

Theoretical and Mathematical Physics

Anatoli Babin
Alexander Figotin

Neoclassical Theory of Electromagnetic Interactions

A Single Theory for Macroscopic and
Microscopic Scales

 Springer

Neoclassical Theory of Electromagnetic Interactions

Theoretical and Mathematical Physics

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To our parents
Vladimir Shvarts and Varvara Babina
Lev and Valentina Figotin

Preface

It is hard to overestimate the importance of the theory of electromagnetic (EM) interactions in the body of our knowledge. It is the only interaction that is equally significant at all spatial scales: gigantic cosmic scales which are relevant to EM radiation propagating through the Universe, macroscopic scales covering our life on Earth, and microscopic atomic scales. The significance of EM interactions at all spatial scales poses great challenges to their fundamental theory. The best theoretical minds worked tirelessly through the span of several centuries to develop and perfect the electromagnetic theory which we have today. The EM theory was advanced many times by remarkable discoveries during its historical development. These discoveries include the wave nature of light and its polarization, the Maxwell prediction of EM waves, the unification of light, electric and magnetic phenomena, special relativity, and quantum theory. The phenomena at cosmic and macroscopic scales are described remarkably well by classical electrodynamics, whereas the understanding of phenomena at atomic scales requires quantum mechanics (QM). The classical and quantum-mechanical approaches are fundamentally different. Many atomic phenomena do not find explanation in the classical framework, and within quantum mechanics the micro- and macroscales are related rather loosely by the “correspondence principle”.

The purpose of this monograph is to present our recently developed neoclassical theory of electromagnetic interactions. We demonstrate that the classical EM theory can be extended down to atomic scales so that many phenomena at atomic scales, usually explained in the quantum-mechanical framework, can be explained in our neoclassical framework. The proposed extension bridges the classical and quantum-mechanical approaches, so they are not separated by a gap but rather overlap in a large common domain. Our theory, though similar to QM in some respects, is markedly different from it. In particular, (i) there is no need, in our theory, for the correspondence principle and consequent quantization procedure to obtain the wave equation; (ii) the Heisenberg uncertainty principle, though quite often applicable, is not a universal principle; (iii) there is no configuration space; (iv) there is no probabilistic interpretation of the wave function.

Our neoclassical theory models the EM interactions between elementary charges based on the Lagrangian framework. The theory is manifestly relativistic, and it is self-consistent at all spatial scales. At the center of the theory is the concept of an elementary charge which is not a point but an entity distributed in space, propagating as a wave; nevertheless it can be well localized in relevant situations, exhibiting point-like features. The behavior of the charge in different regimes is described by different approximations in the same framework. In particular, the theory accounts for the de Broglie phase wave mechanics, the Schrödinger wave mechanics, including the hydrogen atom spectrum, the EM radiation phenomena, and the classical theory of point charges interacting with the EM field through the Lorentz forces.

We gratefully acknowledge the support of Dr. A. Nachman through the U.S. Air Force Office of Scientific Research (AFOSR) and Dr. R. Albanese (AFRL—retired). It is due to their unwavering interest in and support of our studies over many years that we were able to complete numerous projects that provided the basis for our neoclassical theory of electromagnetic interactions. A. Babin is very grateful to his wife Lioudmila Babina for her patience, understanding, and encouragement during the long work on this book.

Irvine, CA, USA

Anatoli Babin
Alexander Figotin

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Mathematical Symbols

A^μ	Four-potential
a	Size parameter
a_C	Reduced Compton wavelength, $a_C = \frac{\hbar}{m\mathcal{C}}$ or $a_C = \frac{\lambda}{m\mathcal{C}}$
a_B	The Bohr radius, $a_B = \frac{\hbar^2}{q^2 m}$ or $a_B = \frac{\lambda^2}{q^2 m}$
α	Fine structure constant, $\alpha = \frac{q^2}{\hbar\mathcal{C}}$ or $\alpha = \frac{q^2}{\lambda\mathcal{C}}$
$\mathbf{A}$	Vector potential of EM field
$\mathbf{A}_{\text{ex}}$	Vector potential of the external EM field
$\tilde{A}_\ell^\mu$	Actual field potential for the ℓ -th charge, formulas (11.8) and (11.9)
$\mathbf{B}$	Magnetic field
c	Speed of light
C^1	Space of continuously differentiable functions
C^2	Space of twice continuously differentiable functions
χ	The elementary action parameter which is equal to (or is close to) the reduced Planck constant $\hbar$
∂_t	Partial derivative with respect to time, sometimes the notation is used as a complete derivative with respect to time when it is clear from the context
∂_i	Partial derivative with respect to x^i , $i = 1, 2, 3$ when the relativistic notation is used; in non-relativistic cases we may use the notation x_i for coordinates and ∂_i is a partial derivative with respect to x_i
$\mathrm{d}x$	Four-dimensional volume element, $\mathrm{d}x = \mathrm{d}x^0 \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3$
$\mathrm{d}\mathbf{x}$	three-dimensional volume element, $\mathrm{d}\mathbf{x} = \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3$
$\Delta = \nabla^2$	Laplace operator, $\Delta\psi = \nabla^2\psi = \nabla \cdot (\nabla\psi)$
$\mathbf{E}$	Electric field
H^1	Sobolev space with norm (35.3)
$\tilde{\partial}_t$	Covariant derivative with respect to time, $\tilde{\partial}_t = \partial_t + \frac{iq}{\chi}\varphi_{\text{ex}}$ or $\tilde{\partial}_t = \partial_t + \frac{iq}{\chi}\tilde{\varphi}$
$\mathcal{E}$	Total energy, $\mathcal{E} = cP^0 = \int_{\mathbb{R}^3} T^{00}(x)\mathrm{d}\mathbf{x}$

$F^{\mu\nu}$	Electromagnetic field tensor
G'	The nonlinearity
G	The nonlinear potential
$\hbar$	The reduced Planck constant, $\hbar = \frac{h}{2\pi}$
J^ν	4-current density in the relativistic notation, $J^\mu = (c\rho, \mathbf{J})$
$\mathbf{J}$	Current density
$\bar{\mathbf{J}}$	Total current, $\bar{\mathbf{J}} = \int_{\mathbb{R}^3} \mathbf{J}(t, \mathbf{x}) d\mathbf{x}$
ℓ	The label of the ℓ -th charge in the system
$\bar{\ell}$	The label of $\bar{\ell}$ -th variable of a general Lagrangian system
$\ \cdot\ $	L^2 -norm, formula (14.25)
m	Mass parameter for the charge
∇	Gradient, $\nabla\psi = (\partial_1\psi, \partial_2\psi, \partial_3\psi)$
$\tilde{\nabla}$	Covariant gradient, $\tilde{\nabla} = \nabla - \frac{iq}{\hbar c} \mathbf{A}_{\text{ex}}$ or $\tilde{\nabla} = \nabla - \frac{iq}{\hbar c} \tilde{\mathbf{A}}$
$\mathbf{P}$	Momentum density
P	Total momentum, $P = P^j = \int_{\mathbb{R}^3} T^{0j}(x) d\mathbf{x}$
φ	Scalar potential of EM field
φ_{ex}	Scalar potential of the external EM field
ψ	Charge distribution
ψ_ℓ	Charge distribution of ℓ -th charge
q	Value of the charge
ρ	Charge density
$\bar{\rho}$	Total charge
$\mathbb{R}^3$	3-dimensional space
$\overset{\circ}{T}^{\mu\nu}$	Canonical energy-momentum tensor of charge field
$T^{\mu\nu}$	Symmetric energy-momentum tensor
$\Theta(\omega)$	Mass correction coefficient in (15.34) given by (14.57)
$\overset{\circ}{\Theta}^{\mu\nu}$	Canonical energy-momentum tensor of EM field
$\Theta^{\mu\nu}$	Symmetric energy-momentum tensor of EM field
u	Energy density
x	Spacetime 4-vector, $x = x^\mu = (x^0, x^1, x^2, x^3) = (ct, \mathbf{x})$
$\mathbf{x}$	Space vector, $\mathbf{x} = (x^1, x^2, x^3)$ or $\mathbf{x} = (x_1, x_2, x_3)$ in non-relativistic treatment

Abbreviations

BCT	Balanced charges theory
BEM	Balanced charges electromagnetic (theory)
CEM	Classical electromagnetic (theory)
DCT	Dressed charges theory
EL	Euler–Lagrange (equations)
EM	Electromagnetic
EnMT	Energy-momentum tensor
EQS	Electroquasistatic(s)
GA	Geometric algebra
KG	Klein–Gordon (equation)
MQS	Magnetoquasistatic(s)
NKG	Nonlinear Klein–Gordon (equation)
NLS	Nonlinear Schrödinger (equation)
QED	Quantum electrodynamics
QM	Quantum mechanics
STA	Spacetime algebra
WCM	Wave-corpuscle mechanics

Chapter 1

Introduction

Based on Faraday's field ideas, Maxwell in a stroke of genius wrote his celebrated Maxwell equations unifying almost all electromagnetic (EM) phenomena known in his day. The Maxwell theory was a continuum field theory which related EM fields to continuously distributed currents and charges. The physical origin of the charges and currents was rather mysterious at that time. But as experimental and theoretical knowledge of charges and currents was growing, new ideas on their origin emerged, pointing to microscopic scales. Some of those developments inspired Lorentz to come up with a very ambitious microscopic theory which was to explain all known macroscopic EM phenomena. The Lorentz theory was based on a clear separation of Maxwell's theory of ether from Clausius's material charges, the motion of which constituted electric currents. Lorentz also made the concept of small charged particles (electrons) acted upon by the Lorentz force an integral part of his microscopic EM theory. This microscopic Maxwell–Lorentz theory became the foundation of the classical electromagnetic (CEM) theory with elementary charges treated ideally as points.

The CEM theory was very successful in describing EM phenomena at macroscopic length scales. But it was realized that at the fundamental microscopic level the theory had a problem mathematically integrating the concept of a point charge into the continuous Maxwell field theory. Namely, the electric field generated by a point charge diverges exactly at its location, causing the so-called infinite self-energy problem. Lorentz knew about this problem and proposed his model of electron which is not a point but a charge distribution (charged insulator), [345, 2.1]. This extended electron model and its modifications by Abraham and Poincaré remedied some of the problems, in particular the infinite self-energy problem, but some difficulties persisted. Referring to these internal problems with the CEM theory, R. Feynman wrote, [121]:

“Classical mechanics is a mathematically consistent theory, it just doesn't agree with experience. It is interesting, though, that the classical theory of electromagnetism is an unsatisfactory theory all by itself.”

In addition to this, the CEM theory does not provide an explanation of quantum electromagnetic phenomena at the atomic scales such as de Broglie waves and spectroscopic data of the Hydrogen atom.

We advance in this book a neoclassical EM theory that describes EM phenomena at all spatial scales—microscopic and macroscopic. The mathematical framework of our theory is conceived to satisfy a number of critical requirements presented below in the relativistic case.

The neoclassical theory aims to describe major phenomena at atomic scales including the infinite series of discrete energy levels and frequencies of the Hydrogen atom satisfying the Rydberg formula. To achieve this, we arrive at our first requirement:

(i) *An elementary charge is described by a continuous distribution (field) over the four dimensional space-time continuum.*

Assuming an elementary charge to be a continuous distribution, we want its dynamics to be consistent with the principles of locality and relativity, leading us to the second requirement:

(ii) *The charge field evolution equations have to be relativistically covariant partial differential equations.*

To ensure that a system of many moving charges interacting with the EM field satisfies well tested fundamental laws such as energy and momentum conservation laws, we impose our third requirement:

(iii) *A system of many charges and their EM fields is governed by a relativistic covariant Lagrangian field theory.*

We would also like to preserve critical features of the classical EM theory occurring in the macroscopic scale, which is reflected in our next requirement:

(iv) *The field equations for the EM field must be exactly the Maxwell equations with source terms determined by the charge distributions.*

In many experimental situations elementary charges such as electrons or protons behave as being spatially localized. The classical theory represents such states as solutions to the Maxwell equations with ideal point sources described by Dirac's delta-functions. This leads to our final requirement:

(v) *The field equations for a single charge distribution in the absence of EM fields must possess a localized solution with a spatial scale described by a size parameter a . The concept of point charge is recovered then as a limit when the size parameter a approaches zero. In this limit the charge motion in an external EM field is governed by the relativistic version of Newton's equation with the Lorentz force.*

It is by no means obvious that there exists a theory satisfying all of the above requirements, but we demonstrate the possibility of such a neoclassical theory by constructing it, and our goal is also to construct the simplest one satisfying all the above requirements. A concise presentation of the primary version of the neoclassical theory is provided in Chap. 3, whereas its detailed exposition is given in Part III. The spinorial version of the neoclassical theory is provided in Part IV. The proposed theory embraces elements of the Maxwell–Lorentz theory as well as quantum mechanics (QM). A justification for the name “neoclassical” is that the theory is a classical field theory based on the Lagrangian framework, and that charge distributions are interpreted as material waves rather than probability amplitudes as in QM. Many

significant aspects of our theory are studied with full mathematical rigor and in great detail. Those aspects are presented in Part V.

With the above requirements in mind, one can start off by searching for a possible field equation for a single elementary charge. The simplest relativistic covariant partial differential equation is the wave equation, and its modification with the mass parameter is the Klein–Gordon (KG) equation. The KG equation is commonly used in quantum electrodynamics (QED) to describe relativistic particles. But the wave and KG equations don't have localized solutions. *Hence the equations have to be modified to admit localized solutions. This is a critical turning point in the construction of our theory. The desired modification is achieved by introducing into the KG equation a nonlinearity that represents new internal cohesive forces of non-electromagnetic origin.* These forces can provide for the localization of an elementary charge in relevant regimes. It would be appropriate here to recall the non-electromagnetic cohesive forces suggested by Poincaré in 1905–1906 to be added to the Lorentz–Abraham model. The nonlinear KG (NKG) equation is the simplest relativistic covariant equation involving the mass parameter which has localized solutions. Its non-relativistic approximation turns out to be a nonlinear Schrödinger (NLS) equation which is derived from the relativistic NKG as a limit when Sommerfeld's fine structure constant tends zero.

The NKG equation is used in our theory within the framework of the classical field theory. The Lagrangian structure is constructed so that the field equations have the NKG equations for the material charge fields and the Maxwell equations for the relevant EM fields. The EM interaction between the material charge field and the EM field is introduced by implementing the minimal coupling, well known in QM. This choice of coupling is supported by further analysis showing in particular that the minimal coupling in the localization limit $a \rightarrow 0$ yields the Lorentz forces acting on the limit point charges. Note that *the validity of Newton's equations in point-like regimes and the expression for the Lorentz force is not prescribed but is derived from the analysis of the field equations.*

We would like to point out that a number of important properties of EM interactions are derived in our field theory rather than postulated as in QM. For instance, we prove that an accelerating charge represented by an exact localized solution to the field equations—which we call a *wave-corpuscle*—naturally possesses wave properties as in the de Broglie theory. The Planck–Einstein energy-frequency relation $E_1 - E_2 = \hbar(\omega_1 - \omega_2)$ for transitions is derived as exact for time-harmonic regimes in the non-relativistic version of our theory. Notice though that this celebrated relation is only an approximate one in the full relativistic version of our theory. The validity of Einstein's famous formula $E = Mc^2$ is also derived based on the field equations for an accelerated motion of an elementary charge.

Importantly, our theory accounts for the macroscopic and microscopic behavior of an elementary charge such as an electron within the same framework. In particular, in the case of a Hydrogen atom, where electron is in a microscopic regime, our theory yields discrete energy levels given by the Rydberg formula. In macroscopic regimes, when the charge size a is small compared with the spatial macroscopic scale R of variation of the external EM field, the Newtonian dynamics is derived as the

asymptotic localization limit of the field equation when $a/R \rightarrow 0$. We would like to stress though that, even in macroscopic regimes, the charge is not reduced entirely to just a point. It still possesses a wave structure and has a well defined intrinsic frequency and a wave number in accordance with the de Broglie wave theory.

Not surprisingly, different mathematical techniques are needed to analyze different regimes of the same fundamental theory. In particular, we derive for a macroscopic regime the relativistic version of Newton's law yielding the rest mass of an electron that differs slightly from the electron mass parameter in the NKG equation which enters the Rydberg formula. Interestingly, the value of the electron rest mass which is measured in Penning trap measurements differs slightly from the recommended value of the electron mass which enters the Rydberg formula. We show in Sect. 17.6.2 that this mass difference can be explained by the non-zero size of a free electron. As to particle-like regimes, we demonstrate that the theory implies in the localization limit the classical particle mechanics governed by the Newton equations with the Lorentz forces. In Part III of the book we use two methods to derive the point charge mechanics from the field theory: (i) via averaged quantities in the spirit of the well known Ehrenfest Theorem from quantum mechanics; (ii) by construction of approximate solutions (wave-corpuscles) to the field equations.

Below we provide a concise description of the context of the Parts, Chapters and some Sections of the book.

The first two chapters are introductory. In Chap. 2 we review some steps of the historical development of concepts and approaches related to our theory. In Chap. 3 we give a concise self-contained exposition of our main results and approaches. The purpose of this chapter is to provide a relatively concise presentation of the entire theory comparing and contrasting it with classical electrodynamics and quantum mechanics. Yet another purpose of this section is to provide the reader with a guidance to detailed exposition by presenting critical elements of the theory and their integration into a bigger picture.

The next two parts I and II are devoted to classical electromagnetic theory and classical field theory. They are also introductory and contain fundamental facts, terminology and concepts which are used in the consequent treatment. In Part I we present basic elements of classical EM theory including the Maxwell equations and the corresponding Green functions. The section contains the basics of special relativity and describes the Lagrangian formalism, including the energy-momentum tensor and conservation laws. We also provide in this Part a detailed treatment of the Helmholtz decomposition of electromagnetic fields and their potentials as well as their quasistatic, non-relativistic approximations.

Part II is devoted to classical field theory. We introduce there the Lagrangian variational framework, the derivation of the Euler–Lagrange equations, and the Noether theorem relating symmetries and conservation laws. We pay special attention to the Lorentz and the Galilean invariant Lagrangians and consider the canonical and symmetric energy-momentum tensors with the corresponding conservation laws. As the primary application of the general theory, we consider in Chap. 11 the Lagrangian field formalism for distributed charges interacting with the EM field. Their Lagrangians are gauge-invariant, and we derive, in particular, the charge con-

servations laws for the corresponding field equations. We also give a detailed treatment of many basic properties of the field equations, in particular energy-momentum tensors and conservation laws which are used later in the exposition. As particular examples, we consider in detail the Lagrangian formalism for the NLS and NKG equations. We also consider the Lagrangian formalism for two types of distributed charges, namely balanced and dressed charges. We consider in this part Lagrangians for the balanced and the dressed charges as examples and treat only formal aspects of their field theory. Application of the introduced concepts and their discussion is left for consequent parts of the book.

The central part of this monograph is Part III, devoted to the construction of the neoclassical theory of charged matter. This theory, though significantly different from both classical electrodynamics and quantum mechanics, accounts for many of their features as approximations, and it is entirely free of “infinities problems” including the self-energy infinity. We present in Part III the basics of the neoclassical theory of charges and consider there two variants of the theory: (i) balanced charges and (ii) dressed charges. We use in the exposition the material from Part II, in particular, the explicit expressions for currents, for entries of the energy-momentum tensor, etc. In Chap. 14 we consider the rest states of the charges and describe the nonlinearities which allow for localized charges. In Chap. 15 we treat uniform motion of charges, and the corresponding solutions to the field equations are obtained by application of an appropriate group transformations to the rest solutions. For special localized solutions in the form of “wave-corpuscles”, we evaluate their energies, momenta, velocities and group velocities. We study in Chap. 16 wave-corpuscle solutions of the non-relativistic field equations (the NLS equations) in accelerating regimes. We describe there classes of external EM fields allowing for wave-corpuscles as exact solutions. We also prove that the wave-corpuscles provide approximate solutions in the case of general EM fields. The structure of the wave-corpuscles allows us to introduce the de Broglie phase function, and relations between the wave properties and the particle properties are discussed in Sect. 16.2. We also provide in Sect. 18.1 a comparison with the Schrödinger wave theory.

In Chap. 17 we study the interaction theory for the balanced charges. We consider there a system of many charges interacting via their EM fields. Since the field equations for the system of charges split into a system of equations for every involved charge, and they are coupled through their EM fields, an important part of the study is the theory of a single charge in an external field. Using the conservation laws for the NLS equations, we derive in Sect. 17.2.3, in the non-relativistic case, Newton’s equations of motion for a system of localized well-separated balanced charges. In Sect. 17.3 we consider charges that may be in close proximity. We derive the Planck–Einstein formula for transitions between multi-harmonic states of the system. In Sect. 17.5 we consider a system of two charges which can serve as a model of the Hydrogen atom and derive there the Rydberg formula. In Chap. 17.6 we consider the relativistic theory of balanced charges. In particular, we develop in Sect. 17.6.2 a relativistic analog of the Ehrenfest theorem for the treatment of the dynamics of localized relativistic balanced charges. The relativistic point mechanics, including Einstein’s formula, is derived there from the field equation. In Sect. 17.6.4 we

consider a relativistic model of the Hydrogen atom in the framework of balanced charges.

Chapter 19 is devoted to the EM interaction of dressed charges. We derive there the Ehrenfest theorem and Newton's law for a system of interacting dressed charges. In Sect. 19.5 we discuss the Hydrogen atom model in the framework of dressed charges. In Chap. 20 we compare the aspects of the theory of dressed charges and balanced charges concerning EM fields.

In Part IV we introduce a theory of distributed charges with spin $1/2$. We develop Lagrangian aspects of the theory based on the spacetime algebra (STA), in other words the Clifford algebra of the Minkowski vector space. Thanks to the introduced nonlinearity, we obtain localized charges with spin $1/2$. As a part of the theory, we calculate the energy of the charge and the gyromagnetic ratio $g = 2$.

Part V treats mathematical aspects of the neoclassical theory of charges. In this part we formulate and prove mathematically rigorous statements concerning a number of aspects of the developed theory. We introduce in Chap. 34 the concept of the trajectory of concentration. This concept allows us to give a precise mathematical meaning to the concept of localization of the charge used in our studies of macroscopic regimes. We prove in Sect. 34.1 that the trajectory of concentration in the localization limit must satisfy Newton's equations. This property is obtained in the non-relativistic case for charges described by the NLS equation. We also prove there that every classical point trajectory of a charge can be represented by concentrating wave-corpuscle asymptotic solutions to the NLS equation. In Sect. 34.5 we consider the relativistic case described by the nonlinear KG equation in the EM field and derive there (i) the relativistic version of Newton's law and (ii) Einstein's formula for the inertial mass of a charge in accelerating regimes. Examples of relativistic concentrating solutions are constructed in Sect. 34.6. In particular, they show that Einstein's formula is more robust in accelerating regimes than the Lorentz contraction.

In Chap. 35 we treat mathematical aspects of the close interaction of balanced charges. We consider there properties of the logarithmic nonlinearity and, in particular, the orbital stability of solutions to the NLS equations. We prove in Sect. 35.1.3 that, if the nonlinearity is logarithmic, the Planck–Einstein frequency-energy relation for transitions between multi-harmonic states holds exactly. In Sect. 35.2 we prove that the energy levels of the non-relativistic balanced charge in the Coulomb field with logarithmic nonlinearity converge to the energy levels of the Hydrogen atom described by the Rydberg formula if the ratio a_B/a of the Bohr radius a_B to the free electron size parameter a tends to zero.

We use standard relativistic notation and the common convention for summation over repeated indices. We try our best to use different letters for different quantities, but the same letter may be used to denote different quantities in different parts of the book. We hope that the meaning of the notation will be clear from its context.

When we refer to books, typical references look like: [4, III.1], [4, (12.3)] and [4, p. 115]. They respectively refer to Section III.1, equation (12.3) and page 115 of book [4]. A similar convention is used for references to papers.

Chapter 2

The History of Views on Charges, Currents and the Electromagnetic Field

“It is of great advantage to the student of any subject to read the original memoirs on that subject, for science is always most completely assimilated when it is found in its nascent state...”

*J.C. Maxwell.*¹

The history of the concepts of charge, current and electromagnetic field is a fascinating and enormously vast area of research, see [6], [55], [24], [70], [183], [210], [209], [329], [216], [341], [342], [343], [288], [312] and references therein. Our modest goal here is to explore some of “old” ideas which stimulated and influenced our views on electromagnetic interactions. Many generations of distinguished scholars developed and advanced these ideas, and we find it to be instructive and inspiring to see the evolution of those ideas having acquired more experimental knowledge. In the following sections, the reader can find a *very sketchy review* of some of those ideas subjectively selected by us. These remind us of the dramatic and revolutionary changes which occurred in views on the concepts of electromagnetism and light. Nearly every time such a change brought about a simpler, more complete and more accurate theory, and we think that future advances in the electromagnetic theory can greatly benefit from knowing this history. In this part of the book we present a review of the period where the fundamentals of the modern EM theory were formed. More recent works are referred to in our exposition in corresponding parts of the book, and for a review of a more recent history we refer the reader to [195], [269], [310], [278].

To preserve views of the authors in their most original form, we prefer exact quotations from their original papers and monographs or from historical monographs. As in the case of any new theory, a historical perspective is important from many points of view, and it helps to elucidate those components of the theory that are “under construction”. We would like to acknowledge that we benefited from the works of many scholars in the history of physics who collected and analyzed numerous

¹[239, p. xiii–ix].

manuscripts and letters of prominent authors who contributed to the development of electromagnetic theory, quantum mechanics and physics in general. They made accessible to all of us a treasure of ideas and their evolution that is often hard to get directly from original sources, especially those not written in English. Their work goes far beyond a merely accurate description of the views of distinguished authors, and provides deep analysis and insights helpful for a better understanding of the discussed subjects.

2.1 The PreMaxwellian Era, Luminiferous Ether and Action-at-a-Distance

“‘Give me matter and motion,’ he [Descartes] cried,
‘and I will construct the universe.’”
*E. Whittaker.*²

We consider here a brief history of the development of a few selected subjects from the electromagnetic theory before Maxwell. We have relied on the monographs [6], [50, p. xxv–xxxiii], [341, p. 3], [342, I–VII], [70, 1–3], [306, 1]. The subject selection is motivated, in particular, by our interest in two different approaches to describing electromagnetic interactions: (i) the existence of a medium called “ether” or “aether” (“upper sky” in Greek mythology) that fills all space and is capable of transmitting forces and excitations; (ii) “action-at-a-distance” that allows for simpler models accounting for interactions.

2.1.1 Ether and Action-at-a-Distance

The “ether” approach rests on the doctrine that a force can be transmitted only by pressure or impact, and it goes back to the ancient Greek atomists and Aristotle, [342, p. 5]. As to “action-at-a-distance”, Roger Cotes (1682–1716) expressed its essence in a preface to the second edition (1713) of the *Principia* as follows, [342, p. 30]:

“The Newtonian law of action at a distance is championed as being the only formulation of the facts of experience which does not introduce unverifiable and useless suppositions. The principle which Cotes now affirmed was that the aim of theoretical physics is simply the prediction of future events, and that everything which is not strictly needed for this purpose, and which is not directly deducible from observed facts, should be pruned away.”

Until Newton’s discovery of gravitation, the only known influence capable of passing from star to star was that of light. The vehicle to pass those influences has been given the name “ether” (or “aether”), [341, p. 1]. According to E. Whittaker, it

²[342, p. 8].

was Rene Descartes (1596–1650) who introduced the concept of ether in the 17-th century, [342, p. 5, 6]:

“Space is thus, in Descartes’ view, a *plenum*, being occupied by a medium which, though imperceptible to the senses, is capable of transmitting force, and exerting effects on material bodies immersed in it – the aether, as it is called.

... Descartes was the originator of the Mechanical Philosophy, i.e. of the doctrine that the external inanimate world may for scientific purposes be regarded as an automatic mechanism, and that it is possible and desirable to imagine a mechanical model of every physical phenomenon.

...It was at this juncture that Descartes put forward his revolutionary suggestion, that the cosmos might be thought of as an immense machine. This entailed the general principle that all happenings in the material world can be predicted by mathematical calculation, and this principle has proved the element of greatest value in the Cartesian philosophy of nature.”

Since Descartes’ system assumes the world to be an “immense machine”, it has to reject all forms of “action-at-a-distance” and faces a challenge of providing an explicit mechanism for any force/interaction, [341, p. 3]:

“Descartes regarded the world as an immense machine, operating by the motion and pressure of matter. “Give me matter and motion,” he cried,” and I will construct the universe.” A peculiarity which distinguished his system from that which afterwards sprang from its decay was the rejection of all forms of action at a distance; he assumed that force cannot be communicated except by actual pressure or impact. By this assumption he was compelled to provide an explicit mechanism in order to account for each of the known forces of nature – a task evidently much more difficult than that which lies before those who are willing to admit action at a distance as an ultimate property of matter.”

As to light, Descartes considered it “to be essentially a pressure transmitted through a perfectly elastic medium (the aether) which fills all space, and he attributed the diversity of colors to rotary motions with different velocities of the particles in this medium”, [50, xxv].

The aim of a theory of the aether according to J. Larmor is as follows, [217, p. 207]:

“... The aim of a theory of the aether is not the impossible one of setting down a system of properties in which everything that may hereafter be discovered in physics shall be virtually included, but rather the practical one of simplifying and grouping relations and of reconciling apparent discrepancies in existing knowledge.

2. It would be an unwarranted restriction to assume that the properties of the aether must be the same as belong to material media.”

The two manifestly different approaches, namely the ether and action-at-a-distance, are not mutually exclusive and can be intimately related.

2.1.2 *Corpuscular and Wave Theories*

Another subject of our interest is the wave theory of electromagnetic interactions versus the corpuscular theory.

One of the most vivid manifestations of the wave origin of light was the *interference phenomenon* discovered independently by Robert Boyle (1627–1691) and Robert Hooke (1635–1703). “Hooke was the first to advocate the view that light consists of rapid vibrations propagated instantaneously, or with a very great speed, over any distance, and believed that in an homogeneous medium every vibration will generate a sphere which will grow steadily.”, [50, xxvi]. Another manifestation of the interference known today as “Newton’s rings” was discovered by Isaac Newton (1642–1727) in 1717. Newton also developed his corpuscular theory of light based on an idea that light consists of minute particles propagating from a luminous body. Interestingly, Newton’s position on the nature of light also admitted an ethereal medium as is stated by Thomas Young in his *Course of Lectures on Natural Philosophy*, [348, p. 477]:

“With respect to the nature of light, the theory which Newton adopted was materially different from the opinions of most of his predecessors. He considered indeed the operation of an ethereal medium as absolutely necessary to the production of the most remarkable effects of light, but he denied that the motions of such a medium actually constituted light; he asserted, on the contrary, that the essence of light consisted in the projection of minute particles of matter from the luminous body; and maintained that this projection was only accompanied by the vibration of a medium as an accidental circumstance, which was also renewed at the surface of every refractive or reflective substance.”

Christian Huygens (1629–1695) greatly advanced many aspects of the wave theory pioneered by Hooke. In particular, he formulated the *Huygens principle*, namely that “every point of a wave-front may be considered as a centre of a secondary disturbance which gives rise to spherical wavelets, and the wave-front at any later instant may be regarded as the envelope of these wavelets”, [50, 8.2].

Thomas Young (1773–829) pointed out great difficulties of the corpuscular theory to account for the constancy of the light velocity whereas such difficulties do not exist in the wave theory of elastic fluids. He also showed the superiority of the wave theory to explain reflection and refraction, [341, p. 105]:

“The reluctance which some philosophers felt to filling all space with an elastic fluid he [Young] met with an argument which strangely foreshadows the electric theory of light: ‘That a medium resembling in many properties that which has been denominated ether does really exist, is undeniably proved by the phenomena of electricity. The rapid transmission of the electrical shock shows that the electric medium is possessed of an elasticity as great as is necessary to be supposed for the propagation of light. Whether the electric ether is to be considered the same with the luminous ether, if such a fluid exists, may perhaps at some future time be discovered by experiment: hitherto I have not been able to observe that the refractive power of a fluid undergoes any change by electricity.’ Young then proceeds to show the superior power of the wave-theory to explain reflexion and refraction. In the corpuscular theory it is difficult to see why part of the light should be reflected and another part of the same beam refracted; but in the undulatory theory there is no trouble, as is shown by analogy with the partial reflexion of sound from a cloud or denser stratum of air: ‘Nothing more is necessary than to suppose all refracting media to retain, by their attraction, a greater or less quantity of the luminous ether, so as to make its density greater than that which it possesses in a vacuum, without increasing its elasticity.’ This is precisely the hypothesis adopted later by Fresnel and Green.”

2.1.3 *Electromagnetism*

In 1820, Hans Christian Oersted (1777–1851), a Danish physicist and chemist, experimenting with a galvanic source, a connecting wire, and a rotating magnetic needle, discovered that electric currents create magnetic fields. A fundamental implication of Oersted's discovery was the existence of a new circular force distributed in space which was different from the push-pull forces known previously in mechanics, electricity, and magnetism. The existence of such a circular force initiated research on the directionality and spatial arrangement of electrical and magnetic forces and their spatial distribution. This research was going on until the end of the 19th century and produced two different schools in electromagnetic theory: the Continental and the British schools stemming respectively from the work of Andre-Marie Ampere and the work of Michael Faraday.

Andre-Marie Ampere (1775–1836) found a way to reduce the Oersted circular forces to the familiar inverse-square central forces. Maxwell referred to Ampere as the “Newton of electricity” for his achievement in unifying the force laws in celestial mechanics and electromagnetism. Ampere's theory assumes the existence of molecular circulating electric currents, and magnetization was associated with a certain orientation of those molecular currents. Consequently, the interaction of a magnet with an electric current in a wire can be described in terms of forces exerted between the molecular currents in the magnet and the macroscopic current in the wire. Interestingly, [306, p. 7]

“...whereas both Newton and Ampere were seen by some of their followers as believers in unmediated action at a distance – between mass elements or current elements, respectively – both in fact conceptualized such interactions as mediated by an intervening medium – an ‘ether.’ Ampere viewed the ethereal substance as arising from the combination of positive and negative electrical fluids, and he considered the ether to be involved in the propagation of optical and thermal effects as well as electrical and magnetic effects. Ampere thus took seriously the imperative of Oersted's work toward general connectedness and unification, and he also agreed in an emphasis on mediated action. This, however, did not compromise Ampere's basic commitment to inverse-square central forces, and he continued to view these forces as originating in imponderable electrical fluids, positive and negative. Ampere's approach, and the approach of the Continental school that followed him, can be characterized in general terms as a mathematical approach, based on a central-force law, with the latter viewed as expressing the mediated interaction of current elements or moving charges. The traditional designation of this school as the “action-at-a-distance” school is thus a bit oblique; one might more appropriately designate it the charge-interaction school, as the emphasis is clearly on the interaction of charges according to a mathematical force law, but this is not necessarily a distance interaction.”

Michael Faraday (1791–1867), an English physicist and chemist, following Oersted's discovery, came up with a very different approach compared to that of Ampere and the Continental school. Faraday was skeptical of “electrical fluids” or other theoretical concepts that can't be related directly to experimental observations. He wrote to Ampere, [306, p. 8]

“I am naturally skeptical in the matter of theories and therefore you must not be angry with me for not admitting the one you have advanced immediately. Its ingenuity and applications

are astonishing and exact but I cannot comprehend how the currents are produced and particularly if they be supposed to exist round each atom or particle and I wait for further proofs of their existence before I finally admit them.”

Faraday not only refined experiments of Oersted showing that magnetism can be produced by currents, he also showed in 1831 that varying magnetic fields induce electric currents. He conceptualized the results of his experiments that established relations between magnets, electric currents and charges in *lines of force* distributed in space and visualized them using iron filings. In 1845 Faraday, stimulated by William Thomson, added to the list of his discoveries a magneto-optical phenomenon known today as the *Faraday effect* or *Faraday rotation*. After several attempts, he observed that light propagating in a special kind of heavy glass, when being exposed to a magnetic field, experiences a small but detectable rotation of its polarization plane.

Faraday’s view on the origin of magnetic action was that it is a function of ether rather than merely action at distance, [116, 3075, p. 330-331]:

“The general conclusion of philosophers seems to be that such cases are by far the most numerous, and for my own part, considering the relation of a vacuum to the magnetic force and the general character of magnetic phenomena external to the magnet, I am more inclined to the notion that in the transmission of the force there is such an action, external to the magnet, than that the effects are merely attraction and repulsion at a distance. Such an action may be a function of the aether; for it is not at all unlikely that, if there be an aether, it should have other uses than simply the conveyance of radiations.”

Faraday’s original theory of lines of forces became a basis for the British field school of the electromagnetic theory. The concept of “field”, defined loosely, refers to physical or mathematical entities residing in the space between electric and magnetic sources. “Faraday’s lines of force, however, provided the first precise and quantitative concept of a field. Moreover, Faraday advocated a pure field theory in which electric charge and current were derivative concepts. Thomson was first to introduce mathematical field concepts and to seek their foundation on a dynamical ether theory”, [70, p. 78]. William Thomson did that based on analogies between the mathematical laws of electricity and magnetism and the dynamics of continuous media, [70, p. 78]. In the 1850s, Faraday further sharpened the concept of lines of magnetic force giving them three operational definitions by the orientation of a compass needle, through the currents induced in a moving wire, and through the magnetocrystalline effect of bismuth.

The efforts of Faraday and W. Thomson prepared the grounds for a general unifying theory of all electromagnetic phenomena including electricity, magnetism, currents, charges and light based on the field concept. The construction of such theory was an enormous challenge, and it was taken in 1954 by Maxwell.

An alternative theory based on action-at-distance principles was proposed by M.W. Weber in 1846. His force law formulation was as follows, [306, p. 10]

$$F = \frac{ee'}{r^2} \left(1 - \frac{v^2}{c_w^2} + 2 \frac{ra}{c_w^2} \right) \quad (2.1)$$

where F is the magnitude of the central force acting between the two particles, e and e' are their electric charges, r is the distance between them, v is their relative velocity, a is their relative acceleration, and c_W is the ratio between electrodynamic and electrostatic units (larger than the modern ratio of units by a factor of $\sqrt{2}$). The terms in Weber's formula (2.1) have the following interpretation, [306, p. 11]: "... the first accounts for electrostatic forces; the second, involving the relative velocity, yields the phenomena of electromagnetism, and also the phenomena of magnetism per se (the latter by way of the assumption of Amperian molecular currents); the third term, involving the relative acceleration, accounts for electromagnetic induction. Together, the various terms add up to a complete and unified account of the phenomena of electricity and magnetism. The formula was open to criticism on physical grounds – the velocity-dependent and acceleration-dependent terms gave rise to much discussion." Maxwell, though he thought very highly of Weber's theory, did not consider it to be an ultimate one, [237, p. 527]:

"This theory, as developed by M.W. Weber and C. Neumann is exceedingly ingenious, and wonderfully comprehensive in its application to the phenomena of static electricity, electromagnetic attractions, induction of currents and diamagnetic phenomena; and it comes to us with the more authority, as it has served to guide the speculations of one who has made so great an advance in the practical part of electric science, both by introducing a consistent system of units in electrical measurement, and by actually determining electrical quantities with an accuracy hitherto unknown.

(2) The mechanical difficulties, however, which are involved in the assumption of particles acting at a distance with forces which depend on their velocities are such as to prevent me from considering this theory as an ultimate one, though it may have been, and may yet be useful in leading to the coordination of phenomena."

2.2 Maxwell's Field Theory

"In the theory which I propose to develop, the mathematical methods are founded upon the smallest possible amount of hypothesis, ..."
*J.C. Maxwell.*³

There are many studies on the history of development of Maxwell's electromagnetic theory, see [70, 4], [54], [216], [306] and references therein. Here we touch upon only fragments of these studies which we find relevant for our neoclassical theory. Most of us are familiar with Maxwell's theory through its presentation in modern textbooks and monographs. H. Hertz stated in 1892: [306, p. 167]: "To the question, 'What is Maxwell's theory?' I know of no shorter or more definite answer than the following: – Maxwell's theory is Maxwell's system of equations." Maxwell might have disagreed with that characterization of his theory, [70, 4.5.4]:

³[239, p. 65].

“In general, Maxwell’s use of mathematical symbolism differed essentially from continental or modern practice. For him the equations were always subordinated to the physical picture. He sought consistency, completeness, and simplicity in the picture, not necessarily in the equations. The latter were symbolic transcriptions of partial aspects of the picture, and therefore could not be safely used without keeping the underlying picture in mind.”

Maxwell views on the electromagnetic theory evolved during his life time 1831–1879. William Thomson (Lord Kelvin) (1824–1907) and his views and ideas played particularly important role in Maxwell’s research. Communication between them continued throughout Maxwell’s life from 1854 until his death in 1879, [70, 4], [306], [216]. Though W. Thomson encouraged Maxwell’s work on a unifying electromagnetic theory, he maintained a rather sceptical attitude throughout his life to Maxwell’s theory, considering it as only a provisional one, [216, p. 2].

By the mid-1850s, when Maxwell began his work in electromagnetic theory, many of its important elements were already known, including, in particular, electrostatics, magnetostatics, electromagnetism, electromagnetic induction, light, currents and charges. What was missing was a general unifying theory. The challenges in trying to unify all these elements were enormous. Maxwell saw a promising basis for such a general theory in the field concept suggested by the works of Faraday and W. Thomson. In his letter to M. Faraday in 1857 Maxwell wrote, [56, p. 202–203]:

“Now, as far as I know, you are the first person in whom the idea of bodies acting at a distance by throwing the surrounding medium into a state of constraint has arisen, as a principle to be actually believed in. We have had streams of hooks and eyes flying around magnets, and even pictures of them so beset; but nothing is clearer than your descriptions of all sources of force keeping up a state of energy in all that surrounds them, which state by its increase or diminution measures the work done by any change in the system. You seem to see the lines of force curving round obstacles and driving plump at conductors, and swerving towards certain directions in crystals, and carrying with them everywhere the same amount of attractive power, spread wider or denser as the lines widen or contract.”

Interestingly, Maxwell found in Faraday’s work not only physical but also mathematical insights that were expressed in an unconventional form, [239, p. X–XI]:

“As I proceeded with the study of Faraday, I perceived that his method of conceiving the phenomena was also a mathematical one, though not exhibited in the conventional form of mathematical symbols. I also found that these methods were capable of being expressed in the ordinary mathematical forms, and thus compared with those of the professed mathematicians.

For instance, Faraday, in his mind’s eye, saw lines of force traversing all space where the mathematicians saw centres of force attracting at a distance: Faraday saw a medium where they saw nothing but distance: Faraday sought the seat of the phenomena in real actions going on in the medium, they were satisfied that they had found it in a power of action at a distance impressed on the electric fluids.

When I had translated what I considered to be Faraday’s ideas into a mathematical form, I found that in general the results of the two methods coincided, so that the same phenomena were accounted for, and the same laws of action deduced by both methods, but that Faraday’s methods resembled those in which we begin with the whole and arrive at the parts by analysis, while the ordinary mathematical methods were founded on the principle of beginning with the parts and building up the whole by synthesis.

I also found that several of the most fertile methods of research discovered by the mathematicians could be expressed much better in terms of ideas derived from Faraday than in their original form."

The *theory of the molecular vortices* favored by W. Thomson became the focus of Maxwell's research in the late 1850s and early 1860s. The two significant innovations of that period introduced by Maxwell were the displacement current and a unification of electromagnetism and optics based on molecular vortices. Here is a description of molecular vortices model in Maxwell's own words in his letter to W. Thomson, 1861, Dec. 10, [216, p. 34]:

"...I suppose that the "magnetic medium" is divided into small portions or cells, the divisions or cell-walls being composed of a single stratum of spherical particles these particles being "electricity". The substance of the cells I suppose to be highly elastic both with respect to compression and distortion and I suppose the connexion between the cells and the particles in the cell walls to be such that there is perfect rolling without slipping between them and that they act on each other tangentially.

...

I have calculated the relation between the force and the displacement on the supposition that the cells are spherical and that their cubic and linear elasticities are connected as in a "perfect" solid. I have found from this the attraction between two bodies having given quantities of free electricity on their surfaces, and then by comparison with Weber's value of the statical measure of a unit of electrical current I have deduced the relation between the elasticity and density of the cells. The velocity of transverse undulations follows from this directly and is equal to 193088 miles per second, very nearly that of light."

In many of his works, Maxwell used and praised the Lagrange method as the most suitable and flexible one for constructing a new field theory, see for instance the second volume of his *Treatise* [240, V]. In his paper "On the Proof of the Equations of Motion of a Connected System" published in 1876, Maxwell formulated very clearly his views on the aim and power of the Lagrange method, [238, p. 302-303]:

"Lagrange's investigation may be regarded from a mathematical point of view as a method of reducing the dynamical equations, of which there are originally three for every particle of the system, to a number equal to that of the degrees of freedom of the system. In other words it is a method of eliminating certain quantities called reactions from the equations. The aim of Lagrange was, as he tells us himself, to bring dynamics under the power of the calculus, and therefore he had to express dynamical relations in terms of the corresponding relations of numerical quantities.

... But the importance of these equations does not depend on their being useful in solving problems in dynamics. A higher function which they must discharge is that of presenting to the mind in the clearest and most general form the fundamental principles of dynamical reasoning."

In the second volume of his *Treatise*, Maxwell stated unambiguously that the *use of a mechanical model, constructed by him, involving an "imaginary mechanism" is "merely to assist the imagination"*, [240, p. 200]:

"To fix our ideas we may conceive the system connected by means of suitable mechanism with a number of moveable pieces, each capable of motion along a straight line, and of no other kind of motion. The imaginary mechanism which connects each of these pieces with

the system must be conceived to be free from friction, destitute of inertia, and incapable of being strained by the action of the applied forces. The use of this mechanism is merely to assist the imagination in ascribing position, velocity, and momentum to what appear, in Lagrange's investigation, as pure algebraical quantities."

As to the energy of the electromagnetic field and where it resides, Maxwell wrote in his seminal paper "A Dynamical Theory of the Electromagnetic Field", [237, p. 564]:

"In speaking of the Energy of the field, however, I wish to be understood literally. All energy is the same as mechanical energy, whether it exists in the form of motion or in that of elasticity, or in any other form. The energy in electromagnetic phenomena is mechanical energy. The only question is, Where does it reside? On the old theories it resides in the electrified bodies, conducting circuits, and magnets, in the form of an unknown quality called potential energy, or the power of producing certain effects at a distance. On our theory it resides in the electromagnetic field, in the space surrounding the electrified and magnetic bodies, as well as in those bodies themselves, and is in two different forms, which may be described without hypothesis as magnetic polarization and electric polarization, or, according to a very probable hypothesis, as the motion and the strain of one and the same medium."

The critique of the use of mechanical "explanations" in physics is well known, and it is interesting to see Maxwell's views on the role of mechanical models from his statements above. Importantly, these views relied on the Lagrangian method allowing for the introduction of mechanical concepts without a "mechanism" which he called "imaginary" and used "merely to assist the imagination". An interested reader can find more on the use of mechanical explanations in the 19-th century in [199].

2.3 Lorentz's Theory

"... we shall be obliged to have recourse to some hypothesis about the mechanism that is at the bottom of the phenomena."

*H.A. Lorentz.*⁴

In our concise review of the seminal contribution to the electromagnetic theory by Hendrik Antoon Lorentz (1853–1928), we have relied primarily on its analysis in [341, XII], [342, XIII], [70, 8.4], [326, I], [278] and Lorentz's own monograph [227, I].

Lorentz's theory of electromagnetic phenomena in a nutshell is a combination of Clausius' theory of electricity with Maxwell's theory of the ether with matter being clearly separated from ether. Electrons in his theory do not interact directly but via the electromagnetic field, that is to say, electrons and their motion determine the electromagnetic field, and the electromagnetic field acts upon the electrons. A significant innovation brought by Lorentz was the microscopic treatment of polarization in contrast to Maxwellian physicists treating polarization as a macroscopic property of the

⁴[227, p. 8].

ether not extended to the atomic scales. Lorentz's intentions were very ambitious, [278, p. 11]:

"Like others before him, he constructed a particle electrodynamics. But his theory went far beyond that of his predecessors, because he meant to account for all the macroscopic phenomena of electrodynamics and optics in terms of the microscopic behavior of electrons and ions."

In 1878 Lorentz successfully applied his ideas to dispersion theory, demonstrating the great potential of the microscopic approach, [70, p. 325]:

"This work of Lorentz anticipated essential features of the future electron theory: the separation of ether from matter, the idea of an electromagnetic coupling between them, and the focus on microscopic processes."

As theoretical physicists tried to extend the electromagnetic theory to the case of moving "ponderable bodies", new concepts have emerged. Some of those efforts, including that of Lorentz, made the concept of charges (electrons) a foundation of a microscopic, atomic theory of electromagnetic phenomena, [341, p. 418–419]:

"The first memoir in which the new conceptions were unfolded was published by H.A. Lorentz in 1892. The theory of Lorentz was, like those of Weber, Riemann, and Clausius, a theory of electrons; that is to say, all electro-dynamical phenomena were ascribed to the agency of moving electric charges, which were supposed in a magnetic field to experience forces proportional to their velocities, and to communicate these forces to the ponderable matter with which they might be associated.

... Moreover, the discoveries of Hertz had shown that a molecule which is emitting light must contain some system resembling a Hertzian vibrator; and the essential process in a Hertzian vibrator is the oscillation of electricity to and fro. Lorentz himself from the outset of his career had supposed the interaction of ponderable matter with the electric field to be effected by the agency of electric charges associated with the material atoms."

E. Whittaker then continues to characterize the original features of the Lorentz theory and its relation to the theory of Maxwell, [341, p. 419–420]:

"The principal difference by which the theory now advanced by Lorentz is distinguished from the theories of Weber, Riemann, and Clausius and from Lorentz' own earlier work, lies in the conception which is entertained of the propagation of influence from one electron to another. In the older writings, the electrons were assumed to be capable of acting on each other at a distance, with forces depending on their charges, mutual distances and velocities; in the present memoir, on the other hand, the electrons were supposed to interact not directly with each other, but with the medium in which they were embedded. To this medium were ascribed the properties characteristic of the aether in Maxwell's theory.

The only respect in which Lorentz' medium differed from Maxwell's was in regard to the effects of the motion of bodies. Impressed by the success of Fresnel's beautiful theory of the propagation of light in moving transparent substances, Lorentz designed his equations so as to accord with that theory, and showed that this might be done by drawing a distinction between matter and aether, and assuming that a moving ponderable body cannot communicate its motion to the aether which surrounds it, or even to the aether which is entangled in its own particles; so that no part of the aether can be in motion relative to any other part. The general plan of Lorentz' investigation was, therefore, to reduce all the complicated cases of electromagnetic action to one simple and fundamental case, in which the field contains only free aether with solitary electrons dispersed in it; the theory which he adopted in this fundamental case was a combination of Clausius' theory of electricity with Maxwell's theory of the aether."

In the later edition of his historical studies, E. Whittaker expanded the above by contrasting the Lorentz theory with the theory of Hertz, [342, p. 393]:

“... Such an aether is simply space endowed with certain dynamical properties; its introduction was the most characteristic and most valuable feature of Lorentz’ theory, which differs completely from, for example, the theory of Hertz so far as concerns the electrodynamics of bodies in motion. ...”

In his course of lectures on the theory of electrons, Lorentz introduces four fields: the electric force $\mathbf{E}$, the magnetic force $\mathbf{H}$, the current of electricity $\mathbf{C}$ and the magnetic induction $\mathbf{B}$ satisfying the following equations, [227, Ch. I.4]

$$\begin{aligned} \operatorname{div} \mathbf{C} &= 0, & \operatorname{div} \mathbf{B} &= 0, \\ \operatorname{rot} \mathbf{E} &= \frac{1}{c} \mathbf{C}, & \operatorname{rot} \mathbf{B} &= -\frac{1}{c} \partial_t \mathbf{B}. \end{aligned} \quad (2.2)$$

Then he introduces the “dielectric displacement” $\mathbf{D}$ satisfying

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad (2.3)$$

where ε is the dielectric constant, and identifies the current $\mathbf{C}$ with the Maxwell displacement current, i.e.

$$\mathbf{C} = \partial_t \mathbf{D}. \quad (2.4)$$

In “conducting bodies” there is a current of conduction, given by

$$\mathbf{J} = \sigma \mathbf{E}, \quad (2.5)$$

where σ is the conductivity constant. The total current then is

$$\mathbf{C} = \partial_t \mathbf{D} + \mathbf{J} = \varepsilon \mathbf{E} \partial_t + \sigma \mathbf{E}. \quad (2.6)$$

The magnetic force and induction are related in simpler cases by the formula

$$\mathbf{B} = \mu \mathbf{H}, \quad (2.7)$$

where μ is the magnetic permeability. The equations (2.2)–(2.7) combined constitute what we call today the Maxwell or the Maxwell–Lorentz equations for dielectric conducting media with no impressed currents, see for instance [314, 1.1–1.6].

As to the constitutive relations (2.3), (2.5) and (2.7), Lorentz comments that they cannot apply in all cases, and a more sophisticated theory is needed “to obtain deeper insight into the nature of the phenomena”, and that the theory requires the introduction of the concept of electrons. Referring to the constitutive relations (2.3), (2.5) and (2.7), he wrote in his “Theory of electrons”, [227, p. 8]:

“Moreover, even if they were so, this general theory, in which we express the peculiar properties of different ponderable bodies by simply ascribing to each of them particular

values of the dielectric constant ε , the conductivity σ and the magnetic permeability μ , can no longer be considered as satisfactory, when we wish to obtain a deeper insight into the nature of the phenomena. If we want to understand the way in which electric and magnetic properties depend on the temperature, the density, the chemical constitution or the crystalline state of substances, we cannot be satisfied with simply introducing for each substance these coefficients, whose values are to be determined by experiment; we shall be obliged to have recourse to some hypothesis about the mechanism that is at the bottom of the phenomena.

It is by this necessity, that one has been led to the conception of electrons, i.e. of extremely small particles, charged with electricity, which are present in immense numbers in all ponderable bodies, and by whose distribution and motions we endeavor to explain all electric and optical phenomena that are not confined to the free ether."

Then Lorentz continues boldly with his fundamental assumptions regarding the structure of an electron in its relation to the ether: [227, p. 11] :

"In the first place, we shall ascribe to each electron certain finite dimensions, however small they may be, and we shall fix our attention not only on the exterior field, but also on the interior space, in which there is room for many elements of volume and in which the state of things may vary from one point to another. As to this state, we shall suppose it to be of the same kind as at outside points. Indeed, one of the most important of our fundamental assumptions must be that the ether not only occupies all space between molecules, atoms or electrons, but that, it pervades all these particles. We shall add the hypothesis that, though the particles may move, the *ether always remains at rest*. We can reconcile ourselves with this, at first sight, somewhat startling idea, by thinking of the particles of matter as of some local modifications in the state of the ether. These modifications may of course very well travel onward while the volume-elements of the medium in which they exist remain at rest.

Now, if within an electron there is ether, there can also be an electromagnetic field, and all we have got to do is to establish a system of equations that may be applied as well to the parts of the ether: where there is an electric charge, i.e. to the electrons, as to those where there is none. As to the distribution of the charge, we are free to make any assumption we like. ..."

Yet another key element of the Lorentz theory was what we know now as the "*Lorentz force*", [227, p. 14]:

"However this may be, we must certainly speak of such a thing as the force acting on a charge, or on an electron, on charged matter, whichever appellation you prefer. Now, in accordance with the general principles of Maxwell's theory, we shall consider this force as caused by the state of the ether, and even, since this medium pervades the electrons, as exerted by the ether on all internal points of these particles where there is a charge. If we divide the whole electron into elements of volume, there will be a force acting on each element and determined by the state of the ether nesting within it. We shall suppose that this force is proportional to the charge of the element, so that we only want to know the force acting per unit charge. This is what we can now properly call the *electric force*. We shall represent it by $\mathbf{f}$. The formula by which it is determined, and which is the one we still have to add to (17)–(20), is as follows:

$$\mathbf{f} = \mathbf{d} + \frac{1}{c} [\mathbf{v} \cdot \mathbf{h}].$$

Like our former equations, it is got by generalizing the results of electromagnetic experiments."

2.4 Problems with Elementary Charge Treated as a Point

The concept of a point mass is evidently an idealization which can be traced to yet another idealization of a rigid body, [207, p. 5]. In quantum mechanics, the particle-wave duality might obscure the fact that an elementary charge is a point. This issue was important enough for many distinguished contributors to the development of quantum mechanics to articulate their positions.

R. Feynman stresses that the electron wave function should not obscure the fact that it is a point charge, [122, p. 21–6]:

“The wave function $\psi(\mathbf{r})$ for an electron in an atom does not, then, describe a smeared-out electron with a smooth charge density. The electron is either here, or there, or somewhere else, but wherever it is, it is a point charge.”

A clarification on the same matter is provided by H. Kramers [207, p. 48–49]:

“One often interprets the probability density J as follows. The particle is no longer localized at a well defined point of space, but it is, so to say, extended over the whole of space according to the density function J . If it possesses a charge e , one says that this charge is smeared out over space in a definite way, and one speaks of a continuous charge cloud with a charge density eJ and a current density $e\mathbf{S}$.

Although this picture often fits naturally into the mathematical description of the action of the particle on other particles or on external systems (compare §52), it is tempting to introduce too easily a much too simplistic interpretation of quantum mechanics. This interpretation uses models which can be described in classical terms, but does not do justice to the basic characteristics of quantum theory. Especially we want to warn against the idea that one must abandon the corpuscular nature of elementary particles and that according to quantum theory one must instead consider a theory with continuous charge and current densities.”

R. Feynman carries out a detailed analysis of a number of problems of the classical EM theory in [121, 28] where he wrote, in particular, [121, p. 28–1]:

“It is interesting, though, that the classical theory of electromagnetism is an unsatisfactory theory all by itself. There are difficulties associated with the ideas of Maxwell’s theory which are not solved by and not directly associated with quantum mechanics.”

A more recent analysis of problems with classical EM theory is provided by H. Spohn in his monograph [310, I].

The point charge concept is also fundamentally inconsistent with relativity theory which is deeply rooted in the four-dimensional continuum. On this, Einstein points out that, [295, p. 61]:

We now shall inquire into the insights of definite nature which physics owes to the special theory of relativity.

(1) There is no such thing as simultaneity of distant events; consequently there is also no such thing as immediate action at a distance in the sense of Newtonian mechanics. Although the introduction of actions at a distance, which propagate with the speed of light, remains thinkable, according to this theory, it appears unnatural; for in such a theory there could be no such thing as a reasonable statement of the principle of conservation of energy. It therefore appears unavoidable that physical reality must be described in terms of continuous

functions in space. The material point, therefore, can hardly be conceived anymore as the basic concept of the theory.

(2) The principles of the conservation of momentum and of the conservation of energy are fused into one single principle. The inert mass of a closed system is identical with its energy, thus eliminating mass as an independent concept.

He also wrote, [106, p. 167]:

“The physical world is represented as a four-dimensional continuum. ... it seems to me certain that we have to give up the notion of an absolute localization of the particles in a theoretical model.”

Yet another argument, due to von Laue, on the incompatibility of relativity with any finite-dimensional mechanical system was articulated by W. Pauli, [265, p. 131–132]:

“The final clarification was brought about in a paper by Laue, who showed by quite elementary arguments that the number of kinematic degrees of freedom of a body cannot be limited, according to the theory of relativity. For, since no action can be propagated with a velocity greater than that of light, an impulse which is given to the body simultaneously at n different places, will, *to start off with*, produce a motion to which at least n degrees of freedom must be ascribed.”

Insistence of QM on the point charge concept as fundamentally exact is a definite source of its serious difficulties. In particular, Dirac wrote on the necessity to get away from the point model of the electron, [86, p. 11]:

“The difficulty becomes most apparent when one takes into account the interaction between, let us say, electrons and the electromagnetic field. If one supposes this interaction to arise from a point model of the electron, one gets infinities occurring in the equation. These infinities, of course, are not to be tolerated. One has to remove them in some way, and the natural way to remove them is to say that the electron is not a point charge, but the charge is distributed over a certain region.

Many physicists assume point charges and remove the difficulty of the infinities just by means of working rules. They say, let us depart from ordinary mathematics. Let us neglect infinities occurring in our equations when we don't want them. This formalism does sometimes lead to results in good agreement with observation, and many physicists are happy with this state of affairs. But I am most unhappy about it. I feel that we do not have definite physical concepts at all if we just apply working mathematical rules; that's not what the physicist should be satisfied with. ...

Again we have to get away from the point model of the electron, because in that case the mass of the Coulomb field around it would be infinitely great.”

2.5 The Concept of an Extended Charge

The problems with point charges discussed above motivated continuing efforts to overcome them. A natural step to resolve those problems is to replace the point charge with a continuously distributed density of charge which is termed an “*extended*

charge". There are two well known models of such an extended charge: the semi-relativistic Abraham rigid charge model (a rigid sphere with spherically symmetric charge distribution, 1903, 1904), [310, 2.4, 3.1, 4.1, 10.2, 13], [278, 2.2], [70, 9.3.1] and the Lorentz relativistically covariant model (1904), [310, 2.5, 3.1, 4.2], [278, 2.2], [70, 9.3.2]. These two models are based on an assumption of the electron rigidity (or relativistic rigidity) on one hand and, on the other hand, on an assumption of the electromagnetic interaction between different parts of the electron as described by Maxwell's theory. Consequently, these extended charge models treat an electron essentially as a small macroscopic charged body, [278, 6–1]:

"This was the problem faced by Abraham, Lorentz, and Poincaré in the first few years of this century. The most obvious model of a charged particle is a sphere carrying a spherically symmetrical charge wave function. While such a model is meant (and was indeed proposed) as a picture of a charged elementary particle (an electron, for example), it is obvious that it is basically a macroscopic charged body, only much smaller. There is nothing "elementary" about it."

Lorentz himself analyzed thoroughly his and Abraham's models and their possible extensions in his monograph [227, Sect. 26–38, 178–194]. In particular, he considers there a necessity to introduce forces of nonelectromagnetic nature such as "Poincaré stresses" to maintain electron stability, [227, p. 213–214]:

"The nature of this new energy and the mechanism of the contraction are made much clearer by the remark, first made by Poincaré, that the electron will be in equilibrium, both in its original and in its flattened form, if it has the properties of a very thin, perfectly flexible and extensible shell, whose parts are drawn inwards by a normal stress, ..."

Adding such non-electromagnetic cohesive forces to the Lorentz–Abraham model was suggested by Poincaré in 1905–1906, [273], [70, 9.3.3] (see also [179, 16.4–16.6], [278, 2.3, 6.1–6.3], [265, 63], [301], [345, 4.2] and references therein). Poincaré required that the cohesive pressure of the electron should be Lorentz invariant, and he identified the formal condition of Lorentz invariance with the principle of relativity, [70, p. 365]:

"The reason why we can, without modifying any apparent phenomenon, confer to the whole system a common translation, is that the equations of an electromagnetic medium are not changed under certain transformations which I shall call the Lorentz transformations; two systems, one at rest, the other in translation, thus become exact images of one another."

As to the nature of forces involved in an electron Lorentz made the following critical remark, [227, p. 215]:

"In speculating on the structure of these minute particles we must not forget that there may be many possibilities not dreamt of at present; it may very well be that other internal forces serve to ensure the stability of the system, and perhaps, after all, we are wholly on the wrong track when we apply to the parts of an electron our ordinary notion of force."

Leaving aside the special mechanism that has been imagined by Poincaré, we are offered the following alternative. Either a spherical electron must be regarded as a material system between whose parts there are certain forces ensuring the constancy of its size and, form, or we must simply assume this constancy as a matter of fact which we have not to analyze any further."

When making “adventurous” assumptions on details of the interior of an electron, Lorentz expressed some hesitations about them, [227, p. 16]:

“While I am speaking so boldly of what goes on in the interior of an electron, as if I had been able to look into these small particles, I fear one will feel inclined to think I had better not try to enter into all these details. My excuse must be that one can scarcely refrain from doing so, if one wishes to have a perfectly definite system of equations; moreover, as we shall see later on, our experiments can really teach us something about the dimensions of the electrons. In the second place, it may be observed that in those cases in which the internal state of the electrons can make itself felt, speculations like those we have now entered upon, are at all events interesting, be they right or wrong, whereas they are harmless so soon as we may consider the internal state as a matter of little importance.”

The Lorentz model and its extensions were studied and advanced in different directions by a number of authors, [5], [179, 16], [255], [269], [278, 2, 6], [301], [310, 2.5, 3.1, 4.2, 10.1], [345]. A critical review and analysis of extended charge models as well as a more general approach to account for elementary charges was carried out by W. Pauli in [265, 63].

2.6 Poincaré’s Contribution

“... If one of them [theories] taught us a true relation, this relation is definitively acquired, and it will be found again under a new disguise in the other theories which will successively come to reign in place of the old.”

*H. Poincaré.*⁵

Poincaré’s contributions to the electromagnetic theory and physics are significant and original, [70, 9.1-9.3]. In particular, we owe to Poincaré the modern form of the Lorentz transformations and the idea that all theories must be Lorentz-invariant in order to satisfy the relativity postulate.

Poincaré identified very clearly the key elements of special relativity in his influential book “Science and Hypothesis” published first in French in 1902, [274]. He stated that there is no absolute space, absolute time or even absolute simultaneity, [275, p. 92–93], [70, p. 381]:

“1. There is no absolute space and we can conceive only of relative motions; yet usually the mechanical facts are enunciated as if there were an absolute space to which to refer them.

2. There is no absolute time; to say two durations are equal is an assertion which has by itself no meaning and which can acquire one only by convention.

3. Not only have we no direct intuition of the equality of two durations, but we have not even direct intuition of the simultaneity of two events occurring in different places: this I have explained in an article entitled *La mesure du temps*.

4. Finally, our Euclidean geometry is itself only a sort of convention of language; mechanical facts might be enunciated with reference to a non-Euclidean space which would be a

⁵[275, p. 351].

guide less convenient than, but just as legitimate as, our ordinary space; the enunciation would thus become much more complicated, but it would remain possible.

Thus absolute space, absolute time, geometry itself, are not conditions which impose themselves on mechanics; all these things are no more antecedent to mechanics than the French language is logically antecedent to the verities one expresses in French.”

Interestingly, Einstein read the above text before 1905 as is attested in Solovine 1956, [70, p. 381].

The view that mechanics is a fundamental basis for all of physics was dominant at the end of the 19th century. Poincaré, being an authority in mechanics, provided a deep insight into the role of mechanics in physics. In particular, when discussing Maxwell’s views on mechanical explanations of electromagnetic phenomena, Poincaré commented as follows, [275, p. 176]:

“Maxwell does not give a mechanical explanation of electricity and magnetism; he confines himself to demonstrating that such an explanation is possible.”

He also made a more general statement on the possibility of a mechanical explanation based on the principle of least action and the use of it by Maxwell, [275, p. 180–181]:

“Thus if the principle of least action can not be satisfied, no mechanical explanation is possible; if it can be satisfied, there is not only one, but an infinity, whence it follows that as soon as there is one there is an infinity of others. ...

And then Maxwell asked himself whether he could make this choice and that of the two energies T and U , in such a way that the electrical phenomena would satisfy this principle. Experiment shows us that the energy of an electromagnetic field is decomposed into two parts, the electrostatic energy and the electrodynamic energy. Maxwell observed that if we regard the first as representing the potential energy U , the second as representing the kinetic energy T ; if, moreover, the electrostatic charges of the conductors are considered as parameters q and the intensities of the currents as the derivatives of other parameters $\dot{q}$; under these conditions, I say, Maxwell observed that the electric phenomena satisfy the principle of least action. Thenceforth he was certain of the possibility of a mechanical explanation.”

2.7 Planck’s Insights on Black-Body Radiation and Energy Quanta

“... on the very day when I formulated this law,
I began to devote myself to the task
of investing it with a true physical meaning.”
*M. Planck.*⁶

Max Planck’s seminal contribution to the electromagnetic theory was his famous black-body radiation formula that reads, [183, p. 21]:

⁶[272, p. 41].

$$u_\nu = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1} \quad (2.8)$$

where u_ν is the radiation density as a function of its frequency ν , h is the Planck constant, k is the Boltzmann constant and T is the temperature. Planck's derivation of formula (2.8) was based on concepts of entropy and probability championed by L. Boltzmann, [272, p. 41]:

"But even if the absolutely precise validity of the radiation formula is taken for granted, so long as it had merely the standing of a law disclosed by a lucky intuition, it could not be expected to possess more than a formal significance. For this reason, on the very day when I formulated this law, I began to devote myself to the task of investing it with a true physical meaning. This quest automatically led me to study the interrelation of entropy and probability – in other words, to pursue the line of thought inaugurated by Boltzmann."

Importantly, when using entropy and probability, Planck boldly postulated energy quanta $h\nu$. In an unpublished letter to R.W. Wood written in 1931, [183, p. 22], he characterized psychological motives leading to energy quanta as:

"an act of desperation, done because a theoretical explanation had to be supplied at all cost, whatever the price".

In the course of his studies Planck considered spherical waves emitted by an oscillator and their relation to thermodynamical aspects of the "natural radiation" including the irreversibility of the radiation process, [272, p. 36-37]:

"Moreover, my suggestion that the oscillator was capable of exerting a unilateral, in other words irreversible, effect on the energy of the surrounding field, drew a vigorous protest from Boltzmann, who, with his wider experience in this domain, demonstrated that according to the laws of classical dynamics, each of the processes I considered could also take place in the opposite direction; and indeed in such a manner, that a spherical wave emitted by an oscillator could reverse its direction of motion, contract progressively until it reached the oscillator and be reabsorbed by the latter, so that the oscillator could then again emit the previously absorbed energy in the same direction from which the energy had been received. To be sure, I could exclude such odd phenomena as inwardly directed spherical waves, by the introduction of a specific stipulation – the hypothesis of a natural radiation, which plays the same part in the theory of radiation as the hypothesis of molecular disorder in the kinetic theory of gases, in that it guarantees the irreversibility of the radiation processes. But the calculations showed ever more clearly that an essential link was still missing, without which the attack on the core of the entire problem could not be undertaken successfully."

Planck critically analyzed the usage of light quanta in Einstein's paper "On the Development of our Views Concerning the Nature and Constitution of Radiation", [102, Doc. 60]. He pointed out that Einstein's conclusions on the quantization of electromagnetic radiation are based on the assumption that the interactions between the radiation in vacuum and the motion of matter are completely known, when that does not seem to be the case, [102, Doc. 61, p. 395-396]:

"According to the latest considerations of Mr. Einstein, it would be necessary to conceive the free radiation in vacuum, and thus the light waves themselves, as atomistically constituted, and hence to give up Maxwell's equations. This seems to me a step which in my opinion is not yet necessary. I will not go into details, but will rather note the following. In the latest consideration by Mr. Einstein he inferred the fluctuations of free radiation in pure vacuum

from the motion of matter. This inference seems to me absolutely irreproachable only in the case that the interactions between the radiation in vacuum and the motion of matter are completely known; if this is not the case, then the bridge necessary to cross from the motion of the mirror to the intensity of the incident light is missing. However, it seems to me that we know very little about this interaction between the free electrical energy in vacuum and the motion of the atoms of matter. This interaction is essentially based on the emission and absorption of light. Essentially this is also the case for radiation pressure, at least according to the generally accepted theory of dispersion, which also reduces reflection to absorption and emission. However, it is just emission and absorption which are the obscure points about which we know very little. We may know a little about absorption, but what about emission? We imagine that it is produced by the acceleration of electrons. But this is the weakest point in the entire theory of electrons. One imagines that the electron possesses a certain volume and a certain finite charge density, whether due to a volume or surface charge, one cannot manage without that; this, however, conflicts in a certain sense with the atomistic conception of electricity. These are not impossibilities but difficulties, and I am almost surprised that this has not met with more opposition.

This is the point, I believe, at which the quantum theory can be employed with advantage. We can stipulate the laws for large time intervals only. But for small time intervals and great accelerations we still face a gap whose filling requires new hypotheses. Perhaps we may be allowed to assume that an oscillating resonator does not have a continuously variable energy, but that its energy is a simple multiple of an elementary quantum instead. I believe that by using this theory one can arrive at a satisfactory theory of radiation. The question is, then: How does one visualize something like that? That is to say, one asks for a mechanical or electrodynamic model of such a resonator. But mechanics and current electrodynamics do not provide for discrete elements of action, and hence we cannot produce a mechanical or electrodynamic model. Thus, mechanically this seems impossible, and we will have to get used to that. After all, our attempts to mechanically represent the luminiferous ether also have failed completely.”

2.8 Einstein’s Insights

“Physics up to now is naturally in its essence
a continuum physics, in spite of the use
of the material point, which looks like
a discontinuous conceptual element.”
*A. Einstein.*⁷

Einstein’s monumental contributions to the electromagnetic theory are well known. They include a critical analysis of the ether concept, material properties of electric currents, his revolutionary discoveries of the relativity of time in its relations to space, the relation between mass and energy, quantum properties of electromagnetic radiation and more. We present in this section a few of Einstein’s ideas which we found to be particularly important and inspiring for our constructions of the neoclassical electromagnetic theory.

⁷[312, p. 142].

From the earliest stages of Einstein's studies, he relied on thermodynamics as a theory that will never be obsolete. He wrote out in his autobiographical Notes, [295, p. 87]:

"Therefore the deep impression which classical thermodynamics made upon me. It is the only physical theory of universal content concerning which I am convinced that, within the framework of the applicability of its basic concepts, it will never be overthrown (for the special attention of those who are skeptics on principle)."

As to the relation between matter and field Einstein emphasized that they cannot be distinctly separated, [105, p. 257]:

"There is no sense in regarding matter and field as two qualities quite different from each other. We cannot imagine a definite surface separating distinctly field and matter.

The same difficulty arises for the charge and its field. It seems impossible to give an obvious qualitative criterion for distinguishing between matter and field or charge and field.

Our structure laws, that is, Maxwell's laws and the gravitational laws, break down for very great concentrations of energy or, as we may say, where sources of the field, that is electric charges or matter, are present. But could we not slightly modify our equations so that they would be valid everywhere, even in regions where energy is enormously concentrated?

We cannot build physics on the basis of the matter concept alone. But the division into matter and field is, after the recognition of the equivalence of mass and energy, something artificial and not clearly defined. Could we not reject the concept of matter and build a pure field physics? What impresses our senses as matter is really a great concentration of energy into a comparatively small space. We could regard matter as the regions in space where the field is extremely strong. In this way a new philosophical background could be created. Its final aim would be the explanation of all events in nature by structure laws valid always and everywhere."

In the following subsections we consider Einstein's ideas on a few specific subjects.

2.8.1 *Ether and Action-at-Distance*

It is well known that Einstein's studies on the electromagnetic properties of moving bodies and the special relativity brought him to denounce the ether. But this rejection of the ether must be taken in the context of treating it in a crudely materialistic fashion, including attributing to it the velocity, [70, p. 373–374]. In particular, in his letter to Mileva Maric, [70, p. 373], Einstein rejects the Maxwellian concept of the electric current as the decay of electric polarization, viewing it rather as a motion of "true electric masses":

"I am more and more convinced that the electrodynamics of moving bodies, as it is presented today, does not agree with the truth, and that it should be possible to present it in a simpler way. The introduction of the name 'ether' into the electric theories has led to the notion of a medium of whose motion one could speak of without being able, I believe, to associate a physical meaning to this statement. I believe that electric forces can be directly defined only for empty space, [which is] also emphasized by Hertz. Further, electric currents

will have to be regarded not as ‘the vanishing of electric polarization in time’ but as motion of true electric masses, whose physical reality seems to result from the electrochemical equivalents [...]. Electrodynamics would then be the science of the motions in empty space of moving electricities and magnetisms.”

But in a deeper analysis, Einstein argues for the necessity to have an ether of a certain kind. The seeming contradiction lies in a more precise definition of what constitutes an “ether”. Here is a historical argument for a “luminiferous ether”, [107, p. 6]:

“It appeared beyond question that light must be interpreted as a vibratory process in an elastic, inert medium filling up universal space. It also seemed to be a necessary consequence of the fact that light is capable of polarisation that this medium, the ether, must be of the nature of a solid body, because transverse waves are not possible in a fluid, but only in a solid. Thus the physicists were bound to arrive at the theory of the “quasi-rigid” luminiferous ether, the parts of which can carry out no movements relatively to one another except the small movements of deformation which correspond to light-waves.”

The evolving history of the “ether” concept took a new turn in Lorentzian ether without mechanical properties as a component of the electromagnetic theory, [107, p. 8]:

“Such was the state of things when H.A. Lorentz entered upon the scene. He brought theory into harmony with experience by means of a wonderful simplification of theoretical principles. He achieved this, the most important advance in the theory of electricity since Maxwell, by taking from ether its mechanical, and from matter its electromagnetic qualities. As in empty space, so too in the interior of material bodies, the ether, and not matter viewed atomistically, was exclusively the seat of electromagnetic fields. According to Lorentz the elementary particles of matter alone are capable of carrying out movements; their electromagnetic activity is entirely confined to the carrying of electric charges. Thus Lorentz succeeded in reducing all electromagnetic happenings to Maxwell’s equations for free space.”

A very important consideration in favor of a properly defined “ether” is that it is an alternative to action at distance, [107, p. 6]:

“... or by assuming that the Newtonian action at a distance is only apparently immediate action at a distance, but in truth is conveyed by a medium permeating space, whether by movements or by elastic deformation of this medium. Thus the endeavour toward a unified view of the nature of forces leads to the hypothesis of an ether. This hypothesis, to be sure, did not at first bring with it any advance in the theory of gravitation or in physics generally, so that it became customary to treat Newton’s law of force as an axiom not further reducible. But the ether hypothesis was bound always to play some part in physical science, even if at first only a latent part.”

Einstein clarifies that an ether that does not consist of “particles” is not in conflict with the special theory of relativity, [107, p. 10]:

“There may be supposed to be extended physical objects to which the idea of motion cannot be applied. They may not be thought of as consisting of particles which allow themselves to be separately tracked through time. In Minkowski’s idiom this is expressed as follows:—Not every extended conformation in the four-dimensional world can be regarded as composed of world-threads. The special theory of relativity forbids us to assume the ether to consist of

particles observable through time, but the hypothesis of ether in itself is not in conflict with the special theory of relativity. Only we must be on our guard against ascribing a state of motion to the ether.”

The picture of *ether as space endowed with special physical properties* goes back to Drude's phenomenological picture of ether processes, [70, p. 373]. Einstein takes this view on Newton's absolute space, [107, p. 11]:

“Certainly, from the standpoint of the special theory of relativity, the ether hypothesis appears at first to be an empty hypothesis. In the equations of the electromagnetic field there occur, in addition to the densities of the electric charge, only the intensities of the field. The career of electromagnetic processes in vacuo appears to be completely determined by these equations, uninfluenced by other physical quantities. The electromagnetic fields appear as ultimate, irreducible realities, and at first it seems superfluous to postulate a homogeneous, isotropic ether-medium, and to envisage electromagnetic fields as states of this medium.

But on the other hand there is a weighty argument to be adduced in favour of the ether hypothesis. To deny the ether is ultimately to assume that empty space has no physical qualities whatever. The fundamental facts of mechanics do not harmonize with this view. For the mechanical behaviour of a corporeal system hovering freely in empty space depends not only on relative positions (distances) and relative velocities, but also on its state of rotation, which physically may be taken as a characteristic not appertaining to the system in itself. In order to be able to look upon the rotation of the system, at least formally, as something real, Newton objectivises space. Since he classes his absolute space together with real things, for him rotation relative to an absolute space is also something real. Newton might no less well have called his absolute space “Ether”; what is essential is merely that besides observable objects, another thing, which is not perceptible, must be looked upon as real, to enable acceleration or rotation to be looked upon as something real.”

Einstein continues on with endowing the ether with the effects of inertia as understood by Mach, [107, p. 11–12]:

“It is true that Mach tried to avoid having to accept as real something which is not observable by endeavouring to substitute in mechanics a mean acceleration with reference to the totality of the masses in the universe in place of an acceleration with reference to absolute space. But inertial resistance opposed to relative acceleration of distant masses presupposes action at a distance; and as the modern physicist does not believe that he may accept this action at a distance, he comes back once more, if he follows Mach, to the ether, which has to serve as medium for the effects of inertia. But this conception of the ether to which we are led by Mach's way of thinking differs essentially from the ether as conceived by Newton, by Fresnel, and by Lorentz. Mach's ether not only conditions the behaviour of inert masses, but is also conditioned in its state by them.”

2.8.2 *Problems with Maxwell's Theory and Quantization of Electromagnetic Radiation*

It is well known that Einstein in early 1900 had concerns with the prevailing theory at that time (Maxwell's wave theory), and it was one of his motivation for introducing his “light quanta”. It is interesting and instructive to find out his thoughts on where exactly the problem was with the wave theory. In his paper on the constitution of

radiation written in 1909, Einstein identifies the root of the problem in *that the radiation of an oscillating ion has no inverse elementary process*, [102, p. 387]:

“The basic property of the wave theory that gives rise to these difficulties seems to me to lie in the following. While in the kinetic theory of matter there exists an inverse process for every process in which only a few elementary particles take part. e.g., for every molecular collision, according to the wave theory this is not the case for elementary radiation processes. According to the prevailing theory, an oscillating ion produces an outwardly propagated spherical wave. The opposite process does not exist as an elementary process. It is true that the inwardly propagated spherical wave is mathematically possible; however, its approximate realization requires an enormous amount of emitting elementary structures. Thus, the elementary process of light radiation as such does not possess the character of reversibility. Here, I believe, our wave theory is off the mark. Concerning this point the Newtonian emission theory of light seems to contain more truth than does the wave theory, since according to the former the energy imparted at emission to a particle of light is not scattered throughout the infinite space but remains available for an elementary process of absorption. Keep in mind the laws of production of secondary cathode rays by X-rays.”

Einstein proceeds further and makes an argument that the radiated energy is not scattered as a spherical wave according the wave theory but rather it is directed, [102, p. 388]:

“In other words, the elementary radiation process seems to proceed such that it does not, as the wave theory would require, distribute and scatter the energy of the primary electron in a spherical wave propagating in all directions. Rather, it seems that at least a large part of this energy is available at some location of P_2 or somewhere else. The elementary process of radiation seems to be directed. Furthermore, one gets the impression that the process of X-ray production in P_1 and the process of secondary cathode ray production in P_2 are essentially inverse processes.”

Based on his analysis of black-body radiation and its consistency with Maxwell’s theory, Einstein comes with a bold conclusion that somehow the energy of electromagnetic radiation has to be quantized, [210, p. 186–187]:

“He asked, in particular, whether those discontinuities could be restricted to the interaction of radiation and matter – thus preserving the validity of Maxwell’s equations for propagation in empty space – and he concluded that they could not. One must, he insisted, accept the particle-like behavior of high frequency radiation, a position he defended by a considerable extension and generalization of the fluctuation argument he had developed in 1905.”

Interestingly, in his letter to Einstein (2 June, 1906), von Laue objected to the necessity of Einstein’s conclusion about the quantization of radiation, attributing it rather to the energy exchange with matter, [210, pp. 189, 300]:

“If, at the beginning of your last reply, you state your heuristic viewpoint [i.e., the light-particle hypothesis) in the form, radiant energy can only be emitted and absorbed in certain finite quanta, then I know nothing to which to object; also all your applications correspond to this mode [of conceiving your theory]. Only, this is not a characteristic of electromagnetic processes in a vacuum, but of the absorbing or emitting material. Radiation does not consist of light-quanta, as [you say] in § 6 of the first [i.e. the light-particle] paper but only behaves during energy exchange with matter as though it did.”

2.8.3 *Ghost Field*

When thinking of how to reconcile the wave properties of EM radiation with light quanta, Einstein introduced a heuristic idea of a “ghost-field” [“Gespenster-feld”]. Einstein did not publish anything on this idea but expressed it on different occasions. In particular, this idea is discussed in the letter from H. Lorentz to A. Einstein, November 13, 1921, [312, p. 382]:

“Basic ideas. In emission of light two things are radiated. There is namely:

1. An interference radiation, which occurs according to the ordinary laws of optics, but still carries no energy. One can, for example, imagine that this radiation consists of ordinary electromagnetic waves, but with vanishingly small amplitudes. As a consequence they cannot themselves be observed; they serve only to prepare the way for the radiation of energy. It is like a dead pattern, that is first brought to life by the energy radiation. [In the book, he says: “On the screen you will have something like an undeveloped photographic image.”]

2. The energy radiation. This consists of indivisible quanta of magnitude $h\nu$. Their path is prescribed by the (vanishingly small) energy flux in the interference radiation, and they can never reach places where this flux is zero (dark interference bands). In an individual act of radiation the full interference radiation arises, but only a single quantum is radiated, which therefore can only reach one place on a screen placed in the radiation. However, this elementary act is repeated innumerable many times, with as good as identical interference radiation (the same pattern). The different quanta now distribute themselves statistically over the pattern, in the sense that the average number of them at each point of the screen is proportional to the intensity of the interference radiation reaching that point. In this way the observed interference phenomena arise, corresponding to the classical results.”

Lorentz explicitly attributes the ideas quoted to Einstein, [312, p. 382].

“Interestingly, Einstein’s ghost field idea paved a road for Born’s probabilistic interpretation of the Schrödinger wave function. M. Born wrote in his letter to Einstein, November 30, 1926, [312, p. 383]: “To report about myself, I am quite satisfied as far as physics goes, since my idea to conceive the Schrödinger wave field as a ghost field in your sense, is constantly proving to be better”. According to Baggot, [24, p. 74]: “Born subsequently claimed that he had been influenced by a remark that Einstein had made in one of his unpublished papers. In the context of light quanta interpreted using de Broglie’s wave-particle ideas, Einstein had suggested that the waves represented a *Gespensterfeld*, a kind of ‘ghost field’, which determines the probability for the light quantum to follow a specific path. Born had therefore chosen to reject Schrödinger’s attempts to provide a literal interpretation of the wavefunction as a real wave disturbance and, following Einstein’s logic, instead regarded the wavefunction as a measure of the probability of realizing specific outcomes in a quantum transition, such as a collision.”

2.8.4 *Light Quanta*

In this subsection we present several of Einstein’s statements regarding light quanta and their particle and wave properties.

From a letter from Einstein to Paul Bonofield, September 18, 1939, [312, pp. 389]:

"I do not believe that the light-quanta have reality in the same immediate sense as the corpuscles of electricity. Likewise I do not believe that the particle-waves have reality in the same sense as the particles themselves. The wave-character of particles and the particle-character of light will – in my opinion – be understood in a more indirect way, not as immediate physical reality."

From a letter from A. Einstein to H. Lorentz, May 23, 1909, [312, p. 378]:

"I am not at all of the opinion that one should think of light as composed of quanta which are independent of each other, and localized in relatively small regions. For the explanation of the Wien limit of the radiation formula this would indeed be the easiest way. But even the splitting of a light ray at the surface of a refracting medium completely forbids this approach. A light ray divides, but a light quantum indeed cannot divide without an alteration of frequency."

From the last interview given by Einstein, [63]:

"I asked Einstein whether Planck had ever fully accepted the "theory of photons," or whether he had continued to restrict his interest to the absorption or emission of light without regard to its transmission. Einstein stared at me for a moment or two in silence. Then he smiled and said: "No, not a theory. Not a theory of photons," and again his deep laughter enveloped us both – and the question was never answered. I remembered that Einstein's 1905 paper, for which (nominally) he had been awarded the Nobel prize, did not contain the word "theory" in the title, but referred instead to considerations from a "heuristic viewpoint.""

2.8.5 *Material Points Versus Continuous Fields*

Concerning the role of the concepts of material points and fields, Einstein wrote [104, p. 266–267]:

"According to Newton's system, physical reality is characterized by the concepts of space, time, material point, and force (reciprocal action of material points). Physical events, in Newton's view, are to be regarded as the motions, governed by fixed laws, of material points in space. The material point is our only mode of representing reality when dealing with changes taking place in it, the solitary representative of the real, in so far as the real is capable of change. Perceptible bodies are obviously responsible for the concept of the material point; people conceived it as an analogue of mobile bodies, stripping these of the characteristics of extension, form, orientation in space, and all "inward" qualities, leaving only inertia and translation and adding the concept of force. The material bodies, which had led psychologically to our formation of the concept of the "material point," had now themselves to be regarded as systems of material points. It should be noted that this theoretical scheme is in essence an atomistic and mechanistic one. All happenings were to be interpreted purely mechanically – that is to say, simply as motions of material points according to Newton's law of motion."

Einstein also wrote in his autobiographical Notes, [295, p. 61]:

"It therefore appears unavoidable that physical reality must be described in terms of continuous functions in space. The material point, therefore, can hardly be conceived anymore as the basic concept of the theory."

In the introduction to [236, p. 30] Einstein wrote [104, p. 267–268]:

“... before Maxwell people conceived of physical reality – in so far as it is supposed to represent events in nature – as material points, whose changes consist exclusively of motions, which are subject to total differential equations. After Maxwell they conceived physical reality as represented by continuous fields, not mechanically explicable, which are subject to partial differential equations. This change in the conception of reality is the most profound and fruitful one that has come to physics since Newton; but it has at the same time to be admitted that the program has by no means been completely carried out yet. The successful systems of physics which have been evolved since rather represent compromises between these two schemes, which for that very reason bear a provisional logically incomplete character, although they may have achieved great advances in certain particulars.”

He concludes:

“... the continuous field appeared side by side with the material point as the representative of physical reality. This dualism has to this day not disappeared, disturbing as it must be to any systematic mind.”

2.9 De Broglie's Theory of Phase Waves

“We are then inclined to admit that any moving body may be accompanied by a wave and that it is impossible to disjoin motion of body and propagation of wave.”

*L. de Broglie.*⁸

De Broglie's seminal contribution to physics was to realize coexisting wave and particle properties of light and matter, and to develop the concept of “*matter waves*”. Here is de Broglie's main idea in his own words, [79, p. 3]:

“The idea which, in my 1923–1924 works, served as the point of departure for Wave Mechanics was the following: Since for light there exists a corpuscular aspect and a wave aspect united by the relationship $\text{Energy} = h \text{ times frequency}$, where h , Planck's constant, enters in, it is natural to suppose that, for matter as well, there exists a corpuscular and a wave aspect, the latter having been hitherto unrecognized. These two aspects must be united by the general formulas in which Planck's constant figures, and must contain as special cases those relationships applicable to light.

In order to elaborate this idea, it seemed to me in 1923 that it was necessary to associate a periodic element to the corpuscular concept.”

De Broglie singled out the concept of phase wave to be of a fundamental importance, relating it to the nature of time, [183, p. 240]:

“This difference between the relativistic variations of the frequency of a clock and the frequency of a wave,” he wrote “is fundamental; it had greatly attracted my attention, and thinking over this difference determined the whole trend of my research.”

⁸[78, p. 450].

The key elements of de Broglie's approach are as follows, [78]. The point of departure is the relativistic formulas for the energy W and the momentum $\mathbf{p}$ of a point particle of rest mass m_0 , i.e.

$$W = mc^2, \quad \mathbf{p} = m\mathbf{v}, \quad \text{where } m = \frac{m_0}{\sqrt{1 - \beta^2}}, \quad \beta = \frac{v}{c}, \quad v = |\mathbf{v}|. \quad (2.9)$$

De Broglie then associates with the particle a “periodic phenomenon” whose frequency ω_0 is naturally defined in the particle's own system through the Planck-Einstein relation, namely

$$\omega_0 = \frac{W_0}{\hbar} = \frac{m_0 c^2}{\hbar}. \quad (2.10)$$

The corresponding frequency for the moving particle is then

$$\omega = \frac{W}{\hbar} = \frac{mc^2}{\hbar} = \frac{\omega_0}{\sqrt{1 - \beta^2}}. \quad (2.11)$$

Observe that the above formula readily yields the following representation for the particle velocity:

$$v = \beta c = c \sqrt{1 - \frac{\omega_0^2}{\omega^2}}. \quad (2.12)$$

Notice also that an observer looking at the internal periodic phenomenon sees the frequency

$$\omega_1 = \omega_0 \sqrt{1 - \beta^2} = \omega (1 - \beta^2), \quad (2.13)$$

which is manifestly different than the frequency ω in (2.11). This observer sees this phenomenon to vary as $\sin \omega_1 t$, and since at time t the particle is at distance $x = vt$, we have

$$\sin \omega_1 t = \sin \omega_1 \frac{x}{v}. \quad (2.14)$$

On other hand, the wave of frequency ω associated with the particle is described at time t and position $x = vt$ by

$$\sin \omega \left(t - \frac{\beta x}{c} \right) = \sin \omega (1 - \beta^2) \frac{x}{v}. \quad (2.15)$$

Relations (2.13)–(2.15) imply the following fundamental identity between a moving particle and the associated wave

$$\sin \omega_1 \frac{x}{v} = \sin \omega x \left(\frac{1}{v} - \frac{\beta}{c} \right). \quad (2.16)$$

De Broglie refers to the above identity as “*phase harmony*”, stating it as the following theorem: [78, p. 449]:

“If, at the beginning, the internal phenomenon of the moving body is in phase with the wave, this harmony of phase will always persist.”

He concludes his reflections on phase harmony with a remark, [78, p. 450]:

“We are then inclined to admit that any moving body may be accompanied by a wave and that it is impossible to disjoin motion of body and propagation of wave.”

Notice that the wave described by equality (2.15) has wave number k , wavelength λ and phase velocity V defined by the following relations:

$$k = \omega \frac{\beta}{c} = \frac{mv}{\hbar} = \frac{p}{\hbar}, \quad \lambda = \frac{2\pi}{k}, \quad V = \frac{c}{\beta}. \quad (2.17)$$

The above quantities are often referred to as the *de Broglie wave number* k and the *de Broglie wavelength* λ .

The particle-wave relation has another side to it. The frequency expression (2.12) for β and relations (2.17) imply the following representation for the group velocity u

$$\frac{1}{u} = \frac{dk}{d\omega} = \frac{1}{c} \frac{d(\omega\beta)}{d\omega} = \frac{1}{c} \frac{d\sqrt{\omega^2 - \omega_0^2}}{d\omega} = \frac{1}{c} \frac{\omega}{\sqrt{\omega^2 - \omega_0^2}} = \frac{1}{\beta c} = \frac{1}{v}. \quad (2.18)$$

The above identity shows that the particle velocity is identical to its wave group velocity.

De Broglie also showed that, in view of (2.17), the Sommerfeld quantum condition (also known as the selection rule)

$$\oint p \, dq = 2\pi n \hbar$$

can be written in the form

$$\oint \frac{1}{\lambda} \, dq = n,$$

that is to say, [78, p. 451]:

“The motion can only be stable if the phase wave is tuned with the length of the path.”

2.10 Schrödinger Wave Mechanics

“It is hardly necessary to emphasize how much more congenial it would be to imagine that at a quantum transition the energy changes over from one form of vibration to another, than to think of a jumping electron.”
*E. Schrödinger.*⁹

Erwin Schrödinger’s original and monumental contribution to the quantum theory of matter was his wave mechanics, including what we know today as the *Schrödinger equation*. The Schrödinger wave theory, [296], was inspired by de Broglie ideas on phase waves, [77], [27, Sect. 2.1]. He acknowledges this and underlines the difference between his interpretation and that of de Broglie, [296, p. 9]:

“I was led to these deliberations in the first place by the suggestive papers of M. Louis de Broglie, and by reflecting over the space distribution of those “phase waves”, of which he has shown that there is always a whole number, measured along the path, present on each period or quasi-period of the electron. The main difference is that de Broglie thinks of progressive waves, while we are led to stationary proper vibrations if we interpret our formulae as representing vibrations.”

