{x^2}{2} - \sum_{l=3}^{d} t_l\,\frac{x^l}{l}$, we have:

$$\frac{\partial}{\partial t_l}\Big(\operatorname*{Res}_{\infty} V(x)\frac{dz}{z}\Big) = -\operatorname*{Res}_{\infty}\frac{x^l}{l}\frac{dz}{z} + \operatorname*{Res}_{\infty} V'(x)\,\frac{dz}{z}\,\Big(\frac{\partial\alpha}{\partial t_l} + \frac{\partial\gamma}{\partial t_l}(z+\frac{1}{z})\Big).$$

Since we have $V'(x(z)) = \sum_{i=0}^{d-1} u_i(z^i + z^{-i})$, we have:

$$0 = 2\,u_0 = -\operatorname*{Res}_{\infty} V'(x)\,\frac{dz}{z} \quad , \quad \frac{t}{\gamma} = u_1 = -\operatorname*{Res}_{\infty} V'(x)\,dz = -\operatorname*{Res}_{\infty} V'(x)\,\frac{dz}{z^2}$$

therefore:

$$\frac{\partial}{\partial t_l}\Big(\operatorname*{Res}_{\infty} V(x)\frac{dz}{z}\Big) = -\operatorname*{Res}_{\infty}\frac{x^l}{l}\frac{dz}{z} - \frac{2t}{\gamma}\frac{\partial\gamma}{\partial t_l}$$

i.e.

$$\frac{\partial}{\partial t_l}\Big(\operatorname*{Res}_{\infty} V(x)\frac{dz}{z} + t\ln\gamma^2\Big) = -\operatorname*{Res}_{\infty}\frac{x^l}{l}\frac{dz}{z} = r_0$$

where we have defined

$$\frac{1}{l}x^l = r_0 + \sum_{j=1}^{l} r_j(z^j + z^{-j}) = r_0 + r_+(z) + r_-(z) \quad , \quad r_\pm(z) = \sum_{j=1}^{l} r_j z^{\pm j} = r_\mp(1/z).$$

For $k = 1$ and $g = 0$, Eq. (3.4.2), together with $\omega_2^{(0)}(z_1,z_2) = 1/(z_1-z_2)^2$, becomes:

$$\begin{aligned}\left.\frac{\partial W_1^{(0)}(x)}{\partial t_l}\right|_x &= -\frac{1}{l}\operatorname*{Res}_{x_2\to\infty} W_2^{(0)}(x,x_2)\,x_2^l\,dx_2\\ &= -\frac{1}{l}\operatorname*{Res}_{x_2\to\infty}\left(\frac{1}{x'(z)x'(z_2)(z-z_2)^2} - \frac{1}{(x-x_2)^2}\right)x_2^l\,dx_2\\ &= -\frac{1}{l}\sum_{j=1}^{\infty} j\operatorname*{Res}_{x_2\to\infty}\left(\frac{1}{x'(z)x'(z_2)}\,\frac{z^{j-1}}{z_2^{j+1}} - \frac{x^{j-1}}{x_2^{j+1}}\right)x_2^l\,dx_2\\ &= -x^{l-1} - \frac{1}{l\,x'(z)}\operatorname*{Res}_{z_2\to\infty}\sum_j j\,\frac{z^{j-1}}{z_2^{j+1}}\,x(z_2)^l\,dz_2\end{aligned}$$

Since $x'(z)W_1^{(0)}(x)$ contains only negative powers of z, we have:

$$x'(z)\,\left.\frac{\partial W_1^{(0)}(x)}{\partial t_l}\right|_x = -(x(z)^{l-1}x'(z))_- = -\frac{1}{l}\,((x(z))^l)'_- = -r_-(z)'. \quad (3.4.6)$$

That implies:

$$\begin{aligned}
& \frac{\partial}{\partial t_l}\ \operatorname*{Res}_{\infty}\ V(x)\,W_1^{(0)}(x)\,dx + 2\ \operatorname*{Res}_{\infty}\ \frac{x^l}{l}\,W_1^{(0)}(x)\,dx \\
&= \operatorname*{Res}_{\infty}\ V(x)\,\frac{\partial W_1^{(0)}(x)}{\partial t_l}\,dx + \operatorname*{Res}_{\infty}\ \frac{x^l}{l}\,W_1^{(0)}(x)\,dx \\
&= \operatorname*{Res}_{\infty}\ V'(x)\,r_-(z)\,x'(z)dz + \operatorname*{Res}_{\infty}\,(r_0 + r_+(z))\,W_1^{(0)}(x)\,dx \\
&= \operatorname*{Res}_{\infty}\ V'(x)\,r_-(z)\,x'(z)dz + \operatorname*{Res}_{\infty}\ r_+(z)\,W_1^{(0)}(x)\,dx - tr_0 \\
&= \operatorname*{Res}_{\infty}\,(V'(x) - W_1^{(0)}(x))\,r_-(z)\,x'(z)dz + \operatorname*{Res}_{\infty}\ r_+(z)\,W_1^{(0)}(x)\,dx - tr_0 \\
&= \operatorname*{Res}_{\infty}\ W_1^{(0)}(x(1/z))\,r_-(z)\,x'(z)dz + \operatorname*{Res}_{\infty}\ r_+(z)\,W_1^{(0)}(x)\,dx - tr_0 \\
&= \operatorname*{Res}_{0}\ W_1^{(0)}(x(z))\,r_+(z)\,x'(z)\,dz + \operatorname*{Res}_{\infty}\ r_+(z)\,W_1^{(0)}(x)\,dx - tr_0 \\
&= \operatorname*{Res}_{0}\ W_1^{(0)}(x(z))\,r_+(z)\,dx + \operatorname*{Res}_{\infty}\ r_+(z)\,W_1^{(0)}(x)\,dx - tr_0 \\
&= -tr_0
\end{aligned}$$

indeed it gives $-tr_0$, since $W_1^{(0)}(z)r_+(z)x'(z)$ has no other poles than 0 and ∞.

That implies:

$$\frac{\partial}{\partial t_l}(\operatorname*{Res}_{\infty}\ V(x)W_1^{(0)}dx + t\ \operatorname*{Res}_{\infty}\ V\frac{dz}{z} + t^2\ln\gamma^2) = -2\ \operatorname*{Res}_{\infty}\ \frac{x^l}{l}W_1^{(0)}dx$$

which is the result we sought.

- This shows that F_0 is given by Eq. (3.4.5), up to an integration constant independent of t_k's. The constant can be computed with $t_k = 0$ for $k \geq 3$, i.e. a quadratic potential $V(x) = \frac{-t_2}{2}x^2$, which we have already studied in Sect. 3.1.9, and for which we should have $F_0 = -\frac{t^2}{2}\ln(-t_2)$. In that case we have (see the example Sect. 3.1.9):

$$u_0 = 0 \quad , \quad u_1 = \frac{t}{\gamma} \quad , \quad \gamma = \sqrt{-t/t_2} \quad , \quad x(z) = \gamma(z+\frac{1}{z}) \quad , \quad y(z) = \frac{t}{2\gamma}(z-\frac{1}{z})$$

The expression Eq. (3.4.5) gives as expected:

$$\frac{1}{2}\Big(\frac{\gamma^2}{2}u_1^2 - 2t\gamma u_1 + \frac{3t^2}{2} + t^2 \ln(\gamma^2/t)\Big) = -\frac{t^2}{2}\ln(-t_2).$$

□

There are also nice expressions for the derivatives of F_0 with respect to t:

Theorem 3.4.5 *Derivatives of F_0 with respect to t:*

$$\begin{aligned}\frac{\partial F_0}{\partial t} &= \operatorname*{Res}_{z\to\infty} V(x(z))\frac{dz}{z} + t + t\,\ln\left(\frac{\gamma^2}{t}\right)\\ &= t\,\ln\left(\frac{\gamma^2}{t}\right) + t - 2v_0\end{aligned}$$

$$\frac{\partial^2 F_0}{\partial t^2} = \ln\left(\frac{\gamma^2}{t}\right)$$

$$\frac{1}{t} + \frac{\partial^3 F_0}{\partial t^3} = \operatorname*{Res}_{z\to\pm 1} \frac{dz}{z^3\,x'(z)\,y'(z)} = \frac{1}{2\gamma}\left(\frac{1}{y'(1)} + \frac{1}{y'(-1)}\right).$$

Proof Specialize Eq. (3.1.12) of Lemma 3.1.4 to $z = 1$ and $z = -1$, at which $x'(z)$ vanishes, this implies:

$$\frac{\partial\alpha}{\partial t} \pm 2\frac{\partial\gamma}{\partial t} = \frac{\partial x(\pm 1)}{\partial t} = \frac{\pm 1}{y'(\pm 1)},$$

from which we get

$$\frac{\partial\gamma}{\partial t} = \frac{1}{4}\left(\frac{1}{y'(1)} + \frac{1}{y'(-1)}\right) \quad , \quad \frac{\partial\alpha}{\partial t} = \frac{1}{2}\left(\frac{1}{y'(1)} - \frac{1}{y'(-1)}\right).$$

This also implies that $\dot{x} = \partial x/\partial t$ at fixed z is $\dot{x} = \dot{\alpha} + \dot{\gamma}(z + 1/z)$ (we denote $^{\cdot} = \partial/\partial t$ and $' = \partial/\partial z$), and applying the chain rule, we find that at fixed x we have

$$\frac{\partial z}{\partial t} = -\frac{1}{x'(z)}\,(\dot{\alpha} + \dot{\gamma}(z + 1/z)) = -\frac{1}{x'(z)}\,(\dot{\alpha} + \frac{\dot{\gamma}}{\gamma}\,(x(z) - \alpha)),$$

and

$$\frac{\partial \ln z}{\partial t} = -\frac{1}{z\,x'(z)}\,(\dot{\alpha} + \dot{\gamma}(z + 1/z)).$$

Then, use the first expression of F_0, namely:

$$2\,F_0 = \operatorname*{Res}_{z\to\infty} V(x) W_1^{(0)}(x) dx + t \operatorname*{Res}_{z\to\infty} V(x)\, d\ln z + \frac{3t^2}{2} + t^2 \,\ln\left(\frac{\gamma^2}{t}\right)$$

and take a derivative of each term with respect to t at fixed x, that gives

$$2\,\frac{\partial F_0}{\partial t} = 2 \operatorname*{Res}_{z\to\infty} V(x)\frac{dz}{z} + t \operatorname*{Res}_{z\to\infty} V(x)\, d\left(\frac{\partial \ln z}{\partial t}\right) + 2t + 2t\ln\frac{\gamma^2}{t} + 2t^2\frac{\dot\gamma}{\gamma}.$$

Let us integrate the second term by parts, and get

$$\begin{aligned}
2\,\frac{\partial F_0}{\partial t} &= 2 \operatorname*{Res}_{z\to\infty} V(x)\frac{dz}{z} + t \operatorname*{Res}_{z\to\infty} V'(x)\,\frac{dz}{z}(\dot\alpha + \frac{\dot\gamma}{\gamma}(x(z)-\alpha)) \\
&\qquad +2t + 2t\ln\frac{\gamma^2}{t} + 2t^2\frac{\dot\gamma}{\gamma} \\
&= 2 \operatorname*{Res}_{z\to\infty} V(x)\frac{dz}{z} + 2t + 2t\ln\frac{\gamma^2}{t} \\
&\qquad + \frac{t}{\gamma}\,(\gamma\dot\alpha - \alpha\dot\gamma) \operatorname*{Res}_{z\to\infty} V'(x)\,\frac{dz}{z} + \frac{t\,\dot\gamma}{\gamma}\left(\operatorname*{Res}_{z\to\infty} x\,V'(x)\,\frac{dz}{z} + 2t\right).
\end{aligned}$$

We have

$$\operatorname*{Res}_{z\to\infty} V'(x)\,\frac{dz}{z} = \frac{\partial}{\partial t} \operatorname*{Res}_{x\to\infty} V'(x)\,W_1^{(0)}(x)\,dx = 0,$$

$$\operatorname*{Res}_{z\to\infty} xV'(x)\,\frac{dz}{z} = \frac{\partial}{\partial t} \operatorname*{Res}_{x\to\infty} xV'(x)\,W_1^{(0)}(x)\,dx = -\frac{\partial}{\partial t}\,t^2 = -2t.$$

This implies the result for $\partial F_0/\partial t$.

Then, one easily finds the last equality that $\partial^2 F_0/\partial t^2 = \ln\gamma^2/t$. □

3.4.4 Genus 1 Maps

Genus 1 closed maps are elements of $\mathbb{M}_0^{(1)}$. Their generating function is given by the following theorem.

Theorem 3.4.6 *The generating function of genus 1 maps is*

$$F_1 = -\frac{1}{24}\ln\left(\gamma^2\; y'(1)\,y'(-1)/t^2\right).$$

This result was derived many times, in particular in [4, 22, 31].

Proof Let us denote $\partial/\partial_{t_k} = \dot{}$, and $\partial/\partial z = '$. From Eq. (3.1.13) of Lemma 3.1.4, we have

$$\dot{x}\, y' - \dot{y}\, x' = H_k' \qquad , \qquad H_k(z) = \frac{1}{2k}\left((x(z)^k)_+ - (x(z)^k)_- \right).$$

Taking the first derivative, we have

$$\dot{x}'\, y' + \dot{x}\, y'' - \dot{y}'\, x' - \dot{y}\, x'' = H_k''$$

and the second derivative

$$\dot{x}''\, y' + 2\dot{x}' y'' + \dot{x}\, y''' - \dot{y}''\, x' - 2\dot{y}'\, x'' - \dot{y}\, x''' = H_k'''.$$

At $z = 1$, we have $x'(z) = 0$ and $\dot{x}'(1) = 0$, therefore we get

$$\dot{x}(1)\, y''(1) - \dot{y}(1)\, x''(1) = H_k''(1)$$

i.e.

$$\dot{y}(1) = \dot{x}(1)\, \frac{y''(1)}{x''(1)} - \frac{H_k''(1)}{x''(1)},$$

and

$$\dot{x}''(1)\, y'(1) + \dot{x}(1)\, y'''(1) - 2\dot{y}'(1)\, x''(1) - \dot{y}(1)\, x'''(1) = H_k'''(1)$$

i.e.

$$\frac{\dot{y}'(1)}{y'(1)} = -\frac{H_k'''(1)}{2x''(1)\, y'(1)} + \dot{x}''(1)\, \frac{1}{2x''(1)} + \dot{x}(1)\, \frac{y'''(1)}{2x''(1)\, y'(1)} - \dot{y}(1)\, \frac{x'''(1)}{2x''(1)\, y'(1)}$$

$$\begin{aligned}\frac{\dot{y}'(1)}{y'(1)} = & -\frac{H_k'''(1)}{2x''(1)\, y'(1)} + \frac{H_k''(1)\, x'''(1)}{2x''(1)^2\, y'(1)} + \dot{x}''(1)\, \frac{1}{2x''(1)} \\ & + \dot{x}(1)\left(\frac{y'''(1)}{2x''(1)\, y'(1)} - \frac{x'''(1) y''(1)}{2x''(1)^2\, y'(1)} \right).\end{aligned}$$

Beside, we have

$$\dot{x}(1) = \dot{\alpha} + 2\dot{\gamma} = \frac{H_k'(1)}{y'(1)} \qquad , \qquad \frac{\dot{x}''(1)}{x''(1)} = \frac{\dot{\gamma}}{\gamma} = \frac{1}{4\gamma}\left(\frac{H_k'(1)}{y'(1)} - \frac{H_k'(-1)}{y'(-1)} \right).$$

Therefore

$$\frac{1}{24}\,\frac{\dot{y}'(1)}{y'(1)} = \frac{\dot{\gamma}}{48\gamma} - \frac{H_k'''(1)}{96\gamma\, y'(1)} - \frac{H_k''(1)}{32\gamma\, y'(1)} + H_k'(1)\,\left(\frac{y'''(1)}{96\gamma\, y'(1)^2} + \frac{y''(1)}{32\gamma\, y'(1)^2}\right)$$

and, similarly at $z = -1$ we obtain

$$\begin{aligned}\frac{1}{24}\,\frac{\dot{y}'(-1)}{y'(-1)} = \frac{\dot{\gamma}}{48\gamma} + \frac{H_k'''(-1)}{96\gamma\, y'(-1)} - \frac{H_k''(-1)}{32\gamma\, y'(-1)}\\ +H_k'(-1)\,\left(-\frac{y'''(-1)}{96\gamma\, y'(-1)^2} + \frac{y''(-1)}{32\gamma\, y'(-1)^2}\right).\end{aligned}$$

From Eq. (3.3.4), we have

$$\begin{aligned}\omega_1^{(1)}(z) = \frac{1}{16\gamma y'(1)}\left(\frac{1}{(z-1)^4} + \frac{1}{(z-1)^3} - \frac{1+\frac{y''(1)}{y'(1)}+\frac{y'''(1)}{3y'(1)}}{2(z-1)^2}\right)\\ -\frac{1}{16\gamma y'(-1)}\left(\frac{1}{(z+1)^4} - \frac{1}{(z+1)^3} - \frac{1-\frac{y''(-1)}{y'(-1)}+\frac{y'''(-1)}{3y'(-1)}}{2(z+1)^2}\right)\end{aligned}$$

and thus

$$\begin{aligned}&\operatorname*{Res}_{z\to\pm 1} H_k(z)\,\omega_1^{(1)}(z)\,dz\\ &= \frac{1}{16\gamma y'(1)}\left(\frac{H_k'''(1)}{6} + \frac{H_k''(1)}{2} - H_k'(1)\,\frac{1+\frac{y''(1)}{y'(1)}+\frac{y'''(1)}{3y'(1)}}{2}\right)\\ &\quad -\frac{1}{16\gamma y'(-1)}\left(\frac{H_k'''(-1)}{6} - \frac{H_k''(-1)}{2} - H_k'(-1)\,\frac{1-\frac{y''(-1)}{y'(-1)}+\frac{y'''(-1)}{3y'(-1)}}{2}\right)\end{aligned}$$

i.e.

$$\begin{aligned}\operatorname*{Res}_{z\to\pm 1} H_k(z)\,\omega_1^{(1)}(z)\,dz &= -\frac{1}{24}\,\frac{\dot{y}'(1)}{y'(1)} - \frac{1}{24}\,\frac{\dot{y}'(-1)}{y'(-1)} + \frac{\dot{\gamma}}{24\gamma}\\ &\quad -\frac{1}{32\,\gamma}\,\left(\frac{H_k'(1)}{y'(1)} - \frac{H_k'(-1)}{y'(-1)}\right)\\ &= -\frac{1}{24}\,\frac{\dot{y}'(1)}{y'(1)} - \frac{1}{24}\,\frac{\dot{y}'(-1)}{y'(-1)} + \frac{\dot{\gamma}}{24\gamma} - \frac{\dot{\gamma}}{8\,\gamma}\\ &= -\frac{1}{24}\,\frac{\dot{y}'(1)}{y'(1)} - \frac{1}{24}\,\frac{\dot{y}'(-1)}{y'(-1)} - \frac{2\,\dot{\gamma}}{24\gamma}.\end{aligned}$$

Eventually, we arrive at

$$\frac{1}{24}\frac{\partial}{\partial t_k}\ \ln \gamma^2\, y'(1)\, y'(-1) = -\operatorname*{Res}_{z\to\pm 1} H_k(z)\,\omega_1^{(1)}(z)\,dz.$$

The only poles of $H_k(z)\,\omega_1^{(1)}(z)$ are at $z = 1, -1, 0, \infty$, and thus, moving the integration contour we have:

$$\frac{1}{24}\frac{\partial}{\partial t_k}\ \ln \gamma^2\, y'(1)\, y'(-1) = \operatorname*{Res}_{\infty} H_k(z)\,\omega_1^{(1)}(z)\,dz + \operatorname*{Res}_{0} H_k(z)\,\omega_1^{(1)}(z)\,dz.$$

Near $z = 0$, we use the symmetry $H_k(1/z) = -H_k(z)$ and $\omega_1^{(1)}(1/z) = z^2\omega_1^{(1)}(z)$, and thus

$$\operatorname*{Res}_{0} H_k(z)\,\omega_1^{(1)}(z)\,dz = \operatorname*{Res}_{\infty} H_k(1/z)\,\omega_1^{(1)}(1/z)\,d(1/z) = \operatorname*{Res}_{\infty} H_k(z)\,\omega_1^{(1)}(z)\,dz.$$

Then, near $z = \infty$, $2k\,H_k(z) = (x(z)^k)_+ - (x(z)^k)_-$, and it is clear that $(x(z)^k)_-\,\omega_1^{(1)}(z)$ has no pole at $z = \infty$, thus:

$$2\operatorname*{Res}_{\infty} H_k(z)\,\omega_1^{(1)}(z)\,dz = \frac{1}{k}\ \operatorname*{Res}_{\infty} x(z)^k\,\omega_1^{(1)}(z)\,dz.$$

This implies that

$$\frac{1}{24}\frac{\partial}{\partial t_k}\ \ln \gamma^2\, y'(1)\, y'(-1) = \frac{1}{k}\ \operatorname*{Res}_{\infty} x(z)^k\,\omega_1^{(1)}(z)\,dz = -\frac{\partial F_1}{\partial t_k}$$

where we used Eq. (3.4.1). This proves that $F_1 + \frac{1}{24}\ \ln \gamma^2\, y'(1)\, y'(-1)$ is independent of t_k, and can be computed when all $t_k = 0\ \ \forall\, k$, i.e. for the Gaussian matrix model, and we find that it is worth $\frac{1}{24}\ln t^2$. □

Example: Quadrangulations

For quadrangulations, that gives ($r = \sqrt{1-12tt_4}$):

$$\begin{aligned} F_1 &= \frac{1}{12}\ \ln \frac{1+r}{2r} \\ &= \frac{tt_4}{4} + \frac{15\,t^2t_4^2}{8} + \frac{33\,t^3t_4^3}{2} + \frac{2511\,t^4t_4^4}{16} + \dots \\ &= \frac{1}{12}\sum_{n\geq 1}\frac{3^n}{n}\left(2^{2n-1} - \frac{(2n-1)!}{n!\,(n-1)!}\right)\ (tt_4)^n. \end{aligned}$$

3.4.5 Derivatives of F_g's

The following derivative formulae are very useful:

Theorem 3.4.7

$$\frac{\partial F_g}{\partial t_k} = -\frac{1}{k} \operatorname*{Res}_{z\to\infty} \omega_1^{(g)}(z)\ x(z)^k\, dz = \frac{1}{k} \operatorname*{Res}_{z\to 0} \omega_1^{(g)}(z)\ x(z)^k\, dz$$
$$= \frac{1}{2k} \operatorname*{Res}_{z\to\pm 1} \omega_1^{(g)}(z)\ \left((x(z)^k)_+ - (x(z)^k)_-\right)\, dz$$

$$\left.\frac{\partial \omega_n^{(g)}(z_1,\dots,z_n)}{\partial t_k}\right|_{x(z_i)} = -\frac{1}{k} \operatorname*{Res}_{z\to\infty} \omega_{n+1}^{(g)}(z,z_1,\dots,z_n)\ x(z)^k\, dz$$
$$= \frac{1}{k} \operatorname*{Res}_{z\to 0} \omega_{n+1}^{(g)}(z,z_1,\dots,z_n)\ x(z)^k\, dz$$
$$= \frac{1}{2k} \operatorname*{Res}_{z\to\pm 1} \omega_{n+1}^{(g)}(z,z_1,\dots,z_n)\ \left((x(z)^k)_+ - (x(z)^k)_-\right)\, dz.$$

This theorem can be seen as a consequence of a general property of the topological recursion, the form-cycle duality deformation Theorem 7.3.2. For completeness we give the direct proof for maps:

Proof Using Theorem 3.4.1, we have

$$\frac{\partial F_g}{\partial t_k} = \frac{1}{k}\mathcal{T}_k^{(g)} = -\frac{1}{k} \operatorname*{Res}_{x\to\infty} x^k\ W_1^{(g)}(x)\, dx = -\frac{1}{k} \operatorname*{Res}_{z\to\infty} x(z)^k\ \omega_1^{(g)}(z)\, dz.$$

Since $\omega_1^{(g)}(z) = O(1/z^2)$ at large z, we can replace $x(z)^k \to (x(z)^k)_+$ by keeping only positive powers of z:

$$\frac{\partial F_g}{\partial t_k} = -\frac{1}{k} \operatorname*{Res}_{z\to\infty} (x(z)^k)_+\ \omega_1^{(g)}(z)\, dz,$$

and we can even add any negative powers of z, for example:

$$\frac{\partial F_g}{\partial t_k} = -\frac{1}{k} \operatorname*{Res}_{z\to\infty} \left((x(z)^k)_+ - (x(z)^k)_-\right)\ \omega_1^{(g)}(z)\, dz.$$

Using the symmetry $z \to 1/z$, we also have

$$\frac{\partial F_g}{\partial t_k} = -\frac{1}{k} \operatorname*{Res}_{z\to 0} \left((x(z)^k)_+ - (x(z)^k)_-\right)\ \omega_1^{(g)}(z)\, dz.$$

and thus adding the 2:

$$\frac{\partial F_g}{\partial t_k} = -\frac{1}{2} \operatorname*{Res}_{z\to 0,\infty} \frac{(x(z)^k)_+ - (x(z)^k)_-}{k}\, \omega_1^{(g)}(z)\, dz.$$

$x(z)$ has poles only at 0 and ∞, and so do $x(z)^k_\pm$, and $\omega_1^{(g)}(z)$ has poles only at $z = \pm 1$, and the sum of all residues of a rational fraction has to vanish, therefore:

$$\frac{\partial F_g}{\partial t_k} = \frac{1}{2} \operatorname*{Res}_{z\to \pm 1} \frac{(x(z)^k)_+ - (x(z)^k)_-}{k}\, \omega_1^{(g)}(z)\, dz.$$

we have proved the third equation of the theorem.

Theorem 3.4.1, also applies to $W_n^{(g)}$ and gives in the same manner, the fourth equation. □

There is a similar theorem for the derivatives with respect to t:

Theorem 3.4.8

$$g \geq 1 \;, \qquad \frac{\partial F_g}{\partial t} = \int_0^\infty \omega_1^{(g)}(z)\;\; dz = -\operatorname*{Res}_{z\to\pm 1} \omega_1^{(g)}(z)\, dz \;\; \ln z,$$

and more generally for $2g - 2 + n \geq 0$ *we have*

$$\left.\frac{\partial \omega_n^{(g)}(z_1,\ldots,z_n)}{\partial t}\right|_{x(z_i)} = \int_0^\infty \omega_{n+1}^{(g)}(z, z_1,\ldots,z_n)\;\; dz$$

$$= -\operatorname*{Res}_{z\to\pm 1} \omega_{n+1}^{(g)}(z, z_1,\ldots,z_n)\, dz \;\; \ln z.$$

We shall admit it. This theorem is also a consequence of the form-cycle duality deformation of topological recursion: Theorem 7.3.2.

3.4.6 Summary Closed Maps

Finally, we have just proved that the F_g's, are the symplectic invariants (see Chap. 7) of the spectral curve $\mathcal{E}$ of Theorem 3.3.1:

Theorem 3.4.9 *The* F_g*'s, are the symplectic invariants of the spectral curve* $\mathcal{E} = (\mathbb{C} \cup \{\infty\}, x, y)$ *of Theorem 3.3.1:*

$$F_g = \mathcal{F}_g(\mathcal{E}) + \frac{B_{2g}\, t^{2-2g}}{2g\,(2-2g)}$$

$$F_1 = \mathcal{F}_1(\mathcal{E}) + \frac{1}{24} \ln t^2$$

$$F_0 = \mathcal{F}_0(\mathcal{E}) + \frac{3t^2}{4} - \frac{t^2}{2} \ln t^2.$$

Again, this theorem gives an efficient way of computing explicitly the F_g's. It can be represented diagrammatically as in Sect. 7.4 of Chap. 7. Also, this theorems allows to compute easily the asymptotic numbers of large maps, as we shall see in Chap. 5.

3.5 Structure Properties

So far, we had defined $W_n^{(g)}$'s and $W_0^{(g)} = F_g$'s to be formal series of t, whose coefficients are polynomials of $t_3, t_4, \dots, t_d$, and polynomials of $1/x_i$:

$$W_n^{(g)}(x_1, \dots, x_n) \;\in\; \mathbb{Q}[\{1/x_i\}, t_3, t_4, \dots, t_d][[t]].$$

We didn't consider the question of convergency of the formal series.

Now, from the explicit solution, and from the topological recursion, we see that the $W_n^{(g)}$'s, are algebraic combinations of γ, α, and the Zhukovski variables z_i'z, and all of them are algebraic functions of t.

This shows that $W_n^{(g)}$'s are algebraic functions of t, the series are thus convergent in some disk. Let us be more precise.

Since the topological recursion of Theorem 3.3.1 only amounts to computing residues at $z = \pm 1$, all the $\omega_{g,n}$'s will be polynomials of the Taylor series coefficients of the function $y(z)$ at $z = \pm 1$. We write the Taylor expansion at $z = 1$

$$y(z) - y(1/z) = 2\, y'(1)\, (z - 1 + \sum_{k=2}^{\infty} y_{+,k}\, (z-1)^k)$$

or at $z = -1$

$$y(z) - y(1/z) = 2\, y'(-1)\, (z + 1 + \sum_{k=2}^{\infty} y_{-,k}\, (z+1)^k).$$

Then, the topological recursion of Theorem 3.3.1 can be written:

$$\omega_{n+1}^{(g)}(z_0, L) = \sum_{\epsilon=\pm 1} \frac{-1}{4\gamma\, y'(\epsilon)} \operatorname*{Res}_{z\to\epsilon} \frac{dz}{(z_0 - z)(zz_0 - 1)\,(z-\epsilon)}\, \frac{1}{1 + \sum_{k=1}^{\infty} y_{\epsilon,k+1}\, (z-\epsilon)^k}$$
$$\Big[\sum_{h=0}^{g} \sum_{J\subset L}{}' \omega_{1+\#J}^{(h)}(z, J)\omega_{1+n-\#J}^{(g-h)}(\frac{1}{z}, L\setminus J) + \omega_{n+2}^{(g-1)}(z, \frac{1}{z}, L)\Big].$$

By an easy recursion, this says that

Proposition 3.5.1 *The stable* ($2g-2+n>0$) $\omega_n^{(g)}$*'s are polynomials with rational coefficients, of the* $1/(z_i \pm 1)$*, and of the Taylor series coefficients* $y_{+,j}$*'s and the* $y_{-,j}$*'s, and of* $1/4\gamma y'(\pm 1)$*:*

$$\gamma^{2g-2+n}\,\omega_n^{(g)}(z_0,L) \quad \in \quad \mathbb{Q}[1/(z_i-1), 1/(z_i+1),\ 1/y'(1), 1/y'(-1), y_{+,j}, y_{-,j}].$$

There is a lot of redundancy in the parameters $y_{\pm,j}$'s, for example $y''(1) = -3y'(1)$. The moments of $M(x)$, see Definition 3.1.1, are better suited.

The method of moments was introduced by Ambjørn, Chekhov, Kristjansen, Makeenko in 1993 [5], as the coefficients of the Taylor series expansion of $M(x)$ rather than $y(z) = -\frac{1}{2}M(x)\sqrt{(x-a)(x-b)}$, near $z = \pm 1$, i.e. near $x = a, b$:

$$M(x) = M_{+,0}\,(1 - \sum_{k=1}^{\infty} M_{+,k}\ (x-a)^k)$$
$$= M_{-,0}\,(1 - \sum_{k=1}^{\infty} M_{-,k}\ (x-b)^k).$$

We then use the topological recursion in the form of Theorem 3.3.2:

$$\begin{aligned}
&\sqrt{(x_0-a)(x_0-b)}\,W_{n+1}^{(g)}(x_0,x_1,\dots,x_n)\\
&= \operatorname*{Res}_{x\to a,b} \frac{dx}{x_0-x}\,\frac{1}{M(x)}\ \mathcal{W}_{k+1}^{(g)}(x;x_1,\dots,x_k)\\
&= \frac{1}{M_{+,0}} \operatorname*{Res}_{x\to a} \frac{dx}{x_0-x}\,\frac{1}{1-\sum_{k\ge 1} M_{+,k}(x-a)^k}\ \mathcal{W}_{n+1}^{(g)}(x;x_1,\dots,x_n)\\
&\quad + \frac{1}{M_{-,0}} \operatorname*{Res}_{x\to b} \frac{dx}{x_0-x}\,\frac{1}{1-\sum_{k\ge 1} M_{-,k}(x-b)^k}\ \mathcal{W}_{n+1}^{(g)}(x;x_1,\dots,x_n).
\end{aligned}$$

From it, it we have (first claimed by [5]) that:

Theorem 3.5.1 *If* $2g-2+n>0$*, the* $\omega_n^{(g)}(z_1,\dots,z_n)$ *are rational functions of the* z_i*'s, with poles only at* $z_i = \pm 1$*, of degree* $d_i + 2$*, such that* $d_i \ge 0$ *and* $\sum_i d_i \le 6g-6+2n$*.*

Moreover, they are rational fractions of γ *and the* $3g-3+n$ *first moments* $M_{\pm,k}$*'s:*

$$\begin{aligned}
&(16\gamma)^{4g-4+2n}\,W_n^{(g)}(x_1,\dots,x_n)\prod_{i=1}^{n}\sqrt{(x_i-a)(x_i-b)}\\
&\in\ \mathbb{Z}\left[\frac{4\gamma}{x_i-a},\frac{4\gamma}{x_i-b}, 1/M_{+,0}, 1/M_{-,0}, \{(4\gamma)^k M_{+,k}\}_{k\le 3g-3+n}, \{(4\gamma)^k M_{-,k}\}_{k\le 3g-3+n}\right]
\end{aligned}$$

and more precisely

$$(16\gamma)^{4g-4+2n}\, W_n^{(g)}(x_1,\dots,x_n)\, \prod_{i=1}^n \sqrt{(x_i-a)(x_i-b)}$$
$$= \sum_{\epsilon_i=\pm 1} \sum_{d_i,\, \sum d_i \leq 3g-3+n} \bar{P}^{(g,n)}_{\epsilon,\mathbf{d}}(1/M_{+,0}, 1/M_{-,0}; \{(4\gamma)^k M_{+,k}\}, \{(4\gamma)^k M_{-,k}\})$$
$$\frac{(4\gamma)^{d_i+1}}{\prod_i (x_i - \frac{a+b}{2} - 2\gamma\epsilon_i)^{d_i+1}},$$

where $\bar{P}^{(g,n)}_{\epsilon,\mathbf{d}}(1/M_{+,0}, 1/M_{-,0}; \{(4\gamma)^k M_{+,k}\}_{k\leq 3g-3+n}, \{(4\gamma)^k M_{-,k}\}_{k\leq 3g-3+n})$ *is a universal polynomial with integer coefficients.*

Or equivalently

$$4^{4g-4+2n}\, (4\gamma)^{g-1}\, W_n^{(g)}(x_1,\dots,x_n)\, \prod_{i=1}^n \sqrt{(x_i-a)(x_i-b)}$$
$$= \sum_{\epsilon_i=\pm 1} \sum_{d_i,\, \sum d_i \leq 3g-3+n} P^{(g,n)}_{\epsilon,\mathbf{d}}(1/M_{+,0}, 1/M_{-,0}; 1/4\gamma, \{M_{+,k}\}, \{M_{-,k}\})$$
$$\frac{(4\gamma)^{d_i+1}}{\prod_i (x_i - \frac{a+b}{2} - 2\gamma\epsilon_i)^{d_i+1}},$$

where $P^{(g,n)}_{\epsilon,\mathbf{d}}(1/M_{+,0}, 1/M_{-,0}; 1/4\gamma, \{M_{+,k}\}_{k\leq 3g-3+n}, \{M_{-,k}\}_{k\leq 3g-3+n})$ *is a universal homogeneous polynomial with integer coefficients:*

- *it is homogeneous of degree* $2g-2+n$ *of* $1/M_{+,0}$ *and* $1/M_{-,0}$*,*
- *homogeneous of degree* $(3g-3+n-\sum_i d_i)$ *in all the other variables, where each* $M_{\epsilon,k}$ *is considered to be of degree k, and* $1/\gamma$ *is of degree 1. This explain that it depends only on the first* $3g-3+n$ *moments* $M_{\pm,k}$*'s with*

$$k \leq 3g-3+n.$$

- *It is invariant if we change* $\epsilon_i \to -\epsilon_i$*, and* $(4\gamma)^k M_{\pm,k} \to (-4\gamma)^k M_{\mp,k}$*:*

$$\bar{P}^{(g,n)}_{-\epsilon,\mathbf{d}}(1/M_{+,0}, 1/M_{-,0}; \{(4\gamma)^k M_{+,k}\}, \{(4\gamma)^k M_{-,k}\})$$
$$= (-1)^{n+\sum_i d_i}\, \bar{P}^{(g,n)}_{\epsilon,\mathbf{d}}(1/M_{-,0}, 1/M_{+,0}; \{(-4\gamma)^k M_{-,k}\}, \{(4\gamma)^k M_{+,k}\}) \quad (3.5.1)$$

$$P^{(g,n)}_{-\epsilon,\mathbf{d}}(1/M_{+,0}, 1/M_{-,0}; 1/4\gamma, \{M_{+,k}\}, \{M_{-,k}\})$$
$$= (-1)^{n+\sum_i d_i}\, P^{(g,n)}_{\epsilon,\mathbf{d}}(1/M_{-,0}, 1/M_{+,0}; -1/4\gamma, \{(-)^k M_{-,k}\}, \{(-1)^k M_{+,k}\}) \quad (3.5.2)$$

Proof Let us define the variables $\bar{x}_i$ such that

$$\bar{x}_i = \left(x_i - \frac{a+b}{2}\right)/4\gamma,$$

and define $\bar{M}_{\pm,k} = (4\gamma)^k M_{\pm,k}$.

Define, for $(g,n) \neq (0,2)$:

$$U_n^{(g)}(\bar{x}_1,\dots,\bar{x}_n) = (16\gamma)^{4g-4+2n}\, W_n^{(g)}(x_1,\dots,x_n) \prod_{i=1}^{n} \sqrt{(x_i-a)(x_i-b)},$$

and

$$U_2^{(0)}(x_1,x_2) = \frac{2x_1x_2 - \frac{1}{2}}{4(x_1-x_2)^2}.$$

Define also, for $(g,n) \neq (1,0)$:

$$\begin{aligned}\bar{\mathcal{W}}_{n+1}^{(g)}(x;x_1,\dots,x_n) &= U_{n+2}^{(g-1)}(x,x,x_1,\dots,x_n)\\ &\quad + \sum_{h+h'=g,\, I\sqcup I'=\{x_1,\dots,x_n\}}' U_{1+\#I}^{(h)}(x,I)\, U_{1+\#I'}^{(h')}(x,I')\end{aligned}$$

and

$$\bar{\mathcal{W}}_1^{(1)}(x) = \frac{1}{16\ (x-1/2)(x+1/2)}.$$

The topological recursion can be written

$$\begin{aligned}&U_{n+1}^{(g)}(x_0,x_1,\dots,x_n)\\ &= \frac{16}{M_{+,0}} \operatorname*{Res}_{x\to 1/2} \frac{dx}{x_0-x}\ \frac{1}{1-\sum_{k\geq 1}\bar{M}_{+,k}(x-1/2)^k}\ \frac{\bar{\mathcal{W}}_{n+1}^{(g)}(x;x_1,\dots,x_n)}{(x-1/2)(x+1/2)}\\ &\quad + \frac{16}{M_{-,0}} \operatorname*{Res}_{x\to -1/2} \frac{dx}{x_0-x}\ \frac{1}{1-\sum_{k\geq 1}\bar{M}_{-,k}(x+1/2)^k}\ \frac{\bar{\mathcal{W}}_{n+1}^{(g)}(x;x_1,\dots,x_n)}{(x-1/2)(x+1/2)}.\end{aligned}$$

First notice that γ, a or b no longer appear.

We can expand:

$$\frac{1}{1-\sum_{k\geq 1}\bar{M}_{\pm,k}(x\mp 1/2)^k} = 1 + \sum_{l=1}^{\infty} \sum_{k_1,\dots,k_l} \left(\prod_{i=1}^{l} \bar{M}_{\pm,k_i}\right)\ (x\mp 1/2)^{\sum_{i=1}^{l} k_i}$$

which involves positive powers of $(x \mp 1/2)$, whose coefficients are polynomials of the $\bar{M}_{\pm,k_i}$'s with positive integer coefficients.

The Laurent series expansion of $16\,\bar{\mathcal{W}}^{(g)}_{n+1}(x; x_1, \ldots, x_n)$ near $x = \pm 1/2$, involves negative powers of $(x \mp 1/2)$, whose coefficients are, by recursion hypothesis, polynomials of the $\bar{M}_{\pm,k_i}$'s with integer coefficients. Notice that we need the factor 16, for the case $\bar{\mathcal{W}}^{(1)}_1(x)$, and also when we have some $U^{(0)}_2(x, x_j)$ (and there can be 2 of them in a product), so indeed the coefficients are integers.

The Taylor series expansion of $(x \pm 1/2)^{-d}$ near $x \to \pm 1/2$, is:

$$(x \pm 1/2)^{-d} = (\mp 1)^d \sum_{m=0}^{\infty} \frac{(d+m-1)!}{m!\,(d-1)!}\,(x \mp 1/2)^m$$

which also involves only integer coefficients.

We also have

$$\frac{1}{x_0 - x} = \sum_{d_0=0}^{\infty} \frac{(x \mp 1/2)^{d_0}}{(x_0 \mp 1/2)^{d_0+1}}$$

$$\frac{2xx_j - 1/2}{(x - x_j)^2} = \pm \sum_{d'=0}^{\infty} (d'+1)\,\frac{(x \mp 1/2)^{d'}}{(x_j \mp 1/2)^{d'+1}} + (x_j - 1/2 + x_j + 1/2) \sum_{d'=1}^{\infty} d'\,\frac{(x \mp 1/2)^{d'}}{(x_j \mp 1/2)^{d'+1}}$$

All of them contain powers of $(x_j \pm 1/2)$, with integer coefficients.

The residue picks the coefficient of the term $(x \mp 1/2)^{-1}$.

For a given term in $\bar{\mathcal{W}}^{(g)}_{n+1}(x; x_1, \ldots, x_n)$, not containing any $U^{(0)}_2$, we have, by recursion hypothesis, terms of the form:

$$(x \mp 1/2)^{-(d+d'+2)} \prod_{j=1}^{n} (x_j \pm 1/2)^{-d_j - 1} \prod_l \bar{M}_{\pm,k'_l}$$

with integer coefficients and where

$$d + d' = (3g - 3 + n - 1) - \sum_{j=1}^{n} d_j - \sum_l k'_l - m' \quad , \qquad m' \geq 0.$$

To that, the term $1/(x_0 - x)$ adds a power d_0, and $1/M(x)$ adds a power k_l for each $\bar{M}_{\pm,k_l}$, and $1/(x - 1/2)(x + 1/2)$ adds a power $-1 + m$. The residues thus keeps the term where

$$-(d + d' + 2) + d_0 + m - 1 = -1$$

i.e.

$$3g-3+n+1 = d_0 + \sum_{j=1}^{n} d_j + \sum_l k'_l + \sum_l k_l + m + m'$$

with $m \geq 0$ and $m' \geq 0$.

A similar equality is found for terms involving some $U_2^{(0)}(x, x_j)$.

This proves the theorem.

□

For closed maps, we have the $n = 0$ counterpart of that theorem, except that the coefficients are not necessarily integers:

Theorem 3.5.2 *For $g > 1$, the generating function F_g of closed maps of genus g, is a homogeneous polynomial of the moments*

$$\gamma^{g-1} F_g \quad \in \quad \mathbb{Q}[1/M_{+,0}, 1/M_{-,0}; 1/\gamma, \{M_{+,k}\}_{k\leq 3g-3}, \{M_{-,k}\}_{k\leq 3g-3}].$$

More precisely

$$4^{4g-4}\ (4\gamma)^{g-1}\ F_g = P^{(g,0)}(1/M_{+,0}, 1/M_{-,0}; 1/4\gamma, \{M_{+,k}\}_{k\leq 3g-3}, \{M_{-,k}\}_{k\leq 3g-3})$$

where $P^{(g,0)}(1/M_{+,0}, 1/M_{-,0}; 1/4\gamma, \{M_{+,k}\}_{k\leq 3g-3}, \{M_{-,k}\}_{k\leq 3g-3})$ is a homogeneous polynomial with rational coefficients

- *of degree $2g-2$ of $1/M_{+,0}$ and $1/M_{-,0}$,*
- *of degree $3g-3$ in all the other variables, where each $M_{\epsilon,k}$ is considered to be of degree k, and $1/4\gamma$ is of degree 1.*

Proof This is easily proved by integrating $W_1^{(g)}(x)$. Integration does not preserve integer coefficients, we get rational coefficients.

□

It is also very convenient to rewrite it in the following form:

Corollary 3.5.1 *For $g \geq 2$*

$$F_g = t^{2-2g}\ P^{(g,0)}(t/M_{+,0}\gamma^2, t/M_{-,0}\gamma^2; 1, \{\gamma^k M_{+,k}\}_{k\leq 3g-3}, \{\gamma^k M_{-,k}\}_{k\leq 3g-3}).$$

The reason to write it this way, is because $\gamma^2 M_{\pm,0}/t$ and $\gamma^k M_{\pm,k}$ are dimensionless, which make them more interesting. See the example of quadrangulations below.

Proof Just use homogeneity. □

Then, notice that the derivatives of $y(z)$, and the moments $M_{\pm,k}$, are linear combinations of the coefficients u_k, i.e. they are polynomials of α, γ and the t_k's. Since α and γ are algebraic functions of the t_k's and t, we see that the coefficients of $\omega_n^{(g)}$ and F_g are algebraic functions of t and the t_k's.

Theorem 3.5.3 *When $(g, n) \neq (0,0),(1,0)$, $\omega_n^{(g)}$ and $W_n^{(g)}$, are formal series of t, which are in fact algebraic. The formal series are convergent in a disk of finite radius R independent of g and n.*

F_0 and F_1 are not algebraic, they have in addition logarithmic terms, but they are also convergent in a disk of the same finite radius.

Proof We have already mentioned that the $\omega_n^{(g)}$'s and the $W_n^{(g)}$'s are rational functions of α and γ. Their denominator are powers of $M_{\pm 0}$ and γ, or in other words, the denominators are powers of $y'(1)$ and $y'(-1)$.

It is easy to see, that, for generic t_k's, α and γ are analytical at $t = 0$, and they are algebraic, therefore they are convergent series of t, within a disk of a certain radius R. The radius R is necessarily finite $R < \infty$, because an algebraic system of equation has a singularity at the zeros of its discriminant, which is itself a polynomial, and thus it has zeros.

Since the only singularities of $\omega_n^{(g)}$'s and $W_n^{(g)}$'s can come from the singularities of α, γ or the zeros of $y'(\pm 1)$, the radius of convergence is independent of g and n. □

3.5.1 Singularities

Therefore, the $W_n^{(g)}$'s have algebraic singularities, of the form:

$$(t_c - t)^{\alpha_{g,n}}$$

where $|t_c| = R$ is the radius of convergence, and $\alpha_{g,n} \in \mathbb{Q}$ is called a "critical exponent".

Chapter 5 is entirely devoted to the study of those algebraic singularities, and what they imply for the asymptotic counting of large maps.

3.5.2 Examples

- $W_3^{(0)}$:

$$\omega_3^{(0)}(z_0, z_1, z_2) = \frac{-1}{2\gamma y'(1)} \frac{1}{(z_0-1)^2} \frac{1}{(z_1-1)^2} \frac{1}{(z_2-1)^2} + \frac{1}{2\gamma y'(-1)} \frac{1}{(z_0+1)^2} \frac{1}{(z_1+1)^2} \frac{1}{(z_2+1)^2},$$

is of the form of Theorem 3.5.1, i.e. $\prod_i 1/(z_i - \epsilon_i)^{d_i+2}$ with $\sum d_i \leq 6g - 6 + 2n = 0$, i.e. all $d_i = 0$, i.e. only double poles.

In terms of $W_3^{(0)}$ and the moments:

$$4^2(4\gamma)^{-1}\prod_{i=0}^{2}\sqrt{(x_i-a)(x_i-b)}\;W_3^{(0)}(x_0,x_1,x_2)$$
$$=2\Big(\frac{1}{M_{+,0}}\,\frac{1}{(x_0-a)(x_1-a)(x_2-a)}-\frac{1}{M_{-,0}}\,\frac{1}{(x_0-b)(x_1-b)(x_2-b)}\Big),$$

is of the form of Theorem 3.5.1, i.e. $\prod_i 1/(x_i{-}a_i)^{d_i+1}$ with $\sum d_i \leq 3g{-}3{+}n=0$, i.e. all $d_i=0$, i.e. only simple poles.

In the notation of Theorem 3.5.1:

$$P^{(0,3)}_{+,0;+,0;+,0}=\frac{2}{M_{+,0}}\qquad,\qquad P^{(0,3)}_{-,0;-,0;-,0}=\frac{-2}{M_{-,0}}.$$

- $W_1^{(1)}$:
  For example we have from Eq. (3.3.4):

$$\omega_1^{(1)}(z)=\frac{-1}{16\gamma y'(1)\ (z-1)^4}-\frac{1}{16\gamma y'(1)\ (z-1)^3}+\frac{y'''(1)}{96\gamma y'(1)^2\ (z-1)^2}$$
$$+\frac{1}{16\gamma y'(-1)\ (z+1)^4}-\frac{1}{16\gamma y'(-1)\ (z+1)^3}-\frac{y'''(-1)}{96\gamma y'(-1)^2\ (z+1)^2}.$$

In terms of moments:

$$16\ \sqrt{(x-a)(x-b)}\,W_1^{(1)}(x)=\frac{1}{M_{+,0}}\left(\frac{1}{x-a}+M_{+,1}-\frac{1}{2\gamma}\right)\frac{1}{x-a}$$
$$+\frac{1}{M_{-,0}}\left(\frac{1}{x-b}+M_{-,1}+\frac{1}{2\gamma}\right)\frac{1}{x-b}.$$

It is indeed a polynomial homogeneous of degree $2g-2+n=1$ in the $M_{\pm,0}$, it has poles of degree at most $1+3g-3+n=2$ at $x=a$ and $x=b$, and the terms inside the brackets are homogenous of degree $3g-3+n=1$ in $1/(x-a), 1/(x-b), M_{\pm,1}, 1/\gamma$.

In the notation of Theorem 3.5.1:

$$P^{(1,1)}_{+,1}=\frac{1}{M_{+,0}}\qquad,\qquad P^{(1,1)}_{+,0}=\frac{M_{+,1}-\frac{2}{4\gamma}}{M_{+,0}}$$
$$P^{(1,1)}_{-,1}=\frac{1}{M_{-,0}}\qquad,\qquad P^{(1,1)}_{-,0}=\frac{M_{-,1}+\frac{2}{4\gamma}}{M_{-,0}}.$$

- $W_4^{(0)}$:

$$\begin{aligned}
& W_4^{(0)}(x_1,x_2,x_3,x_4)\prod_{i=1}^{4}\sqrt{(x_i-a)(x_i-b)} \\
&= \frac{3}{M_{+,0}^2}\prod_{i=1}^{4}\frac{1}{x_i-a}\Big(\frac{1}{16}+\frac{\gamma\,M_{+,1}}{4}+\sum_{j=1}^{4}\frac{\gamma}{4(x_j-a)}\Big) \\
&\quad +\frac{3}{M_{-,0}^2}\prod_{i=1}^{4}\frac{1}{x_i-b}\Big(\frac{1}{16}-\frac{\gamma\,M_{-,1}}{4}-\sum_{j=1}^{4}\frac{\gamma}{4(x_j-b)}\Big) \\
&\quad -\frac{1}{16\,M_{+,0}\,M_{-,0}}\Big(\frac{1}{(x_1-a)(x_2-a)(x_3-b)(x_4-b)}+5\,\text{symmetric terms}\Big).
\end{aligned}$$

In the notation of Theorem 3.5.1:

$$P^{(0,4)}_{+,1,+,0,+,0,+,0}=\frac{3}{M_{+,0}^2}\quad,\qquad P^{(0,4)}_{+,0,+,0,+,0,+,0}=\frac{\frac{3}{4\gamma}+3\,M_{+,1}}{M_{+,0}^2}$$

$$P^{(0,4)}_{-,1,-,0,-,0,-,0}=\frac{-3}{M_{-,0}^2}\quad,\qquad P^{(0,4)}_{-,0,-,0,-,0,-,0}=\frac{\frac{3}{4\gamma}-3\,M_{-,1}}{M_{-,0}^2}$$

$$P^{(0,4)}_{+,0,+,0,-,0,-,0}=\frac{-1}{M_{+,0}\,M_{-,0}}.$$

- F_2:

$$\begin{aligned}
4^2\;4\gamma\,F_2 &= \Big(\frac{21M_{-,1}^3}{10M_{-,0}^2}+\frac{29M_{-,2}M_{-,1}}{8M_{-,0}^2}+\frac{35M_{-,3}}{24M_{-,0}^2}-\frac{21M_{+,1}^3}{10M_{+,0}^2} \\
&\qquad -\frac{29M_{+,1}M_{+,2}}{8M_{+,0}^2}-\frac{35M_{+,3}}{24M_{+,0}^2}\Big) \\
&\quad +\frac{1}{4\gamma}\left(-\frac{M_{-,1}M_{+,1}}{4M_{-,0}M_{+,0}}+\frac{22M_{-,1}^2}{5M_{-,0}^2}+\frac{43M_{-,2}}{12M_{-,0}^2}+\frac{22M_{+,1}^2}{5M_{+,0}^2}+\frac{43M_{+,2}}{12M_{+,0}^2}\right) \\
&\quad +\frac{1}{(4\gamma)^2}\left(\frac{3M_{-,1}}{4M_{-,0}M_{+,0}}-\frac{3M_{+,1}}{4M_{-,0}M_{+,0}}+\frac{181M_{-,1}}{30M_{-,0}^2}-\frac{181M_{+,1}}{30M_{+,0}^2}\right) \\
&\quad +\frac{1}{(4\gamma)^3}\left(\frac{5}{M_{-,0}M_{+,0}}+\frac{181}{30M_{-,0}^2}+\frac{181}{30M_{+,0}^2}\right)
\end{aligned}$$

which is of the form of Theorem 3.5.2. It involves the $M_{\pm,k}$ up to $k=3$.

3.6 Examples of Higher Topologies Computations

3.6.1 Quadrangulations

In Sect. 3.1.7, we enumerated planar rooted quadrangulations. Let us now enumerate quadrangulations of higher topologies.

We have $V(x) = x^2/2 - t_4\, x^4/4$. Let us define:

$$r = \sqrt{1 - 12\, t\, t_4}\ .$$

The disk amplitude $W_1^{(0)}$ is given by Theorem 3.1.1, with the Zhukovsky change of variable

$$x(z) = \gamma(z + 1/z) \qquad , \qquad \gamma^2 = \frac{2t}{1+r},$$

we have

$$u_1 = \frac{t}{\gamma} \qquad , \qquad u_3 = -\, t_4\, \gamma^3.$$

Theorem 3.1.1 gives

$$W_1^{(0)}(x(z)) = \frac{u_1}{z} + \frac{u_3}{z^3} = \frac{t}{\gamma z} - \frac{t_4 \gamma^3}{z^3} = \frac{t}{\gamma}\left(\frac{1}{z} - \frac{1-r}{3\,(1+r)\, z^3}\right).$$

Written in the form $W_1^{(0)} = \frac{1}{2}(V'(x) - M(x)\sqrt{(x-a)(x-b)}$ this gives

$$M(x) = -t_4\,(x^2 - \gamma^2 - \frac{t}{t_4 \gamma^2}) = r - \frac{1-r}{6}\left(\frac{x^2}{\gamma^2} - 4\right).$$

The moments are

$$M_{+,0} = M_{-,0} = r,$$

$$M_{+,1} = -M_{-,1} = \frac{1-r}{3\, r}\ \frac{1}{\gamma},$$

$$M_{+,2} = M_{-,2} = \frac{1-r}{3\, r}\ \frac{1}{\gamma^2}.$$

For planar quadrangulations without marked faces, applying Theorem 3.4.4, we find:

$$F_0 = \frac{1}{2}\left(\frac{\gamma^2}{2}(u_3 - 1_1)^2 + \frac{\gamma^2}{4} u_3^2 - 2t\gamma(u_1 - u_3) + t\gamma\, u_3 + \frac{3t^2}{2} + t^2 \ln \frac{\gamma^2}{t}\right)$$

$$
\begin{aligned}
&= \frac{\gamma^4}{24} - \frac{5\,t\,\gamma^2}{12} + \frac{3t^2}{8} + \frac{t^2}{2}\,\ln\frac{\gamma^2}{t} \\
&= \frac{t^2}{2}\left(\frac{1}{3(1+r)^2} - \frac{5}{3(1+r)} + \frac{3}{4} - \ln\frac{1+r}{2}\right) \\
&= t^2 \sum_n \frac{3^n\,(2n-1)!}{n!\,(n+2)!}\,(t\,t_4)^n \\
&= t^2\left(\frac{1}{2}\ln 2 - \frac{5}{24} - \frac{r^2}{12} + \frac{r^4}{8} - \frac{4r^5}{15} + O(r^6)\right). \qquad (3.6.1)
\end{aligned}
$$

For genus 1 quadrangulations without marked faces, applying Theorem 3.4.6, with

$$y'(1) = y'(-1) = -\frac{t}{\gamma}\,\frac{2r}{1+r},$$

we find:

$$F_1 = \frac{1}{12}\,\ln\frac{1+r}{2r} = \frac{1}{24}\sum_{n\geq 1}\frac{3^n}{n}\,\left(2^{2n} - \frac{(2n)!}{n!\,n!}\right)\,(t\,t_4)^n. \qquad (3.6.2)$$

We also have

$$
\begin{aligned}
\omega_1^{(1)}(z) &= \frac{z(1-r^2) + z^3(8r^2+6r-2) + z^5(r^2-1)}{12\,t\,r^2\,(1-z^2)^4} \\
\mathcal{T}_4^{(1)} &= \frac{2t}{r^2\,(1+r)} = \frac{t}{24}\sum_{n\geq 1} 3^n\,\left(2^{2n} - \frac{(2n)!}{n!\,n!}\right)\,(t\,t_4)^{n-1}.
\end{aligned}
$$

For genus 2 quadrangulations without marked faces, applying Theorem 3.4.3, we find (with the Bernoulli number $B_4 = -1/30$):

$$
\begin{aligned}
F_2 &= t^{-2}\,\left(\frac{-89r^5 + 20r^4 + 130r^3 - 100r^2 - 65r + 56}{5 * 9 * 2^8\,r^5} - \frac{B_4}{8}\right) \\
&= \frac{15}{4}\,t\,t_4^3 + \frac{2007}{16}\,t^2\,t_4^4 + \frac{28323}{10}\,t^3\,t_4^5 + \dots \\
&= \frac{1}{5 * 3^3 * 2^8\,t^2}\,\sum_{n\geq 1}(12tt_4)^{n+2}\left(\frac{(2n+3)!}{2^{2n}\,n!\,(n+2)!}\,(28n+65) - 195(n+1)\right), \qquad (3.6.3)
\end{aligned}
$$

We also have

$$\omega_1^{(2)}(z) = \Big(z(1+r)^3(-(1+z^{16})(r-1)^3(14+r)$$
$$-2(z^2+z^{14})(r-1)^2(7r^2+94r-56)$$
$$+4(z^4+z^{12})(7r^4+320r^3-642r^2+413r-98)$$
$$+(z^6+z^{10})(922r^4-6292r^3+6630r^2-3556r+784)$$
$$+2z^8(4717r^4-5161r^3+4497r^2-2275r+490)\Big)/(576\,t^3\,r^7\,(1-z^2)^{10})$$

$$\mathcal{T}_4^{(2)} = \frac{(r-1)^2\,(r+1)\,(r+14)}{24t\,r^7}.$$

For genus 3 quadrangulations without marked faces, applying Theorem 3.4.3, we find (with the Bernoulli number $B_6 = 1/42$):

$$F_3 = \frac{B_6}{24\,t^4}\,\Big(1+\frac{(1+r)^4}{3^2\,2^8\,r^{10}}\Big(13720-73752r+163275r^2-190340r^3$$
$$+123450r^4-42828r^5+6619r^6\Big)\Big)$$
$$= \frac{1}{t^4}\sum_{n\geq 5}\frac{(12\,t\,t_4)^n}{3^4\,2^{10}\,n}\Big(\frac{n!}{4!\,(n-4)!}\,(781+490\,n)$$
$$-\frac{(2n-1)!!}{2^{n-4}\,(n-4)!\,7!!}\,(45+674\,n)\Big) \tag{3.6.4}$$

$$\mathcal{T}_4^{(3)} = (1-r^2)^3\,\frac{2450-3033r+291r^2+292r^3}{3^3\,2^6\,t^3\,r^{12}}$$
$$= t_4^3\,\Big(\frac{2450}{r^{12}}-\frac{3033}{r^{11}}+\frac{291}{r^{10}}+\frac{292}{r^9}\Big)$$
$$= t_4^3\sum_{n\geq 1}(12tt_4)^n\,\Big(\frac{(n+4)!}{4!\,n!}\,(2741+490\,n)-\frac{(2n+7)!!}{2^n\,n!\,7!!}\,(2741+674\,n)\Big).$$

And so on …

Using Theorem 3.5.2, or more precisely Corollary 3.5.1, we see that for every $g \geq 2$, we shall have:

$$F_g = t^{2-2g}\,\frac{(1+r)^{2g-2}}{r^{2g-2}}\,P_g\Big(\frac{1-r}{3r}\Big)$$

where P_g is a polynomial of degree at most $3g-3$. This implies that

$$F_g = t^{2-2g}\,\frac{Q_g(r)}{r^{5g-5}}$$

where Q_g is a polynomial of degree at most $5g-5$. Since we know that in the limit $t \to 0$, we have $r \to 1$ and $t^{2g-2} F_g$ should vanish, this implies that $P_g(0) = 0$ and $Q_g(1) = 0$.

In other words, $t^{2g-2}\,F_g$ is a polynomial of $1/r$, of degree $5g-5$.

$$F_g = t^{2-2g}\,\sum_{k=0}^{5g-5} \frac{Q_{g,k}}{r^{5g-5-k}}.$$

To anticipate on Chap. 5, we notice that F_g is singular at $r = 0$, i.e. at $t = -1/12t_4$, and it has an algebraic singularity of the type:

$$(t-1/12t_4)^{5/4(2-2g)}.$$

Example: Triangulations

We use the computations and notations from Sect. 3.1.8, i.e. we write $8tt_3^2 = r(1-r^2)$, which has the expansion

$$r = \frac{-1}{2}\sum_n (8tt_3^2)^n\,\frac{\Gamma((3n-1)/2)}{n!\Gamma((n+1)/2)} = (1-4tt_3^2 - 24t^2t_3^4 + \dots).$$

That gives

$$\gamma^2 = \frac{t}{r} \qquad , \qquad \alpha t_3 = \frac{1-r}{2}.$$

Theorem 3.4.4 gives

$$F_0 = t^2\left(\frac{1-r^2}{12r^2} - \frac{1}{2}\ln r\right) = t^2\sum_{n\geq 1}\frac{(8\,t\,t_3^2)^n}{3\,n}\,\frac{\Gamma(1+\frac{3n}{2})}{(n+2)!\,\Gamma(1+\frac{n}{2})}$$

where the last expansion is obtained by the Lagrange inversion as in Sect. 3.1.8.

Theorem 3.4.6 gives

$$F_1 = -\frac{1}{24}\,\ln\frac{3r^2-1}{2\,r^2}. \tag{3.6.5}$$

Also, doing Lagrange inversion like in Sect. 3.1.8, we compute the expansion $F_1 = \sum_n c_n\,(8tt_3^2)^n$ by writing

$$
\begin{aligned}
c_n &= \operatorname*{Res}_{r\to 1} F_1\, \frac{(1-3r^2)}{(r(1-r^2))^{n+1}}\,dr \\
&= -\frac{1}{24}\, \operatorname*{Res}_{r\to 1} \ln\left(\frac{3r^2-1}{2r^2}\right)\, \frac{(1-3r^2)}{(r(1-r^2))^{n+1}}\,dr \\
&= \frac{1}{24\,n}\, \operatorname*{Res}_{r\to 1} \ln\left(\frac{3r^2-1}{2r^2}\right)\, d\left(\frac{1}{(r(1-r^2))^n}\right) \\
&= -\,\frac{1}{24\,n}\, \operatorname*{Res}_{r\to 1} \frac{1}{(r(1-r^2))^n}\, \frac{2r}{r^2\,(3r^2-1)}\,dr \quad \text{[integration by parts]} \\
&= -\,\frac{1}{24\,n}\, \operatorname*{Res}_{u\to 1} \frac{1}{u^{n/2}\,(1-u)^n}\, \frac{1}{u\,(3u-1)}\,du \quad \text{[change variable } r^2=u\text{]} \\
&= \frac{(3/2)^{n-1}}{48\,n}\, \operatorname*{Res}_{v\to 0} \frac{1}{(1-2v/3)^{1+n/2}\,v^n}\, \frac{1}{(1-v)}\,dv \quad \text{[change } u=1-2v/3\text{]} \\
&= \frac{(3/2)^{n-1}}{48\,n}\, \operatorname*{Res}_{v\to 0} \sum_{k=0}^{\infty} \frac{(k+n/2)_k}{k!}\,(2/3)^k\, \frac{v^k}{v^n}\, \frac{1}{(1-v)}\,dv \\
&= \frac{(3/2)^{n-1}}{48\,n}\, \sum_{k=0}^{n-1} \frac{(k+n/2)_k}{k!}\,(2/3)^k
\end{aligned}
$$

where $(a)_k = a(a+1)(a+2)\ldots(a+k-1)$ is the Pochhammer symbol. That gives

$$
F_1 = \frac{1}{3*24}\,\sum_n (3/2)^n\,(8tt_3^2)^n\, \sum_{k=0}^{n-1} \frac{(k+n/2)_k}{k!}\,(2/3)^k. \tag{3.6.6}
$$

$$
M(x) = -t_3 x + t_3\alpha + \frac{t}{\gamma^2} = -t_3 x + \frac{1+r}{2}.
$$

Its moments are

$$
M_{\pm,0} = r \mp \sqrt{\frac{1-r^2}{2}} \quad , \quad M_{+,0}M_{-,0} = \frac{3r^2-1}{2}.
$$

$$
M_{\pm,1} = \frac{t_3}{M_{\pm,0}} = \frac{\pm 1}{\gamma}\, \frac{1}{3r^2-1}\, \left(\frac{1-r^2}{2} \pm r\sqrt{\frac{1-r^2}{2}}\right).
$$

According to Theorem 3.5.2, we have for $g \geq 2$:

$$F_g = \gamma^{1-g}\, Q_g(1/M_{+,0}, 1/M_{-,0}; 1/\gamma, M_{+,1}, M_{-,1})$$

where Q_g is homogeneous of degree $2g - 2$ in the first two variables, and homogeneous of degree $3g - 3$ in the last three variables. Using this homogeneity, we rewrite:

$$\begin{aligned} F_g &= \frac{\gamma^{1-g}}{(M_{+,0}M_{-,0})^{5g-5}}\; Q_g(M_{-,0}, M_{+,0}; M_{+,0}M_{-,0}/\gamma, M_{+,0}M_{-,0}M_{+,1}, M_{+,0}M_{-,0}M_{-,1}) \\ &= \frac{\gamma^{1-g}}{(M_{+,0}M_{-,0})^{5g-5}}\; Q_g(M_{-,0}, M_{+,0}; M_{+,0}M_{-,0}/\gamma, t_3 M_{-,0}, t_3 M_{+,0}). \end{aligned}$$

We then change variables $M_{+,0}, M_{-,0} \to \frac{M_{+,0}+M_{-,0}}{2} = r$, $\frac{M_{+,0}-M_{-,0}}{2} = \sqrt{\frac{1-r^2}{2}}$, which conserves the homogeneity:

$$F_g = \frac{\gamma^{1-g}}{(M_{+,0}M_{-,0})^{5g-5}}\; \tilde{Q}_g\left(r, \sqrt{\frac{1-r^2}{2}}; M_{+,0}M_{-,0}/\gamma, t_3 r, t_3\sqrt{\frac{1-r^2}{2}}\right)$$

again using the homogeneity, we may write

$$F_g = \frac{\gamma^{1-g}\, r^{5g-5}\, t_3^{3g-3}}{(M_{+,0}M_{-,0})^{5g-5}}\; \tilde{Q}_g\left(1, \sqrt{\frac{1-r^2}{2r^2}}; M_{+,0}M_{-,0}/\gamma r t_3, 1, \sqrt{\frac{1-r^2}{2r^2}}\right).$$

Let us introduce a reduced variable $v = \sqrt{\frac{1-r^2}{2r^2}}$, we thus have

$$F_g = \frac{\gamma^{1-g}\, r^{5g-5}\, t_3^{3g-3}}{(M_{+,0}M_{-,0})^{5g-5}}\; \tilde{Q}_g\left(1, v; 2\,\frac{1-v^2}{v}, 1, v\right).$$

This can be rewritten

$$\begin{aligned} F_g &= (t/2)^{2-2g}\; \frac{r^{10g-10}}{(3r^2-1)^{5g-5}}\; v^{3g-3}\, \tilde{Q}_g\left(1, v; 2\,\frac{1-v^2}{v}, 1, v\right) \\ &= (t/2)^{2-2g}\; \frac{r^{10g-10}}{(3r^2-1)^{5g-5}}\; \tilde{Q}_g\left(1, v; 2(1-v^2), v, v^2\right). \end{aligned}$$

Moreover, the polynomial Q_g has the symmetries of Theorem 3.5.2, i.e. $\tilde{Q}_g$ has to satisfy:

$$\tilde{Q}_g(a, -b; c, d, -e) = (-1)^{3g-3}\, \tilde{Q}_g(a, b; c, d, e).$$

This implies that $v^{3g-3}\,\tilde{Q}_g\left(1,v;2\,\frac{1-v^2}{v},1,v\right) = \tilde{Q}_g\left(1,v;2(1-v^2),v,v^2\right)$ is an even function of v. We may write it as a polynomial of v^2, or also as a polynomial of $u = 1-v^2$.

Therefore, there exists a polynomial P_g of degree at most $4g-4$ such that

$$F_g = t^{2-2g}\,\frac{1}{(1-v^2)^{5g-5}}\,P_g(1-v^2).$$

If we write $u = 1-v^2 = \frac{3r^2-1}{2r^2}$, this gives the following theorem for triangulations:

Theorem 3.6.1 *The generating functions F_g for closed triangulations of genus g, are polynomials of $1/u$ of the form:*

$$F_g = t^{2-2g}\,\left(\frac{P_g(u)}{u^{5g-5}} + \frac{B_{2g}}{2g(2-2g)}\right) \quad , \quad P_g \in \mathbb{Q}[u] \quad , \quad \deg P_g \leq 4g-4.$$

where

$$u = \frac{3r^2-1}{2r^2} \quad , \quad \text{with } r - r^3 = 8tt_3^2.$$

And we have $P_g(1) = -\frac{B_{2g}}{2g(2-2g)}$.

Example:

$$F_2 = t^{-2}\,\left(\frac{21}{160u^5} - \frac{55}{128u^4} + \frac{191}{384u^3} - \frac{29}{128u^2} + \frac{3}{128u} + \frac{1}{240}\right)$$

$$F_3 = t^{-4}\Big(\frac{2205}{256u^{10}} - \frac{13365}{256u^9} + \frac{68625}{512u^8} - \frac{4055053}{21504u^7} + \frac{1443995}{9216u^6} - \frac{39311}{512u^5} + \frac{7925}{384u^4} - \frac{22765}{9216u^3} + \frac{63}{1024u^2} - \frac{1}{1008}\Big)$$

3.7 Summary, Generating Functions of Maps

Define $V'(x) = x - \sum_{k=3}^{d} t_k x^{k-1}$.

Let $\mathbb{M}_k^{(g)}(n_3,\ldots,n_d;l_1,\ldots,l_k)$ denote the (finite) set of maps of genus g, with k boundaries (i.e. k marked faces of degrees $l_1,\ldots,l_k$ with one oriented edge marked on each marked face), obtained by gluing n_3 triangles, n_4 squares, ..., n_d d-gones.

The purpose of this chapter was to compute the generating functions:

$$W_k^{(g)}(x_1,\ldots,x_k) = \sum_{l_1,\ldots,l_k}\sum_{n_3,\ldots,n_d} \frac{t_3^{n_3}\ldots t_d^{n_d}}{x_1^{l_1+1}\ldots x_k^{l_k+1}} \sum_{\Sigma\in\mathbb{M}_{k+1}^{(g)}(n_3,\ldots,n_d;l_1,\ldots,l_k)} \frac{t^v}{\#\mathrm{Aut}(\Sigma)}$$

which is a formal series of t:

$$k>0 \qquad W_k^{(g)} \in \mathbb{Z}[t_3,\ldots,t_d,1/x_1,\ldots,1/x_k]\,[[t]],$$
$$k=0 \qquad F_g = W_0^{(g)} \in \mathbb{Q}[t_3,\ldots,t_d]\,[[t]].$$

- **Spectral curve**
  Let α and γ be the unique solutions of the two algebraic equations

$$\begin{cases} 0 = u_0 = \alpha - \sum_{l=1}^{d-1}\sum_{j=0}^{l} t_{l+j+1}\frac{(l+j)!}{j!j!(l-j)!}\gamma^{2j}\alpha^{l-j} \\ \frac{t}{\gamma} = u_1 = \gamma - \sum_{l=2}^{d}\sum_{j=1}^{l} t_{l+j}\frac{(l+j-1)!}{j!(j-1)!(l-j)!}\gamma^{2j-1}\alpha^{l-j} \end{cases}$$

  which behave like

$$\alpha = O(t) \qquad , \qquad \gamma = \sqrt{t} + O(t).$$

  Then define

$$u_k = \alpha\delta_{k,0} + \gamma\delta_{k,1} - \sum_{l=2}^{d-1}\sum_{j=k}^{(l+k)/2} t_{l+1}\frac{l!}{j!j-k!l+k-2j!}\gamma^{2j-k}\alpha^{l+k-2j}.$$

  They are such that

$$V'(x(z)) = \sum_{k=0}^{d-1} u_k(z^k+z^{-k}) \quad \text{with } x(z) = \alpha+\gamma(z+\frac{1}{z}).$$

  The pair of functions $(x(z),y(z))$

$$\begin{cases} x(z) = \alpha + \gamma(z+\frac{1}{z}) \\ y(z) = \frac{-1}{2}\sum_{j=1}^{d-1} u_j(z^j - z^{-j}) \end{cases}$$

  is called the “spectral curve”.
- **Moments**
  We define the polynomial $M(x)$ such that

$$y(z) = -\frac{1}{2}M(x(z))\,\sqrt{(x(z)-a)(x(z)-b)}$$

i.e.

$$M(x(z)) = \frac{1}{\gamma} \sum_{k=1}^{d-1} u_k \, U_{k-1}\left(\frac{x-\alpha}{2\gamma}\right)$$

where U_k is the kth second kind Chebyshev polynomial (defined by $U_k(\cosh\phi) = \frac{\sinh(k+1)\phi}{\sinh\phi}$). The moments $M_{\pm,k}$ are the Taylor series expansion of $M(x)$ near $x = a, b$:

$$\begin{aligned} M(x) &= M_{+,0}\left(1 - \sum_{k=1}^{\infty} M_{+,k}(x-a)^k\right) \\ &= M_{-,0}\left(1 - \sum_{k=1}^{\infty} M_{-,k}(x-b)^k\right). \end{aligned}$$

- **Even case**

  Even maps, are maps whose unmarked faces have an even perimeter. We mention that even planar maps are bipartite maps.

  The even case corresponds to all $t_{2j+1} = 0$, in that case we have

$$\alpha = 0 \qquad , \qquad t = \gamma^2 - \sum_{l=2}^{d/2} t_{2l} \, \frac{(2l-1)!}{l!(l-1)!} \, \gamma^{2l}$$

whose solution is

$$\begin{aligned} \gamma^2 &= t + \sum_{k=1}^{\infty} t^{k+1} \Big(\sum_{n=1}^{k} \frac{(k+n)!}{(k+1)!\, n!} \sum_{a_1+\cdots+a_n=k,\, a_i>0} \; \prod_{i=1}^{n} \tilde{t}_{a_i} \Big) \\ &= t + t^2\, \tilde{t}_1 + t^3\,(\tilde{t}_2 + 2\,\tilde{t}_1^2) + t^4\,(\tilde{t}_3 + 5\,\tilde{t}_1\tilde{t}_2 + 5\,\tilde{t}_1^3) + O(t^5), \end{aligned}$$

where

$$\tilde{t}_k = \frac{(2k+1)!}{k!\,(k+1)!} \, t_{2k+2}.$$

The Lagrange inversion formula gives

$$\gamma^{2m} = t^m \Big(1 + m \sum_{k\geq 1} t^k \sum_{n=1}^{k} \frac{(k+n+m-1)!}{(k+m)!\, n!} \sum_{a_1+\cdots+a_n=k,\, a_i>0} \; \prod_{i=1}^{n} \tilde{t}_{a_i} \Big).$$

We have $u_{2k} = 0$ and

$$u_{2k+1} = \gamma\,\delta_{k,0} - \sum_{j>2k} t_{2j-2k} \, \frac{(2j-2k-1)!}{j!(j-2k-1)!} \, \gamma^{2j-2k-1}.$$

and the spectral curve is

$$\begin{cases} x(z) = \gamma(z + \frac{1}{z}) \\ y(z) = \frac{-1}{2} \sum_j u_{2j+1}(z^{2j+1} - z^{-2j-1}). \end{cases}$$

For the even case, the moments satisfy:

$$M_{-,k} = (-1)^k M_{+,k}.$$

- **Disks, rooted planar maps**

  Then the generating function for counting maps with the topology of a disk (genus 0, and one boundary) is:

$$W_1^{(0)}(x) = \sum_{l=0}^{\infty} \sum_{n_3,\dots,n_d} \frac{t^v}{x^{l+1}} t_3^{n_3} \dots t_d^{n_d} \ \#\mathbb{M}_1^{(0)}(n_3,\dots,n_d;l)$$

$$W_1^{(0)}(x(z)) = \sum_{j=1}^{d-1} u_j z^{-j} = \frac{1}{2} V'(x(z)) + y(z)$$

$$W_1^{(0)}(x) = \frac{1}{2}\left(V'(x) - M(x)\sqrt{(x-a)(x-b)}\right).$$

- **Cylinders, annulus**

$$W_2^{(0)}(x_1,x_2) = \sum_{l_1,l_2=0}^{\infty} \sum_{n_3,\dots,n_d} \frac{t_3^{n_3}\dots t_d^{n_d}}{x_1^{l_1+1} x_2^{l_2+1}} \sum_{\Sigma \in \mathbb{M}_2^{(0)}(n_3,\dots,n_d;l_1,l_2)} \frac{t^v}{\#\mathrm{Aut}(\Sigma)}$$

$$W_2^{(0)}(x(z_1),x(z_2)) = \frac{1}{x'(z_1)x'(z_2)} \frac{1}{(z_1-z_2)^2} - \frac{1}{(x(z_1)-x(z_2))^2}.$$

The following differential form

$$B(z_1,z_2) = \left(W_2^{(0)}(x(z_1),x(z_2))\, x'(z_1)x'(z_2) + \frac{x'(z_1)\, x'(z_2)}{(x(z_1)-x(z_2))^2}\right) dz_1\, dz_2$$

is universal (independent of the type of maps):

$$B(z_1,z_2) = \frac{dz_1\, dz_2}{(z_1-z_2)^2}.$$

It is the fundamental form of the second kind (see [37]) on the spectral curve: the unique (1,1) symmetric meromorphic form, having a simple pole at $z_1 = z_2$ and no other pole, and normalized.

- **Stable maps with boundaries**

$$W_{k+1}^{(g)}(x_0, x_1, \ldots, x_k) = \sum_{l_0, l_1, \ldots, l_k} \sum_{n_3, \ldots, n_d} \frac{t_3^{n_3} \ldots t_d^{n_d}}{x_0^{l_0+1} \ldots x_k^{l_k+1}} \sum_{\Sigma \in \mathbb{M}_{k+1}^{(g)}(n_3, \ldots, n_d; l_0, \ldots, l_k)} \frac{t^v}{\#\mathrm{Aut}(\Sigma)}$$

$$\omega_{k+1}^{(g)}(z_0, \ldots, z_k) = W_{k+1}^{(g)}(x(z_0), x(z_1), \ldots, x(z_k))\, x'(z_0)x'(z_1) \ldots x'(z_k) + \delta_{k,1}\delta_{g,0} \frac{x'(z_0)x'(z_1)}{(x(z_0) - x(z_1))^2} - \frac{1}{2}\delta_{k,0}\delta_{g,0} V'(x(z_0))\, x'(z_0).$$

If $2g - 2 + k \geq 0$, we have recursively:

$$\omega_{k+1}^{(g)}(z_0, L) = \operatorname*{Res}_{z \to \pm 1} K(z_0, z) \Big[\omega_{k+2}^{(g-1)}(z, \frac{1}{z}, L) + \sum_{h=0}^{g} \sum_{I \subset L}' \omega_{1+|I|}^{(h)}(z, I) \omega_{1+k-|I|}^{(g-h)}(\frac{1}{z}, L/I) \Big] dz$$

where $L = \{z_1, \ldots, z_k\}$ and

$$K(z_0, z) = \frac{1}{2} \Big(\frac{1}{z_0 - z} - \frac{1}{z_0 - \frac{1}{z}} \Big) \frac{1}{(y(z) - y(1/z))\, x'(1/z)}. \tag{3.7.1}$$

This guarantees that the $\omega_k^{(g)}$'s are the symplectic invariants in the sense of Chap. 7 for the spectral curve $(x(z), y(z))$.

Equivalently we have for $(g, k) \neq (1, 0)$:

$$W_{k+1}^{(g)}(x_0, x_1, \ldots, x_k) \prod_{i=0}^{k} \sqrt{(x_i - a)(x_i - b)} = \operatorname*{Res}_{x \to a,b} \frac{dx}{x_0 - x} \frac{1}{M(x)} \Big[W_{k+2}^{(g-1)}(x, x, x_1, \ldots, x_k) + \sum_{h=0}^{g} \sum_{I \sqcup I' = \{x_2, \ldots, x_k\}}' W_{1+|I|}^{(h)}(x, I) W_{1+|I'|}^{(g-h)}(x, I') \Big]$$

and

$$W_1^{(1)}(x_0) \sqrt{(x_0 - a)(x_0 - b)} = \operatorname*{Res}_{x \to a,b} \frac{dx}{x_0 - x} \frac{1}{M(x)} \frac{(a - b)^2}{16\ (x - a)(x - b)}.$$

- **Stable maps without boundaries**

$$F_g = \sum_{n_3,\dots,n_d} t_3^{n_3} \dots t_d^{n_d} \sum_{\Sigma \in \mathbb{M}_0^{(g)}(n_3,\dots,n_d)} \frac{t^v}{\#\mathrm{Aut}(\Sigma)}.$$

For $g \geq 2$ we have:

$$F_g = \frac{1}{2-2g} \operatorname*{Res}_{z\to\pm 1} \Phi(z)\, \omega_1^{(g)}(z)\, dz + \frac{B_{2g}\, t^{2-2g}}{2g(2-2g)}$$

where $\Phi'(z) = -y(z)\, x'(z)$.

- **Torus**

$$F_1 = \sum_{n_3,\dots,n_d} t_3^{n_3} \dots t_d^{n_d} \sum_{\Sigma \in \mathbb{M}_0^{(1)}(n_3,\dots,n_d)} \frac{t^v}{\#\mathrm{Aut}(\Sigma)}$$

$$F_1 = \frac{1}{24} \ln\left(\gamma^2\, y'(1) y'(-1)/t^2\right).$$

- **Sphere**

$$F_0 = \sum_{n_3,\dots,n_d} t_3^{n_3} \dots t_d^{n_d} \sum_{\Sigma \in \mathbb{M}_0^{(0)}(n_3,\dots,n_d)} \frac{t^v}{\#\mathrm{Aut}(\Sigma)}$$

$$\begin{aligned} F_0 &= \frac{1}{2}\Big(\operatorname*{Res}_{z\to\infty} V(x(z)) y(z) x'(z) - t \operatorname*{Res}_{z\to\infty} V(x(z)) \frac{1}{z} \\ &\qquad - \frac{3t^2}{2} - t^2 \ln\left(\gamma^2\right)\Big) \\ &= \frac{1}{2}\Big(- \sum_{j\geq 1} \frac{\gamma^2}{j} (u_{j+1} - u_{j-1})^2 - \frac{2t\gamma}{j} (-1)^j (u_{2j-1} - u_{2j+1}) \\ &\qquad - \frac{3t^2}{2} - t^2 \ln(\gamma^2)\Big). \end{aligned}$$

- **Some structure properties**

  The generating functions of maps are of the form:

$$\begin{aligned} &\omega_n^{(g)}(z_1,\dots,z_n) \\ &= \frac{1}{\gamma^{2g-2+n}\, (y'(1)\, y'(-1))^{5g-5+2n}} \sum_{\epsilon_i = \pm 1} \sum_{d_i,\, \sum d_i \leq 6g+2n-4} \frac{P_{\boldsymbol{\epsilon},\mathbf{d}}^{(g,n)}(u_1,\dots,u_{d-1})}{\prod_i (z_i - \epsilon_i)^{d_i+2}} \end{aligned}$$

where $P^{(g,n)}_{\boldsymbol{\epsilon},\mathbf{d}} \in \mathbb{Q}[u_1,\ldots,u_{d-1}]$ is some universal polynomial. For $g \geq 2$, F_g is a rational function of α and γ of the form:

$$F_g = \gamma^{2-2g}\,(y'(1)y'(-1))^{(5-5g)}\; P_g(u_1,\ldots,u_{d-1})$$

where $P_g \in \mathbb{Q}[u_1,\ldots,u_{d-1}]$ is some universal polynomial.

- **Some structure properties in terms of moments**

  For $n \geq 1$ and $2g-2+n > 0$ we have

$$\begin{aligned} &4^{4g-4+2n}\,(4\gamma)^{g-1}\,W_n^{(g)}(x_1,\ldots,x_n)\,\prod_{i=1}^{n}\sqrt{(x_i-a)(x_i-b)} \\ &= \sum_{\epsilon_i=\pm 1}\;\sum_{d_i,\,\sum d_i\leq 3g-3+n} P^{(g,n)}_{\boldsymbol{\epsilon},\mathbf{d}}(1/M_{+,0},1/M_{-,0};1/4\gamma,\{M_{+,k}\},\{M_{-,k}\}) \\ &\qquad \frac{(4\gamma)^{d_i+1}}{\prod_i (x_i-\frac{a+b}{2}-2\gamma\epsilon_i)^{d_i+1}}, \end{aligned}$$

where $P^{(g,n)}_{\boldsymbol{\epsilon},\mathbf{d}}(1/M_{+,0},1/M_{-,0};1/4\gamma,\{M_{+,k}\}_{k\leq 3g-3+n},\{M_{-,k}\}_{k\leq 3g-3+n})$ is a universal homogeneous polynomial with integer coefficients:

- it is homogeneous of degree $2g-2+n$ of $1/M_{+,0}$ and $1/M_{-,0}$,
- it is homogeneous of degree $(3g-3+n-\sum_i d_i)$ in all the other variables, where each $M_{\epsilon,k}$ is considered to be of degree k, and $1/\gamma$ is of degree 1. This explain that it depends only on the first $3g-3+n$ moments $M_{\pm,k}$'s with

$$k \leq 3g-3+n.$$

- It is invariant if we change $\epsilon_i \to -\epsilon_i$, and $(4\gamma)^k M_{\pm,k} \to (-4\gamma)^k M_{\mp,k}$:

$$\begin{aligned} &\bar{P}^{(g,n)}_{-\boldsymbol{\epsilon},\mathbf{d}}(1/M_{+,0},1/M_{-,0};\{(4\gamma)^k M_{+,k}\},\{(4\gamma)^k M_{-,k}\}) \\ &= (-1)^{n+\sum_i d_i}\;\bar{P}^{(g,n)}_{\boldsymbol{\epsilon},\mathbf{d}}(1/M_{-,0},1/M_{+,0};\{(-4\gamma)^k M_{-,k}\},\{(4\gamma)^k M_{+,k}\}) \end{aligned}$$

$$\begin{aligned} &P^{(g,n)}_{-\boldsymbol{\epsilon},\mathbf{d}}(1/M_{+,0},1/M_{-,0};1/4\gamma,\{M_{+,k}\},\{M_{-,k}\}) \\ &= (-1)^{n+\sum_i d_i}\;P^{(g,n)}_{\boldsymbol{\epsilon},\mathbf{d}}(1/M_{-,0},1/M_{+,0};-1/4\gamma,\{(-)^k M_{-,k}\},\{(-1)^k M_{+,k}\}). \end{aligned}$$

- F_g's in terms of moments ($g \geq 2$)

  The generating function F_g of closed maps of genus g, is a homogeneous polynomial of the moments

$$F_g = (t/\gamma^2)^{2-2g}\;P^{(g,0)}(t/M_{+,0},t/M_{-,0};1,\{\gamma^k M_{+,k}\}_{k\leq 3g-3},\{\gamma^k M_{-,k}\}_{k\leq 3g-3})$$

where $P^{(g,0)}(u, v; w, \{M_{+,k}\}_{k\leq 3g-3}, \{M_{-,k}\}_{k\leq 3g-3})$ is a homogeneous polynomial with rational coefficients

- of degree $2g-2$ of the first two variables u and v,
- of degree $3g-3$ in all the other variables, where each $M_{\epsilon,k}$ is considered to be of degree k, and w is of degree 1.
- It has the symmetry:

$$
\begin{aligned}
&P^{(g,0)}(v, u; -w, \{M_{-,k}\}_{k\leq 3g-3}, \{M_{+,k}\}_{k\leq 3g-3}) \\
&= (-1)^{3g-3}\ P^{(g,0)}(u, v; w, \{M_{+,k}\}_{k\leq 3g-3}, \{M_{-,k}\}_{k\leq 3g-3}).
\end{aligned}
$$

- **Some derivative formulae**
  We have

$$
g \geq 1 \qquad \frac{\partial F_g}{\partial t} = \int_0^\infty \omega_1^{(g)}(z)\, dz
$$

and for $n \geq 1$

$$
\left.\frac{\partial \omega_n^{(g)}(z_1, \ldots, z_n)}{\partial t}\right|_{x(z_i)} = \int_0^\infty \omega_{n+1}^{(g)}(z, z_1, \ldots, z_n)\, dz.
$$

$$
\left.\frac{\partial \omega_n^{(g)}(z_1, \ldots, z_n)}{\partial t_k}\right|_{x(z_i)} = -\frac{1}{k} \operatorname*{Res}_{z\to\infty} \omega_{n+1}^{(g)}(z, z_1, \ldots, z_n)\ x(z)^k\, dz.
$$

- **Some Integration formulae**

$$
\operatorname*{Res}_{z\to\pm 1} \omega_{k+1}^{(g)}(z_1, \ldots, z_k, z)\ \Phi(z) = (2-2g-k)\, \omega_k^{(g)}(z_1, \ldots, z_k)
$$

where $\Phi'(z) = -y(z)\, x'(z)$.

3.8 Exercises

Exercise 1 (This exercise will be useful for Chap. 5). Consider maps with quadrangles and hexagons, $t_4 \neq 0$ and $t_6 \neq 0$. We have $V(x) = x^2/2 - t_4 x^4/4 - t_6 x^6/6$. Parametrize the times t_4 and t_6 in terms of two other variables u and v as:

$$
t_4 = \frac{1}{9t}(1 - v/3) \qquad , \qquad t_6 = -\frac{1}{270 t^2}(1 - v + 8u^3 - 2vu).
$$

Prove that γ^2 is given by

$$
\gamma^2 = \frac{3t}{1+2u}.
$$

Then prove that

$$W_1^{(0)}(x) = \frac{t}{\gamma\, z}\left(1 + \frac{1}{6\, z^2}\, \frac{24u^3 - 2uv - 12u - v - 3}{(1+2u)^3} + \frac{1}{10\, z^4}\, \frac{1 - v + 8u^3 - 2uv}{(1+2u)^3}\right).$$

Compute also:

$$y'(\pm 1) = \frac{t}{\gamma}\, \frac{12u^2 - v}{(1+2u)^2}.$$

Conclude that

$$F^{(1)} = -\frac{1}{12}\, \ln \frac{12u^2 - v}{(1+2u)^2}.$$

Exercise 2 Equation (3.3.5) says that the number of rooted quadrangulations of genus 1 is:

$$\frac{3^n}{6}\left(2^{2n} - \frac{(2n)!}{n!\, n!}\right).$$

With $n = 2$ that gives 15. Find the 15 genus 1 rooted quadrangulations with two quadrangles (one is marked with a marked edge).

Exercise 3 Count rooted triangulations of genus 1, i.e. $\mathcal{T}_3^{(1)}$ and $F^{(1)}$.

Answer:

$$F^{(1)} = \frac{-1}{24}\, \ln 1 - v^2$$

where $8tt_3^2 = r - r^3$, and

$$v = \sqrt{\frac{1 - r^2}{2r^2}}$$

.

$$\mathcal{T}_3^{(1)} = \frac{1}{2\, t_3}\, \frac{1 - r^2}{(3r^2 - 1)^2} = \frac{1}{4\, t_3}\, \frac{v^2\,(1 + 2v^2)}{(1 - v^2)^2}.$$

Chapter 4
Multicut Case

Readers only interested in the combinatorics of maps can easily skip this chapter and move directly to Chap. 5.

The goal of the present chapter is only to better understand the "1-cut" assumption, by providing examples which are not 1-cut, and thus which don't count maps in the usual sense, or at least not the same types of maps.

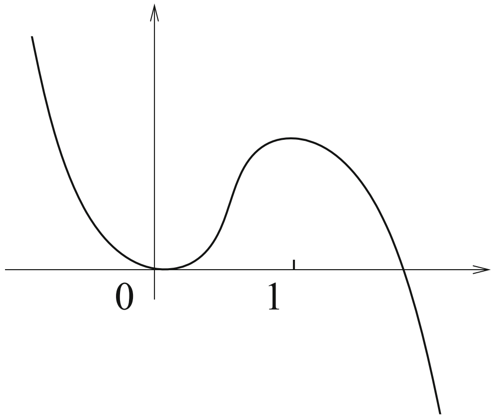
In the previous chapter, we have seen that generating functions of maps are special solutions of loop equations, namely 1-cut solutions. However, loop equations have other solutions, which are also formal power series. In this chapter, we explore the other solutions, and their combinatorial meaning.

Beside mathematical curiosity, those other multicut solutions play an important role in physics (e.g. particularly in string theory) and mathematics (e.g. asymptotics of orthogonal polynomials and asymptotic expansions of large random matrices).

B. Eynard, *Counting Surfaces*, Progress in Mathematical Physics 70,
DOI 10.1007/978-3-7643-8797-6_4

4.1 Formal Integrals and Extrema of V

Let us start by an example: the cubic potential:



$$V(M) = \frac{M^2}{2} - \frac{M^3}{3} \qquad ,$$

So far, we have defined formal series using the Taylor expansion of $e^{-\frac{N}{t}\operatorname{Tr} V(M)}$ near the extremum of $V(M)$ at $M = 0$, so that $V(M)$ has no linear term.

However, there is another extremum $V'(1) = 0$, so there is another possibility of having no linear term in the Taylor expansion:

$$V(M) = \frac{1}{6} - \frac{1}{2}(M-1)^2 - \frac{1}{3}(M-1)^3,$$

where $1 = 1_N$ denotes the $N \times N$ identity matrix.