Schrödinger’s approach to the construction of wave mechanics is rooted in a deep inner connection of the Hamilton theory and propagation of waves, [296, p. 13]:

“The inner connection between Hamilton’s theory and the process of wave propagation is anything but a new idea. It was not only well known to Hamilton, but it also served him as the starting-point for his theory of mechanics, which grew out of his Optics of Nonhomogeneous Media. Hamilton’s variation principle can be shown to correspond to Fermat’s Principle for a wave propagation in configuration space (*q*-space), and the Hamilton–Jacobi equation expresses Huygens’ Principle for this wave propagation. Unfortunately this powerful and momentous conception of Hamilton is deprived, in most modern reproductions, of its beautiful raiment as a superfluous accessory, in favour of a more colourless representation of the analytical correspondence.”

In his quest for a theory that unifies wave and particle properties to account for quantum effects, Schrödinger just as de Broglie departs from the Hamilton analogy of mechanics to geometric optics. He makes then an observation, critical to his entire theory, that geometrical optics by itself is just an approximation to the undulatory (wave) theory, and states that a proper wave equation in the configuration space must replace the fundamental equations of mechanics, [296, p. IX]:

“Hamiltonian analogy of mechanics to optics (pp. 13–18) is an analogy to *geometrical optics*, since to the path of the representative point in configuration space there corresponds on the optical side the light ray, which is only rigorously defined in terms of geometrical optics. The undulatory elaboration of the optical picture (pp. 19–30) leads to the surrender of the idea of the *path of the system*, as soon as the dimensions of the path are not great in comparison with the wave-length (pp. 25–26). Only when they are so does the idea of the path remain, and with it classical mechanics as an approximation (pp. 20–24, 41–44); whereas for “micro-mechanical” motions the fundamental equations of mechanics are just as

⁹[296, p. 10].

useless as geometrical optics is for the treatment of diffraction problems. In analogy with the latter case, a wave equation in configuration space must replace the fundamental equations of mechanics.”

Schrödinger’s grand vision of the integration of quantum phenomena into the wave theory was that these phenomena, including line spectra, should arise naturally as “proper” states and “proper” values (resonances, eigenvalues) of a certain wave equation with boundary conditions, [296, p. IX]:

“In the first instance, this equation is stated for purely periodic vibrations sinusoidal with respect to time (p. 27 et seq.); it may also be derived from a “Hamiltonian variation principle” (p. 1 et seq., pp. 11–12). It contains a “proper value parameter” E , which corresponds to the mechanical energy in macroscopic problems, and which for a single time-sinusoidal vibration is equal to the frequency multiplied by Planck’s quantum of action h . In general the wave or vibration equation possesses no solutions, which together with their derivatives are one-valued, finite, and continuous throughout configuration space, except for certain special values of E , the proper values. These values form the “proper value spectrum” which frequently includes continuous parts (the “band spectrum”, not expressly considered in most formulae: for its treatment see p. 112 et seq.) as well as discrete points (the “line spectrum”). The proper values either turn out to be identical with the “energy levels” (=spectroscopic “term”-value multiplied by h) of the quantum theory as hitherto developed, or differ from them in a manner which is confirmed by experience.”

Schrödinger implements his ideas for a passage from the Hamiltonian geometrical optics to a wave theory by considering first a stationary, time-harmonic state of the energy E . He introduces the corresponding wave function ψ and writes his first equation relating ψ to the action function S , [296, pp. 1-2]:

$$H\left(q, \frac{\partial S}{\partial q}\right) = E, \quad S = K \ln \psi. \quad (2.19)$$

Applying the equation to the specific Hamiltonian H corresponding to an electron in the Coulomb field of the proton, he obtains his first famous equation

$$\nabla^2 \psi + \frac{2m}{K^2} \left(E + \frac{e}{r^2}\right) \psi = 0, \quad (2.20)$$

and recovers the Hydrogen atom spectrum treating (2.20) with $K = \hbar$ with proper boundary conditions as an eigenvalue problem. Schrödinger then comments, [296, p. 9]:

“It is, of course, strongly suggested that we should try to connect the function ψ with some *vibration process* in the atom, which would more nearly approach reality than the electronic orbits, the real existence of which is being very much questioned to-day. I originally intended to find the new quantum conditions in this more intuitive manner, but finally gave them the above neutral mathematical form, because it brings more clearly to light what is really essential. The essential thing seems to me to be, that the postulation of “whole numbers” no longer enters into the quantum rules mysteriously, but that we have traced the matter a step further back, and found the “integralness” to have its origin in the finiteness and single-valuedness of a certain space function.”

Schrödinger's strong adherence to the Hamilton analogy of mechanics to geometric optics made him reject his earlier attempt to construct a relativistic theory based on what is known today as the Klein–Gordon equation, for it was not of the first order in time as the Hamilton–Jacobi equation (39.15) is. The insistence on the Hamilton analogy for a system of several interacting particles leads Schrödinger to introduce a single wave function ψ for the entire system, and this wave function ψ *has to be defined over a multidimensional configuration space and not over the real space*. As to the physical significance and interpretation of such a wave function ψ , he writes, [296, pp. 120–121]:

“This new interpretation may shock us at first glance, since we have often previously spoken in such an intuitive concrete way of the “ ψ -vibrations” as though of something quite real. But there is something tangibly real behind the present conception also, namely, the very real electrodynamically effective fluctuations of the electric space density.

The ψ -function is to do no more and no less than permit of the totality of these fluctuations being mastered and surveyed mathematically by a single partial differential equation. We have repeatedly called attention to the fact that the ψ -function itself cannot and may not be interpreted directly in terms of three-dimensional space – however much the one-electron problem tends to mislead us on this point – because it is in general a function in configuration space, not real space.”

For an insightful analysis of Schrödinger's original interpretation of his equation, see the paper by Jon Dorling in [197, pp. 16–40] and [183, 5.3]. A detailed treatment of aspects of Schrödinger wave mechanics related to the Hamilton–Jacobi theory is provided in [287, 2.13].

Another significant feature of Schrödinger's wave mechanics was its use of the number $i = \sqrt{-1}$ and complex-valued wave functions ψ . Schrödinger's famous equation for the complex-valued wave functions ψ appeared first in his sixth paper, [296, pp. 102–123]

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi$$

where $i = \sqrt{-1}$ and V is the potential that can be space and time dependent. Complex-valued wave function are instrumental in modelling diffraction and interference phenomena. Chen Ning Yang wrote on this subject in [197, pp. 53–64]:

“Classical physics, that is the physics before 1925, used exclusively real quantities. This was true for mechanics, thermodynamics, electrodynamics – the whole of classical physics. To be sure, complex numbers were used in many places. For example, in solving a linear alternating current problem complex numbers were used. But after a solution had been found, one always took the real or imaginary part of the solution in order to obtain the true physical answer. So the use of complex numbers was as a computational aid, i.e. the physics was conceptually in terms of real numbers.

With matrix mechanics and wave mechanics, however, the situation dramatically changed. Complex numbers became a conceptual element of the very foundation of physics: the fundamental equations of matrix mechanics and of wave mechanics:

$$pq - qp = -i\hbar \tag{2.21}$$

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi \quad (2.22)$$

both explicitly contain the imaginary unit $i = \sqrt{-1}$. It is to be emphasized that the very meaning of these equations would be totally destroyed if one tries to get rid of i by writing (2.21) and (2.22) in terms of real and imaginary parts.

....

In the subsequent two-man paper (Born and Jordan, 1925), (2.21) explicitly appeared for the first time in history. That was also the first time that the imaginary i entered physics in a fundamental way.

....

It was in this paper that the concept was first stated that ψ is a complex function of space-time and satisfies the complex time evolution equation (2.22) which Schrödinger called the true* wave equation, in contrast to $H\psi = E\psi$ which he called the vibration or amplitude equation.”

An original analysis of the role of complex numbers and the imaginary unit $i = \sqrt{-1}$ in Schrödinger’s theory as it relates to the Pauli and the Dirac theories was carried out by D. Hestenes and R. Gurtler in [173] and [163]. In particular the authors argue that, [173, p. 1029]:

“It is significant that in Schrödinger theory i and $\hbar$ always appear together in the product $i\hbar$. The significance is made manifest in the study of the Pauli theory to follow. There it is shown that Planck’s constant $\hbar$ enters the theory only as twice the magnitude of the electron’s spin. This strongly suggests that there is a general connection of spin to the appearance both of Planck’s constant and of “complex numbers” in quantum theory. It should especially be noted that this idea has arisen only from insistence on the internal consistency of quantum theory as it exists today; it has not been imposed on the theory by external considerations.”

An important part of the Schrödinger approach was his views on transition between states, [296, p. 10–11]:

“It is hardly necessary to emphasize how much more congenial it would be to imagine that at a quantum transition the energy changes over from one form of vibration to another, than to think of a jumping electron. The changing of the vibration form can take place continuously in space and time, and it can readily last as long as the emission process lasts empirically (experiments on canal rays by W. Wien); nevertheless, if during this transition the atom is placed for a comparatively short time in an electric field which alters the proper frequencies, then the beat frequencies are immediately changed sympathetically, and for just as long as the field operates. It is known that this experimentally established fact has hitherto presented the greatest difficulties. See the well-known attempt at a solution by Bohr, Kramers and Slater.”

Schrödinger was not satisfied with “quantum jumps” and always wanted a constructive theory with transitions which are not instantaneous, [297, p. 113]:

“Bohr’s theory held the ground for about a dozen of years, scoring a grand series of so marvellous and genuine successes, that we may well claim excuses for having shut our eyes to its one great deficiency: while describing minutely the so-called ‘stationary’ states which the atom had normally, i.e. in the comparatively uninteresting periods when nothing happens, the theory was silent about the periods of transition or ‘quantum jumps’ (as one then began to call them). Since intermediary states had to remain disallowed, one could not but regard the transition as instantaneous; but on the other hand, the radiating of a coherent wave train

of 3 or 4 feet length, as it can be observed in an interferometer, would use up just about the average interval between two transitions, leaving the atom no time to ‘be’ in those stationary states, the only ones of which the theory gave a description.”

Trying to advance his views on the fundamentals of quantum mechanics, Schrödinger emphasized certain limitations of the Planck-Einstein energy frequency relation $E = \hbar\omega$ insisting on treating “frequencies just as frequencies”, [297, p. 115]:

“One ought at least to try, and look upon atomic frequencies just as frequencies and drop the idea of energy-parcels. I submit that the word ‘energy’ is at present used with two entirely different meanings, macroscopic and microscopic. Macroscopic energy is a ‘quantity-concept’ (*Quantitätsgrösse*). Microscopic energy meaning $h\nu$ is a ‘quality-concept’ or ‘intensity-concept’ (*Intensitätsgrösse*); it is quite proper to speak of high-grade and low-grade energy according to the value of the frequency ν . True, the macroscopic energy is, strangely enough, obtained by a certain weighted summation over the frequencies, and in this relation the constant h is operative. But this does not necessarily entail that in every single case of microscopic interaction a whole portion $h\nu$ of macroscopic energy is exchanged. I believe one is allowed to regard microscopic interaction as a continuous phenomenon without losing either the precious results of Planck and Einstein on the equilibrium of (macroscopic) energy between radiation and matter, or any other understanding of phenomena that the parcel-theory affords.

...the interaction between two microscopic physical systems is controlled by a peculiar law of resonance. This law requires that the difference of two proper frequencies of the one system be equal to the difference of two proper frequencies of the other.”

Being consistent with his views on the particle as a material wave, Schrödinger viewed a *wave packet* as a model for a spatially localized charge. While his theory was very successful in describing quantum phenomena in the Hydrogen atom, it had great difficulties in treating the elementary charge as the material wave as it moves and interacts with other elementary charges. M. Born commented on this, [48, IV.7]:

“To begin with, Schrödinger attempted to interpret corpuscles and particularly electrons, as *wave packets*. Although his formulae are entirely correct, his interpretation cannot be maintained, since on the one hand, as we have already explained above, the wave packets must in course of time become dissipated, and on the other hand the description of the interaction of two electrons as a collision of two wave packets in ordinary three-dimensional space lands us in grave difficulties.”

An interested reader can find more reflections on the above subject in [98, 7.2].

2.11 De Broglie–Bohm Theory

“Is it not clear from the smallness of the scintillation on the screen that we have to do with a particle?”
J. Bell.¹⁰

Here we concisely present some of the key ideas of the de Broglie–Bohm theory also known as Bohmian mechanics or the “causal interpretation” of quantum mechanics.

¹⁰[36, p. 191].

A detailed exposition of the theory with its relation to quantum mechanics as well as historical aspects can be found in [36], [45], [79], [77], [96], [98], [68], [69], [176].

De Broglie conceived in 1927 an idea that “The wave-particle synthesis is to be achieved by representing the particle as a kind of singularity embedded in an extended wave phenomenon.”, [79, p. 98]. He then put forward his “Double Solution” program, later characterizing its key ideas and their subsequent advancements as follows, [79, p. 99]:

“Equipped with these general ideas, I ventured to assume the following principle, to which I gave the name of “the principle of the Double Solution”.

To every continuous solution $\Psi = ae^{\frac{i}{\hbar}\varphi}$ of the equation of propagation of Wave Mechanics there must correspond a singularity solution $u = fe^{\frac{i}{\hbar}\varphi}$ having the same phase φ as Ψ , but with an amplitude f involving a generally mobile singularity.

As I conceived it, the u function was the true representation of the physical entity “particle”, which would be an extended wave phenomenon centered around a point (or an almost point-like region), which would constitute the particle in the strict sense of the word. I then considered the particle in the strict sense of the word as being defined by a true mathematical singularity, that is, by a point where the function f would become infinite. Considerations that will be referred to later lead me today to believe that the particle must be represented, not by a true point singularity of u , but by a very small singular region in space where u would take on a very large value and would obey a non-linear equation, of which the linear equation of Wave Mechanics would be only an approximate form valid outside the singular region. The idea that the equation of propagation of u , unlike the classical equation of Ψ , is in principle non-linear now strikes me as absolutely essential.”

De Broglie illustrated his ideas, [76], [79, p. 100–101], by a simple example considering the relativistic equation for the propagation of the Ψ wave for a rectilinearly moving particle of rest mass m_0

$$\square\Psi + \frac{m_0c^2}{\hbar^2}\Psi = 0, \text{ where } \square = \frac{1}{c^2}\partial_t^2 - \nabla^2. \quad (2.23)$$

This equation has solutions in the form of plane monochromatic waves $\Psi = ae^{\frac{i}{\hbar}\varphi}$ where

$$\varphi = Wt - \mathbf{p} \cdot \mathbf{r}, \quad W = \frac{m_0c^2}{\sqrt{1-\beta^2}}, \quad \mathbf{p} = \frac{m_0\mathbf{v}}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c}. \quad (2.24)$$

Applying then his principle of the Double solutions, de Broglie introduces the following singular spherically symmetric solution to (2.23)

$$u(x, y, z, t) = \frac{C}{\sqrt{x^2 + y^2 + \frac{(z-vt)^2}{1-\beta^2}}} e^{\frac{i}{\hbar}(Wt - \mathbf{p} \cdot \mathbf{r}), \quad (2.25)$$

pointing out that this solution “would then be valid everywhere except in the interior of a very small ellipsoid surrounding the point $x = y = 0, z = vt$ flattened by the effect of the Lorentz contraction.”, [79, p. 101]. He then noticed that the partial

derivatives of the function u are evidently also singular solutions of equation (2.27) which are of N -polar type, including, in particular, a dipolar type solution.

De Broglie presented at the Solvay Congress on Physics in 1927 a truncated version of the “Double Solution” program called the “pilot-wave” theory with a key idea that the velocity $\mathbf{v}$ of the particle is defined by the “guidance formula”, [79, p. 90]

$$\mathbf{v} = -\frac{1}{m}\nabla\varphi, \quad (2.26)$$

where $\frac{\varphi}{\hbar}$ is wave-phase of the singularity. He commented later on that, [79, p. 90–91]:

“This watered-down version of my original conception happened to coincide exactly with the one put forward at the same date by Madelung in his hydrodynamical interpretation of Wave Mechanics, but this simplified version had far less interest and profundity than my initial ideas on the Double Solution. My presentation at the Solvay Congress was received unfavorably, and the purely probabilistic interpretation of Bohr, Born and Heisenberg supported by Pauli, Dirac and others, was very clearly the one preferred by most of the scientists present. Pauli, in particular, criticized my theory by citing the example of Fermi’s quantized rigid rotator.”

Though de Broglie’s seminal contribution to physics was widely recognized, he thought that his ideas were misinterpreted as he did away with the particle itself. In fact, he advanced the hypothesis of a “particle localized inside its wave”. This is what de Broglie wrote on this issue in 1973, [77, xlii–xliii]:

“Following the appearance of my Thesis, my ideas have often been misinterpreted by those saying that, according to me, the electron was a wave, which, of course, did away with the particle itself. It seems that adoption of this idea motivated Schrödinger in his very elegant 1926 papers to write down for the first time the propagation equation of a wave, which he called the ψ wave. This he did for the electron, but only in the Newtonian approximation and without taking the spin into account. In this manner, Schrödinger was able to compute exactly the wave processes corresponding to the quantified states of an atomic system, as conceived classically in the works of Bohr and his followers. It is quite certain that Schrödinger then viewed his ψ wave as a physical wave. But he abandoned completely any idea of localizing the particle in this wave, so that the picture which he formed of the atom and, more generally, of the ψ waves, made no provision for localized particles. This had very grave consequences and made Schrödinger’s use of configuration space paradoxical in the case of particle systems. Soon afterwards, Born introduced the normalization of the ψ wave, by which he modified arbitrarily the amplitude of the wave and hence deprived it of all physical reality. The normalized ψ wave was thus transformed into a simple probabilistic representation which, while leading to a large number of exact predictions, provided no understandable picture of the coexistence between waves and particles. ...

But, contrary to what is usually admitted, quantum mechanics does not have the right to postulate $W = h\nu$ and $p = h/\lambda$, because the energy W and the momentum p of a particle are properties which are associated with the concept of a localized object that moves through space along a trajectory. The reason I was able to establish these formulas was that I advanced the hypothesis of a particle localized inside its wave. ...

But in about the last 20 years I have again convinced myself that one should return to the idea that a particle is a very small object that is localized and moves along a trajectory.”

In 1952 David Bohm proposed in his two original papers, [46], [47], a “hidden variables” theory which was essentially de Broglie’s pilot-wave theory carried to its logical conclusion, [176, 1.5.1]. He stated in his first paper [46, p. 167]:

“After this article was completed, the author’s attention was called to similar proposals for an alternative interpretation of the quantum theory made by de Broglie in 1926, but later given up by him partly as a result of certain criticisms made by Pauli and partly because of additional objections raised by de Broglie himself. As we shall show in Appendix B of Paper II, however, all of the objections of de Broglie and Pauli could have been met if only de Broglie had carried his ideas to their logical conclusion.”

We provide below a very concise presentation of the Bohm theory following closely [46, 4], [45, 3.1.3.3], [96, 2.3, 5.3], [176, 3.2]. The starting point of Bohm’s theory consists of the following steps: (i) take the Schrödinger equation for a complex-valued wave function ψ

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi, \quad (2.27)$$

(ii) represent ψ in the polar form $\psi = R \exp\{i\frac{S}{\hbar}\}$ where $R = \sqrt{\psi^*\psi} = R(t, \mathbf{x})$ and $S = \frac{1}{2i} \ln \frac{\psi}{\psi^*} = S(t, \mathbf{x})$ are respectively its real amplitude and real phase functions of time and space, and (iii) recast the Schrödinger equation (2.27) as the following system of equations for R and S

$$\partial_t S + \frac{(\nabla S)^2}{2m} + V + Q = 0, \quad Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R}, \quad (2.28)$$

$$\partial_t R^2 + \nabla \cdot \left(\frac{R^2 \nabla S}{m} \right) = 0, \quad (2.29)$$

where Q is termed “quantum potential”. Notice that the Hamilton–Jacobi equation (2.28) for the phase function S differs from its classical counterpart (41.3) by a modification of the classical potential V by adding to it the quantum potential Q .

Then assuming an initial value $\psi(0, \mathbf{x}) = \psi_0(\mathbf{x})$ or, equivalently, initial values

$$R(0, \mathbf{x}) = R_0(\mathbf{x}), \quad S(0, \mathbf{x}) = S_0(\mathbf{x}), \quad (2.30)$$

we find a unique solution $\psi(t, \mathbf{x})$ to the Schrödinger equation (2.27) subject to the initial conditions (2.30).

Having found $\psi(t, \mathbf{x})$ interpreted as a physical wave, one associates with it a point particle of mass m pursuing a trajectory $\mathbf{x}(t)$. Consequently, such a pair of a wave function and a trajectory constitutes a particular configuration of a single physical system. That is, the wave and particle are treated as aspects of this physical system. The connection between the wave and particle aspects is facilitated through the Hamilton–Jacobi theory as in Sect. 39 with the phase S satisfying the Hamilton–Jacobi equation (2.28) subject to the initial condition (2.30). Importantly, the Hamilton–Jacobi equation (2.28) involves the potential $V + Q$ where V is the

classical potential and Q defined in (2.28) is the quantum potential. Recall that, according to the Hamilton–Jacobi theory discussed in Sect. 39, all particle trajectories $\mathbf{x}(t)$ are orthogonal to the surfaces $S = \text{constant}$, and they can be found by integrating the ordinary differential equations

$$m \frac{d\mathbf{x}}{dt} = \mathbf{v}(t, \mathbf{x})|_{\mathbf{x}=\mathbf{x}(t)}, \text{ where } \mathbf{v}(t, \mathbf{x}) = \frac{\mathbf{p}(t, \mathbf{x})}{m} = \frac{\nabla S(t, \mathbf{x})}{m}. \quad (2.31)$$

The fields $\mathbf{v}(t, \mathbf{x})$ and $\mathbf{p}(t, \mathbf{x})$ in the above equation are, respectively, the velocity and the momentum fields of the particle. The kinetic energy of the particle estimated for a trajectory $\mathbf{x}(t)$ is

$$\frac{m}{2} \left(\frac{d\mathbf{x}}{dt} \right)^2 = \frac{1}{2m} (\nabla S(t, \mathbf{x}))^2|_{\mathbf{x}=\mathbf{x}(t)}, \quad (2.32)$$

and its total potential energy is the sum of its classical and quantum components, i.e.

$$V(t, \mathbf{x}) + Q(t, \mathbf{x})|_{\mathbf{x}=\mathbf{x}(t)}. \quad (2.33)$$

Notice that, unlike the classical potential, the quantum potential is not a preassigned function of time and space coordinates, but rather it depends on the “quantum state” described by $R(t, \mathbf{x}) = \sqrt{\psi^* \psi}(t, \mathbf{x})$.

To see that the quantum potential Q defined in (2.28) can be treated on the same footing as the classical potential V for the particle motion, we apply the operator ∇ to the Hamilton–Jacobi equation (2.28) yielding after elementary transformation the following field equation:

$$\left[\partial_t + \frac{1}{m} \nabla S \cdot \nabla \right] \nabla S = -\nabla (V + Q). \quad (2.34)$$

According to (2.31), $\frac{1}{m} \nabla S = \mathbf{v}$, and the equation (2.34) evaluated at the trajectory $\mathbf{x}(t)$ takes the form

$$\frac{d}{dt} (m\mathbf{v}) = -\nabla (V + Q)|_{\mathbf{x}=\mathbf{x}(t)} \quad (2.35)$$

where

$$\frac{d}{dt} = \partial_t + \mathbf{v} \cdot \nabla \quad (2.36)$$

is the so-called *convective derivative* representing the time rate of change for a point moving with the particle. Evidently, equation (2.35) has the form of Newton’s second law for a point particle subjected to the quantum force $-\nabla Q$ in addition to the classical force $-\nabla V$. Interestingly, dependence on the Plank constant $\hbar$ in Bohm’s quantum force comes entirely from the spin, [166, 4].

Comparing Bohm’s theory with de Broglie double solution program, Bohm and Hiley wrote, [45, p. 38-39]:

“The idea of a ‘pilot wave’ that guides the movement of the electron was first suggested by de Broglie [8] in 1927, but only in connection with the one-body system. De Broglie presented this idea at the 1927 Solvay Congress where it was strongly criticised by Pauli [9]. His most important criticism was that, in a two-body scattering process, the model could not be applied coherently. In consequence de Broglie abandoned his suggestion. The idea of a pilot wave was proposed again in 1952 by Bohm [10] in which an interpretation for the many-body system was given. This latter made it possible to answer Pauli’s criticism and indeed opened the way to a coherent interpretation including a theory of measurement which was applicable over a wide range of quantum phenomena. As a result de Broglie [11] took up his original ideas again and continued to develop them in various ways.”

The key points of Bohm’s theory of ontology can be summarized as follows, [45, p. 29–30]:

1. The electron actually is a particle with a well-defined position $x(t)$ which varies continuously and is causally determined.
2. “This particle is never separate from a new type of quantum field that fundamentally affects it”. This field is given by R and S or alternatively by $\psi = R \exp \{i \frac{S}{\hbar}\}$ satisfying the Schrödinger equation (2.27).
3. The particle equation of motion is (2.35)–(2.36) with the “quantum” force $-\nabla Q$ integrated into it.
4. The particle momentum $\mathbf{p}$ is restricted by relation (2.31) where the phase function S satisfies the quantum version (2.28) of the Hamilton–Jacobi equation.
5. In a statistical ensemble of particles, selected so that all have the same quantum field ψ , the probability density is $P = R^2$. This density satisfies the conservation (2.29) which in view of (2.31) can be recast as

$$\partial_t P + \nabla \cdot (P\mathbf{v}) = 0. \quad (2.37)$$

2.12 Continuum Theories and Atomicity

Interestingly, doubts have been raised against the very idea to use continuum theories to account for atomicity, where the atomicity is understood in the sense that there are only a few possible values for elementary charges and all of them are multiples of the electron charge. Here are points made by W. Pauli, [265, p. 205]:

“Each of the theories which we discussed has its particular advantages and drawbacks. Their joint failure prompts us, however, to summarize specifically those shortcomings and difficulties which are common to them all.

It is the aim of all continuum theories to derive the atomic nature of electricity from the property that the differential equations expressing the physical laws have only a discrete number of solutions which are everywhere regular, static, and spherically symmetric. In particular, one such solution should exist for each of the positive and negative kinds of electricity. It is clear that differential equations which have this property must be of a particularly complicated structure. It seems to us that this complexity of the physical laws in itself already speaks against the continuum theories. For it should be required, from a physical point of view, that the existence of atomicity, in itself so simple and basic, should also be interpreted

in a simple and elementary manner by theory and should not, so to speak, appear as a trick in analysis.

Furthermore, we have seen that the continuum theories are forced to introduce special forces which keep the Coulomb repulsive forces in the interior of the electrical elementary particles in equilibrium.”

Another doubt raised by W. Pauli was related to whether electric forces as we know them act in the interior of an electron, [265, p. 206]:

“Finally, a conceptual doubt should be mentioned. The continuum theories make direct use of the ordinary concept of electric field strength, even for the fields in the interior of the electron. This field strength is however defined as the force acting on a test particle, and since there are no test particles smaller than an electron or a hydrogen nucleus, the field strength at a given point in the interior of such a particle would seem to be unobservable, by definition, and thus be fictitious and without physical meaning.

Whatever may be one’s attitude in detail towards these arguments, this much seems fairly certain: new elements which are foreign to the continuum concept of the field will have to be added to the basic structure of the theories developed so far, before one can arrive at a satisfactory solution of the problem of matter.”

2.13 Quantum Electrodynamics (QED)

The development of quantum electrodynamics (QED) was a natural step in advancing ideas of quantum mechanics to the treatment of electromagnetic phenomena, including radiation. There is an extensive literature on QED including the following monographs: [3], [28], [64], [101], [123], [224], [229], [244], [262, 16, 18], [303], [290, 0], [334], [335], [336]. The term *quantum electrodynamics (QED)* was first introduced by P. Dirac in 1927 in his paper on the quantum theory of the emission and absorption of radiation, [89]. In this paper, published shortly after Schrödinger’s seminal papers on the quantum wave mechanics, P. Dirac advanced his new ideas on the quantum treatment of the electromagnetic radiation. In particular, he points to fundamental difficulties in integrating the relativity principles into existing non-relativistic quantum theory, [89, p. 243–244]:

“On the other hand, hardly anything has been done up to the present on quantum electrodynamics. The questions of the correct treatment of a system in which the forces are propagated with the velocity of light instead of instantaneously, of the production of an electromagnetic field by a moving electron, and of the reaction of this field on the electron have not yet been touched. In addition, there is a serious difficulty in making the theory satisfy all the requirements of the restricted principle of relativity, since a Hamiltonian function can no longer be used.”

Twenty years later, R. Feynman, I. Tomonaga and J. Schwinger proposed their versions of a fully relativistic QED theory for which they were awarded the 1965 Nobel Prize in physics. These theories predicted with an astonishing accuracy experimentally observed Lamb shift and the anomalous magnetic moment of the electron. For instance, here is the third order QED approximation of the *anomaly* $a = \frac{g-2}{2}$ of the electron, where g is the gyromagnetic ratio, [178, p. 348],

$$a_3 = \frac{\alpha}{2\pi} - 0.328478445 \left(\frac{\alpha}{\pi}\right)^2 + 1.183 (11) \left(\frac{\alpha}{\pi}\right)^3 = 1159652359(282) \times 10^{-12}. \quad (2.38)$$

F. Dyson has shown that the three approaches were essentially equivalent and that the infinities arising there can be treated by a certain renormalization procedure for the charge and the mass, [99]. Shortly after that, he also showed that the elements of the relevant S -matrix can be calculated consistently by a perturbation theory to any desired order in the fine-structure constant α , and that divergences arising from higher order radiative corrections can be removed from the S -matrix by a “consistent use of the ideas of mass and charge renormalization”, [100].

The treatment of infinities and divergences in QED continued to be a concern for many physicists. Their great efforts to get rid of infinities produced many important insights into the nature of quantum features of electromagnetic phenomena, but, unfortunately, they were not successful in turning the QED into a mathematically consistent theory. R. Feynman acknowledged that there is no satisfactory quantum electrodynamics, [120]:

“That is, I believe there is really no satisfactory quantum electrodynamics, but I’m not sure. And, I believe, that one of the reasons for the slowness of present-day progress in understanding the strong interactions is that there isn’t any relativistic theoretical model, from which you can really calculate everything. Although, it is usually said, that the difficulty lies in the fact that strong interactions are too hard to calculate, I believe, it is really because strong interactions in field theory have no solution, have no sense they’re either infinite, or, if you try to modify them, the modification destroys the unitarity. I don’t think we have a completely satisfactory relativistic quantum-mechanical model, even one that doesn’t agree with nature, but, at least, agrees with the logic that the sum of probability of all alternatives has to be 100%. Therefore, I think that the renormalization theory is simply a way to sweep the difficulties of the divergences of electrodynamics under the rug. I am, of course, not sure of that.”

I. Tomonaga in his Nobel lecture discusses at length different infinities arising in QED, and, when acknowledging spectacular achievements of the QED, he recognizes that the mass and the charge are infinite, [323]:

“It is a very pleasant thing that no divergence is involved in the theory except for the two infinities of electronic mass and charge. We cannot say that we have no divergences in the theory, since the mass and charge are in fact infinite.”

P. Dirac expressed repeatedly his deep dissatisfaction with the QED as a fundamental theory. In particular, he wrote, [91, p. 195]:

“... Hamiltonian is substituted into the fundamental equations of motion of the Heisenberg theory, the result is definitely wrong. It is not only wrong – it is not a sensible result at all. It is a result which has infinities in it. It is really a wrong theory, but still physicists like to use this Hamiltonian which is suggested by classical mechanics.

How then do they manage with these incorrect equations? These equations lead to infinities when one tries to solve them; these infinities ought not to be there. They remove them artificially. That means they are departing from the Heisenberg equations of motion.

...

Just because the results happen to be in agreement with observation does not prove that one's theory is correct. After all, the Bohr theory was correct in simple cases. It gave very good answers, but still the Bohr theory had the wrong concepts."

In spite of the fundamental problems involving infinities discussed above, QED has been successful in different areas, particularly in quantum optics. While many phenomena of quantum optics can be understood based on the so-called semi-classical theory (SCT), [303, 5], [150], there are fundamental phenomena which can be treated at a deeper level when EM radiation is quantized according to the QED. Such phenomena include, for instance, the Lamb shift, the gyromagnetic anomaly, spontaneous emission of light by an excited atom, and parametric fluorescence by a nonlinear crystal subjected to pumping radiation, [28], [303, 1.4] [150, 6]. The QED approach provides a unified framework for the treatment of a full range of matter-radiation interaction phenomena, [150, p. 457]:

"The fully quantum approach also has the merit of allowing a simple interpretation in terms of photons for the various matter-radiation interaction processes, such as absorption, stimulated emission, scattering, and also the basic processes of nonlinear optics. Indeed, it provides a unified framework for both stimulated and spontaneous processes. Finally, it can be used to tackle completely new situations where matter and radiation interact, which lie outside the scope of any semi-classical description, such as cavity quantum electrodynamics or the production of single-photon or entangled-photons states."

Chapter 3

The Neoclassical Field Theory of Charged Matter: A Concise Presentation

“I do not believe in micro- and macro-laws,
but only in (structure) laws of general rigorous validity.
And I believe that these laws are logically simple,
and that reliance on this logical simplicity is our best guide.”
*A. Einstein.*¹

The theory of electromagnetic phenomena known as electrodynamics is one of the major theories in science; some fragments of its rich history are reflected in Part 2. The EM interactions play a fundamental role at all spatial scales ranging from atomic to astronomical. At macroscopic scales the interaction of the EM field with matter is described by classical electrodynamics based on the Maxwell–Lorentz theory. Many electromagnetic phenomena at microscopic scales are covered by the so-called semiclassical theory that treats the matter according to quantum mechanics, whereas the EM field is treated classically, [150, Part I], [303, 5]. Consequently, the semiclassical theory is based on a coupled system of the Maxwell and the Schrödinger equations, [204, 4.2]. But there are also electromagnetic phenomena at microscopic scales that require quantum treatment of both the matter and the EM field, constituting the subject of quantum electrodynamics (QED) which we have very briefly reviewed in Sect. 2.13. The borderlines between the classical, the semiclassical and QED theories are blurred, and domains of their applicability overlap. Certainly, classical electrodynamics in its traditional form is not applicable to atomic scales, where such non-classical effects as wave-particle duality and the Hydrogen atom energy spectrum do not fit into its framework. The nanoscales are already in the domain where quantum effects play a role. Continuing advances in experimental studies of EM phenomena at the nanoscales lead to new areas of research such as plasmonics which combines classical and quantum effects, raising challenges to the fundamental EM theory, see Sect. 18.2.3.

¹From A. Einstein to David Bohm, November 24, 1954, [312, p. 377].

The goal we pursue here is to modify classical electrodynamics into a theory that applies to all spatial scales including atomic and nanoscales. The neoclassical theory advanced here is conceived as one theory for all spatial scales in which classical and quantum aspects are naturally unified and emerge as approximations. We refer to this theory as the *balanced charges theory* (BCT). The BCT is a classical Lagrangian field theory, and consequently it is a local and deterministic theory. Probabilistic aspects of the theory may arise in it effectively through complex nonlinear dynamical evolution.

C. Lanczos made the following insightful points on necessary features of a mathematically sound field theory, [212, p. 395]:

“Most of the field theories with which we are familiar are of a linear nature. This means that they permit the principle of superposition and thus exclude any dynamical interaction. At the same time they lead to singularities, i.e. to points in space at which the field quantities become infinite. The fact of dynamical interaction between fields demonstrates that nature cannot be purely linear, nor can it tolerate singular points.”

The main purpose of this Part is to give (i) a concise presentation of significant features of the BCT with minimal technical details and (ii) to compare it with the well established and experimentally verified classical EM theory and quantum mechanics. A detailed presentation of the same is provided in Parts III and Part IV. We put an effort to make this concise exposition self-contained, while freely using common notation and concepts from classical electrodynamics and classical field theory with details available for the reader in Parts I and II.

The key element of the BCT is the *concept of a balanced charge*, which is conceived to model an elementary charge such as an electron or proton. To keep the presentation as simple as possible, we consider in this Part a spinless balanced charge, leaving its spinorial version to Part IV. When constructing the BCT, we insist on the following requirements. First of all, the BCT must imply as a valid approximation classical Maxwell–Lorentz theory at macroscopic scales. Namely, the dynamics of charges when they are well separated is governed approximately by Newton’s equations with the Lorentz forces, and the evolution of the EM fields is governed by the Maxwell equations. Second, the BCT should describe basic quantum effects at atomic scales, including, in particular, the discreteness of the energy levels in the Hydrogen atom and the main features of the de Broglie wave theory.

Material features of a spinless balanced charge are accounted for by its individual complex-valued wave function $\psi(t, \mathbf{x})$. Notice that *in the BCT every elementary charge has its own individual wave function over the 4-dimensional space-time continuum. This is in contrast to quantum mechanics which is based fundamentally on the concept of multidimensional configuration space.* One can think of an individual wave function associated with every elementary charge as a material wave which describes a spatially distributed charge.

Electromagnetic features of a spinless balanced charge are accounted for by its *elementary four-potential* $A^\mu(t, \mathbf{x})$ and the corresponding *elementary EM field* $F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$ satisfying the Maxwell equations. The EM interaction of a given elementary charge with all other elementary charges is facilitated by the EM field which is the sum of the elementary EM fields of all other charges.

Consequently, the BCT setup explicitly excludes the EM self-interaction for every elementary charge. This is in contrast to classical EM theory that features the concept of a single EM field for all charges with implied self-interaction.

An important and critical property of the balanced charge is the presence of an internal force of non-electromagnetic origin. This internal non-EM force enters the BCT through a nonlinearity which ultimately facilitates the ability of a balanced charge to be localized and to move as a point-like object. In the latter case, the charge evolution can be described approximately by Newton's law with the Lorentz force.

The BCT field equations consist of two sets of Euler–Lagrange equations derived from the Lagrangian: (i) the nonlinear Klein Gordon (NKG) or the nonlinear Schrödinger (NLS) equation in the relativistic and non-relativistic cases, respectively, for the charge's wave functions; (ii) the Maxwell equations for the charge's elementary EM fields. An important feature of the proposed theory is that a free elementary charge is described by a special closed form solution referred to as *wave-corpuscule*. The wave-corpuscule, being an exact solution to a nonlinear evolution equation, is similar to a soliton of the kind entertained by Louis de Broglie in his pursuit of a nonlinear wave mechanics, [79]. G. Lochak wrote in his preface to the de Broglie's monograph, [77, p. XXXIX]:

“...The first idea concerns the solitons, which we would call *ondes à bosses* (humped waves) at the *Institut Henri Poincaré*. This idea of de Broglie's used to be considered as obsolete and too classical, but it is now quite well known, as I mentioned above, and is likely to be developed in the future, but only provided we realize what the obstacle is and has been for twenty-five years: It resides in the lack of a general principle in the name of which we would be able to choose one nonlinear wave equation from among the infinity of possible equations. If we succeed one day in finding such an equation, a new microphysics will arise.”

We agree with G. Lochak that there has to be a sound physical principle allowing us to choose the nonlinearity. In our theory an attractive choice is the logarithmic nonlinearity which is singled out by its unique property of exactly preserving the Planck–Einstein frequency-energy relation $E = \hbar\omega$, see Sect. 17.4.1.

We do not assume that there is a mass distribution associated with the charge distribution, the mass emerges as a characteristic of the dynamics which describes the elementary charge as a whole and is determined from the analysis of the point-like behavior of the charge. It coincides with the mass parameter that enters the NLS equation in the non-relativistic regimes, and in the relativistic regimes the mass is determined by the energy of the wave function in accordance with Einstein's formula $E = mc^2$. This setup is consistent with the following point made by Einstein and Infeld, [105, p. 257]:

“What impresses our senses as matter is really a great concentration of energy into a comparatively small space”.

The BCT setup allows for two different charge distributions to be at the same location in the space-time. *An elementary charge in the BCT is always a material wave that can acquire particle properties when its energy is localized.* The coexistence of wave and particle properties in the BCT is manifested through different regimes for the charge wave function. Namely, an elementary charge is in a particle-like state

when its energy is well localized, and it is in a wave state when its energy is well spread out in the space. Consequently, the balanced charge is neither a point charge as in classical EM theory and in quantum mechanics, nor is it a distributed charge with a fixed size and geometry as in the Lorentz–Abraham model.

This Part is devoted to the presentation of the main steps in the implementation of the above mentioned program. First, we formulate basic concepts of the BCT, and then we proceed with an analysis of different features of the theory including:

- in non-relativistic regimes at macroscopic scales, when elementary charges are well separated and localized, we recover the point charge dynamics described by Newton’s law with the Lorentz forces as a limit;
- in relativistic regimes we demonstrate that the energy of an elementary charge affects its accelerated motion in a full agreement with Einstein’s formula $E = mc^2$;
- we show that the wave-corpuscle state of an elementary charge describes its point-like motion and accounts explicitly for the de Broglie phase waves;
- we demonstrate that the BCT version of the Hydrogen atom yields a frequency spectrum matching the same for the Schrödinger Hydrogen atom with a desired accuracy.

3.1 Point Charges in Classical Electromagnetic Theory

“For me, an electron is a corpuscle which at any given instant is situated at a determinate point of space, ...”
A. Lorentz.²

For motivation and comparison purposes, let us recall the dynamical setup of a classical system of point charges q_ℓ , $\ell = 1, \dots, N$, interacting with the EM field governed by the Maxwell equations. The position and the velocity of the ℓ -th charge are denoted respectively by $\mathbf{r}_\ell(t)$ and $\mathbf{v}_\ell(t)$. The relativistically manifest 4×4 tensor presentation $F^{\mu\nu}(x)$ of the EM field is

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x), \quad (3.1)$$

with $A^\nu = A^\nu(x)$ being its *four-vector potential*,

$$A^\nu = (\varphi, \mathbf{A}), \quad \mathbf{A} = (A_1, A_2, A_3), \quad (3.2)$$

where φ and $\mathbf{A}$ are respectively classical *scalar and vector potentials*. The corresponding relativistically manifest representation of the Maxwell equations is

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu, \quad J^\nu = (c\rho, \mathbf{J}), \quad (3.3)$$

²[1, p. 108].

where J^ν is the total four-vector current with ρ and $\mathbf{J}$ being respectively its charge density and the current density. Under the Lorentz gauge

$$\partial_\nu A^\nu = 0, \quad (3.4)$$

the Maxwell equations (3.3) are reduced to the wave equations

$$\frac{1}{c^2} \partial_t^2 A^\nu - \nabla^2 A^\nu = \frac{4\pi}{c} J^\nu. \quad (3.5)$$

According to classical EM theory, the system of charges interacts with the EM field as follows. First, every point charge with charge value q_ℓ generates its four-current

$$J_\ell^\nu = (c\rho_\ell, \mathbf{J}_\ell), \quad \rho_\ell = q_\ell \delta(\mathbf{x} - \mathbf{r}_\ell(t)), \quad \mathbf{J}_\ell = q_\ell \delta(\mathbf{x} - \mathbf{r}_\ell(t)) \mathbf{v}_\ell(t), \quad (3.6)$$

where $\delta(\mathbf{x} - \mathbf{r}(t))$ is Dirac's delta-function. Then the total current J^ν that determines the total four-vector potential A^ν and EM tensor $F^{\mu\nu}$ through the Maxwell equations (3.3), (3.5) is the sum of all point charge currents:

$$J^\nu = (c\rho, \mathbf{J}) = \sum_\ell J_\ell^\nu = \sum_\ell (c\rho_\ell, \mathbf{J}_\ell). \quad (3.7)$$

Since the Maxwell equations are linear, the total EM field potential equals the sum of elementary fields generated by the elementary currents, that is

$$A^\nu = \sum_\ell A_\ell^\nu, \quad F^{\mu\nu} = \sum_\ell F_\ell^{\mu\nu}, \quad (3.8)$$

where $F_\ell^{\mu\nu}$ are elementary fields, and A_ℓ^ν are their potentials,

$$\partial_\mu F_\ell^{\mu\nu} = \frac{4\pi}{c} J_\ell^\nu, \quad \frac{1}{c^2} \partial_t^2 A_\ell^\nu - \nabla^2 A_\ell^\nu = \frac{4\pi}{c} J_\ell^\nu. \quad (3.9)$$

To complete the setup of a fully interacting system of point charges with the total EM field $F^{\mu\nu}$, the action of the latter on the ℓ -th point charge is defined through the Lorentz four-vector force

$$f_\ell^\mu = \frac{1}{c} q_\ell F^{\mu\nu} V_{\ell\nu} = (f_\ell^0, \mathbf{f}_{\text{Lor}\ell}), \quad V_{\ell\nu} = (c, \mathbf{v}_\ell), \quad (3.10)$$

where $\mathbf{f}_{\text{Lor}\ell}$ is the Lorentz force:

$$\mathbf{f}_{\text{Lor}\ell} = q_\ell \left[\mathbf{E}(t, \mathbf{r}_\ell(t)) + \frac{1}{c} \mathbf{v}_\ell(t) \times \mathbf{B}(t, \mathbf{r}_\ell(t)) \right], \quad (3.11)$$

with the electric field $\mathbf{E}(t, \mathbf{x})$ and the magnetic field $\mathbf{B}(t, \mathbf{x})$ defined by the standard expressions

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\partial_t\mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (3.12)$$

In the non-relativistic case with EM radiation effects being neglected, the dynamics of the ℓ -th point charge is then governed by Newton's law

$$\partial_t [m_\ell \mathbf{v}_\ell(t)] = \mathbf{f}_{\text{Lor}\ell} \quad (3.13)$$

where $\mathbf{f}_{\text{Lor}\ell}$ is the Lorentz force defined by (3.11).

The system of Eqs. (3.5)–(3.13) as a closed system has the following “self-interaction” problem, [98, 2.4], [310, 2.3]. The solutions of the Maxwell equations with point sources have singularities that diverge exactly at the locations of the point charges resulting in infinite values of the corresponding Lorentz forces. That can be readily seen from the Coulomb potential $\varphi_\ell = \frac{q_\ell}{|\mathbf{x} - \mathbf{r}_\ell|}$ generated by the point source $\rho_\ell = q_\ell \delta(\mathbf{x} - \mathbf{r}_\ell)$. There are two well-known ways to deal with such a divergence problem. One way is to replace the point charge by a smeared out “charge cloud”, for example by a charged sphere of a very small radius as in the Lorentz–Abraham model, see Sect. 2.5 and references therein. Then the corresponding EM potential is not singular anymore, and the charge energy in its own field is finite. An alternative way to treat the problem is to exclude the self-interaction of a charge with its own EM field as in the Wheeler–Feynman theory, [339], [340]. Namely, it is assumed that the EM field acting upon the ℓ -th point charge is generated only by other charges with the four-vector current

$$J_{\neq\ell}^\nu = (c\rho_{\neq\ell}, \mathbf{J}_{\neq\ell}) = \sum_{\ell' \neq \ell} (c\rho_{\ell'}, \mathbf{J}_{\ell'}). \quad (3.14)$$

The corresponding EM field $F_{\neq\ell}^{\mu\nu}$ and the four-vector potential $A_{\neq\ell}^\nu$ acting on the ℓ -th charge are then defined by

$$F_{\neq\ell}^{\mu\nu} = \sum_{\ell' \neq \ell} F_{\ell'}^{\mu\nu}, \quad A_{\neq\ell}^\nu = (\varphi_{\neq\ell}, \mathbf{A}_{\neq\ell}) = \sum_{\ell' \neq \ell} A_{\ell'}^\nu = \sum_{\ell' \neq \ell} (\varphi_{\ell'}, \mathbf{A}_{\ell'}). \quad (3.15)$$

This setup explicitly excludes the charge self-action from the Lorentz force. Consequently, the Wheeler–Feynman EM theory disposes the single EM field as a primitive concept, [339], [340, p. 426]:

- “(1) There is no such concept as “the” field, an independent entity with degrees of freedom of its own.
- (2) There is no action of an elementary charge upon itself and consequently no problem of an infinity in the energy of the electromagnetic field.”

The balanced charge theory advanced here disposes the single EM field as a primitive concept just as the Wheeler–Feynman theory does, furnishing instead every elementary charge with its own elementary EM field. Consequently, there is no EM self-interaction for elementary charges.

3.2 The Concept of Balanced Charge, the First Glimpse of the Theory

“The fact of dynamical interaction between fields demonstrates that nature cannot be purely linear, nor can it tolerate singular
C. Lanczos.³

The concept of balanced charge is a key element of the neoclassical theory of balanced charges (BCT) proposed here. The BCT has been designed to integrate into its fundamental features of classical EM fields as well as basic quantum phenomena. Still there are marked differences from the outset between the BCT and classical EM theory as well quantum mechanics (QM).

First of all, there is no point charge as a primitive concept in the BCT, and there is no EM self-interaction for an elementary charge such as an electron or proton. *A state of every balanced charge is described by two fields: its complex-valued wave-function $\psi(t, \mathbf{x})$ and its elementary four-potential $A^\mu(t, \mathbf{x})$.* We refer to the field $\psi(t, \mathbf{x})$ as a “wave function” due to its resemblance to the wave function in QM. However, a very important difference compared to QM is that in the BCT *every elementary charge is assigned its individual wave function $\psi(t, \mathbf{x})$* over the 4-dimensional space-time continuum, and consequently there is no configuration space as in QM.

The wave function $\psi(t, \mathbf{x})$ is complex-valued for a spinless charge and spinor-valued for a charge with spin. In this Part, for simplicity’s sake, we focus on a spinless charge, whereas a spinorial version of our theory is constructed in Part [IV]. The wave function $\psi(t, \mathbf{x})$ can be regarded as a spatially distributed charge. It has features similar to the Schrödinger material wave as in the Schrödinger wave mechanics reviewed in Sect. 2.10. The concept of a wave function is instrumental for providing quantum features such as the discrete spectrum of the Hydrogen atom. The concept of a point charge exists in the BCT only as an approximation in the case of states with spatially localized energy. An important example of such a particle-like state with well localized energy is a *free elementary charge*. In particular, *a free electron is spatially localized, and its size $a = a_e$ becomes a new fundamental spatial scale special to the BCT*, as we discuss in Sect. 3.3.

The individual elementary four-potential $A^\mu(t, \mathbf{x})$ associated with an elementary charge accounts for the EM interactions. To describe properties of the EM interactions in the BCT, let us consider a system of N elementary charges with the corresponding states $(\psi_\ell(t, \mathbf{x}), A_\ell^\mu(t, \mathbf{x}))$ and charge values q_ℓ , $\ell = 1, \dots, N$. A full Lagrangian treatment of such a system is provided in Sect. 3.4, whereas here we only take a look at the Euler–Lagrange field equations and discuss their basic properties. The field equations are separated into two sets of equations:

- (i) the nonlinear Klein–Gordon (NKG) equations for the wave functions ψ_ℓ ;
- (ii) the Maxwell equations for the elementary EM fields $F_\ell^{\mu\nu}(x) = \partial^\mu A_\ell^\nu(x) - \partial^\nu A_\ell^\mu(x)$.

³[212, p. 395].

The EM action of all elementary charges acting upon the ℓ -th elementary charge is facilitated through the 4-potential $A_{\neq\ell}^\nu$ and the corresponding EM field $F_{\neq\ell}^{\mu\nu}$, namely

$$A_{\neq\ell}^\nu(x) = \sum_{\ell' \neq \ell} A_{\ell'}^\nu(x), \quad F_{\neq\ell}^{\mu\nu}(x) = \partial^\mu A_{\neq\ell}^\nu(x) - \partial^\nu A_{\neq\ell}^\mu(x).$$

The above formula similar to (3.15) indicates very clearly that the ℓ -th elementary charge does not interact with itself electromagnetically, for its potential A_ℓ^ν is explicitly excluded from the sum of potentials acting upon it, justifying the subindex “ $\neq \ell$ ” in the actual potential $A_{\neq\ell}^\nu$. To account for elementary charges outside of the system of N elementary charges, we also introduce an external potential $A_{\text{ex}}^\nu = (\varphi_{\text{ex}}, \mathbf{A}_{\text{ex}})$.

The nonlinear Klein–Gordon (NKG) equations for the wave functions ψ_ℓ are of the form

$$-\frac{1}{c^2} \tilde{\partial}_t^2 \psi_\ell + \tilde{\nabla}^2 \psi_\ell - \frac{m_\ell^2 c^2}{\chi^2} \psi_\ell - G'_\ell(\psi_\ell^* \psi_\ell) \psi_\ell = 0 \quad (3.16)$$

where c is the velocity of light. In the non-relativistic case, the above NKG equations are replaced with the corresponding nonlinear Schrödinger (NLS) equations

$$i \tilde{\partial}_t \psi_\ell = \frac{\chi}{2m_\ell} \left[-\tilde{\nabla}^2 \psi_\ell + G'_\ell(\psi_\ell^* \psi_\ell) \psi_\ell \right]. \quad (3.17)$$

In the above equations, ψ_ℓ^* is complex conjugate to ψ_ℓ , and the covariant differentiation operators are defined by the expressions

$$\tilde{\partial}_t = \tilde{\partial}_{t\ell} = \partial_t + \frac{iq_\ell}{\chi} (\varphi_{\neq\ell} + \varphi_{\text{ex}}), \quad \tilde{\nabla} = \tilde{\nabla}_\ell = \nabla - \frac{iq_\ell}{\chi c} (\mathbf{A}_{\neq\ell} + \mathbf{A}_{\text{ex}}). \quad (3.18)$$

The parameter q_ℓ is the total value of the ℓ -th elementary charge, m_ℓ is the mass parameter, and $\chi > 0$ is a constant similar to the Planck constant $\hbar = \frac{h}{2\pi}$. For every ℓ , the *nonlinearity* $G'_\ell(\psi_\ell^* \psi_\ell)$ for the ℓ -th elementary charge accounts for forces of a non-electromagnetic nature internal to the charge that provide for particle-like states/regimes with spatially localized energy. The size a_ℓ of the free ℓ -th elementary charge is an important spatial scale incorporated into the nonlinearity G'_ℓ . The particle-like states and their relations to the nonlinearities G'_ℓ are considered below in Sect. 3.3 and in even more detail in Sect. 14.3.

Most of the time, we assume the value of the constant χ (χ_ℓ) to coincide with the Planck constant $\hbar$, $\chi_\ell = \hbar$. The reason for introducing the constant χ , which may differ somewhat from the Planck constant $\hbar$, is that the role played by the constant χ is not completely the same as the role of $\hbar$ in quantum mechanics. For example, the Planck–Einstein formula $E = \hbar\omega$, which is postulated in quantum mechanics, holds in our theory in the form $E_1 - E_2 = \chi(\omega_1 - \omega_2)$ exactly only for the non-relativistic regimes (see Sect. 3.7.1), whereas in the full relativistic version of the BCT the Planck–Einstein energy frequency relation $E_1 - E_2 = \chi(\omega_1 - \omega_2)$ holds only as an approximation, as discussed in Sect. 14.4.

One can see, looking at the nonlinear wave equations (3.16) and (3.17), that every wave function ψ_ℓ obeys its own evolution equation, and the EM interaction between charges is facilitated through the corresponding actual potentials $A_{\neq\ell}^\nu$ which appear in the covariant derivatives defined by formulas (3.18). This implies in particular that there is no EM self-action for every elementary charge.

The second set of Euler–Lagrange equations is the set of equations for the elementary potentials A_ℓ^ν which are exactly the classical Maxwell equations (3.3) or the corresponding wave equations (3.5). The expressions for the current densities J_ℓ^ν are different of course from the classical particle form (3.11) and are determined in terms of the corresponding Lagrangian L_ℓ for ψ_ℓ by the formula

$$J_\ell^\nu = -c \frac{\partial L_\ell}{\partial A_{\neq\ell}^\nu}, \quad J_\ell^\nu = (c\rho_\ell, \mathbf{J}_\ell). \quad (3.19)$$

In particular, in the non-relativistic case, when the wavefunction ψ_ℓ is a solution of the nonlinear Schrödinger equation (3.17), the above formula yields the following expressions for the charge and current densities

$$\rho_\ell = q_\ell \psi_\ell \psi_\ell^*, \quad \mathbf{J}_\ell = -i \frac{q_\ell \chi}{2m_\ell} \left(\psi_\ell^* \tilde{\nabla} \psi_\ell - \psi_\ell \tilde{\nabla}^* \psi_\ell^* \right), \quad (3.20)$$

which are identical to the corresponding expressions in QM.

The system of Eqs. (3.16), (3.9) is derived from a relativistically and gauge invariant Lagrangian, whereas the non-relativistic system (3.17), (3.9) is derived from a Galilean and gauge invariant Lagrangian. Importantly, thanks to the gauge invariance, the current densities J_ℓ^ν associated with solutions of the NKG or NLS equations (3.16) and (3.17) are conserved and satisfy the charge conservation/continuity equation

$$\partial_t \rho_\ell + \nabla \cdot \mathbf{J}_\ell = 0. \quad (3.21)$$

The fulfillment of the continuity equations allows us to impose the Lorentz gauge on the EM fields and to write the Maxwell equations in the form of wave equations (3.9). Integrating the continuity equation, we observe that the total charge $\bar{\rho}_\ell$ for every elementary ℓ -th charge is conserved:

$$\bar{\rho}_\ell = \int \rho_\ell \, d\mathbf{x} = \text{constant}, \quad (3.22)$$

which is a basic property one would expect of an elementary charge.

According to (3.20), the magnitude of the total charge is proportional to q_ℓ , and that allows us to impose the *normalization condition* which in the non-relativistic case takes the form

$$\int \rho_\ell \, d\mathbf{x} = q_\ell, \quad \int \psi_\ell \psi_\ell^* \, d\mathbf{x} = 1, \quad (3.23)$$

where q_ℓ can be interpreted as the value of the charge.

3.3 Localization of Balanced Charges and the Nonlinearity

The BCT is fundamentally a wave field theory, where every elementary charge is represented by a wave with its energy distributed over the 4-dimensional space-time continuum. Since elementary charges often behave as particles, an immediate question rises: how does the BCT recover particle-like regimes similar to those of classical electrodynamics of point charges?

As formulas (3.6) for point charges suggest, there has to be a localization mechanism providing for the charge density ρ_ℓ to converge to Dirac's delta function. Such a localization mechanism does exist in the BCT, and it is provided by the nonlinear term $G'_\ell(\psi_\ell^* \psi_\ell)$ (called the *nonlinearity* for short) that enters the NKG and the NLS equations. This nonlinearity represents forces of a non-electromagnetic origin internal to the ℓ -th elementary charge, and it explicitly involves the *size parameter* a . To get an insight into the localization mechanism, let us take a look at the resting charge in the non-relativistic case. The dependence of the nonlinearity $G'(|\psi|^2)$ on the size parameter a is introduced as follows:

$$G'(|\psi|^2) = G'_a(|\psi|^2) = a^{-2} G'_1(a^3 |\psi|^2). \quad (3.24)$$

A ground state $\psi = \mathring{\psi}(|\mathbf{x}|)$ that describes a resting charge in the absence of external EM fields solves the following equilibrium equation which is a consequence of the NLS equation (3.17):

$$-\nabla^2 \psi + G'_a(\psi^* \psi) \psi = 0. \quad (3.25)$$

The dependence of $\mathring{\psi}$ on the size parameter a is then as follows:

$$\mathring{\psi}(r) = \mathring{\psi}_a(r) = a^{-3/2} \mathring{\psi}_1(a^{-1}r), \quad (3.26)$$

and evidently this is compatible with the normalization condition (3.23).

One very important example of the nonlinearity is the following *logarithmic nonlinearity*:

$$G'_a(|\psi|^2) = -a^{-2} \ln(a^3 |\psi|^2 / C_g^2) - 3a^{-2}, \quad C_g = \pi^{-3/4}, \quad (3.27)$$

with the corresponding ground state $\mathring{\psi}$ of the Gaussian form, namely

$$\mathring{\psi}(|\mathbf{x}|) = \mathring{\psi}_a(|\mathbf{x}|) = C_g a^{-3/2} e^{-|\mathbf{x}|^2 / 2a^2}. \quad (3.28)$$

It can be verified that this Gaussian state is a global minimum point of the energy constrained to functions satisfying the normalization condition (3.23), and that ensures its stability. *The importance and the significance of the logarithmic nonlinearity (3.27)*

rests on the fact that it is singled out uniquely from a wide class of nonlinearities by its exact compliance with the Planck-Einstein energy-frequency relation

$$\Delta E = \chi \Delta \omega.$$

The exact meaning of this compliance and a detailed analysis of the logarithmic nonlinearity is provided in Sect. 17.4.1.

Observe now that, if the function ψ° is defined by (3.26), and it satisfies the normalization condition (3.23), then the charge density $\rho = q |\psi^\circ_a(\mathbf{x} - \mathbf{r})|^2$ converges to the delta function $q \delta(\mathbf{x} - \mathbf{r}(t))$ in (3.6) as $a \rightarrow 0$. The above introduced *size parameter* a can be naturally interpreted as the size of the free charge, and it determines an intrinsic length scale special to our theory. The size $a = a_e$ of a free electron becomes a new fundamental spatial scale. Our current assessed value for this scale is $a_e \approx 100a_B \approx 5$ nm where a_B is the Bohr radius. The free electron size a_e is much larger than the classical electron radius $r_e = \alpha^2 a_B$ or the Compton wavelength $\lambda_e = 2\pi\alpha a_B$ for the electron, but is much smaller than macroscopic length scales.

The particle-like behavior of an elementary charge of size a can be formally considered as the limit of vanishingly small a when $a \rightarrow 0$. Interestingly, many important features of the BCT are not affected by a specific form of the nonlinearity G'_a . When it comes to the localization, there can be many nonlinearities providing for localized solutions of the equilibrium equation (3.25). Indeed, we can pick as a ground state $\psi^\circ(r)$ any decreasing positive function of the radius $r > 0$ and then define the nonlinearity by the formula

$$G' \left(|\psi^\circ|^2(r) \right) = \frac{\nabla^2 \psi^\circ(r)}{\psi^\circ(r)}, \quad (3.29)$$

using the possibility to express r in terms of $|\psi^\circ|^2(r)$ for a monotone ψ° . Several examples realizing this approach of introducing nonlinearities are analyzed in Sect. 14.3.2.

Notice that the right-hand side of the equality (3.29) that defines the nonlinearity G' equals up to a factor $-\frac{\chi^2}{2m}$ the quantum potential $Q(\psi^\circ(r))$ in (2.28)—the key ingredient of the de Broglie-Bohm theory discussed in Sect. 2.11.

Being given the ground state ψ° for the resting charge, we can readily obtain from it a uniformly moving charge by simply applying the Galilean transformation. Namely, the NLS equation (3.17) has a solution of the form

$$\psi = \psi(t, \mathbf{x}) = e^{i \frac{S(t, \mathbf{x})}{\chi}} \psi^\circ(|\mathbf{x} - \mathbf{r}(t)|), \quad (3.30)$$

where $\mathbf{r}(t)$ is the charge position and $S = S(t, \mathbf{x})$ is the phase function. We refer to a solution of the special form (3.30) as a *wave-corpuscle*. In the case of uniform motion with velocity $\mathbf{v}$, the position and the phase functions take the following simple form

$$\mathbf{r}(t) = \mathbf{v}t, \quad S(t, \mathbf{x}) = \frac{m}{2} \mathbf{v}^2 t + m \mathbf{v} \cdot (\mathbf{x} - \mathbf{r}(t)). \quad (3.31)$$

In the relativistic case, the rest solution to the NKG equation (3.16) with the same nonlinearity has the form

$$\psi = \psi(t, \mathbf{x}) = e^{-i\omega_0 t} \psi(|\mathbf{x}|), \text{ where } \omega_0 = \frac{mc^2}{\chi}, \quad (3.32)$$

with the same ψ . Evidently, the solution (3.32) involves an oscillatory factor $e^{-i\omega_0 t}$, but $|\psi|$ remains time-independent. Wave-corpuscles describing uniformly moving charges in the relativistic case can be obtained by the Lorentz transformations applied to the wave function of the resting charge. The relativistic version of the wave-corpuscle is similar to expression (3.30) but involves the Lorentz contraction, for details see Sect. 15.2.

The very possibility of the wave-corpuscle to be an exact solution to the nonlinear field equations manifests the natural coexistence of wave and particle properties. The concept of a particle in the BCT arises as a special regime of a wave being spatially localized in the course of time evolution. While in QM “particle-wave duality” is the standard characterization of the relations between particle and wave, the word “coexistence” seems to better characterize wave-particle relations in the BCT with the wave being the primary fundamental concept.

We would like to observe that the wave-corpuscle representing the particle in the BCT markedly differs from the wave packet representing a particle in the Schrödinger wave mechanics. Indeed, it is well known that a wave packet governed by the linear Schrödinger equation rapidly disperses with consequent unlimited growth of its size with time. Rather remarkably, the wave-corpuscle as an exact solution not only describes a uniform motion of an elementary charge but its accelerated motion as well. This factor allows one to analyze analytically fine details of interactions between a charge and the EM field.

An elaborate analysis of wave-corpuscles is carried out in Chap. 16. The ways wave-corpuscles naturally combine wave and particle properties are discussed in Sect. 3.5.

3.4 Lagrangian, Field Equations and Conservation Laws for Interacting Balanced Charges

“Mathematics is the tool specially suited for dealing
with abstract concepts of any kind and
there is no limit to its power in this field.
For this reason a book on the new physics,
if not purely descriptive of experimental work,
must be essentially mathematical.”
*P. Dirac.*⁴

To assure that BCT is a field theory that possesses such important attributes as the energy and momentum conservation laws, it is constructed based on the Lagrangian

⁴[87, p. viii].

framework treated in detail in Chap. 11. The Lagrangian approach allows us to apply Noether's theorem, which yields expressions for the currents, the energy-momentum tensor as well as the fundamental conservation laws. The Lagrangian of a system of N elementary balanced charges is conceived to implement the following requirements:

- charges interact only through their elementary EM potentials and fields;
- the field equations for the elementary EM fields are exactly the Maxwell equations with conserved currents as sources;
- the field equations for the charge distributions are the NKG or the NLS equations in the relativistic and non-relativistic cases respectively.

With the above requirements in mind, we consider a system of N elementary charges modeled by N pairs (ψ_ℓ, A_ℓ^μ) , $1 \leq \ell \leq N$, with elementary electromagnetic fields $F_\ell^{\mu\nu}$ defined by the standard formula (3.1), namely

$$F_\ell^{\mu\nu}(x) = \partial^\mu A_\ell^\nu(x) - \partial^\nu A_\ell^\mu(x). \quad (3.33)$$

Then a manifestly relativistic Lagrangian for the system of N balanced charges in an external EM field with the 4-potential A_{ex} is defined by the following expression:

$$\begin{aligned} \mathcal{L}(\{\psi_\ell, \psi_{\ell;\mu}\}, \{\psi_\ell^*, \psi_{\ell;\mu}^*\}, A_\ell^\mu) \\ = \sum_{\ell=1}^N L_\ell(\psi_\ell, \psi_{\ell;\mu}, \psi_\ell^*, \psi_{\ell;\mu}^*) + \mathcal{L}_{\text{BEM}} \end{aligned} \quad (3.34)$$

where: (i) ψ^* is complex conjugate to ψ ; (ii) the EM part $\mathcal{L}_{\text{BEM}}$ of the Lagrangian is

$$\mathcal{L}_{\text{BEM}} = -\frac{1}{16\pi} \sum_{\{\ell, \ell': \ell' \neq \ell\}} F_\ell^{\mu\nu} F_{\ell'\mu\nu}; \quad (3.35)$$

(iii) the “material” part L_ℓ of the Lagrangian is associated with the ℓ -th charge, and in the relativistic case, where it generates the NKG equation (3.16), it has the following form:

$$L_\ell = L_{\text{KG}\ell} = \frac{\chi^2}{2m_\ell} \left\{ \psi_{\ell;\mu}^* \dot{\psi}_\ell^\mu - G_\ell(\psi_\ell \psi_\ell^*) \right\} - \frac{m_\ell c^2}{2} \psi_\ell \psi_\ell^*; \quad (3.36)$$

in the non-relativistic case, $L_\ell = L_{\text{NLS}\ell}$ where the $L_{\text{NLS}\ell}$ is the Lagrangian of the NLS equation

$$L_{\text{NLS}\ell} = i\frac{\chi}{2} \left[\psi_\ell^* \tilde{\partial}_t \psi_\ell - \psi_\ell \tilde{\partial}_t^* \psi_\ell^* \right] - \frac{\chi^2}{2m_\ell} \left\{ \tilde{\nabla} \psi_\ell \tilde{\nabla}^* \psi_\ell^* + G(\psi_\ell \psi_\ell^*) \right\}, \quad (3.37)$$

where we use the notation (3.18). The covariant derivatives $\psi_{\ell;\mu}$ of ψ_ℓ in (3.36) are defined by the following relations:

$$\psi_{\ell;\mu} = \tilde{\partial}_\ell^\mu \psi_\ell = \partial^\mu \psi_\ell + \frac{iq_\ell}{\chi c} \left(A_{\neq \ell}^\mu + A_{\text{ex}}^\mu \right) \psi_\ell, \quad (3.38)$$

with $A_{\neq\ell}^\mu$ defined by (3.15), and

$$F_{\neq\ell}^{\mu\nu} = \partial^\mu A_{\neq\ell}^\nu - \partial^\nu A_{\neq\ell}^\mu. \quad (3.39)$$

Notice that the action on the ℓ -th charge by all other charges is described by the potential field $A_{\neq\ell}^\mu = (\varphi_{\neq\ell}, \mathbf{A}_{\neq\ell})$, and obviously there is no EM self-interaction. The classical *total EM potential* $A^\mu = (\varphi, \mathbf{A})$ is defined as the sum of all elementary EM potentials, namely

$$A^\mu = (\varphi, \mathbf{A}) = \sum_{\ell=1}^N A_\ell^\mu, \quad \varphi = \sum_{\ell=1}^N \varphi_\ell, \quad \mathbf{A} = \sum_{\ell=1}^N \mathbf{A}_\ell. \quad (3.40)$$

Notice that the field $F_{\neq\ell}^{\mu\nu}$ which acts on the ℓ -th charge evidently is not the same as the total field $F^{\mu\nu}$, and it is different for different ℓ . The total potential A^μ can be interpreted as the potential which acts on a test charge. The test charge can be viewed as the $(N+1)$ -th charge, and consequently $A^\mu = A_{\neq N+1}^\mu$.

Notice that the interaction between elementary charges enters the Lagrangians L_ℓ in (3.36) through the covariant derivatives $\psi_{\ell;\mu}$ defined by (3.38). Those specific expressions represent a gauge invariant electromagnetic coupling known in the literature as the *minimal coupling*, [331, 1.1.1], [143, 1.9, 1.10].

The *Euler–Lagrange field equations* for the Lagrangian defined by (3.34)–(3.38) split naturally into two sets: (i) *material equations* which are nonlinear Klein–Gordon (NKG) equations for the wave functions ψ_ℓ ; (ii) the Maxwell equations for the elementary fields $F_\ell^{\mu\nu}$. The NKG equations for the wave functions ψ_ℓ can be written in the manifestly relativistically covariant form as follows:

$$\frac{\chi^2}{2m_\ell} \left[\tilde{\partial}_{\ell\mu} \tilde{\partial}_\ell^\mu + G'_\ell (\psi_\ell \psi_\ell^*) \right] \psi_\ell + \frac{m_\ell c^2}{2} \psi_\ell = 0 \quad (3.41)$$

with the covariant derivative $\tilde{\partial}_\ell^\mu$ defined by (3.38). We call the NKG equations *material equations* to distinguish them from the Maxwell field equations for the EM fields. We would like to stress that the complex-valued wave function ψ_ℓ describes the charge distribution and is as material as the EM potential A_ℓ^μ ; in particular, it does not directly describes such matter characteristic as mass.

The Maxwell equations for the elementary EM fields are the Euler–Lagrange field equations which have the following manifestly relativistic covariant form:

$$\partial_\mu F_\ell^{\mu\nu} = \frac{4\pi}{c} J_\ell^\nu, \quad J_\ell^\nu = (c\rho_\ell, \mathbf{J}_\ell), \quad \ell = 1, \dots, N, \quad (3.42)$$

where the EM tensor $F_\ell^{\mu\nu}$ is given by equality (3.33). Under the Lorentz gauge condition

$$\partial_\mu A_\ell^\mu = 0, \quad (3.43)$$

the Maxwell equations for the potentials A_ℓ^ν take the form of the wave equations with sources, namely

$$\partial_\mu \partial^\mu A_\ell^\nu = \frac{4\pi}{c} J_\ell^\nu, \quad (3.44)$$

which coincide with (3.5). The 4-vector currents $J_\ell^\nu = (c\rho_\ell, \mathbf{J}_\ell)$ that appear in the Maxwell equations (3.42) are defined in terms of the Lagrangians L_ℓ by the following expressions

$$J_\ell^\nu = -i \frac{q_\ell}{\chi} \left(\frac{\partial L_\ell}{\partial \psi_{\ell;\nu}} \psi_\ell - \frac{\partial L_\ell}{\partial \psi_{\ell;\nu}^*} \psi_\ell^* \right) = -c \frac{\partial L_\ell}{\partial A_{\ell\nu}}. \quad (3.45)$$

For the Lagrangian L_ℓ defined by (3.36), the 4-current J_ℓ^ν representation (3.45) turns into the following expression

$$J_\ell^\nu = -\frac{\chi q_\ell}{2m_\ell} i \left(\psi_\ell \tilde{\partial}_\ell^{\nu*} \psi_\ell^* - \psi_\ell^* \tilde{\partial}_\ell^\nu \psi_\ell \right) \quad (3.46)$$

which is evidently defined in terms of the charge distribution ψ_ℓ . The EM potentials A_ℓ^ν are determined by ψ_ℓ through wave equations (3.44) as well.

The 4-current J_ℓ^ν satisfies the conservation law (3.21), which can also be written in 4-vector form:

$$\partial_\nu J_\ell^\nu = 0, \quad J_\ell^\nu = (\rho_\ell c, \mathbf{J}_\ell). \quad (3.47)$$

The above charge conservation law can be derived based on Noether's theorem using the gauge invariance of the NKG and NLS Lagrangians or directly from NLS and NKG equations. The first expression in the 4-current J_ℓ^ν representation (3.45) is a consequence of the gauge invariance of the Lagrangian L_ℓ defined by (3.36). The second expression defines the source currents in the Maxwell equations (3.42). The equality of those two different (in the case of a general Lagrangian) expressions for the 4-current J_ℓ^ν in (3.45) signifies that *the currents as sources in the Maxwell equations are precisely Noether's charge currents. Importantly, the equality of the two different expressions in the 4-current J_ℓ^ν representation (3.45) is a direct consequence of our set up of the EM interactions in the material Lagrangian L_ℓ through the covariant derivatives $\psi_{\ell;\mu}$, namely by means of the minimal coupling.*

We can naturally assign for every charge a symmetric and gauge invariant energy momentum tensor (EnMT) $T_\ell^{\mu\nu}$ that satisfies the conservation law

$$\partial_\mu T_\ell^{\mu\nu} = f_\ell^\nu. \quad (3.48)$$

The importance of the EnMT conservation laws can be illustrated by the fact that the source term in the conservation law (3.48) is the Lorentz force density

$$f_\ell^\nu = \frac{1}{c} J_{\ell\mu} F_{\ell}^{\nu\mu} = (f_\ell^0, \mathbf{f}_\ell) = \left(\frac{1}{c} \mathbf{J}_\ell \cdot \mathbf{E}_{\ell}, \rho \mathbf{E}_{\ell} + \frac{1}{c} \mathbf{J}_\ell \times \mathbf{B}_{\ell} \right).$$

Let us take a look at the complete set of field equations (3.41)–(3.42) for the system of charge fields (ψ_ℓ, A_ℓ^μ) . The material and the Maxwell equations are coupled in the following physically transparent way. The wave functions ψ_ℓ determine the 4-currents J_ℓ^ν by expressions (3.46) which in turn determine the EM elementary fields $F_\ell^{\mu\nu}$ and the potentials A_ℓ^μ as solutions to, respectively, the Maxwell equations (3.42) and the wave equations (3.44). The potentials A_ℓ^μ then enter the material equations (3.41) through the covariant derivatives $\psi_{\ell;\mu}$ involving the potentials $A_{\neq\ell}^\mu$, and the latter affect the evolution of wave functions ψ_ℓ .

3.5 The Concept of Wave-Corpuscle

When analyzing solutions to the NLS equation (3.17), we have found exact solutions of a special form referred to as *wave-corpuscles*. A remarkable feature of the wave-corpuscles is the manifest coexistence of particle and wave properties that can be seen readily from their representation. Indeed, a *wave-corpuscle* is an exact solution to the NLS equation (3.17) of the form (3.30), namely

$$\psi(t, \mathbf{x}) = e^{i\mathfrak{S}(t, \mathbf{x})} \psi^\circ(|\mathbf{x} - \mathbf{r}(t)|), \quad \mathfrak{S}(t, \mathbf{x}) = \frac{1}{\chi} S(t, \mathbf{x}), \quad (3.49)$$

where $\mathbf{r}(t)$ is its *center* and $S(t, \mathbf{x})$ is its *phase*. We can expand the phase function in powers of $\mathbf{y} = \mathbf{x} - \mathbf{r}(t)$, namely

$$S(t, \mathbf{x}) = m\hat{\mathbf{v}}(t) \cdot \mathbf{y} + \sigma_p(t) + \sigma_{p2}(t, \mathbf{y}), \quad \mathbf{y} = \mathbf{x} - \mathbf{r}(t), \quad (3.50)$$

where σ_{p2} involves terms of order two and higher in $\mathbf{y}$. We assume now for simplicity that there is no magnetic field, and then the NLS equation (3.17) takes the form

$$i\chi\partial_t\psi = -\frac{\chi^2}{2m}\nabla^2\psi + q\varphi_{\text{ex}}\psi + \frac{\chi^2}{2m}G'(|\psi|^2)\psi. \quad (3.51)$$

We also assume that the electric field $\mathbf{E}_{\text{ex}} = \mathbf{E}_{\text{ex}}(t)$ is spatially homogeneous and can be time dependent. Then its potential $\varphi_{\text{ex}}(t, \mathbf{x})$ is a linear function of $\mathbf{x}$ of the form

$$\varphi_{\text{ex}}(t, \mathbf{x}) = \varphi_0(t) - \mathbf{E}_{\text{ex}}(t) \cdot \mathbf{x}. \quad (3.52)$$

One can verify that the wave-corpuscle $\psi(t, \mathbf{x})$ defined by formula (3.49) is indeed an exact solution to the NLS equation (3.51) if its center $\mathbf{r}(t)$ satisfies Newton's law of motion

$$m\partial_t^2\mathbf{r} = -q\nabla\varphi_{\text{ex}}(t, \mathbf{r}) \quad (3.53)$$

for a point charge in the electric field $\mathbf{E}_{\text{ex}}(t) = -\nabla\varphi_{\text{ex}}(t, \mathbf{r})$, and its phase function $S(t, \mathbf{x})$ has the following representation:

$$S(t, \mathbf{x}) = \sigma_p(t) + m\mathbf{v}(t) \cdot (\mathbf{x} - \mathbf{r}(t)), \quad \mathbf{v}(t) = \partial_t \mathbf{r}(t), \quad (3.54)$$

where

$$\sigma_p(t) = \frac{m}{2} \int_0^t \mathbf{v}^2(t') dt' - q \int_0^t \varphi_{\text{ex}}(t, \mathbf{r}(t')) dt', \quad (3.55)$$

see Sect. 16.1 for details.

Evidently, the above defined wave-corpuscle $\psi(t, \mathbf{x})$ describes an accelerated motion of a charge, it is a combination of the translation $\hat{\psi}(|\mathbf{x} - \mathbf{r}(t)|)$ of the form factor and oscillations described by the phase S . Observe that the wave-corpuscle center $\mathbf{r}(t)$ satisfies Newton's point particle equation (3.53), and its phase function $S(t, \mathbf{x})$ solves exactly the Hamilton–Jacobi equation for the point charge, see Chap. 40. The phase function $S(t, \mathbf{x})$ also captures wave properties as we will discuss below in Sect. 3.5.3.

3.5.1 The Wave-Corpuscle Versus the WKB Quasiclassical Approximation

Note that formula (3.49) defining the wave-corpuscle is similar to the quasiclassical ansatz commonly used in QM. This ansatz is also known as the WKB quasiclassical asymptotics after Wentzel, Kramers and Brillouin who introduced it in 1926. Let us recall briefly the basics of the WKB method based on the Cauchy problem for the linear Schrödinger equation

$$i\chi\partial_t\psi = -\frac{\chi^2}{2m}\nabla^2\psi + q\varphi_{\text{ex}}(t, \mathbf{x})\psi, \quad (3.56)$$

with oscillatory initial data

$$\psi(0, \mathbf{x}) = e^{iS_0(\mathbf{x})/\chi}\psi_0(\mathbf{x}), \quad \chi \rightarrow 0. \quad (3.57)$$

The general theory related to the WKB quasiclassical asymptotics is a well studied subject, see, for instance, [176, 6.4], [138, 4.5], [146, 8], [243, 7], [235], [204, 3.2], [118].

Notice that the linear Schrödinger equation (3.56) can be readily obtained from the NLS equation (16.43) by omitting there the nonlinearity G' , i.e. setting $G' = 0$. Equation (3.56) can be recast in the operator form

$$i\chi\partial_t\psi = H(-i\chi\nabla, t, \mathbf{x})\psi, \quad (3.58)$$

where the symbol $H(\mathbf{p}, t, \mathbf{x})$ of the differential operator H is of the form

$$H(\mathbf{p}, t, \mathbf{x}) = \frac{1}{2m} \mathbf{p}^2 + q\varphi_{\text{ex}}(t, \mathbf{x}). \quad (3.59)$$

The objective of the WKB method is to find an asymptotic solution to equation (3.58) as $\chi \rightarrow 0$ in the following *quasiclassical* form, [215, p. 20, 51]

$$\psi_{\text{qc}}(t, \mathbf{x}) = e^{iS/\chi} \bar{\psi}(t, \mathbf{x}). \quad (3.60)$$

More precisely, with the representation (3.60) in mind, we seek functions $S(t, \mathbf{x})$ and $\bar{\psi}(t, \mathbf{x})$ such that $\psi_{\text{qc}}(t, \mathbf{x})$ satisfies Eq. (3.58) with discrepancy of order χ^2 :

$$i\chi \partial_t \psi_{\text{qc}} = H(-i\chi \nabla, t, \mathbf{x}) \psi_{\text{qc}} + O(\chi^2). \quad (3.61)$$

Plugging expression (3.60) for ψ_{qc} into Eq. (3.58), we collect terms at the leading powers χ^0 and χ^1 as $\chi \rightarrow 0$. This yields equations for S and $\bar{\psi}$. Namely, the leading term at χ^0 yields the *Hamilton–Jacobi equation*

$$\partial_t S + H(\nabla_{\mathbf{x}} S, t, \mathbf{x}) = 0, \quad (3.62)$$

whereas the term at χ^1 yields the *transport equation*:

$$\partial_t \bar{\psi} + \frac{1}{m} \nabla S \cdot \nabla \bar{\psi} + \frac{1}{2m} \bar{\psi} \nabla^2 S = 0. \quad (3.63)$$

Since the wave-corpuscle is an exact solution for any χ , and the nonlinearity enters the NLS with the coefficient χ^2 , the equation for the phase in the leading term of the asymptotics for the wave-corpuscle must coincide with the Hamilton–Jacobi equation (3.62).

It is instructive to look directly at the relation of the phase function $S(t, \mathbf{x})$ defined by (3.54), (3.55) to the Hamilton–Jacobi theory, [8, 45.D], [287, 2.11], [264, 15.5], see also Sect. 39. Note first that the point charge Lagrangian L_p associated with Newton’s law (3.53) has the form

$$L_p(\partial_t \mathbf{r}, t, \mathbf{r}) = \frac{m}{2} (\partial_t \mathbf{r}(t))^2 - q\varphi_{\text{ex}}(t, \mathbf{r}(t)) \quad (3.64)$$

with the corresponding Hamiltonian

$$H_p(\mathbf{p}, \mathbf{x}) = \frac{\mathbf{p}^2}{2m} + q\varphi_{\text{ex}}(t, \mathbf{x}), \quad \mathbf{p} = m\mathbf{v}, \quad (3.65)$$

which evidently coincides with the symbol $H(\mathbf{p}, t, \mathbf{x})$ in (3.59). The phase function $S(t, \mathbf{x})$ defined by (3.54), (3.55) satisfies the Hamilton–Jacobi equation (3.62), and which can also be verified by direct computation. Notice also that, according to

representation (3.54) for the phase function $S(t, \mathbf{x})$, we have

$$\nabla S(t, \mathbf{x}) = m\mathbf{v}(t), \quad (3.66)$$

and, consequently, the transport equation (3.63) takes here the form

$$\partial_t \bar{\psi} + \mathbf{v} \cdot \nabla \bar{\psi} = 0. \quad (3.67)$$

This equation describes the transport along trajectories $\mathbf{r}(t)$ with velocity field $\mathbf{v}$, and it is obviously satisfied by $\bar{\psi}(t, \mathbf{x}) = \psi_0(\mathbf{x} - \mathbf{r}(t))$.

Formula (3.55) for $\sigma_p(t)$ can be written in terms of the Lagrangian (3.64), namely

$$\sigma_p(t) = \int_0^t \left[\frac{m}{2} \mathbf{v}^2(t') - q\varphi_{\text{ex}}(t', \mathbf{r}) \right] dt' = \int_0^t L_p dt', \quad (3.68)$$

and obviously coincides with the classical action for a point charge, see Chap. 40.

In conclusion, we would like to point out important differences between the WKB quasiclassical approximation and the wave-corpuscle. First of all, the quasiclassical approximation, unlike the wave-corpuscle, does not yield an exact solution to the linear Schrödinger equation but it only provides the principal term of the asymptotic approximation. In particular, it does not capture such an important property of a wave packet governed by the linear Schrödinger equation as its dispersion with time. In contrast to the quasiclassical approximation, the wave-corpuscle is an exact solution to the NLS equation, and the Hamilton–Jacobi equation and the transport equation give exact rather than approximate information about their dynamics. The reason why the Hamilton–Jacobi equation and the transport equation perfectly describe the dynamics of wave-corpuscles and only approximately the dynamics of wave packets is the nonlinearity which exactly balances dispersive effects. The nonlinearity selects a special form factor ψ in (3.49) for which the balance holds.

There is yet another important difference between the quasiclassical approximation and the wave-corpuscle, discussed in more detail in Sect. 3.5.3. In the quasiclassical approximation, the parameter $\chi \rightarrow 0$, and therefore the de Broglie wavelength strictly speaking cannot be consistently treated as a fixed quantity. For a wave-corpuscle, the action parameter χ is fixed. It remains fixed as well as the de Broglie wavelength even for general EM fields, where a wave-corpuscle provides an approximate solution, since the error of approximation is controlled by the size parameter in the point localization limit $a \rightarrow 0$. Hence the concept of wave-corpuscle allows us to consistently introduce the de Broglie wavelength of a particle which moves according to the laws of classical mechanics. The wave-corpuscle seamlessly combines properties of a wave and a particle demonstrating beyond any doubt that these properties can naturally coexist.

3.5.2 The Wave-Corpuscle as an Approximation

Let us assume here as above that there is no magnetic field for simplicity. The general case of a non-zero magnetic field is considered in detail in Sect. 16.3. Equation (3.51) describes the dynamics of a charge in an external electric field which is not necessarily homogeneous. We consider a macroscopic case, where the spatial variation of the electric potential can be characterized by a macroscopic spatial scale R_{ex} of the order 1. Namely, the potential can be written in the form $\varphi_{\text{ex}}(t, \mathbf{x}) = \bar{\varphi}_{\text{ex}}(t, \mathbf{x}/R_{\text{ex}})$, where the function $\bar{\varphi}_{\text{ex}}(t, \mathbf{y})$ is assumed to be bounded together with its derivatives uniformly with respect to all parameters of the problem. In the case of a general electric potential, a wave-corpuscle is not an exact solution to the NLS equation (3.51), but it provides an asymptotic approximation with a discrepancy of the order a^2 as the size parameter $a \rightarrow 0$. We provide below a concise argument to demonstrate this statement. Let us start by selecting a trajectory $\mathbf{r}(t)$ that solves Newton's equation (3.53) and evaluating the linear part φ_{aux} of the potential φ_{ex} at the trajectory, namely

$$\varphi_{\text{aux}}(t, \mathbf{x}) = \varphi_{\text{ex}}(t, \mathbf{r}(t)) + \nabla \varphi_{\text{ex}}(t, \mathbf{r}(t)) \cdot (\mathbf{x} - \mathbf{r}(t)). \quad (3.69)$$

Obviously, this potential is linear as in (3.52), and formulas (3.30), (3.54), (3.55) determine an exact wave-corpuscle solution $\psi_{\text{wc}}(t, \mathbf{x})$ to the auxiliary NLS equation obtained by replacing $\varphi_{\text{ex}}(t, \mathbf{x})$ by $\varphi_{\text{aux}}(t, \mathbf{x})$ in (3.51):

$$i\chi \partial_t \psi_{\text{wc}} = -\frac{\chi^2}{2m} \nabla^2 \psi_{\text{wc}} + q\varphi_{\text{aux}} \psi_{\text{wc}} + \frac{\chi^2}{2m} G'(|\psi_{\text{wc}}|^2) \psi_{\text{wc}}. \quad (3.70)$$

Obviously, the discrepancy obtained by substituting ψ_{wc} into the original Eq. (3.51) equals $q(\varphi_{\text{ex}} - \varphi_{\text{aux}}) \psi_{\text{wc}}$. Note that $|\varphi_{\text{ex}} - \varphi_{\text{aux}}| \leq C R_{\text{ex}}^2 |\mathbf{x} - \mathbf{r}|^2$, and, according to (3.28),

$$|(\varphi_{\text{ex}} - \varphi_{\text{aux}}) \psi_{\text{wc}}| \leq C C_g \frac{a^2}{R_{\text{ex}}^2} \frac{|\mathbf{x} - \mathbf{r}|^2}{a^2} \exp\left(\frac{-|\mathbf{x} - \mathbf{r}|^2}{2a^2}\right) a^{-3/2}.$$

Consequently, the norm of the discrepancy in the Hilbert space L^2 of square integrable functions can be estimated as follows:

$$\begin{aligned} \|(\varphi_{\text{ex}} - \varphi_{\text{aux}}) \psi_{\text{wc}}\|_{L^2} &= \left(\int |(\varphi_{\text{ex}} - \varphi_{\text{aux}}) \psi_{\text{wc}}|^2 d\mathbf{x} \right)^{1/2} \\ &\leq \frac{a^2}{R_{\text{ex}}^2} C C_g \left(\int |\mathbf{y}|^4 \exp(-|\mathbf{y}|^2) d\mathbf{y} \right)^{1/2} = C_1 \frac{a^2}{R_{\text{ex}}^2}. \end{aligned} \quad (3.71)$$

Hence, ψ_{wc} provides an asymptotic solution to (3.51), namely

$$i\chi \partial_t \psi_{\text{wc}} = -\frac{\chi^2}{2m} \nabla^2 \psi_{\text{wc}} + q\varphi_{\text{ex}} \psi_{\text{wc}} + \frac{\chi^2}{2m} G'(|\psi_{\text{wc}}|^2) \psi_{\text{wc}} + O\left(\frac{a^2}{R_{\text{ex}}^2}\right), \quad (3.72)$$

where we use the big O notation. It is natural to estimate the discrepancy in the L^2 -norm since, according to the normalization condition (3.23), the principal terms in the NLS equations (3.70) and (3.51) are of the order 1 in the L^2 -norm.

Note that the above estimate (3.71) does not use the smallness of χ in contrast to the WKB method as in (3.61). In the presence of a general magnetic field, an estimate of the discrepancy, independent of χ , is of the order a , see Remark 16.3.1 for details. As we have already pointed out, the fact that the wave-corpuscle provides an asymptotic solution independently of the smallness of χ shows that the concept of the de Broglie wave is entirely consistent with point-like behavior of particles.

3.5.3 Coexistence of Wave and Particle Properties in a Wave-Corpuscle

The concept of a wave corpuscle (3.30) is truly remarkable because wave and particle properties naturally coexist in it. To see this, let us again consider the case of a purely electric external field, assuming the external magnetic field to be zero. A more general case of the non-zero external EM field is treated in Sect. 16.4.

The motion of the modulus $|\psi(|\mathbf{x} - \mathbf{r}|)|^2$ of a wave-corpuscle is described by the spatial translation of the form factor $|\mathring{\psi}(|\mathbf{x} - \mathbf{r}|/a)|$. This observation is completely consistent with the field concepts defined for general fields. Namely, the total current and momentum can be obtained by integration of corresponding densities defined in terms of the NLS Lagrangian, and a direct computation in Sect. 3.5.3 produces the following expression for the total current and momentum:

$$\bar{\mathbf{J}} = \int \mathbf{J} \, d\mathbf{x} = q\mathbf{v}, \quad (3.73)$$

$$\mathbf{P} = \int \mathbf{P} \, d\mathbf{x} = m\mathbf{v}, \quad (3.74)$$

where $\mathbf{v} = \partial_t \mathbf{r}$ is the translational velocity of the wave-corpuscle. Evidently, the above formula coincides with the expression for the current and the momentum of a point charge if we identify the location of the charge with the center $\mathbf{r}(t)$ of the wave corpuscle. Since the center $\mathbf{r}(t)$ satisfies Newton's law (3.53), we find that a wave-corpuscle features all mechanical properties of a point charge.

Let us turn now to the wave properties of a wave-corpuscle (3.30) with amplitude $\mathring{\psi}(|\mathbf{x} - \mathbf{r}(t)|)$ and the dimensionless phase function

$$\mathfrak{S}(t, \mathbf{y}) = \frac{1}{\chi} S(t, \mathbf{y}), \quad \mathbf{y} = \mathbf{x} - \mathbf{r}(t). \quad (3.75)$$

The oscillatory factor $e^{i\mathfrak{S}}$ in the case of the phase function (3.54) evidently describes a plane wave with wavevector $\bar{\mathbf{k}} = \frac{m}{\chi} \mathbf{v}(t)$. In a more general case (3.50), taking into account that the wave is centered at the point $\mathbf{r}(t)$, it is natural to introduce the de Broglie wavevector by the formula

$$\bar{\mathbf{k}} = \nabla \mathfrak{S}(t, \mathbf{y})_{\mathbf{y}=\mathbf{0}}, \quad (3.76)$$

with the corresponding wavelength

$$\lambda = \frac{2\pi}{|\bar{\mathbf{k}}|}. \quad (3.77)$$

We can then use expansion (3.50) of the phase function $\mathfrak{S}(t, \mathbf{y})$ in powers of $\mathbf{y}$ to obtain

$$\bar{\mathbf{k}} = \nabla \mathfrak{S}(t, \mathbf{y})_{\mathbf{y}=\mathbf{0}} = \frac{1}{\chi} m \hat{\mathbf{v}}. \quad (3.78)$$

In the simplest case of an electric external field, according to (3.54), (3.55),

$$\hat{\mathbf{v}} = \mathbf{v} = \partial_t \mathbf{r}, \quad \bar{\mathbf{k}} = \frac{1}{\chi} m \mathbf{v}. \quad (3.79)$$

Consequently, the wavevector can be expressed in terms of the momentum (3.74):

$$\bar{\mathbf{k}} = \frac{1}{\chi} m \mathbf{v} = \frac{1}{\chi} \mathbf{P}, \quad (3.80)$$

and the corresponding wavelength is given by the *de Broglie formula*

$$\lambda = \frac{2\pi\chi}{|\mathbf{P}|} \quad (3.81)$$

(note that if $\chi = \hbar$ then $2\pi\chi = h$ where h is Planck constant).