The formal integral definition of Chap. 2 requires to Taylor expand the potential near a zero of $V'(M)$ so that there is no linear term, thus we may chose another zero of $V'(M)$, and for instance Taylor expand near $M = 1_N$. One could also expand around any other zero of $V'(M)$, for instance:

$$M_0 = \operatorname{diag}(\overbrace{0, 0, \ldots, 0}^{n \text{ times}}, \overbrace{1, 1, \ldots, 1}^{N-n \text{ times}})$$

and write $M = M_0 + A$:

$$e^{-\frac{N}{t}\operatorname{Tr} V(M)} = e^{-\frac{N}{t}\operatorname{Tr}(\frac{1}{2}M_0^2 - \frac{1}{3}M_0^3)}\, e^{-\frac{N}{t}\operatorname{Tr}(\frac{1}{2}-M_0)A^2}\, e^{-\frac{N}{t}\operatorname{Tr}\frac{A^3}{3}}.$$

In that case, the quadratic form for A is not definite, it has eigenvalues $+1, -1, 0$:

$$\operatorname{Tr}(\frac{1}{2} - M_0)A^2 = \frac{1}{2}\sum_{i,j}(1 - (M_0)_{i,i} - (M_0)_{j,j})\, A_{i,j} A_{j,i}$$

$1-(M_0)_{i,i}-(M_0)_{j,j}\in\{-1,0,1\}$, and it indeed takes the value 0 if there exists i and j such that $(M_0)_{i,i}=0$ and $(M_0)_{j,j}=1$, i.e. unless $n=0$ or $n=N$.

A Gaussian integral $\int dA\ \mathrm{e}^{-\frac{N}{t}\operatorname{Tr}(\frac{1}{2}-M_0)A^2}$ is ill–defined when the quadratic form has null eigenvalues, therefore here it is ill–defined unless $n=0$ or $n=N$.

So, for arbitrary $n\in[0,N]$, we cannot define the formal integral as the exchange of the gaussian integral in A, and the Taylor series, as we did in Chap. 2. However, there exists another way of defining a formal integral around M_0, it is described below.

4.1.1 A Digression on Convergent Normal Matrix Integrals

Consider the convergent integral:

$$\int_{\gamma^N} dx_1\dots dx_N\ \prod_{i<j}(x_j-x_i)^2\ \prod_i \mathrm{e}^{-\frac{N}{t}V(x_i)}$$

where we choose a path γ such that the integral is absoutely convergent, i.e. γ approaches ∞ in directions such that $\operatorname{Re}V(x)\to+\infty$.

We define the set of normal matrices constrained on γ:

$$H_N(\gamma)=\big\{M\in M_N(\mathbb{C})\mid\exists\,U\in U(N),\ \exists(x_1,\dots,x_N)\in\gamma^n\\ ,\ M=U\operatorname{diag}(x_1,\dots,x_N)\,U^\dagger\big\}.$$

Notice that $H_N(\mathbb{R})=H_N$ is the set of hermitian matrices.

$H_N(\gamma)$ is equipped with the measure (not necessarily real positive or normalized):

$$dM=\frac{\operatorname{Vol}(U(N)/U(1)^N)}{N!}\prod_{j<i}(x_j-x_i)^2\ \ dU\ dx_1\dots dx_N$$

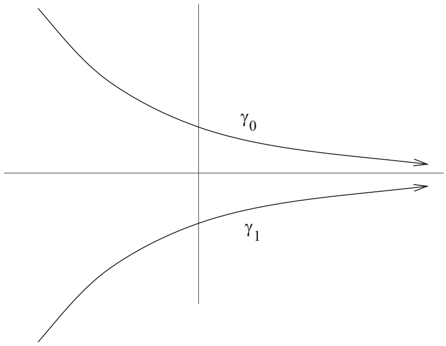
where dU is the Haar measure on $U(N)$, and dx_i is the curvilinear measure along γ. When $\gamma=\mathbb{R}$, $H_N(\mathbb{R})=H_N$, and the measure dM coincides (see [63]) with the $U(N)$ invariant measure $dM=\prod_i dM_{i,i}\prod_{i<j}d\mathrm{Re}M_{i,j}\,d\mathrm{Im}M_{i,j}$, on H_N, with which we computed Wick's theorem in Sect. 2.2.2. The normalization factor is (see [63]):

$$N!\,V_N=\operatorname{Vol}(U(N)/U(1)^N)=\pi^{N(N-1)/2}\prod_{k=1}^{N-1}\frac{1}{k!}.$$

Then, by definition we have:

$$\int_{H_N(\gamma)} dM \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)} = V_N \int_{\gamma^N} dx_1 \ldots dx_N \prod_{i<j}(x_j - x_i)^2 \prod_i \mathrm{e}^{-\frac{N}{t}V(x_i)}. \qquad (4.1.1)$$

In our cubic potential example, we may choose two paths $\gamma = \gamma_0$ or $\gamma = \gamma_1$ for which the integral is convergent:



More generally, the following integral is absolutely convergent:

$$\int_{\gamma_0^n \times \gamma_1^{N-n}} dx_1 \ldots dx_n \, dx_{n+1} \ldots dx_N \prod_{i<j}(x_j - x_i)^2 \prod_i \mathrm{e}^{-\frac{N}{t}V(x_i)} \qquad (4.1.2)$$

which makes sense for any $n \in [0, N]$, but is not a matrix integral of the form of Eq. (4.1.1), although it is nearly. Let us rewrite it as:

$$\begin{aligned}
&\int_{\gamma_0^n \times \gamma_1^{N-n}} dx_1 \ldots dx_n \, dx_{n+1} \ldots dx_N \prod_{i<j}(x_j - x_i)^2 \prod_i \mathrm{e}^{-\frac{N}{t}V(x_i)} \\
&= \int_{\gamma_0^n \times \gamma_1^{N-n}} dx_1 \ldots dx_n \, dx_{n+1} \ldots dx_N \prod_{i<j\leq n}(x_j - x_i)^2 \prod_{n<i<j\leq N}(x_j - x_i)^2 \\
&\qquad \prod_{i=1}^{n} \prod_{j=n+1}^{N} (x_j - x_i)^2 \prod_i \mathrm{e}^{-\frac{N}{t}V(x_i)} \\
&= \frac{1}{V_n \, V_{N-n}} \int_{H_n(\gamma_0)\times H_{N-n}(\gamma_1)} dA \, \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(A)} dB \, \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(B)} \det(A \otimes 1_{N-n} - 1_n \otimes B)^2
\end{aligned}$$

i.e. we have rewritten it as a 2-matrix integral. 1_n denotes the $n \times n$ identity matrix.

Now, it is appropriate to Taylor expand near $A = 0$ and near $B = 1_{N-n}$, for that purpose we change $B \to 1_{N-n} + \mathrm{i}\, B$:

$$\int_{H_n(\gamma_0)\times H_{N-n}(\gamma_1)} dA\, \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(A)} dB\, \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(1+\mathrm{i}B)}\, \det(1_N - A\otimes 1_{N-n} + \mathrm{i}\, 1_n \otimes B)^2$$

$$= \mathrm{e}^{-\frac{N(N-n)}{6t}} \int_{H_n(\gamma_0)\times H_{N-n}(\gamma_1)} dA\, \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} \frac{A^2}{2}-\frac{A^3}{3}} dB\, \mathrm{e}^{-\frac{N}{t}\operatorname{Tr}\frac{B^2}{2}+\mathrm{i}\,\frac{B^3}{3}}$$
$$\det(1_N - A\otimes 1_{N-n} + \mathrm{i}\, 1_n\otimes B)^2.$$

The last integral can be defined as a formal matrix integral as in Chap. 2.

4.1.2 *Definition of Formal Cubic Integrals*

By analogy with what precedes, we are led to the following definition: Let $C_{j,k}$ be the following well-defined gaussian integral:

$$C_{j,k}(t) = \left(\frac{N}{3}t\right)^{j+k} \int_{H_n} dA \mathrm{e}^{-\frac{N}{t}\operatorname{Tr}\frac{A^2}{2}} \int_{iH_{N-n}} dB \mathrm{e}^{\frac{N}{t}\operatorname{Tr}\frac{B^2}{2}}$$
$$(\operatorname{Tr} A^3)^j\ (\operatorname{Tr} B^3)^k\ \det(1_n\otimes 1_{N-n} - A\otimes 1_{N-n} + 1_n\otimes B)^2.$$

Although t appears in a denominator, $C_{j,k}(t)$ is in fact a polynomial in t, and we have:

$$C_{j,k}(t) = \sum_{m=(j+k)/2}^{m_{\max}} C_{j,k,m}\, t^m.$$

The important feature is the lower bound $m = (j+k)/2$ in this sum. It comes from the fact that we have to compute the expectation value of gaussian polynomial moments with the help of Wick's theorem. For each Feynman graph, the number of propagators is half the degree of the polynomial whose moment we wish to compute, that is #propag $\geq 3(k+j)/2$. On the other hand, the power of t is #propag $-(k+j)$, which is thus $\geq (k+j)/2$.

Let:

$$C_m = \sum_{j+k\leq 2m} \frac{1}{j!k!}\, C_{j,k,m}.$$

Then we define the following formal power series in t:

$$Z_{n,N-n}(t) = \mathrm{e}^{-\frac{N^2}{6t}} \sum_{m=0}^{\infty} C_m\, t^m.$$

This definition may look complicated and unnatural. However, one should keep in mind that all what we have done is mimicking the Taylor expansion and exchanging sum and integral in Eq. (4.1.2), i.e. $Z_{n,N-n}(t)$ is nothing but the matrix integral:

$$\boxed{Z_{n,N-n}(t) \quad = \quad \int_{\text{formal } n,N-n} dM\ \mathrm{e}^{-\frac{N}{t}\,\mathrm{Tr}\left(\frac{M^2}{2}-\frac{M^3}{3}\right)}}$$

where n eigenvalues are Taylor expanded near 0, and $N-n$ eigenvalues are expanded near 1, and then we exchange Taylor series expansion and integral. For instance if $n = N$, it coincides with the formal integral we introduced in Chap. 2 and which is the generating function of maps.

The main reason for such a complicated definition, is that $Z_{n,N-n}$ satisfies the same loop equations independently of n. Indeed, the loop equations are independent of the integration path, and are independent of the order of sum and integral. In other words, for every n, we have another solution of the same set of loop equations. We will discuss that point in Sect. 4.3 below. Those solutions do not satisfy Brown's lemma, i.e. they correspond to multicut solutions of the loop equations.

4.1.3 General Definition of Formal Multicut Integrals

We generalize the previous construction for any potential $V(M)$ of degree d.

Let $X_1, \ldots, X_{d-1}$ be the zeros of $V'(x)$ (supposed distinct for simplicity). Let us choose $d-1$ positive integers $(n_1, n_2, \ldots, n_{d-1})$ whose sum is N:

$$n_1 + n_2 + \cdots + n_{d-1} = N.$$

Then we define:

$$U_k(x) = V(x+X_k) - V(X_k) - \frac{V''(X_k)}{2}\,x^2 = -\sum_{j=3}^{d} \frac{t_{j,k}}{j}\,x^j. \tag{4.1.3}$$

For any set of integers $p_{j,k}$, the following well defined gaussian integral:

$$\begin{aligned}
& C_{\{p_{j,k}\}}(t) \\
&= \prod_{k=1}^{d-1}\prod_{j=3}^{d} \frac{1}{p_{j,k}!} \\
& \int_{\frac{H_{n_1}}{\sqrt{V''(X_1)}}} dM_1 \, \mathrm{e}^{-\frac{NV''(X_1)}{2t} M_1^2} \left(\frac{Nt_{3,1}}{3t} \operatorname{Tr} M_1^3\right)^{p_{3,1}} \dots \left(\frac{Nt_{d,1}}{dt} \operatorname{Tr} M_1^d\right)^{p_{d,1}} \\
& \int_{\frac{H_{n_2}}{\sqrt{V''(X_2)}}} dM_2 \, \mathrm{e}^{-\frac{NV''(X_2)}{2t} M_2^2} \left(\frac{Nt_{3,2}}{3t} \operatorname{Tr} M_2^3\right)^{p_{3,2}} \dots \left(\frac{Nt_{d,2}}{dt} \operatorname{Tr} M_2^d\right)^{p_{d,2}} \\
& \int_{\frac{H_{n_{d-1}}}{\sqrt{V''(X_{d-1})}}} dM_{d-1} \, \mathrm{e}^{-\frac{NV''(X_{d-1})}{2t} M_{d-1}^2} \left(\frac{Nt_{3,d-1}}{3t} \operatorname{Tr} M_{d-1}^3\right)^{p_{3,d-1}} \dots \\
& \dots \left(\frac{Nt_{d,d-1}}{dt} \operatorname{Tr} M_{d-1}^d\right)^{p_{d,d-1}} \\
& \qquad \prod_{j<k} \det\left(1_{n_k} \otimes 1_{n_j} + \frac{1}{X_k - X_j}(M_k \otimes 1_{n_j} - 1_{n_k} \otimes M_j)\right)^2
\end{aligned} \tag{4.1.4}$$

is a polynomial in t of the form:

$$C_{\{p_{j,k}\}}(t) = \sum_{m=\frac{1}{2}\sum p_{j,k}}^{m_{\max}} C_{\{p_{j,k}\},m} \, t^m.$$

Thus we define:

$$C_m = \sum_{\sum_{j,k} p_{j,k} \leq 2m} C_{\{p_{j,k}\},m}$$

and

Definition 4.1.1 the formal matrix integral is:

$$Z_{n_1,\dots,n_{d-1}}(t) = \mathrm{e}^{-\frac{N}{t}\sum_k n_k V(X_k)} \prod_{j<k} (X_k - X_j)^{2n_k n_j} \sum_{m=0}^{\infty} C_m \, t^m.$$

Again, this definition may look complicated, but it ensures that $Z_{n_1,\dots,n_{d-1}}(t)$ satisfies the same loop equations as $Z_{N,0,0,\dots,0}(t)$ which we have considered in Chap. 3.

4.2 What Are Multicut Formal Integrals Counting?

The coefficients defined in Eq. (4.1.4), and used to define Z, are gaussian expectation values of matrices $M_1, \dots, M_{d-1}$. As we have seen with the Wick's theorem in Chap. 2, Sect. 2.2.2, we should associate to each $\mathrm{Tr}\,(M_k)^j$ in the exponential, a vertex with j half-edges, or equivalently a j-gon. Since k may take $d-1$ values, we will assign a **color** k to the corresponding j-gon.

To complete the diagrammatic interpretation, we still have to expand the polynomial $\prod_{j<k} \det\left(1_{n_k} \otimes 1_{n_j} - \frac{1}{X_j - X_k}(M_k \otimes 1_{n_j} - 1_{n_k} \otimes M_j)\right)^2$ into powers of traces.

Let us write:

$$\begin{aligned}
&\det\left(1_{n_k} \otimes 1_{n_j} - \frac{1}{X_j - X_k}(M_k \otimes 1_{n_j} - 1_{n_k} \otimes M_j)\right)^2 \\
&= \exp\left(2\,\mathrm{Tr}\,\ln\left(1_{n_k} \otimes 1_{n_j} - \frac{1}{X_j - X_k}(M_k \otimes 1_{n_j} - 1_{n_k} \otimes M_j)\right)\right) \\
&= \exp\left(-\sum_{l=1}^{\infty} \frac{2}{l}\, \frac{1}{(X_j - X_k)^l}\, \mathrm{Tr}\,(M_k \otimes 1_{n_j} - 1_{n_k} \otimes M_j)^l\right) \\
&= \exp\left(-\sum_{l=1}^{\infty} \sum_{m=0}^{l} \frac{2(-1)^m\, l-1!}{m!\, l-m!}\, \frac{1}{(X_j - X_k)^l}\, \mathrm{Tr}\, M_k^{l-m}\, \mathrm{Tr}\, M_j^m\right).
\end{aligned}$$

In other words we have a multimatrix model

$$Z_{n_1,\dots,n_d}(t) \propto \int_{\text{formal}} \prod_{k=1}^{d-1} \mathrm{e}^{-\frac{NV''(X_k)}{2t}\, \mathrm{Tr}\, M_k^2} dM_k \quad \exp \mathcal{S}(M_1, \dots, M_{d-1})$$

with:

$$\begin{aligned}
&\mathcal{S}(M_1, \dots, M_{d-1}) \\
&= \frac{N}{t} \sum_{k=1}^{d-1} \sum_{j=3}^{d} \frac{t_{j,k}}{j}\, \mathrm{Tr}\, M_k^j \\
&\quad -2 \sum_{k} \sum_{j \neq k} \sum_{l=1}^{\infty} \frac{n_j}{(X_j - X_k)^l}\, \frac{1}{l}\, \mathrm{Tr}\, M_k^l \\
&\quad -2 \sum_{j<k} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{l+m-1!}{m-1!\, l-1!}\, \frac{(-1)^m}{(X_j - X_k)^{l+m}}\, \frac{1}{l}\, \mathrm{Tr}\, M_k^l\, \frac{1}{m}\, \mathrm{Tr}\, M_j^m.
\end{aligned}$$

Now, we have to interpret each term as a "face" of a "map".

4.2.1 Discrete Surfaces Made of Di-polygons

Let us define the following object:

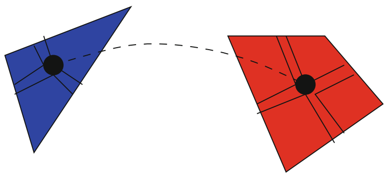
Definition 4.2.1 A colored di-polygon of degree (l_1, l_2) and color (c_1, c_2) (with $c_1 \neq c_2$), is a pair of two polygons, one of degree $l_1 \geq 0$ and color c_1, and one of degree $l_2 \geq 0$ and color $c_2 \neq c_1$.

It can be represented as two polygons glued by their centers, and which can rotate independently.

A di-polygon is invariant under cyclic permutations of each of its two polygons.

Comparing with the Definition 1.1.2 of maps from permutations in Chap. 1, we can say that dipolygons are permutations with two cycles (faces of usual maps are permutations with one cycle).

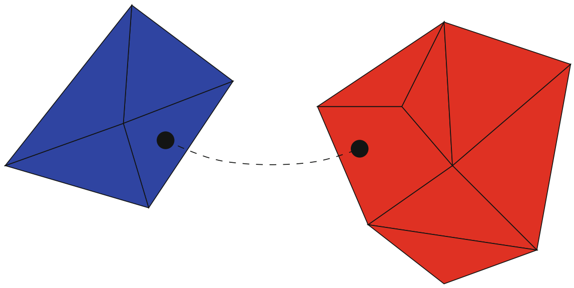
Example: A di-polygon of degree $(3, 4)$ and color (blue,red):



The dotted line should be contracted to only a point, it is drawn only for readability.

Consider discrete surfaces obtained by gluing together polygons and/or di-polygons by their sides.

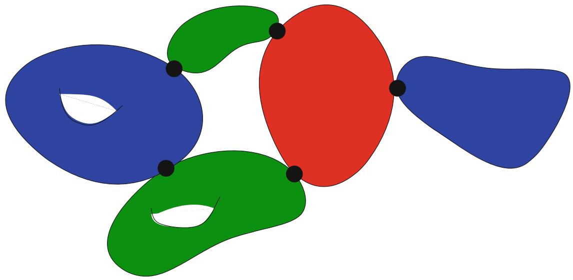
Example: A map obtained by gluing together three blue triangles, six red triangles, and one di-polygon of degree $(3, 4)$ and color (blue-red):



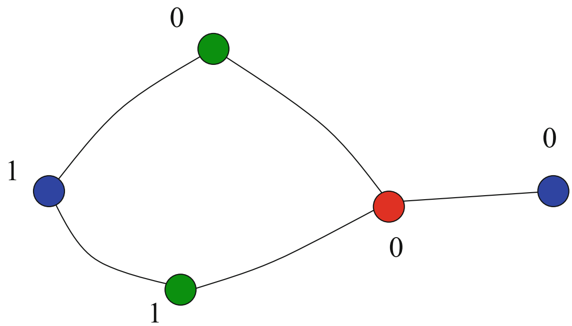
Again, the dotted line should be contracted to only a point. The Euler characteristics of this map is: $\chi = 0$ (it is a cylinder).

More generally, multicut formal matrix integrals count "nodal maps" on **nodal surfaces**.

Example: A nodal-surface with five nodal points, (i.e. maps with five di-polygons can be drawn on it). It is the gluing of three spheres and two tori. Its genus is $g = 3$, and its Euler characteristics is $\chi = -4$.



It is encoded by the following graph:



where each edge represents a nodal point, and vertices represent the surfaces glued by nodal points, the figure at each vertex is the genus of the corresponding surface. The genus of the total nodal surface is the sum of the geni at vertices, plus the number of loops of the graph. In this example, the graph has one loop, and there are two vertices with genus 1, therefore the total genus is 3.

4.2.1.1 Ensemble of Nodal Surfaces

Let $\mathbb{M}_g(\{n_j^i\};\{n_{j,k}^{i,l}\})$ be the set of all connected oriented nodal surfaces (not necessarily stable) obtained by gluing together n_j^i j-gones of color i, and $n_{j,k}^{i,l}$ di-polygons of degree (j,k) and color (i,l), and of total genus g. It is a finite set.

Define the generating function:

$$F_g(\epsilon_1,\dots,\epsilon_{d-1}) = \sum_{\{n_j^i\},\{n_{j,k}^{i,l}\}} \sum_{\Sigma\in\mathbb{M}_g(\{n_j^i\};\{n_{j,k}^{i,l}\})} \frac{\prod_i \epsilon_i^{\#\text{vertices color}\, i}}{\#\text{Aut}(\Sigma)\ \prod_i t_{2,i}^{\#\text{edges color}\, i}} \prod_{i,j} t_{j,i}^{n_j^i} \quad \prod_{i<l}\prod_{j,k} T_{(i,l),(j,k)}^{n_{j,k}^{i,l}}$$

where the weights $t_{j,i}$ were defined in Eq. (4.1.3), and

$$T_{(i,l),(j,k)} = -2\,\frac{j+k-1!}{j-1!\,k-1!}\,\frac{(-1)^k}{(X_i - X_l)^{j+k}}.$$

4.2.2 Formal Multicut Matrix Integrals and Nodal Surfaces

We define the **filling fractions**:

$$\epsilon_i = t\frac{n_i}{N}$$

We have (in the sense of formal series in t):

Theorem 4.2.1 *Multicut formal matrix integrals are generating functions for counting nodal maps:*

$$\boxed{\ln Z_{n_1,\ldots,n_{d-1}} = \sum_{g=0}^{\infty} (N/t)^{2-2g}\, F_g(\epsilon_1,\ldots,\epsilon_{d-1})}$$

Proof Just an application of Wick's theorem.

4.3 Solution of Loop Equations

In this section, we compute explicitly the generating functions of nodal maps $F_g(\epsilon_1,\ldots,\epsilon_d)$. The one cut case involved only rational functions of a variable z, whereas the multicut case involves higher genus hyperelliptical functions. The tool kit for this section is **algebraic geometry**, and we refer the reader to classical textbooks on theta functions and algebraic geometry, for instance [36, 37].

The multicut formal matrix integrals satisfy the same loop equations as one-cut formal matrix integrals. The only difference lies in the 1-cut Brown's Lemma 3.1.1.

4.3.1 Multicut Lemma and Cycle Integrals

The loop equation for $W_1^{(0)}$ is Tutte's equation [cf Eq. (3.1.2)]:

$$W_1^{(0)}(x)^2 = V'(x)W_1^{(0)}(x) - P_1^{(0)}(x)$$

i.e.

$$W_1^{(0)}(x) = \frac{1}{2}\Big(V'(x) - \sqrt{V'(x)^2 - 4P_1^{(0)}(x)}\ \Big)$$

where $P_1^{(0)}(x)$ is a polynomial of degree $d-2$, and whose leading coefficient is the same as that of $V'(x)$. In Chap. 3, this polynomial was determined through Brown's Lemma 3.1.1, by requiring that $V'(x)^2 - 4P_1^{(0)}(x)$ had only one pair of odd zeros.

Here, instead, we have:

Lemma 4.3.1 *Let $\mathcal{C}$ be a closed contour independent of t. Order by order in powers of t we have*

$$-\frac{1}{2i\pi}\oint_{\mathcal{C}} W_1^{(0)}(x)dx = \begin{cases} \epsilon_i = t\frac{n_i}{N} & \text{if } \mathcal{C} \text{ surrounds the point } X_i \\ 0 & \text{otherwise.} \end{cases}$$

and

$$-\frac{1}{2i\pi}\oint_{\mathcal{C}} W_k^{(g)}(x_1,\ldots,x_k)dx_1 = 0 \qquad \text{if } (g,k) \neq (0,1).$$

Proof The proof is very similar to that of Lemma 3.1.1. Indeed

$$\operatorname{Tr}\frac{1}{x-M} = \sum_{i=1}^{d-1}\operatorname{Tr}\frac{1}{x-X_i-M_i} = \sum_{i=1}^{d-1}\sum_{k=0}^{\infty}\frac{\operatorname{Tr} M_i^k}{(x-X_i)^{k+1}}.$$

so that to any order in the t expansion, the number of maps is finite, and thus, to this order, $W_k^{(g)}$ is a rational fraction of $x_1,\ldots,x_k$ with poles at the X_i's. As soon as $k+g \geq 2$, there is no simple poles, and for $g=0, k=1$, the first terms in the expansion is $W_1^{(0)}(x) \sim \sum_i \frac{\epsilon_i}{x-X_i}$+higher order poles (remember that $\epsilon_i/t = n_i/N = O(1)$). □

From this lemma, we get:

Theorem 4.3.1 (Multicut Solution) *The polynomial $V'(x)^2 - 4P_1^{(0)}(x)$ has as many pairs of odd zeros as the number of non-vanishing filling fractions $\epsilon_i \neq 0$.*

If we label the zeros X_i so that $\epsilon_i \neq 0$ for $i = 1,\ldots,\bar{g}+1$ and $\epsilon_i = 0$ for $i > \bar{g}+1$, at small t, we have:

$$\sqrt{V'(x)^2 - 4P_1^{(0)}(x)} = M(x)\sqrt{\prod_{i=1}^{2\bar{g}+2}(x-a_i)}$$

where $M(x)$ is some polynomial whose coefficients are formal power series in t, and a_i are power series in $\sqrt{t}$. To the first few orders in t we have

$$\begin{cases} a_{2i-1} = X_i + \sqrt{\frac{2\epsilon_i}{V''(X_i)}} + o(t) \\ a_{2i} = X_i - \sqrt{\frac{2\epsilon_i}{V''(X_i)}} + o(t) \\ M(x) = \frac{V'(x)}{\prod_{i=1}^{g+1}(x-X_i)} + O(t) \end{cases}.$$

We also have:

$$P_1^{(0)}(x) = \sum_{i=1}^{d-1} \epsilon_i \frac{V'(x) - V'(X_i)}{x - X_i} \quad + o(t) \tag{4.3.1}$$

(remember that $\epsilon_i/t = n_i/N = O(1)$)

Proof Similar to that of Lemma 3.1.1. Indeed, if there is an odd zero away from all X_i's of non-vanishing filling fraction, then it is easy to find a contour $\mathcal{C}$ which surrounds that odd zero and contradicts Lemma 4.3.1. To the first orders in t, we have:

$$W_1^{(0)}(x) = \sum_i \frac{\epsilon_i}{x - X_i} + o(t)$$

which implies

$$\sqrt{V'(x)^2 - 4P_1^{(0)}(x)} = V'(x) - 2\sum_i \frac{\epsilon_i}{x - X_i} + o(t)$$

i.e.

$$V'(x)^2 - 4P_1^{(0)}(x) = V'(x)^2 \left(1 - 4\sum_i \frac{\epsilon_i}{(x - X_i)\, V'(x)} + o(t)\right)$$

i.e., since $V'(X_i) = 0$:

$$P_1^{(0)}(x) = \sum_i \frac{\epsilon_i\, V'(x)}{(x - X_i)} + o(t) = \sum_i \epsilon_i \frac{V'(x) - V'(X_i)}{(x - X_i)} + o(t).$$

Then, if a_i is a zero of $V'^2 - 4P_1^{(0)}$, it cannot be a zero of V', and thus we must have

$$0 = 1 - 4\sum_j \frac{\epsilon_j}{(a_i - X_j)\, V'(a_i)} + o(t)$$

which is possible only if $a_i = X_i \pm \sqrt{2\epsilon_i/V''(X_i)} + O(t)$. □

This theorem determines $P_1^{(0)}(x)$ uniquely. Indeed, $P_1^{(0)}(x)$ is of degree $d-2$, with its leading coefficient fixed ($\lim_{x\to\infty} x P_1^{(0)}(x)/V'(x) = t$), therefore it has $d-2$ unknown coefficients. $M(x)$ is of degree $d-\bar{g}-2$, and its leading coefficient is known, so it has $d-\bar{g}-2$ unknown coefficients, and $a_i, i = 1,\ldots,2\bar{g}+2$ are unknown. The total number of unknowns is thus:

$$d-2+d-\bar{g}-2+2\bar{g}+2 = 2d-2+\bar{g}.$$

The equation

$$V'(x)^2 - 4P_1^{(0)}(x) = M(x)^2 \prod_{i=1}^{2\bar{g}+2}(x-a_i)$$

is of degree $2d-2$ in x, therefore it gives $2d-1$ equations for the coefficients of powers of x, but the highest coefficient gives a tautological equation, so we have $2d-2$ equations. We also have $\bar{g}$ equations saying that

$$\forall\, i = 1,\ldots,\bar{g}, \qquad \oint_{\mathcal{A}_i} \sqrt{V'(x)^2 - 4P_1^{(0)}(x)}\, dx = -4i\pi\, \epsilon_i$$

where $\mathcal{A}_i$ is a counterclockwise contour surrounding $[a_{2i-1}, a_{2i}]$.

In other words the number of equations given by Lemma 4.3.1, matches the number of unknown coefficients of $P_1^{(0)}$.

The solutions for $P_1^{(0)}$ form a discrete set, and only one of them has the small t behaviour of the form of Eq. (4.3.1).

With a little more algebraic geometry, we can write the relationship between the coefficients of $P_1^{(0)}$ and the filling fractions

Let $\mathcal{A}_i$ be a small circle independent of t surrounding X_i and no other X_j, oriented counterclockwise. Order by order in t, it surrounds the segment $[a_{2i-1}, a_{2i}]$.

Algebraic geometry tells us (see for instance [36, 37]), that for each $i = 1,\ldots,\bar{g}$, there exists a unique polynomial $L_i(x)$ of degree $\leq \bar{g}-1$

$$L_i(x) = \sum_{k=0}^{\bar{g}-1} L_{i,k}\, x^k,$$

such that

$$\forall\, j = 1,\ldots,\bar{g} \qquad \oint_{\mathcal{A}_j} \frac{L_i(x)}{\sqrt{\prod_{k=1}^{2\bar{g}+2}(x-a_k)}}\, dx = \delta_{i,j}.$$

The differential form $v_i(x) = L_i(x)dx/\sqrt{\prod_{k=1}^{2\bar{g}+2}(x-a_k)}$ is called the "normalized holomorphic differential".

Theorem 4.3.2 *We have*

$$\frac{\partial P_1^{(0)}(x)}{\partial \epsilon_i} = 2i\pi \; M(x) \; L_i(x),$$

and, if we write $P_1^{(0)}(x) = \sum_k p_k x^k$, *we have*

$$\frac{\partial \epsilon_i}{\partial p_k} = \frac{1}{2i\pi} \oint_{\mathcal{A}_i} \frac{x^k \; dx}{M(x) \sqrt{\prod_{k=1}^{2\bar{g}+2} (x - a_k)}}.$$

In other words, the map $P_1^{(0)} \to \{\epsilon_i\}$ *is locally analytical and invertible.*

4.3.2 Disc Generating Function

We thus have found that:

$$W_1^{(0)}(x) = \frac{1}{2} V'(x) + y = \frac{1}{2} \left(V'(x) - M(x) \sqrt{\prod_{i=1}^{2\bar{g}+2} (x - a_i)} \right)$$

where

$$y^2 = \frac{1}{4} V'(x)^2 - P_1^{(0)}(x) = \frac{1}{4} M(x)^2 \prod_{i=1}^{2\bar{g}+2} (x - a_i)$$

Any algebraic equation of the form y^2 =polynomial(x) is called an **hyperelliptical** curve. The disc amplitude $W_1^{(0)}(x)$ is thus an **hyperelliptical function** of **genus** $\bar{g}$. The points $a_i, i = 1, \ldots, 2\bar{g} + 2$ are called the **branch-points**:

$$a_{2i-1} \sim X_i - \sqrt{2\epsilon_i / V''(X_i)} + o(\sqrt{t}) \quad , \quad a_{2i} \sim X_i + \sqrt{2\epsilon_i / V''(X_i)} + o(\sqrt{t}) \quad , \; i = 1, \ldots, \bar{g} + 1$$

$M(x)$ is a polynomial of degree $d - 2 - \bar{g}$, and its zeros are called **double points**:

$$M(x) \sim \frac{V'(x)}{\prod_{i=1}^{\bar{g}+1} (x - X_i)} + O(t).$$

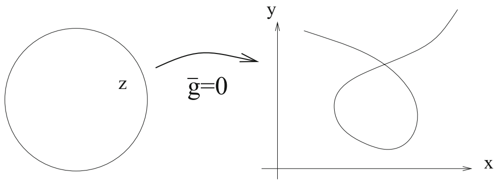
4.3.3 Higher Genus Algebraic Equations

Theorem 4.3.1 implies that $W_1^{(0)}$ is an algebraic function, whose genus is the number of non-vanishing filling fractions minus 1.

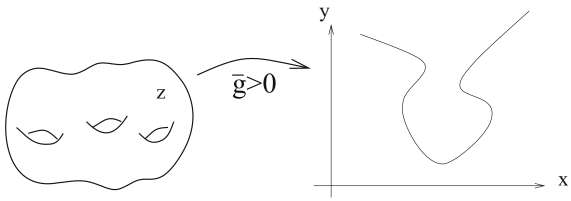
$$\text{genus} = \bar{g} = \#\{\epsilon_i \neq 0\} - 1.$$

For instance if there is only one non-vanishing filling fraction, we have a 1-cut solution, which corresponds to a genus zero algebraic curve $\bar{g} = 0$.

Any $\bar{g} = 0$ algebraic curve is conformally equivalent to the Riemann sphere $\mathbb{C} \cup \{\infty\}$, and can be parametrized by rational functions of a complex variable z (role played by the Zhukovsky map $x(z)$ in Chap. 3).



Similarly, any genus $\bar{g}$ algebraic curve can be parametrized by a variable z which lives on a "standard" genus $\bar{g}$ compact Riemann surface.



Therefore, there exists a compact Riemann surface $\mathcal{L}$ of genus $\bar{g} \leq d-2$, as well as $d-1$ cycles $\mathcal{A}_i$, and two meromorphic functions x and y defined on it, such that:

$$\forall z \in \mathcal{L}, \qquad W_1^{(0)}(x(z)) = \frac{1}{2}V'(x(z)) + y(z)$$

$$\frac{1}{2i\pi}\oint_{\mathcal{A}_i} ydx = \epsilon_i.$$

The branch-points $a_1, \ldots, a_{2\bar{g}+2}$ are the zeros of the differential form dx, i.e. the points at which the tangent is vertical, i.e. the points at which y behaves like a square-root.

4.3.4 *Geometry of the Spectral Curve*

The curve y as a function of x, can be written parametrically as:

$$\{(x,y) \in \mathbb{C}^2 \mid y^2 = \frac{1}{4}V'^2(x) - P_1^{(0)}(x)\} = \{(x(z), y(z)) \mid z \in \mathcal{L}\}.$$

It is called the **spectral curve**, it is an **hyperelliptical curve**[1].

Let us study some of its properties (a more general framework is presented in Chap. 7).

4.3.4.1 Fundamental Second Kind Form

On any compact Riemann surface $\mathcal{L}$ equipped with a symplectic basis (not unique) of non-contractible cycles

$$\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j} \quad , \quad \mathcal{A}_i \cap \mathcal{A}_j = 0 \quad , \quad \mathcal{B}_i \cap \mathcal{B}_j = 0 \quad \forall\ i,j = 1, \ldots, \bar{g},$$

is defined the "fundamental second kind form":

$$B(z_1, z_2)$$

as the unique bilinear differential of the second kind, having one double pole at $z_1 = z_2$ and no other pole, and such that:

$$B(z_1, z_2) \sim_{z_1 \to z_2} \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \text{reg} \quad , \quad \oint_{\mathcal{A}_i} B(z_1, z_2) = 0.$$

One should keep in mind that the fundamental second kind form depends only on $\mathcal{L}$, and not on the functions x and y.

It is easy to see that the fundamental second kind form is unique, because the difference of two of them would be a meromorphic form, with no pole, and with vanishing $\mathcal{A}$-cycle integrals, i.e. it must vanish.

Examples

- if $\mathcal{L} = \mathbb{C} \cup \{\infty\}$ =the Riemann Sphere, the fundamental second kind form is the rational fraction:

$$B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$$

[1] Any algebraic equation of the form $y^2 = \text{Pol}(x)$ is called hyperelliptical. It is called elliptical if $\deg \text{Pol} = 3$ or 4, and it is rational if $\deg \text{Pol} \leq 2$.

- if $\mathcal{L} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ =Torus of modulus τ, the fundamental second kind form is an elliptical function:

$$B(z_1, z_2) = (\wp(z_1 - z_2, \tau) + \frac{\pi^2 E_2}{3})\, dz_1 dz_2$$

where $\wp$ is the Weierstrass elliptical function:

$$\wp(z, \tau) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2\backslash(0,0)} \frac{1}{(z + n + \tau m)^2} - \frac{1}{(n + \tau m)^2} \tag{4.3.2}$$

and E_2 is the second Eisenstein series

$$E_2 = 1 - 24 \sum_{n=1}^{\infty} (\sum_{d\,|\,n} d)\, e^{2i\pi n\tau} \tag{4.3.3}$$

- if $\mathcal{L}$ is a compact Riemann surface of genus $\bar{g} \geq 1$, of Riemann matrix of periods τ, the fundamental second kind form is a second derivative of the log of a Theta function:

$$B(z_1, z_2) = d_{z_1} d_{z_2}\, \ln\left(\theta(u(z_1) - u(z_2) - c, \tau)\right)$$

where $u(z)$ is the Abel map, c is a regular odd characteristic, and θ is the hyperelliptical Riemann theta function of genus $\bar{g}$ (cf [36, 37] for details).
- It can be written as follows: Let

$$\sigma(x) = \prod_{i=1}^{2\bar{g}+2} (x - a_i), \quad , \quad Q(x) = (\sqrt{\sigma(x)})_+ \quad , \quad R(x) = \sigma(x) - Q(x)^2,$$

so that $\deg Q = \bar{g} + 1$ and $\deg R \leq \bar{g}$. We have

$$\begin{aligned} B(x_1, x_2) = &\frac{dx_1\, dx_2}{2\,\sqrt{\sigma(x_1)\,\sigma(x_2)}}\, \frac{Q(x_1)Q(x_2) + \frac{R(x_1)+R(x_2)}{2}}{(x_1 - x_2)^2} + \frac{dx_1\, dx_2}{2\,(x_1 - x_2)^2} \\ &+ \frac{P(x_1, x_2)\, dx_1\, dx_2}{\sqrt{\sigma(x_1)\,\sigma(x_2)}} \end{aligned}$$

where $P(x_1, x_2)$ is the unique symmetric polynomial $P(x_2, x_1) = P(x_1, x_2)$, of degree $\bar{g} - 1$ in each variable, such that

$$\oint_{\mathcal{A}_i} B(z_1, z_2) = 0.$$

- For genus $\bar{g} = 1$, it was written by Akemann [2] as follows:

$$B(x_1, x_2) = \frac{dx_1\, dx_2}{4(x_1 - x_2)^2} \Big(\sqrt{\frac{(x_1 - a_1)(x_1 - a_4)(x_2 - a_2)(x_2 - a_3)}{(x_1 - a_2)(x_1 - a_3)(x_2 - a_1)(x_2 - a_4)}}$$
$$+ \sqrt{\frac{(x_1 - a_2)(x_1 - a_3)(x_2 - a_1)(x_2 - a_4)}{(x_1 - a_1)(x_1 - a_4)(x_2 - a_2)(x_2 - a_3)}} \Big)$$
$$+ \frac{dx_1\, dx_2}{2\,(x_1 - x_2)^2}$$
$$+ \frac{dx_1\, dx_2}{4\sqrt{\sigma(x_1)\sigma(x_2)}} \frac{E(k)}{K(k)} \Big(\frac{a_2 a_3 a_4 (a_2 - a_4)}{(a_1 - a_4)(a_2 - a_1)} + \frac{a_1 a_3 a_4 (a_1 - a_3)}{(a_1 - a_2)(a_2 - a_3)}$$
$$+ \frac{a_1 a_2 a_4 (a_4 - a_2)}{(a_2 - a_3)(a_3 - a_4)} + \frac{a_1 a_2 a_3 (a_1 - a_3)}{(a_1 - a_4)(a_3 - a_4)} \Big)$$

where

$$k^2 = \frac{(a_1 - a_4)(a_2 - a_3)}{(a_1 - a_3)(a_2 - a_4)}$$

is the biratio of the branchpoints, and $K(k)$ is the complete Legendre Elliptic integral of the 1st kind, and $E(k)$ is the complete Legendre elliptic integral of the second kind.

4.3.4.2 Branchpoints and Conjugated Points

Branchpoints are the points with a vertical tangent, i.e. they are the **zeros of d**x. Let us write them a_i, $i = 1, \ldots, 2\bar{g} + 2$.

$$\forall i, \quad dx(a_i) = 0.$$

For any z away from branchpoints, there is a unique point $\bar{z} \neq z$ such that:

$$x(\bar{z}) = x(z) \qquad , \qquad y(\bar{z}) = -y(z).$$

$\bar{z}$ is called the **conjugated point** of z.

The branchpoints are the points where $\bar{z} = z$.

4.3.5 Cylinder Generating Function

Theorem 4.3.3 *The cylinder generating function is the fundamental second kind form:*

$$\omega_2^{(0)}(z_1,z_2) = \left(W_2^{(0)}(x(z_1),x(z_2)) + \frac{1}{(x(z_1)-x(z_2))^2}\right) dx(z_1)dx(z_2) = B(z_1,z_2)$$

where $B(z_1,z_2)$ is the ***fundamental second kind form*** *on $\mathcal{L}$.*

Proof The proof is very similar to that of Theorem 3.2.6 in Chap. 3. We first show that this expression is a meromorphic function, and it can have a pole only at $z_1 = z_2$ and no other pole, and then that its $\mathcal{A}_i$ cycle integral vanish. The only differential form having those properties is the fundamental second kind form. □

4.3.6 Higher Topologies

It can be seen recursively from the loop equations, that

Lemma 4.3.2 *If $2g-2+k>0$,*

$$\omega_k^{(g)}(z_1,\ldots,z_k) = W_k^{(g)}(x(z_1),\ldots,x(z_k))dx(z_1)\ldots dx(z_k)$$

is an hyperelliptical meromorphic form on $\mathcal{L}$ with poles only at the branch-points.

Proof The proof is quite easy, it proceeds by recursion on $2g-2+k$. The loop equations says that

$$y(z)\, W_k^{(g)}(z,z_2,\ldots,z_n) = \text{R.H.S}$$

where the Right hand side RHS is a meromorphic form on $\mathcal{L}$ by recursion hypothesis, having poles only at the branchpoints. Then dividing by $y(z)$ gives poles at branchpoints and possibly simple poles at the double zeros [i.e. the zeros of $M(x(z))$]. However, from Lemma 4.3.1, we see that the residues of $W_k^{(g)}$ at those simple poles must vanish, and therefore, the only poles of $W_k^{(g)}$ are at branchpoints. □

Let us define the "third kind differential form":

$$dS_{z_1,z_2}(z_0) = \int_{z_2}^{z_1} B(z',z_0)$$

which is a meromorphic differential form in the variable z_0, with a simple pole of residue $+1$ at $z_0 = z_1$, and a simple pole of residue -1 at $z_0 = z_2$, and it is normalized on $\mathcal{A}$-cycles:

$$\oint_{\mathcal{A}_i} dS_{z_1,z_2} = 0.$$

On the other hand, regarded as a function of z_1, $dS_{z_1,z_2}(z_0)$ is a scalar, and it is only defined on a fundamental domain (i.e. $\mathcal{L}/(\cup_i \mathcal{A}_i \cup_i \mathcal{B}_i)$ which is simply connected).

Let $o \in \mathcal{L}$ be an arbitrarily fixed origin on the spectral curve. Since $dS_{z,o}(z_0)$ has a simple pole at $z = z_0$, we can write Cauchy theorem

$$\omega_{k+1}^{(g)}(z_0, J) = -\operatorname*{Res}_{z\to z_0} dS_{z,o}(z_0)\ \omega_{k+1}^{(g)}(z, J) = -\frac{1}{2i\pi} \oint_{\mathcal{C}_{z_0}} dS_{z,o}(z_0)\ \omega_{k+1}^{(g)}(z, J)$$

where $\mathcal{C}_{z_0}$ is a small circle surrounding z_0.

We can move the integration contour in the fundamental domain. The only poles of the integrand are at $z = z_0$ or $z = a_i$, and the Riemann bilinear identity (see for instance [36, 37]) says that if the $\mathcal{A}$-cycle integrals of $\omega_{k+1}^{(g)}$ and B vanish, then the boundaries of the fundamental domain don't contribute, therefore:

$$\omega_{k+1}^{(g)}(z_0, J) = \sum_i \operatorname*{Res}_{z\to a_i} dS_{z,o}(z_0)\ \omega_{k+1}^{(g)}(z, J).$$

Now, we use the loop equation (3.3.1), i.e.

$$\omega_{k+1}^{(g)}(z_0, J) = -\sum_i \operatorname*{Res}_{z\to a_i} \frac{dS_{z,o}(z_0)}{2y(z)dx(z)} \Big(\hat{\omega}_{n+2}^{(g-1)}(z, z, J) + \sum_{h=0}^{g} \sum_{I\subset J}' \tilde{\omega}_{1+|I|}^{(h)}(z, I)\, \tilde{\omega}_{1+k-|I|}^{(g-h)}(z, J/I)\Big)$$

where

$$\tilde{\omega}_k^{(g)} = \omega_k^{(g)} - \frac{1}{2}\,\delta_{k,2}\delta_{g,0}\,\frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2}$$

and

$$\hat{\omega}_k^{(g)} = \omega_k^{(g)} - \delta_{k,2}\delta_{g,0}\,\frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2}.$$

One should notice that we have the parity property:

$$\tilde{\omega}_{k+1}^{(g)}(z, J) = -\tilde{\omega}_{k+1}^{(g)}(\bar{z}, J)$$

where $\bar{z} \neq z$ is solution of $x(z) = x(\bar{z})$. This allows to symmetrize the integrand, and get:

$$\omega_{k+1}^{(g)}(z_0, J) = \frac{1}{2} \sum_i \operatorname*{Res}_{z \to a_i} \frac{dS_{z,\bar{z}}(z_0)}{2y(z)dx(z)} \Big(\omega_{n+2}^{(g-1)}(z, \bar{z}, J) + \sum_{h=0}^{g} \sum_{I \subset J}' \tilde{\omega}_{1+|I|}^{(h)}(z, I)\, \tilde{\omega}_{1+k-|I|}^{(g-h)}(\bar{z}, J/I) \Big).$$

Because of parity, it is possible to change $\tilde{\omega} \to \omega$, and all the generating functions $\omega_k^{(g)}$ with $2 - 2g - k < 0$ are given by:

Theorem 4.3.4 *The generating functions of nodal maps of genus g with k boundaries, obey the topological recursion:*

$$\omega_{k+1}^{(g)}(z_0, J) = \frac{1}{2} \sum_i \operatorname*{Res}_{z \to a_i} \frac{dS_{z,\bar{z}}(z_0)}{2y(z)dx(z)} \Big(\omega_{n+2}^{(g-1)}(z, \bar{z}, J) + \sum_{h=0}^{g} \sum_{I \subset J}' \omega_{1+|I|}^{(h)}(z, I)\, \omega_{1+k-|I|}^{(g-h)}(\bar{z}, J/I) \Big). \tag{4.3.4}$$

This theorem shows that the generating functions of nodal maps are a special case of the symplectic invariants of [34], presented in Chap. 7.

4.4 Maps Without Boundaries

The generating functions of maps with no boundaries are given by the analogous of Theorem 3.4.3:

Theorem 4.4.1

$$\forall\ g \geq 2 \qquad , \qquad F_g = \frac{1}{2-2g} \sum_i \operatorname*{Res}_{z \to a_i} \Phi(z)\, \omega_1^{(g)}(z) + \sum_i \frac{B_{2g}}{2g(2-2g)}\, \epsilon_i^{2-2g}$$

where $d\Phi = -ydx$, and B_n is the nth *Bernoulli number.*

Expressions for F_0 and F_1 can be found in Chap. 7.

Proof Since the multicut case is not so relevant for the combinatorics of maps, we let the reader find the proof in the literature [30]. □

Again, the generating functions of nodal maps are a special case of the symplectic invariants of [34], presented in Chap. 7.

4.5 Exercises

Exercise 1 Consider the 2 cuts case, for even maps ($V(x)$ is even) and with filling fraction $1/2$.

Prove that the disc amplitude is of the form:

$$W_1^{(0)}(x) = \frac{1}{2}\left(V'(x) - M(x)\sqrt{(x^2-a^2)(x^2-b^2)}\right)$$

where a^2 and b^2 are determined as follows:

write a Zhukowski map for x^2:

$$x^2 = \frac{a^2+b^2}{2} + \frac{a^2-b^2}{4}\left(z+\frac{1}{z}\right)$$

and expand $V'(x)$ as:

$$\frac{V'}{x} = \sum_{k=0}^{d/2-1} u_k\left(z^k + z^{-k}\right).$$

Determine a and b by:

$$u_0 = 0 \qquad , \qquad u_1 = \frac{4t}{a^2-b^2}.$$

Exercise 2 Consider again the 2 cuts case for even maps, and with filling fraction $1/2$. From Exercise 1, the 2 cuts are then symmetric:

$$[-a,-b] \cup [a,b].$$

Define

$$\sigma(x) = (x^2-a^2)(x^2-b^2).$$

Prove that the cylinder amplitude is

$$W_2^{(0)}(x_1,x_2) = \frac{1}{2\,(x_1-x_2)^2}\left(-1 + \frac{x_1^2x_2^2 - \frac{a^2+b^2}{2}\,(x_1^2+x_2^2) + a^2b^2}{\sqrt{\sigma(x_1)\,\sigma(x_2)}}\right) + \frac{C}{\sqrt{\sigma(x_1)\,\sigma(x_2)}}$$

where C is a constant to be computed by requiring

$$\oint_{[a,b]} W_2^{(0)}(x_1,x_2)\;dx_1 = 0.$$

Akemann [2] writes it

$$W_2^{(0)}(x_1, x_2) = \frac{1}{4\,(x_1 - x_2)^2} \left(\sqrt{\frac{(x_1^2 - a^2)\,(x_2^2 - b^2)}{(x_1^2 - b^2)\,(x_2^2 - a^2)}} + \sqrt{\frac{(x_1^2 - b^2)\,(x_2^2 - a^2)}{(x_1^2 - a^2)\,(x_2^2 - b^2)}} \right)$$
$$- \frac{1}{2\,(x_1 - x_2)^2} + \frac{a^2}{4\,\sqrt{\sigma(x_1)\,\sigma(x_2)}}\,\frac{E(k)}{K(k)}\,\frac{a+b}{a-b}$$

where

$$k = \frac{2\,\sqrt{ab}}{a+b}.$$

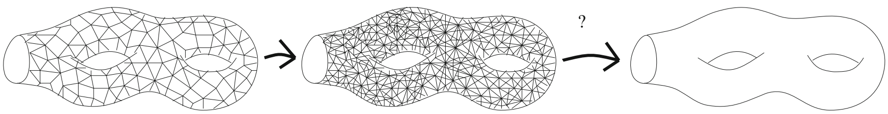
and $K(k)$ and $E(k)$ are the elliptical integrals, see [1].

Chapter 5
Counting Large Maps

Initially, in quantum gravity and string theory, the problem of counting maps, i.e. surfaces made of polygons, was introduced only as a discretized approximation for counting continuous surfaces. The physical motivation is the following: in string theory, particles are 1-dimensional loops called strings, and under time evolution their trajectories in space-time are surfaces. Quantum mechanics amounts to averaging over all possible trajectories between given initial and final states, i.e. all possible surfaces between given boundaries. However, trajectories should be counted only once modulo their symmetries, in particular conformal reparametrizations, in other words, trajectories are in fact Riemann surfaces (equivalence class of surfaces modulo conformal reparametrizations).

The set of all Riemann surfaces with a given topology and given boundaries, is called the moduli space, and string theory amounts to "counting" Riemann surfaces, i.e. measuring the "volume" of the moduli space.

Physicists made the guess that in some appropriate limit, the counting function of discrete surfaces (maps) should tend towards the counting function of Riemann surfaces. In some sense, **surfaces made of a very large number of very small polygons should be a good approximation of Riemann surfaces in quantum gravity**!



In this chapter, we are going to explain how to find the asymptotic generating functions of large maps, and then compare with Liouville conformal field theory of quantum gravity, and in the next chapter we are going to compare it to the enumeration of Riemann surfaces.

B. Eynard, *Counting Surfaces*, Progress in Mathematical Physics 70,
DOI 10.1007/978-3-7643-8797-6_5

5.1 Introduction to Large Maps and Double Scaling Limit

The idea is to count maps made of a very large number of polygons, and send the size of polygons (the mesh) to zero so that the average area remains finite.

5.1.1 Large Size Asymptotics and Singularities

Let us start with general considerations about large order behaviors.

It is a standard knowledge that there is a relationship between the large order behavior of a sequence, and singularities of the corresponding generating series. Consider a sequence $\{A_k\}_{k\in\mathbb{N}}$, and the formal series:

$$A(t) = \sum_{k=0}^{\infty} A_k\, t^k.$$

Imagine that $A(t)$ is convergent in a disc $|t| < |t_c|$, for instance assume that it is an algebraic function of t (which is indeed the case for generating functions for maps).

The basic example is:

$$A(t) = C\ (t_c - t)^{-\alpha} = C\ t_c^{-\alpha} \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha)}{k!\,\Gamma(\alpha)}\, (t/t_c)^k.$$

The large order behavior is obtained from Stirling's asymptotic formula:

$$A_k = C\ t_c^{-\alpha-k}\, \frac{\Gamma(k+\alpha)}{k!\,\Gamma(\alpha)} \underset{k\to\infty}{\sim} C\ \frac{t_c^{-\alpha}}{\Gamma(\alpha)}\ t_c^{-k}\ k^{\alpha-1}. \tag{5.1.1}$$

More generally, if $A(t)$ is an analytical function with several algebraic singularities $t_{c1}, t_{c2}, t_{c3}, \dots$ with exponents $\alpha_1, \alpha_2, \alpha_3, \dots$, the large order behavior of A_k is dominated by the singularity(ies) t_{ci} closest to the origin, those for which $|t_{ci}|$ is minimal.

$$A_k \underset{k\to\infty}{\sim} \sum_{|t_{ci}|=\min\{|t_{cj}|\}} C_i\ \frac{t_{ci}^{-\alpha_i}}{\Gamma(\alpha_i)}\ t_{ci}^{-k}\ k^{\alpha_i-1}.$$

Conversely, if a sequence A_k has a large order behavior of type Eq. (5.1.1) with α rational, then its generating series $A(t)$ has a singularity of algebraic type.

There is also an intuitive approach to understand the link between singularities and large order behaviors. The expectation value of k is:

$$< k >= \frac{\sum_k k\ A_k\, t^k}{\sum_k A_k\, t^k} = \frac{t\, A'(t)}{A(t)}$$

thus, if we want large values of k to dominate the expectation values, i.e. if we want $< k >$ to become very large, we need to choose t such that tA'/A diverges, that is we need to choose t close to a point where $\ln A(t)$ is not analytical.

A weaker statement would be to require that some moment of k diverges, for instance:

$$< k^p >= \frac{1}{A(t)} \sum_k k^p \; A_k \, t^k = \frac{1}{A(t)} \left(t \frac{d}{dt} \right)^p A(t).$$

In other words we want to choose $t = t_c$ such that some derivative of $A(t)$ diverges.

Let us now illustrate those general considerations on some examples.

5.1.2 Example: Quadrangulations

The generating function of quadrangulations of genus g with n_4 quadrangular unmarked faces, and thus $v = n_4 + 2 - 2g$ vertices is:

$$F_g(t_4) = t^{2-2g} \sum_{n_4} (t\, t_4)^{n_4} \sum_{\Sigma \in \mathbb{M}_0^{(g)}(n_4+2-2g)} \frac{1}{\#\mathrm{Aut}(\Sigma)}.$$

The average number of faces is thus:

$$< n_4 >= t_4 \frac{\partial \ln F_g}{\partial t_4} =< v > +2g - 2 = t \frac{\partial \ln F_g}{\partial t} + 2g - 2$$

where $< v >$ is the average number of vertices.

In order to have $< n_4 >$ or $< v >$ very large, one must chose t in the vicinity of a singularity of F_g. We have seen in Chap. 3, that all the F_g's (except F_0 and F_1) are rational fractions of $\gamma^2 = \frac{1-\sqrt{1-12tt_4}}{6t_4}$, and thus F_g is singular when γ^2 is singular, that is at $t = t_c = 1/12\, t_4$. For instance, with the notation $r = \sqrt{1-12tt_4}$, we have according to Eqs. (3.6.1)–(3.6.3):

$$\begin{aligned}
F_0 &= \frac{t^2}{2} \left(\frac{1}{3(1+r)^2} - \frac{5}{3(1+r)} + \frac{3}{4} - \ln \frac{1+r}{2} \right) \\
&= \frac{t^2}{2} \left(\ln 2 - \frac{1}{3} - 4tt_4 + 36t^2t_4^2 \right) - \frac{4}{15} t^2 \, (1 - t/t_c)^{5/2} + O((1-t/t_c)^3), \\
F_1 &= \frac{1}{12} \ln \frac{1+r}{2r} = -\frac{1}{24} \ln (1 - 12tt_4) - \frac{\ln 2}{12} + O((1-t/t_c)^{1/2}), \\
F_2 &= t^{-2} \; \left(\frac{-89r^5 + 20r^4 + 130r^3 - 100r^2 - 65r + 56}{5 * 9 * 2^8 \, r^5} - \frac{B_4}{8} \right) \\
&= \frac{7}{10 \; (12\, t_c)^2} \, (1 - t/t_c)^{-5/2} + O(1 - t/t_c)^{-2}.
\end{aligned}$$

Below, we will prove in Theorem 5.3.1 that in general, for quadrangulations, F_g is singular at $t = t_c = 1/12\, t_4$, and behaves (for $g \geq 2$) like:

$$F_g \sim \tilde{F}_g\; t_c^{2-2g}\, (1 - t/t_c)^{\frac{5}{4}(2-2g)} + \dots \text{subleading}$$

the constant prefactor $\tilde{F}_g$ is called the "**double scaling limit**" of F_g, and our main goal from now on, is to compute it, not only for quandrangulations, but for all sorts of maps. We address that problem below, and the answer is given in Theorem 5.3.1.

For F_1 and F_0, to leading order at $t \to t_c$, only the derivatives diverge as a power law:

$$\frac{\partial^3 F_0}{\partial t^3} = \frac{1}{2\, t_c}\; (1 - t/t_c)^{-1/2} + o((1 - t/t_c)^{-1/2})$$
$$\frac{\partial F_1}{\partial t} = \frac{1}{24\, t_c}\; (1 - t/t_c)^{-1} + o((1 - t/t_c)^{-1}).$$

Let us compute $2\, u_g$ = coefficient of the highest power of $(1 - t/t_c)$ in $\partial^2 F_g / \partial t^2$ (except $2u_0$ computed from the highest power of $(1 - t/t_c)$ in $\partial^3 F_0 / \partial t^3$), we have

$$u_0 = -\frac{1}{2} \quad , \quad u_1 = \frac{1}{48\, t_c^2} \quad , \quad u_2 = \frac{49}{3^2 * 2^8\, t_c^4} \quad , \quad \dots$$
$$\text{for } g \geq 2\,, \qquad u_g = \frac{\tilde{F}_g}{t_c^{2g}}\; \frac{5}{4}\, (2-2g)\, (\frac{5}{4}(2-2g) - 1),$$

and define the formal series

$$u(s) = \sum_g u_g t_c^{2g}\, s^{(1-5g)/2} = -\frac{1}{2}\, s^{1/2} + \frac{1}{48}\, s^{-2} + \frac{49}{3^2 * 2^8}\, s^{-9/2} + \dots$$

The values that we have found for u_0, u_1, u_2 indicate that $u(s)$ seems to satisfy the Painlevé I equation to the first few orders

$$3u^2 + u''/2 = \frac{3}{4}\, s + O(s^{-13/2}).$$

Our goal in this chapter, is to prove that indeed $u(s)$ satisfies Painlevé I equation to all orders:

$$3u^2 + u''/2 = \frac{3}{4}\, s.$$

This Painlevé equation determines all the coefficients u_g, and thus $\tilde{F}_g$, i.e. it gives the asymptotic numbers of large maps.

The Liouville minimal model of conformal field theory coupled to quantum gravity, predicts that the generating function of "number of surfaces", should satisfy the Painlevé I equation, so what we find is an agreement between the asymptotic number of large maps, and the Liouville conformal field theory of gravity.

5.1.2.1 Mesh Size

The average number of quadrangles is $< n_4 >= t_4 \frac{\partial \ln F_g}{\partial t_4}$, and thus, if we say that all quadrangles have the same area ϵ^2 (we call mesh size the side of each quadrangle, that is ϵ), the average area is:

$$< \text{Area} >= \epsilon^2 \ < n_4 >\sim \frac{5}{4}(2-2g)\ \frac{\epsilon^2}{\frac{t}{t_c}-1}.$$

If we want to have a good continuous limit of random surfaces, we require the area to remain finite, and it means that we should choose:

$$\epsilon^2 \sim t_c - t.$$

Therefore, the distance to critical point $t_c - t$ can be interpreted as the mesh area, i.e. the area of elementary quadrangles.

5.1.3 About Double Scaling Limits and Liouville Quantum Gravity

5.1.3.1 Origin of the Name "Double Scaling Limit"

Remember that we have defined $\ln Z = \sum_g N^{2-2g} F_g$, where Z is the generating function of all maps of all genus not necessarily connected. Anticipating on Theorem 5.3.1, we notice that $F_g \sim \tilde{F}_g\ t_c^{2-2g}\ (1-t/t_c)^{(2-2g)\mu}$ with the exponent of $(1-t/t_c)$ proportional to $2-2g$. Thus, it is possible to define a rescaled parameter $\tilde{N} = N\, t_c\, (1-t/t_c)^{\mu}$, and a series:

$$\ln \tilde{Z} = \sum_{g=0}^{\infty} \tilde{N}^{2-2g}\, \tilde{F}_g$$

such that $\tilde{Z}$ is the "limit" of Z, in the "double scaling limit" (double because we take a limit on both N and t):

$$\begin{cases} t \to t_c \\ N \to \infty \end{cases} \qquad N\, t_c\, (1-t/t_c)^{\mu} = \tilde{N} = \text{finite} \qquad \longrightarrow\ Z \sim \tilde{Z}.$$

This double scaling limit $\tilde{Z}$ is to be viewed as the generating series of the continuous limit of maps.

5.1.3.2 From Large Maps to Liouville Gravity

$\tilde{F}_g$ is the generating function of asymptotic numbers of large maps of genus g, rescaled by a power of the mesh size.

In a similar manner, one is also interested in the double scaling limits of $W_n^{(g)}$'s counting asymptotic numbers of large maps of genus g with n asymptotically large marked faces.

The guess made by physicists working in quantum gravity in the 80's and 90's, was that those double scaling limit generating functions $\tilde{F}_g$ and $\tilde{W}_n^{(g)}$, should coincide with correlation functions of Liouville conformal field theory coupled to gravity. This guess was supported by heuristic asymptotics of convergent matrix integrals, hoped to be valid for formal integrals.

On the conformal field theory side, due to conformal invariance, the correlation functions of a conformal field theory, must have the symmetry of some representations of the conformal group, that is they are given in terms of representations of the Virasoro algebra.

Finite representations of the conformal group were classified (in the famous Kacs table [41]) and are called minimal models, they are labeled by two integers (p, q). For the minimal models, the partial differential equations imply that the partition function has to satisfy a non-linear ordinary differential equation. For example, the minimal model $(3, 2)$ is called pure gravity, and its generating function satisfies the Painlevé I equation.

The minimal models are also related to finite reductions of the KP (Kadamtsev-Petviashvili) integrable hierarchy.

If the asymptotics generating functions $\tilde{F}_g$ of large maps were related to Liouville gravity, that would mean that $\tilde{Z}$ would be a tau-function for the KP (Kadamtsev-Petviashvili) hierarchy of integrable equations, and in particular $\tilde{Z}$ should satisfy some non-linear differential equations with the Painlevé property. We shall derive these differential equations below in Sect. 5.4.

Thus, in principle, if we want to compare large maps to Liouville quantum gravity, we have to check that the generating function of the $\tilde{F}_g$ and $\tilde{W}_n^{(g)}$'s, satisfy the differential equations of some (p, q) minimal model. In particular, we have to check that $\tilde{Z}$ is indeed the tau-function of a minimal model reduction of the KP hierarchy

$$\tilde{Z} \overset{?}{=} \text{Tau} - \text{function of}\,(p, q)\ \text{reduction of KP hierarchy}.$$

We also have to check that the scaling exponents of large maps, are those computed by KPZ (Khniznik Polyakov Zamolodchikov) [55]

$$\text{KPZ exponent } \gamma = \frac{-2}{p+q-1}, \qquad F_g \overset{?}{\sim} \tilde{F}_g \ (1-t/t_c)^{(2-2g)(1-\gamma/2)}.$$

All this was done at a heuristic level by physicists in the 90's. We provide a mathematical proof below in this chapter.

5.2 Critical Spectral Curve

Here we study what special happens at $t = t_c$? Why generating functions diverge?

5.2.1 Spectral Curves with Cusps

In Chap. 3, we have seen that the F_g's for $g \geq 2$ are rational fractions of α and γ^2 (F_0 and F_1 also contain logarithms of rational fractions of α and γ^2). α and γ themselves are obtained by solving an algebraic equation, and thus they may have singularities. One can compute (see Theorem 3.4.5, Sect. 3.4.3):

$$\frac{d\gamma}{dt} = \frac{1}{4}\left(\frac{1}{y'(1)} + \frac{1}{y'(-1)}\right)$$

and $y'(1)$ and $y'(-1)$ are themselves algebraic functions of t. Therefore we see that γ is singular whenever $y'(1) = 0$ or $y'(-1) = 0$. Without loss of generality, let us consider that $y'(1)$ vanishes at $t = t_c$.

We are thus led to study the behavior of $y(z)$ near $z = 1$. For any t, let us compute the Taylor expansion of $x(z)$ and $y(z)$ at $z = 1$.

Since $x(z) = \alpha + \gamma(z + 1/z)$ we always have $x'(1) = 0$, and to the order $(z-1)^2$ we have

$$x(z) \sim x(1) + \gamma(z-1)^2 + O((z-1)^3),$$

and thus

$$z - 1 \sim \sqrt{\frac{x-a}{\gamma}}.$$

And $y(z) \sim (z-1)y'(1) + \frac{1}{2}(z-1)^2 y''(1) + \frac{1}{6}(z-1)^3 y'''(1) + \dots$. Generically y behaves like a square root near its branchpoints:

$$y \sim y'(1)\sqrt{\frac{x-a}{\gamma}} + O((x-a)^{\frac{3}{2}}).$$

At $t = t_c$, however, since $y'(1)$ vanishes, y no longer behaves as a square root, it has a cusp singularity of the form $y \sim (x-a)^{3/2}$, and if more derivatives of y vanish, it has a cusp singularity of the form:

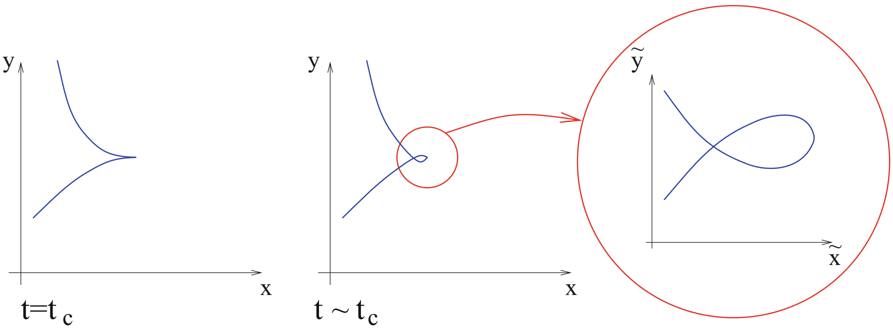
$$y \sim (x-a)^{p/q}.$$

Here, for maps, y is always the square root of some polynomial, so that p/q must be half–integer, i.e. $q = 2$ and $p = 2m+1$ where m corresponds to the first non-vanishing derivative of y at $z = 1$, that is $y(z) \sim O((z-1)^{2m+1})$.

Remark 5.2.1 In more general maps, for instance colored maps carrying an Ising model (see Chap. 8), or a $O(\mathfrak{n})$ model, other exponents p/q are possible. The Ising model allows to reach any rational p/q singularity. The $O(\mathfrak{n})$ model allows to reach all p/q singularities (not necessarily rational) such that $\mathfrak{n} = -2\cos(\frac{p}{q}\pi)$.

The integers p and q are going to be related to the (p,q) minimal model.

If t is close to t_c, the curve $y(x)$ is not singular, but it approaches a singularity. So, let us zoom into a small region near the branchpoint.



For example, consider that the branchpoint which becomes singular is the one at $z = 1$ (in case both branchpoints become singular there are extra factors of 2 in some formulae, this is the case for even maps).

5.2.1.1 Example: Quadrangulations

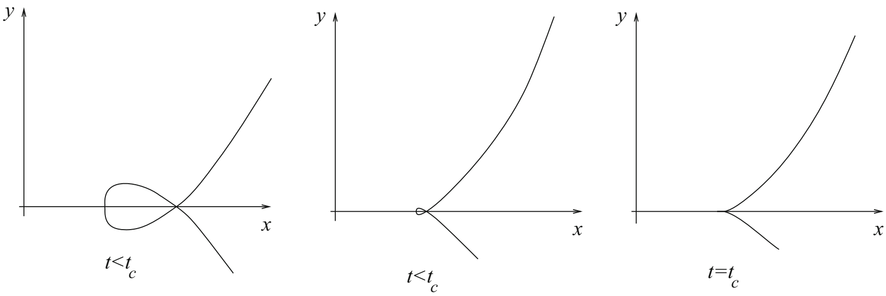
If one plots the spectral curve y versus x, one sees that at $t \neq t_c$, the curve (x, y) is regular, it behaves generically like a square root near its branch points $x = \pm 2\gamma$, it has everywhere a tangent (at the branchpoints the tangent is vertical). At $t = t_c = \frac{1}{12t_4}$: the curve (x, y) ceases to be regular, it has a cusp singularity, it has no tangent at $z = 1$. Indeed, we have [from Eq. (3.1.16)]:

$$y = -\frac{t_4}{2}\,(x^2 - 4\gamma^2 + 3\gamma^2\frac{\gamma^2 - 2t}{\gamma^2 - t})\,\sqrt{x^2 - 4\gamma^2} \qquad , \qquad \gamma^2 = \frac{1 - \sqrt{1 - 12tt_4}}{6t_4}.$$

At $t = t_c = 1/12t_4$ we have $\gamma^2 = 2t$ and thus:

$$t = t_c \quad \Rightarrow \qquad y = -\frac{t_4}{2}\,(x^2 - 8t)^{3/2}.$$

At $t = t_c$, the square root singularity at $x = 2\gamma$ is replaced by a power $3/2$ singularity.



In a vicinity of the critical point, we parametrize t_4 as:

$$tt_4 = \frac{1 - \epsilon^2}{12}$$

where ϵ is the "mesh size".

In the small ϵ limit we have the Taylor expansion:

$$\gamma^2 \sim 2t\,(1 - \epsilon) \quad + O(\epsilon^2),$$

and we reparametrize x in a vicinity of the branch-point $x - 2\gamma = O(\epsilon)$, with an auxillary variable $\zeta \sim O(1)$ as:

$$x = \sqrt{8t}\,(1 + \frac{1}{4}\epsilon(\zeta^2 - 2))$$

we find that y behaves like:

$$y \sim -\frac{\sqrt{t}}{3}\,\epsilon^{\frac{3}{2}}\,(\zeta^3 - 3\zeta) \quad + O(\epsilon^{\frac{5}{2}}).$$

This corresponds to having reparametrized the Zhukovsky's variable near $z = 1$ as

$$z = 1 + \sqrt{\frac{\epsilon}{2}}\,\zeta + O(\epsilon).$$

Let us define the parametric curve $(\tilde{x}, \tilde{y})$ defined by keeping only the leading non-trivial behaviors of x and y at small ϵ:

$$\begin{cases} \tilde{x}(\zeta) = \sqrt{\frac{t}{2}}\,(\zeta^2 - 2) \\ \tilde{y}(\zeta) = -\frac{\sqrt{t}}{3}\,(\zeta^3 - 3\zeta) \end{cases}$$

it is called the "blow up" of the curve (x, y) near its singularity.

This blown up curve is going to play an important role below.

5.2.2 *Multicritical Points*

The previous example of just quadrangulations is in some way too simple, as it does not contain any "multicritical point". The reason is that it depends only on one variable tt_4.

5.2.2.1 Example: Quadrangles + Hexagons

In order to illustrate a more general type of mutlicritical behaviour, consider maps with both quadrangles (weighted by t_4), and hexagons (weighted by t_6). Notice that these are even maps. We have:

$$V'(x) = x - t_4 x^3 - t_6 x^5.$$

The spectral curve is easily computed with Theorem 3.1.1:

$$y = \frac{1}{2}\left(t_6(x^2 - 4\gamma^2)^2 + (t_4 + 10t_6\gamma^2)(x^2 - 4\gamma^2) + 3t_4\gamma^2 + 20t_6\gamma^4 - \frac{t}{\gamma^2}\right)\sqrt{x^2 - 4\gamma^2}$$

where γ^2 is the unique solution of the following algebraic equation, that behaves like $t + O(t^2)$ at small t:

$$t = \gamma^2 - 3t_4\gamma^4 - 10t_6\gamma^6, \tag{5.2.1}$$

i.e. according to Chap. 3, using the Lagrange interpolation method

$$\gamma^2 = \sum_{k,l} t^{k+l+1} \frac{(2k+3l)!}{(k+2l+1)!\, k!\, l!} (3t_4)^k\, (10\, t_6)^l.$$

We have now 2-parameters t_4 and t_6. For each t_4, there is a critical value of t_6 at which y has a $\frac{3}{2}$ cusp $y \sim (x-2\gamma)^{3/2}$. This happens when $3t_4\gamma^2 + 20t_6\gamma^4 - \frac{t}{\gamma^2} = 0$, i.e.

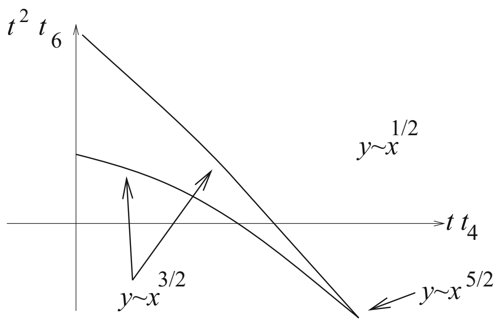
$$t_6 = \frac{2 - 27\, tt_4 \pm 2(1-9\, tt_4)^{\frac{3}{2}}}{270\, t^2}. \tag{5.2.2}$$

This draws two critical lines in the (t_4, t_6) plane.

Then, if in addition to Eq. (5.2.2), we have $t_4 + 10t_6\gamma^2 = 0$, we have a point (at the intersection of the two critical lines) where $y \sim (x-2\gamma)^{5/2}$. This point is at

$$t_4 = \frac{1}{9\, t} \qquad , \qquad t_6 = -\frac{1}{270\, t^2}.$$

This is best represented on a phase diagram:



for generic t_4 and t_6, y has a $\frac{1}{2}$ edge, along the two critical lines, y has a $\frac{3}{2}$ cusp, and at the critical point, y has a $\frac{5}{2}$ cusp.

Now, our goal is to consider t_4 and t_6 a little bit away from the critical point, and study the limit of generating functions of maps, as we approach the critical point.

Of course, depending on how we approach the critical point, we find different asymptotic behaviors. The asymptotics for the F_g's are going to be different if we approach the critical point along a critical line, or from a generic direction.

Let us consider a small vicinity of the critical point, parametrized as:

$$t_4 = \frac{1}{9t}\,(1-\epsilon^2\,\frac{\tilde{t}_0}{3}) \qquad , \qquad t_6 = -\frac{1}{270t^2}\,(1-\epsilon^2\,\tilde{t}_0 + \epsilon^3\, s)$$

where ϵ is small (it is the mesh size), and $s, \tilde{t}_0$ are of order $O(1)$.

It will be more convenient to use a variable u_0 instead of s:

$$s = 8\,u_0^3 - 2\tilde{t}_0\,u_0.$$

In some sense u_0 measures the distance to critical point along the critical line, and $\tilde{t}_0 - 12u_0^2$ measures the "distance" transverse to the critical line.

The Eq. (5.2.1) for γ gives:

$$\frac{3t}{\gamma^2} = 1 + 2\epsilon u_0.$$

Then, reparametrizing x in a vicinity of the branch-point 2γ with an auxillary variable ζ of order $O(1)$ as:

$$x \sim \sqrt{3t}\,(2 + \epsilon\,(\zeta^2 - 2u_0) + O(\epsilon^2))$$

we find that y behaves like:

$$y \sim \sqrt{\frac{t}{3}}\ \epsilon^{\frac{5}{2}}\ \left(-\frac{8\,\zeta^5}{5} + 8\,u_0\,\zeta^3 + (\tilde{t}_0 - 12\,u_0^2)\zeta\right)(1 + O(\epsilon)).$$

The parametric curve $(\tilde{x}, \tilde{y})$

$$\begin{cases} \tilde{x}(\zeta) = \zeta^2 - 2u_0 \\ \tilde{y}(\zeta) = -\frac{8}{5}\,(\zeta^5 - 5\,u_0\zeta^3 + \frac{15\,u_0^2}{2}\,\zeta) + \tilde{t}_0\,\zeta \end{cases}$$

is called the "blown up" of the curve (x, y) near its singularity. Again, anticipating on Sect. 5.4, we notice that the exponents 5 and 2, are a hint that this spectral curve has to do with the $(5, 2)$ minimal model.

The differential form ydx plays a key role in the recursive computations of $W_k^{(g)}$'s, and it scales like:

$$ydx \sim\ t\,\epsilon^{7/2}\ \tilde{y}\,d\tilde{x} + O(\epsilon^{9/2}).$$

Remark 5.2.2 If we would compare the formal matrix model for maps to a convergent matrix integral, then the large N limit of the density of eigenvalues would be $\rho(x)dx = \frac{N}{2\pi i t}\,y\,dx$. Thus, we see that if we choose

$$\epsilon \sim N^{-2/7},$$

then a region of size of order ϵ near the edge, contains a finite number of eigenvalues of the random matrix. This is a hint that the double scaling limit to be considered will be $t \to t_c$ and $N \to \infty$ and $N(1 - t/t_c)^{7/4} = O(1)$.

5.2.2.2 Multicritical Points, General Case

More generally, when we consider maps, we have a spectral curve (x, y) depending on some parameters $t_3, t_4, \dots t_d$ and t. As we have already noticed, the spectral curve depends only on the rescaled parameters $t^{\frac{i}{2}-1}\, t_i$, and the parameter t is redundant, but for further convenience we prefer to keep it.

In the space of parameters t_is, there exists critical sub-manifolds, corresponding to various singular behaviours for the spectral curves (x, y), of the form $y \sim (x - a)^{p/q}$, where $q = 2$ and $p = 2m + 1$.

Consider a critical point $t_i = t_{ic}$, at which we have $y \sim (x - a)^{m+\frac{1}{2}}$.

When we move away from this point, we may move along various directions, for instance along a submanifold where $y \sim (x - a)^{m'+\frac{1}{2}}$ with $m' < m$, or we can also move into a non critical direction $m' = 0$.

Therefore, it is better to reparametrize our parameters t, t_i's as functions of more appropriate parameters $\epsilon, \tilde{t}_i$'s:

$$t_i = t_i(\epsilon, \tilde{t}_1, \dots, \tilde{t}_m) \qquad \text{where} \quad \epsilon^2 = t_c - t$$

and in such a way that the spectral curve can be written in the regime $\epsilon \to 0$ and $\tilde{t}_i = O(1)$ as:

$$\begin{cases} x(\zeta) \sim a_c + \gamma_c \epsilon(\zeta^2 - 2u) \quad + O(\epsilon^2) \\ y(\zeta) \sim \frac{t_c}{\gamma_c}\, \epsilon^{m+\frac{1}{2}} (\sum_{m'=0}^{m} \tilde{t}_{m'} Q_{m'}(\zeta)) \quad + O(\epsilon^{m+\frac{3}{2}}) \end{cases}$$

where

$$Q_{m'}(\zeta) = \sum_{j=0}^{m'} \frac{(-u)^j}{j!} \, \frac{(2m'+1)!!}{(2m'-2j+1)!!}\, \zeta^{2m'-2j+1} = \left((\zeta^2 - 2u)^{m'+\frac{1}{2}}\right)_+ \tag{5.2.3}$$

is a polynomial of ζ of degree $2m' + 1$ (it is the polynomial part of the large ζ Laurent series expansion of $(\zeta^2 - 2u)^{m'+\frac{1}{2}}$).

The first few are

$$Q_0(\zeta) = \zeta \quad , \quad Q_1(\zeta) = \zeta^3 - 3u\zeta, \quad , \quad Q_2(\zeta) = \zeta^5 - 5u\,\zeta^3 + \frac{15\,u^2}{2}\,\zeta.$$

The spectral curve now depends on the parameters ϵ, u, and $\tilde{t}_i$, $i = 1, \dots, m$.

x, y and the $Q_{m'}$ are defined with an extra parameter u, but we shall see below, that u has to be a certain function of the $\tilde{t}_i$'s.

At $\epsilon \neq 0$, the spectral curve is regular, its branchpoints are of square root type. The curve becomes singular in the $\epsilon \to 0$ limit, and depending on the $\tilde{t}_i$'s, it may become critical or multicritical along some critical submanifolds.

Our goal is to study how the F_g's diverge in the limit $\epsilon \to 0$ (i.e. $t - t_c \to 0$). We are going to prove in Theorem 5.3.1 below, that (remember that $\epsilon^2 = t_c - t$):

$$F_g \sim (1 - t/t_c)^{(2-2g)\mu}\ t_c^{2-2g}\ \tilde{F}_g(\tilde{t}_i) \quad (1 + o(1))$$

the scaling exponent $\mu = \frac{2m+3}{2m+2}$, and the values of $\tilde{F}_g$ are computed in Theorem 5.3.1 below, and we shall find that the coefficients $\tilde{F}_g$ are the symplectic invariants (see Chap. 7) of the blown up spectral curve:

$$\begin{cases} \tilde{x}(\zeta) = \zeta^2 - 2u \\ \tilde{y}(\zeta) = \sum_{m'=0}^{m} \tilde{t}_{m'} Q_{m'}(\zeta). \end{cases}$$

Then, we shall show that the symplectic invariants of that curve, are related to the $(2m + 1, 2)$ minimal model, and their generating function satisfies the $(m + 1)$th Painlevé I equation.

5.3 Computation of the Asymptotic $W_n^{(g)}$'s

Here, we compute how the functions $W_n^{(g)}(x_1, \ldots, x_n)$ behave in a small region of size δ around a branchpoint ($z = +1$ for instance). We shall study this behavior independently of being close to a critical point or not, i.e. whether the curve behaves like a square root $y \sim \sqrt{x-a}$ or like any other power $y \sim (x-a)^{p/q}$.

Also here, we choose a small size δ on the spectral curve (i.e. in the z variable), independently of any mesh size ϵ. It is only later that we shall relate the two.

We thus reparametrize the Zhukovsky variables z_i's with some auxillary variables ζ_is as

$$z_i = 1 + \delta\, \zeta_i$$

and thus $x_i = x(z_i) = \alpha + \gamma(z_i + 1/z_i)$ gives:

$$x_i = x(1) + \gamma\, \delta^2 \zeta_i^2 + O(\delta^3).$$

Our goal is to study the asymptotic behavior of $W_n^{(g)}(x_1, \ldots, x_n)$ in the limit $\delta \to 0$, with all ζ_is of order $O(1)$.

For latter purposes, we will also be interested in situations where the size δ may depend (or not depend) on the times t, t_k, and thus $x(1)$ and γ may also have a small δ expansion.

For example, if we are near a critical point, we may want to choose the scale δ of the form $\delta \sim (t_c - t)^{\nu}$ with some appropriate exponent ν ($\nu = 0$ if δ is independent of t).

However, for the moment, we do not assume any particular relationship, in fact we allow any arbitrary relationship. Thus we find, by doing a Taylor expansion in powers of δ:

$$\begin{cases} x(z) \sim x(1) + \gamma\, \delta^q \tilde{x}(\zeta) + o(\delta^2) \\ y(z) \sim \frac{t}{\gamma}\, \delta^p \tilde{y}(\zeta) + o(\delta^p) \end{cases} \quad , \quad \tilde{x}(\zeta) = \zeta^2 - 2u \quad , \quad q = 2$$

where p is the leading exponent in powers of δ, and $\tilde{y}$ is, for the moment, an almost arbitrary function of ζ. For example, if we assume that y would behave locally like $(x-a)^{p/q}$ then $\tilde{y}(\zeta)$ would be a polynomial of ζ of degree p.

The coefficient u comes from the $O(\delta^2)$ term in the expansion of $x(1) = x_0 + x_1\delta - 2\gamma\, u\delta^2 + O(\delta^3)$, it is related to the choice of relationship between δ and t, t_i's, and this choice will depend on the kind of critical point under consideration.

We call the curve $(\tilde{x}, \tilde{y})$ the blown up of the curve (x, y) in the region of size δ:

$$\begin{cases} \tilde{x}(\zeta) \\ \tilde{y}(\zeta) \end{cases} .$$

All the generating functions F_g and $W_n^{(g)}$ are given by Theorem 3.3.1 and Theorem 3.4.3, i.e. by residue formulae in the vicinity of $z = \pm 1$. For the residue at $z = +1$, we write $z = 1 + \delta\zeta$, and for the residue at $z = -1$, we have $z + 1 = 2 + O(\delta)$. Let us study how each term behaves in the small δ limit. The fundamental second kind differential $B(z_0, z) = 1/(z_0 - z)^2$ behaves like:

$B(z_0, z) \sim$

	z near $+1$	z near -1
z_0 near $+1$	$\delta^{-2}\,\tilde{B}(\zeta_0, \zeta)$	$O(1)$
z_0 near -1	$O(1)$	$O(1)$

$\times\,(1 + O(\delta))$,

where $\tilde{B}(\zeta_0, \zeta)$ is the fundamental second kind differential of the curve $(\tilde{x}, \tilde{y})$:

$$\tilde{B}(\zeta_0, \zeta) = \frac{1}{(\zeta - \zeta_0)^2}.$$

Similarly, the kernel K (see Eq. (3.7.1) in Chap. 3)

$$K(z_0, z) = \frac{1}{2}\left(\frac{1}{z_0 - z} - \frac{1}{z_0 - \frac{1}{z}}\right)\frac{1}{2y(z)\,x'(1/z)}$$

behaves like:

$K(z_0, z) \sim$

	z near $+1$	z near -1
z_0 near $+1$	$\frac{1}{t}\,\delta^{-(p+q)}\tilde{K}(\zeta_0, \zeta)$	$O(1)$
z_0 near -1	$O(\delta^{-(p+q-1)})$	$O(1)$

$\times\,(1 + O(\delta))$,

where $\tilde{K}(\zeta_0, \zeta)$ is the recursion kernel (see Chap. 7) of the spectral curve $(\tilde{x}, \tilde{y})$:

$$\tilde{K}(\zeta_0, \zeta) = \frac{1}{2} \left(\frac{1}{\zeta_0 - \zeta} - \frac{1}{\zeta_0 + \zeta} \right) \frac{1}{2\tilde{y}(\zeta)\, \tilde{x}'(\zeta)}.$$

Therefore, we see that the leading contribution to $\omega_{n+1}^{(g)}(1 + \delta\zeta_0, \dots, 1 + \delta\zeta_n)$ is given by the case where all residues are taken near $+1$, and can be computed only in terms of $\tilde{B}$ and $\tilde{K}$. By an easy recursion on $2g + n - 2$, we obtain:

Theorem 5.3.1 *Double scaling limits of correlation functions*

$$\omega_n^{(g)}(1+\delta\zeta_1, \dots, 1+\delta\zeta_n) \sim t^{2-2g-n}\, \delta^{(2-2g-n)(p+q)}\, \delta^{-n}\; \tilde{\omega}_n^{(g)}(\zeta_1, \dots, \zeta_n)\; (1+O(\delta))$$

and $\tilde{\omega}_n^{(g)}$ *are determined by the recursion relation:*

$$\tilde{\omega}_2^{(0)}(\zeta_1, \zeta_2) = \frac{1}{(\zeta_1 - \zeta_2)^2}$$

$$\tilde{\omega}_{n+1}^{(g)}(\zeta_0, J) = \operatorname*{Res}_{\zeta \to 0} \tilde{K}(\zeta_0, \zeta) \Big[\tilde{\omega}_{n+2}^{(g-1)}(\zeta, -\zeta, J) + \sum_{h=0}^{g} \sum_{I \subset J} \tilde{\omega}_{1+|I|}^{(h)}(\zeta, I) \tilde{\omega}_{1+n-|I|}^{(g-h)}(-\zeta, J/I) \Big] \tag{5.3.1}$$

where

$$\tilde{K}(\zeta_0, \zeta) = \frac{1}{2} \left(\frac{1}{\zeta_0 - \zeta} - \frac{1}{\zeta_0 + \zeta} \right) \frac{1}{(\tilde{y}(\zeta) - \tilde{y}(-\zeta))\tilde{x}'(-\zeta)}.$$

Therefore, we have found the scaling limit of $W_n^{(g)}$ in a small region of size δ.

Remark 5.3.1 Notice that the recursion relation Eq. (5.3.1) for the $\tilde{\omega}_n^{(g)}$'s, is very similar to the recursion relation of Theorem 3.3.1 for the $\omega_n^{(g)}$'s themselves. In fact both are special cases of the general "Topological recursion" introduced in [34], which is presented in Chap. 7 in this book. In some sense, the topological recursion commutes with taking limits.

Then, one could be tempted to apply the same method to the computation of F_g (with $g \geq 2$), from Theorem 3.4.3:

$$(2 - 2g)\, F_g = \operatorname*{Res}_{z \to +1} \Phi(z)\omega_1^{(g)}(z)dz + \operatorname*{Res}_{z \to -1} \Phi(z)\omega_1^{(g)}(z)dz\,. \tag{5.3.2}$$

Indeed, we have seen that $\omega_1^{(g)}(1 + \delta\zeta) \sim \delta^{(1-2g)(p+q)-1}\, \tilde{\omega}_1^{(g)}(\zeta)$, whereas near $z = -1$ (if $z = -1$ is not critical) we typically have $\omega_1^{(g)}(z) = o(\delta^{(1-2g)(p+q)-1})$. Thus, naively, one is tempted to write that the leading behavior of F_g would be:

$$F_g \sim \delta^{(2-2g)(p+q)}\, t^{2-2g}\, \tilde{F}_g \quad (1 + o(1))$$

where

$$\tilde{F}_g = \frac{1}{2-2g} \operatorname*{Res}_{\zeta\to 0} \tilde{\Phi}(\zeta)\tilde{\omega}_1^{(g)}(\zeta)d\zeta$$

with $\tilde{\Phi}'(\zeta) = \tilde{y}(\zeta)\tilde{x}'(\zeta)$.

However, this formula can be valid only if $\tilde{F}_g \neq 0$, otherwise this means that in fact F_g is given by subdominant contributions and all what we get is in that case

$$\tilde{F}_g = 0 \qquad \Leftrightarrow \qquad F_g = o(\delta^{(2-2g)(p+q)}).$$

This is not surprising, because F_g is not a function of δ, it is a function of the t_i's and so far we have not considered the relationship between δ and the t_i's. For instance if one chooses δ independent of the t_i's, then in that case F_g should clearly not depend on δ.

Remark 5.3.2 In case where both $z = -1$ and $z = +1$ are critical points of the curve (x, y), it may happen that the two terms of Eq. (5.3.2) are of the same order of magnitude.

For instance this is the case for even maps, where all functions $\omega_n^{(g)}$ have a symmetry $z \to -z$, and in that case, we get an overall prefactor 2:

$$F_g \sim 2\, \delta^{(2-2g)(p+q)}\, t^{2-2g}\, \tilde{F}_g \quad (1 + O(\delta))\,.$$

5.3.1 Double Scaling Limit of F_g

In the case of the spectral curve (x, y) of the enumeration of maps, which has near a critical point a cusp singularity of type $y \sim (x-a)^{p/q}$ (with $p = 2m+1$, $q = 2$) near its branchpoint, we choose a scale $\delta = (1 - t/t_c)^{\nu}$, and the blow up is of the form

$$\begin{cases} x(z) \sim x(1) + \delta^q\, \gamma(t_c)\tilde{x}(\zeta) + o(\delta^q) & , \qquad \deg \tilde{x} = q = 2 \\ y(z) \sim \delta^p \frac{t_c}{\gamma(t_c)}\, \tilde{y}(\zeta) + o(\delta^p) & , \qquad \deg \tilde{y} = p = 2m+1 \end{cases}$$

where $\tilde{y}(\zeta)$ is a polynomial of ζ of degree p. We parametrize the blown up spectral curve as:

$$\begin{cases} \tilde{x}(\zeta) = \zeta^2 - 2u \\ \tilde{y}(\zeta) = \sum_{k=0}^{m} \tilde{t}_k Q_k(\zeta) \end{cases}$$

where we decompose the polynomial $\tilde{y}(\zeta)$ onto the basis of the Q_k's defined in Eq. (5.2.3).

Moreover, we choose the t_k close to their critical value, and we define the $\tilde{t}_j$ to be the distance from the critical point, measured in eigendirections, i.e. in the form:

$$t_k = t_{k,c} + \sum_j C_{k,j}\, \delta^{\nu_j}\, \tilde{t}_j.$$

It can thus also be written $t_k = t_{k,c} + \sum_j C_{k,j}\,(1-t/t_c)^{\nu\nu_j}\,\tilde{t}_j$. The jth exponent $\nu\nu_j$ is called the "dressed exponent" of the flow which moves from the $(2m+1,2)$ singularity to the $(2j+1,2)$ singularity (indeed $\tilde{t}_j$ is associated to $Q_j(\zeta)$):

$$\text{dressed exponents} \qquad \nu\nu_j.$$

It remains to determine the exponents ν and ν_j (and check that they match with the KPZ (Knizhnik–Polyakov–Zamolodchikov) formula [55]).

In this purpose, we recall Lemma 3.1.4, we have (at fixed t_k):

$$\frac{\partial x(z)}{\partial z}\,\frac{\partial y(z)}{\partial t} - \frac{\partial y(z)}{\partial z}\,\frac{\partial x(z)}{\partial t} = \frac{1}{z}.$$

which can be rewritten, in the regime $z = 1+\delta\,\zeta$, and $\delta \sim (1-t/t_c)^{\nu}$, as:

$$\sum_k \tilde{t}_k((p-\nu_k)\,\tilde{x}'(\zeta)\,Q_k(\zeta) - q\,Q_k'(\zeta)\,\tilde{x}(\zeta)) = \frac{-1}{\nu}\,\delta^{\frac{1}{\nu}-(p+q-1)}\,(1+o(1)). \tag{5.3.3}$$

From their definition [see Eq. (5.2.3)], one sees that the Q_k satisfy

$$(2k+1)\tilde{x}'Q_k - 2\tilde{x}Q_k' = -2(2k+3)(-u/2)^{k+1}\,\frac{(2k+2)!}{(k+1)!\,(k+1)!}.$$

Since the Q_k form a basis of odd polynomials of degree $\leq 2m+1$, the only possibility for the right-hand-side of Eq. (5.3.3) to be a constant, is to choose $p - \nu_k = 2k+1$.

Also, since the left-hand-side of Eq. (5.3.3) is independent of δ, we must have $1/\nu = p+q-1$:

$$\nu = \frac{1}{p+q-1} \qquad , \qquad \nu_k = p-(2k+1) = 2(m-k).$$

We also find that u is solution of a polynomial equation:

$$\sum_k \tilde{t}_k\,\frac{(2k+3)!}{(k+1)!^2}\,(-u/2)^k = \frac{p+q-1}{2}. \tag{5.3.4}$$

Therefore, the generating functions of large maps are asymptotically given by

Theorem 5.3.2 *Double scaling limit of the F_g's enumerating functions of maps, at a (p,q) critical point ($p = 2m+1$, $q = 2$), for $g \geq 2$:*

$$F_g \sim (1 - t/t_c)^{(2-2g)\frac{p+q}{p+q-1}}\, t_c^{2-2g}\, \tilde{F}_g \quad + O((1 - t/t_c)^{\nu+(2-2g)\frac{p+q}{p+q-1}})$$

where

$$\tilde{F}_g = \frac{C}{2-2g} \operatorname*{Res}_{\zeta\to 0} \tilde{\Phi}(\zeta)\, \tilde{\omega}_1^{(g)}(\zeta) \tag{5.3.5}$$

and where

$$\tilde{\Phi}'(\zeta) = \tilde{y}(\zeta)\tilde{x}'(\zeta)$$

and where generically $C = 1$. For cases where the two branchpoints are critical, we may have $C \neq 1$, in particular for even maps we have $C = 2$.

Therefore, we have computed the double scaling limit $\tilde{F}_g$ of F_g.

Remark 5.3.3 If $p = 1, q = 2$, i.e. if the spectral curve has a regular branchpoint $y \sim \sqrt{x-a}$, the Blown up spectral curve is simply $\tilde{y} = \sqrt{\tilde{x} + 2u}$, and one may check that this spectral curve has $\tilde{F}_g = 0$, which is expected since F_g is not divergent when the spectral curve is regular. In that case, F_g is given by the subdominant contributions. Therefore Theorem 5.3.2 is useful only when $p \geq 3$.

Remark 5.3.4 The recursion relations [Eqs. (5.3.1) and (5.3.5)] are very similar to the ones for $W_n^{(g)}$ and F_g of Theorems 3.3.1 and 3.4.3 in Chap. 3. We will show in Chap. 7, that it is possible to define a common framework for both F_g and its double scaling limit $\tilde{F}_g$, namely the notion of a family of "symplectic invariants" attached to any spectral curve (x,y). The counting functions of maps as well as their scaling limits are special cases of those invariants.

In other words, if F_g is the gth symplectic invariant of the spectral curve (x,y), then:

Theorem 5.3.3 *$\tilde{F}_g$ is the gth symplectic invariant of the blown up spectral curve $(\tilde{x},\tilde{y})$.*

The notion of symplectic invariants of a spectral curve is explained in Chap. 7.

5.3.2 Critical Exponents and KPZ

In this subsection, we mention very briefly the link to KPZ. Readers can easily skip to the next section. We just sketch without details the link to conformal field theory, and refer the readers to reference books and reviews on the subject [40, 41].

Definition 5.3.1 The critical exponents in quantum gravity are defined as:

- The "**string susceptibility exponent**" γ (often denoted γ_{string} in the physics literature) is such that $\gamma = \gamma_0$ and γ_g are related to how the generating function F_g (generating function for genus g surfaces) diverges when the mesh size $(1 - t/t_c)$ tends to 0 (or equivalently, how F_g diverges at large area):

$$F_0 \sim (1 - t/t_c)^{2-\gamma}\ t_c^2\, \tilde{F}_0 + \text{regular}$$

  and for higher genus

$$F_g \sim (1 - t/t_c)^{2-\gamma_g}\ t_c^{2-2g}\ \tilde{F}_g.$$

- The "**dressing exponents**" $\Delta_{j,1}$ are related to the scaling behaviors when one moves away from the $(2m+1,2)$ critical point along a critical submanifold of codimension r (i.e. a $(2r+1,2)$ critical submanifold), measured in mesh size, and normalized so that $\Delta_{1,1} = 0$ for $j = 1$. In other words it is related to the scalings

$$t_k = t_{k,c} + \sum_j C_{k,j}\,(1 - t/t_c)^{\frac{\Delta_{j,1}-\Delta_{m,1}}{1-\Delta_{m,1}}}\ \tilde{t}_j.$$

  We have thus proved that

Theorem 5.3.4 *The critical exponents are:*

$$2 - \gamma_g = (2-2g)\,(p+q)\nu = (2-2g)\,\frac{p+q}{p+q-1}.$$

In particular at genus $g = 0$*:*

$$\gamma = \frac{-2}{p+q-1}.$$

The exponents $\Delta_{j,1}$ *are related to* $\nu_j = p - (2j+1) = 2(m-j)$ *by:*

$$\frac{\Delta_{j,1} - \Delta_{m,1}}{1 - \Delta_{m,1}} = \nu\nu_j = \frac{2(m-j)}{p+q-1}$$

and since $\Delta_{1,1} = 0$*:*

$$\Delta_{j,1} = \frac{2-2j}{4} = \frac{|p-qj| - |p-q|}{p+q-|p-q|}$$

They are those predicted by the Kac's table [41] and the KPZ formula [55].

5.3.2.1 Kac's Table

We refer the reader to literature on Conformal Field Theory, for example [41].

Finite representations of the conformal group in 2 dimensions, are classified as the (p, q) minimal models. The (p, q) minimal model has central charge

$$c = 1 - 6\,\frac{(p-q)^2}{pq} = 1 - 6\left(\frac{\sqrt{\kappa}}{2} - \frac{2}{\sqrt{\kappa}}\right)^2,$$

where we introduced the parameter $\kappa = \frac{4q}{p}$. This parameter κ is the one that appears in the famous SLE$_\kappa$ processes, see the literature [28, 79].

Minimal models have a finite number of possible highest weights. The highest weights of the (p, q) minimal models are labeled by two integers (r, s) with $0 < r < p$ and $0 < s < q$, and with the identification $(r, s) \equiv (p - r, q - s)$. Their highest weights are given by the famous Kac's formula:

$$h_{r,s} = \frac{(ps - qr)^2 - (p-q)^2}{4pq}.$$

The weights $h_{r,s}$ are the exponents that control how the corresponding fields change under dilatations.

- The field $(1, 1)$ has weight 0, it is called the "identity operator":

$$(1,1)\ \text{field} \ = \text{Identity} \qquad , \qquad h_{1,1} = 0.$$

- The value of (r, s) which gives the minimum of $|ps - qr|$, is called the "most relevant operator", it has the smallest weight $h_{r,s}$.
- The unitary minimal models are those for which $|p - q| = 1$, and for them, the "most relevant operator" is the Identity $(1, 1)$.
- Case $(p, q) = (2m + 1, 2)$.

  In that case, the central charge is

$$c = 1 - 3\,\frac{(2m-1)^2}{2m+1}.$$

There are m highest weights corresponding to $s = 1$ and $1 \le r \le m$, their weights are

$$h_{r,1} = \frac{(r-1)\ (r-2m)}{2(2m+1)}.$$

In that case, the most relevant operator is $(r, s) = (m, 1)$, its weight is:

$$h_{m,1} = \frac{-m\,(m-1)}{2(2m+1)}.$$

The only unitary models among the $(2m + 1, 2)$ models, are the $(3, 2)$ model (pure gravity), with central charge $c = 0$, and the $(1, 2)$ model (Airy model) with central charge $c = -2$.

5.3.2.2 KPZ

Polyakov understood in 1981 [76], that conformal Field theories can be coupled to gravity, in a way preserving conformal invariance, by adding a new field: the Liouville field.

The Liouville field is constructed from the Gaussian free field, see [28], and was recently constructed in probability theory [24].

There are also exponents controlling how the fields change with a dilatation, however, the coupling to gravity means that the metric itself changes under dilatations, and thus the exponents get "dressed" by gravity.

It is customary to measure the behavior under dilatations by measuring how the fields scale in powers of the area of the surface when the area becomes large, or equivalently how they scale in powers of the mesh size at small mesh.

Recall that for us the mesh size is $(1 - t/t_c)$.

The exponent γ_g controls the scaling of the partition function of genus g. In Liouville theory, the topology enters only through the integral of the curvature, which is proportional to the Euler characteristics $\chi = 2-2g$, and thus γ_g is expected to be a polynomial of degree 1 of the genus. We write it:

$$2 - \gamma_g = (1 - g)(2 - \gamma)$$

with $\gamma = \gamma_0$. In other words, the exponent γ should be such that

$$F_g \sim (1 - t/t_c)^{2-\gamma_g}\ t_c^{2-2g}\, \tilde{F}_g \sim (1 - t/t_c)^{(1-g)(2-\gamma)}\ t_c^{2-2g}\, \tilde{F}_g.$$

The KPZ formula, due to Knizhnik, Polyakov, Zamolodchikov [55], computes the dressing exponents $\Delta_{r,s}$ of the weights $h_{r,s}$. They claim that:

$$\frac{\kappa}{4}\,\Delta_{r,s}^2 + \left(1 - \frac{\kappa}{4}\right)\Delta_{r,s} = h_{r,s},$$

where $\kappa = \frac{4q}{p}$ is the SLE parameter.

For (p, q) minimal models, this gives:

$$\Delta_{r,s} = \frac{|ps - qr| - |p - q|}{p + q - |p - q|}.$$

Notice that the identity operator $(r, s) = (1, 1)$, is undressed:

$$\Delta_{1,1} = 0.$$

The most relevant operator $(m, 1)$ has the dressing:

$$\Delta_{m,1} = \frac{1 - |p - q|}{p + q - |p - q|}.$$

They also found the string exponent γ, associated to the most relevant operator (r, s):

$$\gamma_{r,s} = -\frac{2\,|ps - qr|}{p + q - |ps - qr|}.$$

5.3.2.3 KPZ Formulae for the $(2m + 1, 2)$ Minimal Model

In that case we have:

$$\Delta_{r,s} = \frac{2 - 2r}{p + q - 1} = \frac{1 - r}{m + 1},$$

and

$$\gamma = \gamma_{m,1} = -\frac{2}{p + q - 1} = -\frac{1}{m + 1}.$$

This is in agreement with our direct proof from the generating functions of maps.

5.3.3 Example: Triangulations and Pure Gravity

Consider the generating function for triangulations. The potential is:

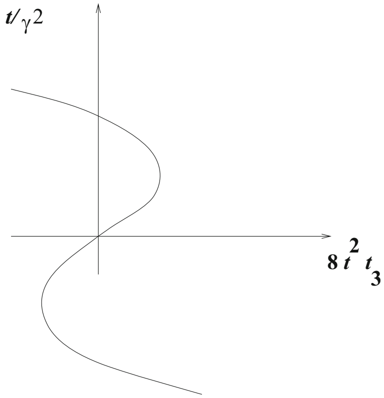
$$V(x) = \frac{x^2}{2} - t_3 \frac{x^3}{3}$$

whose spectral curve was computed in Sect. 3.1.8 of Chap. 3:

$$\begin{cases} x(z) = \alpha + \gamma(z + 1/z) \\ y(z) = \frac{1}{\gamma}(z - 1/z) - t_3\gamma^2(z^2 - z^{-2}) \end{cases}$$

where α, γ are determined by

$$r - r^3 = 8tt_3^2 \quad , \quad \gamma^2 = \frac{t}{r} \quad , \quad \alpha = \frac{1 - r}{2t_3}.$$



The equation for γ becomes singular at $\sqrt{t}\, t_3 = t_c$, where

$$t_c = \frac{1}{2}\, 3^{-3/4} \qquad , \qquad r_c = \frac{1}{\sqrt{3}},$$

and one can check that at this point, the spectral curve has a $(3/2)$ cusp $y \sim (x - x(1))^{3/2}$. This is the $(3,2)$ critical point, $p = 3 = 2m + 1$ with $m = 1$, also called "pure gravity".

Near t_c we parametrize with a scaling δ:

$$\sqrt{t}\, t_3 = t_c(1 - \frac{3}{4}\delta^4),$$

so that we obtain

$$\gamma \sim \gamma(t_c)(1 - \frac{1}{2}\delta^2) + O(\delta^3) \qquad , \qquad \alpha \sim \alpha(t_c) - \gamma(t_c)\, \delta^2 + O(\delta^3)$$

where

$$\gamma(t_c) = 3^{1/4}\, \sqrt{t} \qquad , \qquad \alpha(t_c) = 3^{1/4} \sqrt{t}\, (\sqrt{3} - 1).$$

If we choose

$$z = 1 + \delta\ \zeta$$

we have:

$$\begin{cases} x(z) \sim \alpha(t_c) + 2\gamma(t_c) + 3^{1/4}\sqrt{t}\, \delta^2\, (\zeta^2 - 2) + o(\delta^2) \\ y(z) \sim \frac{\sqrt{t}}{3^{1/4}}\, \delta^3\, (\zeta^3 - 3\zeta) + o(\delta^3) \end{cases}$$

i.e. the blown up curve is

$$\begin{cases} \tilde{x}(\zeta) = \zeta^2 - 2 \\ \tilde{y}(\zeta) = \zeta^3 - 3\zeta. \end{cases}$$

Not surprisingly, we recognize the polynomial $Q_1(\zeta) = \zeta^3 - 3\zeta$ of Eq. (5.2.3). Applying Theorem 5.3.1, for example, we find for the first few n and g:

$$\tilde{\omega}_3^{(0)}(\zeta_1, \zeta_2, \zeta_3) = -\frac{1}{6}\,\frac{1}{\zeta_1^2\zeta_2^2\zeta_3^2} \tag{5.3.6}$$

$$\tilde{\omega}_1^{(1)}(\zeta) = -\frac{1}{(12)^2}\,\frac{\zeta^2+3}{\zeta^4} \tag{5.3.7}$$

$$\tilde{\omega}_2^{(1)}(\zeta_1, \zeta_2) = \frac{15\zeta_1^4 + 15\zeta_2^4 + 9\zeta_1^2\zeta_2^2 + 6\zeta_1^4\zeta_2^2 + 6\zeta_1^2\zeta_2^4 + 2\zeta_1^4\zeta_2^4}{2^5\,3^3\,\zeta_1^6\,\zeta_2^6} \tag{5.3.8}$$

$$\tilde{\omega}_1^{(2)}(\zeta) = -7\,\frac{135 + 87\zeta^2 + 36\zeta^4 + 12\zeta^6 + 4\zeta^8}{2^{10}\,3^5\,\zeta^{10}} \tag{5.3.9}$$

$$\tilde{\omega}_4^{(0)}(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = \frac{1}{9\,\zeta_1^2\zeta_2^2\zeta_3^2\zeta_4^2}\left(1 + 3\sum_i \frac{1}{\zeta_i^2}\right)$$

$$\tilde{\omega}_5^{(0)}(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5) = \frac{1}{9\,\zeta_1^2\zeta_2^2\zeta_3^2\zeta_4^2\zeta_5^2}\left(1 + 3\sum_i \frac{1}{\zeta_i^2} + 6\sum_{i<j}\frac{1}{\zeta_i^2\zeta_j^2} + 5\sum_i \frac{1}{\zeta_i^4}\right)$$

etc…

Using Theorem 3.4.7, we have

$$\frac{\partial F_g}{\partial t} = -\operatorname*{Res}_{z\to\pm 1} \omega_1^{(g)}(z)\,dz\ \ \ln z.$$

To leading order in δ, only the residue at $z = +1$ contributes, and writing $\ln z = \ln(1+\delta\zeta) = \delta\zeta + O(\delta)^2$, we get

$$\frac{\partial F_g}{\partial t} \sim -\,t^{1-2g}\,\delta^{5(1-2g)+1}\operatorname*{Res}_{\zeta\to 0}\tilde{\omega}_1^{(g)}(\zeta)\,\zeta\,d\zeta.$$

Similarly, taking a second derivative gives

$$\frac{\partial^2 F_g}{\partial t^2} \sim t^{-2g}\,\delta^{2-10g}\operatorname*{Res}_{\zeta_1\to 0}\operatorname*{Res}_{\zeta_2\to 0}\tilde{\omega}_2^{(g)}(\zeta_1, \zeta_2)\,\zeta_1\,d\zeta_1\,\zeta_2\,d\zeta_2.$$

and a third derivative

$$\frac{\partial^3 F_g}{\partial t^3} \sim -t^{-1-2g}\,\delta^{3-5(1+2g)} \operatorname*{Res}_{\zeta_1\to 0}\operatorname*{Res}_{\zeta_2\to 0}\operatorname*{Res}_{\zeta_3\to 0} \tilde{\omega}_3^{(g)}(\zeta_1,\zeta_2,\zeta_3)\,\zeta_1\,d\zeta_1\,\zeta_2\,d\zeta_2\,\zeta_3\,d\zeta_3.$$

From Eq. (5.3.6), we thus get

$$\frac{\partial^3 F_0}{\partial t^3} \sim -\frac{\delta^{-2}}{6t} \quad\longrightarrow\quad \frac{\partial^2 F_0}{\partial t^2} \sim -\frac{\delta^2}{2},$$

and using Eq. (5.3.8):

$$\frac{\partial^2 F_1}{\partial t^2} \sim \frac{\delta^{-8}}{2^4\,3^3\,t^2}$$

as well as using Eq. (5.3.9):

$$\frac{\partial F_2}{\partial t} \sim \frac{7\,\delta^{-14}}{2^8\;3^5\;t^3} \quad\longrightarrow\quad \frac{\partial^2 F_2}{\partial t^2} \sim \frac{49\,\delta^{-18}}{2^8\,3^6\,t^4}.$$

We define u_g such that

$$\frac{\partial^2 F_g}{\partial t^2} \sim u_g\,\frac{\delta^{2-10g}}{t^{2g}}$$

i.e.

$$u_0 = -\frac{1}{2}, \quad u_1 = \frac{1}{2^4\;3^3}, \quad u_2 = \frac{49}{2^8\;3^6}, \;\ldots$$

We may thus verify that the second derivative of the free energy:

$$u(s) = \sum_{g=0}^{\infty} s^{(1-5g)/2}\,u_g$$

satisfies the Painlevé I equation to the first orders:

$$2u^2 + \frac{1}{3^3}u'' = \frac{1}{2}s + o(s^{-4}).$$

Our goal now, is to prove that $u(s)$ satisfies Painlevé I to all orders.

5.4 Minimal Models

The goal of this section is to prove that the following formal series

$$\ln \tau = \sum_g N^{2-2g}\, \tilde{F}_g$$

whose coefficients $\tilde{F}_g$ are the generating functions of large maps, is a formal **Tau-function** for the mth reduction of the Kordeweg-De-Vries (**KdV**) hierarchy of integrable equations. That reduction of KdV is also called the $(2m+1, 2)$ minimal model in the context of conformal field theory. It can be obtained from Liouville conformal field theory coupled to 2D gravity.

In some sense, we obtain an argument towards the idea that large maps should be related to Liouville gravity.

5.4.1 Introduction to Minimal Models

There exists several equivalent definitions of minimal models coupled to gravity. Here we shall adopt the approach of Douglas and Shenker in 1990 [27]. Minimal models correspond to representations of the conformal group in 2 dimensions. They are classified by two integers (p, q), and their central charge is:

$$c = 1 - 6\frac{(p-q)^2}{pq}.$$

Some of them have received special names (see [40]):

- $(1, 2)$ = Airy, $c = -2$ (related to Tracy-Widom law [82])
- $(3, 2)$ = pure gravity, $c = 0$
- $(5, 2)$ = Lee-Yang edge singularity, $c = -\frac{22}{5}$
- $(4, 3)$ = Ising, $c = \frac{1}{2}$
- $(6, 5)$ = Potts-3, $c = \frac{4}{5}$

Minimal models can also be viewed as finite reductions of the Kadamtsev-Petviashvili (KP) integrable hierarchy of partial differential equations [8, 53].

The case $q = 2$ is a little bit simpler to address, and is a reduction of the Korteweg de Vries (KdV) hierarchy [8, 47, 58].

The KdV hierarchy, and the minimal models $(p, 2)$ have generated a huge amount of works, and have been presented in many different (but equivalent) formulations. For instance in terms of a string equation for differential operators, in terms of a Lax pair, in terms of commuting hamiltonians, in terms of Schrödinger equation, in terms of Hirota equations, in terms of isomonodromic systems, in terms of Riemann Hilbert problems, in terms of tau functions, in terms of Grasman manifolds, in terms of Yang-Baxter equations, ...etc, see [8] for a comprehensive lecture.

All those formulations are equivalent, and let us recall some of the well known features of the $(p, 2)$ reduction of KdV (see [8, 41]), presented in a way convenient for our purposes.

5.4.2 String Equation

The KdV minimal model $(p, 2)$ with $p = 2m+1$, coupled to gravity, was formulated in terms of a "string equation" by Douglas and Shenker in 1990 [27]. Let P, Q two differential operators of respective orders p and 2, satisfying the so-called "string equation":

$$[P, Q] = \frac{1}{N}\,\mathrm{Id} \tag{5.4.1}$$

$$Q = d^2 - 2u(s) \qquad , \qquad P = d^p - p\,u\,d^{p-2} + \dots \qquad , \qquad d = \frac{1}{N}\,\frac{d}{ds}$$

$\frac{1}{N}$ is a redundant parameter, which can be absorbed by a redefinition of s and u, but we prefer to keep it to play the role of a scaling parameter which can be sent to zero to get the "classical limit".

In all this chapter, we shall denote with a dot the derivative with respect to s: $df/ds = \dot{f}$ in order to shorten notations. The prime will be reserved to derivatives with respect to the spectral parameter $df/dx = f'$.

5.4.2.1 Solution of the String Equation

The general solution of the string equation (5.4.1) is known. Let us describe it below.

Definition 5.4.1 Let $(Q^{j+1/2})_+$ be the unique differential operator of order $2j + 1$, such that:

$$\mathrm{order}[((Q^{j+1/2})_+)^2 - Q^{2j+1}] \leq 2j.$$

For example:

$$(Q^{1/2})_+ = d \qquad , \qquad (Q^{3/2})_+ = d^3 - 3ud - \frac{3\dot{u}}{2N},$$

$$(Q^{5/2})_+ = d^5 - 5ud^3 + \frac{15}{2}\,u^2 d - \frac{15\dot{u}}{2N}\,d^2 - \frac{25\ddot{u}}{4N^2}\,d - \frac{15}{8N^3}\,\dddot{u} + \frac{15u\dot{u}}{2N} \qquad , \qquad \dots$$

Lemma 5.4.1 *It is a classical result (see [40]) that it satisfies:*

$$[(Q^{j-1/2})_+, Q] = \frac{1}{N}\frac{d}{ds}(R_j(u(s))) \tag{5.4.2}$$

where the right hand side is a function (a differential operator of order 0*).*

Proof We propose the proof of this lemma as an exercise at the end of this chapter, and we give some hints of how to do it. □

The coefficients $R_j(u)$ are called the Gelfand-Dikii differential polynomials [40]. They can be obtained by a recursion.

Definition 5.4.2 (Gelfand-Dikii Polynomials) The Gelfand-Dikii differential polynomials are defined by the recursion:

$$R_0 = 2 \qquad , \qquad \dot{R}_{j+1} = -2u\dot{R}_j - \dot{u}R_j + \frac{1}{4N^2}\dddot{R}_j. \tag{5.4.3}$$

and by the condition that R_j is homogenous of degree j in u with the grading convention that $\dot{} = \partial/\partial s$ has the same grading as $\sqrt{u}$.

The first few of them are:

$$\begin{aligned} R_0 &= 2 \\ R_1 &= -2u \\ R_2 &= 3\,u^2 - \tfrac{1}{2N^2}\ddot{u} \\ R_3 &= -5u^3 + \frac{5}{2N^2}u\ddot{u} + \frac{5}{4N^2}\dot{u}^2 - \frac{1}{8N^4}\ddddot{u} \\ &\vdots \end{aligned}$$

and in general:

$$\begin{aligned} R_j(u) = \frac{2\ (-1)^j\ (2j-1)!!}{j!} &\left[u^j - \frac{j(j-1)}{12N^2}\,u^{j-2}\ddot{u}\right. \\ &\left. -\frac{j(j-1)(j-2)}{24N^2}\,u^{j-3}\dot{u}^2\right] + \quad \dots \quad - \frac{2}{(2N)^{2j-2}}\,u^{(2j-2)}. \end{aligned} \tag{5.4.4}$$

Lemma 5.4.2 *Any solution of the string equation*

$$[P, Q] = \frac{1}{N}\,\mathrm{Id}$$

where $Q = d^2 - 2u$ *and* $P = d^{2m+1} + \dots$*, can be written:*

$$P = \sum_{j=0}^{m} \tilde{t}_j (Q^{j+1/2})_+ + \sum_{j=0}^{m-1} c_j Q^j \qquad , \qquad \tilde{t}_m = 1$$

where c_j, $\tilde{t}_j$ are constants (independent of s) and $u(s)$ is a solution of the non-linear differential equation:

$$\boxed{\sum_{j=0}^{m} \tilde{t}_j R_{j+1}(u) = s.} \tag{5.4.5}$$

This equation has the Painlevé property

Proof The proof that the solution takes that form is obvious from Lemma 5.4.1. The fact that the equation satisfies the Painlevé property is beyond the scope of this book, and we shall not use it here. We refer the reader to [23] for more details about the Painlevé property. □

The coefficients c_j associated to Q^j will play no role in what follows, because $[Q^j, Q] = 0$, so from now on, we shall choose $c_j = 0$.

Remark 5.4.1 Since $R_0 = 2$, we see that we can identify s with $s = -2\tilde{t}_{-1}$.

Examples

- For Airy $p = 1$, the equation for u is:

$$-2u = s. \tag{5.4.6}$$

- For pure gravity $p = 3$, this is the Painlevé I equation:

$$3\,u^2 - \frac{1}{2N^2}\ddot{u} - 2\tilde{t}_0 u = s. \tag{5.4.7}$$

- For Lee-Yang $p = 5$, we have:

$$-5u^3 + \frac{5}{2N^2}u\ddot{u} - \frac{5}{4N^2}\dot{u}^2 - \frac{1}{8N^4}\ddddot{u} + \tilde{t}_1(3\,u^2 - \frac{1}{2N^2}\ddot{u}) - 2\tilde{t}_0 u = s. \tag{5.4.8}$$

5.4.3 Lax Pair

Consider the following matrices:

Definition 5.4.3

$$\mathcal{R}(x, s) = \begin{pmatrix} 0 & 1 \\ x + 2u(s) & 0 \end{pmatrix},$$

and for any integer k:

$$\mathcal{D}_k(x,s) = \begin{pmatrix} A_k & B_k \\ C_k & -A_k \end{pmatrix},$$

where $A_k(x,s), B_k(x,s), C_k(x,s)$ are polynomials of respective degree $k-1, k, k+1$ in x, which are defined by:

$$B_k(x,s) = \frac{1}{2}\sum_{j=0}^{k} x^{k-j}\, R_j(u) \qquad , \qquad A_k = -\frac{1}{2N}\dot{B}_k \qquad , \qquad C_k = (x+2u)\, B_k + \frac{1}{N}\dot{A}_k.$$

The recursion relation Eq. (5.4.3) implies that B_k satisfies the equation:

$$2\dot{u}B_k + 2(x+2u)\dot{B}_k - \frac{1}{2N^2}\dddot{B}_k = -\dot{R}_{k+1}(u)$$

and we see that

Lemma 5.4.3 *the matrix $\mathcal{D}_k(x,t)$ satisfies:*

$$\frac{1}{N}\frac{\partial}{\partial s}\mathcal{D}_k(x,s) + [\mathcal{D}_k(x,s), \mathcal{R}(x,s)] = -\frac{1}{N}\dot{R}_{k+1}(u)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{5.4.9}$$

the right hand side is independent of x, and is proportional to $\frac{\partial}{\partial x}\mathcal{R}(x,s)$.

This equation is called a "Lax equation".

5.4.4 Lax Equation

Therefore we have obtained that, if u is a solution of the string equation (5.4.5), then, the matrix:

$$\mathcal{D}(x,s) = \sum_{j=0}^{m} \tilde{t}_j \mathcal{D}_j(x,s) \qquad , \qquad \tilde{t}_m = 1$$

satisfies the Lax equation:

Proposition 5.4.1 *The matrices $\mathcal{D}(x,s)$ and $\mathcal{R}(x,s)$ form a Lax pair, they satisfy the Lax equation*

$$\frac{1}{N}\frac{\partial}{\partial s}\mathcal{D}(x,s) + [\mathcal{D}(x,s), \mathcal{R}(x,s)] = -\frac{1}{N}\frac{\partial}{\partial x}\mathcal{R}(x,s) \tag{5.4.10}$$

which can also be written as

$$\left[\frac{1}{N}\frac{\partial}{\partial x} + \mathcal{D}(x,s), \mathcal{R}(x,s) - \frac{1}{N}\frac{\partial}{\partial s}\right] = 0. \tag{5.4.11}$$

This relation means that the operator $\frac{1}{N}\frac{\partial}{\partial x} + \mathcal{D}(x,s)$ *is a Lax operator [8].*

5.4.5 The Linear ψ System

The Lax equation (5.4.11) is the compatibility condition, which says that the following two differential systems have a common solution $\Psi(x,s)$:

$$\frac{1}{N}\frac{\partial}{\partial x}\Psi(x,s) = -\mathcal{D}(x,s)\,\Psi(x,s) \qquad , \qquad \frac{1}{N}\frac{\partial}{\partial s}\Psi(x,s) = \mathcal{R}(x,s)\,\Psi(x,s) \tag{5.4.12}$$

and $\Psi(x,s)$ is a matrix such that:

$$\Psi(x,s) = \begin{pmatrix} \psi & \phi \\ \tilde{\psi} & \tilde{\phi} \end{pmatrix}. \tag{5.4.13}$$

In particular this implies the Schrödinger equation for ψ:

$$\frac{1}{N^2}\,\ddot{\psi}(x,s) = (x + 2u(s))\,\psi(x,s) \tag{5.4.14}$$

where s can be interpreted as the space variable, $u(s)$ is the potential, and x the energy. This is why x is often called the "**spectral parameter**". $\hbar = 1/N$ can be interpreted as the Planck constant and this is why the limit $N \to \infty$ is called the "**classical limit**".

Definition 5.4.4 The wronskian $w(x,s)$ of $\Psi(x,s)$ is defined as the determinant

$$w(x,s) = \det \Psi(x,s)^{-1}.$$

It satisfies

$$\frac{\partial}{\partial x}\log w(x,s) = N\ \mathrm{Tr}\,\mathcal{D}(x,s) = 0$$

$$\frac{\partial}{\partial s}\log w(x,s) = -N\ \mathrm{Tr}\,\mathcal{R}(x,s) = 0$$

Remark 5.4.2 Up to right multiplication $\Psi(x,s) \to \Psi(x,s)C$ by a constant matrix C independent of x and s, it is thus possible to normalize $w(x,s) = \det \Psi(x,s)^{-1} = 1$. Most often we shall make that convenient choice, and assume that the wronskian is normalized to 1, except when we shall consider the insertion operator in Definition 5.4.9 and in the proof of Lemma 5.4.5 below.

5.4.6 Kernel and Correlators

In the following sections we chose to normalize $\det \Psi(x,s) = 1$, and we define

Definition 5.4.5 the (generalized) Christoffel-Darboux kernel associated to the system $\mathcal{D}(x,s)$ is defined as

$$\begin{aligned} K(x_1,x_2,s) &= \frac{1}{x_1-x_2} \left(\Psi(x_1,s)^{-1} \ \Psi(x_2,s)\right)_{2,2} \\ &= \frac{\psi(x_1,s)\tilde{\phi}(x_2,s) - \tilde{\psi}(x_1,s)\phi(x_2,s)}{x_1-x_2}. \end{aligned} \tag{5.4.15}$$

Most often the s dependence will be implicitly assumed and we shall denote:

$$K(x_1,x_2,s) \equiv K(x_1,x_2).$$

Remark 5.4.3 In fact, the actual Christoffel-Darboux kernel usually considered in the literature, is often the $\left(\Psi(x_1,s)^{-1} \ \Psi(x_2,s)\right)_{2,1}$. It turns out that the two are related, and this one is more convenient for our purposes.

Remark 5.4.4 If we would not assume $w(x,s) = \det \Psi(x,s)^{-1}$ to be normalized to 1, the formula for K would be:

$$\begin{aligned} K(x_1,x_2,s) &= \frac{1}{x_1-x_2} \left(\Psi(x_1,s)^{-1} \ \Psi(x_2,s)\right)_{2,2} \\ &= w(x_1,s) \ \frac{\psi(x_1,s)\tilde{\phi}(x_2,s) - \tilde{\psi}(x_1,s)\phi(x_2,s)}{x_1-x_2}. \end{aligned}$$

Definition 5.4.6 We define the "**connected correlators**" by the "determinantal formulae":

$$\hat{W}_1(x) = \lim_{x'\to x} K(x,x') - \frac{1}{x-x'} = \psi'(x,s)\tilde{\phi}(x,s) - \tilde{\psi}'(x,s)\phi(x,s) \tag{5.4.16}$$

and for $n \geq 2$:

$$\hat{W}_n(x_1,\dots,x_n) = -\frac{\delta_{n,2}}{(x_1-x_2)^2} + (-1)^{n-1} \sum_{\sigma=\text{cyles}} \prod_{i=1}^{n} K(x_i,x_{\sigma(i)}) \tag{5.4.17}$$

where we take the sum over all cyclic permutations (i.e. σ has only one cycle).

For example:

$$\hat{W}_2(x_1,x_2) = -K(x_1,x_2)K(x_2,x_1) - \frac{1}{(x_1-x_2)^2},$$

$$\hat{W}_3(x_1,x_2,x_3) = K(x_1,x_2)K(x_2,x_3)K(x_3,x_1) + K(x_1,x_3)K(x_3,x_2)K(x_2,x_1).$$

Although we have not written it explicitly, the kernel K and the correlators $\hat{W}_n$ depend on s.

Remark 5.4.5 Our goal in this section will be to prove that the correlators $\hat{W}_n$ defined from the minimal model, coincide with the correlators $\tilde{W}_n$ of Sect. 5.3 defined from the double scaling limit of generating functions of large maps:

$$\hat{W}_n \overset{?}{=} \tilde{W}_n.$$

The $\tilde{W}_n$ were defined in Sect. 5.3 as formal power series of $1/N$, and here, we shall consider a formal solution of $\partial_x\Psi(x,s) = -N\mathcal{D}(x,s)\Psi(x,s)$, so that $\hat{W}_n$ are formal power series of $1/N$. The equality $\hat{W}_n = \tilde{W}_n$, will then be an equality of formal series in $\mathbb{C}[[1/N]]$.

Definition 5.4.7 The *non-connected* correlators are defined by:

$$\hat{W}_n(x_1,\dots,x_n)_{n.c.} = \sum_{\mu\vdash\{x_1,\dots,x_n\}} \prod_{i=1}^{\ell(\mu)} \hat{W}_{|\mu_i|}(\mu_i),$$

where the sum is over all partitions $\mu = (\mu_1,\dots,\mu_{\ell(\mu)})$ of $\{x_1,\dots,x_n\}$ into non-empty disjoint subsets. In other words, the connected $\hat{W}_n$'s are the cumulants of the non-connected ones.

For instance:

$$\hat{W}_2(x_1,x_2)_{n.c.} = \hat{W}_2(x_1,x_2) + \hat{W}_1(x_1)\hat{W}_1(x_2),$$

$$\begin{aligned}\hat{W}_3(x_1,x_2,x_3)_{n.c.} = {}& \hat{W}_3(x_1,x_2,x_3) + \hat{W}_1(x_1)\hat{W}_2(x_2,x_3) + \hat{W}_1(x_2)\hat{W}_2(x_1,x_3) \\ & + \hat{W}_1(x_3)\hat{W}_2(x_1,x_2) + \hat{W}_1(x_1)\hat{W}_1(x_2)\tilde{W}_1(x_3). \end{aligned} \tag{5.4.18}$$

The formula Eq. (5.4.17) is called "determinantal formula", because for the non-connected correlators, the sum over cyclic permutations in Eq. (5.4.17) gets replaced by a sum over all permutations, with their signature:

$$\hat{W}_n(x_1,\dots,x_n)_{n.c.} = \det{}'(K(x_i,x_j)) = \sum_\sigma{}' (-1)^\sigma \prod_i K(x_i,x_{\sigma(i)})$$

where $\det'$ and $\sum'$ signify that whenever the permutation σ has a fixed point if $\sigma(i) = i$ we must replace the ill-defined $K(x_i, x_i)$ by $\hat{W}_1(x_i)$, and whenever the permutation σ has a cycle of length 2, i.e. $\sigma(i) = j$ and $\sigma(j) = i$, we replace $K(x_i, x_j)K(x_j, x_i)$ by $-\hat{W}_2(x_i, x_j)$, see [10].

For instance $\hat{W}_{3,\,n.c.}$ is the sum of six terms coming from the six permutations:

$$\begin{aligned}\hat{W}_{3,\,n.c.}(x_1, x_2, x_3) &= \det{}' \begin{pmatrix} K(x_1, x_1) & K(x_1, x_2) & K(x_1, x_3) \\ K(x_2, x_1) & K(x_2, x_2) & K(x_2, x_3) \\ K(x_3, x_1) & K(x_3, x_2) & K(x_3, x_3) \end{pmatrix} \\ &= \hat{W}_1(x_1)\hat{W}_1(x_2)\hat{W}_1(x_3) + \hat{W}_1(x_1)\hat{W}_2(x_2, x_3) + \hat{W}_1(x_2)\hat{W}_2(x_1, x_3) \\ &\quad + \hat{W}_1(x_3)\hat{W}_2(x_1, x_2) + K(x_1, x_2)K(x_2, x_3)K(x_3, x_1) \\ &\quad + K(x_1, x_3)K(x_3, x_2)K(x_2, x_1) \end{aligned} \tag{5.4.19}$$

which coincides with Eq. (5.4.18).

5.4.6.1 Alternative Definition of the Correlators

Notice that:

$$K(x, x') = \frac{(\psi(x, s)\tilde{\phi}(x', s) - \tilde{\psi}(x, s)\phi(x', s))}{x - x'} = \frac{1}{x - x'} \left(\Psi(x, s)^{-1}\, \Psi(x', s)\right)_{2,2}$$

and thus

$$K(x, x')K(x', x'') = \frac{1}{(x - x')(x' - x'')} \left(\Psi(x, s)^{-1}\, \Psi(x', s) E\, \Psi(x', s)^{-1}\, \Psi(x'', s)\right)_{2,2}$$

where E is a matrix which projects on the $(2, 2)$ coefficient:

$$E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

This leads to define

Definition 5.4.8 The projector $M(x, s)$:

$$M(x, s) = \Psi(x, s)\, E\, \Psi(x, s)^{-1} = \begin{pmatrix} -\tilde{\psi}(x, s)\phi(x, s) & \psi(x, s)\phi(x, s) \\ -\tilde{\psi}(x, s)\tilde{\phi}(x, s) & \psi(x, s)\tilde{\phi}(x, s) \end{pmatrix}.$$

The matrix $M(x, s)$ is a projector, it satisfies

$$M(x, s)^2 = M(x, s) \qquad , \qquad \mathrm{Tr}\, M(x, s) = 1 \qquad , \qquad \det M(x, s) = 0.$$

Thanks to that matrix $M(x, s)$, we can rewrite any cyclic product of $K(x_i, x_{\sigma(i)})$ as a cyclic product of matrices $M(x, s)$:

$$\prod_i K(x_i, x_{\sigma(i)}) = \frac{\mathrm{Tr}\ \prod_i M(x_{\sigma^i(1)}, s)}{\prod_i (x_i - x_{\sigma(i)})},$$

where $\sigma^i(1)$ means the image of 1 by the ith power of σ.

For example:

$$\hat{W}_2(x, x') = -K(x, x')K(x', x) - \frac{1}{(x-x')^2} = \frac{\mathrm{Tr}\, M(x, s)M(x', s)}{(x-x')^2} - \frac{1}{(x-x')^2}$$

and

$$K(x, x')K(x', x'')K(x'', x) = \frac{\mathrm{Tr}\, M(x, s)M(x', s)M(x'', s)}{(x-x')(x'-x'')(x''-x)}.$$

It follows that

$$\begin{aligned}\hat{W}_3(x, x', x'') &= \frac{\mathrm{Tr}\ (M(x, s)M(x', s)M(x'', s) - M(x, s)M(x'', s)M(x', s))}{(x-x')(x'-x'')(x''-x)} \\ &= \frac{\mathrm{Tr}\, M(x, s)\, [M(x', s), M(x'', s)]}{(x-x')(x'-x'')(x''-x)}.\end{aligned}$$

And in general the correlators are:

Theorem 5.4.1

$$\hat{W}_1(x) = N\, \mathrm{Tr}\, \mathcal{D}(x, s)\, M(x, s)$$

$$\hat{W}_2(x_1, x_2) = \frac{\mathrm{Tr}\, M(x_1, s)M(x_2, s)}{(x_1 - x_2)^2} - \frac{1}{(x_1 - x_2)^2}$$

and for $n \geq 3$

$$\hat{W}_n(x_1, \ldots, x_n) = (-1)^{n-1} \sum_{\sigma=\text{cyclic}} \frac{\mathrm{Tr}\ \prod_{i=0}^{n-1} M(x_{\sigma^i(1)}, s)}{\prod_{i=1}^{n} (x_i - x_{\sigma(i)})}.$$

5.4.6.2 Loop Insertion

We shall define a "loop insertion operator" δ_x acting as a derivation (i.e. satisfying Leibniz's chain rule) on the functions $\psi, \tilde{\psi}, \phi, \tilde{\phi}, K, \hat{W}_n, \ldots,$ and turning them into functions of more variables, in particular it will be defined so that

$\delta_{x_{n+1}} \hat{W}_n(x_1, \ldots, x_n) = \hat{W}_{n+1}(x_1, \ldots, x_n, x_{n+1})$, whence the name "insertion operator".

In that purpose we first need to define formally the derivative $\partial/\partial s$ acting on a set of functions, without evaluating the derivatives. In other words we shall define some symbols like $\psi(x, s)$, $\partial_s \psi(x, s)$ and so on, assumed to be independent elements in a ring, and consider differential equations as linear subspaces in that ring. This is the notion of the "Picard-Vessiot" differential ring.

In this section, we no longer assume that $w(x, s) = \det \Psi(x, s)^{-1}$ be normalized to 1, because we will find that $\delta_y w(x, s) \neq 0$.

let $\mathbb{F}_n = \mathbb{C}(x_1, \ldots, x_n)$ be the field of rational functions of n variables, and $\mathbb{F}_\infty$ its projective limit $n \to \infty$.

Definition 5.4.9 Let $\mathcal{A}_1$ be the ring over $\mathbb{F}_1$ freely generated by the symbols $1, \psi(x_1, s), \tilde{\psi}(x_1, s), \phi(x_1, s), \tilde{\phi}(x_1, s), w(x_i, s), u(s)$, and their multiple derivatives $\partial^k/\partial s^k$, and quotiented by the relations

$$
\begin{aligned}
& w(x_1, s)\,(\psi(x_1, s)\tilde{\phi}(x_1, s) - \tilde{\psi}(x_1, s)\phi(x_1, s)) = 1 \\
& \partial_s \psi(x_1, s) = N\tilde{\psi}(x_1, s) \\
& \partial_s \phi(x_1, s) = N\tilde{\phi}(x_1, s) \\
& \partial_s \tilde{\psi}(x_1, s) = N(x_1 + 2u(s))\,\psi(x_1, s) \\
& \partial_s \tilde{\phi}(x_1, s) = N(x_1 + 2u(s))\,\phi(x_1, s) \\
& \partial_s w(x_1, s) = 0.
\end{aligned}
$$

(We leave the reader to check that these relations are compatible, in fact there is just to check that the last relation is compatible with the others, which is trivial).

$\mathcal{A}_1$ is called a "Picard-Vessiot differential ring".

We also define its n-variables analogue, $\mathcal{A}_n$ to be the Picard-Vessiot differential ring with n variables $x_1, \ldots, x_n$.

It is the differential ring over $\mathbb{F}_n$ freely generated by the symbols $\psi(x_i, s)$, $\tilde{\psi}(x_i, s), \phi(x_i, s), \tilde{\phi}(x_i, s), w(x_i, s)$, $i = 1, \ldots, n$, and $u(s)$, and their multiple derivatives with respect to s, and quotiented by the relations

$$
\begin{aligned}
& w(x_i, s)\,(\psi(x_i, s)\tilde{\phi}(x_i, s) - \tilde{\psi}(x_i, s)\phi(x_i, s)) = 1 \\
& \partial_s \psi(x_i, s) = N\tilde{\psi}(x_i, s) \\
& \partial_s \phi(x_i, s) = N\tilde{\phi}(x_i, s) \\
& \partial_s \tilde{\psi}(x_i, s) = N(x_i + 2u(s))\,\psi(x_i, s) \\
& \partial_s \tilde{\phi}(x_i, s) = N(x_i + 2u(s))\,\phi(x_i, s) \\
& \partial_s w(x_i, s) = 0.
\end{aligned}
$$

Let $\mathcal{A}_\infty$ its $n \to \infty$ projective limit.

In that ring, we define as before

$$\Psi(x_i,s) = \begin{pmatrix} \psi(x_i,s) & \phi(x_i,s) \\ \tilde\psi(x_i,s) & \tilde\phi(x_i,s) \end{pmatrix} \quad , \qquad \Psi(x_i,s)^{-1} = w(x_i,s) \begin{pmatrix} \tilde\phi(x_i,s) & -\phi(x_i,s) \\ -\tilde\psi(x_i,s) & \psi(x_i,s) \end{pmatrix}$$

$$M(x_i,s) = \Psi(x_i,s) E \Psi(x_i,s)^{-1} = w(x_i,s) \begin{pmatrix} -\phi(x_i,s)\tilde\psi(x_i,s) & \psi(x_i,s)\phi(x_i,s) \\ -\tilde\psi(x_i,s)\tilde\phi(x_i,s) & \psi(x_i,s)\tilde\phi(x_i,s) \end{pmatrix}$$

$$\mathcal{R}(x_i,s) = \frac{1}{N}\partial_s \Psi(x_i,s)\, \Psi(x_i,s)^{-1} = \begin{pmatrix} 0 & 1 \\ x_i + 2u(s) & 0 \end{pmatrix}.$$

Then, we define a loop insertion operator, as an operator $\delta : \mathcal{A}_n \to \mathcal{A}_{n+1}$, by:

Definition 5.4.10 We say that δ, acting in $\mathcal{A}_\infty$ is a "loop insertion operator" if it satisfies:

- δ sends $\mathcal{A}_n$ into $\mathcal{A}_{n+1}$ (we shall write it $\delta_{x_{n+1}}$).
- δ annihilates $\mathbb{F}_\infty$, i.e. $\forall\, n$, $\mathbb{F}_n \subset \operatorname{Ker}\delta$.
- δ is a derivation, it satisfies the Leibniz rule $\delta(fg) = f\delta g + g\delta f$.
- its action on the generators of $\mathcal{A}_n$ is

$$\delta_{x_{n+1}} \Psi(x_i,s) = \frac{M(x_{n+1},s)}{x_i - x_{n+1}}\, \Psi(x_i,s) + U(x_{n+1},s)\, \Psi(x_i,s)$$

$$\begin{aligned} \delta_x\, u(s) &= w(x,s)\,(\psi(x,s)\tilde\phi(x,s) + \tilde\psi(x,s)\phi(x,s)) \\ &= \frac{1}{N}\frac{\partial}{\partial s}\, w(x,s)\, \psi(x,s)\phi(x,s). \end{aligned} \tag{5.4.20}$$

where

$$U(x,s) = w(x,s) \begin{pmatrix} 0 & 0 \\ \psi(x,s)\phi(x,s) & 0 \end{pmatrix}$$

- it commutes with the derivation $\partial/\partial s$:

$$\left[\delta_{x_{n+1}}, \frac{\partial}{\partial s}\right] = 0,$$

this is equivalent to requiring

$$\delta_{x_{n+1}} \mathcal{R}(x_i,s) = \left[M(x_{n+1},s), \frac{\mathcal{R}(x_i,s) - \mathcal{R}(x_{n+1},s)}{x_i - x_{n+1}}\right] + [U(x_{n+1},s), \mathcal{R}(x_i,s)] + \frac{1}{N}\,\partial_s U(x_{n+1},s).$$

- The δ_{x_i}'s commute together:

$$[\delta_{x_i}, \delta_{x_j}] = 0$$

This last requirement is equivalent to demand that

$$\delta_x U(y) - \delta_y U(x) = [U(x), U(y)]$$

and

$$\delta_x \delta_y u(s) = \delta_y \delta_x u(s).$$

For an arbitrary ODE, the existence of an insertion operator is not automatic and not trivial. However, in our case, such an operator exists:

Proposition 5.4.2 *The insertion operator exists and is well defined*

Proof In order for the insertion operator to be well defined over $\mathcal{A}_n$, we need to know how it acts on a basis. So far we know the action of δ only on $\psi, \phi, \tilde{\psi}, \tilde{\phi}$ and u. First, we derive that

$$\delta_{x_{n+1}} w(x_i, s) = \frac{-1}{x_i - x_{n+1}} w(x_i, s).$$

In order to define $\delta_{x_{n+1}}$ over $\mathcal{A}_n$, we need to know how it acts on the derivatives of $\psi, \phi, \tilde{\psi}, \tilde{\phi}, u$ with respect to s. This can be done by commuting $\delta_{x_{n+1}}$ and ∂_s. Therefore all what we need to check is that ∂_s and δ commute.

The fact that ∂_s and δ commute when acting on $\Psi(x_i, s)$, is equivalent to verifying that

$$\delta_{x_{n+1}} \mathcal{R}(x_i, s) = \left[M(x_{n+1}, s), \frac{\mathcal{R}(x_i, s) - \mathcal{R}(x_{n+1}, s)}{x_i - x_{n+1}} \right] + [U(x_{n+1}, s), \mathcal{R}(x_i, s)] + \frac{1}{N} \partial_s U(x_{n+1}, s).$$

The right hand side is equal to:

$$\begin{aligned} &\left[M(x_{n+1}, s), \frac{\mathcal{R}(x_i, s) - \mathcal{R}(x_{n+1}, s)}{x_i - x_{n+1}} \right] \\ &+ [U(x_{n+1}, s), \mathcal{R}(x_i, s)] + \frac{1}{N} \partial_s U(x_{n+1}, s) \\ = &\left[M(x_{n+1}, s), \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right] \end{aligned}$$

$$+[U(x_{n+1},s),\begin{pmatrix}0&1\\0&0\end{pmatrix}]+\frac{1}{N}\,\partial_s U(x_{n+1},s)$$

$$=w(x_{n+1},s)\psi(x_{n+1},s)\phi(x_{n+1},s)\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$+w(x_{n+1},s)(\psi(x_{n+1},s)\tilde{\phi}(x_{n+1},s)+\tilde{\psi}(x_{n+1},s)\phi(x_{n+1},s))\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

$$-w(x_{n+1},s)\psi(x_{n+1},s)\phi(x_{n+1},s)\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\frac{1}{N}\,\partial_s U(x_{n+1},s)$$

$$=2\,w(x_{n+1},s)(\psi(x_{n+1},s)\tilde{\phi}(x_{n+1},s)+\tilde{\psi}(x_{n+1},s)\phi(x_{n+1},s))\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

which indeed coincides with

$$\delta_{x_{n+1}}\mathcal{R}(x_i,s)=2\,\delta_{x_{n+1}}u(s)\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

We also leave to the reader to verify that $[\delta_{x_i},\delta_{x_j}]=0$, which is equivalent to verify that

$$\delta_x U(y)-\delta_y U(x)=[U(x),U(y)]=0$$

and

$$\delta_x\delta_y u(s)=\delta_y\delta_x u(s).$$

□

The main properties of the insertion operator are

Proposition 5.4.3 *The kernel K is self–reproducing:*

$$\delta_{x'}K(x,x'')=-K(x,x')K(x',x'').$$

This implies that

$$\delta_{x_{n+1}}\hat{W}_n(x_1,\dots,x_n)=\hat{W}_{n+1}(x_1,\dots,x_n,x_{n+1})+\frac{\delta_{n,1}}{(x_1-x_2)^2}.$$

We also have that:

$$\delta_{x'}M(x)=\frac{[M(x'),M(x)]}{x-x'}+[U(x'),M(x)]$$

$$\delta_{x'}\mathcal{D}(x)=\frac{[M(x'),\mathcal{D}(x)]}{x-x'}+[U(x'),\mathcal{D}(x)]+\frac{1}{N}\,\frac{M(x')}{(x-x')^2}. \tag{5.4.21}$$

Proof Those relations are easy to derive from the definition of δ, we leave it as an exercise for the reader. □

5.4.6.3 Loop Equations

Theorem 5.4.2 (Loop Equations) *(Proved in [10]): the quantity*

$$\begin{aligned} & P_n(x; x_1, \ldots, x_n) \\ = \; & \hat{W}_{n+2,\,n.c.}(x, x, x_1, \ldots, x_n) \\ & + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{\hat{W}_n(x, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) - \hat{W}_n(x_1, \ldots, x_n)}{x - x_j} \end{aligned} \tag{5.4.22}$$

is a polynomial (over the ring $\mathcal{A}_n$) of the variable x:

$$P_n(x; x_1, \ldots, x_n) \in \mathcal{A}_n[x].$$

This assertion is highly non trivial because none of the terms in the right hand side are polynomials of x, they involve functions $\psi(x)$ for instance. Only this very combination is polynomial.

Proof The full proof can be found in [10, 11]. Let us give a hint of the proof.

The case $n = 0$ is very easy, one can explicitly compute:

$$\begin{aligned} P_0(x) &= \hat{W}_2(x, x) + \hat{W}_1(x)^2 \\ &= \lim_{x' \to x} \operatorname{Tr} \frac{M(x)M(x')}{(x - x')^2} - \frac{1}{(x - x')^2} + \hat{W}_1(x)^2 \\ &= \lim_{x' \to x} \frac{-1 + \operatorname{Tr} M(x)^2}{(x - x')^2} + \frac{\operatorname{Tr} M(x)M'(x)}{x' - x} + \frac{1}{2} \operatorname{Tr} M(x)M(x)'' + \hat{W}_1(x)^2. \end{aligned}$$

Observe that $\operatorname{Tr} M(x)^2 = \operatorname{Tr} M(x) = 1$ and by acting with ∂_x, that $\operatorname{Tr} M(x)M'(x) = 0$, therefore

$$P_0(x) = \frac{1}{2} \operatorname{Tr} M(x)M(x)'' + \hat{W}_1(x)^2.$$

Then, use that

$$M''(x) = -N[\mathcal{D}(x, s)', M(x)] + N^2[\mathcal{D}(x, s), [\mathcal{D}(x, s), M(x)]],$$

and thus

$$P_0(x) = \frac{N^2}{2} \operatorname{Tr} M(x)[\mathcal{D}(x,s), [\mathcal{D}(x,s), M(x)]] + \hat{W}_1(x)^2$$
$$= N^2 \left(\operatorname{Tr} M(x)^2 \mathcal{D}(x,s)^2 - \operatorname{Tr} (\mathcal{D}(x,s)M(x))^2 + (\operatorname{Tr} M(x)\mathcal{D}(x,s))^2 \right).$$

Observe that any 2×2 matrix A satisfies:

$$A^2 = A \operatorname{Tr} A - \mathrm{Id} \det A,$$

Applying it to $A = \mathcal{D}(x,s)$ gives (since $\operatorname{Tr} \mathcal{D}(x,s) = 0$)

$$\mathcal{D}(x,s)^2 = -\mathrm{Id} \det \mathcal{D}(x,s),$$

and thus

$$\operatorname{Tr} M(x)^2 \mathcal{D}(x,s)^2 = -2 \det \mathcal{D}(x,s) \operatorname{Tr} M(x)^2 = -2 \det \mathcal{D}(x,s).$$

And applying it to $A = M(x)\mathcal{D}(x,s)$ gives (since $\det M(x) = 0$)

$$(M(x)\mathcal{D}(x,s))^2 = M(x)\mathcal{D}(x,s) \operatorname{Tr} M(x)\mathcal{D}(x,s),$$

and thus

$$\operatorname{Tr} (M(x)\mathcal{D}(x,s))^2 = (\operatorname{Tr} M(x)\mathcal{D}(x,s))^2.$$

Eventually:

$$P_0(x) = -N^2 \det \mathcal{D}(x,s) = \frac{N^2}{2} \operatorname{Tr} \mathcal{D}(x,s)^2$$

which is indeed a polynomial of x.

The cases $n \geq 1$ can be obtained from $n = 0$ by recursively applying δ_{x_i}. Indeed, we have:

$$P_{n+1}(x; x_1, \ldots, x_{n+1}) = \delta_{x_{n+1}} P_n(x; x_1, \ldots, x_n) - \frac{\partial}{\partial x_{n+1}} \frac{\hat{W}_n(x_{n+1}, x_1, \ldots, x_n)}{x - x_{n+1}}.$$

Thus, observing from Eq. (5.4.21) that $\delta_{x_{n+1}} \mathcal{D}(x)$ is a rational fraction of x, containing only coefficients in $\mathcal{A}_n$, we obtain by recursion on n that $P_n(x; x_1, \ldots, x_n) \in \mathcal{A}_n(x)$ is a rational function of x. Moreover P_n can have no other pole than $x = \infty$, so it must be a polynomial of x. □

For example, we have that

$$\begin{aligned}
P_1(x;x_1) &= \delta_{x_1} P_0(x) - \frac{\partial}{\partial x_1}\,\frac{\hat{W}_1(x_1)}{x-x_1} \\
&= \frac{N^2}{2}\,\delta_{x_1}\left(\mathrm{Tr}\,\mathcal{D}(x,s)^2\right) - N\frac{\partial}{\partial x_1}\,\frac{1}{x-x_1}\;\mathrm{Tr}\,\mathcal{D}(x_1,s)\,M(x_1) \\
&= N^2\;\mathrm{Tr}\,\mathcal{D}(x,s)\,\delta_{x_1}\mathcal{D}(x,s) - N\,\frac{\mathrm{Tr}\,\mathcal{D}(x_1,s)\,M(x_1)}{(x-x_1)^2} \\
&\quad -N\,\frac{\mathrm{Tr}\,\mathcal{D}'(x_1,s)\,M(x_1)}{x-x_1} - N\,\frac{\mathrm{Tr}\,\mathcal{D}(x_1,s)\,M'(x_1)}{x-x_1} \\
&= N^2\;\mathrm{Tr}\,\mathcal{D}(x,s)\,\left(\frac{[M(x_1),\mathcal{D}(x,s)]}{x-x_1} + [U(x_1),\mathcal{D}(x,s)] + \frac{1}{N}\,\frac{M(x_1)}{(x-x_1)^2}\right) \\
&\quad -N\,\frac{\mathrm{Tr}\,\mathcal{D}(x_1,s)\,M(x_1)}{(x-x_1)^2} - N\,\frac{\mathrm{Tr}\,\mathcal{D}'(x_1,s)\,M(x_1)}{x-x_1} \\
&\quad +N^2\,\frac{\mathrm{Tr}\,\mathcal{D}(x_1,s)\,[\mathcal{D}(x_1,s),M(x_1)]}{x-x_1} \\
&= N\;\mathrm{Tr}\,\mathcal{D}(x,s)\;\frac{M(x_1)}{(x-x_1)^2} - N\,\frac{\mathrm{Tr}\,\mathcal{D}(x_1,s)\,M(x_1)}{(x-x_1)^2} - N\,\frac{\mathrm{Tr}\,\mathcal{D}'(x_1,s)\,M(x_1)}{x-x_1}
\end{aligned}$$

and one observes that $(\mathcal{D}(x,s) - \mathcal{D}(x_1,s) - (x-x_1)\mathcal{D}'(x_1,s))/(x-x_1)^2$ is indeed a polynomial of x.

5.4.7 Example: (1,2) Minimal Model, the Airy Kernel

Let us write the $(1,2)$ model, i.e. $m=0$. We have:

$$P = d \qquad , \qquad Q = d^2 - 2u$$

the string equation is:

$$[P,Q] = -\frac{2}{N}\,\dot{u} = \frac{1}{N}$$

i.e.

$$u(s) = -\frac{s}{2} = \tilde{t}_{-1}$$

(where we have reabsorbed a possible integration constant in a redefinition of s and $\tilde{t}_{-1}$).

The Lax pair is:

$$\mathcal{D}(x,s) = \begin{pmatrix} 0 & 1 \\ x-s & 0 \end{pmatrix} \quad , \quad \mathcal{R}(x,s) = \begin{pmatrix} 0 & 1 \\ x-s & 0 \end{pmatrix}$$

The differential system is:

$$\frac{1}{N}\frac{d}{dx}\Psi(x,s) = -\begin{pmatrix} 0 & 1 \\ x-s & 0 \end{pmatrix}\Psi(x,s)$$

i.e.

$$\psi'' = N^2(x-s)\psi$$

whose solution is the Airy function (The Airy function is solution of $\mathrm{Ai}''(x) = x\,\mathrm{Ai}(x)$, see textbooks on classical functions [1]) rescaled by $N^{2/3}$:

$$\psi(x,s) = \mathrm{Ai}(N^{\frac{2}{3}}(x-s)) \quad , \quad \tilde{\psi}(x,s) = -N^{-1/3}\,\mathrm{Ai}'(N^{\frac{2}{3}}(x-s))$$

and the other independent solution is the “BAiry” function [1]:

$$\phi(x,s) = -\pi\,N^{1/3}\mathrm{Bi}(N^{\frac{2}{3}}(x-s)) \quad , \quad \tilde{\phi}(x,s) = \pi\,\mathrm{Bi}'(N^{\frac{2}{3}}(x-s))$$

where in the literature, Bi is normalized so that $\mathrm{Ai}\,\mathrm{Bi}' - \mathrm{Ai}'\,\mathrm{Bi} = 1/\pi$.

The Christoffel–Darboux kernel is thus (up to rescalings by factors $N^{-2/3}$) the famous Airy kernel [82]:

$$N^{-2/3}\,K(s+N^{-2/3}x_1, s+N^{-2/3}x_2) = \pi\,\frac{\mathrm{Ai}(x_1)\mathrm{Bi}'(x_2) - \mathrm{Ai}'(x_1)\mathrm{Bi}(x_2)}{x_1 - x_2} = K_{\mathrm{Airy}}(x_1, x_2)$$

and this is why the $(1,2)$ minimal model coupled to gravity, is sometimes called the “Airy model”.

Remark 5.4.6 The Airy kernel plays a very important role in many problems, in particular in the universal laws of extreme eigenvalues, related to the Tracy-Widom law [82]. We mention this, not as a mere coincidence, but because, as we have seen in Chap. 2 counting maps is closely related to random matrices, and the asymptotic limit is closely related to the eigenvalue statistics at the end of the spectrum, i.e. to the extreme eigenvalues.

So it is very natural that large maps can be related to Tracy-Widom law of extreme eigenvalues.

Let us parametrize

$$\mathrm{Ai}(x) = \frac{\sqrt{f(x)}}{2\sqrt{\pi}}\,\mathrm{e}^{\frac{-2}{3}x^{3/2} - \int_\infty^x \left(\frac{1}{f(x')} - \sqrt{x'}\right)dx'} \quad , \quad \mathrm{Bi}(x) = \frac{\sqrt{f(x)}}{\sqrt{\pi}}\,\mathrm{e}^{\frac{2}{3}x^{3/2} + \int_\infty^x \left(\frac{1}{f(x')} - \sqrt{x'}\right)dx'}.$$

The Airy equation $\mathrm{Ai}''(x) = x\,\mathrm{Ai}(x)$, implies that $f(x)$ satisfies the differential equation

$$xf^2 + \frac{f'^2}{4} - 1 = \frac{1}{2} ff''$$

and taking the derivative again, and after dividing by f'

$$\frac{1}{2} f''' = 2xf' + f.$$

One easily finds from this linear equation and from the leading behavior $f \sim 1/\sqrt{x}$, that:

$$f(x) = \frac{1}{\sqrt{x}} + \sum_{k=1}^{\infty} \frac{(6k-1)!!}{2^{5k}\,3^k\,k!}\,x^{-3k-\frac{1}{2}}.$$

Then, from Eq. (5.4.16) we compute the 1-point function

$$\begin{aligned}
\hat{W}_1(s + N^{-2/3}x) &= \pi\, N^{2/3}\, \left(\mathrm{Ai}'(x)\mathrm{Bi}'(x) - x\,\mathrm{Ai}(x)\mathrm{Bi}(x)\right) \\
&= N^{2/3}\, \frac{1}{2\,f(x)} \left(\frac{f'^2(x)}{4} - 1 - x f(x)^2\right) \\
&= N^{2/3} \left(\frac{1}{4} f''(x) - x f(x)\right).
\end{aligned}$$

Taking the derivative again implies $N^{-4/3}\,\hat{W}_1'(s + N^{-2/3}x) = -f(x)/2$, and therefore

$$\hat{W}_1(x) = -N\sqrt{x-s} + \sum_{k=1}^{\infty} \frac{(6k-3)!!}{2^{5k}\,3^k\,k!}\,(x-s)^{-3k+1/2}\,N^{1-2k}.$$

We may write it:

$$\hat{W}_1(x) = \sum_{g=0}^{\infty} N^{1-2g}\,\hat{W}_1^{(g)}(x)$$

with

$$\hat{W}_1^{(0)}(x) = -\sqrt{x-s} \qquad , \qquad \hat{W}_1^{(g)}(x) = \frac{(6g-3)!!}{2^{5g}\, 3^g\, g!}\,(x-s)^{-3g+1/2}.$$

For the Airy system, the polynomial of Theorem 5.4.2 is simply:

$$P_n(x) = (x-s)\,\delta_{n,0}.$$

5.4.8 *Tau Function*

The notion of Isomonodromic Tau-function was defined for any Lax pair by Jimbo-Miwa [51, 52]. In this book we shall not study in details why the Tau-function is a useful notion, we just mention that indeed it encodes most of the properties of an integrable system, it is a very fundamental notion. We refer the reader to literature on integrable systems for learning more about Tau–functions and their utility, see for instance [8, 49, 59, 60].

Let us describe how it is defined in our case. In order to define the Tau-function, we need to consider the large x formal asymptotic expansion of $\Psi(x)$.

First we define

$$T(x) = \left(\int^x Y(x')\,dx'\right)_+$$

where $Y(x) = \sqrt{A^2(x) + B(x)C(x)}$ is (up to a sign) the eigenvalue of $\mathcal{D}(x)$, and $()_+$ means the strictly positive part of the Laurent series in $\sqrt{x}$ (and thus it is independent of a choice of integration constant). By an easy induction, one sees that the large x formal asymptotics of $\Psi(x)$ is of the form

$$\Psi(x) \sim \frac{1}{\sqrt{2}}\begin{pmatrix} x^{-1/4} & -x^{-1/4} \\ x^{1/4} & x^{1/4}\end{pmatrix}\tilde{\psi}(x)\,\mathrm{e}^{-N\sigma_3 T(x)} \qquad , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix},$$

and where $\tilde{\psi}(x) = \mathrm{Id} + O(1/\sqrt{x})$ is an analytical function of $\sqrt{x}$ near ∞:

$$\tilde{\psi}(x) = \mathrm{Id} + \frac{v}{\sqrt{x}}\sigma_3 + \frac{v^2}{2x}\mathrm{Id} + \frac{u}{2x}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} + O(x^{-3/2}) \qquad , \qquad \frac{1}{N}\,\dot{v} = u, \tag{5.4.23}$$

Miwa-Jimbo [51, 52] define the Tau-function $\tau(s)$ and its log, the free energy function $\mathcal{F}(s) = \ln \tau(s)$ such that:

$$\frac{\partial \mathcal{F}}{\partial s} = -N \operatorname*{Res}_{x\to\infty} \mathrm{Tr}\,\left(\Psi(x)^{-1}\,\Psi'(x)\,\sigma_3\right)\,\frac{\partial T(x)}{\partial s}\,dx.$$

First notice that

$$\begin{aligned}\mathrm{Tr}\,\left(\Psi(x)^{-1}\,\Psi'(x)\,\sigma_3\right) &= \mathrm{Tr}\,\begin{pmatrix}\psi'\tilde{\phi}+\phi'\tilde{\psi} & -\psi'\phi-\phi'\psi\\ \tilde{\psi}'\tilde{\phi}+\tilde{\phi}'\tilde{\psi} & -\psi\tilde{\phi}'-\phi\tilde{\psi}'\end{pmatrix}\\ &= \psi'\tilde{\phi}-\phi\tilde{\psi}'+\phi'\tilde{\psi}-\psi\tilde{\phi}'\\ &= 2\,\hat{W}_1(x)\end{aligned}$$

which is a Laurent series in $\sqrt{x}$.

Therefore the definition of the Tau function is equivalent to

$$\dot{\mathcal{F}} = -2N\,\operatorname*{Res}_{x\to\infty}\,\hat{W}_1(x)\,\dot{T}(x)\,dx.$$

Then, write $Y(x)=\sqrt{-\det\mathcal{D}}=\sqrt{\frac{1}{2}\,\mathrm{Tr}\,\mathcal{D}^2}$, so that

$$2Y(x)\,\frac{\partial Y(x)}{\partial s} = \mathrm{Tr}\,\mathcal{D}\dot{\mathcal{D}} = \mathrm{Tr}\,\mathcal{D}\,(-\mathcal{R}'-N\,[\mathcal{D},\mathcal{R}]) = -\,\mathrm{Tr}\,\mathcal{D}\,\mathcal{R}' = -B(x)$$

i.e.

$$\begin{aligned}\frac{\partial Y(x)}{\partial s} = -\frac{B(x)}{2\,Y(x)} &= -\frac{B}{2\,\sqrt{B\,C+A^2}}\\ &= -\frac{1}{2\,\sqrt{(x+2u)-\frac{1}{2N^2}\frac{\ddot{B}}{B}+\frac{1}{4N^2}\frac{\dot{B}^2}{B^2}}}\\ &= -\frac{1}{2\,\sqrt{x+2u}}\,(1+O(1/x^2)).\end{aligned}$$

Then, since $T(x)=\left(\int^x Y(x')dx'\right)_+$, by integration we find

$$\frac{\partial T(x)}{\partial s} = -\,\sqrt{x}.$$

In our case this leads to

$$N^{-1}\,\partial\mathcal{F}/\partial s = 2\,\operatorname*{Res}_{x\to\infty}\,\hat{W}_1(x)\,\sqrt{x}\,dx$$

where $\hat{W}_1(x)=\psi'(x)\tilde{\phi}(x)-\tilde{\psi}'(x)\phi(x)$. Taking another derivative with respect to s, and using the $\partial/\partial s$ equation satisfied by $\Psi(x,s)$, we get

$$\begin{aligned}N^{-1}\,\partial\hat{W}_1(x)/\partial s &= \tilde{\psi}'(x)\tilde{\phi}(x)+\psi'(x)(x+2u(s))\phi(x)\\ &\quad -((x+2u(s))\psi(x))'\phi(x)-\tilde{\psi}'(x)\tilde{\phi}(x)\\ &= -\psi(x)\phi(x),\end{aligned}$$

and therefore:

$$N^{-2}\,\partial^2\mathcal{F}/\partial s^2 = -2\,\operatorname*{Res}_{x\to\infty} \psi(x)\phi(x)\sqrt{x}\,dx.$$

From the asymptotic expansion Eq. (5.4.23), one has

$$\psi(x)\phi(x) \sim -\frac{1}{2\sqrt{x}}\,(1-\frac{u}{x}+O(x^{-3/2}))$$

and thus

$$N^{-2}\,\partial^2\mathcal{F}/\partial s^2 = \operatorname*{Res}_{x\to\infty}(1-\frac{u}{x}+O(x^{-3/2}))\;dx = u.$$

And therefore we find

$$N^{-2}\,\partial^2\mathcal{F}/\partial s^2 = u(s).$$

Therefore we have just recovered the Its-Matveev's equation [49]:

Theorem 5.4.3 *The Tau-function of the integrable system defined by the Lax pair* $(\mathcal{D},\mathcal{R})$*, is such that* $u(s)$ *is the second derivative of* $\ln\tau$*:*

$$\tau(s) = \mathrm{e}^{N^2\,h(s)} \qquad , \quad \frac{\partial^2\;h(s)}{\partial\,s^2} = u(s),$$

and $u(s)$ *is solution of the Gelfan-Dikii equation (5.4.5) of Lemma 5.4.2:*

$$\sum_{j=0}^{m}\tilde{t}_j R_{j+1}(u) = s.$$

The Tau–function has many properties, which can be found in textbooks and classical works on integrable systems [8, 49, 59, 60], but which are beyond the scope of the present book. In some sense, the Tau–function is the most fundamental function characterizing an integrable system, it contains all the information about the integrable system.

Here, for the integrable system satisfied by the $(2m+1,2)$ minimal model, the Tau function can be computed by integrating twice the function $u(s)$ solution of a Painlevé type equation.

5.4.9 Large N Limit

Our goal is to compare the minimal model's Tau function with the generating function of large maps introduced in Sect. 5.1.3.1, which is by definition a formal power series of $1/N$, $\ln \tilde{Z} = \sum_g N^{2-2g}\, \tilde{F}_g$, where $\tilde{F}_g$ is the asymptotic generating function of large maps. The conjecture of topological gravity (proved below) is that:

$$\tau(s) \quad \overset{?}{=} \quad \tilde{Z}.$$

Therefore, we need to study the formal large N expansion of the minimal model (p, q).

The large N limit for minimal models, is also called "dispersionless" limit. The parameter $1/N$, which we introduced as the coefficient of the identity in the commutator $[P, Q] = \frac{1}{N}\mathrm{Id}$, is called the **"dispersion" parameter**. In the large N limit P and Q tend to commute, $1/N$ plays the role of $\hbar$ in quantum mechanics, and the large N limit is a "classical limit".

Intuitively, in this limit, the operators P and Q will be replaced by functions, also the operator d will be replaced by a function z, and thus P and Q will be replaced by some functions of z and s.

Taking those observations as a guideline, in analogy with $Q = d^2 - 2u(s)$, and $P = d^p + \ldots$, we define:

Definition 5.4.11 We define two functions $x(z, s)$ and $y(z, s)$ (which will be, as we shall see later, in some sense the large N limit of Q and P), polynomials in z, of respective degree 2 and p, of the form:

$$x(z, s) = z^2 - 2u_0(s) \qquad , \qquad y(z, s) = z^p + O(z^{p-2}) \ .$$

which we require to satisfy the following Poisson bracket equation (the "classical limit" of the string equation $[P, Q] = 1/N$):

$$\{y, x\} \overset{\text{def}}{=} \frac{\partial y}{\partial z}\frac{\partial x}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial x}{\partial z} = 1\,. \tag{5.4.24}$$

Proposition 5.4.4 *the general solution of this Poisson equation is:*

$$\begin{aligned} x(z, s) &= z^2 - 2u_0(s) \\ y(z, s) &= \sum_{j=0}^{m} \tilde{t}_j \left(z^{2j+1} \left(1 - \frac{2u_0(s)}{z^2}\right)^{j+1/2} \right)_+ + \sum_{j=0}^{m-1} c_j\; x(z, s)^j, \\ &= \sum_{j=0}^{m} \tilde{t}_j\, Q_j(z) + \sum_{j=0}^{m-1} c_j\; x(z, s)^j, \end{aligned} \tag{5.4.25}$$

where $()_+$ means the positive part of the large z Laurent series expansion, where $Q_j(z)$ was introduced in Eq. (5.2.3), and where the function $u_0(s)$ has to satisfy the algebraic equation

$$\mathcal{P}(u_0(s)) = \sum_{j=0}^{m} \tilde{t}_j \, (-u_0(s)/2)^{j+1} \, \frac{(2j+1)!}{j! \, (j+1)!} = \frac{s}{4}. \tag{5.4.26}$$

From now on, we shall always consider $c_j = 0$.

Proof It is very similar to the proof of Lemma 5.4.2, we leave it as an exercise for the reader.

We just mention that once we have seen that the function $y(z, s)$ must be of the form Eq. (5.4.25), the Poisson equation $\{y, x\} = 1$, written at $z = 0$ reduces to:

$$\dot{u}_0(s) \, y'(0, s) = \frac{-1}{2},$$

i.e.

$$\sum_{j=0}^{m} \tilde{t}_j \, \dot{u}_0(s) \, (-u_0(s)/2)^j \, \frac{(2j+1)!}{(j!)^2} = -\frac{1}{2}$$

which can be integrated with respect to s and gives a polynomial equation for $u_0(s)$:

$$\mathcal{P}(u_0(s)) = \sum_{j=0}^{m} \tilde{t}_j \, (-u_0(s)/2)^{j+1} \, \frac{(2j+1)!}{j! \, (j+1)!} = \frac{s}{4}$$

which is clearly the classical limit of Eq. (5.4.5) (i.e. it coincides with Eq. (5.4.5) by removing all derivative terms). In other words, formally in the classical limit, the non-linear differential equation (5.4.5) for $u(t)$, becomes an algebraic equation for $u_0(s)$.

This is the same equation which we encountered for large maps in Eq. (5.3.4). □

For example, for pure gravity $m = 1$ we have the classical limit of Eq. (5.4.7):

$$4\mathcal{P}(u_0) = 3 \, u_0^2 - 2\tilde{t}_0 \, u_0 = s. \tag{5.4.27}$$

5.4.10 Topological Expansion

In order to compare minimal models with large maps, we now look for a function $u(s)$ which is a formal series in $1/N$.

Proposition 5.4.5 *The formal series in $1/N$ solution $u(s)$ to the string equation (5.4.5), can be expanded as an N^{-2} power series starting with u_0 (solution of $\mathcal{P}(u_0) = s/4$) as a leading order:*

$$u(s) = u_0(s) + \sum_k N^{-2k}\, u_k(s)$$

and where all coefficients u_k are rational functions of u_0 (their denominator is a power of $\mathcal{P}'(u_0)$):

$$u_k \in \mathbb{C}(u_0).$$

Proof One notices that the string equation (5.4.5) involves only N^2 and therefore the expansion is in powers of N^2 instead of N. Almost by definition of u_0, we see that $u_0(s)$ satisfies the string equation (5.4.5) at $N = \infty$, and therefore is the first term of $u(s)$.

Since

$$\mathcal{P}(u_0) = s/4$$

we have

$$\dot{u}_0 = \frac{1}{4\mathcal{P}'(u_0)} \quad , \; \ddot{u}_0 = \frac{-\mathcal{P}''(u_0)}{16\,(\mathcal{P}'(u_0))^3} \quad , \; \dddot{u}_0 = \frac{3\mathcal{P}''(u_0)^2 - \mathcal{P}'(u_0)\mathcal{P}'''(u_0)}{64\,(\mathcal{P}'(u_0))^5} \quad , \; \dots$$

and in general, any derivative of u_0 with respect to s can be written as a rational function of u_0, whose denominator is a power of $\mathcal{P}'(u_0)$. Solving the string equation recursively involves derivatives of u_0, and thus each u_k is a rational function of u_0 whose denominator is a power of $\mathcal{P}'(u_0)$. □

Using the expression of Gelfand-Dikii polynomials Eq. (5.4.4), the equation satisfied by u to order $O(1/N^4)$ is

$$\frac{s}{4} = \mathcal{P}(u) - \frac{\ddot{u}}{12\,N^2}\,\mathcal{P}''(u) - \frac{\dot{u}^2}{24\,N^2}\,\mathcal{P}'''(u) + O(1/N^4)$$

and thus we get

$$u_1 = \frac{\ddot{u}_0}{12}\,\frac{\mathcal{P}''(u_0)}{\mathcal{P}'(u_0)} + \frac{\dot{u}_0^2}{24}\,\frac{\mathcal{P}'''(u_0)}{\mathcal{P}'(u_0)} = \frac{1}{24}\,\left(\frac{\ddot{u}_0^2}{\dot{u}_0^2} - \frac{u_0{}^{\dddot{}}}{\dot{u}_0}\right).$$

We could easily obtain $u_2, u_3, \dots$ by expanding to further orders.

5.4.10.1 Topological Expansion for the Tau-Function

Proposition 5.4.6 *We have:*

From the $1/N^2$ *expansion of* $u(s)$*, we get that the free energy* $\mathcal{F}(s) = \ln\tau(s)$ *such that* $u = \frac{1}{N^2}\ddot{\mathcal{F}}$*, also has a* $1/N^2$ *expansion:*

$$\ln\tau = \mathcal{F} = \sum_{g=0}^{\infty} N^{2-2g}\mathcal{F}_g(u_0) \qquad , \qquad \ddot{\mathcal{F}}_g = u_g. \tag{5.4.28}$$

In particular we have

$$\mathcal{F}_0 = -4\sum_{j,k}\tilde{t}_j\,\tilde{t}_k\,(-u_0/2)^{k+j+3}\,\frac{j+k+4}{j+k+3}\,\frac{(2j+2)!}{j!\,(j+2)!}\,\frac{(2k+2)!}{k!\,(k+1)!}$$

$$\mathcal{F}_1 = \frac{-1}{24}\,\ln(-2\,\dot{u}_0) = \frac{1}{24}\,\ln(\,y'(0,s)).$$

Proof We propose it as an exercise at the end of this chapter. $\mathcal{F}_1$ can be easily derived from the expression of u_1 above, and for $\mathcal{F}_0$, see the hints in the exercise. □

5.4.10.2 Topological Expansion for the Differential Systems

Since the coefficients of the Lax matrix $\mathcal{D}(x,s)$ depend on $u(s)$ and its derivatives, it has a formal $1/N$ expansion:

$$\mathcal{D}(x,s) = \begin{pmatrix} A(x,s) & B(x,s) \\ C(x,s) & -A(x,s) \end{pmatrix} = \sum_g N^{-k}\mathcal{D}^{(k)}(x,s)$$

where

$$B(x,s) = \sum_k N^{-2k}\,B_{2k}(x,s)$$

$$C(x,s) = (z^2+2u-2u_0)B(x,s) - \frac{1}{2N^2}\,\ddot{B}(x,s) = \sum_k N^{-2k}\,C_{2k}(x,s)$$

$$A(x,s) = \frac{-1}{2N}\,\dot{B}(x,s) = \sum_k N^{-2k-1}\,A_{2k+1}(x,s),$$

and notice that B_{2k}, and thus C_{2k} and A_{2k+1} are polynomials of x, i.e. polynomials of $z^2 = x + 2u_0$.

To leading order we have:

$$\mathcal{D}^{(0)}(x,s) = \begin{pmatrix} 0 & \overline{B}(x,u_0) \\ (x+2u_0)\,\overline{B}(x,u_0) & 0 \end{pmatrix} \tag{5.4.29}$$

$$\overline{B}(x,u_0) = \sum_{j=0}^{m}\sum_{k=0}^{j} \tilde{t}_j x^{j-k}\, u_0^k\, \frac{(-1)^k\ (2k-1)!!}{k!}.$$

The determinant of $\mathcal{D}^{(0)}(x,s)$ is:

$$\det \mathcal{D}^{(0)}(x,s) = -\,(z\,\overline{B}(z^2-2u_0,u_0))^2.$$

This means that the eigenvalues of $\mathcal{D}^{(0)}(x,s)$ are $\pm z\,\overline{B}(z^2-2u_0,u_0)$.

Notice that $z\,\overline{B}(z^2-2u_0,u_0)$ is precisely the function $y(z,s)$ of Proposition 5.4.4, in Eq. (5.4.25).

Definition 5.4.12 The "classical spectral curve" is the eigenvalue locus of the classical limit $\mathcal{D}^{(0)}(x)$ of the Lax matrix.

If we parametrize x as $x = z^2 - 2u_0$, the eigenvalues of $\mathcal{D}^{(0)}(x,s)$ are:

$$y = \pm\, y(z,s)$$

where $y(z,s)$ is the function defined in Eq. (5.4.25).

Written in a parametric form where $u_0 = u_0(s)$, the classical spectral curve is thus:

$$\mathcal{E}_{(2m+1,2)} = \begin{cases} x(z,s) = z^2 - 2u_0 \\ y(z,s) = \sum_j \tilde{t}_j Q_j(z) = \sum_j \sum_l \tilde{t}_j\, z^{2j+1-2l}\, (-u_0/2)^l\ \frac{(2j+1)!}{j!}\ \frac{(j-l)!}{l!\,(2j+1-2l)!} \end{cases}. \tag{5.4.30}$$

Remark 5.4.7 It is important to notice that it is a genus 0 hyperelliptical curve, which is equivalent to saying that it can be parametrized by a complex variable z (higher genus would be parametrized by a variable z living on a Riemann surface), and which is equivalent to saying that the polynomial y^2, written as a polynomial in x, has only one simple zero, located at $x = -2u_0$, all the other zeroes are double zeroes:

$$y^2 = z^2\,(\overline{B}(x,u_0))^2 = (x+2u_0)\,(\overline{B}(x,u_0))^2.$$

Remark 5.4.8 It is also the same curve as the blown up spectral curve considered in Sect. 5.2. This is of course not an accident, this is an indication that indeed, large maps are related to the Tau-function of the $(p,2)$ minimal model. Our goal is to show that not only the large N limits coincide, but the full expansion.

5.4.11 WKB Expansion

Similarly, we can look for a formal large N asymptotic expansion of the solutions $\psi(x,s)$ of the differential system. To leading order, it takes the WKB form:

$$\psi(x,s) \sim \frac{\mathrm{e}^{-N\int_{-2u_0}^{x} y dx}}{\sqrt{2}\,(-x-2u_0)^{\frac{1}{4}}} \left(1+\sum_k N^{-k}\psi_k(x,s)\right)$$

$$\tilde{\psi}(x,s) \sim \frac{1}{\sqrt{2}}\mathrm{e}^{-N\int_{-2u_0}^{x} y dx}\,(x+2u_0)^{\frac{1}{4}} \left(1+\sum_k N^{-k}\tilde{\psi}_k(x,s)\right)$$

and we recall that $z=(x+2u_0)^{\frac{1}{2}}$. The BKW expansion of the other solutions ϕ and $\tilde{\phi}$, are obtained by changing $N \to -N$. For the matrix Ψ, we have:

$$\Psi(x,s) \sim \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{z}} & -\frac{1}{\sqrt{z}} \\ \sqrt{z} & \sqrt{z} \end{pmatrix} \hat{\Psi}(x,s)\,\mathrm{e}^{-N\sigma_3\int_{-2u_0}^{x} y dx}$$

where $\sigma_3 = \mathrm{diag}(1,-1)$, and

$$\hat{\Psi}(x,s) = \mathrm{Id} + \sum_{k=1}^{\infty} N^{-k}\,\Psi_k(x,s).$$

where each $\Psi_k(x,s)$ is a square matrix independent of N:

$$\Psi_k(x,s) = \begin{pmatrix} \psi_k(x,s) & \phi_k(x,s) \\ \tilde{\psi}_k(x,s) & \tilde{\phi}_k(x,s) \end{pmatrix}.$$

The fact that Ψ satisfies the differential systems $\Psi' = -N\mathcal{D}\Psi$ and $\dot{\Psi} = N\mathcal{R}\Psi$ imply for $\hat{\Psi}$:

$$\hat{\Psi}' = Ny\,\hat{\Psi}\,\sigma_3 - \frac{N}{2z}\begin{pmatrix} Bz^2+C & Bz^2-C-2Az \\ C-Bz^2-2Az & -Bz^2-C \end{pmatrix}\hat{\Psi} - \frac{1}{4z^2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\Psi}$$

$$\dot{\hat{\Psi}} = -Nz\,\hat{\Psi}\,\sigma_3 + \frac{N}{z}\begin{pmatrix} z^2+u-u_0 & u_0-u \\ u-u_0 & -z^2+u_0-u \end{pmatrix}\hat{\Psi} - \frac{\dot{u}_0}{2z^2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\Psi}.$$

Let us expand it into powers of N, we have:

$$B(x,s) = \sum_k N^{-2k}\,B_{2k}(x,s)$$

$$C(x,s) = (z^2 + 2u - 2u_0)B(x,s) - \frac{1}{2N^2}\ddot{B}(x,s) = \sum_k N^{-2k}\, C_{2k}(x,s)$$

$$A(x,s) = \frac{-1}{2N}\dot{B}(x,s) = \sum_k N^{-2k-1}\, A_{2k+1}(x,s),$$

and notice that B_{2k}, and thus C_{2k} and A_{2k+1} are polynomials of x, i.e. polynomials of z^2. Notice that

$$C_0(x,s) = z^2\, B_0(x,s) = z\, y.$$

that gives

$$\begin{aligned}
\psi_k' &= -\frac{1}{2z}\sum_{j\geq 1}(z^2 B_{2j} + C_{2j})\psi_{k+1-2j} - \frac{1}{2z}\sum_{j\geq 1}(z^2 B_{2j} - C_{2j})\tilde{\psi}_{k+1-2j} \\
&\quad + \sum_{j\geq 0} A_{2j+1}\tilde{\psi}_{k-2j} - \frac{1}{4z^2}\tilde{\psi}_k \\
\tilde{\psi}_k' &= 2y\tilde{\psi}_{k+1} + \sum_{j\geq 0} A_{2j+1}\psi_{k-2j} + \frac{1}{2z}\sum_{j\geq 1}(z^2 B_{2j} - C_{2j})\psi_{k+1-2j} \\
&\quad + \frac{1}{2z}\sum_{j\geq 1}(z^2 B_{2j} + C_{2j})\tilde{\psi}_{k+1-2j} - \frac{1}{4z^2}\psi_k \\
\dot{\psi}_k &= \frac{1}{z}\sum_{j\geq 1} u_j\,(\psi_{k+1-2j} - \tilde{\psi}_{k+1-2j}) - \frac{\dot{u}_0}{2z^2}\tilde{\psi}_k \\
\dot{\tilde{\psi}}_k &= -2z\tilde{\psi}_{k+1} + \frac{1}{z}\sum_{j\geq 1} u_j\,(\psi_{k+1-2j} - \tilde{\psi}_{k+1-2j}) - \frac{\dot{u}_0}{2z^2}\psi_k
\end{aligned} \tag{5.4.31}$$

and we have similar equations for ϕ_k and $\tilde{\phi}_k$:

We have the following Lemma:

Lemma 5.4.4 *$\forall\, k \geq 0$, $\psi_k(x,s) - \delta_{k,0}$ and $\tilde{\psi}_k(x,s)$ are polynomials of $1/z$ of the same parity as k and which behave like $O(1/z)$ at large z.*

Proof We proceed by recursion. We have $\psi_0 = 1$ and $\tilde{\psi}_0 = 0$, so the recursion hypothesis holds for $k = 0$.

Assume the recursion hypothesis at rank k.

Since $\dot{z} = \dot{u}_0/z$, we have that $\dot{\tilde{\psi}}_k$ is a polynomial of $1/z$, and thus from the fourth equation of Eq. (5.4.31), that

$$\tilde{\psi}_{k+1} = \frac{1}{z^2}\left(\text{Polynomial of } 1/z\right)$$

i.e. $z^2\tilde{\psi}_{k+1}$ is also a polynomial in $1/z$, and it has the parity of $k+1$.

Then, the first equation of Eq. (5.4.31) written at rank $k+1$ implies that ψ'_{k+1} is a Laurent polynomial of $1/z$ of parity $k+1$ (remember that B_{2j}, C_{2j}, A_{2j+1} are polynomials of z^2 and thus contain positive powers of z). After integrating with respect to $x = z^2 - 2u_0$, this implies that ψ_{k+1} must be a Laurent polynomial of $1/z$ of parity $k+1$, plus possibly a term proportional to $\ln z$ when $k+1$ is even:

$$\psi_{k+1} = \sum_{j\geq 0} a_{k+1,j} z^j + \; c_{k+1} \, \ln z + \sum_{j\geq 1} b_{k+1,j} z^{-j}.$$

However, from the large x behavior Eq. (5.4.23) we know that at large z, we must have $\psi_{k+1}(z) = o(1)$ and thus the Log term must vanish, and thus $z\psi_{k+1}$ is a polynomial in $1/z$, and the parity is clearly k. We have proved the recursion hypothesis to rank $k+1$.

□

Examples:
to the first few orders

$$\psi_0 = 1 \qquad , \qquad \tilde{\psi}_0 = 0$$

$$\psi_1 = -\frac{1}{24}\,(\frac{\ddot{u}_0}{\dot{u}_0\, z} + \frac{\dot{u}_0}{z^3}) \qquad , \qquad \tilde{\psi}_1 = -\frac{\dot{u}_0}{4z^3}.$$

5.4.11.1 Topological Expansion of the Kernel

The Christoffel Darboux kernel $K(x_1, x_2)$ can be rewritten as:

$$K(x_1,x_2) = \frac{\mathrm{e}^{-N\int_{z_2}^{z_1} ydx}}{2\,\sqrt{z_1 z_2}} \left(\frac{\hat{\psi}(z_1)\hat{\tilde{\phi}}(z_2) - \hat{\tilde{\psi}}(z_1)\hat{\phi}(z_2)}{z_1 - z_2} + \frac{\hat{\tilde{\psi}}(z_1)\hat{\tilde{\phi}}(z_2) - \hat{\psi}(z_1)\hat{\phi}(z_2)}{z_1 + z_2} \right),$$

and since each term has an expansion in $1/N$, whose coefficients are polynomials of $1/z_1$ and $1/z_2$, we have:

$$K(x_1,x_2) = \frac{\mathrm{e}^{-N\int_{z_2}^{z_1} ydx}}{2\,\sqrt{z_1 z_2}} \left(\frac{1}{z_1 - z_2} + \sum_{k=1}^{\infty} N^{-k} K_k(x_1,x_2) \right)$$

where each $K_k(x_1, x_2)$ is a polynomial in $1/z_1$ and in $1/z_2$.

This implies that the correlators also have a $1/N$ expansion:

$$\hat{W}_1(x) = -N\,y + \frac{1}{2z}\sum_{k=1}^{\infty} N^{-k}\, K_k(x,x).$$

$$\hat{W}_2(x_1,x_2) = \frac{1}{4z_1 z_2\,(z_1 - z_2)^2} - \frac{1}{(x_1 - x_2)^2} + O(N^{-1}).$$

5.4.11.2 Topological Expansion of the Projectors $M(x)$

The projector $M(x)$ defined in Eq. (5.4.8) also has a large N expansion:

$$M(x) = \sum_k N^{-k} M^{(k)}(x) = \frac{1}{2}\,\mathrm{Id} - \frac{1}{2}\begin{pmatrix} 0 & 1/z \\ z & 0 \end{pmatrix} + O(1/N)$$

Notice that we have

$$\forall\, x_1, x_2\,, \qquad \left[\frac{M^{(0)}(x_2)}{x_1 - x_2} + \frac{1}{2z_2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,,\, M^{(0)}(x_1)\right] = 0$$

and thus

$$\forall\, x_1, x_2\,, \qquad \left[\frac{M(x_2)}{x_1 - x_2} + \begin{pmatrix} 0 & 0 \\ \frac{1}{2z_2} & 0 \end{pmatrix}\,,\, M(x_1)\right] = O(1/N).$$

Lemma 5.4.5 (Topological Expansion) $N^{n-2}\hat{W}_n$ *is a formal power series in powers of* $1/N^2$

$$\hat{W}_n(x_1, \ldots, x_n) = \sum_{g=0}^{\infty} N^{2-2g-n}\, \hat{W}_n^{(g)}(x_1, \ldots, x_n)$$

where each $\hat{W}_n^{(g)}$ *is a rational function of the* $z_i = \sqrt{x_i + 2u_0}$*, with poles only at* $z_i = 0$*, except* $\hat{W}_2^{(0)}$ *and* $\hat{W}_1^{(0)}$ *which are:*

$$\hat{W}_1^{(0)} = -\, y(z, s)$$

$$\hat{W}_2^{(0)} = \frac{1}{4z_1 z_2}\;\frac{1}{(z_1 - z_2)^2} \;-\; \frac{1}{(z_1^2 - z_2^2)^2} = \frac{1}{4z_1 z_2 (z_1 + z_2)^2}.$$

This Lemma makes some non-trivial claims, first that there is no odd power of $1/N$, second that $\hat{W}_n$ starts as N^{2-n}, and third that the coefficients are polynomials of $1/z_i$.