As to the energy of the charge, note that the matter field energy $\mathring{\mathcal{E}}$ of the charge can be derived by integration of its energy density, resulting in the expression

$$\mathring{\mathcal{E}} = \frac{1}{2} m \mathbf{v}^2 + q \varphi_{\text{ex}}(t, \mathbf{r}(t)) + mc^2 \frac{a_C^2}{a^2} \Theta_0, \quad (3.82)$$

where $a_C = \frac{\chi}{mc}$ is the Compton wavelength and $\Theta_0 = \frac{1}{3} \left\| \nabla \psi_1 \right\|^2 = 1/2$ for the logarithmic nonlinearity G' . Observe that it is only the mass parameter m that enters the non-relativistic Newton's law (3.53), and it coincides with the inertial mass. The internal charge energy $\mathring{\mathcal{E}}$ does not affect the Newtonian dynamics of the wave-corpuscle. This sharply contrasts with the relativistic case considered in Sect. 3.6.2

(see details in Sects. 17.6.2 and 34.3) where the internal energy directly affects the point dynamics of the charge according to the Einstein formula $E = Mc^2$.

It is natural to introduce the wave instantaneous frequency by the formula

$$\omega(t) = -\frac{d}{dt} \mathfrak{S}(t, \mathbf{x} - \mathbf{r}(t))_{\mathbf{x}=\mathbf{r}}, \quad (3.83)$$

where the minus sign is chosen based on the standard form of the time oscillation as in $e^{-i\omega t}$. For the phase function (3.50), the above defined instantaneous frequency takes the form

$$\omega(t) = -\frac{1}{\chi} \partial_t \sigma_p(t) + \frac{1}{\chi} m \hat{\mathbf{v}} \cdot \mathbf{v}. \quad (3.84)$$

In the case of the external electric field (3.54), (3.55), we obtain from (3.83)–(3.84) that

$$\omega(t) = \frac{m}{2\chi} \mathbf{v}^2 + q \frac{1}{\chi} \varphi_{\text{ex}}(t, \mathbf{r}(t)), \quad (3.85)$$

where evidently the right-hand side is the energy of a point charge in the electric field times $\frac{1}{\chi}$. In the accelerated motion of a wave-corpuscle, its velocity and frequency $\mathring{\mathcal{E}} = \mathring{\mathcal{E}}(t)$, $\omega = \omega(t)$ vary with time. Comparing (3.85) and (3.82), we see that *the change of energy and frequency between two states of the charge at time t_1 and t_2 is described by the Planck–Einstein formula*

$$\mathring{\mathcal{E}}(t_2) - \mathring{\mathcal{E}}(t_1) = \chi (\omega(t_2) - \omega(t_1)). \quad (3.86)$$

Note that $\mathring{\mathcal{E}} \neq \chi\omega$ due to the term $mc^2 \frac{a_c^2}{a^2} \Theta_0$, and consequently the Planck–Einstein formula should be written in our framework in the form $\Delta \mathring{\mathcal{E}} = \chi \Delta \omega$ as in (3.86). This is one of the reasons to use the letter χ instead of $\hbar$ for the action parameter to indicate the difference with QM. The term $mc^2 \frac{a_c^2}{a^2} \Theta_0$ describes the internal “elastic” part of the energy of the balanced charge associated with the nonlinearity G . Its magnitude is controlled by the ratio $\frac{a_c}{a}$ of the reduced Compton wavelength a_c and the size a of a free charge. Note that a typical atomic scale is given by the Bohr radius a_B , and $a_c = \alpha a_B$ where $\alpha \approx \frac{1}{137}$ is the Sommerfeld fine structure constant.

An analysis of the BCT-based Hydrogen atom model, carried out in Sect. 3.7.2 (see also Sects. 17.5 and 35.2 for more details), shows a quantitative agreement with the Hydrogen atom spectrum if the size parameter $a = a_e$ for the electron is of order $100a_B$. For such a , the factor $\frac{a_c^2}{a^2}$ is of order 10^{-8} , and the elastic energy provides a small though non-zero contribution to the charge energy.

Since a_B is a typical scale for the lattice constant of a crystal, our current estimate of the free electron size of $100a_B$ is evidently much larger than the typical lattice constant. Consequently, in the case of electron diffraction on a crystal as in the Davisson–Germer experiment, the electron behaves as a wave, [73], [74].

The wave concepts introduced above allow us to define the *dispersion relation* $\omega(\mathbf{k})$ and the group velocity $\mathbf{v}_{\text{gr}}$ for an accelerating wave-corpuscle. Indeed, we determine the dispersion relation for the wave-corpuscle by expressing the frequency ω in (3.85) in terms of the wavevector $\mathbf{k}$ using formula (3.80):

$$\omega = \omega(\mathbf{k}) = \frac{\chi}{2m} \mathbf{k}^2 + q \frac{1}{\chi} \varphi_{\text{ex}}(t, \mathbf{r}). \quad (3.87)$$

We then find the *group velocity* by applying the standard formula

$$\mathbf{v}_{\text{gr}} = \nabla_{\mathbf{k}} \omega(\mathbf{k}) = \frac{\chi}{m} \bar{\mathbf{k}}. \quad (3.88)$$

Using (3.80), we also find that

$$\mathbf{v}_{\text{gr}} = \partial_t \mathbf{r}(t), \quad (3.89)$$

hence *the group velocity exactly coincides with the translational velocity of the wave-corpuscle*.

To summarize, we can state that the wave-corpuscle seamlessly combines the properties of a wave and a particle. The wave attributes such as group velocity and frequency perfectly match the corresponding to them particle attributes, namely the translational velocity and the energy, in accordance with the Newton law and the Planck–Einstein relation $\Delta E = \hbar \Delta \omega$. Importantly, the nonlinearity is “stealthy” in the sense that it doesn’t alter the point and wave features of the dynamics and the kinematics when providing for the localization properties of soliton-like waves in relevant regimes.

3.5.4 A Hypothetical Scenario for the Davisson–Germer Experiment

In the neoclassical field theory proposed here, an elementary charge is always fundamentally a wave, and particle-like states are defined as spatially localized waves. To make a distinction between a particle state and a wave state, we can think of a “wave” as a state having its energy not localized but rather well spread out in space. Taking into account this distinction, it is expected that a charge in the course of its time evolution can experience transitions from wave states to particle states and the other way around.

An interesting and important example of such particle-to-wave and wave-to-particle transitions is the individual electron diffraction on a crystal as in the Davisson–Germer experiment [73], [74]. We have not yet carried out its detailed analysis, but, based on what we have already studied in the BCT, we can offer a hypothetical picture of this important experiment with its consequent phases, which are as follows.

- 1 **Initial particle phase.** The balanced charge is initially located far away from the crystal lattice; being free, it is in a wave-corpuscle (particle-like) state. Its center $\mathbf{r}_{\text{in}}(t)$ moves essentially as the classical point charge in the direction of the crystal.
- 2 **Schrödinger wave phase.** In this phase, the significant part of the charge distribution is located near the lattice. The crystal influence becomes dominant compared to the nonlinearity. The wave propagation is effectively described by the linear Schrödinger equation with the potential corresponding to the lattice. The lattice parameters are of the order of several Bohr radii and are much smaller than the free electron size. Note that the wave function of an electron approaching the lattice has the form of a wave-corpuscle and is effectively represented by the de Broglie wave with an envelope, see Sect. 3.5.3; since the ratio of the lattice parameter and free electron size is small, the envelope during the interaction can be approximated by a constant and the wave corpuscle by a plane wave. The dynamics of the wave-corpuscle at this stage can be approximated by the diffraction of the de Broglie wave on the lattice.
- 3 **Final particle phase.** At this phase, most of the charge wave energy has already reflected from the crystal and is at a macroscopic distance from it, hence the charge becomes essentially free. Consequently, cohesive action of the nonlinearity is dominant again, and it controls the final transformation of the charge wave function into a wave-corpuscle with center $\mathbf{r}_{\text{out}}(t)$.

Notice that the transition from the initial particle-like state with center $\mathbf{r}_{\text{in}}(t)$ before the passage of the barrier into a similar particle-like state with center $\mathbf{r}_{\text{out}}(t)$ after the passage is entirely deterministic. Recall that QM treatment of a diffraction pattern in the experiment is based on the probabilistic interpretation of the amplitude of the wave function fundamental to QM. The same experiment can be explained in the BCT by regular statistics associated with an ensemble of charges having slightly different initial conditions for the wave function ψ and, in particular, its center $\mathbf{r}_{\text{in}}(t)$. Indeed, suppose that there is a small and experimentally difficult to control initial uncertainty in the charge distribution. When it undergoes the Schrödinger phase of evolution, it is described by the wave function $\psi(t, \mathbf{x})$ with a diffraction pattern which is mainly determined by the de Broglie wave and only slightly depends on the initial uncertainty. When most of the charge energy is reflected, the charge nonlinear intrinsic forces take over and localize its wave function into a wave-corpuscle state with center $\mathbf{r}_{\text{out}}(t)$; the value of $\mathbf{r}_{\text{out}}$ is determined by the wave pattern at the end of the Schrödinger phase and is very sensitive to small variations of the pattern and even more sensitive to the initial uncertainty. The above described statistical scenario is similar to the statistics of the particle-to-particle transition from $\mathbf{r}_{\text{in}}(t)$ to $\mathbf{r}_{\text{out}}(t)$ described by the de Broglie–Bohm theory, where the transition is mediated by the “quantum potential”. The consistency of the de Broglie–Bohm theory statistics with the typical diffraction pattern has been confirmed by an analysis and numerical simulations, [176, 5.1]. The last point made above raises the interesting question of the possibility of deriving the de Broglie–Bohm theory as an approximation to the neoclassical theory proposed here in the case where the charge initial and final states are expected to be particle-like.

3.6 Particle-Like Dynamics

Since charges in many situations behave like particles, that is to say spatially localized objects, it is very important to see how the corresponding particle-like regimes arise in our fundamentally field theory. With that in mind, we analyze in this section particle-like dynamics as an approximation applicable in certain regimes.

3.6.1 *Derivation of Newton's Law from the Field Conservation Laws*

It is obvious that the NKG or NLS equations, which determine the dynamics of the balanced charges, look very different from Newton's law (3.13) which determines the dynamics of the point charges. Therefore, it is very important to prove that in macroscopic regimes the dynamics of the localized charges is described by Newton's equations (3.13). The localization limit and the macroscopic regimes can be characterized as regimes where the charge distributions can be approximated by delta-functions and the distributed currents (3.20) by point currents (3.6). We already demonstrated that the motion of wave-corpuscles that provide exact and approximate solutions to the NLS and NKG material equations is described by Newton's law with Lorentz force. Now we show how the point dynamics in the localization limit can be derived without specific assumptions on the form of localized material waves ψ . First, we consider the non-relativistic case where the dynamics of $\psi(t, \mathbf{x})$ is governed by the NLS equation. For brevity, we again assume that the external magnetic field is absent. Our derivation is similar to the argument for the Ehrenfest theorem.

First, we use the charge normalization condition (3.23) and introduce as in the Ehrenfest theorem the position of the charge by the formula

$$\mathbf{r}(t) = \int \mathbf{x} |\psi|^2 d\mathbf{x} = \frac{1}{q} \int \mathbf{x} \rho d\mathbf{x}. \quad (3.90)$$

Then integrating the conservation law (3.48), we obtain an equation for the total momentum $\mathbf{P}$:

$$\partial_t \mathbf{P} = q \int \mathbf{E}_{\text{ex}} |\psi_\ell|^2 d\mathbf{x} \quad (3.91)$$

where

$$\mathbf{P} = \int \mathbf{P}(t, \mathbf{x}) d\mathbf{x}.$$

Now we use the continuity equation which involves the current density $\mathbf{J}$. A direct computation shows that for the NLS equation the momentum density $\mathbf{P}$ and the current density $\mathbf{J}$ are proportional:

$$\mathbf{P}(t, \mathbf{x}) = \frac{m}{q} \mathbf{J}(t, \mathbf{x}). \quad (3.92)$$

Multiplying the continuity equation (3.21) by $\mathbf{x}$ and integrating over space, we arrive to the identities

$$\begin{aligned} \partial_t \mathbf{r}(t) &= \frac{1}{q_\ell} \int \mathbf{x} \partial_t \rho \, d\mathbf{x} \\ &= -\frac{1}{q} \int \mathbf{x} \nabla \cdot \mathbf{J} \, d\mathbf{x} = \frac{1}{q} \int \mathbf{J} \, d\mathbf{x} = \frac{1}{m} \int \mathbf{P} \, d\mathbf{x} = \frac{1}{m} \mathbf{P}. \end{aligned} \quad (3.93)$$

Taking the time derivative and using (3.91), we obtain the following equation:

$$m \partial_t^2 \mathbf{r} = q \int \mathbf{E}_{\text{ex}} |\psi|^2 \, d\mathbf{x}. \quad (3.94)$$

In the localization limit $a \rightarrow 0$, the charge density $|\psi|^2$ converges to Dirac's delta-function $\delta(\mathbf{x} - \mathbf{r}(t))$, implying

$$\int \mathbf{E}_{\text{ex}} |\psi|^2 \, d\mathbf{x} \rightarrow \mathbf{E}_{\text{ex}}(t, \mathbf{r}(t)) = -\nabla \varphi(t, \mathbf{r}(t)).$$

Newton's law (3.53) then follows from (3.94). The derivation in the case of the presence of the magnetic field is given in Sect. 17.2.2 with complete mathematical details available in Sect. 34.1.

Note that we have used in the above derivation only the momentum conservation law, the continuity equation and the momentum-current proportionality (3.92), and those equations do not involve explicitly the nonlinearity and the size parameter a . The only assumption we used concerning the wave function ψ is the convergence of $|\psi|^2$ to the delta-function. An explicit example of the convergence of $|\psi|^2$ to $\delta(\mathbf{x} - \mathbf{r})$ as $a \rightarrow 0$ is provided by the wave corpuscle with

$$|\psi|^2 = a^{-3} \left| \psi(|\mathbf{x} - \mathbf{r}|/a) \right|^2 = \pi^{-3/2} a^{-3} e^{-|\mathbf{x} - \mathbf{r}|^2/a^2}.$$

3.6.2 Derivation of the Relativistic Law of Motion and Einstein's Formula $E = Mc^2$

Let us recall first the basics of the relativistic dynamics of a mass point, [265, 29]. The relativistic dynamics of an accelerating charged point in an external EM field $\mathbf{E}_{\text{ex}}, \mathbf{B}_{\text{ex}}$ is described by the following equation:

$$\partial_t (M \mathbf{v}) = \mathbf{f}_{\text{Lor}}(t, \mathbf{r}), \quad M = m_0 \gamma, \quad \gamma = \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-1/2}, \quad (3.95)$$

where $\mathbf{r} = \mathbf{r}(t)$ is the mass point trajectory, $\mathbf{v}(t) = \partial_t \mathbf{r}(t)$ is its velocity, m_0 is the rest mass of the point, $\mathbf{f}_{\text{Lor}}$ is the Lorentz force (3.11) and γ is the Lorentz factor. Note that in the above equation the rest mass m_0 of a point is an intrinsic property of a point and is simply prescribed. Since equation (3.95) for a single mass point has the form of Newton's law, one can determine as in Newtonian mechanics the mass M as a measure of inertia from the known force $\mathbf{f}_{\text{Lor}}$ and the acceleration $d^2 \mathbf{r}/dt^2$ (the variability of γ can be neglected for mild accelerations). Since the charge dynamics in the BCT is based on the NKG field equation (3.16), we have to derive the relativistic law of motion (3.95) in the localization limit. To this end, we use a relativistic version of the Ehrenfest theorem. We present here only the main points of the analysis leaving the details to Sects. 17.6.2 and 34.5.

Since the relativistic Newton's equation (3.95) deals with a point trajectory $\mathbf{r}(t)$, we have to introduce a concept of a point trajectory $\mathbf{r}(t)$ related to the wave function $\psi(t, \mathbf{x})$ when the dynamics asymptotically can be described by the point dynamics. With that in mind, we consider a sequence of wave functions $\psi_n(t, \mathbf{x})$ assuming that they are localized about a trajectory $\mathbf{r}(t)$ in the following sense. First of all, we assume that the corresponding size parameters a_n approach 0, that is $a_n \rightarrow 0$. Then, for such a sequence, we assume that there exists a "trajectory of concentration" $\mathbf{r}(t)$ with the concentration formulated in terms of the energy densities $u_n(t, \mathbf{x})$ associated with $\psi_n(t, \mathbf{x})$. The energy densities $u_n(t, \mathbf{x})$ are evaluated based on the NKG Lagrangian, and the concentration at the trajectory $\mathbf{r}(t)$ is understood as follows. We introduce a sequence of small balls $\Omega_n = \{\mathbf{x} - \mathbf{r}(t) \mid |\mathbf{x} - \mathbf{r}(t)| \leq R_n\}$ with $R_n \rightarrow 0$ and $\frac{a_n}{R_n} \rightarrow 0$, and define "concentration" of the energy as the existence of the following non-zero limit

$$\int_{\Omega_n} u_n(t, \mathbf{x}) \, d\mathbf{x} \rightarrow \bar{\mathcal{E}}_\infty(t) \quad \text{as } n \rightarrow \infty, \quad (3.96)$$

where $\bar{\mathcal{E}}_\infty(t)$ is called the energy concentrated at $\mathbf{r}(t)$. The concentration assumes also that the fields $\psi_n(t, \mathbf{x})$ converge to zero on the boundaries $\partial\Omega_n = \{\mathbf{x} - \mathbf{r}(t) \mid |\mathbf{x} - \mathbf{r}(t)| = R_n\}$. Obviously, the above conditions can be understood as a specification of the assumption $u(t, \mathbf{x}) \rightarrow \bar{\mathcal{E}}_\infty(t) \delta(\mathbf{x} - \mathbf{r}(t))$.

Using the conservation laws for the nonlinear Klein–Gordon equation, we prove that *if there exists a trajectory of concentration* $\mathbf{r}(t)$, then the value of the charge restricted to Ω_n converges to a time-independent constant ρ_∞ :

$$\int_{\Omega_n} \rho(t, \mathbf{x}) \, d\mathbf{x} \rightarrow \rho_\infty \quad \text{as } n \rightarrow \infty.$$

We then prove also that the trajectory and the concentrated energy $\bar{\mathcal{E}}_\infty(t)$ must satisfy the following equations:

$$\partial_t \left(\frac{1}{c^2} \bar{\mathcal{E}}_\infty(t) \partial_t \mathbf{r} \right) = \mathbf{f}_\infty(t, \mathbf{r}), \quad (3.97)$$

$$\partial_t \bar{\mathcal{E}}_\infty = \partial_t \mathbf{r} \cdot \mathbf{f}_\infty(t, \mathbf{r}), \quad (3.98)$$

where $\mathbf{f}_\infty$ is the Lorentz force acting on a charge that has the value ρ_∞ . The Lorentz force is evaluated on the trajectory $\mathbf{r}(t)$ in terms of the external EM potentials φ_{ex} , $\mathbf{A}_{\text{ex}}$ which enter the NKG equation (3.16), (3.18). Equation (3.97) obviously has the form of Newton's law where the mass is determined by the formula

$$M(t) = \frac{1}{c^2} \bar{\mathcal{E}}_\infty(t), \quad (3.99)$$

in which we readily recognize *Einstein's formula*. Multiplying (3.97) by $2M\partial_t \mathbf{r}$ and using (3.98), we obtain the following equation for the mass:

$$\partial_t (M\partial_t \mathbf{r})^2 = c^2 \partial_t M^2.$$

This equation after integration implies the following relation:

$$M^2 - c^{-2} M^2 (\partial_t \mathbf{r})^2 = M_0^2,$$

where the constant of integration M_0^2 is naturally interpreted as the rest mass squared. Using the Lorentz factor γ , we can rewrite the above formula in the familiar form

$$M = M_0 \gamma.$$

Observe that in the above derivation *the rest mass is not postulated but it is derived as an integral of motion*, and it may have different values for different regimes with different values of the internal energy, see Remark 17.6.1 in Sect. 17.6.2 for details. The rest mass can be determined in terms of the concentrated energy $\bar{\mathcal{E}}_\infty(t)$ using (3.99). If the charge is at rest in a ground state for $t < 0$ and is subjected to external fields only for $t > 0$, we can evaluate its energy and find the rest mass. Assuming that the Compton wavelength a_C and free electron size a satisfy the following relation

$$a_C^2/a^2 \rightarrow \zeta^2,$$

we observe that at $t < 0$ the restricted energy and charge converge respectively to $\bar{\mathcal{E}}_\infty$ and $\bar{\rho}_\infty$, and we obtain for the resting charge in a ground state (3.32) the following equalities:

$$\bar{\mathcal{E}}_\infty = mc^2 (1 + \Theta_0 \zeta^2), \quad \bar{\rho}_\infty = q, \quad (3.100)$$

where m is the mass parameter in the NKG equation, Θ_0 depends on the nonlinearity, and $\Theta_0 = 1/2$ for the logarithmic nonlinearity. The above implies the following value of the rest mass:

$$M_0 = m (1 + \Theta_0 \zeta^2). \quad (3.101)$$

Since the rest mass is constant, we have the same value of the rest mass given by (3.101) for $t > 0$ in the accelerated relativistic motion. Note that, in the case of an electron, the correction factor $\zeta^2 = a_C^2/a^2$ is of order 10^{-8} for the physical

value of the Compton wavelength a_C and $a = a_e \sim 100a_B$. The mass parameter m can be identified with the value of the electron mass obtained from spectroscopic data. The rest mass M_0 describes the inertial mass of the electron. The Penning trap measurements [117], which can be considered as the most direct measurement of the electron mass as the inertial one, gives a mass value which is greater than m by an amount of order $m10^{-8}$. Such a difference can be explained by our formula (3.101), which also gives the inertial mass greater than m by an amount of order $m10^{-8}$, see Sect. 17.6.2 for more details.

Observe that the concept of balanced charge provides a model of a relativistic object with an internal energy, and the mass of this object is found based on the Newtonian definition in an accelerated motion. Alternatively, if we consider a uniform motion of a free balanced charge, we can apply standard arguments from the relativistic field theory. Indeed, the total energy momentum $\mathbf{P}$ of a uniformly moving charge is a 4-vector, as it should be for a closed relativistic invariant system. This total momentum is $\mathbf{P} = M\mathbf{v}$ where the constant velocity $\mathbf{v}$ originates from the corresponding parameter of the Lorentz group, and the coefficient M can be naturally identified with the mass. The Lorentz invariance of the energy-momentum then implies Einstein's formula. Remarkably, the Newtonian definition of mass for accelerated motion and the definition based on the Lorentz invariance for the uniform motion produce the same result when both can be applied.

We have constructed non-trivial examples of accelerated motion of wave-corpuscles in an electric field, to which the above considerations can be applied, see Sect. 34.6 for details. In such a motion one can observe phenomena which are not present in the uniform motion. In particular, the examples of the accelerated motion demonstrate that the Einstein formula is very robust. It holds during transitions from uniform motion with one velocity to uniform motion with another velocity, whereas the resulting motion cannot be obtained by the Lorentz transformation from the initial motion.

In the BCT, the EM self-interaction is completely absent, therefore the equation (3.95) does not describe radiation effects. We consider in [10] a modification of the BCT which allows some EM self-interaction. In that framework, we derive the Lorentz–Abraham–Dirac equation which takes the radiative response of accelerating electron into account. The derivation is in the spirit of the Ehrenfest theorem. It can be considered as a modification and clarification of Dirac's [85] mass renormalization approach.

3.7 Quantum Phenomena

We take a look here at several basic quantum phenomena covered by the proposed neoclassical theory. These phenomena include the Planck–Einstein energy-frequency relation $E = \hbar\omega$ fundamental to QM as well as discrete frequency spectra associated with the neoclassical model of the Hydrogen atom, which turns out to be similar to the Schrödinger Hydrogen atom. Importantly, we show here that the Planck–Einstein

energy-frequency relation holds exactly in the non-relativistic version of our theory provided that the nonlinearity G is chosen to be logarithmic as defined by Eq. (3.27).

3.7.1 The Planck–Einstein Formula and the Logarithmic Nonlinearity

We have discussed in Sect. 3.3 a very important role played by the nonlinearity G in our theory. It turns out that some significant features brought into the theory by the nonlinearity G , including particle-like regimes, do not depend on a particular choice of the nonlinearity G but rather on some of its fundamental properties. But it is natural to expect that a justified unique choice for G is necessary or at least very desirable for the completeness of the theory and its ability to make precise evaluations of physical quantities which can be experimentally verified. It turns out that, as we have already pointed out in Sect. 3.3, there is a physically sound requirement that singles out uniquely the logarithmic nonlinearity defined by equation (3.27). This physical requirement is the fulfilment of the Planck–Einstein energy-frequency relations $E = \hbar\omega$ in the non-relativistic version of our theory, and we give in this chapter a concise line of argument leading to the logarithmic nonlinearity (3.27) as the unique choice to provide for that.

Having in mind a system of charges in an equilibrium, in particular the Hydrogen atom, we consider the case when every wave function $\psi_\ell(t, \mathbf{x})$ varies harmonically in time, that is

$$\psi_\ell(t, \mathbf{x}) = e^{-i\omega_\ell t} \hat{\psi}_\ell(\mathbf{x}). \quad (3.102)$$

We refer to states of the system of the form (3.102) as *multiharmonic*. Then the corresponding time-independent charge densities are $q_\ell |\psi_\ell(\mathbf{x})|^2$. Applying to these states the NLS equations (3.17), we arrive at the following nonlinear eigenvalue problem:

$$\chi\omega_\ell \hat{\psi}_\ell + \frac{\chi^2}{2m_\ell} \tilde{\nabla}_\ell^2 \hat{\psi}_\ell - q_\ell \varphi_{\neq \ell} \hat{\psi}_\ell - \frac{\chi^2}{2m_\ell} G'_\ell(|\hat{\psi}_\ell|^2) \hat{\psi}_\ell = 0, \quad \ell = 1, \dots, N, \quad (3.103)$$

where the potentials φ_ℓ , $\mathbf{A}_\ell$ are determined by the Maxwell equations (3.9).

The canonical energy $\hat{\mathcal{E}}$ of the system of charges is found from the corresponding non-relativistic Lagrangian evaluated on time-harmonic solutions (3.102). This energy is

$$\hat{\mathcal{E}} = \sum_\ell E_{0\ell} + \hat{\mathcal{E}}_{\text{BEM}}, \quad (3.104)$$

where the ℓ -th charge energy has the form

$$E_{0\ell} = \int q_\ell |\psi_\ell|^2 \varphi_{\neq \ell} d\mathbf{x} + \int \frac{\chi^2}{2m_\ell} \left\{ |\tilde{\nabla} \psi_\ell|^2 + G_\ell(|\psi_\ell|^2) \right\} d\mathbf{x}, \quad (3.105)$$

and $\mathring{\mathcal{E}}_{\text{BEM}}$ is the energy of the EM field. The particular multiharmonic state is characterized by the frequencies $\{\omega_\ell\}_{\ell=1}^N$ and energies $\{E_{0\ell}\}_{\ell=1}^N$. Now we consider possible transitions between one multiharmonic state $\{\hat{\psi}_\ell\}_{\ell=1}^N$ and another one $\{\hat{\psi}'_\ell\}_{\ell=1}^N$ with respective frequencies $\{\omega_\ell\}_{\ell=1}^N$, $\{\omega'_\ell\}_{\ell=1}^N$ and energies $\{E_{0\ell}\}_{\ell=1}^N$, $\{E'_{0\ell}\}_{\ell=1}^N$.

It turns out that, if the nonlinearities G'_ℓ are logarithmic, the following *Planck–Einstein frequency-energy relation* holds:

$$\chi(\omega_\ell - \omega'_\ell) = E_{0\ell} - E'_{0\ell}, \quad \ell = 1, \dots, N. \quad (3.106)$$

In fact, the requirement to fulfil the Planck–Einstein relation singles out the logarithmic nonlinearity from all possible nonlinearities. Indeed, to derive (3.106), we multiply Eq. (3.103) by $\hat{\psi}_\ell^*$ and integrate the result over the entire space. Using integration by parts and the normalization condition (3.23), we arrive at the equation

$$\begin{aligned} \chi\omega_\ell = & \int \left[\frac{\chi^2}{2m_\ell} |\tilde{\nabla}\hat{\psi}_\ell|^2 \, dy + q_\ell \varphi_{\neq\ell} |\hat{\psi}_\ell|^2 \right] d\mathbf{x} \\ & + \int \frac{\chi^2}{2m_\ell} G'_\ell(|\hat{\psi}_\ell|^2) |\hat{\psi}_\ell|^2 d\mathbf{x}. \end{aligned} \quad (3.107)$$

Comparing the above with (3.105), we find that

$$\chi\omega_\ell - E_{0\ell} = \frac{\chi^2}{2m_\ell} \int \left[G'_\ell(|\hat{\psi}_\ell|^2) |\hat{\psi}_\ell|^2 - G_\ell(|\hat{\psi}_\ell|^2) \right] d\mathbf{x}. \quad (3.108)$$

Notice that the logarithmic nonlinearities G_ℓ satisfy the determining relation $G'_\ell(s)s - G_\ell(s) = sC_\ell$ with constants C_ℓ . Consequently, two different multiharmonic states satisfying the normalization condition (3.23) must satisfy the following identity:

$$\chi\omega_\ell - E_{0\ell} = \chi\omega'_\ell - E'_{0\ell} \quad (3.109)$$

implying the Planck–Einstein relation (3.106).

Notice that the Planck–Einstein relation (3.106) involves the difference of two frequencies associated with a transition between two multiharmonic states. Transitions between multiharmonic states may be caused by resonance interactions which may occur when the EM field has a component with the resonant frequency $\omega = \omega_\ell - \omega'_\ell$. Since for any transition the total energy $\mathring{\mathcal{E}}$ in (3.104) is preserved, there has to be an energy transfer between the energies of charges $E_{0\ell}$ and the EM energy $\mathring{\mathcal{E}}_{\text{BEM}}$, and for resonant transitions the energy of the component of the EM field with frequency ω is changed by a quantum of energy $E = \chi\omega$. *This property can be interpreted as an interchange of quanta of energy $\chi\omega$ between the system charges and the EM field.*

Notice that a possibility to have multiharmonic states with wave functions (3.102) is provided by an important feature of our theory that assigns to every balanced charge its individual wave function. Note also that a transition from one multiharmonic

state to another takes some transition time. If this transition time is sufficiently short compared to a typical observation time, it might be interpreted as an instantaneous “jump” as in QM.

3.7.2 Hydrogen Atom

With the Hydrogen atom in mind, we consider in this section a system of two bound charges—the first one is negative (electron), and the second is positive (proton). We explain here the main points of the analysis leaving the details for Sects. 17.5 and 35.2. Let us apply the general system (3.103) to the case of two charges neglecting the magnetic field. The Maxwell equations for the potentials

$$\varphi_{\neq 1} = \varphi_2, \quad \varphi_{\neq 2} = \varphi_1$$

turn into the following Poisson equations

$$\frac{1}{4\pi} \nabla^2 \varphi_\ell = -q_\ell \left| \hat{\psi}_\ell \right|^2, \quad \ell = 1, 2. \quad (3.110)$$

It is convenient to introduce parameters

$$\hat{a}_\ell = \frac{\chi^2}{q^2 m_\ell}, \quad \ell = 1, 2. \quad (3.111)$$

The parameter $\hat{a}_1$ turns into *the Bohr radius* a_B if χ coincides with the Planck constant $\hbar$, and m_1, m_2 and q are respectively the electron mass, the proton mass and the electron charge. Then we introduce the following changes of variables in equations (3.103) for the two charges:

$$\mathbf{x} = \hat{a}_\ell \mathbf{y}_\ell, \quad \ell = 1, 2; \quad (3.112)$$

$$\varphi_\ell = \frac{q_\ell}{\hat{a}_\ell} \phi_\ell(\mathbf{y}_\ell), \quad \hat{\psi}_\ell(\mathbf{x}) = \frac{1}{\hat{a}_\ell^{3/2}} \Psi_\ell(\mathbf{y}_\ell), \quad \omega'_\ell = \frac{\chi \hat{a}_\ell}{q^2} \omega_\ell. \quad (3.113)$$

In the new variables, the equations (3.103) can be rewritten as the following equations for the electron and the proton wave functions respectively:

$$\omega'_1 \Psi_1 + \frac{1}{2} \nabla^2 \Psi_1 + \frac{1}{b} \phi_2 \left(\frac{\mathbf{y}}{b} \right) \Psi_1 = \frac{1}{2} G'_1 (|\Psi_1|^2) \Psi_1, \quad (3.114)$$

$$\omega'_2 \Psi_2 + \frac{1}{2} \nabla^2 \Psi_2 + b \phi_1(b\mathbf{y}) \Psi_2 = \frac{1}{2} G'_2 (|\Psi_2|^2) \Psi_2. \quad (3.115)$$

Here b is a small parameter defined by

$$b = \frac{\hat{a}_2}{\hat{a}_1} = \frac{m_1}{m_2} \simeq \frac{1}{1800} \ll 1, \quad (3.116)$$

and the potentials ϕ_ℓ are determined from the Poisson equations

$$\nabla^2 \phi_1 = -4\pi |\Psi_1|^2, \quad \nabla^2 \phi_2 = -4\pi |\Psi_2|^2. \quad (3.117)$$

In the proton equation, the potential $b\phi_1$ (**by**) is small and almost constant, and can naturally be approximated by the constant $b\phi_1(\mathbf{0})$. Observing that the ground state is a solution of the resulting proton equation if $\omega'_2 + b\phi_1(\mathbf{0}) = 0$, we see that the ground state is a good approximation for Ψ_2 . An analysis (see Sects. 14.4.2 and 17.5.2) shows that the potential $\frac{1}{b}\phi_2(\frac{1}{b}r)$ in the electron equation (3.114) can be replaced by the Coulomb potential with a resulting error of order b^2 in a corresponding variational problem. Hence, the problem of finding frequencies with lower energy levels can be approximated by the following nonlinear eigenvalue problem for Ψ_1 with the Coulomb potential:

$$\omega'_1 \Psi_1 + \frac{1}{2} \nabla^2 \Psi_1 + \frac{1}{|\mathbf{y}|} \Psi_1 = \frac{1}{2} G'_1 (|\Psi_1|^2) \Psi_1. \quad (3.118)$$

Now observe that the nonlinearity G'_1 obtained by rescaling from the original nonlinearity depends on the size parameter a as follows:

$$G'_1(s) = G'_{1\kappa}(s) = \kappa^2 G'_{11}(\kappa^{-3}s), \quad \kappa = \frac{\hat{a}_1}{a}.$$

The emerged parameter $\kappa = \frac{\hat{a}_1}{a}$ is the ratio of the electron Bohr radius $\hat{a}_1$ to the size parameter $a_1 = a$. To assure the proximity of our Hydrogen atom model as described by equation (3.118) to the well tested Schrödinger equation with the frequencies $\omega'_{1n} = -\frac{1}{2n^2}$, we assume the parameter κ to be small. Consequently, the nonlinearity G'_1 is small, therefore equation (3.118) is a small perturbation of the linear Schrödinger equation. The frequencies $\omega'_{1n} = -\frac{1}{2n^2}$, according to the change of variables (3.113), correspond to the frequencies

$$\omega'_{1n} = -\frac{q^2}{\chi \hat{a}_1} \frac{1}{2n^2}, \quad (3.119)$$

and the corresponding energy levels for the electron are given by the formula

$$E_n = \chi \omega_{1n} = -\frac{q^2}{\hat{a}_1} \frac{1}{2n^2} = -2\pi \chi c R_\infty \frac{1}{n^2}, \quad n = 1, 2, \dots, \quad (3.120)$$

where $R_\infty = \frac{q^4 m}{4\pi \chi^3 c}$ coincides with the Rydberg constant if $\chi = \hbar = \frac{h}{2\pi}$.

The above frequencies (3.119) and energies (3.120) are approximations to the solutions of the nonlinear eigenvalue problem, and we have proved in Sect. 35.2 that the error of approximation is of order $\kappa^2 \ln \kappa$. We assume that $\kappa = \frac{\hat{a}_1}{a} \sim 10^{-2}$, that is, the size of a free electron is 100 times larger than its size in the Hydrogen atom. It seems quite natural for the electron to shrink under the attractive action of the proton's Coulomb field.

We conclude this section by pointing out that the Schrödinger Hydrogen atom energy spectrum naturally emerges in the BCT.

3.8 Comparison with Quantum Mechanics and Classical Electrodynamics

One of the key ingredients of any theory is the set of its primitive concepts. In the case of classical electrodynamics (CEM) these concepts are: (i) elementary charges such as electron and positron and (ii) the electromagnetic fields. When defining an elementary charge both CEM and quantum mechanics (QM) assume it to be a point-like object. As R. Feynman put it, [122, pp. 21–6]:

“The wave function $\psi(\mathbf{r})$ for an electron in an atom does not, then, describe a smeared-out electron with a smooth charge density. The electron is either here, or there, or somewhere else, but wherever it is, it is a point charge.”

In contrast, *in the BCT an elementary balanced charge is always a spatially distributed object*. Under certain conditions, for instance when it is free, it localizes and can be treated approximately as a point particle. In particular, a free electron has localization size a_e that becomes a *new fundamental spatial scale which is special to our theory*. Our currently assessed value for this scale is $a_e \approx 100a_B \approx 5 \text{ nm}$ where a_B is the Bohr radius. An experimental evidence of emergence of spatial scales of order $10^2 a_B$ in several areas of physics is provided in Sect. 18.2. The size of the free electron $a_e \approx 100a_B$ might seem at first sight too large, but its electric field at macroscopic distances is extremely close to the field of a point charge, and such a free balanced charge at atomic scales of order a_B looks like a de Broglie wave, as it should according to the Davisson–Germer experiment, [73], [74]. The balanced charge reproduces the Hydrogen atom spectrum with an error of approximation controlled by the ratio $\frac{a_B^2}{a_e^2} \sim 10^{-4}$. On the other hand, when we use wave-corpuscles or the Ehrenfest theorem in Sects. 16.3, 16.4 to obtain Newtonian mechanics in the macroscopic regimes, the error of approximation is estimated in terms of $\frac{a^2}{R_{\text{ex}}^2} \sim 10^{-9}$, where R_{ex} is a typical macroscopic length scale of the order 10^{-4} m defined as the length scale accessible to the naked eye. The possibility of constructing a neoclassical BCT, which is in an agreement with both classical electrodynamics and quantum mechanics, relies on the very small value of the ratio $\frac{a_B^2}{R_{\text{ex}}^2} \sim 10^{-13}$.

The idea of treating elementary charges as spatially extended objects is not new, of course. In the early days of the development of EM theory, under pressure of

theoretical difficulties attributed directly to the concept of a point charge, physicists introduced and studied a number of different models of a structured spatially distributed elementary charge. The first and the most well known models of an extended charge are the Abraham rigid charge model and the Lorentz relativistically covariant model, see Sect. 2.5. In contrast to those models, we do not prescribe to an elementary charge any fixed geometry, but instead the elementary charge has a complex-valued (or spinor-valued) wave function governed by the nonlinear Klein–Gordon or the nonlinear Schrödinger equation in the relativistic and non-relativistic cases, respectively. The size and shape of a balanced charge may significantly vary depending on the EM field acting upon it. For example, our current assessment of the free electron size is of order $100a_B$, whereas when the electron is bound to a proton as in the Hydrogen atom it has a much smaller size of order a_B .

An important feature of the balanced charge as a model of an “elementary” charge is that it does not interact with itself electromagnetically. In other words, different parts of the balanced charge do not interact with each other making it an elementary object with respect to EM interactions. From this point of view, the charge in the Abraham–Lorentz model is not elementary, for its different parts do interact electromagnetically by the Lorentz forces. As to the possibility of introducing and studying different models for an electron, H. Lorentz (1853–1928) wrote in 1927 in Sect. 42 “Structure of the Electron” of his monograph [228, pp. 125–126]:

“Therefore physicists are absolutely free to form any hypotheses on the properties and size of electrons that may best suit them. You can, for instance, choose the old electron (a small sphere with charge uniformly distributed over the surface) or Parson’s ring-shaped electron, endowed with rotation and therefore with a magnetic field; you can also make different hypotheses about the size of the electron. In this connection I may mention that A.H. Compton’s experiments on the scattering of γ rays by electrons have led him to ascribe to the electron a size considerably greater than it was formerly supposed to have.”

Replacing the point charge with an extended one is not the only difference of our theory as compared with the standard version of CEM. Similarly to the Wheeler–Feynman theory, the BCT disposes the single EM field as a primitive concept furnishing instead every elementary charge with its own elementary EM field, and there is no EM self-interaction for elementary charges. Still every elementary EM field satisfies exactly the Maxwell equations, and it is not so surprising that the BCT recovers the CEM as a good approximation in relevant cases.

The BCT covers a number of quantum phenomena, for its evolution equations are closely related to the fundamental equations of QM. Indeed, the material equations in the BCT in the non-relativistic and the relativistic cases are respectively the NLS and the NKG equations. These equations are obtained from the linear Schrödinger and the Klein–Gordon equations, respectively, by adding to them nonlinearities special to our theory. These nonlinearities are relatively small compared with the EM potentials in relevant cases, and consequently the BCT describes many phenomena similarly to QM. With that said, the BCT differs fundamentally from QM. First of all, as we repeatedly emphasized, an elementary charge in the BCT is a spatially distributed object and it is not a point as in QM. Second of all, every elementary charge has its individual wave function $\psi_\ell(t, \mathbf{x})$ over the 4-dimensional space-time continuum.

This wave function does not have a probabilistic interpretation as in QM, but it can be interpreted as a charge “cloud”, that is, a charge distribution for the ℓ -th charge. *Consequently, in sharp contrast to QM, there is no configuration space in our theory. The system of N interacting charges is described by N wave functions of the three spatial variables and time. Recall that in QM the same system has one wave function of time and $3N$ spatial variables known as the configuration space.*

In particle-like regimes, the balanced charge described by wave-corpuscles is not subjected to dispersion. In contrast, QM wave-function in similar circumstances disperses. Also, in the BCT the charge energy is spatially distributed, and the energy affects the charge dynamics according to Einstein’s energy-mass relation. This property does not have a direct analog in QM.

In the BCT, the energy and the frequency are two independent physical quantities, which for certain regimes are related according to the Planck–Einstein energy-frequency relation $E = \hbar\omega$. In QM, the Planck–Einstein energy-frequency relation $E = \hbar\omega$ is universal and fundamentally exact, and the Schrödinger equation for the entire system is in fact an operator form of the Planck–Einstein energy-frequency relation.

The BCT is a classical Lagrangian field theory with the nonlinear field equations for the charge wave functions and the Maxwell equations for the elementary EM fields. In the BCT, as we have discussed in previous sections of this Part, regimes that exhibit quantum phenomena coexist naturally with regimes for charges in particle-like states. Consequently, there is no quantization and correspondence principle in the BCT, whereas they are absolutely instrumental in QM to generate wave equations from the classical particle theory. The Heisenberg uncertainty principle does not hold as a universal principal in the BCT. Since a balanced charge is not a point, there is an uncertainty in its location, but it is not probabilistic in nature. In the case of the wave-corpuscle, the total momentum of the charge is defined without any uncertainty, whereas the wavevector and position of the charge allow uncertainties which satisfy the uncertainty principle.

We explore further similarities and differences between the neoclassical theory advanced here and quantum mechanics in Chap. 18.

Part I

Classical Electromagnetic Theory and Special Relativity

“The most fascinating subject at the time that
I was a student was Maxwell’s theory.
What made this theory appear revolutionary was
the transition from forces at a distance
to fields as fundamental variables.”
*A. Einstein.*¹

We provide basic information on classical electromagnetic theory and special relativity in this part. We also present the Lagrangian formalism, the Green functions, the Helmholtz decomposition for the Maxwell equations and their quasistatic, non-relativistic approximations here. We describe the notation, choice for units, setup of coordinates, and the metric in the Minkowski four-dimensional spacetime.

¹The Einstein quotation is from his Autobiographical notes [295, Autobiographical notes].

Chapter 4

The Maxwell Equations

“Nothing beautiful can be produced by Man
except by the laws of mind acting in him
as those of Nature do without him ...”
*J.C. Maxwell.*¹

The electromagnetic (EM) field is described by a combination of the electric field $\mathbf{E}(t, \mathbf{x})$ and the magnetic field (induction) $\mathbf{B}(t, \mathbf{x})$. The Maxwell equations describe the dynamics of electric and magnetic fields. For the EM fields in the CGS units, which we use throughout the book, the Maxwell equations in vacuum have the form

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \text{Gauss's law,} \quad (4.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad \text{Absence of magnetic monopoles,} \quad (4.2)$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\partial_t \mathbf{E} = \frac{4\pi}{c}\mathbf{J}, \quad \text{Ampere–Maxwell's law,} \quad (4.3)$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\partial_t \mathbf{B} = \mathbf{0}, \quad \text{Faraday's induction law,} \quad (4.4)$$

where c is the speed of light, see, for instance, [179, 11.9], [214, 23, 30], [146, 7.4, 11.2], [332, 2]. Here $\rho(t, \mathbf{x})$ is the charge density distribution, and $\mathbf{J}(t, \mathbf{x})$ is the current density distribution; they are assumed to satisfy the charge conservation (or continuity) equation

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0. \quad (4.5)$$

¹[56, p. 343].

Fulfillment of the continuity equation is necessary for Eqs.(4.1) and (4.3) to be compatible.

A very important concept in the EM theory is the concept of electromagnetic potentials, which involve the *scalar potential* φ and the *vector potential* $\mathbf{A}$. The EM potentials not only allow us to simplify the Maxwell equations, but also, as we will see later, they play an important role at microscopic scales. The EM potentials are introduced by the following standard formulas, [179, 11.9]:

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\partial_t\mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (4.6)$$

Obviously, the fields determined in terms of the EM potentials by (4.6) always satisfy homogeneous equations (4.2) and (4.4). We can write the remaining Eqs.(4.1) and (4.3) in terms of the EM potentials as follows:

$$\nabla \cdot \left(\frac{1}{c}\partial_t\mathbf{A} + \nabla\varphi \right) = -4\pi\rho, \quad (4.7)$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c}\partial_t \left(\frac{1}{c}\partial_t\mathbf{A} + \nabla\varphi \right) = \frac{4\pi}{c}\mathbf{J}. \quad (4.8)$$

Though the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ are determined uniquely in terms of the potentials φ and $\mathbf{A}$, the converse is not true. If one needs to determine $\mathbf{A}$ and φ in terms of $\mathbf{E}$ and $\mathbf{B}$, so called gauge conditions have to be imposed. Most commonly used are the Lorentz and Coulomb gauges. The *Lorentz gauge* is the following condition:

$$\frac{1}{c}\partial_t\varphi + \nabla \cdot \mathbf{A} = 0, \quad (4.9)$$

and the *Coulomb gauge* has the form

$$\nabla \cdot \mathbf{A} = 0. \quad (4.10)$$

We would like to point out that, in addition to the commonly used Lorentz and Coulomb gauge conditions, there are also other gauge conditions which are summarized below, [182, IV.B], [180]

$\partial_\mu A^\mu = 0,$	Lorentz gauge	
$\nabla \cdot \mathbf{A} = \partial_j A^j = 0,$	Coulomb or radiation or transverse gauge	
$n_\mu A^\mu = 0, \quad n_\mu n^\mu = 0,$	light-cone gauge	
$A_0 = \varphi = 0,$	Hamiltonian or temporal gauge	(4.11)
$A_3 = 0,$	axial gauge	
$x_\mu A^\mu = 0,$	Fock–Schwinger gauge	
$\mathbf{x} \cdot \mathbf{A} = x_j A^j = 0$	Poincaré gauge	

Some of the above gauge conditions are written in the relativistic notation, see Sects.6.1 and 4.1.

4.1 The Maxwell Equations in Tensorial Form

To represent the Maxwell equations in a manifestly Lorentz invariant form, it is common to introduce the EM four-potential A^μ and the four-current density J^μ , $\mu = 0, \dots, 3$:

$$A = A^\mu = (\varphi, \mathbf{A}), \quad (4.12)$$

$$J = J^\mu = (c\rho, \mathbf{J}). \quad (4.13)$$

The concept of a 4-vector and the corresponding notation are discussed in more detail in Sect. 6.1. Then the EM field antisymmetric second-rank tensor (also called “field strength tensor”) is defined by the following formula, [179, 11.9], [332, 2.3]:

$$F^{\mu\nu} = F^{\mu\nu}(A) = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad \mu, \nu = 0, 1, 2, 3. \quad (4.14)$$

Here and below we use the notation

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c} \partial_t, \nabla \right), \quad \partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c} \partial_t, -\nabla \right), \quad (4.15)$$

$$A^\mu = (\varphi, \mathbf{A}), \quad A_\mu = (\varphi, -\mathbf{A}). \quad (4.16)$$

The above definitions are evidently in agreement with the relativistic notation for the space-time four-vector

$$x = x^\mu = (x^0, x^1, x^2, x^3) = (ct, \mathbf{x}) \quad (4.17)$$

as in (6.1).

According to (4.6), the components of the tensor relate to the components of the vector EM fields as follows, [179, 11.9], [332, 2.3]:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix}, \quad F_{\mu\nu} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix}. \quad (4.18)$$

Notice that the above is consistent with expressions (4.6) for the EM fields $\mathbf{E}$ and $\mathbf{B}$ in terms of the EM potentials φ and $\mathbf{A}$.

As previously mentioned, the EM field $F^{\mu\nu}$ defined by (4.14) does not determine the potential A^ν uniquely. Specifically, the potential $A^\nu + \partial^\nu \gamma$ yields for any scalar field $\gamma = \gamma(x)$ the same field $F^{\mu\nu}$ as the potential A^ν . This important transformation

$$A^\nu \rightarrow A^\nu + \partial^\nu \gamma, \quad \varphi \rightarrow \varphi + \frac{\partial \gamma}{\partial t}, \quad \mathbf{A} \rightarrow \mathbf{A} - \nabla \gamma, \quad (4.19)$$

is known as a *gauge transformation*. The invariance of the EM field $F^{\mu\nu}$ with respect to the gauge transformation (4.19) is known as *gauge invariance*—a subject with a rich and interesting history, [182].

Evidently, the EM field $F^{\mu\nu}$ determines the potential A^ν only up to the gauge transformation (4.19). A common way to eliminate the ambiguity in the potential is by imposing the *Lorentz gauge (condition)* as in (4.9), which can be written in the relativistic notation in the form

$$\partial_\mu A^\mu = 0. \quad (4.20)$$

We use here and below the common convention on summation over repeating indices. Another common convention concerning spatial components of 4-vectors such as A^μ is to use a Latin letter index for components of 3-vector $\mathbf{A}$, that is,

$$\mathbf{A} = (A^1, A^2, A^3) = A^j, \quad j = 1, 2, 3; \quad \partial_j A^j = \nabla \cdot \mathbf{A}. \quad (4.21)$$

As previously mentioned, the two homogeneous equations (4.2) and (4.4) are satisfied as long as we use the EM potentials. Their fulfillment can also be derived from the elementary identity

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0, \quad \alpha, \beta, \gamma = 0, 1, 2, 3. \quad (4.22)$$

The remaining Maxwell equations (4.1)–(4.3) (or (4.7), (4.8)) written in terms of the tensor $F^{\mu\nu}$ take the following manifestly relativistic form

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu, \quad (4.23)$$

where J^ν is the four-vector current.

As we already mentioned, it follows from the anti-symmetry of $F^{\mu\nu}$ combined with the Maxwell equation (4.23) and (4.12)–(4.14) that, for the system to be consistent, the four-vector current J^ν must satisfy the *continuity equation (charge conservation equation)*

$$\partial_\mu J^\mu = 0 \quad \text{or} \quad \partial_t \rho + \nabla \cdot \mathbf{J} = 0. \quad (4.24)$$

The Maxwell equations (4.23) can be written in the form of equations for the four-vector potential A^ν :

$$\square A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{4\pi}{c} J^\nu, \quad (4.25)$$

where $\square$ is the d'Alembertian operator,

$$\square = \partial_\mu \partial^\mu = \frac{1}{c^2} \partial_t^2 - \nabla^2. \quad (4.26)$$

An alternative form of the potential equations (4.25) in terms of the scalar and vector potentials (similar to Eqs. (4.7), (4.8)) is, [332, 2.2],

$$\begin{aligned} \nabla^2 \varphi + \frac{1}{c} \partial_t \nabla \cdot \mathbf{A} &= -4\pi \rho, \\ \nabla^2 \mathbf{A} - \frac{1}{c^2} \partial_t^2 \mathbf{A} - \nabla \left(\frac{1}{c} \partial_t \varphi + \nabla \cdot \mathbf{A} \right) &= -\frac{4\pi}{c} \mathbf{J}. \end{aligned} \quad (4.27)$$

If the potentials A^ν satisfy the Lorentz gauge condition (4.20), that is, $\partial_\mu A^\mu = 0$, then the Maxwell equations (4.25) evidently turn into the wave equations

$$\square A^\nu = \frac{4\pi}{c} J^\nu, \quad \text{or} \quad \frac{1}{c^2} \partial_t^2 A^\nu - \nabla^2 A^\nu = \frac{4\pi}{c} J^\nu. \quad (4.28)$$

Hence, the wave equations (4.28) for the EM potentials are equivalent to the Maxwell equations under the Lorentz gauge, and Eqs. (4.25) and (4.28) *preserve the Lorentz gauge as long as the current J^ν satisfies the continuity equation*. Note that, in contrast to the Maxwell equations, fulfillment of the continuity equation is not necessary for the consistency of the system of wave equations, but it is required for fulfillment of the Lorentz gauge and for the equivalence of the wave equations to the Maxwell equations.

4.1.1 Frame Transformation Formulas

The electric and magnetic fields are transformed from one frame to another one that moves relatively with the velocity $\mathbf{v}$ by the following formulas, [179, Section 11.10]:

$$\begin{aligned} \mathbf{E}' &= \gamma (\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) - \frac{\gamma^2}{\gamma + 1} (\boldsymbol{\beta} \cdot \mathbf{E}) \boldsymbol{\beta}, \\ \mathbf{B}' &= \gamma (\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) - \frac{\gamma^2}{\gamma + 1} (\boldsymbol{\beta} \cdot \mathbf{B}) \boldsymbol{\beta}, \\ \boldsymbol{\beta} &= \frac{\mathbf{v}}{c}, \quad \beta = |\boldsymbol{\beta}|, \quad \gamma = \frac{1}{\sqrt{1 - (\frac{v}{c})^2}}. \end{aligned} \quad (4.29)$$

The above formulas can also be recast as, [142, Section 22],

$$\begin{aligned} \mathbf{E}'_\perp &= \gamma (\mathbf{E}_\perp + \boldsymbol{\beta} \times \mathbf{B}), & \mathbf{E}'_\parallel &= \mathbf{E}_\parallel, \\ \mathbf{B}'_\perp &= \gamma (\mathbf{B}_\perp - \boldsymbol{\beta} \times \mathbf{E}), & \mathbf{B}'_\parallel &= \mathbf{B}_\parallel, \end{aligned} \quad (4.30)$$

where the subindices $\perp$ and $\parallel$ stand for vector components respectively parallel and perpendicular to $\mathbf{v}$. Observe that, for $\beta \ll 1$, the formulas (4.29), (4.30) yield the

following approximations with an error proportional to β^2 :

$$\begin{aligned} \mathbf{E}'_{\perp} &\cong \mathbf{E}_{\perp} + \boldsymbol{\beta} \times \mathbf{B}, & \mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel}, \\ \mathbf{B}'_{\perp} &\cong \mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E}, & \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel}. \end{aligned} \quad (4.31)$$

4.2 The Green Functions for the Maxwell Equations

In this section we describe EM fields $F^{\mu\nu}$ generated by impressed (external) currents J^{ν} , following mostly [179, 6.4, 6.5, 12.11] and [332, 2]. Namely, the EM field $F^{\mu\nu}$ satisfies the inhomogeneous Maxwell equation (4.23). If the EM potential A^{μ} satisfies the Lorentz gauge condition (4.20), it is a solution of the wave equation (4.28). To write down the solution of the inhomogeneous wave equation (4.28), one has to find a Green function $G(x, x')$ for the equation

$$\square G(z) = \delta^{(4)}(z), \quad G(x, x') = G(x - x'), \quad (4.32)$$

where $\delta^{(4)}(z) = \delta(z_0) \delta(\mathbf{z})$ is a four-dimensional delta function. One can introduce then the so-called *retarded or causal Green function* solving the above Eq. (4.32), namely

$$G^{(+)}(x - x') = \frac{\theta(x_0 - x'_0) \delta(x_0 - x'_0 - R)}{4\pi R}, \quad R = |\mathbf{x} - \mathbf{x}'|, \quad (4.33)$$

where $\theta(x_0)$ is the Heaviside step function. The name causal or retarded is justified by the fact that the source-point time x'_0 is always earlier than the observation-point time x_0 . Similarly, one can introduce the *advanced Green function*

$$G^{(-)}(x - x') = \frac{\theta[-(x_0 - x'_0)] \delta(x_0 - x'_0 + R)}{4\pi R}, \quad R = |\mathbf{x} - \mathbf{x}'|. \quad (4.34)$$

These Green functions can be written in the following covariant form:

$$\begin{aligned} G^{(+)}(x - x') &= \frac{1}{2\pi} \theta(x_0 - x'_0) \delta[(x - x')^2], \\ G^{(-)}(x - x') &= \frac{1}{2\pi} \theta(x'_0 - x_0) \delta[(x - x')^2], \end{aligned} \quad (4.35)$$

where

$$(x - x')^2 = (x_0 - x'_0)^2 - |\mathbf{x} - \mathbf{x}'|^2,$$

$$\delta[(x - x')^2] = \frac{1}{2R} [\delta(x_0 - x'_0 - R) + \delta(x_0 - x'_0 + R)]. \quad (4.36)$$

The more explicit form of the Green functions $G^{(\pm)}$ in terms of space-time variables is

$$G^{(\pm)}(\tau, R) = \frac{1}{R} \delta\left(\tau \mp \frac{R}{c}\right) \quad (4.37)$$

where

$$R = |\mathbf{x} - \mathbf{x}'|, \quad \tau = t - t', \quad (4.38)$$

or

$$G^{(\pm)}(t, \mathbf{x}; t', \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} \delta\left(t - \left[t' \mp \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right]\right). \quad (4.39)$$

The solution to the wave equation (4.28) can be written in terms of the Green functions as follows:

$$A^\nu(x) = A_{\text{in}}^\nu(x) + \frac{4\pi}{c} \int G^{(+)}(x - x') J^\nu(x') dx', \quad (4.40)$$

or

$$A^\nu(x) = A_{\text{out}}^\nu(x) + \frac{4\pi}{c} \int G^{(-)}(x - x') J^\nu(x') dx', \quad (4.41)$$

where $A_{\text{in}}^\nu(x)$ and $A_{\text{out}}^\nu(x)$ are solutions to the homogeneous wave equation. In (4.40), the retarded Green function is used. If the sources are localized in space and time, the integral involving the sources vanishes in the limit $x_0 \rightarrow -\infty$ because of the retarded nature of the Green function, and $A_{\text{in}}^\nu(x)$ can be interpreted as an “incident” or “incoming” potential specified at $x_0 \rightarrow -\infty$. Similarly, in (4.41) with the advanced Green function, the homogeneous solution $A_{\text{out}}^\nu(x)$ is the asymptotic “outgoing” potential specified at $x_0 \rightarrow +\infty$.

A more explicit form based on the retarded Green function $G^{(+)}$ of the potential $A^\nu(x)$ solving the inhomogeneous wave equation (4.28) with $A_{\text{in}}^\nu(x) = 0$ is

$$\varphi(t, \mathbf{x}) = \int \frac{\rho(t_r, \mathbf{x}')}{R} d\mathbf{x}', \quad \mathbf{A}(t, \mathbf{x}) = \frac{1}{c} \int \frac{\mathbf{J}(t_r, \mathbf{x}')}{R} d\mathbf{x}', \quad (4.42)$$

where

$$\mathbf{R} = \mathbf{x} - \mathbf{x}', \quad R = |\mathbf{x} - \mathbf{x}'|, \quad (4.43)$$

and t_r is the so-called *retarded time*:

$$t_r = t - \frac{R}{c} = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}. \quad (4.44)$$

Note that in many cases, where $\rho(t_r, \mathbf{x}')$ slowly varies in time, and R is not large, the retardation $\frac{R}{c}$ can be neglected. Then $\rho(t_r, \mathbf{x}') \approx \rho(t, \mathbf{x}')$, $\mathbf{J}(t_r, \mathbf{x}') \approx \mathbf{J}(t, \mathbf{x}')$ and formula (4.42) turns into

$$\varphi(t, \mathbf{x}) = \int \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}', \quad \mathbf{A}(t, \mathbf{x}) = \frac{1}{c} \int \frac{\mathbf{J}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'. \quad (4.45)$$

Such a representation is commonly used in electroquasistatics and magnetoquasistatics, [146, 10.2.2]. The formulas (4.45) provide exact solutions not to the wave equations (4.28) but to the Poisson equations, that is,

$$\nabla^2 \varphi = -4\pi\rho, \quad \nabla^2 \mathbf{A} = -\frac{4\pi}{c} \mathbf{J}. \quad (4.46)$$

We provide a rather detailed analysis of quasistatic non-relativistic approximation of the Maxwell equations in Chap. 8.

The EM fields defined by (4.6) in terms of the potentials (4.42) can be represented by the *Jefimenko formulas*, [187, 15.7], [179, 6.5], [146, 10.2.2]

$$\begin{aligned} \mathbf{E}(t, \mathbf{x}) = & \int \frac{\rho(t_r, \mathbf{x}')}{R^2} d\mathbf{x}' + \int \frac{\partial_t \rho(t_r, \mathbf{x}')}{cR} \hat{\mathbf{R}} d\mathbf{x}' \\ & - \int \frac{\partial_{tt} \mathbf{J}(t_r, \mathbf{x}')}{c^2 R} d\mathbf{x}', \end{aligned} \quad (4.47)$$

$$\mathbf{B}(t, \mathbf{x}) = \int \left\{ \frac{\mathbf{J}(t_r, \mathbf{x}')}{cR^2} + \frac{\partial_t \mathbf{J}(t_r, \mathbf{x}')}{c^2 R} \right\} \times \hat{\mathbf{R}} d\mathbf{x}', \quad (4.48)$$

where $\hat{\mathbf{R}} = \frac{\mathbf{R}}{|\mathbf{R}|}$. An essentially equivalent form of the Jefimenko equation (4.47) for the electric field $\mathbf{E}$ is due to Panofsky and Phillips, [263, 14.3]:

$$\begin{aligned} \mathbf{E}(t, \mathbf{x}) = & \int \frac{\rho(t_r, \mathbf{x}')}{R^2} d\mathbf{x}' + \\ & + \int \frac{(\mathbf{J}(t_r, \mathbf{x}') \cdot \hat{\mathbf{R}}) \hat{\mathbf{R}} + (\mathbf{J}(t_r, \mathbf{x}') \times \hat{\mathbf{R}}) \times \hat{\mathbf{R}}}{cR^2} d\mathbf{x}' + \\ & + \int \frac{(\partial_{tt} \mathbf{J}(t_r, \mathbf{x}') \times \hat{\mathbf{R}}) \times \hat{\mathbf{R}}}{c^2 R} d\mathbf{x}'. \end{aligned} \quad (4.49)$$

It was pointed out by McDonald [241] that the combination of Eqs. (4.48) and (4.49) has a certain advantage, since it “manifestly displays the mutually transverse character of the radiation fields (those that vary as $1/R$)”.

The radiation fields are defined as follows, [85],

$$A_{\text{rad}}^\nu(x) = \frac{4\pi}{c} \int G(x - x') J^\nu(x') dx', \quad (4.50)$$

where

$$G(x - x') = G^{(+)}(x - x') - G^{(-)}(x - x'),$$

and $G^{(+)}$ is the retarded and $G^{(-)}$ is the advanced Green function (4.39). The radiation field defined by the above Dirac's formula relates to the part of the field generated by an accelerating charge which affects the motion of the accelerating charge, [85]. The action of the EM fields on other charges is usually described in terms of the retarded Green functions as in (4.42)–(4.49); sometimes the half-sum of retarded and advanced Green functions is used, [339]. The reader can find a detailed discussion of related questions in [85], [339], [340].

If the sources are localized, the leading terms at large distances from the source decay as $1/R$ for large R . Then the corresponding *radiation fields* $\mathbf{E}_{\text{rad}}$ and $\mathbf{B}_{\text{rad}}$ can be extracted from the expressions (4.49), (4.48) for the entire EM fields as follows:

$$\mathbf{E}_{\text{rad}}(t, \mathbf{x}) = \int \frac{(\partial_{t_r} \mathbf{J}(t_r, \mathbf{x}') \times \hat{\mathbf{R}}) \times \hat{\mathbf{R}}}{c^2 R} d\mathbf{x}', \quad (4.51)$$

$$\mathbf{B}_{\text{rad}}(t, \mathbf{x}) = \int \partial_{t_r} \mathbf{J}(t_r, \mathbf{x}') \times \frac{\hat{\mathbf{R}}}{c^2 R} d\mathbf{x}'. \quad (4.52)$$

4.2.1 Point Charges and the Liénard–Wiechert Potential

If the particle is a point charge with the charge value q , whose position and velocity in an inertial frame are respectively $\mathbf{r}(t)$ and $\mathbf{v}(t) = \partial_t \mathbf{r}(t)$, the corresponding charge and current densities in that frame are, [179, Section 12.11, (12.138)]

$$\rho(t, \mathbf{x}) = q \delta(\mathbf{x} - \mathbf{r}(t)) \quad \text{and} \quad \mathbf{J}(t, \mathbf{x}) = q \mathbf{v}(t) \delta(\mathbf{x} - \mathbf{r}(t)). \quad (4.53)$$

Using the formulas (4.42) for the charge density and the current as in the Eqs. (4.53), we obtain the Liénard–Wiechert Potential, [179, Section 14.1]:

$$\varphi(t, \mathbf{x}) = \left[\frac{q}{(1 - \beta \cdot \hat{\mathbf{R}})} \right]_{\text{ret}}, \quad \mathbf{A}(t, \mathbf{x}) = \left[\frac{q\beta}{(1 - \beta \cdot \hat{\mathbf{R}})} \right]_{\text{ret}}, \quad (4.54)$$

where: (i)

$$\mathbf{R} = \mathbf{x} - \mathbf{r}(t), \quad R = |\mathbf{x} - \mathbf{r}(t)|, \quad \hat{\mathbf{R}} = \frac{\mathbf{R}}{R}, \quad \beta = \frac{\mathbf{v}(t)}{c}, \quad (4.55)$$

(ii) $[(\cdot)]_{\text{ret}}$ means that the expression $(\cdot)$ is evaluated at the retarded time t_r , and (iii) the retarded time $t_r = t_r(\mathbf{x}, t)$ is defined implicitly by the following equation

$$t_r = t - \frac{|\mathbf{x} - \mathbf{r}(t_r)|}{c}. \quad (4.56)$$

Then with the help of the Jefimenko formulas (4.47), (4.48) applied to the point charge density and the current (4.53), one can derive the Heaviside–Feynman formulas (first discovered by Heaviside (1902) and rediscovered by Feynman (1950)), [153], [154, Subsection 510], [125, Vol. I, Section 28; Vol II, Section 21], [179, Section 6.5], [184], [249]:

$$\mathbf{E}(t, \mathbf{x}) = q \left\{ \left[\frac{\hat{\mathbf{R}}}{R^2} \right]_{\text{ret}} + \frac{[R]_{\text{ret}}}{c} \partial_t \left[\frac{\hat{\mathbf{R}}}{R^2} \right]_{\text{ret}} + \frac{1}{c^2} \partial_t^2 [\hat{\mathbf{R}}]_{\text{ret}} \right\}, \quad (4.57)$$

$$\begin{aligned} \mathbf{B}(t, \mathbf{x}) &= \frac{q}{c} \left\{ \left[\frac{\mathbf{v} \times \hat{\mathbf{R}}}{\varkappa^2 R^2} \right]_{\text{ret}} + \frac{1}{c [R]_{\text{ret}}} \partial_t \left[\frac{\mathbf{v} \times \hat{\mathbf{R}}}{\varkappa} \right]_{\text{ret}} \right\} \\ &= \frac{q}{c} \left\{ \left[\frac{\hat{\mathbf{R}}}{R} \right]_{\text{ret}} \times \partial_t [\hat{\mathbf{R}}]_{\text{ret}} + \frac{[\hat{\mathbf{R}}]_{\text{ret}}}{c} \times \partial_t^2 [\hat{\mathbf{R}}]_{\text{ret}} \right\} \\ &= \mathbf{E}(t, \mathbf{x}) \times [\hat{\mathbf{R}}]_{\text{ret}}, \end{aligned} \quad (4.58)$$

where

$$\varkappa = 1 - \frac{\mathbf{v} \cdot \hat{\mathbf{R}}}{c}. \quad (4.59)$$

In view of the implicit relations (4.56) between the retarded time t_r and the space-time variables $(\mathbf{x}, t)$, it is important to keep in mind that there is evidently a difference between $\partial_t [(\cdot)]_{\text{ret}}$ and $[\partial_t (\cdot)]_{\text{ret}}$.

An alternative way to represent the EM fields $\mathbf{E}$ and $\mathbf{B}$ is as follows, [332, 2.4.2]:

$$\begin{aligned} \mathbf{E}(t, \mathbf{x}) &= q \left[\frac{(\hat{\mathbf{R}} - \boldsymbol{\beta})(1 - \beta^2)}{R^2 (1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^3} \right]_{\text{ret}} \\ &\quad + \frac{q}{c} \left[\frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \partial_t \boldsymbol{\beta}]}{R (1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^3} \right]_{\text{ret}}, \end{aligned} \quad (4.60)$$

$$\mathbf{B}(t, \mathbf{x}) = \hat{\mathbf{R}} \times \mathbf{E}(t, \mathbf{x}). \quad (4.61)$$

The total radiated power for the point charge is then, [332, 2.4.2]:

$$P = \frac{2q^2}{3c} \gamma^6 [(\partial_t \beta)^2 - (\beta \times \partial_t \beta)^2], \quad \beta = \frac{\mathbf{v}(t)}{c}, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}. \quad (4.62)$$

4.2.2 Radiation Fields and Radiated Energy

Since the radiation fields decay as $1/R$ for large R , we can extract from the Jefimenko formulas (4.47), (4.48) the corresponding terms and obtain the following formulas for the radiation fields:

$$\mathbf{E}_{\text{rad}}(t, \mathbf{x}) = \int \frac{1}{c^2 R} \left(\partial_{t_r} \mathbf{J}(t_r, \mathbf{x}') \times \hat{\mathbf{R}} \right) \times \hat{\mathbf{R}} d\mathbf{x}' \quad (4.63)$$

$$\mathbf{B}_{\text{rad}}(t, \mathbf{x}) = \int \frac{1}{c^2 R} \partial_{t_r} \mathbf{J}(t_r, \mathbf{x}') \times \hat{\mathbf{R}} d\mathbf{x}', \quad t_r = t - \frac{|\mathbf{x} - \mathbf{r}(t_r)|}{c}. \quad (4.64)$$

By the same argument, the Heaviside–Feynman formulas (4.57), (4.58) imply the following formulas for the radiation fields of the moving point charge:

$$\mathbf{E}_{\text{rad}}(t, \mathbf{x}) = \frac{q}{c^2} \partial_t^2 \left[\hat{\mathbf{R}} \right]_{\text{ret}}, \quad \mathbf{B}_{\text{rad}}(t, \mathbf{x}) = \frac{q}{c^2 [R]_{\text{ret}}} \partial_t \left[\frac{\mathbf{v} \times \hat{\mathbf{R}}}{\varkappa} \right]_{\text{ret}}. \quad (4.65)$$

Another important representation of the EM fields of a moving charge is their decomposition into the *velocity and acceleration fields*, [180, Section 14.1]. Namely, for the electric field,

$$\mathbf{E}(t, \mathbf{x}) = \mathbf{E}_v(t, \mathbf{x}) + \mathbf{E}_a(t, \mathbf{x}), \quad (4.66)$$

where

$$\mathbf{E}_v(t, \mathbf{x}) = q \left[\frac{(\hat{\mathbf{R}} - \beta)(1 - \beta^2)}{\varkappa^3 R^2} \right]_{\text{ret}}$$

with

$$\beta = \frac{\mathbf{v}}{c}, \quad \varkappa = 1 - \beta \cdot \hat{\mathbf{R}},$$

$$\mathbf{E}_a(t, \mathbf{x}) = \mathbf{E}_{\text{rad}}(t, \mathbf{x}) = \frac{q}{c} \left[\frac{\hat{\mathbf{R}}}{\varkappa^3 R} \times \left\{ (\hat{\mathbf{R}} - \beta) \times \dot{\beta} \right\} \right]_{\text{ret}}.$$

For the magnetic field,

$$\begin{aligned}\mathbf{B}(t, \mathbf{x}) &= \hat{\mathbf{R}} \times \mathbf{E}(t, \mathbf{x}) = \mathbf{B}_v(t, \mathbf{x}) + \mathbf{B}_a(t, \mathbf{x}), \\ \mathbf{B}_v(t, \mathbf{x}) &= \hat{\mathbf{R}} \times \mathbf{E}_v(t, \mathbf{x}), \\ \mathbf{B}_a(t, \mathbf{x}) &= \mathbf{B}_{\text{rad}}(t, \mathbf{x}) = \hat{\mathbf{R}} \times \mathbf{E}_a(t, \mathbf{x}).\end{aligned}\tag{4.67}$$

The *velocity fields* are *essentially static fields* falling off as R^{-2} , and the *acceleration fields* are *typical radiation fields*, both $\mathbf{E}_a$ and $\mathbf{B}_a$ being transverse to the radius vector $\mathbf{R}$ and decaying as R^{-1} . The formulas (4.66), (4.67) for low velocities turn into the following asymptotic expressions:

$$\begin{aligned}\mathbf{E}_{\text{rad}}(t, \mathbf{x}) &= \frac{q}{c} \left[\frac{\hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \dot{\mathbf{v}})}{|\mathbf{R}|} \right]_{\text{ret}} (1 + O(|\beta|)), \\ \mathbf{B}_{\text{rad}}(t, \mathbf{x}) &= \frac{q}{c} \left[\frac{\dot{\mathbf{v}} \times \hat{\mathbf{R}}}{|\mathbf{R}|} \right]_{\text{ret}} (1 + O(|\beta|)), \quad |\beta| \ll 1.\end{aligned}\tag{4.68}$$

Chapter 5

Dipole Approximation for Localized Distributed Charges

We describe here electromagnetic fields generated by localized distributed charges $\rho(t, \mathbf{x})$ and currents $\mathbf{J}(t, \mathbf{x})$ satisfying the continuity equation (4.24). The EM fields are given by (4.42):

$$\begin{aligned}\varphi(t, \mathbf{X}) &= \frac{1}{4\pi} \int \frac{\rho(t - |\mathbf{X} - \mathbf{x}|/c, \mathbf{x})}{|\mathbf{X} - \mathbf{x}|} d\mathbf{x}, \\ \mathbf{A}(t, \mathbf{X}) &= \frac{1}{4\pi c} \int \frac{\mathbf{J}(t - |\mathbf{X} - \mathbf{x}|/c, \mathbf{x})}{|\mathbf{X} - \mathbf{x}|} d\mathbf{x}.\end{aligned}\tag{5.1}$$

Below we study the fields generated by localized sources based on multipole expansions and the corresponding moments.

5.1 Dipole Fields

Assume that charges $\rho(t, \mathbf{x})$ and currents $\mathbf{J}(t, \mathbf{x})$ are localized about the origin, namely $x = |\mathbf{x}|$ is small, and $\mathbf{X}$ is a general observation position. Then, under the localization condition $x = |\mathbf{x}| \ll |\mathbf{X}| = X$, the following expansion is instrumental to the multipole theory, [281, 1.1]:

$$\begin{aligned}\frac{1}{|\mathbf{X} - \mathbf{x}|} &= \frac{1}{(X^2 - 2\mathbf{X} \cdot \mathbf{x} + x^2)^{1/2}} = \frac{1}{X} \left[1 + \frac{x^2 - 2\mathbf{x} \cdot \mathbf{X}}{X} \right]^{-1/2} \\ &= \frac{1}{X} + \frac{\mathbf{X} \cdot \mathbf{x}}{X^3} + \frac{3(\mathbf{X} \cdot \mathbf{x})^2 - X^2 x^2}{2X^5} + \frac{5(\mathbf{X} \cdot \mathbf{x})^2 - 3X^2(\mathbf{X} \cdot \mathbf{x})^2 x^2}{2X^7} + \dots\end{aligned}\tag{5.2}$$

We introduce the total charge

$$q = \int \rho(\mathbf{x}) d\mathbf{x} \quad (5.3)$$

and the electric and magnetic dipole moments respectively by formulas

$$\mathbf{p}(t) = \int \mathbf{x} \rho(t, \mathbf{x}) d\mathbf{x}, \quad \mathbf{m}(t) = \frac{1}{2c} \int \mathbf{x} \times \mathbf{J}(t, \mathbf{x}) d\mathbf{x}. \quad (5.4)$$

Then the following expansions hold for the potentials, [245, 1.5.1], [281, 1.1]:

$$\varphi(t, \mathbf{X}) = \frac{1}{4\pi} \left[\frac{q}{X} + \nabla \cdot \frac{\mathbf{p}(t - X/c)}{X} + \dots \right], \quad (5.5)$$

$$\mathbf{A}(t, \mathbf{X}) = \frac{1}{4\pi} \left[\frac{1}{c} \frac{\partial_t \mathbf{p}(t - X/c)}{X} + \nabla \times \frac{\mathbf{m}(t - X/c)}{X} + \dots \right]. \quad (5.6)$$

For stationary charge distributions where $\rho(t, \mathbf{x}) = \rho(\mathbf{x})$ and $\mathbf{A}(t, \mathbf{X}) = \mathbf{A}(\mathbf{X})$, the expansions (5.5), (5.6) are time-independent:

$$\varphi(t, \mathbf{X}) = \varphi(\mathbf{X}) = \frac{1}{4\pi} \left[\frac{q}{X} + \frac{\mathbf{p} \cdot \mathbf{X}}{X^3} + \dots \right], \quad (5.7)$$

$$\mathbf{A}(t, \mathbf{X}) = \mathbf{A}(\mathbf{X}) = \frac{1}{4\pi} \left[\frac{\mathbf{m} \times \mathbf{X}}{X^3} + \dots \right]. \quad (5.8)$$

Effective point charge and current distributions associated with the dipole moments $\mathbf{p}(t)$ and $\mathbf{m}(t)$ are of the form

$$\begin{aligned} \rho_{\text{eff}}(t, \mathbf{X}) &= -\nabla \cdot [\mathbf{p}(t) \delta(\mathbf{X})], \\ \mathbf{J}_{\text{eff}}(t, \mathbf{X}) &= \partial_t [\mathbf{p}(t) \delta(\mathbf{X})] + c \nabla \times [\mathbf{m}(t) \delta(\mathbf{X})], \end{aligned} \quad (5.9)$$

where $\delta(\mathbf{X})$ is Dirac's delta function. Notice that higher order multipole moments and effective charge and currents distributions involve higher order derivatives of Dirac's delta function $\delta(\mathbf{X})$, [245, 2.3], [325, 2.9-10].

The *Hertz vectors* Π_e and Π_m and the corresponding potentials are as follows:

$$\Pi_e(t, \mathbf{X}) = \frac{\mathbf{p}(t - X/c)}{4\pi X}, \quad \Pi_m(t, \mathbf{X}) = \frac{\mathbf{m}(t - X/c)}{4\pi X}, \quad (5.10)$$

$$\varphi_{\text{eff}}(t, \mathbf{X}) = -\nabla \cdot \Pi_e(t, \mathbf{X}), \quad (5.11)$$

$$\mathbf{A}_{\text{eff}}(t, \mathbf{X}) = \frac{1}{c} \partial_t \Pi_e(t, \mathbf{X}) + \nabla \times \Pi_m(t, \mathbf{X}).$$

Using the identity $-\nabla^2 1/r = 4\pi \delta(\mathbf{r})$, one can verify that the Hertz vectors satisfy the following wave equations with the effective point sources

$$\begin{aligned} \left(\nabla^2 - \frac{1}{c^2} \partial_t^2 \right) \Pi_e(t, \mathbf{X}) &= -\mathbf{p}(t) \delta(\mathbf{X}), \\ \left(\nabla^2 - \frac{1}{c^2} \partial_t^2 \right) \Pi_m(t, \mathbf{X}) &= -\mathbf{m}(t) \delta(\mathbf{X}). \end{aligned} \quad (5.12)$$

5.2 Dipole Elementary Currents

Based on (5.9), we can define an ideal electric dipole source concentrated at a point $\mathbf{x}_0$ with the charge and current densities $\mathbf{J}_d$ and ρ_d as follows, [325, Section 7.10, (7.151), Appendix 8]:

$$\begin{aligned} \mathbf{J}_d(t, \mathbf{x}) &= \partial_t [\mathbf{p}(t) \delta(\mathbf{x} - \mathbf{x}_0)] = \dot{\mathbf{p}}(t) \delta(\mathbf{x} - \mathbf{x}_0), \\ \rho_d(t, \mathbf{x}) &= -\nabla \cdot [\mathbf{p}(t) \delta(\mathbf{x} - \mathbf{x}_0)] = -\mathbf{p}(t) \cdot \nabla \delta(\mathbf{x} - \mathbf{x}_0) \end{aligned} \quad (5.13)$$

where $\delta(\mathbf{x} - \mathbf{x}_0)$ is Dirac's delta-function, and $\mathbf{p}(t)$ and $\partial_t \mathbf{p}(t)$ satisfy

$$p_j(t) = \int x'_j \rho(t, \mathbf{x}') d\mathbf{x}', \quad j = 1, 2, 3, \quad \mathbf{p} = (p_1, p_2, p_3), \quad (5.14)$$

$$\dot{\mathbf{p}}(t) = \partial_t \mathbf{p}(t) = \int \mathbf{J}(t, \mathbf{x}') d\mathbf{x}'. \quad (5.15)$$

It readily follows from (5.13), (4.42) that the potentials of the ideal electric dipole are given by the following expressions:

$$\varphi(t, \mathbf{x}) = 0, \quad \mathbf{A}(t, \mathbf{x}) = \frac{\dot{\mathbf{p}}(t_0)}{c |\mathbf{R}|}, \quad (5.16)$$

where

$$\mathbf{R} = \mathbf{x} - \mathbf{x}_0, \quad \hat{\mathbf{R}} = \frac{\mathbf{R}}{|\mathbf{R}|}, \quad t_0 = t - \frac{|\mathbf{x} - \mathbf{x}_0|}{c}. \quad (5.17)$$

Then, applying the Jefimenko and the Panofsky–Phillips formulas (4.48) and (4.49) for the ideal electric dipole sources $\rho(t, \mathbf{x})$ and $\mathbf{J}(t, \mathbf{x})$ defined by formulas (5.13), we obtain the following formulas for the EM field:

$$\mathbf{E}(t, \mathbf{x}) = \frac{3(\hat{\mathbf{R}} \cdot \mathbf{p}(t_0))\hat{\mathbf{R}} - \mathbf{p}(t_0)}{|\mathbf{R}|^3} + \frac{3(\dot{\mathbf{p}}(t_0) \cdot \hat{\mathbf{R}})\hat{\mathbf{R}} - \dot{\mathbf{p}}(t_0)}{c|\mathbf{R}|^2} + \frac{(\ddot{\mathbf{p}}(t_0) \times \hat{\mathbf{R}}) \times \hat{\mathbf{R}}}{c^2|\mathbf{R}|}, \quad (5.18)$$

$$\mathbf{B}(t, \mathbf{x}) = \left[\frac{\dot{\mathbf{p}}(t_0)}{cR^2} + \frac{\ddot{\mathbf{p}}(t_0)}{c^2R} \right] \times \hat{\mathbf{R}} \text{ where } \ddot{\mathbf{p}} = \partial_t^2 \mathbf{p}. \quad (5.19)$$

When we derived formula (5.18), the vector identity (43.14) was used.

In a simpler case, when the dipole function $\mathbf{p}(t)$ is time-harmonic and complex-valued of the form

$$\mathbf{p}(t) = \mathbf{p}_\omega e^{-i\omega t}, \quad \mathbf{p}(t_0) = \mathbf{p}_\omega e^{ik|\mathbf{R}|-i\omega t}, \quad \text{where } k = \frac{\omega}{c}, \quad (5.20)$$

(here the real part has to be taken to obtain physical quantities) the general formulas (5.18), (5.19) yield the well known formulas for the ideal electric dipole fields, [180, Section 9.2]:

$$\mathbf{E}(t, \mathbf{x}) = -\frac{k^2 e^{ik|\mathbf{R}|-i\omega t}}{|\mathbf{R}|} (\mathbf{p} \times \hat{\mathbf{R}}) \times \hat{\mathbf{R}} + \frac{k^2 e^{ik|\mathbf{R}|-i\omega t}}{|\mathbf{R}|} \left[3(\hat{\mathbf{R}} \cdot \mathbf{p})\hat{\mathbf{R}} - \mathbf{p} \right] \left(\frac{1}{k^2|\mathbf{R}|^2} - \frac{i}{k|\mathbf{R}|} \right), \quad (5.21)$$

$$\mathbf{B}(t, \mathbf{x}) = -k^2 e^{ik|\mathbf{R}|-i\omega t} \frac{\mathbf{p} \times \hat{\mathbf{R}}}{|\mathbf{R}|} \left(1 + \frac{i}{k|\mathbf{R}|} \right), \quad (5.22)$$

implying in the radiation zone $k|\mathbf{R}| \gg 1$ the following asymptotic formulas:

$$\mathbf{E}(t, \mathbf{x}) = k^2 e^{ik|\mathbf{R}|-i\omega t} \frac{\hat{\mathbf{R}} \times (\mathbf{p}_\omega \times \hat{\mathbf{R}})}{|\mathbf{R}|} \left[1 + O\left(\frac{1}{k|\mathbf{R}|}\right) \right], \quad (5.23)$$

$$\mathbf{B}(t, \mathbf{x}) = k^2 e^{ik|\mathbf{R}|-i\omega t} \frac{\hat{\mathbf{R}} \times \mathbf{p}_\omega}{|\mathbf{R}|} \left[1 + O\left(\frac{1}{k|\mathbf{R}|}\right) \right]. \quad (5.24)$$

For the case of a multi-frequency dipole function $\mathbf{p}(t)$ with the frequency spectrum Λ_p , we can introduce

$$k_{\min} = \min_{\omega \in \Lambda_p} \left\{ \frac{\omega}{c} \right\} > 0. \quad (5.25)$$

Then radiation components decaying as $|\mathbf{R}|^{-1}$ dominate for $k_{\min}|\mathbf{R}| \gg 1$ in formulas (5.18), (5.19) and imply the following asymptotic expressions for the radiation fields,

see [146, Section 11.1.4]:

$$\begin{aligned}\mathbf{E}(t, \mathbf{x}) &= \frac{(\ddot{\mathbf{p}}(t_0) \times \hat{\mathbf{R}}) \times \hat{\mathbf{R}}}{c^2 |\mathbf{R}|} \left[1 + O\left(\frac{1}{k_{\min} |\mathbf{R}|}\right) \right], \\ \mathbf{B}(t, \mathbf{x}) &= \frac{\ddot{\mathbf{p}}(t_0) \times \hat{\mathbf{R}}}{c^2 |\mathbf{R}|} \left[1 + O\left(\frac{1}{k_{\min} |\mathbf{R}|}\right) \right], \text{ for } k_{\min} |\mathbf{R}| \gg 1.\end{aligned}\tag{5.26}$$

Chapter 6

The Minkowski Four-Dimensional Spacetime and Relativistic Kinematics

“The special theory of relativity owes its origin to Maxwell’s equations of the electromagnetic field.”
A. Einstein.¹

6.1 The Minkowski Four-Dimensional Spacetime

Here we provide basic facts and the notation related to the Minkowski four-dimensional geometry and relativistic kinematics following [27, 1], [137, 7], [179, 11.3], [213, 1.1-1.4, 2], [277]. There are a few popular conventions in setting up coordinates and the metric in the Minkowski four-dimensional space-time. We pick the one which seems to be dominant nowadays as in [27, 1], [213, 1.1-1.4,2], [178, 1-1-1].

The space-time four-vector in its contravariant x^μ and covariant x_μ forms is represented as follows:

$$x = x^\mu = (x^0, x^1, x^2, x^3) = (ct, \mathbf{x}), \quad (6.1)$$

$$x_\mu = g_{\mu\nu}x^\nu = (x^0, -x^1, -x^2, -x^3) = (x_0, x_1, x_2, x_3), \quad (6.2)$$

$$\mu = 0, 1, 2, 3,$$

with the common convention on the summation of the repeated indices. The metric tensor $g_{\mu\nu} = g^{\mu\nu}$ is given by the 4×4 matrix

¹The Einstein quotation is from his Autobiographical notes [295, Autobiographical notes].

$$\{g_{\mu\nu}\} = \{g^{\mu\nu}\} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (6.3)$$

We also use another common notation for the space 3-vector

$$x^i = (x^1, x^2, x^3) = \mathbf{x}, \quad i = 1, 2, 3. \quad (6.4)$$

Notice also that, [179, 11.6], [299, 6.1],

$$g_{\nu}^{\mu} = g^{\mu\sigma} g_{\sigma\nu} = \delta_{\nu}^{\mu} \quad (6.5)$$

where δ_{ν}^{μ} is Kronecker's delta,

$$\delta_{\nu}^{\mu} = \begin{cases} 1 & \text{if } \mu = \nu, \\ 0 & \text{if } \mu \neq \nu \end{cases}.$$

It can also be written in the form of the identity matrix. The covariant derivatives ∂_{μ} and ∂^{μ} defined by (4.15) satisfy the relation

$$\partial^{\mu} = \frac{\partial}{\partial x_{\mu}} = g_{\mu\nu} \partial_{\nu} = g_{\mu\nu} \frac{\partial}{\partial x^{\nu}}. \quad (6.6)$$

6.2 The Lorentz Transformation

The elementary Lorentz transformation to a frame moving with velocity $\mathbf{v}$ is determined by the following:

$$\begin{aligned} x^{0'} &= \gamma (x^0 - \boldsymbol{\beta} \cdot \mathbf{x}), \quad \mathbf{x}' = \mathbf{x} + \frac{\gamma - 1}{\beta^2} (\boldsymbol{\beta} \cdot \mathbf{x}) \boldsymbol{\beta} - \gamma \boldsymbol{\beta} x^0, \\ \boldsymbol{\beta} &= \frac{\mathbf{v}}{c}, \quad \beta = |\boldsymbol{\beta}|, \quad \gamma = \frac{1}{\sqrt{1 - (\frac{v}{c})^2}}. \end{aligned} \quad (6.7)$$

If we introduce for a space vector $\mathbf{x}$ its components $\mathbf{x}_{\parallel}$ and $\mathbf{x}_{\perp}$ that are, respectively, parallel and perpendicular to the velocity $\mathbf{v}$, i.e. $\mathbf{x} = \mathbf{x}_{\parallel} + \mathbf{x}_{\perp}$, then (6.7) can be recast as

$$x^{0'} = \gamma (x^0 - \boldsymbol{\beta} \cdot \mathbf{x}), \quad \mathbf{x}'_{\parallel} = \gamma (\mathbf{x}_{\parallel} - \boldsymbol{\beta} x^0), \quad \mathbf{x}'_{\perp} = \mathbf{x}_{\perp}. \quad (6.8)$$

The above formula in the case when $\mathbf{v}$ is parallel to the axis x^1 turns into

$$x^{0'} = \gamma (x^0 - \beta x^1), \quad x^{1'} = \gamma (x^1 - \beta x^0), \quad x^{2'} = x^2, \quad x^{3'} = x^3. \quad (6.9)$$

The Lorentz invariance of a 4-vector x under the above transformation reduces to the equation

$$(x^0)^2 - |\mathbf{x}'|^2 = (x^0)^2 - |\mathbf{x}|^2. \quad (6.10)$$

The general *infinitesimal* form of the *inhomogeneous* Lorentz transformations which form the *Poincaré group* is, [299, 6.1-6.2], [248, 6.1],

$$x'^\mu = x^\mu + \xi^{\mu\nu} x_\nu + a^\mu, \quad \xi^{\mu\nu} = -\xi^{\nu\mu}, \quad (6.11)$$

with ten parameters that include six parameters $\xi^{\mu\nu}$ which parametrize generators of the 4-rotations and four parameters a^μ which parametrize generators of the translations.

A *proper, orthochronous Lorentz transformation*, also called a *Lorentz rotation*, has the form, [299, 6.2]

$$x'^\mu = \Lambda^\mu_\nu x^\nu. \quad (6.12)$$

The rotation Λ has the form

$$\Lambda = e^{\mathcal{E}}, \quad (6.13)$$

where $\mathcal{E}$ is an anti-symmetric 4×4 matrix

$$\mathcal{E} = -\mathcal{E}^T \text{ with entries } \mathcal{E}^{\mu\nu} = \xi^{\mu\nu}, \quad (6.14)$$

which relates to the infinitesimal form of the rotation, [248, 6.1], [299, 6.1-6.2]:

$$x'^\mu = x^\mu + \xi^{\mu\nu} x_\nu, \quad \xi^{\mu\nu} = -\xi^{\nu\mu}; \quad (6.15)$$

the matrix involves six independent parameters $\xi^{\mu\nu}$ (*rotation angles* in the Minkowski space). The Lorentz rotations form a special group often called the *restricted Lorentz group*.

Notice that, [331, A.1],

$$[\Lambda^{-1}]^\mu_\nu = g_{\mu\alpha} \Lambda^\alpha_\beta g^{\beta\nu} = \Lambda^\mu_\nu, \quad \Lambda^\mu_\alpha \Lambda^\alpha_\nu = \delta^\mu_\nu, \quad (6.16)$$

implying the following transformational behavior of the differentiation operators

$$\partial_\mu = \Lambda^\nu_\mu \partial'_\nu, \quad \partial'_\mu = [\Lambda^{-1}]^\nu_\mu \partial_\nu = \Lambda^\nu_\mu \partial_\nu. \quad (6.17)$$

The infinitesimal Lorentz transformations are often represented in terms of generators relevant to the Lie theory, [143, 16.2], [27, I.5E], namely

$$x'^\mu = \left(\delta^\mu_\nu + \frac{1}{2} (I^{(\alpha\beta)})^\mu_\nu \xi_{\alpha\beta} \right) x^\nu, \quad (6.18)$$

where *generators* $I^{(\alpha\beta)}$ are 4×4 matrices with the following entries

$$(I^{(\alpha\beta)})_{\mu\nu} = -(I^{(\alpha\beta)})_{\nu\mu} = g_\mu^\alpha g_\nu^\beta - g_\nu^\alpha g_\mu^\beta. \quad (6.19)$$

Notice that every generator $I^{(\alpha\beta)}$ is associated with a rotation in a corresponding plane and, according to (6.19), is represented by a skew-symmetric matrix. For instance, we have the boost generator $I^{(10)}$

$$(I^{(10)})_\nu^\mu = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (I^{(10)})_{\mu\nu} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and similar representations following from (6.19) hold for other generators $I^{(\alpha\beta)}$.