Proof Notice that in the products $\prod_i K(z_{\sigma(i)}, z_{\sigma(i+1)})$, all the exponentials cancel, and the square roots $1/\sqrt{z_i}$ appear only by pairs, so the result is, order by order in N^{-k}, a rational fraction of the z_i's having poles at $z_i = 0$, or possibly at $z_i = z_j$. Except for $\hat{W}_1^{(0)}$ and $\hat{W}_2^{(0)}$, the poles at $z_i = z_j$ are at most simple poles, and it is easy to see that in the sum over permutations, the residues cancel, therefore there is no pole at $z_i = z_j$. Thus each $\hat{W}_n^{(g)}$ is a rational function of the z_i's having poles only at $z_i = 0$. The cases of $\hat{W}_2$ and $\hat{W}_1$ need to be treated separately, and are easy.

The fact that $\hat{W}_n$ has a $1/N^2$ expansion instead of $1/N$ comes from a simple symmetry argument. In the expression of $\hat{W}_n$, changing $\psi \to \phi$ and $\tilde{\psi} \to \tilde{\phi}$, can also be obtained by permuting the x_i's, and since we take a symmetric sum, only the terms which are invariant under the exchange $\psi \to \phi$ and $\tilde{\psi} \to \tilde{\phi}$ contribute to $\hat{W}_n$. Exchanging the two solutions $\psi \to \phi$ and $\tilde{\psi} \to \tilde{\phi}$, is also equivalent to changing $N \to -N$, and therefore $\hat{W}_n$ has the parity $(-1)^n$, in N.

It remains to prove that the leading order is N^{2-n}. This is obvious for $n = 1$ or $n = 2$. For $n \geq 3$, we shall proceed by induction, by applying the insertion operator defined in Sect. 5.4.6.2, which has the property that

$$\delta_{x_{n+1}} \hat{W}_n(x_1, \ldots, x_n) = \hat{W}_{n+1}(x_1, \ldots, x_n, x_{n+1}).$$

Let us write:

$$M(x) = xU(x) + A(x) - \frac{1}{N} \frac{\partial M(x)}{\partial s} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

where

$$U(x) = w(x)\psi(x)\phi(x) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$A(x) = w(x)\psi(x)\tilde{\phi}(x)\, \mathrm{Id} + w(x)\psi(x)\phi(x) \begin{pmatrix} 0 & 1 \\ 2u & 0 \end{pmatrix}.$$

Observe that

$$\forall\, x, y, \alpha \ , \qquad [A(x) + \alpha U(x), A(y) + \alpha U(y)] = 0.$$

This implies that the insertion operator δ_y acts on $M(x)$ like

$$\begin{aligned} \delta_y M(x) &= [\frac{M(y)}{x-y} + U(y), M(x)] \\ &= -\frac{1}{N} \frac{1}{x-y} \left([xU(y) + A(y), \dot{M}(x) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}] - [xU(x) + A(x), \dot{M}(y) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}] \right) \\ &\quad + \frac{1}{N^2} \frac{1}{x-y} \left(\dot{M}(x), [\dot{M}(y), \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}] - \dot{M}(y)[\dot{M}(x), \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}] \right) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

and it acts on $u(s)$ by Eq. (5.4.20), i.e.

$$\delta_y u(s) = \frac{w(y)}{N} \frac{\partial}{\partial s}\, \psi(y)\phi(y)$$

therefore the action of the operator δ_y brings a factor $1/N$, and the result is again expressed in terms of $M(x)$, $M(y)$, and $u(s)$ and their $\partial/\partial s$ derivatives, and we recall that the insertion operator commutes with $\partial/\partial s$.

Since

$$\hat{W}_2(x_1, x_2) = \frac{\operatorname{Tr} M(x_1)M(x_2)}{(x_1 - x_2)^2} - \frac{1}{(x_1 - x_2)^2}$$

is of order $O(1)$ and is expressed only in terms of M, and since for $n \geq 3$

$$\hat{W}_n(x_1, \ldots, x_n) = \delta_{x_n} \hat{W}_{n-1}(x_1, \ldots, x_{n-1})$$

we see that by recursion:

$$\hat{W}_n = O(N^{2-n}).$$

□

We mention that this theorem is far from being true for any Lax matrix. It holds because our Lax matrix is related to the $(p, 2)$ minimal model.

5.4.12 Link with Symplectic Invariants

We have found that the minimal model correlators $\hat{W}_n$ have a formal large N expansion of the form

$$\hat{W}_n(x_1, \ldots, x_n) = \sum_g N^{2-2g-n} \hat{W}_n^{(g)}(x_1, \ldots, x_n)$$

where each $\hat{W}_n^{(g)}$ with $2 - 2g - n < 0$ is a rational function of the $z_i = \sqrt{x_i + 2u_0}$, with poles only at $z_i = 0$. And we have found that they satisfy loop equations in Theorem 5.4.2.

Let us define:

$$\hat{\omega}_n^{(g)}(z_1, \ldots, z_n) = \hat{W}_n^{(g)}(x(z_1), \ldots, x(z_n)) \prod_{i=1}^{n} x'(z_i) + \frac{\delta_{n,2}\delta_{g,0} x'(z_1)x'(z_2)}{(x(z_1) - x(z_2))^2}.$$

The first few are easily computed from the BKW expansion, and one finds:

$$\hat{\omega}_1^{(0)}(z) = -y(z, s)\, x'(z)$$

$$\hat{\omega}_2^{(0)}(z_1, z_2) = \frac{1}{(z_1 - z_2)^2}$$

and all the other $\hat{\omega}_n^{(g)}(z_1, \dots, z_n)$ with $2-2g-n < 0$ are symmetric polynomials of $1/z_i$.

Then, since they satisfy loop equations, we have:

Theorem 5.4.4 *The $\hat{\omega}_n^{(g)}$ can be computed by the "topological recursion"*

$$\hat{\omega}_{n+1}^{(g)}(z_0, z_1, \dots, z_n) = -\operatorname*{Res}_{z\to 0} \frac{\frac{1}{z_0-z} - \frac{1}{z_0+z}}{2y(z,t)\,x'(z)} \Big[\hat{\omega}_{n+2}^{(g-1)}(z, -z, z_1, \dots, z_n) + \sum'_{h+h'=g,\, I \uplus I' = \{z_1,\dots,z_n\}} \hat{\omega}_{1+\#I}^{(h)}(z, I)\, \hat{\omega}_{1+\#I'}^{(h')}(-z, I')\Big]$$

where we recall that $\sum'$ means the sum over all h, h', I, I' excluding $(h, I) = (0, \emptyset)$ and $(h', I') = (0, \emptyset)$. In other words, the differentials $\hat{\omega}_n^{(g)}(z_1, \dots, z_n)\prod_i dz_i$, are the symplectic invariant correlators for the spectral curve of Eq. (5.4.30) (see Chap. 7 for the definition of symplectic invariants of the spectral curve).

Proof Notice that, since $\hat{\omega}_{n+1}^{(g)}(z_0, z_1, \dots, z_n)$ is a polynomial in $1/z_0$, we have the Cauchy identity:

$$\begin{aligned}
\hat{\omega}_{n+1}^{(g)}(z_0, z_1, \dots, z_n) &= -\operatorname*{Res}_{z\to z_0} \frac{dz}{z_0 - z}\, \hat{\omega}_{n+1}^{(g)}(z, z_1, \dots, z_n) \\
&= \operatorname*{Res}_{z\to 0} \frac{dz}{z_0 - z}\, \hat{\omega}_{n+1}^{(g)}(z, z_1, \dots, z_n) \\
&= -\operatorname*{Res}_{z\to 0} \frac{dz}{(z_0 - z) y(z) x'(z)}\, 2\hat{\omega}_1^{(0)}(z)\, \hat{\omega}_{n+1}^{(g)}(z, z_1, \dots, z_n).
\end{aligned}$$

Then, the loop equations (Theorem 5.4.2) imply that the quantity

$$2\hat{\omega}_1^{(0)}(z)\, \hat{\omega}_{n+1}^{(g)}(z, z_1, \dots, z_n) + \sum'_{h+h'=g,\, I \uplus I' = \{z_1,\dots,z_n\}} \hat{\omega}_{1+\#I}^{(h)}(z, I)\, \hat{\omega}_{1+\#I'}^{(h')}(z, I') + \hat{\omega}_{n+2}^{(g-1)}(z, z, z_1, \dots, z_n)$$

is equal to $x'(z)^2$ times a rational function of $x(z)$, with no pole at $z = 0$ (in fact it is a polynomial of $x(z)$ plus a rational function of $x(z)$ with poles at $z = \pm z_i$), in other words it cannot contribute to the residue. This shows that

$$\hat{\omega}_{n+1}^{(g)}(z_0, z_1, \dots, z_n) = \operatorname*{Res}_{z\to 0} \frac{1}{(z_0 - z)\, y(z,t)\, x'(z)} \Big[\hat{\omega}_{n+2}^{(g-1)}(z, z, z_1, \dots, z_n) + \sum'_{h+h'=g,\, I \uplus I' = \{z_1,\dots,z_n\}} \hat{\omega}_{1+\#I}^{(h)}(z, I)\, \hat{\omega}_{1+\#I'}^{(h')}(z, I')\Big]$$

Then, using the fact that each $\hat{\omega}_n^{(g)}$ has a given parity in the z_i's, it is easy to complete the proof.

A special care is needed for $\hat{\omega}_1^{(1)}(z_0)$ because $\hat{\omega}_2^{(0)}(z,z)$ is ill-defined, but we leave to the reader to check that the theorem also holds for that case. □

As an immediate consequence we have that:

Corollary 5.4.1 *The correlation function $\hat{\omega}_n^{(g)}$ of the minimal model $(2m+1,2)$, coincide with the generating function of large maps $\tilde{\omega}_n^{(g)}$ of genus g and $n \geq 1$ boundaries (defined in Theorem 5.3.1):*

$$\hat{\omega}_n^{(g)} = \tilde{\omega}_n^{(g)}.$$

Proof The topological recursion Theorem 5.4.4 for the minimal model $(2m+1,2)$, is identical to the topological recursion of Theorem 5.3.1 for the generating functions of large maps.

Therefore, the $\hat{\omega}_n^{(g)}$ of the minimal model $(2m+1,2)$ and the generating function of large maps $\tilde{\omega}_n^{(g)}$ are both equal to the symplectic invariants of the spectral curve $(x(z,s), y(z,s))$, they satisfy the same topological recursion with the same initial condition. □

5.4.13 Tau Function

Here, we prove that the double scaling limits $\tilde{F}_g$ of the large maps generating functions (see Sect. 5.1.3.2), which coincide with the symplectic invariants $\mathcal{F}_g$ of our spectral curve (see Theorem 5.3.3), do also coincide with the coefficients of the topological expansion of the minimal model Tau-function introduced in Sect. 5.4.8, i.e. Proposition 5.4.6, Eq. (5.4.28):

$$\ln\tau = \sum_g N^{2-2g}\hat{F}_g \qquad , \; \partial^2\hat{F}_g/\partial s^2 = u_g(s).$$

From the Poisson equation (5.4.24), it is easy to see that our spectral curve has the property that

$$\left.\frac{\partial y(z,s)}{\partial s}\right|_{x(z,s)} = -\frac{1}{2z}$$

and thus

$$x'(z)\left.\frac{\partial}{\partial s}\right|_x y(z,s) = -1 = \mathop{\mathrm{Res}}_{z'\to\infty}\frac{z'\,dz'}{(z-z')^2} = \mathop{\mathrm{Res}}_{z'\to\infty} z'\,\hat{\omega}_2^{(0)}(z,z')\,dz'.$$

Knowing that, it follows from general property of symplectic invariants $\mathcal{F}_g$ of a spectral curve (see Chap. 7), that:

$$\frac{\partial}{\partial s}\tilde{F}_g = \operatorname*{Res}_{z\to\infty} z\,\hat{\omega}_1^{(g)}(z)\,dz = 2\operatorname*{Res}_{x\to\infty} \sqrt{x+2u_0}\;\hat{W}_1^{(g)}(x)\,dx.$$

In other words

$$\frac{\partial}{\partial s}\tilde{F}_g = 2\operatorname*{Res}_{x\to\infty} \hat{W}_1^{(g)}(x)\,\sqrt{x}\,dx$$

and summing over g:

$$\frac{1}{N}\frac{\partial \tilde{F}}{\partial s} = 2\operatorname*{Res}_{x\to\infty} \hat{W}_1(x)\,\sqrt{x}\,dx = \frac{1}{N}\frac{\partial \hat{F}}{\partial s}$$

where the last equality holds by definition of the τ-function in Sect. 5.4.8.

This proves:

Theorem 5.4.5 *Near a* mth *order critical point, the coefficients of the double scaling limit of large maps* $\tilde{F}_g$ *such that* $F_g \sim (t-t_c)^{(2-2g)\frac{2m+3}{2m+2}}\tilde{F}_g$*, are the symplectic invariants of the classical spectral curve Eq. (5.4.30), and are such that the generating series:*

$$\tau = \exp\sum_g N^{2-2g}\tilde{F}_g$$

is the Tau-function of the $(2m+1,2)$ *minimal model, or also,* $u(s) = d^2\ln\tau/ds^2$ *satisfies the* $m+1$th *Gelfand Dikii equation:*

$$R_{m+1}(u(s)) = s.$$

We have thus seen, that the asymptotic generating function which counts large maps near a critical point of order m, is the Tau-function for the $(2m+1,2)$ reduction of the KdV hierarchy. In particular, its second derivative satisfies the $(m+1)$th Gelfand-Dikii equation.

5.4.14 Large N and Large s

5.4.14.1 Rescaling N

We have introduced the parameter N as a scaling parameter in order to define formal power series.

But notice that N is redundant, it can be absorbed by the change of variable $s = N^{-\frac{p+1}{p+2}}\,\tilde{s}$ and $u(s) = N^{\frac{-2}{p+2}}\,\tilde{u}(\tilde{s})$. We have

$$Q = N^{\frac{-2}{p+2}}\,\tilde{Q} \qquad , \qquad P = N^{\frac{-p}{p+2}}\,\tilde{P}$$

with

$$\tilde{Q} = \tilde{d}^2 - 2\tilde{u}(\tilde{s}) \qquad , \qquad \tilde{P} = \tilde{d}^p - p\tilde{u}\tilde{d}^{p-2} + \dots \qquad , \qquad \tilde{d} = \frac{d}{d\tilde{s}},$$

and they satisfy the string equation without $1/N$:

$$[\tilde{P}, \tilde{Q}] = \mathrm{Id}.$$

5.4.14.2 Homogeneous Case

A case particularly interesting is when all $\tilde{t}_j$'s with $j < m$ vanish. In that case, the equation for $u_0(s)$ is simply:

$$\frac{s}{4} = \mathcal{P}(u_0) = \tilde{t}_m\,\frac{(2m+1)!}{m!\,(m+1)!}\,(-u_0/2)^{m+1},$$

i.e. $\mathcal{P}(u_0)$ is a homogeneous polynomial of u_0.

This implies that the BKW expansion of $u(s)$ has only homogeneous terms:

$$u(s) = u_0 + \sum_{g=1}^{\infty} N^{-2g}\; c_g\; u_0^{1-g(2m+3)}$$

where c_g are some complex coefficients.

Using the reparametrization $s = N^{-\frac{p+1}{p+2}}\,\tilde{s}$ and $u(s) = N^{\frac{-2}{p+2}}\,\tilde{u}(\tilde{s})$, this amounts to writing a large $\tilde{s}$ expansion for $\tilde{u}$:

$$\tilde{u}(\tilde{s}) = \sum_{g=0}^{\infty} \tilde{u}_g\;\tilde{s}^{\frac{2}{p+1}\,(1-g(p+2))} \quad , \quad \tilde{u}_g = c_g\,\left(\frac{-2\,\tilde{t}_m\;(-1)^m\;(2m+1)!!}{(m+1)!}\right)^{-\frac{2}{p+1}\,(1-g(p+2))}.$$

The coefficients $\tilde{u}_g$ can be found by inserting this expansion into the Gelfand Dikii equation $R_{m+1}(u) = s$, or also, since $F = \sum_g N^{2-2g}\,\tilde{F}_g(s)$ and $\ddot{F} = u(s)$, we have just shown that, for $g \geq 2$:

$$\frac{(1-g)\,(m+1)^2}{(2m+3)\,(m+2-g(2m+3))}\;\tilde{u}_g = \tilde{F}_g = \mathcal{F}_g(\{x(z,s), y(z,s)\}).$$

We thus formulate the theorem:

Theorem 5.4.6 *If $\tilde{u}(\tilde{s})$ written as a large $\tilde{s}$ series*

$$\tilde{u}(\tilde{s}) = \sum_{g=0}^{\infty} \tilde{u}_g \; \tilde{s}^{\frac{2}{p+1}\,(1-g(p+2))}$$

is solution of the $m+1$th Gelfand Dikii equation (here we choose $N=1$)

$$\tilde{t}_m \; R_{m+1}(\tilde{u}) = \tilde{s},$$

then

$$(-\tilde{u}_0/2)^{m+1} = \frac{s}{4\,\tilde{t}_m} \; \frac{m!\,(m+1)!}{(2m+1)!}$$
$$\tilde{u}_1 = -\frac{m}{24\,(m+1)}$$

and for $g \geq 2$, the coefficients $\tilde{u}_g$ of the expansion, are related to the symplectic invariants $\mathcal{F}_g$ of the spectral curve $\tilde{\mathcal{S}}$

$$\tilde{\mathcal{S}} = \begin{cases} x(z) = z^2 - 2\tilde{u}_0 \\ y(z) = \tilde{t}_m Q_m(z) = \tilde{t}_m \; \sum_{j=0}^{m} z^{2m+1-2j} \, (-\tilde{u}_0/2)^j \; \frac{(2m+1)!}{m!} \; \frac{(m-j)!}{j!\,(2m+1-2j)!} \end{cases}$$

as:

$$\frac{(1-g)\,(m+1)^2}{(2m+3)\,(m+2-g(2m+3))}\,\tilde{u}_g = \mathcal{F}_g(\tilde{\mathcal{S}}).$$

and as an immediate corollary:

Theorem 5.4.7 *Near a* mth *order critical point, the double scaling limit of the generating functions of large maps of genus g:*

$$\tilde{F}_g = \lim_{\epsilon \to 0} \; \epsilon^{(2g-2)\,\frac{2m+3}{2m+2}} \; t^{2g-2} \, F_g$$

are related to the coefficients $\tilde{u}_g(\tilde{s})$ of the large $\tilde{s}$ expansion of the solution of the $m+1$th *Gelfand-Dikii equation:*

$$\frac{(1-g)\,(m+1)^2}{(2m+3)\,(m+2-g(2m+3))}\,\tilde{u}_g = \tilde{F}_g.$$

This theorem is an indication that large maps are related to Liouville conformal quantum field theory coupled to the $(2m+1, 2)$ minimal model.

5.4.15 Example: Pure Gravity Case

Let us illustrate all this on the important example of pure gravity case, $m = 1$, the $(3, 2)$ minimal model.

We have:

$$Q = d^2 - 2u \qquad , \qquad P = d^3 - 3ud - \frac{3}{2N^2}\,\dot{u}.$$

The string equation $[P, Q] = \frac{1}{N}\,\mathrm{Id}$ gives the Painlevé I equation for $u(s)$:

$$3u^2 - \frac{1}{2N^2}\,\ddot{u} = s.$$

There is a formal solution of this equation with an expansion in powers of $1/N^2$:

$$u(s) = -\sqrt{\frac{s}{3}} - \frac{1}{48\,N^2\,s^2} + \frac{49}{32\,N^4\,\sqrt{3}\,s^{\frac{9}{2}}} + O(1/N^6)$$

which can be written

$$u(s) = \sum_{g=0}^{\infty} c_g\; N^{-2g}\; u_0^{1-5g} \qquad , \qquad u_0 = -\sqrt{\frac{s}{3}}.$$

With the rescaling

$$u = N^{\frac{-2}{5}}\,\tilde{u} \qquad , \qquad s = N^{\frac{-4}{5}}\,\tilde{s}$$

we have

$$\tilde{u} = \sum_g \tilde{u}_g\; \tilde{s}^{\frac{1}{2}-\frac{5}{2}g}.$$

The free energy $\mathcal{F}(s)$ such that $N^{-2}\ddot{\mathcal{F}} = u(s)$ has an expansion:

$$\mathcal{F}(s) = -\frac{4}{15\sqrt{3}}\,N^2\,s^{\frac{5}{2}} + \frac{\ln s}{48} + \frac{7}{40\sqrt{3}\,N^2\,s^{\frac{5}{2}}} + \sum_{g\geq 3} (N\,s^{\frac{5}{4}})^{2-2g}\;\tilde{F}_g.$$

For example, the first few correlators computed from the topological recursion are

$$\tilde{\omega}_3^{(0)}(z_1, z_2, z_3) = \frac{1}{6u_0}\,\frac{dz_1}{z_1^2}\,\frac{dz_2}{z_2^2}\,\frac{dz_3}{z_3^2}$$

$$\tilde{\omega}_1^{(1)}(z) = \frac{dz}{3^2\;2^4}\left(\frac{3}{z^4\,u_0} + \frac{1}{z^2\,u_0^2}\right).$$

5.5 Summary: Large Maps and Liouville Gravity

We have seen that

- Large maps are obtained when the weights t_k of k-gons are tuned to some critical, or multi-critical values $t_k \to t_{k,c}$ (the subscript $_c$ stands for "critical"). At those critical values, the disc amplitude $W_1^{(0)}(x)$ has cusps of the form $W_1^{(0)}(x) \sim \frac{V'(x)}{2} + C\,(x-a)^{\frac{p}{q}}$, with $q = 2$ and $p = 2m+1$.
- The tuning of the t_k's

$$t_k = t_{k,c} + \sum_j C_{k,j}\ (1 - t/t_c)^{\nu\nu_j}\, \tilde{t}_j$$

comes with some critical exponents

$$\nu = \frac{1}{p+q-1} = \frac{1}{2m+2} \qquad , \qquad \nu_j = 2(m-j).$$

We then have the scalings

$$F_g(t, \{t_k\}) \sim (1 - t/t_c)^{(2-2g)(1-\gamma/2)}\, t_c^{2-2g}\, \tilde{F}_g(\{\tilde{t}_j\})$$

with the exponent (called "**string susceptibility exponent**" by physicists)

$$\gamma = \frac{-2}{p+q-1} = \frac{-1}{m+1}.$$

Those exponents agree with the KPZ formula.

- The asymptotic generating functions of large maps, are obtained by the topological recursion, corresponding to the spectral curve:

$$\mathcal{E}_{(2m+1,2)} = \begin{cases} x(z,s) = z^2 - 2u_0 \\ y(z,s) = \sum_j \tilde{t}_j Q_j(z) = \sum_j \sum_l \tilde{t}_j\, z^{2j+1-2l}\, (-u_0/2)^l\ \frac{(2j+1)!}{j!}\ \frac{(j-l)!}{l!\,(2j+1-2l)!} \end{cases}$$

which is the blow up of the cusp singularity of $W_1^{(0)}(x) \sim (x-a)^{p/q}$
- This means that when $t \to t_c$

$$F_g \sim (1 - t/t_c)^{(2-2g)\frac{2m+2}{2m+3}}\, t_c^{2-2g} \tilde{F}_g = (1 - t/t_c)^{(2-2g)\frac{2m+2}{2m+3}}\, t_c^{2-2g}\, \mathcal{F}_g(\mathcal{E}_{(2m+1,2)}).$$

and $1 - t/t_c = \epsilon^2$ is the "mesh–size".
- The asymptotic generating functions of large maps, $\tilde{F}_g$, are such that

$$\tau = \mathrm{e}^{\sum_g N^{2-2g}\, \tilde{F}_g}$$

is the Tau–function of the mth reduction of the KdV hierarchy of integrable equations, called the $(2m+1,2)$ minimal model coupled to gravity.

- This means that the second derivative u of $\ln\tau$, satisfies a non-linear differential equation of Painlevé type, namely the $m+1$th Gelfand-Dikii equation:

$$R_{m+1}(u) = s.$$

- This means that the asymptotic generating functions of large maps coincide with those of the Liouville conformal field theory coupled to gravity.

5.6 Exercises

Exercise 1 Prove Proposition 5.4.6, i.e. that

$$\mathcal{F}_0 = -4\sum_{j,k}\tilde{t}_j\,\tilde{t}_k\,\,(-u_0/2)^{k+j+3}\,\frac{j+k+4}{j+k+3}\,\frac{(2j+2)!}{j!\,(j+2)!}\,\frac{(2k+2)!}{k!\,(k+1)!}.$$

Hint: first look for a polynomial $S(u_0)$ such that $\frac{d}{ds}\,S(u_0) = 4\,u_0$, and show that

$$S'(u_0) = 16u_0\,\mathcal{P}'(u_0).$$

From there, and from the explicit expression of $\mathcal{P}(u_0)$, deduce $S(u_0)$.

Then look for a polynomial $\Xi\,(u_0)$, such that $\frac{d}{ds}\,\Xi\,(u_0) = S(u_0)$, and show that

$$\Xi'(u_0) = \mathcal{P}'(u_0)\,\,S(u_0).$$

From there, deduce the expression of $\Xi\,(u_0)$. It satisfies $d^2/ds^2\,\Xi = u_0$, and thus $\Xi = \mathcal{F}_0$.

Exercise 2 Prove Lemma 5.4.1 and the recursion for the Gelfand-Dikii polynomials Eq. (5.4.3).

Hint: To prove Lemma 5.4.1, show that

$$[\left((Q^{j-\frac{1}{2}})_+\right)^2, Q] = (Q^{j-\frac{1}{2}})_+\,[(Q^{j-\frac{1}{2}})_+, Q] + [(Q^{j-\frac{1}{2}})_+, Q]\,(Q^{j-\frac{1}{2}})_+$$

is an operator of order at most $2j-1$, and this implies that $[(Q^{j-\frac{1}{2}})_+, Q]$ must be an operator of order 0, i.e. a function of s.

Using $(Q^{1/2})_+ = d$ find $R_1 = -2u$, and then proceed by recursion on j.

First show (using the recursion hypothesis) that it is possible to choose two functions $\alpha_j(s)$ and $\beta_j(s)$ such that

$$\left(Q\,(Q^{j-\frac{1}{2}})_+ + \alpha_j d + \beta_j\right)^2 = Q^{2j+1} + O(d^{2j})$$

i.e. that

$$(Q^{j+\frac{1}{2}})_+ = Q\,(Q^{j-\frac{1}{2}})_+ + \alpha_j d + \beta_j = (Q^{j-\frac{1}{2}})_+\,Q + (\alpha_j + \dot{R}_j)d + \beta_j.$$

Then, writing that $[(Q^{j+\frac{1}{2}})_+, Q]$ must be an operator of degree 0, find the coefficients α_j, β_j, and find the recursion relation for R_j.

Chapter 6
Counting Riemann Surfaces

In the previous chapter, we have computed the asymptotic generating functions of large maps, and we have seen that they are related to the (p, q) minimal model.

Now, in this chapter, we compute generating functions for "counting" Riemann surfaces directly. The set of all Riemann surfaces (modulo holomorphic reparametrizations) of a given topology, called **moduli space**, is a finite dimensional complex variety (it is not a manifold because it is not smooth, instead it is called an orbifold), which can be endowed with some "volume form" which allows to define "volumes" of moduli spaces, i.e. in some sense the "number of Riemann surfaces".

In the physics literature, this approach is often called "topological gravity", and it was conjectured by Witten [87], and later proved by Kontsevich [57], that the limit of large maps, is (in some sense which we make precise below) equivalent to topological gravity. We shall reprove this theorem in this chapter, using again the topological recursion.

6.1 Moduli Spaces of Riemann Surfaces

Riemann surfaces are 2-dimensional manifolds, equipped with a complex structure. They are thus 1-dimensional complex manifolds, and are also called "**complex curves**". They are defined modulo conformal reparametrization, and since the group of conformal reparametrizations is very large, there are not so many different Riemann surfaces, they can be parametrized by a finite number of complex parameters called "moduli".

In all this chapter we shall denote

$$\chi_{g,n} = 2 - 2g - n \qquad , \qquad d_{g,n} = 3g - 3 + n.$$

B. Eynard, *Counting Surfaces*, Progress in Mathematical Physics 70,
DOI 10.1007/978-3-7643-8797-6_6

The classification theorem of surfaces says that the topology of compact orientable surfaces, is entirely characterized by their genus, i.e. their number of handles.

Definition 6.1.1 (Moduli Space) The Moduli space of orientable compact Riemann surfaces of genus g, with n distinct labeled marked points is denoted

$$\mathcal{M}_{g,n} = \{(C, p_1, \ldots, p_n)\} \,/\, \text{automorphisms}$$

where C is a connected smooth orientable compact Riemann surface of genus g, and $p_1, \ldots, p_n$ are n distinct labeled points on C.

Notice that $C \setminus \{p_1, \ldots, p_n\}$ is topologically a surface of genus g with n points removed, it has Euler characteristics

$$\chi = \chi_{g,n} = 2 - 2g - n.$$

We shall see below that, if $2g - 2 + n > 0$, i.e. $\chi_{g,n} < 0$ then $\mathcal{M}_{g,n}$ is locally a complex manifold of dimension

$$\dim \mathcal{M}_{g,n} = d_{g,n} = 3g - 3 + n$$

i.e. is locally parametrized by $d_{g,n} = 3g - 3 + n$ complex numbers (called **moduli**). $\mathcal{M}_{g,n}$ is not a manifold, it is an orbifold because surfaces with automorphisms are divided by their automorphism group.

6.1.1 Examples of Moduli Spaces

6.1.1.1 Example $\mathcal{M}_{0,3}$ Sphere with Three Marked Points

We shall admit, that there is only one (up to conformal bijections) simply connected (i.e. genus 0) compact Riemann surface, it is called the Riemann sphere, or also the projective complex plane $\mathbb{C}P^1$, and it is a compactification $\bar{\mathbb{C}}$ of the complex plane with a point at ∞ added:

$$\overline{\mathbb{C}} = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} = \text{Riemann sphere}.$$

More precisely, a surface is a manifold, defined by an atlas of charts, that is a collection of open connected sets in $\mathbb{R}^2$ (the charts) together with a set of continuous transition functions from charts to charts. A compact surface can be realized by an atlas with a finite set of charts. Orientability requires that the Jacobians of all transition functions be positive. For defining a Riemann surface, the transition functions from charts to charts are required to be holomorphic.

The Riemann sphere can be realized with two charts U_+, U_-, each being a copy of a disc of radius $R_\pm$ centered at the origin in $\mathbb{C}$, and we assume $R_+ R_- > 1$:

$$U_\pm = \{z \in \mathbb{C} \mid |z| < R_\pm\} \qquad , \; R_+ R_- > 1 \,. \tag{6.1.1}$$

A point $z \in U_+$ and $\tilde{z} \in U_-$ are identified if $\tilde{z} = 1/z$. The transition function is:

$$f_{+-} : U_+ \cap \{z \mid 1/R_- < |z| < R_+\} \to U_- \cap \{\tilde{z} \mid 1/R+- < |\tilde{z}| < R_-\}$$
$$z \mapsto 1/z$$

it is holomorphic and bijective and its inverse is holomorphic.

The Riemann sphere is then the equivalence class of all atlases equivalent to that one. In particular it is independent of the choice of R_+ and R_- provided $R_+R_- > 1$.

The automorphisms of the Riemann sphere, are analytic bijective functions whose inverse is analytic, from the Riemann sphere to itself. We leave to the reader to prove[1] that an automorphism of the Riemann sphere is necessarily a Moebius transformation:

$$f : z \mapsto \frac{az+b}{cz+d} \qquad , \; ad - bc = 1 \,, \; (a,b,c,d) \in \mathbb{C}^4$$

in other words $\mathrm{Aut}(\mathbb{C}P^1) \sim Sl_2(\mathbb{C})$, the group of Moebius transformations.

Consider a genus 0 Riemann surface with three marked points p_1, p_2, p_3. As a Riemann surface, it can always be conformally mapped to the Riemann sphere. And up to composition by some Moebius transformation, we can always assume that the three points (p_1, p_2, p_3) are mapped to $(0, 1, \infty)$.

This means, that there is only one Riemann surface of genus 0 with three marked points, modulo conformal reparametrization.

$$\mathcal{M}_{0,3} = \text{singleton} = \{\, (\overline{\mathbb{C}}, 0, 1, \infty)\}.$$

$\mathcal{M}_{0,3}$ is a point, it is a dimension 0 manifold:

$$\dim \mathcal{M}_{0,3} = d_{0,3} = 0.$$

6.1.1.2 Example $\mathcal{M}_{0,4}$ Genus 0 with Four Marked Points

Let $(C, p_1, p_2, p_3, p_4) \in \mathcal{M}_{0,4}$. Since C is a genus zero Riemann surface, it can be mapped to the Riemann sphere, and up to a Moebius transformations, the three

[1] Hint: notice that if f is bijective, then exactly one point is sent to ∞, and the fact that f is analytic and bijective means that f can only have one simple pole, and is analytic everywhere else. Then, use that a holomorphic function with no pole on a compact surface can only be a constant.

points p_1, p_2, p_3 can be mapped to $0, 1, \infty$, and then the fourth point p_4 is mapped to a point of the Riemann sphere different from $0, 1, \infty$, i.e. to the complex plane without 0 and 1:

$$\mathcal{M}_{0,4} \sim \overline{\mathbb{C}} \setminus \{0, 1, \infty\}.$$

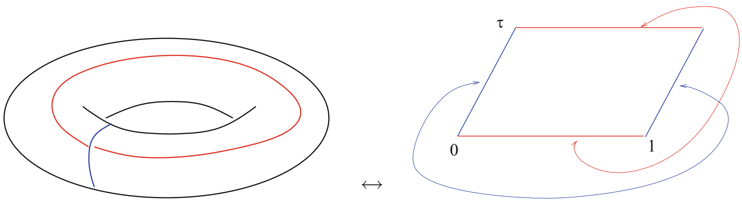
It is locally isomorphic to $\mathbb{C} \sim \mathbb{R}^2$, thus it is a surface, with real dimension 2, and it is also a complex manifold of complex dimension

$$\dim \mathcal{M}_{0,4} = d_{0,4} = 1.$$

Observe that it is not compact.

6.1.1.3 Example $\mathcal{M}_{1,1}$ Torus with a Marked Point

We shall admit that every genus one Riemann surface (torus) can be conformally mapped to a parallelogram of modulus τ (with $\operatorname{Im} \tau > 0$) with opposite sides identified, in other words the complex plane quotiented by the relationships $z \equiv z + 1 \equiv z + \tau$, and we set the marked point at the origin:



Two such representations are equivalent (they represent the same Riemann surface up to a conformal reparametrization which conserves the marked point) if and only if they have the same modulus τ modulo an $Sl_2(\mathbb{Z})$ modular transformation (see proof as Exercise 1):

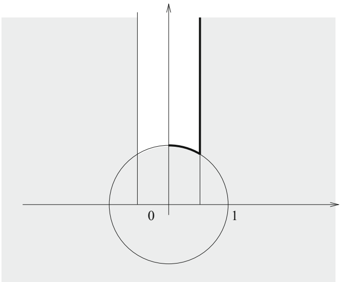
$$\tau \equiv \tau' = \frac{a\tau + b}{c\tau + d} \quad , \quad (a, b, c, d) \in \mathbb{Z}^4 \quad , ad - bc = 1$$

$Sl_2(\mathbb{Z})$ is generated by

$$\tau \mapsto \tau + 1 \quad , \quad \tau \mapsto \frac{-1}{\tau}.$$

Therefore, the fundamental domain for values of τ is:

$$\mathcal{M}_{1,1} = \Big\{ -\frac{1}{2} < \operatorname{Re} \tau \leq \frac{1}{2} \Big\} \cap \Big\{ \operatorname{Im} \tau > 0 \Big\} \cap \Big\{ |\tau| > 1 \Big\} \cup \Big\{ \tau = e^{i\theta} \, , \, \theta \in [\pi/3, \pi/2] \Big\}$$



Each point in that domain corresponds to exactly one Riemann surface of genus one with a marked point, and that domain is called the moduli space of surfaces of genus 1 with one marked point, and denoted:

$$\mathcal{M}_{1,1} = \mathbb{C}_+/Sl_2(\mathbb{Z})$$

We see that it is a dimension 1 complex orbifold (it inherits its complex structure from that of $\mathbb{C}_+$, and is quotiented by a group, here the upper half complex plane $\mathbb{C}_+$ quotiented by $Sl_2(\mathbb{Z})$). It has a non-trivial topology because of the identifications $\tau \equiv \tau + 1 \equiv -1/\tau$. For instance it has conic singularities at $\tau = e^{i\pi/3}$ and at $\tau = i$.

The upper half plane $\mathbb{C}_+$ is known as the hyperbolic plane, it is endowed with a metric of constant curvature $= -1$, whose geodesics are circles or straight lines orthogonal to the real axis. We thus see that $\mathcal{M}_{1,1}$ is an hyperbolic triangle whose three boundaries are geodesics. Its three angles are $\pi/3, \pi/3, 0$. It is well known in hyperbolic geometry, that the area of a triangle is its deficit angle, that is π minus the sum of its angles. Here:

$$\text{Hyperbolic Area}(\mathcal{M}_{1,1}) = \pi - (\pi/3 + \pi/3 + 0) = \pi/3$$

Moreover, Gauss-Bonnet theorem says that the average curvature is related to the Euler characteristics by:

$$\int_{\mathcal{M}_{1,1}} \text{curvature} = 2\pi\ \chi(\mathcal{M}_{1,1}).$$

Here, the curvature is constant $= -1$, and thus:

$$2\pi\ \chi(\mathcal{M}_{1,1}) = -\text{Area}(\mathcal{M}_{1,1}) = -\pi/3$$

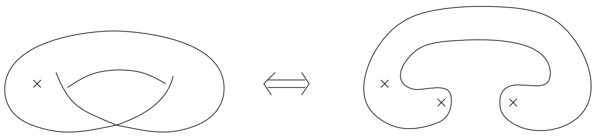
i.e.

$$\chi(\mathcal{M}_{1,1}) = -\frac{1}{6}.$$

One may find this result directly from the Euler formula, indeed a cellular decomposition of $\mathcal{M}_{1,1}$ is made of a dimension 2 contractible open set $\left\{-\frac{1}{2} < \mathrm{Re}\,\tau < \frac{1}{2}\right\} \cap \left\{\mathrm{Im}\,\tau > 0\right\} \cap \left\{|\tau| > 1\right\}$, 2 dimension one contractible open sets the open half-line $\{\mathrm{Re}\,\tau = 1/2\} \cap \{\mathrm{Im}\,\tau > \sqrt{3}/2\}$ and the open arc $\left\{\tau = \mathrm{e}^{i\theta}\,,\ \theta \in]\pi/3, \pi/2[\right\}$, and two conical points $\tau = \mathrm{i}$ with automorphism of order 2, and the point $\tau = \mathrm{e}^{i\pi/3}$ with automorphism of order 3, i.e. finally:

$$\chi(\mathcal{M}_{1,1}) = 1 - 2 + \frac{1}{2} + \frac{1}{3} = -\frac{1}{6}.$$

Remark 6.1.1 $\mathcal{M}_{1,1}$ is not compact. It can be compactified by adding a degenerate torus where a cycle has been pinched.



We shall identify this degenerate torus with the point $\tau = \mathrm{i}\infty$ we add to the hyperbolic upper complex plane $\mathbb{C}_+$. This degenerate torus can also be constructed as a sphere with three marked points, where we identify two marked points together (they correspond to the pinched cycle), and the third marked point is simply the initial marked point on the torus. In other words it is an element of $\mathcal{M}_{0,3}$.

We can thus define $\overline{\mathcal{M}}_{1,1}$ as the compactification of $\mathcal{M}_{1,1}$, obtained by adding this degenerate torus to $\mathcal{M}_{1,1}$:

$$\overline{\mathcal{M}}_{1,1} = \mathcal{M}_{1,1} \cup \mathcal{M}_{0,3} \qquad , \qquad \partial\mathcal{M}_{1,1} = \mathcal{M}_{0,3}.$$

From now on, our goal will be to "count" the number of Riemann surfaces of a given genus, or in other words measure the volume of the moduli space $\mathcal{M}_{g,n}$. In that purpose we have to define a "volume form" on it.

6.1.2 Stability and Unstability

Consider a Riemann surface of genus g, with n marked points (or n boundaries). It is said to be **stable** if it has a finite group of automorphisms, and **unstable** if the group of automorphisms is infinite.

The reason why stability matters, is because we wish to define a volume form, invariant under automorphisms, and if the automorphism group were infinite, the volume would unavoidably be infinite.

For instance the Riemann sphere has genus 0, and no marked point, its Euler characteristics is $\chi = 2$. It can be represented as the complex plane with an added point at ∞. It is clear that any Moebius transformation $z \mapsto \frac{az+b}{cz+d}$ with $(a, b, c, d) \in \mathbb{C}^4$, $ad - bc = 1$, maps the Riemann sphere bijectively onto itself, and is thus an automorphism. The group of automorphisms in $\mathcal{M}_{0,0}$ is $Sl_2(\mathbb{C})$, it is an infinite group.

If we consider the sphere with one marked point (topologically a disc $\chi = 1$), and we choose the marked point to be at ∞, then the automorphisms that conserve the marked point, are bijective maps of the form $z \mapsto az + b$. The group of automorphisms in $\mathcal{M}_{0,1}$ is the set of affine maps, this is still an infinite group of automorphisms.

If we consider the sphere with two marked points (topologically a cylinder $\chi = 0$), let us say the marked points are at 0 and ∞, all the linear transformations $z \mapsto az$ or inversions $z \mapsto a/z$ are automorphisms of the sphere with these two marked points. The group of automorphisms in $\mathcal{M}_{0,2}$ is still an infinite group.

Then, if we consider the Riemann sphere with three (or more) marked points, there can be at most a finite number of automorphisms preserving the marked points. Only the maps $z \mapsto \frac{az+b}{cz+d}$ that map marked points to marked points can be automorphisms. If the number of marked points is ≥ 3, that fixes the coefficients a, b, c, d. The group of automorphisms of $\mathcal{M}_{0,n}$ with $n \geq 3$ is then a subgroup of the permutation group of the marked points.

Therefore, the sphere ($g = 0$) has a finite number of automorphisms only if it has at least $n \geq 3$ marked points, i.e. $\chi = 2 - 2g - n < 0$.

Similarly, the torus can be represented as a parallelogram with identified opposite sides, i.e. the complex plane $\mathbb{C}$ quotiented by the lattice $\mathbb{Z} + \tau\mathbb{Z}$, however, the origin is arbitrary, i.e. the complex plane is invariant by translations, so $\mathcal{M}_{1,0}$ has an infinite number of automorphisms.

A torus with one (or more) marked point is no longer invariant by arbitrary translations, it has only a finite group of automorphisms.

We shall admit that every Riemann surfaces of genus $g \geq 2$ can be represented as a polygon with identified sides, embedded in the hyperbolic plane (the upper half complex plane, where geodesics are lines or half-circles orthogonal to the real axis), and using the properties of hyperbolic geometry, it is possible to prove that surfaces of genus $g \geq 2$ always have only a finite group of automorphisms, even if they have no marked points. To summarize:

- **Unstable surfaces:** the sphere $\chi = 2$, the disc $\chi = 1$, the cylinder $\chi = 0$, and the torus $\chi = 0$. They all have non-negative Euler characteristics $\chi = 2-2g-n \geq 0$
- **Stable surfaces:** all the others. They all have strictly negative Euler characteristics $\chi = 2 - 2g - n < 0$.

Examples

- $\mathcal{M}_{0,3}$. Every Riemann surface of genus 0 is conformally equivalent to the Riemann sphere, i.e. the complex plane with an added point at ∞. Then, by a suitable Moebius map $z \mapsto \frac{az+b}{cz+d}$, the three marked points can always be sent to $0, 1, \infty$. In other words, there is a unique element in $\mathcal{M}_{0,3}$.

$\mathcal{M}_{0,3}$ is a point, it has dimension $d_{0,3} = 0$.

- $\mathcal{M}_{1,1}$. See Sect. 6.1.1.3. Every Riemann surface of genus 1 can be mapped to a parallelogram of modulus τ, with identified opposite sides. The marked point can be mapped to the bottom-left corner of the parallelogram. We have

$$\mathcal{M}_{1,1} = \mathbb{C}_+/Sl_2(\mathbb{Z})$$

where $\mathbb{C}_+$ is the upper half complex plane. Locally (except near the points $\tau = i, e^{i\pi/3}$) $\mathcal{M}_{1,1}$ looks like a domain of $\mathbb{C}$, it can be described by a complex number τ, therefore

$$\dim \mathcal{M}_{1,1} = d_{1,1} = 1.$$

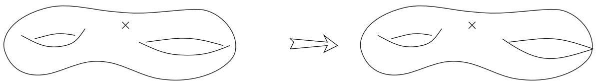
The point $\tau = e^{i\pi/3}$ is a conical singularity with a $\mathbb{Z}_3$ automorphism, and $\tau = i$ is a conical singularity with a $\mathbb{Z}_2$ automorphism.

6.1.3 Compactification

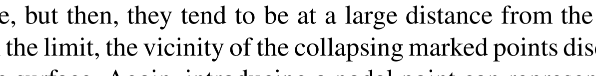
Let us consider $2 - 2g - n < 0$.

Similarly to Sect. 6.1.1.3, the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points can be compactified by adding degenerate surfaces to it.

$\mathcal{M}_{g,n}$ is not compact because the limit of a family of smooth Riemann surfaces of genus g with n marked points, might not be in $\mathcal{M}_{g,n}$. Either the limit is not smooth, because a cycle gets pinched, or also, two (or more) marked points might collapse in the limit.



In order to compactify $\mathcal{M}_{g,n}$, we need to add some "degenerate" surfaces that correspond to those limits. Pinched cycles naturally tend to nodal points. For collapsing marked points, we may, by a suitable conformal transformation, magnify the vicinity of those marked points, so that they become mutually separated by a finite distance, but then, they tend to be at a large distance from the other marked points, and in the limit, the vicinity of the collapsing marked points disconnects from the rest of the surface. Again, introducing a nodal point can represent this singular limit.



The degenerate surfaces we need to add, are thus "nodal" Riemann surfaces, i.e. Riemann surfaces with pinched cycles. Nodal Riemann surfaces can also be obtained by gluing together smooth Riemann surfaces at nodal points. In other words, a nodal surface is an union of ℓ smooth Riemann surfaces, each having genus g_i and n_i marked points and k_i nodal points.

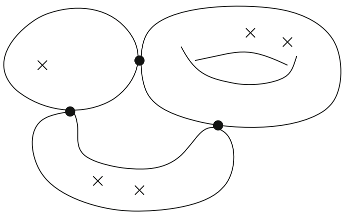
A stable nodal surface, is a nodal surface, whose each component is stable (if a component is a sphere, it must have at least three marked or nodal points, if it is a torus, it must have at least one marked or nodal point), i.e.

$$\forall\, i = 1, \dots, \ell\,, \qquad \chi_i = 2 - 2g_i - n_i - k_i < 0.$$

We must have $n = \sum_i n_i$, and $\sum_i k_i$ is even. The total Euler characteristics is:

$$\chi_{g,n} = 2 - 2g - n = \sum_{i=1}^{\ell} (2 - 2g_i - n_i - k_i).$$

Example of a nodal surface of $\overline{\mathcal{M}}_{2,5}$:



In this example the nodal surface has three components, one torus and two spheres, glued by three nodal points. The first sphere has one marked point and two nodal points, so that it is stable it has $\chi = -1$, the second sphere has two marked points and two nodal points i.e. $\chi = -2$, and the torus has two marked points and two nodal points $\chi = -4$, i.e. each component is stable $\chi_i < 0$. The total Euler characteristics is $-1 - 2 - 4 = -7 = 2 - 2 * 2 - 5$, it corresponds to genus $g = 2$ with $n = 5$ marked points, so it belongs to $\overline{\mathcal{M}}_{2,5}$.

Definition 6.1.2 (Deligne-Mumford Compactification) A stable curve $(C, p_1, \dots, p_n)$ is the data of a stable nodal Riemann surface C, with n smooth non-nodal marked points $p_1, \dots, p_n$.

The set of all stable curves $(C, p_1, \dots, p_n)$, modulo automorphisms, is called the compact moduli space $\overline{\mathcal{M}}_{g,n}$.

We shall admit here, that $\overline{\mathcal{M}}_{g,n}$ is compact. Let us check this on examples:

6.1.3.1 Example: $\overline{\mathcal{M}}_{0,4}$

An element (C, p_1, p_2, p_3, p_4) of $\mathcal{M}_{0,4}$ is a genus zero smooth Riemann surface (thus the Riemann sphere), with four labeled marked points.

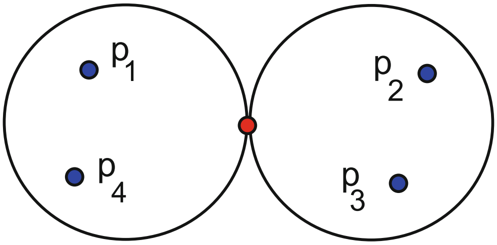
By a Moebius transformation, we can always assume that $p_1 = 0, p_2 = 1$, $p_3 = \infty$, and we call $p = p_4$. We must have $p \neq 0, 1, \infty$, and each value of p corresponds uniquely to a Riemann sphere with four distinct marked points. Therefore $\mathcal{M}_{0,4}$ is isomorphic to a Riemann sphere $\overline{\mathbb{C}}$ with three points removed:

$$\mathcal{M}_{0,4} \sim \overline{\mathbb{C}} \setminus \{0, 1, \infty\}.$$

It is a complex manifold of dimension 1, and it is not compact.

Its boundary consists of three limiting cases, $p \to 0$, $p \to 1$ and $p \to \infty$.

The limit $p \to 0$ i.e. $p_4 \to p_1$ corresponds to (C, p_1, p_2, p_3, p_4) getting split into two spheres glued at a nodal point, one sphere containing p_1, p_4 and the nodal point, and the other containing p_2, p_3 and the nodal point.



Same thing for $p_4 \to p_2$ and $p_4 \to p_3$.

We thus have:

$$\partial\mathcal{M}_{0,4} = (\mathcal{M}_{0,3} \times \mathcal{M}_{0,3}) \cup (\mathcal{M}_{0,3} \times \mathcal{M}_{0,3}) \cup (\mathcal{M}_{0,3} \times \mathcal{M}_{0,3}).$$

Each $\mathcal{M}_{0,3} \times \mathcal{M}_{0,3}$ is a point, which we shall identify respectively with the points $p = 0, 1, \infty$ of the Riemann sphere.

Finally we have:

$$\overline{\mathcal{M}}_{0,4} \sim (\bar{\mathbb{C}} \setminus \{0, 1, \infty\}) \ \cup \{0\} \cup \{1\} \cup \{\infty\} \sim \overline{\mathbb{C}},$$

i.e. $\overline{\mathcal{M}}_{0,4}$ is isomorphic to the full Riemann sphere, it is compact, and it is a smooth complex manifold of dimension $d_{0,4} = 1$.

6.1.3.2 Example: $\overline{\mathcal{M}}_{0,5}$

An element $(C, p_1, p_2, p_3, p_4, p_5)$ of $\mathcal{M}_{0,5}$ is a genus zero Riemann surface (thus the Riemann sphere), with five labeled marked points.

By a Moebius transformation, we can always assume that $p_1 = 0, p_2 = 1$, $p_3 = \infty$, and we call $p = p_4$, $q = p_5$. We must have $p, q \neq 0, 1, \infty$ and $p \neq q$, and each value of (p, q) corresponds uniquely to a Riemann sphere with five marked points. Therefore:

$$\mathcal{M}_{0,5} \sim (\overline{\mathbb{C}} \setminus \{0, 1, \infty\}) \times (\overline{\mathbb{C}} \setminus \{0, 1, \infty\}) \setminus \{(p, p) \,|\, p \in \overline{\mathbb{C}} \setminus \{0, 1, \infty\}\}.$$

$\mathcal{M}_{0,5}$ is thus a complex manifold of dimension $d_{0,5} = 2$, and it is not compact.

We see that

$$\mathcal{M}_{0,5} \subset \overline{\mathbb{C}} \times \overline{\mathbb{C}}.$$

One may wrongly think that the compactification $\overline{\mathcal{M}}_{0,5}$ would consist in completing the missing pieces of $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$. The missing pieces consist of:

- Seven dimension 1 sub-manifolds $(p = 0, q \in \mathbb{C} \setminus \{0, 1, \infty\})$, $(p = 1, q \in \mathbb{C} \setminus \{0, 1, \infty\})$, $(p = \infty, q \in \mathbb{C} \setminus \{0, 1, \infty\})$, $(p \in \mathbb{C} \setminus \{0, 1, \infty\}, q = 0)$, $(p \in \mathbb{C} \setminus \{0, 1, \infty\}, q = 1)$, $(p \in \mathbb{C} \setminus \{0, 1, \infty\}, q = \infty)$, $(p \in \mathbb{C} \setminus \{0, 1, \infty\}, q = p)$,
- and nine points $(p, q) = (0, 0), (1, 0), (\infty, 0), (0, 1), (1, 1), (\infty, 1), (0, \infty), (1, \infty), (\infty, \infty)$.

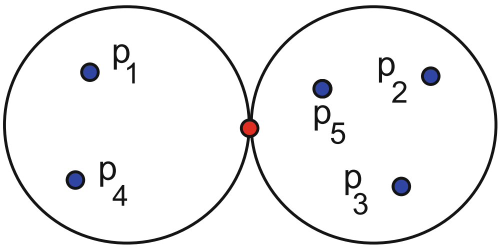
However, this is wrong. Indeed, let us now study the boundary of $\mathcal{M}_{0,5}$ in more details:

- A codimension 1 boundary, occurs when two marked points collapse, and the other points remain distinct. In that limit, the surface C splits into a sphere with the two collapsing marked points and a nodal point, and a sphere with the three other marked points and the nodal point, i.e. a codimension 1 boundary is isomorphic to

$$\mathcal{M}_{0,3} \times \mathcal{M}_{0,4} \quad \sim \mathbb{C} \setminus \{0, 1, \infty\}.$$

Notice that $\mathcal{M}_{0,4}$ itself is not compact and has three boundaries which are points.

For example the boundary $p_4 \to p_1$ with p_2, p_3, p_5 distinct, corresponds to $p \to 0$ and $q \neq 0, 1, \infty$, it can naturally be glued to the corresponding missing $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ of $\mathcal{M}_{0,5}$.



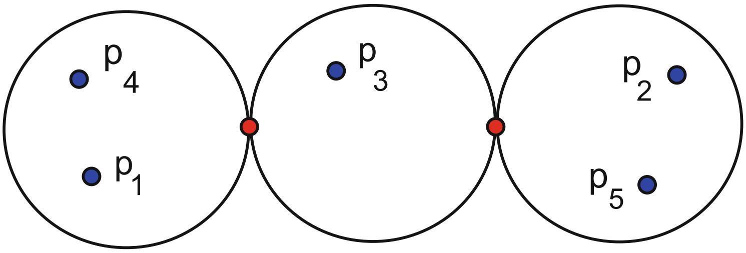
However, observe that there are ten dimension 1 boundaries, because there are ten possibilities of choosing a pair of collapsing points among five points. Therefore it is not possible to identify each of them to the seven missing codimension 1 pieces of $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$.

Seven of the dimension 1 boundaries can be easily identified with the seven dimension 1 missing sub–manifolds of $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$.

The three remaining dimension 1 boundaries are more subtle, they cannot be well described in the coordinates (p, q), and don't match with missing lines of $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$. For example $p_1 \to p_2$ with p_3, p_4, p_5 distinct, corresponds to $p \to \infty, q \to \infty$, and thus we can only glue it to the point (∞, ∞), we can not glue it to a line of $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$.

- Codimension 2 boundaries occur when two pairs of points collapse together. There are 15 possibilities of choosing two pairs of points among five points, so there are $15 = 9 + 6$ dimension 0 boundaries. Some of those points can be naturally glued to the nine missing points of $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$.

For instance, when $p_4 \to p_1$ and $p_5 \to p_2$, but p_1, p_2, p_3 remain distinct, the surface C gets split into three spheres, one with p_1, p_4 and a nodal point, one with p_5, p_2 and a nodal point, both glued by their nodal points to a sphere with p_3 and two nodal points. This corresponds to $p \to 0$ and $q \to 1$.

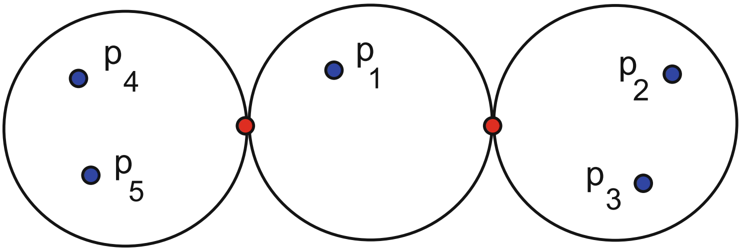


Such a boundary is thus:

$$\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \sim \text{ point}$$

and must be identified with the point $(0, 1) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}}$.

Another example is when $p_2 \to p_3$ and $p_5 \to p_4$, but p_1, p_2, p_4 remain distinct, the surface C gets split into three spheres, one with p_2, p_3 and a nodal point, one with p_4, p_5 and a nodal point, both glued by their nodal points to a sphere with p_1 and two nodal points. This corresponds to $p \to 0$ and $q \to 0$.



However, there are three ways to obtain a point corresponding to $p \to 0$ and $q \to 0$, namely: $p_2 \to p_3$, and then two of the points p_1, p_4, p_5 collapsing together.

Finally we have:

$$\partial \mathcal{M}_{0,5} = \overbrace{\mathcal{M}_{0,3} \times \mathcal{M}_{0,4}}^{10\,\text{times}} \cup \overbrace{\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}}^{15\,\text{times}} \sim \overbrace{\overline{\mathbb{C}} \setminus \{0,1,\infty\}}^{10\,\text{copies}} \cup \overbrace{\text{point}}^{15\,\text{copies}} .$$

By gluing seven of the ten $\mathcal{M}_{0,3} \times \mathcal{M}_{0,4}$ to the seven missing 1-dimensional pieces in the (p, q) plane, and nine of the 15 $\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}$ to the nine missing points in the (p, q) plane, we complete the (p, q) plane into $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$. There remains three $\mathcal{M}_{0,3} \times \mathcal{M}_{0,4} \sim \overline{\mathbb{C}} \setminus \{0, 1, \infty\}$, and six $\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}$ points. For each $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$, we can glue two $\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}$ points into the corresponding $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$, i.e. we get three copies of $\overline{\mathbb{C}} \setminus \{0, 1, \infty\} \cup \{\text{point}\} \cup \{\text{point}\} \sim \mathbb{C}$. Therefore, we finally get that:

$$\overline{\mathcal{M}}_{0,5} \sim \overline{\mathbb{C}} \times \overline{\mathbb{C}} \ \cup \ \mathbb{C} \ \cup \ \mathbb{C} \ \cup \ \mathbb{C}$$

where the three copies of $\mathbb{C} = \bar{\mathbb{C}} \setminus \{\text{point}\}$ i.e. three Riemann spheres with a point removed, are attached to $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ at the points $(0, 0), (1, 1), (\infty, \infty)$.

The topology of $\overline{\mathcal{M}}_{0,5}$ is already quite non-trivial. Also, this shows that $\overline{\mathcal{M}}_{0,5}$ is not a manifold of constant dimension, it has singular points, and it has subsets of smaller dimensions. It is called a "stack".

6.2 Informal Introduction to Intersection Numbers

Some information about the topology of $\mathcal{M}_{g,n}$ (resp. $\overline{\mathcal{M}}_{g,n}$) is provided by characteristic classes, which, in some sense, generalize the Euler characteristics. Remember that for a surface, the Euler characteristics is the integral of the curvature (for any metric) of the surface:

$$2\pi \chi = \int\int d^2x \ R(x),$$

(for example for the sphere in $\mathbb{R}^3$ of radius r, with the canonical metric of $\mathbb{R}^3$ the curvature is constant $R = 1/r^2$, the area is $4\pi r^2$, which gives $\chi = 2$). It is a topological invariant independent of the choice of a metric on the surface, and it is worth

$$\chi = 2 - 2g$$

where g is the genus, i.e. the number of holes of the surface. The Euler characteristics, i.e. the integral of the curvature, thus gives some information about the topology.

Chern classes generalize this idea, they are curvatures of connections over some fibre bundles, and again, the integrals of curvatures are topological invariants, called Chern numbers.

6.2.1 Informal Introduction to Chern Classes

Let us consider a complex manifold X of dimension n, with local coordinates x^μ, $\mu = 1, \dots, n$, and a complex line bundle $\mathcal{L}$ over X, i.e. to every point $x \in X$, we associate a copy of the complex plane $\mathbb{C}_x$. A non-vanishing section of $\mathcal{L}$, associates to every $x \in X$, a point $z(x) \in \mathbb{C}_x$, with $z(x) \neq 0$. In order to study the notion of analyticity, we need to define analytic invertible transition maps $f_{x\to x'} : \mathbb{C}_x \to \mathbb{C}_{x'}$, $z(x) \mapsto z(x')$, and such that $f_{x\to x'}$ depends analytically on x and x'. If x and $x' = x + dx$ are infinitesimally close to each other, the transition map has to be infinitesimally close to identity, and thus belongs to the cotangent space of the bundle:

$$f : \mathbb{C}_x \to \mathbb{C}_{x+dx} \quad , \qquad z \mapsto z\,(1 + 2\mathrm{i}\pi \sum_\mu A_\mu(x) dx^\mu) + O(dx^2).$$

The differential forms $A_\mu(x)dx^\mu$ belong to the cotangent space of X.

If we have an analytic non-vanishing section $z(x)$, we can compute its derivative, which contains two types of terms, those coming from the transition between fibres, and those which compute the derivative of the function $z(x)$ with respect to the local coordinates x^μ in one fibre, i.e. the total derivative is:

$$z(x + dx) - z(x) = \sum_\mu \frac{\partial z}{\partial x^\mu}\, dx^\mu + 2\mathrm{i}\pi\, z \sum_\mu A_\mu(x)\, dx^\mu + O(dx^2)$$

i.e. we can define the 1-form over the fibre bundle

$$\mathcal{D}z = dz + 2\mathrm{i}\pi\, z \sum_\mu A_\mu(x) dx^\mu = (d + 2\mathrm{i}\pi \sum_\mu A_\mu(x) dx^\mu)\, z$$

We emphasize that the notation $dz = \sum_\mu \frac{\partial z}{\partial x^\mu} dx^\mu$ depends on a choice of local coordinates and an explicit realization of $\mathbb{C}_x$ at fixed x, it is not intrinsically defined, only the sum of the 2-terms, which includes the transition map, i.e. $\mathcal{D}z$ has an intrinsic meaning. $A = d + 2i\pi \sum_\mu A_\mu(x)dx^\mu$ is called a connection on $\mathcal{L}$.

Since z is nowhere vanishing, we can divide by z and consider the 1-form:

$$\alpha = \frac{\mathcal{D}z}{2i\pi\ z} = \frac{dz}{2i\pi\ z} + \sum_\mu A_\mu(x)dx^\mu$$

which is analytic and well defined over the total space of the line bundle. It has the property that if we integrate it in a fibre at fixed x, around a non-contractible cycle of $\mathbb{C}_x^*$, i.e. over the unit circle $\mathcal{S}_x^1$ oriented in the trigonometric direction, we have:

$$\oint_{\mathcal{S}_x^1} \alpha = 1.$$

The curvature of α is the 2-form $d\alpha$:

$$d\alpha = \sum_{\mu,\nu} \frac{\partial A_\mu}{\partial x^\nu} dx^\nu \wedge dx^\mu$$

notice that $d^2 z = 0$ so that the coordinate along the fibre has disappeared in the curvature. The curvature $d\alpha$ is thus independent of a choice of section $z(x)$. Also, one can symmetrize over μ and ν and write:

$$d\alpha = \frac{1}{2}\sum_{\mu,\nu}\left(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}\right) dx^\nu \wedge dx^\mu.$$

It can be proved that the cohomology class of the 2-form $d\alpha$ is independent of a choice of connection $A = \sum_\mu A_\mu dx^\mu$, i.e. a contour integral $\oint_C d\alpha$ depends only on the homology class of the contour $C \subset X$, and not on the choice of connection A, but this is beyond the scope of this book.

What we would like the reader to retain, is that in order to compute the Chern class of a complex line bundle $\mathcal{L}$ over a manifold X, one has to find a 1-form α well-defined everywhere on the total space of $\mathcal{L}$, whose integral along a circle in a fibre, is:

$$\oint_{\text{fibre } S_x^1} \alpha = 1$$

and then the Chern class (or more precisely a representative) is defined as its curvature:

$$c_1(\mathcal{L}) = d\alpha$$

and $d\alpha$ is a 2-form on T^*X, whose cohomology class is independent of a choice of α. The Chern class is a topological invariant of the bundle $\mathcal{L}$.

Remark (Computation of the Chern Class of the Trivial Bundle) The trivial bundle is the bundle $\mathcal{L}$ whose fibre $\mathbb{C}_x$ is the same for all x, i.e. $\mathcal{L} = X \times \mathbb{C}$. The transition map can be chosen as the identity and we can chose $A_\mu = 0$, and thus $c_1(\mathcal{L}) = 0$. Vice-versa, if one finds that the Chern class $c_1(\mathcal{L}) \neq 0$, this means that the bundle $\mathcal{L}$ is not homeomorphic to a trivial bundle.

The converse is not true, the vanishing of $c_1(\mathcal{L})$ doesn't imply that the bundle is trivial.

Remark (Computation of the Chern Class of Product Bundles) Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two line bundles over a manifold X. Then we can define the line bundle $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ over X by simply taking the Cartesian product of each fibres. Imagine that we have a connection A on $\mathcal{L}_1$ and $\tilde{A}$ on $\mathcal{L}_2$, then $A + \tilde{A}$ is a connection on the product $\mathcal{L}$, and thus the Chern classes add:

$$c_1(\mathcal{L}) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2).$$

In particular if $\mathcal{L}_2$ is a trivial bundle then

$$c_1(\mathcal{L}) = c_1(\mathcal{L}_1).$$

This remark will be very useful for us, we shall consider line bundles over $\overline{\mathcal{M}}_{g,n}$, that we shall extend to line bundles over $\overline{\mathcal{M}}_{g,n} \times \mathbb{R}_+^n$, this will not change their Chern class.

6.2.2 Intersection Numbers of Cotangent Bundles

Let $\overline{\mathcal{M}}_{g,n}$ be the compact moduli space of stable curves of genus g, with n marked points. Since each point p_i is smooth, we have a natural line bundle $\mathcal{L}_i$ over $\overline{\mathcal{M}}_{g,n}$, whose fibre, for each point $(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$, is $T^*_{p_i}C$ the cotangent space of C at p_i. We can consider its first Chern class $c_1(\mathcal{L}_i)$, that is the curvature form (in fact its cohomology class) of an arbitrary connection on that line bundle. $c_1(\mathcal{L}_i)$ is a 2-form on $\overline{\mathcal{M}}_{g,n}$, and it is a topological invariant, independent of the choice of connection.

We usually denote

$$\psi_i = c_1(\mathcal{L}_i).$$

Since ψ_i is a 2–form on $\overline{\mathcal{M}}_{g,n}$, the wedge product of $d_{g,n} = \dim \overline{\mathcal{M}}_{g,n}$ such 2–forms, is a top dimensional symplectic volume form on $\overline{\mathcal{M}}_{g,n}$, and thus one can compute its integral on $\overline{\mathcal{M}}_{g,n}$.

Definition 6.2.1 (Intersection Numbers) Let $\mathcal{L}_i \to \overline{\mathcal{M}}_{g,n}$ the cotangent bundle to the ith marked point p_i, and $\psi_i = c_1(\mathcal{L}_i)$ its first Chern class. The intersection numbers are defined as:

$$< \psi_1^{k_1} \dots \psi_n^{k_n} >_{g,n} := \begin{cases} \int_{\overline{\mathcal{M}}_{g,n}} c_1(\mathcal{L}_1)^{k_1} \wedge \dots \wedge c_1(\mathcal{L}_n)^{k_n} & \text{if } \sum_i k_i = d_{g,n} \\ 0 & \text{if } \sum_i k_i \neq d_{g,n}. \end{cases}$$

Very often, one uses Witten's notation:

$$< \tau_{k_1} \dots \tau_{k_n} >_g := < \psi_1^{k_1} \dots \psi_n^{k_n} >_{g,n} .$$

(we don't need to write the subscript n, since n is seen as the number of τ factors).

Intersection numbers are topological invariants of $\overline{\mathcal{M}}_{g,n}$.

We recall that the moduli space $\overline{\mathcal{M}}_{g,n}$ is not a manifold. It contains non-smooth points, corresponding to Riemann surfaces with non-trivial automorphisms, and the moduli space is defined by quotienting with the automorphisms group, this means that the intersection numbers can be rational numbers instead of integers (denominators correspond to the order of automorphism groups). Also, since $\overline{\mathcal{M}}_{g,n}$ may contain pieces of different dimensions, the notion of cycle is not exactly the notion of sub-manifolds, instead it is related to the notion of cycles and chains in DeRham cohomology. However, those notions being beyond the scope of this book, we shall stay at the intuitive level.

6.2.2.1 Witten's Conjecture and Kontsevich Integral

In order to compute the intersection numbers, Maxim Kontsevich in 1991 [57], used an explicit foliation of the space $\mathcal{M}_{g,n}$, already introduced by Harer–Mumford–Strebel–Thurston–Zagier–Penner [45, 75], with an explicit coordinate system, and he found an explicit connection α_i, and thus an explicit represent of each Chern class $d\alpha_i = \psi_i = c_1(\mathcal{L}_i)$ in this coordinate system. In practice that means finding a 1-form α_i on $\mathcal{L}_i$ whose integral around a circle in each fibre is 1, and then its curvature form $d\alpha_i$ is a representant of the Chern class ψ_i.

The explicit foliation of the space $\mathcal{M}_{g,n}$, is based on graphs, and thus, using his coordinate system, Kontsevich could reduce the computation of intersection numbers to combinatorics of graphs, and, using Wick's theorem again (see Chap. 2), showed that they can be put together to form a generating series which is a formal matrix integral. He used that to show that the generating series is a Tau-function for the KdV hierarchy, thus proving Witten's conjecture.

Witten's conjecture came from the problem of enumeration of maps: if maps could be seen as a good "discretization" of Riemann surfaces, then the "discretized" intersection numbers would just be the number of maps with given boundaries and given topology, and thus the double scaling limit of the generating function for the number of large maps (see Chap. 5), should coincide with the generating function for intersection numbers. And it was already known from heuristic asymptotic approximations in matrix models, that the double scaling limits of matrix models had good chances to be a Tau-function for the KdV hierarchy (proved in Chap. 5). This led Witten to conjecture [87] that the generating function of intersection numbers had to be a KdV Tau–function.

The physical idea was clear, the mathematical proof came with the work of Kontsevich in 1991. Since then, Witten's conjecture has received many other proofs.

In some sense, this Witten-Kontsevich theorem, is the claim that the limit of large maps, is "topological gravity".

What was surprising in Kontsevich's proof, was that the matrix integral he used, was in fact very different from the formal matrix integrals of Brezin–Itzykson–Parisi–Zuber seen in Chap. 2 for counting discretized surfaces. He used a formal matrix integral which directly corresponds to Riemann surfaces, not using a discretization and sending a mesh to 0.

6.3 Parametrizing Surfaces

A point in the moduli space $\mathcal{M}_{g,n}$ is a Riemann surface of genus g with n marked points, and a Riemann surface is an equivalence class modulo bijective conformal reparametrizations. In order to describe the moduli space, one needs to find a unique canonical represent of a Riemann surface for each point in the moduli space. In other words, if the moduli space is a finite dimensional manifold, parametrized by $d_{g,n}$ complex moduli, or $2d_{g,n}$ real moduli, we need to generate a unique surface out of $2d_{g,n}$ real numbers. The idea is to cut the surface into slices, this is called a foliation of the surface.

Several methods of foliations have been invented, and here we present two of them.

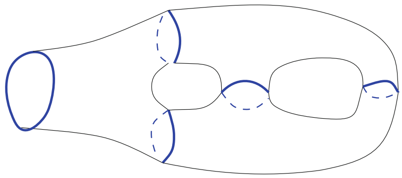
6.3.1 Teichmüller Hyperbolic Foliation

Consider a smooth surface of genus g, with n boundaries (instead of n marked points), and assume $2-2g-n<0$, which implies that its average total curvature is negative:

$$\text{curvature} = 2\pi\chi = 2\pi(2-2g-n) < 0.$$

It is possible to find on that surface, a Riemannian metric of constant negative curvature -1, such that the boundaries are geodesics of prescribed lengths $L_1,\dots,L_n$. This is called the Poincaré metric. The surface can then naturally be embedded into the hyperbolic plane $\mathbb{H}$, i.e. the upper complex plane $\mathbb{C}_+$ endowed with the hyperbolic geometry (whose geodesics are half-circles or straight lines, orthogonal to the real axis).

It is then possible to find closed geodesics, cutting the surface into "**pairs of pants**".



Therefore, any surface of genus g with n boundaries, is conformally equivalent to the gluing of pairs of pants along circles. The number of pairs of pants and circles can be computed by the Euler characteristics. Each pair of pants has Euler characteristics -1, therefore the number of pairs of pants is

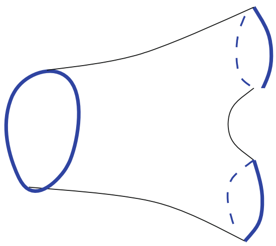
$$\#\text{pairs of pants} = 2g - 2 + n.$$

Moreover, each pair of pants has three boundaries, which implies $3(2g - 2 + n) = n + 2\#\text{inner circles}$, and thus the number of inner circles is

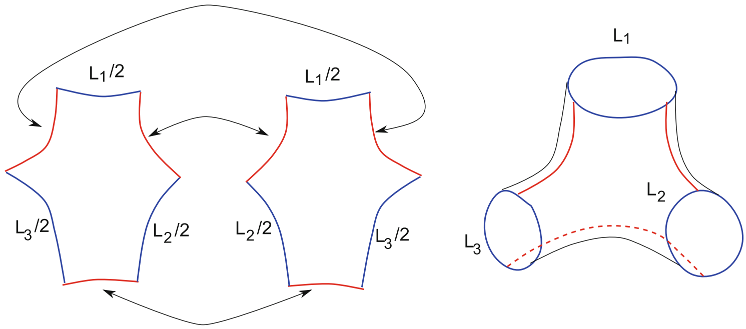
$$\#\text{inner circles} = 3g - 3 + n.$$

Every surface of genus g with n boundaries can be obtained by gluing $2g - 2 + n$ pairs of pants along $3g - 3 + n$ circles.

A classical result in hyperbolic geometry, is that an hyperbolic pair of pants is uniquely characterized by the three lengths of its three geodesic boundaries, which are positive real numbers.



This comes from the fact that in the hyperbolic plane, there is a unique (up to isometries) right angles hexagon with geodesic boundaries with three given lengths,



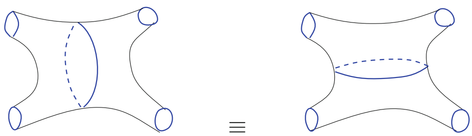
and a pair of pant is obtained by gluing two identical hexagons.

Two pairs of pants can be glued together conformally, if and only if the geodesic boundaries to be glued together have the same length. However, the boundaries can be rotated by an arbitrary twist angle before gluing. In other words, a genus g Riemann surface with n boundaries is entirely characterized by $3g-3+n$ positive real lengths, together with $3g-3+n$ gluing angles, i.e. in total by $6g-6+2n$ real parameters. This shows that:

$$\frac{1}{2}\dim_{\mathbb{R}} \mathcal{M}_{g,n} = d_{g,n} = 3g-3+n$$

in agreement with a complex dimension $\dim_{\mathbb{C}} \mathcal{M}_{g,n} = d_{g,n} = 3g-3+n$.

However, this description of $\mathcal{M}_{g,n}$ is valid only locally, indeed the decomposition into pants is not unique because of Dehn twists and pants flops, for example:



We don't have a global bijection between $\mathcal{M}_{g,n}$ and $R_+^{3g-3+n} \times [0, 2\pi[^{3g-3+n}$, the bijection is only valid locally, and $\mathcal{M}_{g,n}$ has a non-trivial topology.

Nevertheless one can show that $\prod_{i=1}^{3g-3+n} dl_i \wedge d\theta_i$ is a symplectic volume form well-defined globally (it is invariant under Dehn twists and pants flops, i.e. it is independent of a choice of cutting into pairs of pants), and which can be used to define the Weil-Petersson volumes

$$V_{g,n}(L_1, \ldots, L_n) = \int_{\overline{\mathcal{M}}_{g,n}(L_1,\ldots,L_n)} \prod_{i=1}^{3g-3+n} dl_i \wedge d\theta_i$$

of the moduli spaces of curves with n boundaries of fixed geodesic lengths $L_1, \ldots, L_n$.

We are going to compute those volumes in Sect. 6.6. For instance, one easily gets that:

$$V_{0,3}(L_1, L_2, L_3) = 1,$$

(indeed $\overline{\mathcal{M}}_{0,3}$ is a point, the integral is trivial), and with some efforts using hyperbolic geometry:

$$V_{1,1}(L_1) = \frac{1}{48}\,(4\pi^2 + L_1^2).$$

It was proved by Wolpert [88], that this symplectic volume form is a topological class. The Weil-Petersson metric form $\sum_i dl_i \wedge d\theta_i$, is nothing but the κ_1 Mumford class (see Sect. 6.6 below):

$$\sum_i dl_i \wedge d\theta_i = 4\pi^2\kappa_1$$

which implies, by raising it to the power $d_{g,n} = 3g - 3 + n$:

$$(4\pi^2\kappa_1)^{d_{g,n}} = d_{g,n}!\,\prod_{i=1}^{d_{g,n}} dl_i \wedge d\theta_i.$$

We shall admit here, that the boundaries can be encoded into Chern classes, and we admit the identity:

$$V_{g,n}(L_1,\ldots,L_n) = 2^{-d_{g,n}} \sum_{d_0+d_1+\cdots+d_n=d_{g,n}} \frac{(4\pi^2)^{d_0}}{d_0!} \prod_{i=1}^{n} \frac{L_i^{2d_i}}{d_i!} \; < \kappa_1^{d_0}\,\psi_1^{d_1}\ldots\psi_n^{d_n} >_{g,n}$$

which is a polynomial in the L_i's, and which we can rewrite as:

$$V_{g,n}(L_1,\ldots,L_n) = \frac{1}{d_{g,n}!} \; < (2\pi^2\kappa_1 + \frac{1}{2}\sum_i L_i^2\psi_i)^{d_{g,n}} >_{g,n} .$$

Using the fact that intersection numbers of classes whose dimension is not the expected dimension $d_{g,n}$ are defined to be vanishing, we may rewrite this as:

$$V_{g,n}(L_1,\ldots,L_n) = \left\langle \mathrm{e}^{2\pi^2\kappa_1 + \frac{1}{2}\sum_i L_i^2\psi_i} \right\rangle_{g,n}.$$

We are going to show how to compute it below in Sect. 6.6. In 2004, M. Mirzakhani found a recursion, based on hyperbolic geometry, and in particular the Mac-Shane relation among lengths of geodesics, to compute recursively the volumes $V_{g,n}(L_1,\dots,L_n)$. Mirzakhani's recursion is the Laplace transform of the topological recursion which we derive in Sect. 6.6 below. It earned her the Fields medal in 2014 [65].

6.3.2 Strebel Foliation

Instead of Teichmüller decomposition into pants, Kontsevich used the "Strebel" foliation, in his famous article of 1992 [57]. Given n marked points on a Riemann surface of genus g, and given n positive real numbers $L_1,\dots,L_n$ (called perimeters), Strebel's theorem [80] (Theorem 6.3.1 below) asserts that there exists a unique Strebel quadratic differential Ω with double poles at the marked points with residues equal to $-L_i^2$.

Definition 6.3.1 Let C be a compact Riemann surface of genus g. Ω is a quadratic differential, if in every chart $U\subset C$, with local coordinate z, Ω is of the form:

$$\Omega(z)=f(z)\ dz^2$$

where $f(z)$ is meromorphic in U.

If $\Omega(z)$ has a double pole at p, of the form:

$$\Omega(z)\sim\frac{R}{(z-p)^2}\ dz^2\ (1+O(z-p)),$$

the coefficient $R\in\mathbb{C}$ is independent of the choice of a local coordinate, and is called the "residue" of Ω at p.

A quadratic differential Ω is such that $\sqrt{\Omega}$ is locally a 1-form on C, but not globally (indeed it is not analytic at the zeroes or poles of Ω). $\sqrt{\Omega}$ can be used to compute integrals along paths.

Definition 6.3.2 Let Ω be a quadratic differential. Horizontal trajectories of Ω are defined as lines

$$\mathrm{Im}\left(\int^x\sqrt{\Omega}\right)=\text{constant}.$$

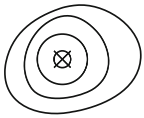
Let Ω be a quadratic differential with a double pole at $p_i\in C$, with negative residue $R_i=-L_i^2\in\mathbb{R}_-$. We have:

$$\sqrt{\Omega(z)}\sim \mathrm{i}\,L_i\,\frac{dz}{z-p_i}\,(1+O(z-p_i))$$

and thus

$$\int^z \sqrt{\Omega} \underset{z\to p_i}{\sim} \mathrm{i}\, L_i \,\ln(z-p_i) \,+\, \text{analytic}$$

and thus horizontal trajectories of Ω near the pole p_i, are topologically circles encircling p_i:



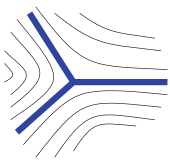
Near a simple zero of Ω

$$\Omega(z) \sim c_a\,(z-a)\,dz^2\;(1+O(z-a)) \quad , \quad \sqrt{\Omega(z)} \sim \sqrt{c_a}\,\sqrt{z-a}\,dz\;(1+O(z-a))$$

and thus

$$\int^z \sqrt{\Omega} \underset{z\to a}{\sim} \sqrt{c_a}\,(z-a)^{3/2}\;(1+O(z-a))$$

and thus three horizontal trajectories (called critical) meet at a, at angles $2\pi/3$:



If a is a higher order zero of Ω, i.e. $\Omega \sim (z-a)^k\,dz^2$, one has $\int^z \sqrt{\Omega} \sim (z-a)^{1+k/2}$ and thus $k+2$ horizontal trajectories meet at a.

Definition 6.3.3 (Strebel Differential) We say that Ω is a Strebel differential, if Ω is a quadratic differential with at most double poles, with negative residues, and if the union of all circle trajectories surrounding double poles

$$U = \cup\, \text{circle trajectories around double poles}$$

is such that

$$\overline{U} = C,$$

In that case, $C \setminus U$ which is the set of horizontal trajectories which are nor circles (called critical trajectories) is a graph on C, called the "Strebel graph".

Another way to say that, is that the graph of critical trajectories, is a cellular graph (all the faces are homeomorphic to discs).

Strebel's theorem is that:

Theorem 6.3.1 (Strebel's Theorem [80]) *If* $(C, p_1, \dots, p_n) \in \mathcal{M}_{g,n}$*, and* $L_1, \dots, L_n \in \mathbb{R}_+^n$*, there exists a unique Strebel differential with double poles at* p_i*'s with residues* $-L_i^2$*.*

The critical horizontal trajectories of the Strebel differential Ω *form a unique ribbon graph drawn on the Riemann surface C, whose n faces are topological discs surrounding the marked points* p_i*'s, and the perimeter (measured with the metric* $\frac{1}{2\pi}|\sqrt{\Omega}|$*) of the* ith *face is* L_i*.*

Example: Consider $\mathcal{M}_{0,3}$, i.e. the Riemann sphere with three marked points $0, 1, \infty$. Choose three positive perimeters L_0, L_1, L_∞.

The Strebel differential is:

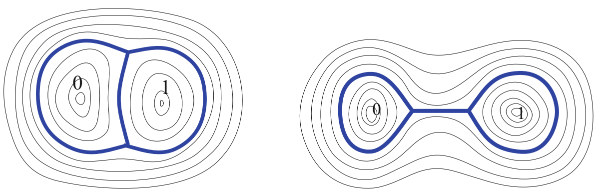
$$\Omega(z) = -\frac{L_\infty^2 z^2 - (L_\infty^2 + L_0^2 - L_1^2)z + L_0^2}{z^2(z-1)^2}\, dz^2 = -L_\infty^2 \frac{(z-a)(z-b)}{z^2(z-1)^2}\, dz^2,$$

indeed, it has three poles at $z = 0, 1, \infty$ and behaves like $-L_0^2 dz^2/z^2$ near $z = 0$, like $-L_1^2 dz^2/(z-1)^2$ near $z = 1$ and like $-L_\infty^2 dz^2/z^2$ near $z = \infty$, and we leave to the reader to check that this is the unique quadratic differential having those properties.

The vertices are located at the zeroes a, b of Ω:

$$a, b = \frac{1}{2L_\infty^2}\left(L_\infty^2 + L_0^2 - L_1^2 \pm \sqrt{L_0^4 + L_1^4 + L_\infty^4 - 2L_0^2L_1^2 - 2L_0^2L_\infty^2 - 2L_1^2L_\infty^2}\right).$$

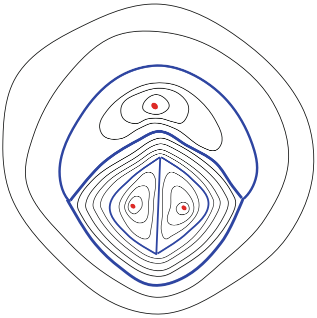
Here are the horizontal trajectories and ribbon graphs, for the cases ($L_\infty \le L_0 + L_1$ and $L_\infty \ge L_0 + L_1$):



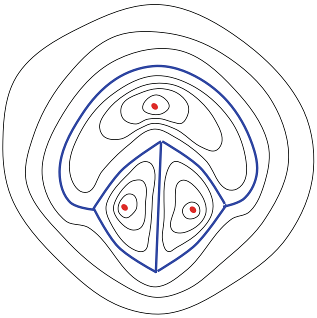
Consider now $\mathcal{M}_{0,4}$, i.e. the Riemann sphere with four marked points $0, 1, \infty, q$. Choose four positive perimeters L_0, L_1, L_∞, L_q. Any quadratic differential with double poles with residues $-L_i^2$ must be of the form:

$$\Omega(z) = \frac{-dz^2}{z(z-1)(z-q)}\left(L_\infty^2 z + \frac{qL_0^2}{z} + \frac{(1-q)L_1^2}{z-1} + \frac{q(q-1)L_q^2}{z-q} + c\right)$$

where c is a constant. For arbitrary values of c, the horizontal trajectories may be circles which don't surround one pole but two poles, i.e. the complement of the graph of critical trajectories has a non-simply connected face (a face with the topology of a cylinder), as follows:



Strebel's theorem says that there is a unique value of $c \in \mathbb{C}$, function of q and of the L_i's (in general this is not an analytic function, it depends separately on $\operatorname{Re} q$ and $\operatorname{Im} q$) such that the union of all circle trajectories surrounding double poles is dense on the surface, i.e. the graph of critical trajectories is cellular, and we get the Strebel graph:



Let us return to the general case.

Introduce the length l_e of each edge e of the ribbon graph, measured with the metric $\frac{1}{2\pi}|\sqrt{\Omega}|$. Since a generic ribbon graph has only 3-valent vertices (graphs with higher valency vertices can be viewed as trivalent graphs with some edges of vanishing lengths), the number of edges and the number of vertices are related by:

$$2\#\text{edges} = 3\#\text{vertices}.$$

And since the Euler characteristics is

$$\chi = 2 - 2g = \#\text{faces} - \#\text{edges} + \#\text{vertices}$$

the number of edges is:

$$\#\text{edges} = 3(2g-2+n) = n + 2(3g-3+n).$$

Therefore, to a Riemann surface with n marked points, and n perimeters $L_1, \dots, L_n$, we can associate a unique metric ribbon graph with $n + 2(3g-3+n)$ edge lengths (positive real numbers) $l_e > 0$.

The converse is true as well, i.e. given a ribbon graph of genus g with n faces, and given the $n + 2(3g-3+n)$ lengths of its edges, we can reconstruct a unique Riemann surface with n marked points, by gluing conformally n discs along the edges of the ribbon graph, as well as n positive real numbers $L_1, \dots, L_n$ which are the perimeters of the discs $L_i = \sum_{e\mapsto i} l_e$.

Therefore, because of the uniqueness of the Strebel differential, we have a bijection:

Theorem 6.3.2 *We have the isomorphism of orbifolds:*

$$\mathcal{M}_{g,n} \times \mathbb{R}_+^n \quad \sim \bigcup_{\text{Ribbon graphs}} \mathbb{R}_+^{n+2(3g-3+n)}.$$

This is an isomorphism of orbifolds, i.e. modulo automorphisms. This means that for curves in $\mathcal{M}_{g,n}$ which have a non-trivial automorphism group, the corresponding ribbon graph has the same automorphism group.

We say that we have a decomposition of our moduli space $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$ into cells (each cell is isomorphic to $\mathbb{R}_+^{n+2(3g-3+n)}$) labeled by ribbon graphs.

This bijection shows again that the complex dimension of the manifold $\mathcal{M}_{g,n}$ is

$$d_{g,n} = \dim_{\mathbb{C}} \mathcal{M}_{g,n} = \frac{1}{2} \dim_{\mathbb{R}} \mathcal{M}_{g,n} = 3g - 3 + n.$$

In each cell (for each ribbon graph), $\prod_e dl_e$ is a top-dimensional symplectic volume form on $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$.

6.3.3 Chern Classes

Thanks to the Strebel foliation, we have for each cell (i.e. each ribbon graph) an explicit set of coordinates on $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$, given by the lengths l_e of edges e of the ribbon graph.

We can also represent the line bundle $\mathcal{L}_i$ whose fibre is the cotangent space at the marked point p_i. Remember that the point p_i is the point at the center of face i of the graph. Face i has a total perimeter L_i:

$$L_i = \sum_{e \text{ around face } i} l_e.$$

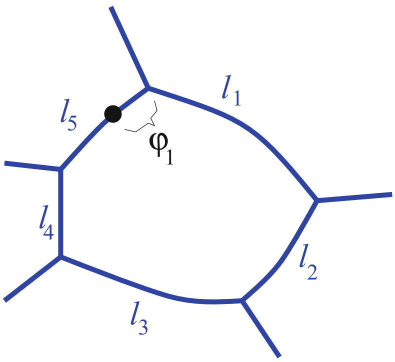
A section of a $U(1)$ connection on the cotangent bundle on $\tilde{\mathcal{L}}_i \to \mathcal{M}_{g,n} \times \mathbb{R}_+^n$, consists in choosing an angle $e^{2i\pi\varphi} \in U(1)$, or equivalently choosing a point on a circle, or also choosing a marked point on the boundary of the face:

$$\tilde{\mathcal{L}}_i \quad \sim \bigcup_{\text{Ribbon graphs with marked point on boundary of face } i} \mathbb{R}_+^{\text{edges}}.$$

Such a marked point being chosen for each graph, let us define the distances of vertices of the face to that point. We chose arbitrarily a labeling of vertices $v_1, v_2, v_3, \dots, v_{\deg \text{face } i}$ around face p_i, with a clockwise ordering, in other words we arbitrarily choose a "first" vertex v_1. Then we define the distances of the marked point:

- the vertex number 1, is at distance φ_1 from the marked point,
- the vertex number 2, is at distance $\varphi_2 = \varphi_1 + l_1$ from the marked point,
- the vertex number 3, is at distance $\varphi_3 = \varphi_1 + l_1 + l_2$ from the marked point,
- and so on, vertex number k is at distance $\varphi_k = \varphi_1 + l_1 + \cdots + l_{k-1}$ from the marked point.

And all those distances are computed modulo L_i, $\varphi_k \equiv \varphi_k + L_i$.



Each $d\varphi_i$ is a $U(1)$ connection on the fibre, but is not defined globally on $\tilde{\mathcal{L}}_i$, indeed, the labels of vertices are not globally defined, they depend on our arbitrary choice of v_1. Only quantities which are symmetric in the vertices of a face, can be globally defined.

The following 1-form

$$\alpha_i = \sum_{e \text{ around face } i} \frac{l_e}{L_i}\, d\,\frac{\varphi_e}{L_i} = d\frac{\varphi_1}{L_i} + \sum_{e'<e} \frac{l_e}{L_i}\, d\,\frac{l_{e'}}{L_i}$$

is well defined on $\tilde{\mathcal{L}}_i$, indeed all vertices around the face now play a symmetric role, it is independent of a choice of v_1.

When we integrate α_i along the fibre of the $U(1)$ bundle, i.e. when the marked point goes around the face counterclockwise, the lengths l_e and L_i are unchanged, we only integrate φ_e from 0 to L_i, and for each vertex we have $\int d\varphi_e = -L_i$, therefore

$$\int \alpha_i = -1.$$

α_i is thus a 1-form globally defined on $\tilde{\mathcal{L}}_i$, and it is a $U(1)$ connection in each fibre. This implies that its curvature is the Chern class of $\tilde{\mathcal{L}}_i$:

$$\tilde{\psi}_i = c_1(\tilde{\mathcal{L}}_i) = d\alpha_i = \sum_{e'<e} d\,\frac{l_e}{L_i} \wedge d\,\frac{l_{e'}}{L_i}.$$

Notice that $\mathbb{R}_+^n$ is a trivial bundle, on which we can easily define a trivial connection whose curvature is zero, and therefore, under the projection $\mathcal{M}_{g,n} \times \mathbb{R}_+^n \to \mathcal{M}_{g,n}$, the line bundle $\tilde{\mathcal{L}}_i$ is pushed to the cotangent bundle $\mathcal{L}_i$, and the Chern class $\tilde{\psi}_i = c_1(\tilde{\mathcal{L}}_i)$ is pushed to $\psi_i = c_1(\mathcal{L}_i)$. By abuse of language we shall write that

$$\psi_i = \tilde{\psi}_i = c_1(\mathcal{L}_i) = c_1(\tilde{\mathcal{L}}_i).$$

Theorem 6.3.3 (Kontsevich 1991 [57]) *The 2-form ψ_i on $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$, defined in each cell (each ribbon graph) by:*

$$\psi_i = \sum_{e'<e} d\,\frac{l_e}{L_i} \wedge d\,\frac{l_{e'}}{L_i}.$$

is the first Chern class of the bundle whose fibre is the cotangent bundle at the ith *marked point:*

$$\psi_i = c_1(\mathcal{L}_i \times \mathbb{R}_+^n) = c_1(\tilde{\mathcal{L}}_i).$$

Moreover, since $\tilde{\mathcal{L}}_i$ is a direct product $\tilde{\mathcal{L}}_i = \mathcal{L}_i \times \mathbb{R}_+^n$, and since $\mathbb{R}_+^n$ is flat (one can obviously find a constant connection whose curvature vanishes), the Chern class of $\mathcal{L}_i$ is the push forward of the Chern class of $\tilde{\mathcal{L}}_i$, by the projection $\mathcal{M}_{g,n} \times \mathbb{R}_+^n \to \mathcal{M}_{g,n}$.

Our goal now, is to compute "intersection numbers", i.e. integrals of Chern classes.

Remember that we have defined **Intersection numbers** in Definition 6.2.1 above:

$$< \psi_1^{k_1} \dots \psi_n^{k_n} >_{g,n} := \begin{cases} \int_{\overline{\mathcal{M}}_{g,n}} \quad c_1(\mathcal{L}_1)^{k_1} \dots c_1(\mathcal{L}_n)^{k_n} & \text{if } k_1 + \dots + k_n = d_{g,n} \\ 0 & \text{if } k_1 + \dots + k_n \neq d_{g,n}. \end{cases}$$

And very often, one uses Witten's notation:

$$< \tau_{k_1} \dots \tau_{k_n} >_g := < \psi_1^{k_1} \dots \psi_n^{k_n} >_{g,n} .$$

In Witten's notation, the index n is not needed, it is encoded as the number of τ factors.