Consequently, the orthochronous Lorentz transformation Λ defined by (6.13) can also be written as

$$\Lambda = \exp \left\{ \frac{1}{2} \xi_{\alpha\beta} I^{(\alpha\beta)} \right\}. \quad (6.20)$$

The generators $I^{(\alpha\beta)}$ are commonly split into two 3-component matrix-valued “vectors” $\mathbf{J}$ and $\mathbf{N}$ based on whether they represent *spatial rotations* or *boosts* (*pure Lorentz transformations*), namely

$$\begin{aligned} \mathbf{J} &= (J_1, J_2, J_3) = (I^{(23)}, I^{(31)}, I^{(12)}), \\ \mathbf{N} &= (N_1, N_2, N_3) = (I^{(10)}, I^{(20)}, I^{(30)}). \end{aligned}$$

6.2.1 Spinorial Form of the Lorentz Transformations

Let us consider two inertial frames I and I' with respective space-time coordinates x and x' and the spinor wave functions ψ and ψ' . The *inhomogeneous Lorentz transformation* from the frame I and I' , also called the *Poincaré transformation*, is of the form

$$x' = \Lambda x + a, \text{ or } x'^\mu = \Lambda_\nu^\mu x^\nu + a^\mu, \quad (6.21)$$

where $\Lambda = \Lambda_\nu^\mu$ is the homogeneous Lorentz transformation.

The general *infinitesimal form of the Poincaré transformations* is given by (6.11), (6.17):

$$\begin{aligned} x'^\mu &= x^\mu + \xi^{\mu\nu} x_\nu + a^\mu, & \xi^{\mu\nu} &= -\xi^{\nu\mu}, \\ \partial_\mu &= \Lambda_\mu^\nu \partial'_\nu, & \Lambda_\mu^\nu &= \delta_\mu^\nu + \xi_\mu^\nu \end{aligned} \quad (6.22)$$

The so-called *spinorial form* $S(\Lambda)$ of the Lorentz transformation Λ is defined by the following identities, [299, 6.2.1, 6.2.2], [143, 3.2, 3.3], [331, 2.1.2]

$$S(\Lambda)^{-1} \gamma^\mu S(\Lambda) = \Lambda^\mu_\nu \gamma^\nu = \gamma'^\mu, \quad (6.23)$$

where $S(\Lambda)$ is a 4×4 matrix and γ^μ are Dirac's matrices. *Dirac's gamma matrices* have the following form, [299, 5.3.5.4, 6.2.5], [143, 3, p. 129]:

$$\gamma^0 = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}, \quad i = 1, 2, 3, \quad (6.24)$$

where σ^i are the Hermitian *Pauli matrices* defined by

$$\sigma^0 = \mathbb{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (6.25)$$

The operator $S = S(\Lambda)$ also satisfies the following relation, [299, 6.2.3]:

$$S^{-1} = \gamma^0 S^\dagger \gamma^0, \quad \gamma^5 S = S \gamma^5. \quad (6.26)$$

The infinitesimal form for the matrix $S(\Lambda)$ is

$$S(\Lambda) = \mathbb{I} - \frac{i}{4} \sigma_{\mu\nu} \xi^{\mu\nu} + \text{higher order terms}. \quad (6.27)$$

The gamma matrices satisfy the following fundamental relations:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbb{I}_4.$$

The matrices $\sigma_{\mu\nu}$ can be defined in terms of Dirac's γ^μ matrices as follows, [299, 6.2.2.1], [143, 3.2, 3.3], [331, 2.2.1]:

$$\sigma_{\mu\nu} = \frac{i}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu). \quad (6.28)$$

The following explicit representation holds for the matrix $S(\Lambda)$ associated with the *proper orthochronous Lorentz transformation* Λ :

$$S(\Lambda) = e^{-\frac{i}{4} \sigma_{\mu\nu} \xi^{\mu\nu}}. \quad (6.29)$$

The transformational properties of Dirac's spinor wave function are defined by the formula

$$\psi'(x') = S(\Lambda) \psi(x) = S(\Lambda) \psi(\Lambda^{-1}(x' - a)), \quad (6.30)$$

which can be written in the following component form:

$$\psi'_\mu(x') = \sum_{\nu=1}^4 S_{\mu\nu}(\Lambda) \psi_\nu(\Lambda^{-1}(x' - a)). \quad (6.31)$$

To summarize, the complete set of spinorial transformations described by (6.23) and (6.30) between two different frames is

$$\psi'(x') = S(\Lambda) \psi(x), \quad \gamma'^\mu = S(\Lambda)^{-1} \gamma^\mu S(\Lambda) = \Lambda^\mu_\nu \gamma^\nu. \quad (6.32)$$

The transformations (6.32) are sometimes called *bispinor*, [331, 1.2], and they are *passive*, for they refer to the very same physical state represented in two different coordinate frames, [169, IV], [331, 1.2]. For a detailed treatment of the Lorentz group representations we refer to [147, 5.8, 5.9], [248, 6.1], [304, 3, 6-9], [299, 6.1-6.2], [331, 2.2].

6.3 Relativistic Kinematics

Let us consider the relativistic treatment of the simplest mechanical object, namely a mass point moving in an inertial frame with velocity $\mathbf{v}$. The first important step is to realize that the concepts fundamental to the relativistic theory are the invariant *spacetime time* ds and *proper time* $d\tau$, [137, 7.1], [277, 23]. They are defined as follows:

$$(ds)^2 = dx_\mu dx^\mu = c^2 dt^2 - (d\mathbf{x})^2, \quad (6.33)$$

$$d\tau = \frac{ds}{c} = \frac{dt}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma dt.$$

Similarly, the four-velocity u^μ and four-acceleration a^μ are defined by

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma \frac{dx^\mu}{dt} = (u^0, \mathbf{u}) = \gamma(c, \mathbf{v}), \quad (6.34)$$

$$a^\mu = \frac{du^\mu}{d\tau} = \gamma \frac{du^\mu}{dt} = \gamma \left(\frac{d\gamma}{dt} c, \frac{d\gamma}{dt} \mathbf{u} + \gamma \frac{d\mathbf{v}}{dt} \right). \quad (6.35)$$

The fundamental feature of the relativistic treatment is that the square uu of the four-velocity u^μ is fixed, namely

$$uu = u_\mu u^\mu = c^2, \quad (6.36)$$

following readily from (6.34). Notice that

$$aa = a_\mu a^\mu = -\gamma^6 \left(\frac{d\mathbf{u}}{dt} \right)^2 \frac{\mathbf{a}^2}{c^2} - \gamma^4 \mathbf{a}^2$$

and also that the four-velocity u^μ and four-acceleration a^μ are orthogonal:

$$u_\mu a^\mu = 0 \quad (6.37)$$

according to (6.35) and (6.36).

6.4 Point Charges in an External Electromagnetic Field

“For me, an electron is a corpuscle which at any given instant is situated at a determinate point of space ...
A. Lorentz.²

The non-relativistic motion of a point charge with mass m and charge q in an external EM field is described by Newton’s law with the Lorentz force $\mathbf{f}_{\text{Lor}}$:

$$m \partial_t^2 \mathbf{r}(t) = \mathbf{f}_{\text{Lor}}(t, \mathbf{r}). \quad (6.38)$$

The *Lorentz force* which acts on the point charge with time-dependent position $\mathbf{r}$ and velocity

$$\mathbf{v} = \dot{\mathbf{r}} = \partial_t \mathbf{r}$$

is determined by the EM field according to the formula

$$\mathbf{f}_{\text{Lor}}(t, \mathbf{r}) = q\mathbf{E}(t, \mathbf{r}) + \frac{q}{c} \mathbf{v} \times \mathbf{B}(t, \mathbf{r}), \quad (6.39)$$

where $\mathbf{E}(t, \mathbf{r})$ and $\mathbf{B}(t, \mathbf{r})$ are respectively the electric field and the magnetic induction. The EM fields can be expressed in terms of the EM potentials by formula (4.6). According to (43.7),

$$\mathbf{v} \times \mathbf{B} = \mathbf{v} \times (\nabla \times \mathbf{A}) = \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla) \mathbf{A}, \quad (6.40)$$

hence the Lorentz force $\mathbf{f}_{\text{Lor}}$ in (6.39) written in terms of the EM potentials takes the form

$$\mathbf{f}_{\text{Lor}} = -q\nabla\varphi - \frac{q}{c} \partial_t \mathbf{A} + \frac{q}{c} \nabla(\mathbf{v} \cdot \mathbf{A}) - \frac{q}{c} (\mathbf{v} \cdot \nabla) \mathbf{A}. \quad (6.41)$$

²[1, p. 108].

The *relativistic version* of Eq. (6.38) is described by the equation

$$\partial_t [\gamma m \mathbf{v}(t)] = \mathbf{f}_{\text{Lor}}(t, \mathbf{r}), \quad (6.42)$$

where

$$\gamma = 1/\sqrt{1 - \mathbf{v}^2(t)/c^2} \quad (6.43)$$

is the Lorentz factor.

The Eq. (6.38) and its relativistic generalization can be derived from corresponding Lagrangians as we show below.

The relativistic Lagrangian L_p of a point charge q with mass m in an external EM field is defined by the following formula, [179, 12.1]:

$$L_p = -mc^2 \sqrt{1 - \left(\frac{\mathbf{v}}{c}\right)^2} - q\varphi + \frac{q}{c} \mathbf{v} \cdot \mathbf{A}, \quad (6.44)$$

where φ is the electric potential and $\mathbf{A}$ is the vector potential. For this Lagrangian, the ordinary kinetic momentum $\mathbf{p}$, the canonical (conjugate) momentum $\mathring{\mathbf{p}}$, and the Hamiltonian H_p are defined by the following relations:

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \left(\frac{\mathbf{v}}{c}\right)^2}}, \quad \mathring{\mathbf{p}} = \mathbf{p} + \frac{q}{c} \mathbf{A}, \quad (6.45)$$

$$H_p = \mathring{\mathbf{p}} \cdot \mathbf{v} - L = \sqrt{(\mathbf{c}\mathbf{p} - q\mathbf{A})^2 + m^2c^4} + q\varphi. \quad (6.46)$$

The EM potentials $\mathbf{A}$, φ are related to the EM fields by formula (4.6). The relativistic Euler–Lagrange equations are

$$\partial_t \mathbf{p} = \mathbf{f}_{\text{Lor}}(t, \mathbf{r}), \quad (6.47)$$

where $\mathbf{f}$ is the Lorentz force (6.39).

The *non-relativistic* version of the above Lagrangian (6.44) is, [137, 1.5],

$$L_p = \frac{m\dot{\mathbf{r}}^2}{2} - q\varphi(\mathbf{r}) + \frac{q}{c} \mathbf{A}(\mathbf{r}) \cdot \dot{\mathbf{r}}, \quad \dot{\mathbf{r}} = \partial_t \mathbf{r}. \quad (6.48)$$

The corresponding ordinary kinetic momentum $\mathbf{p}$, the canonical (conjugate) momentum $\mathring{\mathbf{p}}$, and the Hamiltonian H_p are defined by the following relations:

$$\mathbf{p} = m\dot{\mathbf{r}}, \quad \mathring{\mathbf{p}} = \mathbf{p} + \frac{q}{c} \mathbf{A}, \quad H_p = \mathring{\mathbf{p}} \cdot \dot{\mathbf{r}} - L = \frac{m\dot{\mathbf{r}}^2}{2} + q\varphi. \quad (6.49)$$

The canonical Euler–Lagrange equations for the non-relativistic Lagrangian (6.48) take the form

$$\frac{d}{dt} \dot{\mathbf{p}} = -q \nabla \varphi + \frac{q}{c} \dot{\mathbf{r}} \cdot \nabla \mathbf{A}, \quad (6.50)$$

or, if we use the identity

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \dot{\mathbf{r}} \cdot \nabla \mathbf{A}, \quad (6.51)$$

we can recast (6.50) as (6.38); this is equivalent to writing the Lorentz force in the form (6.41).

The canonical Euler–Lagrange equation (6.50) involves the canonical momentum $\dot{\mathbf{p}}$ and the canonical force $-q \nabla \varphi + \frac{q}{c} \dot{\mathbf{r}} \cdot \nabla \mathbf{A}$ which manifestly depend on the EM potentials φ and $\mathbf{A}$ rather than the EM fields $\mathbf{E}$ and $\mathbf{B}$. Consequently, the Eq. (6.50) involves quantities which in classical electrodynamics are not directly measurable in contrast to the equivalent Eq. (6.38) which is gauge invariant and involves measurable quantities, namely the kinematic momentum $\mathbf{p}$ and the Lorentz force $q\mathbf{E} + \frac{q}{c} \dot{\mathbf{r}} \times \mathbf{B}$.

In the absence of external fields, the Lagrangian L_p takes the form

$$L_p = -mc^2 \sqrt{1 - \left(\frac{\mathbf{v}}{c}\right)^2}. \quad (6.52)$$

The momentum (the ordinary kinetic momentum) and the energy of the point mass for the relativistic Lagrangian L_p defined by (6.52) can easily be found, [213, 2.9],

$$\begin{aligned} \mathbf{p} &= \frac{m\mathbf{v}}{\sqrt{1 - \left(\frac{\mathbf{v}}{c}\right)^2}}, \\ \mathcal{E} = p^0 c = \mathbf{p} \cdot \mathbf{v} - L &= \frac{mc^2}{\sqrt{1 - \left(\frac{\mathbf{v}}{c}\right)^2}} = c\sqrt{\mathbf{p}^2 + m^2 c^2}, \end{aligned} \quad (6.53)$$

implying for $\left|\frac{\mathbf{v}}{c}\right| \ll 1$ the following non-relativistic approximation

$$\mathcal{E} \cong mc^2 + \frac{m\mathbf{v}^2}{2}.$$

6.4.1 Point Charges and the Lorentz–Abraham Model

The non-relativistic dynamics of a point charge q of mass m in an external electromagnetic (EM) field is governed by the Eq. (6.38). If the position and velocity of a moving point charge are $\mathbf{r}$ and $\mathbf{v}$ respectively, then, according to classical electromagnetism, it generates the EM field described by the Eqs. (4.1)–(4.4) with point sources, namely

$$\rho = q\delta(\mathbf{x} - \mathbf{r}(t)), \quad \mathbf{J} = q\delta(\mathbf{x} - \mathbf{r}(t)) \mathbf{v}(t), \quad (6.54)$$

$$\begin{aligned} \frac{1}{c} \partial_t \mathbf{E} - \nabla \times \mathbf{B} &= -\frac{4\pi}{c} q\delta(\mathbf{x} - \mathbf{r}(t)) \mathbf{v}(t), \\ \nabla \cdot \mathbf{E} &= 4\pi q\delta(\mathbf{x} - \mathbf{r}(t)), \quad \mathbf{v}(t) = \dot{\mathbf{r}}(t), \end{aligned} \quad (6.55)$$

where δ is Dirac's delta-function. But if, naturally, we would like to consider the Eq. (6.38) and (6.55) as a closed system “charge-EM field”, there is a problem. The origin of the problem is in the divergence of the EM field exactly at the position of the point charge, as, for instance, for the electrostatic field $\mathbf{E}$ with Coulomb's potential $\frac{q}{|\mathbf{x}-\mathbf{r}|}$ with a singularity at $\mathbf{x} = \mathbf{r}$. If (6.38) is replaced by the relativistic equation (6.42), the system constituted by (6.42) and (6.55) becomes Lorentz invariant and has a Lagrangian that yields it via the variational principle [27, (4.21)], [310, (2.36)], but the problem still persists. Some studies indicate that, [346], “a fully consistent classical equation of motion for a point charge, unlike that of an extended charge, does not exist”. If one wants to stay within classical theory of electromagnetism, a possible remedy is the introduction of an extended charge which, though very small, is not a point. There are two well known models for such an extended charge: the semi-relativistic Abraham rigid charge model (a rigid sphere with spherically symmetric charge distribution), [310, 2.4, 4.1, 10.2, 13], [278, 2.2], and the Lorentz relativistically covariant model which was studied and advanced in [5], [179, Sections 16], [255], [269], [278, Sections 2, 6], [301], [310, 2.5, 4.2, 10.1], [345]. Importantly, Poincaré suggested in 1905-1906, [273] (see also [179, 16.4-16.6], [278, 2.3, 6.1-6.3], [265, 63], [301], [310, 4.2] and references therein), to add to the Lorentz–Abraham model non-electromagnetic cohesive forces which balance the charge internal repulsive electromagnetic forces and remarkably also restore the covariance of the entire model. This idea of Poincaré is developed in Part III of the book.

6.4.2 Forces and Torques Exerted on Localized Distributed Charges

The forces and torques exerted upon localized charges and currents in an external inhomogeneous EM field can be expressed in terms of their electric and magnetic multipole moments. With that in mind, suppose that there are charges and steady currents localized about some point $\mathbf{r}_0$ and subject to the EM field $\mathbf{E}(\mathbf{r})$, $\mathbf{B}(\mathbf{r})$. Then the force $\mathbf{f}$ and the torque $\boldsymbol{\tau}$ exerted upon them have the following representations,

$$\mathbf{f}(\mathbf{r}_0) = q\mathbf{E}(\mathbf{r}_0) + (\mathbf{p} \cdot \nabla) \mathbf{E}(\mathbf{r}_0) + \nabla(\mathbf{m} \cdot \mathbf{B})(\mathbf{r}_0) + \dots, \quad (6.56)$$

$$\boldsymbol{\tau}(\mathbf{r}_0) = \mathbf{p} \times \mathbf{E}(\mathbf{r}_0) + \mathbf{m} \times \mathbf{B}(\mathbf{r}_0) + \dots, \quad (6.57)$$

where the center $\mathbf{r}_0$ of the charge distribution and its total charge are defined by

$$\mathbf{r}_0 = \frac{1}{q} \int \rho(\mathbf{r}) \mathbf{r} \, d\mathbf{r}, \quad q = \int \rho(\mathbf{r}) \, d\mathbf{r},$$

and the electric and magnetic dipoles are represented by the following formulas

$$\mathbf{p} = \int \rho(\mathbf{r}) (\mathbf{r} - \mathbf{r}_0) \, d\mathbf{r}, \quad (6.58)$$

$$\mathbf{m} = \frac{q}{2c} \int \frac{\rho(\mathbf{r})}{q} (\mathbf{r} - \mathbf{r}_0) \times (\mathbf{v}(\mathbf{r}) - \mathbf{v}(\mathbf{r}_0)) \, d\mathbf{r}. \quad (6.59)$$

Notice that the above formulas for the dipole moments are alternative representations of the same quantities in (5.4). Indeed, notice that the current $\mathbf{J}(\mathbf{r})$ is the density of charge $\rho(\mathbf{r})$ times its velocity $\mathbf{v}(\mathbf{r})$, namely

$$\rho(\mathbf{r}) \mathbf{v}(\mathbf{r}) = \mathbf{J}(\mathbf{r}). \quad (6.60)$$

For a detailed discussion of the areas of applicability of the formulas (6.56)–(6.60) and their variations we refer the reader to [245, 2.2, 2.4], [256, 2-4, 11-1], [302, 4.1], [263, 1-8], [205, 4.2, 5.3], [327, 2.2], [281, 1, App. C, D].

6.4.3 Angular Momentum and Gyromagnetic Ratio

Let us consider a system of localized currents of negligible size with charge density $\rho(\mathbf{r})$, and suppose that it carries a magnetic dipole moment $\mathbf{m}$ defined by (6.59) and a mass density $\rho_m(\mathbf{r})$ with total mass m and that $\mathbf{r}_0$ is the common center of both distributions $\rho(\mathbf{r})$ and $\rho_m(\mathbf{r})$. An external magnetic field $\mathbf{B}$ exerts on this system a torque $\boldsymbol{\tau}$ defined by (6.57), and its angular momentum $\mathbf{L}$ satisfies the following evolution equation

$$\partial_t \mathbf{L} = \mathbf{m} \times \mathbf{B}. \quad (6.61)$$

Obviously, the angular momentum

$$\mathbf{L} = m \int \frac{\rho_m(\mathbf{r})}{m} (\mathbf{r} - \mathbf{r}_0) \times (\mathbf{v}(\mathbf{r}) - \mathbf{v}(\mathbf{r}_0)) \, d\mathbf{r} \quad (6.62)$$

and the magnetic dipole moment (6.59) are represented by similar expressions, [205, 4.2], [170, 9-5], [137, 5.9], [263, 7-11]. It turns out that in many interesting situations $\mathbf{m}$ and $\mathbf{L}$ defined by (6.62), (6.59) are proportional, that is

$$\mathbf{m} = \gamma \mathbf{L}, \quad \gamma = g \frac{q}{2mc}, \quad (6.63)$$

where γ is called the *gyromagnetic ratio* and the dimensionless g is called the *gyromagnetic factor* or just *g-factor*. Though expressions (6.59) and (6.62) are evidently similar, they do not imply, of course, the proportionality of $\mathbf{m}$ and $\mathbf{L}$. But under certain symmetry conditions, for instance, if both distributions can be treated as rigid “centrosymmetric” bodies, the relation (6.63) does hold, [170, 7-2, p. 449].

For localized charge distributions, the gyromagnetic ratio can vary depending on the densities of charge $\rho(\mathbf{r})$ and mass $\rho_m(\mathbf{r})$. If the distribution has a constant ratio of charge density to mass density, it is easy to show that $g = 1$. *However, other assumptions about the structure of the particle will give almost any desired value for g* , [170, 7-2, p. 449], [161, 1]. In the Dirac theory the electron gyromagnetic ratio $g = 2$, whereas according to the QED the electron gyromagnetic ratio $g = 2(1 + \frac{\alpha}{2\pi})$, [178, 7.2.1]. Spin 1/2 particles may have very different values of g . For instance, the proton has $g = 5.59$ and the neutron has $g = -3.83$, they have different g values due to their internal structure, [178, 1.1.3]. There is a natural classical explanation for the gyromagnetic ratio $g = 2$, [161, 1], [164, 3], [169, V].

The gyromagnetic relation (6.63) allows us to transform the angular momentum evolution equation (6.61) into the following simple form:

$$\partial_t \mathbf{L} = \boldsymbol{\omega}_L \times \mathbf{L}, \quad \boldsymbol{\omega}_L = -g \frac{q}{2mc} \mathbf{B}. \quad (6.64)$$

The evolution described by Eq. (6.64) is known as *Larmor precession* with *Larmor frequency* $\omega_L = |\boldsymbol{\omega}_L|$, [205, 5.5], [263, 7-11], [170, 9-5].

Chapter 7

Longitudinal and Transversal Fields

The Helmholtz field decomposition $\mathbf{F} = \mathbf{F}_L + \mathbf{F}_T$ of a field $\mathbf{F}$ on $\mathbb{R}^3$ into the sum of a longitudinal (irrotational, curl-free) component $\mathbf{F}_L$ and transversal (solenoidal, divergence-free) component $\mathbf{F}_T$ is treated in Chap. 44. It provides important insights into properties of the Maxwell equations (4.1)–(4.4). One of these insights points to the transversal components as the origin of radiated fields phenomena and the longitudinal components as the origin of the quasistatic Coulomb fields, [324, 7.3]. Yet another important insight is that the EM fields gauging is associated entirely with the longitudinal component $\mathbf{E}_L$ of the electric field $\mathbf{E}$.

To avoid confusion on issues related to relativity and causality, we would like to stress that by its very nature *the Helmholtz field decomposition into the longitudinal and the transversal components is instantaneous and involves only spatial variables. It is also nonlocal.*

7.1 The Helmholtz Decomposition of the Potential Form of the Maxwell Equations

In this section we consider the Helmholtz decomposition of the potential form (4.27) of the Maxwell equations, namely

$$\begin{aligned} \nabla^2 \varphi + \frac{1}{c} \partial_t \nabla \cdot \mathbf{A} &= -4\pi\rho, \\ \nabla^2 \mathbf{A} - \frac{1}{c^2} \partial_t^2 \mathbf{A} - \nabla \left(\frac{1}{c} \partial_t \varphi + \nabla \cdot \mathbf{A} \right) &= -\frac{4\pi}{c} \mathbf{J}, \end{aligned} \quad (7.1)$$

and the potential representation (4.6) of the EM fields:

$$\mathbf{E} = -\nabla \varphi - \frac{1}{c} \partial_t \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (7.2)$$

Importantly, we assume that the charge continuity (conservation) equation (4.5) holds:

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0. \quad (7.3)$$

Using the defining properties $\nabla \times \mathbf{F}_L = 0$ and $\nabla \cdot \mathbf{F}_T = 0$ of the Helmholtz decomposition (44.1) $\mathbf{F} = \mathbf{F}_L + \mathbf{F}_T$, we readily obtain from (7.1)–(7.3) that

$$-\nabla \cdot \left(\nabla \varphi + \frac{1}{c} \partial_t \mathbf{A}_L \right) = 4\pi \rho, \quad (7.4)$$

$$\partial_t \left(\nabla \varphi + \frac{1}{c} \partial_t \mathbf{A}_L \right) = 4\pi \mathbf{J}_L,$$

$$\nabla^2 \mathbf{A}_T - \frac{1}{c^2} \partial_t^2 \mathbf{A}_T = -\frac{4\pi}{c} \mathbf{J}_T,$$

$$\partial_t \rho + \nabla \cdot \mathbf{J}_L = 0, \quad (7.5)$$

$$\mathbf{E} = -\nabla \varphi - \frac{1}{c} \partial_t \mathbf{A}_L - \frac{1}{c} \partial_t \mathbf{A}_T, \quad \mathbf{B} = \nabla \times \mathbf{A}_T. \quad (7.6)$$

Notice that the first equation in (7.6) implies the following representation for the longitudinal and the transversal components of the electric field:

$$\mathbf{E}_L = -\nabla \varphi - \frac{1}{c} \partial_t \mathbf{A}_L, \quad \mathbf{E}_T = -\frac{1}{c} \partial_t \mathbf{A}_T. \quad (7.7)$$

In view of equations (7.7), the first two equations (7.4) can be recast as

$$\nabla \cdot \mathbf{E}_L = 4\pi \rho, \quad \partial_t \mathbf{E}_L = -4\pi \mathbf{J}_L. \quad (7.8)$$

Observe now that fulfillment of equations (7.8) readily implies the charge conservation law (7.5) and that the longitudinal component $\mathbf{E}_L$ of the electric field $\mathbf{E}$ is determined entirely by the charge density ρ and the longitudinal component $\mathbf{J}_L$ of the current $\mathbf{J}$. Notice also that combining the third equation in (7.4) with Eqs. (7.6) and (7.7), we obtain the following system:

$$\nabla^2 \mathbf{A}_T - \frac{1}{c^2} \partial_t^2 \mathbf{A}_T = -\frac{4\pi}{c} \mathbf{J}_T, \quad (7.9)$$

$$\mathbf{B} = \mathbf{B}_T = \nabla \times \mathbf{A}_T, \quad \mathbf{E}_T = -\frac{1}{c} \partial_t \mathbf{A}_T. \quad (7.10)$$

The system of equations (7.9), (7.10) shows that the magnetic field $\mathbf{B}$ and the transversal component $\mathbf{E}_T$ of the electric field $\mathbf{E}$ are determined entirely by the transversal component $\mathbf{J}_T$ of the current $\mathbf{J}$. Namely, the transversal vector potential field $\mathbf{A}_T$ is driven by the transversal current $\mathbf{J}_T$ according to Eq. (7.9), and it determines the

transversal magnetic field $\mathbf{B} = \mathbf{B}_T$ and electric field $\mathbf{E}_T$ as described by equations (7.10).

Note also that in the case of the Coulomb gauge where $\mathbf{A}_L = 0$, the system of equations (7.4) turns into the system

$$\nabla^2 \varphi = -4\pi\rho, \quad (7.11)$$

$$\nabla^2 \mathbf{A}_T - \frac{1}{c^2} \partial_t^2 \mathbf{A}_T = -\frac{4\pi}{c} \mathbf{J}_T, \quad (7.12)$$

where we have omitted the equation $\partial_t \nabla \varphi = 4\pi \mathbf{J}_L$ for it follows from (7.11) and the charge conservation law (7.5).

A number of issues related to the Helmholtz decomposition of the potential form of the Maxwell equations is treated in [142, 20].

7.1.1 Scalar Potentials of Longitudinal Fields

For a field on $\mathbb{R}^3$, its longitudinal field component is the gradient of a scalar field as in (44.1), and we can introduce the following representations:

$$\mathbf{A}_L = -\nabla \alpha_L, \quad \mathbf{J}_L = -\nabla \xi_L, \quad (7.13)$$

where α_L and ξ_L are the scalar fields defining respectively $\mathbf{A}_L$ and $\mathbf{J}_L$. The first two equations in (7.4) and Eqs. (7.7) suggest to introduce the following potential, which we shall refer to as the *Coulomb potential*:

$$\varphi_C = \varphi - \frac{1}{c} \partial_t \alpha_L. \quad (7.14)$$

The Coulomb potential φ_C is naturally the scalar part of the pair $\{\varphi_C, \mathbf{A}_T\}$ where the corresponding vector transversal potential $\mathbf{A}_T$ satisfies evidently the Coulomb condition $\nabla \cdot \mathbf{A}_T = 0$ justifying the name “Coulomb” for the potential φ_C . *To avoid any confusion, we would like to stress that the introduction of the Coulomb potential φ_C does not fix the gauge. More discussions on this subject are provided in the following Sect. 7.1.2.*

Notice that, using equalities (7.13) and (7.14), we can recast $\mathbf{E}_L$ represented by Eq. (7.7) as follows:

$$\mathbf{E}_L = -\nabla \varphi - \frac{1}{c} \partial_t \mathbf{A}_L = -\nabla \varphi_C. \quad (7.15)$$

Utilizing equalities (7.13), (7.14) and (7.15), we can recast equations (7.4), (7.5) and the first equation in (7.7) into the following system:

$$\nabla^2 \varphi_C = -4\pi\rho, \quad \partial_t \varphi_C = -4\pi\xi_L, \quad (7.16)$$

$$\left(\nabla^2 - \frac{1}{c^2} \partial_t^2 \right) \mathbf{A}_T = -\frac{4\pi}{c} \mathbf{J}_T, \quad (7.17)$$

$$\partial_t \rho - \nabla^2 \xi_L = 0, \quad (7.18)$$

with the following representation for the electric and the magnetic fields:

$$\mathbf{E} = -\nabla \varphi_C - \frac{1}{c} \partial_t \mathbf{A}_T, \quad \mathbf{B} = \nabla \times \mathbf{A}_T. \quad (7.19)$$

Observe that Eqs. (7.16) and (7.17) are respectively for longitudinal and transversal field components, Eq. (7.18) is the potential form of the charge conservation law (7.3), and Eq. (7.19) constitute a potential representation of the EM fields in terms of the longitudinal and transverse components.

Obviously, (7.18) follows from (7.16). Conversely, the second equation in (7.16) can be derived as a consequence of remaining equations of the system. Indeed, assuming that ρ and $\mathbf{J}$ decay sufficiently fast at infinity, and using (i) the Poisson integral representation for the potential φ_C and (ii) formulas (4.45) for the vector potential $\mathbf{A}_T$, we readily obtain from Eqs. (7.16)–(7.17) that

$$\varphi_C(t, \mathbf{x}) = \int \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}', \quad (7.20)$$

$$\mathbf{A}_T(t, \mathbf{x}) = \int \frac{\mathbf{J}_T(t_r, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}', \quad t_r = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}. \quad (7.21)$$

According to formulas (7.20) and (7.21) for the potentials φ_C and $\mathbf{A}_T$, the transverse current $\mathbf{J}_T$ is a source of radiated fields, whereas the longitudinal current $\mathbf{J}_L$, which is related to ρ by the equation of charge conservation (7.5), generates quasistatic fields, see also [324, 7.3].

Equations (7.20) and (7.18) imply in turn that

$$\partial_t \varphi_C(\mathbf{x}) = \int \frac{\partial_t \rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}' = \int \frac{\nabla^2 \xi_L(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}' = -4\pi \xi_L(t, \mathbf{x}), \quad (7.22)$$

which are exactly the second equations in (7.16). Hence we may omit the second equation in (7.16) from the system of equations (7.16)–(7.18), arriving at the following equivalent system:

$$\nabla^2 \varphi_C = -4\pi\rho, \quad (7.23)$$

$$\nabla^2 \mathbf{A}_T - \frac{1}{c^2} \partial_t^2 \mathbf{A}_T = -\frac{4\pi}{c} \mathbf{J}_T, \quad (7.24)$$

$$\partial_t \rho - \nabla^2 \xi_L = 0, \quad (7.25)$$

where the scalar fields φ_C and ξ_L are defined by equalities (7.13), (7.14). Observe that, applying ∇ to both sides of (7.22) in view of (7.13) and (7.15), we obtain the identity

$$\partial_t \mathbf{E}_L(t, \mathbf{x}) = -4\pi \mathbf{J}_L. \quad (7.26)$$

Notice also that equalities (7.25) and (7.13) readily imply that

$$\xi_L(t, \mathbf{x}) = -\frac{1}{4\pi} \int \frac{\partial_t \rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}', \quad (7.27)$$

$$\mathbf{J}_L = -\nabla \xi_L = -\frac{1}{4\pi} \partial_t \nabla \int \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}', \quad (7.28)$$

showing that for a conserved charge-current system its longitudinal current component $\mathbf{J}_L$ is determined entirely by the charge density ρ . *Consequently, under the assumed charge conservation law (7.25), the charge density ρ and the transversal current component $\mathbf{J}_T$ can be considered as a set of independent variables representing the original variables ρ and $\mathbf{J}$.*

The EM fields $\mathbf{E}$ and $\mathbf{B}$ are then determined by equations (7.19), implying that

$$\mathbf{E}_L = -\nabla \varphi_C, \quad \mathbf{E}_T = -\frac{1}{c} \partial_t \mathbf{A}_T, \quad (7.29)$$

$$\mathbf{B}_L = \mathbf{0}, \quad \mathbf{B}_T = \nabla \times \mathbf{A}_T, \quad (7.30)$$

where φ_C and $\mathbf{A}_T$ satisfy Eqs. (7.20) and (7.21). Notice also that Eq. (7.22) combined with (7.13) imply that

$$\partial_t \mathbf{E}_L = -\partial_t \nabla \varphi_C(\mathbf{x}) = 4\pi \nabla \xi_L(\mathbf{x}) = -4\pi \mathbf{J}_L. \quad (7.31)$$

One might be concerned with formula (7.20) for the potential φ_C , since it involves “instantaneous” dependence of the potential on the sources, and that can raise questions on its compatibility with the finite speed of propagation of EM perturbations as well as causality. One can also raise similar questions for the vector potential $\mathbf{A}_T$ represented by (7.21) since the expression of $\mathbf{J}_T$ in terms of the original current $\mathbf{J}$ involves a spatially nonlocal transformation (44.4). An answer to these concerns is that a judgment on the finite speed of propagation and causality has to be made based on expressions for the fields $\mathbf{E}$ and $\mathbf{B}$ rather than on gauge dependent potentials φ_C and $\mathbf{A}_T$. A somewhat laborious analysis of the issue shows that after proper transformations the fields $\mathbf{E}$ and $\mathbf{B}$ can be represented by the manifestly causal Jefimenko expressions (4.47)–(4.49) with “retarded time”, [187, 15.7], [188], which are valid in any gauge including the Coulomb one, [181, V.A].

7.1.2 Gauge Transformations in Scalar Potential Form

Let us recall that the gauge transformation (4.19) for EM fields is defined by

$$\varphi = \varphi' + \frac{\partial \gamma}{\partial t}, \quad \mathbf{A} = \mathbf{A}' - \nabla \gamma, \quad (7.32)$$

where $\gamma = \gamma(t, \mathbf{x})$ is a real-valued scalar field. We recall that the significance of the gauge transformation lies in the fact that the EM fields $\mathbf{E}$ and $\mathbf{B}$ determine the potentials φ and $\mathbf{A}$ only up to the gauge transformation (7.32). So choosing a gauge is equivalent to choosing a pair of the potentials φ and $\mathbf{A}$ to describe the EM fields $\mathbf{E}$ and $\mathbf{B}$. The gauge choice can be facilitated by picking up a fixed pair $\{\varphi', \mathbf{A}'\}$ of the potentials as reference point and then choosing the scalar field $\gamma = \gamma(t, \mathbf{x})$. Fixing a gauge is commonly accomplished by imposing a condition such as the Coulomb and the Lorentz conditions in the form of an equation for γ , see Section 4.

Notice that somewhat similarly to the charge-current system, potentials φ_C and $\mathbf{A}_T$ form a complete set of independent variables that determine the EM fields $\mathbf{E}$ and $\mathbf{B}$ by Eq. (7.31). Observe also that gauging of the original potentials φ and $\mathbf{A}$, when it comes to the determination of the fields $\mathbf{E}$ and $\mathbf{B}$, can be traced to the first equation (7.31) representing the electric field $\mathbf{E}$. Indeed, according to Eqs. (7.19) and (7.31)–(7.30), the electric field longitudinal component $\mathbf{E}_L = -\nabla \varphi_C$ is determined by $\varphi_C = \varphi - \frac{1}{c} \partial_t \alpha_L$ rather than by two independent fields φ and α_L . This observation suggests that the invariance of the EM fields $\mathbf{E}$ and $\mathbf{B}$ under the gauge transformation (7.32) for the potentials φ and $\mathbf{A}$ is equivalent to the gauge invariance of the Coulomb potential $\varphi_C = \varphi - \frac{1}{c} \partial_t \alpha_L$ defined in (7.14) under the following gauge transformations

$$\varphi = \varphi' + \frac{1}{c} \partial_t \gamma, \quad \alpha_L = \alpha'_L + \gamma. \quad (7.33)$$

One can see from (7.33) that in fact *the freedom to choose a gauge is reduced entirely to choosing the scalar field α_L , and that completely determines the gauge.*

Notice also that, in view of equations (7.20), (7.21), the standard *Coulomb gauge potentials* are represented by the following equations

$$\varphi_C = \varphi, \quad \alpha_L = 0, \quad \mathbf{A}_C = \mathbf{A}_T, \quad (7.34)$$

since for this choice the defining equation $\nabla \cdot \mathbf{A}_C = 0$ for the Coulomb gauge evidently holds. The above consideration suggests that the Coulomb potentials $\{\varphi_C, \mathbf{A}_C\}$ defined by formulas (7.34) represent a convenient choice as a *reference potentials*.

It is straightforward to verify using relation (7.13) for $\mathbf{A}_L = -\nabla \alpha_L$ that the Lorentz gauge condition (4.9), namely $\frac{1}{c} \partial_t \varphi + \nabla \cdot \mathbf{A} = 0$, corresponds to the following choice of α_L and the corresponding expressions of the potentials:

$$\nabla^2 \alpha_L = \frac{1}{c} \partial_t \varphi, \quad (7.35)$$

$$\varphi = \varphi_C + \frac{1}{c} \partial_t \alpha_L, \quad \mathbf{A} = \mathbf{A}_T - \nabla \alpha_L. \quad (7.36)$$

In fact, α_L in the Lorentz gauge can be related to the longitudinal current $\mathbf{J}_L = -\nabla \xi_L$ by the following equation:

$$\left(\nabla^2 - \frac{1}{c^2} \partial_t^2 \right) \alpha_L = -\frac{4\pi}{c} \xi_L. \quad (7.37)$$

Using (7.35), (7.36) and (7.23), we see that under the Lorentz gauge the potential φ satisfies the wave equation

$$\left(\nabla^2 - \frac{1}{c^2} \partial_t^2 \right) \varphi = -4\pi \rho \quad (7.38)$$

which also follows from (4.28).

7.2 Maxwell's Equations Decomposition

The Maxwell equations (4.1)–(4.4) together with the charge conservation equations (4.5), when decomposed into longitudinal and transversal components, naturally split into two sets of equations: the equations for the longitudinal components (7.5), (7.8), namely

$$\partial_t \rho + \nabla \cdot \mathbf{J}_L = 0, \quad (7.39)$$

$$\nabla \cdot \mathbf{E}_L = 4\pi \rho, \quad -\frac{1}{c} \partial_t \mathbf{E}_L = \frac{4\pi}{c} \mathbf{J}_L, \quad (7.40)$$

and the equations for the transversal components (7.9)–(7.10) written in the form

$$\nabla \times \mathbf{B}_T - \frac{1}{c} \partial_t \mathbf{E}_T = \frac{4\pi}{c} \mathbf{J}_T, \quad (7.41)$$

$$\nabla \times \mathbf{E}_T + \frac{1}{c} \partial_t \mathbf{B}_T = \mathbf{0}, \quad (7.42)$$

where, in view of equations (7.7), we have

$$\mathbf{E}_L = -\nabla \varphi - \frac{1}{c} \partial_t \mathbf{A}_L, \quad \mathbf{E}_T = -\frac{1}{c} \partial_t \mathbf{A}_T, \quad \mathbf{B} = \nabla \times \mathbf{A}_T. \quad (7.43)$$

Observe that the first set of equations (7.39)–(7.40) evidently involves only longitudinal fields $\mathbf{E}_L$ and $\mathbf{J}_L$ as well as a charge density ρ , whereas the second set

(7.41)–(7.42) evidently involves only transversal fields $\mathbf{E}_T$, $\mathbf{B}_T$ and $\mathbf{J}_T$. Notice that the two equations (7.40) imply the charge conservation (7.39), or, alternatively, the second equation in (7.40) follows from two remaining equations of this set as we have shown in the previous section.

Equations (7.39)–(7.40) suggest that it is the longitudinal field $\mathbf{E}_L$ that is associated with the charge density ρ and the longitudinal component $\mathbf{J}_L$ of the original current density $\mathbf{J}$, whereas the current transversal component $\mathbf{J}_T$ determines through Eqs. (7.41)–(7.42) all transversal EM field components, namely $\mathbf{E}_T$ and $\mathbf{B}_T$. Equations (7.41)–(7.42) also suggest that the radiated fields effects are associated exclusively with the transversal fields in the chosen (fixed) space-time frame. With that said, we have to remember though that the Helmholtz decomposition into longitudinal transversal fields depends manifestly on the space-time frame as well as being spatially nonlocal, see Chap. 44.

To summarize, we can state that the system of the Maxwell equations under the Helmholtz decomposition into longitudinal and transversal components splits up into two subsystems. The first system of equations (7.39)–(7.40) for the longitudinal fields describes the so-called Coulombian fields. The second subsystems of equations (7.41)–(7.42) for the transversal fields describes the so-called radiated fields.

Chapter 8

Non-relativistic Quasistatic Approximations

Non-relativistic (quasistatic) approximations are of great importance for the corresponding regimes for at least two reasons. First of all, they are supported experimentally in relevant regimes. Second, their theoretical and computational analysis is significantly simpler compared to fully relativistic analysis. Non-relativistic (quasistatic) regimes occur when typical velocities are much smaller than the speed of light and consequently EM fields can be treated as if they propagate instantaneously.

It turns out that the limit of instantaneous propagation is by no means a trivial procedure and its details have to be specified, [179, 5.18], [222], [80], [81], [208]. In particular, depending on specific features of quasistatic regimes, there can be different approximations such as the *electroquasistatic (EQS)*, the *magnetoquasistatic (MQS)*, the *Darwin approximation* and more. One natural approach to constructing quasistatic approximations is by modifying the Maxwell equations to remove time retardation effects. Consequently, any non-relativistic quasistatic approximation provides only a limited representation of the EM phenomena dependent upon the approximation and chosen gauge. Usually an approximation assumes: (i) a truncated version of the Maxwell equations, [222, 2.2-3], [81, II], (ii) a representation of the EM fields $\mathbf{E}$ and $\mathbf{B}$ in terms of the scalar and vector potentials φ and $\mathbf{A}$, [222, 2.2]. Needless to say that some relations that are exact for Maxwell's theory do not have to be exact for an approximation. For instance, in the case of the magnetoquasistatics the charge continuity/conservation does not hold exactly.

A natural way to assess the accuracy and limitations of a non-relativistic approximation is by comparing its solutions with the exact solutions to the Maxwell equations. As to the intrinsic theoretical consistency of the approximation, it can be judged by the laws that hold for it exactly, i.e. the charge and the energy conservation laws. A way to assure such a consistency is to provide the approximation with the Lagrangian structure.

8.1 Galilean Electromagnetism

A possible natural way to arrive at a consistent non-relativistic approximation is to require it to be perfectly consistent with the Galilean transformations, as much as the Maxwell equations are consistent with the Lorentz transformations. This approach was advanced by Le Bellac and Levy–Leblond, and it is often referred to as “Galilean Electromagnetism”, [222], [286].

The starting point of Galilean Electromagnetism is to approximate the Lorentz transformations (6.7) with Galilean transformations, and it turns out that there can be two different Galilean limits, [222, 2]. Namely, if $u = (u_0, \mathbf{u})$ is a four-vector, for instance the current four-vector ($c\rho, \mathbf{J}$), these two limits are

$$u'_0 = u_0, \quad \mathbf{u}' = \mathbf{u} - \frac{\mathbf{v}}{c}u_0, \quad (8.1)$$

and

$$u'_0 = u_0 - \frac{\mathbf{v} \cdot \mathbf{u}}{c}, \quad \mathbf{u}' = \mathbf{u}, \quad (8.2)$$

where $\mathbf{v}$ is velocity of the transformation. Equations (8.1) are valid if $|\mathbf{v}|/c \ll 1$ and $|\mathbf{u}| \ll |u_0|$, that is, when u is “largely time-like”. For example, the standard Galilean transformation

$$c\Delta t' = c\Delta t, \quad \Delta \mathbf{x}' = \Delta \mathbf{x} - \mathbf{v}\Delta t, \quad (8.3)$$

is similar to the transformation in (8.1) and it holds only if $|\Delta \mathbf{x}| \ll c|\Delta t|$. Equations (8.2) are valid if $|\mathbf{v}|/c \ll 1$ and $|\mathbf{u}| \gg |u_0|$, that is, when u is “largely space-like”. Observe that the space-time gradient transforms according to Eq. (8.2):

$$\frac{1}{c}\partial_{t'} = \frac{1}{c}\partial_t - \frac{1}{c}\mathbf{v} \cdot \nabla, \quad \nabla' = \nabla. \quad (8.4)$$

Applying the described ideas to the EM fields, Le Bellac and Levy–Leblond have found two well defined Galilean limits. The first limit referred to as the *electric limit* or *electroquasistatic approximation (EQS)* applies when $c\rho \gg |\mathbf{J}|$ implying $|\mathbf{E}| \gg c|\mathbf{B}|$. *EQS applies to dielectrics and to electrohydrodynamics, where the Faraday induction can be neglected.* The second limit referred to as the magnetic limit or *magnetoquasistatic approximation (MQS)* when $c\rho \ll |\mathbf{J}|$ implying $|\mathbf{E}| \ll c|\mathbf{B}|$. *MQS applies to ohmic conductors and to magnetohydrodynamics, where the displacement current can be neglected.*

To summarize, in quasistatic regimes we always assume $|\mathbf{v}|/c \ll 1$ where $\mathbf{v}$ is a relevant velocity, and this condition should be interpreted as the velocity $|\mathbf{v}|$ being small compared to c rather than c approaching infinity. Then, according to the Galilean Electromagnetism, the condition $|\mathbf{v}|/c \ll 1$ alone does not determine

a unique limit. More exactly, there can be two limits—electric (EQS) and magnetic (MQS)—as determined by the following relations, [222, 1, 2]

$$c\rho \gg |\mathbf{J}|, \quad |\mathbf{E}| \gg c|\mathbf{B}| \quad \text{in EQS}, \quad (8.5)$$

$$c\rho \ll |\mathbf{J}| \quad |\mathbf{E}| \ll c|\mathbf{B}| \quad \text{in MQS}, \quad (8.6)$$

where the second inequalities for the fields $\mathbf{E}$ and $\mathbf{B}$ in each line (8.5), (8.6) follow from the corresponding first inequalities for the sources ρ and $\mathbf{J}$ (see representations (8.35), (8.36)).

When analyzing non-relativistic quasistatic approximation, it is useful to introduce the following two significant dimensionless parameters, [81, III]

$$\epsilon = \frac{L}{cT}, \quad \xi = \frac{J}{c\rho}, \quad (8.7)$$

where L , T , J , and ρ represent the orders of magnitude of length, time, current density, and charge density, respectively. For any quasistatic approximation $\epsilon \ll 1$, suggesting the EM fields can be treated as if they propagate instantaneously. Then, according to (8.5)–(8.7), small values of $\xi \ll 1$ correspond to the EQS, whereas large values $\xi \gg 1$ correspond to the MQS:

$$\epsilon \ll 1 \text{ in quasistatics; } \xi \ll 1 \text{ in EQS; } \xi \gg 1 \text{ in MQS.} \quad (8.8)$$

Under the Lorentz gauge, the exact wave equations for φ and $\mathbf{A}$ in quasistatic approximation $\epsilon \ll 1$ can be approximated by the Poisson equations,

$$\nabla^2 \varphi = -4\pi\rho, \quad \nabla^2 \mathbf{A} = -\frac{4\pi}{c} \mathbf{J}. \quad (8.9)$$

Based on the potential equations (8.9), the magnitudes of the charge density ρ and the current density $\mathbf{J}$ can be roughly related to the resulting EM potentials as follows, [81, III]

$$\frac{A}{\varphi} \simeq \xi, \quad \frac{|\partial_t \mathbf{A}|}{c|\nabla \varphi|} \simeq \epsilon \xi. \quad (8.10)$$

The Galilean invariant EQS and MQS versions of the Maxwell equations are as follows, [222, 2]

$$\nabla \cdot \mathbf{E}_e = 4\pi\rho_e, \quad \nabla \times \mathbf{E}_e = 0, \quad \text{EQS}, \quad (8.11)$$

$$\nabla \cdot \mathbf{B}_e = 0, \quad \nabla \times \mathbf{B}_e - \frac{1}{c} \partial_t \mathbf{E}_e = \frac{4\pi}{c} \mathbf{J}_e, \quad (8.12)$$

$$\nabla \cdot \mathbf{E}_m = 4\pi\rho_m, \quad \nabla \cdot \mathbf{B}_m = 0, \quad \text{MQS}, \quad (8.13)$$

$$\nabla \times \mathbf{E}_m + \frac{1}{c}\partial_t \mathbf{B}_m = 0, \quad \nabla \times \mathbf{B}_m = \frac{4\pi}{c}\mathbf{J}_m. \quad (8.14)$$

We distinguished in the above equations the EM fields $\mathbf{E}$ and $\mathbf{B}$ and sources ρ and $\mathbf{J}$ associated with the EQS and the MQS approximations by indices “e” and “m” respectfully. As to the charge conservation, the following equations hold:

$$\partial_t \rho_e + \nabla \cdot \mathbf{J}_e = 0 \quad \text{in EQS}, \quad (8.15)$$

$$\nabla \cdot \mathbf{J}_m = 0 \quad \text{in MQS}. \quad (8.16)$$

Thus we can see from (8.15) and (8.16) that in the case of EQS the charge conservation has exactly the same form as in the case of the Maxwell equations, whereas in the case of MQS this is not the case, but instead the current $\mathbf{J}_m$ is required to be exactly transversal (solenoidal).

To be consistent with the Galilean transformations, the Lorentz force density expression (9.18), (9.19) has to be modified in the EQS and MQS cases, respectively, as follows, [222, 2]:

$$\mathbf{F}_e = \rho_e \mathbf{E}_e \quad \text{in EQS}, \quad (8.17)$$

$$\mathbf{F}_m = \frac{1}{c}\mathbf{J}_m \times \mathbf{B}_m \quad \text{in MQS}. \quad (8.18)$$

It is evident from the EQS form (8.11), (8.12) of the Maxwell equations and the expression (8.17) for the EQS form of the Lorentz force density that in this approximation the magnetic field $\mathbf{B}_e$ exists, but it has no effect at all on a test point charge. Similarly in the MQS case, the electric field $\mathbf{E}_e$ exists, but it has no effect.

The relations between fields and potentials are different in the EQS and in the MQS, namely, [222, 2], [81, II]

$$\mathbf{E}_e = -\nabla\varphi_e, \quad \mathbf{B}_e = \nabla \times \mathbf{A}_e \text{ in EQS}, \quad (8.19)$$

$$\mathbf{E}_m = -\nabla\varphi_m - \frac{1}{c}\partial_t \mathbf{A}, \quad \mathbf{B}_e = \nabla \times \mathbf{A}_m \text{ in MQS}. \quad (8.20)$$

Comparing the relations between fields and potentials (8.19), (8.20) with the same for the Maxwell equations (4.6), we find the following. In the EQS case where $\epsilon, \xi \ll 1$, the relation (8.19) between the electric field $\mathbf{E}$ does not have the term $-\frac{1}{c}\partial_t \mathbf{A}$, as it can be justified by the second inequality in (8.10). In the MQS case, the relations (8.20) between fields and potentials are the same as for the Maxwell equations.

Notice also that in view of estimate (8.10), the difference between the exact expression of the electric field $\mathbf{E}$ and its EQS approximation $\mathbf{E}_e$ is relatively small, namely

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\partial_t\mathbf{A} = -\nabla\varphi(1 + O(\epsilon\xi)) = \mathbf{E}_e(1 + O(\epsilon\xi)). \quad (8.21)$$

There exist improved and richer versions of the EQS and MQS which take into account more effects remaining perfectly consistent with Galilean transformations. In these versions one introduces two kinds of charges and currents $(\rho_e, \mathbf{J}_e)$ and $(\rho_m, \mathbf{J}_m)$ but only one kind of EM fields $\mathbf{E}$ and $\mathbf{B}$, namely, [222, 3, chap. 37]:

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e, \quad \nabla \times \mathbf{E} = 0 \quad \text{improved EQS}, \quad (8.22)$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} - \frac{1}{c}\partial_t\mathbf{E} = \frac{4\pi}{c}(\mathbf{J}_e + \mathbf{J}_m), \quad (8.23)$$

$$\mathbf{F} = (\rho_e + \rho_m)\mathbf{E} + \frac{1}{c}\mathbf{J}_m \times \mathbf{B}, \quad (8.24)$$

$$\nabla \cdot \mathbf{E} = 4\pi(\rho_e + \rho_m), \quad \nabla \times \mathbf{E} = 0 \quad \text{improved MQS}, \quad (8.25)$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} - \frac{1}{c}\partial_t\mathbf{E} = \frac{4\pi}{c}\mathbf{J}_m, \quad (8.26)$$

$$\mathbf{F} = (\rho_e + \rho_m)\mathbf{E} + \frac{1}{c}(\mathbf{J}_e + \mathbf{J}_m) \times \mathbf{B}. \quad (8.27)$$

8.2 Electroquasistatics (EQS)

Here we discuss the electroquasistatics approximation (EQS) introduced and discussed in Sect. 8.1 in more detail. We show that the *EQS, as an approximation to Maxwell's theory, is an intrinsically consistent theory all by itself, and it can be furnished with a Lagrangian* as we show in Sect. 9.4.3. The foundational elements of the EQS are: (i) the EQS system of the field equations which is a modification of the Maxwell system; (ii) modified relations between the EQS fields $\mathbf{E}_e$ and $\mathbf{B}_e$ and the potentials φ_e and $\mathbf{A}_e$; (iii) the EQS expressions for solutions to the EQS system of the field equations; (iv) relations between the EQS and Maxwell's theory.

As was already pointed out in Sect. 8.1, the *EQS modification of the Maxwell equations* is obtained by neglecting the inductive term $\frac{1}{c}\partial_t\mathbf{B}$ in the Faraday equation yielding, [222, 2], [286, 4], Eqs. (8.11), (8.12):

$$\nabla \cdot \mathbf{E}_e = 4\pi\rho, \quad \nabla \times \mathbf{E}_e = 0, \quad (8.28)$$

$$\nabla \cdot \mathbf{B}_e = 0, \quad \nabla \times \mathbf{B}_e - \frac{1}{c}\partial_t\mathbf{E}_e = \frac{4\pi}{c}\mathbf{J}. \quad (8.29)$$

The charge conservation (continuity) equation (4.5), just as in the case of Maxwell's theory, is assumed to hold exactly for the EQS

$$\partial_t\rho + \nabla \cdot \mathbf{J} = 0. \quad (8.30)$$

There are several ways to analyze solutions to the EQS system of Eqs. (8.28)–(8.30). One of them is as follows. Armed with the idea that the EQS regime can be obtained by removing retardation effects associated with the finite speed of propagation, we can try to modify the Jefimenko formulas (4.47), (4.48) for the fields $\mathbf{E}$ and $\mathbf{B}$ by removing from them the retardation, [146, 10.2.2]. We proceed by recalling the expression for the retarded time t_r defined by (4.55), (4.55), which is

$$t_r = t - \frac{R}{c} = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \text{ where} \quad (8.31)$$

$$\mathbf{R} = |\mathbf{x} - \mathbf{x}'|, \quad R = |\mathbf{R}|, \quad \hat{\mathbf{R}} = \frac{\mathbf{R}}{R}. \quad (8.32)$$

Then plugging approximations

$$\rho(t_r, \mathbf{x}') = \rho(t, \mathbf{x}') - \partial_t \rho(t, \mathbf{x}') \frac{R}{c} + O\left(\frac{R^2}{c^2}\right), \quad (8.33)$$

$$\mathbf{J}(t_r, \mathbf{x}') = \mathbf{J}(t, \mathbf{x}') - \partial_t \mathbf{J}(t, \mathbf{x}') \frac{R}{c} + O\left(\frac{R^2}{c^2}\right), \quad (8.34)$$

into the Jefimenko formulas (4.47), (4.48) we obtain the following expression for the EQS fields

$$\mathbf{E}_e(t, \mathbf{x}) = \int \frac{\rho(t, \mathbf{x}') \hat{\mathbf{R}}}{R^2} d\mathbf{x}' = -\nabla \int \frac{\rho(t, \mathbf{x}')}{R} d\mathbf{x}', \quad (8.35)$$

$$\mathbf{B}_e(t, \mathbf{x}) = \int \frac{\mathbf{J}(t, \mathbf{x}')}{cR^2} \times \hat{\mathbf{R}} = \nabla \times \frac{1}{c} \int \frac{\mathbf{J}(t, \mathbf{x}')}{R} d\mathbf{x}', \quad (8.36)$$

which provide, as we show below, exact solutions to the EQS system of Eqs. (8.28)–(8.30).

Notice that the above expressions (8.35), (8.36) for the EQS fields $\mathbf{E}_e$ and $\mathbf{B}_e$ are consistent with the following expressions for the EQS potentials φ_e and $\mathbf{A}_e$:

$$\mathbf{E}_e = -\nabla \varphi_e, \quad \varphi_e = \int \frac{\rho(t, \mathbf{x}')}{R} d\mathbf{x}', \quad (8.37)$$

$$\mathbf{B}_e = \nabla \times \mathbf{A}_e, \quad \mathbf{A}_e = \frac{1}{c} \int \frac{\mathbf{J}(t, \mathbf{x}')}{R} d\mathbf{x}'. \quad (8.38)$$

The potentials φ_e and $\mathbf{A}_e$ then satisfy the Poisson equations

$$\nabla^2 \varphi_e = -4\pi\rho, \quad \nabla^2 \mathbf{A}_e = -\frac{4\pi}{c} \mathbf{J}, \quad (8.39)$$

which can be considered as the *potential form of the EQS field equations* (8.28)–(8.30). It is straightforward to verify using the charge conservation equation (8.30)

that the pair of potentials φ_e and $\mathbf{A}_e$ defined by Eqs. (8.37), (8.38) satisfy the Lorentz gauge condition (4.9), namely

$$\frac{1}{c} \partial_t \varphi_e + \nabla \cdot \mathbf{A}_e = 0. \quad (8.40)$$

Notice also that Eqs. (8.37), (8.38) for the fields $\mathbf{E}_e$ and $\mathbf{B}_e$ involve the vector potential $\mathbf{A}_e$ only in $\mathbf{B}_e = \nabla \times \mathbf{A}_e \equiv \nabla \times (\mathbf{A}_e)_T$. Therefore, we can replace in the second Poisson equation (8.39) $(\mathbf{A}_e)_T$ and $\mathbf{J}_T$ for, respectively, $\mathbf{A}_e$ and $\mathbf{J}$ resulting in equations

$$\nabla^2 \varphi_e = -4\pi\rho, \quad \nabla^2 (\mathbf{A}_e)_T = -\frac{4\pi}{c} \mathbf{J}_T, \quad (8.41)$$

and this system results in the same $\mathbf{E}_e$ and $\mathbf{B}_e$. The longitudinal component of $\mathbf{A}$ then has to be determined from the equation

$$\nabla^2 (\mathbf{A}_e)_L = -\frac{4\pi}{c} \mathbf{J}_L.$$

The relations between the potentials and the EM fields can then be expressed as follows:

$$\mathbf{E}_e = -\nabla \varphi_e, \quad \mathbf{B}_e = \nabla \times (\mathbf{A}_e)_T. \quad (8.42)$$

Let us verify now that *the EQS expressions (8.37), (8.38) solve exactly the EQS system of Eqs. (8.28)–(8.30)*. Indeed, relations (8.37) readily imply that the EQS electric field $\mathbf{E}_e$ satisfies both equations in (8.28). It is also obvious that the EQS magnetic field $\mathbf{B}_e = \nabla \times \mathbf{A}_e$ satisfies the first equation in (8.29). To verify the remaining second equation in (8.29), we notice that Eqs. (8.37), (8.38) defining the fields $\mathbf{E}_e$ and $\mathbf{B}_e$ when combined with the charge conservation equation (8.30) imply the following identities:

$$\begin{aligned} \frac{1}{c} \partial_t \mathbf{E}_e(t, \mathbf{x}) &= -\nabla \frac{1}{c} \int \frac{\partial_t \rho(t, \mathbf{x}')}{R} d\mathbf{x}' \\ &= \nabla \frac{1}{c} \int \frac{\nabla \cdot \mathbf{J}(t, \mathbf{x}')}{R} d\mathbf{x}' = -\frac{4\pi}{c} \mathbf{J}_L(t, \mathbf{x}), \end{aligned} \quad (8.43)$$

$$\nabla \times \mathbf{B}_e = \nabla \times \nabla \times \frac{1}{c} \int \frac{\mathbf{J}(t, \mathbf{x}')}{R} d\mathbf{x}' = \frac{4\pi}{c} \mathbf{J}_T(t, \mathbf{x}), \quad (8.44)$$

where we use formulas (44.1), (44.4) for the Helmholtz decomposition $\mathbf{J} = \mathbf{J}_L + \mathbf{J}_T$ into the longitudinal $\mathbf{J}_L$ and transversal $\mathbf{J}_T$ components. Equalities (8.43), (8.44) readily imply the second equation in (8.29) completing the verification of equations (8.37), (8.38) to be exact solutions to the EQS system of Eqs. (8.28)–(8.30).

It is instructive to compare the elements of the EQS theory with their counterpart for the Maxwell equations. As to similarities, in the EQS the charge conservation equation (8.30) is exact just as it is Maxwell's theory. Let us take a look now at the differences between the two theories.

We have already noticed that a key difference between the EQS and the Maxwell system is the omission of the inductive term $\frac{1}{c}\partial_t \mathbf{B}$ in the Faraday equation that triggers in turn all other modifications. The first evident and significant modification can be seen in the Eqs. (8.35)–(8.39). *According to these equations in the EQS all the fields follow the sources ρ and J instantaneously with no retardation justifying the name quasistatics.*

The equation $\nabla \times \mathbf{E}_e = 0$ in (8.28) for the electric field $\mathbf{E}_e$ as well as its explicit forms (8.35), (8.37) clearly shows that in the EQS theory the electric field is always entirely longitudinal and has no transversal component, that is,

$$\mathbf{E}_e = \mathbf{E}_{eL}, \quad \mathbf{E}_{eT} = \mathbf{0}. \quad (8.45)$$

This readily implies that Maxwell's theory relation $\mathbf{E} = -\nabla\varphi - \frac{1}{c}\partial_t \mathbf{A}$ from (4.6) representing the electric field through its potentials in the EQS is violated, for the right-hand side of the relation evidently has a generically nonzero transversal component $\frac{1}{c}\partial_t \mathbf{A}_T$, and since $\mathbf{B} = \nabla \times \mathbf{A}_T$ in (4.6), $\partial_t \mathbf{A}_T$ cannot be zero if $\mathbf{B}$ is a time varying quantity.

8.3 Darwin's Quasistatics Approximation

Darwin's approximation to fully relativistic motion of charged particles was introduced by its author in [72]. It is accurate to the second order $|\mathbf{v}|^2/c^2$ inclusive, and it is based on the Darwin particle Lagrangian, [179, 12.6], [213, 65]

$$\begin{aligned} L_{\text{Dp}} = & \sum_{\ell} \left(\frac{m_{\ell} \dot{\mathbf{r}}_{\ell}^2}{2} + \frac{m_{\ell} \dot{\mathbf{r}}_{\ell}^4}{8c^2} \right) - \frac{1}{2} \sum_{\ell \neq \ell'} \frac{q_{\ell} q_{\ell'}}{|\mathbf{r}_{\ell\ell'}|} \\ & + \frac{1}{2} \sum_{\ell \neq \ell'} \frac{q_{\ell} q_{\ell'}}{2c^2 |\mathbf{r}_{\ell\ell'}|} \left[\dot{\mathbf{r}}_{\ell} \cdot \mathbf{r}_{\ell'} + (\dot{\mathbf{r}}_{\ell} \cdot \hat{\mathbf{r}}_{\ell\ell'}) (\dot{\mathbf{r}}_{\ell'} \cdot \hat{\mathbf{r}}_{\ell\ell'}) \right], \end{aligned} \quad (8.46)$$

where

$$\mathbf{r}_{\ell\ell'} = \mathbf{r}_{\ell} - \mathbf{r}_{\ell'}, \quad \hat{\mathbf{r}}_{\ell\ell'} = \frac{\mathbf{r}_{\ell\ell'}}{|\mathbf{r}_{\ell\ell'}|}. \quad (8.47)$$

The Darwin Lagrangian L_{Dp} suggest the following expressions for the scalar and vector potentials for the system of charges

$$\varphi(t, \mathbf{x}) = \sum_{\ell} \frac{q_{\ell}}{|\mathbf{x} - \mathbf{r}_{\ell}|}, \quad \mathbf{A}(t, \mathbf{x}) = \sum_{\ell} \frac{q_{\ell} [\dot{\mathbf{r}}_{\ell} + (\dot{\mathbf{r}}_{\ell} \cdot \hat{\mathbf{r}}_{\ell 0}) \hat{\mathbf{r}}_{\ell 0}]}{2c |\mathbf{x} - \mathbf{r}_{\ell}|}, \quad (8.48)$$

where $\hat{\mathbf{r}}_{\ell 0} = \frac{\mathbf{x} - \mathbf{r}_{\ell}}{|\mathbf{x} - \mathbf{r}_{\ell}|}$.

The reason to keep the second order terms in $|\mathbf{v}|/c$ when neglecting the third order terms and higher is that the radiation of electromagnetic waves by moving charges occurs starting with the third order in $|\mathbf{v}|/c$, [213, 65], (see also formula (4.62) for the radiated power of a moving charge).

Following Darwin's approach, one can construct Darwin's quasistatic approximation for the EM fields, [208, II.B],

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \nabla \cdot \mathbf{B} = 0, \quad (8.49)$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \partial_t \mathbf{B} = 0, \quad \nabla \times \mathbf{B} - \frac{1}{c} \partial_t \mathbf{E}_L = \frac{4\pi}{c} \mathbf{J}, \quad (8.50)$$

with the charge conservation law of the form (4.5)

$$\partial_t \rho + \nabla \cdot \mathbf{J}_L = 0. \quad (8.51)$$

Evidently, the difference between Darwin's approximation system of Eqs. (8.49)–(8.50) and the Maxwell equations (4.1)–(4.4) is that the Ampere–Maxwell's law is modified by substituting $\partial_t \mathbf{E}_L$ in place of $\partial_t \mathbf{E}$ omitting the transversal term $\frac{1}{c} \partial_t \mathbf{E}_T$. Notice that, using the identity (7.26), that is,

$$\mathbf{J}_L = -\frac{1}{4\pi} \partial_t \mathbf{E}_L = \frac{1}{4\pi} \partial_t \nabla \varphi_C, \quad (8.52)$$

the second equation in (8.50) can be recast as

$$\nabla \times \mathbf{B}_T = \frac{4\pi}{c} \mathbf{J}_T, \quad \mathbf{J}_T = \mathbf{J} - \mathbf{J}_L = \mathbf{J} - \frac{1}{4\pi} \partial_t \nabla \varphi_C. \quad (8.53)$$

Consequently, using the Helmholtz decomposition into a longitudinal and transversal field, one can readily recast the Darwin's approximation equations (8.49)–(8.50) as follows, [208, II.B]:

$$\nabla \cdot \mathbf{E}_L = 4\pi\rho, \quad \mathbf{B} = \mathbf{B}_T, \quad (8.54)$$

$$\nabla \times \mathbf{E}_T + \frac{1}{c} \partial_t \mathbf{B}_T = 0, \quad \nabla \times \mathbf{B}_T = \frac{4\pi}{c} \mathbf{J}_T. \quad (8.55)$$

The relations between the fields and the potentials in Darwin's approximation are exactly the same as in Maxwell's theory, namely (4.6), and the corresponding Helmholtz decomposition is (7.19):

$$\mathbf{E}_L = -\nabla\varphi_C, \quad \mathbf{E}_T = -\frac{1}{c}\partial_t\mathbf{A}_T, \quad \mathbf{B} = \nabla \times \mathbf{A}_T, \quad (8.56)$$

$$\mathbf{E} = \mathbf{E}_L + \mathbf{E}_T = -\nabla\varphi_C - \frac{1}{c}\partial_t\mathbf{A}_T, \quad (8.57)$$

where the vector potential $\mathbf{A}$ is defined by

$$\mathbf{A} = \frac{1}{c} \int \frac{\mathbf{J}(t, \mathbf{x}')}{R} d\mathbf{x}', \quad (8.58)$$

$$-\Delta\mathbf{A} = \frac{4\pi}{c}\mathbf{J}. \quad (8.59)$$

Darwin's approximation equations (8.54)–(8.55) can be written in terms of the potentials in the form

$$\Delta\varphi_C = -4\pi\rho, \quad (8.60)$$

$$\Delta\mathbf{A}_T = -\frac{4\pi}{c}\mathbf{J}_T = -\frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\partial_t\nabla\varphi_C, \quad (8.61)$$

where ρ and $\mathbf{J}$ satisfy the charge conservation law (8.51).

Observe that the potential form of Darwin's approximation is identical to the alternative form (8.41) for the EQS approximation. The difference is in the relations between the fields and the potentials. Namely, the electric field in EQS is represented by the formula $\mathbf{E} = \mathbf{E}_L = -\nabla\varphi_C$, whereas in Darwin's approximation it is given by (8.57), implying different approximations for the Faraday equation:

$$\nabla \times \mathbf{E} = 0 \text{ for EQS approximation,} \quad (8.62)$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\partial_t\mathbf{B} = 0 \text{ for Darwin's approximation.} \quad (8.63)$$

One can establish by direct computation the following representation for the longitudinal component $\mathbf{J}_L$ of the current $\mathbf{J}$, [208, II]:

$$\mathbf{J}_L = \frac{1}{4\pi} \int \frac{\left(\mathbf{J}(t, \mathbf{x}') - 3\hat{\mathbf{R}}(\hat{\mathbf{R}} \cdot \mathbf{J}(t, \mathbf{x}'))\right)}{R^3} d\mathbf{x}', \text{ where} \quad (8.64)$$

$$\mathbf{R} = |\mathbf{x} - \mathbf{x}'|, \quad R = |\mathbf{R}|, \quad \hat{\mathbf{R}} = \frac{\mathbf{R}}{R},$$

readily implying

$$\mathbf{J}_T = \mathbf{J} - \mathbf{J}_L = \mathbf{J} - \frac{1}{4\pi} \int \frac{\left(\mathbf{J}(t, \mathbf{x}') - 3\hat{\mathbf{R}}(\hat{\mathbf{R}} \cdot \mathbf{J}(t, \mathbf{x}'))\right)}{R^3} d\mathbf{x}'. \quad (8.65)$$

Then solving the Poisson equations (8.60), (8.61), one can obtain the following representations for scalar and vector potentials:

$$\varphi_C = \int \frac{\rho}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}', \quad \mathbf{A}_T = \frac{1}{2c} \int \frac{\mathbf{J} + (\mathbf{J} \cdot \hat{\mathbf{R}}) \hat{\mathbf{R}}}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}', \quad (8.66)$$

where

$$\rho = \rho(t, \mathbf{x}'), \quad \mathbf{J} = \mathbf{J}(t, \mathbf{x}'), \quad \hat{\mathbf{R}} = \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|}.$$

Notice that the potential representation (8.66) is perfectly consistent with the potential expressions in (8.48). To get the indicated representation (8.66) for the transversal component $\mathbf{A}_T$ of the vector potential $\mathbf{A}$ defined by (8.58)–(8.59), we solve first the Poisson equation (8.60) for $\mathbf{A}_T$ using representation (8.65) for the transversal current $\mathbf{J}_T$, resulting in

$$\mathbf{A}_T = \frac{4\pi}{c} (-\Delta)^{-1} \mathbf{J}_T = \frac{4\pi}{c} [(-\Delta)^{-1} \mathbf{J} + \nabla (\nabla \cdot (-\Delta)^{-2} \mathbf{J})]. \quad (8.67)$$

Then using the following Green functions (Riesz potentials) (37.11), (37.12)

$$(-\Delta)^{-1}(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}'|}, \quad (-\Delta)^{-2}(\mathbf{x}, \mathbf{x}') = -\frac{1}{8\pi} |\mathbf{x} - \mathbf{x}'|, \quad (8.68)$$

and the operator representations (8.67), we obtain that

$$\begin{aligned} \mathbf{A}_T &= \frac{4\pi}{c} \left[\frac{1}{4\pi} \int \frac{\mathbf{J}(t, \mathbf{x}')}{R} d\mathbf{x}' - \frac{1}{8\pi} \nabla_{\mathbf{x}} \left(\nabla_{\mathbf{x}} \cdot \int R \mathbf{J}(t, \mathbf{x}') d\mathbf{x}' \right) \right] \\ &= \frac{1}{2c} \int \frac{\mathbf{J}(t, \mathbf{x}') + \hat{\mathbf{R}} (\hat{\mathbf{R}} \cdot \mathbf{J}(t, \mathbf{x}'))}{R} d\mathbf{x}', \end{aligned} \quad (8.69)$$

which is the desired presentation for $\mathbf{A}_T$ in (8.66).

8.4 The First Non-relativistic Approximation

We introduce the term *almost electrostatic* referring to an approximation when the vector potential and the magnetic field are neglected entirely. We also refer to this approximation as the *first non-relativistic approximation*. This approximation is based on the Lagrangian $L_{\text{em1}}(A)$ introduced in Sect. 9.4.2.

In this approximation, the only field that matters is the electric field $\mathbf{E}$ satisfying the following equations:

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \nabla \times \mathbf{E} = 0. \quad (8.70)$$

The relation between the field and the potential is

$$\mathbf{E} = -\nabla\varphi, \quad (8.71)$$

and then

$$\Delta\varphi = -4\pi\rho. \quad (8.72)$$

The Lorentz force density expression (9.19) then is reduced to

$$\mathbf{F} = \rho\mathbf{E}. \quad (8.73)$$

8.5 The Second Non-relativistic Approximation

Our *second non-relativistic approximation* is based on the Lagrangian $L_{\text{em}2}(\mathbf{A})$ introduced in Sect. 9.4.3, and it is in many respects similar to the EQS approximation discussed in Sect. 8.2 and to Darwin's approximation discussed in Sect. 8.3. In particular, the EL equations for the potentials φ and $\mathbf{A}$ are the following Poisson equations

$$-\nabla^2\varphi = 4\pi\rho, \quad -\nabla^2\mathbf{A} = \frac{4\pi}{c}\mathbf{J}. \quad (8.74)$$

The charge conservation law (continuity equation) (4.5), namely $\partial_t\rho + \nabla \cdot \mathbf{J} = 0$, is always fulfilled in our models both in the relativistic and non-relativistic cases. Therefore, the potentials defined by (8.74) satisfy the Lorentz gauge condition (4.9), namely

$$\frac{1}{c}\partial_t\varphi + \nabla \cdot \mathbf{A} = 0. \quad (8.75)$$

We preserve the relation between the potentials and the EM fields exactly as in (4.6) for Maxwell equations, namely

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\partial_t\mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (8.76)$$

According to our analysis in Sects. 16.1, 17.2.1, exactly this relation between the EM potentials and the EM fields correctly determines the Lorentz forces which act on the charges in point-like regimes.

Observe now that the Poisson equations (8.74) for the potentials φ and $\mathbf{A}$ are identical to the Poisson equations (8.39) for the EQS approximation considered in Sects. 8.1, 8.2 and to the Poisson equations (8.60), (8.59) for Darwin's approximation in Sect. 8.3. But expression (8.76) for the electric field differs both from the formula (8.17) for the EQS approximation and from (8.57) for Darwin's approximation.

The reason we use (8.76) is that we will use the second non-relativistic approximation not for EM fields alone, but for EM fields coupled with the charge distributions governed by the NLS equations. The Lorentz force density is derived from the NLS analysis, and is given by expression (9.19), namely

$$\mathbf{F} = \rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B}$$

where $\mathbf{E}$ and $\mathbf{B}$ are given by (8.76), hence $\mathbf{E}$ and $\mathbf{B}$ should be given by the same expression in our EM theory. Our second non-relativistic approximation produces the same order of approximation of relativistic regimes as EQS but allows us to preserve exactly expressions for the Lorentz force density and the classical definition (4.6) of EM fields in terms of the EM potentials.

Chapter 9

Electromagnetic Field Lagrangians

The history of use of the Lagrange method for modeling electromagnetic interactions goes back to the 1800s, being associated with names of R. Clausius (1877, 1880), O. Heaviside (1889), J. Larmor (1900) and K. Schwarzschild (1903), [182, II.D]. The standard Maxwellian Lagrangian for the EM field can be written in terms of the field strength tensor $F(A) = F^{\mu\nu}(A^\alpha)$ (4.14) as follows, [179, 12.7], [27, III.4.D, IV.1], [332, 2.8.2]:

$$L_{\text{em}}(A, J) = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}J_\mu A^\mu = \frac{\mathbf{E}^2 - \mathbf{B}^2}{8\pi} - \rho\varphi + \frac{1}{c}\mathbf{J} \cdot \mathbf{A}, \quad (9.1)$$

where J_μ is an external (impressed) current satisfying the continuity condition (4.24), namely

$$\partial_\mu J^\mu = 0 \quad \text{or} \quad \partial_t \rho + \nabla \cdot \mathbf{J} = 0. \quad (9.2)$$

Direct computation shows that the Maxwell equations (4.23) coincide with the Euler–Lagrange equations derived from the Lagrangian (9.1). Note also that the above Lagrangian can be recast as

$$L_{\text{em}}(A, J) = \frac{1}{8\pi} \left[\left(\nabla\varphi + \frac{1}{c}\partial_t \mathbf{A} \right)^2 - (\nabla \times \mathbf{A})^2 \right] - \rho\varphi + \frac{1}{c}\mathbf{J} \cdot \mathbf{A} \quad (9.3)$$

for which the Euler–Lagrange equations take the vector form of the Maxwell equations (4.7), (4.8).

Instead of the standard EM Lagrangian $L_{\text{em}}(A, J)$, which is manifestly gauge invariant, an alternative Lagrangian can be written in terms of the decomposition of the vector potentials $\mathbf{A}$ and the current $\mathbf{J}$ into its longitudinal and transverse components. Using the EM fields representations (7.31) where φ_C is defined by (7.14) and the charge conservation law (7.25), we combine them with evident identities $\nabla \cdot \mathbf{A}_T = 0$ and $\nabla \cdot \mathbf{J}_T = 0$ and obtain after elementary transformations the following identities:

$$\begin{aligned}
& -\rho\varphi + \frac{1}{c}\mathbf{J} \cdot \mathbf{A} \\
& = -\rho\varphi_C + \frac{1}{c}\mathbf{J}_T\mathbf{A}_T + \frac{1}{c}\partial_t(\rho\alpha_L) + \frac{1}{c}\nabla \cdot (\nabla\xi_L - \xi_L\mathbf{A}_T - \alpha_L\mathbf{J}_T),
\end{aligned} \tag{9.4}$$

$$\begin{aligned}
& \left(\nabla\varphi + \frac{1}{c}\partial_t\mathbf{A} \right)^2 - (\nabla \times \mathbf{A}_T)^2 \\
& = (\nabla\varphi_C)^2 + \left(\frac{1}{c}\partial_t\mathbf{A}_T \right)^2 - (\nabla \times \mathbf{A}_T)^2 + 2\frac{1}{c}\nabla \cdot (\varphi_C\partial_t\mathbf{A}_T).
\end{aligned} \tag{9.5}$$

The above identities imply, in turn, the following representation of the standard EM Lagrangian:

$$\begin{aligned}
L_{\text{em}}(A, J) &= \frac{1}{8\pi} \left[\left(\nabla\varphi + \frac{1}{c}\partial_t\mathbf{A} \right)^2 - (\nabla \times \mathbf{A})^2 \right] - \rho\varphi + \mathbf{J} \cdot \mathbf{A} \\
&= \frac{1}{8\pi} \left[(\nabla\varphi_C)^2 + \left(\frac{1}{c}\partial_t\mathbf{A}_T \right)^2 - (\nabla \times \mathbf{A}_T)^2 \right] - \rho\varphi_C + \frac{1}{c}\mathbf{J}_T\mathbf{A}_T + \partial_\mu D^\mu,
\end{aligned} \tag{9.6}$$

where

$$D^\mu = \left(\rho\alpha_L, \frac{1}{c}(\alpha_L\nabla\xi_L - \xi_L\mathbf{A}_T - \alpha_L\mathbf{J}_T + 2\varphi_C\partial_t\mathbf{A}_T) \right). \tag{9.7}$$

Since two Lagrangians differing by a 4-divergence $\partial_\mu D^\mu$ have equivalent Euler–Lagrange equations, by omitting $\partial_\mu D^\mu$ in Eq.(9.6) we obtain the following form of the EM Lagrangian as a function of the pair of potentials $(\varphi_C, \mathbf{A}_T)$:

$$\begin{aligned}
L_{\text{em}}(\varphi_C, \mathbf{A}_T) &= \frac{1}{8\pi} \left[(\nabla\varphi_C)^2 + \left(\frac{1}{c}\partial_t\mathbf{A}_T \right)^2 - (\nabla \times \mathbf{A}_T)^2 \right] \\
&\quad - \rho\varphi_C + \frac{1}{c}\mathbf{J}_T\mathbf{A}_T,
\end{aligned} \tag{9.8}$$

where the scalar potential φ_C and the transverse vector potential $\mathbf{A}_T$ are treated as independent variables. The Euler–Lagrange equations for the Lagrangian $L_{\text{em}}(\varphi_C, \mathbf{A}_T)$ defined above are

$$\nabla^2\varphi_C = -4\pi\rho, \tag{9.9}$$

$$-\frac{1}{c^2}\partial_t^2\mathbf{A}_T + \nabla^2\mathbf{A}_T = -\frac{4\pi}{c}\mathbf{J}_T. \tag{9.10}$$

Notice that the above equations match the potential form (7.11)–(7.12) for the Maxwell equations in the Coulomb gauge.

9.1 Energy-Momentum Tensor for Electromagnetic Field

The canonical energy-momentum (stress, power-momentum) tensor (EnMT) $\hat{\Theta}^{\mu\nu}$ for the EM field based on the Lagrangian (9.1) in the case of the presence of external currents J_μ is as follows, [179, 12.10], [27, III.4.D, IV.1], [332, 2.3.5, 2.8.2]:

$$\hat{\Theta}^{\mu\nu} = -\frac{1}{4\pi} F^{\mu\gamma} \partial^\nu A_\gamma + \frac{1}{16\pi} g^{\mu\nu} F^{\xi\gamma} F_{\xi\gamma} + g^{\mu\nu} \frac{1}{c} J_\alpha A^\alpha. \quad (9.11)$$

The above formula agrees with the general definitions (10.54) of the canonical energy-momentum as Noether's conserved current associated with the space-time translations symmetries. The components of the EnMT $\hat{\Theta}^{\mu\nu}$ (A, J) are given by the following expressions:

$$\begin{aligned} \hat{\Theta}^{00} &= -\frac{1}{8\pi} (\mathbf{E}^2 - \mathbf{B}^2) + \rho\varphi - \frac{1}{c} \mathbf{J} \cdot \mathbf{A} - \frac{1}{4\pi} \partial_0 \mathbf{A} \cdot \mathbf{E}, \\ \hat{\Theta}^{0i} &= -\frac{\partial_i \mathbf{A} \cdot \mathbf{E}}{4\pi}, \quad \hat{\Theta}^{i0} = -\frac{E_i \partial_0 \varphi}{4\pi} + \frac{(\mathbf{B} \times \partial_0 \mathbf{A})_i}{4\pi}, \\ \hat{\Theta}^{ij} &= -\frac{E_i \partial_j \varphi}{4\pi} + \frac{(\mathbf{B} \times \partial_j \mathbf{A})_i}{4\pi} + \left(\frac{\mathbf{E}^2 - \mathbf{B}^2}{8\pi} - \rho\varphi + \frac{1}{c} \mathbf{J} \cdot \mathbf{A} \right) \delta^{ij}, \end{aligned} \quad (9.12)$$

where $i, j = 1, 2, 3$. Notice that the canonical energy-momentum $\hat{\Theta}^{\mu\nu}$ is not gauge invariant, for it depends on the potentials and it is not symmetric. Let us recall though that the EnMT is not defined uniquely but up to a term of zero divergence, Sect. 10.5. This flexibility in choosing the EnMT can be used to define a *symmetric and gauge invariant energy-momentum* $\Theta^{\mu\nu}$ for the EM field as follows, [179, 12.10C], [27, III.4.D], [332, 2.3.5, 2.8.2]:

$$\Theta^{\mu\nu}(F) = \frac{1}{4\pi} \left(g^{\mu\alpha} F_{\alpha\beta} F^{\beta\nu} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right). \quad (9.13)$$

The above definition implies the following formulas for the EM field energy density u_{em} , the momentum density $\mathbf{P}_{\text{em}}$ and the Maxwell stress tensor τ_{ij} :

$$u_{\text{em}} = \Theta^{00} = \frac{1}{8\pi} (\mathbf{E}^2 + \mathbf{B}^2), \quad cP_{\text{emi}} = \Theta^{0i} = \Theta^{i0} = \frac{1}{4\pi} \mathbf{E} \times \mathbf{B}, \quad (9.14)$$

$$\Theta^{ij} = -\tau_{ij} = -\frac{1}{4\pi} \left[E_i E_j + B_i B_j - \frac{1}{2} \delta_{ij} (\mathbf{E}^2 + \mathbf{B}^2) \right], \quad (9.15)$$

$$\begin{aligned} \Theta^{\mu\nu} &= \begin{bmatrix} u_{\text{em}} & c\mathbf{P}_{\text{em}} \\ c\mathbf{P}_{\text{em}} & -\tau_{ij} \end{bmatrix}, \quad \Theta_{\mu\nu} = \begin{bmatrix} u_{\text{em}} & -c\mathbf{P}_{\text{em}} \\ -c\mathbf{P}_{\text{em}} & -\tau_{ij} \end{bmatrix}, \\ \Theta^\mu{}_\nu &= \begin{bmatrix} u_{\text{em}} & -c\mathbf{P}_{\text{em}} \\ c\mathbf{P}_{\text{em}} & -\tau_{ij} \end{bmatrix}, \quad \Theta_\mu{}^\nu = \begin{bmatrix} u_{\text{em}} & c\mathbf{P}_{\text{em}} \\ -c\mathbf{P}_{\text{em}} & -\tau_{ij} \end{bmatrix}. \end{aligned} \quad (9.16)$$

It is instructive to observe a substantial difference between the expressions for Θ^{00} , which is the Hamiltonian density of the EM field, and the energy density Θ^{00} defined by (9.14).

9.2 Conservation Laws

One can deduce from the Maxwell equations that the energy-momentum $\Theta^{\mu\nu}$ satisfies the following *conservation laws*, [179, 12.10C], [27, III.4.D], [332, 2.3.5, 2.8.2]:

$$\partial_\mu \Theta^{\mu\nu} = -f^\nu, \quad (9.17)$$

where the 4-vector f^μ is given by the expression

$$f^\mu = \frac{1}{c} F^{\mu\nu} J_\nu \quad (9.18)$$

and is known as the *Lorentz force density*.

Note that, using the matrix form (4.18) of the tensor $F^{\mu\nu}$ and (4.13), we can express the force density f^μ by the following formula:

$$f^\mu = \frac{1}{c} F^{\mu\nu} J_\nu = \left(\frac{1}{c} \mathbf{J} \cdot \mathbf{E}, \rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} \right). \quad (9.19)$$

The time and space components of the Eq.(9.17) imply the conservation laws for the energy u_{em} and momentum $\mathbf{P}_{\text{em}}$. The energy conservation law can be recast as follows:

$$\frac{1}{c} (\partial_t u_{\text{em}} + \nabla \cdot \mathbf{S}) = -\frac{1}{c} \mathbf{J} \cdot \mathbf{E}, \quad (9.20)$$

where $\mathbf{S}$ is the Poynting vector, namely

$$\mathbf{S} = c^2 \mathbf{P}_{\text{em}} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}.$$

The energy conservation law (9.20) is often called Poynting's theorem, [179, 6.7]. The momentum conservation law takes the form

$$\partial_t P_{\text{emi}} - \sum_{j=1}^3 \frac{\partial}{\partial x^j} \tau_{ij} = - \left[\rho E_i + \frac{1}{c} (\mathbf{J} \times \mathbf{B})_i \right]. \quad (9.21)$$

For completeness, we provide, following [27, III.4.D], a direct verification of the conservation law (9.17). We differentiate (9.13):

$$\begin{aligned}\partial_\mu \Theta^{\mu\nu} &= \frac{1}{4\pi} \left(\partial^\alpha (F_{\alpha\beta} F^{\beta\nu}) + \frac{1}{4} \partial^\nu (F_{\alpha\beta} F^{\alpha\beta}) \right) \\ &= \frac{1}{4\pi} \left(\partial^\alpha F_{\alpha\beta} F^{\beta\nu} + F_{\alpha\beta} \partial^\alpha F^{\beta\nu} + \frac{1}{2} F_{\alpha\beta} \partial^\nu F^{\alpha\beta} \right).\end{aligned}$$

To simplify the right-hand side, we use formula (4.22) written in the form

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0$$

and the Maxwell equation (4.23) and arrive at the following equation:

$$\begin{aligned}\partial_\mu \Theta^{\mu\nu} &= \frac{1}{4\pi} \left(\frac{4\pi}{c} J_\beta F^{\beta\nu} + F_{\alpha\beta} F^{\beta\nu,\alpha} - \frac{1}{2} F_{\alpha\beta} (\partial^\alpha F^{\beta\nu} + \partial^\beta F^{\nu\alpha}) \right) \\ &= \frac{1}{c} J_\beta F^{\beta\nu} + \frac{1}{8\pi} F_{\alpha\beta} \partial^\alpha F^{\beta\nu} - \frac{1}{8\pi} F_{\alpha\beta} \partial^\beta F^{\nu\alpha}.\end{aligned}$$

The last two terms cancel, and we obtain (9.17).

The conservation law for the canonical EnMT $\dot{\Theta}^{\mu\nu}$ has the form (11.83). If there are no external currents, the conservation law takes a simpler form

$$\partial_\mu \Theta^{\mu\nu} = 0, \quad \partial_\mu \dot{\Theta}^{\mu\nu} = 0. \quad (9.22)$$

In particular, we obtain the energy conservation law

$$\frac{1}{c} (\partial_t u_{\text{em}} + \nabla \cdot \mathbf{S}) = 0. \quad (9.23)$$

9.3 The Fermi Lagrangian

Notice that the classical wave equations (4.28) for the EM potentials can be obtained as the Euler–Lagrange (EL) equations for the following Lagrangian:

$$L_{\text{em}0}(A, J) = -\frac{1}{8\pi} \partial_\mu A_\nu \partial^\mu A^\nu - \frac{1}{c} J^\nu A_\nu \quad (9.24)$$

where J^ν is an external current satisfying the continuity equation

$$\partial_\nu J^\nu = 0. \quad (9.25)$$

The vector form of the above Lagrangian is

$$L_{\text{em}0} = -\frac{1}{8\pi} \left[\frac{(\partial_t \varphi)^2}{c^2} - (\nabla \varphi)^2 - \frac{(\partial_t \mathbf{A})^2}{c^2} + (\nabla \mathbf{A})^2 \right] - \rho \varphi + \frac{1}{c} \mathbf{J} \cdot \mathbf{A}. \quad (9.26)$$

Recall that the potential wave equations (4.28) are equivalent to the Maxwell equations provided that the external current J^ν satisfies the continuity equation (9.25) and the 4-potential A^μ satisfies the Lorentz gauge condition (4.9), (4.20)

$$\partial_\mu A^\mu = 0. \quad (9.27)$$

The Lagrangian $L_{\text{em}0}$ defined by (9.24) is known in the literature on quantum field theory as the *Fermi Lagrangian or Fock–Podolsky Lagrangian*, [27, III.3], [33], [43, §4, 5], [44, 4.4], [119], [115, 32.4], [231, 5.2], [260], [266, II.2(a)], [337, IV.16]. It is the Fermi Lagrangian rather than the standard gauge invariant Lagrangian $L_{\text{em}}(A, J)$ defined by (9.1) that happens to be the most convenient for field quantization. We are interested in the Fermi Lagrangian since the classical wave equations are useful for our non-relativistic approximations. Note that the EM fields are defined in terms of EM potentials by the standard expressions (4.6), namely

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\partial_t\mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (9.28)$$

The first important point when using the Fermi Lagrangian (9.24) for EM fields is that *the Lorentz gauge condition (9.27) must be imposed to complement the Lagrangian formalism*. To avoid any confusion, we stress that the Lorentz gauge condition (9.27) is not treated as a “constraint” to be satisfied by field variations. Instead it is imposed to complement the Euler–Lagrange equations, selecting only those solutions that satisfy the Lorentz gauge condition (9.27).

Then an important question arises: does the Fermi Lagrangian $L_{\text{em}0}$ complemented by the Lorentz gauge condition and compared with the standard Lagrangian L_{em} defined by (9.1) have the same or different (i) the Euler–Lagrange equations and (ii) the energy-momentum and angular momentum tensors? The answer to the first part of the question is that the EL equations for the Fermi Lagrangian $L_{\text{em}0}$ with the added Lorentz gauge condition are equivalent to Maxwell’s. Namely, they are exactly the classical wave equations which are equivalent to the Maxwell equations as discussed at the end of Sect. 4.1. The same conclusion can be obtained alternatively based on general grounds, see Sect. 10.1. Indeed, in view of identity (9.32) below, the difference $L_{\text{em}0} - L_{\text{em}}$ can be represented as a divergence after the Lorentz gauge condition is taken into account.

As to the second question, the answer is that the expressions for the energy-momentum and angular momentum tensors are different for the two Lagrangians $L_{\text{em}0}$ and L_{em} . But we recall now that the energy-momentum is not defined uniquely but up to a zero divergence term, see Sect. 10.5, and the same applies to the angular momentum tensor. In this case, the difference between the two energy-momentum tensors has zero divergence after taking into account the Lorentz gauge conditions and the EL equations, and a similar statement holds for the angular-momentum tensor. This fact is well known, [43, 4.1, p. 37], and we verify it below for the energy-momentum tensor.