6.3.4 Computing Intersection Numbers

Consider the differential form on $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$:

$$\Omega = \left(\sum_{i=1}^{n} L_i^2 \, \psi_i \right)^{d_{g,n}} \wedge dL_1 \wedge dL_2 \wedge \dots \wedge dL_n.$$

Since $\sum_i L_i^2 \psi_i$ is a 2-form, Ω is of order $2d_{g,n} + n = 3(2g-2+n) = \dim_{\mathbb{R}} \mathcal{M}_{g,n} \times \mathbb{R}_+^n$, i.e. it is a top dimensional form on $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$.

Since $d\frac{l_e}{L_i} = \frac{dl_e}{L_i} - l_e \frac{dL_i}{L_i^2}$ and since we multiply by $\prod_i dL_i$ and $dL_i \wedge dL_i = 0$, we may drop the $-l_e \frac{dL_i}{L_i^2}$ term and replace $d\frac{l_e}{L_i}$ by $\frac{1}{L_i} dl_e$ in the product, i.e. replace $L_i^2 \psi_i$ by $\sum_{e'<e \text{ , in face } i} dl_e \wedge dl_{e'}$, and thus, in each cell given by a ribbon graph, we have

$$\Omega = \left(\sum_i \sum_{e'<e \text{ , in face } i} dl_e \wedge dl_{e'} \right)^{d_{g,n}} \prod_i dL_i$$

so that Ω is a top-dimensional volume form on $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$, and has constant coefficients in the coordinates l_e. Therefore, up to a proportionality factor we have:

$$\Omega \propto \prod_{e=\text{edges}} dl_e.$$

The proportionality factor was computed by Kontsevich [57] (and it is really the hard part of Kontsevich's computation, see also [21]), and it turns out that it doesn't depend on the graph, it depends only on g and n.

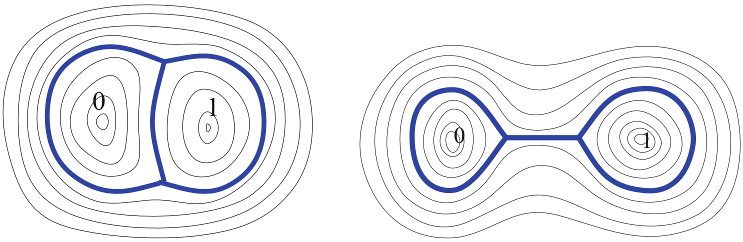
Theorem 6.3.4 (Kontsevich 1991)

$$\prod_{e=\text{edges}} dl_e = \frac{2^{\chi_{g,n}-d_{g,n}}}{d_{g,n}!} \left(\sum_{i=1}^{n} L_i^2\, \psi_i\right)^{d_{g,n}} \prod_{i=1}^{n} dL_i,$$

where

$$\chi_{g,n} = 2-2g-n \qquad , \qquad d_{g,n} = 3g-3+n.$$

Example: for $\mathcal{M}_{0,3}$, we have $\chi_{0,3} = -1$ and $d_{0,3} = 0$, and we have three edge lengths l_1, l_2, l_3.



In the first graph we have $L_0 = l_1 + l_2$, $L_1 = l_2 + l_3$ and $L_\infty = l_1 + l_3$, and thus:

$$\Omega = dL_0 \wedge dL_1 \wedge dL_\infty = (dl_1 + dl_2) \wedge (dl_2 + dl_3) \wedge (dl_1 + dl_3) = 2\, dl_1 \wedge dl_2 \wedge dl_3$$

and for the second graph we have $L_0 = l_1$, $L_1 = l_2$, $L_\infty = l_1 + l_2 + 2l_3$ and thus

$$\Omega = dL_0 \wedge dL_1 \wedge dL_\infty = dl_1 \wedge dl_2 \wedge (dl_1 + dl_2 + 2dl_3) = 2\, dl_1 \wedge dl_2 \wedge dl_3.$$

In both cases we get the same factor $2 = 2^{d_{0,3}-\chi_{0,3}}$.

6.3.4.1 Compactification

The Strebel foliation we have described, exists only in $\mathcal{M}_{g,n}$. However, intersection numbers should be computed on its compactification $\bar{\mathcal{M}}_{g,n}$, and one needs to see how the Strebel differentials behave at the boundaries of $\mathcal{M}_{g,n}$.

In Kontsevich's work, this question was ignored, and some have argued that it was a hole in the proof of Witten's conjecture. Other proofs not using Strebel differentials have later been found, in particular Okounkov and Pandaripande [69–71], or Looijenga [62], and also, the continuation of Strebel's differentials to the boundary of $\mathcal{M}_{g,n}$ have been studied by [89], and it was proved that Kontsevich's argument can indeed be made rigorous.

Here, in this book, we shall work with the same level of rigor as Kontsevich, we leave to the motivated reader to check that all the computations extend nicely to the boundary as in [89].

The boundary of $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$, corresponds to ribbon graphs, with some edge lengths vanishing, or some edge lengths going to ∞. The volume form $\prod dl_e$ is well behaved at vanishing lengths, but is not integrable at large lengths. In other words we face the problem that $\mathbb{R}_+^{6g-6+3n}$ is not compact.

One way Kontsevich used to circumvent that difficulty, is to compute Laplace transforms, i.e. we shall integrate

$$\prod_{i=1}^{n} \mathrm{e}^{-\lambda_i L_i} \prod_e dl_e$$

with $\mathrm{Re}\,\lambda_i > 0$, and thus the integral converges also at large lengths, even though we integrate over a non-compact space. In some (heuristic) sense, using the Laplace transform allows us to ignore the boundary.

6.3.5 Generating Function for Intersection Numbers

In order to compute intersection numbers, we introduce a generating function through Laplace transform in the L_i's:

$$\begin{aligned}
&A_{g,n}(\lambda_1, \ldots, \lambda_n) \\
&= \frac{1}{d_{g,n}!} \int_0^\infty dL_1 \mathrm{e}^{-\lambda_1 L_1} \ldots \int_0^\infty dL_n \mathrm{e}^{-\lambda_n L_n} \int_{\overline{\mathcal{M}}_{g,n}} (\sum_i L_i^2\, \psi_i)^{d_{g,n}} \\
&= \sum_{d_1+\cdots+d_n=d_{g,n}} \int_0^\infty dL_1 \frac{L_1^{2d_1}\, \mathrm{e}^{-\lambda_1 L_1}}{d_1!} \ldots \int_0^\infty dL_n \frac{L_n^{2d_n}\, \mathrm{e}^{-\lambda_n L_n}}{d_n!} \quad < \psi_1^{d_1} \ldots \psi_n^{d_n} >_{g,n} \\
&= \sum_{d_1+\cdots+d_n=d_{g,n}} \frac{(2d_1)!}{d_1!\, \lambda_1^{2d_1+1}} \cdots \frac{(2d_n)!}{d_n!\, \lambda_n^{2d_n+1}} \quad < \psi_1^{d_1} \ldots \psi_n^{d_n} >_{g,n} .
\end{aligned}$$

or using Witten's notations $\psi_i^d = \tau_d$,

$$A_{g,n}(\lambda_1, \ldots, \lambda_n) = \sum_{d_1+\cdots+d_n=d_{g,n}} \frac{(2d_1)!}{d_1!\, \lambda_1^{2d_1+1}} \cdots \frac{(2d_n)!}{d_n!\, \lambda_n^{2d_n+1}} \quad < \tau_{d_1} \ldots \tau_{d_n} >_g .$$

From the preceding section, we can rewrite it as:

$$A_{g,n}(\lambda_1,\ldots,\lambda_n) = 2^{d_{g,n}-\chi_{g,n}} \sum_{\text{ribbon graphs}} \frac{1}{\#\text{Aut}} \int \prod_{e=\text{edges}} dl_e \prod_{i=1}^{n} \mathrm{e}^{-\lambda_i L_i}.$$

Since every edge e borders two (possibly not distinct) faces:

$$e = (i,j)$$

we have that:

$$\sum_{\text{faces } i} \lambda_i L_i = \sum_{\text{edges } (i,j)} (\lambda_i + \lambda_j) l_{(i,j)}$$

and thus:

$$\begin{aligned} A_{g,n}(\lambda_1,\ldots,\lambda_n) &= 2^{d_{g,n}-\chi_{g,n}} \sum_{\text{ribbon graphs}} \frac{1}{\#\text{Aut}} \prod_{(i,j)=\text{edges}} \int_0^\infty dl_{(i,j)}\ \mathrm{e}^{-\sum_{(i,j)}(\lambda_i+\lambda_j)l_{(i,j)}} \\ &= 2^{d_{g,n}-\chi_{g,n}} \sum_{\text{ribbon graphs}} \frac{1}{\#\text{Aut}} \prod_{(i,j)=\text{edges}} \frac{1}{\lambda_i+\lambda_j} \end{aligned}$$

where the sum is over all ribbon graphs of genus g with n faces, and to each face is associated a variable λ_i. Notice that $2^{d_{g,n}} = \prod_i 2^{d_i}$, and

$$\frac{(2d)!}{2^d\, d!} = (2d-1)!!,$$

this gives:

Theorem 6.3.5 (Kontsevich) *The generating function of intersection numbers can be computed as a weighted sum of graphs:*

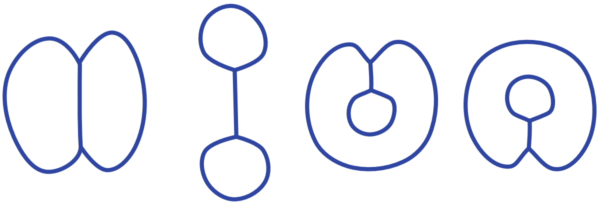
$$\begin{aligned} 2^{-d_{g,n}} A_{g,n}(\lambda_1,\ldots,\lambda_n) &= \sum_{d_1+\ldots+d_n=d_{g,n}} \frac{(2d_1-1)!!}{\lambda_1^{2d_1+1}} \cdots \frac{(2d_n-1)!!}{\lambda_n^{2d_n+1}} < \tau_{d_1}\ldots\tau_{d_n} >_g \\ &= 2^{-\chi_{g,n}} \sum_{\text{ribbon graphs}} \frac{1}{\#\text{Aut}} \prod_{(i,j)=\text{edges}} \frac{1}{\lambda_i+\lambda_j} \end{aligned}$$

where the sum is over all labeled ribbon graphs of genus g with n faces, and to the ith *face is associated the variable* λ_i.

Remark 6.3.1 Remark that the two expressions in the right hand side, are rational functions of the λ_i's. In the first line, we have poles only at $\lambda_i = 0$, whereas in the second line each term has poles at $\lambda_i = -\lambda_j$. There are terms such that $i = j$, so the second line also has poles at $\lambda_i = 0$. What is remarkable, is that after performing the summation over all graphs, all poles at $\lambda_i = -\lambda_j$ with $i \neq j$ should cancel. This is very non trivial from the graph point of view.

Example:

- For $\mathcal{M}_{0,3}$, we have four different Strebel graphs, three graphs where one of perimeters is larger that the sum of the two others $L_i \geq L_j + L_k$, and one graph where the three triangular inequalities are satisfied $L_i \leq L_j + L_k$.



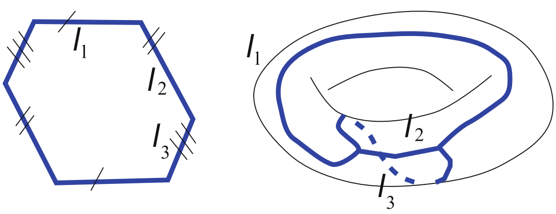
  This gives:

$$\frac{1}{2} A_{0,3}(\lambda_0, \lambda_1, \lambda_\infty) = \frac{1}{(\lambda_0+\lambda_1)(\lambda_1+\lambda_\infty)(\lambda_\infty+\lambda_0)} + \frac{1}{2\lambda_0\,(\lambda_0+\lambda_1)(\lambda_0+\lambda_\infty)}$$
$$+\frac{1}{2\lambda_1\,(\lambda_1+\lambda_0)(\lambda_1+\lambda_\infty)} + \frac{1}{2\lambda_\infty\,(\lambda_\infty+\lambda_0)(\lambda_\infty+\lambda_1)}$$
$$= \frac{1}{2\,\lambda_0\,\lambda_1\,\lambda_\infty}$$

  i.e. the intersection number

$$< \tau_0 \tau_0 \tau_0 >_0 = 1.$$

  We could have found this result easily, knowing that $\mathcal{M}_{0,3} = \{\text{point}\}$ and $\psi_i^0 = 1$.
- For $\mathcal{M}_{1,1}$, there is only one Strebel graph, it is an hexagon with opposite sides identified, it has a $\mathbb{Z}_6$ rotation symmetry and thus a symmetry factor of 6.



It has only one face with label λ_1, and three edges whose weight is $1/2\lambda_1$, this gives:

$$\frac{1}{4} A_{1,1}(\lambda_1) = \frac{1}{2} \frac{1}{\lambda_1^3} < \tau_1 >_1$$
$$= \frac{1}{6} \frac{1}{(2\lambda_1)^3}$$

this yields the intersection number

$$< \tau_1 >_1 = \frac{1}{24}.$$

6.3.6 Generating Function and Kontsevich Integral

Kontsevich's theorem gives a sum of graphs, where each graph is weighted by its symmetry factor and by a product of edge weights. This is typically the kind of graphs obtained from Wick's theorem, and therefore, exactly like in Chap. 2 of this book, it can be obtained with a Gaussian Hermitian matrix measure, namely:

$$d\mu_0(M) = \mathrm{e}^{-N \operatorname{Tr} \Lambda M^2} \; dM \qquad , \qquad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N).$$

Indeed, writing the quadratic form

$$\operatorname{tr} \Lambda \, M^2 = \frac{1}{2} \sum_{i,j} (\lambda_i + \lambda_j) M_{i,j} M_{j,i},$$

Wick's theorem says that the propagator is:

$$< M_{i,j} M_{k,l} >_{d\mu_0} = \frac{1}{\lambda_i + \lambda_j} \frac{1}{N} \delta_{i,l} \delta_{j,k}.$$

The trivalent ribbon graphs are generated by a cubic formal matrix integral

$$Z_{\text{Kontsevich}} = \frac{1}{\int d\mu_0(M)} \int_{\text{formal}} d\mu_0(M) \; \mathrm{e}^{N \operatorname{Tr} \frac{M^3}{3}}$$

with the normalization factor

$$\int d\mu_0(M) = (\pi/N)^{N^2/2} \prod_{i,j} (\lambda_i + \lambda_j)^{-1/2}.$$

Exactly like in Chap. 2, Wick's theorem decomposition of $Z_{\text{Kontsevich}}$, is the sum over all trivalent ribbon graphs. Each face f carries a matrix index $a_f \in [1, \dots, N]$ where N is the size of the matrix. We thus have

$$Z_{\text{Kontsevich}} = \sum_n \frac{1}{n!} \sum_{\substack{\text{ribbon graphs,}\\ n\,\text{faces}}} \sum_{a_1,\dots,a_n} \frac{1}{\#\text{Aut}} N^{\#\text{vertex}-\#\text{edges}} \prod_{(i,j)=\text{edges}} \frac{1}{(\lambda_{a_i} + \lambda_{a_j})}$$

where the faces are labeled (whence the $1/n!$ automorphism prefactor) and a_i is the index running around the ith face. In that sum, all graphs are included, connected or not, and with any number n of faces, and any genus g.

Like for maps, since the weights are multiplicative (the weight of a disconnected graph is the product of weights of its connected components) the logarithm generates only connected ribbon graphs. For a connected ribbon graph of genus g with n faces, we have

$$\#\text{vertex} - \#\text{edges} + n = 2 - 2g$$

therefore

$$\ln Z_{\text{Kontsevich}} = \sum_n \frac{1}{n!} \sum_{\substack{\text{connected graphs,}\\ n\,\text{faces}}} \sum_{a_1,\dots,a_n} \frac{1}{\#\text{Aut}} N^{2-2g-n} \prod_{(i,j)=\text{edges}} \frac{1}{(\lambda_{a_i} + \lambda_{a_j})}.$$

Keeping only graphs of genus g we write (in the sense of formal power series in powers of Λ^{-1})

$$\ln Z_{\text{Kontsevich}} = \sum_{g=0}^{\infty} N^{2-2g} F_g$$

where

$$F_g = \sum_n \frac{1}{n!} \sum_{\substack{\text{connected graphs,}\\ n\,\text{faces, genus}\,g}} \sum_{a_1,\dots,a_n} \frac{1}{\#\text{Aut}} N^{-n} \prod_{(i,j)=\text{edges}} \frac{1}{(\lambda_{a_i} + \lambda_{a_j})}$$

here the sum is over all labeled graphs (faces are labeled) with all possible labelings $a_1, \dots, a_n$, we recognize the generating function $A_{g,n}(\lambda_{a_1}, \dots, \lambda_{a_n})$ introduced earlier:

$$F_g = \sum_n \frac{1}{n!} N^{-n} \sum_{a_1,\dots,a_n} 2^{\chi_{g,n}-d_{g,n}} A_{g,n}(\lambda_{a_1}, \dots, \lambda_{a_n}).$$

Thanks to Kontsevich's Theorem 6.3.5, we thus have

$$F_g = \sum_n \frac{2^{\chi_{g,n}}}{n!} N^{-n} \sum_{a_1,\dots,a_n} \sum_{d_1,\dots,d_n} \frac{(2d_1-1)!!}{\lambda_{a_1}^{2d_1+1}} \dots \frac{(2d_n-1)!!}{\lambda_{a_n}^{2d_n+1}} \quad < \tau_{d_1} \dots \tau_{d_n} >_g .$$

The sum over labels $a_i \in [1, \dots, N]$ can be performed, we denote (our notation differs slightly from Kontsevich's):

$$t_j = \frac{1}{N} \operatorname{Tr} \Lambda^{-j}$$

and thus:

$$F_g = \sum_n \frac{2^{\chi_{g,n}}}{n!} \sum_{d_1+\dots+d_n=d_{g,n}} \prod_{i=1}^n (2d_i-1)!!\ t_{2d_i+1} \quad < \tau_{d_1} \dots \tau_{d_n} >_g .$$

Notice that for $g = 0$ there is no planar trivalent graph with $n = 1$ or $n = 2$ faces, so the sum starts at $n \geq 3$, for which intersection numbers are indeed defined.

So, this shows that the generating functions F_g for intersection numbers of moduli space of genus g, coincides with the topological expansion of the Kontsevich matrix integral:

Theorem 6.3.6 (Kontsevich) *Let the **Kontsevich integral***

$$Z_{\text{Kontsevich}}(\Lambda) = \prod_{i,j} \sqrt{\lambda_i + \lambda_j} (\pi/N)^{N^2/2} \int dM\ e^{-N \operatorname{Tr} \Lambda M^2} \ e^{N \operatorname{Tr} \frac{M^3}{3}}$$

be defined as a formal power series at large $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_N)$. *Then, in the sense of formal series at large* Λ *one has:*

$$\ln Z_{\text{Kontsevich}}(\Lambda) = \sum_{g=0}^{\infty} N^{2-2g} F_g(\{t_k\})$$

where $F_g(\{t_k\})$ *is the generating function of intersection numbers of genus* g*:*

$$F_g(\{t_k\}) = \sum_n \frac{2^{\chi_{g,n}}}{n!} \sum_{d_1+\dots+d_n=d_{g,n}} \prod_{i=1}^n (2d_i-1)!!\ t_{2d_i+1} \quad < \tau_{d_1} \dots \tau_{d_n} >_g, \tag{6.3.1}$$

and where

$$t_k = \frac{1}{N} \operatorname{Tr} \Lambda^{-k}.$$

It is also common to write:

$$F_g = 2^{2-2g} \left\langle e^{\frac{1}{2}\sum_{d=0}^{\infty}(2d-1)!!\, t_{2d+1}\, \tau_d} \right\rangle_g$$

where the right hand side means exactly Eq. (6.3.1) after expanding the exponential, and using the fact that we can ignore the constraint $\sum d_i = d_{g,n}$ because terms which don't satisfy the constraint are vanishing by definition.

6.3.6.1 Renormalizing Time t_1

Notice that if $t_1 = 0$, only intersection numbers with $d_i > 0$ would contribute. In the next sections we shall always assume $t_1 = 0$ for simplicity. However, here, for the sake of completeness, let us show how one can reduce $t_1 \neq 0$ to a situation without t_1.

When $t_1 \neq 0$, there can be terms with $d_i = 0$. Let us say that there are l of them, with $n = l + k$ (there are $n!/k!l!$ ways of choosing l among n), and we rewrite:

$$F_g = \sum_k \sum_l \frac{2^{\chi_{g,k+l}}}{k!\, l!} \sum_{d_1+\dots+d_k = l+d_{g,k},\, d_i>0} t_1^l \prod_i (2d_i-1)!!\, t_{2d_i+1} \;< \tau_0^l\, \tau_{d_1} \dots \tau_{d_k} >_{\mathcal{M}_{g,k+l}} .$$

The characteristic class $\tau_0 = (c_1(\mathcal{L}_i))^0 = 1$ is trivial, i.e. we don't compute the Chern class of l marked points among the $k + l$ marked points of a curve in $\mathcal{M}_{g,k+l}$, in other words we can forget those l marked points, and reduce to an integral in $\mathcal{M}_{g,k}$. Intersection theory tells us that under the forgetful map, we have:

$$< \tau_0\ \tau_{d_1} \dots \tau_{d_k} >_{\mathcal{M}_{g,k+1}} = \sum_j \; < \tau_{d_1} \dots \tau_{d_j-1} \dots \tau_{d_k} >_{\mathcal{M}_{g,k}} ,$$

(with the convention that $\tau_d = 0$ for $d < 0$), and by an easy induction

$$< \tau_0^l\, \tau_{d_1} \dots \tau_{d_k} >_{\mathcal{M}_{g,k+l}} = \sum_{\sum_i j_i = l,\, j_i \leq d_i} \frac{j!}{\prod_{i=1}^k j_i!} \; < \tau_{d_1-j_1} \dots \tau_{d_k-j_k} >_{\mathcal{M}_{g,k}} . \tag{6.3.2}$$

This implies

$$\begin{aligned} F_g &= \sum_k \frac{2^{\chi_{g,k}}}{k!} \sum_l \sum_{\sum_i j_i = l} \frac{t_1^{j_i}}{2^{j_i}\, j_i!} \\ &\qquad \sum_{d_1+\dots+d_k = d_{g,k},\, d_i \geq 0} \prod_i (2d_i + 2j_i - 1)!!\ t_{2d_i+2j_i+1} \quad < \tau_{d_1} \dots \tau_{d_k} >_{\mathcal{M}_{g,k}} \\ &= \sum_k \frac{2^{\chi_{g,k}}}{k!} \sum_{d_1+\dots+d_k = d_{g,k}} \prod_i (2d_i - 1)!!\ \dot{t}_{2d_i+1} \quad < \tau_{d_1} \dots \tau_{d_k} >_{\mathcal{M}_{g,k}}, \end{aligned}$$

where we have defined for every $d \geq 0$:

$$\dot{t}_{2d+1} = \sum_{j=0}^{\infty} \frac{(2d+2j-1)!!}{(2d-1)!!\ 2^j j!}\ t_1^j\ t_{2d+2j+1} - \delta_{d,0}\, t_1.$$

We see that $\dot{t}_1 \neq 0$, and thus we still have terms with $d_i = 0$.

We can cure this problem, by splitting t_1 into two parts:

$$t_1 = c + \check{t}_1.$$

We write:

$$\begin{aligned} F_g &= \sum_k \sum_l \frac{2^{\chi_{g,k+l}}}{k!\, l!} \sum_{d_1+\dots+d_k = l+d_{g,k},\, d_i>0} \check{t}_1^l \prod_i (2d_i-1)!!((1-\delta_{d_i,0})t_{2d_i+1} + \delta_{d_i,0}\, c) \\ &\qquad < 1^l\, \tau_{d_1} \dots \tau_{d_k} >_{\mathcal{M}_{g,k+l}} \\ &= \sum_k \frac{2^{\chi_{g,k}}}{k!} \sum_l \sum_{\sum_i j_i = l} \frac{\check{t}_1^{j_i}}{2^{j_i} j_i!} \sum_{d_1+\dots+d_k = d_{g,k},\, d_i \geq 0} \\ &\qquad \prod_i (2d_i+2j_i-1)!!\ ((1-\delta_{d_i,0})t_{2d_i+1} + \delta_{d_i,0}\, c) \quad < \tau_{d_1} \dots \tau_{d_k} >_{\mathcal{M}_{g,k}} \\ &= \sum_k \frac{2^{\chi_{g,k}}}{k!} \sum_{d_1+\dots+d_k = d_{g,k},\, d_i>0} \prod_i (2d_i-1)!!\ \check{t}_{2d_i+1} \quad < \tau_{d_1} \dots \tau_{d_k} >_{\mathcal{M}_{g,k}}, \end{aligned}$$

where for $d > 0$:

$$\check{t}_{2d+1} = \sum_{j=0}^{\infty} \frac{(2d+2j-1)!!}{(2d-1)!!\ 2^j j!}\ \check{t}_1^j\ t_{2d+2j+1},$$

and now we may chose $\check{t}_1$ such that the coefficients corresponding to $d_i = 0$ vanish, i.e. $\check{t}_1$ must be solution of:

$$0 = c + \sum_{j=1}^{\infty} \frac{(2j-1)!!}{2^j j!}\ \check{t}_1^j\ t_{2j+1},$$

i.e., using $c = t_1 - \check{t}_1$:

$$\check{t}_1 = \sum_{j=0}^{\infty} \frac{(2j-1)!!}{2^j j!}\ \check{t}_1^j\ t_{2j+1}.$$

This equation has a unique solution (as a formal powers series of Λ^{-1} as in all this chapter), and to the first few orders it is given by:

$$\check{t}_1 = \frac{2}{2-t_3}\, t_1 + \frac{6}{(2-t_3)^3}\, t_1^2\, t_5 + \dots .$$

Then, we also chose to define the even times (they can be chosen arbitrarily since they don't appear in the generating function of intersection numbers) by

$$\check{t}_{2d} = \sum_{j=0}^{\infty} \frac{(d+j-1)!!}{(d-1)!!\, j!}\, \check{t}_1^{\,j}\, t_{2d+2j},$$

we see that we have for any $k \geq 1$:

$$\check{t}_k = \sum_{j=0}^{\infty} \frac{\Gamma(1-k/2)}{j!\,\Gamma(1-k/2-j)}\, (-\check{t}_1)^j\, t_{k+2j} = \frac{1}{N}\, \mathrm{Tr}\, (\Lambda^2 - \check{t}_1)^{-k/2},$$

in other words, we have replaced the matrix $\Lambda = \mathrm{diag}(\lambda_i)$ with a matrix $\check{\Lambda}$:

$$\check{\Lambda} = \sqrt{\Lambda^2 - \check{t}_1}.$$

We have thus obtained that:

Theorem 6.3.7 *The generating function F_g of intersection numbers*

$$F_g = \sum_k \frac{2^{\chi_{g,k}}}{k!} \sum_{d_1+\dots+d_k = d_{g,k},\, d_i \geq 0} \prod_i (2d_i-1)!!\, t_{2d_i+1} \quad < \tau_{d_1} \dots \tau_{d_k} >_g,$$

can be rewritten without τ_0 terms by changing Λ to:

$$\check{\Lambda} = \sqrt{\Lambda^2 - \check{t}_1}\,, \qquad \check{t}_1 = \frac{1}{N}\, \mathrm{Tr}\, \check{\Lambda}^{-1}$$

i.e. the t_k's to

$$\forall\, k > 1, \qquad \check{t}_k = \frac{1}{N}\, \mathrm{Tr}\, \check{\Lambda}^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(1-k/2)}{j!\,\Gamma(1-k/2-j)}\, (-\check{t}_1)^j\, t_{k+2j},$$

i.e. one has

$$F_g = \sum_k \frac{2^{\chi_{g,k}}}{k!} \sum_{d_1+\dots+d_k = d_{g,k},\, d_i > 0} \prod_i (2d_i-1)!!\, \check{t}_{2d_i+1} \quad < \tau_{d_1} \dots \tau_{d_k} >_{\mathcal{M}_{g,k}},$$

This concludes that, up to a renormalization of the λ_i's, we can always choose:

$$t_1 = 0.$$

This is what we shall most often assume in the next sections.

6.3.7 Generating Functions with Marked Points

We now assume $t_1 = 0$. In computing F_g's, we summed over all possibilities to mark points, i.e. we have a sum over $n = 0, \ldots, \infty$, and we have integrated over all possible perimeters $L_1, \ldots, L_n$ for the n faces of the Strebel graphs.

One may also be interested in enumerating Strebel graphs, where some faces are marked and have fixed given perimeters, and other faces are unmarked and summed over. This can be in principle recovered from the generating function F_g by taking derivatives with respect to some λ_i's.

But we find it more convenient to encode those intersection numbers into another generating function.

Consider the following expectation value from the Kontsevich integral:

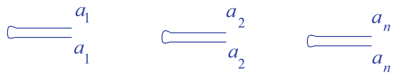
$$< M_{a_1,a_1}\, M_{a_2,a_2}\, \ldots M_{a_n,a_n} >_c = \frac{1}{Z_{\text{Kontsevich}}} \int_{\text{formal}} d\mu_0(M)\ \mathrm{e}^{N\,\mathrm{Tr}\,\frac{M^3}{3}}\ M_{a_1,a_1}\, \ldots M_{a_n,a_n}$$

where we assume that $a_1, \ldots, a_n$ are distinct integers between 1 and N, and the subscript c means cumulant, for instance

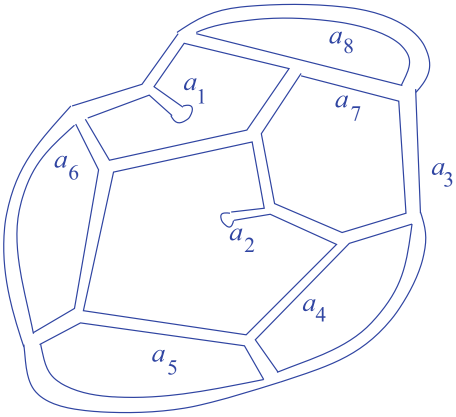
$$< M_{a_1,a_1} M_{a_2,a_2} >_c = < M_{a_1,a_1} M_{a_2,a_2} > - < M_{a_1,a_1} > < M_{a_2,a_2} > .$$

Again, Wick's theorem allows to write this expectation value as a sum of ribbon graphs. The fact that we divide by $Z_{\text{Kontsevich}}$ and take cumulants ensures that we get only connected graphs, as usual when weights are multiplicative.

The only addition compared to the previous section's computation, is that we need to add n new vertices, which are 1-valent, and which ensure that the lines arriving on them must have given matrix index a_j, with $j = 1, \ldots, n$.



Every ribbon graph must contain each such 1-valent vertex exactly once, and may contain an arbitrary number of trivalent vertices, and an arbitrary number of edges. A typical ribbon graph then looks like that:



In this example, we have $n = 2$, therefore two faces contain the two 1-valent vertices and have a fixed index a_1 and a_2 running around them, and the other faces, labeled from 3 to 8, have some index $a_j, j > n$, which can take any value in $[1, \dots, N]$.

We have from Wick's theorem:

$$< M_{a_1,a_1} M_{a_2,a_2} \dots M_{a_n,a_n} >_c$$
$$= \sum_k \frac{1}{k!} \sum_{\substack{\text{connected graphs,} \\ n+k \text{ faces}}} \sum_{a_{n+1},\dots,a_{n+k}} \frac{N^{\#\text{tri.vertex}-\#\text{edges}}}{\#\text{Aut}} \prod_{(i,j)=\text{edges}} \frac{1}{(\lambda_{a_i} + \lambda_{a_j})}$$

where we consider graphs with labeled faces (whence the $1/k!$). Each 1-valent vertex is connected to an edge whose both sides have the same label a_i, and thus it contributes a factor

$$\frac{1}{2\lambda_{a_i}}.$$

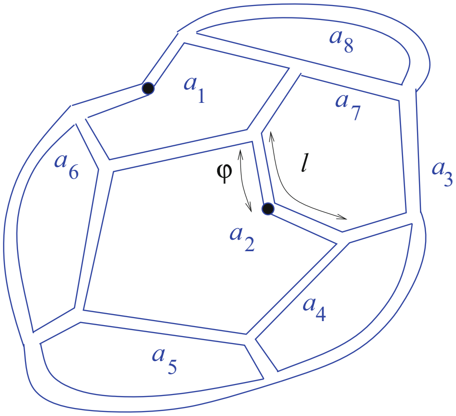
Notice that for a graph of genus g, we have

$$\#\text{tri.vertex} + n - \#\text{edges} + n + k = 2 - 2g.$$

Thus:

$$< M_{a_1,a_1} \dots M_{a_n,a_n} >_c$$
$$= \frac{N^{-n}}{2^n \prod_{i=1}^n \lambda_{a_i}} \sum_k \frac{1}{k!} \sum_{\substack{\text{connected graphs}', \\ n+k \text{ faces}}} \sum_{a_{n+1},\dots,a_{n+k}} \frac{N^{2-2g-n-k}}{\#\text{Aut}} \prod_{(i,j)=\text{edges}} \frac{1}{(\lambda_{a_i} + \lambda_{a_j})}$$

where graphs′ means graphs where we have shrinked the edge of the 1-valent vertices.



Now, we use that

$$\frac{1}{(\lambda_{a_i}+\lambda_{a_j})}=\int_0^\infty dl_{(i,j)}\ \mathrm{e}^{-l_{(i,j)}\,(\lambda_{a_i}+\lambda_{a_j})},$$

and each time there is a marked point on an edge, the two half edges separated by the marked point bear the same indices and we have a factor

$$\frac{1}{(\lambda_{a_i}+\lambda_{a_j})^2}=\int_0^\infty dl_{(i,j)}\ \mathrm{e}^{-l_{(i,j)}\,(\lambda_{a_i}+\lambda_{a_j})}\ \int_0^{l_{(i,j)}} d\varphi_{(i,j)}\ .$$

This shows that the product of propagators can be realized by a Laplace transform of graphs with lengths on their edges, and some marked points around the marked faces. $l_{(i,j)}$ is the length of edge (i,j) between face i and face j of the graph, and $\varphi_{(i,j)}$ is the distance of the marked point from the previous vertex along the edge, i.e. the position of the marked point around the marked face.

Therefore we have:

$$\begin{aligned}&< M_{a_1,a_1}\ \ldots M_{a_n,a_n} >_c\\ &=\frac{N^{-n}}{2^n\,\prod_{i=1}^n\lambda_{a_i}}\sum_k\frac{1}{k!}\sum_{\substack{\text{connected graphs}',\\ n+k\text{ faces}}}\ \sum_{a_{n+1},\ldots,a_{n+k}}\frac{N^{2-2g-n-k}}{\#\mathrm{Aut}}\\ &\qquad\prod_{(i,j)=\text{edges}}\int_0^\infty dl_{(i,j)}\ \mathrm{e}^{-l_{(i,j)}\,(\lambda_{a_i}+\lambda_{a_j})}\prod_{e=\text{edges with marked point}}\int_0^{l_e}d\varphi_e\end{aligned}$$

$$= \frac{N^{-n}}{2^n \prod_{i=1}^n \lambda_{a_i}} \sum_k \frac{1}{k!} \sum_{\substack{\text{connected graphs}',\\ n+k \text{ faces}}} \sum_{a_{n+1},\dots,a_{n+k}} \frac{N^{2-2g-n-k}}{\#\text{Aut}}$$

$$\prod_{(i,j)=\text{edges}} \int_0^\infty dl_{(i,j)} \prod_{i=1}^{n+k} \mathrm{e}^{-\sum_i L_i \lambda_{a_i}} \prod_{e=\text{edges with marked point}} \int_0^{l_e} d\varphi_e .$$

If we consider the subset of all graphs where the marked point is on one of the edges along a given marked face, the integral over the position of the marked point around a marked face can be performed, it is simply L_i:

$$\sum_{\text{graphs}} \sum_{e=\text{edge around } i} \int_0^{l_e} d\varphi_e = L_i .$$

Therefore, we can integrate out marked points, by considering graphs with no marked points, and with an additional L_i factor for each marked face, we thus have:

$$< M_{a_1,a_1} \dots M_{a_n,a_n} >_c$$
$$= \frac{N^{-n}}{2^n \prod_{i=1}^n \lambda_{a_i}} \sum_k \frac{1}{k!} \sum_{\substack{\text{connected graphs},\\ n+k \text{ faces}}} \sum_{a_{n+1},\dots,a_{n+k}} \frac{N^{2-2g-n-k}}{\#\text{Aut}}$$
$$\prod_{(i,j)=\text{edges}} \int_0^\infty dl_{(i,j)} \prod_{i=1}^{n+k} \mathrm{e}^{-\sum_i L_i \lambda_{a_i}} \prod_{1=1}^{n} L_i$$

where now, $l_{(i,j)}$ refers to the length of edges without marked points.

As before, using Kontsevich's Theorem 6.3.4 we have

$$< M_{a_1,a_1} \dots M_{a_n,a_n} >_c$$
$$= \frac{N^{-n}}{2^n \prod_{i=1}^n \lambda_{a_i}} \sum_k \frac{1}{k!} \sum_{\substack{\text{connected graphs},\\ n+k \text{ faces}}} \sum_{a_{n+1},\dots,a_{n+k}} \frac{2^{\chi_{g,n+k}-d_{g,n+k}}\, N^{2-2g-n-k}}{d_{g,n+k}!\ \#\text{Aut}}$$
$$\int (\sum_i L_i^2 \psi_i)^{d_{g,n+k}} \prod_{i=n+1}^{n+k} dL_i\, \mathrm{e}^{-\sum_i L_i \lambda_{a_i}} \prod_{i=1}^{n} L_i\, dL_i\, \mathrm{e}^{-\sum_i L_i \lambda_{a_i}}$$
$$= \frac{N^{-n}}{2^n \prod_{i=1}^n \lambda_{a_i}} \sum_k \frac{1}{k!} \sum_{\substack{\text{connected graphs},\\ n+k \text{ faces}}} \sum_{a_{n+1},\dots,a_{n+k}} \sum_{d_1+\dots+d_{n+k}=d_{g,n+k}}$$
$$\frac{2^{\chi_{g,n+k}-d_{g,n+k}}\, N^{2-2g-n-k}}{\#\text{Aut}} \int \prod_{i=1}^{n+k} \frac{\psi_i^{d_i}}{d_i!} \prod_{i=n+1}^{n+k} L_i^{2d_i}\, dL_i\, \mathrm{e}^{-\sum_i L_i \lambda_{a_i}}$$
$$\prod_{i=1}^{n} L_i^{2d_i+1}\, dL_i\, \mathrm{e}^{-\sum_i L_i \lambda_{a_i}}$$

$$= 2^{-n} N^{-n} \sum_g \sum_k \frac{1}{k!} \sum_{a_{n+1},\dots,a_{n+k}} \sum_{d_1+\dots+d_{n+k}=d_{g,n+k}} 2^{\chi_{g,n+k}-d_{g,n+k}} N^{2-2g-n-k}$$

$$\int_{\bar{\mathcal{M}}_{g,n+k}} \prod_{i=1}^{n+k} \psi_i^{d_i} \prod_{i=n+1}^{n+k} \frac{(2d_i)!}{d_i!\,\lambda_{a_i}^{2d_i+1}} \prod_{i=1}^{n} \frac{(2d_i+1)!}{d_i!\,\lambda_{a_i}^{2d_i+3}}$$

$$= 2^{-n} N^{-n} \sum_g \sum_k \frac{1}{k!} \sum_{d_1+\dots+d_{n+k}=d_{g,n+k}} 2^{\chi_{g,n+k}} N^{2-2g-n}$$

$$< \prod_{i=1}^{n+k} \tau_{d_i} >_g \prod_{i=n+1}^{n+k} \frac{(2d_i)!\, t_{2d_i+1}}{2^{d_i} d_i!} \prod_{i=1}^{n} \frac{(2d_i+1)!}{2^{d_i} d_i!\,\lambda_{a_i}^{2d_i+3}}$$

$$= 2^{-n} N^{-n} \sum_g \sum_k \frac{1}{k!} \sum_{d_1+\dots+d_{n+k}=d_{g,n+k}} 2^{\chi_{g,n+k}} N^{2-2g-n}$$

$$< \prod_{i=1}^{n+k} \tau_{d_i} >_g \prod_{i=n+1}^{n+k} (2d_i-1)!!\, t_{2d_i+1} \prod_{i=1}^{n} \frac{(2d_i+1)!!}{\lambda_{a_i}^{2d_i+3}}.$$

In fact, the relationship to intersection numbers holds only if $n + k + 2g - 2 > 0$. For $g = 0$ and $n = 1, 2$, we have to treat separately the first few values of k, which have no interpretation as intersection numbers, namely:

$$< M_{a_1,a_1} \dots M_{a_n,a_n} >_c$$

$$= \delta_{g,0}\delta_{n,1} \frac{N^{-1}}{2\lambda_{a_1}} \sum_a \frac{1}{\lambda_{a_1} + \lambda_a}$$

$$+\delta_{g,0}\delta_{n,2} \frac{N^{-2}}{4\lambda_{a_1}\lambda_{a_2}} \frac{1}{(\lambda_{a_1} + \lambda_{a_2})^2}$$

$$+2^{-n} N^{-n} \sum_g \sum_k \frac{1}{k!} \sum_{d_1+\dots+d_{n+k}=d_{g,n+k}} 2^{\chi_{g,n+k}} N^{2-2g-n}$$

$$< \prod_{i=1}^{n+k} \tau_{d_i} >_g \prod_{i=n+1}^{n+k} (2d_i-1)!!\, t_{2d_i+1} \prod_{i=1}^{n} \frac{(2d_i+1)!!}{\lambda_{a_i}^{2d_i+3}}.$$

We thus have

Lemma 6.3.1 *If $a_1 \neq a_2 \neq \dots \neq a_n$, the expectation values $< M_{a_1,a_1} \dots M_{a_n,a_n} >_c$ computed with the formal Kontsevich's integral matrix measure, are the following* *generating functions of intersection numbers:*

$$< M_{a_1,a_1}\dots M_{a_n,a_n} >_c = 2^{-n} N^{-n} \sum_g \sum_k \frac{1}{k!} \sum_{d_1+\dots+d_{n+k}=d_{g,n+k}} 2^{\chi_{g,n+k}} N^{2-2g-n}$$
$$< \prod_{i=1}^{n+k} \tau_{d_i} >_g \prod_{i=n+1}^{n+k} (2d_i-1)!!\, t_{2d_i+1} \prod_{i=1}^{n} \frac{(2d_i+1)!!}{\lambda_{a_i}^{2d_i+3}}$$
$$+\delta_{g,0}\delta_{n,1}\frac{N^{-1}}{2\lambda_{a_1}}\sum_a \frac{1}{\lambda_{a_1}+\lambda_a}$$
$$+\delta_{g,0}\delta_{n,2}\frac{N^{-2}}{4\lambda_{a_1}\lambda_{a_2}}\frac{1}{(\lambda_{a_1}+\lambda_{a_2})^2}.$$

One can also write:

$$< M_{a_1,a_1}\dots M_{a_n,a_n} >_c = 2^{-n} N^{-n} \sum_g \sum_{d_1,\dots,d_n} 2^{\chi_{g,n}} N^{2-2g-n}$$
$$\left\langle \tau_{d_1}\dots\tau_{d_n}\, \mathrm{e}^{\frac{1}{2}\sum_d (2d-1)!!\, t_{2d+1}\,\tau_d} \right\rangle_g \prod_{i=1}^{n} \frac{(2d_i+1)!!}{\lambda_{a_i}^{2d_i+3}}$$
$$+\delta_{g,0}\delta_{n,1}\frac{N^{-1}}{2\lambda_{a_1}}\sum_a \frac{1}{\lambda_{a_1}+\lambda_a}$$
$$+\delta_{g,0}\delta_{n,2}\frac{N^{-2}}{4\lambda_{a_1}\lambda_{a_2}}\frac{1}{(\lambda_{a_1}+\lambda_{a_2})^2}.$$

6.3.7.1 Kappa Classes

Cotangent bundle Chern classes ψ_i are associated to marked points on a Riemann surface.

It is also possible to define classes for unmarked points by summing over their moduli, somehow forgetting them. The $\boldsymbol{\kappa}$ **classes**, introduced by Mumford, are defined as the pushforward of $\tau_d = \psi^d$ classes through the forgetful map.

Let $k = k_1 + k_2$ be a number of marked points. Let $\pi_{k_1+k_2\to k_1} : \overline{\mathcal{M}}_{g,k_1+k_2} \to \overline{\mathcal{M}}_{g,k_1}$ be the forgetful projection, which forgets k_2 points. Arbarello and Cornalba showed [6, 7, 87] that the push forward of the classes $\tau_{d_i} = \psi_i^{d_i}$, can then be rewritten in terms of Mumford's classes $\kappa_0, \kappa_1, \kappa_2, \dots$ on $\overline{\mathcal{M}}_{g,k_1}$, by the relation:

$$\left\langle \psi_{k_1+1}^{\hat{d}_1+1}\dots\psi_{k_1+k_2}^{\hat{d}_{k_2}+1}\prod_{i=1}^{k_1}\psi_i^{d_i}\right\rangle_{g,k_1+k_2} = \sum_{\sigma\in\mathfrak{S}_{k_2}} \left\langle \prod_{c=\text{cycles of }\sigma} \kappa_{\sum_{i\in c}\hat{d}_i} \prod_{i=1}^{k_1}\psi_i^{d_i}\right\rangle_{g,k_1}$$

or with Witten's notations

$$\left\langle \tau_{\hat{d}_1+1} \dots \tau_{\hat{d}_{k_2}+1} \prod_{i=1}^{k_1} \tau_{d_i} \right\rangle_{g,k_1+k_2} = \sum_{\sigma\in\mathfrak{S}_{k_2}} \left\langle \prod_{c=\text{cycles of}\,\sigma} \kappa_{\sum_{i\in c}\hat{d}_i} \prod_{i=1}^{k_1} \tau_{d_i} \right\rangle_{g,k_1}.$$

This relationship can be used as a definition of κ classes, in terms of ψ classes.

Examples:

$$k_2 = 1 \ : \qquad \tau_{d+1} \textstyle\prod_{i=1}^{k_1} \tau_{d_i} \overset{\pi_{k_1+1\to k_1}}{\longrightarrow} \kappa_d \textstyle\prod_{i=1}^{k_1} \tau_{d_i}$$

$$k_2 = 2 \ : \qquad \tau_{d+1}\tau_{d'+1} \textstyle\prod_{i=1}^{k_1} \tau_{d_i} \overset{\pi_{k_1+2\to k_1}}{\longrightarrow} (\kappa_d\kappa_{d'} + \kappa_{d+d'}) \textstyle\prod_{i=1}^{k_1} \tau_{d_i}$$

$$k_2 = 3 \ : \tau_{d+1}\tau_{d'+1}\tau_{d''+1} \textstyle\prod_{i=1}^{k_1} \tau_{d_i} \overset{\pi_{k_1+3\to k_1}}{\longrightarrow} (\kappa_d\kappa_{d'}\kappa_{d''} + \kappa_{d+d'}\kappa_{d''} + \kappa_{d+d''}\kappa_{d'} + \kappa_{d'+d''}\kappa_d + 2\,\kappa_{d+d'+d''}) \textstyle\prod_{i=1}^{k_1} \tau_{d_i}.$$

In some sense it takes into account all possibilities of grouping forgotten points into clusters.

This definition of κ classes from τ classes can be conveniently written with generating series, by summing over k_2, with some formal parameters s_d, i.e.

$$\mathrm{e}^{-\sum_d s_d\tau_{d+1}} = 1-\sum_d s_d\tau_{d+1}+\sum_{d,d'}\frac{s_ds_{d'}}{2}\tau_{d+1}\tau_{d'+1}-\sum_{d,d',d''}\frac{s_ds_{d'}s_{d''}}{6}\tau_{d+1}\tau_{d'+1}\tau_{d''+1}+\dots$$

which become under the forgetful map:

$$\begin{aligned}\pi_*\,\mathrm{e}^{-\sum_d s_d\tau_{d+1}} &\to 1-\sum_d s_d\kappa_d+\frac{1}{2}\sum_{d,d'} s_ds_{d'}(\kappa_{d+d'}+\kappa_d\kappa_{d'}) \\ &\quad -\frac{1}{6}\sum_{d,d',d''} s_ds_{d'}s_{d''}(2\kappa_{d+d'+d''}+3\kappa_d\kappa_{d'+d''}+\kappa_d\kappa_{d'}\kappa_{d''})+\dots \\ &= \mathrm{e}^{-\sum_d s_d\kappa_d+\frac{1}{2}\sum_d(\sum_{j=0}^d s_js_{d-j})\kappa_d-\frac{1}{3}\sum_d(\sum_{j+j'+j''=d} s_js_{j'}s_{j''})\kappa_d+\dots}\end{aligned}$$

i.e. by defining new times $\hat{s}_d$ as:

$$\hat{s}_d = -s_d+\frac{1}{2}(\sum_{j+j'=d} s_js_{j'})-\frac{1}{3}\sum_{j+j'+j''=d} s_js_{j'}s_{j''}+\dots$$

we have

$$\pi_*\,\mathrm{e}^{-\sum_d s_d\tau_{d+1}} \to \mathrm{e}^{\sum_d \hat{s}_d\kappa_d}.$$

More generally we have:

Lemma 6.3.2 *The formal series of κ classes $\mathrm{e}^{\sum_d \hat{t}_d \kappa_d}$ is the forgetful push forward of* $\mathrm{e}^{-\sum_d s_d \tau_{d+1}}$, *i.e.*

$$\left\langle \mathrm{e}^{\sum_d \hat{t}_d \kappa_d} \prod_{i=1}^{n} \tau_{d_i} \right\rangle_g = \left\langle \pi_* \ \mathrm{e}^{-\sum_d s_d \tau_{d+1}} \prod_{i=1}^{n} \tau_{d_i} \right\rangle_g$$

iff the formal times $\hat{t}_d$ are the Schur transforms of the formal times s_d, i.e. they are related by

$$\sum_d \hat{t}_d u^{-d} = -\ln\left(1 + \sum_d s_d u^{-d}\right).$$

For example:

$$\hat{t}_0 = -\ln(1+s_0) \quad , \quad \hat{t}_1 = \frac{-s_1}{1+s_0} \quad , \quad \hat{t}_2 = \frac{s_1^2}{2(1+s_0)^2} - \frac{s_2}{1+s_0}, \ \dots$$

Proof By definition we have

$$\pi_* \ \mathrm{e}^{-\sum_d s_d \tau_{d+1}} = 1 + \sum_{k\geq 1} \frac{(-1)^k}{k!} \sum_{d_1,\dots,d_k} s_{d_1} \dots s_{d_k} \sum_{\sigma \in \mathfrak{S}_k} \prod_{c=\text{cycles of } \sigma} \kappa_{\sum_{i\in c} d_i}$$

Let us decompose the sum over permutations $\sigma \in \mathfrak{S}_k$ as a sum over conjugacy classes, indexed by partitions $\lambda = (\lambda_1 \leq \dots \leq \lambda_m)$, of weight $|\lambda| = \sum_i \lambda_i$, i.e.

$$\pi_* \mathrm{e}^{-\sum_d s_d \tau_{d+1}} = 1 + \sum_m \sum_{\lambda_1 \leq \dots \leq \lambda_m} \frac{(-1)^{|\lambda|}}{|\lambda|!} \sum_{\sigma \in \lambda} \sum_{b_1,\dots,b_m} \prod_{j=1}^{m} \kappa_{b_j} \left(\sum_{d_1+\dots+d_{\lambda_j}=b_j} s_{d_1} \dots s_{d_{\lambda_j}} \right).$$

The size of a conjugacy class is

$$\#\lambda = \frac{|\lambda|!}{\prod_j \lambda_j \ \prod_j \#\{i, \lambda_i = j\}!}$$

and thus

$$\pi_* \ \mathrm{e}^{-\sum_d s_d \tau_{d+1}} = 1 + \sum_m \sum_{\lambda_1 \leq \dots \leq \lambda_m} \frac{(-1)^{|\lambda|}}{\prod_j \lambda_j \ \prod_j \#\{i, \lambda_i = j\}!} \sum_{b_1,\dots,b_m} \prod_{j=1}^{m} \kappa_{b_j} \left(\sum_{d_1+\dots+d_{\lambda_j}=b_j} s_{d_1} \dots s_{d_{\lambda_j}} \right)$$

Since the summand is symmetric in all λ_j's, we may relax the constraint $\lambda_1 \leq \cdots \leq \lambda_m$, by dividing by $m!/\prod_j \#\{i\,,\lambda_i = j\}!$ and write:

$$\pi_* \mathrm{e}^{-\sum_d s_d \tau_{d+1}} = 1 + \sum_m \frac{1}{m!} \sum_{\lambda_1,\ldots,\lambda_m} \frac{(-1)^{\sum_j \lambda_j}}{\prod_j \lambda_j} \sum_{b_1,\ldots,b_m} \prod_{j=1}^m \kappa_{b_j} \left(\sum_{d_1+\cdots+d_{\lambda_j}=b_j} s_{d_1} \ldots s_{d_{\lambda_j}} \right)$$
$$= \mathrm{e}^{\sum_\lambda \frac{(-1)^\lambda}{\lambda} \sum_b \kappa_b \left(\sum_{d_1+\cdots+d_\lambda=b} s_{d_1}\ldots s_{d_\lambda}\right)}$$
$$= \mathrm{e}^{\sum_b \hat{t}_b \kappa_b}$$

with

$$\hat{t}_b = \sum_\lambda \frac{(-1)^\lambda}{\lambda} \left(\sum_{d_1+\cdots+d_\lambda=b} s_{d_1} \ldots s_{d_\lambda} \right)$$

whose generating function $\sum_b \hat{t}_b u^{-b}$ is as announced

$$\sum_b \hat{t}_b u^{-b} = -\ln(1 + \sum_d s_d u^{-d}).$$

□

In our case, we should chose

$$s_d = -\frac{1}{2}(2d+1)!!\, t_{2d+3},$$

that allows to rewrite Lemma 6.3.1 (remember that we chose $t_1 = 0$):

$$< M_{a_1,a_1} \ldots M_{a_n,a_n} >_c = 2^{-n} N^{-n} \sum_g \sum_{d_1,\ldots,d_n} 2^{\chi_{g,n}} N^{2-2g-n}$$
$$\left\langle \tau_{d_1} \ldots \tau_{d_n}\ \mathrm{e}^{\frac{1}{2}\sum_d (2d+1)!!\ t_{2d+3}\,\tau_{d+1}} \right\rangle_g \prod_{i=1}^n \frac{(2d_i+1)!!}{\lambda_{a_i}^{2d_i+3}}$$
$$= 2^{-n} N^{-n} \sum_g \sum_{d_1,\ldots,d_n} 2^{\chi_{g,n}} N^{2-2g-n}$$
$$\left\langle \tau_{d_1} \ldots \tau_{d_n}\ \mathrm{e}^{\sum_d \hat{t}_d \kappa_d} \right\rangle_g \prod_{i=1}^n \frac{(2d_i+1)!!}{\lambda_{a_i}^{2d_i+3}}$$

where the times $\hat{t}_d$'s are related to the times t_k's by:

$$\sum_d \hat{t}_d u^{-d} = -\ln\left(1 - \frac{1}{2}\sum_d (2d+1)!! t_{2d+3} u^{-d}\right).$$

For example the first few of them are

$$\hat{t}_0 = -\ln(1 - \frac{t_3}{2})$$
$$\hat{t}_1 = \frac{3\, t_5}{2 - t_3}$$
$$\hat{t}_2 = \frac{15\, t_7}{2 - t_3} + \frac{9\, t_5^2}{2(2 - t_3)^2}$$

and so on…

In Theorem 6.3.9 below, we shall see how the times $\hat{t}_d$ are related to the spectral curve through the Laplace transform.

We have thus found that:

Theorem 6.3.8 *the formal expectation values $< M_{a_1,a_1} \dots M_{a_n,a_n} >_c$ with the Kontsevich integral measure, are the generating functions for the mixed κ and ψ intersection numbers:*

$$< M_{a_1,a_1} \dots M_{a_n,a_n} >_c = 2^{-n} N^{-n} \sum_g \sum_{d_1,d_2,\dots,d_n} 2^{\chi_{g,n}} N^{2-2g-n} \left\langle \prod_{i=1}^n \psi_i^{d_i}\, e^{\sum_b \hat{t}_b \kappa_b} \right\rangle_{\mathcal{M}_{g,n}} \prod_{i=1}^n \frac{(2d_i+1)!!}{\lambda_{a_i}^{2d_i+3}}$$

and also when $n = 0$

$$F_g = 2^{\chi_{g,0}} \left\langle e^{\frac{1}{2}\sum_d (2d+1)!!\, t_{2d+3}\, \tau_{d+1}} \right\rangle_g = 2^{\chi_{g,0}} \left\langle e^{\sum_d \hat{t}_d \kappa_d} \right\rangle_g$$

where the times $\hat{t}_k$ are related to the times $t_k = \frac{1}{N} \operatorname{Tr} \Lambda^{-k}$ by

$$e^{-\sum_{k\geq 0} \hat{t}_k x^k} = 1 - \frac{1}{2} \sum_{k\geq 0} (2k+1)!!\, t_{2k+3}\, x^k.$$

6.3.7.2 Simplification κ_0 and t_3

The Mumford class κ_k is a $2k$-form on $\overline{\mathcal{M}}_{g,n}$, and in particular for $k = 0$, the Mumford class κ_0 is a scalar $\in \mathbb{C}$, it is in fact the Euler class, up to a sign:

$$\kappa_0 = -\chi = 2g - 2 + n.$$

It can thus be factored out, and we get:

Corollary 6.3.1

$$< M_{a_1,a_1} \dots M_{a_n,a_n} >_c = 2^{-n} N^{-n} \sum_g \sum_{d_1,d_2,\dots,d_n} (2-t_3)^{\chi_{g,n}} N^{2-2g-n} \left\langle \prod_{i=1}^n \psi_i^{d_i} \, e^{\sum_{b>0} \hat{t}_b \kappa_b} \right\rangle_{\mathcal{M}_{g,n}} \prod_{i=1}^n \frac{(2d_i+1)!!}{\lambda_{a_i}^{2d_i+3}}$$

where

$$e^{-\sum_{k>0} \hat{t}_k x^k} = 1 - \frac{1}{2-t_3} \sum_{k>0} (2k+1)!! \, t_{2k+3} \, x^k.$$

In particular when $n = 0$ we have:

$$F_g = (2-t_3)^{2-2g} \int_{\overline{\mathcal{M}}_{g,0}} e^{\sum_{k>0} \hat{t}_k \kappa_k}.$$

6.3.7.3 Other Rewritings

We can rewrite this expression in many other ways. Write that

$$\begin{aligned}
\sum_d \psi^d \frac{(2d+1)!}{2^d \, d! \, \lambda^{2d+2}} &= \sum_d \psi^d \frac{1}{2^d \, d!} \int_0^\infty dL \, L^{2d+1} \, e^{-\lambda L} \\
&= \int_0^\infty L \, dL \, e^{-\lambda L} \sum_d \frac{\psi^d L^{2d}}{2^d \, d!} \\
&= \int_0^\infty L \, dL \, e^{-\lambda L} \, e^{\frac{1}{2} L^2 \psi}
\end{aligned}$$

and thus:

Corollary 6.3.2

$$\begin{aligned}
&< M_{a_1,a_1} \dots M_{a_n,a_n} >_c \\
&= \frac{2^{-n} N^{-n}}{\prod_{i=1}^n \lambda_{a_i}} \sum_g (2-t_3)^{\chi_{g,n}} N^{2-2g-n} \\
&\quad \int_0^\infty \prod_{i=1}^n L_i dL_i \, e^{-\lambda_{a_i} L_i} \int_{\overline{\mathcal{M}}_{g,n}} e^{\frac{1}{2} \sum_{i=1}^n L_i^2 \psi_i + \sum_{k>0} \hat{t}_k \kappa_k}.
\end{aligned}$$

We may also write

$$\begin{aligned}\sum_d \psi^d\, \frac{(2d+1)!!}{\lambda^{2d+3}} &= \frac{1}{\Gamma(3/2)}\, \sum_d \psi^d\, \frac{\Gamma(d+3/2)\, 2^d}{\lambda^{2d+3}} \\ &= \frac{2}{\sqrt{\pi}} \sum_d \int_0^\infty d\mu\,\, \mu^{d+1/2}\, \mathrm{e}^{-\mu\,\lambda^2}\,\, 2^d\,\, \psi^d \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty d\mu\,\, \mu^{1/2}\, \mathrm{e}^{-\mu\,\lambda^2}\,\, \frac{1}{1-2\mu\,\psi} \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty d\mu\,\, \mu^{1/2}\, \mathrm{e}^{-\frac{1}{2}\,\mu\,\lambda^2}\,\, \frac{1}{1-\mu\,\psi}\end{aligned}$$

It follows the following corollary:

Corollary 6.3.3 *The expectation values* $< M_{a_1,a_1} \ldots M_{a_n,a_n} >_c$ *are the Laplace transforms of classes* $1/(1-\mu_i\psi_i)$*:*

$$\begin{aligned}< M_{a_1,a_1} \ldots M_{a_n,a_n} >_c \,=\, (8\pi)^{-n/2}\, N^{-n}\, \sum_g\, (2-t_3)^{\chi_{g,n}}\, N^{2-2g-n} \\ \int_0^\infty d\mu_1 \ldots d\mu_n\, \prod_{i=1}^n \mathrm{e}^{-\frac{1}{2}\mu_i\,\lambda_{a_i}^2} \left\langle \prod_{i=1}^n \frac{\sqrt{\mu_i}}{1-\mu_i\psi_i}\,\, \mathrm{e}^{\sum_{b>0} \hat{t}_b\kappa_b} \right\rangle_{\mathcal{M}_{g,n}}.\end{aligned}$$

Remark 6.3.2 For specialists, we mention that this formula is very similar to the famous ELSV formula (Ekedahl Lando Shapiro Wainshtein [29]). It is of the same nature, it expresses combinatorial objects $< M_{a_1,a_1} \ldots M_{a_n,a_n} >_c$ counting graphs, in terms of intersection numbers, involving the classes $1/(1-\mu_i\psi_i)$, as well as a bulk class $\mathrm{e}^{\sum_{b>0} \hat{t}_b\kappa_b}$.

6.3.7.4 Spectral Curve and Laplace Transform

All the expectation values in Kontsevich integral involve the class $\mathrm{e}^{\sum_k \hat{t}_k\kappa_k}$, where the times $\hat{t}_k$ are computed out of the t_{2k+3}, i.e. out of the coefficients appearing in the spectral curve

$$y(z) = z - \frac{1}{2} \sum_k t_{2k+3}\, z^{2k+1}.$$

Let us see how to express directly the generating function $\hat{f}(1/u) = \sum_k \hat{t}_k u^{-k}$ from the spectral curve.

Let the generating function of the $\hat{t}_k$'s

$$f(1/u) = 1 - \mathrm{e}^{-\hat{f}(1/u)} = \frac{1}{2} \sum_k (2k+1)!!\, t_{2k+3}\, u^{-k}.$$

Then write

$$y(z) - y(-z) = 2z - \sum_k t_{2k+3} z^{2k+1}.$$

Compute the Laplace transform of the spectral curve with $x(z) = z^2$:

$$\begin{aligned}
\int_{z=-\infty}^{z=\infty} y(z)\, dx(z)\ \mathrm{e}^{-\frac{u}{2}x(z)} &= \int_{z=0}^{z=\infty} (y(z) - y(-z))\, dx(z)\ \mathrm{e}^{-\frac{u}{2}x(z)} \\
&= \frac{1}{2} \int_{-\infty}^{\infty} (y(z) - y(-z)) 2z dz\ \mathrm{e}^{-z^2 u/2} \\
&= -\sum_k (t_{2k+3} - 2\delta_{k,0}) \int_{-\infty}^{\infty} z^{2k+2}\, dz\ \mathrm{e}^{-z^2 u/2} \\
&= -\sqrt{\frac{2\pi}{u}} \left[\frac{t_3 - 2}{u} + \sum_{k\geq 1} t_{2k+3} \frac{(2k+1)!!}{u^{k+1}} \right] \\
&= \frac{2\sqrt{2\pi}}{u^{3/2}} \left[1 - f(1/u) \right] \\
&= \frac{2\sqrt{2\pi}}{u^{3/2}}\ \mathrm{e}^{-\hat{f}(1/u)}.
\end{aligned}$$

It follows:

Theorem 6.3.9 *The spectral curve's class* $\mathrm{e}^{\sum_k \hat{t}_k \kappa_k}$*, is generated by the Laplace transform of the spectral curve:*

$$\mathrm{e}^{-\sum_k \hat{t}_k u^{-k}} = \frac{u^{3/2}}{2\sqrt{2\pi}} \int_\gamma y dx\ \mathrm{e}^{-\frac{u}{2}x}$$

where γ *is the "steepest descent path" going through the branchpoint* $z = 0$*, i.e. the contour of equation* $\operatorname{Im} x(z) = 0$*,* $\operatorname{Re} x(z) > 0$.

This theorem is very useful for more complicated examples of topological recursion, and it is a hint of the deep link between mirror symmetry and Laplace transform.

6.4 Combinatorics of Graphs and Recursions

The purpose of this section is to use graph combinatorics, to derive recursion relations among intersection numbers, and in particular the *topological recursion*.

Definition 6.4.1 Let $\mathcal{G}_{g,n}(z_1,\dots,z_n)$ be the set of connected ribbon graphs of genus g, with n labeled marked faces, and with n one-valent vertices and v trivalent vertices, and such that:

- each face has a label. unmarked faces don't contain a one-valent vertex, and they carry a label $\in \{\lambda_1,\dots,\lambda_N\}$,
- the ith marked face contains a unique one-valent vertex, and carries the label z_i.

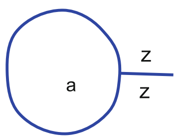
To a graph $G \in \mathcal{G}_{g,n}(z_1,\dots,z_n)$, we associate a weight:

$$w(G) = \frac{N^{-\#\text{unmarked faces}}}{\#\text{Aut}\,(G)} \prod_{(i,j)=\text{edges}} \frac{1}{\text{label}(i)+\text{label}(j)}$$

and we define the following formal series for weighted graphs (graded by inverse powers of λ's and z's, i.e. graded by number of edges)

$$\Omega_{g,n}(z_1,\dots,z_n) = -\delta_{g,0}\delta_{n,1}z_1 + \sum_{G\in\mathcal{G}_{g,n}(z_1,\dots,z_n)} \frac{N^{-\#\text{unmarked faces}}}{\#\text{Aut}\,(G)} \prod_{(i,j)=\text{edges}} \frac{1}{\text{label}(i)+\text{label}(j)}.$$

For example, $\mathcal{G}_{0,1}(z)$ contains only one type of graphs of degree 2 (i.e. with two edges), it is made of one trivalent vertex, one one-valent vertex, two faces (one with label z, one with a label λ_a, two edges (one is a $(z+z)$ edge, the other a $(z+\lambda_a)$ edge):



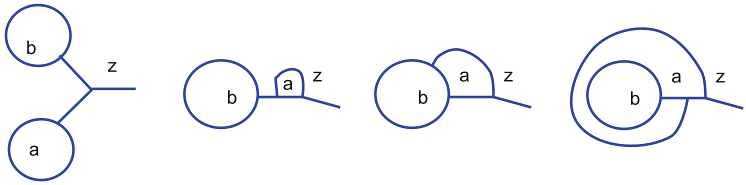
Its weight is

$$\frac{1}{N}\frac{1}{2z\,(z+\lambda_a)}$$

and thus contributes to the generating function as

$$\Omega_{0,1}(z) = -z + \frac{1}{N}\sum_a \frac{1}{2z\,(z+\lambda_a)} + O(\deg \geq 5).$$

Then $\mathcal{G}_{0,1}(z)$ contains four graphs of degree 5:



whose total weight is

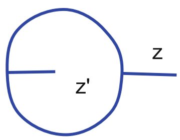
$$\frac{1}{N^2}\sum_{a,b}\frac{1}{(2z)^3(z+\lambda_a)(z+\lambda_b)}+\frac{1}{(2z)^2(z+\lambda_a)^2(z+\lambda_b)}$$
$$+\frac{1}{2z\,(z+\lambda_a)^2(z+\lambda_b)(\lambda_a+\lambda_b)}+\frac{1}{2z\,(z+\lambda_a)^2 2\lambda_a(\lambda_a+\lambda_b)}$$
$$=\frac{1}{N^2}\sum_{a,b}\frac{1}{(2z)^3\,\lambda_a\,\lambda_b}$$
$$=\frac{t_1^2}{8\,z^3}$$

and thus:

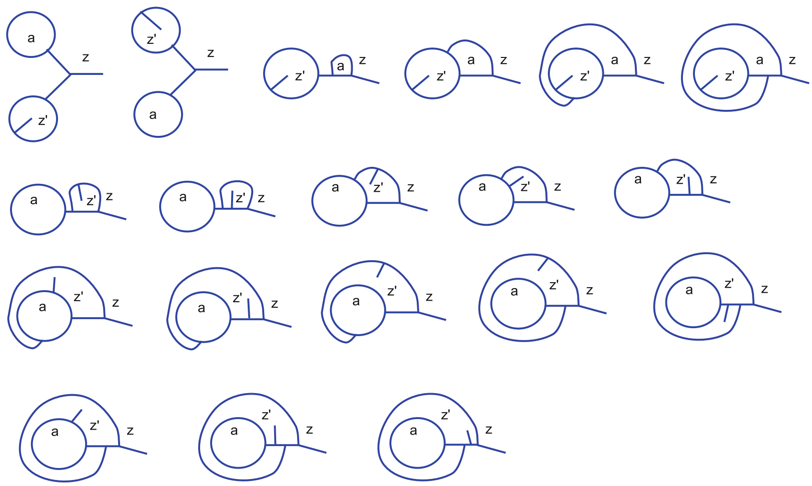
$$\Omega_{0,1}(z)=-z+\frac{1}{N}\sum_a\frac{1}{2z\,(z+\lambda_a)}+\frac{t_1^2}{8\,z^3}+O(\deg\geq 8)$$

Remark 6.4.1 Observe that the series $\Omega_{g,n}$ is a series whose coefficients are rational functions of the z_i's and λ's, and there is not a unique way of writing a rational function, and cancellations may occur. In particular, $\Omega_{g,n}$ is NOT a generating series of graphs in the combinatorial sense, it contains less information than the set of graphs. This can be best seen on the example of $\Omega_{0,2}$:

The lowest degree for graphs in $\mathcal{G}_{0,2}(z,z')$ is 4, and there is only one graph of degree 4 (i.e. with four edges), it is made of two trivalent vertices, two one-valent vertices, two faces (one with label z, one with a label z', four edges (one $(z+z)$ edge, one $(z'+z')$ edge, and two $(z+z')$ edges):



Then, the next order consists of 19 graphs of degree 7:



i.e.

$$
\left.
\begin{aligned}
\Omega_{0,2}(z,z') = \; & \frac{1}{4zz'\,(z+z')^2} \\
& +2\,\frac{1}{4zz'(z+z')^2}\;\frac{1}{N}\sum\nolimits_a \frac{1}{2z\lambda_a(z+\lambda_a)} \\
& +\frac{1}{N}\sum\nolimits_a \Bigg(\frac{1}{4zz'(z+\lambda_a)^2(z+z')^2 2z} + \frac{1}{4zz'(z+\lambda_a)^2(z+z')^2(z'+\lambda_a)} \\
& + \frac{1}{4zz'(z+\lambda_a)^2(z+z')(z'+\lambda_a)^2} + \frac{1}{4zz'(z+\lambda_a)^2(z'+\lambda_a)^2 2\lambda_a} \Bigg) \\
& +\frac{1}{N}\sum\nolimits_a \Bigg(\frac{1}{4zz'(z+\lambda_a)(z+z')^3 2z} + \frac{1}{4zz'(z+\lambda_a)(z+z')^3 2z} \\
& + \frac{1}{4zz'(z+\lambda_a)(z+z')^3(z'+\lambda_a)} + \frac{1}{4zz'(z+\lambda_a)(z+z')^2(z'+\lambda_a)^2} \\
& + \frac{1}{4zz'(z+\lambda_a)(z+z')^3(z'+\lambda_a)} + \frac{1}{4zz'(z+\lambda_a)(z+z')^2(z'+\lambda_a)^2} \\
& + \frac{1}{4zz'(z+\lambda_a)(z+z')^3(z'+\lambda_a)} + \frac{1}{4zz'(z+\lambda_a)(z+z')^3(z'+\lambda_a)} \\
& + \frac{1}{4zz'(z+\lambda_a)(z+z')^3(z'+\lambda_a)2z'} + \frac{1}{4zz'(z+\lambda_a)(z+z')^2(z'+\lambda_a)(2z')^2} \\
& + \frac{1}{4zz'(z+\lambda_a)(z+z')^2(z'+\lambda_a)^2 2z'} + \frac{1}{4zz'(z+\lambda_a)(z+z')^2(z'+\lambda_a)(2z')^2} \\
& + \frac{1}{4zz'(z+\lambda_a)(z+z')^3(z'+\lambda_a)2z'} + \frac{1}{4zz'(z+\lambda_a)(z+z')^3(z'+\lambda_a)2z'} \Bigg)
\end{aligned}
\right\} = 0
$$

$$+O(\deg \geq 10)$$

and observe that the sum of weights of degree 7 graphs cancels! In other words, the function $\Omega_{0,2}$ doesn't encode graphs of degree 7 in this example.

By construction we have:

Proposition 6.4.1 *The generating functions of intersections numbers, i.e. the expectation values of type* $< M_{a_1,a_1} \dots M_{a_n,a_n} >$, *with* $a_1, \dots, a_n$ *all distinct, with Kontsevich integral's measure, equals the functions* $\Omega_{g,n}$ *with arguments* $z_i = \lambda_{a_i}$*:*

$$< M_{a_1,a_1} \dots M_{a_n,a_n} >_c = \sum_g N^{2-2g-n}\, \Omega_{g,n}(\lambda_{a_1}, \dots, \lambda_{a_n})$$

i.e., for $2-2g-n<0$

$$\Omega_{g,n}(\lambda_{a_1},\dots,\lambda_{a_n}) = 2^{-n}2^{2-2g-n}\sum_{d_1,\dots,d_n} < \tau_{d_1}\dots\tau_{d_n} \mathrm{e}^{\sum_b \hat{t}_b\kappa_b} >_g \prod_{i=1}^n \frac{(2d_i+1)!!}{\lambda_{a_i}^{2d_i+3}}. \tag{6.4.1}$$

The purpose of defining those functions $\Omega_{g,n}$, is that one can easily write Tutte–like recursion relations, which determine them.

6.4.1 Edge Removal and Tutte's Equations

Like in Chap. 1, by recursively removing edges, we get Tutte–like recursions:

Theorem 6.4.1 *If* $n\geq 0$ *and* $2g-2+(n+1)>0$*, and* $J=\{z_1,\dots,z_n\}$*, we have the Tutte's equations:*

$$\begin{aligned}\delta_{g,0}\delta_{n,0}\, z^2 &= \Omega_{g-1,n+2}(z,z,z_1,\dots,z_n)\\ &\quad+\sum_{h+h'=g,\, I\uplus I'=\{z_1,\dots,z_n\}} \Omega_{h,|I|+1}(z,I)\,\Omega_{h',|I'|+1}(z,I')\\ &\quad-\frac{1}{N}\sum_{a=1}^N \frac{\Omega_{g,n+1}(z,z_1,\dots,z_n)-\Omega_{g,n+1}(\lambda_a,z_1,\dots,z_n)}{z^2-\lambda_a^2}\\ &\quad+\sum_{j=1}^n \frac{1}{2z_j}\frac{d}{dz_j}\frac{\Omega_{g,n}(z,z_1,\dots,\not{z_j},\dots,z_n)-\Omega_{g,n}(z_1,\dots,z_n)}{z^2-z_j^2}.\end{aligned} \tag{6.4.2}$$

Proof Let us denote $J=\{z_1,\dots,z_n\}$. Consider a graph in $\mathcal{G}_{g,n+1}(z,z_1,\dots,z_n)$. Consider its first marked face, it has label z, and it has a one-valent vertex. Attached to the one-valent vertex is an edge, necessarily with weight $1/(z+z)$, and at the end of that edge, there is necessarily a trivalent vertex (see the figure below).

Consider the edge of this trivalent vertex, located to the right of the edge coming from the 1-valent vertex.

If we cut that edge, several situations may occur:

– the face on the other side of that edge is again the first marked face. Then we have two possibilities:

- cutting that edge disconnects the ribbon graph into two graphs, whose external face carries the label z. The two subgraphs belong to $\mathcal{G}_{h,|I|+1}(z,I)\times\mathcal{G}_{h',|I'|+1}(z,I')$ with complementary genus $h+h'=g$ and complementary subsets of marked faces $I\uplus I'=J$.

Also pay attention that we have added a term $-\delta_{g,0}\delta_{n,0}z$ within the definition of $\Omega_{g,n+1}(z,J)$, which doesn't correspond to the weight of a graph.

The corresponding equality of generating functions is thus:

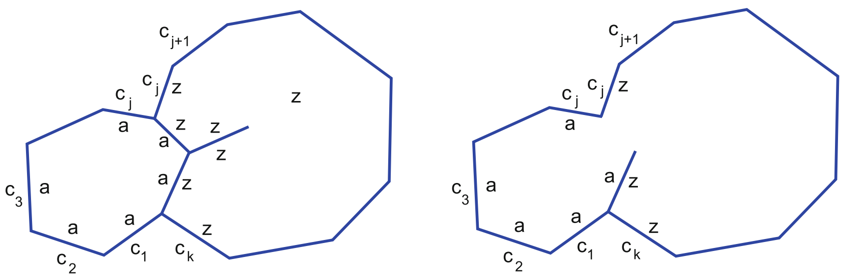
$$2z\Omega_{g,n+1}(z,J) = \sum_{h+h'=g,\, I\uplus I'=J} (\Omega_{h,|I|+1}(z,I) + \delta_{h,0}\delta_{I,\emptyset}z)\,(\Omega_{h',|I'|+1}(z,I') + \delta_{h',0}\delta_{I',\emptyset}z)$$
$$+\text{other possibilities}$$

- cutting that edge doesn't disconnect the ribbon graph, this is possible only if there was a handle relating the subgraphs on the two sides of the edge, i.e. we diminish the genus by one, and create two marked faces with label z, i.e. we get a graph in $\mathcal{G}_{g-1,n+2}(z,z,J)$.

  The corresponding equality of generating functions is thus:

$$2z\Omega_{g,n+1}(z,J) = \Omega_{g-1,n+2}(z,z,J) + \text{other possibilities}$$

– on the other side of that edge, there is an unmarked face, whose label is some λ_a with $a \in [1,N]$. Cutting the edge, creates a bi-labeled face with two labels z and λ_a, i.e. some edges have a z, followed by edges with a λ_a. Let c_i be the set of labels of edges adjacent to this bi–labeled face, so that $c_1,\ldots,c_j$ are adjacent to label λ_a and $c_j,\ldots,c_k$ are adjacent to label z (notice that label c_j appears twice, because it is adjacent to the trivalent vertex).



The weight associated to all edges of that bi–labeled face is:

$$\frac{1}{z+\lambda_a}\prod_{i=1}^{j}\frac{1}{\lambda_a+c_i}\prod_{i=j}^{k}\frac{1}{z+c_i}.$$

Observe the following equality of rational functions:

Lemma 6.4.1 *Let* $F(z) = \prod_{i=1}^{k} 1/(z+c_i)$*, then we have*

$$\sum_{j=1}^{k} \prod_{1\le i\le j} \frac{1}{z+c_i} \prod_{j\le i\le k} \frac{1}{z'+c_i} = -\frac{F(z)-F(z')}{z-z'}.$$

This (very simple to prove) lemma implies that, the weighted sum over all possibilities of bi-labeling a face (i.e. the sum over j), can be recovered as a divided difference of weighted graphs with uni–labeled faces:

$$\sum_{j=1}^{k} \frac{1}{z+\lambda_a} \prod_{i=1}^{j} \frac{1}{\lambda_a+c_i} \prod_{i=j}^{k} \frac{1}{z+c_i} = -\frac{1}{\lambda_a+z}\frac{1}{z-\lambda_a}\left(\prod_{i=1}^{k}\frac{1}{z+c_i} - \prod_{i=1}^{k}\frac{1}{\lambda_a+c_i}\right)$$

Observe that if $g=0$ and $n=0$, it may happen that there is no other edge, and after removing the edge, the graph is empty. This corresponds to the degree 2 term $\frac{1}{N}\sum_a \frac{1}{2z(z+\lambda_a)}$ in $\Omega_{0,1}(z)$.

The corresponding equality of generating functions is thus:

$$2z\Omega_{g,n+1}(z,J) = -\frac{1}{N}\sum_{a=1}^{N} \frac{(\Omega_{g,n+1}(z,J)+\delta_{g,0}\delta_{n,0}z) - (\Omega_{g,n+1}(\lambda_a,J)+\delta_{g,0}\delta_{n,0}\lambda_a)}{z^2-\lambda_a^2}$$
$$+\frac{2z}{N}\sum_a \frac{\delta_{g,0}\delta_{n,0}}{2z(z+\lambda_a)} + \text{other possibilities}$$

i.e. (this is the reason why we conveniently added a $-z\delta_{g,0}\delta_{n,0}$ term):

$$2z\Omega_{g,n+1}(z,J) = -\frac{1}{N}\sum_{a=1}^{N} \frac{\Omega_{g,n+1}(z,J)-\Omega_{g,n+1}(\lambda_a,J)}{z^2-\lambda_a^2} + \text{other possibilities}$$

– on the other side of that edge, there is another marked face, whose label is some $z_j \in J$. We can repeat the same reasoning as for the case of an unmarked face. We just need to add the 1-valent vertex of face z_j in all possible ways, this is done by taking a derivative.

$$2z\Omega_{g,n+1}(z,J) = \sum_{z_j\in J} \frac{1}{2z_j}\frac{d}{dz_j}\frac{\Omega_{g,n}(z,J\setminus\{z_j\})-\Omega_{g,n}(J)}{z^2-z_j^2} + \text{other possibilities.}$$

Finally, one has to pay attention to boundary cases, i.e. the case where after removing the edge the graph is empty, this contributes a factor z^2 for the only case where this happens, namely $g=0$ and $n=0$.

In the end, the equality of generating functions is:

$$
\begin{aligned}
2z\Omega_{g,n+1}(z,J) = & \sum_{h+h'=g,\, I\uplus I'=J} \Omega_{h,|I|+1}(z,I)\,\Omega_{h',|I'|+1}(z,I') + 2z\Omega_{g,n+1}(z,J) \\
& + \Omega_{g-1,n+2}(z,z,J) \\
& - \frac{1}{N}\sum_{a=1}^{N} \frac{\Omega_{g,n+1}(z,J) - \Omega_{g,n+1}(\lambda_a,J)}{z^2-\lambda_a^2} \\
& + \sum_{z_j\in J} \frac{1}{2z_j}\frac{d}{dz_j}\frac{\Omega_{g,n}(z,J\setminus\{z_j\}) - \Omega_{g,n}(J)}{z^2-z_j^2} \\
& - \delta_{g,0}\delta_{n,0}\, z^2 .
\end{aligned}
$$

□

6.4.2 Disc Amplitude (Rooted Planar Strebel Graphs)

With $g = 0$ and $n = 1$, the Tutte's equation reduces to

$$z^2 = \Omega_{0,1}(z)^2 - \frac{1}{N}\sum_{a=1}^{N}\frac{\Omega_{0,1}(z) - \Omega_{0,1}(\lambda_a)}{z^2-\lambda_a^2}$$

and we must look for a solution of that equation which is a formal series of $1/z$ and $1/\Lambda$ which behaves at large z, Λ as:

$$\Omega_{0,1}(z) = -z + \frac{1}{N}\sum_a \frac{1}{2z(z+\lambda_a)} + O(\deg \geq 3).$$

Remark 6.4.2 (Unicity) There is a unique formal series of $1/z$, $1/\Lambda$ solution of this Tutte's equation. This can be seen by solving the equation degree by degree, and in fact this is related to the fact that adding edges recursively constructs all graphs in a unique way. This is similar to the Brown's 1-cut lemma of Chap. 3.

The solution (unique) can be explicitly found:

Theorem 6.4.2

$$\Omega_{0,1}(z) = -\sqrt{z^2-\check{t}_1} - \frac{1}{N}\sum_a \frac{1}{2\check{\lambda}_a(\sqrt{z^2-\check{t}_1}+\check{\lambda}_a)}$$

where the diagonal matrix $\check\Lambda = \mathrm{diag}(\check\lambda_1, \dots, \check\lambda_N)$ *is the same matrix introduced in Theorem 6.3.7:*

$$\check\Lambda = \sqrt{\Lambda^2 - \check t_1} \quad , \quad \check t_1 = \frac{1}{N} \mathrm{Tr}\, \check\Lambda^{-1} \quad , \quad \check\Lambda = \Lambda + O(1/\Lambda)$$

and the sign of the square root is chosen such that $\sqrt{z^2 - \check t_1} = z + O(1/z)$ *at large* z*.*

In particular, if $t_1 = 0$*, we have* $\check\Lambda = \Lambda$*, i.e.*

$$\Omega_{0,1}(z) = -z + \frac{1}{N} \sum_a \frac{1}{2z(z+\lambda_a)}$$

Proof Consider the function

$$f(z) = -z - \frac{1}{N} \sum_a \frac{1}{2\check\lambda_a (z + \check\lambda_a)}$$

we have

$$f(z) + f(-z) = -\frac{1}{N} \sum_a \frac{1}{2\check\lambda_a(z+\check\lambda_a)} - \frac{1}{2\check\lambda_a(z-\check\lambda_a)} = \frac{1}{N} \sum_a \frac{1}{z^2 - \check\lambda_a^2}$$

and the product $f(z)f(-z)$ is an even rational function, with only simple poles at $z = \pm\check\lambda_a$, and which behaves like $-z^2$ at large z. Therefore it is of the form:

$$f(z)f(-z) = -z^2 + C + \sum_a \frac{C_a}{z^2 - \check\lambda_a^2}$$

where C, and the C_a's are yet to be determined.

At large z we have $f(z) \sim -z - \check t_1/2z + O(1/z^2)$ and thus $f(z)f(-z) \sim -z^2 - \check t_1 + O(1/z^2)$, i.e. $C = -\check t_1$. The coefficient C_a is related to the residue at the pole of $f(z)f(-z)$ at $z = \check\lambda_a$, i.e.

$$\frac{C_a}{2\check\lambda_a} = \operatorname*{Res}_{z \to \check\lambda_a} f(z)f(-z) = \frac{1}{2N\check\lambda_a} f(\check\lambda_a)$$

i.e. $C_a = \frac{1}{N} f(\check\lambda_a)$, and thus we get the equation for $f(z)$:

$$f(z)\left(\frac{1}{N}\sum_a \frac{1}{z^2 - \check\lambda_a^2} - f(z)\right) = -z^2 - \check t_1 + \frac{1}{N}\sum_a \frac{f(\check\lambda_a)}{z^2 - \check\lambda_a^2}$$

i.e. $f(z)$ satisfies:

$$f(z)^2 - \frac{1}{N}\sum_a \frac{f(z)-f(\check{\lambda}_a)}{z^2-\check{\lambda}_a^2} = z^2 + \check{t}_1$$

and using $\check{\lambda}_a^2 = \lambda_a^2 - \check{t}_1$:

$$f(z)^2 - \frac{1}{N}\sum_a \frac{f(z)-f(\check{\lambda}_a)}{z^2+\check{t}_1-\lambda_a^2} = z^2 + \check{t}_1.$$

We thus see that $f(\sqrt{z^2-\check{t}_1})$ satisfies the same equation as $\Omega_{0,1}(z)$, and is a power series in powers of $1/\lambda$ and $1/z$, which behaves at large λ and z like

$$-z + \frac{1}{N}\sum_a \frac{1}{2z(z+\lambda_a)} + \dots$$

therefore (using Remark 6.4.2 about unicity) we find that $f(\sqrt{z^2-\check{t}_1}) = \Omega_{0,1}(z)$.

When $t_1 = 0$, we have $\check{t}_1 = 0$ and $\check{\lambda}_a = \lambda_a$, and thus

$$\begin{aligned}\Omega_{0,1}(z) = f(z) &= -z - \frac{1}{N}\sum_a \frac{1}{2\lambda_a(z+\lambda_a)}\\ &= -z + \frac{1}{N}\sum_a \frac{1}{2z(z+\lambda_a)}\end{aligned}$$

where the last equality comes from:

$$\sum_a \frac{1}{\lambda_a(z+\lambda_a)} + \frac{1}{z(z+\lambda_a)} = \sum_a \frac{z+\lambda_a}{z\lambda_a(z+\lambda_a)} = \sum_a \frac{1}{z\lambda_a} = \frac{Nt_1}{z} = 0.$$

□

6.4.3 Cylinder Amplitude

The Tutte equation for $\Omega_{0,2}$ reads:

$$0 = 2\Omega_{0,1}(z)\Omega_{0,2}(z,z') - \frac{1}{N}\sum_a \frac{\Omega_{0,2}(z,z')-\Omega_{0,2}(\lambda_a,z')}{z^2-\lambda_a^2} + \frac{1}{2z'}\frac{d}{dz'}\frac{\Omega_{0,1}(z)-\Omega_{0,1}(z')}{z^2-z'^2}$$

and $\Omega_{0,2}(z,z')$ behaves like

$$\Omega_{0,2}(z,z') = \frac{1}{4zz'(z+z')^2} + O(\deg \geq 5).$$

Using the expression of $\Omega_{0,1}$, rewrite the equation as:

$$2z\Omega_{0,2}(z,z')\left(1+\frac{1}{N}\sum_a \frac{1}{2\lambda_a(z^2-\lambda_a^2)}\right) = \frac{1}{2z'\,(z+z')^2} + \frac{1}{N}\sum_a \frac{\Omega_{0,2}(\lambda_a,z')}{z^2-\lambda_a^2}$$
$$+\frac{1}{N}\sum_a \frac{1}{2z'}\frac{d}{dz'}\frac{1}{2\lambda_a(z+\lambda_a)(z'+\lambda_a)(z+z')}.$$

To the lowest degrees we get

$$\Omega_{0,2}(z,z') = \frac{1}{4zz'(z+z')^2} - \frac{1}{N}\sum_a \frac{1}{4zz'(z+z')^2}\frac{1}{2\lambda_a(z^2-\lambda_a^2)}$$
$$+\frac{1}{N}\sum_a \frac{1}{8zz'\lambda_a(z'+\lambda_a)^2\,(z^2-\lambda_a^2)}$$
$$+\frac{1}{N}\sum_a \frac{1}{4zz'}\,\frac{d}{dz'}\,\frac{1}{2\lambda_a(z+\lambda_a)(z'+\lambda_a)(z+z')}$$
$$+\dots$$

Theorem 6.4.3 *The solution is*

$$\Omega_{0,2}(z,z') = \frac{1}{4\sqrt{z^2+\check{t}_1}\,\sqrt{z'^2+\check{t}_1}\,(\sqrt{z^2+\check{t}_1}+\sqrt{z'^2+\check{t}_1})^2}.$$

In particular, if $t_1 = 0$ *we have*

$$\Omega_{0,2}(z,z') = \frac{1}{4\,zz'(z+z')^2} = \frac{-1}{4zz'}\,\frac{d}{dz}\frac{d}{dz'}\,\ln\frac{z^2-z'^2}{z-z'}.$$

Proof One can easily check that the expression above does satisfy Tutte's equation, with the correct highest lowest degree term, i.e. it is the unique solution. □

It is remarkable that for $\Omega_{0,2}$, contributions of all graphs of higher degrees exactly cancel when $t_1 = 0$!

6.4.4 *The Pair of Pants (0,3)*

From now on, we shall assume

$$t_1 = 0.$$

For $\Omega_{0,3}$, the Tutte's equation is

$$\begin{aligned}0 &= 2\ \Omega_{0,1}(z)\,\Omega_{0,3}(z,z_1,z_2) + 2\ \Omega_{0,2}(z,z_1)\,\Omega_{0,2}(z,z_2)\\ &\quad + \frac{1}{2z_1}\frac{d}{dz_1}\frac{\Omega_{0,2}(z,z_2)-\Omega_{0,2}(z_1,z_2)}{z^2-z_1^2} + \frac{1}{2z_2}\frac{d}{dz_2}\frac{\Omega_{0,2}(z,z_1)-\Omega_{0,2}(z_1,z_2)}{z^2-z_2^2}\\ &\quad - \frac{1}{N}\sum_{a=1}^{N}\frac{\Omega_{0,3}(z,z_1,z_2)-\Omega_{0,3}(\lambda_a,z_1,z_2)}{z^2-\lambda_a^2}.\end{aligned}$$

Using that $\Omega_{0,2}(z,z') = 1/4zz'(z+z')^2$, one computes that:

$$\begin{aligned}&2\ \Omega_{0,2}(z,z_1)\,\Omega_{0,2}(z,z_2)\\ &\quad + \frac{1}{2z_1}\frac{d}{dz_1}\frac{\Omega_{0,2}(z,z_2)-\Omega_{0,2}(z_1,z_2)}{z^2-z_1^2} + \frac{1}{2z_2}\frac{d}{dz_2}\frac{\Omega_{0,2}(z,z_1)-\Omega_{0,2}(z_1,z_2)}{z^2-z_2^2}\\ &= \frac{1}{8z^2\,z_1^3\,z_2^3}\end{aligned}$$

and thus Tutte's equation for $\Omega_{0,3}$ can be rewritten

$$0 = 2\ \Omega_{0,1}(z)\,\Omega_{0,3}(z,z_1,z_2) + \frac{1}{8z^2\,z_1^3\,z_2^3} - \frac{1}{N}\sum_{a=1}^{N}\frac{\Omega_{0,3}(z,z_1,z_2)-\Omega_{0,3}(\lambda_a,z_1,z_2)}{z^2-\lambda_a^2}$$

i.e.

$$2z\left(1+\frac{1}{N}\sum_a\frac{1}{2\lambda_a(z^2-\lambda_a^2)}\right)\Omega_{0,3}(z,z_1,z_2) = \frac{1}{8z^2\,z_1^3\,z_2^3} + \frac{1}{N}\sum_{a=1}^{N}\frac{\Omega_{0,3}(\lambda_a,z_1,z_2)}{z^2-\lambda_a^2}. \tag{6.4.3}$$

This equation determines uniquely $\Omega_{0,3}$ as a formal series in powers of Λ^{-1} and z_i^{-1}.

For instance to leading order we have:

$$\Omega_{0,3}(z,z_1,z_2) = \frac{1}{16\,z^3\,z_1^3\,z_2^3} + O(\deg_{\geq 12}).$$

Proposition 6.4.2 *We have:*

$$\Omega_{0,3}(z, z_1, z_2) = \frac{1}{8\,(2-t_3)\ z^3\, z_1^3\, z_2^3}$$

Proof First, notice that Tutte equation (6.4.3) determines a unique solution which is a formal series in powers of Λ^{-1} and z_i^{-1}. Therefore, if we can exhibit a solution of Eq. (6.4.3) which is a formal series in powers of Λ^{-1} and z_i^{-1}, then it will be $\Omega_{0,3}$.

One can try $\Omega_{0,3}$ in the form

$$\Omega_{0,3}(z, z_1, z_2) = \frac{\alpha}{z^3\, z_1^3\, z_2^3}$$

and insert this into Eq. (6.4.3), that gives:

$$2\alpha = \frac{1}{8} + \alpha t_3$$

i.e. $\alpha = 1/8(2-t_3)$. □

Using Eq. (6.4.1)

$$\begin{aligned}\Omega_{0,3}(z_1, z_2, z_3) &= \frac{1}{8\,(2-t_3)\ z_1^3\, z_2^3\, z_3^3}\\ &= \frac{1}{2^3}\,(2-t_3)^{-1} \sum_{\{d_i\}} < \prod_{i=1}^{3} \tau_{d_i}\ \mathrm{e}^{\sum_{k>0} \hat{t}_k \kappa_k} >_0\ \prod_{i=1}^{3} \frac{(2d_i+1)!!}{z_i^{2d_i+3}}\end{aligned}$$

this gives (which we already knew):

$$< \tau_0^3 >_0 = 1.$$

6.4.5 *The Lid (1,1)*

For $\Omega_{1,1}$, the Tutte's equation is

$$0 = 2\ \Omega_{0,1}(z)\,\Omega_{1,1}(z) + \Omega_{0,2}(z, z) - \frac{1}{N} \sum_{a=1}^{N} \frac{\Omega_{1,1}(z) - \Omega_{1,1}(\lambda_a)}{z^2 - \lambda_a^2}$$

and using that $\Omega_{0,2}(z,z') = 1/4zz'(z+z')^2$

$$0 = 2\ \Omega_{0,1}(z)\,\Omega_{1,1}(z) + \frac{1}{16\ z^4} - \frac{1}{N}\sum_{a=1}^{N} \frac{\Omega_{1,1}(z) - \Omega_{1,1}(\lambda_a)}{z^2 - \lambda_a^2}$$

i.e.

$$2z\left(1 + \frac{1}{N}\sum_a \frac{1}{2\lambda_a(z^2-\lambda_a^2)}\right)\Omega_{1,1}(z) = \frac{1}{16z^4} + \frac{1}{N}\sum_{a=1}^{N}\frac{\Omega_{1,1}(\lambda_a)}{z^2-\lambda_a^2}$$

and thus to leading order:

$$\Omega_{1,1}(z) = \frac{1}{32\,z^5} + O(\deg_{\geq 8}).$$

Proposition 6.4.3 *We have (we assume $t_1 = 0$):*

$$\Omega_{1,1}(z) = \frac{1}{16\,(2-t_3)\,z^5} + \frac{t_5}{16\,(2-t_3)^2\ z^3}$$

Proof It is easy to see that this expression satisfies the Tutte equation, and has the correct leading degree behavior, therefore it is the unique solution of Tutte's equation which is a formal series in powers of Λ^{-1} and z^{-1}. □

Using Eq. (6.4.1)

$$\begin{aligned}\Omega_{1,1}(z) &= \frac{1}{16\,(2-t_3)\,z^5} + \frac{t_5}{16\,(2-t_3)^2\ z^3} \\ &= \frac{1}{2}\,(2-t_3)^{-1}\sum_{\{d\}} <\tau_d\ \mathrm{e}^{\sum_{k>0}\hat{t}_k\kappa_k}>_0\ \frac{(2d+1)!!}{z^{2d+3}}\end{aligned}$$

where we had $\hat{t}_1 = 3t_5/(2-t_3)$ in Theorem 6.3.8, this gives (which we already knew):

$$<\tau_1>_1 = \frac{1}{24}.$$

$$\hat{t}_1 <\tau_0\kappa_1>_1 = \frac{1}{24}\ \frac{3t_5}{2-t_3} \qquad ,\ \text{i.e.} \quad <\tau_0\kappa_1>_1 = \frac{1}{24}.$$

6.4.6 Stable Topologies

We have so far computed $\Omega_{0,1}$ $\Omega_{0,2}$, $\Omega_{1,1}$ and $\Omega_{0,3}$, now we shall compute $\Omega_{g,n}$ for all $n > 0$ and $2 - 2g - n < 0$.

First, let us rewrite Tutte's equations Eq. (6.4.2) for $2 - 2g - n < 0$ as (with $J = \{z_1, \dots, z_n\}$):

$$\begin{aligned}
& 2z \left(1 + \frac{1}{N} \sum_a \frac{1}{2\lambda_a(z^2 - \lambda_a^2)}\right) \Omega_{g,n+1}(z, J) \\
&= \Omega_{g-1,n+2}(z, z, J) \\
&\quad + \sum_{h+h'=g,\, I \uplus I' = J}^{\text{stable}} \Omega_{h,|I|+1}(z, I)\, \Omega_{h',|I'|+1}(z, I') \\
&\quad + \frac{1}{N} \sum_{a=1}^{N} \frac{\Omega_{g,n+1}(\lambda_a, J)}{z^2 - \lambda_a^2} \\
&\quad + 2 \sum_{z_j \in J} \Omega_{0,2}(z, z_j)\, \Omega_{g,n}(z, J \setminus \{z_j\}) \\
&\quad + \sum_{z_j \in J} \frac{1}{2z_j} \frac{d}{dz_j} \frac{\Omega_{g,n}(z, J \setminus \{z_j\}) - \Omega_{g,n}(J)}{z^2 - z_j^2}
\end{aligned} \tag{6.4.4}$$

which shows that there is a unique solution which is a formal large Λ, z power series. In fact, since Tutte's equation is a recursion on the number of edges, which expresses the generating function of graphs with n edges in terms of those with $n-1$ edges, it necessarily determines uniquely all the generating functions.

Lemma 6.4.2 (Symmetry) *If $n > 0$ and $2 - 2g - n < 0$, the functions $\Omega_{g,n}$ are odd:*

$$\Omega_{g,n}(-z, z_2, \dots, z_n) = -\,\Omega_{g,n}(z, z_2, \dots, z_n)$$

Proof We know that it is true for $\Omega_{0,3}$ and $\Omega_{1,1}$, i.e. for $2g + n - 2 = 1$. We shall proceed by recursion.

Assume that it is already proved for all (g', n') such that $0 < 2g' + n' - 2 \leq 2g + n - 2$, we shall prove it for $(g, n + 1)$.

Rewrite Eq. (6.4.4) as

$$\begin{aligned}
&2z\left(1+\frac{1}{N}\sum_a \frac{1}{2\lambda_a(z^2-\lambda_a^2)}\right)\Omega_{g,n+1}(z,J)\\
&=\Omega_{g-1,n+2}(z,z,J)\\
&\quad+\sum_{h+h'=g,\, I\uplus I'=J}^{\text{stable}} \Omega_{h,|I|+1}(z,I)\,\Omega_{h',|I'|+1}(z,I')\\
&\quad+\frac{1}{N}\sum_{a=1}^{N}\frac{\Omega_{g,n+1}(\lambda_a,J)}{z^2-\lambda_a^2}\\
&\quad+\sum_{z_j\in J}\frac{z^2+z_j^2}{2zz_j(z^2-z_j^2)^2}\,\Omega_{g,n}(z,J\setminus\{z_j\})\\
&\quad-\sum_{z_j\in J}\frac{1}{2z_j}\frac{d}{dz_j}\frac{\Omega_{g,n}(J)}{z^2-z_j^2}
\end{aligned}$$

where in the RHS, the symbol $\sum^{\text{stable}}$ means that we sum only over (h,h',I,I') such that $2-2h-|I|<0$ and $2-2h'-|I'|<0$. In particular, by recursion hypothesis, all terms in the RHS are even functions of z, and thus $\Omega_{g,n+1}$ is odd. □

Before going further, let us define:

Definition 6.4.2 We define:

$$\omega_{g,n}(z_1,\dots,z_n)=2^n\prod_{i=1}^{n} z_i\left(\Omega_{g,n}(z_1,\dots,z_n)+\frac{\delta_{g,0}\delta_{n,2}}{(z_1^2-z_2^2)^2}\right)$$

and

$$y(z)=\Omega_{0,1}(-z)=z+\frac{1}{N}\sum_{a=1}^{N}\frac{1}{2\lambda_a(z-\lambda_a)}$$

and

$$x(z)=z^2.$$

For example we have:

$$\omega_{0,1}(z)=-2z^2+\frac{1}{N}\sum_{a=1}^{N}\frac{1}{z+\lambda_a}=2z\,y(-z)$$

$$\omega_{0,2}(z_1,z_2)=\frac{1}{(z_1-z_2)^2}$$

$$\omega_{0,3}(z_1,z_2,z_3) = \frac{1}{(2-t_3)\ z_1^2\, z_2^2\, z_3^2}$$

$$\omega_{1,1}(z) = \frac{1}{8(2-t_3)\ z^4} + \frac{t_5}{8(2-t_3)^2\ z^2}$$

Proposition 6.4.4 *Using the symmetry lemma, Tutte's equation (6.4.2) for* $2g-2+n>1$ *becomes:*

$$\begin{aligned}0 =\ & \omega_{g-1,n+2}(z,-z,J) + \sum_{h+h'=g,\, I\uplus I'=J}' \omega_{h,|I|+1}(z,I)\, \omega_{h',|I'|+1}(-z,I') \\ & + 2z(y(z)-y(-z))\ \omega_{g,n+1}(z,J) \\ & + \frac{4z^2}{N}\sum_{a=1}^{N} \frac{\omega_{g,n+1}(\lambda_a,J)}{2\lambda_a(z^2-\lambda_a^2)} - 2z^2 \sum_{z_j\in J} \frac{d}{dz_j}\, \frac{\omega_{g,n}(J)}{z_j\,(z^2-z_j^2)}.\end{aligned} \tag{6.4.5}$$

Proof When $2g-2+n>0$, the left hand side of Tutte's equation is vanishing, and Tutte's equation (6.4.2) is:

$$\begin{aligned}0 =\ & \Omega_{g-1,n+2}(z,z,J) + \sum_{h+h'=g,\, I\uplus I'=J} \Omega_{h,|I|+1}(z,I)\, \Omega_{h',|I'|+1}(z,I') \\ & - \frac{1}{N}\sum_{a=1}^{N} \frac{\Omega_{g,n+1}(z,J) - \Omega_{g,n+1}(\lambda_a,J)}{z^2-\lambda_a^2} \\ & + \sum_{z_j\in J} \frac{1}{2z_j}\, \frac{d}{dz_j}\, \frac{\Omega_{g,n}(z,J\setminus\{z_j\}) - \Omega_{g,n}(J)}{z^2-z_j^2}\end{aligned}$$

after multiplying by $4z^2\prod_i(2z_i)$ we get:

$$\begin{aligned}0 =\ & \omega_{g-1,n+2}(z,z,J) + \sum_{h+h'=g,\, I\uplus I'=J} \omega_{h,|I|+1}(z,I)\, \omega_{h',|I'|+1}(z,I') \\ & - \frac{2z}{N}\sum_{a=1}^{N} \frac{\omega_{g,n+1}(z,J) - \frac{z}{\lambda_a}\omega_{g,n+1}(\lambda_a,J)}{z^2-\lambda_a^2} \\ & + 2z \sum_{z_j\in J} \frac{d}{dz_j}\, \frac{\omega_{g,n}(z,J\setminus\{z_j\}) - \frac{z}{z_j}\,\omega_{g,n}(J)}{z^2-z_j^2} \\ & - 2\sum_{z_j\in J} \frac{4zz_j}{(z^2-z_j^2)^2}\, \omega_{g,n}(z,J\setminus\{z_j\}).\end{aligned}$$

In this equation, all $\omega_{h,m}$ with $(h, m) \neq (0, 1), (0, 2)$ are even and z can be changed to $-z$. Only the factors containing $\omega_{0,1}$ or $\omega_{0,2}$ require some attention.

The factors containing $\omega_{g,n+1}$ come with $\omega_{0,1}$, they are:

$$2\omega_{0,1}(z)\,\omega_{g,n+1}(z,J) - \frac{2z}{N}\sum_{a=1}^{N} \frac{\omega_{g,n+1}(z,J) - \frac{z}{\lambda_a}\omega_{g,n+1}(\lambda_a,J)}{z^2-\lambda_a^2}$$

which we would like to compare with

$$\omega_{0,1}(z)\,\omega_{g,n+1}(-z,J) + \omega_{0,1}(-z)\,\omega_{g,n+1}(z,J) + \frac{4z^2}{N}\sum_{a=1}^{N}\frac{\omega_{g,n+1}(\lambda_a,J)}{2\lambda_a(z^2-\lambda_a^2)}.$$

The difference is

$$\begin{aligned}
&2\omega_{0,1}(z)\,\omega_{g,n+1}(z,J) - \frac{2z}{N}\sum_{a=1}^{N}\frac{\omega_{g,n+1}(z,J)}{z^2-\lambda_a^2}\\
&-\omega_{0,1}(z)\,\omega_{g,n+1}(-z,J) - \omega_{0,1}(-z)\,\omega_{g,n+1}(z,J)\\
&= \left(\omega_{0,1}(z) - \omega_{0,1}(-z) - \frac{2z}{N}\sum_{a=1}^{N}\frac{1}{z^2-\lambda_a^2}\right)\omega_{g,n+1}(z,J)\\
&= 0.
\end{aligned}$$

We also need to compare terms involving $\omega_{0,2}$, i.e. factors of $\omega_{g,n}$. We thus need to compare

$$\sum_{z_j\in J} 2\omega_{0,2}(z,z_j)\omega_{g,n}(z,J\setminus\{z_j\}) + 2z\frac{d}{dz_j}\,\frac{\omega_{g,n}(z,J\setminus\{z_j\}) - \frac{z}{z_j}\,\omega_{g,n}(J)}{z^2-z_j^2} - \frac{8zz_j}{(z^2-z_j^2)^2}\,\omega_{g,n}(z,J\setminus\{z_j\})$$

with

$$\sum_{z_j\in J}\omega_{0,2}(z,z_j)\omega_{g,n}(-z,J\setminus\{z_j\}) + \omega_{0,2}(-z,z_j)\omega_{g,n}(z,J\setminus\{z_j\}) - 2z^2\sum_{z_j\in J}\frac{d}{dz_j}\,\frac{\omega_{g,n}(J)}{z_j\,(z^2-z_j^2)}.$$

The difference is

$$\sum_{z_j\in J}\left((\omega_{0,2}(z,z_j) - \omega_{0,2}(-z,z_j)) + 2z\frac{d}{dz_j}\,\frac{1}{z^2-z_j^2} - \frac{8zz_j}{(z^2-z_j^2)^2}\right)\omega_{g,n}(z,J\setminus\{z_j\}) = 0.$$

This concludes the proof. □

Theorem 6.4.4 *For any* $n > 0$ *and* $2g - 2 + n > 0$*, we have that* $\omega_{g,n}(z_1, \ldots, z_n)$ *is an even rational function of the* z_i*'s, with poles only at* $z_i = 0$*, and which satisfies the "topological recursion":*

$$\omega_{g,n+1}(z_0, J) = \operatorname*{Res}_{z \to 0} K(z_0, z)\, dz \Big[\omega_{g-1,n+2}(z, -z, J) + \sum'_{h+h'=g,\, I \uplus I'=J} \omega_{h,1+\#I}(\tilde{\mathcal{E}}_K; z, I)\, \omega_{g-h,1+n-\#I}(\tilde{\mathcal{E}}_K; -z, J \setminus I)\Big] \tag{6.4.6}$$

where $\sum'$ *means that* $(h, I) = (0, \emptyset)$ *and* $(h, I) = (g, J)$ *are excluded from the sum, where* K *is the kernel*

$$K(z_0, z) := -\frac{\int_{z'=-z}^{z} \omega_{0,2}(z_0, z')}{2(y(z) - y(-z))\, x'(-z)}$$

and $y(z)$ *is the formal power series in powers of* Λ*:*

$$y(z) = z + \frac{1}{N} \sum_{a=1}^{N} \frac{1}{2\lambda_a (z - \lambda_a)} = z - \frac{1}{2} \sum_{k=0}^{\infty} t_{k+2} z^k \quad , \quad x(z) = z^2$$

and we have:

$$\omega_{g,n}(z_1, \ldots, z_n) = (2 - t_3)^{2-2g-n} \sum_{\{d_i\}} \left\langle \prod_{i=1}^{n} \tau_{d_i}\, \mathrm{e}^{\sum_{k>0} \hat{t}_k \kappa_k} \right\rangle_g \prod_{i=1}^{n} \frac{(2d_i + 1)!!}{z_i^{2d_i+2}}$$

Proof The proof proceeds more or less like in Chap. 3. We shall replace the 1-cut Brown's lemma by the following observation:

Tutte equations have a unique solution which is a formal power series in inverse powers of Λ and z_i'z. Therefore, exhibiting a solution is enough to claim that it is the solution.

We proceed by recursion. The theorem holds true for $\omega_{0,3}$ and $\omega_{1,1}$. Choose (g, n) such that $2g - 2 + n > 0$, and define $\hat{\omega}_{g,n+1}(z_0, z_1, \ldots, z_n)$ as the right hand side of Eq. (6.4.6) with $J = \{z_1, \ldots, z_n\}$:

$$\hat{\omega}_{g,n+1}(z_0, J) = \operatorname*{Res}_{z \to 0} K(z_0, z)\, dz \Big[\omega_{g-1,n+2}(z, -z, J) + \sum'_{h+h'=g,\, I \uplus I'=J} \omega_{h,1+\#I}(\tilde{\mathcal{E}}_K; z, I)\, \omega_{g-h,1+n-\#I}(\tilde{\mathcal{E}}_K; -z, J \setminus I)\Big]$$

with

$$K(z_0, z) = \frac{1}{2\,(z_0^2 - z^2)\,(y(z) - y(-z))}.$$

By recursion hypothesis, we see that $\hat{\omega}_{g,n+1}(z_0, z_1, \ldots, z_n)$ is an even rational function of the z_i's with $i \geq 1$, and with poles only at $z_i = 0$. It is also an even function of z_0 because $K(z_0, z)$ is, and since $K(z_0, z)$ has a pole at $z_0^2 = z^2$, it can diverge only if z_0 or $-z_0$ pinches the integration contour for z, i.e. only if $z_0 \to 0$. At $z_0 \to 0$, the singularity is a pole, so that $\hat{\omega}_{g,n+1}(z_0, z_1, \ldots, z_n)$ is an even rational function of z_0 with poles only at $z_0 = 0$.

Moreover, $\hat{\omega}_{g,n+1}(z_0, z_1, \ldots, z_n)$ is clearly a power series in inverse powers of Λ and of the z_i's.

It remains to prove that it satisfies Tutte's equation, i.e. Eq. (6.4.4).

Consider the following even function of z:

$$\begin{aligned} f(z) = {} & \omega_{g-1,n+2}(z, -z, J) + \sum_{h+h'=g,\, I \uplus I' = J}' \omega_{h,|I|+1}(z, I)\, \omega_{h',|I'|+1}(-z, I') \\ & + 2z(y(-z) - y(z))\; \hat{\omega}_{g,n+1}(z, J) \\ & + \frac{4z^2}{N} \sum_{a=1}^{N} \frac{\hat{\omega}_{g,n+1}(\lambda_a, J)}{2\lambda_a (z^2 - \lambda_a^2)} - 2z^2 \sum_{z_j \in J} \frac{d}{dz_j} \frac{\omega_{g,n}(J)}{z_j\,(z^2 - z_j^2)} \end{aligned}$$

We have $f(z) = f(-z)$. It seems that $f(z)$ may have poles at $z = \pm\lambda_a$, $z = \pm z_i$ and $z = 0$. The only possible poles at $z = \lambda_a$ are simple poles and their residue is

$$\begin{aligned} \operatorname*{Res}_{z \to \lambda_a} f(z) &= \operatorname*{Res}_{z \to \lambda_a} -2zy(z)\, \hat{\omega}_{g,n+1}(z, J) + \frac{4z^2}{N} \sum_{a=1}^{N} \frac{\hat{\omega}_{g,n+1}(\lambda_a, J)}{2\lambda_a (z^2 - \lambda_a^2)} \\ &= -\frac{1}{N}\, \hat{\omega}_{g,n+1}(\lambda_a, J) + \frac{4\lambda_a^2}{N}\, \frac{\hat{\omega}_{g,n+1}(\lambda_a, J)}{4\lambda_a^2} \\ &= 0, \end{aligned}$$

i.e. $f(z)$ has no pole at $z = \lambda_a$, and since $f(-z) = f(z)$ it also has no pole at $z = -\lambda_a$.

The only possible pole of $f(z)$ at $z = z_j$ are at most double poles and

$$\begin{aligned} f(z) &\sim_{z \to z_j} \omega_{0,2}(z, z_j)\, \omega_{g,n}(z, J \setminus \{z_j\}) - 2z^2 \frac{d}{dz_j} \frac{\omega_{g,n}(J)}{z_j\,(z^2 - z_j^2)} + O(1) \\ &\sim_{z \to z_j} \frac{d}{dz_j} \left(\frac{1}{(z - z_j)}\, \omega_{g,n}(z, J \setminus \{z_j\}) - 2z^2 \frac{\omega_{g,n}(J)}{z_j\,(z^2 - z_j^2)} \right) + O(1) \end{aligned}$$

$$\sim_{z\to z_j} \frac{d}{dz_j}\,\frac{1}{(z-z_j)}\left(\omega_{g,n}(z,J\setminus\{z_j\}) - 2z^2\,\frac{\omega_{g,n}(J)}{z_j\,(z+z_j)}\right) + O(1)$$
$$\sim_{z\to z_j} O(1)$$

i.e. there is no pole at $z = \pm z_j$.

For the pole at $z = 0$, let us compute $\operatorname{Res}_{z\to 0} K(z_0,z)f(z)$, for that notice that the definition of $\hat{\omega}_{g,n+1}$ amounts to:

$$\hat{\omega}_{g,n+1}(z_0,J) = \operatorname*{Res}_{z\to 0} K(z_0,z)\Big(f(z) - 2z(y(-z)-y(z))\hat{\omega}_{g,n+1}(z,J)$$
$$-\frac{4z^2}{N}\sum_{a=1}^{N}\frac{\hat{\omega}_{g,n+1}(\lambda_a,J)}{2\lambda_a\,(z^2-\lambda_a^2)} + 2z^2\sum_{z_j\in J}\frac{d}{dz_j}\,\frac{\omega_{g,n}(J)}{z_j\,(z^2-z_j^2)}\Big)$$

and the last two terms have no pole at $z = 0$ so don't contribute to the residue. This implies that

$$\operatorname*{Res}_{z\to 0} K(z_0,z)f(z) = \hat{\omega}_{g,n+1}(z_0,J) - \operatorname*{Res}_{z\to 0} K(z_0,z)\,2z(y(z)-y(-z))\,\hat{\omega}_{g,n+1}(z,J)$$
$$= \hat{\omega}_{g,n+1}(z_0,J) - \operatorname*{Res}_{z\to 0}\frac{z}{(z_0^2-z^2)}\,\hat{\omega}_{g,n+1}(z,J).$$

In the last term, $\hat{\omega}_{g,n+1}(z,J)$ is a rational function of z whose only poles are at $z = 0$, and thus we can move the integration contour:

$$\operatorname*{Res}_{z\to 0} K(z_0,z)f(z) = \hat{\omega}_{g,n+1}(z_0,J)$$
$$+ \operatorname*{Res}_{z\to z_0}\frac{z}{(z_0^2-z^2)}\,\hat{\omega}_{g,n+1}(z,J) + \operatorname*{Res}_{z\to -z_0}\frac{z}{(z_0^2-z^2)}\,\hat{\omega}_{g,n+1}(z,J)$$

which implies

$$\forall\, z_0\,,\qquad \operatorname*{Res}_{z\to 0} K(z_0,z)f(z) = 0$$

The fact that this residue vanishes for every z_0, means (by expanding around $z_0 = \infty$), that $f(z)$ must have no pole at $z = 0$.

Finally, we have found that $f(z)$ is a rational function with no pole at all, it must be a constant, and it vanishes at $z \to \infty$, so that:

$$f(z) = 0.$$

This implies that $\hat{\omega}_{g,n+1}$ satisfies the Tutte's equation. And since $\hat{\omega}_{g,n+1}(z_0, z_1, \dots, z_n)$ is a power series in inverse powers of Λ and of the z_i's, we must have

$$\hat{\omega}_{g,n+1}(z_0, z_1, \dots, z_n) = \omega_{g,n+1}(z_0, z_1, \dots, z_n).$$

Then, since we now know that $\omega_{g,n}(z_1, \dots, z_n)$ is a rational function of the z_i's with poles only at $z_i = 0$, we can say that Proposition 6.4.1 holds not only at $z_i = \lambda_{a_i}$, but holds for every z_i.

This concludes our proof. □

Remark 6.4.3 It is quite remarkable, that $\omega_{g,n}$ which is a sum of graphs weighted with denominators of the form $1/(z_i + \lambda_a)$ or $1/(z_i + z_j)$, ends up having no pole at $z_i = -\lambda_a$ or at $z_i = -z_j$, and eventually it has poles only at $z_i = 0$.

This shows that there are lots of cancellations among graphs. This also means that $\omega_{g,n}$ is not a proper "generating function of graphs", it is only a sum of weighted graphs, but it looses information about the graphs, it does not even encode the number of graphs.

However, $\omega_{g,n}$ is a good generating function of intersection numbers, it encodes them completely without any loss of information, it was in fact designed for that purpose.

6.4.7 *Topological Recursion for Intersection Numbers*

The topological recursion translates for intersection numbers (by expanding at λ_as$\to \infty$) into:

$$\begin{aligned}
&(2d_0+1)!! \left\langle \tau_{d_0}\ \tau_{d_1} \dots \tau_{d_n} \right\rangle_g \\
= &\frac{1}{2} \sum_{d+d'=d_0-2} (2d+1)!!(2d'+1)!! \Bigg[\left\langle \tau_d \tau_{d'}\ \tau_{d_1} \dots \tau_{d_n} \right\rangle_{g-1} \\
&+ \sum_{h+h'=g,\ I\sqcup I'=\{1,\dots,n\}}^{\text{stable}} \left\langle \tau_d \prod_{i\in I} \tau_{d_i} \right\rangle_h \left\langle \tau_{d'} \prod_{i\in I'} \tau_{d_i} \right\rangle_{g-h} \Bigg] \\
&+ \sum_{j=1}^{n} \frac{(2d_j+2d_0-1)!!}{(2d_j-1)!!} \left\langle \tau_{d_0+d_j-1} \prod_{i\neq j} \tau_{d_i} \right\rangle_g .
\end{aligned}$$

Remark 6.4.4 Those equations can be interpreted as a set of Virasoro constraint, as in Sect. 2.6 in Chap. 2. We shall not elaborate on this, and refer the interested reader to the works of [67].

6.4.8 Examples

We assume $t_1 = 0$. Let us show application of Theorem 6.4.4 for the first few values of n and g:

We have

$$K(z_0, z) = \frac{1}{2(z_0^2 - z^2)\,(y(z) - y(-z))}$$
$$= \frac{1}{2z(2-t_3)\,z_0^2}\left(1 + \frac{z^2}{z_0^2} + \frac{z^4}{z_0^4} + O(z^6)\right)$$
$$\left(1 + \frac{t_5\,z^2}{2-t_3} + \frac{t_7\,z^4}{2-t_3} + \frac{t_5^2\,z^4}{(2-t_3)^2} + O(z^6)\right)$$

Theorem 6.4.4 easily gives

$$\omega_{0,3}(z_1, z_2, z_3) = \frac{1}{2-t_3}\,\frac{1}{z_1^2\,z_2^2\,z_3^2}$$

$$\omega_{1,1}(z) = \frac{1}{8(2-t_3)}\left(\frac{1}{z^4} + \frac{t_5}{(2-t_3)\,z^2}\right)$$

$$\omega_{0,4}(z_1, z_2, z_3, z_4) = \frac{1}{(2-t_3)^2\,z_1^2 z_2^2 z_3^2 z_4^2}\left(\frac{3t_5}{2-t_3} + 3\sum_i \frac{1}{z_i^2}\right)$$

$$\omega_{1,2}(z_1, z_2) = \frac{dz_1 \otimes dz_2}{8(2-t_3)\,z_1^6\,z_2^6}\Big[(2-t_3)^2\,(5z_1^4 + 5z_2^4 + 3z_1^2 z_2^2)$$
$$+6t_5^2 z_1^4 z_2^4 + (2-t_3)(6t_5 z_1^4 z_2^2 + 6t_5 z_1^2 z_2^4 + 5t_7 z_1^4 z_2^4)\Big]$$

$$\omega_{2,1}(z) = -\frac{dz}{128(2-t_3)^7\,z^{10}}\Big[252 t_5^4 z^8 + 12 t_5^2 z^6 (2-t_3)(50 t_7 z^2 + 21 t_5)$$
$$+z^4(2-t_3)^2(252 t_5^2 + 348 t_5 t_7 z^2 + 145 t_7^2 z^4 + 308 t_3 t_9 z^4)$$
$$+z^2(2-t_3)(203 t_5 + 145 z^2 t_7 + 105 z^4 t_9 + 105 z^6 t_{11}) + 105(2-t_3)^4\Big].$$

The topological recursion for computing the $\omega_{g,n}$'s can easily be implemented on a computer, and gives tables of intersection numbers. For the lowest values of g and n we get:

Genus 0 :

$n =$	3	4	5	6
	$< \tau_0^3 >_0 = 1$	$< \tau_0^3 \tau_1 >_0 = 1$	$< \tau_0^4 \tau_2 >_0 = 1$ $< \tau_0^3 \tau_1^2 >_0 = 2$	$< \tau_0^5 \tau_3 >_0 = 1$ $< \tau_0^4 \tau_1 \tau_2 >_0 = 3$ $< \tau_0^3 \tau_1^3 >_0 = 6$
$\kappa' s$		$< \tau_0^4 \kappa_1 >_0 = 1$	$< \tau_0^5 \kappa_2 >_0 = 1$ $< \tau_0^4 \tau_1 \kappa_1 >_0 = 3$ $< \tau_0^5 \kappa_1^2 >_0 = 5$	$< \tau_0^6 \kappa_3 >_0 = 1$ $< \tau_0^6 \kappa_1 \kappa_2 >_0 = 9$ $< \tau_0^6 \kappa_1^3 >_0 = 61$ $< \tau_0^5 \tau_1 \kappa_2 >_0 = 4$ $< \tau_0^5 \tau_1 \kappa_1^2 >_0 = 26$ $< \tau_0^5 \tau_2 \kappa_1 >_0 = 6$ $< \tau_0^4 \tau_1^2 \kappa_1 >_0 = 12$

In fact for genus zero, all intersection numbers are known by a general formula:

$$< \prod_{i=1}^{n} \tau_{d_i} >_0 = \frac{(n-3)!}{\prod_i d_i!}.$$

Genus 1:

$n =$	1	2	3	4
	$< \tau_1 >_1 = \frac{1}{24}$	$< \tau_0 \tau_2 >_1 = \frac{1}{24}$ $< \tau_1^2 >_1 = \frac{1}{24}$	$< \tau_0^2 \tau_3 >_1 = \frac{1}{24}$ $< \tau_0 \tau_1 \tau_2 >_1 = \frac{1}{12}$ $< \tau_1^3 >_1 = \frac{1}{12}$	$< \tau_0^3 \tau_4 >_1 = \frac{1}{24}$ $< \tau_0^2 \tau_1 \tau_3 >_1 = \frac{1}{8}$ $< \tau_0^2 \tau_2^2 >_1 = \frac{1}{6}$ $< \tau_0 \tau_1^2 \tau_2 >_1 = \frac{1}{4}$ $< \tau_1^4 >_1 = \frac{1}{4}$
$\kappa' s$	$< \tau_0 \kappa_1 >_1 = \frac{1}{24}$	$< \tau_0^2 \kappa_2 >_1 = \frac{1}{24}$ $< \tau_0^2 \kappa_1^2 >_1 = \frac{1}{8}$ $< \tau_0 \tau_1 \kappa_1 >_1 = \frac{1}{12}$	$< \tau_0^3 \kappa_3 >_1 = \frac{1}{24}$ $< \tau_0^3 \kappa_1 \kappa_2 >_1 = \frac{1}{4}$ $< \tau_0^3 \kappa_1^3 >_1 = \frac{7}{6}$ $< \tau_0^2 \tau_1 \kappa_2 >_1 = \frac{1}{8}$ $< \tau_0^2 \tau_1 \kappa_1^2 >_1 = \frac{13}{24}$ $< \tau_0^2 \tau_2 \kappa_1 >_1 = \frac{1}{6}$ $< \tau_0 \tau_1^2 \kappa_1 >_1 = \frac{1}{4}$	$< \tau_0^4 \kappa_4 >_1 = \frac{1}{24}$ $< \tau_0^4 \kappa_1 \kappa_3 >_1 = \frac{5}{12}$ $< \tau_0^4 \kappa_2^2 >_1 = \frac{13}{24}$ $< \tau_0^4 \kappa_1^2 \kappa_2 >_1 = \frac{83}{24}$ $< \tau_0^4 \kappa_1^4 >_1 = \frac{529}{24}$ $< \tau_0^3 \tau_1 \kappa_3 >_1 = \frac{1}{6}$ $< \tau_0^3 \tau_1 \kappa_1 \kappa_2 >_1 = \frac{31}{24}$ $< \tau_0^3 \tau_1 \kappa_1^3 >_1 = \frac{187}{24}$ $< \tau_0^3 \tau_2 \kappa_2 >_1 = \frac{7}{24}$ $< \tau_0^3 \tau_2 \kappa_1^2 >_1 = \frac{41}{24}$ $< \tau_0^3 \tau_3 \kappa_1 >_1 = \frac{7}{24}$ $< \tau_0^2 \tau_1^2 \kappa_2 >_1 = \frac{1}{2}$ $< \tau_0^2 \tau_1^2 \kappa_1^2 >_1 = \frac{17}{6}$ $< \tau_0^2 \tau_1 \tau_2 \kappa_1 >_1 = \frac{2}{3}$ $< \tau_0 \tau_1^3 \kappa_1 >_1 = 1$

In fact for genus 1, all intersection numbers are known by a general formula:

$$< \prod_{i=1}^{n} \tau_{d_i} >_1 = \frac{n!}{24 \ \prod_i d_i!}.$$

Genus 2 :

$n =$	1	2	3
	$< \tau_4 >_2 = \frac{1}{1152}$	$< \tau_0 \tau_5 >_2 = \frac{1}{1152}$	$< \tau_0^2 \tau_6 >_2 = \frac{1}{1152}$
		$< \tau_1 \tau_4 >_2 = \frac{1}{384}$	$< \tau_0 \tau_1 \tau_5 >_2 = \frac{1}{288}$
		$< \tau_2 \tau_3 >_2 = \frac{29}{5760}$	$< \tau_0 \tau_2 \tau_4 >_2 = \frac{11}{1440}$
			$< \tau_1^2 \tau_4 >_2 = \frac{1}{96}$
			$< \tau_0 \tau_3^2 >_2 = \frac{29}{2880}$
			$< \tau_1 \tau_2 \tau_3 >_2 = \frac{29}{1440}$
			$< \tau_2^3 >_2 = \frac{7}{240}$

Genus 3 :

$n =$	1	2	3
	$< \tau_7 >_3 = \frac{1}{82944}$	$< \tau_0 \tau_8 >_3 = \frac{1}{82944}$	$< \tau_0^2 \tau_9 >_3 = \frac{25889}{19155502080}$
		$< \tau_1 \tau_7 >_3 = \frac{5}{82944}$	$< \tau_0 \tau_1 \tau_8 >_3 = \frac{1597}{100818432}$
		$< \tau_2 \tau_6 >_3 = \frac{77}{414720}$	$< \tau_0 \tau_2 \tau_7 >_3 = \frac{12097}{148262400}$
		$< \tau_3 \tau_5 >_3 = \frac{503}{1451520}$	$< \tau_1^2 \tau_7 >_3 = \frac{721}{5930496}$
		$< \tau_4^2 >_3 = \frac{607}{1451520}$	$< \tau_0 \tau_3 \tau_6 >_3 = \frac{32269}{138378240}$
			$< \tau_1 \tau_2 \tau_6 >_3 = \frac{49}{99840}$
			$< \tau_0 \tau_4 \tau_5 >_3 = \frac{373}{887040}$
			$< \tau_1 \tau_3 \tau_5 >_3 = \frac{9059}{7983360}$
			$< \tau_2^2 \tau_5 >_3 = \frac{923}{570240}$
			$< \tau_1 \tau_4^2 >_3 = \frac{1201}{725760}$
			$< \tau_2 \tau_3 \tau_4 >_3 = \frac{443}{145152}$
			$< \tau_3^3 >_3 = \frac{317}{69120}$

6.4.9 *Computation of $F_g = \omega_{g,0}$*

The previous theorem computes $\omega_{g,n}$ with $n \geq 1$. It remains to compute $F_g = \omega_{g,0}$ given by

$$F_g = (2 - t_3)^{2-2g} \left\langle e^{\sum_{k>0} \hat{t}_k \kappa_k} \right\rangle_g .$$

Theorem 6.4.5 *F_g is determined by*

$$\frac{\partial F_g}{\partial \lambda_a} = -\omega_{g,1}(\lambda_a) + \delta_{g,0} \left(-2\lambda_a^2 + \frac{1}{N} \sum_b \frac{1}{\lambda_a + \lambda_b} \right).$$

and by the fact that it vanishes at large Λ.

More generally we have:

$$\frac{\partial\,\omega_{g,n}(z_1,\ldots,z_n)}{\partial\lambda_a} = -\omega_{g,n+1}(\lambda_a,z_1,\ldots,z_n) + \delta_{g,0}\delta_{n,0}\left(-2\lambda_a^2 + \frac{1}{N}\sum_b \frac{1}{\lambda_a+\lambda_b}\right).$$

Proof We have:

$$\omega_{g,n} = (-1)^n N^{-n} \sum_k \frac{1}{k!} \sum_{d_1+\cdots+d_{n+k}=d_{g,n+k}} 2^{\chi_{g,n+k}} \left\langle \prod_{i=1}^{n+k} \tau_{d_i} \right\rangle_g \prod_{i=n+1}^{n+k} (2d_i-1)!!t_{2d_i+1}$$
$$\prod_{i=1}^{n} \frac{(2d_i+1)!!}{z_i^{2d_i+2}}$$

Then notice that $t_d = \frac{1}{N}\sum_a \lambda_a^{-d}$ satisfies

$$N\,\frac{\partial t_d}{\lambda_a\,\partial\lambda_a} = \frac{-d}{\lambda_a^{d+2}}$$

that immediately gives

$$\frac{\partial\omega_{g,n}(z_1,\ldots,z_n)}{\partial\lambda_a} = -\Big(\omega_{g,n+1}(\lambda_a,z_1,\ldots,z_n)) - \delta_{g,0}\delta_{n,0}(-2\lambda_a^2 + \frac{1}{N}\sum_b \frac{1}{\lambda_a+\lambda_b})\Big)$$

□

Remark 6.4.5 (Heuristic Derivation from the Matrix Integral) Observe from Theorem 6.3.6, that Kontsevich's integral

$$Z_{\text{Kontsevich}}(\Lambda) = \prod_{i,j}\sqrt{\lambda_i+\lambda_j}(\pi/N)^{N^2/2} \int d\tilde{M}\ \mathrm{e}^{-N\,\mathrm{Tr}\,\Lambda^2\tilde{M}}\ \ \mathrm{e}^{N\,\mathrm{Tr}\,\frac{\tilde{M}^3}{3}}$$

is such that:

$$\frac{1}{N}\,\frac{d\,\ln Z_{\text{Kontsevich}}}{d\lambda_a} = -2\lambda_a < M_{a,a} > + \frac{1}{N}\sum_b \frac{1}{\lambda_a+\lambda_b}$$

and thus, using that

$$< M_{a,a} >= \sum_g N^{1-2g}\ \left(\Omega_{g,1}(\lambda_a) + \delta_{g,0}\lambda_a\right) = N\lambda_a + \sum_g N^{1-2g}\ \frac{1}{2\lambda_a}\,\omega_{g,1}(\lambda_a)$$

we get that

$$\frac{\partial F_g}{\partial \lambda_a} = -\omega_{g,1}(\lambda_a) + \delta_{g,0}\left(-2\lambda_a^2 + \frac{1}{N}\sum_b \frac{1}{\lambda_a+\lambda_b}\right).$$

Before going further, observe that most often, the solution of Tutte equations involve $\check{t}_d$ instead of t_d, and they reduce to t_d only when $t_1 = 0$, thus most often we like to work with the constraint $t_1 = 0 = \sum_a 1/\lambda_a$, but then the λ_a's are not independent, so it is better to work with $t_1 \neq 0$, and come back to $\check{\lambda}_a$'s. So we shall first need the lemma:

Lemma 6.4.3 *for* $k \neq 0$

$$\frac{N}{k}\,\frac{\partial \check{t}_k}{\lambda_a\,\partial\lambda_a} = \frac{\check{t}_{k+2}}{(\check{t}_3-2)\,\check{\lambda}_a^3} - \frac{1}{\check{\lambda}_a^{k+2}}$$

$$\check{\lambda}_b\,\frac{\partial\check{\lambda}_b}{\partial\lambda_a} = \lambda_a\left(\delta_{a,b} - \frac{1}{N\,(\check{t}_3-2)\,\check{\lambda}_a^3}\right).$$

In particular when we take $t_1 = 0$ *after evaluating the derivative:*

$$\frac{N}{k}\,\frac{\partial \check{t}_k}{\lambda_a\,\partial\lambda_a}\bigg|_{t_1=0} = \frac{t_{k+2}}{(t_3-2)\,\lambda_a^3} - \frac{1}{\lambda_a^{k+2}}.$$

Proof From $\check{\lambda}_b^2 = \lambda_b^2 - \check{t}_1$ we have

$$2\check{\lambda}_b\,\frac{\partial\check{\lambda}_b}{\partial\lambda_a} + \frac{\partial\check{t}_1}{\partial\lambda_a} = 2\delta_{a,b}\lambda_a,$$

then multiplying by $\check{\lambda}_b^{-k}$ and summing over b gives

$$\frac{2}{2-k}\,\frac{\partial\check{t}_{k-2}}{\partial\lambda_a} + \check{t}_k\,\frac{\partial\check{t}_1}{\partial\lambda_a} = \frac{2\,\lambda_a}{N\,\check{\lambda}_a^k}.$$

In particular with $k = 3$ we get:

$$\frac{\partial\check{t}_1}{\partial\lambda_a} = \frac{2\,\lambda_a}{N\,(\check{t}_3-2)\,\check{\lambda}_a^3}$$

and thus

$$\frac{\partial\check{t}_{k-2}}{\partial\lambda_a} = (k-2)\left(\check{t}_k\,\frac{\lambda_a}{N\,(\check{t}_3-2)\,\check{\lambda}_a^3} - \frac{\lambda_a}{N\,\check{\lambda}_a^k}\right)$$

□

Theorem 6.4.6 (Genus 1)

$$F_1 = \frac{1}{24}\ \ln(1 - t_3/2)$$

Proof Recall that

$$\omega_{1,1}(z) = -\frac{1}{8(2-t_3)}\left(\frac{1}{z^4} + \frac{t_5}{(2-t_3)\,z^2}\right).$$

Using Lemma 6.4.3 we have

$$\frac{\partial t_3}{\partial \lambda_a} = \frac{3}{N}\left(\frac{t_5}{(t_3-2)\lambda_a^2} - \frac{1}{\lambda_a^4}\right) = 24\,(2-t_3)\,\omega_{1,1}(\lambda_a).$$

This shows that $F_1 - \frac{1}{24}\ln(1 - t_3/2)$ is a constant independent of Λ, which is 0 because F_1 is a sum of weighted graphs whose weight goes to zero as $\Lambda \to \infty$, i.e. as $t_3 \to 0$. □

Theorem 6.4.7 (Genus 0) *When t_1 is arbitrary*

$$\frac{1}{N}\ F_0 = -\frac{1}{2N^2}\sum_{i,j}\ln\left(\frac{\check{\lambda}_i + \check{\lambda}_j}{\lambda_i + \lambda_j}\right) + \frac{2}{3}(\check{t}_{-3} - t_{-3}) + \check{t}_1\check{t}_{-1}$$

And in particular, if $t_1 = 0$ we find

$$F_0 = 0.$$

Proof Recall that

$$\omega_{0,1}(z) = 2z\Omega_{0,1}(z) = -2z\sqrt{z^2 - \check{t}_1} - \frac{1}{N}\sum_j \frac{z}{\check{\lambda}_j(\sqrt{z^2 - \check{t}_1} + \check{\lambda}_j)}.$$

i.e. we want to solve

$$\frac{\partial F_0}{\lambda_a \partial \lambda_a} = 2(\check{\lambda}_a - \lambda_a) + \frac{1}{N}\sum_j \frac{1}{\check{\lambda}_j(\check{\lambda}_a + \check{\lambda}_j)} + \frac{1}{\lambda_a(\lambda_a + \lambda_j)}$$

Recall from Lemma 6.4.3:

$$\check{\lambda}_b\,\frac{\partial \check{\lambda}_b}{\partial \lambda_a} = \lambda_a\left(\delta_{a,b} - \frac{1}{N\,(\check{t}_3 - 2)\,\check{\lambda}_a^3}\right).$$

This gives

$$\begin{aligned}
\frac{\partial}{2\lambda_a\,\partial\lambda_a}\sum_{i,j}\ln(\check\lambda_i+\check\lambda_j) &= 2\sum_{i,j}\frac{1}{\check\lambda_i+\check\lambda_j}\,\frac{\partial\check\lambda_i}{2\lambda_a\,\partial\lambda_a}\\
&= 2\sum_{i,j}\frac{1}{\check\lambda_i+\check\lambda_j}\,\frac{1}{2\check\lambda_i}\left(\delta_{a,i}+\frac{1}{N(2-\check t_3)\check\lambda_a^3}\right)\\
&= \sum_j\frac{1}{\check\lambda_a+\check\lambda_j}\,\frac{1}{\check\lambda_a}\\
&\quad+\sum_{i,j}\frac{1}{\check\lambda_i+\check\lambda_j}\left(\frac{1}{2\check\lambda_i}+\frac{1}{2\check\lambda_j}\right)\frac{1}{N(2-\check t_3)\check\lambda_a^3}\\
&= \sum_j\frac{1}{\check\lambda_a+\check\lambda_j}\,\frac{1}{\check\lambda_a}+\frac{N^2\check t_1^2}{2}\,\frac{1}{N(2-\check t_3)\check\lambda_a^3}\\
&= \frac{N\check t_1}{\check\lambda_a}-\sum_j\frac{1}{\check\lambda_a+\check\lambda_j}\,\frac{1}{\check\lambda_j}+\frac{N^2\check t_1^2}{2}\,\frac{1}{N(2-\check t_3)\check\lambda_a^3}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{\partial}{\lambda_a\,\partial\lambda_a}\left(F_0+\frac{1}{2N}\sum_{i,j}\ln\left(\frac{\check\lambda_i+\check\lambda_j}{\lambda_i+\lambda_j}\right)\right)\\
&= 2(\check\lambda_a-\lambda_a)+\frac{\check t_1}{\check\lambda_a}+\frac{\check t_1^2}{2}\,\frac{1}{(2-\check t_3)\check\lambda_a^3}\\
&= \frac{2N}{3}\frac{\partial(\check t_{-3}-t_{-3})}{\lambda_a\partial\lambda_a}+\frac{2\check t_{-1}}{(\check t_3-2)\,\check\lambda_a^3}+\frac{\check t_1}{\check\lambda_a}+\frac{\check t_1^2}{2}\,\frac{1}{(2-\check t_3)\check\lambda_a^3}.
\end{aligned}$$

Notice that

$$\frac{\check t_1}{\check\lambda_a}+\frac{2\check t_{-1}}{(\check t_3-2)\check\lambda_a^3}=\frac{\check t_1^2}{(\check t_3-2)\check\lambda_a^3}+N\frac{\partial\,\check t_1\check t_{-1}}{\lambda_a\partial\lambda_a}$$

i.e.

$$\begin{aligned}
&\frac{\partial}{\lambda_a\,\partial\lambda_a}\left(F_0+\frac{1}{2N}\sum_{i,j}\ln\left(\frac{\check\lambda_i+\check\lambda_j}{\lambda_i+\lambda_j}\right)\right)\\
&= \frac{2N}{3}\frac{\partial(\check t_{-3}-t_{-3})}{\lambda_a\partial\lambda_a}+N\frac{\partial\,\check t_1\check t_{-1}}{\lambda_a\partial\lambda_a}
\end{aligned}$$

and thus

$$\frac{1}{N}\, F_0 = -\frac{1}{2N^2}\sum_{i,j}\ln\left(\frac{\check\lambda_i+\check\lambda_j}{\lambda_i+\lambda_j}\right) + \frac{2}{3}(\check t_{-3} - t_{-3}) + \check t_1 \check t_{-1}.$$

□

Theorem 6.4.8 (Higher Genus) *For $g \geq 2$ we have*

$$F_g = \frac{1}{2-2g}\ \operatorname*{Res}_{z\to 0}\ \omega_{g,1}(z)\,\Phi(z)\,dz$$

where $d\Phi = \omega_{0,1}$, i.e. when $t_1 = 0$:

$$\Phi(z) = -z^2 + \frac{1}{N}\sum_j \ln(z+\lambda_j) = -z^2 + \frac{1}{N}\ln\det(z+\Lambda)$$

and more generally if $2g-2+n > 0$

$$(2g-2+n)\,\omega_{g,n}(z_1,\dots,z_n) = \operatorname*{Res}_{z\to 0}\ \omega_{g,n+1}(z_1,\dots,z_n,z)\,\Phi(z)\,dz. \tag{6.4.7}$$

Proof We shall first prove Eq. (6.4.7) for $n > 0$ by recursion on $k = -\chi_{g,n} = 2g-2+n$.

For $k = 0$ the only case is $(g,n) = (0,2)$, and we have:

$$\omega_{0,3}(z_1,z_2,z_3) = \frac{1}{2-t_3}\,\frac{1}{z_1^2\,z_2^2\,z_3^2}$$

which means that

$$\begin{aligned}
\operatorname*{Res}_{z\to 0}\ \omega_{0,3}(z_1,z_2,z)\,\Phi(z)\,dz &= \frac{1}{2-t_3}\ \operatorname*{Res}_{z\to 0}\ \frac{1}{z_1^2\,z_2^2}\,\frac{\Phi(z)}{z^2}\,dz \\
&= \frac{1}{2-t_3}\,\frac{1}{z_1^2\,z_2^2}\,\Phi'(0) \\
&= \frac{1}{2-t_3}\,\frac{1}{z_1^2\,z_2^2}\,y(0)\,x'(0) \\
&= 0.
\end{aligned}$$

For $k > 0$, and $n \geq 1$, we have from Theorem 6.4.4

$$\begin{aligned}
&\omega_{g,n+1}(z_0,\overbrace{z_1,\dots,z_n}^{J}) \\
&= \operatorname*{Res}_{z\to 0}\ K(z_0,z)\Big(\omega_{g-1,n+2}(z,z,J) + \sum_{h+h'=g,\, I\uplus I'=J}^{\prime} \omega_{h,1+\#I}(z,I)\,\omega_{h',1+\#I'}(z,I')\Big)
\end{aligned}$$

and thus

$$\begin{aligned}
&\operatorname*{Res}_{z_1\to 0} \Phi(z_1)\,\omega_{g,n+1}(z_0,z_1,\overbrace{z_2,\dots,z_n}^{J}) \\
&= \operatorname*{Res}_{z_1\to 0}\operatorname*{Res}_{z\to 0} K(z_0,z)\Big(\omega_{g-1,n+2}(z,z,z_1,J) \\
&\quad + \sum_{h+h'=g,\, I\uplus I'=J}{}' \omega_{h,2+\#I}(z,z_1,I)\omega_{h',1+\#I'}(z,I') \\
&\quad + \sum_{h+h'=g,\, I\uplus I'=J}{}' \omega_{h,1+\#I}(z,I)\omega_{h',2+\#I'}(z,z_1,I')\Big)\,\Phi(z_1).
\end{aligned}$$

Except for the term with a $\omega_{0,2}(z,z_1)$ factor, there is no pole at $z = z_1$ and we can exchange the order of residues. Then by the recursion hypothesis we have:

$$\begin{aligned}
&\operatorname*{Res}_{z_1\to 0} \Phi(z_1)\,\omega_{g,n+1}(z_0,z_1,\overbrace{z_2,\dots,z_n}^{J}) \\
&= \operatorname*{Res}_{z\to 0} K(z_0,z)\Big(\big(2(g-1)-2+(n+1)\big)\,\omega_{g-1,n+1}(z,z,J) \\
&\quad + \sum_{h+h'=g,\, I\uplus I'=J}{}' (2h-2+1+\#I)\omega_{h,1+\#I}(z,I)\omega_{h',1+\#I'}(z,I') \\
&\quad + \sum_{h+h'=g,\, I\uplus I'=J}{}' (2h'-2+1+\#I')\omega_{h,1+\#I}(z,I)\omega_{h',1+\#I'}(z,I')\Big) \\
&\quad +2\operatorname*{Res}_{z_1\to 0}\operatorname*{Res}_{z\to 0} K(z_0,z)\Big(\omega_{0,2}(z,z_1)\omega_{g,n+1}(z,J)\Big)\,\Phi(z_1).
\end{aligned}$$

The first three lines add up to make (using Theorem 6.4.4)

$$(2g-3+n)\omega_{g,n}(z_0,J).$$

In the last line, we move the integration contour of z_1, i.e. a small circle around 0, through that of z, and thus we pick a residue at $z = z_1$:

$$\operatorname*{Res}_{z_1\to 0}\operatorname*{Res}_{z\to 0} = \operatorname*{Res}_{z\to 0}\operatorname*{Res}_{z_1\to 0} + \operatorname*{Res}_{z\to 0}\operatorname*{Res}_{z_1\to z}.$$

Notice that $\omega_{0,2}(z,z_1)\Phi(z_1)$ has no pole at $z_1 = 0$, and we have:

$$\operatorname*{Res}_{z_1\to z} \omega_{0,2}(z,z_1)\Phi(z_1) = d\Phi(z)/dz = y(z)x'(z)$$

so it remains

$$\operatorname*{Res}_{z_1\to 0}\ \operatorname*{Res}_{z\to 0} K(z_0,z)\left(\omega_{0,2}(z,z_1)\omega_{g,n+1}(z,J)\right)\Phi(z_1) = \operatorname*{Res}_{z\to 0}\ K(z_0,z)\,\omega_{g,n+1}(z,J)\,y(z)x'(z).$$

We have that $K(z_0,z) = \frac{1}{z_0^2-z^2}\frac{1}{2(y(z)-y(-z))}$ and, using the parity of $\omega_{g,n}(-z,J) = \omega_{g,n}(z,J)$, we have

$$\begin{aligned}
&\operatorname*{Res}_{z_1\to 0}\ \operatorname*{Res}_{z\to 0} K(z_0,z)\left(\omega_{0,2}(z,z_1)\omega_{g,n+1}(z,J)\right)\Phi(z_1)\\
&= \frac{1}{2}\operatorname*{Res}_{z\to 0}\ K(z_0,z)\,\omega_{g,n+1}(z,J)\,(y(z)-y(-z))\,x'(z)\\
&= \frac{1}{2}\operatorname*{Res}_{z\to 0}\ \frac{z}{z_0^2-z^2}\,\omega_{g,n+1}(z,J)
\end{aligned}$$

the integrand's only poles are $z=0$ and $z=\pm z_0$, so we can move the integration contour:

$$\begin{aligned}
&\operatorname*{Res}_{z_1\to 0}\ \operatorname*{Res}_{z\to 0} K(z_0,z)\left(\omega_{0,2}(z,z_1)\omega_{g,n+1}(z,J)\right)\Phi(z_1)\\
&= \frac{-1}{2}\operatorname*{Res}_{z\to z_0,-z_0}\ \frac{z}{z_0^2-z^2}\,\omega_{g,n+1}(z,J)\\
&= \frac{1}{4}\omega_{g,n+1}(z_0,J) + \frac{1}{4}\,\omega_{g,n+1}(-z_0,J)\\
&= \frac{1}{2}\omega_{g,n+1}(z_0,J)
\end{aligned}$$

i.e. it remains, as announced for every $n \geq 1$ and $2g-2+n > 0$:

$$\begin{aligned}
&\operatorname*{Res}_{z_1\to 0}\ \Phi(z_1)\,\omega_{g,n+1}(z_0,z_1,z_2,\dots,z_n)\\
&= (2g-3+n)\omega_{g,n}(z_0,z_2,\dots,z_n) + 2\,\frac{\omega_{g,n+1}(z_0,J)}{2}\\
&= (2g-2+n)\omega_{g,n}(z_0,z_2,\dots,z_n).
\end{aligned}$$

We have thus proved Eq. (6.4.7) for $n > 0$.

Then it remains to prove the case $n = 0$. Let us define for $g \geq 2$:

$$\tilde{F}_g = \frac{1}{2-2g}\ \operatorname*{Res}_{z\to 0}\ \Phi(z)\,\omega_{g,1}(z)$$

and let us compute $\partial\tilde{F}_g/\partial\lambda_a$, we have:

$$(2-2g)\,\frac{\partial\tilde{F}_g}{\partial\lambda_a} = \operatorname*{Res}_{z\to 0}\,\frac{\partial\Phi(z)}{\partial\lambda_a}\,\omega_{g,1}(z) + \operatorname*{Res}_{z\to 0}\,\frac{\partial\omega_{g,1}(z)}{\partial\lambda_a}\,\Phi(z).$$

From Theorem 6.4.5 we have

$$\frac{\partial\omega_{g,1}(z)}{\partial\lambda_a} = -\omega_{g,2}(z,\lambda_a)$$

and

$$\frac{\partial\Phi(z)}{\partial\lambda_a} = -\int^z \omega_{0,2}(z',\lambda_a)\,dz' = \frac{1}{z-\lambda_a}.$$

Therefore

$$(2-2g)\,\frac{\partial\tilde{F}_g}{\partial\lambda_a} = \operatorname*{Res}_{z\to 0}\,\frac{1}{z-\lambda_a}\,\omega_{g,1}(z) - \operatorname*{Res}_{z\to 0}\,\omega_{g,2}(z,\lambda_a)\,\Phi(z).$$

In the first term, the poles of the integrand are at $z=0$ or at $z=\lambda_a$, thus moving the integration contour we trade the residue at $z=0$ to a residue at $z=\lambda_a$

$$\operatorname*{Res}_{z\to 0}\,\frac{1}{z-\lambda_a}\,\omega_{g,1}(z) = -\operatorname*{Res}_{z\to\lambda_a}\,\frac{1}{z-\lambda_a}\,\omega_{g,1}(z) = -\omega_{g,1}(\lambda_a),$$

and meanwhile for the second term, we use Eq. (6.4.7) for $\omega_{g,2}$

$$\operatorname*{Res}_{z\to 0}\,\omega_{g,2}(z,\lambda_a)\,\Phi(z) = (2-2g-1)\omega_{g,1}(\lambda_a).$$

This gives

$$(2-2g)\,\frac{\partial\tilde{F}_g}{\partial\lambda_a} = -\omega_{g,1}(\lambda_a)-(1-2g)\omega_{g,1}(\lambda_a) = -(2-2g)\omega_{g,1}(\lambda_a) = (2-2g)\,\frac{\partial F_g}{\partial\lambda_a}$$

in other words it implies that $F_g-\tilde{F}_g$ is independent of Λ, and since it should vanish at large Λ, it must be identically zero:

$$F_g = \tilde{F}_g = \frac{1}{2-2g}\,\operatorname*{Res}_{z\to 0}\,\Phi(z)\,\omega_{g,1}(z).$$

□

As an example of application, let us compute some intersection numbers. The topological recursion computation of F_2 gives

$$F_2 = \frac{21\, t_5^3}{160\,(2-t_3)^{\,5}} + \frac{29\, t_7 t_5}{128\,(2-t_3)^{\,4}} + \frac{35\, t_9}{384\,(2-t_3)^{\,3}}.$$

We write it as

$$F_2 = (2-t_3)^{-2} \;\; < \hat{t}_3\kappa_3 + \hat{t}_1\hat{t}_2\kappa_1\kappa_2 + \frac{\hat{t}_1^3}{6}\kappa_1^3 >_{\mathcal{M}_{2,0}}$$

with

$$3\,t_5 = (2-t_3)\,\hat{t}_1 \quad , \quad 15\,t_7 = (2-t_3)\,(\hat{t}_2 - \frac{\hat{t}_1^2}{2}) \quad , \quad 105\,t_9 = (2-t_3)\,(\hat{t}_3 - \hat{t}_1\hat{t}_2 + \frac{\hat{t}_1^3}{6})$$

this gives that

$$< \kappa_3 >_{\mathcal{M}_{2,0}} = \frac{1}{2^7\;3^2} \quad , \quad < \kappa_1\kappa_2 >_{\mathcal{M}_{2,0}} = \frac{1}{240} \quad , \quad < \kappa_1^3 >_{\mathcal{M}_{2,0}} = \frac{43}{2880}.$$

Proposition 6.4.5 (Link with Symplectic Invariants) *Theorems 6.4.4 and 6.4.8 mean that F_g and the $\omega_{g,n}(z_1,\dots,z_n)dz_1\dots dz_n$ are the symplectic invariants (in the sense of Chap. 7) of the spectral curve:*

$$\begin{cases} x(z) = z^2 + \check{t}_1 \\ y(z) = z + \frac{1}{N}\sum_a \frac{1}{2\check{\lambda}_a\,(z-\check{\lambda}_a)} = z - \frac{1}{2}\sum_{k=0}^{\infty} \check{t}_{k+2}\, z^k \end{cases}$$

i.e. when $t_1 = 0$:

$$\begin{cases} x(z) = z^2 \\ y(z) = z + \frac{1}{N}\sum_a \frac{1}{2\lambda_a\,(z-\check{\lambda}_a)} = z - \frac{1}{2}\sum_{k=0}^{\infty} t_{k+2}\, z^k \end{cases} .$$

An important remark is the following: the topological recursion Theorem 6.4.4, shows that only $y(z) - y(-z)$ appears in the computations, this is a special case of the general symplectic invariance of Chap. 7,and thus:

Corollary 6.4.1 *F_g and the $\omega_{g,n}(z_1,\dots,z_n)dz_1\dots dz_n$ are the symplectic invariants (in the sense of Chap. 7) of the spectral curve (we assume $t_1 = 0$):*

$$\begin{cases} x(z) = z^2 \\ y(z) = z - \frac{1}{2}\sum_{k=0}^{\infty} t_{2k+3}\, z^{2k+1} \end{cases}$$

6.5 Large Maps, Liouville Gravity and Topological Gravity

We assume $t_1 = 0$. From their recursive definition as residues, the symplectic invariants $\mathcal{F}_g$ depend only on a finite number of terms of the Taylor expansion of $y(z)$ near the branchpoint $z = 0$, namely, $\mathcal{F}_g$ depends only on:

$$\mathcal{F}_g = \mathcal{F}_g(t_3, \dots, t_{6g-3}).$$

This can also be seen from the definition of F_g in terms of intersection numbers $F_g = (2 - t_3)^{2-2g} \left\langle e^{\sum_{k>0} \hat{t}_k \kappa_k} \right\rangle$.

Therefore, for each g, one can compute $\mathcal{F}_g$ with only a finite number of t_i's non-vanishing. Choose

$$t_k = 0 \quad \text{when} \;\; k > 2m + 3.$$

The spectral curve is then (symplectic invariance allows to add an arbitrary constant to x without changing the $\mathcal{F}_g$'s, and we may ignore the even powers of z for $y(z)$):

$$\hat{\mathcal{E}}_K = \begin{cases} x(z) = z^2 - 2u_0 \\ y(z) = z - \frac{1}{2} \sum_{k=0}^{m} t_{2k+3}\, z^{2k+1}. \end{cases}$$

We can identify it with the spectral curve of Sect. 5.4, upon the identification

$$y(z) = z - \frac{1}{2} \sum_{k=0}^{m-1} t_{2k+3}\, z^{2k+1} = \sum_{j=0}^{m} \tilde{t}_j \, Q_j(z).$$

Using the expression of the polynomial $Q_j(z)$ in Eq. (5.2.3) this gives

$$t_{2k+3} = 2\delta_{k,0} - 2 \sum_{j=k}^{m} \tilde{t}_j \, \frac{(-u_0)^{j-k}}{(j-k)!} \, \frac{(2j+1)!!}{(2k+1)!!}.$$

For example

$$t_{2m+3} = -2\,\tilde{t}_m \qquad , \qquad t_{2m+1} = 2\,(2m+1)\,u_0\,\tilde{t}_m - 2\,\tilde{t}_{m-1} \qquad , \qquad \dots$$

The generating function of the times t_k's is:

$$e^{\sum_k \hat{t}_k u^{-k}} = 1 - f(1/u) = 1 - \frac{1}{2} \sum_k \frac{(2k+1)!!\, t_{2k+3}}{u^k} = \sum_j \frac{(2j+1)!!\, \tilde{t}_j}{u^j} \sum_{k=0}^{j} \frac{(-u_0\, u)^k}{k!}$$

thus

$$1 - f(1/u) = \left(\tilde{f}(u)\ \mathrm{e}^{-u_0\, u}\right)_{-}$$

where the subscript $()_{-}$ means that we keep only negative powers of u in the Laurent series expansion at $u \to 0$, and where

$$\tilde{f}(u) = \sum_j \tilde{t}_j\ \frac{(2j+1)!!}{u^j}.$$

Finally, we see that the spectral curve $\hat{\mathcal{E}}_K$ is identical to the spectral curve $\mathcal{E}_{(2m+1,2)}$ (see Eq. (5.4.30) in Sect. 5.4) of the minimal model $(2m+1, 2)$ encountered in the asymptotics of large maps in Sect. 5.4.

Theorem 6.5.1 *The asymptotic generating function of large maps $\tilde{F}_g$ near an mth order critical point, coincides with the topological expansion of the Tau-function of the minimal model $(2m+1, 2)$, and with the generating function of intersection numbers:*

$$\tilde{F}_g(\tilde{t}_i) = \mathcal{F}_g(\mathcal{E}_{(2m+1,2)}) = \mathcal{F}_g(\hat{\mathcal{E}}_K) = \left\langle \mathrm{e}^{\sum_k \hat{t}_k \kappa_k} \right\rangle_{\mathcal{M}_{g,0}}$$

provided that we identify the Kontsevich times t_k's and $(2m+1, 2)$-model times $\tilde{t}_j$ as:

$$t_{2k+3} = 2\delta_{k,0} - 2\sum_{j=k}^{m} \tilde{t}_j\ \frac{(-u_0)^{j-k}}{(j-k)!}\ \frac{(2j+1)!!}{(2k+1)!!}.$$

In other words "the limit partition function of large maps, agrees with Liouville conformal field theory coupled to the minimal model $(2m+1, 2)$, and agrees with topological gravity".

6.6 Weil-Petersson Volumes

Here, we come back to the description of moduli spaces in terms of hyperbolic geometry, which we evoked in Sect. 6.3.1.

It turns out, that Mumford's class κ_1, is closely related to the Weil-Petersson 2-form on $\mathcal{M}_{g,n}$, by Wolpert's relation [88]:

$$\sum_i dl_i \wedge d\theta_i = 2\pi^2 \kappa_1.$$

The Weil-Petersson volume form is

$$\prod_i dl_i \wedge d\theta_i = \frac{1}{d_{g,n}!}\,(\sum_i dl_i \wedge d\theta_i)^{d_{g,n}} = \frac{1}{d_{g,n}!}\,(2\pi^2\kappa_1)^{d_{g,n}}.$$

Thus, the idea is to choose a spectral curve which will lead to intersection numbers of κ_1 only.

Let us chose $t_1 = 0$ and:

$$t_{2k+3} = 2\delta_{k,0} - 2\,\frac{(-1)^k\,(2\pi)^{2k}}{(2k+1)!},$$

It induces:

$$\begin{aligned} y(z) &= z - \frac{1}{2}\sum_{k=0}^{\infty} t_{2k+3} z^{2k+1} \\ &= z - z + \sum_{k=0}^{\infty}(-1)^k\,\frac{(2\pi)^k\,z^{2k+1}}{(2k+1)!} \\ &= \frac{1}{2\pi}\,\sin 2\pi z \end{aligned}$$

or in other words the spectral curve $\hat{\mathcal{E}}_K$ is (we denote it $\mathcal{E}_{WP}$):

$$\mathcal{E}_{WP} = \begin{cases} x(z) = z^2 \\ y(z) = \frac{1}{2\pi}\,\sin(2\pi\,z). \end{cases}$$

The Schur transformed times $\hat{t}_k$ are obtained from Theorem 6.3.8 (or also Theorem 6.3.9), and are such that:

$$f(1/u) = \frac{1}{2}\sum_k (2k+1)!!\,t_{2k+3}\,u^{-k} = 1 - \sum_k \frac{(-1)^k\,(2\pi)^{2k}}{2^k\,k!}\,u^{-k} = 1 - \mathrm{e}^{-2\pi^2\,u^{-1}}$$

and

$$\hat{f}(1/u) = \sum_k \hat{t}_k u^{-k} = -\ln\left(1 - f(1/u)\right) = 2\pi^2\,u^{-1}.$$

This implies

$$\hat{t}_k = 2\pi^2\;\delta_{k,1}$$

and therefore

$$\mathrm{e}^{\sum_k \hat{t}_k \kappa_k} = \mathrm{e}^{2\pi^2 \kappa_1} = \sum_{d_0=0}^{\infty} \frac{(2\pi^2)^{d_0}}{d_0!} \kappa_1^{d_0}.$$

We thus have:

$$\omega_n^{(g)}(\mathcal{E}_{WP}; z_1, \ldots, z_n) = (-2)^{\chi_{g,n}} \sum_{d_0, d_1, \ldots, d_n} \left\langle \kappa_1^{d_0} \prod_{i=1}^{n} \psi_i^{d_i} \right\rangle_{\mathcal{M}_{g,n}} \frac{(2\pi^2)^{d_0}}{d_0!} \prod_{i=1}^{n} \frac{(2d_i+1)!!}{z_i^{2d_i+2}}$$

Notice that the intersection number is non-vanishing only if $d_0 + d_1 + \cdots + d_n = d_{g,n}$.

Then, observe that

$$\int_0^{\infty} L\, dL\, L^{2d}\, \mathrm{e}^{-zL} = \frac{(2d+1)!}{z^{2d+2}}.$$

This allows to rewrite:

$$\begin{aligned}
&\omega_n^{(g)}(\mathcal{E}_{WP}; z_1, \ldots, z_n) \\
&= (-2)^{\chi_{g,n}} \int_0^{\infty} L_1 dL_1 \mathrm{e}^{-z_1 L_1} \ldots \int_0^{\infty} L_n dL_n \mathrm{e}^{-z_n L_n} \\
&\qquad \sum_{d_0+d_1+\cdots+d_n=d_{g,n}} \left\langle \kappa_1^{d_0} \prod_{i=1}^{n} \psi_i^{d_i} \right\rangle_{\mathcal{M}_{g,n}} \frac{(2\pi^2)^{d_0}}{d_0!} \prod_{i=1}^{n} \frac{L_i^{2d_i}}{2^{d_i}\, d_i!} \\
&= \frac{(-2)^{\chi_{g,n}}}{d_{g,n}!} \int_0^{\infty} L_1 dL_1 \mathrm{e}^{-z_1 L_1} \ldots \int_0^{\infty} L_n dL_n \mathrm{e}^{-z_n L_n} \\
&\qquad \sum_{d_0+d_1+\cdots+d_n=d_{g,n}} \frac{d_{g,n}!}{d_0! d_1! \ldots d_n!} \left\langle (2\pi^2 \kappa_1)^{d_0} \prod_{i=1}^{n} (\frac{L_i^2}{2}\, \psi_i)^{d_i} \right\rangle_{\mathcal{M}_{g,n}} \\
&= \frac{(-2)^{\chi_{g,n}}}{d_{g,n}!} \int_0^{\infty} L_1 dL_1 \mathrm{e}^{-z_1 L_1} \ldots \int_0^{\infty} L_n dL_n \mathrm{e}^{-z_n L_n} \\
&\qquad \left\langle (2\pi^2 \kappa_1 + \frac{1}{2} \sum_{i=1}^{n} L_i^2 \psi_i)^{d_{g,n}} \right\rangle_{\mathcal{M}_{g,n}}.
\end{aligned}$$

The right hand side

$$\mathrm{Vol}(\mathcal{M}_{g,n}(L_1, \ldots, L_n)) = \frac{1}{d_{g,n}!} \left\langle (2\pi^2 \kappa_1 + \frac{1}{2} \sum_{i=1}^{n} L_i^2 \psi_i)^{d_{g,n}} \right\rangle_{\mathcal{M}_{g,n}}$$

is the Weil-Petersson volume of $\mathcal{M}_{g,n}(L_1, \ldots, L_n)$, see Sect. 6.3.1 for more details.

We thus get the theorem:

Theorem 6.6.1 *The symplectic invariant correlators of the spectral curve*

$$\mathcal{E}_{WP} = \begin{cases} x(z) = z^2 \\ y(z) = \frac{1}{2\pi}\,\sin(2\pi\, z) \end{cases}$$

are the Laplace transforms of Weil-Petersson volumes:

$$\frac{\omega_n^{(g)}(\mathcal{E}_{WP};\Lambda_1,\ldots,\Lambda_n)}{(-2)^{\chi_{g,n}}\prod_i d\Lambda_i} = \int_0^\infty L_1 dL_1 \mathrm{e}^{-\Lambda_1 L_1}\ldots\int_0^\infty L_n dL_n \mathrm{e}^{-\Lambda_n L_n}\,\mathrm{Vol}(\mathcal{M}_{g,n}(L_1,\ldots,L_n)).$$

It is an immediate corollary, by performing the Laplace transform of the topological recursion, that:

Corollary 6.6.1 *The Weil-Petersson volumes satisfy Mirzakhani's topological recursion:*

$$\begin{aligned} &2LV_{g,n+1}(L,L_K)\\ &= \int_0^L dt\int_0^\infty xdx\int_0^\infty ydyK_M(x+y,t)\Big[V_{g-1,n+2}(x,y,L_K)\\ &\qquad + \textstyle\sum_{h=0}^g\sum_{J\in K}V_{h,1+|J|}(x,L_J)V_{g-h,n+1-|J|}(y,L_{K/J})\Big]\\ &+\sum_{m=1}^n\int_0^L dt\int_0^\infty xdx(K_M(x,t+L_m)+K_M(x,t-L_m))V_{g,n-1}(x,L_K\setminus\{L_m\}) \end{aligned}$$

with Mirzakhani's recursion kernel given by:

$$K_M(x,t) = \frac{1}{1+\mathrm{e}^{\left(\frac{x+t}{2}\right)}} + \frac{1}{1+\mathrm{e}^{\left(\frac{x-t}{2}\right)}}.$$

M. Mirzakhani, fields medalist 2014, first discovered that recursion relation in 2004 [65] from the Mac-Shane identity on geodesic lengths in hyperbolic geometry. Here, we rederived the same recursion from the combinatorics of intersection numbers, or more precisely, from the fact that Kontsevich's matrix integral can be written in terms of symplectic invariants.

6.7 Summary: Riemann Surfaces and Topological Gravity

We have seen that

- The space (moduli space) of genus g Riemann surfaces with n labeled marked points $\mathcal{M}_{g,n}$, is of dimension $2(3g-3+n) = 2d_{g,n}$. $\mathcal{M}_{g,n}$ is not a manifold, it is an orbifold, with singular points quotiented by a group of automorphisms.

$\mathcal{M}_{g,n}$ is not compact, and can be compactified by adding nodal surfaces. $\bar{\mathcal{M}}_{g,n}$ is the Deligne–Mumford compactification of $\mathcal{M}_{g,n}$. $\bar{\mathcal{M}}_{g,n}$ is not a manifold neither an orbifold, it is a stack, it contains pieces of different dimensions, and singular points quotiented by some group.

- $\mathcal{M}_{g,n}$ is stable iff $\chi_{g,n} = 2-2g-n < 0$ and unstable otherwise. The only unstable cases are $(g, n) = (0, 0), (0, 1), (0, 2), (1, 0)$.
- Using Strebel's theorem, we have a bijection (an orbifold homeomorphism) between $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$ with a combinatorial set of metric ribbon graphs:

$$\mathcal{M}_{g,n} \times \mathbb{R}_+^n \quad \sim \quad \bigcup_{G \in \mathcal{G}_{g,n}} \mathbb{R}_+^{\#\text{edges of } G} \;/\; \text{Aut}\, G.$$

The lengths l_e of the $6g - 6 + 3n$ edges of G provide a set of coordinates on $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$. This is an orbifold bijection, it respects the symmetries.

- The Chern class $c_1(\tilde{\mathcal{L}}_i)$ of the cotangent line bundle $\tilde{\mathcal{L}}_i \to \mathcal{M}_{g,n} \times \mathbb{R}_+^n$ whose fibre is the cotangent plane at the ith marked point, can be written explicitly in the edge lengths coordinates:

$$\psi_i = c_1(\mathcal{L}_i) = \sum_{e' < e \text{ along face } i} d\left(\frac{l_e}{L_i}\right) \wedge d\left(\frac{l_{e'}}{L_i}\right) \qquad \text{where } L_i = \sum_{e \mapsto i} l_e.$$

Intersection numbers are integrals of product of Chern classes:

$$< \tau_{d_1} \dots \tau_{d_n} >_g \stackrel{\text{def}}{=} \begin{cases} \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^n \psi_i^{d_i} & \text{if } \sum_i d_i = d_{g,n} = 3g - 3 + n \\ 0 & \text{otherwise.} \end{cases}$$

- The Mumford's kappa classes κ_d are push-forwards of Chern classes ψ^{d+1} under the forgetful projection $\mathcal{M}_{g,n+m} \to \mathcal{M}_{g,n}$, forgetting m marked points.

  An easy way to relate intersection numbers of kappa classes to those of ψ classes, is by writing generating functions:

$$\left\langle e^{\sum_k \hat{t}_k \kappa_k} \prod_{i=1}^n \tau_i^{d_i} \right\rangle_{g,n} = \left\langle e^{\frac{1}{2} \sum_k (2k+1)!! \, t_{2k+3} \tau_{k+1}} \prod_{i=1}^n \tau_i^{d_i} \right\rangle_g$$

with the times $\hat{t}_k$ related to the t_k's by writing that:

$$e^{-\sum_k \hat{t}_k u^{-k}} = 1 - \frac{1}{2} \sum_d (2d+1)!! t_{2d+3} u^{-d} = \frac{u^{3/2}}{2\sqrt{2\pi}} \int_\gamma y dx \, e^{-\frac{u}{2} x}.$$

For example the first few are

$$\hat{t}_0 = -\ln(1 - \frac{t_3}{2}) \quad , \quad \hat{t}_1 = \frac{3\, t_5}{2 - t_3} \quad , \quad \hat{t}_2 = \frac{15\, t_7}{2 - t_3} + \frac{9\, t_5^2}{2(2 - t_3)^2} \quad , \quad \dots$$

- the generating functions of intersection numbers can be written as a sum over weighted graphs (Theorem 6.3.5):

$$2^{\chi_{g,n}} \sum_{d_1+\cdots+d_n=d_{g,n}} \frac{(2d_1-1)!!}{\lambda_1^{2d_1+1}} \cdots \frac{(2d_n-1)!!}{\lambda_n^{2d_n+1}} < \tau_{d_1} \ldots \tau_{d_n} >_g$$
$$= \sum_{\text{ribbon graphs}} \frac{1}{\#\text{Aut}} \prod_{(i,j)=\text{edges}} \frac{1}{\lambda_i + \lambda_j}$$

where the sum is over all labeled ribbon graphs of genus g with n faces, and to the ith face is associated the variable λ_i.
- The sum of weighted graphs can be obtained from a Wick theorem, and can be written as a formal matrix integral, the **Kontsevich integral**:

$$Z(\Lambda) = \frac{1}{Z_0} \int_{\text{formal}} dM e^{N \operatorname{Tr} \frac{M^3}{3} - M^2 \Lambda} \quad , \quad Z_0 = (\pi/N)^{N^2/2} \prod_{i,j} (\lambda_i + \lambda_j)^{-1/2}.$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N)$.

In the sense of formal series (of λ_i^{-1}) we have

$$\ln Z(\Lambda) = \sum_{g=0}^{\infty} N^{2-2g} F_g(\mathbf{t}) \quad , \quad t_k = \frac{1}{N} \operatorname{Tr} \Lambda^{-k} \quad , \quad \mathbf{t} = (t_1, t_2, t_3, \ldots)$$

where

$$F_g(\mathbf{t}) = 2^{2-2g} \left\langle e^{\frac{1}{2} \sum_k (2k+1)!! \, t_{2k+3} \, \tau_{k+1}} \right\rangle_g$$
$$= \left\langle e^{\sum_k \hat{t}_k \kappa_k} \right\rangle_g$$
$$= \sum_n \frac{2^{2-2g-n}}{n!} \sum_{d_1,\ldots,d_n} \prod_{i=1}^{n} (2d_i - 1)!! \, t_{2d_i+1} \quad < \tau_{d_1} \ldots \tau_{d_n} >_g .$$

- We define the following weighted sums of graphs:

$$\Omega_{g,n}(z_1, \ldots, z_n) = -\delta_{g,0}\delta_{n,1} z + \sum_{G \in \mathcal{G}_{g,n}(z_1,\ldots,z_n)} \frac{N^{-\#\text{unmarked faces}}}{\#\text{Aut}\,(G)}$$
$$\prod_{(i,j)=\text{edges}} \frac{1}{\text{label}(i) + \text{label}(j)}$$

summed over all ribbon graphs of genus g, with n labeled faces having a 1-valent vertex with respective label z_i, $i = 1, \ldots, n$, and an arbitrary number of unmarked faces having labels $\in [\lambda_1, \dots, \lambda_N]$. They are worth:

$$\begin{aligned}
&\Omega_{g,n}(z_1, \dots, z_n) + \delta_{g,0}\delta_{n,1} z_1 \\
&= 2^{2-2g-n} \sum_{d_1,\dots,d_n} \left\langle \tau_{d_1} \dots \tau_{d_n} \, \mathrm{e}^{\frac{1}{2}\sum_d (2d-1)!!\, t_{2d+1}\, \tau_d} \right\rangle_g \prod_{i=1}^{n} \frac{(2d_i+1)!!}{z_i^{2d_i+3}} \\
&+\delta_{g,0}\delta_{n,1} \frac{N^{-1}}{2z_1} \sum_{a=1}^{N} \frac{1}{z_1 + \lambda_a} \\
&+\delta_{g,0}\delta_{n,2} \frac{1}{4z_1 z_2} \frac{1}{(z_1+z_2)^2}.
\end{aligned}$$

They are equal to the expectation values of diagonal entries of M, if $a_1, \dots, a_n$ are all distinct:

$$\Omega_{g,n}(\lambda_{a_1}, \dots, \lambda_{a_n}) + \delta_{g,0}\delta_{n,1}\lambda_{a_1} \quad = \quad N^n 2^n < M_{a_1,a_1} \dots M_{a_n,a_n} >_c^{(g)} .$$

- Tutte's recursion and Virasoro constraints.

By recursively removing edges from the ribbon graphs, we get the identities: If $n \geq 0$ and $2g - 2 + (n+1) > 0$, and $J = \{z_1, \dots, z_n\}$, we have the Tutte's equations:

$$\begin{aligned}
\delta_{g,0}\delta_{n,0}\, z^2 = \Omega_{g-1,n+2}(z, z, J) &+ \sum_{h+h'=g,\, I \uplus I' = J} \Omega_{h,|I|+1}(z, I)\, \Omega_{h',|I'|+1}(z, I') \\
&-\frac{1}{N} \sum_{a=1}^{N} \frac{\Omega_{g,n+1}(z, J) - \Omega_{g,n+1}(\lambda_a, J)}{z^2 - \lambda_a^2} \\
&+\sum_{z_j \in J} \frac{1}{2z_j} \frac{d}{dz_j} \frac{\Omega_{g,n}(z, J \setminus \{z_j\}) - \Omega_{g,n}(J)}{z^2 - z_j^2}.
\end{aligned} \tag{6.7.1}$$

This translates into Virasoro constraints for intersection numbers

$$\begin{aligned}
&(2d_0+1)!! \left\langle \tau_{d_0}\ \tau_{d_1} \dots \tau_{d_n} \right\rangle_g \\
&= \frac{1}{2} \sum_{d+d'=d_0-2} (2d+1)!!(2d'+1)!! \Big[\left\langle \tau_d \tau_{d'}\ \tau_{d_1} \dots \tau_{d_n} \right\rangle_{g-1} \\
&\quad + \sum_{h+h'=g,\ I \sqcup I' = \{1,\dots,n\}}^{\text{stable}} \left\langle \tau_d \prod_{i \in I} \tau_{d_i} \right\rangle_h \left\langle \tau_{d'} \prod_{i \in I'} \tau_{d_i} \right\rangle_{g-h} \Big] \\
&\quad + \sum_{j=1}^{n} \frac{(2d_j + 2d_0 - 1)!!}{(2d_j - 1)!!} \left\langle \tau_{d_0+d_j-1} \prod_{i \neq j} \tau_{d_i} \right\rangle_g.
\end{aligned}$$

- The disc amplitude is (we assume $t_1 = 0$)

$$\Omega_{0,1}(z) = y(z) = -z + \frac{1}{N}\sum_{a=1}^{N} \frac{1}{2z(z+\lambda_a)} = -z - \frac{1}{2}\sum_k (-1)^k\, t_{k+2}\, z^k.$$

- The cylinder amplitude is ($t_1 = 0$ assumed)

$$\Omega_{0,2}(z,z') = \frac{1}{4zz'(z+z')^2}.$$

It is independent of the t_k's.

- Topological recursion ($t_1 = 0$ assumed)
  We define the amplitudes

$$\omega_{g,n}(z_1,\dots,z_n) = 2^n \left(\Omega_{g,n}(z_1,\dots,z_n) + \frac{\delta_{g,0}\delta_{n,2}}{(z_1^2 - z_2^2)^2}\right) \prod_{i=1}^{n} z_i.$$

If $2g-2+n>0$, they are odd rational functions of each z_i, with poles only at $z_i = 0$, and they satisfy the topological recursion

$$\omega_{g,n+1}(z_0,J) = \operatorname*{Res}_{z\to 0} K(z_0,z)\,dz \Big[\omega_{g-1,n+2}(z,-z,J) + \sum_{h+h'=g,\, I\uplus I'=J} \omega_{h,1+\#I}(\tilde{\mathcal{E}}_K; z, I)\, \omega_{h',1+\#I'}(\tilde{\mathcal{E}}_K; -z, I')\Big]. \tag{6.7.2}$$

where $\sum'$ means that $(h,I) = (0,\emptyset)$ and $(h,I) = (g,J)$ are excluded from the sum, and where K is the kernel

$$K(z_0,z) := \frac{\int_{z'=-z}^{z} \omega_{0,2}(z_0,z')}{2(\Omega_{0,1}(z) - \Omega_{0,1}(-z))\, x'(-z)}$$

- F_g's ($t_1 = 0$ assumed)

$$F_0 = 0$$

$$F_1 = \frac{1}{24}\ln\left(1 - \frac{t_3}{2}\right)$$

and for $g \geq 2$

$$F_g = \frac{1}{2-2g} \operatorname*{Res}_{z\to 0} \omega_{g,1}(z)\, \Phi(z)\, dz \qquad , \qquad d\Phi/dz = \omega_{0,1}(z).$$

- Comparison between topological gravity, $(2m+1,2)$ minimal model and large maps:

  The asymptotic generating function of large maps $\tilde{F}_g$ near an mth order critical point, coincides with the topological expansion of the Tau-function of the minimal model $(2m+1,2)$, and with the generating function of intersection numbers:

$$\tilde{F}_g(\tilde{t}_i) = \mathcal{F}_g(\mathcal{E}_{(2m+1,2)}) = \mathcal{F}_g(\hat{\mathcal{E}}_K) = \left\langle e^{\sum_k \hat{t}_k \kappa_k} \right\rangle_{\mathcal{M}_{g,0}}$$

  provided that we identify the Kontsevich times t_k's and $(2m+1,2)$-model times $\tilde{t}_j$ as:

$$t_{2k+3} = 2\delta_{k,0} - 2\sum_{j=k}^{m} \tilde{t}_j \, \frac{(-u_0)^{j-k}}{(j-k)!} \, \frac{(2j+1)!!}{(2k+1)!!}.$$

- Weil-Petersson volumes

  The choice

$$t_{2k+3} = 2\delta_{k,0} - 2\,\frac{(-1)^k\,(2\pi)^{2k}}{(2k+1)!}$$

  gives $y(z) = \frac{1}{2\pi}\sin 2\pi z$, and it computes the Laplace transforms of Weil-Petersson volumes

$$\omega_{g,n}(z_1,\dots,z_n) = (-2)^{\chi_{g,n}} \int_0^\infty L_1 dL_1 e^{-z_1 L_1} \dots \int_0^\infty L_n dL_n e^{-z_n L_n} \,\mathrm{Vol}(\mathcal{M}_{g,n}(L_1,\dots,L_n)).$$

  where

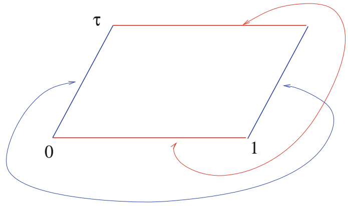
$$\mathrm{Vol}(\mathcal{M}_{g,n}(L_1,\dots,L_n)) = \frac{1}{d_{g,n}!}\left\langle (2\pi^2\kappa_1 + \frac{1}{2}\sum_{i=1}^{n} L_i^2\psi_i)^{d_{g,n}} \right\rangle_{\overline{\mathcal{M}}_{g,n}}.$$

  The fact that $\omega_{g,n}$ satisfy the topological recursion, implies the Mirzakhani's recursion for Weil-Petersson volumes.

6.8 Exercises

Exercise 1 (Moduli Space of Genus 1 Surfaces) Consider a torus T_τ of modulus τ, with $\mathrm{Im}\,\tau > 0$, i.e. a parallelogram in the complex plane with opposite sides identified:

$$T_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}).$$



Prove that two Torus T_τ and $T_{\tau'}$ are in conformal bijection if and only if $\tau' \equiv \tau \bmod Sl_2(\mathbb{Z})$, i.e.

$$\tau' = \frac{a\tau + b}{c\tau + d} \quad , \quad ad - bc = 1 \quad , \quad (a, b, c, d) \in \mathbb{Z}^4.$$

Hint:

- sufficient condition: prove that it is possible to find a piecewise affine function $f : \mathbb{C} \to \mathbb{C}$, of the form $f(z) = \alpha z + \beta$, such that $f(z+1) \equiv f(z) \bmod \mathbb{Z} + \tau'\mathbb{Z}$ and $f(z+\tau) \equiv f(z) \bmod \mathbb{Z} + \tau'\mathbb{Z}$. In other words determine α and β.
- converse, necessary condition: assume that there exists a conformal bijection $f : T_\tau \to T_{\tau'}$. It can be lifted to a piecewise analytic function $f : \mathbb{C} \to \mathbb{C}$, which has to satisfy $f(z+1) \equiv f(z) \bmod \mathbb{Z} + \tau'\mathbb{Z}$ and $f(z+\tau) \equiv f(z) \bmod \mathbb{Z} + \tau'\mathbb{Z}$. Show that this implies that $f'(z)$ has to be bi-periodic. Since any bi-periodic function without pole must be a constant, deduce that $f'(z)$ has to be a constant, and thus f is piecewise affine. Then by studying $f(0), f(1), f(\tau), f(1+\tau)$, show that $\tau' = (a\tau + b)/(c\tau + d)$.

Exercise 2 (The Strebel Differential on $\mathcal{M}_{0,4}$) Let $p \in \overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ and let L_0, L_1, L_∞, L_p be four positive real numbers.

Find the general form of a quadratic differential on $\overline{\mathbb{C}}$ with double poles at $z = 0, 1, \infty, p$ and respective residues $-L_0^2, -L_1^2, -L_\infty^2, -L^2$.

Answer:

$$\Omega(z) = \frac{-1}{z(z-1)(z-p)} \left(L_\infty^2 z + \frac{p\, L_0^2}{z} + \frac{(1-p)\, L_1^2}{(z-1)} + \frac{p(p-1)\, L^2}{(z-p)} - \alpha \right) dz^2$$

where $\alpha \in \mathbb{C}$ is an arbitrary constant. Ω has four zeroes a, b, c, d, we have $\alpha = a + b + c + d$. Then chose α such that

$$\mathrm{Im} \int_a^b \sqrt{\Omega(z)} = 0 \quad , \quad \mathrm{Im} \int_a^c \sqrt{\Omega(z)} = 0$$

Exercise 3 (Number of Triangulations in Terms of Intersection Numbers) Choose the matrix $\Lambda = \lambda\, \mathrm{Id}_N$, i.e. $t_k = \lambda^{-k}$, and relate Kontsevich's matrix integral to the cubic formal matrix integral which enumerates triangulations. Then show that the number of rooted triangulations (where all faces including the marked one are triangles) of genus g with v vertices is

$$\#\{\text{rooted triangulations, genus } g, \#\text{vertices} = v\}$$
$$= 6\,(2g-2+v)\,\frac{2^{4g-4+2v}}{v!} \sum_{d_1+\cdots+d_v=3g-3+v} < \tau_{d_1} \ldots \tau_{d_v} >_{g,v}$$

Exercise 4 For $\overline{\mathcal{M}}_{1,1}$, there is only one Strebel graph. Write the Chern class in terms of edge lengths, and compute directly the integral of the Chern class. Recover $< \tau_1 >_1 = \frac{1}{24}$.

Exercise 5 Prove that all intersection numbers of genus 0 are given by:

$$< \tau_{d_1} \ldots \tau_{d_n} >_0 = \frac{(n-3)!}{\prod_{i=1}^{n} d_i!} \; \delta_{\sum_i d_i, n-3},$$

Hint: use the fact that if $g = 0$, necessarily some $d_i = 0$, and one can use equation Eq. (6.3.2).

Exercise 6 Prove that all intersection numbers of genus 1 are given by:

$$< \tau_{d_1} \ldots \tau_{d_n} >_1 = \frac{1}{24} \, \frac{n!}{\prod_{i=1}^{n} d_i!} \; \delta_{\sum_i d_i, n}.$$

Hint: Use the fact that if $g = 1$, then either some $d_i = 0$ and one can use equation Eq. (6.3.2), or some $d_i = 1$, and the forgetful pushforward of τ_1 is κ_0, and κ_0 is the Euler class $\kappa_0 = 2g - 2 + n$.

Chapter 7
Topological Recursion and Symplectic Invariants

We have seen, in almost all previous chapters, that symplectic invariants and topological recursion play an important role. They give the solution to Tutte's recursion equation for maps, they give the formal expansion of various matrix integrals, including Kontsevich integral, and they also give the asymptotics of large maps.

The goal of this chapter is to give their general definition, which is an algebraic geometry notion, and exists beyond the context of combinatorics, and beyond matrix models.

7.1 Symplectic Invariants of Spectral Curves

Building on works in matrix models, an axiomatic definition of the symplectic invariants was first introduced in [34]. At that time, the goal was to have a common framework for the solution of loop equations of several matrix models: 1-matrix, 2-matrix, matrix with external field (in particular Kontsevich integral), chain of matrices,..., as well as their scaling limits. Then it was discovered that they have many nice properties, in particular symplectic invariance (whence their name), and that they appear in other problems of enumerative geometry.

Here we only briefly summarize the construction of [34], and we refer the reader to the original article for more details.

7.1.1 Spectral Curves

Definition 7.1.1 A spectral curve $\mathcal{E} = (\mathcal{L}, x, y, B)$, is the data of a Riemann surface $\mathcal{L}$ (not necessarily compact nor connected), and two analytic functions x and y from

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some open domain of $\mathcal{L}$ to $\mathbb{C}$, and a symmetric meromorphic 2-form B on $\mathcal{L} \times \mathcal{L}$ having a double pole on the diagonal (in any choice of local coordinates):

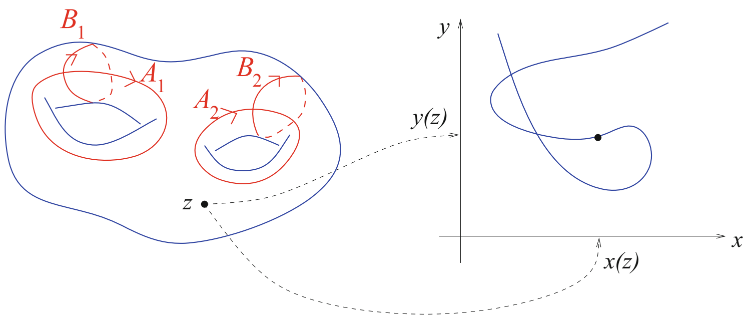
$$B(p,q) \sim \frac{dp \otimes dq}{(p-q)^2} + \text{analytic}.$$

Remark 7.1.1 In fact, the most general definition of symplectic invariants needs only that $\mathcal{L}$ be a collection of formal neighborhoods of some points, and y and B be germs of analytic functions on those formal neighborhoods.[1] However, the symplectic invariants will obey more properties when $\mathcal{L}$ is compact, connected, and y and B are globally meromorphic. This is the case for all spectral curves considered in this book, related to maps.