Indeed, notice first that, using the definition of the EM field tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, one can verify the following elementary identities:

$$\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \partial_\mu A^\nu \partial^\mu A_\nu - \partial_\mu A_\nu \partial^\nu A^\mu, \quad (9.29)$$

$$\partial_\mu (A^\nu \partial_\nu A^\mu - A^\mu \partial_\nu A^\nu) = \partial^\mu A^\nu \partial_\nu A_\mu - (\partial_\mu A^\mu)^2. \quad (9.30)$$

The identities (9.29)–(9.30) imply in turn

$$\begin{aligned} \partial_\mu A^\nu \partial^\mu A_\nu &= \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \partial_\mu A_\nu \partial^\nu A^\mu \\ &= \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu A^\mu)^2 + \partial_\mu (A^\nu \partial_\nu A^\mu - A^\mu \partial_\nu A^\nu), \end{aligned} \quad (9.31)$$

$$\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \partial_\mu A^\nu \partial^\mu A_\nu - (\partial_\mu A^\mu)^2 - \partial_\mu (A^\nu \partial_\nu A^\mu - A^\mu \partial_\nu A^\nu). \quad (9.32)$$

We can infer from the identity (9.32) that the difference between the Fermi and standard Lagrangian after taking into account the Lorentz gauge condition (9.27) is a divergence, and consequently the EL equations for the Lagrangians must be equivalent, a fact we already know from the discussion at the end of Sect. 4.1.

Let us consider now the difference between two energy-momentum tensors. Recall that general formula (10.54) for the canonical energy-momentum tensor $\hat{T}^{\mu\nu}$ for the Lagrangian $\mathcal{L}$ involving field variables $\bar{\ell}$, μ reads

$$\hat{T}^{\mu\nu} = \hat{T}^{\mu\nu}(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell},\mu}} \cdot q_{\bar{\ell}}^\nu - g^{\mu\nu} \mathcal{L}. \quad (9.33)$$

In our case, the variables $q_{\bar{\ell},\mu}$ are $\partial_\mu A^\alpha$ where $\bar{\ell}$ is α . Following the results discussed at the end of Sect. 10.5, we would like to show that the difference between the two canonical EnMT $\hat{T}^{\mu\nu}(L_{\text{em}0})$ and $\hat{T}^{\mu\nu}(L_{\text{em}})$ has zero divergence:

$$\partial_\mu \left[\hat{T}^{\mu\nu}(L_{\text{em}0}) - \hat{T}^{\mu\nu}(L_{\text{em}}) \right] = 0. \quad (9.34)$$

Using definitions (9.24) and (9.1) for Lagrangians $L_{\text{em}0}$ and L_{em} and the first equality in (9.31), we obtain that

$$L_{\text{em}0} - L_{\text{em}} = -\frac{1}{8\pi} \left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \partial_\mu A^\nu \partial^\mu A_\nu \right) = \frac{1}{8\pi} \partial_\mu A_\nu \partial^\nu A^\mu. \quad (9.35)$$

Applying general formula (9.33) to the Lagrangian $\mathcal{L} = \partial_\mu A_\nu \partial^\nu A^\mu$, we get

$$\hat{T}^{\mu\nu}(\partial_\mu A_\nu \partial^\nu A^\mu) = 2\partial_\alpha A^\mu \partial^\nu A^\alpha - g^{\mu\nu} \partial_\alpha A^\beta \partial_\beta A^\alpha. \quad (9.36)$$

Then taking the divergence of the above EnMT and using the Lorentz gauge condition (9.27), we obtain after elementary transformations that

$$\begin{aligned} \partial_\mu \dot{T}^{\mu\nu} (\partial_\mu A_\nu \partial^\nu A^\mu) &= \partial_\mu (2\partial_\alpha A^\mu \partial^\nu A^\alpha - g^{\mu\nu} \partial_\alpha A^\beta \partial_\beta A^\alpha) \\ &= 2\partial_\alpha A^\mu (\partial_\mu \partial^\nu A^\alpha) - (\partial_\alpha \partial^\nu A^\beta) \partial_\beta A^\alpha - \partial_\alpha A^\beta (\partial^\nu \partial_\beta A^\alpha) \\ &= 2\partial_\alpha A^\mu (\partial_\mu \partial^\nu A^\alpha) - \partial_\alpha A^\mu (\partial_\mu \partial^\nu A^\alpha) - \partial_\alpha A^\mu (\partial^\nu \partial_\mu A^\alpha) = 0. \end{aligned} \quad (9.37)$$

Combining Eqs. (9.35)–(9.37), we conclude that Eq. (9.34) is correct. This completes the proof of the statement that the difference between the two energy-momentum tensors has zero divergence after taking into account the Lorentz gauge conditions and the EL equations.

9.4 Non-relativistic Quasistatic EM Lagrangians and the Field Equations

Having in mind non-relativistic quasistatic approximations, we introduce two non-relativistic Lagrangians $L_{\text{em1}}(A)$ and $L_{\text{em2}}(A)$ for the EM field. We also discuss Electrostatics and Darwin's Lagrangians.

9.4.1 Electroquasistatics and Darwin's Lagrangians

If we set $\frac{1}{c} \partial_t \mathbf{A} = \mathbf{0}$ in the Lagrangian (9.3), we obtain the EQS Lagrangian L_{EQS} :

$$L_{\text{EQS}} = \frac{1}{8\pi} [(\nabla\varphi)^2 - (\nabla \times \mathbf{A})^2] - \rho\varphi + \frac{1}{c} \mathbf{J} \cdot \mathbf{A}. \quad (9.38)$$

A straightforward application of the variational approach described in Chap. 10 to this Lagrangian may lead to an inconsistent system of equations, in particular to the equation

$$\nabla \times (\nabla \times \mathbf{A}) = \frac{1}{c} \mathbf{J}$$

which does not have solutions if $\nabla \cdot \mathbf{J} \neq \mathbf{0}$. Therefore, certain modifications are required to produce electroquasistatic equations (8.11), (8.12). We assume that $\rho, \mathbf{J}$ satisfy the continuity equation (4.5), and we also assume that the EL equations should be derived from the action integral involving the Lagrangian L_{EQS} by variation under the constraint

$$\nabla \cdot \mathbf{A} = \mathbf{0}. \quad (9.39)$$

As a result, we obtain EL equations of the form

$$\nabla \times (\nabla \times \mathbf{A}) = \frac{4\pi}{c} \mathbf{J} + \nabla \alpha, \quad (9.40)$$

$$-\nabla^2 \varphi_C = 4\pi \rho. \quad (9.41)$$

The gradient (longitudinal) field $\nabla \alpha$ emerges in (9.40) since the restriction of variations of $\mathbf{A}$ to divergence-free (transversal) fields allows the presence in the Euler–Lagrange equation of a field orthogonal to the variations, and according to the Helmholtz decomposition such fields are longitudinal (see Chap. 44). Evaluating the divergence of both sides of (9.40) and taking into account the continuity equation (4.5), we see that

$$\nabla^2 \alpha = -\frac{4\pi}{c} \nabla \cdot \mathbf{J} = 4\pi \partial_t \rho.$$

Comparing with (9.41) and (8.19), we see that

$$\alpha = -\partial_t \varphi, \quad \nabla \alpha = \partial_t \mathbf{E},$$

and (9.40)–(9.41) is equivalent to (8.11), (8.12). Hence the Lagrangian (9.38) describes the electroquasistatic approximation.

We have already pointed out in Sect. 8.3 that the potential form (8.60) of the Darwin approximation is identical to the alternative form (8.41) for the EQS approximation with the difference being in the relations between the fields and potentials. Consequently, Darwin’s approximation Lagrangian L_D can be defined by the same formula (9.38):

$$L_D = L_{\text{EQS}} = \frac{1}{8\pi} [(\nabla \varphi)^2 - (\nabla \times \mathbf{A})^2] - \rho \varphi + \frac{1}{c} \mathbf{J} \cdot \mathbf{A},$$

with the constraint (9.39) added to the Lagrangian formalism, as in the case of the Lagrangian L_{EQS} .

9.4.2 *The First Non-relativistic EM Field Lagrangian and the Field Equations*

We use the term the first non-relativistic approximation for the case when the magnetic field is neglected entirely by assuming that

$$\mathbf{A} = 0, \quad A^\mu = (\varphi, 0, 0, 0). \quad (9.42)$$

The first non-relativistic approximation is associated with the following EM Lagrangian:

$$L_{\text{em1}}(A, \rho) = -\frac{1}{8\pi} \partial_i \varphi \partial^i \varphi - \rho \varphi = \frac{1}{8\pi} (\nabla \varphi)^2 - \rho \varphi, \quad (9.43)$$

where $\rho = \rho(t, \mathbf{x})$ is an external charge density. This approximation can be referred to as almost electrostatic, for the potential $\varphi = \varphi(t, \mathbf{x})$ can depend on time following the charge density $\rho = \rho(t, \mathbf{x})$. Observe that the Lagrangian $L_{\text{em1}}(A)$ can be obtained from the EM Lagrangian $L_{\text{em1}}(A)$ defined by (9.3) by setting there $\mathbf{A} = 0$.

Though the above Lagrangian does not involve the current $\mathbf{J}$, we still assume that there is a current $\mathbf{J} = \mathbf{J}(t, \mathbf{x})$ and that the charge continuity equation (4.5) holds, that is,

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0. \quad (9.44)$$

In this case the electric field $\mathbf{E}$ and the magnetic inductions $\mathbf{B}$ defined by (4.6) turn into

$$\mathbf{E} = -\nabla \varphi, \quad \mathbf{B} = \mathbf{0}. \quad (9.45)$$

The EL equations in this case turn into the Poisson equation for the potential:

$$-\nabla^2 \varphi = 4\pi \rho. \quad (9.46)$$

Then, in view of (9.45) and (9.46), the Maxwell equation (4.1), that is,

$$\nabla \cdot \mathbf{E} = 4\pi \rho \quad (9.47)$$

is satisfied. The other two Maxwell equations (4.2) and (4.4) are evidently satisfied as well. As to the Ampere equation (4.3), for the fields $\mathbf{E}$ and $\mathbf{B}$ defined by (9.45) it is reduced to the equation

$$\partial_t \mathbf{E} = 4\pi \mathbf{J} \quad (9.48)$$

which is consistent with the assumed charge conservation (9.44).

9.4.3 *The Second Non-relativistic EM Field Lagrangian and the Field Equations*

We introduce here the second non-relativistic type Lagrangian $L_{\text{em2}}(A)$, used in Sect. 12.2.1, which is obtained from the Fermi–Fock–Podolsky Lagrangian (9.26) by omitting there the time derivatives $\frac{1}{c} \partial_t \varphi$ and $\frac{1}{c} \partial_t \mathbf{A}$, namely

$$\begin{aligned}
L_{\text{em}2}(A, J) &= -\frac{1}{8\pi} \partial_i \varphi \partial^i \varphi - \frac{1}{8\pi} \partial_i A_j \partial^i A^j - \frac{1}{c} J^\nu A_\nu \\
&= \frac{1}{8\pi} (\nabla \varphi)^2 - \frac{1}{8\pi} (\nabla \mathbf{A})^2 - \rho \varphi + \frac{1}{c} \mathbf{J} \cdot \mathbf{A}.
\end{aligned} \tag{9.49}$$

It is assumed that ρ and $\mathbf{J}$ satisfy the continuity equation (4.24).

The EL equations for the Lagrangian $L_{\text{em}2}(A)$ are the following Poisson equations:

$$-\nabla^2 A^\nu = \frac{4\pi}{c} J^\nu, \quad \text{or} \quad -\nabla^2 \varphi = 4\pi \rho, \quad -\nabla^2 \mathbf{A} = \frac{4\pi}{c} \mathbf{J}. \tag{9.50}$$

Evidently, the above equations can alternatively be obtained from the wave equations (4.28) by neglecting the time derivatives $\partial_0 A^\alpha$, just as in the case of obtaining the Lagrangian $L_{\text{em}2}(A)$ from the Lagrangian (9.24), see Sect. 9.3. Note also that solutions to the Poisson equations (9.50) can be obtained from expressions (4.42) for solutions of the wave equations (4.28) derived from the Fermi–Fock–Podolsky Lagrangian by neglecting the time retardation. *Just as in the case of the Fermi–Fock–Podolsky Lagrangian, an additional Lorentz gauge condition (4.9) is imposed on the solutions of the EL equations, namely*

$$\frac{1}{c} \frac{\partial \varphi}{\partial t} + \nabla \cdot \mathbf{A} = 0. \tag{9.51}$$

For bounded solutions of the Poisson equations (9.50) fulfillment of the Lorentz gauge condition (9.51) is equivalent to the charge continuity equation (4.24):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{9.52}$$

The corresponding expressions (4.6) for $\mathbf{E}$ and $\mathbf{B}$ are

$$\mathbf{E} = -\nabla \varphi - \frac{1}{c} \partial_t \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{9.53}$$

For a discussion of the relations between our second non-relativistic approximation and EQS and Darwin’s approximations, see Sect. 8.5.

Part II

Classical Field Theory

“Finally, there is the Principle of Least Action. Now, Least Action has no more to do with the matter than the man in the moon, so far as I can see. It is quite unnecessary, to begin with. Next, it obscures and complicates the matter, so much so as sometimes to lead to serious error. I make this remark advisedly, remembering previous applications of the Principle of Least Action to electromagnetics, which is much clearer without it.”

*O. Heaviside.*¹

Classical field theory is a mathematical foundation of many aspects of the theory we advance here. Important concepts of field theory are systematically used in our studies of distributed charge models, in particular for the derivation of Newtonian dynamics for localized distributed charges. To make the presentation self-contained, we provide its condensed exposition, relying on many excellent monographs written on this subject, to name a few [27], [137], [213], [250, Sect. 3.4], [266].

The exposition includes the basics of relativity theory, variational principles including the Principle of Least Action, the Lagrangian formalism with symmetries, gauge invariance, and conservation laws. Whenever possible, we provide multiple references to monographs and papers, so the reader can benefit from different and complementary points of view leading to a deeper understanding. We also provide a concise derivation of formulas whenever possible. The list of considered subjects includes symmetries, conservation laws, and energy-momentum tensors for the Lagrangians of the distributed charge theories. We do not discuss in this part the physical aspects of the charge theories; they are saved for Part III.

In contrast to the views of O. Heaviside shown in the above epigraph, J.C. Maxwell appreciated the Lagrange method, for it “brings dynamics under the power of the calculus,” see page 15 for the complete quotations. The power to concisely express a dynamical problem in a single formula was stressed by J. Larmor, [218, p. 83]:

¹[155, p. 509].

“It is now a well-trying resource to utilize the principle that every dynamical problem can be enunciated, in a single formula, as a variation problem, in order to help in the reduction to dynamics of physical theories in which the intimate dynamical machinery is more or less hidden from direct inspection.”

Chapter 10

Variational Principles, Lagrangians, Field Equations and Conservation Laws

“Amid the more or less general laws which mark the achievements of physical science during the course of the last centuries, the principle of least action is perhaps that which, as regards form and content, may claim to come nearest to that ideal final aim of theoretical research.”
*M. Planck.*¹

The important role of the variational principles and the Lagrangian formalism for the foundations of physics is well recognized. H. Helmholtz referred to the principle of least action as a unifying natural law dominating all of physics, [349, p. 143]:

“... the general validity of the principle-of least action seems to me assured, since it may claim a higher place as a heuristic and guiding principle in our endeavour to formulate the laws governing new classes of phenomena.”

The variational principles and the principle of least action were not always appreciated by everyone. For instance, O. Heaviside, who made significant contributions to the development of the EM theory, had very different opinion about the place of the principle of least action in physics, [156, p. 175]:

“Now at Cambridge, or somewhere else, there is a golden or brazen idol called the Principle of Least Action. Its exact locality is kept secret, but numerous copies have been made and distributed amongst the mathematical tutors and lecturers at Cambridge, who make the young men fall down and worship the idol.”

In this section we collect well known basic facts from the classical Lagrangian field formalism and the Principle of Least Action following [27, Section III.3], [250, Section 3.4], [266] and other classical sources.

¹[349, p. 144].

10.1 The Action Integral and the Euler–Lagrange Field Equations

“Among all the possible ways of passing from one position to another, there is evidently one for which the average value of the action is less than for any other. There is, moreover, only one; and it results from this that the principle of least action suffices to determine the path followed and consequently the equations of motion. Thus we obtain what are called the equations of Lagrange.”

*H. Poincaré.*²

Let us recall here the very basics of the Lagrangian variational formalism that yields the Euler–Lagrange field equations. Suppose that a state of a physical system of interest is described by real-valued fields $q_{\bar{\ell}}(x)$, $\bar{\ell} = 1, \dots, N$, where x is the space-time 4-vector as in (6.1). The variables $q_{\bar{\ell}}(x)$ can be multi-component, they can be scalars, vectors or spinors. In any case, the value of $q_{\bar{\ell}}(x)$ can be considered as an element of $N_{\bar{\ell}}$ -dimensional real space $\mathbb{R}^{N_{\bar{\ell}}}$.

The initial step in the Lagrangian formalism is to introduce the system *Lagrangian density*

$$\mathcal{L}(q_{\bar{\ell}}(x), q_{\bar{\ell},\mu}(x), x) \quad (10.1)$$

that depends on the fields $q_{\bar{\ell}}(x)$ and their first-order derivatives

$$q_{\bar{\ell},\mu}(x) = \partial_{\mu} q_{\bar{\ell}}(x), \quad \mu = 0, 1, 2, 3. \quad (10.2)$$

Based on the Lagrangian density $\mathcal{L}$, we define for any allowed state $\{q_{\bar{\ell}}\}$ of the system and any 4-dimensional region D with a regular enough three-dimensional boundary ∂D the so-called *action integral*

$$W = \int_D \mathcal{L}(q_{\bar{\ell}}(x), q_{\bar{\ell},\mu}(x), x) \, dx, \quad dx = dx^0 dx^1 dx^2 dx^3. \quad (10.3)$$

To determine the variation of the action integral, we consider *infinitesimal field variations* $\delta q_{\bar{\ell}}(x)$ that vanish on the boundary ∂D :

$$q'_{\bar{\ell}}(x) = q_{\bar{\ell}}(x) + \delta q_{\bar{\ell}}(x), \quad \delta q_{\bar{\ell}}(x)|_{\partial D} = 0. \quad (10.4)$$

The variation of the action δW which is defined as the linear (with respect to $\delta q_{\bar{\ell}}$) part of the difference of actions $W(\{q'_{\bar{\ell}}\}) - W(\{q_{\bar{\ell}}\})$. An elementary computation using integration by parts and the fact that $\delta q_{\bar{\ell}}$ vanish on the boundary ∂D yields the following representation for the variation:

²[275, p. 179].

$$\delta W = \int_D \delta \mathcal{L} \, dx = \int_D \left[\frac{\partial \mathcal{L}}{\partial q_{\bar{\ell}}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial q_{\bar{\ell},\mu}} \right) \right] \cdot \delta q_{\bar{\ell}}(x) \, dx \quad (10.5)$$

where $\cdot$ denotes the dot product in the space $\mathbb{R}^{N_{\bar{\ell}}}$. In the case where $q_{\bar{\ell}}$ is multi-component, the definition of the derivatives $\frac{\partial \mathcal{L}}{\partial q_{\bar{\ell}}}$ and $\frac{\partial \mathcal{L}}{\partial q_{\bar{\ell},\mu}}$ may depend on the dot product; we will always assume that the standard dot product in $\mathbb{R}^{N_{\bar{\ell}}}$ is used. If $q_{\bar{\ell}}$ in (10.1) has several components $q_{\bar{\ell}}^i$, $i = 1, \dots, N_{\bar{\ell}}$, the variation of the Lagrangian takes the form

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell}}^i} \delta q_{\bar{\ell}}^i + \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell},\mu}^i} \delta q_{\bar{\ell},\mu}^i = \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell}}} \cdot \delta q_{\bar{\ell}} + \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell},\mu}} \cdot \delta q_{\bar{\ell},\mu},$$

where, as usual, we use the summation convention. Hence,

$$\frac{\partial \mathcal{L}}{\partial q_{\bar{\ell}}} = \left\{ \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell}}^i}, \quad i = 1, \dots, N_{\bar{\ell}} \right\} \quad (10.6)$$

is an element of $\mathbb{R}^{N_{\bar{\ell}}}$; the same is true for $\frac{\partial \mathcal{L}}{\partial q_{\bar{\ell},\mu}}$.

The principle of least action (or stationary action to be precise) demands that the variation δW must vanish for a realizable configuration of the system fields $q_{\bar{\ell}}(x)$, namely

$$\delta W = 0 \text{ for any } D \text{ and any } \delta q_{\bar{\ell}} \text{ such that } \delta q_{\bar{\ell}}(x)|_{\partial D} = 0. \quad (10.7)$$

The principle of least action applied to the representation (10.5) for δW implies that the following *Euler–Lagrange equations* must hold:

$$\frac{\partial \mathcal{L}}{\partial q_{\bar{\ell}}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell},\mu}} = 0, \quad \bar{\ell} = 1, \dots, N. \quad (10.8)$$

The left-hand side of the Eq. (10.8) is the so-called *Eulerian*

$$\Lambda_{\bar{\ell}} = \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell}}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell},\mu}}, \quad (10.9)$$

which takes values in the space $\mathbb{R}^{N_{\bar{\ell}}}$.

In our considerations, the fields $q_{\bar{\ell}}(x)$ may describe the states of elementary charges and can be complex-valued, and some other $q_{\bar{\ell}}$ may also describe the 4-potentials A_μ of the classical EM fields. The details of the extension of the Lagrangian formalism to complex-valued fields are provided below.

Note that every multi-component variable $q_{\bar{\ell}}$ may have an additional algebraic structure, it can be a scalar, a vector, or a spinor; such a structure does not affect the derivation of the Euler–Lagrange equations. For example, a complex-valued variable ψ can be considered as a pair of two real variables and treated as above, but we can

also use the complex number structure to write down the equations. With that in mind, we define the dot product in the complex plane by the formula

$$\psi_1 \cdot \psi_2 = \operatorname{Re} \{ \psi_1^* \psi_2 \} = \frac{1}{2} (\psi_1^* \psi_2 + \psi_1 \psi_2^*). \quad (10.10)$$

This dot product coincides with the standard dot product in $\mathbb{R}^2$ but is written using the complex structure of $\mathbb{R}^2$. The Lagrangian formalism can be directly applied to complex-valued fields as long as the Lagrangian is real-valued. Very often we consider Lagrangians which involve a complex variable ψ together with its conjugate ψ^* . Namely, we consider the Lagrangians of the NKG and NLS equations, the Lagrangians are real-valued:

$$\mathcal{L}(\psi_\ell, \psi_{\ell,\mu}, \psi_\ell^*, \psi_{\ell,\mu}^*) = \mathcal{L}(\psi_\ell, \psi_{\ell,\mu}, \psi_\ell^*, \psi_{\ell,\mu}^*)^*. \quad (10.11)$$

The NLS Lagrangian and the NKG Lagrangian also satisfy the structural condition:

$$\frac{\partial \mathcal{L}}{\partial \psi_\ell^*} = \left(\frac{\partial \mathcal{L}}{\partial \psi_\ell} \right)^*, \quad \frac{\partial \mathcal{L}}{\partial \psi_{\ell,\mu}^*} = \left(\frac{\partial \mathcal{L}}{\partial \psi_{\ell,\mu}} \right)^*. \quad (10.12)$$

This structure allows us to write down the Euler–Lagrange (EL) equations in a simple form. Namely, the variation of the Lagrangian in this case takes the form

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \psi_\ell} \delta \psi_\ell + \frac{\partial \mathcal{L}}{\partial \psi_\ell^*} \delta \psi_\ell^* + \frac{\partial \mathcal{L}}{\partial \psi_{\ell,\mu}} \partial_\mu \delta \psi_\ell + \frac{\partial \mathcal{L}}{\partial \psi_{\ell,\mu}^*} \partial_\mu \delta \psi_\ell^*.$$

According to (10.12) and (10.10)

$$\delta \mathcal{L} = 2 \frac{\partial \mathcal{L}}{\partial \psi_\ell}^* \cdot \delta \psi_\ell + 2 \frac{\partial \mathcal{L}}{\partial \psi_{\ell,\mu}}^* \cdot \partial_\mu \delta \psi_\ell, \quad (10.13)$$

and the complex conjugate

$$\delta \mathcal{L}^* = 2 \frac{\partial \mathcal{L}}{\partial \psi_\ell} \cdot \delta \psi_\ell^* + 2 \frac{\partial \mathcal{L}}{\partial \psi_{\ell,\mu}} \cdot \partial_\mu \delta \psi_\ell^*. \quad (10.14)$$

Therefore, the EL equations takes the form

$$\frac{\partial \mathcal{L}}{\partial \psi_\ell} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \psi_{\ell,\mu}} = 0, \quad (10.15)$$

or the equivalent equations for the complex conjugate

$$\frac{\partial \mathcal{L}}{\partial \psi_\ell^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \psi_{\ell,\mu}^*} = 0. \quad (10.16)$$

Everywhere in this book, both the conditions (10.11) and (10.12) are fulfilled, and we usually write the EL equation in the form of the Eq. (10.16). Obviously, *Eqs. (10.15) and (10.16) have exactly the same form as the Euler–Lagrange equations obtained under the assumption that ψ_ℓ and ψ_ℓ^* are independent variables.*

Notice that the EL field equations (10.8) do not determine the Lagrangian uniquely, [27, III.3.A]. Indeed let us add to $\mathcal{L}$ a term

$$\partial_\mu \Gamma^\mu (q_\ell(x), x), \quad (10.17)$$

and consider the action variation δW over a domain D . Notice that functions Γ^μ depend on the fields but do not depend of the fields derivatives. Then using the fact that the field variations are required to vanish on the domain boundary ∂D as in (10.7), we obtain

$$\begin{aligned} \delta W &= \delta \int_D (\mathcal{L} + \partial_\mu \Gamma^\mu) \, dx \\ &= \delta \int_D (\mathcal{L} + \partial_\mu \Gamma^\mu) \, dx + \delta \int_{\partial D} \partial_\mu \Gamma^\mu \, ds = \delta \int_D \mathcal{L} \, dx. \end{aligned} \quad (10.18)$$

That is, the modified Lagrangian $\mathcal{L} + \partial_\mu \Gamma^\mu$ yields the same EL equations as the Lagrangian $\mathcal{L}$.

10.2 Symmetry Transformations of a Lagrangian and Its Action Integral

Symmetries and conservation laws play a fundamental role in modern physics and particular in particle physics. As R. Feynman put it, [124, p. XV]:

“A great unifying theme among particle physicists has been the role of symmetry and conservation laws in bringing order to the subatomic zoo.”

In this section, we consider relations between physical system symmetries and conservation laws, particularly the energy, momentum and angular momentum conservation. We follow Barut’s approach which is based on Belinfante’s method, [27, III.3, III.4], [206, 5.1, 5.2]. There are other excellent monographs on symmetries and conservation laws, see for instance, [137, 13.7], [145, 2.4, 2.5], [304, 10.1, 10.2].

We start with the celebrated Noether theorem which allows us to obtain conservation laws based on Lagrangian invariance (symmetry) with respect to a general Lie group of transformations. Let us consider a Lie group of transformations in a vicinity of the identity transformation parametrized by parameters $\{\epsilon^k, k = 1, \dots, K\}$ as follows:

$$x' = X(x, \{\epsilon^k\}), \quad q'_\ell(x') = \Psi_\ell(x, q_\ell(x), \{\epsilon^k\}), \quad (10.19)$$

with

$$X(x, \{0\}) = x, \quad \Psi_\ell(x, q_\ell(x), 0) = q_\ell(x). \quad (10.20)$$

The corresponding infinitesimal coordinate transformation has the form

$$x'^\mu = x^\mu + \delta x^\mu, \quad (10.21)$$

where the infinitesimal variation δx^μ is of the first order with respect to ϵ^k . The transformation also involves a *local* change $\bar{\delta} q_\ell(x)$ of $q_\ell(x)$ at a point x :

$$q'_\ell(x') = q_\ell(x) + \delta q_\ell(x). \quad (10.22)$$

The *total field variation* $\bar{\delta}$ is caused by the *total* change of q_ℓ due to both x and q_ℓ and is defined by

$$q'_\ell(x') = q_\ell(x) + \bar{\delta} q_\ell(x). \quad (10.23)$$

It relates to δx^μ and δq_ℓ according to the formula

$$\bar{\delta} q_\ell(x) = \delta q_\ell(x) + q_{\ell,\mu} \delta x^\mu, \quad q_{\ell,\mu} = \partial_\mu q_\ell. \quad (10.24)$$

The infinitesimal form of (10.19) for small ϵ^k is

$$\delta x^\mu = \Gamma_k^\mu(x) \epsilon^k, \quad \bar{\delta} q_\ell = \Phi_{\ell k}(x, q_\ell) \epsilon^k. \quad (10.25)$$

Notice that Ψ_ℓ and $\Phi_{\ell k}$ in the above formulas (10.19), (10.25) take values in the space $\mathbb{R}^{N_\ell}$. From (10.25) we obtain that

$$\bar{\delta} q_{\ell,\mu} = \partial_\mu \Phi_{\ell k}(x, q_\ell) \epsilon^k = \frac{\partial \Phi_{\ell k}}{\partial x^\mu}(x, q_\ell) \epsilon^k + \frac{\partial \Phi_{\ell k}}{\partial q_\ell}(x, q_\ell) q_{\ell,\mu} \epsilon^k. \quad (10.26)$$

10.2.1 Symmetry Transformations for the Poincaré Group

A particularly important case of the coordinate symmetry transformations is the inhomogeneous Lorentz (Poincaré) group of transformations with the infinitesimal form (6.15), namely

$$x'^\mu = x^\mu + \delta x^\mu, \quad \delta x^\mu = \xi^{\mu\nu} x_\nu + a^\mu, \quad \xi^{\mu\nu} = -\xi^{\nu\mu}, \quad (10.27)$$

where $\xi^{\mu\nu} = g^{\sigma\nu} \xi_\sigma^\mu$. The action of the Poincaré group on the field variables q_ℓ , which can be considered elements of the space $\mathbb{R}^{N_\ell}$ and in particular cases can be scalars,

vectors or spinors, results in a representation of this group which is different in particular cases. The associated infinitesimal field transformations can be written in the form

$$\bar{\delta} q_{\bar{\ell}} = \frac{1}{2} \xi_{\mu\nu} S_{\bar{\ell}}^{\mu\nu} q_{\bar{\ell}}. \quad (10.28)$$

For every $\bar{\ell}$, μ , and ν the quantity $S_{\bar{\ell}}^{\mu\nu}$ in (10.28) is a linear operator (matrix) acting on the space $\mathbb{R}^{N_{\bar{\ell}}}$, and it is determined by the particular group representation. In particular, [27, III.4(A)],

$$S_{\bar{\ell}}^{\mu\nu} = 0 \text{ if } q \text{ is a scalar field,} \quad (10.29)$$

$$S_{\beta}^{\alpha\mu\nu} = g^{\alpha\mu} g_{\beta}^{\nu} - g^{\alpha\nu} g_{\beta}^{\mu} \text{ if } q \text{ is a vector field.} \quad (10.30)$$

In all the cases, [27, III.4(A)],

$$S_{\bar{\ell}}^{\mu\nu} = -S_{\bar{\ell}}^{\nu\mu}. \quad (10.31)$$

Comparing with (10.25), we see that the group parameters ϵ^k now have a specific form

$$\{\epsilon^k\} = \{a^{\nu}, \xi^{\sigma\gamma}\}.$$

Transformations (10.27), (10.28) can be written in the form (10.25) as follows:

$$\Gamma_k^{\mu}(x)(a^{\nu}, \xi^{\sigma\gamma}) = \xi^{\mu\sigma} x_{\sigma} + a^{\mu} = \delta_{\sigma}^{\mu} \xi^{\sigma\gamma} g_{\gamma\alpha} x^{\alpha} + \delta_{\nu}^{\mu} a^{\nu}, \quad (10.32)$$

and $\Phi_{\bar{\ell}k}$ for every $\bar{\ell}$, k only acts on $\xi^{\sigma\gamma}$:

$$\Phi_{\bar{\ell}k}(x, q_{\bar{\ell}})(\xi^{\sigma\gamma}) = \frac{1}{2} \xi_{\mu\nu} S_{\bar{\ell}}^{\mu\nu} q_{\bar{\ell}} = \frac{1}{2} \xi^{\sigma\gamma} g_{\sigma\mu} g_{\gamma\nu} S_{\bar{\ell}}^{\mu\nu} q_{\bar{\ell}}. \quad (10.33)$$

10.2.2 Invariance of the Action Integral

Noether's theorem infers conservation laws from the system symmetry transformations. More precisely, the derivation of the conservation laws is based on two factors:

- (i) invariance of the system Lagrangian with respect to group symmetry transformations;
- (ii) fulfillment of the Euler–Lagrange equations.

Let us take a look at the variation of the action integral under the symmetry transformations. Notice that

$$\int_{D'} \mathcal{L}(q'_{\bar{\ell}}, q'_{\bar{\ell},\mu}, x') dx' = \int_D \mathcal{L}(q'_{\bar{\ell}}, q'_{\bar{\ell},\mu}, x') \det\left(\frac{\partial x'}{\partial x}\right) dx, \quad (10.34)$$

suggesting the following *definition of the transformed Lagrangian density* $\mathcal{L}'$:

$$\mathcal{L}' = \mathcal{L}'(q'_{\bar{\ell}}, q'_{\bar{\ell},\mu}, x') = \mathcal{L}(q'_{\bar{\ell}}, q'_{\bar{\ell},\mu}, x') \det\left(\frac{\partial x'}{\partial x}\right). \quad (10.35)$$

Observe that the variational treatment of symmetry transformations assumes no conditions on the boundary values of the fields $q_{\bar{\ell}}$. This is in contrast to the derivation of the field equations (10.8). We say that the Lagrangian is invariant under transformations of the form (10.21)–(10.25) if they preserve the action integral, namely

$$W' = W \text{ or } \int_{D'} \mathcal{L}(\psi^{\ell'}, \psi^{\ell'}_{,\mu}, x') dx' = \int_D \mathcal{L}(q_{\bar{\ell}}, q_{\bar{\ell},\mu}, x) dx \quad (10.36)$$

for any four-dimensional region D with a regular enough boundary.

Notice then that, according to (10.34), the difference of actions $W' - W$ can be recast into

$$W' - W = \int_D \left(\mathcal{L}'(q'_{\bar{\ell}}, q'_{\bar{\ell},\mu}, x') - \mathcal{L}(q_{\bar{\ell}}, q_{\bar{\ell},\mu}, x) \right) dx.$$

Since the domain D is arbitrary, the symmetry assumption $W' - W = 0$ is equivalent to the assumption

$$\mathcal{L}' - \mathcal{L} = 0. \quad (10.37)$$

The terms of the first order with respect to ϵ^k of the differences $W' - W$ and $\mathcal{L}' - \mathcal{L}$ are denoted by $\bar{\delta}W$ and $\bar{\delta}\mathcal{L}$ respectively. The symmetry (10.37) implies

$$\bar{\delta}W = \int_D \bar{\delta}\mathcal{L} dx = 0. \quad (10.38)$$

The above identity can consequently be expressed as the following *infinitesimal symmetry invariance condition*

$$\bar{\delta}\mathcal{L} = 0, \quad (10.39)$$

with the variation $\bar{\delta}\mathcal{L}$ being defined by

$$\mathcal{L}(\psi^{\ell'}, \psi^{\ell'}_{,\mu}, x') \det\left(\frac{\partial x'}{\partial x}\right) - \mathcal{L}(\psi^{\ell}, \psi^{\ell}_{,\mu}, x) = \bar{\delta}\mathcal{L} + \text{higher order terms}, \quad (10.40)$$

where the expansion is with respect to the parameters ϵ^k . The infinitesimal conditions (10.38) and (10.39) are obviously necessary for the invariance conditions (10.36) and (10.37); thanks to the group properties, they are in fact equivalent to them.

To find an explicit form of the symmetry conditions for a given Lie group, we consider the variation $\bar{\delta}\mathcal{L}$. According to (10.20), the determinant in (10.40) can be expanded as follows:

$$\det \left(\frac{\partial x'}{\partial x} \right) = 1 + \partial_\mu \delta x^\mu + \text{higher order terms.} \quad (10.41)$$

It follows from (10.25), (10.26) that

$$\begin{aligned} \bar{\delta} \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell}, \mu}} \cdot \bar{\delta} q_{\bar{\ell}, \mu} + \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell}}} \cdot \bar{\delta} q_{\bar{\ell}} + \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu + \mathcal{L} \partial_\mu \delta x^\mu \\ &= \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell}, \mu}} \left(\frac{\partial \Phi_{\bar{\ell}k}}{\partial x^\mu} + \frac{\partial \Phi_{\bar{\ell}k}}{\partial q_{\bar{\ell}}} q_{\bar{\ell}, \mu} \right) \epsilon^k + \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell}}} \Phi_{\bar{\ell}k} \epsilon^k \\ &\quad + \frac{\partial \mathcal{L}}{\partial x^\mu} \Gamma_k^\mu(x) \epsilon^k + \mathcal{L} \partial_\mu \Gamma_k^\mu(x) \epsilon^k. \end{aligned} \quad (10.42)$$

Therefore, the symmetry condition on the Lagrangian (10.39) takes the following explicit form:

$$\frac{\partial \mathcal{L}}{\partial q_{\bar{\ell}, \mu}} \left(\frac{\partial \Phi_{\bar{\ell}k}}{\partial x^\mu} + \frac{\partial \Phi_{\bar{\ell}k}}{\partial q_{\bar{\ell}}} q_{\bar{\ell}, \mu} \right) + \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell}}} \Phi_{\bar{\ell}k} + \frac{\partial \mathcal{L}}{\partial x^\mu} \Gamma_k^\mu(x) + \mathcal{L} \partial_\mu \Gamma_k^\mu(x) = 0 \quad (10.43)$$

where $k = 1, \dots, K$; as always, we use summation over repeated indices, in particular $\bar{\ell}$ and μ .

10.3 Conservation Laws for Noether's Currents

Assuming that the Lagrangian satisfies the invariance condition (10.39), let us derive conservation laws for solutions of the Euler–Lagrange equations (10.8). Observe that the total variation $\bar{\delta} \mathcal{L}$ is the sum of three variations: $\delta_\psi \mathcal{L}$ associated with field variations only, $\delta_x \mathcal{L}$ associated with the coordinate variations only, and the term $\mathcal{L} \partial_\mu \delta x^\mu$ which comes from (10.41):

$$\bar{\delta} \mathcal{L} = \delta_\psi \mathcal{L} + \delta_x \mathcal{L} + \mathcal{L} \partial_\mu \delta x^\mu \quad (10.44)$$

where

$$\delta_\psi \mathcal{L} = \Lambda_{\bar{\ell}} \cdot \bar{\delta} q_{\bar{\ell}} + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial q_{\bar{\ell}, \mu}} \cdot \bar{\delta} q_{\bar{\ell}} \right), \quad (10.45)$$

$$\delta_x \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu,$$

and $\Lambda_{\bar{\ell}}$ is the Eulerian (10.9). Consequently, combining (10.44) and (10.45), and taking into account that $\Lambda_{\bar{\ell}} = 0$ for fields satisfying the Euler–Lagrange fields equations (10.8), we obtain that

$$\bar{\delta}\mathcal{L} = \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial q_{\bar{\ell},\mu}} \cdot \bar{\delta}q_{\bar{\ell}} + \mathcal{L}\delta x^\mu \right). \quad (10.46)$$

Since the Lagrangian satisfies the symmetry condition (10.39), we have

$$\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial q_{\bar{\ell},\mu}} \cdot \bar{\delta}q_{\bar{\ell}} + \mathcal{L}\delta x^\mu \right) = 0. \quad (10.47)$$

Using (10.24), we can rewrite the above expressions as follows:

$$\frac{\partial\mathcal{L}}{\partial q_{\bar{\ell},\mu}} \cdot \delta q_{\bar{\ell}} + \mathcal{L}\delta x^\mu = \frac{\partial\mathcal{L}}{\partial q_{\bar{\ell},\mu}} \cdot \bar{\delta}q_{\bar{\ell}} - \left(\frac{\partial\mathcal{L}}{\partial q_{\bar{\ell},\mu}} \cdot q_{\bar{\ell},\nu} - \delta_\nu^\mu \mathcal{L} \right) \delta x^\nu. \quad (10.48)$$

The factors before the variations $\bar{\delta}q_{\bar{\ell}}$ and δx^ν in formula (10.48) are known as the *conjugate momentum* $\pi_{\bar{\ell}}^\mu$ and the *canonical energy-momentum tensor* $\dot{T}^{\mu\nu}$ and are defined by the formulas

$$\pi_{\bar{\ell}}^\mu = \frac{\partial\mathcal{L}}{\partial q_{\bar{\ell},\mu}}, \quad \dot{T}^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial q_{\bar{\ell},\mu}} \cdot q_{\bar{\ell},\nu} - \delta_\nu^\mu \mathcal{L}. \quad (10.49)$$

Note that the above expressions are solely defined in terms of the Lagrangian $\mathcal{L}$. Collecting terms at ϵ^k after substituting into (10.47) $\delta q_{\bar{\ell}}$ and δx^ν given by (10.25), we obtain the *conservation law*

$$\partial_\mu J_k^\mu = 0 \quad (10.50)$$

where J_k^μ is Noether's *current*:

$$J_k^\mu = \left[\frac{\partial\mathcal{L}}{\partial q_{\bar{\ell},\mu}} \cdot q_{\bar{\ell},\nu} - \mathcal{L}\delta_\nu^\mu \right] \Gamma_k^\nu - \frac{\partial\mathcal{L}}{\partial q_{\bar{\ell},\mu}} \Phi_{\bar{\ell}k}. \quad (10.51)$$

If we integrate (10.50) over a domain D bounded by two space surfaces $\Sigma_1 = \{t = t_1\}$ and $\Sigma_2 = \{t = t_2\}$, we obtain

$$\int_{\Sigma_1} J_k^0 d\sigma = \int_{\Sigma_2} J_k^0 d\sigma = \text{const.} \quad (10.52)$$

Hence the integral does not depend on time, hence the *integrated Noether's current is a conserved quantity*. Obviously, (10.52) can be generalized to any two space-like surfaces Σ_1 and Σ_2 , i.e. surfaces where time can be represented as a function of spatial variables.

In the case of the group of inhomogeneous Lorentz transformations defined by (6.11), there are ten parameters a^μ and $\xi^{\mu\nu}$, and consequently there are ten corresponding conserved quantities P^ν and $J^{\nu\gamma} = -J^{\gamma\nu}$ defined by (10.66). Another

important example is the group of gauge transformation of the first kind defined by (11.40) below. For this group there is a conserved current $J_{\bar{\ell}}^{\nu}$ for every $\bar{\ell}$ defined by (11.50).

We are particularly interested in the following transformation groups:

- (i) translation groups;
- (ii) the Galilean group in the non-relativistic case;
- (iii) the Lorentz and the Poincaré group in the relativistic case.

The Lagrangians we are interested in are also assumed to be invariant with respect to the relevant gauge transformations, see Sect. 11.2.1 for details.

10.4 Canonical Energy-Momentum Tensor

A fundamental role is played by the group of translations $x_{\nu} \rightarrow x_{\nu} + \xi_{\nu}$ with its standard action on fields

$$q_{\bar{\ell}}(x_{\nu}) \rightarrow q_{\bar{\ell}}(x_{\nu} + \xi_{\nu}).$$

In this case the parameters $\epsilon^k = \xi_{\nu}$, and the corresponding Noether's current J_k^{μ} (10.51) with $k = \nu$ turn into the *canonical energy-momentum tensor* (EnMT) denoted by $\hat{T}_{\nu}^{\mu}$ which takes the form

$$\hat{T}_{\nu}^{\mu} = \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell},\mu}} \cdot q_{\bar{\ell},\nu} - \delta_{\nu}^{\mu} \mathcal{L}. \quad (10.53)$$

The energy-momentum tensor is associated with the group of translations which acts on space-time, and that explains its fundamental significance. The fulfillment of the Lagrangian invariance condition (10.36) for the translation group is obvious for any Lagrangian $\mathcal{L}(q_{\bar{\ell}}(x), q_{\bar{\ell},\mu}(x))$ which does not explicitly depend on x , therefore corresponding energy-momentum tensors always satisfy the conservation law (10.50) called the *energy-momentum conservation law*.

In the relativistic notation, the canonical energy momentum (10.53) is represented in a slightly different form as follows. Namely, the *canonical energy-momentum tensor* (also called *stress-energy tensor* or *stress-tensor*) is defined by the following formula, [27, (3.63)], [213, 32, (32.5)], [137, 13.3, (13.30)], [299, 12.4, (12.4.1)]

$$\hat{T}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell},\mu}} \cdot q_{\bar{\ell}}^{\cdot\nu} - g^{\mu\nu} \mathcal{L}. \quad (10.54)$$

(Notice that the notation in [250, (3.4.2)], [137, Section 13.3] Goldstein is slightly different compared with the above, namely the canonical energy-momentum tensor is defined as matrix-transposed to $\hat{T}^{\mu\nu}$ defined by (10.54), namely $\hat{T}^{\mu\nu} \rightsquigarrow \hat{T}^{\nu\mu}$; also, as everywhere in this section, we use the summation convention with respect to $\bar{\ell}$.)

The energy-momentum conservation laws (10.50) for the energy-momentum (10.54) in the case when the Lagrangian $\mathcal{L}$ involves explicit dependence on x can be written in the form, [27, (3.94)], [137, (13.28)]

$$\partial_\mu \dot{T}^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial x_\nu}, \quad (10.55)$$

where according to (6.6) $\partial/\partial x_\nu = \partial^\nu = g_{\mu\nu}\partial_\mu$. To show that the above conservation law is based solely on the Euler–Lagrange equation (10.8), we give an elementary direct verification:

$$\begin{aligned} \partial_\mu \dot{T}^{\mu\nu} &= \partial_\mu \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell},\mu}} q_{\bar{\ell}}^{\nu} + \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell},\mu}} \partial_\mu q_{\bar{\ell}}^{\nu} - g^{\mu\nu} \partial_\mu (\mathcal{L}) \\ &= \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell}}} q_{\bar{\ell}}^{\nu} + \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell},\mu}} \partial_\mu q_{\bar{\ell}}^{\nu} - \partial^\nu (\mathcal{L}) \\ &= \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell}}} \partial^\nu q_{\bar{\ell}} + \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell},\mu}} \partial^\nu q_{\bar{\ell},\mu} - \partial^\nu (\mathcal{L}) = -\partial^\nu \mathcal{L} \end{aligned} \quad (10.56)$$

confirming (10.55). Notice that in the above calculation $\partial^\nu (\mathcal{L})$ is the complete derivative of $\mathcal{L}$, whereas $\partial^\nu \mathcal{L}$ is the partial derivative.

If the Lagrangian $\mathcal{L}$ does not depend explicitly on x_ν , then $\frac{\partial \mathcal{L}}{\partial x_\nu} = 0$, and the conservation laws (10.55) turn into the following form,

$$\partial_\mu \dot{T}^{\mu\nu} = 0. \quad (10.57)$$

A typical situation when the general conservation law (10.55) applies rather than (10.57) is when there are external (driving) forces.

In any classical field theory over the four-dimensional space-time continuum, the energy-momentum tensor is of fundamental importance. It provides the density of the energy and the momentum; corresponding conservation laws govern the energy and momentum transport. It is worth pointing out that it is the differential form of the energy-momentum conservation that involves the densities of energy, momentum and forces rather than the original field equations. Therefore the conservation laws are more directly related to corpuscular properties of the fields. In particular, for the charge models we study here, the Lorentz force density arises in the differential form of the energy-momentum conservation equations and not in the original field equations. The energy momentum tensors are useful in our studies because they satisfy conservation laws of the form (10.55), and this makes them instrumental in the study of the macroscopic dynamics of the charges. Namely, we are able to derive in the spirit of the Ehrenfest theorem the Newtonian mechanics of the centers of localized charges from the conservation laws for the related field EnMT in relativistic and non-relativistic settings, see Sects. 17.6.2, 34.1.

10.5 The Symmetric Energy-Momentum Tensor

We would like to point out that the canonical energy-momentum tensor (EnMT) defined by (10.54) is not the only one that satisfies the conservation laws (10.55). Indeed, any tensor $T^{\mu\nu}$ such that

$$\partial_\mu \left(T^{\mu\nu} - \dot{T}^{\mu\nu} \right) = 0 \quad (10.58)$$

satisfies the energy-momentum conservation laws (10.55) if the canonical EnMT $\dot{T}^{\mu\nu}$ satisfies it. For instance, for any tensor $\dot{f}^{\mu\gamma\nu}$ such that

$$\dot{f}^{\mu\gamma\nu} = -\dot{f}^{\gamma\mu\nu}, \quad (10.59)$$

the following elementary identity holds

$$\partial_\mu \partial_\gamma \dot{f}^{\mu\gamma\nu} = -\partial_\mu \partial_\gamma \dot{f}^{\gamma\mu\nu} = -\partial_\gamma \partial_\mu \dot{f}^{\mu\gamma\nu} = -\partial_\mu \partial_\gamma \dot{f}^{\mu\gamma\nu} = 0.$$

Then any EnMT $T^{\mu\nu}$ of the form, [27, (3,73)], [213, (32.7)], [266, (14)],

$$T^{\mu\nu} = \dot{T}^{\mu\nu} - \partial_\gamma \dot{f}^{\mu\gamma\nu} \quad (10.60)$$

evidently satisfies (10.58) and consequently the conservation laws (10.55) if EnMT $\dot{T}^{\mu\nu}$ satisfies it.

The flexibility in choosing the energy-momentum can be used to properly define $\dot{f}^{\mu\gamma\nu}$ and construct a *symmetric EnMT* $T^{\mu\nu}$, i.e. a tensor which satisfies the symmetry relation

$$T^{\mu\nu} = T^{\nu\mu}.$$

The above symmetry condition on the EnMT is also necessary and sufficient for the field angular momentum density to be represented by the usual formula in terms of the field momentum density, [213, 32], [27, III.4]. We provide explicit expressions for the EnMT for the relativistic Lagrangians of interest, see Sect. 11.5.2, see also (9.13) for the EM fields.

There are well understood approaches, due to Belinfante and Rosenfeld, [213, 32], [27, III.4], [212, 22], for constructing the symmetric energy-momentum tensor (EnMT) $T^{\mu\nu}$ from a general relativistic invariant Lagrangian $\mathcal{L}$ based on its invariance with respect to the Poincaré group, see Sect. 10.7 for more details. Note that the symmetric EnMT can be obtained from the canonical EnMT by adding a more general expression than the one in (10.60), see Sect. 10.7. Importantly, the added terms must have divergence form, so that the conserved quantities obtained after density integration produce the same values as for canonical energy and momentum.

Since it is the symmetric EnMT $T^{\mu\nu}$ that is used in most of the cases, we often refer to it just as the *energy-momentum tensor*, while the tensor $\dot{T}^{\mu\nu}$ defined by (10.54) is referred to as the *canonical energy-momentum tensor*.

The interpretation of the symmetric EnMT $T^{\mu\nu}$ entries is as follows, [213, 32], [137, 13.2, 13.3], [250, 3.4]:

$$T^{\mu\nu} = \begin{bmatrix} u & cp^1 & cp^2 & cp^3 \\ c^{-1}s^1 & -\sigma^{11} & -\sigma^{12} & -\sigma^{13} \\ c^{-1}s^2 & -\sigma^{21} & -\sigma^{22} & -\sigma^{23} \\ c^{-1}s^3 & -\sigma^{31} & -\sigma^{32} & -\sigma^{33} \end{bmatrix}, \quad (10.61)$$

with

$u = T^{00}$	field energy density,	(10.62)
$p^j = c^{-1}T^{0j}$	field momentum density,	
$s^j = cT^{j0}$	field energy flux density,	
$\sigma^{ij} = \sigma^{ji} = T^{ij}$	field symmetric stress tensor,	

where $i, j = 1, 2, 3$.

We provide an explicit form of the symmetric energy-momentum tensors for the following two cases of interest: (i) scalar fields governed by the Klein–Gordon field equation, see the derivation of the conservation laws and the discussion in Sects. 11.5.2 and 11.6.1; (ii) electromagnetic fields with the symmetric EnMT being given by (9.13). We also consider cases which can be reduced to the above mentioned, see Chap. 11. The symmetric EnMT for the spinorial version is studied in Part IV.

Remark 10.5.1 As a consequence of the symmetry of the EnMT $T^{\mu\nu}$, we have the following relation between the field energy flux and the field momentum densities:

$$\mathbf{s} = c^2 \mathbf{p}. \quad (10.63)$$

W. Pauli calls the identity (10.63) a theorem and makes a comment, [265, p.125]:

“This is the theorem of the momentum of the energy current, first expressed by Planck according to which a momentum is associated with each energy current. This theorem can be considered as an extended version of the principle of the equivalence of mass and energy. Whereas the principle only refers to the total energy, the theorem has also something to say on the localization of momentum and energy.”

There is an intimate relation between the concept of particle and the field concept of the *symmetric energy-momentum tensor* (EnMT) $T^{\mu\nu}$. In particular, the fundamental Einstein mass-energy relation $\mathbf{E}_0 = m_0 c^2$ can be interpreted as the symmetry of the energy-momentum tensor, a point stressed by C. Lanczos, [212, p. 394]:

“It was Planck in 1909 who pointed out that the field theoretical interpretation of Einstein’s principle can only be the symmetry of the energy-momentum tensor. If the T_{i4} ($i = 1, 2, 3$) (i.e. the momentum density) and the T_{4i} , the energy current, did not agree, then the conservation of mass and energy would follow different laws and the principle $m = E$ could not be maintained. Nor could a non-symmetric energy-momentum tensor guarantee the law of inertia, according to which the centre of mass of an isolated system moves in a straight line with constant velocity.”

10.6 Conserved Quantities

In view of (10.58), the EnMT $T^{\mu\nu}$ defined by (10.60) satisfies the same conservation law (10.55) as $\hat{T}^{\mu\nu}$, namely

$$\partial_\mu T^{\mu\nu} = f, \quad f = -\frac{\partial \mathcal{L}}{\partial x_\nu}. \quad (10.64)$$

Note that in the cases when the Lagrangian explicitly depends on space and time variables, the expression (10.60) may be replaced by a more general expression, and the forcing term f in the right-hand side in (10.64) may take a different form. If the Lagrangian does not depend explicitly on x_ν , the above conservation law turns into

$$\partial_\mu T^{\mu\nu} = 0. \quad (10.65)$$

The corresponding total conserved quantities are, [27, (3.76)–(3.77)], [266, (6), (12), (15b)], [304, 10.2],

$$P^\nu = \int_\sigma T^{\mu\nu} d\sigma_\mu, \quad J^{\nu\gamma} = \int_\sigma M^{\mu\nu\gamma} d\sigma_\mu, \quad (10.66)$$

where σ is any space-like surface, for instance $x_0 = \text{const}$. Here

$$M^{\mu\nu\gamma} = x^\nu T^{\mu\gamma} - x^\gamma T^{\mu\nu}$$

is the *angular momentum density tensor* (consistent with the angular momentum $x^\nu p^\gamma - p^\nu x^\gamma$ of a point mass), P^ν is the *four-vector of the total energy-momentum*, and $J^{\nu\gamma} = -J^{\gamma\nu}$ is the *total angular momentum tensor*. The differential form of the conservation laws is (10.65) together with

$$\partial_\mu M^{\mu\nu\gamma} = T^{\gamma\nu} - T^{\nu\gamma} = 0. \quad (10.67)$$

Observe that, according to (10.67), the conservation of the angular momentum $M^{\mu\nu\gamma}$ implies the symmetry of the energy-momentum tensor and, in view of (10.60), the following identities, [27, (3.81')],

$$\frac{\partial \mathcal{L}}{\partial q_{\bar{t},\mu}} q_{\bar{t}}^{\nu} - \partial_\gamma f^{\mu\gamma\nu} = \frac{\partial \mathcal{L}}{\partial q_{\bar{t},\nu}} q_{\bar{t}}^{\mu} - \partial_\gamma f^{\nu\gamma\mu}. \quad (10.68)$$

We would like to stress that the symmetry of the energy-momentum tensor and the corresponding identities (10.68) are nontrivial relations, which hold provided that the involved fields satisfy the field equations (10.8).

If we introduce the 4-momentum and 4-flux densities by the formulas

$$p^\nu = \frac{1}{c} T^{0\nu} = \left(\frac{1}{c} u, \mathbf{p} \right), \quad s^\nu = c T^{\nu 0} = (cu, \mathbf{s}),$$

then the conservation laws (10.64) can be recast as follows:

$$\partial_t p^\nu + \partial_j T^{j\nu} = - \frac{\partial \mathcal{L}}{\partial x_\nu}.$$

In other words, the momentum conservation law takes the form

$$\partial_t p^i = \partial_j \sigma^{ji} - \frac{\partial \mathcal{L}}{\partial x_i}$$

with

$$p^i = \frac{1}{c} T^{0i}, \quad \sigma^{ji} = -T^{ji}, \quad i, j = 1, 2, 3,$$

and the energy conservation law

$$\partial_t u + c \partial_j T^{j0} = - \frac{\partial \mathcal{L}}{\partial t},$$

with

$$u = T^{00}, \quad s^i = c T^{i0} = c^2 p^i, \quad i = 1, 2, 3.$$

The *total energy-momentum* 4-vector takes the form

$$P^\nu = \frac{1}{c} \int_{\mathbb{R}^3} T^{0\nu}(x) \, d\mathbf{x}; \quad (10.69)$$

it is *conserved*:

$$\partial_t P^\nu = 0 \quad \text{if} \quad \frac{\partial \mathcal{L}}{\partial x_\nu} = 0.$$

Its components, the total energy and momentum, are respectively

$$\mathcal{E} = c P^0 = \int_{\mathbb{R}^3} T^{00}(x) \, d\mathbf{x}, \quad (10.70)$$

$$\mathbf{P} = P^j = \int_{\mathbb{R}^3} T^{0j}(x) \, d\mathbf{x}, \quad j = 1, 2, 3. \quad (10.71)$$

Evidently, the formulas (10.69), (10.70) are particular cases of the formulas (10.66) for the special important choice of the surface

$$\sigma = \{x = (x_0, \mathbf{x}) : \mathbf{x} \in \mathbb{R}^3\}. \quad (10.72)$$

Importantly, for closed Lorentz-invariant systems, the conserved total energy-momentum P^ν and the angular momentum $J^{\nu\gamma}$ as defined by formulas (10.69) and (10.66) transform respectively as a 4-vector and 4-tensor under the Lorentz transformation, and that is directly related to the conservation laws (10.65), (10.67), [248, 6.2], [179, 12.10A]. But for open (not closed) systems, the total energy-momentum P^ν and $J^{\nu\gamma}$ angular momentum do not transform generally as respectively a 4-vector and 4-tensor, [248, 7.1, 7.2], [179, 12.10 A, 16.4].

The symmetry of the energy-momentum tensor for matter fields is also important in the theory of gravity, since it is a source for the gravitational field, [253, 3.8], [246, 5.7].

To summarize, a general source of conservation laws in Lagrangian theories is Noether's theorem, [137, 13.7], which canonically yields conservation laws based on the Lagrangian symmetries, in other words on the invariance of the Lagrangian with respect to continuous groups of transformations. *We would like to stress that the conservation laws are not independent equations. They hold only for solutions of the Euler–Lagrange equations and can be deduced from them.* The energy-momentum and charge conservation laws are two important laws in any EM theory, and they have special significance in our studies for several reasons. First of all, the conservation of the energy-momentum describes its transport in space and is directly related to the point charge approximations. Second, the Lorentz force density, which is one of the most important components of any EM theory, arises in the energy-momentum conservation laws (9.17) and not in the field Maxwell equations.

10.7 Symmetries and Conservation Laws Revisited

Here we discuss in more detail general Lagrangians which are invariant with respect to the Lorentz transformations and, in particular, the construction of the symmetric energy-momentum tensor. Note that the related questions for the cases of scalar charge distributions interacting with electromagnetic fields are studied independently in the corresponding sections of the book, and the purpose of this section is to provide for additional insights.

The unique form of the symmetric energy-momentum can be derived based on a variational principle involving a varied boundary, [27, Section III.3(B)], and under the following assumptions: (i) the Lagrangian does not depend explicitly on x ; (ii) the fields $q_{\bar{\ell}}(x)$ satisfy the field equations (10.8); (iii) the fields vanish at spatial infinity sufficiently fast. The result is the symmetric Belinfante–Rosenfeld EnMT $T^{\mu\nu}$, [27, (3.73)–(3.75)], [266, (13a), (13b), (13c), (14)], [31], [32], [279], namely

$$T^{\mu\nu} = \dot{T}^{\mu\nu} - \partial_\gamma f^{\mu\gamma\nu} = \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell},\mu}} q_{\bar{\ell}}^{\cdot\nu} - g^{\mu\nu} \mathcal{L} - \partial_\gamma f^{\mu\nu\gamma}, \quad (10.73)$$

where

$$\begin{aligned} \bar{f}^{\mu\gamma\nu} = -\bar{f}^{\gamma\mu\nu} &= \frac{1}{2} \left[\sum_{\bar{\ell}} \left(\pi_{\bar{\ell}}^{\mu} S_{\bar{\ell}}^{\gamma\nu} + \pi_{\bar{\ell}}^{\gamma} S_{\bar{\ell}}^{\nu\mu} - \pi_{\bar{\ell}}^{\nu} S_{\bar{\ell}}^{\mu\gamma} \right) q_{\bar{\ell}} \right], \\ \pi_{\bar{\ell}}^{\mu} &= \frac{\partial \mathcal{L}}{\partial q_{\bar{\ell},\mu}}. \end{aligned} \quad (10.74)$$

The tensor $S_{\bar{\ell}}^{\mu\nu}$ in the above equation describes the infinitesimal transformation of the involved fields $q_{\bar{\ell}}(x) \rightarrow q'_{\bar{\ell}}(x')$ related to the infinitesimal Lorentz transformations (6.11) as in (10.28).

For an alternative insightful derivation of the symmetric energy-momentum tensor based on kinosthenic (ignorable) variables and Noether's method as a way to generate such variables, we refer to [211, 3.5, 3.6, 3.10]. Interestingly, under this approach the conservation laws take the form of the Euler–Lagrange equations for those kinosthenic variables.

10.7.1 Symmetry Transformations and Noether's Conserved Currents

Expressions for Noether's conserved currents can be obtained as follows. Using Gauss' theorem, we obtain from (10.46) to (10.48) that

$$\bar{\delta} W = \int_D \bar{\delta} \mathcal{L} \, dx = \mathring{G}(\Sigma_2) - \mathring{G}(\Sigma_1), \quad (10.75)$$

where Σ_1, Σ_2 are two space-like surfaces extending to infinity, and $\mathring{G}(\Sigma)$ is the *canonical symmetry transformation generator* defined by the surface integral

$$\mathring{G}(\Sigma) = \int_{\Sigma} \left[\pi_{\bar{\ell}}^{\mu} \cdot \bar{\delta} q_{\bar{\ell}} - \mathring{T}^{\mu\nu} \delta x_{\nu} \right] d\sigma_{\mu}. \quad (10.76)$$

Now, if the Eulerian $\Lambda_{\bar{\ell}}$ defined by (10.9) satisfies the Euler–Lagrange equation

$$\Lambda_{\bar{\ell}} = 0,$$

it follows that

$$\bar{\delta} \mathcal{L} = \partial_{\mu} \left(\pi_{\bar{\ell}}^{\mu} \cdot \bar{\delta} q_{\bar{\ell}} - \mathring{T}^{\mu\nu} \delta x_{\nu} \right) = 0, \quad (10.77)$$

and

$$\bar{\delta} W = \int_D \bar{\delta} \mathcal{L} \, dx = \mathring{G}(\Sigma_2) - \mathring{G}(\Sigma_1) = 0. \quad (10.78)$$

The differential form of the conservation law (10.77) is the continuity equation

$$\partial_\mu J^\mu = 0, \quad J^\mu = \pi_\ell^\mu \cdot \bar{\delta} q_\ell - \mathring{T}^{\mu\nu} \delta x_\nu, \quad (10.79)$$

where J^μ is *Noether's current*, [27, III.4], [206, 5.1], [145, 2.4, 2.5], [304, 10.2]. A more detailed form of the conserved Noether's current based on (10.19), (10.25) is

$$J_k^\mu = \pi_\ell^\mu \cdot \Phi_k^{\bar{\ell}} - \mathring{T}^{\mu\nu} \Gamma_{k\nu}, \quad \partial_\mu J_k^\mu = 0, \quad (10.80)$$

that is, for every parameter ϵ^k there is a *conserved Noether's current* J_k^μ .

Let us consider an important case, where the Lagrangian density $\mathcal{L}$ defined by (10.1) does not depend explicitly on x , and consequently it is invariant with respect to the spacetime translations. This is evidently a particular case of the invariance with respect to the Poincaré transformation (10.27) where $\xi^{\mu\nu} = 0$, $\delta x^\mu = a^\mu$, and $\bar{\delta} q_\ell = 0$ in (10.28), hence $\psi^{\ell'}(x') = q_\ell(x)$. The canonical action generator $\mathring{G}(\Sigma)$ then takes the form

$$\mathring{G}(\Sigma) = -P^\nu(\Sigma) a_\nu, \quad P^\nu(\Sigma) = \int_\Sigma \mathring{T}^{\mu\nu} d\sigma_\mu. \quad (10.81)$$

The quantity $P^\mu(\Sigma)$ is the energy-momentum 4-vector which is a conserved quantity, that is,

$$P^\nu(\Sigma) \text{ is a constant for any space-like surface } \Sigma \text{ extending to infinity.} \quad (10.82)$$

The differential form of the above conservation law is

$$\partial_\mu \mathring{T}^{\mu\nu} = 0. \quad (10.83)$$

Suppose now that the Lagrangian density $\mathcal{L}$ defined by (10.1) depends explicitly on x , hence the system is not translation invariant. Then the energy-momentum conservation law (10.83) does not hold, but it can be modified. Indeed, in this case we consider exactly the same spacetime translations, that is, $\xi^{\mu\nu} = 0$, $\delta x^\mu = a^\mu$ and $\bar{\delta} q_\ell = 0$. Then using the relation (10.46), where we set $\bar{\delta} \mathcal{L} = (\partial^\nu \mathcal{L}) \delta x_\nu$ and $\bar{\delta} q_\ell = 0$, we obtain the following generalization of the law (10.83):

$$\partial_\mu \mathring{T}^{\mu\nu} = -\partial^\nu \mathcal{L}. \quad (10.84)$$

Evidently, if $\mathcal{L}$ does not depend explicitly on x , then $\partial^\nu \mathcal{L} = 0$, and relation (10.84) turns into the conservation law (10.83). The relation (10.84) can also be verified directly by using the expression (10.49) for EnMT $\mathring{T}^{\mu\nu}$ and the Euler–Lagrange fields equations (10.8), see [27, (3.94)], [137, (13.28)].

10.7.2 The Symmetric Energy-Momentum Tensor (EnMT) and Angular Momentum

The canonical EnMT $\dot{T}^{\mu\nu}$ often is not symmetric, whereas there are several important reasons to insist on the energy-momentum tensor (EnMT) $T^{\mu\nu}$ symmetry. A satisfactory resolution to this problem comes through a proper modification of the canonical EnMT $\dot{T}^{\mu\nu}$ and its symmetry generator $\dot{G}(\Sigma)$ to another EnMT $T^{\mu\nu}$ with its symmetry generator $G(\Sigma)$, so that the key relation (10.78) is preserved, namely

$$\bar{\delta}W = \bar{\delta} \int_D \mathcal{L} dx = G(\Sigma_2) - G(\Sigma_1) = 0. \quad (10.85)$$

To introduce such a modification, let us pick a field C^μ satisfying

$$\partial_\mu C^\mu = 0, \quad (10.86)$$

and introduce the modified generator

$$G(\Sigma) = \int_\Sigma \left[\pi_{\bar{\ell}}^\mu \cdot \bar{\delta}q_{\bar{\ell}} - \dot{T}^{\mu\nu} \delta x_\nu + C^\mu \right] d\sigma_\mu = \dot{G}(\Sigma) + \int_\Sigma C^\mu d\sigma_\mu. \quad (10.87)$$

Then the following relation holds:

$$G(\Sigma_2) - G(\Sigma_1) = \dot{G}(\Sigma_2) - \dot{G}(\Sigma_1) + \int_D \partial_\mu C^\mu dx = \dot{G}(\Sigma_2) - \dot{G}(\Sigma_1), \quad (10.88)$$

implying that the modified generator $G(\Sigma)$ defined by (10.87) satisfies the desired relation (10.85). This suggests a method of constructing a symmetric EnMT $T^{\mu\nu}$ by relating it to the generator $G(\Sigma)$ satisfying (10.86), (10.87). This method of constructing a symmetric EnMT $T^{\mu\nu}$ is essentially due to Belinfante with some particular features that are due to Barut, [27, III.3-4], [206, 5.1, 5.2].

Notice that, according to expressions (10.77) and (10.78), the total variation of the Lagrangian density $\bar{\delta}\mathcal{L}$ and consequently the total variation of the action integral $\bar{\delta}W$ involve both the field variations $\bar{\delta}q_{\bar{\ell}}$ and coordinate variations δx^ν . And this is why the conserved current J^μ defined by (10.79) involves both the variations $\bar{\delta}q_{\bar{\ell}}$ and δx^ν together with the corresponding factors: conjugate momentum $\pi_{\bar{\ell}}^\mu$ and the canonical EnMT $\dot{T}^{\mu\nu}$. Since the Poincaré symmetry group is related entirely to spacetime transformation, we may naturally want to have only the energy-momentum tensor to represent this symmetry group. For this to happen, we obviously need to express somehow the field total variations $\bar{\delta}q_{\bar{\ell}}$ in terms of δx^ν . A way to do that is proposed by Barut in [27, III.3(A)]. Suppose the Lorentz–Poincaré coordinate transformations together with relevant field transformations are defined by (10.27) and (10.28). We seek a particular representation for the generator $G(\Sigma)$ of a form similar to (10.76), namely

$$G(\Sigma) = - \int_{\Sigma} T^{\mu\nu} \delta x_{\nu} d\sigma_{\mu} = - \int_{\Sigma} T^{\mu\nu} (\xi_{\nu\rho} x^{\rho} + a_{\nu}) d\sigma_{\mu}, \quad (10.89)$$

and consider the relation (10.89) as a requirement for a suitable choice of the vector field C^{μ} in (10.87).

If that is accomplished, we readily obtain

$$G(\Sigma) = -P^{\nu}(\Sigma) a_{\nu} - \frac{1}{2} J^{\nu\rho}(\Sigma) \xi_{\nu\rho}, \quad (10.90)$$

where the 4-vector energy-momentum $P^{\mu}(\Sigma)$ and the angular momentum $J^{\nu\rho}(\Sigma)$ are defined by

$$P^{\nu}(\Sigma) = \int_{\Sigma} T^{\mu\nu} d\sigma_{\mu}, \quad J^{\nu\rho}(\Sigma) = \int_{\Sigma} J^{\mu\nu\rho} d\sigma_{\mu}, \quad (10.91)$$

$$J^{\mu\nu\rho} = T^{\mu\nu} x^{\rho} - T^{\mu\rho} x^{\nu}.$$

Consequently, in view of (10.85), the following conservation law holds:

$$P^{\nu}(\Sigma) \text{ and } J^{\nu\rho}(\Sigma) \text{ are constant} \quad (10.92)$$

for any space-like surface Σ extending to infinity. The differential form of the above conservation laws is

$$\partial_{\mu} T^{\mu\nu} = 0, \quad \partial_{\mu} J^{\mu\nu\rho} = T^{\rho\nu} - T^{\nu\rho} = 0. \quad (10.93)$$

As we can see from the identities (10.93), the *differential form of the angular momentum conservation is equivalent to the symmetry of the EnMT* $T^{\mu\nu}$. An additional satisfactory feature of the approach is a *particle-like form* $T^{\mu\nu} x^{\rho} - T^{\mu\rho} x^{\nu}$ of the *density of the angular momentum* $J^{\mu\nu\rho}$ in (10.91). It adds up to the interpretation of $T^{\mu\nu}$ as the “real” density of the energy-momentum since, as we know, it cannot be defined uniquely.

Our goal now is to obtain the representation (10.89) for the generator $G(\Sigma)$ from $\mathring{G}(\Sigma)$ based on (10.49), (10.76) and (10.87) by an intelligent choice of C^{μ} . Taking into account the particular form of the coordinate and field transformation (10.27)–(10.28), and comparing expression (10.76) for the canonical generator $\mathring{G}(\Sigma)$ with the desired expressions (10.87) and (10.89) for the generator $G(\Sigma)$, we see that the transformation parameters $\xi_{\mu\nu}$ have to be expressed somehow in terms of the coordinate variations δx_{ν} . This can be accomplished by observing the following simple relation between them:

$$\xi_{\mu\nu} = \frac{\partial \delta x_{\mu}}{\partial x^{\nu}} = - \frac{\partial \delta x_{\nu}}{\partial x^{\mu}}. \quad (10.94)$$

To represent $\pi_{\bar{\ell}}^{\mu} \bar{\delta} \psi^{\ell}$ in (10.76) in terms involving δx_{ν} , we use the above relations (10.94), (10.28) and notice first that

$$\pi_{\bar{\ell}}^{\mu} \cdot \bar{\delta} q_{\bar{\ell}} = \xi_{\mu\nu} \frac{1}{2} \pi_{\bar{\ell}}^{\mu} \cdot \left(S_{\bar{\ell}}^{\mu\nu} q_{\bar{\ell}} \right) = \xi_{\mu\nu} \frac{1}{2} \left(S_{\bar{\ell}}^{\mu\nu*} \pi_{\bar{\ell}}^{\mu} \right) \cdot q_{\bar{\ell}}, \quad (10.95)$$

where S^* is an operator adjoint to S , defined by the identity $a \cdot (Sb) = (S^*a) \cdot b$. Using the antisymmetry of $\xi_{\mu\nu}$, we now want to recast the right-hand side of the relation (10.95) suppressing the index ℓ so that

$$\frac{1}{2} S^{\rho\nu*} \pi^{\mu} \xi_{\rho\nu} = f^{\mu\rho\nu} \xi_{\rho\nu}, \quad (10.96)$$

where we want $f^{\mu\rho\nu}$ to be antisymmetric in μ and ρ , that is,

$$f^{\mu\rho\nu} = -f^{\rho\mu\nu}. \quad (10.97)$$

One can verify that a solution to this problem is

$$f^{\mu\rho\nu} = \frac{1}{2} [S^{\rho\nu*} \pi^{\mu} + S^{\nu\mu*} \pi^{\rho} - S^{\mu\rho*} \pi^{\nu}]. \quad (10.98)$$

Indeed, the antisymmetries $S^{\rho\nu} = -S^{\nu\rho}$ and $\xi_{\rho\nu} = -\xi_{\nu\rho}$ imply the desired antisymmetry (10.97) as well as the identities

$$(S^{\nu\mu*} \pi^{\rho} - S^{\mu\rho*} \pi^{\nu}) \xi_{\rho\nu} = (S^{\nu\mu*} \pi^{\rho} + S^{\rho\mu*} \pi^{\nu}) \xi_{\nu\rho} = 0,$$

which, in turn, imply the desired relation (10.96).

Using (10.94) and (10.96), we recast the canonical generator $\dot{G}(\Sigma)$ defined by (10.76) as follows:

$$\begin{aligned} \dot{G}(\Sigma) &= \int_{\Sigma} \left[-f_{\bar{\ell}}^{\mu\rho\nu} \cdot q_{\bar{\ell}} \frac{\partial \delta x_{\nu}}{\partial x^{\rho}} - \dot{T}^{\mu\nu} \delta x_{\nu} \right] d\sigma_{\mu} \\ &= - \int_{\Sigma} \left[\dot{T}^{\mu\nu} - \frac{\partial}{\partial x^{\rho}} \left(f_{\bar{\ell}}^{\mu\rho\nu} \cdot q_{\bar{\ell}} \right) \right] \delta x_{\nu} d\sigma_{\mu} \\ &\quad - \int_{\Sigma} \left[\frac{\partial}{\partial x^{\rho}} \left(f_{\bar{\ell}}^{\mu\rho\nu} \cdot q_{\bar{\ell}} \delta x_{\nu} \right) \right] d\sigma_{\mu}, \end{aligned} \quad (10.99)$$

where $f^{\ell\mu\rho\nu}$ are of the form (10.98) with the obvious substitution $\pi^{\mu} = \pi_{\bar{\ell}}^{\mu}$. Observe now that if we set

$$C^{\mu} = \frac{\partial}{\partial x^{\rho}} \left(f_{\bar{\ell}}^{\mu\rho\nu} \cdot q_{\bar{\ell}} \delta x_{\nu} \right), \quad (10.100)$$

then, in view of the antisymmetry property (10.97) of $f^{\ell\mu\rho\nu}$, we readily obtain $\partial_{\mu} C^{\mu} = 0$.

Consequently, we obtain the following *symmetric energy-momentum tensor* $T^{\mu\nu}$ together with the corresponding generator

$$T^{\mu\nu} = \hat{T}^{\mu\nu} - \frac{\partial}{\partial x^\rho} \left(f_{\tilde{\ell}}^{\mu\rho\nu} \cdot q_{\tilde{\ell}} \right), \quad G(\Sigma) = - \int_{\Sigma} T^{\mu\nu} \delta x_\nu \, d\sigma_\mu, \quad (10.101)$$

where $f^{\ell\mu\rho\nu}$ are defined by expressions (10.98). Notice that, though according to the above Eq. (10.101) the symmetric and canonical EnMT densities can differ, the relations (10.101) combined with (10.97) imply that the system total energy $\mathcal{E}(t)$ can be represented (in the absence of external forces) in any frame in terms of any of them, namely

$$\mathcal{E}(t) = \int_{\mathbb{R}^3} T^{00} \, d\mathbf{x} = \int_{\mathbb{R}^3} \hat{T}^{00} \, d\mathbf{x}, \quad (10.102)$$

under an assumption that the fields decay at infinity.

In the case when the system is not closed, the EnMT satisfies *energy-momentum conservation laws with forces*:

$$\partial_\mu T^{\mu\nu} = f^\nu$$

where f^ν is the force density. The *angular momentum conservation law* has the form [304, Sect. 10.2], [27, Sect. II.1, III.4],

$$\partial_\mu M^{\mu\nu\gamma} = N^{\nu\gamma} \quad (10.103)$$

where

$$N^{\nu\gamma} = x^\nu f^\gamma - f^\nu x^\gamma$$

is the *torque density tensor*.

10.8 Examples of the Classical Field Theories

In this section we consider two simple examples of field theories, and more examples are given in Chap. 11.

10.8.1 Compressional Waves in Non-viscous Compressible Fluid

As an example, following [250, 3.3] and [251, 6.2], we consider in this section compressional waves in a non-viscous and compressible fluid at equilibrium. The waves are described by the pressure field p and velocity field $\mathbf{v}$ and are governed by the following system of equations:

$$\rho \partial_t \mathbf{v} = -\nabla p, \quad \kappa \partial_t p = -\nabla \cdot \mathbf{v}, \quad c^2 = \frac{1}{\rho \kappa}. \quad (10.104)$$

Here ρ and κ are, respectively, the uniform constant mass density and the compressibility (adiabatic) of the fluid at equilibrium, which are assumed to be given, and c is velocity of wave propagation. If the velocity potential ψ satisfies the classical wave equation

$$\frac{1}{c^2} \partial_t^2 \psi - \nabla^2 \psi = 0, \quad (10.105)$$

and we define the pressure and velocity fields by

$$p = \rho \partial_t \psi, \quad \mathbf{v} = -\nabla \psi, \quad (10.106)$$

then equations (10.104) hold.

The Lagrangian density for the wave equation (10.105) can be written as follows:

$$L = \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} - \frac{1}{2} \kappa p^2 = \frac{1}{2} \rho \left[\frac{1}{c^2} (\partial_t \psi)^2 - (\nabla \psi)^2 \right]. \quad (10.107)$$

Since

$$T = \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \text{ is the kinetic energy and } V = \frac{1}{2} \kappa p^2 \text{ is the potential energy,} \quad (10.108)$$

the Lagrangian density L equals $T - V$ as expected.

The canonical energy-momentum tensor which corresponds to the wave equation has the form

$$\hat{T}^{\mu\nu} = \begin{bmatrix} \hat{T}^{00} & \rho \partial_0 \psi \partial_1 \psi & \rho \partial_0 \psi \partial_2 \psi & \rho \partial_0 \psi \partial_3 \psi \\ \rho \partial_1 \psi \partial_0 \psi & \hat{T}^{11} & -\rho \partial_1 \psi \partial_2 \psi & -\rho \partial_1 \psi \partial_3 \psi \\ \rho \partial_2 \psi \partial_0 \psi & -\rho \partial_2 \psi \partial_1 \psi & \hat{T}^{22} & -\rho \partial_2 \psi \partial_3 \psi \\ \rho \partial_3 \psi \partial_0 \psi & -\rho \partial_3 \psi \partial_1 \psi & -\rho \partial_3 \psi \partial_2 \psi & \hat{T}^{33} \end{bmatrix}, \quad (10.109)$$

where

$$\begin{aligned} \partial_0 &= \frac{1}{c} \partial_t, \quad \hat{T}^{00} = \frac{\rho}{2} [(\partial_0 \psi)^2 + (\nabla \psi)^2], \\ \hat{T}^{jj} &= \frac{\rho}{2} [(\nabla \psi)^2 - 2(\partial_j \psi)^2 - (\partial_0 \psi)^2]. \end{aligned} \quad (10.110)$$

Now let us consider a slightly more general case, where we have an external force with the force density $\mathbf{f} = \rho \partial_t \mathbf{F}$, and Eq. (10.104) take the form

$$\rho \partial_t \mathbf{v} = -\nabla p + \mathbf{f}, \quad \kappa \partial_t p = -\nabla \cdot \mathbf{v}. \quad (10.111)$$

Similarly to the above, if ψ is a solution of the forced wave equation

$$\frac{1}{c^2} \partial_t^2 \psi = \nabla^2 \psi - \frac{1}{c^2} \nabla \cdot \mathbf{F}, \quad (10.112)$$

then a direct computation shows that the velocity field and pressure p defined by the formula

$$p = \rho \partial_t \psi, \quad \mathbf{v} = \mathbf{F} - \nabla \psi \quad (10.113)$$

satisfy (10.111).

The Lagrangian for the forced wave equation has the form

$$L = \frac{1}{2} \rho \left[\frac{1}{c^2} (\partial_t \psi)^2 - (\nabla \psi)^2 - \frac{1}{c^2} \psi \nabla \cdot \mathbf{F} \right]. \quad (10.114)$$

Note that the Lagrangian force density in the right-hand side of the conservation law (10.55) has the form

$$-\frac{\partial L}{\partial x_\nu} = \frac{\rho}{2} \frac{1}{c^2} \psi \frac{\partial}{\partial x_\nu} \nabla \cdot \mathbf{F}. \quad (10.115)$$

Clearly, the above Lagrangian field force density is different from the Newtonian force density $\mathbf{f} = \rho \partial_t \mathbf{F}$ in the original setting (10.111) and also is different from the “forcing term” $c^{-2} \nabla \cdot \mathbf{F}$ in the forced wave equation (10.112).

10.8.2 The Lagrangian for an Abstract Schrödinger Equation

Let us consider an abstract Schrödinger equation

$$i\hbar \partial_t \psi = H\psi, \quad (10.116)$$

where ψ belongs to a Hilbert space $\mathcal{H}$ with a scalar product (ψ_1, ψ_2) and H is a self-adjoint operator in $\mathcal{H}$. Let us represent the scalar product in the following form $(\psi_1, \psi_2) = \langle \psi_1^*, \psi_2 \rangle$ where $\langle \cdot, \cdot \rangle$ is a bilinear form and ψ^* is complex conjugate to ψ . Then the following Lagrangian

$$L = i\frac{\hbar}{2} (\langle \psi^*, \partial_t \psi \rangle - \langle \psi, \partial_t \psi^* \rangle) - \langle \psi^*, H\psi \rangle \quad (10.117)$$

under the assumption of independence of ψ and ψ^* yields the Euler–Lagrange equations

$$i\hbar \partial_t \psi = H\psi, \quad -i\hbar \partial_t \psi^* = (H\psi)^*. \quad (10.118)$$

Chapter 11

Lagrangian Field Formalism for Charges Interacting with EM Fields

“The basic Lagrangian can certainly not be purely quadratic, but must contain higher-order terms which become of decisive importance near the core of the material particles. In ignorance of these higher-order terms we put something on the right side of our linear field equations as a substitute for the non-linearity which is hidden from our knowledge (“asylum ignorantiae” in Einstein’s terminology). Fortunately, these domains are very small and become of importance only if we inquire into the structure of elementary particles.”
*C. Lanczos.*¹

The Lagrangian formalism for EM fields governed by the Maxwell equations is presented in Chap. 9. In this chapter, we introduce Lagrangians for systems of distributed charges interacting with EM fields. The corresponding wave equations are the nonlinear Klein–Gordon (NKG) equations for relativistic systems or the nonlinear Schrödinger (NLS) equations for non-relativistic ones. We apply the framework of Chap. 10 to implement general Lagrangian constructions for specific Lagrangians used in our studies. In particular, we analyze basic properties of the EnMT and conservation laws for those specific Lagrangians.

Suppose that we have a system of N spinless distributed charges described by complex-valued scalar fields $\psi_\ell = \psi_\ell(x^\mu) = \psi_\ell(t, \mathbf{x})$, $\ell = 1, \dots, N$. Suppose also that the interaction between the charges is described by real-valued (multi-component) fields $\{V_\sigma\}$, $\sigma = 1, \dots, N_\alpha$ which in cases of interest can be identified with electromagnetic fields. The system Lagrangian is assumed to be of the form

$$\mathcal{L} = \mathcal{L}(\{\psi_\ell, \psi_{\ell,\mu}, \psi_\ell^*, \psi_{\ell,\mu}^*\}, \{V_\sigma, V_{\sigma,\mu}\}, x^\mu), \quad (11.1)$$

¹[212, p. 395].

where ψ_ℓ^* is complex conjugate to ψ_ℓ , and the fields ψ_ℓ , their conjugates ψ_ℓ^* and V_σ are treated as independent, see Sect. 10.1. Notice that the Lagrangian (11.1) is of the form of (10.1) where the variables q_ℓ are ψ_ℓ , ψ_ℓ^* and V_σ . Hence the corresponding Euler–Lagrange field equations (10.8) are, [250, 3.3], [337, II.3, (3.3)]:

$$\frac{\partial \mathcal{L}}{\partial \psi_\ell} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \psi_{\ell,\mu}} \right) = 0, \quad (11.2)$$

$$\frac{\partial \mathcal{L}}{\partial \psi_\ell^*} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \psi_{\ell,\mu}^*} \right) = 0, \quad \ell = 1, \dots, N,$$

$$\frac{\partial \mathcal{L}}{\partial V_\sigma} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial V_{\sigma,\mu}} \right) = 0, \quad \sigma = 1, \dots, N_\alpha, \quad (11.3)$$

where we use the following abbreviations

$$\partial^\mu \psi = \psi^{\cdot\mu}, \quad \partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \partial^\mu = \frac{\partial}{\partial x_\mu}, \quad (11.4)$$

that are consistent with (4.15). As we explain in Sect. 10.1, the Eqs.(11.2) in the cases we study in the book are the Euler–Lagrange equations even if ψ_ℓ and ψ_ℓ^* are not independent but are complex conjugates of one another.

The canonical energy-momentum tensor for the Lagrangian (11.1) is readily obtained from the general formula (10.54), namely

$$\hat{T}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \psi_{\ell,\mu}} \psi_\ell^{\cdot\nu} + \frac{\partial \mathcal{L}}{\partial \psi_{\ell,\mu}^*} \psi_\ell^{\cdot\nu*} + \frac{\partial \mathcal{L}}{\partial V_{\sigma,\mu}} V_\sigma^{\cdot\nu} - g^{\mu\nu} \mathcal{L}. \quad (11.5)$$

In particular, in the case when the Lagrangian $\mathcal{L}$ depends only on complex-valued fields ψ_ℓ and ψ_ℓ^* , the canonical stress tensor is of the form:

$$\hat{T}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \psi_{\ell,\mu}} \psi_\ell^{\cdot\nu} + \frac{\partial \mathcal{L}}{\partial \psi_{\ell,\mu}^*} \psi_\ell^{\cdot\nu*} - g^{\mu\nu} \mathcal{L}. \quad (11.6)$$

11.1 Many Charges Interacting with Electromagnetic Fields: General Aspects

This section is devoted to the analysis of different aspects of the Lagrangian framework concerning interaction between charges and EM fields, in particular the EM self-interaction issue.

11.1.1 Lagrangian and Field Equations

A special case of the Lagrangian (11.1) important to us is the one where several charges described by their complex-valued fields ψ_ℓ and ψ_ℓ^* , $\ell = 1, \dots, N$, interact through their elementary EM four-potentials A_ℓ^μ . We also introduce the system *total field potential* by the formula

$$A^\mu = \sum_\ell A_\ell^\mu. \quad (11.7)$$

When entertaining the issue of the EM self-interaction, we would like to explore different possibilities. With that in mind, we introduce the *actual field potential* acting upon the ℓ -th charge by the following formula

$$\tilde{A}_\ell^\mu = A_{\text{ex}}^\mu + A^\mu + \theta A_\ell^\mu, \quad (11.8)$$

where $-1 \leq \theta \leq 0$ is a real parameter that accounts for the relative magnitude of the self-interaction, and A_{ex}^μ is the 4-potential of an EM field external to the system. We are particularly interested in the following two important particular cases:

- $\theta = 0$, the case of the *classical single EM* field acting on every ℓ -th charge; in this case we refer to charges as *dressed charges*;
- $\theta = -1$, the case when there is *no EM self-interaction*, and there are N elementary EM fields A_ℓ^μ generated by the corresponding charges; in this case we refer to charges as *balanced charges*.

The reason for introducing the parameter θ is to observe the effects of the charge self-interaction on ultimate properties of the EM interactions for the entire system. For example, the Hydrogen atom spectrum depends on θ , and, as we have found, the case of balanced charges without self-interaction where $\theta = -1$ provides for a better matching with the Schrödinger Hydrogen atom spectrum, see Sect. 19.5.

For balanced charges, the actual potential acting on the ℓ -th charge takes the form

$$\tilde{A}_\ell^\mu = A_{\text{ex}}^\mu + A_{\neq \ell}^\mu \quad A_{\neq \ell}^\mu = \sum_{\ell' \neq \ell} A_{\ell'}^\mu \quad (11.9)$$

showing that the EM self-interaction is manifestly excluded.

The EM interaction is described in terms of so-called *covariant derivatives* $\psi_{\ell;\mu}$ and $\psi_{\ell;\mu}^*$ by the following formulas:

$$\begin{aligned} \dot{\psi}_\ell^\mu &= \tilde{\partial}_\ell^\mu \psi_\ell, & \psi_{\ell;\mu} &= \tilde{\partial}_{\ell\mu} \psi_\ell, \\ \dot{\psi}_\ell^{\mu*} &= \tilde{\partial}_\ell^{\mu*} \psi_\ell^*, & \psi_{\ell;\mu}^* &= \tilde{\partial}_{\ell\mu}^* \psi_\ell^*, \end{aligned} \quad (11.10)$$

where

$$\tilde{\partial}_\ell^\mu = \partial^\mu + \frac{iq_\ell}{\chi c} \tilde{A}_\ell^\mu, \quad \tilde{\partial}_\ell^{\mu*} = \partial^\mu - \frac{iq_\ell}{\chi c} \tilde{A}_\ell^\mu. \quad (11.11)$$

The real number q_ℓ in the above formulas is interpreted as the charge value of the ℓ -th elementary charge, and $\tilde{\partial}_\ell^\mu$ and $\tilde{\partial}_\ell^{\mu*}$ are called the *covariant differential operators* (or *covariant differentiations*). The index ℓ in the above notation stresses the dependence of the differentiation operator on the charge q_ℓ and $\tilde{A}_\ell^\mu$. In agreement with (4.15), we also use the notation

$$\tilde{\partial}_\mu = \left(\frac{1}{c} \tilde{\partial}_t, \tilde{\nabla} \right), \quad (11.12)$$

$$\tilde{\partial}_t = \partial_t + \frac{iq}{\chi} \tilde{\varphi}, \quad \tilde{\nabla} = \nabla - \frac{iq}{\chi c} \tilde{\mathbf{A}}, \quad (11.13)$$

where $\tilde{\varphi}$, $\tilde{\mathbf{A}}$ are the scalar and vector components of $\tilde{A}^\mu$ or $\tilde{A}_\ell^\mu$.

Now we apply the formalism developed in the previous section to the Lagrangian (11.1) with

$$V_\sigma = A_\ell.$$

Namely, we introduce the Lagrangian of the form

$$\begin{aligned} & \mathcal{L}(\{\psi_\ell, \psi_{\ell;\mu}, \psi_\ell^*, \psi_{\ell;\mu}^*\}, A_\ell^\mu) \\ &= \sum_\ell L_\ell(\psi_\ell, \psi_{\ell;\mu}, \psi_\ell^*, \psi_{\ell;\mu}^*) - \tilde{L}_{\text{em}}(A_\ell), \end{aligned} \quad (11.14)$$

where the electromagnetic part of the Lagrangian is of the form

$$\tilde{L}_{\text{em}}(A_\ell) = L_{\text{em}}(A) + \theta \sum_\ell L_{\text{em}}(A_\ell). \quad (11.15)$$

Note that the potential A is defined by (11.7) in terms of A_ℓ and is not an independent variable. The external field 4-potential A_{ex}^μ which enters the covariant derivatives is given and, therefore, is not subject to variation, and the charge Lagrangians L_ℓ may involve explicit dependence on space and time variables through A_{ex}^μ . We use the semicolon in the covariant derivative notation $\psi_{\ell;\mu}$ to point to the specific form of the dependence on the actual field $\tilde{A}^\mu$ of the Lagrangian L_ℓ . Obviously, the covariant derivatives (11.10) and (11.4) satisfy the relation

$$\psi_{\ell;\mu}^\mu = \psi_\ell^{\cdot\mu} + \frac{iq_\ell}{\chi c} \tilde{A}_\ell^\mu \psi_\ell. \quad (11.16)$$

In the cases of interest, the Lagrangians L_ℓ are either the NKG Lagrangians (11.124) in the relativistic case or the NLS Lagrangians (11.155) in the non-relativistic case. The standard Lorentz and gauge invariant EM Lagrangian L_{em} is defined by expression (9.1), namely

$$L_{\text{em}}(A) = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}, \quad L_{\text{em}}(A_\ell) = -\frac{1}{16\pi} F_\ell^{\mu\nu} F_{\ell\mu\nu} \quad (11.17)$$

with

$$F_\ell^{\mu\nu} = \partial^\mu A_\ell^\nu - \partial^\nu A_\ell^\mu, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \sum_\ell F_\ell^{\mu\nu}. \quad (11.18)$$

We also consider non-relativistic approximations where the EM Lagrangians are given by expressions (9.43) or (9.49). In all the cases, the EM Lagrangians $L_{\text{em}}(A_\ell)$, $L_{\text{em}}(A)$ do not involve explicit dependence on space and time variables, they coincide with the Lagrangians for free EM fields.

For the Lagrangian $\mathcal{L}$ defined by (11.14), (11.10) and (11.11), the charge field equations (or material equations) (11.2) for ψ_ℓ according to (10.16) have the form

$$\frac{\partial L_\ell}{\partial \psi_\ell^*} - \tilde{\partial}_{\ell\mu} \left[\frac{\partial L_\ell}{\partial \psi_{\ell;\mu}^*} \right] = 0, \quad \ell = 1, \dots, N. \quad (11.19)$$

Note that the above equations are exactly the same (though might look different) as the Eq. (11.2) which are derived under the assumptions that $\psi_{\ell;\mu}$ are the variables and not $\psi_{\ell;\mu}$. To see this, one only has to take into account that the partial derivative $\frac{\partial L_\ell}{\partial \psi_\ell^*}$ is evaluated differently in the cases where $\psi_{\ell;\mu}$ and $\psi_{\ell;\mu}$ are constant.

We refer to the field equations (11.19) as *material equations* to distinguish them from the Maxwell field equations for the EM fields. We would like to stress that ψ_ℓ describes only the charge distribution, and we do not assume it is related to such “matter” properties as mass, though some matter properties such as energy and momentum distribution can be expressed in terms of the energy-momentum density tensor.

The EL equations (11.3) for the EM potentials A_ℓ^μ obviously can be written in the form

$$-\partial_\mu \left(\frac{\partial (L_{\text{em}}(A) + \theta L_{\text{em}}(A_\ell))}{\partial A_{\ell\nu,\mu}} \right) = \hat{J}_{\ell A}^\nu \quad (11.20)$$

where

$$\hat{J}_\ell^\nu = \hat{J}_{\ell A}^\nu = -c \sum_{\ell'} \frac{\partial L_{\ell'}}{\partial A_{\ell\nu}} = -c \sum_{\ell'} \frac{\partial L_{\ell'}}{\partial A_\nu} - c\theta \frac{\partial L_\ell}{\partial A_\nu}. \quad (11.21)$$

The factor c in the above formula for the currents is introduced for notational consistency with the Maxwell equations (4.23).

Looking at the definition (11.14) of the Lagrangian, one might think that the field equations (11.14) are uncoupled. But this is not the case, for they are coupled through the expressions (11.16) of the covariant derivatives which, according to (11.9), involve the EM fields generated by all charges of the system. This important type of coupling through the covariant derivatives is called the *minimal coupling*.

11.1.2 Field Equations for Elementary EM Fields

We consider in this section a reduction of the system of equations (11.20) in the case $N > 1$ to an equivalent system where A_ℓ are not coupled. Notice that according to (11.7),

$$\frac{\partial L_{\text{em}}(A)}{\partial A_{\ell\nu,\mu}} = \frac{\partial L_{\text{em}}(A)}{\partial A_{\nu,\mu}}, \quad (11.22)$$

hence Eq. (11.20) can be written in the following equivalent form:

$$-\partial_\mu \frac{\partial L_{\text{em}}(A)}{\partial A_{\nu,\mu}} - \theta \partial_\mu \frac{\partial L_{\text{em}}(A_\ell)}{\partial A_{\ell\nu,\mu}} = \hat{J}_{\ell A}^\nu. \quad (11.23)$$

Notice that

$$\sum_\ell \hat{J}_\ell^\nu = -c \sum_\ell \sum_{\ell'} \frac{\partial L_{\ell'}}{\partial A_\nu} - c\theta \sum_\ell \frac{\partial L_\ell}{\partial A_\nu} = -c(N + \theta) \sum_{\ell'} \frac{\partial L_{\ell'}}{\partial A_\nu}. \quad (11.24)$$

Observe also that, since the EM Lagrangians (11.17), (9.43) and (9.49) are quadratic functions of their variables, their derivatives are linear functions. Consequently, according to (11.7), we have

$$\sum_\ell \frac{\partial L_{\text{em}}(A_\ell)}{\partial A_{\ell\nu,\mu}} = \frac{\partial L_{\text{em}}(A)}{\partial A_{\nu,\mu}}, \quad (11.25)$$

and summing the expressions in (11.23) with respect to ℓ , we obtain

$$-(N + \theta) \partial_\mu \left(\frac{\partial (L_{\text{em}}(A))}{\partial A_{\nu,\mu}} \right) = -c(N + \theta) \sum_{\ell'} \frac{\partial L_{\ell'}}{\partial A_\nu}. \quad (11.26)$$

Since $N + \theta \neq 0$ for $N > 1$ and $-1 \leq \theta \leq 0$, this implies

$$-\partial_\mu \left(\frac{\partial (L_{\text{em}}(A))}{\partial A_{\nu,\mu}} \right) = -c \sum_{\ell'} \frac{\partial L_{\ell'}}{\partial A_\nu}. \quad (11.27)$$

We infer from (11.23) and (11.27) that

$$-\theta \partial_\mu \frac{\partial L_{\text{em}}(A_\ell)}{\partial A_{\ell\nu,\mu}} = -c\theta \frac{\partial L_\ell}{\partial A_\nu}. \quad (11.28)$$

If $\theta \neq 0$, we obtain then an equation for an elementary field A_ℓ :

$$-\partial_\mu \frac{\partial L_{\text{em}}(A_\ell)}{\partial A_{\ell\nu,\mu}} = J_\ell^\nu, \quad (11.29)$$

where the ℓ -th charge four-vector current density $J_{\ell A}^\nu$ is given according to (11.11) by the formula

$$J_\ell^\nu = J_{\ell A}^\nu = -c \frac{\partial L_\ell}{\partial A_{\ell\nu}} = -i \frac{q_\ell}{\chi} \left(\frac{\partial L_\ell}{\partial \psi_{\ell;\nu}} \psi_\ell - \frac{\partial L_\ell}{\partial \psi_{\ell;\nu}^*} \psi_\ell^* \right). \quad (11.30)$$

Obviously, if $\theta \neq 0$, the system (11.29) is equivalent to (11.23).

We define the total current density J^ν naturally as the sum of elementary current densities J_ℓ^ν , that is,

$$J^\nu = \sum_\ell J_\ell^\nu = \sum_\ell J_{\ell A}^\nu. \quad (11.31)$$

In view of (11.23) and (11.28), the total field A then satisfies the following equation

$$-\partial_\mu \frac{\partial L_{\text{em}}(A)}{\partial A_{\nu,\mu}} = J^\nu \quad (11.32)$$

in all cases of interest including the case $\theta = 0$.

Let us take a look at the classical case of the single EM field $\theta = 0$, which can be considered as the limit as $\theta \rightarrow 0$. Note that for $\theta = 0$ all Eq. (11.20) with different ℓ are equivalent to a single Eq. (11.32). But since the above equation is linear, its solution is reduced to solution of equations (11.29) for elementary charges even in the limit case $\theta = 0$. This allows us to set the field A_ℓ into correspondence with the current $J_{\ell A}^\nu$.

In the relativistic case with the standard gauge invariant Lagrangian (11.17), Eqs. (11.32) and (11.29) take the form of the Maxwell equations for the total EM field

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu, \quad (11.33)$$

and for individual EM fields

$$\partial_\mu F_\ell^{\mu\nu} = \frac{4\pi}{c} J_\ell^\nu, \quad (11.34)$$

with the source currents J^ν and J_ℓ^ν being defined respectively by (11.31), (11.30).

In the non-relativistic case (9.43), we obtain the Poisson equations

$$\nabla^2 \varphi = -\frac{4\pi}{c} J^0, \quad (11.35)$$

$$\nabla^2 \varphi_\ell = -\frac{4\pi}{c} J_\ell^0. \quad (11.36)$$

In the case (9.49), we also obtain the Poisson equations for every component

$$\nabla^2 A_\ell^\nu = -\frac{4\pi}{c} J_\ell^\nu \quad (11.37)$$

complemented with the Lorentz gauge condition

$$\partial_\mu A_\ell^\mu = 0. \quad (11.38)$$

Importantly, the currents J_ℓ^ν in all the cases are defined by the same formula (11.30), and the charge conservation law holds for all of them, that is,

$$\partial_\mu J_\ell^\mu = 0. \quad (11.39)$$

Observe that we might consider the total field A^μ in the Lagrangian (11.14) as an independent variable. Then we would obtain (11.32) and (11.29) as the Euler–Lagrange equations. Formula (11.7) then would follow from (11.31) and the uniqueness of solution of (11.32). The desired uniqueness can be provided by fixing a Green function which determines solutions to the Maxwell equations.

As we already mentioned concerning the charge Lagrangians L_ℓ , we consider in this book two principal cases: the relativistic case where the field equations (11.19) are the nonlinear Klein–Gordon (NKG) equations and the non-relativistic case where the field equations are the nonlinear Schrödinger (NLS) equations. Notice that for the multi-charge Lagrangian $\mathcal{L}$ defined by (11.14) the material equations have the following basic structural properties: (i) every charge interacts with the EM field described by the four-potentials A_ℓ^μ ; (ii) different charges don’t interact directly with each other, but they interact only indirectly through the EM potential A_ℓ^μ . The charge is coupled to the EM field only through the covariant derivatives, and such a coupling is widely used and is called the *minimal coupling*.

11.2 Gauge Invariance and Symmetric Energy-Momentum Tensors

The gauge invariance of Lagrangians, the energy-momentum tensors and their symmetry are very important to the entire theory of EM interactions. In this section, we analyze those properties and constructive ways to establish them.

11.2.1 Symmetries of the Lagrangian

The multiparticle Lagrangian $\mathcal{L}$ in the particular form (11.14) and its ℓ -th charge components L_ℓ are subjected to the very important condition of *gauge invariance*. In

particular, we consider *gauge transformations of the first or the second kind* (known also as, respectively, *global and local gauge transformations*) for the fields ψ_ℓ and ψ_ℓ^* . These transformations are described, respectively, by the following formulas [266, (17), (23a), (23b)], [337, Section 11, (11.4)], [332, 2.8.4]:

$$\psi_\ell \rightarrow e^{-\frac{iq_\ell \lambda_\ell}{\hbar c}} \psi_\ell, \quad \psi_\ell^* \rightarrow e^{\frac{iq_\ell \lambda_\ell}{\hbar c}} \psi_\ell^*, \quad (11.40)$$

where λ_ℓ is an arbitrary real constant, and

$$\psi_\ell \rightarrow e^{-\frac{iq_\ell \lambda(x)}{\hbar c}} \psi_\ell, \quad \psi_\ell^* \rightarrow e^{\frac{iq_\ell \lambda(x)}{\hbar c}} \psi_\ell^*, \quad A^\mu \rightarrow A^\mu + \partial^\mu \lambda, \quad (11.41)$$

where $\lambda(x)$ is an arbitrary real function.

We will always assume that the Lagrangian is invariant with respect to the gauge transformations (11.40), (11.41). In particular, the invariance with respect to the gauge transformations (11.40) means that for every ℓ and any real γ

$$L_\ell(e^{i\gamma} \psi_\ell, e^{i\gamma} \psi_{\ell;\mu}, e^{-i\gamma} \psi_\ell^*, e^{-i\gamma} \psi_{\ell;\mu}^*) = L_\ell(\psi_\ell, \psi_{\ell;\mu}, \psi_\ell^*, \psi_{\ell;\mu}^*). \quad (11.42)$$

Note that the invariance with respect to more general local gauge transformations (11.41) follows from the invariance with respect to (11.40) and the definition of the covariant derivatives $\psi_{\ell;\mu}$ and $\psi_{\ell;\mu}^*$ by (11.10), (11.11), that is,

$$\psi_{\ell;\mu} = \partial_\mu \psi_\ell + \frac{iq_\ell}{\hbar c} \tilde{A}_{\ell\mu} \psi_\ell, \quad \psi_{\ell;\mu}^* = \partial_\mu \psi_\ell^* - \frac{iq_\ell}{\hbar c} \tilde{A}_{\ell\mu} \psi_\ell^*. \quad (11.43)$$

If we take the derivative of the above condition (11.42) with respect to γ at $\gamma = 0$, we obtain the following structural restriction on every Lagrangian L_ℓ :

$$\frac{\partial L_\ell}{\partial \psi_{\ell;\mu}} \psi_{\ell;\mu} - \frac{\partial L_\ell}{\partial \psi_{\ell;\mu}^*} \psi_{\ell;\mu}^* + \frac{\partial L_\ell}{\partial \psi_\ell} \psi_\ell - \frac{\partial L_\ell}{\partial \psi_\ell^*} \psi_\ell^* = 0. \quad (11.44)$$

In fact, this structural condition is equivalent to (11.42). We show in the following subsection that this condition implies the continuity (charge conservation) equation.

We also introduce the following structural symmetry condition on every Lagrangian L_ℓ :

$$\frac{\partial L_\ell}{\partial \psi_{\ell;\mu}} \psi_\ell^{;\nu} + \frac{\partial L_\ell}{\partial \psi_{\ell;\mu}^*} \psi_\ell^{*\nu} = \frac{\partial L_\ell}{\partial \psi_{\ell;\nu}} \psi_\ell^{;\mu} + \frac{\partial L_\ell}{\partial \psi_{\ell;\nu}^*} \psi_\ell^{*\mu}. \quad (11.45)$$

As we show below, the symmetry condition (11.45) implies that the energy-momentum assigned to every individual charge is symmetric and gauge invariant. A simple sufficient condition for the symmetry condition (11.45) is the requirement for the Lagrangians L_ℓ to depend on the field covariant derivatives only through their combination $\psi_\ell^{;\mu} \psi_{\ell;\mu}^*$, in other words if there exist such functions $K_\ell(\psi_\ell, \psi_\ell^*, b)$ that

$$L_\ell(\psi_\ell, \psi_{\ell;\mu}, \psi_\ell^*, \psi_{\ell;\mu}^*) = K_\ell(\psi_\ell, \psi_\ell^*, \psi_\ell^{\dot{\mu}} \psi_{\ell;\mu}^*). \quad (11.46)$$

Indeed, in this case

$$\frac{\partial L_\ell}{\partial \psi_{\ell;\mu}} \psi_\ell^{\dot{\nu}} + \frac{\partial L_\ell}{\partial \psi_{\ell;\mu}^*} \psi_\ell^{\dot{\nu}*} = \frac{\partial K_\ell}{\partial b}(\psi_\ell^{\dot{\mu}*} \psi_\ell^{\dot{\nu}} + \psi_\ell^{\dot{\mu}} \psi_\ell^{\dot{\nu}*}), \quad (11.47)$$

readily implying that the symmetry condition (11.45) does hold.

We would like to point out that even if a Lagrangian of the form (11.14) is not invariant with respect to the entire Lorentz group of transformations, it may satisfy a reduced version of the symmetry condition (11.45) which holds for the space indices only, namely

$$\frac{\partial L_\ell}{\partial \psi_{\ell;i}} \psi_{\ell;j} + \frac{\partial L_\ell}{\partial \psi_{\ell;i}^*} \psi_{\ell;j}^* = \frac{\partial L_\ell}{\partial \psi_{\ell;j}} \psi_{\ell;i} + \frac{\partial L_\ell}{\partial \psi_{\ell;j}^*} \psi_{\ell;i}^*, \quad i, j = 1, 2, 3. \quad (11.48)$$

Under the reduced symmetry condition (11.48), the space part of the energy-momenta $T_\ell^{\mu\nu}$, known as the stress tensor, is symmetric, namely

$$T_\ell^{ij} = T_\ell^{ji}, \quad i, j = 1, 2, 3. \quad (11.49)$$

We recall that the symmetry of the stress tensor is an important property equivalent to the space angular momentum conservation, see Sect. 10.6 and, for instance, [248, Sects. 6.1, 6.2].

The Lagrangian of the nonlinear Klein–Gordon (NKG) equation satisfies both conditions (11.42) and (11.46), see Sect. 11.6. The Lagrangian of the NLS equation satisfies (11.42) and (11.48), but does not satisfy (11.45), see Sect. 11.8.

11.2.2 The Continuity Equation and Preservation of the Lorentz Gauge

Under the gauge invariance condition (11.40) for the Lagrangian $\mathcal{L}$, one can use Noether's theorem and formula (10.51) to introduce for every charge ψ_ℓ the following 4-vector current, [266, (19)], [337, (3.11)–(3.13)]:

$$J_\ell^\nu = -i \frac{q_\ell}{\chi} \left(\frac{\partial L_\ell}{\partial \psi_{\ell;\nu}} \psi_\ell - \frac{\partial L_\ell}{\partial \psi_{\ell;\nu}^*} \psi_\ell^* \right). \quad (11.50)$$

Introducing the charge density ρ and the current $\mathbf{J}$,

$$J^\nu = (c\rho, \mathbf{J}), \quad (11.51)$$

we rewrite (11.50) in the form

$$\rho_\ell = -i \frac{q_\ell}{\chi} \left(\frac{\partial L_\ell}{\partial \psi_{\ell;0}} \psi_\ell - \frac{\partial L_\ell}{\partial \psi_{\ell;0}^*} \psi_\ell^* \right), \quad (11.52)$$

$$\mathbf{J}_{\ell j} = -i \frac{q_\ell}{\chi} \left(\frac{\partial L_\ell}{\partial \psi_{\ell;j}} \psi_\ell - \frac{\partial L_\ell}{\partial \psi_{\ell;j}^*} \psi_\ell^* \right), \quad j = 1, 2, 3. \quad (11.53)$$

To show that the above definition agrees with the definition of Noether's currents, let us derive the above formula (11.50) from (10.51). On taking ψ_ℓ and ψ_ℓ^* as $q_{\bar{\ell}_1}$ and $q_{\bar{\ell}_2}$, we observe that for transformation (11.40) the variations in (10.25) take the form

$$\delta x^\mu = 0, \quad \delta \psi_\ell = \frac{i q_\ell}{\chi c} \psi_\ell \delta \lambda_\ell, \quad \delta \psi_\ell^* = -\frac{i q_\ell}{\chi c} \psi_\ell^* \delta \lambda_\ell,$$

producing, respectively,

$$\Gamma_k^\mu = 0, \quad \Phi_{\bar{\ell}_1 k} = \frac{i q_\ell}{\chi c} \psi_\ell, \quad \Phi_{\bar{\ell}_2 k} = -\frac{i q_\ell}{\chi c} \psi_\ell^*.$$

Note that the variations are independent for different ℓ , and different k correspond to different ℓ . Hence we see that the formula for the current (11.50) is a particular case of Noether's current (10.51) where the sum over $\bar{\ell}$ involves only two terms, and k is determined by ℓ .

Under the assumption (11.44), for every ℓ the 4-vector current formed for a solution of (11.19) satisfies the charge conservation/continuity equations

$$\partial_\nu J_\ell^\nu = 0. \quad (11.54)$$

According to (11.51), the continuity equation can also be written in the form

$$\partial_t \rho_\ell + \nabla \cdot \mathbf{J}_\ell = 0. \quad (11.55)$$

This equation is a particular case of conservation law (10.50). To verify the continuity equation directly, we evaluate $\partial_\nu J_\ell^\nu$ as follows:

$$\begin{aligned} \partial_\nu J_\ell^\nu &= -i \frac{q_\ell}{\chi} \left(\frac{\partial L_\ell}{\partial \psi_{\ell;\nu}} \partial_\nu \psi_\ell - \frac{\partial L_\ell}{\partial \psi_{\ell;\nu}^*} \partial_\nu \psi_\ell^* \right) \\ &\quad - i \frac{q_\ell}{\chi} \left(\left(\frac{\partial L_\ell}{\partial \psi_\ell} + \frac{i q_\ell}{\chi c} \tilde{A}_{\ell\nu} \frac{\partial L_\ell}{\partial \psi_{\ell;\nu}} \right) \psi_\ell - \left(\frac{\partial L_\ell}{\partial \psi_\ell^*} - \frac{i q_\ell}{\chi c} \tilde{A}_{\ell\nu} \frac{\partial L_\ell}{\partial \psi_{\ell;\nu}^*} \right) \psi_\ell^* \right), \end{aligned} \quad (11.56)$$

where we used the expression for $\partial_\nu \frac{\partial L_\ell}{\partial \psi_{\ell;\nu}}$ obtained from the Euler–Lagrange equation (11.19). After that we use (11.44) to readily obtain (11.54) from (11.56). Note that the above derivation of the continuity equation is universal, it does not use equations for the EM fields. It uses only the ℓ -th charge equation and the structural property (11.44) (or the gauge invariance (11.42)) of the charge Lagrangian L_ℓ for which it is derived.

The NKG Lagrangian and NLS Lagrangians satisfy the gauge invariance condition (11.42) and (11.44). Consequently, the continuity equations are satisfied when the field equations (11.19) coincide with the nonlinear Klein Gordon equations or the nonlinear Schrödinger equations.

Under the Lorentz gauge, the Maxwell equations (11.33) are equivalent to wave equations (4.28) for 4-potential components. Since the continuity equations are satisfied, the wave equations preserve the Lorentz gauge. *Using this property, we prescribe the Lorentz gauge for the EM fields in all our models.* Note that any choice of gauge produces the same EM fields, and we show that the macroscopic dynamics of charges in our models is affected only by the EM fields. Hence the choice of gauge does not affect macroscopic effects. But at the atomic scales the potentials themselves affect the charge behavior since they enter the material equations.

11.2.3 Source Currents in Maxwell's Equations and Charge Conserved Currents

The Maxwell equations (11.33), (11.34) involve the source currents $J_{\ell A}^\nu$ defined by (11.30). At the same time, in the previous section we introduced in (11.50) the charge currents J_ℓ^ν based on the gauge invariance of the charge Lagrangian L_ℓ . Comparing expressions (11.30) and (11.50) for the currents, we notice that the source terms in the Maxwell equations (11.33), (11.34), namely the currents $J_{\ell A}^\nu$ defined by (11.30), coincide with the charge currents J_ℓ^ν defined by (11.50):

$$J_\ell^\nu = -c \frac{\partial L_\ell}{\partial A_\nu} = J_{\ell A}^\nu, \quad \ell = 1, \dots, N. \quad (11.57)$$

We would like to emphasize here the physical significance of the “two-way” representation (11.57) equating two complementary views on the elementary current: (i) as a source current in the Maxwell equations (11.33); (ii) as the conserved Noether's elementary current (11.50). The equality (11.57) originates from the particular form of the coupling between the EM field and charges in the Lagrangian (11.14), namely the minimal coupling through the covariant derivatives (11.10). One may also view the electric currents identity (11.57) as a physical rationale for introducing the coupling exactly as it is done in the expressions (11.10), (11.14). Note that, thanks to (11.57), the notation (11.51) is consistent with (4.13). Importantly, formula (11.52) is applicable in every particular case of the EM Lagrangian, namely (11.17), (9.43) or (9.49).

11.2.4 The Additivity Property of Currents and Fields

Formula (11.31) signifies the *additivity property of the currents*, namely the current, which is the source in Maxwell's equations (11.33) for the total field, equals the sum of currents J_ℓ^ν where the ℓ -th current is determined exclusively by the ℓ -th charge distribution ψ_ℓ . Since the Lagrangian depends on the derivatives of electromagnetic potentials quadratically, the EM field equations, namely the Maxwell equations, are linear with respect to the electromagnetic fields. Solutions $F^{\mu\nu}$ of the Maxwell equations can be written in terms of the corresponding Green functions. For example, one can take retarded potentials G^+ or advanced potentials G^- , or one can take a half-sum $\frac{1}{2}(G^+ + G^-)$ of the retarded and advanced potentials as in the Wheeler–Feynman theory, see Sect. 4.2. We assume that a particular Green function is chosen, and, therefore, the solution operator $\mathcal{G}$ for the Maxwell equations (17.103) is defined. Since the equation is linear, we deduce from (11.31) that

$$F^{\mu\nu} = \sum_{\ell} F_{\ell}^{\mu\nu} \quad F_{\ell}^{\mu\nu} = \frac{4\pi}{c} \mathcal{G} J_{\ell}^{\nu}, \quad (11.58)$$

where $F_{\ell}^{\mu\nu}$ satisfies Maxwell's equation

$$\partial_{\mu} F_{\ell}^{\mu\nu} = \frac{4\pi}{c} J_{\ell}^{\nu}. \quad (11.59)$$

Formula (11.58) implies the field partition (11.7), namely

$$A^{\mu} = \sum_{\ell} A_{\ell}^{\mu} \quad (11.60)$$

even in the case $\theta = 0$ in (11.7); for $\theta \neq 0$ the above formula is just a definition of A . Hence, the total EM field is represented in all cases as a sum of elementary fields A_{ℓ}^{μ} generated by the charges.

Since the operator ∇^2 is also linear, the field partition (11.60) also holds in the cases (9.43) and (9.49). Note that the solution operator for the Poisson equation is determined uniquely by the requirement to produce for a localized source a potential which decays at infinity, in particular the Coulomb potential for a point source, namely the solution is given by (12.19).

Notice also that every individual charge satisfies its field equation (11.19) with the EM field entering it via the potential $\tilde{A}^{\mu}$ in the covariant derivatives (11.10). Separating the field generated by the charge itself, we can always represent the actual field potential (11.8) which acts on the ℓ -th charge in the form

$$\tilde{A}_{\ell}^{\mu} = (1 + \theta) A_{\ell}^{\mu} + A_{\neq \ell}^{\mu} + A_{\text{ex}}^{\mu}, \quad (11.61)$$

$$A_{\neq \ell}^{\mu} = \sum_{\ell' \neq \ell} A^{\ell' \mu}, \quad (11.62)$$

where A_ℓ^μ represents the charge EM self-interaction and $A_{\neq\ell}^\mu$ represents the field generated by all the remaining charges. If $\theta = -1$, there is no self-interaction, and we call the corresponding theory the *balanced charges theory* (BCT), see Chap. 12.

11.3 Partition of Energy-Momentum for Many Interacting Fields

The Lagrangian (11.14) involves many interacting charges and corresponding EM fields. Therefore, the system of field equations (11.19)–(11.20) is composed of many equations for the fields involved. As we have shown in Sect. 11.2.2, there are individual charge conservation laws for every single charge. At the same time, there is a single energy-momentum conservation law derived from the total Lagrangian $\mathcal{L}$. In this subsection we define individual Lagrangians and derive the corresponding individual conservation laws. Observe that the Lagrangian (11.14) has the following structure:

$$\mathcal{L} = \sum_{\ell} L_{\ell} (\psi_{\ell}, \psi_{\ell;\mu}; \psi_{\ell}^*, \psi_{\ell;\mu}^*) + L_{\text{em}} (A) + \sum_{\ell} \theta L_{\text{em}} (A_{\ell}), \quad (11.63)$$

where the EM field Lagrangians $L_{\text{em}} = L_{\text{em}} (A, 0)$ are defined by (11.17), (9.43) or (9.49) with $J = 0$. The Lagrangians L_{ℓ} are either the NLS Lagrangians or the NKG Lagrangians in the non-relativistic and relativistic cases respectively. One can easily see that the coupling between the field equations (11.19)–(11.20) comes only from the covariant derivatives (11.11).

11.4 Partition of Canonical Energy-Momentum

We show that, for the Lagrangian of the form (11.63), the evaluation of the EnMT can be reduced to the evaluation of the EnMT for individual Lagrangians involved in (11.63).

With the above discussed decomposition in mind, we consider the canonical EnMT as a function of a Lagrangian:

$$\hat{T}^{\mu\nu} = \hat{T}^{\mu\nu} (\mathcal{L}).$$

Obviously, $\hat{T}^{\mu\nu} (\mathcal{L})$ depends on $\mathcal{L}$ linearly:

$$\hat{T}^{\mu\nu} (\mathcal{L}_1 + \mathcal{L}_2) = \hat{T}^{\mu\nu} (\mathcal{L}_1) + \hat{T}^{\mu\nu} (\mathcal{L}_2), \quad \hat{T}^{\mu\nu} (C\mathcal{L}) = C\hat{T}^{\mu\nu} (\mathcal{L}) \quad (11.64)$$

as long as $\mathcal{L}_1$ and $\mathcal{L}_2$ depend on the same variables. If $\mathcal{L}_1$ and $\mathcal{L}_2$ depend on different variables, we can obviously unite all the variables, and then (11.64) continues to hold.

According to (11.63), the energy-momenta for the Lagrangian $\mathcal{L}$ can be derived from the expressions obtained for the EM fields, the NLS Lagrangians and the NKG Lagrangians. Namely, the canonical total energy-momentum can be expanded as follows:

$$\begin{aligned} \dot{T}^{\mu\nu} &= \dot{T}^{\mu\nu}(\mathcal{L}) \\ &= \sum_{\ell} \dot{T}^{\mu\nu}(L_{\ell}) + \dot{T}^{\mu\nu}(L_{\text{em}}(A)) + \theta \sum_{\ell} \dot{T}^{\mu\nu}(L_{\text{em}}(A_{\ell})). \end{aligned} \quad (11.65)$$

Notice that the covariant derivatives which enter L_{ℓ} depend only on the EM potentials and not on their derivatives, and the energy momentum tensor involves derivatives of $\mathcal{L}$ only with respect to the derivatives of the fields. Hence, *the canonical energy-momenta tensors which enter (11.65) can be evaluated independently*. In other words, the canonical energy-momentum tensor for the Lagrangian (11.14) can be naturally decomposed into the energy momenta for the EM fields and the energy momenta for the charge distributions ψ_{ℓ} . The individual EnMT $\dot{T}_{\ell}^{\mu\nu}$ of the ℓ -th charge is represented by (11.6) as follows:

$$\dot{T}_{\ell}^{\mu\nu} = \dot{T}^{\mu\nu}(L_{\ell}) = \frac{\partial L_{\ell}}{\partial \psi_{\ell;\mu}} \psi_{\ell}^{\nu} + \frac{\partial L_{\ell}}{\partial \psi_{\ell;\mu}^*} \psi_{\ell}^{\nu*} - g^{\mu\nu} L_{\ell}. \quad (11.66)$$

We denote by $\dot{\Theta}^{\mu\nu}$, $\dot{\Theta}_{\ell}^{\mu\nu}$ the canonical energy-momenta of the EM fields,

$$\dot{\Theta}^{\mu\nu} = \dot{T}^{\mu\nu}(L_{\text{em}}(A)), \quad \dot{\Theta}_{\ell}^{\mu\nu} = \dot{T}^{\mu\nu}(L_{\text{em}}(A_{\ell})). \quad (11.67)$$

Since the Lagrangians $L_{\text{em}}(A_{\ell})$ do not involve dependence on x , *the EnMT $\dot{\Theta}^{\mu\nu}$, $\dot{\Theta}_{\ell}^{\mu\nu}$ are derived from the Lagrangians for free fields*. They are directly given by the general definition (11.5), and in the relativistic case (11.17) are given by formula (9.11) with $J_{\alpha} = 0$.

In the non-relativistic case (9.43),

$$\begin{aligned} \dot{\Theta}_{\text{em}1}^{\mu\nu} &= \dot{T}^{\mu\nu}(L_{\text{em}1}(A)) \\ &= -\frac{1}{4\pi} \partial^{\nu} \varphi \partial_i \varphi + \frac{1}{8\pi} g^{\mu\nu} \partial_i \varphi \partial^i \varphi = -\frac{1}{4\pi} \partial^{\nu} \varphi \partial_i \varphi - \frac{1}{8\pi} g^{\mu\nu} (\nabla \varphi)^2 \end{aligned}$$

or in the matrix notation

$$\dot{\Theta}_{\text{em}1}^{\mu\nu} = \begin{bmatrix} \dot{u}_{\text{em}} & c\dot{P}_{\text{em}1} & c\dot{P}_{\text{em}2} & c\dot{P}_{\text{em}3} \\ c^{-1}\dot{S}_1 & -\dot{T}_{11} & -\dot{T}_{12} & -\dot{T}_{13} \\ c^{-1}\dot{S}_2 & -\dot{T}_{21} & -\dot{T}_{22} & -\dot{T}_{23} \\ c^{-1}\dot{S}_3 & -\dot{T}_{31} & -\dot{T}_{32} & -\dot{T}_{33} \end{bmatrix} \quad (11.68)$$

with

$$\begin{aligned} \dot{u}_{\text{em}} &= -\frac{|\nabla\varphi|^2}{8\pi}, \quad \dot{p}_{\text{em}j} = 0, \quad \dot{s}_j = c \frac{\partial_j \varphi \partial_0 \varphi}{4\pi}, \\ \dot{r}_{jj} &= \frac{\partial_j^2 \varphi}{4\pi} - \frac{|\nabla\varphi|^2}{8\pi}, \quad \dot{r}_{ij} = \frac{\partial_i \varphi \partial_j \varphi}{4\pi}. \end{aligned} \quad (11.69)$$

For the non-relativistic Lagrangian $L_{\text{em}2}$ from (9.49) we have

$$L_{\text{em}2}(A) = -\frac{1}{8\pi} \partial_i A_\alpha \partial^i A^\alpha = -\frac{1}{8\pi} (\nabla\varphi)^2 + \frac{1}{8\pi} (\nabla\mathbf{A})^2, \quad (11.70)$$

$$\dot{\Theta}_{\text{em}2}^{\mu\nu} = \dot{T}^{\mu\nu}(L_{\text{em}2}(A)) = \frac{\partial L_{\text{em}2}}{\partial A_{,\mu}^\alpha} A^{\alpha,\nu} - g^{\mu\nu} L_{\text{em}2} \quad (11.71)$$

where $\frac{\partial L_{\text{em}2}}{\partial A_{,0}^\alpha} = 0$, hence

$$\dot{\Theta}_{\text{em}2}^{00} = -L_{\text{em}2} = \frac{1}{8\pi} (\nabla\varphi)^2 - \frac{1}{8\pi} (\nabla\mathbf{A})^2, \quad (11.72)$$

and for $i = 1, 2, 3$

$$\dot{\Theta}_{\text{em}2}^{i\nu} = \frac{1}{4\pi} A_{\alpha,i} A^{\alpha,\nu} - g^{i\nu} L_{\text{em}2}. \quad (11.73)$$

11.5 Partition of the EnMT Conservation Law

The partition (11.63) suggests a corresponding partition of the EnMT and corresponding conservation laws. The possibility of a partition of the total EnMT conservation law is based on the fact that the field equation for every field variable can be considered from two viewpoints: (i) it can be derived as the EL equation from the total Lagrangian $\mathcal{L}$; (ii) it can be derived from its own Lagrangian. The second possibility allows us to define an EnMT and derive corresponding individual conservation laws for every field variable.

The canonical total EnMT admits a canonical partition (11.65), and the EnMT conservation law for the canonical EnMT in all cases of interest can be obtained by summation of corresponding individual conservation laws. In addition to the canonical EnMT, we consider symmetrized partial energy-momenta for field variables and corresponding conservation laws. We define the total symmetrized energy momentum as the sum of partial energy-momenta. The total EnMT satisfies a conservation law which can be obtained by summing up the partial conservation laws.

11.5.1 Partition of the Conservation Law for the Total Canonical EnMT

According to (10.55), the total canonical EnMT satisfies the conservation law

$$\partial_\mu \dot{T}^{\mu\nu}(\mathcal{L}) = -\frac{\partial \mathcal{L}}{\partial x_\nu} \quad (11.74)$$

where, according to (11.30) and (11.57),

$$-\frac{\partial \mathcal{L}}{\partial x_\nu} = \frac{1}{c} J^\mu \partial^\nu A_{\text{ex}\mu}, \quad J^\mu = \sum_\ell J_\ell^\mu, \quad (11.75)$$

and the currents J_ℓ^μ are defined by (11.57). Now we show that (11.74) can be split into a sum of conservation laws for individual charges and their fields. According to (11.65)

$$\partial_\mu \dot{T}^{\mu\nu}(\mathcal{L}) = \sum_\ell \partial_\mu \dot{T}^{\mu\nu}(L_\ell) + \partial_\mu \dot{\Theta}^{\mu\nu} + \sum_\ell \theta \partial_\mu \dot{\Theta}_\ell^{\mu\nu} \quad (11.76)$$

where, in agreement with (11.67), $\dot{\Theta}_\ell^{\mu\nu} = T^{\mu\nu}(L_{\text{em}}(A_\ell))$, and $L_{\text{em}}(A_\ell)$ are EM Lagrangians which correspond to *free fields*. In this section L_{em} can be a relativistic Lagrangian (11.17) or (9.24), or non-relativistic (9.43) or (9.49). Hence the total canonical EnMT conservation law (11.74) takes the form

$$\sum_\ell \partial_\mu \dot{T}^{\mu\nu}(L_\ell) + \partial_\mu \dot{\Theta}^{\mu\nu} + \sum_\ell \theta \partial_\mu \dot{\Theta}_\ell^{\mu\nu} = \frac{1}{c} J^\mu \partial^\nu A_{\text{ex}\mu}. \quad (11.77)$$

If the external field is absent, we can write the above conservation law in the form

$$\sum_\ell \partial_\mu \dot{T}^{\mu\nu}(L_\ell) = -\partial_\mu \dot{\Theta}^{\mu\nu} - \sum_\ell \theta \partial_\mu \dot{\Theta}_\ell^{\mu\nu}. \quad (11.78)$$

This relation can be interpreted as fulfillment of Newton's Third Law "every action has an equal and opposite reaction" for the interaction of all charges of the system with the EM field.

Note that the Lagrangian L_ℓ of the ℓ -th charge depends on the variables $\psi_\ell, \psi_\ell^*, A_\ell^\alpha, A^\alpha$ and the external field A_{ex}^μ . Along the lines of (10.56) we derive conservation laws for individual terms in (11.76):

$$\begin{aligned} \partial_\mu \dot{T}^{\mu\nu}(L_\ell) = & -g^{\mu\nu} \partial_\mu(L_\ell) + \partial_\mu \frac{\partial L_\ell}{\partial \psi_{\ell,\mu}} \psi_\ell^{\prime\nu} + \frac{\partial L_\ell}{\partial \psi_{\ell,\mu}} \partial_\mu \psi_\ell^{\prime\nu} \\ & + \partial_\mu \frac{\partial L_\ell}{\partial \psi_{\ell,\mu}^*} \psi_\ell^{*\prime\nu} + \frac{\partial L_\ell}{\partial \psi_{\ell,\mu}^*} \partial_\mu \psi_\ell^{*\prime\nu}. \end{aligned}$$

As in (10.56), we use EL equations (11.2) for ψ_ℓ , but now we have to take into account the fact that L_ℓ depends not only on ψ_ℓ, ψ_ℓ^* but also on A_ℓ^α, A^α :

$$\begin{aligned}\partial_\mu \hat{T}^{\mu\nu}(L_\ell) &= -\partial^\nu(L_\ell) + \frac{\partial L_\ell}{\partial \psi_\ell} \psi_\ell^{\nu} + \frac{\partial L_\ell}{\partial \psi_{\ell,\mu}} \partial^\nu \psi_{\ell,\mu} \\ &\quad + \frac{\partial L_\ell}{\partial \psi_\ell^*} \partial^\nu \psi_\ell^* + \frac{\partial L_\ell}{\partial \psi_{\ell,\mu}^*} \partial^\nu \psi_{\ell,\mu}^* \\ &= -\left(\theta \frac{\partial L_\ell}{\partial A_\ell^\alpha} A_\ell^{\alpha,\nu} + \frac{\partial L_\ell}{\partial A^\alpha} A^{\alpha,\nu} + \frac{\partial L_\ell}{\partial A^\alpha} A_{\text{ex}}^{\alpha,\nu} \right).\end{aligned}$$

Taking into account the definition (11.57) of the current J_ℓ^ν and the form of dependence of L_ℓ on A_ℓ^α, A^α through covariant derivatives (11.11) determined by (11.8), we obtain that

$$\partial_\mu \hat{T}^{\mu\nu}(L_\ell) = -\frac{\partial L_\ell}{\partial A^\alpha} (\theta A_\ell^{\alpha,\nu} + A^{\alpha,\nu} + A_{\text{ex}}^{\alpha,\nu}).$$

Using the definition of the current J_ℓ^α , we rewrite the above equation as follows:

$$\partial_\mu \hat{T}^{\mu\nu}(L_\ell) = \frac{1}{c} J_\ell^\alpha A_\alpha^{\nu} + \theta \frac{1}{c} J_\ell^\alpha A_{\alpha\ell}^{\nu} + \frac{1}{c} J_\ell^\alpha A_{\text{ex}\ell}^{\nu}, \quad (11.79)$$

which is the conservation law for the individual canonical EnMT $\hat{T}^{\mu\nu}(L_\ell)$. We obtain from (11.74), (11.76) and (11.79) that

$$\frac{1}{c} J^\alpha A_\alpha^{\nu} + \theta \sum_\ell \frac{1}{c} J_\ell^\alpha A_{\alpha\ell}^{\nu} + \partial_\mu \hat{\Theta}^{\mu\nu} + \sum_\ell \theta \partial_\mu \hat{\Theta}_\ell^{\mu\nu} = 0. \quad (11.80)$$

Now we consider the EnMT for electromagnetic fields. As we have already mentioned, the tensors $\hat{\Theta}^{\mu\nu}$ and $\hat{\Theta}_\ell^{\mu\nu}$ in (11.80) are the free EM field tensors, but the fields are found as solutions of Maxwell equations with nonzero currents, and the corresponding Lagrangians $L_{\text{em}}(A, J)$ or $L_{\text{em}}(A_\ell, J_\ell)$ for the equations have the form (9.1). According to the definition (10.54) of the canonical EnMT,

$$\hat{T}^{\mu\nu}(L_{\text{em}}(A, J)) = \hat{T}^{\mu\nu}(L_{\text{em}}(A, 0)) + g^{\mu\nu} \frac{1}{c} J_\alpha A^\alpha \quad (11.81)$$

where the energy-momentum tensor $\hat{T}^{\mu\nu}(L_{\text{em}}(A, 0)) = \hat{\Theta}^{\mu\nu}$ for the free Maxwell equations is defined by (9.11) with $J_\mu = 0$. Since $L_{\text{em}}(A^\mu, J_\mu)$ explicitly depends on x through J_μ , according to (10.55)

$$\partial_\mu \hat{T}^{\mu\nu}(L_{\text{em}}(A_\ell, J_\ell)) = \frac{1}{c} A_\ell^\mu \partial^\nu J_{\ell\mu},$$

and, using (11.81), we conclude that

$$\partial_\mu \dot{\Theta}_\ell^{\mu\nu} = -\frac{1}{c} J_{\ell\alpha} \partial^\nu A_\ell^\alpha, \quad (11.82)$$

$$\partial_\mu \dot{\Theta}^{\mu\nu} = -\frac{1}{c} J_\alpha \partial^\nu A^\alpha. \quad (11.83)$$

Obviously, (11.79), (11.82) imply (11.74) for the Lagrangian (11.65). Therefore we conclude that the total EnMT conservation law (11.74) can be partitioned into individual conservation laws (11.79), (11.82).

Adding (11.82) and (11.83) to (11.79), we observe that

$$\partial_\mu \dot{T}^{\mu\nu}(L_\ell) + \theta \partial_\mu \dot{\Theta}_\ell^{\mu\nu} + \partial_\mu \dot{\Theta}^{\mu\nu} = \frac{1}{c} J_\ell^\alpha A_{\text{ex}\alpha}^{\nu}. \quad (11.84)$$

If the external field is absent, this formula takes the form of the partial balance

$$\partial_\mu \dot{T}^{\mu\nu}(L_\ell) = -\theta \partial_\mu \dot{\Theta}_\ell^{\mu\nu} - \partial_\mu \dot{\Theta}^{\mu\nu} \quad (11.85)$$

and can be interpreted as fulfillment of Newton's Third Law for the interaction of an individual charge with the EM field. We stress that the conservation laws (11.77) and (11.84) are fulfilled for any Lagrangian of the form (11.63). In particular, the Lagrangians L_ℓ can be NLS Lagrangians or NKG Lagrangians, and the EM free field Lagrangians $L_{\text{em}} = L_{\text{em}}(A, 0)$ can be of the form (9.1), (9.24), (9.43) or (9.49).

11.5.2 Symmetrized Energy-Momenta and Conservation Laws for Every Charge

In the previous section we discussed the canonical EnMT. Here we consider symmetrized EnMT and derive conservation laws for individual symmetrized EnMT for every charge distribution ψ_ℓ , $\ell = 1, \dots, N$ in relativistic and non-relativistic cases. One of the main reasons of our interest in symmetrized EnMT is that the partial conservation laws for them involve the Lorentz force densities, and this is instrumental for our analysis of point-like regimes. Note that the Lagrangians L_ℓ for charge fields and the Lagrangians $L_{\text{em}}(A_\ell)$ for EM fields may behave differently under the action of the Galilean or Poincaré groups, in particular, L_ℓ may be either relativistic invariant or Galilean invariant, as well as L_{em} . We can always consider every individual charge based on its Euler–Lagrange (EL) equation, and to write its individual Lagrangian, then we derive the symmetric or symmetrized EnMT for every charge based on this individual Lagrangian, and the partial EnMT satisfies a corresponding conservation law. We can also do the same for EM fields of the charges based on the Lagrangians $L_{\text{em}}(A_\ell, J)$. But we have to take into account that even if all components of $\mathcal{L}$ are relativistic invariant, the tensors $S_\ell^{\mu\nu}$ in the definition (10.73)–(10.74) of the symmetric

EnMT for charges may differ from the tensors $S_\ell^{\mu\nu}$ for their EM fields, therefore, in the general case, we define a symmetric EnMT for the total Lagrangian $\mathcal{L}$ as the sum of partial EnMT and not directly. Fulfillment of partial balance similar to (11.85) is verified in every case of interest. Below we also check the fulfillment of the EnMT conservation laws which involves the Lorentz force densities.

The EL equation for the ℓ -th charge wave function ψ_ℓ derived from the total Lagrangian $\mathcal{L}$ and from the charge Lagrangian L_ℓ obviously coincide. Importantly, the EM fields generated by charges enter the equation and the Lagrangian through covariant derivatives, and when we treat the Lagrangian L_ℓ and the equation separately from the remaining charges and fields, the EM fields should be considered as external. In this case, the conservation law (10.55) for the canonical EnMT takes the form

$$\partial_\mu \hat{T}^{\mu\nu}(L_\ell) = -\frac{\partial L_\ell}{\partial x_\nu} = \frac{1}{c} J_\ell^\mu \partial^\nu \tilde{A}_{\ell\mu}, \quad (11.86)$$

where we used (11.30) and (11.57). This conservation law can be derived from the Euler–Lagrange equation (11.19) exactly as in (10.56). We introduce now the following “symmetrized” EnMT by the formula

$$T_\ell^{\mu\nu} = T_\ell^{\mu\nu}(L_\ell) = \hat{T}_\ell^{\mu\nu} - \frac{1}{c} J_\ell^\mu \tilde{A}_\ell^\nu. \quad (11.87)$$

We derive from (11.66) using (11.11) the following representation for $T_\ell^{\mu\nu}$:

$$T_\ell^{\mu\nu} = \frac{\partial L_\ell}{\partial \psi_{\ell;\mu}} \psi_{\ell;\nu} + \frac{\partial L_\ell}{\partial \psi_{\ell;\mu}^*} \psi_{\ell;\nu}^* - g^{\mu\nu} L_\ell. \quad (11.88)$$

If the symmetry condition (11.45) is satisfied, the symmetrized tensor $T_\ell^{\mu\nu}$ is symmetric; but when (11.45) does not hold, the symmetrized tensor may be non-symmetric. The symmetrized tensor is symmetric for the NKG Lagrangian and is not symmetric for the NLS Lagrangian. In particular, using interpretation (10.62), we obtain the following expressions for the ℓ -th charge energy density:

$$u_\ell = T_\ell^{00} = \frac{\partial L_\ell}{\partial \psi_{\ell;0}} \psi_{\ell;0} + \frac{\partial L_\ell}{\partial \psi_{\ell;0}^*} \psi_{\ell;0}^* - L_\ell, \quad (11.89)$$

and the momentum densities

$$p_\ell^j = T_\ell^{0j} = \frac{\partial L_\ell}{\partial \psi_{\ell;0}} \psi_{\ell;j} + \frac{\partial L_\ell}{\partial \psi_{\ell;0}^*} \psi_{\ell;j}^*. \quad (11.90)$$

Let us derive now the conservation law for the tensor $T_\ell^{\mu\nu}$. We differentiate (11.87) using the conservation law (11.86) and the continuity equation (11.54):

$$\begin{aligned}\partial_\mu T_\ell^{\mu\nu} &= \frac{1}{c} J_\ell^\mu \partial^\nu \tilde{A}_{\ell\mu} - \frac{1}{c} \partial_\mu \left(J_\ell^\mu \tilde{A}_\ell^\nu \right) = \frac{1}{c} J_\ell^\mu \left(\partial^\nu \tilde{A}_{\ell\mu} - \partial_\mu \tilde{A}_\ell^\nu \right) \\ &= \frac{1}{c} J_{\ell\mu} \left(\partial^\nu \tilde{A}_\ell^\mu - \partial^\mu \tilde{A}_\ell^\nu \right) = \frac{1}{c} J_{\ell\mu} F_\ell^{\nu\mu},\end{aligned}\quad (11.91)$$

where $F_\ell^{\nu\mu} = F^{\nu\mu} \left(\tilde{A}_\ell \right)$ is defined by (4.14) with $\tilde{A}_\ell$ given by (11.8). Consequently, we obtain the following *individual EnMT conservation law*

$$\partial_\mu T_\ell^{\mu\nu} = f_\ell^\nu, \quad (11.92)$$

with f_ℓ^ν defined by the formula

$$f_\ell^\nu = \frac{1}{c} J_{\ell\mu} F_\ell^{\nu\mu} \left(\tilde{A}_\ell \right) \quad (11.93)$$

where $J_{\ell\mu} = J_{\ell A\mu}$ are determined by (11.30) or (11.50). We readily recognize in expressions (11.93) for f_ℓ^ν the well known *Lorentz force density* expression (9.18). Using the matrix form (4.18) of $F^{\mu\nu}$ and (11.51), we can express f_ℓ^ν by formula (9.19). This formula turns into the formula (6.39) for the Lorentz force if we set for the point charge $\rho = q$, $\mathbf{J} = q\mathbf{v}$, providing a justification for calling f_ℓ^ν the Lorentz force density.

Note also that the Lorentz force density f_ℓ^ν involves the field $F_\ell^{\nu\mu} \left(\tilde{A}_\ell \right)$ where $\tilde{A}_\ell^\mu$ is the actual field potential which acts on the ℓ -th charge. If we use the additivity property (11.60) of the fields, we see that the actual field potential which acts on the ℓ -th charge can be written in the form

$$\tilde{A}_\ell^\mu = A_{\text{ex}}^\mu + \sum_{\ell'} A_{\ell'}^\mu + \theta A_\ell^\mu \quad (11.94)$$

where $A_{\ell'}^\mu$ is the field potential generated by the ℓ' -th charge.

The individual EnMT $T_\ell^{\mu\nu}$ are instrumental when we study macroscopic motion of charges, see Sects. 17.2, 17.6.2, 19.1, 19.3.

11.5.3 The Energy-Momentum Tensor for EM Fields

The Maxwell equations (11.33)–(11.34) evidently have the form of (4.23) with the corresponding Lagrangian $L_{\text{em}} \left(A^\mu, J_\mu \right)$ or $L_{\text{em}} \left(A_\ell^\mu, J_{\ell\mu} \right)$ of the form (9.1). Note that the Lagrangians for the Euler–Lagrange equations involve currents and are not the free field Lagrangians as in (11.65). Still, as in conservation laws (11.82), (11.83)

for the canonical EnMT, we write the partial conservation laws in terms of the EnMT $\Theta^{\mu\nu}(A) = T^{\mu\nu}(L_{\text{em}}(A, 0))$ which formally correspond to free fields.

The symmetric energy-momentum tensors $\Theta^{\mu\nu} = \Theta^{\mu\nu}(A)$, $\Theta_\ell^{\mu\nu} = \Theta^{\mu\nu}(A_\ell)$ for the Maxwell equations are defined by (9.13). The tensors $\Theta^{\mu\nu}$, $\Theta_\ell^{\mu\nu}$ satisfy the conservation laws (9.17), namely

$$\partial_\mu \Theta^{\mu\nu} = -f^\nu, \quad (11.95)$$

$$\partial_\mu \Theta_\ell^{\mu\nu} = -f_{\ell\ell}^\nu, \quad (11.96)$$

where f^μ and $f_{\ell\ell}^\mu$ are the Lorentz force densities defined by

$$f^\nu = \frac{1}{c} J_\mu F^{\nu\mu}(A), \quad f_{\ell\ell}^\nu = \frac{1}{c} J_{\ell\mu} F^{\mu\nu}(A_\ell). \quad (11.97)$$

Obviously, conservation laws (11.95)–(11.96) are similar to (11.82)–(11.83). The Lorentz force density $f_{\ell\ell}^\nu$ in the formula (11.92) is related to the above densities by the formula

$$f_{\ell\ell}^\nu = \frac{1}{c} F^{\nu\mu}(\tilde{A}_\ell) J_{\ell\mu} = \frac{1}{c} (F^{\nu\mu}(A) + F^{\nu\mu}(A_{\text{ex}})) J_{\ell\mu} + \theta f_{\ell\ell}^\nu. \quad (11.98)$$

Note that the Lorentz force densities enter the charge EnMT conservation laws (11.92), where they can be interpreted as the Lorentz forces exerted by the EM fields on the ℓ -th charge. Also, the Lorentz force densities enter the EM field conservation laws (11.95), (11.96), where they can be interpreted as the forces exerted by the charges on the EM field, and the densities enter with opposite sign. If the external field is absent, we obtain the partial conservation laws

$$\partial_\mu T^{\mu\nu}(L_\ell) = -\theta \partial_\mu \Theta_\ell^{\mu\nu} - \partial_\mu \Theta^{\mu\nu}, \quad (11.99)$$

which, similarly to (11.85), can be interpreted as fulfillment of Newton's Third Law, but now it is written in terms of the symmetrized EnMT.

Now we provide an instructive derivation of the conservation law (11.95) which can also be applied to the Fermi Lagrangian (9.24) and, after a proper modification, to the non-relativistic Lagrangian $L_{\text{em}2}$. Direct evaluation shows that the symmetric energy-momentum tensor $\Theta^{\mu\nu}(F)$ defined by (9.13) is related to the canonical energy-momentum tensor $\hat{\Theta}^{\mu\nu}(F)$ defined by (9.11) with $J = 0$ as follows:

$$T^{\mu\nu}(L_{\text{em}}(A, J)) = \Theta^{\mu\nu}(A) = \hat{\Theta}^{\mu\nu} + \frac{1}{4\pi} F^{\mu\gamma} \partial_\gamma A^\nu. \quad (11.100)$$

This formula can be taken as a definition of the symmetrized tensor $T^{\mu\nu}(L_{\text{em}}(A))$. Now we check the fulfillment of the individual conservation law (11.95). Using (11.83) and Maxwell equation (4.23) (which also holds for solutions of wave

equations (4.28) subjected to the Lorentz gauge), we derive a conservation law for the EnMT given by (11.100):

$$\begin{aligned}\partial_\mu \Theta^{\mu\nu} &= \partial_\mu \dot{\Theta}^{\mu\nu} + \frac{1}{4\pi} \partial_\mu (F^{\mu\gamma} \partial_\gamma A^\nu) \\ &= -\frac{1}{c} J_\alpha \partial^\nu A^\alpha + \frac{1}{4\pi} \partial_\mu \partial_\gamma (F^{\mu\gamma} A^\nu) + \frac{1}{c} \partial_\mu (J^\mu A^\nu).\end{aligned}\quad (11.101)$$

Taking into account the identity

$$\partial_\mu \partial_\gamma (F^{\mu\gamma} A^\nu) = -\partial_\mu \partial_\gamma (F^{\gamma\mu} A^\nu) = -\partial_\gamma \partial_\mu (F^{\gamma\mu} A^\nu) = -\partial_\mu \partial_\gamma (F^{\mu\gamma} A^\nu) = 0,$$

the continuity equation $\partial_\mu J^\mu = 0$ and the definition of $F^{\mu\gamma}$, we conclude that

$$\begin{aligned}\partial_\mu \Theta^{\mu\nu} &= -\frac{1}{c} J_\alpha \partial^\nu A^\alpha + \frac{1}{c} J^\mu \partial_\mu A^\nu \\ &= -\frac{1}{c} J_\mu \partial^\mu A^\alpha + \frac{1}{c} J_\mu \partial^\mu A^\nu = -\frac{1}{c} J_\mu F^{\mu\nu},\end{aligned}$$

and (11.95) is fulfilled.

In the non-relativistic cases, we define a “symmetrized” EnMT modifying the canonical EnMT similarly to the relativistic case where the Belinfante–Rosenfeld approach was applied to obtain the symmetric EnMT. The symmetrized EnMT in the non-relativistic cases are not necessarily symmetric. Our goal is to obtain a conservation law which involves the Lorentz force density in the right-hand side. This, together with (11.92), ensures fulfillment of Newton’s Third Law for every charge-field interaction written in terms of the symmetrized EnMT.

For the non-relativistic Lagrangian L_{eml} defined by (9.43), we define $\Theta_{\text{eml}}^{\mu\nu} = T^{\mu\nu}(L_{\text{eml}}(A))$ by setting $\mathbf{A} = \mathbf{0}$ in (9.14)–(9.16), namely by setting

$$\mathbf{E} = -\nabla\varphi, \quad \mathbf{A} = \mathbf{0}, \quad \mathbf{B} = \mathbf{0}. \quad (11.102)$$

We see that formulas (9.14)–(9.15) for the components of the EnMT (9.16) turn into

$$\begin{aligned}\Theta_{\text{eml}}^{00} &= u_{\text{em}} = \frac{1}{8\pi} \mathbf{E}^2, \quad \Theta^{0i} = \Theta^{i0} = c P_{\text{emi}} = 0, \\ \Theta_{\text{eml}}^{ij} &= -\tau_{ij} = -\frac{1}{4\pi} \left[E_i E_j - \frac{1}{2} \delta_{ij} \mathbf{E}^2 \right].\end{aligned}\quad (11.103)$$

We can also see that the conservation law (11.95) holds, that is,

$$\partial_\mu T^{\mu\nu}(L_{\text{eml}}(A)) = -f^\nu, \quad \partial_\mu T^{\mu\nu}(L_{\text{eml}}(A_\ell)) = -f_{\ell\ell}^\nu, \quad (11.104)$$

where, taking into account (11.102) and (4.18), we recover the following well known expressions for the Lorentz force density components:

$$\begin{aligned} f^v &= \frac{1}{c} F^{v\mu} (A) J_\mu = \left(\frac{1}{c} F^{0j} (A) J_j, \frac{1}{c} F^{i0} (A) J_0 \right) \\ &= \left(\frac{1}{c} \mathbf{J} \cdot \mathbf{E}, \rho \mathbf{E} \right). \end{aligned} \quad (11.105)$$

Now we consider the second non-relativistic approximation with the Lagrangian L_{em2} given by (9.49). According to (10.54) applied to the Lagrangian (9.49), the canonical energy-momentum tensor $\hat{\Theta}_{\text{em2}}^{\mu\nu}$ is given by (11.71)–(11.73). Now we introduce a “symmetrized” EnMT so that a conservation law similar to (11.82)–(11.83) involves the Lorentz force density in the right-hand side. To this end, we modify (11.100) and set

$$T^{\mu\nu} (L_{\text{em2}} (A)) = \hat{\Theta}_{\text{em2}}^{\mu\nu} + \frac{1}{4\pi} F^{\mu\gamma} \partial_\gamma A^\nu - \frac{1}{4\pi} A^\nu \partial_0^2 A^\mu \quad (11.106)$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. An explicit expression for $\hat{\Theta}_{\text{em2}}^{\mu\nu}$ is given in (11.71)–(11.73). Similarly to (11.101),

$$\begin{aligned} \partial_\mu \Theta_{\text{em2}}^{\mu\nu} &= \partial_\mu \hat{\Theta}_{\text{em2}}^{\mu\nu} + \frac{1}{4\pi} \partial_\mu (F^{\mu\gamma} \partial_\gamma A^\nu - \partial_0^2 A^\mu A^\nu) \\ &= -\frac{1}{c} J_\alpha \partial^\nu A^\alpha + \frac{1}{4\pi} \partial_\mu \partial_\gamma (F^{\mu\gamma} A^\nu) - \frac{1}{4\pi} \partial_\mu (\partial_\gamma F^{\mu\gamma} A^\nu + \partial_0^2 A^\mu A^\nu) \\ &= -\frac{1}{c} J_\alpha \partial^\nu A^\alpha - \frac{1}{4\pi} \partial_\mu (\partial_\gamma F^{\mu\gamma} A^\nu + \partial_0^2 A^\mu A^\nu). \end{aligned} \quad (11.107)$$

Under the Lorentz gauge we have

$$\partial_\gamma F^{\mu\gamma} = \partial_\gamma (\partial^\mu A^\gamma - \partial^\gamma A^\mu) = -\partial_\gamma \partial^\gamma A^\mu,$$

and using Poisson equation (8.74), namely $\partial_j \partial^j A^\mu = \frac{4\pi}{c} J^\mu$, combined with the continuity equation $\partial_\mu J^\mu = 0$ we obtain that

$$\begin{aligned} \partial_\mu \Theta_{\text{em2}}^{\mu\nu} &= -\frac{1}{c} J_\alpha \partial^\nu A^\alpha - \frac{1}{4\pi} \partial_\mu (-\partial_\gamma \partial^\gamma A^\mu A^\nu + \partial_0^2 A^\mu A^\nu) \\ &= -\frac{1}{c} J_\alpha \partial^\nu A^\alpha + \frac{1}{4\pi} \partial_\mu (\partial_j \partial^j A^\mu A^\nu) = -\frac{1}{c} J_\alpha \partial^\nu A^\alpha + \frac{1}{c} \partial_\mu (J^\mu A^\nu) \\ &= -\frac{1}{c} J_\alpha \partial^\nu A^\alpha + \frac{1}{c} J^\mu \partial_\mu A^\nu = -\frac{1}{c} J_\alpha F^{\alpha\nu}. \end{aligned} \quad (11.108)$$

Hence,

$$\partial_\mu \Theta_{\text{em2}}^{\mu\nu} = -f^\nu, \quad \partial_\mu \Theta^{\mu\nu} (L_{\text{em2}} (A_\ell)) = -f_{\ell\ell}^\nu \quad (11.109)$$

where f^ν , $f_{\ell\ell}^\nu$ are the Lorentz forces given by (11.97). We see that, *if the symmetrized EnMT for EM fields are properly defined, the conservation laws for them in all cases of interest involve the same Lorentz force densities (with the opposite sign) as in individual EnMT conservation laws (11.92) for every charge, and Newton's Third Law for the interaction of an individual charge with the EM field is fulfilled.*

11.5.4 Total Symmetrized Energy Momentum

Now we consider the total symmetrized energy momentum tensor corresponding to the complete Lagrangian $\mathcal{L}$ (the tensor is symmetric if the symmetry condition (11.45) holds for all involved components of the Lagrangian). Similarly to (11.65), we set

$$T^{\mu\nu} = T^{\mu\nu}(\mathcal{L}) = \sum_{\ell} T^{\mu\nu}(L_{\ell}) + T^{\mu\nu}(L_{\text{em}}(A)) + \theta \sum_{\ell} T^{\mu\nu}(L_{\text{em}}(A_{\ell})). \quad (11.110)$$

Now we show that the so defined total EnMT satisfies a conservation law which is similar to (11.77). According to (11.92), (11.95), (11.96), (11.98) and (11.31)

$$\begin{aligned} \partial_{\mu} T^{\mu\nu}(\mathcal{L}) &= \sum_{\ell} f_{\ell}^{\nu} - f^{\nu} - \sum_{\ell} \theta f_{\ell\ell}^{\nu} \\ &= \frac{1}{c} (F^{\nu\mu}(A) + F^{\nu\mu}(A_{\text{ex}})) J_{\mu} - f^{\nu} = \frac{1}{c} J_{\mu} F^{\nu\mu}(A_{\text{ex}}). \end{aligned}$$

Hence the *total energy-momentum conservation law* takes the form

$$\partial_{\mu} T^{\mu\nu}(\mathcal{L}) = f_{\text{ex}}^{\nu} \quad (11.111)$$

where

$$f_{\text{ex}}^{\nu} = \frac{1}{c} J_{\mu} F^{\nu\mu}(A_{\text{ex}}) \quad (11.112)$$

is the external Lorentz force density. Using (11.104) and (11.109), we also obtain that in both non-relativistic cases the tensors

$$T_{\text{em}1}^{\mu\nu} = \sum_{\ell} T^{\mu\nu}(L_{\ell}) + T^{\mu\nu}(L_{\text{em}1}(A)) + \theta \sum_{\ell} T^{\mu\nu}(L_{\text{em}1}(A_{\ell})), \quad (11.113)$$

$$T_{\text{em}2}^{\mu\nu} = \sum_{\ell} T^{\mu\nu}(L_{\ell}) + T^{\mu\nu}(L_{\text{em}2}(A)) + \theta \sum_{\ell} T^{\mu\nu}(L_{\text{em}2}(A_{\ell})) \quad (11.114)$$

satisfy the conservation laws

$$\partial_\mu \mathcal{T}_{\text{em}1}^{\mu\nu} = \frac{1}{c} J_\mu F^{\nu\mu} (A_{\text{ex}}), \quad \partial_\mu \mathcal{T}_{\text{em}2}^{\mu\nu} = \frac{1}{c} J_\mu F^{\nu\mu} (A_{\text{ex}}).$$

Note that the total field $F^{\mu\nu} (A)$ equals the sum of the individual fields according to (11.60), but the situation is more subtle in the case of the corresponding EnMT. Since $\Theta^{\mu\nu}$ is a *quadratic* function of the EM potentials A^β , the EnMT of the total field generally differs from the sum of individual EnMT $\Theta_\ell^{\mu\nu}$:

$$\Theta^{\mu\nu} = \Theta^{\mu\nu} (L_{\text{em}} (A)) \neq \sum_\ell \Theta^{\mu\nu} (L_{\text{em}} (A_\ell)). \quad (11.115)$$

11.5.5 Cancellation of Self-interaction in Energy-Momentum Conservation Laws

The EM field potential A can be naturally partitioned according to (11.60) into the sum of potentials A_ℓ generated by the ℓ -th charge, and there is the corresponding partition (11.58) of the total EM field tensor $F^{\mu\nu}$, namely:

$$F^{\mu\nu} (A) = \sum_\ell F^{\mu\nu} (A_\ell). \quad (11.116)$$

The tensor $\Theta_\ell^{\mu\nu} = \Theta^{\mu\nu} (F (A_\ell))$ satisfies the conservation law (11.96), that is,

$$\partial_\mu \Theta_\ell^{\mu\nu} = -\frac{1}{c} J_{\ell\mu} F_\ell^{\nu\mu}, \quad F_\ell^{\nu\mu} = F^{\nu\mu} (A_\ell). \quad (11.117)$$

Using (11.94), we write the conservation law (11.92) in the form

$$\partial_\mu T_\ell^{\mu\nu} = \frac{1}{c} J_{\ell\mu} F_\ell^{\nu\mu} + \frac{1}{c} J_{\ell\mu} \sum_{\ell' \neq \ell} F_{\ell'}^{\nu\mu} + \theta \frac{1}{c} J_{\ell\mu} F_\ell^{\nu\mu} + \frac{1}{c} J_{\ell\mu} F_{\text{ex}}^{\nu\mu}. \quad (11.118)$$

We introduce now the combined energy-momentum $T_\ell^{\mu\nu}$ of the ℓ -th charge by the formula

$$T_\ell^{\mu\nu} = T_\ell^{\mu\nu} + \Theta_\ell^{\mu\nu} + \theta \Theta_\ell^{\mu\nu}. \quad (11.119)$$

Using conservation laws (11.117) and (11.118), we readily obtain the following conservation laws (which sometimes are called *equations of motion*)

$$\partial_\mu T_\ell^{\mu\nu} = \frac{1}{c} J_\mu^\ell \sum_{\ell' \neq \ell} F_{\ell'}^{\nu\mu} + \frac{1}{c} J_\mu^\ell F_{\text{ex}}^{\nu\mu}, \quad \ell = 1, \dots, N. \quad (11.120)$$

This formula is equivalent to (11.99) if $A_{\text{ex}}^\nu = 0$. Note that *the Lorentz force density in the right-hand of (11.120) manifestly excludes self-interaction* in contrast to the Lorentz force density in the right-hand side of (11.118) which explicitly includes the self-interaction term $\frac{1}{c} J_{\ell\mu} F$ if $\theta \neq -1$. Thus, *we can conclude that when the charge and its EM field are treated as a single entity, there is no self-interaction as signified by the exact equations (11.120)*. The above argument also holds for the EnMT introduced in Sect. 11.5.3 for the non-relativistic Lagrangians $L_{\text{em}1}$, $L_{\text{em}2}$.

11.6 Lagrangian Field Formalism for the Klein–Gordon Equation

The nonlinear Klein–Gordon (NKG) Lagrangian has the form

$$\mathcal{L}_{\text{KG}}(\psi) = \frac{\chi^2}{2m} \left\{ \frac{1}{c^2} \tilde{\partial}_t \psi \tilde{\partial}_t^* \psi^* - \tilde{\nabla} \psi \cdot \tilde{\nabla}^* \psi^* - \kappa_0^2 \psi^* \psi - G(\psi^* \psi) \right\}, \quad (11.121)$$

where $\psi(t, \mathbf{x})$ is a complex-valued wave function over the space-time continuum, and ψ^* is its complex conjugate. In the expression (11.121) $G(s)$ is a nonlinearity, c is the speed of light, and

$$\kappa_0 = \frac{mc}{\chi} = \frac{\omega_0}{c}, \quad (11.122)$$

where ω_0 is the Compton frequency. We call $m > 0$ the *mass parameter*, and the elementary action parameter $\chi > 0$ is a constant which we usually take equal to the Planck constant $\hbar = \frac{h}{2\pi}$. The linear Klein–Gordon equation with $G = 0$ is widely used, [299, 8.1.1], [143, 1.2, 1.9, 1.11], [331, 1].

The Lagrangian $\mathcal{L}_{\text{KG}}$ in (11.121) is a particular case of (11.14) with a single ℓ which can be omitted, $A = 0$, and $\theta = 0$ in (11.15). The covariant derivatives in (11.121) are defined by (11.11) and take for a single charge a simpler form

$$\tilde{\partial}_t = \partial_t + \frac{iq}{\chi} \varphi_{\text{ex}}, \quad \tilde{\nabla} = \nabla - \frac{iq}{\chi c} \mathbf{A}_{\text{ex}}, \quad (11.123)$$

where q is the value of the charge. The Lagrangian can be written in the relativistic notation as follows:

$$\mathcal{L}_{\text{KG}}(\psi) = \frac{\chi^2}{2m} \left\{ \psi_{;\mu} \psi^{;\mu} - \kappa_0^2 \psi^* \psi - G(\psi^* \psi) \right\}. \quad (11.124)$$

The Lagrangian $\mathcal{L}_{\text{KG}}$ defined by the formula (11.124) is obtained from the linear Klein–Gordon Lagrangian, [146, 7.1, 11.2], [27, III.3], by adding the nonlinear term $G(\psi^* \psi)$. The Lagrangian $\mathcal{L}_{\text{KG}}$ is obviously gauge invariant with respect to the gauge transformations (11.40), (11.41) and satisfies the symmetry condition (11.45). The

Lagrangian involves external fields $A_{\text{ex}} = (\varphi_{\text{ex}}(t, \mathbf{x}), \mathbf{A}_{\text{ex}}(t, \mathbf{x}))$ which enter the covariant derivatives. The field equation corresponding to the Lagrangian (11.121) is the nonlinear Klein–Gordon (NKG) equation

$$\left[-\frac{1}{c^2} \tilde{\partial}_t^2 + \tilde{\nabla}^2 - \kappa_0^2 - G'(\psi^* \psi) \right] \psi = 0. \quad (11.125)$$

As a consequence of the gauge invariance (11.42), we obtain (see Sect. 11.2.2) that the four-current J^ν is conserved. We often write the four-current J^ν in the form

$$J^\nu = (c\rho, \mathbf{J}). \quad (11.126)$$

As we mentioned in Sect. 11.2.3, and we will see in Chaps. 17 and 19 that, if the NKG equation is coupled with the Maxwell equations, the notation (11.126) is consistent with (4.13). The conservation of the four-current is equivalent to the continuity equation (11.55), namely

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0. \quad (11.127)$$

The components of the four-current J^ν can be explicitly written in terms of ψ , φ_{ex} and $\mathbf{A}_{\text{ex}}$. Since we often use the expressions, we introduce the following notation

$$\mathcal{J}_0(\psi, \tilde{\varphi}) = -\frac{i}{2} (\psi^* \tilde{\partial}_t \psi - \psi \tilde{\partial}_t^* \psi^*), \quad (11.128)$$

$$\mathcal{J}_\nabla(\psi, \tilde{\mathbf{A}}) = -\frac{i}{2} (\psi^* \tilde{\nabla} \psi - \psi \tilde{\nabla}^* \psi^*), \quad (11.129)$$

(the dependence on $\tilde{\varphi}$, $\tilde{\mathbf{A}}$ comes from the covariant differential operators in (11.13)). Obviously, the above formulas can be written in the form

$$\mathcal{J}_0(\psi, \tilde{\varphi}) = \text{Im} \frac{\tilde{\partial}_t \psi}{\psi} |\psi|^2 = \left(\text{Im} \frac{\partial_t \psi}{\psi} + \frac{q}{\chi} \tilde{\varphi} \right) |\psi|^2, \quad (11.130)$$

$$\mathcal{J}_\nabla(\psi, \tilde{\mathbf{A}}) = \text{Im} \frac{\tilde{\nabla} \psi}{\psi} |\psi|^2 = \left(\text{Im} \frac{\nabla \psi}{\psi} - \frac{q}{\chi c} \tilde{\mathbf{A}} \right) |\psi|^2. \quad (11.131)$$

Using this notation, we obtain from (11.52), (11.53) the expressions for the charge density and the current for the NKG equation:

$$\rho = -\frac{\chi q}{mc^2} \mathcal{J}_0(\psi, \tilde{\varphi}), \quad (11.132)$$

$$\mathbf{J} = \frac{\chi q}{m} \mathcal{J}_\nabla(\psi, \tilde{\mathbf{A}}), \quad (11.133)$$

where $\tilde{\varphi} = \varphi_{\text{ex}}$, $\tilde{\mathbf{A}} = \mathbf{A}_{\text{ex}}$. We can also write the above formula as the 4-current:

$$J^\nu = \frac{\chi q}{m} \left(-\frac{1}{c} \mathcal{J}_0(\psi, \varphi_{\text{ex}}), \mathcal{J}_\nabla(\psi, \mathbf{A}_{\text{ex}}) \right). \quad (11.134)$$

11.6.1 The Energy-Momentum Tensor and Conservation Laws for the NKG Equation

The EnMT $T^{\mu\nu}$ corresponding to the NKG Lagrangian $\mathcal{L}_{\text{KG}}$ defined by (11.121) is given by formula (11.88) where the index ℓ takes only one value and is skipped, namely

$$\begin{aligned} T^{\mu\nu} = & \frac{\chi^2}{2m} [\psi^{;\mu*} \psi^{;\nu} + \psi^{;\mu} \psi^{;\nu*}] \\ & - \frac{\chi^2}{2m} [\psi^{*}_{;\mu} \psi^{;\mu} - \kappa_0^2 \psi^* \psi - G(\psi^* \psi)] g^{\mu\nu}. \end{aligned} \quad (11.135)$$

The covariant derivatives $\psi^{;\mu}$ defined by (11.11) take the form

$$\psi^{;\mu} = \tilde{\partial}^\mu \psi = \left(\partial^\mu + \frac{iq}{\chi c} A_{\text{ex}}^\mu \right) \psi, \quad (11.136)$$

and $\psi^{*;\mu}$ is the complex conjugate to $\psi^{;\mu}$. In agreement with the notation (4.12), we often write the covariant differential operator (13.2) in the form

$$\tilde{\partial}^\mu = \left(\frac{1}{c} \tilde{\partial}_t, -\tilde{\nabla} \right), \quad (11.137)$$

$$\tilde{\partial}_t = \partial_t + \frac{iq\varphi_{\text{ex}}}{\chi}, \quad \tilde{\nabla} = \nabla - \frac{iq}{\chi c} \mathbf{A}_{\text{ex}}. \quad (11.138)$$

The energy-momentum conservation law (11.92) for the EnMT (11.135) turns into

$$\partial_\mu T^{\mu\nu} = f^\nu, \quad (11.139)$$

where, according to (11.93) and (11.9),

$$f^\nu = \frac{1}{c} J_\mu F_{\text{ex}}^{\nu\mu} = (f^0, \mathbf{f}). \quad (11.140)$$

The *Lorentz force density* $(f^0, \mathbf{f})$ generated by the external EM field $F_{\text{ex}}^{\mu\nu}$ acting on a 4-current J^μ is of the form

$$f^\nu = (f^0, \mathbf{f}) = \left(\frac{1}{c} \mathbf{J} \cdot \mathbf{E}_{\text{ex}}, \rho \mathbf{E}_{\text{ex}} + \frac{1}{c} \mathbf{J} \times \mathbf{B}_{\text{ex}} \right) \quad (11.141)$$

with $\mathbf{J}, \rho$ given in (11.132). Using the interpretation of the EnMT entries (10.63) and (11.135), we obtain the following representations:

$$u = \frac{\chi^2}{2m} \left[\frac{1}{c^2} \tilde{\partial}_t \psi \tilde{\partial}_t^* \psi^* + \tilde{\nabla} \psi \tilde{\nabla}^* \psi^* + G(\psi^* \psi) + \kappa_0^2 \psi \psi^* \right] \quad (11.142)$$

for the energy density, and

$$\mathbf{p} = (p^1, p^2, p^3) = -\frac{\chi^2}{2mc^2} \left(\tilde{\partial}_t \psi \tilde{\nabla}^* \psi^* + \tilde{\partial}_t^* \psi^* \tilde{\nabla} \psi \right) \quad (11.143)$$

for the momentum density. Conservation law (11.139) implies in particular the energy conservation law

$$\partial_t u + c \partial_j T^{j0} = f^0. \quad (11.144)$$

The Klein–Gordon equation is a commonly used model for a *relativistic spinless charge*. In particular, in quantum electrodynamics nonlinearities are used to provide for interaction between free modes. We use in Part III of this book the nonlinearity $G(\psi^* \psi)$ to provide for a binding self-force of non-electromagnetic origin. Nonlinear alterations of the Klein–Gordon equations have been considered in literature, see, for instance, [37], [60], [146, 11.7, 11.8], [135] for rigorous mathematical studies.

11.6.2 The Linear Klein–Gordon Equation

If in the NKG equation (11.125) the nonlinearity is set to zero, $G' = 0$, we obtain the linear Klein–Gordon equation. The Klein–Gordon equation is a well known relativistic model for a free spinless charge in quantum mechanics, [267, Section 18]. A modification of the form (11.125), which involves external EM potentials, describes a charge interacting with an external EM field, [299, Section 8.1]. Here we follow [233, Section 1.5.2]. Namely, a freely propagating particle of rest mass m is described by a complex-valued wave function $\psi(t, \mathbf{x})$ satisfying the *Klein–Gordon equation*

$$-\frac{1}{c^2} \partial_t^2 \psi = (-\Delta + \kappa_0^2) \psi, \quad \kappa_0 = \frac{mc}{\hbar}. \quad (11.145)$$

This equation can be obtained from the fundamental relativistic mass-energy relation

$$\frac{E^2}{c^2} = \mathbf{p}^2 + m^2 c^2, \quad (11.146)$$

where E is the particle energy, and $\mathbf{p}$ is the three-dimensional space momentum, by the substitution $E = \hbar \partial_t$ and $\mathbf{p} = -\hbar \nabla$. A static solution V to the Klein–Gordon equation (11.145) with a δ -function source, i.e.

$$\{-\Delta + \mu^2\} V = -g^2 \delta(\mathbf{x}), \quad (11.147)$$

is called the *Yukawa potential*

$$V(|\mathbf{x}|) = -\frac{g^2}{4\pi} \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|}, \quad \mu = \frac{mc}{\hbar}. \quad (11.148)$$

The quantity $\mu^{-1} = \frac{\hbar}{mc}$ is called the *range of the potential* V , and it is also known as the *Compton wavelength* of the relativistic particle of the mass m . The constant g is a so-called coupling constant representing the basic strength of the interaction.

There is an interpretation of the 1D Klein–Gordon equation as a flexible string with additional stiffness forces provided by the medium surrounding it. Namely, if the string is embedded in a thin sheet of rubber or if it is along the axis of a cylinder of rubber whose outside surface is kept fixed, [250, 2.1].

The linear Klein–Gordon equation with electromagnetic field has the form, [299, 8.1.1], [143, 1.2, 1.9, 1.11], [331, 1]

$$\begin{aligned} P_\mu P^\mu \psi &= m^2 c^2 \psi, \\ P_\mu &= i\hbar \partial_\mu - \frac{e}{c} A_\mu, \end{aligned} \quad (11.149)$$

or

$$\left(i\hbar \frac{\partial}{\partial t} - e\varphi \right)^2 \psi = c^2 \left(-i\hbar \nabla - \frac{e}{c} \mathbf{A} \right)^2 \psi + m^2 c^4 \psi. \quad (11.150)$$

Obviously, the above equation can be written in the form of (11.125) with $G = 0$.

11.7 The Frequency Shifted Lagrangian

The mass term $\kappa_0^2 \psi^* \psi$ in (11.121) is responsible for high-frequency oscillations, and sometimes it is useful to change variables and eliminate this term in the NKG Lagrangian L_{KG} . This can be done by introducing the time-harmonic factor $e^{-i\omega_0 t}$. Namely, we introduce the change of variables

$$\psi(t, \mathbf{x}) \rightarrow e^{-i\omega_0 t} \psi(t, \mathbf{x}) \quad (11.151)$$

and substitute it into the Lagrangian L_{KG} defined by (11.121) to obtain the Lagrangian L_{ω_0} , which we call *frequency shifted*, namely

$$\begin{aligned} L_{\text{KG}\omega_0}(\psi, A^\mu) &= \frac{\chi}{2} i \left(\psi^* \tilde{\partial}_t \psi - \psi \tilde{\partial}_t^* \psi^* \right) \\ &+ \frac{\chi^2}{2m} \left\{ \frac{1}{c^2} \tilde{\partial}_t \psi \tilde{\partial}_t^* \psi^* - \tilde{\nabla} \psi \tilde{\nabla}^* \psi^* - G(\psi^* \psi) \right\}, \end{aligned} \quad (11.152)$$

where we use the notation (11.123). The Lagrangian $L_{\text{KG}\omega_0}$ defined by the formula (11.152) is manifestly gauge and space-time translation invariant, it is also invariant with respect to space rotations, but it is not invariant with respect to the entire group of Lorentz transformations. Notice also that time-harmonic with frequency ω_0 states for the field equation (11.125) corresponding to the original Lagrangian defined by (13.13) turn into time-independent rest states for the Lagrangian $L_{\text{KG}\omega_0}$, and that is one of the reasons to introduce it. The relation between the frequency shifted NKG Lagrangian and the NLS Lagrangian is discussed in Sect. 12.2.1.

11.8 Lagrangian Field Formalism for the Nonlinear Schrödinger Equation

The nonlinear Schrödinger (11.153) equation has the form

$$\chi i \tilde{\partial}_t \psi = \frac{\chi^2}{2m} \left[-\tilde{\nabla}^2 \psi + G'(\psi^* \psi) \psi \right]. \quad (11.153)$$

Here $\psi = \psi(t, \mathbf{x})$ is a complex-valued wave function, G' is a nonlinearity, the actual EM field coincides with the external field, namely $\tilde{\varphi} = \varphi_{\text{ex}}$, $\tilde{\mathbf{A}} = \mathbf{A}_{\text{ex}}$, and the covariant differential operators defined by (11.13) take the form

$$\tilde{\partial}_t = \partial_t + \frac{iq}{\chi} \varphi_{\text{ex}}, \quad \tilde{\nabla} = \nabla - \frac{iq}{\chi c} \mathbf{A}_{\text{ex}}, \quad \tilde{\nabla}^2 \psi = \tilde{\nabla} \cdot \tilde{\nabla} \psi. \quad (11.154)$$

The NLS equation is the Euler–Lagrange field equation (together with its conjugate) for the following Lagrangian density:

$$L_{\text{NLS}} = i \frac{\chi}{2} \left(\psi^* \tilde{\partial}_t \psi - \psi \tilde{\partial}_t^* \psi^* \right) - \frac{\chi^2}{2m} \left[\tilde{\nabla} \psi \tilde{\nabla}^* \psi^* + G(\psi^* \psi) \right]. \quad (11.155)$$

Obviously, this Lagrangian has the form (11.14) with a single (omitted) ℓ and with $\theta = -1$. The gauge invariance condition (11.40) is evidently satisfied. Hence we

can use formula (11.52) to define the charge density and current density based on $L_\ell = L_{\text{NLS}}$, namely

$$\rho = q\psi\psi^*, \quad (11.156)$$

$$\mathbf{J} = \frac{q\chi}{m} \mathcal{J}_\nabla(\psi, \mathbf{A}_{\text{ex}}) = -i\frac{q\chi}{2m} \left(\psi^* \tilde{\nabla} \psi - \psi \tilde{\nabla}^* \psi^* \right) \quad (11.157)$$

where we use the notation (11.131). Thanks to the gauge invariance, the so defined current satisfies the continuity equation (11.55), namely

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0. \quad (11.158)$$

Note that a quantum mechanical charged particle in an external EM field with the 4-potential $A^\mu = (\varphi, \mathbf{A})$ is described by the linear Schrödinger equation (11.153) with $G = 0$. It takes the form, [250, (2.6.47)],

$$\hbar i \partial_t \psi = \frac{1}{2m} \left(\frac{\hbar}{i} \nabla - \frac{q}{c} \mathbf{A}_{\text{ex}} \right)^2 \psi + q\varphi_{\text{ex}} \psi. \quad (11.159)$$

11.8.1 The Energy-Momentum Tensor for the NLS

The canonical energy-momentum tensor (EnMT) $\hat{T}^{\mu\nu}$ for the NLS Lagrangian is given by formula (11.66), in particular

$$\hat{T}^{00} = \hat{u} = \frac{\chi^2}{2m} \left[\left| \tilde{\nabla} \psi \right|^2 + G(|\psi|^2) \right] + \psi^2 \varphi_{\text{ex}} \quad (11.160)$$

is the canonical energy density.

We introduce the symmetrized EnMT for the NLS by formula (11.88), namely

$$T^{\mu\nu} = \frac{\partial L_{\text{NLS}}}{\partial \psi_{;\mu}} \psi^{;\nu} + \frac{\partial L_{\text{NLS}}}{\partial \psi^*_{;\mu}} \psi^{;\nu*} - g^{\mu\nu} L_{\text{NLS}}.$$

In particular, the expression (11.89) for the energy density takes the form

$$u = T^{00} = \frac{\chi^2}{2m} \left[\left| \tilde{\nabla} \psi \right|^2 + G(|\psi|^2) \right], \quad (11.161)$$

and the momentum density (11.90)

$$p^j = T^{0j} = i\frac{\chi}{2} \left(\psi \tilde{\partial}_j^* \psi^* - \psi^* \tilde{\partial}_j \psi \right). \quad (11.162)$$

The formula for the momentum density can be written in the form

$$\mathbf{P} = i\frac{\chi}{2} \left[\psi \cdot \tilde{\nabla}^* \psi^* - \psi^* \cdot \tilde{\nabla} \psi \right]. \quad (11.163)$$

The remaining entries of (10.61) take the form

$$s^j = T^{j0} = -\frac{\chi^2}{2m} \left(\tilde{\partial}_t \psi \tilde{\partial}_j^* \psi^* + \tilde{\partial}_t^* \psi^* \tilde{\partial}_j \psi \right), \quad j = 1, 2, 3,$$

$$\sigma^{ii} = T^{ii} = u - \frac{\chi^2}{m} \tilde{\partial}_i \psi \tilde{\partial}_i^* \psi^* + i\frac{\chi}{2} \left(\psi \tilde{\partial}_i^* \psi^* - \psi^* \tilde{\partial}_i \psi \right), \quad (11.164)$$

and

$$\sigma^{ij} = \sigma^{ji} = T^{ij} = -\frac{\chi^2}{2m} \left(\tilde{\partial}_i \psi \tilde{\partial}_j^* \psi^* + \tilde{\partial}_j \psi \tilde{\partial}_i^* \psi^* \right) \quad (11.165)$$

for $i \neq j$, $i, j = 1, 2, 3$. Since $T^{0j} \neq T^{j0}$, the symmetrized EnMT is not symmetric. Comparing (11.162) and (11.157), we see that the current density $\mathbf{J}$ and the momentum density $\mathbf{P}$ for the NLS are proportional:

$$\mathbf{J} = \frac{q}{m} \mathbf{P}. \quad (11.166)$$

This gives us the ground to introduce the velocity density

$$\mathbf{v}(t, \mathbf{x}) = \frac{1}{q} \mathbf{J}(t, \mathbf{x}) \quad (11.167)$$

implying

$$\mathbf{J} = q\mathbf{v}, \quad \mathbf{P} = m\mathbf{v}, \quad (11.168)$$

where the second equality can be viewed as the momentum density kinematic representation. The proportionality of the momentum and the current is known for systems governed by the nonlinear Schrödinger equations, [317, Section 2.3].

The energy-momentum tensor satisfies the conservation law (11.92), where the Lorentz force density is given by (11.141). In particular, the momentum equation has the form

$$\partial_t \mathbf{P} + \partial_i T^{ij} = \rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B}. \quad (11.169)$$

Sometimes it is more convenient to use the canonical EnMT, in particular the energy

$$\hat{\mathcal{E}}(\tilde{\psi}, \varphi_{\text{ex}}, \mathbf{A}_{\text{ex}}) = \int \hat{u} \, d\mathbf{x}$$

with the canonical energy density defined by (11.160). It satisfies the total energy equation obtained from (10.55) with EM potentials $\varphi_{\text{ex}}, \mathbf{A}_{\text{ex}}$ explicitly dependent on t :

$$\partial_t \dot{\mathcal{E}}(\tilde{\psi}, \varphi_{\text{ex}}, \mathbf{A}_{\text{ex}}) = - \int \partial_t L_{\text{NLS}} \, d\mathbf{x} = \frac{1}{c} \int J^\nu \partial_t A_\nu \, d\mathbf{x}. \quad (11.170)$$

11.8.2 Galilean Gauge-Invariance

The Lagrangian L_{NLS} is not invariant with respect to either the Lorentz or the Galilean group of transformations. But a straightforward examination shows that L_{NLS} is invariant with respect to the following *Galilean-gauge group of transformations*

$$t' = t, \quad \mathbf{x}' = \mathbf{x} - \mathbf{v}t, \quad (11.171)$$

or

$$\begin{aligned} x^{0'} &= x^0, & \mathbf{x}' &= \mathbf{x} - \frac{\mathbf{v}}{c} x^0, \\ \psi(t, \mathbf{x}) &= e^{i \frac{m}{2\hbar} (\mathbf{v}^2 t' + 2\mathbf{v} \cdot \mathbf{x}')} \psi'(t', \mathbf{x}'), \end{aligned} \quad (11.172)$$

or

$$\psi'(t', \mathbf{x}') = e^{i \frac{m}{2\hbar} (\mathbf{v}^2 t - 2\mathbf{v} \cdot \mathbf{x})} \psi(t, \mathbf{x}),$$

with $\varphi(t, \mathbf{x}) = \varphi'(t', \mathbf{x}')$. One can also verify that the above transformations form an Abelian (commutative) group of transformations parametrized by the velocity parameter $\mathbf{v}$. It is curious to observe that, according to the Galilean-gauge transformations (11.171), (11.172), the charge wave function does not transform as a scalar as in the relativistic case. These transformations are known, [138, 7.3], and were used, in particular, in studies of nonlinear Schrödinger equations, [317, 2.3].

The above defined Galilean-gauge group is naturally extended to the *general inhomogeneous Galilean-gauge group* by adding to it the group of spatial rotations O and space-time translations a^μ , namely

$$t' = t + \tau, \quad \mathbf{x}' = O\mathbf{x} - \mathbf{v}t + \mathbf{a}, \quad (11.173)$$

$$\psi'(t', \mathbf{x}') = e^{i \frac{m}{2\hbar} [\mathbf{v}^2(t+\tau) - 2\mathbf{v} \cdot (O\mathbf{x} + \mathbf{a})]} \psi(t, \mathbf{x}), \quad \varphi'(t', \mathbf{x}') = \varphi(t, \mathbf{x}). \quad (11.174)$$

The *infinitesimal form* of the above group of transformations is as follows:

$$\begin{aligned} t' &= t + \tau, \quad \mathbf{x}' = \mathbf{x} + \boldsymbol{\xi} \times \mathbf{x} - \mathbf{v}t + \mathbf{a}, \\ a^\mu &= (c\tau, \mathbf{a}), \quad a^0 = c\tau, \quad x^0 = ct, \end{aligned} \quad (11.175)$$

or, equivalently,

$$x'^\mu = x^\mu + \xi^{\mu\nu} x_\nu + a^\mu$$

where

$$\xi^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{v^1}{c} & 0 & -\xi^3 & \xi^2 \\ -\frac{v^2}{c} & \xi^3 & 0 & -\xi^1 \\ -\frac{v^3}{c} & -\xi^2 & \xi^1 & 0 \end{bmatrix}. \quad (11.176)$$

The real number τ and the coordinates of the three-dimensional vectors $\mathbf{v}$, $\boldsymbol{\xi}$, $\mathbf{a}$ provide the total of ten real parameters as in the case of the infinitesimal inhomogeneous Lorentz group defined by (6.11). The infinitesimal form of the transformations (11.174) is

$$\delta\psi = -i\frac{m}{\chi}(x^j \cdot \delta v^j)\psi, \quad \delta\varphi = 0. \quad (11.177)$$

We use the relativistic conventions for upper and lower indices and the summation convention as in Chap. 6, including

$$x^\mu = (x^0, \mathbf{x}), \quad x_\mu = (x^0, -\mathbf{x}), \quad x^0 = ct. \quad (11.178)$$

Carrying out the Noether currents analysis as in Sect. 10.2 for the Lagrangian L_{NLS} , we obtain 10 conservation laws. The conservation laws in the absence of spatial and time dependence can be formulated in terms of the canonical EnMT $\hat{T}^{\mu\nu}$, which is defined by (11.66). From conservation law (10.57) and from the symmetry of $\hat{T}^{ij}$ we obtain the total of ten conservation laws:

$$\partial_\mu \hat{T}^{\mu\nu} = 0\text{-energy-momentum conserv.}, \quad (11.179)$$

$$\hat{T}^{ij} = \hat{T}^{ji}, \quad i, j = 1, 2, 3\text{-space angular momentum conserv.}, \quad (11.180)$$

$$P^i = \hat{T}^{0i} = \frac{m}{q}J^i, \quad i = 1, 2, 3\text{-space-time angular momentum conserv.} \quad (11.181)$$

The first four standard conservation laws (11.179) are associated with Noether's currents with respect to space-time translations a^μ . Three more conservation laws in (11.180) are associated with space rotation parameters $\boldsymbol{\xi}$, and they turn into the partial symmetry of the EnMT $\hat{T}^{\mu\nu}$ for the spatial indices similarly to relations (10.67). The last three conservation laws (11.181) are equivalent to (11.166). *They are special to the NLS Lagrangian L_{NLS} , and are due to the Galilean-gauge invariance*

(11.172), (11.174). Relations (11.181) indicate that the *total momentum density* P^i is proportional with the constant factor $\frac{m}{q}$ to the *current density* J^i defined by (11.156), (11.157). The important identity (11.166) is analogous to the kinematic representation $\mathbf{p} = m\mathbf{v}$ of the momentum $\mathbf{p}$ of a point charge.

Chapter 12

Lagrangian Field Formalism for Balanced Charges

The balanced charge model is an important particular case of a system described by the Lagrangian (11.14) with $\theta = -1$. It involves the charge densities ψ_ℓ and the elementary potentials $A_\ell^\mu = (\varphi_\ell, \mathbf{A}_\ell)$. We will study properties of balanced charges in detail in Part III of the book, and here we only present their basic Lagrangian theory. Since $\theta = -1$, the expression (11.8) for the actual potential which affects the ℓ -th charge takes the form

$$\tilde{A}_\ell^\mu = A_{\text{ex}}^\mu + A^\mu - A_\ell^\mu \quad (12.1)$$

and can be written in the form (11.9). We consider below two major subcases: relativistic and non-relativistic in Sects. 12.1 and 12.2 respectively. In the relativistic case, the system of interacting charges is described by the NKG-Maxwell equations; in the non-relativistic case they are described by the NLS-Poisson equations.