The maps $x : \mathcal{L} \to \mathbb{C}$ and $y : \mathcal{L} \to \mathbb{C}$ provide an immersion of $\mathcal{L} \hookrightarrow \mathbb{C} \times \mathbb{C}$. If $\mathcal{L}$ is compact, and x and y are both meromorphic, they must be related by an algebraic equation $E(x, y) = 0$. The locus in $\mathbb{C} \times \mathbb{C}$ of the solutions of an algebraic equation, is called a plane curve.

x and y thus provide a parametric representation of a plane curve of some equation $E(x, y) = 0$, where the space of the parameter z is a Riemann surface $\mathcal{L}$, i.e.

$$\{(x(z), y(z)) \mid z \in \mathcal{L}\} = \{(x, y) \mid E(x, y) = 0\}$$



Definition 7.1.2 If $\mathcal{L}$ is a compact Riemann surface of genus $\bar{g}$, and x and y are meromorphic functions on $\mathcal{L}$, we say that the spectral curve is algebraic. If $\mathcal{L} = \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere (thus $\bar{g} = 0$), we say that the spectral curve is rational.

Indeed, for a compact Riemann surface $\mathcal{L}$, it is always possible to find a polynomial relationship between any two meromorphic functions x and y, and there

[1] Roughly speaking, y and B are defined as formal series, whose radius of convergency can be vanishing, i.e. any truncation of the formal series is defined in some disk around a point, called a "formal neighbourhood" of the point. We shall not go further, as this notion is beyond the scope of this book.

always exists a polynomial E such that

$$\forall\, z \in \mathcal{L}\,, \qquad E(x(z), y(z)) = 0.$$

Moreover, on the Riemann sphere, meromorphic functions are rational functions, i.e. for a rational spectral curve we can chose $x, y \in \mathbb{C}(z)$.

Definition 7.1.3 A spectral curve $(\mathcal{L}, x, y, B)$ is called regular if:

- the differential form dx has a finite number (non vanishing) of zeros $dx(a_i) = 0$, and all zeros of dx are simple zeros.
- The differential form dy does not vanish at the zeros of dx, i.e. $dy(a_i) \neq 0$.

This means that near $x(a_i)$, $x(z) - x(a_i)$ has a double zero, and thus $\sqrt{x(z) - x(a_i)}$ is a good local coordinate. If dy doesn't vanish it means that

$$y(z) \sim y(a_i) + y'(a_i)\, \sqrt{(x(z) - x(a_i)} + O(x(z) - x(a_i)) \qquad , \qquad y'(a_i) \neq 0,$$

in other words y behaves locally like a square-root, or also the curve $x \mapsto y$ has a vertical tangent at a_i.

From now on, we assume that we are considering only regular spectral curves (the symplectic invariants for non-regular spectral curves can be defined in a similar manner, see [18]). Symplectic invariants defined for regular spectral curves, may diverge when the curve becomes singular. Examples of singular spectral curves appeared in Chap. 5, where they play a central role in the double scaling limit, i.e. the limit of large maps.

Definition 7.1.4 We say that two spectral curves $\mathcal{E} = (\mathcal{L}, x, y, B)$ and $\tilde{\mathcal{E}} = (\tilde{\mathcal{L}}, \tilde{x}, \tilde{y}, \tilde{B})$ are symplectically equivalent if there exists a symplectomorphism ϕ : $\mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$, such that $\mathcal{L} = \phi * \mathcal{L}$, $\tilde{B} = \phi * B$, and $\tilde{x} = \phi * x$ and $\tilde{y} = \phi * y$,

And where the group of symplectomorphisms is defined to be generated by the maps:

- $\phi : (x, y) \mapsto (x, y + R(x))$, where $R(x) \in \mathbb{C}(x)$ is a rational function of x.
- $\phi : (x, y) \mapsto (\frac{ax+b}{cx+d}, (cx + d)^2 y)$, with $ad - bc = 1$,
- $\phi : (x, y) \mapsto (y, -x)$.

all those transformations conserve the symplectic form

$$dx \wedge dy.$$

The main property of the $\mathcal{F}_g$'s that we are going to define, is that they are symplectic invariants, i.e. two regular spectral curves which are symplectically equivalent, have the same $\mathcal{F}_g$'s.

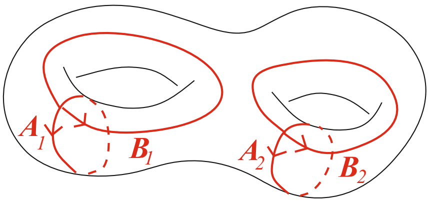
7.1.2 Geometry of the Spectral Curve

7.1.2.1 Topology

Consider a compact Riemann surface $\mathcal{L}$, of genus $\bar{g}$. If it is simply connected, the genus is $\bar{g} = 0$, and $\mathcal{L}$ is the Riemann sphere $\hat{\mathbb{C}}$, i.e. the complex plane compactified with a point at ∞. If it is of genus $\bar{g} \geq 1$, it is not simply connected, and one can find a basis of $2\bar{g}$ non-contractible cycles, that can be normalized in order that their intersections (the sign of the intersection corresponds to the orientation of contours, positive if $\mathcal{A}_i, \mathcal{B}_i$ is direct) are:

$$\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j} \quad , \quad \mathcal{A}_i \cap \mathcal{A}_j = 0 \quad , \quad \mathcal{B}_i \cap \mathcal{B}_j = 0.$$

This choice of basis of non-contractible cycles is called "symplectic", and it is not unique.



If we chose some representants $\mathcal{A}_i$'s and $\mathcal{B}_i$'s of the cycles, then $\overline{\mathcal{L}} = \mathcal{L} \setminus \cup_i \mathcal{A}_i \cup_i \mathcal{B}_i$ is simply connected. It is called a "fundamental domain" .

The universal covering of $\mathcal{L}$ is a (non-compact) Riemann surface, obtained by gluing an infinite number of copies of the fundamental domain $\overline{\mathcal{L}}$, along the corresponding boundaries. It is also the set of all homotopy classes of paths between a given base point, and arbitrary points of $\mathcal{L}$. The universal covering is simply connected, but non-compact.

7.1.2.2 Bergman Kernel = Fundamental Form of the Second Kind

When a symplectic basis of cycles is chosen, one defines the fundamental form of second kind (sometimes called Bergman kernel [12, 13, 37, 56]):

$$B(z_1, z_2)$$

as the unique bilinear differential having one double pole at $z_1 = z_2$ (it is called "second kind") and no other pole, and such that:

$$B(z_1, z_2) \underset{z_1 \to z_2}{\sim} \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2} + \text{analytic} \quad , \quad \forall i = 1, \ldots, \bar{g}, \quad \oint_{z_1 \in \mathcal{A}_i} B(z_1, z_2) = 0.$$

One should keep in mind that B depends only on $\mathcal{L}$, and not on the functions x and y.

We encountered it in Chaps. 3, 4, and 6. In each case, the cylinder amplitude was (up to trivial terms) the fundamental form of the second kind.

Intuitively, the fundamental second kind form can be viewed as the electric field on $\mathcal{L}$ measured in z_1 generated by a small unit dipole located at z_2. Or said differently, the integral

$$\ln E(z_1, z_2) = \int_{z_1'=o_1}^{z_1} \int_{z_2'=o_2}^{z_2} B(z_1', z_2')$$

(where o_1, o_2 are arbitrary base points), is the electric potential measured at z_1, created by a unit charge located at z_2, it satisfies the Poisson equation

$$\Delta_{z_1} \ln |E(z_1, z_2)| = 2\pi\, \delta(z_1 - z_2).$$

Examples

- if $\mathcal{L} = \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ =the Riemann Sphere (genus $\bar{g} = 0$), the fundamental second kind form is

$$B(z_1, z_2) = \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2} = d_{z_1} d_{z_2} \ln (z_1 - z_2)$$

- if $\mathcal{L} = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ =Torus (genus $\bar{g} = 1$) of modulus τ, the fundamental second kind form is

$$B(z_1, z_2) = (\wp(z_1 - z_2, \tau) - \frac{E_2(\tau)}{3})\, dz_1 \otimes dz_2$$

  where $\wp$ is the Weierstrass elliptical function, and E_2 the second Eisenstein's series.
- if $\mathcal{L}$ is a compact Riemann surface of genus $\bar{g} \geq 1$, of Riemann matrix of periods $\tau = [\tau_{i,j}]_{i,j=1,\ldots,\bar{g}}$, the fundamental second kind form is

$$B(z_1, z_2) = d_{z_1} d_{z_2} \ln (\theta(u(z_1) - u(z_2) - c, \tau))$$

  where $u(z)$ is the Abel map, c is an odd characteristic, and θ is the Riemann theta function of genus $\bar{g}$ (cf [36, 37] for theta-functions).

7.1.2.3 Branchpoints

Branchpoints are the points with a vertical tangent, they are the **zeros of d**x. Let us write them a_i, $i = 1, \ldots, \#$branchpoints.

$$\forall i, \quad dx(a_i) = 0.$$

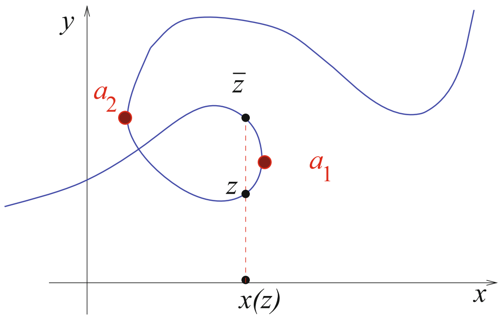
Since we consider a regular spectral curve, all branchpoints are simple zeros of dx, the map $x : \mathcal{L} \to \mathbb{C}$ is locally $2:1$, and thus there are exactly two points z and $\bar{z}$ in the vicinity of a_i such that:

$$x(\bar{z}) = x(z)$$

the involution $z \mapsto \bar{z}$ is called the **local Galois conjugate** of z. It is defined locally near each branchpoint a_i, and it is not necessarily defined globally.

Locally, the map $z \mapsto y(z)$ behaves like a square root as a function of $x(z)$, near a branchpoint a_i:

$$y(z) \sim y(a_i) + y'(a_i)\,\sqrt{x(z) - x(a_i)} + O(x(z) - x(a_i)).$$



Examples of Spectral Curves

- **maps =1-matrix model, 1-cut**. In Chap. 3 we have seen that maps' spectral curves are parametrized by the Zhukovsky variable z. In that case $z \in \mathcal{L}$ =Riemann sphere= $\bar{\mathbb{C}}$, and $x(z)$ and $y(z)$ are rational functions of z. In particular we have

$$x(z) = \alpha + \gamma(z + 1/z) \qquad , \qquad dx(z) = x'(z)dz = \gamma(1 - z^{-2})\,dz.$$

  The zeros of $dx(z)$ are $z = \pm 1$, and we clearly have $\bar{z} = 1/z$:

$$a_1 = 1,\ \ a_2 = -1 \qquad , \qquad \bar{z} = 1/z.$$

  In that case the local Galois involution $z \mapsto 1/z$ is defined globally on $\mathcal{L}$. The spectral curves of maps, are examples of algebraic rational spectral curve.
- **pure gravity** (3, 2). In Chap. 5, we have seen that the pure gravity (3, 2) minimal model, is related to the spectral curve

$$x(z) = z^2 - 2 \qquad , \qquad y(z) = z^3 - 3z \qquad , \qquad z \in \mathcal{L} = \text{Riemann sphere}.$$

We have

$$dx(z) = x'(z)dz = 2z\,dz$$

whose only zero is $z = 0$, and we have $\bar{z} = -z$:

$$a = 0 \qquad , \qquad \bar{z} = -z.$$

In that case the local Galois involution $z \mapsto -z$ is defined globally on $\mathcal{L}$.

- **Ising model** $(4,3)$. The minimal model $(4,3)$ (not studied in this book) with central charge $c = 1/2$, is also called "Ising model". It has the rational spectral curve

$$x(z) = z^3 - 3z \qquad , \qquad y(z) = z^4 - 4z^2 + 2 \qquad , \qquad z \in \mathcal{L} = \text{Riemann sphere},$$

and thus

$$dx(z) = x'(z)dz = 3(z^2 - 1)\,dz$$

whose zeros are $a_i = \pm 1$, and near $a_i = \pm 1$ we have $\bar{z} = -\frac{1}{2}(z - a_i\sqrt{12 - 3z^2})$:

$$a_i = \pm 1 \qquad , \qquad \bar{z} = -\frac{1}{2}(z - a_i\sqrt{12 - 3z^2}).$$

In this case, the local Galois involution $z \mapsto \bar{z}$ is not defined globally, it is defined only in the vicinity of each a_i (the two Galois involutions near $a_i = \pm 1$, differ by the choice of sign of the square–root).

7.1.2.4 Recursion Kernel

Definition 7.1.5 We define the recursion kernel with z in a vicinity of a branch point a:

$$K_a(z_0, z) = \frac{1}{2}\,\frac{\int_{z'=\bar{z}}^{z} B(z_0, z')}{(y(z) - y(\bar{z}))\,dx(z)}$$

where $z \mapsto \bar{z}$ is the local Galois involution at a, and the integration path $z \to \bar{z}$ is chosen in the vicinity of a, in particular it doesn't intersect the $\mathcal{A}_i$–cycles or $\mathcal{B}_i$–cycles.

$K_a(z_0, z)$ is a meromorphic 1-form in the variable z_0, globally defined on $z_0 \in \mathcal{L}$, it has a simple pole at $z_0 = z$ and at $z_0 = \bar{z}$.

On the contrary, with respect to the variable z, $K_a(z_0, z)$ is defined only locally near the branchpoint a, and it is a 1-form raised to the power -1. As we shall see below, $K_a(z_0, z)$ will always be multiplied by a quadratic differential, so that the product will be a 1-form. It is symmetric under the involution

$$K_a(z_0, z) = K_a(z_0, \bar{z}).$$

$K_a(z_0, z)$ has a simple pole at $z = a$, and near $z = a$ it behaves like:

$$K_a(z_0, z) \sim \frac{1}{2} \frac{B(z_0, z)}{dy(z)\, dx(z)} + \text{analytic}.$$

Remember that $dx(z)$ has a simple zero at $z = a$ and $dy(a) \neq 0$.

7.1.2.5 Correlators

Definition 7.1.6 We define recursively the following meromorphic forms:

$$\omega_1^{(0)}(z_1) = y(z_1)\, dx(z_1)$$

$$\omega_2^{(0)}(z_1, z_2) = B(z_1, z_2)$$

and if $2g - 2 + n \geq 0$, and $J = \{z_1, \ldots, z_n\}$:

$$\omega_{n+1}^{(g)}(z_0, J) = \sum_i \operatorname*{Res}_{z \to a_i} K_{a_i}(z_0, z) \left[\omega_{n+2}^{(g-1)}(z, \bar{z}, J) + \sum_{h+h'=g,\, I \uplus I' = J}' \omega_{1+|I|}^{(h)}(z, I) \omega_{1+|I'|}^{(h')}(\bar{z}, I') \right] \tag{7.1.1}$$

where $\sum'$ in the right hand side means that we exclude the terms $(h, I) = (0, \emptyset), (g, J)$.

This definition is indeed a recursive one, because all the terms in the right hand side have a strictly smaller $2g - 2 + n$ than the left hand side.

An important property proved in [34], is that:

Proposition 7.1.1 *$\omega_n^{(g)}(z_1, \ldots, z_n)$ is a meromorphic n-form on $\mathcal{L}^n$, it is a tensor product of meromorphic forms on $\mathcal{L}$ of each variables z_i. It can be proved by recursion, that it is always a symmetric form. Moreover, if $2g - 2 + n > 0$, its only poles are at branchpoints $z_i \to a_j$, and have no residues.*

Those properties can be proved by recursion on $2g - 2 + n$, and we refer the reader to [34].

7.1.2.6 Symplectic Invariants

The previous definition, defines $\omega_n^{(g)}$ with $n \geq 1$. Now, we define $\mathcal{F}_g = \omega_0^{(g)}$ by the following:

7.1.2.7 • $\mathcal{F}_g$ for $g \geq 2$

Definition 7.1.7 (Symplectic Invariants)
We define for $g \geq 2$:

$$\boxed{\mathcal{F}_g = \frac{1}{2-2g} \sum_i \operatorname*{Res}_{z \to a_i} \Phi(z)\, \omega_1^{(g)}(z) \qquad , \qquad d\Phi = ydx} \tag{7.1.2}$$

($\mathcal{F}_g$ is independent of a choice of integration constant for Φ, indeed due to Proposition 7.1.1, $\operatorname{Res}_{a_i} \omega_1^{(g)} = 0$, so the contribution of a constant in Φ vanishes).

7.1.2.8 • $\mathcal{F}_g$ for $g = 1$

Definition 7.1.8 For $g = 1$ we define

$$\boxed{\mathcal{F}_1 = -\frac{1}{24} \ln \left(\tau_B(\{x(a_i)\})^{12} \prod_i y'(a_i) \right)}$$

where

$$y'(a_i) = \lim_{z \to a_i} \frac{y(z) - y(a_i)}{\sqrt{x(z) - x(a_i)}}$$

and τ_B is the Bergman τ-function of Kokotov–Korotkin [56], it depends only on the values of x at branch points $x_i = x(a_i)$, it is defined by:

$$\frac{\partial \ln \tau_B(\{x_i)\})}{\partial x_i} = \operatorname*{Res}_{z \to a_i} \frac{B(z, \bar{z})}{dx(z)}$$

- For example, for maps, we have a rational spectral curve with $B(z, z') = dz\, dz'/(z - z')^2$ and with two branchpoints a, b, parametrized by Zhukovsky map $x(z) = (a + b)/2 + \gamma\,(z + 1/z)$ where $\gamma = (a - b)/4$, and thus the local Galois involution $\bar{z} = 1/z$. In that case we have

$$\frac{\partial \ln \tau_B(a,b)}{\partial a} = \operatorname*{Res}_{z\to 1} \frac{-dz^2/z^2}{(z-1/z)^2} \frac{1}{\gamma(1-1/z^2)\, dz} = \frac{1}{16\gamma} = \frac{1}{4(a-b)}$$

and similarly $\frac{\partial \ln \tau_B(a,b)}{\partial b} = \frac{1}{4(b-a)}$, which leads to

$$\tau_B(a,b) \propto \gamma^{1/4}.$$

i.e.

$$\mathcal{F}_1 = -\frac{1}{24} \ln\left(\gamma^3\, y'(a)\, y'(b)\right).$$

Then, notice that $y'(a) = \lim_{z\to 1} \frac{y(z)-y(1)}{\sqrt{\gamma(z+1/z-2)}} = \frac{1}{\sqrt{\gamma}} \left.\frac{dy}{dz}\right|_{z=1}$, and similarly for $y'(b)$, i.e. finally:

$$\mathcal{F}_1 = -\frac{1}{24} \ln\left(\gamma^2 \left.\frac{dy}{dz}\right|_{z=1} \left.\frac{dy}{dz}\right|_{z=-1}\right),$$

which is what we found in Chap. 3.

- For example, If we have a rational spectral curve with only one branchpoint a, parametrized by $x(z) = a + z^2$, we have

$$\frac{\partial \ln \tau_B(a)}{\partial a} = \operatorname*{Res}_{z\to 0} \frac{-dz^2}{(z+z)^2} \frac{1}{2z\, dz} = 0$$

and thus

$$\tau_B(a) \propto 1,$$

and

$$\mathcal{F}_1 = -\frac{1}{24} \ln\left(\left.\frac{dy}{dz}\right|_{z=0}\right),$$

which is what we found for (p,q) minimal models and Kontsevich integral in Chaps. 5 and 6.

7.1.2.9 • $\mathcal{F}_g$ for $g = 0$

Here we assume that we have an algebraic spectral curve.

Let α_i, $i = 1, \ldots, n_{\text{poles}}$ be the poles of ydx.

– If α_i is a pole of degree d_i of $x(z)$, we define the local coordinate $\xi_i(z) = x(z)^{-1/d_i}$
– If α_i is not a pole of $x(z)$ [it is thus a pole of $y(z)$], we define the local coordinate $\xi_i(z) = x(z) - x(\alpha_i)$

We define the potentials V_i and times t_i:

$$t_i = \operatorname*{Res}_{z\to\alpha_i} y(z)\,dx(z) \qquad , \qquad V_i(z) = \operatorname*{Res}_{z'\to\alpha_i} \ln\left(1 - \frac{\xi_i(z')}{\xi_i(z)}\right) y(z')\,dx(z').$$

Notice that $\sum_i t_i = 0$. They are such that

$$y(z)dx(z) - dV_i(z) - t_i\, d\xi_i(z)/\xi_i(z) = \text{analytic at}\, z \to \alpha_i.$$

Then, given an arbitrary generic base point $o \in \mathcal{L}$, we define:

$$\mu_i = \int_{\alpha_i}^{o} \left(y(z)dx(z) - dV_i(z) - t_i\, d\xi_i(z)/\xi_i(z)\right) + V_i(o) + t_i\, \ln\xi_i(o)$$

Then, we define

Definition 7.1.9

$$\mathcal{F}_0 = \frac{1}{2}\Big[\sum_i \operatorname*{Res}_{\alpha_i} V_i(z)\,y(z)dx(z) + \sum_i t_i\mu_i + \sum_{i=1}^{\bar{g}} \frac{1}{2i\pi}\oint_{\mathcal{A}_i} ydx \oint_{\mathcal{B}_i} ydx\Big].$$

One may check that this quantity is independent of the choice of base point o (this is because $\sum_i t_i = 0$).

Notice, that contrarily to all $\mathcal{F}_g$ with $g \geq 2$, which depend only on the local behavior of x and y near branchpoints, $\mathcal{F}_0$ depends on the full spectral curve, and in particular on its local behavior near the poles.

7.2 Main Properties

So, for every spectral curve $\mathcal{E} = (\mathcal{L}, x, y, B)$, we have defined some meromorphic forms $\omega_n^{(g)}$ and some complex numbers $\mathcal{F}_g = \omega_0^{(g)}$. They have some remarkable properties (proved in [34]):

- $\omega_n^{(g)}$ is symmetric in its n variables (this is proved by recursion).
- If $2 - 2g - n < 0$, then $\omega_n^{(g)}$ is a meromorphic form in each variable, with poles only at the branch-points, of degree at most $6g - 6 + 2n + 2$, and with vanishing residue.

- If $2-2g-n<0$, $\omega_n^{(g)}$ is homogeneous of degree $2-2g-n$:

$$\omega_n^{(g)}((\mathcal{L},x,\lambda y,B);z_1,\dots,z_n) = \lambda^{2-2g-n}\,\omega_n^{(g)}((\mathcal{L},x,y,B);z_1,\dots,z_n),$$

  and in particular for $n=0$

$$\mathcal{F}_g(\mathcal{L},x,\lambda y,B) = \lambda^{2-2g}\,\mathcal{F}_g(\mathcal{L},x,y,B).$$

  In particular, if $\lambda=-1$, we see that $\mathcal{F}_g$ is invariant under $y\to -y$.
- If two spectral curves $\mathcal{E}=(\mathcal{L},x,y,B)$ and $\tilde{\mathcal{E}}=(\mathcal{L},\tilde{x},\tilde{y},B)$ are symplectically equivalent in the sense of Definition 7.1.4, i.e. $dx\wedge dy = d\tilde{x}\wedge d\tilde{y}$, or alternatively $ydx-\tilde{y}d\tilde{x}=$ exact form, then they have the same $\hat{\mathcal{F}}_g=\mathcal{F}_g-\frac{1}{2-2g}\sum_i t_i\int_o^{\alpha_i}\omega_1^{(g)}$'s for $g\geq 2$:

$$dx\wedge dy = d\tilde{x}\wedge d\tilde{y} \qquad \Rightarrow \qquad \forall\, g\geq 2 \quad \hat{\mathcal{F}}_g(\mathcal{E}) = \hat{\mathcal{F}}_g(\tilde{\mathcal{E}}).$$

  This property justifies the name "symplectic invariants" for the $\hat{\mathcal{F}}_g$'s.

  In general they do not have the same $\omega_n^{(g)}$'s, but we have that:

$$\omega_1^{(g)}(\mathcal{E};z_1) - \omega_1^{(g)}(\tilde{\mathcal{E}};z_1) = \text{exact form},$$

  i.e. the cohomology class of $\omega_1^{(g)}$ is a symplectic invariant.

  This property has been proved (at the time this book is being written) only for algebraic spectral curves, where $\mathcal{L}$ is compact, and x and y are meromorphic. It is believed to hold in a more general setting.
- Out of the $\mathcal{F}_g$'s, one can construct a formal tau function, which is believed to obey Hirota's equation. It is of the form $\tau=\exp\left(\sum_{g=0}^{\infty}N^{2-2g}\mathcal{F}_g\right)\Theta$, where we refer the reader to [17] for more details.
- Dilaton equation, for $2g-2+n>0$:

$$\sum_i \operatorname*{Res}_{z_{n+1}\to a_i} \Phi(z_{n+1})\,\omega_{n+1}^{(g)}(z_1,\dots,z_n,z_{n+1}) = (2-2g-n)\,\omega_n^{(g)}(z_1,\dots,z_n)$$

  where $d\Phi=\omega_1^{(0)}=ydx$.
- Their derivatives with respect to any parameter of the spectral curve, are computed below in Sect. 7.3.
- They have many other properties, for instance their modular behaviour satisfies the "holomorphic anomaly equation", known as "BCOV equation". See [15, 35].

7.3 Deformations of Symplectic Invariants

Consider a spectral curve $\mathcal{E} = (\mathcal{L}, x, y, B)$.

Embed it within a C^1 1-parameter family of algebraic spectral curves $\mathcal{E}_t = (\mathcal{L}_t, x_t, y_t, B_t)$, defined for t in a small vicinity of $t = 0$, and such that at $t = 0$ it is the spectral curve $\mathcal{E}$:

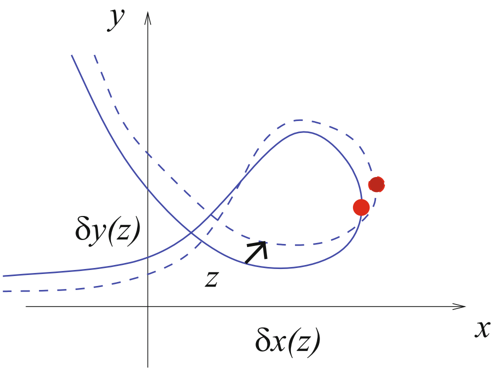
$$\mathcal{E}_0 = \mathcal{E} \qquad , \qquad (\mathcal{L}_0, x_0, y_0, B_0) = (\mathcal{L}, x, y, B).$$

Our goal is to compute derivatives of the symplectic invariants at $t = 0$:

$$\left.\frac{\partial^n \mathcal{F}_g(\mathcal{E}_t)}{\partial t^n}\right|_{t=0} \qquad \text{and} \qquad \left.\frac{\partial^n \omega_n^{(g)}(\mathcal{E}_t)}{\partial t^n}\right|_{t=0} .$$

Here and in all the following sections, we assume that we have an algebraic regular spectral curve for almost each t, and that $\mathcal{L}_t$ is compact and $B = B_t$ is the fundamental 2-form on it.

7.3.1 Spectral Curve Deformation



We would like to compute derivatives like $\dot{x}_t(z) = \partial x_t(z)/\partial t = \lim_{\epsilon\to 0} \frac{x_{t+\epsilon}(z) - x_t(z)}{\epsilon}$. However, this doesn't make sense, because in the term $x_{t+\epsilon}(z)$ we have $z \in \mathcal{L}_{t+\epsilon}$ and in the second term $x_t(z)$ we have $z \in \mathcal{L}_t$, and $\mathcal{L}_{t+\epsilon} \neq \mathcal{L}_t$, are two different Riemann surfaces.

In other words, we first need to chose a common local coordinate on both curves, a common atlas of charts.

We thus locally chose a smooth family of open domains $U_t \subset \mathcal{L}_t$, and a smooth C^1 family of charts, i.e. a C^1 family of coordinates $\zeta_t : U_t \to U \subset \mathbb{C}$. The maps

$x_t \circ \zeta_t^{-1}$ and $y_y \circ \zeta_t^{-1}$ form two C^1 families of meromorphic functions $U \to \mathbb{C}$. We can then use the local coordinate ζ_0 at $t = 0$, to identify $U \subset \mathbb{C}$, with the domain $U_0 \subset \mathcal{L}_0 = \mathcal{L}$.

By abuse of notation, we identify (locally in an open set $U_0 \subset \mathcal{L}$):

$$x_t(z) \equiv x_t \circ \zeta_t^{-1} \circ \zeta_0(z) \qquad , \qquad y_t(z) \equiv y_t \circ \zeta_t^{-1} \circ \zeta_0(z).$$

Now, it makes sense to derive with respect to t.

We write the beginning of the Taylor expansion near $t = 0$

$$\begin{cases} x_t \circ \zeta_t^{-1}(z) = x \circ \zeta_0^{-1}(z) + t\,\dot{x}(z) + O(t^2) \\ y_t \circ \zeta_t^{-1}(z) = y \circ \zeta_t^{-1}(z) + t\,\dot{y}(z) + O(t^2). \end{cases}$$

Here, $\dot{x}$ and $\dot{y}$ are analytic (meromorphic) functions on $\mathcal{L} = \mathcal{L}_0$

There is some arbitrariness in this writing. The meromorphic functions $\dot{x}$ and $\dot{y}$ depend on the choice of coordinates ζ_t and ζ_0.

In particular, one may change the parameter ζ_t to $\tilde{\zeta}_t = f_t(\zeta_t)$ with f_t any analytic bijection $U \to U$. This means that the parameter ζ_t is not intrinsic, and thus $\dot{x}$ and $\dot{y}$ are not intrinsically defined.

Instead, what is intrinsic is the following 1-form:

Proposition 7.3.1 *The meromorphic 1-form on* $\mathcal{L}$

$$\Omega(z) = \dot{x}(z)\; dy(z) - \dot{y}(z)\; dx(z)$$

is independent of a choice of a family of coordinates ζ_t *on* $\mathcal{L}_t$.

Proof Let $\xi_t : U_0 \to U_t$ defined by $\xi_t = \zeta_t^{-1} \circ \zeta_0$. If we change the local coordinates $\zeta_t \to f_t(\zeta_t)$, this changes $\xi_t \to g_t(\xi_t)$ where $g_t = \zeta_t^{-1} \circ f_t^{-1} \circ f_0 \circ \zeta_t$. This changes the time derivative:

$$\dot{x} \to \dot{x} + g_t'\, dx \qquad , \qquad \dot{y} \to \dot{y} + g_t'\, dy$$

and thus it changes $\Omega \to \Omega$, i.e. Ω is independent of a choice of coordinate. $\square$

Remark 7.3.1 An easy choice, is that away from branch–points, we may use $\zeta_t(z) = x_t(z)$ as a local coordinate on $\mathcal{L}_t$. In that case we have $\dot{x} = 0$, and we have

$$\Omega(z) = -\dot{y}(z)\, dx(z) = -\left.\frac{\partial y(z)}{\partial t}\right|_{x(z)=\text{constant}} dx(z).$$

Another way to state this proposition is that:

Proposition 7.3.2 *The tangent space to the space of spectral curves* $\{(\mathcal{L}, x, y)\}$ *(here we don't consider B for the moment), is homeomorphic to the space of* *meromorphic 1-forms on $\mathcal{L}$:*

$$T_*\{(\mathcal{L},x,y)\}|_{(\mathcal{L},x,y)} \sim \mathcal{M}^1(\mathcal{L}) = \{\Omega = \text{meromorphic } 1-\text{forms on } \mathcal{L}\}.$$

In other words the Lie derivative of a flow ∂_t, is equivalent to a meromorphic 1-form Ω on $\mathcal{L}$.

One may notice that the location of branchpoints a_i may be a function of t. Let $X_i = x(a_i)$ be the $x-$projection of the ith branchpoint a_i. Since $dx(a_i) = 0$ by definition, we have:

$$\frac{\partial X_i}{\partial t} = \dot{X}_i = \frac{\Omega(a_i)}{dy(a_i)}.$$

In particular, the projection of a branchpoint in the x-plane is constant only if $\Omega(a_i)$ vanishes.

If $\Omega(a_i) \neq 0$, we have $\dot{X}_i \neq 0$, and the conformal structure of the curve $\mathcal{L}$ changes.

A classical result in algebraic geometry of Riemann surfaces, is the **Rauch variational formula** [37, 56, 68, 78], which tells how the fundamental second kind form B changes under a change of conformal structure:

Proposition 7.3.3 (Rauch Formula)

$$\left.\frac{\partial B(z_1,z_2)}{\partial t}\right|_{x_t(z_1),x_t(z_2)\text{ constant}} = \dot{B}(z_1,z_2) = \sum_i \dot{X}_i \operatorname*{Res}_{z\to a_i} \frac{B(z,z_1)\,B(z,z_2)}{dx(z)}.$$

Moreover, since $\dot{X}_i = \frac{\Omega(a_i)}{dy(a_i)}$, and $dx(z)$ is assumed to have a simple zero at a_i, we may rewrite

$$\dot{B}(z_1,z_2) = \sum_i \operatorname*{Res}_{z\to a_i} \frac{\Omega(z)\,B(z,z_1)\,B(z,z_2)}{dx(z)\,dy(z)}. \tag{7.3.1}$$

Since we now know the variation of the spectral curve, and the variation of the fundamental second kind form B, we may deduce the variation of the recursion kernel $K(z_0,z)$, and by recursion, the variation of every $\omega_n^{(g)}$. This can be understood in a geometrical way as follows.

7.3.2 Form-Cycle Duality

Let us assume that $\Omega \in \mathcal{M}^1(\mathcal{L})$ is a meromorphic 1-form on $\mathcal{L}$. It may have poles α_i of some degrees d_i. It is customary to classify meromorphic forms into four kinds:

- **Exact forms**. $\Omega(z)$ is an exact form if and only if, for any closed contour γ on $\mathcal{C}$ avoiding the poles, we have $\oint_\gamma \Omega = 0$. In that case, there exists a meromorphic function $f(z)$ such that $\Omega(z) = df(z)$. Notice that, since $B(z, z')$ has a double pole at $z = z'$, we have:

$$df(z) = \operatorname*{Res}_{z' \to z} f(z')\, B(z, z') = -\sum_i \operatorname*{Res}_{z' \to \alpha_i} f(z')\, B(z, z').$$

- **First kind differential**. $\Omega(z)$ is said to be 1st kind, if it has no pole. On a Riemann surface $\mathcal{L}$ of genus $\bar{g}$, the vector space of 1st kind forms, is of dimension $\bar{g}$:

$$\mathcal{M}^{1,\,\text{first}} = H^1(\mathcal{L}, \mathbb{C}) \qquad , \qquad \dim\ H^1(\mathcal{L}, \mathbb{C}) = \bar{g}.$$

A basis is given by:

$$v_i(z) = \frac{1}{2\pi \mathrm{i}} \oint_{z' \in \mathcal{B}_i} B(z, z') \qquad , \qquad i = 1, \dots, \bar{g}\,.$$

This choice of basis is dual to the basis of $\mathcal{A}_i$ cycles:

$$\oint_{z \in \mathcal{A}_i} v_j(z) = \delta_{i,j} \qquad , \qquad i,j = 1, \dots, \bar{g}\,.$$

- **Third kind differentials**. $\Omega(z)$ is said to be third kind, if it has only simple poles. We may add any first kind form without changing that property, so, up to adding a first kind form, we will assume that Ω can be normalized on $\mathcal{A}$-cycles: $\oint_{\mathcal{A}_i} \Omega = 0$ for every for every $i = 1, \dots, \bar{g}$. Since the sum of all residues of a differential form must vanish, a third kind differential must have at least two poles. Let us denote by $\{p_i\}$ the set of poles of Ω, and $t_i = \operatorname{Res}_{p_i} \Omega$ the corresponding residues (and thus $\sum_i t_i = 0$). The following formal sum of points of $\mathcal{L}$

$$D = \sum_i t_i\,[p_i] \qquad , \qquad \deg D = \sum_i t_i = 0,$$

is called a divisor on $\mathcal{L}$. The set of divisors of degree zero is denoted

$$\mathrm{Div}_0(\mathcal{L}).$$

We thus have:

$$\mathcal{M}^{1,\,3\text{rd}}(\mathcal{L})/H^1(\mathcal{L}, \mathbb{C}) \sim \mathrm{Div}_0(\mathcal{L}).$$

A basis of $\mathrm{Div}_0(\mathcal{L})$, is the set of 2-points divisors with residues $+1$ and -1:

$$\text{basis of } \mathrm{Div}_0(\mathcal{L}) = \{\ [z_1] - [z_2]\ |\ (z_1, z_2) \in \mathcal{L} \times \mathcal{L}\,,\ z_1 \neq z_2\,\}.$$

The corresponding basis of third kind differentials is:

$$dS_{z_1,z_2}(z) = \int_{\gamma_{z_2 \to z_1}} B(z, z').$$

where the integration path $\gamma_{z_2 \to z_1}$ is the unique homology chain such that:

$$\partial \gamma_{z_2 \to z_1} = [z_1] - [z_2] \qquad , \qquad \forall\, i = 1 \ldots, \bar{g}, \quad \gamma_{z_2 \to z_1} \cap \mathcal{A}_i = 0 = \gamma_{z_2 \to z_1} \cap \mathcal{B}_i.$$

$dS_{z_1,z_2}(z)$ clearly has a simple pole at $z = z_1$, with residue $+1$, and a simple pole at $z = z_2$ with residue -1:

$$\operatorname*{Res}_{z \to z_1} dS_{z_1,z_2}(z) = 1 = - \operatorname*{Res}_{z \to z_2} dS_{z_1,z_2}(z) \qquad , \qquad \oint_{\mathcal{A}_i} dS_{z_1,z_2} = 0.$$

- **Second kind differentials**. They are everything remaining, i.e. $\Omega(z)$ is said to be second kind, if it has poles of degrees ≥ 2, with vanishing residues, and vanishing integrals around $\mathcal{A}_i$ cycles:

$$\operatorname*{Res}_{\alpha_i} \Omega = 0 \qquad , \qquad \oint_{\mathcal{A}_i} \Omega = 0.$$

One can prove that if $\Omega(z)$ is second kind, there always exist an analytic function $f(z)$, locally defined near the poles α_i (not necessary defined globally on $\mathcal{L}$), such that, $\forall\, z \in \mathcal{L}$:

$$\Omega(z) = \sum_i \operatorname*{Res}_{z' \to \alpha_i} B(z, z') f(z').$$

In the end, we see, that for any meromorphic form $\Omega \in \mathcal{M}^1(\mathcal{L})$, there exists an integration contour $\gamma_{\Omega^*} \subset \mathcal{L}$, and a function $f_\Omega(z)$, such that

$$\Omega(z) = \int_{z' \in \gamma_{\Omega^*}} B(z, z') f_\Omega(z').$$

The linear map $\mathcal{M}^1(\mathcal{L}) \to \mathbb{C}$, that associates to any 1-form ω its integral on γ_{Ω^*} with integrand f_{Ω^*}:

$$\omega \mapsto \int_{\gamma_{\Omega^*}} f_{\Omega^*}\ \omega$$

is an element of the dual $\mathcal{M}^1(\mathcal{L})^*$, it is a generalized cycle on $\mathcal{L}$, it is also called a "current".

Definition 7.3.1 A generalized cycle $\Omega^* = (\gamma_{\Omega^*}, f_{\Omega^*})$, is the data of small circle γ_{Ω^*} around a point, and a complex valued function f_{Ω^*} analytic on a vicinity of γ_{Ω^*}. The set of generalized cycles is called

$$\mathcal{M}_1(\mathcal{L})$$

There is a pairing to the space of meromorphic 1-forms $\mathcal{M}^1(\mathcal{L})$, by:

$$(\Omega_*, \omega) = \int_{\Omega^*} \omega \stackrel{\text{def}}{=} \int_{\gamma_{\Omega^*}} f_{\Omega^*}\ \omega.$$

$\mathcal{M}_1(\mathcal{L}) \subset \mathcal{M}^1(\mathcal{L})^*$, i.e. generalized cycles belong to the dual of meromorphic forms, but the dual may be bigger.

Remark 7.3.2 One may get rid of the function f_Ω, by changing the variable $z \to \zeta(z)$, whose jacobian cancels the $f_\Omega(z)$, and then write $\Omega(z) = \int_{z' \in \Omega^*} B(z, z')$ in the new variable ζ. This is why (Ω^*, f_Ω) is indeed a cycle.

This leads to the notion of form-cycle duality:

Definition 7.3.2 (Form-Cycle Duality)
The map

$$\hat{B} \quad : \quad \begin{cases} \mathcal{M}_1(\mathcal{L}) \to \mathcal{M}^1(\mathcal{L}) \\ \qquad \Omega^* \mapsto (\Omega^*, B) = \int_{\Omega^*(z')} B(z, z') \end{cases}$$

realizes an isomorphism

$$\mathcal{M}_1(\mathcal{L})/\mathrm{Ker}\,\hat{B} \sim \mathcal{M}^1(\mathcal{L}).$$

that we call "form cycle duality".

For any meromorphic 1-form $\Omega \in \mathcal{M}^1(\mathcal{L})$, we call $\Omega^* = \hat{B}^{-1}(\Omega) \in \mathcal{M}_1(\mathcal{L})$ modulo $\mathrm{Ker}\,\hat{B}$, its dual cycle.

A vector ∂_t of the tangent space of the space of spectral curves, is thus dual to a meromorphic 1-form Ω and to a cycle Ω^*:

$$\partial_t \in T_*\{(\mathcal{L}, x, y, B)\} \quad \leftrightarrow \quad \Omega \in \mathcal{M}^1(\mathcal{L}) \quad \leftrightarrow \quad \Omega^* \in \mathcal{M}_1(\mathcal{L})/\mathrm{Ker}\,\hat{B}.$$

Remark 7.3.3 There are two notions of form-cycle dualities here. The Poincaré duality induced by the integration pairing, and the one introduced here, induced by $\hat{B}$. Those 2 dualities define a "mixed Hodge structure", this is beyond the scope of this book.

7.3.3 Variation of $\omega_n^{(g)}$

Using the Rauch formula Eq. (7.3.1) for the variation of the fundamental second kind form B, we find that:

$$\begin{aligned}\partial_t B(z_1,z_2) &= \sum_i \operatorname*{Res}_{z\to a_i} \frac{B(z,z_1)\,B(z,z_2)}{dx(z)\,dy(z)} \int_{z'\in\Omega^*} B(z,z') \\ &= \int_{z'\in\Omega^*} \sum_i \operatorname*{Res}_{z\to a_i} \frac{B(z,z')\,B(z,z_1)\,B(z,z_2)}{dx(z)\,dy(z)} \\ &= \int_{z'\in\Omega^*} \omega_3^{(0)}(z',z_1,z_2)\end{aligned}$$

(we recall that the derivative ∂_t is taken with $x(z_i)$ kept constant, i.e. using $x(z_i)$'s as common local coordinates for all t.

We thus get:

Theorem 7.3.1 *The variation of $\omega_2^{(0)}$, is the integral of $\omega_3^{(0)}$ on the dual cycle to ∂_t:*

$$\partial_t \omega_2^{(0)}(z_1,z_2) = \int_{z'\in\Omega^*} \omega_3^{(0)}(z',z_1,z_2).$$

We shall generalize this theorem to any $\omega_n^{(g)}$. First, let us rewrite:

$$\begin{aligned}\partial_t B(z_1,z_2) &= \int_{z'\in\Omega^*} \omega_3^{(0)}(z',z_1,z_2) \\ &= \int_{z'\in\Omega^*} \sum_i \operatorname*{Res}_{z\to a_i} K(z_1,z)\,(B(z,z')\,B(\bar z,z_2) + B(z,z_2)\,B(\bar z,z')).\end{aligned} \tag{7.3.2}$$

Then, since

$$K(z_0,z) = \frac{\int_{z'=\bar z}^{z} B(z_0,z')}{2(y(z)-y(\bar z))\,dx(z)},$$

and since we know the derivatives of B and y and x, we find (we leave it to the reader, or otherwise look in [34]), that for any quadratic differential $Q(z,t)$ defined in the vicinity of branchpoints a_i and such that $Q(\bar{z},t) = Q(z,t)$, that

$$\partial_t \sum_i \operatorname*{Res}_{z\to a_i} K(z_0,z)\, Q(z,t) = \sum_i \operatorname*{Res}_{z\to a_i} K(z_0,z)\, \partial_t Q(z,t) + \sum_{i,j} \operatorname*{Res}_{z\to a_i} \operatorname*{Res}_{z'\to a_j} K(z_0,z')\, K(z',z)\, Q(z,t)\ \Omega(z'). \tag{7.3.3}$$

The topological recursion takes that form with

$$Q(z,t) = \omega_{n+1}^{(g-1)}(z,\bar{z},z_2,\dots,z_n) + \sum'_{h+h'=g,\, I\sqcup I'=\{z_2,\dots,z_n\}} \omega^{(h)}_{1+\|I\|}(z,I)\omega^{(h')}_{1+\|I'\|}(\bar{z},I'),$$

and then, by an easy recursion on $2g-2+n$, we find:

Theorem 7.3.2 (Form-Cycle Duality Variation) *The variation of $\omega_n^{(g)}$, is the integral of $\omega_{n+1}(g)$ on the dual cycle to ∂_t:*

$$\partial_t \omega_n^{(g)}(z_1,\dots,z_n) = \int_{z'\in\Omega^*} \omega_{n+1}^{(g)}(z',z_1,\dots,z_n),$$

where we recall that derivatives are taken with $x(z_i)$'s kept constant.

In particular, for $n=0$:

$$\partial_t \mathcal{F}_g = \int_{z'\in\Omega^*} \omega_1^{(g)}(z').$$

Notice that the case $n=1, g=0$

$$\partial_t \omega_1^{(0)} = \int_{\Omega^*} B = \hat{B}(\Omega^*),$$

is true by definition of Ω^*.

The case $n=0, g=0$ amounts to

$$\partial_t \mathcal{F}_0 = \int_{\Omega^*} y\,dx.$$

Application: Prepotential

Let us specialize this theorem further.

Let us denote

$$\epsilon_i = \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{A}_i} ydx.$$

If we want to vary ϵ_i, while keeping all other ϵ_js unchanged, and all poles of ydx (and all the negative part of their Laurent series expansion near poles) unchanged, we find that $\Omega(z)$ must have no pole, and that

$$\frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{A}_j} \Omega = \frac{\partial}{\partial \epsilon_i} \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{A}_j} ydx = \frac{\partial}{\partial \epsilon_i} \epsilon_j = \delta_{i,j}$$

thus, $\Omega(z) = 2\mathrm{i}\pi v_i(z)$ is a first kind differential, and is dual to the cycle $\mathcal{B}_i$

$$\Omega(z) = 2\pi \mathrm{i}\, v_i(z) = \oint_{z' \in \mathcal{B}_i} B(z, z').$$

Theorem 7.3.2 thus gives

$$\frac{\partial \omega_n^{(g)}(z_1, \ldots, z_n)}{\partial \epsilon_i} = \oint_{z' \in \mathcal{B}_i} \omega_{n+1}^{(g)}(z', z_1, \ldots, z_n) \qquad , \qquad \frac{\partial \mathcal{F}_g}{\partial \epsilon_i} = \oint_{\mathcal{B}_i} \omega_1^{(g)}.$$

And in particular:

$$\frac{\partial \mathcal{F}_0}{\partial \epsilon_i} = \oint_{\mathcal{B}_i} ydx \qquad , \qquad \text{where} \quad \epsilon_i = \frac{1}{2i\pi} \oint_{\mathcal{A}_i} ydx.$$

This property of $\mathcal{F}_0$ is the characterization of the prepotential in Seiberg-Witten theory. This is why, we claim that $\mathcal{F}_0$ **is the prepotential**.

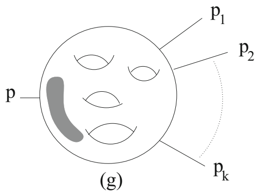
Remark 7.3.4 Theorem 7.3.2 is often referred to as "special geometry" in the context of string theory. As we have just seen, it is a generalization of the Seiberg–Witten equation when we restrict to Ω being first kind forms.

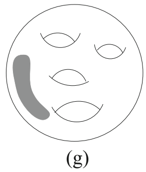
If we restrict to Ω being second kind forms, we would find that it is also a generalization of the Miwa-Jimbo tau function, but all this is beyond the scope of this book.

7.4 Diagrammatic Representation

The recursive definitions of $\omega_k^{(g)}$ and $\mathcal{F}_g$ can be represented **graphically**.

We represent the multilinear form $\omega_k^{(g)}(p_1,\dots,p_k)$ as a blob-like "surface" with g holes and k legs (or punctures) labeled with the variables $p_1,\dots,p_k$, and $\mathcal{F}_g = \omega_0^{(g)}$ with 0 legs and g holes.

$\omega^{(g)}_{k+1}(p,p_1,\dots,p_k) :=$ 

, $\mathcal{F}_g :=$ 

We represent the fundamental second kind form $B(p,q)$ (which is also $\omega_2^{(0)}$, i.e. a blob with two legs and no hole) as a straight non-oriented line between p and q

$$B(p,q) := \text{p} \,\text{———}\, \text{q}\ .$$

We represent the recursion kernel $K(p,q)$ as a straight arrowed line with the arrow from p towards q, and with a planar tri-valent vertex whose left leg is q and right leg is $\overline{q}$

$K(p,q) :=$ 

.

7.4.1 Graphs

Definition 7.4.1 For any $k \geq 0$ and $g \geq 0$ such that $k + 2g \geq 3$, we define:

Let $\mathcal{G}^{(g)}_{k+1}(p,p_1,\dots,p_k)$ be the set of connected trivalent graphs defined as follows:

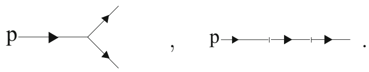
1. there are $2g + k - 1$ tri-valent vertices called vertices.
2. there is one 1-valent vertex labelled by p, called the root.
3. there are k 1-valent vertices labelled with $p_1,\dots,p_k$ called the leaves.
4. There are $3g + 2k - 1$ edges.
5. Edges can be arrowed or non-arrowed. There are $k + g$ non-arrowed edges and $2g + k - 1$ arrowed edges.
6. The edge starting at p has an arrow leaving from the root p.
7. The k edges ending at the leaves $p_1,\dots,p_k$ are non-arrowed.

8. The arrowed edges form a "spanning[2] planar[3] binary skeleton[4] tree" with root p. The arrows are oriented from root towards leaves. In particular, this induces a partial ordering of all vertices.
9. There are k non-arrowed edges going from a vertex to a leaf, and g non arrowed edges joining two inner vertices. Two inner vertices can be connected by a non arrowed edge only if one is the parent of the other along the tree.
10. If an arrowed edge and a non-arrowed inner edge come out of a vertex, then the arrowed edge is the left child. This rule only applies when the non-arrowed edge links this vertex to one of its descendants (not one of its parents).

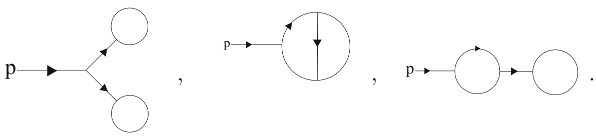
7.4.2 *Example of* $\mathcal{G}_1^{(2)}(p)$

As an example, let us build step by step all the graphs of $\mathcal{G}_1^{(2)}(p)$, i.e. $g = 2$ and $k = 0$.

We first draw all planar binary skeleton trees with one root p and $2g + k - 1 = 3$ arrowed edges:



Then, we draw $g + k = 2$ non-arrowed edges in all possible ways such that every vertex is trivalent, also satisfying rule 9) of Definition 7.4.1. There is only one possibility for the first graph and two for the second one:



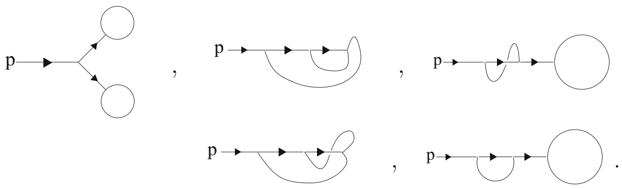
It just remains to specify the left and right children for each vertex. The only possibilities in accordance with rule 10) of Definition 7.4.1 are[5]:

[2]It goes through all vertices.

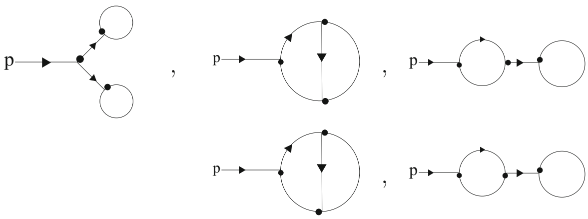
[3]planar tree means that the left child and right child are not equivalent. The right child is marked by a black disk on the outgoing edge.

[4]a binary skeleton tree is a binary tree from which we have removed the leaves, i.e. a tree with vertices of valence 1, 2 or 3.

[5] Note that the graphs are not necessarily planar.



In order to simplify the drawing, we can draw a black dot to specify the right child. This way one gets only planar graphs.



Remark that without the prescriptions 9) and 10), one would get 13 different graphs whereas we only have five.

7.4.3 *Weight of a Graph*

Consider a graph $G \in \mathcal{G}_{k+1}^{(g)}(p, p_1, \dots, p_k)$. Then, to each vertex $i = 1, \dots, 2g+k-1$ of G, we associate a label q_i, and we associate q_i to the beginning of the left child edge, and $\overline{q}_i$ to the right child edge. Thus, each edge (arrowed or not), links two labels which are points on the spectral curve $\mathcal{L}$.

- To an arrowed edge going from q' towards q, we associate a factor $K(q', q)$.
- To a non arrowed edge going between q' and q we associate a factor $B(q', q)$.
- Following the arrows backwards (i.e. from leaves to root), for each vertex q, we take the sum over all branchpoints a_i of residues at $q \to a_i$.

After taking all the residues, we get the weight of the graph:

$$w(G)$$

which is a k-form of $(p, p_1, \dots, p_k) \in \mathcal{L}^{k+1}$.

Similarly, we define weights of linear combinations of graphs by linearity:

$$w(\alpha G_1 + \beta G_2) = \alpha w(G_1) + \beta w(G_2)$$

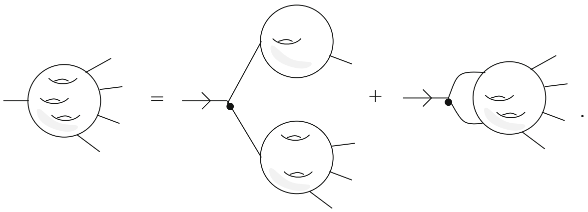
and for a disconnected graph, i.e. a product of two graphs:

$$w(G_1 \cup G_2) = w(G_1)\, w(G_2).$$

Theorem 7.4.1 *We have:*

$$\omega_{k+1}^{(g)}(p, p_1, \ldots, p_k) = \sum_{G \in \mathcal{G}_{k+1}^{(g)}(p,p_1,\ldots,p_k)} w(G) = w\left(\sum_{G \in \mathcal{G}_{k+1}^{(g)}(p,p_1,\ldots,p_k)} G \right)$$

Proof This is a mere rewriting of the definition. This encodes precisely what the recursion equations (7.1.1) of Definition 7.1.6 are doing. Indeed, one can represent them diagrammatically by



□

Such graphical notations are very convenient, and are a good support for intuition and even help proving some relationships. It was immediately noticed after [30] that those diagrams look very much like Feynman graphs, and there was a hope that they could be the Feynman's graphs for the Kodaira–Spencer quantum field theory. But they ARE NOT Feynman graphs, because Feynman graphs can't have non-local restrictions like the fact that non oriented lines can join only a vertex and one of its descendent.

Those graphs are merely a notation for the recursive definition (7.1.1).

Lemma 7.4.1 (Symmetry Factor) *The weight of two graphs differing by the exchange of the right and left children of a vertex are the same. Indeed, the distinction between right and left child is just a way of encoding symmetry factors. We could restrict ourselves to only topologically different graphs, weighted with a symmetry factor which would be a power of* 2.

Proof This property follows directly from the fact that $K(z_0, z) = K(z_0, \bar{z})$. □

7.4.4 *Examples*

Let us present some examples of computations of $\omega_k^{(g)}$ and $\mathcal{F}_g$ for low values of (g,k).

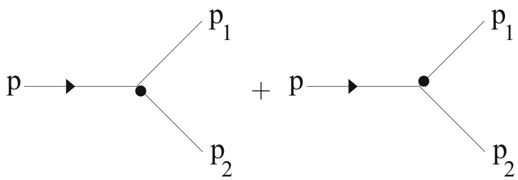
7.4.4.1 Correlators

- $(g,k) = (0,2)$.

$$\omega_2^{(0)}(p,q) = B(p,q).$$

- $(g,k) = (0,3)$.

$\omega_3^{(0)}(p,p_1,p_2) =$

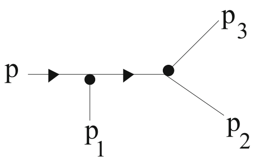


$$\begin{aligned}
&= \operatorname*{Res}_{q\to\mathbf{a}} K(p,q)\ [B(q,p_1)B(\overline{q},p_2) + B(\overline{q},p_1)B(q,p_2)] \\
&= -2 \operatorname*{Res}_{q\to\mathbf{a}} K(p,q)\ [B(q,p_1)B(q,p_2)] \\
&= \operatorname*{Res}_{q\to\mathbf{a}} \frac{B(q,p)\,B(q,p_1)\,B(q,p_2)}{dx(q)\,dy(q)} \\
&= \sum_i \frac{B(a_i,p)\,B(a_i,p_1)\,B(a_i,p_2)}{2\,dy(a_i)\,dz_i(a_i)^2} \\
&= \sum_i \frac{1}{2y'(a_i)}\,\frac{B(a_i,p)}{dz_i(a_i)}\,\frac{B(a_i,p_1)}{dz_i(a_i)}\,\frac{B(a_i,p_2)}{dz_i(a_i)}
\end{aligned}$$

where $z_i(z) = \sqrt{x(z) - z(a_i)}$ is the canonical local coordinate near a_i.

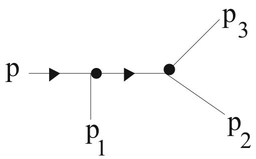
- $(g,k) = (0,4)$.

$$\omega_4^{(0)}(p,p_1,p_2,p_3) \quad =$$



$+ \quad$ perm. (1,2,3)

$+$

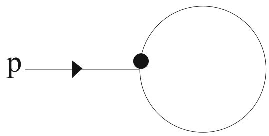


$+ \quad$ perm. (1,2,3)

$$\begin{aligned}
= & \operatorname*{Res}_{q\to\mathbf{a}} \operatorname*{Res}_{r\to\mathbf{a}} K(p,q)\,K(q,r)\ [B(\overline{q},p_1)B(r,p_2)B(\overline{r},p_3) \\
& + B(\overline{q},p_1)B(\overline{r},p_2)B(r,p_3) + B(\overline{q},p_2)B(r,p_1)B(\overline{r},p_3) \\
& + B(\overline{q},p_2)B(\overline{r},p_1)B(r,p_3) + B(\overline{q},p_3)B(r,p_2)B(\overline{r},p_1) \\
& + B(\overline{q},p_3)B(\overline{r},p_2)B(r,p_1)] \\
& + \operatorname*{Res}_{q\to\mathbf{a}} \operatorname*{Res}_{r\to\mathbf{a}} K(p,q)\,K(\overline{q},r)\ [B(q,p_1)B(r,p_2)B(\overline{r},p_3) \\
& + B(q,p_1)B(\overline{r},p_2)B(r,p_3) + B(q,p_2)B(r,p_1)B(\overline{r},p_3) \\
& + B(q,p_2)B(\overline{r},p_1)B(r,p_3) + B(q,p_3)B(r,p_2)B(\overline{r},p_1) \\
& + B(q,p_3)B(\overline{r},p_2)B(r,p_1)]
\end{aligned}$$

- $(g,k) = (1,1)$.

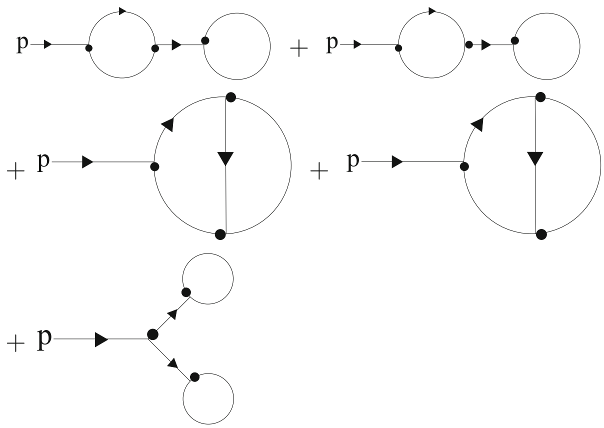
$$\omega_1^{(1)}(p) \quad =$$



$$= \operatorname*{Res}_{q\to\mathbf{a}} K(p,q)\,B(q,\overline{q})$$

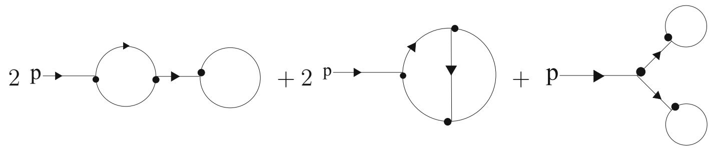
- $(g, k) = (2, 1)$.

$\omega_1^{(2)}(p) =$



$$= \operatorname*{Res}_{q\to\mathbf{a}} \operatorname*{Res}_{r\to\mathbf{a}} \operatorname*{Res}_{s\to\mathbf{a}} K(p,q)K(q,r)K(\overline{q},s)\; B(r,\overline{r})B(s,\overline{s})$$
$$+ \operatorname*{Res}_{q\to\mathbf{a}} \operatorname*{Res}_{r\to\mathbf{a}} \operatorname*{Res}_{s\to\mathbf{a}} K(p,q)K(q,r)K(r,s)\; B(r,\overline{q})B(s,\overline{s})$$
$$+ \operatorname*{Res}_{q\to\mathbf{a}} \operatorname*{Res}_{r\to\mathbf{a}} \operatorname*{Res}_{s\to\mathbf{a}} K(p,q)K(q,r)K(r,s)\; [B(\overline{q},\overline{r})B(s,\overline{s})$$
$$+B(\overline{s},\overline{q})B(s,\overline{r}) + B(s,\overline{q})B(\overline{s},\overline{r})]$$

$=$



where the last expression is obtained using Lemma 7.4.1.

- $(g, k) = (2, 0)$.

$$-2\mathcal{F}_2 = \operatorname*{Res}_{p\to\mathbf{a}} \operatorname*{Res}_{q\to\mathbf{a}} \operatorname*{Res}_{r\to\mathbf{a}} \operatorname*{Res}_{s\to\mathbf{a}} \Phi(p)\, K(p,q)\, K(q,r)K(\overline{q},s)\; B(r,\overline{r})B(s,\overline{s})$$
$$+ \operatorname*{Res}_{p\to\mathbf{a}} \operatorname*{Res}_{q\to\mathbf{a}} \operatorname*{Res}_{r\to\mathbf{a}} \operatorname*{Res}_{s\to\mathbf{a}} \Phi(p)K(p,q)K(q,r)K(r,s)\; B(r,\overline{q})B(s,\overline{s})$$
$$+ \operatorname*{Res}_{p\to\mathbf{a}} \operatorname*{Res}_{q\to\mathbf{a}} \operatorname*{Res}_{r\to\mathbf{a}} \operatorname*{Res}_{s\to\mathbf{a}} \Phi(p)K(p,q)K(q,r)K(r,s)\; [B(\overline{q},\overline{r})B(s,\overline{s})$$
$$+B(\overline{s},\overline{q})B(s,\overline{r}) + B(s,\overline{q})B(\overline{s},\overline{r})]$$

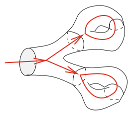
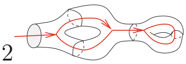
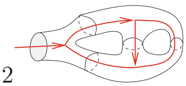
where $d\Phi = y\,dx$.

7.4.5 Remark: Pants Gluings

Every Riemann surface of genus g with k boundaries can be decomposed into $2g-2+k$ pants whose boundaries are $3g-3+k$ closed geodesics (in the Poincaré metric with constant negative curvature -1). The number of ways of gluing $2g-2+k$ pants by their boundaries is clearly the same as the number of diagrams of $\mathcal{G}_k^{(g)}$, and each diagram corresponds to one pant decomposition.

Indeed, consider the first boundary labelled by z_1, and attach a pair of pant to this boundary. Draw an arrowed propagator from the boundary to the first pant. Then, choose one of the other boundaries of the pair of pants (there are thus two choices), it must be glued to another pair of pants (possibly not distinct from the first one). If this pair of pants was never visited, draw an arrowed propagator, and if it was already visited, draw a non-arrowed propagator. In the end, you get a diagram of $\mathcal{G}_k^{(g)}$. This procedure is bijective, and to a diagram of $\mathcal{G}_k^{(g)}$, one may associate a gluing of pants.

Example with $k = 1$ and $g = 2$:

$\omega_1^{(2)} =$  $+\,2$  $+\,2$ 

7.5 Exercises

Exercise 1 Compute $\mathcal{F}_0$ for the Kontsevich's spectral curve:

$$\tilde{\mathcal{E}}_K \ : \qquad \begin{cases} x(z) = z^2 \\ y(z) = z + \frac{1}{2N}\sum_{i=1}^{N} \frac{1}{\Lambda_i\,(z-\Lambda_i)}. \end{cases}$$

Hint: there are $N+1$ poles: $\alpha_0 = \infty$, and $\alpha_i = \Lambda_i$ for $i = 1, \dots, N$. The potentials are $V_0(z) = \frac{2}{3}z^3 = \frac{2}{3}x(z)^{\frac{3}{2}}$, and $V_i(z) = 0$ for $i = 1, \dots, N$.

Exercise 2 Compute $\mathcal{F}_0$ and $\mathcal{F}_1$ for the $(2m+1, 2)$ curve:

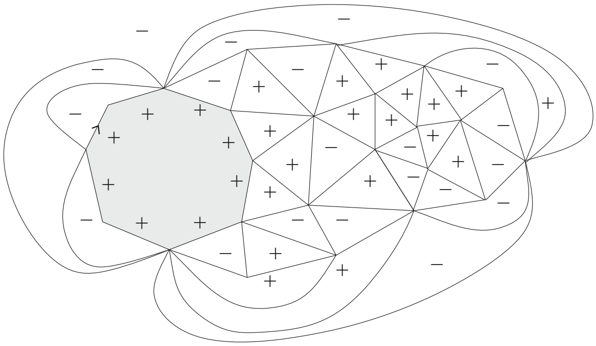
$$\tilde{\mathcal{E}}_K \ : \qquad \begin{cases} x(z) = z^2 - 2u \\ y(z) = \sum_{k=0}^{m} \tilde{t}_k Q_k(z) \end{cases}$$

where $Q_k(z) = (z^2 - 2u)_+^{k+1/2}$, defined in Eq. (5.2.3).

Chapter 8
Ising Model

In statistical physics, the Ising model represents a simplified model for magnetization. Each piece of the surface (here each face of a map) carries a unit of magnetization, pointing either upward $+$ or downward $-$. This can also be represented as a map with bicolored faces black/white, or $+/-$, or any other convenient choice. The color is also called the spin, worth $+$ or $-$.

Our goal is to put the Ising model on a random map, i.e. study the generating functions counting bi-colored maps.



In this chapter, we extend the previous method of Tutte's equations and its solution by topological recursion, to bicolored maps, i.e. Ising model maps. The method is more or less the same: define generating functions as formal series in t^v where v is the number of vertices, then write loop equations (generalization of Tutte's equations), and then solve loop equations.

The loop equation for the disc, is an algebraic equation, and thus the disc amplitude is an algebraic function, called the "spectral curve".

Then, once we know the spectral curve, all the other generating functions (called amplitudes) can be computed by the topological recursion of Chap. 7. It may look surprising that the same topological recursion which solves the loop equations for non-colored maps, also solves the more intricated loop equations of the Ising model.

B. Eynard, *Counting Surfaces*, Progress in Mathematical Physics 70,
DOI 10.1007/978-3-7643-8797-6_8

In fact, this same topological recursion also solves the loop equations of many other enumerative geometry problems, like $O(\mathfrak{n})$-model on random maps, or Potts model on random maps.

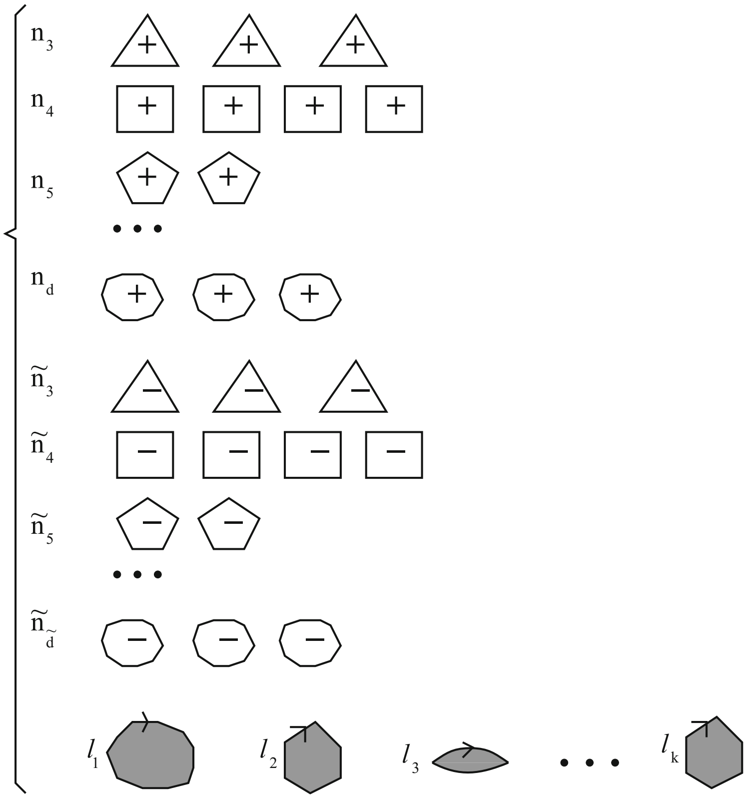
In this chapter, we don't present all computations in details, we just give the definitions of the model, and the Tutte-like equations, and then the solution without detailed proof.

The main new feature compared to maps, is that we also compute generating functions for maps having multi-coloured boundaries. For such boundary generating functions, we merely state the main results, without proofs (proofs are found in the literature).

8.1 Bicolored Maps

The Ising model is a model of maps carrying two possible "colors" or two possible "spins" $+$ or $-$. The unmarked faces can carry a spin $+$ or $-$. Here, spin means color, the spin can take two values $+$ or $-$.

Our maps are constructed by gluing the following sorts of oriented polygonal pieces, marked on unmarked:

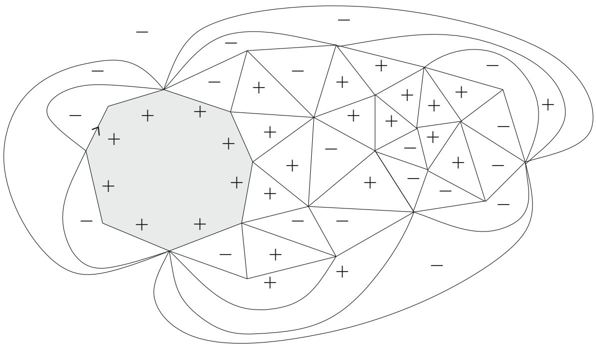


Unmarked faces are required to have degree at least 3. And for the moment, we consider that marked faces carry only spin $+$, and as usual, marked faces may have any degree, and must have a marked edge.

Definition 8.1.1 The set $\mathbb{M}_k^{(g)}(v)$ is defined to be the set of connected oriented Ising maps of given genus g, with given number of marked faces k, and given number of vertices v, which are obtained by gluing those (oriented) pieces together.

Like for uncolored maps, one easily proves, by computing the Euler characteristics, that this is a finite set.

Example of a typical map contributing to $\mathbb{M}_1^{(0)}$, it is a planar triangulation, with only one marked face of perimeter $l_1 = 8$:



We wish to enumerate those maps, recording numbers of all kinds of faces, and also recording the numbers of edges gluing faces of the same color $++$ or $--$, or of different colors $+-$.

Definition 8.1.2 We define the generating function

$$
\begin{aligned}
& W_k^{(g)}(x_1,\dots,x_k;t_3,\dots,t_d,\tilde t_3,\dots,\tilde t_{\tilde d},c_{++},c_{--},c_{+-};t) \\
&= \frac{t\,\delta_{k,1}\delta_{g,0}}{x_1} \\
&\quad + \sum_{v=1}^{\infty} t^v \sum_{\Sigma\in\mathbb{M}_k^{(g)}(v)} \frac{t_3^{n_3(\Sigma)}\,t_4^{n_4(\Sigma)}\ \dots\ t_d^{n_d(\Sigma)}\ \tilde t_3^{\tilde n_3(\Sigma)}\,\tilde t_4^{\tilde n_4(\Sigma)}\ \dots\ \tilde t_{\tilde d}^{\tilde n_{\tilde d}(\Sigma)}}{x_1^{1+l_1(\Sigma)}\ x_2^{1+l_2(\Sigma)}\ \dots\ x_k^{1+l_k(\Sigma)}} \\
&\qquad\qquad \frac{1}{\#\mathrm{Aut}(\Sigma)}\ c_{++}^{n_{++}(\Sigma)}\,c_{--}^{n_{--}(\Sigma)}\,c_{+-}^{n_{+-}(\Sigma)}
\end{aligned}
$$

where $n_{ij}(\Sigma)$ is the number of edges separating two faces of colors i and j.

$W_k^{(g)}(x_1,\dots,x_k;t_3,\dots,t_d,\tilde{t}_3,\dots,\tilde{t}_{\tilde{d}},c_{++},c_{--},c_{+-};t)$ is a formal power series in powers of t:

$$W_k^{(g)} \in \mathbb{Q}[\{1/x_i\},\{t_k\},\{\tilde{t}_k\},c_{++},c_{--},c_{+-}][[t]].$$

As usual, we write only the x_i dependence explicitly, and for short we shall write:

$$W_k^{(g)}(x_1,\dots,x_k;t_3,\dots,t_d,\tilde{t}_3,\dots,\tilde{t}_{\tilde{d}},c_{++},c_{--},c_{+-};t) = W_k^{(g)}(x_1,\dots,x_k)$$

and

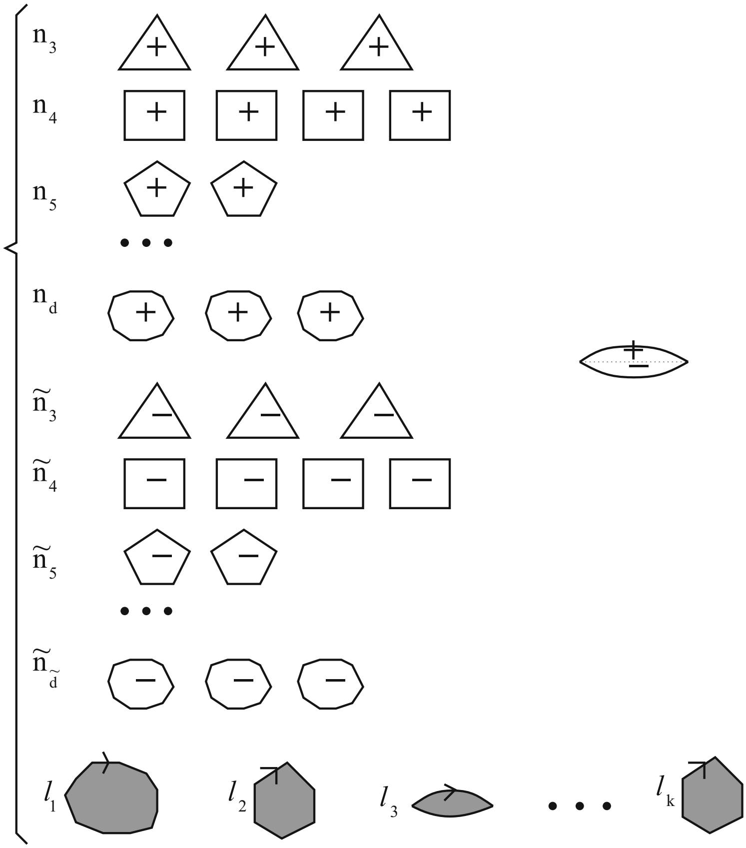
$$F_g = W_0^{(g)}.$$

Notice, that since a face can be glued to itself, the two faces on both sides of an edge, may be not distinct.

It is not so easy to write directly some Tutte-like equations for $W_k^{(g)}$, by removing the marked edge recursively on those maps. In fact, it is much easier to first consider a slightly different set of maps.

8.1.1 Reformulation

Instead of the previous Ising model, let us introduce another model. Consider maps, whose unmarked pieces can be of spin $+$ or $-$, and also with some bicolored pieces of degree 2. We add the constraint that edges can be glued together along an edge only if the spin is the same on both sides of the edge.



Definition 8.1.3 We define a generating function, with a weight c for that new piece, as well as a weight $1/a$ per number of $++$ edges, and $1/b$ per number of $--$ edges:

$$
\begin{aligned}
&\hat{W}_k^{(g)}(x_1,\dots,x_k;t_3,\dots,t_d,\tilde{t}_3,\dots,\tilde{t}_{\tilde{d}},a,b,c;t)\\
&= \frac{t\,\delta_{k,1}\delta_{g,0}}{x_1}\\
&\quad + \sum_{v=1}^{\infty} t^v \sum_{\Sigma\in\hat{\mathbb{M}}_k^{(g)}(v)} \frac{t_3^{n_3(\Sigma)}\,t_4^{n_4(\Sigma)}\,\dots\,t_d^{n_d(\Sigma)}\,\tilde{t}_3^{\tilde{n}_3(\Sigma)}\,\tilde{t}_4^{\tilde{n}_4(\Sigma)}\,\dots\,\tilde{t}_{\tilde{d}}^{\tilde{n}_{\tilde{d}}(\Sigma)}}{x_1^{1+l_1(\Sigma)}\,x_2^{1+l_2(\Sigma)}\,\dots\,x_k^{1+l_k(\Sigma)}}\\
&\qquad \frac{1}{\#\mathrm{Aut}(\Sigma)}\;a^{-n_{++}(\Sigma)}\,b^{-n_{--}(\Sigma)}\,c^{\hat{n}(\Sigma)}
\end{aligned}
$$

where $\hat{n}(\Sigma)$ is the number of bicolored pieces.

In this definition, the coefficient of t^v is a formal power series in c (indeed $\hat{\mathbb{M}}_k^{(g)}(v)$ is not a finite set, because we can glue together as many bicolored pieces as we wish without changing the number of vertices, but for each power of c, there is a finite number of maps):

$$\hat{W}_k^{(g)} \in \mathbb{Q}[\{1/x_i\}, \{t_k\}, \{\tilde{t}_k\}, 1/a, 1/b][[c]][[t]].$$

The reason why we have introduced this model, is because it coincides with the Ising model:

Theorem 8.1.1 *The generating function $\hat{W}_n^{(g)}$ of this model coincides with the generating function $W_n^{(g)}$ of the Ising model,*

$$\begin{aligned} W_k^{(g)}&(x_1, \dots, x_k; t_3, \dots, t_d, \tilde{t}_3, \dots, \tilde{t}_{\tilde{d}}, c_{++}, c_{--}, c_{+-}; t) \\ &= \hat{W}_k^{(g)}(x_1, \dots, x_k; t_3, \dots, t_d, \tilde{t}_3, \dots, \tilde{t}_{\tilde{d}}, a, b, c; t). \end{aligned}$$

with the identification:

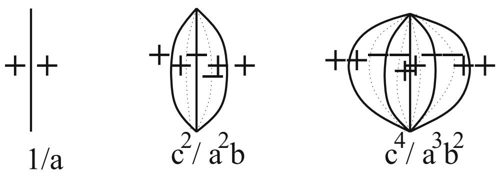
$$\begin{pmatrix} c_{++} & c_{+-} \\ c_{+-} & c_{--} \end{pmatrix}^{-1} = \begin{pmatrix} a & -c \\ -c & b \end{pmatrix}.$$

i.e.

$$c_{++} = \frac{b}{ab - c^2} \quad , \quad c_{--} = \frac{a}{ab - c^2} \quad , \quad c_{+-} = \frac{c}{ab - c^2}.$$

Proof The sum over powers of c, is a geometrical series and can be performed explicitly.

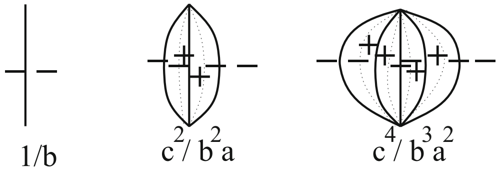
We may glue several bicolored pieces so that both external sides have spin $+$:



that corresponds to an effective $++$ edge gluing weight:

$$c_{++} = \frac{1}{a} \sum_k \frac{c^{2k}}{a^k\, b^k} = \frac{b}{ab - c^2}.$$

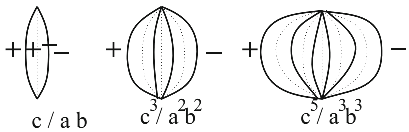
Similarly we may glue such pieces so that both external sides have spin −:



which corresponds to an effective −− edge gluing weight:

$$c_{--} = \frac{1}{b} \sum_k \frac{c^{2k}}{a^k\, b^k} = \frac{a}{ab - c^2}.$$

And Similarly we may glue such pieces so that external sides have spin + and −:



which corresponds to an effective +− edge gluing weight:

$$c_{+-} = \frac{c}{ab} \sum_{k=0}^{\infty} \frac{c^{2k}}{a^k\, b^k} = \frac{c}{ab - c^2}.$$

Finally we recognize the matrix relationship

$$\begin{pmatrix} c_{++} & c_{+-} \\ c_{+-} & c_{--} \end{pmatrix}^{-1} = \begin{pmatrix} a & -c \\ -c & b \end{pmatrix}.$$

□

8.2 Tutte-Like Equations

Definition 8.2.1 We define the generating function of maps of genus g with n marked faces of given perimeters:

$$\mathcal{T}^{(g)}_{l_1,\dots,l_n} = (-1)^n \operatorname{Res}\; x_1^{l_1} \dots x_n^{l_n}\; W_n^{(g)}(x_1,\dots,x_n)\, dx_1 \dots dx_n.$$

We have:

$$W_n^{(g)}(x_1,\dots,x_n)=\sum_{l_1,\dots,l_n}\frac{\mathcal{T}^{(g)}_{l_1,\dots,l_n}}{x_1^{l_1+1}\dots x_n^{l_n+1}},\tag{8.2.1}$$

as usual this equality is an equality of formal power series in t, and for each power of t, the sum over $l_1,\dots,l_n$ is a finite sum.

Definition 8.2.2 Let us also define $\hat{\mathcal{T}}^{(g)}_{l,k;l_1,\dots,l_n}$ to be the generating function of maps of genus g, and $n+1$ marked faces. n of the marked faces are usual marked faces carrying spin $+$, they have degrees l_i, $i=1,\dots,n$, and one marked face, is of degree $l+k$, so that there are l consecutive edges gluing to spin $+$, and k consecutive edges gluing to spin $-$. If $k\geq 1$, the marked edge on that marked face can always be assumed to be the first $+$ edge on the right side of a $-$ edge.

Similarly, we define $G^{(g)}_{n,k}(x;x_1,\dots,x_n)$ to be the generating series where we sum over perimeter of marked faces weighted by $x_i^{-l_i-1}$ and x^{-l-1}. We have:

$$\hat{\mathcal{T}}^{(g)}_{l,k;l_1,\dots,l_n}=(-1)^{n+1}\ \mathrm{Res}\ \ x^l\ x_1^{l_1}\dots x_n^{l_n}\ G^{(g)}_{n,k}(x;x_1,\dots,x_n)\,dx\,dx_1\dots dx_n$$

(notice that we don't sum over k), and

$$G^{(g)}_{n,k}(x;x_1,\dots,x_n)=\sum_{l,l_1,\dots,l_n}\frac{\hat{\mathcal{T}}^{(g)}_{l,k;l_1,\dots,l_n}}{x^{l+1}\ x_1^{l_1+1}\dots x_n^{l_n+1}}.\tag{8.2.2}$$

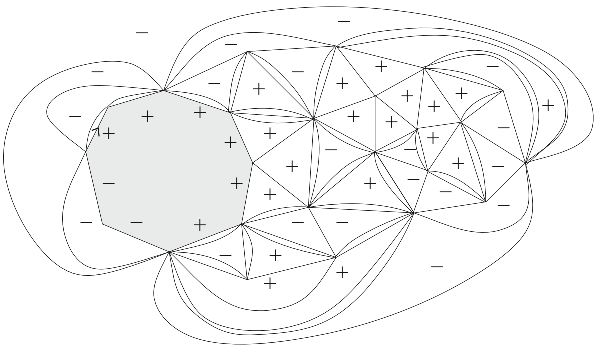
Also, if $k=0$, we recover:

$$\hat{\mathcal{T}}^{(g)}_{l,0;l_1,\dots,l_n}=\mathcal{T}^{(g)}_{l,l_1,\dots,l_n}$$

and

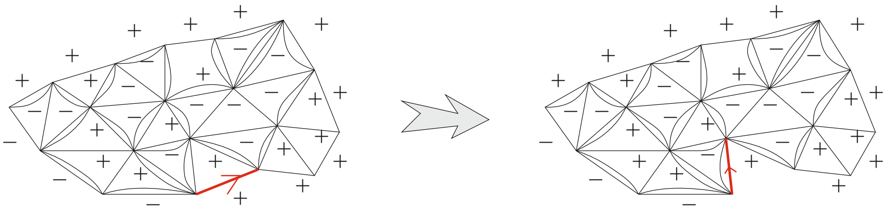
$$G^{(g)}_{n,0}(x;x_1,\dots,x_n)=W^{(g)}_{n+1}(x,x_1,\dots,x_n).$$

For example, here is a typical map contributing to $\hat{\mathcal{T}}^{(0)}_{6,2}$:

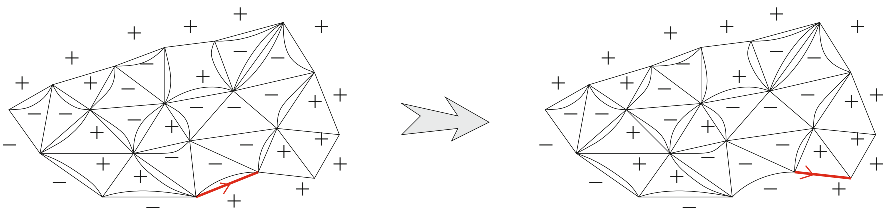


Consider a map contributing to $\hat{\mathcal{T}}^{(g)}_{l+1,k;l_1,\dots,l_n}$. On the other side of the marked edge (which is a $+$ edge), there can be either:

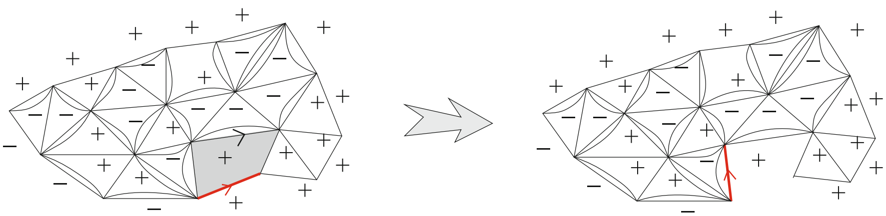
- an unmarked spin $+$ face of degree j, with $3 \le j \le d$, and removing the marked edge gives a map of $\hat{\mathcal{T}}^{(g)}_{l+j-1,k;l_1,\dots,l_n}$ weighted by t_j/a.



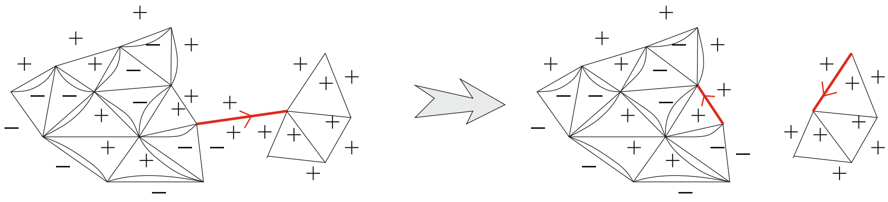
- a bicolored face $(+-)$, and removing the marked edge gives a map of $\hat{\mathcal{T}}^{(g)}_{l,k+1;l_1,\dots,l_n}$ weighted by c/a.



- the ith marked face of degree l_i, and removing the marked edge gives a map of $\hat{\mathcal{T}}^{(g)}_{l+l_i-1,k;l_1,\dots,\not{l_i},\dots,l_n}$ weighted by l_i/a.



- the same marked face. In that case, removing the marked edge either disconnects the map into two maps, or if there was a handle relating the two sides, it gives a map of lower genus $g-1$ and one more boundary.

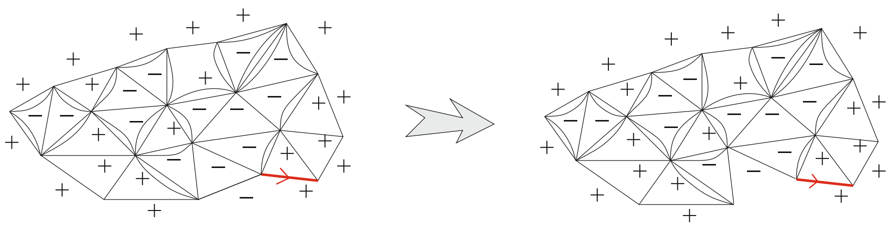


Finally, we see that bijectively removing the marked edge implies the following relationships among generating functions:

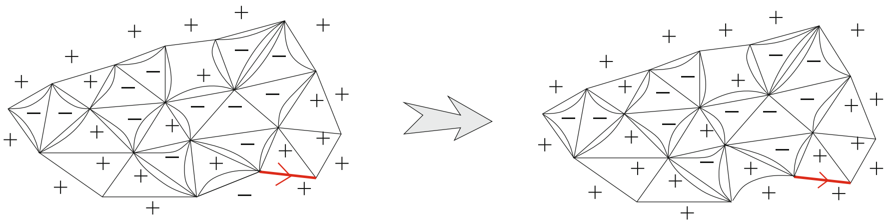
$$
\begin{aligned}
a\,\hat{\mathcal{T}}^{(g)}_{l+1,k;l_1,\dots,l_n} &= \sum_{j=3}^{d} t_j \hat{\mathcal{T}}^{(g)}_{l+j-1,k;l_1,\dots,l_n} \\
&\quad + c\,\hat{\mathcal{T}}^{(g)}_{l,k+1;l_1,\dots,l_n} \\
&\quad + \sum_{i=1}^{n} l_i\,\hat{\mathcal{T}}^{(g)}_{l+l_i-1,k;l_1,\dots,\not{l_i},\dots,l_n} \\
&\quad + \sum_{j=1}^{l-1} \hat{\mathcal{T}}^{(g-1)}_{j,k;l-j-1,l_1,\dots,l_n} \\
&\quad + \sum_{j=1}^{l-1} \sum_{h=0}^{g} \sum_{J\subset\{l_1,\dots,l_n\}} \hat{\mathcal{T}}^{(h)}_{j,k;I}\,\mathcal{T}^{(g-h)}_{l-j-1,J\backslash I}\,.
\end{aligned}
$$

Since those equations may increase k by 1, they can't be closed, and thus we need another equation. For that purpose, consider a map contributing to $\hat{\mathcal{T}}^{(g)}_{l,1;l_1,\dots,l_n}$ with $k=1$. It has a unique $-$ edge, and on the other side of the $-$ edge, there can be either:

- an unmarked spin $-$ face of degree j, with $3 \leq j \leq \tilde{d}$, and removing the $-$ edge gives a map of $\hat{\mathcal{T}}^{(g)}_{l,j-1;l_1,\dots,l_n}$ weighted by $\tilde{t}_j/b$.



- a bicolored face $(+-)$, and removing the $-$ edge gives a map of $\hat{\mathcal{T}}^{(g)}_{l+1,0;l_1,\dots,l_n}$ weighted by c/b.



There cannot be another marked face, neither the same marked face, because there is no other $-$ edge to glue to.

Finally, in terms of generating functions we have

$$b\,\hat{\mathcal{T}}^{(g)}_{l,1;l_1,\dots,l_n} = \sum_{j=3}^{\tilde{d}} \tilde{t}_j \hat{\mathcal{T}}^{(g)}_{l,j-1;l_1,\dots,l_n} + c\,\hat{\mathcal{T}}^{(g)}_{l+1,0;l_1,\dots,l_n}.$$

8.2.1 Equation for Generating Functions

As for uncolored maps, we introduce the following series:

$$V_1'(x) = ax - \sum_{j=3}^{d} t_j\, x^{j-1} \qquad ,\ t_2 = -a$$

$$V_2'(y) = by - \sum_{j=3}^{\tilde{d}} \tilde{t}_j\, y^{j-1} \qquad ,\ \tilde{t}_2 = -b$$

$$c\,Y(x) = V_1'(x) - W_1^{(0)}(x),$$

and:

$$U_n^{(g)}(x,y;x_1,\dots,x_n) = (-cV_2'(y) + c^2 x)\delta_{n,0}\delta_{g,0} - \sum_{j=2}^{\tilde{d}} \sum_{k=0}^{j-2} \tilde{t}_j\, y^{j-2-k}\, G^{(g)}_{n,k}(x;x_1,\dots,x_n) \tag{8.2.3}$$

$$P_n^{(g)}(x,y;x_1,\dots,x_n) = \sum_{j=2}^{d} \sum_{\tilde{j}=2}^{\tilde{d}} \sum_{l=0}^{j-2} \sum_{k=0}^{\tilde{j}-2} \sum_{l_1,\dots,l_n} t_j\, \tilde{t}_{\tilde{j}}\, x^{j-2-l}\, y^{\tilde{j}-2-k}\, \frac{\hat{\mathcal{T}}^{(g)}_{l,k;l_1,\dots,l_n}}{x_1^{l_1+1} \cdots x_n^{l_n+1}}. \tag{8.2.4}$$

Notice that $U_n^{(g)}(x, y; x_1, \ldots, x_n)$ is a polynomial in y, and $P_n^{(g)}(x, y; x_1, \ldots, x_n)$ is a polynomial in both x and y.

In terms of these, the loop equations become:

Theorem 8.2.1 *The generating functions of the Ising model satisfy the following set of equations (called "loop equations" or "Tutte–like equations"):*

$$\begin{aligned} & c\,(y - Y(x))\, U_n^{(g)}(x, y; x_1, \ldots, x_n) + W_{n+1}^{(g)}(x, x_1, \ldots, x_n)\, U_0^{(0)}(x, y) \\ & + U_{n+1}^{(g-1)}(x, y; x, x_1, \ldots, x_n) \\ & + \sum_{h=0}^{g} \sideset{}{'}\sum_{I \subset \{x_1, \ldots, x_n\}} U_{\# I}^{(h)}(x, y; I)\, W_{n+1-\# I}^{(g-h)}(x, J \setminus I) \\ & + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{U_n^{(g)}(x, y; \{x_1, \ldots, x_n\} \setminus \{x_i\}) - U_n^{(g)}(x_i, y; \{x_1, \ldots, x_n\} \setminus \{x_i\})}{x - x_i} \\ = {} & c\,\big((V_1'(x) - cy)(V_2'(y) - cx) + t\,c\big)\, \delta_{n,0}\delta_{g,0} - P_n^{(g)}(x, y; x_1, \ldots, x_n). \end{aligned} \tag{8.2.5}$$

8.3 Solution of Loop Equations

Loop equation (8.2.5) look substantially more intricated than Tutte's equations for maps without Ising spins, however, as we shall see, the symplectic invariants of Chap. 7 still give the solution.