12.1 Relativistic Balanced Charges

Setting $\theta = -1$ in (11.14), we obtain the Lagrangian for a system of N balanced charges:

$$\begin{aligned} \mathcal{L}(\{\psi_\ell, \psi_{\ell;\mu}, \psi_\ell^*, \psi_{\ell;\mu}^*\}, A_\ell^\mu) = \\ + \sum_\ell L_\ell(\psi_\ell, \psi_{\ell;\mu}, \psi_\ell^*, \psi_{\ell;\mu}^*) - \frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{1}{16\pi} \sum_\ell F_\ell^{\mu\nu} F_{\ell\mu\nu} \end{aligned} \quad (12.2)$$

where, in accordance with (11.7),

$$F^{\mu\nu} = \sum_\ell F_\ell^{\mu\nu}, \quad (12.3)$$

and L_ℓ is the nonlinear Klein–Gordon Lagrangian :

$$L_\ell (\psi_\ell, \psi_{\ell;\mu}, \psi_\ell^*, \psi_{\ell;\mu}^*) \quad (12.4)$$

$$= \frac{\chi^2}{2m_\ell} \left[\frac{1}{c^2} |\tilde{\partial}_{t\ell} \psi_\ell|^2 - |\tilde{\nabla}_\ell \psi_\ell|^2 - \kappa_{0\ell}^2 |\psi_\ell|^2 - G_\ell (\psi_\ell \psi_\ell^*) \right].$$

The covariant derivatives are defined by (11.10) with the covariant differential operators written in the form (11.13) with $\tilde{A}_\ell^\mu$ given by (12.1), namely

$$\tilde{\nabla}_\ell = \nabla - \frac{iq_\ell}{\chi c} \mathbf{A}_{\text{ex}} - \frac{iq_\ell}{\chi c} \mathbf{A}_{\neq\ell}, \quad \tilde{\partial}_{t\ell} = \partial_t + \frac{iq_\ell}{\chi} \varphi_{\text{ex}} + \frac{iq_\ell}{\chi} \varphi_{\neq\ell} \quad (12.5)$$

where

$$\varphi_{\neq\ell} = \varphi - \varphi_\ell = \sum_{\ell' \neq \ell} \varphi_{\ell'}, \quad \mathbf{A}_{\neq\ell} = \mathbf{A} - \mathbf{A}_\ell = \sum_{\ell' \neq \ell} \mathbf{A}_{\ell'}, \quad (12.6)$$

$$(\varphi, \mathbf{A}) = \sum_\ell (\varphi_\ell, \mathbf{A}_\ell), \quad (12.7)$$

and $\mathbf{A}_{\text{ex}}(t, \mathbf{x})$ and $\varphi_{\text{ex}}(t, \mathbf{x})$ are the potentials of the external EM fields. As always, ψ^* is the complex conjugate to ψ . The parameter $m_\ell > 0$ is the ℓ -th charge mass parameter, and q_ℓ is the value of the charge. The coefficient $\kappa_{0\ell} = \frac{m_\ell c}{\chi}$ is given by (11.122), and $\chi > 0$ is a constant similar to the Planck constant $\hbar = \frac{h}{2\pi}$; as always, $G_\ell(\psi_\ell^* \psi_\ell)$ is a nonlinearity. The field equations for ψ_ℓ are the NKG equations of the form (11.125), namely

$$-\frac{1}{c^2} \tilde{\partial}_{t\ell} \tilde{\partial}_{t\ell} \psi_\ell + \tilde{\nabla}^{\ell 2} \psi_\ell - G'_\ell (\psi_\ell^* \psi_\ell) \psi_\ell - \kappa_{0\ell}^2 \psi_\ell = 0, \quad \ell = 1, \dots, N. \quad (12.8)$$

(Obviously, the above formula does not involve summation with respect to repeated ℓ).

The total EM potentials $A^\mu = (\varphi, \mathbf{A})$ and the elementary EM potentials $A_\ell^\mu = (\varphi_\ell, \mathbf{A}_\ell)$ satisfy the Maxwell equations (11.33), (11.34) where the 4-currents $J_\ell^\nu = (c\varrho_\ell, \mathbf{J}_\ell)$ are defined by standard expressions (11.30), (11.31). The currents satisfy the continuity equation (11.54)–(11.55), and we assume that the Lorentz gauge is imposed on the potentials. It is important to notice that we determine the field potentials from the Maxwell equations, and it is not sufficient to determine the fields $\mathbf{E}_\ell, \mathbf{B}_\ell$ because the field potentials $\varphi_\ell, \mathbf{A}_\ell$ are directly involved in the NLS equations through the covariant derivatives (12.5). This is in contrast to classical electrodynamics where all quantities of interest can be expressed in terms of the EM fields $\mathbf{E}_\ell, \mathbf{B}_\ell$. At the same time, our analysis of macroscopic regimes in Sects. 17.2, 19.1, where the charges demonstrate point-like behavior, shows that their point-like dynamics is determined by the fields $\mathbf{E}_\ell, \mathbf{B}_\ell$.

It is assumed that the solution operator of the Maxwell equations (given, for example, by the retarded potentials) is fixed, and therefore the field additivity property (11.58) holds. Under the Lorentz gauge (4.9), the Maxwell equations (11.59) turn into the wave equations (4.28):

$$\nabla^2 \varphi_\ell - \frac{1}{c^2} \partial_t^2 \varphi_\ell = -4\pi \rho_\ell, \quad (12.9)$$

$$\frac{1}{c^2} \partial_t^2 \mathbf{A}_\ell - \nabla^2 \mathbf{A}_\ell = \frac{4\pi}{c} \mathbf{J}_\ell, \quad \ell = 1, \dots, N. \quad (12.10)$$

12.2 Non-relativistic Balanced Charges

As in the relativistic case, the charges are described by their distributions ψ_ℓ , $\ell = 1, \dots, N$, and their elementary fields φ_ℓ , $\mathbf{A}_\ell$. The NKG Lagrangians for ψ_ℓ in (12.2) are replaced by the NLS Lagrangians. We also replace the EM Lagrangian L_{em} in (11.14) by the Lagrangian $L_{\text{em}2}$ given by (9.49) and discussed in Sect. 9.4.3, and we finally obtain the Lagrangian which describes the system of non-relativistic balanced charges:

$$\begin{aligned} & \mathcal{L}_0 (\{\psi_\ell\}_{\ell=1}^N, \{(\varphi_\ell, \mathbf{A}_\ell)\}_{\ell=1}^N) \\ &= \sum_\ell \frac{\chi_\ell}{2} \left[\psi_\ell^* \tilde{\partial}_{t\ell} \psi_\ell - \psi_\ell \tilde{\partial}_{t\ell}^* \psi_\ell^* \right] - \sum_\ell \frac{\chi_\ell^2}{2m_\ell} \left[|\tilde{\nabla}_\ell \psi_\ell|^2 + G_\ell (\psi_\ell \psi_\ell^*) \right] \\ &+ \frac{1}{8\pi} [(\nabla \varphi)^2 - (\nabla \mathbf{A})^2] - \sum_\ell \frac{1}{8\pi} [(\nabla \varphi_\ell)^2 - (\nabla \mathbf{A}_\ell)^2], \end{aligned} \quad (12.11)$$

where $A^\mu = (\varphi, \mathbf{A})$ is defined by (11.7):

$$\varphi = \sum_\ell \varphi_\ell, \quad \mathbf{A} = \sum_\ell \mathbf{A}_\ell. \quad (12.12)$$

Here ψ_ℓ^* is the complex conjugate to ψ_ℓ ,

$$(\nabla \mathbf{A})^2 = (\nabla A_1)^2 + (\nabla A_2)^2 + (\nabla A_3)^2, \quad (12.13)$$

and $(\nabla \mathbf{A}_\ell)^2$ is defined by a similar formula. The covariant differential operators $\tilde{\partial}_{t\ell}$, $\tilde{\nabla}_\ell$ are given by the expressions (12.5). The NLS Lagrangian can be derived from the relativistic NKG Lagrangian, see Sect. 12.2.1.

The field equations for charge distributions ψ_ℓ are the NLS equations (11.153), they present *non-relativistic equations for the dynamics of balanced charges*:

$$i\chi \partial_t \psi_\ell = -\frac{\chi^2}{2m_\ell} \tilde{\nabla}_\ell^2 \psi_\ell + q_\ell (\varphi_{\neq \ell} + \varphi_{\text{ex}}) \psi_\ell + \frac{\chi^2}{2m_\ell} G'_\ell (|\psi_\ell|^2) \psi_\ell. \quad (12.14)$$

In the above equation $\ell = 1, \dots, N$, and there is no summation in ℓ . The charge and current densities $\rho_\ell, \mathbf{J}_\ell$ generated by the charge are defined by (11.156)–(11.157):

$$\rho_\ell = q_\ell |\psi_\ell|^2, \quad \mathbf{J}_\ell = \frac{\chi q_\ell}{m_\ell} \mathcal{J}_\nabla (\psi, \mathbf{A}_{\text{ex}} + \mathbf{A}_{\neq \ell}), \quad (12.15)$$

where we use the notation (11.31). The charge and current densities of every charge satisfy the continuity equation (11.55).

Since the Lagrangian (12.11) involves the non-relativistic EM Lagrangian $L_{\text{em}2}$, the Maxwell equations (11.33), (11.34) for EM potentials $\varphi_\ell, \mathbf{A}_\ell$ are replaced by the Poisson equations (11.37), namely

$$\nabla^2 \varphi_\ell = -4\pi \rho_\ell, \quad (12.16)$$

$$-\frac{1}{4\pi} \nabla^2 \mathbf{A}_\ell = \frac{1}{c} \mathbf{J}_\ell, \quad \ell = 1, \dots, N. \quad (12.17)$$

Note that the charge continuity equation (11.55) implies that the above EM potentials $\varphi_\ell, \mathbf{A}_\ell$ satisfy the Lorentz gauge (17.106), namely

$$\frac{1}{c} \frac{\partial \varphi}{\partial t} + \nabla \cdot \mathbf{A} = 0. \quad (12.18)$$

We recall that, to be consistently derived from the Maxwell equations by neglecting retardation as in (4.45), the Poisson equations must be complemented with the Lorentz gauge condition, see also Sects. 9.4, 9.3 and 12.2.1. According to (4.45), solutions of (12.16), (12.17) are given by the following formulas:

$$\varphi_\ell = \int \frac{1}{|\mathbf{x} - \mathbf{y}|} \rho_\ell(t, \mathbf{y}) d\mathbf{y}, \quad (12.19)$$

$$\mathbf{A}_\ell = \frac{1}{c} \int \frac{1}{|\mathbf{x} - \mathbf{y}|} \mathbf{J}_\ell(t, \mathbf{y}) d\mathbf{y}. \quad (12.20)$$

Using (4.6), we obtain from (12.20) the following formula for the magnetic (induction) field

$$\mathbf{B}_\ell = \frac{1}{c} \nabla \times \int \frac{1}{|\mathbf{x} - \mathbf{y}|} \mathbf{J}_\ell(t, \mathbf{y}) d\mathbf{y}, \quad (12.21)$$

which coincides with the known expression in magnetostatics, [180, p. 137], though we allow time dependence. Note also that the expression for the electric potential φ_ℓ coincides with the expression known in electrostatics [180, pp. 8, 13], but the electric field is calculated based on the same formula (4.6) as in the relativistic case:

$$\mathbf{E}_\ell = -\nabla \varphi_\ell - \frac{1}{c} \partial_t \mathbf{A}_\ell. \quad (12.22)$$

The above formula (12.22) exactly agrees with the expression for the electric field derived from the Lorentz force (6.39)–(6.41) acting on a charge by setting charge velocity to zero. Our analysis of solutions of the NLS equation (12.14) in Sects. 17.2, 19.1 shows that the Lorentz force which acts on a non-relativistic balanced charge in point-like regimes is given by (6.39)–(6.41), hence fulfillment of (12.22) is required for the consistency of the theory. Formula (12.22) also provides fulfillment of Newton's third law for interaction of a balanced charge with EM fields, see Sects. 11.5.1, 11.5.3.

Note that formula (12.22) differs from formula (8.19) which is used in electroquasistatics described in Sects. 8.1, 9.4.1 and differs from formula (8.57) which is used in Darwin's approximation in Sect. 8.3. The expression (12.22) turns into the formula (8.19) $\mathbf{E}_\ell = -\nabla\varphi_\ell$ of the electroquasistatics if $\mathbf{A}_\ell$ is time-independent, and the correction $\frac{1}{c}\partial_t\mathbf{A}_\ell$ is negligible if the speed of variation of $\mathbf{A}_\ell$ is small relative to the speed of light. The expression (12.22) turns into the formula (8.57) $\mathbf{E} = -\nabla\varphi_\ell - \frac{1}{c}\partial_t\mathbf{A}_{\ell T}$ of Darwin's approximation if the longitudinal component $\mathbf{A}_{\ell L}$ of $\mathbf{A}_\ell$ is time-independent, and the correction $\frac{1}{c}\partial_t\mathbf{A}_{\ell L}$ is negligible if the speed of variation of $\mathbf{A}_{\ell L}$ is small relative to the speed of light.

According to (10.54) applied to the Lagrangian (9.49), the canonical energy-momentum tensor for the EM field is given by the formula

$$\hat{T}_{\text{em}2}^{\mu\nu} = \frac{\partial L_{\text{em}2}}{\partial A^{\alpha}_{,\mu}} A^{\alpha,\nu} - g^{\mu\nu} L_{\text{em}2} \quad (12.23)$$

where

$$L_{\text{em}2}(A) = -\frac{1}{8\pi}\partial_i A_\alpha \partial^i A^\alpha = \frac{1}{8\pi}(\nabla\varphi)^2 - \frac{1}{8\pi}(\nabla\mathbf{A})^2.$$

In particular, the canonical energy density

$$u_{\text{em}2} = \hat{T}_{\text{em}2}^{00} = -L_{\text{em}2} = -\frac{1}{8\pi}(\nabla\varphi)^2 + \frac{1}{8\pi}(\nabla\mathbf{A})^2. \quad (12.24)$$

This expression differs from the Maxwell equations canonical energy given by (9.12):

$$\begin{aligned} \hat{\Theta}^{00} &= -\frac{1}{8\pi}(\mathbf{E}^2 - \mathbf{B}^2) - \frac{1}{4\pi}\frac{1}{c}\partial_t\mathbf{A} \cdot \mathbf{E} \\ &= -\frac{1}{8\pi}\left(\left(\nabla\varphi + \frac{1}{c}\partial_t\mathbf{A}\right)^2 - (\nabla \times \mathbf{A})^2\right) + \frac{1}{4\pi}\frac{1}{c}\partial_t\mathbf{A} \cdot \left(\nabla\varphi + \frac{1}{c}\partial_t\mathbf{A}\right). \end{aligned} \quad (12.25)$$

As we have shown in Sect. 9.3, the difference $\hat{T}_{\text{em}2}^{\mu\nu} - \hat{T}_{\text{em}}^{\mu\nu}$ between the two EnMT has zero divergence. We present here an alternative analysis of this difference for the energy density in the case when the EM fields vary slowly compared with the speed of light, namely if $\frac{1}{c}\partial_t\mathbf{A} = o(1)$ and $\frac{1}{c}\partial_t\varphi = o(1)$ are small. Assuming this, we get

$$\mathring{\Theta}^{00} = -\frac{1}{8\pi} \left((\nabla\varphi)^2 - (\nabla \times \mathbf{A})^2 \right) + o(1). \quad (12.26)$$

According to the Lorentz gauge (17.106) $\nabla \cdot \mathbf{A} = o(1)$, and we see that according to (43.18)

$$(\nabla \times \mathbf{A})^2 = (\nabla \mathbf{A})^2 + o(1) + \nabla \cdot [\mathbf{A} (\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla) \mathbf{A}], \quad (12.27)$$

hence the non-negligible part of the difference of the energy densities is a divergence:

$$\mathring{\Theta}^{00} - u_{\text{em}2} = \nabla \cdot [\mathbf{A} (\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla) \mathbf{A}] + o(1). \quad (12.28)$$

Therefore the difference of total energies defined in terms of integrals of $\mathring{\Theta}^{00}$ and $u_{\text{em}2}$ is negligibly small for slowly varying fields. The symmetrized EnMT for the non-relativistic EM Lagrangian $L_{\text{em}2}$ is given by formula (11.106) in Sect. 11.5.3.

The canonical energy density for the charges which satisfy the NLS equation (12.14) is given by (11.160), namely

$$\dot{u}_\ell = \dot{T}_\ell^{00} = q_\ell (\varphi_{\neq\ell} + \varphi_{\text{ex}}) \psi_\ell \psi_\ell^* + \frac{\chi^2}{2m_\ell} \left[\tilde{\nabla} \psi_\ell \cdot \tilde{\nabla}^* \psi_\ell^* + G(\psi_\ell^* \psi_\ell) \right]. \quad (12.29)$$

The momentum density is given by

$$\dot{p}_{\ell j} = \frac{\chi \mathbf{i}}{2} \left(\psi \tilde{\partial}_j^* \psi_\ell^* - \psi_\ell^* \tilde{\partial}_j \psi \right), \quad j = 1, 2, 3, \quad (12.30)$$

with the covariant derivatives defined by (12.5). The current-momentum proportionality (11.166) holds for every charge.

In the case of a *single charge*, the covariant derivatives (12.5) take a simpler form which reflects complete canceling of the EM self-interaction:

$$\tilde{\partial}_t = \partial_t + \frac{iq}{\chi} \varphi_{\text{ex}}, \quad \tilde{\nabla} = \nabla - \frac{iq \mathbf{A}_{\text{ex}}}{\chi c}. \quad (12.31)$$

We also omit the label ℓ in (12.14), (12.19) and obtain the equations for the charge density and EM field:

$$\chi \mathbf{i} \partial_t \psi + \frac{\chi^2}{2m} \tilde{\nabla}^2 \psi - \frac{\chi^2}{2m} G'(|\psi|^2) \psi - q \varphi_{\text{ex}} \psi = 0, \quad (12.32)$$

$$\varphi = \int_{\mathbb{R}^3} \frac{\rho(\mathbf{y})}{|\mathbf{y} - \mathbf{x}|} d\mathbf{y}, \quad (12.33)$$

$$\mathbf{A} = \frac{1}{c} \int \frac{1}{|\mathbf{x} - \mathbf{y}|} \mathbf{J}(t, \mathbf{y}) d\mathbf{y}. \quad (12.34)$$

If the magnetic fields generated by the charges are neglected, we use the first non-relativistic approximation from Sect. 9.4.2 and replace the EM Lagrangian L_{em} in (11.14) by $L_{\text{em}1}$ given by (9.43) or, equivalently, set $\mathbf{A}_\ell = \mathbf{0}$. The resulting Lagrangian has the form

$$\begin{aligned} \hat{\mathcal{L}}_0 (\{\psi_\ell\}_{\ell=1}^N, \{\varphi_\ell\}_{\ell=1}^N) &= \frac{1}{8\pi} |\nabla \varphi|^2 + \sum_\ell \hat{L}_\ell (\psi_\ell, \psi_\ell^*, \varphi), \\ \hat{L}_\ell &= \frac{\chi i}{2} [\psi_\ell^* \partial_t \psi_\ell - \psi_\ell \partial_t \psi_\ell^*] - \frac{\chi^2}{2m_\ell} \left\{ |\tilde{\nabla}_{\ell\text{ex}} \psi_\ell|^2 + G_\ell (\psi_\ell^* \psi_\ell) \right\} \\ &\quad - q_\ell (\varphi + \varphi_{\text{ex}} - \varphi_\ell) \psi_\ell \psi_\ell^* - \frac{|\nabla \varphi_\ell|^2}{8\pi}. \end{aligned} \quad (12.35)$$

12.2.1 Derivation of the Non-relativistic Approximation

In this section we describe how the non-relativistic model of the previous section can be naturally derived from the relativistic model described in Sect. 12.1. Namely, we show how the nonlinear Schrödinger (NLS) equation is obtained from the NKG equation, resulting in transformation of the Lagrangians. The derivation is made in the case of charges moving with non-relativistic velocities, that is, the velocities are much smaller than the speed of light, and we set $\varphi_{\text{ex}} = 0$, $\mathbf{A}_{\text{ex}} = 0$ for simplicity. Using the frequency-shifting substitution (11.151)

$$\psi_\ell(t, \mathbf{x}) = e^{-i\omega_0 t} \psi_{\ell\omega}(t, \mathbf{x}), \quad \omega_0 = \frac{mc^2}{\chi} = c\kappa_0, \quad (12.36)$$

we observe that the terms which do not involve spatial derivatives and the nonlinearity in the Klein–Gordon equation (12.8) can be written in the form

$$\begin{aligned} & -\frac{1}{c^2} \tilde{\partial}_{t\ell} \tilde{\partial}_{t\ell} \psi_{\ell\omega} - \kappa_{0\ell}^2 \psi_{\ell\omega} \\ &= \frac{1}{c^2} \left(\partial_t + \frac{i q_\ell}{\chi} \varphi_{\neq \ell} \right)^2 \psi_{\ell\omega} - 2i \frac{m}{\chi} \omega_0 \left(\partial_t + \frac{i q_\ell}{\chi} \varphi_{\neq \ell} \right) \psi_{\ell\omega}. \end{aligned} \quad (12.37)$$

Formally, assuming that typical velocities are of order 1, we derive the non-relativistic approximation assuming that

$$\frac{1}{c} \simeq 0. \quad (12.38)$$

Let us discuss now the regions of validity of such an approximation. We obtain the *non-relativistic approximation* if we neglect the first term in the right-hand side of (12.37). This term may be neglected in regimes with time scales much larger than $1/\omega_0$. The first term is responsible for wave propagation at the speed of light c .

A typical atomic spatial scale is given by the Bohr radius a_B . Let $T_B = \frac{a_B}{c}$ be the time at which light passes the distance which equals the Bohr radius a_B . Note that the frequency ω_0 is expressed in terms of the Bohr radius by the formula $\omega_0 = \frac{mc^2}{\chi} = \frac{c}{\alpha^2 a_B}$ where α is Sommerfeld's fine structure constant. We can apply the non-relativistic approximation at atomic scales if

$$\frac{1}{\omega_0} = \frac{\alpha^2 a_B}{c} \ll T_B = \frac{a_B}{c}, \quad (12.39)$$

that is, if we can neglect terms of order α^2 . Note that $\alpha \simeq \frac{1}{135}$, and assumption (12.38) at atomic length scales corresponds to the following assumption:

$$\alpha^2 \ll 1. \quad (12.40)$$

Hence, using (12.38) in the non-relativistic regime, we substitute

$$-2i \frac{m_\ell}{\chi} \left(\partial_t + \frac{i q_\ell}{\chi} \varphi_{\neq \ell} \right) \psi_{\ell\omega}$$

for the term $-\frac{1}{c^2} \tilde{\partial}_{t\ell} \tilde{\partial}_{t\ell} \psi_{\ell\omega} - \kappa_0^2 \psi_{\ell\omega}$ in Eq. (17.98). Consequently, we obtain from the nonlinear Klein–Gordon equation (17.98) the following nonlinear Schrödinger equation (where we denote the frequency shifted $\psi_{\ell\omega}$ once again by ψ_ℓ):

$$\chi i \partial_t \psi_\ell + \frac{\chi^2}{2m_\ell} \tilde{\nabla}_\ell^2 \psi_\ell - \frac{\chi^2}{2m_\ell} G'_\ell (|\psi_\ell|^2) \psi_\ell - q_\ell \varphi_{\neq \ell} \psi_\ell = 0, \quad (12.41)$$

where $\tilde{\nabla}_\ell$ is given by (12.5).

As to the equations for the EM potentials, we omit the terms with time derivatives in (17.107)–(17.108) since they have factor $\frac{1}{c^2}$ and obtain equations of the form (12.16), (12.17):

$$-\frac{1}{4\pi} \nabla^2 \varphi_\ell = q_\ell \rho_\ell, \quad (12.42)$$

$$-\frac{1}{4\pi} \nabla^2 \mathbf{A}_\ell = \frac{q_\ell}{c} \mathbf{J}_\ell, \quad (12.43)$$

with ρ_ℓ and $\mathbf{J}_\ell$ given by (12.15). Note that the reduction of the wave equations (17.107)–(17.108) to the Poisson equations (12.16), (12.17) is equivalent to neglecting the retardation (4.44) in the Green formulas (4.42) which turn into (12.19), (12.20). The effect of retardation is proportional to time derivatives of the currents multiplied by the retardation time $\frac{R}{c}$ where R is typical distance between the interacting charges. If the currents do not vary too fast, and/or the interaction distances are not too large, it is natural to neglect the retardation to obtain the non-relativistic model.

We can also consider a more radical reduction of the non-relativistic model which leads to the Lagrangian (12.35). Since the magnetic fields enter the equations with the coefficient $\frac{1}{c}$, we can neglect them in the radical non-relativistic approximation. Often it is natural to consider a subsystem of charges of a very large system. The external system of very many charges still may create a magnetic field which should not be neglected even under the assumption (12.38). Therefore, we neglect magnetic fields generated by the charges of a subsystem, but we take into account the magnetic field generated by the remaining charges of a large system. The remaining charges create an external EM field with the potentials $\varphi_{\text{ex}}, \mathbf{A}_{\text{ex}}$ which are assumed to be given functions of time and spatial variables. Hence we replace $\tilde{\nabla}_\ell$ by the covariant gradient $\tilde{\nabla}_{\text{ex}}$ defined by the expression

$$\tilde{\nabla}_{\text{ex}} = \nabla - \frac{iq_\ell}{\chi c} \mathbf{A}_{\text{ex}}. \quad (12.44)$$

12.3 Balanced Charges Gauge Invariance

Now we consider relativistic balanced charges described by (12.2)–(12.8). We can write the Lagrangian (12.2) in the form

$$\begin{aligned} & \mathcal{L}(\{\psi_\ell, \psi_{\ell;\mu}\}, \{\psi_\ell^*, \psi_{\ell;\mu}^*\}, A_\ell^\mu) \\ &= \sum_{1 \leq \ell \leq N} L_\ell(\psi_\ell, \psi_{\ell;\mu}, \psi_\ell^*, \psi_{\ell;\mu}^*) + \mathcal{L}_{\text{BEM}}, \end{aligned} \quad (12.45)$$

$$\mathcal{L}_{\text{BEM}} = \mathcal{L}_{\text{CEM}}(A) - \sum_{1 \leq \ell \leq N} L_{\text{em}}(A_\ell), \quad (12.46)$$

where $\mathcal{L}_{\text{CEM}}$ is the part of the EM Lagrangian which corresponds to the total field A^μ defined by (11.7), namely

$$\mathcal{L}_{\text{CEM}}(A) = L_{\text{em}}(A) = -\frac{1}{16\pi} \sum_{\ell, \ell'} F_\ell^{\mu\nu} F_{\ell'\mu\nu}. \quad (12.47)$$

The charge Lagrangians L_ℓ in (12.45) are the NKG Lagrangians (12.4).

Observe that the EM part $\mathcal{L}_{\text{BEM}}$ of the Lagrangian $\mathcal{L}$, according to (12.46), is obtained by the removal from the classical EM Lagrangian $\mathcal{L}_{\text{CEM}}$ all self-interaction contributions $L_{\text{em}}(A_\ell)$ of the elementary EM fields and it can be recast as

$$\begin{aligned} \mathcal{L}_{\text{BEM}} &= -\frac{1}{16\pi} \sum_{\{\ell, \ell'\}: \ell' \neq \ell} F_\ell^{\mu\nu} F_{\ell'\mu\nu} = -\frac{1}{16\pi} \sum_{1 \leq \ell \leq N} F_\ell^{\mu\nu} F_{\neq \ell \mu\nu}, \\ F_{\neq \ell}^{\mu\nu} &= \sum_{\ell' \neq \ell} F_{\ell'}^{\mu\nu}. \end{aligned} \quad (12.48)$$

The system Lagrangian $\mathcal{L}$ is manifestly Lorentz and gauge invariant with respect to the gauge transformations of the first kind (11.40). The gauge invariance allows us to introduce elementary conserved currents $J_\ell^\nu = (c\rho_\ell, \mathbf{J}_\ell)$ by (11.50), which can be written as (11.132), (11.133). Notice then that the continuity equation (11.54) holds. The total potential A^μ and the corresponding total EM field $F^{\mu\nu}$ are given by the formulas (11.60), (11.58) respectively. Note also that Eq. (11.34) determine the elementary potential A_ℓ^μ generated by the ℓ -th charge.

From the conservation law (11.92) for $T_\ell^{\mu\nu}$, taking into account (11.93) and (11.94) with $\theta = -1$, we obtain the following conservation laws:

$$\partial_\mu T_\ell^{\mu\nu} = \frac{1}{c} J_\xi^\ell \sum_{\ell' \neq \ell} F_{\ell'}^{\nu\xi} = \frac{1}{c} J_\xi^\ell F_{\neq \ell}^{\nu\xi}. \quad (12.49)$$

Let us look at the gauge properties of the system Lagrangian $\mathcal{L}$ defined by (12.45), (12.46). Recall that gauge *transformations of the first or the second kind* (known also as, respectively, *the global and local gauge transformation*, [266, (17), (23a), (23b)], [337, Section 11, (11.4)]) are described, respectively, by formulas (11.40), (11.41). The invariance with respect to the gauge transformation of the first kind (11.40) was used to construct individual currents J_ℓ^μ defined in (11.59). We introduce now another gauge transformation which we call the *elementary gauge transformation or gauge transformation of the third kind*:

$$\begin{aligned} A_\ell^\mu &\rightarrow A_\ell^\mu + \partial^\mu \lambda_\ell(x), \\ \psi_\ell &\rightarrow e^{-\frac{iq_\ell \lambda_\ell(x)}{\hbar c}} \psi_\ell, \quad \psi_\ell^* \rightarrow e^{\frac{iq_\ell \lambda_\ell(x)}{\hbar c}} \psi_\ell^*, \end{aligned} \quad (12.50)$$

where $\lambda_\ell(x)$, $1 \leq \ell \leq N$, are independent real-valued scalar functions of x and

$$\lambda(x) = \sum_{1 \leq \ell \leq N} \lambda_\ell(x), \quad \tilde{\lambda}_\ell(x) = \lambda(x) - \lambda_\ell(x). \quad (12.51)$$

It readily follows from (12.51) that

$$\lambda_\ell = \lambda - \tilde{\lambda}_\ell = \frac{\tilde{\lambda}}{N-1} - \tilde{\lambda}_\ell \text{ where } \tilde{\lambda} = \sum_{1 \leq \ell \leq N} \tilde{\lambda}_\ell = (N-1)\lambda, \quad (12.52)$$

implying the independence of functions $\tilde{\lambda}_\ell(x)$, $1 \leq \ell \leq N$. A straightforward examination shows that for $N \geq 2$ the system Lagrangian $\mathcal{L}$ is invariant with respect to the gauge transformation of the third kind.

Chapter 13

Lagrangian Field Formalism for Dressed Charges

The dressed charges are described by the Lagrangian of the form (11.14) in the particular case $\theta = 0$, that is, in the classical case of the single EM field. In this chapter we apply general constructions of Sect. 11.1 to this particular case.

13.1 Relativistic Lagrangian Formalism for Interacting Dressed Charges

The Lagrangian (11.14) in the particular case $\theta = 0$ takes the form

$$\mathcal{L} = \sum_{\ell} L_{\ell} (\psi_{\ell}, \psi_{\ell;\mu}, \psi_{\ell}^*, \psi_{\ell;\mu}^*) - \frac{F^{\mu\nu} F_{\mu\nu}}{16\pi} \quad (13.1)$$

where the *covariant derivatives* $\psi_{\ell;\mu}$ are defined by (11.10), (11.11), and the *covariant differentiation operators* take the form

$$\tilde{\partial}_{\ell}^{\mu} = \partial^{\mu} + \frac{iq_{\ell}}{\chi c} A^{\mu} + \frac{iq_{\ell}}{\chi c} A_{\text{ex}}^{\mu}. \quad (13.2)$$

We also assume that every Lagrangian L_{ℓ} has the form of the NKG Lagrangian (11.121):

$$L_{\ell} = \frac{\chi^2}{2m_{\ell}} \left\{ \psi_{\ell;\mu}^* \dot{\psi}_{\ell}^{\mu} - \kappa^{\ell 2} \psi_{\ell}^* \psi_{\ell} - G_{\ell} (\psi_{\ell}^* \psi_{\ell}) \right\} \quad (13.3)$$

with the coefficients

$$\kappa_{\ell} = \frac{\omega_{\ell}}{c} = \frac{m_{\ell} c}{\chi}, \quad \omega_{\ell} = \frac{m_{\ell} c^2}{\chi}. \quad (13.4)$$

The parameter m_ℓ is called the ℓ -th charge mass parameter. The parameter q_ℓ is the value of the charge, the term $G(\psi^*\psi)$ corresponds to the nonlinear self-interaction. The Lagrangian $\mathcal{L}$ defined by (13.1) is manifestly Lorentz invariant and gauge invariant with respect to gauge transformations (11.41), (11.40). The Euler–Lagrange field equations (11.19) for charge densities ψ_ℓ take the form of the NKG equations:

$$\left[\tilde{\partial}_{\ell\mu} \tilde{\partial}_\ell^\mu + \kappa_\ell^2 + G'(|\psi_\ell|^2) \right] \psi_\ell = 0, \quad \ell = 1, \dots, N, \quad (13.5)$$

and the complex conjugate equation for ψ_ℓ^* . The EM field satisfies the Maxwell equations (11.33). The ℓ -th charge 4-vector EM current J_ℓ^ν defined by (11.50) takes for the NKG Lagrangian the form (11.134), namely

$$J^\nu = \frac{\chi q}{m} \left(-\frac{1}{c} \mathcal{J}_0(\psi_\ell, \varphi_{\text{ex}} + \varphi), \mathcal{J}_\nabla(\psi_\ell, \mathbf{A}_{\text{ex}} + \mathbf{A}) \right), \quad \ell = 1, \dots, N. \quad (13.6)$$

Using the notation (11.12), we write the covariant differential operator (13.2) in the form

$$\tilde{\partial}_\ell^\mu = \left(\frac{1}{c} \tilde{\partial}_t, \tilde{\nabla}_\ell \right), \quad (13.7)$$

where

$$\tilde{\partial}_t = \partial_t + \frac{i q_\ell}{\chi} \varphi + \frac{i q_\ell}{\chi} \varphi_{\text{ex}}, \quad \tilde{\nabla}_\ell = \nabla - \frac{i q_\ell}{\chi c} \mathbf{A} - \frac{i q_\ell}{\chi c} \mathbf{A}_{\text{ex}}. \quad (13.8)$$

Therefore, the NKG equation (13.5) can be recast as

$$\left[c^{-2} \tilde{\partial}_{\ell t}^2 - \tilde{\nabla}_\ell^2 + \kappa_\ell^2 + G'_\ell(|\psi_\ell|^2) \right] \psi_\ell = 0. \quad (13.9)$$

In view of (13.6), the 4-current density $J^\mu = (\rho c, \mathbf{J})$ in the Maxwell equations (11.33) can be written using the notation (11.130), (11.131) in the form

$$\rho_\ell = -\frac{q_\ell \chi}{m_\ell c^2} \mathcal{J}_0(\psi_\ell, \varphi + \varphi_{\text{ex}}), \quad (13.10)$$

$$\mathbf{J}_\ell = \frac{\chi q_\ell}{m_\ell} |\psi_\ell|^2 \mathcal{J}_\nabla(\psi_\ell, \mathbf{A} + \mathbf{A}_{\text{ex}}). \quad (13.11)$$

The above formulas for the 4-current density $J^\mu = (\rho c, \mathbf{J})$ are often used in the treatment of the KG equations, see for instance, [337, (11.3)], [250, Section 3.3, (3.3.27), (3.3.34), (3.3.35)].

The gauge invariance (11.40) implies that every individual ℓ -th charge 4-vector current J_ℓ^μ defined for a solution of the NKG equation satisfies the continuity equation (11.54)–(11.55), namely

$$\partial_t \rho_\ell + \nabla \cdot \mathbf{J}_\ell = 0.$$

The individual energy-momentum tensor $T_\ell^{\mu\nu}$ for the ℓ -th charge is defined by formula (11.88) which, in the particular case (13.3), takes the form

$$T_\ell^{\mu\nu} = \frac{\chi^2}{2m_\ell} \left(\psi_\ell^{\dot{\mu}*} \psi_\ell^{\dot{\nu}} + \psi_\ell^{\dot{\mu}} \psi_\ell^{\dot{\nu}*} \right) - \left[\psi_{\ell;\mu}^* \psi_\ell^{\dot{\mu}} - \kappa_\ell^2 \psi_\ell^* \psi_\ell - G_\ell (\psi_\ell^* \psi_\ell) \right] \delta^{\mu\nu}. \quad (13.12)$$

Notice that the system Lagrangian $\mathcal{L}$ defined by (13.1)–(13.3) satisfies the symmetry condition (11.45). The energy-momentum tensors $T_\ell^{\mu\nu}$, $\Theta^{\mu\nu}$ satisfy the conservation laws equations (9.17), (11.92) where the Lorentz force density is given respectively by (11.93), (11.97), (9.19).

13.1.1 A Single Relativistic Dressed Charge in an External Electromagnetic Field

A single charge interacting with the EM field is evidently a particular case of a system of many charges considered in Sects. 11.1, 13.1 where ℓ takes only one value. Therefore, we skip the index ℓ for brevity. A single free charge is described by a complex scalar field $\psi = \psi(t, \mathbf{x})$, and it is coupled to the EM field described by its 4-potential $A^\mu = (\varphi, \mathbf{A})$. A dressed charge is always coupled with its EM field, and when we use the term *dressed charge*, we refer to the pair $\{\psi, A^\mu\}$. The Lagrangian (13.1) for a single charge involves only one NKG Lagrangian L_ℓ given by (13.3) or (11.121), namely

$$\mathcal{L}_{\text{KG}} = \frac{\chi^2}{2m} \{ \psi_{;\mu}^* \psi^{\dot{\mu}} - \kappa_0^2 \psi^* \psi - G(\psi^* \psi) \} - \frac{F^{\mu\nu} F_{\mu\nu}}{16\pi}, \quad (13.13)$$

where $F^{\mu\nu}$ is defined by (4.14), and we use the notation (11.10)–(11.11) for the covariant derivatives. The parameter κ_0 is defined by formula (13.4). The term $G(\psi^* \psi)$ corresponds to the nonlinear self-interaction, ψ^* is the complex conjugate to ψ . The Lagrangian $\mathcal{L}$ in (13.13) has the form (11.14) with $\theta = 0$ in (11.8) and satisfies the symmetry condition (11.45). The Euler–Lagrange field equations (11.2) for the Lagrangian (13.13) now take the form of the nonlinear Klein–Gordon equation (13.5)

$$\left[\tilde{\partial}_\mu \tilde{\partial}^\mu + \kappa_0^2 + G'(\psi^* \psi) \right] \psi = 0 \quad (13.14)$$

coupled with the Maxwell equations (11.33)

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu, \quad (13.15)$$

where J^μ is the four-vector current related to the charge by (13.6). Observe that the Lagrangian $\mathcal{L}$ defined by (13.13) is manifestly Lorentz and gauge invariant. The expression for the 4-current density J^μ turns into (13.6) or, in the space-time

variables, into (13.10)–(13.11) with ℓ skipped. Since ψ satisfies the NKG equation, the charge conservation/continuity equation (11.54) holds.

13.2 Non-relativistic Lagrangian Formalism for Interacting Dressed Charges

Here again we consider dressed charges, namely charges described by the Lagrangian (11.14) in the case where $\theta = 0$, but now we study the non-relativistic case. Namely, now the charge Lagrangians L_ℓ are not the NKG Lagrangians of the form (11.121) but rather the NLS Lagrangians of the form (11.155). The non-relativistic Lagrangian L_ℓ describes interaction between charges only through the electric field, namely

$$A^\mu = (\varphi, 0, 0, 0), \quad \mathbf{E} = -\nabla\varphi, \quad \mathbf{B} = \mathbf{0}. \quad (13.16)$$

Consequently, we use here the non-relativistic Lagrangian L_{em1} defined by (9.43), see Sect. 9.4.2 EM. As a result, we obtain from (11.14) the non-relativistic dressed charge Lagrangian in the form

$$\mathcal{L} = \frac{1}{8\pi} |\nabla\varphi|^2 + \sum_{\ell} L_{\ell}(\psi_{\ell}, \psi_{\ell}^*, \varphi), \quad (13.17)$$

where L_{ℓ} is the NLS Lagrangian (11.155), namely

$$\begin{aligned} L_{\ell} = L_{\text{NLS}\ell} &= \frac{\chi}{2} \mathbf{i} \left[\psi_{\ell}^* \tilde{\partial}_t^{\ell} \psi_{\ell} - \psi_{\ell} \tilde{\partial}_t^{\ell*} \psi_{\ell}^* \right] \\ &\quad - \frac{\chi^2}{2m_{\ell}} \left\{ \tilde{\nabla} \psi_{\ell} \tilde{\nabla}^* \psi_{\ell}^* + G_{\ell}(\psi_{\ell}^* \psi_{\ell}) \right\}, \end{aligned} \quad (13.18)$$

and φ is defined by (12.12). The Lagrangian (13.17) corresponds to a system of N charges described by complex-valued fields ψ_{ℓ} , $\ell = 1, \dots, N$. This Lagrangian can be obtained from the Lagrangian (11.63) where the electromagnetic Lagrangian $L_{\text{em}} = L_{\text{em1}}$ is given by (9.43) and $\theta = 0$. Note that the non-relativistic theory can be derived from the relativistic as in Sect. 12.2.1. Here we give a brief derivation directly from the NKG Lagrangian for one charge. The non-relativistic Lagrangian $L_{\text{NLS}\ell}$ is constructed as a non-relativistic modification of the frequency shifted Lagrangian $L_{\omega_0}(\psi, A^\mu)$ introduced in Sect. 11.7. The initial step in the construction of the frequency-shifted Lagrangian L_{ω_0} defined by (11.152) is the change of variables (11.151). Then the gauge invariant non-relativistic Lagrangian L_{ℓ} in (13.18) is obtained from the Lagrangian L_{ω_0} by omitting in (11.152) the term $\frac{\chi^2}{2mc^2} \tilde{\partial}_t \psi \tilde{\partial}_t^* \psi^*$ and setting $\mathbf{A} = \mathbf{0}$.

The covariant differentiation (11.13) which enters (13.18) takes the form

$$\tilde{\partial}_t \ell = \partial_t + \frac{iq_\ell}{\chi} (\varphi + \varphi_{\text{ex}}), \quad \tilde{\nabla}_\ell = \nabla - \frac{iq_\ell}{\chi c} \mathbf{A}_{\text{ex}}, \quad (13.19)$$

where $(\varphi_{\text{ex}}, \mathbf{A}_{\text{ex}})$ is the potential of the external EM field. The EL field equations (11.19) for the Lagrangian (13.17) are the NLS equations of the form (11.153), namely

$$\chi i \tilde{\partial}_t \psi_\ell = \frac{\chi^2}{2m_\ell} \left[-\tilde{\nabla}_\ell^2 \psi_\ell + G'_\ell(|\psi_\ell|^2) \psi_\ell \right], \quad (13.20)$$

coupled with the Poisson equation (11.35) for the electric potential:

$$-\Delta \varphi = 4\pi \rho. \quad (13.21)$$

Here the charge density $\rho = \frac{1}{c} J_A^0$ is given by (11.31), (11.156):

$$\rho = \sum_\ell \rho_\ell, \quad \rho_\ell = q_\ell |\psi_\ell|^2. \quad (13.22)$$

The solution of the Poisson equation is given by Green's formula (4.45), and expansion (11.60) holds, namely

$$\varphi = \sum_\ell \varphi_\ell, \quad (13.23)$$

where φ_ℓ satisfies Eq. (11.36) which takes the form

$$\nabla^2 \varphi_\ell = -4\pi q_\ell |\psi_\ell|^2, \quad (13.24)$$

and we attribute to every ℓ -th charge its potential φ_ℓ by Green's formula (12.19).

The NLS equation (13.20) can be written in the form which explicitly shows the dependence on the EM potentials:

$$i\chi \partial_t \psi_\ell = -\frac{\chi^2}{2m_\ell} \left(\nabla - \frac{iq}{\chi c} \mathbf{A}_{\text{ex}} \right)^2 \psi_\ell + q_\ell (\varphi + \varphi_{\text{ex}}) \psi_\ell + G'_{\ell a}(|\psi_\ell|^2) \psi_\ell, \quad \ell = 1, \dots, N. \quad (13.25)$$

Obviously, the equations with different ℓ are coupled only through the potential φ which is responsible for the charges interaction.

The Lagrangian $\mathcal{L}$ defined by (13.17) is gauge invariant with respect to the first and the second gauge transformations (11.40), (11.41), and, consequently, every ℓ -th charge has a conserved current $J_\ell^\mu = (c\rho_\ell, \mathbf{J}_\ell)$ with formulas (11.156), (11.157) for $\rho_\ell, \mathbf{J}_\ell$. Every current J_ℓ^μ satisfies the conservation/continuity equations (11.54), (11.55), namely

$$\partial_t \rho_\ell + \nabla \cdot \mathbf{J}_\ell = 0. \quad (13.26)$$

The charge conservation equation (13.26) can be derived directly from the NLS equation (13.20) by multiplying it and its complex conjugate by ψ_ℓ^* and ψ_ℓ , respectively, and subtracting from one another.

A single charge is a particular case of the system of charges considered above. It is described by the following NLS-Poisson equations:

$$\chi i \tilde{\partial}_t \psi = \frac{\chi^2}{2m} \left[-\tilde{\nabla}^2 \psi + G'(\psi^* \psi) \psi \right], \quad (13.27)$$

$$-\Delta \varphi = 4\pi q \psi \psi^*. \quad (13.28)$$

The explicit expressions for the charge and current densities are given by formulas (11.156), (11.157).

13.2.1 Energy-Momentum Tensors for Non-relativistic Dressed Charges

A non-relativistic dressed charge involves the charge distribution ψ_ℓ and the corresponding electric potential φ_ℓ ; here we consider their EnMT tensors $T_\ell^{\mu\nu}$ and $\Theta_\ell^{\mu\nu}$. We introduce for the ℓ -th charge its EnMT $T_\ell^{\mu\nu}$ based on the formulas (11.88), (10.61). The formulas (11.161), (11.162) for energy and momentum densities for individual charges have the form

$$u_\ell = \frac{\chi^2}{2m} \left[\tilde{\nabla}_\ell \psi_\ell \cdot \tilde{\nabla}_\ell^* \psi_\ell^* + G_\ell(\psi_\ell^* \psi_\ell) \right], \quad (13.29)$$

$$p_{\ell j} = \frac{\chi^2}{2} \left(\psi_\ell \tilde{\partial}_{\ell j}^* \psi_\ell^* - \psi_\ell^* \tilde{\partial}_{\ell j} \psi_\ell \right), \quad j = 1, 2, 3. \quad (13.30)$$

The EnMT $\Theta^{\mu\nu}$ for the EM field is defined by (11.103). The EnMT $T_\ell^{\mu\nu}$ and $\Theta^{\mu\nu}$ satisfy conservation laws (11.92) and (11.104). Since $\tilde{A}_\ell^\mu = A_{\text{ex}}^\mu + A^\mu$ in (11.92), the conservation laws take the form

$$\partial_\mu T_\ell^{\mu\nu} = f_\ell^\nu + f_{\ell\text{ex}}^\nu, \quad \partial_\mu \Theta^{\mu\nu} = - \sum_\ell f_\ell^\nu, \quad (13.31)$$

where

$$f_\ell^\nu = \frac{1}{c} J_{\ell\mu} F^{\nu\mu} = \left(\frac{1}{c} \mathbf{J}_\ell \cdot \mathbf{E}, \rho_\ell \mathbf{E} \right), \quad (13.32)$$

$$f_{\ell\text{ex}}^\nu = \frac{1}{c} J_{\ell\mu} F_{\text{ex}}^{\nu\mu} = \left(\frac{1}{c} \mathbf{J}_\ell \cdot \mathbf{E}_{\text{ex}}, \rho_\ell \mathbf{E}_{\text{ex}} + \frac{1}{c} \mathbf{J}_\ell \times \mathbf{B}_{\text{ex}} \right). \quad (13.33)$$

We readily recognize in f^ν and f_{ex}^ν which enter Eq. (13.31) the Lorentz force densities for the charge in its own field and the external EM field respectively. The equations in (13.31) also indicate that Newton's principle "action equals reaction" does manifestly hold for all involved densities at every point of space-time.

The energy-momentum conservation equations (13.31) can be viewed as equations of motion in an elastic continuum, [248, Section 6.4, (6.56), (6.57)], similar to the case of the kinetic energy-momentum tensor for a single relativistic particle, [295, Section 37, (3.24)].

According to (11.166), the charge momentum density $\mathbf{P}_\ell$ is proportional to the current density $\mathbf{J}_\ell$ for every charge, namely the following identity holds

$$\mathbf{P}_\ell = \frac{m_\ell}{q_\ell} \mathbf{J}_\ell, \quad \ell = 1, \dots, N. \quad (13.34)$$

One can verify then that the conservation law (13.31) for $\Theta_\ell^{\mu\nu}$ takes here the form

$$\partial_\mu \Theta_\ell^{\mu\nu} = -\frac{1}{c} J_{\ell\mu} F^{\ell\nu\mu}. \quad (13.35)$$

Now for every ℓ -th dressed charge we define its EnMT $T_\ell^{\mu\nu}$ by the formula (11.119), namely

$$T_\ell^{\mu\nu} = T_\ell^{\mu\nu} + \Theta_\ell^{\mu\nu}. \quad (13.36)$$

The tensor $T_\ell^{\mu\nu}$ satisfies conservation law (11.120), which can be written as follows:

$$\begin{aligned} \partial_\mu T_\ell^{\mu\nu} &= f_\ell^\nu + f_{\text{ex}}^\nu, \\ f_\ell^\nu &= \frac{1}{c} J_{\ell\mu} \sum_{\ell' \neq \ell} F^{\ell'\nu\mu} = \sum_{\ell' \neq \ell} \left(\frac{1}{c} \mathbf{J}_\ell \cdot \mathbf{E}_{\ell'}, \rho_\ell \mathbf{E}_{\ell'} \right). \end{aligned} \quad (13.37)$$

Note that the Lorentz force f_ℓ^ν in the right-hand side of (13.37) manifestly excludes the self-interaction in contrast to the Lorentz force acting upon the bare charge as in (13.31), which explicitly includes the self-interaction term $\frac{1}{c} J_\mu^\ell F^{\ell\nu\mu}$. Thus we can conclude that when the charge and its EM field are treated as a single entity, as in the dressed charge model, there is no self-interaction in the EnMT conservation laws, as signified by the Eqs. (11.120) and (13.37).

The canonical EnMT $\hat{\Theta}^{\mu\nu}$ for the EM field is given by formula (9.11) in the particular case (13.16), namely in the matrix form (11.68)–(11.69). The symmetric energy-momentum tensor of the EM field takes the form (9.13) in the same particular case (13.16). The matrix entry $u_{\text{em}} = \Theta^{00}$ (the EM field energy density) in (9.16) satisfies the conservation law (9.17).

Part III

The Neoclassical Theory of Charges

“The electron is a stranger in electrodynamics.”

*A. Einstein.*¹

“In atomic theory we have fields and we have particles.
The fields and the particles are not two different things.

They are two ways of describing the same thing—
two different points of view.”

*P. Dirac.*²

“And yet a theory may perfectly well exist,
which is in a genuine sense an atomistic one
(and not merely on the basis of a particular interpretation),
in which there is no localizing of the particles
in a mathematical model.”

*A. Einstein.*³

In this part we present a self-contained treatment of our neoclassical theory of distributed spinless elementary charges interacting via the electromagnetic field. In Chap. 3 we have presented concisely the main features of the theory for the case of balanced charges with no spin, and this concise presentation can be viewed as an introduction to this part. Here we do not impose simplifying assumptions, in particular we do not exclude magnetic fields, and consider balanced charges and dressed charges. The difference between the balanced charge and the dressed charge is in the treatment of the EM self-interaction of the charge. For the balanced charges the EM self-interaction is absent, and for the dressed charges it is compensated by a special nonlinearity. Though we consider balanced charges as the primary model, the parallel treatment of dressed charges helps to better understand the role of the EM self-interaction in different aspects of EM theory. The idea to introduce an extended charge in EM theory instead of the point charge is not new, and the most well-known models for it are the Abraham rigid charge model and the Abraham–Lorentz relativistically covariant model. These models are studied and advanced in many papers, see [16, 179], [196], [269], [2, 6, 278], [301], [310], [345] and references therein. In contrast to those

¹The Einstein quotation is from Sommerfeld’s book [308, Section 28, p. 236].

²The Dirac quotation is from his lectures on the quantum field theory [88].

³The Einstein quotation is from his paper/lecture [106].

models, we do not prescribe to an elementary charge a certain geometry, but instead the elementary charge has a wave function governed either by a nonlinear Klein–Gordon (NKG) or by a nonlinear Schrödinger (NLS) equation in the relativistic and non-relativistic cases respectively, [15–20]. The idea to eliminate the self-interaction, as we do for balanced charges, is also not new. The most well-known EM theory with this feature is due to J. Wheeler and R. Feynman, [339], [340], but our neoclassical EM theory of distributed charges is very different from it.

The construction of significant elements of our theory is accomplished in several steps. First, we construct rest states and wave-corpuscle solutions for both balanced and dressed charges. Then we derive Newtonian mechanics for the charges in point-like regimes by two methods. The first one is based on the wave-corpuscle representation, and the second one is similar to the Ehrenfest theorem, relying on the Lagrangian formalism described in Chaps. 12 and 13. Finally, we study systems of charges at equilibrium, in particular the Hydrogen atom model for both balanced and dressed charges and compare their properties. At every step we cover both the relativistic and the non-relativistic cases.

Chapter 14

Rest and Time-Harmonic States of a Charge

“However, Maxwell’s equations did not permit the derivations of the equilibrium of the electricity which constitutes a particle. Only other, nonlinear field equations could possibly accomplish such a thing.”

*A. Einstein.*¹

“...it seems to me certain that we have to give up the notion of an absolute localization of the particles in a theoretical model.”

*A. Einstein.*²

The simplest state of a charge is when it is at rest and does not interact with anything else. Naturally, it is the first case to be studied. In this chapter we consider a spinless elementary charge and leave the case of particles with spin 1/2 for Part IV. The rest state assumes that it does not evolve with time. Consequently, its magnetic field has to be zero and its electric field is expected to be described by Coulomb’s law at macroscopic length scales with a very high precision, [344]. To be consistent with such a physical picture, a rest state of the charge is supposed to satisfy the following requirements. The most basic and the simplest rest state can be a time-independent solution to the field equation which is localized in a very small spatial domain. Another option for a rest state next in simplicity is a time-harmonic state, that is, a state with a time-independent shape and a time-harmonic phase factor with a given frequency. Such a solution would correspond to a charge which has an internal (rest) energy associated with a certain internal “rotation” or “oscillation”, but it does not move translationally with respect to the frame where the charge’s electric field is time-independent.

¹The Einstein quotation is from his Autobiographical notes [295, Autobiographical notes].

²The Einstein quotations are from his paper/lecture [106].

Based on the above considerations, we impose two principal requirements on rest and time-harmonic states of an elementary charge:

- (i) **Localization.** *The (time-harmonic) charge at rest has to be localized, with a spatial scale of localization being small compared with macroscopic scales.*
- (ii) **Coulomb's law.** *The EM field generated by a resting (time-harmonic) charge has to be time-independent and arbitrarily close to Coulomb's field as the charge localization size becomes arbitrarily small compared with macroscopic scales.*

Notice that any quadratic Lagrangian invariant with respect to spatial translations yields linear field equations which are linear spatially homogeneous partial differential equations, and such equations do not have localized solutions. Consequently, there has to be a nonlinearity that would provide for localized solutions. This nonlinearity could provide for intrinsic non-electromagnetic cohesive forces similar to the Poincaré forces. An important feature of our approach is the way we choose the nonlinearity. Assuming at this point no specific information about the cohesive forces except that they are sufficient to keep an elementary charge localized, we base the construction of the nonlinearity on the requirement of localization. With that in mind, *we choose a time-independent radially symmetric charge distribution (ground state) with a given profile $|\psi| = |\psi(\mathbf{x})|$ (form factor) and determine then the nonlinearity $G'(|\psi|^2)$ from the requirement that $\psi(\mathbf{x})$ satisfies the field equation.* This approach allows us to avoid technical difficulties in solving a nonlinear partial differential equation and provides immediately for a localized solution. In addition to that, many properties of the charges, in particular their localization, are more transparent than corresponding properties of the nonlinearity. It turns out rather remarkably that many properties of localized charge dynamics, including Newton's law, do not depend on a particular choice of the ground state and consequently the nonlinearity.

In this section we discuss basic properties of rest states and nonlinearities associated with them and provide their detailed treatment in the cases of balanced and dressed charges in both the non-relativistic and relativistic settings. The necessary Lagrangian formalism for balanced and dressed charges is provided in Chaps. 12 and 13.

14.1 Rest States of a Non-relativistic Balanced Charge

Non-relativistic balanced charges are described by the NLS-Poisson equations (12.14)–(12.17) associated with the Lagrangian (12.11). For a single charge, the system is reduced to a single Eq. (12.32). A single charge is free if there are no external EM fields, that is,

$$\varphi_{\text{ex}} = 0, \quad \mathbf{A}_{\text{ex}} = 0. \quad (14.1)$$

It is natural to assume that the electric field generated by a resting spinless charge does not depend on time and that its magnetic field is zero. Based on these assumptions,

we can prove then that a resting state ψ of the charge must be time-harmonic. Indeed, the EM potentials generated by a non-relativistic balanced charge satisfy the Poisson equations (12.16), (12.17). Evidently, the charge electric field and its potential can be time-independent only if their source

$$\rho(t, \mathbf{x}) = q |\psi(t, \mathbf{x})|^2 \quad (14.2)$$

in Eq. (12.16) does not depend on time, that is,

$$|\psi(t, \mathbf{x})| = \check{\psi}(\mathbf{x}). \quad (14.3)$$

By a similar argument, the magnetic field can be zero only if the source current $\mathbf{J}$ in (12.17) equals zero, that is,

$$\mathbf{J}(t, \mathbf{x}) = \mathbf{0}. \quad (14.4)$$

Comparing representation (11.131) for $\mathbf{J}$ with the above equation, we readily obtain

$$\text{Im} \frac{\nabla \psi}{\psi} = \nabla \arg \psi = \mathbf{0} \text{ implying } \arg \psi(t, \mathbf{x}) = s(t). \quad (14.5)$$

It follows then from (14.3) and (14.5) that

$$\psi(t, \mathbf{x}) = \check{\psi}(\mathbf{x}) e^{is(t)} \quad (14.6)$$

implying together with (12.32) the following equation

$$-\chi \check{\psi} \partial_t s(t) = \frac{\chi^2}{2m} \left[-\nabla^2 \check{\psi} + G'(\check{\psi}^2) \check{\psi} \right](\mathbf{x}). \quad (14.7)$$

The above equality implies that $\partial_t s$ must be a constant. This together with (14.7) implies that the state ψ associated with a free charge has to be *time-harmonic*, that is,

$$\psi = \psi(t, \mathbf{x}) = \check{\psi}(\mathbf{x}) e^{-i\omega t} \quad (14.8)$$

where ω is the state frequency.

Based on the above considerations, we can naturally define a *non-relativistic rest state* as a time-harmonic state with zero frequency $\omega = 0$. Equivalently, a non-relativistic rest state is a time-independent solution of the NLS equation (12.32) which turns into the steady-state equation

$$-\nabla^2 \psi + G'(|\psi|^2) \psi = 0. \quad (14.9)$$

Equation (14.9) can be naturally interpreted as the *charge equilibrium condition*. It signifies a complete balance of two forces acting upon the resting charge: (i) internal

“elastic deformation force” associated with the term $-\Delta\psi$; (ii) internal nonlinear self-interaction of the charge associated with the term $G'(|\psi|^2)\psi$.

The electric potential $\varphi(\mathbf{x})$ generated by the charge density $\rho(\mathbf{x})$ defined by Eq. (14.2) is a solution to the Poisson equation (12.16):

$$\nabla^2 \varphi(\mathbf{x}) = -4\pi \rho(\mathbf{x}), \quad (14.10)$$

given by formula (12.33), that is,

$$\varphi(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{\rho(\mathbf{y})}{|\mathbf{y} - \mathbf{x}|} d\mathbf{y} \quad (14.11)$$

with the charge density $\rho(\mathbf{x})$ satisfying Eq. (14.2). The magnetic field associated with the resting charge is zero. In Sect. 14.3 we consider radial solutions of (14.9) of the form

$$\psi = \overset{\circ}{\psi}(|\mathbf{x}|), \quad (14.12)$$

and then the corresponding potential is also radial:

$$\varphi = \overset{\circ}{\varphi}(|\mathbf{x}|).$$

For a discussion of general time-harmonic states of non-zero frequencies see Sect. 14.4 where we consider time-harmonic states of the NKG equation.

14.2 The Charge Normalization Condition

We formulate here a charge normalization condition which ensures that the electric potentials generated by the charge behaves asymptotically as Coulomb’s potential $q/|\mathbf{x}|$ as $|\mathbf{x}| \rightarrow \infty$; *this allows us to interpret the parameter q as the value of the charge*. To study the asymptotic behavior as $|\mathbf{x}| \rightarrow \infty$, we set $\mathbf{x} = \theta\mathbf{z}$ with $|\mathbf{z}| = 1$, $\theta \rightarrow \infty$. We rewrite (14.11) in the form

$$\varphi = \int_{\mathbb{R}^3} \frac{\rho(\mathbf{y})}{|\mathbf{y} - \theta\mathbf{z}|} d\mathbf{y} = \frac{1}{\theta} \int_{\mathbb{R}^3} \frac{\rho(\mathbf{y})}{|\mathbf{y}/\theta - \mathbf{z}|} d\mathbf{y} = \frac{1}{\theta} \int_{\mathbb{R}^3} \frac{\theta^3 \rho(\theta\mathbf{y})}{|\mathbf{y} - \mathbf{z}|} d\mathbf{y}. \quad (14.13)$$

Let $\bar{\rho}$ be the total charge:

$$\bar{\rho} = \int_{\mathbb{R}^3} \rho(\mathbf{y}) d\mathbf{y} = \int_{\mathbb{R}^3} \theta^3 \rho(\theta\mathbf{y}) d\mathbf{y}.$$

Obviously, the rescaled density $\theta^3 \rho(\theta \mathbf{y})$ in (14.13) converges to Dirac's delta-function:

$$\theta^3 \rho(\theta \mathbf{y}) \rightarrow \bar{\rho} \delta(\mathbf{y}) \quad \text{as } \theta \rightarrow \infty, \quad (14.14)$$

implying

$$\varphi = \frac{1}{\theta} \int_{\mathbb{R}^3} \frac{\theta^3 \rho(\theta \mathbf{y})}{|\mathbf{y} - \mathbf{z}|} d\mathbf{y} \rightarrow \frac{\bar{\rho}}{\theta |\mathbf{z}|} = \frac{\bar{\rho}}{|\mathbf{x}|}. \quad (14.15)$$

The above expression matches the Coulomb potential $q/|\mathbf{x}|$ if we impose the following *charge density normalization condition*:

$$\int_{\mathbb{R}^3} \rho(\mathbf{y}) d\mathbf{y} = q. \quad (14.16)$$

Formula (12.15) defines the charge and current densities, and the charge density takes a very simple form (14.2). Therefore, (14.16) implies the following *charge normalization condition*:

$$\int_{\mathbb{R}^3} |\psi|^2(\mathbf{y}) d\mathbf{y} = 1. \quad (14.17)$$

14.3 Ground State and the Nonlinearity

Here we explore the localization requirement imposed on the resting charge. As we have already pointed out, the localization must be provided by a properly defined nonlinearity. To define the nonlinearity through Eq. (14.9), we impose the following conditions on the ground state of the charge distribution. A *ground state* is associated with a positive function $\psi(r)$, $r = |\mathbf{x}|$, called the *form factor*, which is a twice differentiable and monotonically decreasing function, namely

$$\partial_r \psi(r) < 0 \quad \text{for } r > 0, \quad (14.18)$$

and we assume that $\psi(r)$ satisfies the charge normalization condition (14.17). If $\psi(r)$ is a ground state, we can determine and define the nonlinearity G' from the following equilibrium equation obtained from (14.9):

$$\nabla^2 \psi = G'(|\psi|^2) \psi. \quad (14.19)$$

For a radial $\dot{\psi}$, we obtain an expression for the nonlinearity G' :

$$G'(\dot{\psi}^2(r)) = G'(\dot{\psi}^2(r)) = \frac{(\nabla^2 \dot{\psi})(r)}{\dot{\psi}(r)}. \quad (14.20)$$

Since $s = \dot{\psi}^2(r)$ is a *monotonic* function of radius r , we can find its inverse $r = r(s)$, yielding the following expression for the nonlinearity:

$$G'(s) = \frac{\nabla^2 \dot{\psi}(r(s))}{\dot{\psi}(r(s))}, \quad 0 = \dot{\psi}^2(\infty) \leq s \leq \dot{\psi}^2(0). \quad (14.21)$$

Thus, we pick the form factor $\dot{\psi}$, and then the nonlinear self-interaction function G' is determined based on the charge equilibrium equation (14.19). One of the advantages of determining G' in terms of $\dot{\psi}$ is that we use more often properties of $\dot{\psi}$ in our analysis rather than properties of G . Note that after the nonlinearity G' is determined, it is fixed from this point on. In particular, one can consider solutions to the NLS equation (12.32) that may evolve in time and they can differ from $\dot{\psi}$.

Now let us take a look at general properties of the nonlinearity which follow from its defining relation (14.21). Since $\dot{\psi}(r)$ is smooth, and $\partial_r \dot{\psi} < 0$, $G'(|\psi|^2)$ is smooth for $0 < |\psi|^2 < \dot{\psi}^2(0)$. If we do not need $G'(s)$ to be smooth at every point, we can extend $G'(s)$ for $s \geq \dot{\psi}^2(0)$ as a constant, namely

$$G'(s) = G'(\dot{\psi}^2(0)) \text{ if } s \geq \dot{\psi}^2(0). \quad (14.22)$$

The first derivative of such an extension at $s = \dot{\psi}^2(0)$ has a discontinuity point. If $\dot{\psi}(r)$ is a smooth function of class C^n , $n > 2$, (i.e. n times continuously differentiable), we can always define an extension of $G'(s)$ for $s \geq \dot{\psi}^2(0)$ as a bounded function of class C^{n-2} for all $r > 0$ and

$$G'_1(s) = G'_1(\dot{\psi}_1^2(0)) - 1 \text{ if } s \geq \dot{\psi}^2(0) + 1. \quad (14.23)$$

Functions $G'(s)$ which are not constant for large s can also be used, as we show in the examples below. In all those examples the function $G'(s)$ is not differentiable at $s = 0$, but if $\dot{\psi}(r)$ decays exponentially or by a power law, the nonlinearity $g(\psi) = G'(|\psi|^2)\psi$ as it enters the field equation (12.14) is differentiable at zero, hence it satisfies the Lipschitz condition. For a Gaussian ground state $\dot{\psi}(r)$ which decays superexponentially, $G'(|\psi|^2)$ is unbounded at zero and $g(\psi)$ is not differentiable at zero. Since $\dot{\psi}(|\mathbf{x}|) > 0$, the sign of $G'(|\psi|^2)$ coincides with the sign of $\nabla^2 \dot{\psi}(|\mathbf{x}|)$. At the origin $\mathbf{x} = \mathbf{0}$, the function $\dot{\psi}(|\mathbf{x}|)$ has its maximum, and consequently $G'(s) \leq 0$ for s close to $s = \dot{\psi}^2(0)$. The Laplacian applied to the radial function $\dot{\psi}$ takes the form $\frac{1}{r} \frac{\partial^2}{\partial r^2} (r \dot{\psi}(r))$. Consequently, if $r \dot{\psi}(r)$ is convex, we have $\nabla^2 \dot{\psi}(|\mathbf{x}|) \geq 0$. Since $r^2 \dot{\psi}(r)$ must be integrable, we naturally assume that $|\mathbf{x}| \dot{\psi}(|\mathbf{x}|) \rightarrow 0$ as $|\mathbf{x}| \rightarrow \infty$.

Then, if the second derivative of $r\mathring{\psi}(r)$ has a constant sign near infinity, it must be non-negative as well as $G'(s)$ for $s \ll 1$. In the examples we give below, $G'(s)$ has exactly one zero on the half-axis.

14.3.1 Size Parameter

To quantify the degree of localization of the ground state $\mathring{\psi}$, we introduce an explicit dependence on the *size parameter* $a > 0$ as follows:

$$\mathring{\psi}(r) = \mathring{\psi}_a(r) = a^{-3/2} \mathring{\psi}_1(a^{-1}r), \quad r = |x| \geq 0, \quad (14.24)$$

where $\mathring{\psi}_1(r)$, $\mathring{\psi} \geq 0$, is a function of the dimensionless variable $r \geq 0$. The dependence on a is chosen so that the L^2 -norm

$$\|\mathring{\psi}_a\| = \left(\int \left| \mathring{\psi}_a(|x|) \right|^2 d\mathbf{x} \right)^{1/2} \quad (14.25)$$

does not depend on a , hence the function $\mathring{\psi}_a(r)$ satisfies the charge normalization condition (14.17) for every $a > 0$. A common way to assign the spatial size of $\mathring{\psi}_a$ is based on the variance, and it is proportional to a with a coefficient depending on $\mathring{\psi}_1$, namely

$$\left(\int |x|^2 \left| \mathring{\psi}_a(|x|) \right|^2 d\mathbf{x} \right)^{1/2} = a \left(\int |x|^2 \left| \mathring{\psi}_1(|x|) \right|^2 d\mathbf{x} \right)^{1/2}. \quad (14.26)$$

Definition (14.20) then implies that $G'_a(s) = a^{-2} G'_1(a^3 s)$. To extend $G'_a(s)$ to all values of $s \geq 0$, we first extend $G'_1(s)$ as a smooth function for $s \geq \mathring{\psi}_1^2(0)$ and then set the dependence on the size parameter as follows:

$$G'_a(s) = a^{-2} G'_1(a^3 s), \quad s \geq 0. \quad (14.27)$$

The nonlinear potential

$$G_a(s) = \int_0^s a^{-2} G'_a(s') ds' \quad (14.28)$$

depends on a as follows:

$$G_a(s) = a^{-5} G_1(a^3 s), \quad s \geq 0. \quad (14.29)$$

Based on the above, we assume that the nonlinearity G' depends on the size parameter as in Eq. (14.27); consequently, we can find the ground state $\mathring{\psi}_a(r)$ which

satisfies (14.24). Notice that the corresponding electric potential determined by (14.11) depends on the size parameter as follows:

$$\phi_a(\mathbf{x}) = q \int_{\mathbb{R}^3} \frac{|\psi_a|^2(\mathbf{y})}{|\mathbf{y} - \mathbf{x}|} d\mathbf{y} = a^{-1} \phi_1(a^{-1}\mathbf{x}). \quad (14.30)$$

14.3.2 Examples of the Nonlinearity

In this section we provide examples where the construction of G is carried out explicitly.

14.3.2.1 Ground State Decaying by a Power Law

Example 14.3.1 Let the form factor $\psi_1(r)$ decay according to a *power law*, namely

$$\psi_1(r) = c_{\text{norm}} (1 + r^2)^{-p}, \quad p > 3/4. \quad (14.31)$$

Then an elementary computation of $\nabla^2 \psi_1$ shows that (14.21) takes the form

$$G'_1(s) = -2p (s/c_{\text{norm}}^2)^{1/p} \left((s/c_{\text{norm}}^2)^{1/(2p)} + (2p+2) (s/c_{\text{norm}}^2)^{1/p} \right) \quad (14.32)$$

for $0 \leq s \leq c_{\text{norm}}^2$, where c_{norm} is the normalization factor and $s = |\psi|^2$. The same formula (14.32) can be used to obtain an extension of $G'_1(s)$ for all $s \geq 0$. In this example $G'_1(s)$ is differentiable for all $s \geq 0$ if $3/p \geq 2$, but it is not twice differentiable at $s = 0$. In particular, if $p = 5/4$, we obtain formula (14.95). A mild singularity of $G'(\psi^2)$ at $\psi = 0$ cannot be avoided, since ψ^2 , which satisfies (14.17), has to decay faster than r^{-3} as $r \rightarrow \infty$, and $\nabla^2 \psi/\psi$ for a rational function ψ decays as r^{-2} . Consequently, $\nabla^2 \psi/\psi$ cannot be represented as an analytic function of ψ^2 in a vicinity of zero.

Example 14.3.2 Now let us consider a form factor $\psi_1(r)$ which decays as a *power law*, but has a different structure:

$$\psi_1(r) = \frac{c_{\text{norm}}}{1 + r^{2p}} \quad (14.33)$$

with $p > 1$. Here c_{norm} is the normalization factor:

$$4\pi \int_0^\infty \frac{1}{(1 + r^{2p})^2} r^2 dr = c_{\text{norm}}^{-2}.$$

In particular, for $p = 2$:

$$4\pi \int_0^\infty \frac{r^2}{(r^4 + 1)^2} dr = \frac{1}{4} \sqrt{2} \pi^2, \quad c_{\text{norm}} = 2^{3/4} / \pi.$$

The inverse function is

$$r^2 = \left(c_{\text{norm}} / \psi_1 - 1 \right)^{1/p}, \quad \text{for } \psi_1 \leq c_{\text{norm}}.$$

Then an elementary computation of $\nabla^2 \psi_1$ yields

$$\begin{aligned} \nabla^2 \psi_1 &= c_{\text{norm}} \frac{1}{r} \frac{\partial^2}{\partial r^2} \left(\frac{r}{1 + r^{2p}} \right) \\ &= -2p \psi_1 r^{2p-2} \left(4p \frac{\psi_1^2}{c_{\text{norm}}^2} + (1 - 2p) \frac{\psi_1}{c_{\text{norm}}} \right), \end{aligned}$$

and for $0 \leq |\psi| \leq c_{\text{norm}}$ the definition (14.21) takes the form

$$G'_1(|\psi|^2) = -2p (c_{\text{norm}} / |\psi| - 1)^{1-1/p} \left(4p \frac{|\psi|^2}{c_{\text{norm}}^2} + (1 - 2p) \frac{|\psi|}{c_{\text{norm}}} \right).$$

Then we extend G'_1 as follows

$$G'_1(|\psi|^2) = 0 \quad \text{for } |\psi| \geq c_{\text{norm}}.$$

14.3.2.2 Exponentially and Super Exponentially Decaying Ground States

Example 14.3.3 Let a *super exponentially* decaying form factor be defined by

$$\psi_1(r) = c_{\text{norm}} e^{-r^{2p}}, \quad p \geq 1. \quad (14.34)$$

A direct computation yields

$$\nabla^2 \psi_1 = -2p \psi_1 r^{2p-2} (2p - 2p r^{2p} + 1)$$

with

$$r^2(\psi) = \ln^{1/p} (c_{\text{norm}} / \psi) \quad \text{if } 0 < \psi / c_{\text{norm}} \leq 1.$$

Hence,

$$G'(|\psi|^2) = -2p (\ln^{1-1/p} (c_{\text{norm}} / |\psi|)) (2p - 2p \ln (c_{\text{norm}} / |\psi|) + 1) \quad (14.35)$$

if $0 < |\psi|/c_{\text{norm}} \leq 1$. If $p > 1$, then $G'(c_{\text{norm}}^2) = 0$ and we can set

$$G'(|\psi|^2) = 0 \text{ if } |\psi| \geq c_{\text{norm}}.$$

So defined, $G'(|\psi|^2)$ is continuous for $\psi > 0$ but is not differentiable (if $p \neq 1$) at the point $\psi = c_{\text{norm}}$ which corresponds to the maximum of $\dot{\psi}_1$ at $r = 0$. The function $G'(|\psi|^2)$ also has a singularity at $\psi = 0$, but the nonlinearity $G'(|\psi|^2) \psi$ as it enters (14.19) is obviously continuous at $\psi = 0$, but it is not differentiable at this point. As we mentioned before, a mild singularity of $G'(|\psi|^2)$ at $\psi = 0$ cannot be avoided, in the case of exponential decay the singularity is logarithmic.

Example 14.3.4 To make the nonlinearity in the previous example differentiable at the maximum $\dot{\psi}_1(0)$ of $\dot{\psi}_1(r)$, we slightly modify expression (14.36) and set

$$\dot{\psi}_1(r) = c_{\text{norm}} e^{-(r^2+1)^p}, \quad p > 0. \quad (14.36)$$

Evidently, $\dot{\psi}_1(r)$ is positive and monotonically decreasing. A direct computation yields for $c_{\text{norm}}/|\psi| > e$ that

$$\begin{aligned} & G'(|\psi|^2) \\ &= -2p \left((2p+1) \ln^{(p-1)/p} (c_{\text{norm}}/|\psi|) - 2p \ln^{(2p-1)/p} (c_{\text{norm}}/|\psi|) \right) \\ & \quad - 2p \left(2p \ln^{(2p-2)/p} (c_{\text{norm}}/|\psi|) + (2-2p) \ln^{(p-2)/p} (c_{\text{norm}}/|\psi|) \right). \end{aligned} \quad (14.37)$$

In particular, for $p = 1/2$, we get an exponentially decaying form factor $\dot{\psi}_1$ of the form (14.100)

$$\dot{\psi}_1(r) = c_{\text{norm}} e^{-(r^2+1)^{1/2}},$$

and the nonlinearity $G'_1(s) = G'_{\nabla,1}(s)$ for $s \leq c_{\text{norm}}^2 e^{-2}$ is given by (14.104). We can use the same formula (14.37) for larger $|\dot{\psi}|$ if we replace $\ln$ by $|\ln|$, or we can extend it for larger s as follows:

$$G'_1(s) = G'_1(c_{\text{norm}}^2 e^{-2}) = -3 \text{ if } s \geq 2c_{\text{norm}}^2 e^{-2},$$

and in the interval $c_{\text{norm}}^2 e^{-2} \leq s \leq 2c_{\text{norm}}^2 e^{-2}$ we interpolate to obtain a smooth function for $s > 0$. The function $G'_1(s)$ is not differentiable at $s = 0$. At the same time, the function $g(\psi) = G'_1(\psi) \psi$ is continuous if we set $g(0) = 0$, and $g(\psi)$ is continuously differentiable with respect to ψ at zero, hence $g(\psi)$ satisfies the Lipschitz condition. The variance of the exponential form factor $\dot{\psi}_1(r)$ is obviously finite. To find $G'_a(s)$ for arbitrary a , we use its representation (14.27).

Example 14.3.5 In this example we define a *Gaussian* form factor by the formula

$$\dot{\psi}(r) = C_g e^{-r^2/2}, \quad C_g = \frac{1}{\pi^{3/4}}. \quad (14.38)$$

Such a ground state is called a *gausson* in [40]. Elementary computation shows that

$$\frac{\nabla^2 \psi(r)}{\psi(r)} = r^2 - 3 = -\ln(\psi^2(r)/C_g^2) - 3.$$

Consequently, we can define the nonlinearity by the formula

$$G'(|\psi|^2) = -\ln(|\psi|^2/C_g^2) - 3, \quad (14.39)$$

and we refer to it as a *logarithmic nonlinearity*. The corresponding nonlinear potential function (antiderivative) has the form

$$G(s) = \int_0^s (-\ln(s'/C_g^2) - 3) ds' = -s \ln s + s \left(\ln \frac{1}{\pi^{3/2}} - 2 \right). \quad (14.40)$$

The dependence on the size parameter $a > 0$ is given by the formula

$$G'_a(|\psi|^2) = -a^{-2} \ln(a^3 |\psi|^2 / C_g^2) - 3a^{-2} \quad (14.41)$$

with the antiderivative

$$G(s) = G_a(s) = -a^{-2} s [\ln(a^3 s) + \ln \pi^{3/2} + 2], \quad s \geq 0. \quad (14.42)$$

The function $g(\psi) = G'_1(|\psi|^2) \psi$ which enters the NLS equation (12.14) and the equilibrium equation (14.19) is continuous for all $\psi \in \mathbb{C}$ if at zero we set $g(0) = 0$, and it is differentiable for every $\psi \neq 0$, but it is not differentiable at $\psi = 0$ and does not satisfy the Lipschitz condition. The form factor dependence on the size parameter is as follows

$$\psi_a(r) = \frac{1}{\pi^{3/4}} a^{-3/2} e^{-r^2/2a^2}. \quad (14.43)$$

14.4 Relativistic Time-Harmonic States of a Balanced Charge

Relativistic charges and EM fields are described, respectively, by the NKG equation (34.128) and the Maxwell equations (12.9), (12.10). The relativistic charge at rest has to be localized and it has to have a time-independent electric field and zero magnetic field. Since the EM fields satisfy Eqs. (12.9), (12.10), the charge density ρ for a resting charge should be time-independent and the current $\mathbf{J}$ should be zero.

For a time-independent charge and zero current, we derive according to (11.130) and (11.131) the following equations for ψ :

$$\operatorname{Im} \frac{\partial_t \psi}{\psi} = \partial_t \arg \psi \text{ is time-independent,}$$

and

$$\operatorname{Im} \frac{\nabla \psi}{\psi} = \nabla \arg \psi = 0.$$

These two conditions imply that

$$\arg \psi = -\omega t,$$

and ψ is a time-harmonic solution to the NKG equation (34.128) and has the form of a standing wave

$$\psi = e^{-i\omega t} \check{\psi}(x) \quad (14.44)$$

where $\check{\psi}(x)$ is a real-valued function (a form factor). We refer to solutions of the form (14.44) as *time-harmonic states of a balanced charge*.

Substitution of (14.44) into the NKG equation (11.125) yields the following *non-linear eigenvalue problem*:

$$-\nabla^2 \check{\psi} + G'_a \left(|\check{\psi}|^2 \right) \check{\psi} = \left(\frac{\omega^2}{c^2} - \frac{\omega_0^2}{c^2} \right) \check{\psi}. \quad (14.45)$$

We refer to such a solution as ω -static. The solution $\check{\psi}$ must also satisfy the charge normalization condition (17.111), see Sect. 14.2, (17.6.1.1) for details. According to (11.132), (14.1) and (14.44), the normalization condition takes the form

$$q = \int_{\mathbb{R}^3} \rho(t, \mathbf{x}) \, d\mathbf{x} = -\omega \frac{\chi q}{mc^2} \int_{\mathbb{R}^3} |\psi|^2 \, d\mathbf{x}, \quad (14.46)$$

resulting in the constraint

$$\int |\check{\psi}|^2 \, dx = \frac{\omega_0}{\omega}, \quad \omega_0 = c\kappa_0 = \frac{mc^2}{\chi}. \quad (14.47)$$

If the nonlinearity satisfies the equilibrium condition (14.19), the nonlinear eigenvalue problem (14.45) with the constraint (14.47) has a particular solution

$$\omega = \omega_0, \quad \check{\psi} = \check{\psi}, \quad (14.48)$$

where the ground state $\check{\psi}$ is a solution to Eq. (14.19). Then the rest solution (ground state) has the form

$$\check{\psi}(t, \mathbf{x}) = e^{-i\omega_0 t} \check{\psi}(\mathbf{x}), \quad \omega_0 = \frac{mc^2}{\chi} = c\kappa_0. \quad (14.49)$$

We refer to $\check{\psi}(\mathbf{x})$ as the *form factor*

The energy defined by (11.142), (10.70) yields for the standing wave in (14.44) the following expression

$$\bar{\mathcal{E}} = \frac{\chi^2}{2m} \int_{\mathbb{R}^3} \left[\frac{1}{c^2} \omega^2 \check{\psi} \check{\psi}^* + \kappa_0^2 \check{\psi} \check{\psi}^* + \nabla \check{\psi} \nabla \check{\psi}^* + G_a(\check{\psi} \check{\psi}^*) \right] d\mathbf{x}. \quad (14.50)$$

The problem (14.45) with the constraint (14.47) can be reduced to a problem with the following fixed constraint:

$$\int_{\mathbb{R}^3} |\check{\psi}_1|^2 d\mathbf{x} = 1, \quad (14.51)$$

where we used the substitution

$$\check{\psi} = \frac{\omega_0}{\omega} \check{\psi}_1. \quad (14.52)$$

As a result, problem (14.45) can be recast as the following eigenvalue problem:

$$-\nabla^2 \check{\psi}_1 + G'_a \left(\frac{\omega_0^2}{\omega^2} |\check{\psi}_1|^2 \right) \check{\psi}_1 = \xi \check{\psi}_1, \quad (14.53)$$

where $\check{\psi}_1$ satisfies the normalization constraint (14.51), and the eigenvalue ξ relates to the frequency ω by the equation

$$\xi = \frac{\omega^2}{c^2} - \frac{\omega_0^2}{c^2}. \quad (14.54)$$

Notice that solutions to Eqs. (14.53), (14.51) are critical points of the following energy functional:

$$\mathcal{E}(\psi) = \frac{\chi^2}{2m} \int_{\mathbb{R}^3} |\nabla \psi|^2 + G(|\psi|^2) d\mathbf{x} \quad (14.55)$$

subject to the constraint (14.51).

An analysis similar to that in Sect. 35.2.2 (see [38], [39], [41], [58]) shows that the problem (14.45), (14.47) has a sequence of solutions with the corresponding sequence of frequencies ω . Their energies $\bar{\mathcal{E}}_{0\omega}$ are related to the frequency $\omega > 0$ by the formula

$$\bar{\mathcal{E}}_{0\omega} = \chi \omega (1 + \Theta(\omega)), \quad (14.56)$$

$$\Theta(\omega) = \Theta_0 \frac{a_C^2 \omega_0^2}{a^2 \omega^2}, \quad a_C = \frac{\chi}{mc}, \quad (14.57)$$

where

$$\Theta_0 = \frac{1}{3} \left\| \nabla \psi_1 \right\|^2. \quad (14.58)$$

Observe that the coefficient Θ_0 depends on the shape of the rest charge, and the parameter

$$a_C = \frac{\chi}{mc} = \frac{\lambda_C}{2\pi} \quad (14.59)$$

coincides with the *reduced Compton wavelength* of a particle with mass m if $\chi = \hbar$. The evaluation of the energy in (14.56) is based on the Pokhozhaev formula (14.175). In the case of the logarithmic nonlinearity G defined by (14.42), the ground state corresponds to $\omega = \omega_0$. This ground state has the Gaussian shape given by (14.38), it provides for the minimum of energy (14.55), and in this case the coefficient $\Theta_0 = 1/2$.

Notice that ω_0 is the minimal frequency and $\omega \geq \omega_0$. We also assume that $a \gg a_C$ since, according to our Hydrogen model analysis in [14]–[17] and in Sects. 17.5, 35.2, a reasonable value a for an electron satisfies $a_C^2/a^2 \lesssim 10^{-8}$. We can see then that for such values of the size parameter a relation (14.56) differs only slightly from the Planck–Einstein energy-frequency relation $E = \hbar\omega$. Note that in the non-relativistic version of our theory in Sect. 17.4.1, the relation $\Delta E = \hbar\Delta\omega$ is an exact identity for many systems, in particular for hydrogenic atoms.

Using the Lorentz invariance of the system, one can easily obtain a solution to the NKG equation (11.125) representing the charge moving with a constant velocity $\mathbf{v}$ by simply applying the Lorentz transformation to a rest solution of the problem (14.45) with constraint (14.47) (see Sect. 15.2).

14.4.1 Time-Harmonic States for the Logarithmic Nonlinearity

In the case of the logarithmic nonlinearity G defined by (14.42), the problem (14.45), (14.47) has a sequence of solutions with the corresponding sequence of frequencies ω . Their energies $E_{0\omega}$ are related to the frequency ω by the formula (14.56):

$$E_{0\omega} = \chi \omega (1 + \Theta(\omega)), \quad \Theta(\omega) = \frac{a_C^2 \omega_0^2}{2a^2 \omega^2} \quad (14.60)$$

where a_C given by (14.59) is the reduced Compton wavelength of a particle with mass m . The Gaussian wave function

$$\psi(t, \mathbf{x}) = e^{-i\omega_0 t} a^{-3/2} \pi^{-3/4} e^{-|\mathbf{x}|^2/2a^2} \quad (14.61)$$

with $\omega = \omega_0$ is the *ground state* to the problem. This state has the minimal possible energy among all functions satisfying (14.47), hence it is stable.

Notice that, for the logarithmic nonlinearity (14.42), the nonlinear eigenvalue problem (14.53) subject to the normalization constraint (14.51) takes the form

$$-\nabla^2 \check{\psi}_1 + G'_1 \left(|\check{\psi}_1|^2 \right) \check{\psi}_1 = \xi \check{\psi}_1, \quad \int_{\mathbb{R}^3} |\check{\psi}_1|^2 d\mathbf{x} = 1, \quad (14.62)$$

where the parameter ξ is related to the frequency ω by the formula

$$\xi = \frac{a^2}{a_C^2} \left(\frac{\omega^2}{\omega_0^2} - 1 \right) - \frac{1}{2} \ln \frac{\omega^2}{\omega_0^2}. \quad (14.63)$$

The eigenvalue problem (14.62) with the logarithmic nonlinearity has infinitely many solutions (ξ_n, ψ_{1n}) , $n = 0, 1, 2, \dots$, representing localized charge distributions. The energy of ψ_n , $n > 0$, is higher than the energy of the Gaussian ground state which corresponds to $\xi = \xi_0 = 0$ and has the lowest possible energy. These solutions coincide with critical points of the energy functional under the normalization constraint. For mathematical details, see [58], [38], [39]. The next two values of ξ for the radial rest states are approximately 2.17 and 3.41 according to [41].

Putting in (14.63) the values $\xi = \xi_n$, we can find the corresponding values of $\frac{\omega_0}{\omega}$ yielding for $a^2 \gg a_C^2$ the following approximate formula:

$$\frac{\omega_0}{\omega} \simeq 1 - \xi \frac{a_C^2}{2a^2}. \quad (14.64)$$

The difference of the energy of the higher states and the ground state energy is small. If $\chi = \hbar = h/2\pi$, it is of order $\hbar\omega_0 \xi \frac{a_C^2}{2a^2} = \xi \frac{a_B^2}{a^2} \hbar c R_\infty$ where $\hbar c R_\infty = mc^2 \frac{\alpha^2}{2}$ is the Rydberg energy. Since we assume that $a_B^2/a^2 \lesssim 10^{-4}$, the difference is comparable with the effect of the fine structure in the Hydrogen atom which is of order $mc^2 \alpha^4$. This comparison suggests that the cohesive forces associated with the nonlinearity are relatively small.

Remark 14.4.1 The rest states of higher energies corresponding to $\xi = \xi_n$ with $n > 0$ are expected to be unstable. Indeed, since charges are coupled via electromagnetic interactions, there is always a possibility of the energy transfer from charges with higher energies into the EM fields, making such states improbable in normal circumstances. It is conceivable though that, in a system consisting of a very large number of strongly interacting charges, a significant quantity of such states may be present.

14.4.2 Electric Potential Proximity to the Coulomb's Potential

If the ground state charge density $|\dot{\psi}_a|^2$ is replaced by Dirac's delta-function, the corresponding electrostatic potential $\dot{\phi}_a$ given by (14.30) coincides with Coulomb's potential $q/|\mathbf{x}|$. In this subsection we study the proximity of the potential $\dot{\phi}_a(|\mathbf{x}|)$ given by (14.30) to Coulomb's potential for small a . This is an important issue since it is a well known experimental fact [344] that Coulomb's potential $q/|\mathbf{x}|$ represents the electrostatic field of the charge very accurately even for very small values of $|\mathbf{x}|$.

According to (14.30), the dependence of the potential $\dot{\phi}_a(r) = \dot{\phi}_a(r)/q$ on the size parameter a is of the form

$$\dot{\phi}_a(r) = a^{-1} \dot{\phi}_1(a^{-1}r). \quad (14.65)$$

Consequently, its behavior for small a is determined by the behavior of $\phi_1(r)$ for large r . Rather than study the integral (14.30) which gives the solution of the Poisson equation, we study the Poisson equation for radial solutions directly. Using spherical coordinates, we consider the radial solution $\zeta(r) = r\dot{\phi}_1(r)$ to the Poisson equation

$$\frac{1}{r} \left(\frac{d}{dr} \right)^2 \zeta(r) = -4\pi \dot{\psi}_1^2(r), \quad r \geq 0. \quad (14.66)$$

We seek such a solution $\zeta(r)$ to the above equation that the corresponding potential $\phi_1(r)$ is close to Coulomb's potential $1/r$, namely it satisfies the following condition

$$\zeta(r) = r\phi_1(r) \rightarrow 1 \text{ as } r \rightarrow \infty. \quad (14.67)$$

We integrate equation (14.66) twice, and, taking into account (14.67), obtain that

$$\begin{aligned} \zeta(r) &= 1 - 4\pi \int_r^\infty \int_{r_2}^\infty r_1 \dot{\psi}_1^2(r_1) \, dr_1 dr_2 \\ &= 1 - 4\pi \int_r^\infty (r_1 - r) r_1 \dot{\psi}_1^2(r_1) \, dr_1. \end{aligned} \quad (14.68)$$

The second equality in (14.68) is obtained by rewriting the preceding repeated integral as a double integral and changing the order of integration, namely

$$\begin{aligned} \int_r^\infty \int_{r_2}^\infty r_1 \dot{\psi}_1^2(r_1) \, dr_1 dr_2 &= \int_r^\infty \int_r^{r_1} r_1 \dot{\psi}_1^2(r_1) \, dr_2 dr_1 \\ &= \int_r^\infty (r_1 - r) r_1 \dot{\psi}_1^2(r_1) \, dr_1. \end{aligned}$$

In view of the charge normalization condition (14.82), we readily obtain from (14.68) that

$$\zeta(0) = 1 - 4\pi \int_0^\infty r_1^2 \dot{\psi}_1^2(r_1) dr_1 = 1 - \int_{\mathbb{R}^3} \dot{\psi}_1^2(|\mathbf{x}|) d\mathbf{x} = 0. \quad (14.69)$$

The representation (14.68) for $\zeta(r) = r\phi_1(r)$ readily implies the following representation for the potential $\phi_1(r)$:

$$\phi_1(r) = \frac{1}{r} \left[1 - 4\pi \int_r^\infty (r_1 - r) r_1 \dot{\psi}_1^2(r_1) dr_1 \right] = \frac{1}{r} (1 + D'_C). \quad (14.70)$$

Combining (14.70) with (14.69), we conclude that $\phi_1(r)$ is regular for small $r \geq 0$. Using (14.70) once more, we obtain the following expression for the difference $D_C = \frac{1}{r} D'_C$ between $\phi_1(r)$ and $1/r$:

$$D_C = \frac{1}{r} D'_C(\phi_1) = \phi_1(r) - \frac{1}{r} = -\frac{4\pi}{r} \int_r^\infty (r_1 - r) r_1 \dot{\psi}_1^2(r_1) dr_1. \quad (14.71)$$

The relation (14.71) together with (14.65) implies that

$$D'_C(\phi_a) = r\phi_a(r) - 1 = -4\pi \int_{a^{-1}r}^\infty (r_1 - a^{-1}r) r_1 \dot{\psi}_1^2(r_1) dr_1, \quad (14.72)$$

showing in particular that the difference D_C becomes small for small a if $\dot{\psi}_1^2$ decays fast at infinity. Obviously,

$$D_C(\phi_a) \leq 0,$$

and

$$\begin{aligned} 4\pi \int_{a^{-1}r}^\infty (r_1 - a^{-1}r)^2 \dot{\psi}_1^2(r_1) dr_1 &\leq |D'_C(\phi_a)| \\ &\leq 4\pi \int_{a^{-1}r}^\infty r_1^2 \dot{\psi}_1^2(r_1) dr_1. \end{aligned} \quad (14.73)$$

If $\dot{\psi}_1^2$ decays exponentially as in (14.100), then directly from (14.72) we obtain

$$\begin{aligned} |D'_C(\phi_a)(r)| &= 4\pi \int_{a^{-1}r}^\infty (r_1 - a^{-1}r) r_1 c_e^2 e^{-2(r_1^2+1)^{1/2}} dr_1 \\ &= 4\pi \int_0^\infty (r_2 + a^{-1}r) r_2 c_e^2 e^{-2((r_2+a^{-1}r)^2+1)^{1/2}} dr_2 \\ &\leq 4\pi c_e^2 e^{-2a^{-1}r} \int_0^\infty (r_2 + a^{-1}r) r_2 e^{-2r_2} dr_2 = \pi c_e^2 (a^{-1}r + 1) e^{-2a^{-1}r}. \end{aligned} \quad (14.74)$$

In particular for $r/a \geq 10$, the correction coefficient D'_C of Coulomb's potential $1/r$ is very small:

$$|D_C(\phi_a)(r)| = \pi c_e^2 (a^{-1}r + 1) e^{-2a^{-1}r} \lesssim 5 \times 10^{-8}.$$

Notice that if we would take $\psi_1(r) = 0$ for all $r \geq r_0$, as in the Abraham–Lorentz model, then according to formula (14.70) $\phi_a(r)$ would exactly coincide with Coulomb's potential for $r \geq ar_0$. But for such a $\psi_1(r)$ we would not be able to construct the nonlinear self-interaction component G' which would satisfy (14.20) since it requires $\psi^2(r)$ to be strictly positive for all $r \geq 0$.