8.3.1 The Disc: Spectral Curve

The disc corresponds to $n = 0$ and $g = 0$, for which the loop equation reads

$$c\,(y - Y(x))\; U_0^{(0)}(x, y) = c(V_1'(x) - cy)(V_2'(y) - cx) - P_0^{(0)}(x, y) + t\,c^2.$$

The right hand side is a polynomial of both x and y, and we call it $E(x, y)$:

$$E(x, y) = (V_1'(x) - cy)(V_2'(y) - cx) - \frac{1}{c}\, P_0^{(0)}(x, y) + t\,c.$$

Notice that $(V_1'(x) - cy)(V_2'(y) - cx)$ is a polynomial of x of degree d and of y of degree $\tilde{d}$, whereas $P_0^{(0)}(x, y)$ is a polynomial of x of degree $d - 2$ and of y of degree $\tilde{d} - 2$.

The loop equation for the disc can thus be written

$$(y - Y(x))\ U_0^{(0)}(x,y) = E(x,y). \tag{8.3.1}$$

Since $U_0^{(0)}(x,y)$ is a polynomial in y, it has no pole at $y = Y(x)$, and thus, by substituting $y \to Y(x)$ we get

$$E(x, Y(x)) = 0.$$

This equation shows that $Y(x)$ is an algebraic function of x, and therefore $W_1^{(0)}(x) = V_1'(x) - cY(x)$ is an algebraic function of x. Moreover, we leave to the reader a straightforward generalization of the 1-cut Brown's Lemma (see Sect. 3.1.2 in Chap. 3) , which shows that this algebraic equation must be of genus 0, and thus the solution can be parametrized by rational functions. Like Zhukowski variable, we define:

$$\begin{cases} x(z) = \gamma\, z + \sum_{k=0}^{\tilde{d}-1} \alpha_k z^{-k} \\ Y(x(z)) = y(z) = \gamma\, z^{-1} + \sum_{k=0}^{d-1} \beta_k z^k. \end{cases}$$

Writing that this parametrization is solution of $E(x(z), y(z)) = 0$, determines all the coefficients $\gamma, \alpha_k, \beta_k$, as well as all coefficients of the polynomial $P_0^{(0)}(x,y)$, as algebraic functions of t, a, b, c, t_j, $\tilde{t}_j$, (they are thus algebraic power series in t).

Theorem 8.3.1 *The disc amplitude $Y(x) = \frac{1}{c}(V_1'(x) - W_1^{(0)}(x))$ is determined as follows: Let*

$$\begin{cases} x(z) = \gamma\, z + \sum_{k=0}^{\tilde{d}-1} \alpha_k z^{-k} \\ Y(x(z)) = y(z) = \gamma\, z^{-1} + \sum_{k=0}^{d-1} \beta_k z^k \end{cases}$$

where $\gamma, \alpha_k, \beta_k$ are the unique solutions of the system of equations

$$V_1'(x(z)) - cy(z) \underset{z\to\infty}{\sim} \frac{t}{\gamma z} + O(1/z^2)$$

$$V_2'(y(z)) - cx(z) \underset{z\to 0}{\sim} \frac{t\, z}{\gamma} + O(z^2)$$

such that

$$\gamma^2 = \frac{ct}{ab - c^2} + O(t^2) = c_{+-}t + O(t^2).$$

Then the disc amplitude $Y(x)$ is:

$$Y(x(z)) = y(z).$$

Proof First, notice that there is a unique solution of loop equations which is a power series in t (other solutions would have negative or fractional powers of t), indeed, those equations correspond to recursively adding edges, and they uniquely allow to construct every Ising map). One can check that this parametric curve is indeed solution of loop equations, and it is a power series of t, therefore it is the solution needed. □

Definition 8.3.1 The spectral curve $\mathcal{E} = (x, y)$ is the pair of rational functions $x(z)$ and $y(z)$:

$$\begin{cases} x(z) = \gamma z + \sum_{k=0}^{\tilde{d}-1} \alpha_k z^{-k} \\ y(z) = \frac{\gamma}{z} + \sum_{k=0}^{d-1} \beta_k z^k \end{cases}$$

where $\alpha_k, \beta_k, \gamma$ are determined by

$$V_1'(x(z)) = cy(z) + \frac{t}{\gamma z} + O(z^{-2}) \quad , \quad V_2'(y(z)) = cx(z) + \frac{tz}{\gamma} + O(z^2)$$

and we choose the unique solution such that $\gamma^2 = O(t)$ at small t.

Theorem 8.3.2 (Algebraic Equation) *The functions $x(z)$ and $y(z)$ are related by the algebraic equation $E(x(z), y(z)) = 0$, where $E(x, y)$ is the polynomial of two variables given by the resultant:*

$$E(x,y) = \frac{-(-1)^{\tilde{d}}\, c^2}{\gamma^{d+\tilde{d}-2}} \det \begin{pmatrix} \gamma & \alpha_0 - x & \alpha_1 & \dots & \alpha_{\tilde{d}-1} & & & & \\ & \gamma & \alpha_0 - x & \alpha_1 & \dots & \alpha_{\tilde{d}-1} & & & \\ & & \ddots & & & & \ddots & & \\ & & & \ddots & & & & \ddots & \\ & & & & \gamma & \alpha_0 - x & \alpha_1 & \dots & \alpha_{\tilde{d}-1} \\ \beta_{d-1} & \dots & & \beta_1 & \beta_0 - y & \gamma & & & \\ & \beta_{d-1} & \dots & & \beta_1 & \beta_0 - y & \gamma & & \\ & & \ddots & & & & \ddots & & \\ & & & \beta_{d-1} & \dots & & \beta_1 & \beta_0 - y & \gamma \end{pmatrix}$$

Proof The resultant vanishes if and only if there is a z which is a common zero of $x(z)-x = 0$ and $y(z)-y = 0$, indeed observe that the vector $(1, z^{-1}, z^{-2}, z^{-3}, \dots)$ is an eigenvector of that matrix, for the eigenvalue 0. The resultant is thus a polynomial of x and y of the correct degree, which vanishes exactly when $E(x, y)$ vanishes, it is thus proportional to $E(x, y)$. The prefactor is determined by matching the large x behavior of $E(x, y) \sim -cxV_1'(x)$. □

Theorem 8.3.3 (Variational Principle)
One can determine the coefficients $\alpha_k, \beta_k, \gamma$ *by extremizing the functional:*

$$\mu(\gamma, \alpha_k, \beta_k) = \operatorname*{Res}_{z\to\infty} (V_1(x(z)) + V_2(y(z)) - c\, x(z)y(z))\ \frac{dz}{z} + 2t\ \ln\gamma$$

Proof

$$\begin{aligned}\frac{\partial\mu}{\partial\alpha_k} &= \operatorname*{Res}_{z\to\infty} (V_1'(x(z)) - cy(z))\,\frac{dz}{z^{k+1}}\\ &= \operatorname*{Res}_{z\to\infty} \left(\frac{t}{\gamma z} + O(1/z^2)\right)\,\frac{dz}{z^{k+1}}\\ &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial\mu}{\partial\beta_k} &= \operatorname*{Res}_{z\to\infty} (V_2'(y(z)) - cxz))\, z^{k-1}\, dz\\ &= -\operatorname*{Res}_{z\to 0} (V_2'(y(z)) - cxz))\, z^{k-1}\, dz\\ &= -\operatorname*{Res}_{z\to 0} \left(\frac{tz}{\gamma} + O(z^2)\right)\, z^{k-1}\, dz\\ &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial\mu}{\partial\gamma} &= \operatorname*{Res}_{z\to\infty} (V_1'(x(z)) - cy(z))\, dz + \operatorname*{Res}_{z\to\infty} (V_2'(y(z)) - cx(z))\,\frac{dz}{z^2} + \frac{2t}{\gamma}\\ &= \operatorname*{Res}_{z\to\infty} (V_1'(x(z)) - cy(z))\, dz - \operatorname*{Res}_{z\to 0} (V_2'(y(z)) - cx(z))\,\frac{dz}{z^2} + \frac{2t}{\gamma}\\ &= \operatorname*{Res}_{z\to\infty} \left(\frac{t}{\gamma z} + O(1/z^2)\right)\, dz - \operatorname*{Res}_{z\to 0} \left(\frac{tz}{\gamma} + O(z^2)\right)\,\frac{dz}{z^2} + \frac{2t}{\gamma}\\ &= -\frac{t}{\gamma} - \frac{t}{\gamma} + \frac{2t}{\gamma}\\ &= 0.\end{aligned}$$

Reciprocally, if $\partial\mu/\partial\alpha_k = 0$ for all k that means

$$\operatorname*{Res}_{z\to\infty} (V_1'(x(z)) - cy(z))\,\frac{dz}{z^{k+1}} = 0$$

and thus $V_1'(x(z)) - cy(z) = O(1/z)$. Similarly, $\partial\mu/\partial\beta_k = 0$ for all k implies that $V_2'(y(z)) - cx(z) = O(z)$. And then, $\partial\mu/\partial\gamma = 0$ implies that $V_1'(x(z)) - cy(z) \sim t/x(z)$ and $V_2'(y(z)) - cx(z) \sim t/y(z)$. □

8.3.2 Example: Ising Model on Quadrangulations

We chose only t_4 and $\tilde{t}_4$ non-vanishing.

We have $V_1'(x) = ax - t_4 x^3$ and $V_2'(y) = by - \tilde{t}_4 y^3$.

Theorem 8.3.1 says that we should look for two rational functions $x(z)$ and $y(z)$ of the form (we exploit the parity of V_1 and V_2):

$$x(z) = \gamma z + \alpha_1 z^{-1} + \alpha_3 z^{-3} \qquad , \qquad y(z) = \gamma/z + \beta_1 z + \beta_3 z^3.$$

We need to compute:

$$V_1'(x(z)) = a(\gamma z + \alpha_1 z^{-1}) - t_4(\gamma^3 z^3 + 3\alpha_1\gamma^2 z + 3\alpha_3\gamma^2 z^{-1} + 3\alpha_1^2\gamma z^{-1}) + O(z^{-3})$$

and thus:

$$c\,\beta_3 = -t_4\gamma^3 \qquad , \qquad c\beta_1 = a\gamma - 3t_3\alpha_1\gamma^2 \qquad , \qquad a\alpha_1 - 3t_4(\alpha_3\gamma^2 + \alpha_1^2\gamma) - c\gamma = \frac{t}{\gamma},$$

and similarly by computing $V_2'(y(z)) - cx(z)$:

$$c\,\alpha_3 = -\tilde{t}_4\gamma^3 \qquad , \qquad c\alpha_1 = b\gamma - 3\tilde{t}_3\beta_1\gamma^2 \qquad , \qquad b\beta_1 - 3t_4(\beta_3\gamma^2 + \beta_1^2\gamma) - c\gamma = \frac{t}{\gamma}.$$

Let us consider for simplicity the symmetric case, where $a = b$ and $t_4 = \tilde{t}_4$. In that case we shall find $\alpha_i = \beta_i$, and thus:

$$c\,\alpha_3 = -t_4\gamma^3 \qquad , \qquad c\alpha_1 = a\gamma - 3t_4\alpha_1\gamma^2 \qquad , \qquad a\alpha_1 - 3t_4(\alpha_3\gamma^2 + \alpha_1^2\gamma) - c\gamma = \frac{t}{\gamma}.$$

That gives an algebraic equation of degree 5 for γ^2:

$$(c + 3t_4\gamma^2)^2\,(t + c\gamma^2 - 3\frac{t_4^2}{c}\,\gamma^6) - ca^2\gamma^2 = 0.$$

and we chose the unique solution which behaves at small t like

$$\gamma^2 = \frac{c\,t}{a^2 - c^2} + O(t^2).$$

We then have

$$\alpha_1 = \frac{a\,\gamma}{c + 3t_4\gamma^2} \qquad , \qquad \alpha_3 = -\frac{t_4\gamma^3}{c}.$$

8.3.3 All Topologies Generating Functions

Then, knowing this spectral curve we have for Ising maps the equivalent of Theorem 3.3.1 :

Theorem 8.3.4 *The generating functions counting Ising maps, are given by the symplectic invariants of Chap. 7:*

$$F_g = \mathcal{F}_g(\mathcal{E}).$$

For the spectral curve $\mathcal{E} = (\mathbb{C}, x, y, B(z, z') = dz\, dz'/(z - z')^2)$. *The* $\omega_n^{(g)}(\mathcal{E})$*'s of Chap. 7 give the generating functions of maps with n marked faces of spin +:*

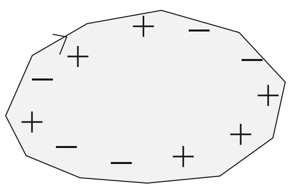
$$W_n^{(g)}(x(z_1), \ldots, x(z_n))\ dx(z_1) \ldots dx(z_n) = \omega_n^{(g)}(z_1, \ldots, z_n) + \delta_{n,1}\, \delta_{g,0}\, V_1'(x(z_1))\, dx(z_1) + \delta_{n,2}\, \delta_{g,0}\, \frac{dx(z_1)\, dx(z_2)}{(x(z_1) - x(z_2))^2}. \tag{8.3.2}$$

We skip the proof of this theorem, the interested reader can read the proof in [73]. We just mention that the proof is much more technical than for uncolored maps, it is not at all a straightforward extension of Chap. 3.

8.4 Mixed Boundary Conditions

So far, we have been considering marked faces, as well as unmarked faces carrying one spin in their center.

Now, let us also consider marked faces having different spins on their sides (unmarked faces will always have only one spin, either + or −). A typical marked face can then be:



Let us construct a good set of generating functions for counting maps with such marked faces with spin boundary conditions.

8.4.1 Maps with Mixed Boundaries

First, consider maps having n marked faces, of respective perimeters $l_1, \dots, l_n$, such that the ith marked face has $2k_i$ changes of boundary conditions:

$$\text{marked face } i = l_{i,1} \text{ spin } +, \tilde{l}_{i,1} \text{ spin } -, l_{i,2} \text{ spin } +, \tilde{l}_{i,2} \text{ spin } -, \dots, l_{i,k_i} \text{ spin } +, \tilde{l}_{i,k_i} \text{ spin } - .$$

$$l_i = \sum_{j=1}^{k_i} l_{i,j} + \tilde{l}_{i,j}.$$

Our goal is to compute the generating function which enumerates such configurations:

$$\sum_{v=1}^{\infty} t^v \sum_{\Sigma \in \mathbb{M}^{(g)}_{n;k_1,\dots,k_n}(v)} \frac{t_3^{n_3(\Sigma)}\, t_4^{n_4(\Sigma)} \dots t_d^{n_d(\Sigma)}\, \tilde{t}_3^{\tilde{n}_3(\Sigma)}\, \tilde{t}_4^{\tilde{n}_4(\Sigma)} \dots \tilde{t}_{\tilde{d}}^{\tilde{n}_{\tilde{d}}(\Sigma)}}{\prod_{i=1}^n \prod_{j=1}^{k_i} x_{i,j}^{1+l_{i,j}(\Sigma)}\, y_{i,j}^{1+\tilde{l}_{i,j}(\Sigma)}}$$
$$\frac{1}{\#\mathrm{Aut}(\Sigma)}\ a^{-n_{++}(\Sigma)}\, b^{-n_{--}(\Sigma)}\, c^{\hat{n}(\Sigma)} .$$

This generating function depends on k parameters of type $x_{i,j}$ (associated to spin $+$ boundaries of length $l_{i,j}$) and k parameters of type $y_{i,j}$ (associated to spin $-$ boundaries of length $\tilde{l}_{i,j}$), where $2k$ is the total number of boundary condition changes:

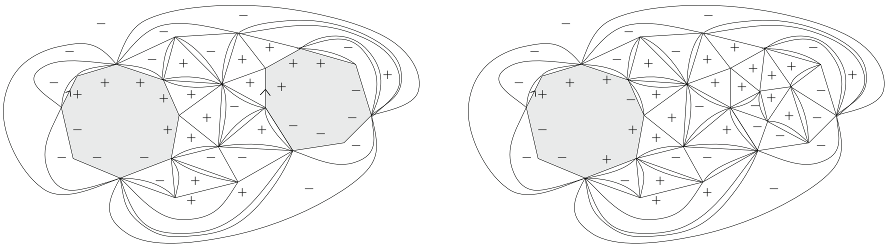
$$k = \sum_{i=1}^{n} k_i.$$

8.4.1.1 Fixed *k*

From now on, it will be better to compute at once all generating functions with a given k, i.e. with an arbitrary number of marked faces, provided that the total number of boundary condition changes is $2k$.

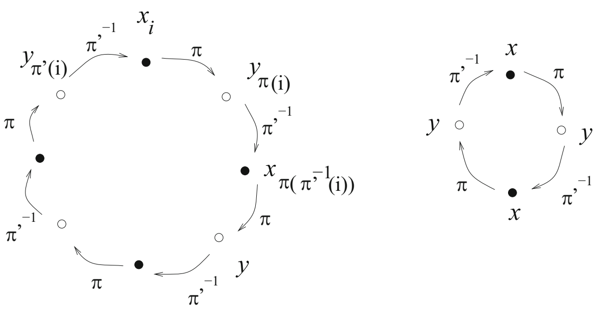
For instance for $k = 2$, we have either two marked faces with $k_1 = k_2 = 1$, or 1 marked face with $k_1 = 2$.

For example the following maps both correspond to $k = 2$. The first one has two marked faces, one with $l_{1,1} = 5, \tilde{l}_{1,1} = 3$ and one with $l_{2,1} = 3, \tilde{l}_{2,1} = 4$, and the second map has only one marked face with $l_{1,1} = 3, \tilde{l}_{1,1} = 1, l_{1,2} = 2, \tilde{l}_{1,2} = 2$:



For that purpose, let us consider k parameters x_i, $i = 1, \ldots, k$ associated to spin $+$ pieces of boundaries of respective lengths l_i, and k parameters y_i, $i = 1, \ldots, k$ associated to spin $-$ pieces of boundaries of respective lengths $\tilde{l}_i$. Let us consider all possible boundary conditions which can be encoded by those $2k$ parameters.

Consider x_1, it is associated to a piece of $+$ boundary of length l_1 of some marked face. Going around the marked face (in the direction defined by the map orientation), it must be followed by a $-$ piece of boundary $y_{\pi(1)}$ of length $\tilde{l}_{\pi(1)}$, where π is some permutation of indices. Then, the $-$ piece of boundary $y_{\pi(1)}$ must be followed by a $+$ piece of boundary, let us call it $x_{\pi'^{-1}(\pi(1))}$, where π' is another permutation. We proceed until we have completed a cycle around a marked face, i.e. until we have completed a cycle of the permutation $\pi'^{-1} \circ \pi$. Then we repeat the same procedure for all cycles of $\pi'^{-1} \circ \pi$.



Considering all pairs of permutations (π, π') exhausts all possible boundary conditions with $2k$ changes of boundary spins.

Definition 8.4.1 Let us define the following generating function:

$$
\begin{aligned}
&\widehat{H}^{(g)}_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k)\\
&= -c\,\delta_{g,0}\,\delta_{k,1}\\
&\quad + \sum_{v=1}^{\infty} t^v \sum_{\Sigma\in\mathbb{M}^{(g)}_{\pi,\pi'}(v)} \frac{t_3^{n_3(\Sigma)}\, t_4^{n_4(\Sigma)}\,\dots\, t_d^{n_d(\Sigma)}\, \tilde{t}_3^{\tilde{n}_3(\Sigma)}\, \tilde{t}_4^{\tilde{n}_4(\Sigma)}\,\dots\, \tilde{t}_{\tilde{d}}^{\tilde{n}_{\tilde{d}}(\Sigma)}}{\prod_{i=1}^k x_i^{1+l_i(\Sigma)}\, y_i^{1+\tilde{l}_i(\Sigma)}}\\
&\qquad \frac{1}{\#\mathrm{Aut}(\Sigma)}\; a^{-n_{++}(\Sigma)}\, b^{-n_{--}(\Sigma)}\, c^{\hat{n}(\Sigma)}
\end{aligned}
$$

where we sum over the set of maps whose boundary corresponds to the permutations π, π'.

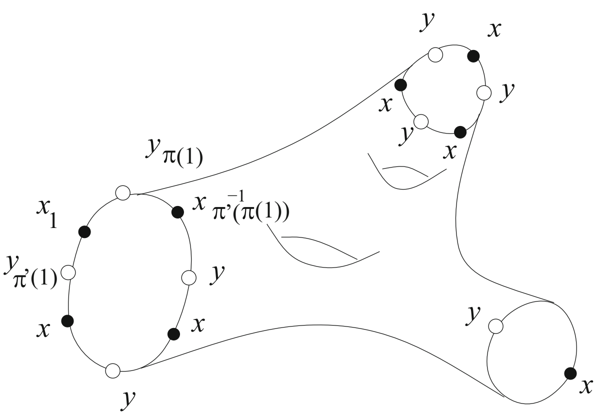
We also define the generating functions summed over the genus:

$$
\widehat{H}_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k) = \sum_{g=0}^{\infty} (N/t)^{2-2g-\ell(\pi'^{-1}\circ\pi)}\; \widehat{H}^{(g)}_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k) \tag{8.4.1}
$$

where $\ell(\pi'^{-1}\circ\pi)$ is the number of cycles of $\pi'^{-1}\circ\pi$. As usual, this equality is to be understood as an equality of formal series in t, and for each power of t, the sum over g is finite.

$\widehat{H}^{(g)}_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k)$ counts maps drawn on surfaces of genus g, with $\ell(\pi'^{-1}\circ\pi)$ boundaries, and whose boundaries are labeled by a sequence of x and y variables according to the cycles of $\pi'^{-1}\circ\pi$.

Example of a surface of genus 2, with $k = 8$, and such that $\pi'^{-1}\circ\pi$ has three cycles:



8.4.1.2 Non Connected Generating Functions

The formulae that will follow are better written using generating function for non-necessarily connected maps. But we require that connected pieces contain at least one boundary.

We define generating functions of non-connected maps, as the product of connected ones.

For example for $k = 1$ there is only one boundary, and the map must be connected, we define

$$H_{\mathrm{Id}_1,\mathrm{Id}_1}(x;y) = \widehat{H}_{\mathrm{Id}_1,\mathrm{Id}_1}(x;y).$$

For $k = 2$, if $(\pi, \pi') = (\mathrm{Id}_2, \mathrm{Id}_2)$, we see that $\pi \circ \pi'^{-1}$ has two cycles, so the maps can either be connected with two boundaries, or disconnected, thus we define:

$$H_{\mathrm{Id}_2,\mathrm{Id}_2}(x_1,x_2;y_1,y_2) = \widehat{H}_{\mathrm{Id}_2,\mathrm{Id}_2}(x_1,x_2;y_1,y_2) + \widehat{H}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_1;y_1)\,\widehat{H}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_2;y_2)$$

and if $(\pi, \pi') = (\mathrm{Id}_2, (1,2))$, we see that $\pi'^{-1} \circ \pi$ has only one cycle, so it must be connected and thus we define

$$H_{\mathrm{Id}_2,(1,2)}(x_1,x_2;y_1,y_2) = \widehat{H}_{\mathrm{Id}_2,(1,2)}(x_1,x_2;y_1,y_2).$$

And so on.

In general,

Definition 8.4.2 $H_{\pi,\pi'}$ is defined as the sum of products of $\widehat{H}_{\pi_i,\pi_i'}$ for all possible ways of decomposing the permutation $\pi'^{-1} \circ \pi$ into a product disjoint permutations $\prod_i \pi_i'^{-1} \circ \pi_i$.

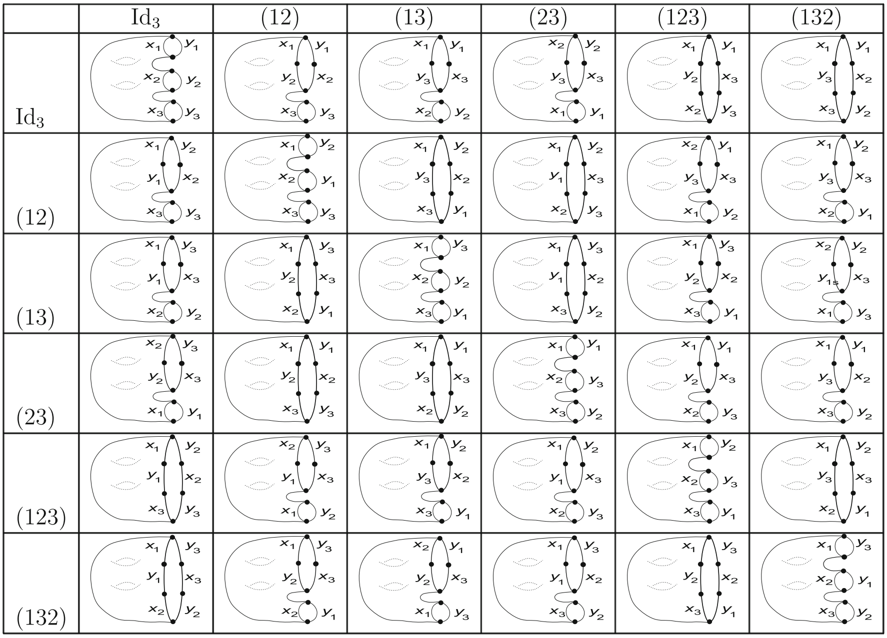
8.4.1.3 The Matrix Generating Function

For every pair of permutations of k variables (π, π'), we have defined a generating function $H_{\pi,\pi'}$. Let us now collect them all together into a $k! \times k!$ matrix:

Definition 8.4.3 The matrix generating function $H(x_1,\dots,x_k;y_1,\dots,y_k)$ is the $k! \times k!$ matrix, whose lines and columns are indexed by permutations π, π' and whose entries are the $H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k)$:

$$H(x_1,\dots,x_k;y_1,\dots,y_k) = \{H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k)\}_{\pi,\pi'\in\mathfrak{S}_k}\,.$$

Example with $k = 3$, we have the 6×6 matrix



here the pictures mean that we sum over all surfaces of all genus, and possibly disconnected, with the corresponding boundaries.

Theorem 8.4.1 *The matrix $H(x_1, \ldots, x_k; y_1, \ldots, y_k)$ is symmetric:*

$$H_{\pi,\pi'}(x_1, \ldots, x_k; y_1, \ldots, y_k) = H_{\pi',\pi}(x_1, \ldots, x_k; y_1, \ldots, y_k) \tag{8.4.2}$$

Proof It just consists in remarking that reversing the orientation of a map, gives another map, with the same number k of boundary conditions, and reversing the boundary just exchanges π and π'. □

Theorem 8.4.2 (Commutation Relations) *We have*

$$[H(x_1, \ldots, x_k; y_1, \ldots, y_k), \mathcal{A}] = 0$$

$$\forall\, i = 1, \ldots, k, \qquad [H(x_1, \ldots, x_k; y_1, \ldots, y_k), \mathcal{A}_i] = 0$$

where $\mathcal{A}_i$ is the following $k! \times k!$ matrix with lines and columns indexed by permutations

$$(\mathcal{A}_i)_{\pi,\pi'} = \begin{cases} y_{\pi(i)} & \text{if } \pi' = \pi \\ \frac{-t}{Nc}\,\frac{1}{x_i - x_j} & \text{if } \pi' = \pi \circ (ij)\,. \\ 0 & \text{otherwise} \end{cases}$$

and $\mathcal{A} = \sum_i x_i \mathcal{A}_i$:

$$\mathcal{A}_{\pi,\pi'} = \begin{cases} \sum_i x_i y_{\pi(i)} & \text{if } \pi' = \pi \\ \frac{-t}{Nc} & \text{if } \pi^{-1} \circ \pi' = \text{transposition} \\ 0 & \text{otherwise} \end{cases}.$$

Example with $k = 3$:

$$\mathcal{A}_1 = \begin{pmatrix} y_1 & \frac{-t}{Nc}\frac{1}{x_1-x_2} & \frac{-t}{Nc}\frac{1}{x_1-x_3} & 0 & 0 & 0 \\ \frac{-t}{Nc}\frac{1}{x_1-x_2} & y_2 & 0 & 0 & 0 & \frac{-t}{Nc}\frac{1}{x_1-x_3} \\ \frac{-t}{Nc}\frac{1}{x_1-x_3} & 0 & y_3 & 0 & \frac{-t}{Nc}\frac{1}{x_1-x_2} & 0 \\ 0 & 0 & 0 & y_1 & \frac{-t}{Nc}\frac{1}{x_1-x_3} & \frac{-t}{Nc}\frac{1}{x_1-x_2} \\ 0 & 0 & \frac{-t}{Nc}\frac{1}{x_1-x_2} & \frac{-t}{Nc}\frac{1}{x_1-x_3} & y_2 & 0 \\ 0 & \frac{-t}{Nc}\frac{1}{x_1-x_3} & 0 & \frac{-t}{Nc}\frac{1}{x_1-x_2} & 0 & y_3 \end{pmatrix}.$$

And we have the corollary.

Theorem 8.4.3 *For every* $\xi, \eta \in \mathbb{C}$, *the matrix* $H(x_1, \ldots, x_k; y_1, \ldots, y_k)$ *commutes with the* $k! \times k!$ *matrix* $\mathcal{M}(x_1, \ldots, x_k; y_1, \ldots, y_k; \xi, \eta)$ *defined by:*

$$\mathcal{M}_{\pi,\pi'}(x_1, \ldots, x_k; y_1, \ldots, y_k; \xi, \eta) = \prod_{i=1}^{k} \left(\delta_{\pi(i),\pi'(i)} - \frac{t}{Nc} \frac{1}{(x_i - \xi)\,(y_{\pi(i)} - \eta)} \right) \tag{8.4.3}$$

(it is a symmetric matrix).

We have

$$\forall\, \xi,\, \eta \quad , \quad [H(x_1, \ldots, x_k; y_1, \ldots, y_k), \mathcal{M}(x_1, \ldots, x_k; y_1, \ldots, y_k; \xi, \eta)] = 0. \tag{8.4.4}$$

Proof We first prove Theorem 8.4.2. We use again some Tutte–like equations.[1]

Consider the boundary containing x_1, and consider the first edge of that boundary, it has a sign $+$.

When we erase this edge, several situations may occur:

- on the other side of the removed edge, we have a j gon of sign $+$, then the corresponding term in Tutte equation will be:

$$a x_1 H_{\pi,\pi'}(x_1, \ldots, x_k; y_1, \ldots, y_k) = (\sum_j t_j x_1^{j-1} H_{\pi,\pi'}(x_1, \ldots, x_k; y_1, \ldots, y_k))_{-}$$
$$+ \text{other possibilities}$$

[1]The proof of these two theorems was done in [72]. The proof presented here, is much simpler, and is due to Luigi Cantini in 2007. It was never published and we thank L. Cantini for that proof.

where the subscript $()_-$ means that we keep only negative powers of x_1.

- on the other side of the removed edge, we have a bicolored $(+-)$ face, i.e. after removing the edge, we get an edge of sign $-$, which thus enters the boundary $y_{\pi'(1)}$, then the corresponding term in Tutte equation will be:

$$ax_1 H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k) = c\,(y_{\pi'(1)}\,H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k))_- \\ +\text{ other possibilities}$$

and the subscript $()_-$ means that we keep only negative powers of $y_{\pi'(1)}$.

- on the other side of the removed edge, we have an edge of the same face or of another marked face, let us say it is x_j for some $j \neq 1$. Erasing the edge either disconnects the boundary into two pieces (and reduces the genus by 1), or on the contrary merges two boundaries. In both cases, the boundary (π,π') becomes $(\pi,\pi'\,(1j))$. Then the corresponding term in Tutte equation will be:

$$ax_1 H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k) = \frac{t}{N}\,\frac{-1}{x_1-x_j}\Big(H_{\pi,\pi'(1,j)}(x_1,\dots,x_k;y_1,\dots,y_k) \\ -H_{\pi,\pi'(1,j)}(x_j,x_2,\dots,x_k;y_1,\dots,y_k)\Big) \\ +\text{other possibilities}$$

- on the other side of the removed edge, we have an edge of the x_1 component of the same boundary. Erasing the edge either disconnects the boundary into two pieces, which either disconnects the surface, or decreases the genus by 1. Then the corresponding term in Tutte equation will be:

$$ax_1 H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k) = W(x_1)\,H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k) \\ +H^{(1)}_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k;x_1) \\ +\text{other possibilities}$$

where $H^{(1)}_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k;x_1)$ gathers all the possibilities of disconnecting the surface or decreasing the genus by 1.

Finally, writing that $ax-\sum_j t_j x^{j-1} = V_1'(x)$, that gives

$$\big((V_1'(x_1)-W(x_1))H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k)\big)_- \\ -H^{(1)}_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k;x_1) \\ -\frac{t}{N}\sum_{j\neq 1}\frac{1}{x_1-x_j}\,H_{\pi,\pi'(1,j)}(x_j,x_2,\dots,x_k;y_1,\dots,y_k)$$

$$= c\, y_{\pi'(1)}\, H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k) - \frac{t}{N}\sum_{j\neq 1}\frac{1}{x_1-x_j}\,H_{\pi,\pi'(1,j)}(x_1,\dots,x_k;y_1,\dots,y_k)$$

$$= c\sum_{\pi''} H_{\pi,\pi''}(x_1,\dots,x_k;y_1,\dots,y_k)_{\pi'',\pi'}\ \ (\mathcal{A}_1)_{\pi'',\pi'}\ .$$

The key is to observe that all the terms in the left hand side are symmetric under transposition $\pi \leftrightarrow \pi'$, and thus, taking the difference of that equation with its transpose gives:

$$0 = H\mathcal{A}_1 - (H\,\mathcal{A}_1)^t = H\mathcal{A}_1 - \mathcal{A}_1 H = [H,\mathcal{A}_1].$$

The proof is similar for the other $\mathcal{A}_j$'s. This ends the proof of Theorem 8.4.2.

We shall not prove Theorem 8.4.3 here. We just give an argument towards it. The full proof is very involved and relies on group theory, so is beyond the scope of this book (more on the properties of matrices $\mathcal{M}$ can be found in [33, 77]).

The argument, is that the algebra of $k!\times k!$ matrices which commute with all $\mathcal{A}_i$'s is generated by the matrices $\mathcal{M}$.

Just observe that all those matrices commute together:

$$\forall\, \xi,\xi',\eta,\eta', \qquad [\mathcal{M}(x_1,\dots,x_k;y_1,\dots,y_k;\xi,\eta),\mathcal{M}(x_1,\dots,x_k;y_1,\dots,y_k;\xi',\eta')] = 0$$

(this commutation relation is not trivial, it relies on group theory of the unitary group $U(k)$ [33]).

Then, expanding $\mathcal{M}$ at large ξ and η, one has

$$\mathcal{A} = \frac{-Nc}{t}\operatorname*{Res}_{\xi\to\infty}\operatorname*{Res}_{\eta\to\infty}\ \xi\eta(\mathcal{M} - (1+\frac{k(k-1)}{2}\,\frac{t^2}{N^2\,c^2})\,\mathrm{Id}_{k!})$$

which implies that $[\mathcal{A},\mathcal{M}] = 0$.

Similarly, expanding at $\eta\to\infty$ and $\xi\to x_i$ we have

$$\mathcal{A}_i = \left(\frac{t}{Nc}\sum_{j\neq i}\frac{1}{x_i-x_j}\right)\mathrm{Id}_{k!} + \frac{Nc}{t}\operatorname*{Res}_{\xi\to x_i}\lim\nolimits_{\eta\to\infty}\eta\,(\mathcal{M}-\mathrm{Id}_{k!}),$$

which implies that $[\mathcal{A}_i,\mathcal{M}] = 0$.

There are also matrices $\mathcal{M}_{i,j}$ defined as

$$\mathcal{M}_{i,j}(x_1,\dots,x_k;y_1,\dots,y_k) = \operatorname*{Res}_{\xi\to x_i}\operatorname*{Res}_{\eta\to y_j}\ \mathcal{M}(x_1,\dots,x_k;y_1,\dots,y_k;\xi,\eta)$$

which also commute with all the others

$$[\mathcal{M}_{i,j},\mathcal{M}] = 0.$$

□

8.4.1.4 Example $k = 2$

Let us see what this theorem tells us for $k = 2$:

We have

$$\mathcal{A} = \begin{pmatrix} x_1 y_1 + x_2 y_2 & \frac{-t}{Nc} \\ \frac{-t}{Nc} & x_1 y_2 + x_2 y_1 \end{pmatrix} = x_1 \mathcal{A}_1 + x_2 \mathcal{A}_2,$$

$$\mathcal{A}_1 = \begin{pmatrix} y_1 & \frac{-t}{Nc}\frac{1}{x_1-x_2} \\ \frac{-t}{Nc}\frac{1}{x_1-x_2} & y_2 \end{pmatrix} \quad , \quad \mathcal{A}_2 = \begin{pmatrix} y_2 & \frac{-t}{Nc}\frac{1}{x_2-x_1} \\ \frac{-t}{Nc}\frac{1}{x_2-x_1} & y_1 \end{pmatrix},$$

and we find that the matrix $\mathcal{M}$ is

$$\mathcal{M}(x_1, x_2; y_1, y_2; \xi, \eta) = \left(1 - \frac{t}{Nc}\,\frac{2\xi\eta - \xi(y_1+y_2) - \eta(x_1+x_2) - \frac{t}{Nc}}{(x_1-\xi)(y_1-\eta)(x_2-\xi)(y_2-\eta)}\right) \mathrm{Id}_2 - \frac{t}{Nc}\,\frac{1}{(x_1-\xi)(y_1-\eta)(x_2-\xi)(y_2-\eta)}\,\mathcal{A},$$

and

$$\mathcal{M}_{1,1} = \frac{-t}{N\,c}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{t^2}{N^2\,c^2}\,\frac{1}{(x_1-x_2)\,(y_1-y_2)}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$
$$= \frac{t}{N\,c}\,\frac{x_1 y_2 + x_2 y_1 + \frac{t}{Nc}}{(x_1-x_2)(y_1-y_2)}\,\mathrm{Id}_2 - \frac{t}{N\,c}\,\frac{\mathcal{A}}{(x_1-x_2)(y_1-y_2)}.$$

One easily verifies that they all commute together.

The eigenvalues of $\mathcal{A}$ are:

$$\lambda = \frac{(x_1+x_2)(y_1+y_2)}{2} \pm \frac{1}{2}\sqrt{(x_1-x_2)^2\,(y_1-y_2)^2 + 4\,\frac{t^2}{N^2\,c^2}}.$$

and the eigenvalues of $\mathcal{A}_1$ and $\mathcal{A}_2$ are

$$\lambda_1 = \frac{y_1+y_2}{2} \pm \frac{1}{2(x_1-x_2)}\sqrt{(x_1-x_2)^2\,(y_1-y_2)^2 + 4\,\frac{t^2}{N^2\,c^2}}$$

$$\lambda_2 = \frac{y_1+y_2}{2} \mp \frac{1}{2(x_1-x_2)}\sqrt{(x_1-x_2)^2\,(y_1-y_2)^2 + 4\,\frac{t^2}{N^2\,c^2}}.$$

The common vectors of all these matrices, normalized so that $\sum_\pi (-1)^\pi v_\pi = 1$ are:

$$v = \frac{1}{2\ x_{12}\, y_{12}} \begin{pmatrix} \frac{2t}{Nc} + x_{12}\, y_{12} + \sqrt{x_{12}^2\, y_{12}^2 + 4\, \frac{t^2}{N^2\, c^2}} \\ \frac{2t}{Nc} - x_{12}\, y_{12} + \sqrt{x_{12}^2\, y_{12}^2 + 4\, \frac{t^2}{N^2\, c^2}} \end{pmatrix}$$

where we have denoted $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$ to shorten the notations.

The matrix V with entries $V_{i,j} = \sum_{\pi,\, \pi(i)=j} (-1)^\pi\, v_\pi$, that we shall consider in the next section is:

$$V = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{\frac{2t}{Nc} + \sqrt{x_{12}^2\, y_{12}^2 + 4\, \frac{t^2}{N^2\, c^2}}}{2\ x_{12}\, y_{12}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then write

$$H(x_1, x_2; y_1, y_2) = \begin{pmatrix} H_{\mathrm{Id}_2,\mathrm{Id}_2}(x_1, x_2; y_1, y_2) & H_{\mathrm{Id}_2,(1.2)}(x_1, x_2; y_1, y_2) \\ H_{(1,2),\mathrm{Id}_2}(x_1, x_2; y_1, y_2) & H_{(1,2),(1.2)}(x_1, x_2; y_1, y_2) \end{pmatrix}.$$

That gives

$$\begin{aligned} &[\mathcal{M}(x_1, x_2; y_1, y_2; \xi, \eta), H(x_1, x_2; y_1, y_2)] \\ &= \Big((x_1 - x_2)(y_2 - y_1) H_{\mathrm{Id}_2,(1.2)}(x_1, x_2; y_1, y_2) + \frac{t}{Nc} (H_{\mathrm{Id}_2,\mathrm{Id}_2}(x_1, x_2; y_1, y_2) \\ &\quad - H_{(1,2),(1.2)}(x_1, x_2; y_1, y_2)) \Big) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

and therefore Theorem 8.4.3 implies

$$H_{\mathrm{Id}_2,(1.2)}(x_1, x_2; y_1, y_2) = \frac{t}{Nc}\, \frac{H_{\mathrm{Id}_2,\mathrm{Id}_2}(x_1, x_2; y_1, y_2) - H_{(1,2),(1,2)}(x_1, x_2; y_1, y_2)}{(x_1 - x_2)\,(y_1 - y_2)}.$$

This can be rewritten in terms of connected $\widehat{H}$, and also written for fixed genus:

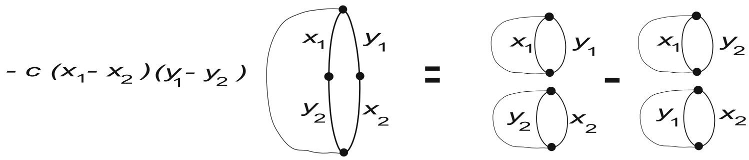
$$\begin{aligned} &c\, \widehat{H}^{(g)}_{\mathrm{Id}_2,(1.2)}(x_1, x_2; y_1, y_2) \\ &= \frac{\widehat{H}^{(g-1)}_{\mathrm{Id}_2,\mathrm{Id}_2}(x_1, x_2; y_1, y_2) - \widehat{H}^{(g-1)}_{(1,2),(1,2)}(x_1, x_2; y_1, y_2)}{(x_1 - x_2)\,(y_1 - y_2)} \\ &\quad + \sum_{h=0}^{g} \frac{\widehat{H}^{(h)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_1; y_1)\, \widehat{H}^{(g-h)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_2; y_2) - \widehat{H}^{(h)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_1; y_2)\, \widehat{H}^{(g-h)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_2; y_1)}{(x_1 - x_2)\,(y_1 - y_2)}. \end{aligned}$$

In particular, for the planar case $g = 0$, that gives:

Corollary 8.4.1

$$c\,\widehat{H}^{(0)}_{\mathrm{Id}_2,(1.2)}(x_1,x_2;y_1,y_2)$$
$$= -\frac{\widehat{H}^{(0)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_1;y_1)\,\widehat{H}^{(0)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_2;y_2) - \widehat{H}^{(0)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_1;y_2)\,\widehat{H}^{(0)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_2;y_1)}{(x_1-x_2)\,(y_1-y_2)}. \tag{8.4.5}$$

Graphically it says that:



At the time of writing of this book, there is no known combinatorial interpretation to that remarkable relationship.

8.4.1.5 Eigenvalues and Eigenvectors of the Commuting Matrices $\mathcal{A}_i, \mathcal{A}, \mathcal{M}, \mathcal{M}_{i,j}$

The $k! \times k!$ matrices $\mathcal{A}_i$ all commute $[\mathcal{A}_i, \mathcal{A}_j] = 0$, and they also commute with $\mathcal{M}$, and thus they have a common basis of eigenvectors.

Finding the eigenvalues of a $k! \times k!$ matrix is not easy, and fortunately, the following theorem allows to find these eigenvalues, only in terms of $k \times k$ matrices, which is much easier:

Theorem 8.4.4 *Let* $Y = \mathrm{diag}(y_1,\dots,y_k)$ *and* Ξ *the* $k \times k$ *antisymmetric matrix* $\Xi_{i,j} = \frac{t}{N}\frac{1}{x_i-x_j}$ *if* $i \neq j$, *and* $\Xi_{i,i} = 0$. *Let* $\Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_k)$ *be a solution (there are* $k!$ *solutions to this algebraic equation) of*

$$\forall\,\eta,\qquad \det(\eta\,\mathrm{Id}_k - \Lambda - \Xi) = \det(\eta\,\mathrm{Id}_k - Y),$$

i.e. find Λ *such that the eigenvalues of* $\Lambda + \Xi$ *are* $y_1,\dots,y_k$

$$\mathrm{sp}(\Lambda + \Xi) = \{y_1,\dots,y_k\}.$$

Then, let $V_{i,j}$ *be the* $k \times k$ *matrix of eigenvectors of* $\Lambda + \Xi$

$$\Lambda + \Xi = V\,Y\,V^{-1}$$

normalized such that it is a stochastic matrix (it is proved below that this is indeed possible):

$$\sum_i V_{i,j} = \sum_j V_{i,j} = 1.$$

Then:

- *eigenvalue of* $\mathcal{A}_i : \lambda_i$
- *eigenvalue of* $\mathcal{A} : \lambda = \sum_i x_i \lambda_i$
- *eigenvalue of* $\mathcal{M} : \mu = 1 - \frac{t}{Nc} \sum_{i,j} \frac{V_{i,j}}{(\xi - x_i)(\eta - y_j)}$.
- *eigenvalue of* $\mathcal{M}_{i,j} : \mu_{i,j} = -\frac{t}{Nc} V_{i,j}$

where $\mathcal{M}_{i,j} = \operatorname{Res}_{\xi \to x_i} \operatorname{Res}_{\eta \to y_j} \mathcal{M}$.

Proof Let $v = (v_\pi)$ be a common eigenvector. Let us define the $k \times k$ matrix

$$V_{i,j} = \sum_{\pi \mid \pi(i) = j} (-1)^\pi \; v_\pi .$$

It satisfies:

$$\sum_i V_{i,j} = \sum_j V_{i,j} = \sum_\pi (-1)^\pi \; v_\pi = e^t.v$$

where e is the vector $e_\pi = (-1)^\pi$, and e^t is its transposition.

Notice that when N is large, or equivalently, when the x_i's and y_i's are large compared to t/Nc, the matrices $\mathcal{A}$ and $\mathcal{A}_i$'s are almost diagonal, with distinct eigenvalues. In this regime, the eigenvectors v tend to the basis vectors, i.e. only one component v_π is non-vanishing, i.e. $e^t.v \to (-1)^\pi v_\pi \neq 0$.

Moreover, the eigenvectors are algebraic functions of the x_i's and y_i's, and thus $e^t.v$ is an algebraic function, and it doesn't vanish in this regime, so it doesn't vanish for generic values of x_i's and y_i's.

We can thus chose to normalize our eigenvector v, for generic values of x_i's and y_i's, so that the matrix V is not identically vanishing, and so that:

$$e^t.v = 1.$$

The equation $\mathcal{A}_i \, v = \lambda_i v$ implies:

$$\forall \; i,j, \qquad \lambda_i V_{i,j} + \sum_{l \neq i} \frac{t}{Nc(x_i - x_l)} \, V_{l,j} = y_j \, V_{i,j}$$

If we define $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_k)$ and $Y = \mathrm{diag}(y_1, \ldots, y_k)$, and $\Xi_{i,j} = \frac{t}{Nc(x_i - x_j)}$ and $\Xi_{i,i} = 0$ we have

$$(\Lambda + \Xi)\, V = V\, Y.$$

If V would be invertible that would mean that the y_i's are the eigenvalues of $\Lambda + \Xi$:

$$\det(y - \Lambda - \Xi) = \prod_{i=1}^{k} (y - y_i),$$

or

$$\forall\ i\,, \qquad \det(y_i - \Lambda - \Xi) = 0.$$

In other words, we have the y_i's as functions of the λ_i's and x_i's. We can invert those relations and deduce the λ_i's as algebraic functions of the y_i's and x_i's.

Let $\tilde{Y} = (\tilde{y}_1, \ldots, \tilde{y}_k)$ be the eigenvalues of $\Lambda + \Xi$, and let $\tilde{V}$ be a matrix whose columns are a basis of eigenvectors of $\Lambda + \Xi$. By definition the matrix $\tilde{V}$ of eigenvectors is invertible. The eigenvectors are defined up to a scalar factor, i.e. $\tilde{V}$ is defined up to right multiplication by an invertible diagonal matrix. For generic choices of x_i's and y_i's, we may normalize $\tilde{V}$ so that:

$$\forall j\,, \qquad \sum_i \tilde{V}_{i,j} = 1.$$

We thus have, by definition:

$$\Lambda + \Xi = \tilde{V}\tilde{Y}\tilde{V}^{-1}.$$

Multiplying by V on the right, and by $\tilde{V}^{-1}$ on the left, we get:

$$\tilde{Y}\tilde{V}^{-1}V = \tilde{V}^{-1}VY,$$

and Y and $\tilde{Y}$ are both diagonal matrices. Let $C = \tilde{V}^{-1}V$, we have:

$$\forall\, i,j\,, \qquad C_{i,j}\,(\tilde{y}_i - y_j) = 0.$$

This implies that either $C_{i,j} = 0$, or $\tilde{y}_i = y_j$. Moreover we have, by our choices of normalization, that

$$\forall j\,, \qquad \sum_i C_{i,j} = 1$$

so that for each j there must exist some i with $C_{i,j} \neq 0$, and thus there must exist some i with $\tilde{y}_i = y_j$. If the y_j's are all distinct, then there is at most one $\tilde{y}_i$ equal to y_j for each j, and thus $C_{i,j} = 0$ for all the others. Up to reordering the eigenvalues of $\tilde{Y}$, we may chose that $\tilde{y}_i = y_i$, and C must be diagonal, and since $\sum_i C_{i,j} = 1$, we must have $C = \text{Id}$:

$$\tilde{Y} = Y \qquad , \qquad V = \tilde{V}.$$

This proves in particular that V is invertible (for generic values of x_i's and y_i's).

So we have proved that V is the matrix of eigenvectors of $\Lambda + \Xi$, and can be chosen to be stochastic.

Then, if we write

$$\mathcal{M}(x_1, \dots, x_k; y_1, \dots, y_k; \xi, \eta) = \text{Id} + \sum_{i,j} \frac{1}{(\xi - x_i)\,(\eta - y_j)}\, \mathcal{M}_{i,j}(x_1, \dots, x_k; y_1, \dots, y_k)$$

where

$$\mathcal{M}_{i,j}(x_1, \dots, x_k; y_1, \dots, y_k) = \operatorname*{Res}_{\xi \to x_i}\ \operatorname*{Res}_{\eta \to y_j}\ \mathcal{M}(x_1, \dots, x_k; y_1, \dots, y_k; \xi, \eta).$$

Let us call μ the eigenvalue of $\mathcal{M}$ and $\mu_{i,j}$ the eigenvalues of $\mathcal{M}_{i,j}$ for the eigenvector v.

$$\mathcal{M}(x_1, \dots, x_k; y_1, \dots, y_k; \xi, \eta)\, v = \mu\, v \qquad , \qquad \mathcal{M}_{i,j}(x_1, \dots, x_k; y_1, \dots, y_k)\, v = \mu_{i,j}\, v$$

we have:

$$\mu = 1 + \sum_{i,j} \frac{\mu_{i,j}}{(\xi - x_i)(\eta - y_j)}.$$

Let us multiply on the left by the vector e of components $e_\pi = (-1)^\pi$, we get

$$e^t\, \mathcal{M}\, v = \mu\, e^t v.$$

Let us compute $e^t \mathcal{M}$:

$$\begin{aligned}
(e^t\, \mathcal{M})_\pi &= \sum_{\pi'} (-1)^{\pi'} \prod_i \left(\delta_{\pi'(i),\pi(i)} - \frac{t}{Nc}\, \frac{1}{(\xi - x_i)(\eta - y_{\pi'(i)})} \right) \\
&= \det \left(\delta_{j,\pi(i)} - \frac{t}{Nc}\, \frac{1}{(\xi - x_i)(\eta - y_j)} \right) \\
&= (-1)^\pi \ \det \left(\delta_{i,j} - \frac{t}{Nc}\, \frac{1}{(\xi - x_i)(\eta - y_{\pi(j)})} \right).
\end{aligned}$$

Notice that the matrix inside the determinant is of the form $\mathrm{Id} - AB^t$, where A and B are vectors, we have:

$$\det(\mathrm{I} - AB^t) = 1 - B^t A,$$

and thus

$$(e^t\,\mathcal{M})_\pi = (-1)^\pi \left(1 - \frac{t}{Nc}\sum_i \frac{1}{(\xi - x_i)(\eta - y_{\pi(i)})}\right),$$

and then, taking the residue at $\xi = x_i$ and $\eta = y_j$ gives

$$(e^t\,\mathcal{M}_{i,j})_\pi = -\frac{t}{Nc}\ (-1)^\pi\ \delta_{\pi(i),j}.$$

Multiplying $\mathcal{M}_{i,j}v = \mu_{i,j}v$ by e^t on the left thus gives

$$-\frac{t}{Nc}\ V_{i,j} = \mu_{i,j}\ e^t.v = \mu_{i,j}\,.$$

In that case, we have that the eigenvalue of $\mathcal{M}_{i,j}$ is $\mu_{i,j} = -\frac{t}{Nc}\,V_{i,j}$.

$$\mathcal{M}_{i,j}\,v = -\frac{t}{Nc}\ V_{i,j}\,v.$$

This ends the proof of the theorem. □

Corollary 8.4.2 *Since the matrices $H(x_1,\dots,x_k;y_1,\dots,y_k)$ commute with $\mathcal{M}$'s, they must have the same basis of eigenvectors. Let $\mathcal{V}_{\pi,\rho}$ be a matrix whose columns are eigenvectors normalized so that:*

$$\mathcal{V}^t\mathcal{V} = \mathrm{Id}$$

(which is possible since all our matrices are symmetric), and so that

$$\mathcal{V}e = e \qquad , \qquad e^t\mathcal{V} = e^t$$

where e is the vector $e_\pi = (-1)^\pi$.

We thus may write:

$$H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k) = \sum_\rho \mathcal{V}_{\pi,\rho}\,\mathcal{V}_{\pi',\rho}\ \ \mathcal{H}_\rho(x_1,\dots,x_k;y_1,\dots,y_k)$$

where the $\mathcal{H}_\rho(x_1,\dots,x_k;y_1,\dots,y_k)$'s are the eigenvalues of $H(x_1,\dots,x_k;y_1,\dots,y_k)$, indexed by a permutation ρ.

This can also be written as:

$$\mathcal{H} = \mathcal{V}^t H \mathcal{V} \qquad \text{is diagonal.}$$

8.4.2 Planar Discs

Here we restrict ourselves to the $g = 0$ planar case, and also to the case where we have only one boundary, i.e. $\pi'^{-1} \circ \pi$ has only one cycle, and up to renaming the variables, we can always choose $\pi = \mathrm{Id}_k$ and π' as the shift $\pi'(i) = i - 1 \bmod k$, i.e. $\pi' = (1 \to k \to k-1 \to k-2 \to \cdots \to 2 \to 1)$. Our goal in this section is to compute explicitly all the generating functions

$$H^{(0)}_{\mathrm{Id}_k,(1\to 2\to\cdots\to k\to 1)}(x_1,\ldots,x_k;y_1,\ldots,y_k).$$

This is achieved by the following theorem, and it uses the knowledge of the spectral curve $E(x,y) = 0$, which we have written parametrically in Sect. 8.3 above as $x = x(z), y = y(z)$, or in Theorem 8.3.2:

Theorem 8.4.5 *for* $k = 1$

$$H^{(0)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x(z);y(z')) = \frac{-1}{c}\,\frac{E(x(z),y(z'))}{(x(z)-x(z'))\,(y(z)-y(z'))}, \tag{8.4.6}$$

and for $k > 1$*:*

$$\begin{aligned} &H^{(0)}_{\mathrm{Id}_k,(1\to 2\to\cdots\to k\to 1)}(x_1,\ldots,x_k;y_1,\ldots,y_k) \\ &= \sum_\sigma C_\sigma(x_1,\ldots,x_k;y_1,\ldots,y_k)\,\prod_{i=1}^k H^{(0)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_i;y_{\sigma(i)}) \end{aligned} \tag{8.4.7}$$

where the coefficients $C_\sigma(x_1,\ldots,x_k;y_1,\ldots,y_k)$ *are some universal rational functions (defined further below) of the* x_i*'s and* y_i*'s, they are independent of the parameters* t_k*'s and* $c_{\pm,\pm}$.

Again, the proof of this theorem is very technical and far from straightforward, and we refer the motivated reader to [72]. Equation (8.4.7) can also be derived from Theorem 8.4.3.

Proof Let us first prove Eq. (8.4.6) by using Tutte's method again.

Recall that we have computed the generating function (see Definition 8.2.2)

$$U^{(0)}_0(x,y) = -V_2'(y) + cx - \sum_{j=2}^{\tilde d}\sum_{k=0}^{j-2-k} G^{(0)}_{0,k}(x)$$

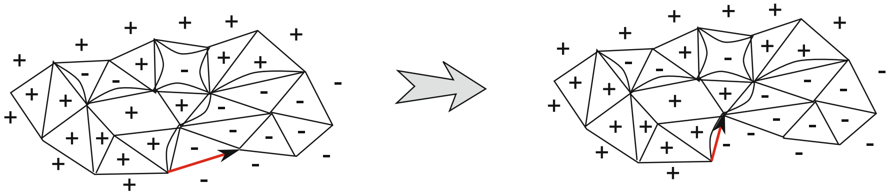
which counts planar maps with 1 mixed boundary, made of a $-$ boundary of length k and a $+$ boundary of arbitrary length weighted by $x^{-1-\text{length}}$, and in Eq. (8.3.1) we have found:

$$U_0^{(0)}(x, y) = \frac{E(x, y)}{c(y - Y(x))}.$$

Then, consider a planar Ising map with a unique marked face with $(+, -)$ boundary with arbitrary lengths.

Chose the first $-$ edge on the boundary, and we shall erase it. Several possibilities may occur:

- on the other side of the removed edge, we have a j gon of sign $-$



then the corresponding term in Tutte equation will be:

$$by(c + H_{\mathbf{1},\mathbf{1}}^{(0)}(x; y)) = \sum_j \tilde{t}_j y^{j-1}(c + H_{\mathbf{1},\mathbf{1}}^{(0)}(x; y)) + \text{other possibilities}.$$

Notice that erasing the edge can be done only if the length is positive, i.e. if the power of y is strictly negative, which we can write

$$b\left(yH_{\mathbf{1},\mathbf{1}}^{(0)}(x; y)\right)_- = \left(\sum_j \tilde{t}_j y^{j-1} H_{\mathbf{1},\mathbf{1}}^{(0)}(x; y)\right)_- + \text{other possibilities}.$$

Recall that by definition of V_2'

$$by - \sum_j \tilde{t}_j y^{j-1} = V_2'(y),$$

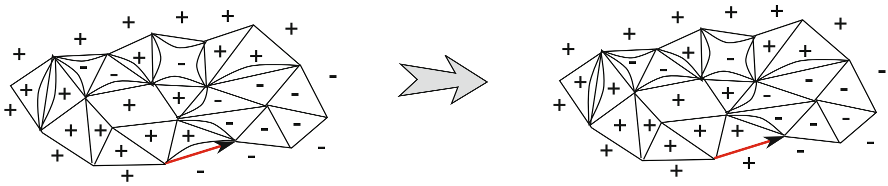
and observe that the positive powers of y in $V_2'(y)H_{\mathbf{1},\mathbf{1}}^{(0)}(x; y)$ is precisely the definition Eq. (8.2.3) in Sect. 8.2.1 of $U_0^{(0)}(x, y) + cV_2'(y) - c^2x$:

$$\left(V_2'(y)(c + H_{\mathbf{1},\mathbf{1}}^{(0)}(x; y))\right)_+ = U_0^{(0)}(x, y) + cV_2'(y) - c^2x,$$

and thus the equation is

$$V_2'(y)(c + H_{\mathbf{1},\mathbf{1}}^{(0)}(x;y)) = U_0^{(0)}(x,y) + cV_2'(y) - c^2 x + \text{other possibilities}$$

- on the other side of the removed edge, we have a bicolored $(+-)$ face, i.e. after removing the edge, we get an edge of sign $+$,



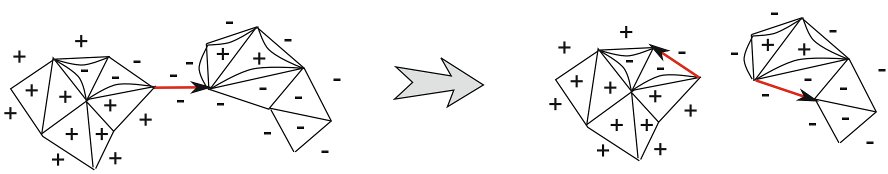
the corresponding term in Tutte equation will be:

$$b\left(y(c + H_{\mathbf{1},\mathbf{1}}^{(0)}(x;y))\right)_- = c\left(x\,(c + H_{\mathbf{1},\mathbf{1}}^{(0)}(x;y))\right)_- + \text{other possibilities}$$

where in the right hand side we need to keep only strictly negative powers of x. Notice that the positive powers of x in $x\,H_{\mathbf{1},\mathbf{1}}^{(0)}(x;y)$ give $\tilde{W}(y) = V_2'(y) - cX(y)$, and thus we get

$$\left(by(c + H_{\mathbf{1},\mathbf{1}}^{(0)}(x;y))\right)_- = cx\,(c+H_{\mathbf{1},\mathbf{1}}^{(0)}(x;y)) - c(V_2'(y) - cX(y)) + \text{other possibilities}$$

- on the other side of the removed edge, we have the same face



then the corresponding term in Tutte equation will be:

$$\begin{aligned}\left(by(c + H_{\mathbf{1},\mathbf{1}}^{(0)}(x;y))\right)_- &= \tilde{W}(y)\,(c + H_{\mathbf{1},\mathbf{1}}^{(0)}(x;y)) + \text{other possibilities}\\ &= (V_2'(y) - cX(y))\,(c + H_{\mathbf{1},\mathbf{1}}^{(0)}(x;y)) + \text{other possibilities.}\end{aligned}$$

Finally, putting all possibilities together we get

$$\begin{aligned}V_2'(y)\,(c + H_{\mathbf{1},\mathbf{1}}^{(0)}(x;y)) = {} & U_0^{(0)}(x,y) + cV_2'(y) - c^2 x\\ & + cx\,(c + H_{\mathbf{1},\mathbf{1}}^{(0)}(x;y)) - c(V_2'(y) - cX(y))\\ & + (V_2'(y) - cX(y))\,(c + H_{\mathbf{1},\mathbf{1}}^{(0)}(x;y))\end{aligned}$$

many terms cancel and it remains

$$c(X(y)-x))\,H^{(0)}_{\mathbf{1},\mathbf{1}}(x;y) = U^{(0)}_0 = \frac{E(x,y)}{y-Y(x)}.$$

This proves the first part of the theorem.

Then we use:

$$[H,\mathcal{A}_1]_{\pi,\pi'} = 0$$

with $\pi = \mathrm{Id}_k$ and $\pi' = S_k$. This gives:

$$\begin{aligned}&\frac{Nc}{t}\,(y_1-y_k)\,H_{\mathrm{Id}_k,S_k}(x_1,\dots,x_k;y_1,\dots,y_k)\\ &=\sum_{j\neq 1}\frac{1}{x_1-x_j}\left(H_{(1,j),S_k}(x_1,\dots,x_k;y_1,\dots,y_k)-H_{\mathrm{Id}_k,S_k(1,j)}(x_1,\dots,x_k;y_1,\dots,y_k)\right)\end{aligned}$$

and keep only planar terms:

$$H_{\mathrm{Id}_k,S_k}(x_1,\dots,x_k;y_1,\dots,y_k)\to\frac{N}{t}\hat{H}^{(0)}_{\mathrm{Id}_k,S_k}(x_1,\dots,x_k;y_1,\dots,y_k)$$

and

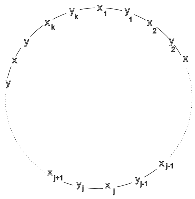
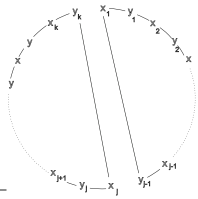
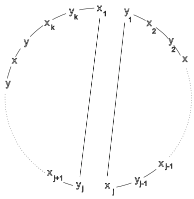
$$\begin{aligned}H_{(1,j),S_k}(x_1,\dots,x_k;y_1,\dots,y_k)\to\frac{N^2}{t^2}&\hat{H}^{(0)}_{\mathrm{Id}_{j-1},S_{j-1}}(x_j,x_2,\dots,x_{j-1};y_1,\dots,y_{j-1})\\ &\hat{H}^{(0)}_{\mathrm{Id}_{k+1-j},S_{k+1-j}}(x_1,x_{j+1},\dots,x_k;y_j,\dots,y_k)\end{aligned}$$

and

$$\begin{aligned}H_{\mathrm{Id}_k,S_k\circ(1,j)}(x_1,\dots,x_k;y_1,\dots,y_k)\to\frac{N^2}{t^2}&\hat{H}^{(0)}_{\mathrm{Id}_{j-1},S_{j-1}}(x_2,\dots,x_j;y_2,\dots,y_j)\\ &\hat{H}^{(0)}_{\mathrm{Id}_{k+1-j},S_{k+1-j}}(x_1,x_{j+1},\dots,x_k;y_1,y_{j+1},\dots,y_k)\end{aligned}$$

$$\begin{aligned}&c\,(y_k-y_1)\,\hat{H}^{(0)}_{\mathrm{Id}_k,S_k}(x_1,\dots,x_k;y_1,\dots,y_k)\\ &=\sum_{j\neq 1}\frac{1}{x_1-x_j}\Big(\hat{H}^{(0)}_{\mathrm{Id}_{j-1},S_{j-1}}(x_1,\dots,x_{j-1};y_1,\dots,y_{j-1})\quad\hat{H}^{(0)}_{\mathrm{Id}_{k+1-j},S_{k+1-j}}(x_j,\dots,x_k;y_j,\dots,y_k)\\ &\quad-\hat{H}^{(0)}_{\mathrm{Id}_{j-1},S_{j-1}}(x_j,x_2,\dots,x_{j-1};y_1,\dots,y_{j-1})\quad\hat{H}^{(0)}_{\mathrm{Id}_{k+1-j},S_{k+1-j}}(x_1,x_{j+1},\dots,x_k;y_j,\dots,y_k)\Big)\end{aligned}$$

This can be illustrated as:

$c(y_k - y_1)$  $= \sum_j \frac{1}{x_1 - x_j}$  $-$ 

We see that at each step we split the set of variables $\{x_i\}$'s and $\{y_i\}$'s into disjoint subsets, by drawing two arcs, which split the circle into two circles. The two arcs can never cross.

By an easy recursion, we shall eventually split the circle by a set of arcs, in order to reach only circles of length 2, i.e. a product of $\hat{H}^{(0)}_{\mathbf{1,1}}(x_i; y_{\sigma(i)})$ with σ a permutation. Moreover, σ must be a "planar" permutation, i.e. it draws a link pattern on the circle, which can never cross itself.

Therefore there exists some coefficients C_σ's which are rational functions of the x_i's and of the y_i's, such that

$$\hat{H}^{(0)}_{\mathrm{Id}_k, S_k}(x_1, \ldots, x_k; y_1, \ldots, y_k) = \sum_{\sigma \in \mathfrak{S}_k} C_\sigma(x_1, \ldots, x_k; y_1, \ldots, y_k) \prod_{i=1}^{k} \hat{H}^{(0)}_{\mathbf{1,1}}(x_i; y_{\sigma(i)}).$$

where $C_\sigma = 0$ if σ is not planar.

By inserting this expression into the equation for $\hat{H}^{(0)}_{\mathrm{Id}_k, S_k}$, one finds that the coefficients C_σ have to satisfy the recursion:

$$C_\sigma(x_1, \ldots, x_k; y_1, \ldots, y_k) = \frac{1}{c\,(x_{\sigma^{-1}(k)} - x_1)} \sum_{j=1}^{k-1} \sum_{\tau \in \mathfrak{S}(1,\ldots,j)} \sum_{\rho \in \mathfrak{S}(j+1,\ldots,k)}$$
$$\delta_{\sigma, \tau\rho} \, \frac{C_\tau(x_1, \ldots, x_j; y_1, \ldots, y_j)\, C_\rho(x_{j+1}, \ldots, x_k; y_{j+1}, \ldots, y_k)}{y_k - y_j}.$$

This recursion determines all the coefficients C_σ. We shall write them explicitly below.

This ends the proof. □

- Example $k = 2$:

$$\hat{H}^{(0)}_{\mathrm{Id}_2, S_2}(x_1, x_2; y_1, y_2) = \frac{\hat{H}^{(0)}_{\mathbf{1,1}}(x_1; y_1) \quad \hat{H}^{(0)}_{\mathbf{1,1}}(x_2; y_2) - \hat{H}^{(0)}_{\mathbf{1,1}}(x_1; y_2) \quad \hat{H}^{(0)}_{\mathbf{1,1}}(x_2; y_1)}{c\, y_{21} x_{12}}.$$

- Example $k = 3$:

$$\begin{aligned}
&\hat{H}^{(0)}_{\mathrm{Id}_3,S_3}(x_1,x_2,x_3;y_1,y_2,y_3)\\
&= \frac{\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_1;y_1)\ \hat{H}^{(0)}_{\mathrm{Id}_2,S_2}(x_2,x_3;y_2,y_3) - \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_2;y_1)\ \hat{H}^{(0)}_{\mathrm{Id}_2,S_2}(x_1,x_3;y_2,y_3)}{c\, x_{12}\, y_{31}}\\
&\quad + \frac{\hat{H}^{(0)}_{\mathrm{Id}_2,S_2}(x_1,x_2;y_1,y_2)\ \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_3;y_3) - \hat{H}^{(0)}_{\mathrm{Id}_2,S_2}(x_3,x_2;y_1,y_2)\ \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_1;y_3)}{c\, x_{13}\, y_{31}}
\end{aligned}$$

which gives:

$$\begin{aligned}
&c^2\, \hat{H}^{(0)}_{\mathrm{Id}_3,S_3}(x_1,x_2,x_3;y_1,y_2,y_3)\\
&= \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_1;y_1)\ \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_2;y_2)\ \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_3;y_3)\ \frac{1}{x_{12}y_{31}}\left(\frac{1}{x_{23}y_{32}} + \frac{1}{x_{13}y_{21}}\right)\\
&\quad + \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_1;y_3)\ \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_2;y_1)\ \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_3;y_2)\ \frac{1}{x_{13}y_{31}}\left(\frac{1}{x_{12}y_{32}} + \frac{1}{x_{32}y_{21}}\right)\\
&\quad + \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_1;y_1)\ \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_2;y_3)\ \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_3;y_2)\ \frac{1}{x_{12}x_{23}y_{23}y_{31}}\\
&\quad + \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_1;y_3)\ \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_2;y_2)\ \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_3;y_1)\ \frac{1}{x_{13}x_{32}y_{21}y_{13}}\\
&\quad + \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_1;y_2)\ \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_2;y_1)\ \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_3;y_3)\ \frac{1}{x_{21}x_{13}y_{32}y_{21}}.
\end{aligned}$$

8.4.2.1 The Planar Link Patterns and the Coefficients C_σ

In the planar case, the boundary generating functions are thus of the form:

$$\hat{H}^{(0)}_{\mathrm{Id}_k,S_k}(x_1,\ldots,x_k;y_1,\ldots,y_k) = \sum_{\sigma\in\mathfrak{S}_k} C_\sigma(x_1,\ldots,x_k;y_1,\ldots,y_k)\ \prod_{i=1}^{k} \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_i;y_{\sigma(i)}).$$

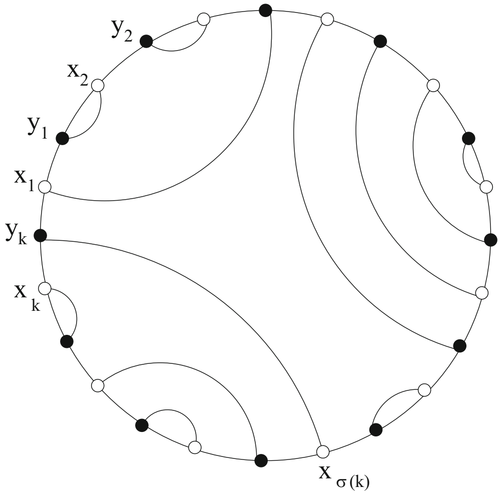
The coefficients C_σ satisfy a recursion relation. The explicit solution of this recursion was found in [72], and we just mention the result.

The coefficients C_σ are determined as follows (we recall that $\pi = \mathrm{Id}_k$ and $\pi' = S_k$ is the shift $\pi'(i) = i-1 \bmod k$, and $\ell(\sigma)$ denotes the number of cycles of a permutation σ):

- C_σ vanishes if $\ell(\sigma^{-1}\circ\pi) + \ell(\sigma^{-1}\circ\pi') - \ell(\pi'^{-1}\circ\pi) \neq k$:

$$C_\sigma \neq 0 \quad \Rightarrow \quad \ell(\sigma^{-1}\circ\pi) + \ell(\sigma^{-1}\circ\pi') - \ell(\pi'^{-1}\circ\pi) = k. \qquad (8.4.8)$$

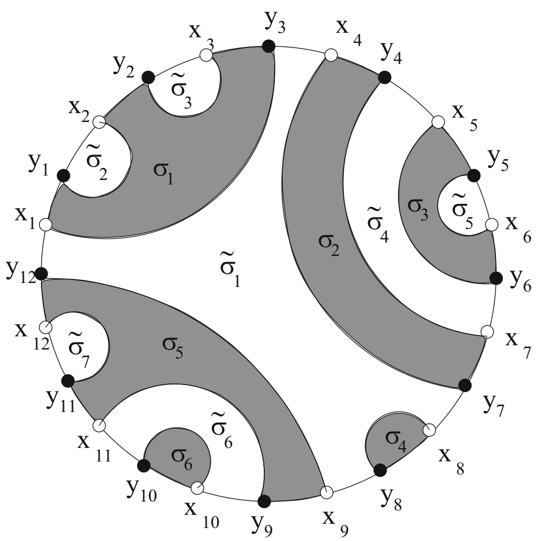
This means that σ must be a planar link pattern drawn on the cycles of $\pi'^{-1} \circ \pi$:



- If condition Eq. (8.4.8) is satisfied, we decompose the cycles of $\sigma^{-1} \circ \pi$ and $\sigma^{-1} \circ \pi'$ as follows:

$$\sigma^{-1} \circ \pi = \prod_{i=1}^{m} \sigma_i \qquad , \qquad \sigma^{-1} \circ \pi' = \prod_{i=1}^{m'} \bar{\sigma}_i.$$

In other words, each σ_i or $\bar{\sigma}_i$ is a face of the link pattern.



If j is in a cycle σ_i of $\sigma^{-1} \circ \pi$, the face is

$$x_j \to y_{\pi(j)} \to x_{\sigma_i(j)} \to y_{\pi(\sigma_i(j))} \to x_{\sigma_i(\sigma_i(j))} \to \cdots \to y_{\sigma^{-1}(j)} \to x_j$$

and we prefer to write σ_i as the ordered set of variables:

$$\sigma_i \equiv (x_j, y_{\pi(j)}, x_{\sigma_i(j)}, y_{\pi(\sigma_i(j))}, x_{\sigma_i(\sigma_i(j))}, \ldots, y_{\sigma^{-1}(j)}).$$

Similarly for $\bar{\sigma}_i$, we write

$$\bar{\sigma}_i \equiv (x_j, y_{\pi'(j)}, x_{\bar{\sigma}_i(j)}, y_{\pi'(\bar{\sigma}_i(j))}, x_{\bar{\sigma}_i(\bar{\sigma}_i(j))}, \ldots, y_{\sigma^{-1}(j)}).$$

With those notations, we shall write C_σ as a product of faces:

$$C_\sigma = \prod_{i=1}^{m} F_{\ell(\sigma_i)}(\sigma_i) \prod_{i=1}^{m'} F_{\ell(\bar{\sigma}_i)}(\bar{\sigma}_i).$$

The "face" functions $F_k(x_1, y_1, x_2, y_2, \ldots, x_k, y_k)$ are defined by the following recursion:

$$F_1(x, y) = 1$$

and

$$F_k(x_1, y_1, \ldots, x_k, y_k) = \sum_{j=1}^{k-1} \frac{F_j(x_1, y_1, \ldots, x_j, y_j)\, F_{k-j}(x_{j+1}, y_{j+1}, \ldots, x_k, y_k)}{c\,(x_1 - x_k)\,(y_k - y_j)}.$$

They have the property to be cyclically invariant.

For instance for $k = 2$ we get

$$F_2(x_1, y_1, x_2, y_2) = \frac{1}{c\,(x_1 - x_2)\,(y_2 - y_1)}.$$

In fact, it is possible to write the functions F_k's as sums over trees, this was done in [72], and we refer the interested reader to that article.

8.5 Summary: Ising Model

Let us summarize the concepts introduced in this chapter:

- The Ising model is the combinatorics of bi-colored maps (colors = $+$ and $-$, also called spin), conditioned on the number of edges separating faces of same or different colors.

We introduce Boltzman weights t_k for the number of $k-$gons of color $+$, $\tilde{t}_k$ for the number of $k-$gons of color $-$, and c_{++} for $++$ edges, c_{--} for $--$ edges, and c_{+-} for $+-$ edges. We define $c = c_{+-}/(c_{++}c_{--} - c_{+-}^2)$, $a = c_{--}/(c_{++}c_{--} - c_{+-}^2)$, $b = c_{++}/(c_{++}c_{--} - c_{+-}^2)$.

These define the potentials:

$$V_1(x) = a\frac{x^2}{2} - \sum_{k\geq 3} \frac{t_k}{k} x^k$$

$$V_2(y) = b\frac{y^2}{2} - \sum_{k\geq 3} \frac{\tilde{t}_k}{k} y^k.$$

- One can write Tutte equations by recursively erasing the marked edge of the first marked face.
- The disc amplitude $W_1^{(0)}(x)$ is an algebraic function, we define $Y(x) = \frac{1}{c}(V_1'(x) - W_1^{(0)}(x))$. It satisfies an algebraic equation:

$$E(x, Y) = 0$$

where $E(x, y)$ is a bivariate polynomial, of degree $\deg_x E = d - 1 = \deg V_1$ and $\deg_y E = \tilde{d} - 1 = \deg V_2$. It can be written

$$E(x, y) = (V_1'(x) - cy)(V_2'(y) - cx) - \frac{1}{c} P_0^{(0)}(x, y) + tc$$

where $P_0^{(0)}(x, y)$ is a polynomial of degree at most $\deg V_1''$ in x and $\deg V_2''$ in y.

The polynomial $P_0^{(0)}(x, y)$ is uniquely determined by requiring that the algebraic equation $E(x, y) = 0$ defines a Riemann surface of genus 0, and by $P_0^{(0)}(x, y) = \frac{V_1'(x)}{x} \frac{V_2'(y)}{y} + O(t)$.

Since the algebraic equation has genus zero, one can find a parametric solution with rational functions:

$$\begin{cases} x = x(z) = \gamma z + \sum_{k=0}^{\deg V_2'} \alpha_k z^{-k} \\ Y = y(z) = \gamma z^{-1} + \sum_{k=0}^{\deg V_1'} \beta_k z^k \end{cases}$$

and the coefficients γ, α_k, β_k's are uniquely determined by requiring that

$$V_1'(x(z)) - cy(z) \underset{z\to\infty}{\sim} \frac{t}{\gamma z} + O(z^{-2})$$

$$V_2'(y(z)) - cx(z) \underset{z\to 0}{\sim} \frac{t\,z}{\gamma} + O(z^2),$$

$$\gamma^2 = c_{+-}t + O(t^2).$$

The disc amplitude is then:

$$W_1^{(0)}(x(z)) = V_1'(x(z)) - cy(z).$$

- The cylinder amplitude $W_2^{(0)}$.

  In the z variables (i.e. writing $x_i = x(z_i)$), we have:

$$W_2^{(0)}(x_1,x_2) = \frac{1}{(z_1-z_2)^2\, x'(z_1)\, x'(z_2)} - \frac{1}{(x_1-x_2)^2}.$$

  The differential form:

$$B(z_1,z_2) = W_2^{(0)}(x_1,x_2)dx_1d_2 + \frac{dx_1dx_2}{(x_1-x_2)^2} = \frac{dz_1dz_2}{(z_1-z_2)^2}$$

  is the fundamental second kind differential.

  The cylinder amplitude is thus universal, in the z variables it is always the fundamental second kind differential.
- Higher topology amplitudes are given by the topological recursion.
- There are also algebraic formulae for enumerating maps with multi-colored boundaries.

8.6 Exercises

Exercise 1 For Ising quadrangulations (only t_4 and $\tilde{t}_4$ non vanishing), count elements of $\mathbb{M}_1^{(0)}(2)$ and $\mathbb{M}_1^{(0)}(3)$, i.e. planar Ising quadrangulations with one boundary (of color $+$), and with two and three vertices.

Answer:

$$W_1^{(0)} = \frac{t}{x} + t^2\frac{c_{++}}{x^3} + t^3\left(\frac{2c_{++}^2}{x^5} + \frac{2t_4c_{++}^3 + 2\tilde{t}_4c_{+-}^2c_{--}}{x^3}\right) + O(t^4).$$

Exercise 2 For Ising quadrangulations (only t_4 and $\tilde{t}_4$ non vanishing), find the disc amplitude. Write the parametrization:

$$x(z) = \gamma z + \alpha_1 z^{-1} + \alpha_3 z^{-3},$$

$$y(z) = \gamma z^{-1} + \beta_1 z + \beta_3 z^3.$$

Find that the algebraic equation satisfied by $R = \gamma^2$ is of degree 7:

$$R = \frac{t}{c} + R\frac{(bc - 3a\tilde{t}_4R)(ac - 3bt_4R)}{(c^2 - 9t_4\tilde{t}_4R^2)^2} + 3\frac{t_4\tilde{t}_4}{c^2}R^3.$$

Consider the special case where $t_4 = \tilde{t}_4 = 1$, $a = b$ and $c = 1$. In that case we shall have $\alpha_1 = \beta_1$ and $\alpha_3 = \beta_3$. Find the equation of degree 5 for $R = \gamma^2$:

$$R = t + \frac{a^2 R}{(1+3R)^2} + 3R^3.$$

Exercise 3 For the same Ising model on quadrangulations with $t_4 = \tilde{t}_4 = 1$, $a = b$ and $c = 1$, find the critical points, i.e. the values of a and t_4 such that $W_1^{(0)}(x)$ has non-square root singularities. This is obtained by requiring that $x'(z)$ and $y'(z)$ have a common zero.

Find the critical lines at which there is a $3/2$ singularity:

$$\frac{9}{2}t = 1 - \frac{3}{8}a^2$$

and

$$\frac{9}{2}t = -1 + 3a \mp 2a\sqrt{a}.$$

If furthermore we require that $x'(z), y'(z), x''(z), y''(z)$ have a common zero, then find $a = 4$ and $t = -10/9$. This corresponds to a $(p, q) = (4, 3)$ critical point as in Chap. 5.

Bibliography

1. M. Abramowitz, I.A Stegun, *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*, vol. 55 (Dover, New York, 1972)
2. G. Akemann, Higher genus correlators for the hermitian matrix model with multiple cuts. Nucl. Phys. B **482**(1–2), 403–430 (1996)
3. A.S. Alexandrov, A. Mironov, A. Morozov, Solving Virasoro constraints in matrix models. Fortschr. Phys. **53**, 512–521 (2005)
4. J. Ambjørn, L. Chekhov, Y. Makeenko, Higher genus correlators from the hermitian one-matrix model. Phys. Lett. B **282**, 341–348 (1992)
5. J. Ambjørn, L. Chekhov, C.F. Kristjansen, Y. Makeenko, Matrix model calculations beyond the spherical limit. Nucl. Phys. B **404**, 127–172 (1993)
6. E. Arbarello, M. Cornalba, Combinatorial and algebro-geometric cohomology classes on the moduli space of curves. J. Algebraic Geom. **5**, 705–709 (1996)
7. E. Arbarello, J.D. Harris, M. Cornalba, P. Griffiths, *Geometry of Algebraic Curves: Volume II with a Contribution by Joseph Daniel Harris*, vol. 268 (Springer, New York, 2011)
8. O. Babelon, D. Bernard, M. Talon, *Introduction to Classical Integrable Systems*. Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 2002)
9. C. Berge, *The Theory of Graphs* (Dover, New York, 2003)
10. M. Bergère, B. Eynard, Determinantal formulae and loop equations. ArXiv e-prints (2009)
11. M. Bergère, G. Borot, B. Eynard, Rational differential systems, loop equations, and application to the q-th reductions of kp. arXiv preprint (2013). arXiv:1312.4237
12. S. Bergmann, Sur la fonction-noyau d'un domaine et ses applications dans la théorie des transformations pseudo-conformes. Memor. Sci. Math. 108 (1948)
13. S. Bergman, M. Schiffer, Kernel functions and conformal mapping. Compos. Math. **8**, 205–249 (1951)
14. S. Bergman, M. Schiffer, et al., A representation of green's and neumann's functions in the theory of partial differential equations of second order. Duke Math. J. **14**(3), 609–638 (1947)
15. M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa, Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes. Commun. Math. Phys. **165**, 311–428 (1994)
16. P. Bleher, A. Its, *Random Matrix Models and Their Applications*, vol. 40 (Cambridge university Press, Cambridge, 2001)
17. G. Borot, B. Eynard, Geometry of spectral curves and all order dispersive integrable system. ArXiv e-prints (2011)
18. V. Bouchard, B. Eynard, Think globally, compute locally. J. High Energy Phys. **2**, 143 (2013)
19. E. Brezin, N. Deo, Smoothed correlators for symmetric double-well matrix models: some puzzles and resolutions. eprint arXiv:cond-mat/9805096 (1998)

B. Eynard, *Counting Surfaces*, Progress in Mathematical Physics 70,
DOI 10.1007/978-3-7643-8797-6

20. W.G. Brown, Enumeration of triangulations of the disk. Proc. Lond. Math. Soc. **14**, 746–768 (1964)
21. K.M. Chapman, M. Mulase, B. Safnuk, The Kontsevich constants for the volume of the moduli of curves and topological recursion. arXiv preprint (2010). arXiv:1009.2055
22. L.O. Chekhov, Genus-one correction to multicut matrix model solutions. Theor. Math. Phys. **141**, 1640–1653 (2004)
23. R. Conte, M. Musette, *The Painlevé Handbook* (Springer, Berlin, 2008)
24. F. David, A. Kupiainen, R. Rhodes, V. Vargas, Liouville quantum gravity on the Riemann sphere. ArXiv e-prints (2014)
25. P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*, vol. 3 (American Mathematical Society, Providence, 2000)
26. P. Deift, D. Gioev, *Random Matrix Theory: Invariant Ensembles and Universality*, vol. 18 (American Mathematical Society, Providence, 2009)
27. M.R. Douglas, S.H. Shenker, Strings in less than one dimension. Nucl. Phys. B **335**, 635–654 (1990)
28. B. Duplantier, S. Sheffield, Liouville quantum gravity and KPZ. ArXiv e-prints (2008)
29. T. Ekedahl, S. Lando, M. Shapiro, A. Vainshtein, On Hurwitz numbers and Hodge integrals. C. R. Acad. Sci. Ser. I Math. **328**(12), 1175–1180 (1999)
30. B. Eynard, Topological expansion for the 1-hermitian matrix model correlation functions. J. High Energy Phys. **11**, 31 (2004)
31. B. Eynard, Formal matrix integrals and combinatorics of maps. ArXiv Mathematical Physics e-prints (2006)
32. B. Eynard, Formal matrix integrals and combinatorics of maps, in *Random Matrices, Random Processes and Integrable Systems*, ed. by J. Harnad. CRM-Springer Series on Mathematical Physics (2011) [ISBN 978-1-4419-9513-1]
33. B. Eynard, A.P. Ferrer, 2-matrix versus complex matrix model, integrals over the unitary group as triangular integrals. Commun. Math. Phys. **264**, 115–144 (2006)
34. B. Eynard, N. Orantin, Invariants of algebraic curves and topological expansion. Commun. Number Theory Phys. **1**(2), 347–452 (2007)
35. B. Eynard, N. Orantin, M. Mariño, Holomorphic anomaly and matrix models. J. High Energy Phys. **6**, 58 (2007)
36. H.M. Farkas, I. Kra, *Riemann Surfaces* (Springer, New York, 1992)
37. J.D. Fay, *Theta Functions on Riemann Surfaces* (Springer, New York, 1973)
38. A.S. Fokas, A. Its, A. Kitaev, The isomonodromy approach to matrix models in 2d quantum gravity. Commun. Math. Phys. **147**, 395–430 (1992)
39. P.J. Forrester, *Log-Gases and Random Matrices (LMS-34)* (Princeton University Press, Princeton, 2010)
40. P.D. Francesco, P. Ginsparg, J. Zinn-Justin, 2d gravity and random matrices. Phys. Rep. **254**(1), 1–133 (1995)
41. P.D. Francesco, P. Mathieu, D. Sénéchal, *Conformal Field Theory* (Springer, New York, 1997)
42. D.J. Gross, T. Piran, S. Weinberg, *Two Dimensional Quantum Gravity and Random Surfaces* (World Scientific, Singapore, 1992)
43. J.L. Gross, T.W. Tucker, *Topological Graph Theory*. Wiley-Interscience Series in Discrete Mathematics and Optimization (Wiley, New York, 1987)
44. A. Guionnet, *Large Random Matrices: Lectures on Macroscopic Asymptotics: Ecole d'Ete de Probabilites de Saint-Flour XXXVI-2006* (Springer, Berlin/Heidelberg, 2008)
45. J. Harer, D. Zagier, The euler characteristic of the moduli space of curves. Invent. Math. **85**(3), 457–485 (1986)
46. J.M. Harris, J.L. Hirst, M.J. Mossinghoff, *Combinatorics and Graph Theory* (Springer, New York, 2008)
47. R. Hirota, Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons. Phys. Rev. Lett. **27**, 1192 (1971)
48. G. 't Hooft, A planar diagram theory for strong interactions. Nucl. Phys. B **72**(3), 461–473 (1974)

49. A.R. Its, V.B. Matveev, Schrödinger operators with finite-gap spectrum and n-soliton solutions of the korteweg-de vries equation. Theor. Math. Phys. **23**(1), 343–355 (1975)
50. D. Jackson, T.I. Visentin, *An Atlas of the Smaller Maps in Orientable and Nonorientable Surfaces* (CRC Press, Boca Raton, 2000)
51. M. Jimbo, T. Miwa, Solitons and infinite dimensional Lie algebras. RIMS **19**, 943–1001 (1983)
52. M. Jimbo, T. Miwa, K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients: I. General theory and τ-function. Phys. D **2**(2), 306–352 (1981)
53. B.B. Kadomtsev, V.I. Petviashvili, On the stability of solitary waves in weakly dispersing media. Sov. Phys. Dokl **15**, 539–541 (1970)
54. É. Brezin, V. Kazakov, D. Serban, P. Wiegmann, A. Zabrodin (eds.), *Applications of Random Matrices in Physics*. Nato Science Series II, vol. 221 (Springer, Berlin, 2006)
55. V.G. Knizhnik, A.M. Polyakov, A.B. Zamolodchikov, Fractal structure of 2D quantum gravity. Mod. Phys. Lett. **A3**, 819 (1988)
56. A. Kokotov, D. Korotkin, Tau-functions on hurwitz spaces. Math. Phys. Anal. Geom. **7**(1), 47–96 (2004)
57. M. Kontsevich, Intersection theory on the moduli space of curves and the matrix airy function. Commun. Math. Phys. **147**(1), 1–23 (1992)
58. D.J. Korteweg, G. de Vries, Xli. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. Philos. Mag. Ser. 5 **39**(240), 422–443 (1895)
59. I. Krichever, Methods of algebraic geometry in the theory of non linear equations. Russ. Math. Surv. **32**, 185–213 (1977)
60. I. Krichever, The Tau function of the universal Whitham hierarchy, matrix models and topological field theories. Commun. Pure Appl. Math. **47**(4), 437–475 (1994)
61. S.K. Lando, A.K. Zvonkin, *Graphs on Surfaces and Their Applications* (Springer, Berlin/Heidelberg, 2003)
62. E. Looijenga, Cellular decompositions of compactified moduli spaces of pointed curves, in *The Moduli Space of Curves* (Springer, New York, 1995), pp. 369–400
63. M.L. Mehta, *Random Matrices*, vol. 142 (Academic, New York, 2004)
64. A.A. Migdal, Loop equations and 1n expansion. Phys. Rep. **102**(4), 199–290 (1983)
65. M. Mirzakhani, Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces. Invent. Math. **167**(1), 179–222 (2006)
66. B. Mohar, C. Thomassen, *Graphs on Surfaces* (Johns Hopkins University Press, Baltimore, 2001)
67. M. Mulase, B. Safnuk, Mirzakhani's recursion relations, Virasoro constraints and the KdV hierarchy (2006), p. 21. arXiv:math/0601194
68. D. Mumford, *The Red Book of Varieties and Schemes: Includes the Michigan Lectures (1974) on Curves and Their Jacobians*, vol. 1358 (Springer, New York, 1999)
69. A. Okounkov, Generating functions for intersection numbers on moduli spaces of curves. Int. Math. Res. Not. **2002**(18), 933–957 (2002)
70. A. Okounkov, R. Pandharipande, Gromov-Witten theory, Hurwitz numbers, and matrix models, I. ArXiv Mathematics e-prints (2001)
71. A. Okounkov, R. Pandharipande, Gromov-Witten theory, Hurwitz numbers, and matrix models, in *I, Proc. Symposia Pure Math*, vol. 80 (2009), pp. 325–414
72. N. Orantin, B. Eynard, Mixed correlation functions in the 2-matrix model, and the Bethe ansatz. J. High Energy Phys. **0508**, 028 (2005)
73. C. Chekhov, B. Eynard, N. Orantin, Free energy topological expansion for the 2-matrix model. J. High Energy Phys. **0612**, 053 (2006)
74. G. Parisi, J.B. Zuber, E. Brezin, C. Itzykson, Planar diagrams. Commun. Math. Phys. **59**, 35 (1978)
75. R.C. Penner, Perturbative series and the moduli space of riemann surfaces. J. Differ. Geom. **27**(1), 35–53 (1988)
76. A.M. Polyakov, Quantum geometry of bosonic strings. Phys. Lett. **B103**, 207–210 (1981)

77. A. Prats Ferrer, B. Eynard, P. di Francesco, J.-B. Zuber, Correlation functions of Harish-Chandra integrals over the orthogonal and the symplectic groups. J. Stat. Phys. **129**, 885–935 (2007)
78. H.E. Rauch, Weierstrass points, branch points, and moduli of riemann surfaces. Commun. Pure Appl. Math. **12**(3), 543–560 (1959)
79. S. Rohde, O. Schramm, Basic properties of SLE. ArXiv Mathematics e-prints (2001)
80. K. Strebel, *Quadratic Differentials* (Springer, New York, 1984)
81. T. Tao, *Topics in Random Matrix Theory*, vol. 132 (American Mathematical Society, Providence, 2012)
82. C.A. Tracy, H. Widom, Level-spacing distributions and the airy kernel. Commun. Math. Phys. **159**(1), 151–174 (1994)
83. W.T. Tutte, A census of planar triangulations. Can. J. Math. **14**, 21 (1962)
84. W.T. Tutte, A census of planar maps. Can. J. Math. **15**, 249–271 (1963)
85. W.T. Tutte, On the enumeration of planar maps. Bull. Am. Math. Soc. **74**(1), 64–74 (1968)
86. G.C. Wick, The evaluation of the collision matrix. Phys. Rev. **80**, 268–272 (1950)
87. E. Witten, Two dimensional gravity and intersection theory on moduli space. Surv. Differ. Geom. **1**, 243–310 (1991)
88. S. Wolpert, On the homology of the moduli space of stable curves. Ann. Math. **118**(3), 491–523 (1983)
89. D. Zvonkine, Strebel differentials on stable curves and Kontsevich's proof of Witten's conjecture. arXiv preprint math/0209071 (2002)

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B. Eynard, *Counting Surfaces*, Progress in Mathematical Physics 70,
DOI 10.1007/978-3-7643-8797-6