14.5 The Rest State of a Non-relativistic Dressed Charge

A dressed charge is constituted of the charge distribution ψ and the corresponding electromagnetic potential φ , and so we refer to the pair $\{\psi, \varphi\}$ as a *dressed charge*. The dressed charge is free when there is no external EM field, and it is described by the NLS-Poisson equations (13.27), (13.28) of the form

$$\chi i \partial_t \psi - q \varphi \psi = -\frac{\chi^2}{2m} [\nabla^2 \psi - G'(\psi^* \psi) \psi], \quad (14.75)$$

$$-\nabla^2 \varphi = 4\pi q \psi \psi^*. \quad (14.76)$$

The resting charge is described by a static, time-independent solution to the Eqs. (14.75)–(14.76). These equations turn into the following charge equilibrium equations for a rest state described by time-independent real-valued radial functions $\psi = \psi(|\mathbf{x}|)$ and $\varphi = \varphi(|\mathbf{x}|)$:

$$-\nabla^2 \psi + \frac{2m}{\chi^2} q \varphi \psi + G'(|\psi|^2) \psi = 0, \quad (14.77)$$

$$-\nabla^2 \varphi = 4\pi q |\psi|^2. \quad (14.78)$$

Here ψ is a ground state, namely the function $\psi = \psi(r)$, $r = |x|$, is a *positive, monotonically decreasing function* which is twice differentiable and satisfies (14.18). We refer to the quantities ψ and φ respectively as the *charge form factor* and *form factor potential*.

In view of the Eq. (14.78), the charge form factor ψ determines the form factor potential $\varphi(|\mathbf{x}|) = \varphi_\psi(|\mathbf{x}|)$ by the formula (14.11), and if we plug the expression into Eq. (14.77), we get the following charge equilibrium equation:

$$-\nabla^2 \dot{\psi} + \frac{2mq}{\chi^2} \dot{\phi} \dot{\psi} + G'(|\dot{\psi}|^2) \dot{\psi} = 0. \quad (14.79)$$

Similarly to the construction for balanced charges in Sect. 14.3, we use this equation to find the nonlinearity $G'(|\dot{\psi}|^2)$ based on the ground state $\dot{\psi}$. The Eq. (14.79) can be interpreted as a complete balance of three forces acting upon the resting charge: (i) internal elastic deformation force associated with the term $-\nabla^2 \dot{\psi}$; (ii) the charge's electromagnetic self-interaction force associated with the term $\frac{2mq}{\chi^2} \dot{\phi} \dot{\psi}$; (iii) internal nonlinear self-interaction of the charge associated with the term $G'(|\dot{\psi}|^2) \dot{\psi}$. We refer to the Eq. (14.79), which establishes an explicit relation between the form factor $\dot{\psi}$ and the self-interaction nonlinearity G , as the *charge equilibrium equation*. Thus, we pick the form factor $\dot{\psi}$, and then the nonlinear self-interaction function G is determined based on the charge equilibrium equation (14.79). Note that after the nonlinearity G is determined, it is fixed forever, and while solutions of Eqs. (14.75)–(14.76) may evolve in time, they do not need to coincide with $\{\dot{\psi}, \dot{\phi}\}$. Details and examples of the construction of the nonlinear self-interaction function G based on the form factor are provided in the following sections.

The charge density takes the form (13.22) which coincides with (12.15) and the *charge normalization condition* which ensures that $\dot{\phi}(|\mathbf{x}|)$ is close to Coulomb's potential with the charge q for large $|\mathbf{x}|$ takes the same form (14.17) as for non-relativistic balanced charges.

14.5.1 Nonlinear Self-Interaction of a Dressed Charge and Its Basic Properties

As we have already explained, the *nonlinear self-interaction function* G' is determined from the *charge equilibrium equation* (14.79) based on the form factor $\dot{\psi}$. In this section we consider the construction of the function G' , study its properties and provide examples for which the construction of G' is carried out explicitly. Throughout this section we have

$$\psi, \dot{\psi} \geq 0, \text{ hence } |\psi| = \psi.$$

We introduce explicitly the *size parameter* $a > 0$ by formula (14.24) for the charge form factor $\dot{\psi}(r)$ and formula (14.30) for the form factor potential $\dot{\phi}(r)$:

$$\dot{\psi}(r) = \dot{\psi}_a(r) = a^{-3/2} \dot{\psi}_1(a^{-1}r), \quad (14.80)$$

$$\dot{\phi}(r) = \dot{\phi}_a(r) = a^{-1} \dot{\phi}_1(a^{-1}r). \quad (14.81)$$

Here $\dot{\psi}_1(r)$ and $\dot{\phi}_1(r)$ are functions of the dimensionless parameter r ; as a consequence of (14.17), the function $\dot{\psi}_a(r)$ satisfies the charge normalization condition

$$\int_{\mathbb{R}^3} \dot{\psi}_a^2(|\mathbf{x}|) d\mathbf{x} = 1 \text{ for all } a > 0. \quad (14.82)$$

The charge equilibrium equation (14.79) can be written in the following form:

$$\frac{\chi^2}{2m} G'_a \left(|\dot{\psi}_a|^2 \right) \dot{\psi}_a = \frac{\chi^2}{2m} \nabla^2 \dot{\psi}_a - q \dot{\phi}_a \dot{\psi}_a, \quad (14.83)$$

$$\nabla^2 \dot{\phi}_a = -4\pi q |\dot{\psi}_a|^2. \quad (14.84)$$

As in Sect. 14.3, the ground state $\dot{\psi}_a(r)$, $r \geq 0$ is a positive, monotonically decreasing function of r which satisfies the charge normalization condition (14.82). Since $\dot{\psi}_a(|\mathbf{x}|)$ and $\dot{\phi}_a(|\mathbf{x}|)$ are radial functions, to solve the Eq. (14.84) for $\dot{\phi}_a$ we can use formula (14.70) which determines $\dot{\phi}_1(r) = q\phi_1(r)$ in terms of $\dot{\psi}_1^2(r)$. Obviously, if $\dot{\psi}_1^2(r)$ decays sufficiently fast as $r \rightarrow \infty$ and a is sufficiently small, then the potential $\dot{\phi}_a(r)$ is very close to the Coulomb potential q/r , as we show in Sect. 14.4.2.

Now let us consider the construction of the nonlinearity. It is done for dressed charges similarly to the construction of the nonlinearity for balanced charges in Sect. 14.3, but there are some differences. Let us take a look first at the case $a = 1$, $\dot{\psi}_a = \dot{\psi}_1$, $\dot{\phi}_a = \dot{\phi}_1$, for which the Eq. (14.83) yields the following representation for $G'(\dot{\psi}_1^2)$ from (14.83):

$$G'(\dot{\psi}_1^2(r)) = \frac{(\nabla^2 \dot{\psi}_1)(r)}{\dot{\psi}_1(r)} - \frac{2m}{\chi^2} q \dot{\phi}_1(r).$$

Since $\dot{\psi}_1^2(r)$ is a monotonic function, we can find its inverse $r = r(\dot{\psi}_1^2)$ yielding

$$G'(s) = \left[\frac{\nabla^2 \dot{\psi}_1}{\dot{\psi}_1} - \frac{2m}{\chi^2} q \dot{\phi}_1 \right] (r(s)), \quad 0 = \dot{\psi}_1^2(\infty) \leq s \leq \dot{\psi}_1^2(0). \quad (14.85)$$

We can then extend $G'(s)$ for $s \geq \dot{\psi}_1^2(0)$ to be a constant, namely

$$G'(s) = G'(\dot{\psi}_1^2(\infty)) \text{ if } s \geq \dot{\psi}_1^2(\infty). \quad (14.86)$$

Observe that the positivity and the monotonicity of the form factor $\dot{\psi}_1$ is instrumental for recovering the function $G'(s)$ from the charge balance equation (14.83).

The representation (14.85) for the nonlinearity $G'(s)$ naturally defines its two components:

$$G'(s) = G'_{\nabla}(s) - \frac{2}{a_B} G'_{\varphi}(s) \text{ where } a_B = \frac{\chi^2}{mq^2}. \quad (14.87)$$

The parameter a_B in the case of an electron coincides with the Bohr radius. The two components balance, respectively, the Laplacian and electrostatic potential in (14.83):

$$G'_{\nabla}(\dot{\psi}_1^2) = \frac{\nabla^2 \dot{\psi}_1}{\dot{\psi}_1}, \quad (14.88)$$

$$G'_{\varphi}(\dot{\psi}_1^2) = \frac{\dot{\varphi}_1}{q}. \quad (14.89)$$

We refer to $G'_{\nabla}(s)$ and $G'_{\varphi}(s)$, respectively, as the *elastic and EM nonlinearity components*. Note that the expression (14.88) for the elastic component G'_{∇} coincides with (14.20). In the case of an arbitrary size parameter a , we observe that the dependence on a is as follows:

$$G'_{\nabla,a}(s) = a^{-2} G'_{\nabla,1}(a^3 s), \quad G'_{\varphi,a}(s) = a^{-1} G'_{\varphi,1}(a^3 s), \quad a > 0. \quad (14.90)$$

Then, combining (14.90) with (14.87) and (14.88), we obtain the following representation for the function $G'_a(s)$:

$$G'_a(s) = \frac{1}{a^2} G'_{\nabla,1}(a^3 s) - \frac{2}{aa_B} G'_{\varphi,1}(a^3 s). \quad (14.91)$$

Let us take a look at general properties of $G'(s)$ and its components $G'_{\nabla}(s)$ and $G'_{\varphi}(s)$ as they follow from the defining relations (14.85)–(14.91). Starting with the EM component $G'_{\varphi}(s)$, we notice that $\dot{\varphi}_1(|\mathbf{x}|)$ is a radial solution to the Eq. (14.84). Using the Maximum principle, we conclude that $\dot{\varphi}_1(|\mathbf{x}|)/q$ is a positive function without local minima, implying that it is a monotonically decreasing function of $|\mathbf{x}|$. Consequently, $G'_{\varphi}(s)$ defined by (14.88) is a monotonically increasing function of s , hence

$$G'_{\varphi}(s) > 0 \text{ for all } \dot{\psi}^2(0) \geq s > 0, \text{ and } G'_{\varphi}(0) = 0. \quad (14.92)$$

Note that $G'_{\varphi}(s)$ is not differentiable at zero, and that can be seen by comparing the behavior of $\dot{\varphi}_1(r)$ and $\dot{\psi}_1(r)$ at infinity. Indeed, $\varphi_1(r)/q \sim r^{-1}$ as $r \rightarrow \infty$ and since $\dot{\psi}^2(|\mathbf{x}|)$ is integrable, it has to decay faster than $|\mathbf{x}|^{-3}$ as $|\mathbf{x}| \rightarrow \infty$. Consequently, $|G'_{\varphi}(s)|$ for small s has to be greater than $s^{1/3}$, and it prohibits differentiability at zero.

The basic properties of the elastic component $G'_{\nabla}(s)$ defined by the relations (14.88) are described in Sect. 14.3, in particular

$$G'_{\nabla}(s) \leq 0 \text{ for } s = \dot{\psi}_1^2(0). \quad (14.93)$$

From the relations (14.87), (14.92) and (14.93) we obtain that

$$G'(s) < 0 \text{ if } s = \dot{\psi}_1^2(0). \quad (14.94)$$

14.5.2 Examples of Nonlinearities for a Dressed Charge

In this section we provide examples of the form factor $\dot{\psi}$ for which the form factor potential $\dot{\phi}$ and the corresponding nonlinear self-interaction function G can be constructed explicitly. Since the expression (14.88) coincides with (14.20), we can use the examples given in Sect. 14.3.2 to present the elastic component G'_{∇} . The EM nonlinearity component G'_{φ} can then be determined by formula (14.89) and the integral formula (14.70) for a radial solution $\phi = \dot{\phi}/q$ of the Poisson equation (14.84) with given $|\psi|^2 = |\dot{\psi}|^2$.

The first example is for the ground state $\dot{\psi}(r)$ decaying at a power law as $r \rightarrow \infty$. In this case both the $\dot{\phi}$ and G are represented by simple explicit formulas, but some properties of these functions are not as appealing. Namely, the variance of the function $\dot{\psi}$ is infinite and the rate of approximation of the exact Coulomb potential by $\dot{\phi}_a(\mathbf{x})$ for small a is not very fast. The second example is the form factor $\dot{\psi}(r)$ decaying exponentially as $r \rightarrow \infty$. In this case representations for $\dot{\phi}_0$ and G are more involved compared with the power law form factor, but all the properties of $\dot{\psi}$ and $\dot{\phi}$ are satisfactory in any regard. The third example is the Gaussian ground state.

14.5.2.1 Nonlinearity for a Ground State Decaying by a Power Law

Example 14.5.1 We consider here the ground state to be of the form (14.31). Since the expression (14.88) for the elastic component G'_{∇} coincides with (14.20), G'_{∇} is given by Eq. (14.32). Let us take a value $p = 5/4$ in (14.31) to obtain a simple presentation of $\dot{\phi}$ and G'_{φ} . Namely, we take the ground state $\dot{\psi}_1(r)$ in the form

$$\dot{\psi}_1(r) = \frac{c_{\text{norm}}}{(1+r^2)^{5/4}}, \quad c_{\text{norm}} = \frac{3^{1/2}}{(4\pi)^{1/2}}. \quad (14.95)$$

This function evidently is positive and monotonically decreasing, and $0 < \dot{\psi}_1(r) \leq c_{\text{norm}}$. Let us find now $G'_{\nabla}(s)$ and G'_{φ} using relations (14.88). An elementary computation shows that

$$\nabla^2 \dot{\psi}_1 = \frac{15}{4c_{\text{norm}}^{4/5}} \left(1 - \frac{3}{c_{\text{norm}}^{4/5}} \dot{\psi}_1^{4/5} \right) \dot{\psi}_1^{1+4/5},$$

implying

$$G'_{\nabla}(s) = \frac{15s^{2/5}}{4c_{\text{norm}}^{4/5}} - \frac{45s^{4/5}}{4c_{\text{norm}}^{8/5}}, \quad G_{\nabla}(s) = \frac{75s^{7/5}}{28c_{\text{norm}}^{4/5}} - \frac{25s^{9/5}}{4c_{\text{norm}}^{8/5}}, \quad (14.96)$$

for $0 \leq s \leq c_{\text{norm}}^2$. To determine G'_{φ} , we find by a straightforward examination that the function

$$\dot{\phi}_1 = q\dot{\phi}_1 = \frac{q}{(1+r^2)^{1/2}} = \frac{q}{c_{\text{norm}}^{2/5}} \dot{\psi}_1^{2/5} \quad (14.97)$$

solves the Poisson equation $\nabla^2 \dot{\phi}_1 = -4\pi q \dot{\psi}_1^2$. Together with (14.88) this yields

$$G'_{\varphi}(s) = \frac{s^{1/5}}{c_{\text{norm}}^{2/5}}, \quad G_{\varphi}(s) = \frac{5s^{6/5}}{6c_{\text{norm}}^{2/5}} \quad \text{for } 0 \leq s \leq c_{\text{norm}}^2. \quad (14.98)$$

Observe that both the components $G'_{\nabla}(s)$ and $G'_{\varphi}(s)$ in (14.96), (14.98) of the nonlinearity $G'(s)$ defined by (14.87) are *not differentiable* at $s = 0$. Notice that the variance of the form factor $\dot{\psi}_1^2(|\mathbf{x}|)$ decaying as a power law (14.95) is infinite, i.e.

$$\int_{\mathbb{R}^3} |\mathbf{x}|^2 \dot{\psi}_1^2(|\mathbf{x}|) \, d\mathbf{x} = 4\pi \int_0^\infty \frac{c_{\text{norm}}^2}{(1+r^2)^{5/2}} r^4 \, dr = \infty. \quad (14.99)$$

14.5.2.2 Exponentially Decaying and Gaussian Ground States

Example 14.5.2 We introduce here an exponentially decaying ground state $\dot{\psi}_1$ of the form (14.36) with the particular value $p = 1/2$:

$$\dot{\psi}_1(r) = c_e e^{-(r^2+1)^{1/2}} \quad (14.100)$$

where

$$c_e = \left(4\pi \int_0^\infty r^2 e^{-2(r^2+1)^{1/2}} \, dr \right)^{-1/2} \simeq 0.79.$$

Obviously, this function is positive and monotonically decreasing. The dependence $r(s)$ defined by relation (14.100) is as follows:

$$r(s) = [\ln^2(c_e/\sqrt{s}) - 1]^{1/2}, \quad \text{if } \sqrt{s} \leq \dot{\psi}_1(0) = c_e e^{-1}. \quad (14.101)$$

An elementary computation shows that

$$\nabla^2 \mathring{\psi}_1 = -W \mathring{\psi}_1 \quad (14.102)$$

where

$$W = \frac{2}{(r^2 + 1)^{\frac{1}{2}}} + \frac{1}{(r^2 + 1)} + \frac{1}{(r^2 + 1)^{\frac{3}{2}}} - 1,$$

implying

$$G'_{\nabla} (\psi_1^2 (r)) = -W (r). \quad (14.103)$$

Combining (14.101) with (14.103), we readily obtain the following function:

$$G'_{\nabla,1} (s) = \left[1 - \frac{4}{\ln (c_e^2/s)} - \frac{4}{\ln^2 (c_e^2/s)} - \frac{8}{\ln^3 (c_e^2/s)} \right] \text{ for } \sqrt{s} \leq c_e e^{-1}, \quad (14.104)$$

this function is evidently a monotonically decreasing one. We extend it for larger s as follows:

$$G'_{\nabla,1} (s) = G'_{\nabla,1} (c_e^2 e^{-2}) = -3 \text{ if } \sqrt{s} \geq c_e e^{-1}. \quad (14.105)$$

The relations (14.104) and (14.105) imply that $G'_{\nabla,1} (s)$ takes values in the interval $[1, -3]$. It also follows from (14.104) that

$$G'_{\nabla,1} (s) \cong 1 - \frac{4}{\ln 1/s} \text{ as } s \rightarrow 0, \quad (14.106)$$

and we can set $G'_{\nabla,1} (0) = 0$ obtaining a continuous function. Obviously, the function $G'_{\nabla,1} (s)$ is *not differentiable* at $s = 0$ and consequently is not analytic. To determine the second component $G'_{\varphi,1} (s)$, we use (14.70) which takes the form

$$\mathring{\phi}_1 (r) = \frac{q}{r} \left[1 - 4\pi c_e^2 \int_r^\infty (r_1 - r) r_1 e^{-2\sqrt{r_1^2 + 1}} dr_1 \right]. \quad (14.107)$$

Then using the relations (14.89) and (14.101), we find consequently

$$G'_{\varphi,1} (|\psi|^2) = \frac{1}{q} \mathring{\phi}_1 (r (|\psi|)), \text{ for } |\psi| \leq c_e e^{-1}, \quad (14.108)$$

$$G'_{\varphi,1} (|\psi|^2) = \frac{q}{r} \frac{\pi c_e^2}{r (|\psi|)} \int_0^{r (|\psi|)} \left[1 + 2 (r_1^2 + 1)^{1/2} \right] e^{-2(r_1^2 + 1)^{1/2}} dr_1 \quad (14.109)$$

where

$$r (|\psi|) = \left[\ln^2 (c_e / |\psi|) - 1 \right]^{1/2} \text{ for } |\psi| \leq c_e e^{-1}.$$

We extend $G'_{\varphi,1}(|\psi|^2)$ for larger values of ψ as a constant:

$$G'_{\varphi,1}(\psi^2) = \lim_{r \rightarrow 0} \frac{\mathring{\varphi}(r)}{q} = 3\pi c_e^2 e^{-2} \text{ for } \psi \geq c_e e^{-1}. \quad (14.110)$$

To find $G'_a(s)$ for arbitrary a , we use its representation (14.27). The variance of the exponential form factor $\mathring{\psi}_1(r)$ is

$$\int_{\mathbb{R}^3} |\mathbf{x}|^2 \mathring{\psi}_1^2(|\mathbf{x}|) d\mathbf{x} = 4\pi c_e^2 \int_0^\infty r^4 e^{-2(r^2+1)^{1/2}} dr \simeq 3.8268. \quad (14.111)$$

Example 14.5.3 The Gaussian ground state is given by (14.38) and formula (14.39) shows that corresponding nonlinearity $G'_{\nabla,1}(s)$ is logarithmic, $G'_{\nabla,1}(s) = -\ln(|\psi|^2/C_g^2) - 3$. To determine the second component $G'_{\varphi,1}(s)$ of the nonlinearity, we use (14.70):

$$\mathring{\varphi}_1(r) = \frac{q}{r} \left[1 - 4\pi C_g^2 \int_r^\infty (r_1 - r) r_1 e^{-r_1^2} dr_1 \right] = \frac{q}{r} \operatorname{erf}(r), \quad (14.112)$$

where

$$\operatorname{erf}(r) = \frac{2}{\pi^{1/2}} \int_0^r e^{-s^2} ds, \quad C_g = \pi^{-3/4}.$$

Hence,

$$G'_{\varphi,1}(|\psi|^2) = \frac{1}{q} \mathring{\varphi}_1[r(|\psi|)] = \frac{1}{\ln^{1/2}(C_g/|\psi|)} \operatorname{erf}(\ln^{1/2}(C_g/|\psi|)).$$

14.5.3 The Energy Related Spatial Scale

An attractive choice for the spatial scale which describes a resting non-relativistic dressed charge can be obtained based on the requirement that the relative energy $\mathcal{E}'_0(\mathring{\psi})$ of the resting relativistic dressed charge defined by the expression (14.125) be exactly 0. This assumption readily reduces to the requirement

$$\mathcal{E}_1(\mathring{\psi}) = \mathcal{E}_2(\mathring{\varphi}), \quad (14.113)$$

where

$$\mathcal{E}_1(\mathring{\psi}) = \frac{\chi^2}{2m} \int_{\mathbb{R}^3} |\nabla \mathring{\psi}|^2 d\mathbf{x}, \quad \mathcal{E}_2(\mathring{\varphi}) = \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla \mathring{\varphi}|^2 d\mathbf{x}.$$

Plugging $\mathring{\psi} = \mathring{\psi}_a$ and $\mathring{\phi} = \mathring{\phi}_a$ defined by (14.24) into the equalities (14.113), we obtain that

$$\mathcal{E}_1\left(\mathring{\psi}_a\right) = a^{-2}\mathcal{E}_1\left(\mathring{\psi}_1\right), \quad \mathcal{E}_2\left(\mathring{\phi}_a\right) = a^{-1}q^2\mathcal{E}_2\left(\mathring{\phi}_1\right) \quad (14.114)$$

where $\mathring{\phi}_1 = q^{-1}\mathring{\phi}_1$. Hence in view of the relations (14.114), the requirement $\mathcal{E}_1\left(\mathring{\psi}\right) = \mathcal{E}_2\left(\mathring{\phi}\right)$ is equivalent to the following choice $a = a_\psi$ of the size parameter a :

$$a_\psi = \frac{\mathcal{E}_1\left(\mathring{\psi}_1\right)}{\mathcal{E}_2\left(\mathring{\phi}_1\right)} = \frac{4\pi\chi^2}{m} \frac{\int_{\mathbb{R}^3} \left|\nabla\mathring{\psi}_1\right|^2 d\mathbf{x}}{\int_{\mathbb{R}^3} \left(\nabla\mathring{\phi}_1\right)^2 d\mathbf{x}} = a_B\theta_\psi, \quad (14.115)$$

with

$$\theta_\psi = \frac{4\pi \int_{\mathbb{R}^3} \left|\nabla\mathring{\psi}_1\right|^2 d\mathbf{x}}{\int_{\mathbb{R}^3} \left|\nabla\mathring{\phi}_1\right|^2 d\mathbf{x}}, \quad a_B = \frac{\chi^2}{mq^2}$$

where a_B is the Bohr radius. Since the functions $\mathring{\psi}_1, \mathring{\phi}_1$ in the above relations are radial, the Dirichlet integrals in (14.115) can be recast as

$$\int_{\mathbb{R}^3} \left|\nabla\mathring{\psi}_1\right|^2 d\mathbf{x} = 4\pi \int_0^\infty \left(\partial_r \left(r\mathring{\psi}_1(r)\right)\right)^2 dr \quad (14.116)$$

with the similar formula for $\mathring{\phi}_1$. We refer to the space scale a_ψ in (14.115) obtained based on the equality $\mathcal{E}_1\left(\mathring{\psi}\right) = \mathcal{E}_2\left(\mathring{\phi}\right)$ as the *energy-based spatial scale*.

The energy-based spatial scale a_ψ for the power law form factor (14.95) is defined by (14.115) with

$$\theta_\psi = \frac{4\pi \int_{\mathbb{R}^3} \left(\nabla\mathring{\psi}_1\right)^2 d\mathbf{x}}{\int_{\mathbb{R}^3} \left(\nabla\mathring{\phi}_1\right)^2 d\mathbf{x}} = \frac{40}{7\pi} \simeq 1.8189. \quad (14.117)$$

For the exponentially decaying form factor, we get $a_\psi = \theta_\psi a_B$ with

$$\theta_\psi \simeq 1.2473.$$

Note that the energy based spatial scales for the power law form factor and the exponential form factor are of the same order, though their variances are absolutely different (infinite variance for the power law as in (14.95)).

14.6 The Rest State of a Relativistic Dressed Charge

The field equations for the dressed charge distribution ψ and 4-potential $A^\mu = (\varphi, \mathbf{A})$ are the NKG-Maxwell equations (13.14)–(13.15). A ground state (centered at the origin $\mathbf{x} = \mathbf{0}$) is a time-harmonic radial solution to the field equations (13.14)–(13.15) which is similar to (14.44) and has the following special form

$$\begin{aligned}\psi(t, \mathbf{x}) &= e^{-i\omega_0 t} \mathring{\psi}(|\mathbf{x}|), & \omega_0 &= \frac{mc^2}{\chi}, \\ \varphi(t, \mathbf{x}) &= \mathring{\varphi}(|\mathbf{x}|), & \mathbf{A}(t, \mathbf{x}) &= \mathbf{0}.\end{aligned}\quad (14.118)$$

As always, the form factor $\mathring{\psi}(r)$ is a twice differentiable, positive and monotonically decreasing function of $r = |\mathbf{x}|$.

As follows from (13.10), the density ρ and the current $\mathbf{J}$ for the 4-current $J^\nu = (\rho c, \mathbf{J})$ for the ω_0 -static solution (14.118) are given by the formulas

$$\rho = \mathring{\rho} = q \left(1 - \frac{q}{mc^2} \mathring{\varphi}\right) \mathring{\psi}^2, \quad \mathbf{J} = \mathbf{0}. \quad (14.119)$$

For the charge at rest as described by relations (14.118), the field equations (13.14)–(13.15) turn into the following system of two equations for the real-valued functions $\mathring{\psi}$ and $\mathring{\varphi}$ which we call *charge equilibrium equations*:

$$-\nabla^2 \mathring{\varphi} = 4\pi \mathring{\rho}, \quad \mathring{\rho} = q \left(1 - \frac{q}{mc^2} \mathring{\varphi}\right) \mathring{\psi}^2, \quad (14.120)$$

$$-\nabla^2 \mathring{\psi} + \frac{qm}{\chi^2} \mathring{\varphi} \left(2 - \frac{q}{mc^2} \mathring{\varphi}\right) \mathring{\psi} + G'(|\mathring{\psi}|^2) \mathring{\psi} = 0. \quad (14.121)$$

We refer to such a solution as ω_0 -static; we also refer to ω_0 -static solutions as a charge at rest. Similarly to solutions of (14.77)–(14.78), we call $\mathring{\psi}$ and $\mathring{\varphi}$, respectively, the *charge form factor* and *form factor potential*. Since the ground state $\mathring{\psi}$ is a given function, we can solve Eq. (14.120) to determine the Coulomb-like potential $\mathring{\varphi} = \mathring{\varphi}_{\mathring{\psi}}$ by the formula

$$\mathring{\varphi} = \mathring{\varphi}_{\mathring{\psi}} = 4\pi q \left(-\nabla^2 + \frac{4\pi q^2}{mc^2} \mathring{\psi}^2\right)^{-1} \mathring{\psi}^2. \quad (14.122)$$

Note that Eq. (14.120) turns into the classical equation for Coulomb's potential if ρ is replaced by $q\delta(\mathbf{x})$, where the delta-function has the standard property $\int \delta(\mathbf{x}) d\mathbf{x} = 1$. Since we want $\mathring{\varphi}$ to behave as Coulomb's electrostatic potential at large distances, and q to be the charge value, (19.52) implies the following *charge normalization condition* imposed on the form factor $\mathring{\psi}$:

$$\int_{\mathbb{R}^3} \left(1 - \frac{q}{mc^2} \mathring{\varphi}(|\mathbf{x}|)\right) \mathring{\psi}^2(|\mathbf{x}|) d\mathbf{x} = 1. \quad (14.123)$$

Plugging the expression (14.122) into the Eq. (14.121), we get the nonlinear equation for $\dot{\psi}$ as follows:

$$-\nabla^2 \dot{\psi} + \frac{qm}{\chi^2} \dot{\phi}_{\dot{\psi}} \left(2 - \frac{q\dot{\phi}_{\dot{\psi}}}{mc^2} \right) \dot{\psi} + G'(|\dot{\psi}|^2) \dot{\psi} = 0. \quad (14.124)$$

The above Eq. (14.124) signifies a complete balance (equilibrium) of the following three forces acting upon the resting charge: (i) the internal elastic deformation force associated with the term $-\nabla^2 \dot{\psi}$; (ii) the charge's electromagnetic self-interaction force associated with the term $\frac{m}{\chi^2} \dot{\phi}_{\dot{\psi}} \left(2q - \frac{q^2}{mc^2} \dot{\phi}_{\dot{\psi}} \right) \dot{\psi}$; (iii) the internal nonlinear self-interaction of the charge associated with the term $G'(|\dot{\psi}|^2) \dot{\psi}$. Importantly, the charge equilibrium equation (14.124) establishes an explicit relation between the form factor $\dot{\psi}$ and the self-interaction nonlinearity G .

Being given the form factor $\dot{\psi}$, we can find from the equilibrium equation (14.124) the self-interaction nonlinearity G which exactly produces this factor under an assumption that $\dot{\psi}(r)$, $r \geq 0$, is a nonnegative, monotonically decaying and sufficiently smooth function, see Sect. 14.5.1.

Let us denote by $\mathcal{E}_0(\psi, A^\mu)$ the energy of the dressed charge $\{\psi, A^\mu\}$ derived from the Lagrangian (13.13). Using now the results of Sect. 14.8 including the relation (14.176), we find that for the ground state pair $\{\psi, \phi\}$ the energy $\mathcal{E}_0$ can be written in the following form

$$\begin{aligned} \mathcal{E}_0(\psi, \phi) &= mc^2 + \mathcal{E}'_0\left(\dot{\psi}\right), \\ \mathcal{E}'_0\left(\dot{\psi}\right) &= \frac{2}{3} \int_{\mathbb{R}^3} \left[\frac{\chi^2}{2m} \nabla \dot{\psi}^* \cdot \nabla \dot{\psi} - \frac{(\nabla \dot{\phi})^2}{8\pi} \right] dx. \end{aligned} \quad (14.125)$$

Note that, according to (14.122), $\dot{\phi}$ is determined by $\dot{\psi}$, and the above energy $\mathcal{E}_0$ is a functional of $\dot{\psi}$ and the model constants only. We refer to the energy $\mathcal{E}'_0(\dot{\psi})$ defined in (14.125) as the relative energy. Note that the representation (14.125) for the energy $\mathcal{E}_0$ does not explicitly involve the nonlinear self-interaction G .

On relations between model parameters. The Lagrangian for a dressed charge involves the mass parameter m , the speed of light c and the Planck parameter χ . The definition of the rest state also involves the fundamental frequency ω_0 . Below we discuss the possibility of choosing the parameters so that the Planck formula holds for the fundamental frequency and the rest energy. Note that in this subsection we do not consider transitions, and this is different from the situation treated in Sect. 17.4.1 where we derive the Planck formula (17.57) for transitions between states with different frequencies. Here we allow χ to vary and be different from the Planck constant $\hbar$.

Note that the fundamental frequency $\omega_0 = \frac{mc^2}{\chi}$ and the energy $\mathcal{E}_0$ of the rest state are defined independently. We demonstrate that we can define the parameter χ so

that they satisfy the Planck–Einstein relation $\mathcal{E}_0 = \hbar\omega_0$. Now we assume that the frequency defined by (11.122) satisfies the Planck–Einstein relation which would determine ω_0 as a function of $\dot{\psi}$, constants c, m, q and, importantly, χ , namely

$$\begin{aligned}\omega_0 &= \omega_0(\dot{\psi}, \chi) \\ &= \frac{1}{\hbar} \left\{ mc^2 + \frac{2}{3} \int_{\mathbb{R}^3} \left[\frac{\chi^2}{2m} \nabla \dot{\psi}^* \cdot \nabla \dot{\psi} - \frac{1}{8\pi} (\nabla \dot{\phi})^2 \right] dx \right\}.\end{aligned}\quad (14.126)$$

Then, to be consistent with the definition of ω_0 in (11.122), we have to set the value of χ so that

$$\chi \omega_0(\dot{\psi}, \chi) = mc^2. \quad (14.127)$$

Note that the above assumptions are made only in this subsection. In view of the representation (14.126) for $\omega_0(\dot{\psi}, \chi)$, the above equation is equivalent to the requirement for $\chi = \chi(\dot{\psi})$ to be a positive solution to the following cubic equation

$$\chi \{c_2 \chi^2 + c_1\} = \hbar, \quad (14.128)$$

where

$$c_1 = 1 - \frac{2}{3mc^2} \int_{\mathbb{R}^3} \frac{(\nabla \dot{\phi})^2}{8\pi} dx, \quad c_2 = \frac{1}{3m^2 c^2} \int_{\mathbb{R}^3} \nabla \dot{\psi}^* \cdot \nabla \dot{\psi} dx. \quad (14.129)$$

Observe that $\dot{\phi}$, in view of its defining Eq. (14.120), depends only on $\dot{\psi}$ and the constants c, m, q . Note also that we can choose $\dot{\psi}$ independently of χ , and then determine G . The cubic equation (14.128) always has a positive solution, and, if $c_1 \geq 0$, the left-hand side of the Eq. (14.128) is a monotonically increasing function for $\chi \geq 0$, implying uniqueness of the solution.

Now we would like to discuss the following question: when does the value of χ determined from (14.128) coincide with $\hbar$? Applying to the energy $\mathcal{E}_0$ Einstein's principle of equivalence of mass and energy, we define the *dressed charge mass* $\tilde{m} = \tilde{m}(\dot{\psi})$ by the equality

$$\mathcal{E}_0(\dot{\psi}) = \tilde{m}c^2 = \tilde{m}(\dot{\psi})c^2. \quad (14.130)$$

Combining the relation (14.130) with (14.125), we readily obtain

$$\begin{aligned}(\tilde{m} - m)c^2 &= \mathcal{E}'_0(\dot{\psi}) \\ &= \frac{2}{3} \int_{\mathbb{R}^3} \left[\frac{\chi^2}{2m} \nabla \dot{\psi}^* \cdot \nabla \dot{\psi} - \frac{1}{8\pi} (\nabla \dot{\phi})^2 \right] dx.\end{aligned}\quad (14.131)$$

The very same relations (14.126) and (14.127) readily imply that the value of χ determined from (14.127) coincides with $\hbar$ if $\tilde{m} = m$.

In the case of a generic form factor $\mathring{\psi}$, the relative energy $\mathcal{E}'_0(\mathring{\psi})$ does not necessarily have to vanish, and the mass $\tilde{m}$ may differ from m . Then it follows from relation (14.126) that $\chi \neq \hbar$. We want to stress that here we consider only the fundamental frequency ω_0 . Note that all transition energies and frequencies considered in Sect. 17.4.1 for balanced charges satisfy the Planck formula with $\chi = \hbar$.

14.6.1 Relativistic and Non-relativistic Resting Charges

In this section we consider a single relativistic dressed charge at rest and consider its dependence on the parameters of the problem, namely on the fine structure constant

$$\alpha = \frac{q^2}{\chi c}. \quad (14.132)$$

We show that the electric potential of the relativistic charge is close to that for the non-relativistic charge if α is small. We use rescaled variables (19.54):

$$\mathbf{x} = a_B \mathbf{y}, \quad \psi(\mathbf{x}) = \frac{1}{a_B^{3/2}} \Psi\left(\frac{\mathbf{x}}{a_B}\right), \quad \varphi(\mathbf{x}) = \frac{q}{a_B} \Phi\left(\frac{\mathbf{x}}{a_B}\right) \quad (14.133)$$

where

$$a_B = \frac{\chi^2}{mq^2}. \quad (14.134)$$

The parameter a_B in the case of an electron coincides with the Bohr radius. In the new variables, we arrive at the following dimensionless version of the resting charge equations (14.120), (14.121) and the charge normalization condition (14.123):

$$-\frac{1}{2} \nabla_y^2 \Psi + \left(\Phi - \frac{\alpha^2 \Phi^2}{2} \right) \Psi + \frac{\hat{G}'(\Psi^2)}{2} \Psi = 0, \quad (14.135)$$

$$-\nabla^2 \Phi = 4\pi (1 - \alpha^2 \Phi) |\Psi|^2 = 0, \quad (14.136)$$

$$\int_{\mathbb{R}^3} [(1 - \alpha^2 \Phi) |\Psi|^2] d\mathbf{x} = 1. \quad (14.137)$$

Setting in the above equations $\alpha = 0$, we obtain the dimensionless form of the non-relativistic equilibrium equations (14.78), (14.77) and the charge normalization condition (14.17), namely

$$-\frac{1}{2}\nabla_y^2\psi + \Phi\psi + \hat{G}'_0(|\psi|^2)\check{\psi} = 0, \quad -\nabla^2\Phi = 4\pi|\psi|^2, \quad (14.138)$$

$$\int_{\mathbb{R}^3} |\psi|^2 d\mathbf{x} = 1. \quad (14.139)$$

Using perturbations analysis, we argue that for small α the solution ψ_α, Φ_α to the Eq. (14.136) is close to the solution ψ_0, Φ_0 of the Eq. (14.138). Indeed, the zero approximation

$$\Phi_0(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{|\psi_0(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y},$$

and the first order approximation Φ_1 is a solution to

$$-\nabla^2\Phi_1 = 4\pi(1 - \alpha^2\Phi_0)|\psi_0|^2.$$

Using the Maximum principle, we see that

$$0 < \Phi_1(\mathbf{x}) < \Phi(\mathbf{x}) < \Phi_0(\mathbf{x}) = \check{\Phi}(\mathbf{x}) \text{ for all } \mathbf{x}. \quad (14.140)$$

Obviously,

$$\Phi_1(\mathbf{x}) = \Phi_0(\mathbf{x}) + \alpha^2\Phi_{01}(\mathbf{x}) \text{ where } \nabla^2\Phi_{01} = 4\pi\Phi_0|\psi_0|^2, \quad (14.141)$$

hence

$$\Phi_{01}(\mathbf{x}) = - \int_{\mathbb{R}^3} \frac{\Phi_0(\mathbf{y})|\psi_0(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y}.$$

Consequently, inequalities (14.140) imply an explicit estimate:

$$\alpha^2\Phi_{01}(\mathbf{x}) < \Phi(\mathbf{x}) - \Phi_0(\mathbf{x}) < 0 \text{ for all } \mathbf{x} \in \mathbb{R}^3. \quad (14.142)$$

Therefore, (14.135) is a small perturbation of (14.138), and one may expect that ψ is a small perturbation of ψ_0 if α is small.

14.6.2 The Energy-Momentum Tensor and Forces at Equilibrium of a Dressed Charge

For detailed considerations of the structure and properties of the energy-momentum tensor including its symmetry, gauge invariance and conservation laws, we refer the reader to Sect. II. Here, using the results of that section, we analyze the energy-momentum tensor for the Lagrangian L_0 defined by the formula (13.13) and for the ω_0 -static state defined by (14.118).

The symmetric energy-momentum tensor $T^{\mu\nu}$ is given by formula (13.12) with skipped index ℓ , and the EM field energy-momentum tensor $\Theta^{\mu\nu}$ is given by (9.13). The tensors satisfy conservation laws (11.92), (11.95), where the Lorentz force density f^ν is given by (11.141):

$$f^\nu = \frac{1}{c} J_\mu F^{\nu\mu} = \left(\frac{1}{c} \mathbf{J} \cdot \mathbf{E}, \rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} \right).$$

Using the interpretation (10.61)–(10.62) of the EnMT $T^{\mu\nu}$ and formulas (11.135), (9.13), we find that EnMT takes the form (10.61) where the energy density u , the momentum and the energy flux components p_j and s_j are as follows:

$$u = \frac{\chi^2}{2m} \left[\left(\nabla \dot{\psi} \right)^2 + G \left(\dot{\psi}^2 \right) \right] + \left(mc^2 - q\dot{\phi} + \frac{q\dot{\phi}^2}{2mc^2} \right) \dot{\psi}^2, \quad (14.143)$$

$$p^j = 0, \quad s^j = 0, \quad j = 1, 2, 3, \quad (14.144)$$

and the stress tensor components σ_{ij} with $i, j = 1, 2, 3$ are represented by the following formulas:

$$\begin{aligned} \sigma_{ij} = & -\frac{\chi^2}{m} \left[\partial_i \dot{\psi} \partial_j \dot{\psi} - \frac{1}{2} \left(\nabla \dot{\psi} \right)^2 \delta_{ij} \right] \\ & + \left[q \left(\dot{\phi} - \frac{q}{2mc^2} \dot{\phi}^2 \right) \dot{\psi}^2 + \frac{\chi^2}{2m} G \left(\dot{\psi}^2 \right) \right] \delta_{ij}. \end{aligned} \quad (14.145)$$

Notice that the vanishing of the momentum $\mathbf{p}$ and the energy flux $\mathbf{s}$ in (14.144) is yet another justification for the name ω_0 -static solution. Observe also that, for the ω_0 -static state defined by (14.118), the EM field is as follows:

$$\mathbf{E} = -\nabla \dot{\phi}, \quad \mathbf{B} = \mathbf{0}. \quad (14.146)$$

Using the representation (9.14)–(9.15) for the EM EnMT $\Theta^{\mu\nu}$ combined with the formulas (14.146) for the EM field, we obtain the following representation of $\Theta^{\mu\nu}$ for the ω_0 -static solution (14.118):

$$\Theta^{\mu\nu} = \begin{bmatrix} \left(\nabla \dot{\phi} \right)^2 / (8\pi) & 0 \\ 0 & -\tau_{ij} \end{bmatrix}, \quad (14.147)$$

where

$$-\tau_{ij} = \Theta^{ij} = -\frac{1}{4\pi} \left[\partial_i \dot{\phi} \partial_j \dot{\phi} - \frac{\left(\nabla \dot{\phi} \right)^2}{2} \delta_{ij} \right] \quad (14.148)$$

for $i, j = 1, 2, 3$. Combining the conservation law (11.92) with the general representation (10.61) of the charge EnMT $T^{\mu\nu}$, we obtain

$$\partial_t p_i = \sum_{j=1,2,3} \partial_j \sigma_{ji} + f_i = 0, \quad i = 1, 2, 3. \quad (14.149)$$

Notice that for the ω_0 -static solution (14.118), in view of (14.146), (14.147), and (14.148), the Eq. (14.149) turns into the equilibrium equations

$$\sum_{j=1,2,3} \partial_j \sigma_{ji} - \rho \partial_i \hat{\phi} = 0. \quad (14.150)$$

Observe now that the stress tensor σ_{ij} defined in (14.145) can be naturally decomposed into the three components which we name as follows:

$$\sigma_{ij} = \sigma_{ij}^{\text{el}} + \sigma_{ij}^{\text{em}} + \sigma_{ij}^{\text{nl}}, \quad (14.151)$$

where

$$\sigma_{ij}^{\text{el}} = -\frac{\chi^2}{m} \left[\partial_i \hat{\psi} \partial_j \hat{\psi} - \frac{1}{2} (\nabla \hat{\psi})^2 \delta_{ij} \right] \quad (14.152)$$

is the elastic deformation stress tensor;

$$\sigma_{ij}^{\text{em}} = -p^{\text{em}} \delta_{ij}, \quad p^{\text{em}} = -q \left(\hat{\phi} - \frac{q \hat{\phi}^2}{2mc^2} \right) \hat{\psi}^2, \quad (14.153)$$

is the EM interaction stress tensor;

$$\sigma_{ij}^{\text{nl}} = -p^{\text{nl}} \delta_{ij}, \quad p^{\text{nl}} = -\frac{\chi^2 G (\hat{\psi}^2)}{2m}, \quad (14.154)$$

is the nonlinear self-interaction stress tensor. Consequently, the respective volume force densities are:

$$\sum_{j=1,2,3} \partial_j \sigma_{ij}^{\text{el}} = f_i^{\text{el}} = -\frac{\chi^2}{m} \Delta \hat{\psi} \partial_i \hat{\psi}, \quad (14.155)$$

$$\sum_{j=1,2,3} \partial_j \sigma_{ij}^{\text{em}} = f_i^{\text{em}} + \rho \partial_i \hat{\phi}, \quad (14.156)$$

$$f_i^{\text{em}} = q \left(2\dot{\phi} - \frac{q}{mc^2} \dot{\phi}^2 \right) \dot{\psi} \partial_i \dot{\psi}, \quad (14.157)$$

$$\sum_{j=1,2,3} \partial_j \sigma_{ij}^{\text{nl}} = f_i^{\text{nl}} = \frac{\chi^2}{m} G'(\dot{\psi}^2) \dot{\psi} \partial_i \dot{\psi}. \quad (14.158)$$

Notice that the volume force density for the electromagnetic interaction stress in (14.156) has two parts: f_i^{em} , which we call the *internal electromagnetic force*, and $\rho \partial_i \dot{\phi}$ which is the negative of the Lorentz force. Observe that the stress tensor σ_{ij}^{el} has a structure similar to that for compressional waves, see Sect. 10.8.1 and (10.109), whereas both the stress tensors σ_{ij}^{em} and σ_{ij}^{nl} have a structure typical for perfect fluids, [248, Section 6.6], with respective hydrostatic pressures p^{em} and p^{nl} defined by the relations (14.153)–(14.154). Notice that formula (14.154) provides an interpretation of the nonlinearity $G(\dot{\psi}^2)$, namely $p^{\text{nl}} = -\chi^2 G(\dot{\psi}^2)/(2m)$ is the hydrostatic pressure when the charge is at rest.

14.7 Variational Characterization of Static and Time-Harmonic States

The rest and time-harmonic charge distributions play an important role in the theory of balanced and dressed charges. In this Section we show that corresponding solutions to the field equations of a general Lagrangian field theory can be obtained as critical points of relevant energy functionals. Such a characterization gives us the possibility to use the Pokhozhaev identity and obtain useful expressions for the energy of rest states.

We consider the *time-harmonic* or ω -*static* solutions ψ_ℓ of the field equations (11.19), (11.33), namely

$$\psi_\ell = e^{-i\omega_\ell t} \tilde{\psi}_\ell, \quad \psi_\ell^* = e^{i\omega_\ell t} \tilde{\psi}_\ell^* \quad (14.159)$$

where $\tilde{\psi}_\ell$ is time-independent, and we also add to the assumption (14.159) the assumption that the EM field is static:

$$A^v = (\varphi, \mathbf{A}); \quad \partial_t \varphi = 0, \quad \partial_t \mathbf{A} = \mathbf{0}. \quad (14.160)$$

If $\omega_\ell = 0$, we call the regime simply *static*. The above conditions are abbreviated by the symbol *stat*:

$$\text{stat} \equiv \left\{ \partial_t \tilde{\psi}_\ell = 0, \quad \partial_t \varphi = 0, \quad \partial_t \mathbf{A} = \mathbf{0}, \quad A^\mu = (\varphi, \mathbf{A}) \right\}. \quad (14.161)$$

Obviously, for the time-harmonic solutions of the field equations

$$\begin{aligned}\psi_\ell^{;0} &= \frac{1}{c} \partial_0 \psi_\ell = -i \frac{1}{c} \omega_\ell e^{-i\omega_\ell t} \tilde{\psi}_\ell = -i \frac{1}{c} \omega_\ell \psi^\ell; \\ \psi_\ell^{;0} &= \frac{1}{c} \left(\frac{iq_\ell \varphi}{\chi} - i\omega_\ell \right) \psi^\ell.\end{aligned}\quad (14.162)$$

The field equations (11.19), (11.33) written in terms of the Lagrangian L_ℓ $(\psi_\ell, \psi_{\ell;0}, \psi_{\ell;j}, \psi_\ell^*, \psi_{\ell;0}^*, \psi_{\ell;j}^*)$ can be written in the form

$$\frac{\partial L_\ell}{\partial \psi_\ell} - \partial_j^{\ell*} \frac{\partial L_\ell}{\partial \psi_{\ell;j}} - \tilde{\partial}_0^{\ell*} \frac{\partial L_\ell}{\partial \psi_{\ell;0}} = 0, \quad \frac{\partial L_\ell}{\partial \psi_\ell^*} - \partial_j^\ell \frac{\partial L_\ell}{\partial \psi_{\ell;j}^*} - \tilde{\partial}_0^\ell \frac{\partial L_\ell}{\partial \psi_{\ell;0}^*} = 0, \quad (14.163)$$

$$\frac{\partial L_\ell}{\partial A^v} - \partial_\mu \frac{\partial L_\ell}{\partial A_{,\mu}^v} - \partial_0 \frac{\partial L_\ell}{\partial A_{,\mu}^v} = 0. \quad (14.164)$$

We assume that the Lagrangian L_ℓ satisfies the following structural assumption. The Lagrangian L_ℓ , when restricted to time-harmonic functions which satisfy (14.159), (14.161), must satisfy the following equalities:

$$\partial_t \left(e^{i\omega_\ell t} \frac{\partial L_\ell}{\partial \psi_{\ell;0}^*} \right) = 0, \quad \partial_t \left(e^{-i\omega_\ell t} \frac{\partial L_\ell}{\partial \psi_{\ell;0}} \right) = 0, \quad \partial_t \frac{\partial L_\ell}{\partial A_{,\mu}^v} = 0. \quad (14.165)$$

Note that the above condition is fulfilled for terms of the form

$$\psi_{\ell;0} \psi_{\ell;0}^*, \quad \psi_\ell \psi_{\ell;0}^*, \quad \psi_{\ell;j} \psi_{\ell;0}^*,$$

their complex conjugates, and time-independent functions of the above expressions and EM fields. Under the structural condition (14.165), the Eqs. (14.163), (14.164) for time-harmonic solutions coincide with the Euler equations for critical points $\tilde{\psi}_\ell, \tilde{\psi}_\ell^*$ of the functional

$$\mathcal{E}_{\omega \text{ stat}} = \int_{\mathbb{R}^3} \mathcal{U}_{\omega \text{ stat}} \, d\mathbf{x}, \quad (14.166)$$

where

$$\begin{aligned}\mathcal{U}_{\omega \text{ stat}} &= \frac{F^{\mu\nu} F_{\mu\nu}}{16\pi} \\ &- \sum_\ell L_\ell \left(\psi_\ell, \left(-\frac{i\omega_\ell}{c} + \frac{iq_\ell \varphi}{\chi c} \right) \psi_\ell, \psi_{\ell;j}, \psi_\ell^*, \left(\frac{i\omega_\ell}{c} - \frac{iq_\ell \varphi}{\chi c} \right) \psi_\ell^*, \psi_{\ell;j}^* \right) \Big|_{\text{stat}}.\end{aligned}\quad (14.167)$$

Now let us compare the above density $\mathcal{U}_{\omega \text{ stat}}$ with the energy component of the canonical energy-momentum given by (11.66), (9.11). Using (11.52), we obtain that

$$\mathring{T}_\ell^{00} = -\frac{\chi \omega_\ell}{q_\ell} \rho_\ell - L_\ell, \quad \mathring{\Theta}^{00} = \frac{F^{\mu\nu} F_{\mu\nu}}{16\pi},$$

and the energy component $\mathcal{E} = \mathcal{T}^{00}$ of the total EnMT $\mathcal{T}^{\mu\nu}$ given by (11.65) (with $\theta = 0$) has the form

$$\mathcal{T}^{00} = \mathring{\mathcal{O}}^{00} + \sum_{\ell} \mathring{T}_{\ell}^{00} = \mathring{\mathcal{U}}_{\omega \text{ stat}} + \sum_{\ell} \frac{\chi \omega_{\ell}}{c q_{\ell}} \rho_{\ell},$$

yielding the following expression for the potential energy functional (14.166):

$$\mathcal{E} \left(e^{-i\omega t} \tilde{\psi}, e^{i\omega t} \tilde{\psi}^*, A^{\mu} \right) = \mathcal{E}_{\omega \text{ stat}} + \sum_{\ell} \frac{\chi \omega_{\ell}}{q_{\ell}} \int_{\mathbb{R}^3} \rho_{\ell} dx. \quad (14.168)$$

If $\omega_{\ell} = 0$, $\mathcal{E}_{\text{stat}}$ equals the energy of rest states canonically defined in terms of the Lagrangian $\mathcal{L}$. Hence, a rest state is a stationary point of the static energy functional $\mathcal{E}_{\text{stat}}$ in complete agreement with the *principle of virtual work* for the state of equilibrium, [212, Section III.1], [309, Section II.8]. The energy $\mathcal{E}_{\text{stat}}$ in a rest regime can be identified with the potential energy.

14.8 Energy Partition for Rest and Time-Harmonic States

The density $\mathring{\mathcal{U}}_{\omega \text{ stat}}$ given by (14.167) is a function of ψ_{ℓ} , $\psi_{\ell,j}$, their complex conjugates and of the time-independent fields A^{μ} and their derivatives $A^{\mu,j}$. We assume now that the potential energy density $\mathring{\mathcal{U}}$ defined by (14.167) is expanded into the series of terms which are homogeneous with respect to the spatial derivatives $\psi_{\ell,j}^*$, $\psi_{\ell,j}$, $A^{\mu,j}$, namely

$$\mathring{\mathcal{U}}_{\omega \text{ stat}} = \sum_{n=0}^{\infty} \mathring{\mathcal{U}}^{(n)}. \quad (14.169)$$

Here $\mathring{\mathcal{U}}^{(n)}$ is n -homogeneous, namely for any real ξ

$$\mathring{\mathcal{U}}^{(n)} (\psi_{\ell}, \xi \psi_{\ell,j}, \psi_{\ell}^*, \xi \psi_{\ell,j}^*, A^{\mu}, \xi A^{\mu,j}) = \xi^n \mathring{\mathcal{U}}^{(n)} (\psi_{\ell}, \psi_{\ell,j}, \psi_{\ell}^*, \psi_{\ell,j}^*, A^{\mu}, A^{\mu,j}).$$

This expansion readily implies via the representation (14.166) for the potential energy $\mathcal{E}_{\text{stat}}$ the corresponding expansion for $\mathcal{E}_{\text{stat}}$:

$$\mathring{\mathcal{U}}_{\omega \text{ stat}} = \sum_{n=0}^{\infty} \mathcal{E}_{\text{stat}}^{(n)} \text{ where } \mathcal{E}_{\text{stat}}^{(n)} = \int_{\mathbb{R}^3} \mathring{\mathcal{U}}^{(n)} dx. \quad (14.170)$$

Obviously, using a change of variables in the above integral, we obtain that

$$\begin{aligned} & \mathcal{E}_{\text{stat}}^{(n)}(\psi_\ell(\xi x), \nabla(\psi_\ell(\xi x)), \psi_\ell^*(\xi x), \nabla(\psi_\ell^*(\xi x)), A^\mu, \nabla(A^\mu(\xi x))) \\ &= \xi^{n-3} \mathcal{E}_{\text{stat}}^{(n)}(\psi_\ell(x), \psi_\ell^*(x), (\nabla\psi_\ell)(x), (\nabla\psi_\ell^*)(x), \nabla(A^\mu(x))). \end{aligned} \quad (14.171)$$

Given a time-harmonic solution $\{\psi_\ell, \psi_\ell^*, A^\mu\}$, we use its property being a stationary point of the functional $\mathcal{U}_{\omega \text{ stat}}$ as defined by formula (14.167). Namely, we introduce the following family of fields

$$\psi_{\ell\xi}(x) = \psi_\ell(\xi x), \quad A_\xi^\mu = A^\mu(\xi x),$$

where ξ is a real parameter. Since $\{\psi_{\ell\xi}, \psi_{\ell\xi}^*, A_\xi^\mu\}$ at $\xi = 1$ is a stationary point of the functional $\mathcal{E}_{\omega \text{ stat}}$ defined by (14.166), we obtain using (14.171) that

$$\left. \frac{d}{d\xi} \mathcal{E}_{\omega \text{ stat}} \right|_{\xi=1} = \left. \frac{d}{d\xi} \sum_{n=0}^{\infty} \xi^{n-3} \mathcal{E}_{\text{stat}}^{(n)} \right|_{\xi=1} = 0. \quad (14.172)$$

Consecutively, the homogeneous energy components $\mathcal{E}_{\text{stat}}^{(n)}$ for a time-harmonic solution always satisfy the identity

$$\sum_{n=0}^{\infty} (n-3) \mathcal{E}_{\text{stat}}^{(n)} = 0. \quad (14.173)$$

Very often the Lagrangian density $\mathcal{L}$ depends quadratically on the spatial derivatives so that

$$\begin{aligned} & \mathring{\mathcal{U}}(\psi_\ell, \psi_\ell^*, A^\mu, \nabla\psi_\ell, \nabla\psi_\ell^*, \nabla A^\mu) \\ &= \mathring{\mathcal{U}}^{(2)}(\psi_\ell, \psi_\ell^*, A^\mu, \nabla\psi_\ell, \nabla\psi_\ell^*, \nabla A^\mu) + \mathring{\mathcal{U}}^{(0)}(\psi_\ell, \psi_\ell^*, A^\mu). \end{aligned} \quad (14.174)$$

In this case the identity (14.173) turns into the following important identity for two constituting components $\mathcal{E}_{\text{stat}}^{(2)}$ and $\mathcal{E}_{\text{stat}}^{(0)}$ of the total potential energy $\mathcal{E}_{\text{stat}} = \mathcal{E}_{\text{stat}}^{(2)} + \mathcal{E}_{\text{stat}}^{(0)}$:

$$\mathcal{E}_{\text{stat}}^{(0)} = -\frac{1}{3} \mathcal{E}_{\text{stat}}^{(2)} \text{ implying } \mathcal{E}_{\text{stat}} = \frac{2}{3} \mathcal{E}_{\text{stat}}^{(2)}. \quad (14.175)$$

The significance of the above identity for our goals is that in the cases of interest the energy component $\mathcal{E}_{\text{stat}}^{(0)}$ accounts for the energy of nonlinear self-interactions and the formula $\mathcal{E}_{\text{stat}} = \frac{2}{3} \mathcal{E}_{\text{stat}}^{(2)}$ shows that *the total energy has a representation that does not depend explicitly on the nonlinear self-interactions. This property agrees with the characterization of the nonlinear self-interactions as stealthy.*

The identity (14.175) for a single field is known as the *Pokhozhaev–Derrick identity*, [276, [84] (see also [193] and [65, Section 2.4]).

Example. Let the Lagrangian to be of the special form

$$L_{\ell\omega} = K_{\ell} \left(\psi_{\ell}^* \psi_{\ell}, \psi_{\ell;\mu}^* \psi_{\ell}^{\dot{\mu}} \right) = k_{\ell 2} \left(\psi_{\ell}^{\ell*} \psi_{\ell}^{\ell} \right) \psi_{\ell;\mu}^* \dot{\psi}_{\ell}^{\mu} + k_{\ell 0} \left(\psi_{\ell}^* \psi_{\ell} \right). \quad (14.176)$$

Let $\mathbf{A} = \mathbf{0}$ for a static solution. In this case $F^{\mu\nu} F_{\mu\nu} = -2\nabla\varphi \cdot \nabla\varphi$, and the expression for the total energy density of the charges and the EM field takes the form

$$\mathcal{U}_{\omega \text{ stat}}^{\circ} = \mathcal{U}_{\text{stat}}^{\circ(2)} + \mathcal{U}_{\text{stat}}^{\circ(0)}$$

where

$$\begin{aligned} \mathcal{U}_{\text{stat}}^{\circ(2)} &= -\frac{(\nabla\varphi)^2}{8\pi} + \sum_{\ell} k_{\ell 2} \left(\tilde{\psi}_{\ell}^* \tilde{\psi}_{\ell} \right) \nabla \tilde{\psi}_{\ell}^* \cdot \nabla \tilde{\psi}_{\ell}, \\ \mathcal{U}_{\text{stat}}^{\circ(0)} &= -\sum_{\ell} \left[k_{\ell 2} \left(\tilde{\psi}_{\ell}^* \tilde{\psi}_{\ell} \right) \left(\frac{\omega_{\ell}}{c} - \frac{q_{\ell}\varphi}{\chi c} \right)^2 \tilde{\psi}_{\ell}^* \tilde{\psi}_{\ell} + k_{\ell 0} \left(\tilde{\psi}_{\ell}^* \tilde{\psi}_{\ell} \right) \right]. \end{aligned}$$

The corresponding expression for the total energy is

$$\mathcal{E}_{\text{stat}} = \mathcal{E}_{\text{stat}}^{(2)} + \mathcal{E}_{\text{stat}}^{(0)} \quad (14.177)$$

where

$$\mathcal{E}_{\text{stat}}^{(2)} = \int_{\mathbb{R}^3} \left[-\frac{(\nabla\varphi)^2}{8\pi} + \sum_{\ell} k_{\ell}^2 \left(\tilde{\psi}_{\ell}^* \tilde{\psi}_{\ell} \right) \nabla \tilde{\psi}_{\ell}^* \cdot \nabla \tilde{\psi}_{\ell} \right] dx, \quad (14.178)$$

$$= -\sum_{\ell} \int_{\mathbb{R}^3} \left[k_{\ell}^2 \left(\tilde{\psi}_{\ell}^* \tilde{\psi}_{\ell} \right) \left(\frac{\omega_{\ell}}{c} - \frac{q_{\ell}\varphi}{\chi c} \right)^2 \tilde{\psi}_{\ell}^* \tilde{\psi}_{\ell} + k_{\ell 0}^{\mathcal{E}_{\text{stat}}^{(0)}} \left(\tilde{\psi}_{\ell}^* \tilde{\psi}_{\ell} \right) \right] dx. \quad (14.179)$$

Applying now (14.175), we obtain the following formula:

$$\mathcal{E}_{\text{stat}}^{(0)} = -\frac{1}{3} \mathcal{E}_{\text{stat}}^{(2)}, \quad (14.180)$$

implying the following representation for the total system energy:

$$\begin{aligned} \mathcal{E}_{\text{stat}} &= \frac{2}{3} \mathcal{E}_{\text{stat}}^{(2)} \\ &= \frac{2}{3} \int_{\mathbb{R}^3} \left[-\frac{(\nabla\varphi)^2}{8\pi} + \sum_{\ell} k_{\ell}^2 \left(\psi_{\ell}^* \psi_{\ell} \right) \nabla \tilde{\psi}_{\ell}^* \cdot \nabla \tilde{\psi}_{\ell} \right] dx. \end{aligned} \quad (14.181)$$

In the case of a single charge, the above formula turns into

$$\mathcal{E}_{\text{stat}} = \frac{2}{3} \mathcal{E}_{\text{stat}}^{(2)} = \frac{2}{3} \int_{\mathbb{R}^3} \left[-\frac{(\nabla \varphi)^2}{8\pi} + k_2 (\psi^* \psi) \nabla \tilde{\psi}^* \cdot \nabla \tilde{\psi} \right] dx. \quad (14.182)$$

We want to emphasize once more the importance of the representation (14.181) in comparison with the original formula (14.177)–(14.179), which shows that the *total energy of the system of charges interacting with the EM field does not explicitly depend on the terms $k_{\ell 0} (\psi_\ell^* \psi_\ell)$ which include the nonlinear self-interactions.*

In the special case (14.176), using the charge normalization condition (14.118), we write the total energy (14.168) in the form

$$\begin{aligned} & \mathcal{E} \left(e^{-i\omega_\ell t} \tilde{\psi}_\ell, e^{i\omega_\ell t} \tilde{\psi}_\ell^*, \varphi \right) \\ &= \sum_\ell \chi \omega_\ell + \frac{2}{3} \int_{\mathbb{R}^3} \left[-\frac{(\nabla \varphi)^2}{8\pi} + \sum_\ell k_2^\ell (\psi_\ell^* \psi_\ell) \nabla \tilde{\psi}_\ell^* \cdot \nabla \tilde{\psi}_\ell \right] dx. \end{aligned} \quad (14.183)$$

The above formula gives a representation for the energy of time-harmonic fields which does not explicitly involve the nonlinear self-interactions.

Chapter 15

Uniform Motion of a Charge

In this chapter we consider a uniformly moving charge, that is, the case where the motion of the charge is translational with a constant velocity. We also assume that the shape of the charge wave function is preserved. Such a motion is possible if the charge is free and there are no external EM fields acting upon it. The properties of free dressed and balanced charges are quite similar. Both can be obtained from a resting charge by applying the relevant group transformation. Namely, in the non-relativistic case one applies the Galilean transformation whereas in the relativistic case the Lorentz transformation. In both cases, the uniformly moving charge wave function has the form of a *wave-corpuscle*, that is, a special closed form solution to the field equations. The properties of wave corpuscles are considered in the following sections. Importantly, the wave-corpuscle involves both localization, which has been already demonstrated by resting charges, and undulatory properties. It turns out quite remarkably that the wave-corpuscles as exact solutions to the field equations describe not only the uniform motion but also the accelerated motion for both the balanced and dressed charges, see Chap. 16. The wave properties of the wave-corpuscles can be interpreted as the de Broglie waves, see Sect. 16.2.1. The relations between the particle properties and wave properties of a wave-corpuscle are considered in Sect. 16.2.

15.1 Freely Moving Non-relativistic Balanced Charges

We can use the invariance of the NLS Lagrangian with respect to the Galilean-gauge transformations (11.171)–(11.172) to obtain a freely moving charge solution to the field equations (12.32), (14.10). Such a freely moving charge solution is obtained from the resting charge solution (14.9)–(14.10) by applying the Galilean transformation. Namely, for a given velocity $\mathbf{v}$, the NLS field equations (12.32) have a wave-corpuscle solution of the form

$$\psi = \psi(t, \mathbf{x}) = e^{i\mathfrak{S}} \hat{\psi}, \quad \hat{\psi} = \hat{\psi}(|\mathbf{x} - \mathbf{r}(t)|), \quad (15.1)$$

$$\mathfrak{S} = \frac{m}{2\chi} \mathbf{v}^2 t + \frac{m}{\chi} \mathbf{v} \cdot (\mathbf{x} - \mathbf{r}(t)), \quad \mathbf{r}(t) = \mathbf{v}t, \quad (15.2)$$

where $\mathbf{r}(t)$ is the center of the wave-corpuscle and $\hat{\psi}$ is the form factor based on the ground state (14.19). Similar form solutions associated with propagation at a constant speed are well-known in the theory of Nonlinear Schrödinger equations, see [317] and references therein. Alternatively, one can directly check by substitution that (15.1)–(15.2) indeed defines a solution to the NLS equation.

The EM potentials satisfying the Poisson equations (12.16)–(12.17) are given by the formulas

$$\varphi(t, \mathbf{x}) = \hat{\varphi}(|\mathbf{x} - \mathbf{v}t|), \quad \mathbf{A} = \frac{1}{c} q \mathbf{v} \hat{\varphi}(|\mathbf{x} - \mathbf{v}t|), \quad (15.3)$$

where $\hat{\varphi}$ is expressed in terms of $|\psi|^2$ by (14.11), (14.2).

In what follows, we refer to a wave function represented by the formulas (15.1) as a *wave-corpuscle*. Below we describe the basic properties of a wave-corpuscle.

15.1.1 Point and Wave Attributes of Wave Corpuscles

The explicit representation of a *freely moving charge* by the exact closed form solution (15.1) to the field equations shows that it harmoniously integrates the features of a point charge and a wave. Indeed, on one hand, the wave amplitude $\hat{\psi}(|\mathbf{x} - \mathbf{v}t|)$ in (15.1) is a soliton-like (particle-like) field that moves exactly as a free point charge described by its position $\mathbf{r} = \mathbf{v}t$. *On the other hand, the exponential factor $e^{i\mathfrak{S}}$ is a plane wave with phase $\mathfrak{S}$ that depends only on the point charge position $\mathbf{r} = \mathbf{v}t$ and its velocity $\mathbf{v}$, and it does not depend on the nonlinear self-interaction.*

The gradient $\nabla \mathfrak{S}$ of the phase $\mathfrak{S}$ is described in terms of the point charge quantities, namely

$$\mathbf{k} = \nabla \mathfrak{S} = \frac{m}{\chi} \mathbf{v}, \quad (15.4)$$

and we readily recognize in $\mathbf{k}$ the *de Broglie wavevector*. Notice that the frequency $\omega(\mathbf{k})$ of the oscillations of the wave-corpuscle for ψ is given by the formula

$$\omega(\mathbf{k}) = \frac{\chi \mathbf{k}^2}{2m}, \quad (15.5)$$

where the right-hand side coincides exactly with the dispersion relation $\omega = \omega(\mathbf{k})$ of the linear part of the field equations (14.75). From the dispersion relation, using the standard formulas, we determine the *group velocity* $\mathbf{v}_{\text{gr}}$ (see Sect. 15.1.2 for a discussion):

$$\mathbf{v}_{\text{gr}} = \nabla_{\mathbf{k}} \omega(\mathbf{k}) = \frac{\chi \mathbf{k}}{m}. \quad (15.6)$$

Combining then the expression (15.6) for the group velocity $\nabla_{\mathbf{k}} \omega(\mathbf{k})$ with the expression (15.4) for the wavevector $\mathbf{k}$, we establish another exact relation:

$$\mathbf{v} = \mathbf{v}_{\text{gr}} = \nabla_{\mathbf{k}} \omega(\mathbf{k}), \quad (15.7)$$

signifying the equality between the point charge translational velocity $\mathbf{v}$ and the group velocity $\mathbf{v}_{\text{gr}}$ representing the wave properties.

Using the relations (11.156)–(11.157) and (11.181), we readily obtain the following representations for the charge and the current densities:

$$\rho(t, \mathbf{x}) = q \left| \dot{\psi} \right|^2 (|\mathbf{x} - \mathbf{v}t|), \quad \mathbf{J}(t, \mathbf{x}) = q \mathbf{v} \left| \dot{\psi} \right|^2 (|\mathbf{x} - \mathbf{v}t|). \quad (15.8)$$

According to (11.166), the momentum and current densities are proportional:

$$\mathbf{P}(t, \mathbf{x}) = \frac{m}{q} \mathbf{J}(t, \mathbf{x}) = m \mathbf{v} \left| \dot{\psi} \right|^2 (|\mathbf{x} - \mathbf{v}t|). \quad (15.9)$$

The above expressions and the charge normalization condition (14.17) readily imply the following representations for the total balanced charge field momentum $\mathbf{P}$ and the total current $\mathbf{J}$ for the solution (15.1) in terms of the point charge quantities:

$$\mathbf{P} = \frac{m}{q} \mathbf{J} = \int_{\mathbb{R}^3} \frac{\chi q}{m} \text{Im} \frac{\nabla \dot{\psi}}{\dot{\psi}} \dot{\psi}^2 d\mathbf{x} = m \mathbf{v}. \quad (15.10)$$

Obviously the above formula coincides with the expression for the momentum and current of a point charge.

15.1.2 Plane Waves, Wave Packets and Dispersion Relations

We discuss on several occasions (see Sects. 15.1.1, 15.4, 16.2.1, 16.2.2, 18.1) the properties of wave-corpuscles which are similar to the properties of plane waves and wave packets. For completeness, we present here the basic terminology and formulate basic facts about their properties.

Recall that plane waves $\psi_{\text{pl}}(t, \mathbf{x})$ have the form

$$\psi_{\text{pl}}(t, \mathbf{x}) = e^{i\mathfrak{S}_{\text{pl}}(t, \mathbf{x})}, \quad \mathfrak{S}_{\text{pl}}(t, \mathbf{x}) = \mathbf{k} \cdot \mathbf{x} - \omega t, \quad (15.11)$$

where ω is the *frequency* and $\mathbf{k}$ is the *wavevector*, and, importantly, the phase function $\mathfrak{S}_{\text{pl}}$ is linear with respect to $\mathbf{x}$ and t . If a plane wave is a solution to a linear partial differential equation of the form

$$P(\partial_t, \nabla) \psi = 0, \quad (15.12)$$

where P is a polynomial of the differential operators ∂_t, ∇ , and the polynomial $P(-i\omega, i\mathbf{k})$ of variables $\omega, \mathbf{k}$ is its symbol, then the representation (15.11) for ψ_{pl} implies the following equation relating the frequency ω and the wavevector $\mathbf{k}$:

$$P(-i\omega, i\mathbf{k}) = 0. \quad (15.13)$$

The above relation is called the *dispersion relation*. Solving it for ω , we obtain

$$\omega = \omega(\mathbf{k}). \quad (15.14)$$

If we set $\mathfrak{S}_{\text{pl}} = \mathbf{k} \cdot \mathbf{x} - \omega(\mathbf{k})t$, the plane wave (15.11) provides a solution to (15.12).

In particular, for the linear Schrödinger equation

$$i\partial_t \psi + \frac{\chi}{2m} \nabla^2 \psi = 0, \quad (15.15)$$

the dispersion relation has the form

$$\omega = \frac{\chi}{2m} \mathbf{k}^2. \quad (15.16)$$

Another example is the wave equation

$$\frac{1}{c^2} \partial_t^2 \psi = \nabla^2 \psi. \quad (15.17)$$

It has the dispersion relation of the form

$$\frac{1}{c^2} \omega^2 = \mathbf{k}^2, \quad \omega = \pm c |\mathbf{k}|. \quad (15.18)$$

Based on the dispersion relation, the group velocity $\omega = \omega(\mathbf{k})$ is introduced by the following standard formula

$$\mathbf{v}_{\text{gr}} = \nabla_{\mathbf{k}} \omega(\mathbf{k}). \quad (15.19)$$

In particular, the group velocity for the linear Schrödinger equation (15.15) is

$$\mathbf{v}_{\text{gr}} = \mathbf{v}_{\text{gr}}(\mathbf{k}) = \frac{\chi}{m} \mathbf{k}, \quad (15.20)$$

and the group velocity for the wave equation (15.17) is

$$\mathbf{v}_{\text{gr}} = \mathbf{v}_{\text{gr}}(\mathbf{k}) = \pm \frac{c}{|\mathbf{k}|} \mathbf{k}. \quad (15.21)$$

Let us now take a look at the concept of a wave packet. A wave packet in the simplest case can be written in terms of the Fourier transform $\check{\psi}(\mathbf{k}) = (\mathcal{F}\psi)(\mathbf{k})$ defined by (16.65). Namely, an integral of the form

$$(2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-i\omega(\mathbf{k})t} e^{i\mathbf{k} \cdot \mathbf{x}} \check{\psi}(\mathbf{k}) d\mathbf{x} \quad (15.22)$$

with a localized $\check{\psi}(\mathbf{k})$ is called a wave packet. Evidently the integral (15.22) defining a wave packet is a linear combination of plane waves of the form $e^{i\omega(\mathbf{k})t} e^{i\mathbf{k} \cdot \mathbf{x}} \check{\psi}(\mathbf{k})$, hence it is a solution to the primary evolution equation (15.12).

It can be shown in certain asymptotic regimes that, if a wave packet is concentrated around a wavevector $\bar{\mathbf{k}}$, it propagates in physical space with velocity $\mathbf{v}_{\text{gr}}(\bar{\mathbf{k}})$. If $\mathbf{v}_{\text{gr}}(\mathbf{k})$ depends on $\mathbf{k}$, different velocities of propagation of different components of the wave packet lead to its dispersion with time.

Now let us discuss very briefly wave-corpuscles and compare them with wave packets (see Chap. 16 for more details). The wave corpuscles considered in Sect. 15.1.1 also have an oscillatory factor $e^{i\mathfrak{S}}$ which is similar to ψ_{pl} defined by (15.11). As we have shown in Sect. 15.1.1, one can define the dispersion relation $\omega(\mathbf{k})$ and the corresponding group velocity $\mathbf{v}_{\text{gr}}$ for the wave corpuscle based on the phase function $\mathfrak{S}$. In addition to that, the wave corpuscle has a naturally defined center $\mathbf{r}(t)$ and velocity $\mathbf{v}(t)$. A fundamental property of the wave-corpuscle is that both the velocities coincide, $\mathbf{v}_{\text{gr}} = \mathbf{v}$. Another important property of the wave corpuscle is the conservation of its shape at all times, facilitated by the nonlinearity that compensates exactly the dispersion. In Chap. 16 we show that the wave and particle properties of the wave corpuscles coexist perfectly in an accelerating motion.

15.2 Uniform Motion of a Relativistic Balanced Charge

Consider now a free relativistic motion of a charge governed by the NKG equation (34.128) where the external fields vanish, that is, $\varphi_{\text{ex}} = 0$, $\mathbf{A}_{\text{ex}} = 0$. Since the NKG equation is relativistic invariant, the solution can be obtained from the rest solution defined by (14.44), (14.45) by simply applying the Lorentz transformation as in [14], [16]. Hence, a solution to the NKG equation (34.128) representing a free particle that moves with velocity $\mathbf{v}$ is given by the formula

$$\psi(t, \mathbf{x}) = \psi_{\text{free}}(t, \mathbf{x}) = e^{-i(\gamma\omega t - \mathbf{k} \cdot \mathbf{x})} \check{\psi}(\mathbf{x}'), \quad (15.23)$$

with $\check{\psi}(\mathbf{x}') = \check{\psi}_a(\mathbf{x}')$ satisfying the Eq. (14.45) and

$$\check{\psi}_a(\mathbf{x}') = a^{-3/2} \check{\psi}_1\left(\frac{\mathbf{x}'}{a}\right), \quad \mathbf{x}' = \mathbf{x} + \frac{(\gamma - 1)}{v^2} (\mathbf{v} \cdot \mathbf{x}) \mathbf{v} - \gamma \mathbf{v}t, \quad (15.24)$$

$$\mathbf{k} = \gamma \omega \frac{\mathbf{v}}{c^2}, \quad (15.25)$$

where γ is the Lorentz factor:

$$\gamma = (1 - \beta^2)^{-1/2}, \quad \beta = \frac{1}{c} \mathbf{v}. \quad (15.26)$$

One can recognize in the oscillatory exponential factor in (15.23) the *de Broglie plane wave* of the frequency ω with the *de Broglie wavevector* $\mathbf{k}$. The second amplitude factor $\check{\psi}_a$ is determined in terms of the form factor $\check{\psi}(r)$ decaying at infinity. Consequently, *the balanced charge remains well localized and does not disperse in space at all times, justifying its characterization as a wave-corpucle*. The above formulas also indicate that the wave function of the balanced charge contracts by the factor γ as it moves with velocity $\mathbf{v}$, as one expects from special relativity principles. The balanced charge position can be naturally identified with the location of its center $\mathbf{r}(t) = \mathbf{v}t$. Notice also that velocity $\partial_t \mathbf{r}$ of the center coincides with the parameter $\mathbf{v}$ of the Lorentz transformation used to obtain the uniformly moving charge from the resting charge.

All basic characteristics of a free charge can be explicitly calculated. Namely, the charge density ρ defined by the relation (11.132) and the total charge are given by the expressions

$$\rho(t, \mathbf{x}) = \gamma q \left| \check{\psi}(\mathbf{x}') \right|^2, \quad \bar{\rho} = \int \rho(t, \mathbf{x}) d\mathbf{x} = q. \quad (15.27)$$

The total charge energy E is obtained by integration of the density (11.142), and its evaluation is reduced to the formula (14.56), resulting in the expression

$$E = \int u(t, \mathbf{x}) d\mathbf{x} = \gamma mc^2 (1 + \Theta(\omega)), \quad (15.28)$$

where $\Theta(\omega)$ is given by (14.57). Note that for a ground state

$$\omega = \omega_0 = \frac{mc^2}{\chi}, \quad \Theta(\omega_0) = \Theta_0 a_C^2 / a^2. \quad (15.29)$$

Using (11.131) and (11.143), we find that the current density $\mathbf{J}$, the total momentum $\mathbf{P}$ and the total current $\bar{\mathbf{J}}$ for the free charge can be represented as follows:

$$\mathbf{J} = \frac{q}{m} \chi \operatorname{Im} \frac{\nabla \psi}{\psi} |\psi|^2 = \gamma \chi \frac{q}{m} \omega \frac{1}{c^2} \mathbf{v} \left| \tilde{\psi} \right|^2 (\mathbf{x}'), \quad (15.30)$$

$$\mathbf{P} = \gamma m \mathbf{v} (1 + \Theta(\omega)), \quad (15.31)$$

$$\bar{\mathbf{J}} = \int_{\mathbb{R}^3} \mathbf{J}(\mathbf{x}) \, d\mathbf{x} = q \mathbf{v}. \quad (15.32)$$

The total momentum formula (15.31) can be written in the form

$$\mathbf{P} = M \mathbf{v}, \quad M = \gamma m (1 + \Theta(\omega)). \quad (15.33)$$

Taking into account the above kinematic representation $\mathbf{P} = M \mathbf{v}$ of the total momentum, it is natural, [248, Sect. 3.3], [265, Sect. 37], [37], to identify the coefficient M as the *mass* and to define the *rest mass* m_0 of the charge by the formula

$$m_0 = m (1 + \Theta(\omega)). \quad (15.34)$$

Then the expression (15.28) for the energy takes the form of Einstein's mass-energy relation $\bar{\mathcal{E}} = M c^2$. The 4-vector $(E, c\mathbf{P})$ is a relativistic energy-momentum 4-vector with the Lorentz invariant, namely

$$E^2 - c^2 \mathbf{P}^2 = (1 + \Theta)^2 m^2 c^4. \quad (15.35)$$

A direct comparison shows that the above definition of the mass based on the Lorentz invariance of a uniformly moving free charge is fully consistent with the definition of the Newtonian inertial mass which is derived from the analysis of the accelerated motion of localized charges in an external EM field in Sect. 17.6.2, see Remark 17.6.1 for a more detailed discussion.

15.3 A Single Free Non-relativistic Dressed Charge

In this section we consider a non-relativistic dressed charge introduced in Sect. 13.2. A free dressed non-relativistic charge satisfies the system of the NLS-Poisson equations which can be written using (14.87) in the form

$$\begin{aligned} \chi i \left(\partial_t + \frac{i q}{\chi} \varphi \right) \psi &= \frac{\chi^2}{2m} \left[-\nabla^2 \psi + G'_{\nabla} (\psi^* \psi) \psi \right] - q G'_{\varphi} \psi, \\ \nabla^2 \varphi &= -4\pi q |\psi|^2. \end{aligned} \quad (15.36)$$

We would like to show that the construction of the solutions of the NLS-Poisson equations for the uniformly moving dressed charge can be reduced to the construction of the previous section for the uniformly moving balanced charge. Suppose ψ is

the wave-corpuscle defined by formulas (15.1)–(15.2), and the charge potential is defined by

$$\varphi = \hat{\phi}(|\mathbf{x} - \mathbf{v}t|), \quad (15.37)$$

where $\hat{\phi}$ is the potential satisfying (14.78). For the wave-corpuscle of the form (15.1), $|\psi|^2 = \left| \hat{\psi} \right|^2$, and

$$\nabla^2 \varphi = -4\pi q |\psi|^2, \quad \varphi = \hat{\phi}(\mathbf{x} - \mathbf{v}t), \quad (15.38)$$

and according to (14.89) the electrostatic self-interaction is balanced by the nonlinearity:

$$q\varphi\psi + qG'_\varphi\psi = 0. \quad (15.39)$$

Hence representation (15.36) is equivalent to the equation

$$\chi i\partial_t \psi = -\frac{\chi^2}{2m} \nabla^2 \psi + \frac{\chi^2}{2m} G'_\nabla (\psi^* \psi) \psi. \quad (15.40)$$

Obviously, the electric potential φ defined by (15.37) does not enter (15.40), and Eq. (15.40) coincides with the NLS equation (12.32) for the balanced charge distribution with $G'_\nabla = G'$. Consequently, formulas (15.1)–(15.2) provides a solution to (15.36) and together with (15.37) determine the free non-relativistic dressed charge.

15.4 A Relativistic Dressed Charge in Uniform Motion

Here we consider a free dressed charge in the relativistic setting introduced in Sect. 13.1. We use the Lorentz invariance of the system to obtain the state of the dressed charge moving with a constant velocity $\mathbf{v}$. Namely, we apply to the rest solution described by (14.118) the Lorentz transformation from the original “rest frame” to the frame in which the “rest frame” moves with the constant velocity $\mathbf{v}$ as described by the formulas (6.7), (4.29). First, we introduce the normalized velocity β and the Lorentz factor γ

$$\beta = \frac{\mathbf{v}}{c}, \quad \beta = |\beta|, \quad \gamma = \left(1 - \left(\frac{v}{c}\right)^2\right)^{-1/2}, \quad (15.41)$$

and introduce

$$\mathbf{x}' = \mathbf{x} + \frac{\gamma - 1}{\beta^2} (\beta \cdot \mathbf{x}) \beta - \gamma \mathbf{v}t, \text{ or } \mathbf{x}'_\parallel = \gamma (\mathbf{x}_\parallel - \mathbf{v}t), \quad \mathbf{x}'_\perp = \mathbf{x}_\perp, \quad (15.42)$$

where $\mathbf{x}_\parallel$ and $\mathbf{x}_\perp$ refer, respectively, to the components of $\mathbf{x}$ parallel and perpendicular to velocity $\mathbf{v}$. We obtain the following representation for the charge distribution and

EM potentials of a dressed charge moving with velocity $\mathbf{v}$:

$$\begin{aligned}\psi(t, \mathbf{x}) &= e^{-i(\gamma\omega_0 t - \mathbf{k} \cdot \mathbf{x})} \dot{\psi}(|\mathbf{x}'|), \\ \varphi(t, \mathbf{x}) &= \gamma \dot{\varphi}(|\mathbf{x}'|), \quad \mathbf{A}(t, \mathbf{x}) = \gamma \boldsymbol{\beta} \dot{\varphi}(|\mathbf{x}'|),\end{aligned}\tag{15.43}$$

where $\dot{\psi}$, $\dot{\varphi}$ satisfy equilibrium equations (14.120)–(14.121), $\omega_0 = mc^2/\chi$,

$$\mathbf{k} = \gamma \boldsymbol{\beta} \frac{\omega_0}{c} = \boldsymbol{\beta} \frac{\omega}{c} = \gamma \frac{m\mathbf{v}}{\chi}, \quad \omega = \gamma\omega_0.\tag{15.44}$$

The EM fields are given by the expressions

$$\begin{aligned}\mathbf{E}(t, \mathbf{x}) &= -\gamma \nabla \dot{\varphi}(|\mathbf{x}'|) + \frac{\gamma^2}{\gamma + 1} (\boldsymbol{\beta} \cdot \nabla \dot{\varphi}(|\mathbf{x}'|)) \boldsymbol{\beta}, \\ \mathbf{B}(t, \mathbf{x}) &= \gamma \boldsymbol{\beta} \times \nabla \dot{\varphi}(|\mathbf{x}'|).\end{aligned}\tag{15.45}$$

The above formulas provide a solution to field equations (13.14)–(13.15). They indicate that the fields of the dressed charge contract by a factor of γ as the charge moves with velocity $\mathbf{v}$. The oscillatory exponential factor in (15.43) is the *de Broglie plane wave* with frequency ω and *de Broglie wavevector* $\mathbf{k}$. The second factor in the formula (15.43) for ψ involves the form factor $\dot{\psi}(r)$, $r \geq 0$, which is a monotonically decreasing function of r decaying at infinity. Consequently, the dressed charge moving with a constant velocity $\mathbf{v}$ as described by Eqs. (15.42)–(15.43) remains well localized and does not disperse in space at all times justifying its characterization as a wave-corpuscle.

15.4.1 Properties of a Free Dressed Charge

The free dressed charge as described by equalities (15.43)–(15.45) has properties which allow an interpretation in terms of point charge concepts.

Based on the Lagrangian (13.13), we find the symmetric energy-momentum tensor (11.119) with $\theta = 0$ which involves two terms given by (13.12), (9.13). Now we consider the total 4-momentum $\mathbf{P}^\nu$ obtained from its density by integration over the space $\mathbb{R}^3$ as in (10.69) in Sect. 10.6. For a resting charge, the 4-momentum $\mathbf{P}^\nu$ has zero momentum, $\mathbf{P}^\nu = (\mathcal{E}_0(\dot{\psi}), \mathbf{0})$ where the rest energy $\mathcal{E}_0(\dot{\psi})$ is given by (14.125). The energy-momentum of a dressed charge moving with a constant velocity $\mathbf{v}$ and described by (15.41)–(15.42) is described by the 4-vector $\mathbf{P}^\nu = (\mathbf{E}, c\mathbf{P})$ with energy $\mathbf{E}$ and momentum $\mathbf{P}$. Since the free dressed charge is a Lorentz covariant system, its total 4-momentum $\mathbf{P}^\nu = (\mathbf{E}, c\mathbf{P})$ is transformed as a 4-vector, and, as an alternative to direct calculation, can be obtained from $(\mathcal{E}_0(\dot{\psi}), \mathbf{0})$ by the Lorentz transformation. Using this vector property and the value $\mathbf{P}^\nu = (\mathcal{E}_0(\dot{\psi}), \mathbf{0})$ for the resting dressed

charge, we find that the dressed charge total 4-momentum $\mathbf{P}^\nu$ satisfies relations

$$\mathbf{P}^\nu = (\mathbf{E}, c\mathbf{P}), \quad \mathbf{E} = \gamma \mathcal{E}_0 \left(\overset{\circ}{\psi} \right), \quad \mathbf{P} = \gamma \mathcal{E}_0 \left(\overset{\circ}{\psi} \right) \mathbf{v} / c^2. \quad (15.46)$$

Notice that the above expression can be written in the familiar form

$$\mathbf{P} = \tilde{m} \mathbf{v}, \quad \mathbf{E} = \tilde{m} c^2 \text{ with } \tilde{m} = \gamma \mathcal{E}_0 \left(\overset{\circ}{\psi} \right) / c^2. \quad (15.47)$$

We would like to point out that, though the above argument used to obtain the relations (15.46) is rather standard, in our case relations (15.46) are deduced rather than rationally imposed. We can also reasonably assign to the dressed charge described by equalities (15.23)–(15.25) a location $\mathbf{r}(t)$ of its center at any instant t of time by setting $\mathbf{x}' = 0$ in (15.25). Not surprisingly,

$$\mathbf{r}(t) = \mathbf{v}t. \quad (15.48)$$

From (15.48) we readily obtain (for the motion without acceleration) another fundamental relation between the Lorentz parameter $\mathbf{v}$ and position $\mathbf{r}$

$$\partial_t \mathbf{r}(t) = \mathbf{v} = \mathbf{P} / \tilde{m}. \quad (15.49)$$

Obviously the above relation for parameters of the distributed charge has the same form as for a point charge.

Now we discuss wave concepts related to the dressed charge (15.43). Formula (15.42) involves the frequency of time oscillations

$$\omega = \gamma \omega_0.$$

Notice that the equalities (15.44) readily imply the following *dispersion relation* between ω and $\mathbf{k}$ which coincides with the group velocity of the linear medium:

$$\omega = \omega(\mathbf{k}) = \sqrt{\omega_0^2 + c^2 \mathbf{k}^2}, \quad \omega_0 = \frac{mc^2}{\chi}. \quad (15.50)$$

In addition to that, the charge velocity $\mathbf{v}$ and its de Broglie wavevector $\mathbf{k}$ satisfy the following relation

$$\mathbf{v} = \nabla_{\mathbf{k}} \omega(\mathbf{k}),$$

signifying that *the velocity of the dressed charge coincides with the group velocity derived from the dispersion relation (15.50)*. This fact clearly points to the wave origin of the charge kinematics as it moves in the three-dimensional space continuum with the dispersion relation (15.50).

Observe that the dispersion relation (15.50), under the assumption that $\chi = \hbar$, is identical to that of a free charge as described by the linear Klein–Gordon equa-

tion, [267, Sections 1, 18] (see also Sect. 11.6.2), but there are significant differences between the two models which are as follows. First, the dressed charge is described by the pair $\{\psi, A^\mu\}$. From the very outset it includes the EM field as its inseparable part whereas the linear Klein–Gordon model describes a free charge by a complex-valued wave function ψ which is not coupled to its own EM field (not to be confused with an external EM field). Second, a free dressed charge, when it moves, evidently preserves its shape up to the natural Lorentz contraction whereas any wave packet satisfying the linear Klein–Gordon equation spreads out in the course of time.

Chapter 16

Accelerating Wave-Corpuscles

“If one views this phase of the development of theory critically, one is struck by the dualism which lies in the fact that the material point in Newton’s sense and the field as continuum are used as elementary concepts side by side.”

*A. Einstein.*¹

In this chapter we show that the non-relativistic wave-corpuscles describe not only the uniform motion of an elementary charge but its accelerated motion as well. In particular, we present exact wave-corpuscle solutions of the field equations for both the non-relativistic balanced and dressed charges. The constructed wave-corpuscles naturally unite particle and wave properties of charges in accelerated motion in the presence of external EM fields.

The dynamics of the charge distribution for a non-relativistic balanced (or dressed) charge obeys the NLS equation (17.8):

$$\chi i \left(\partial_t + \frac{iq}{\chi} \varphi_{\text{ex}} \right) \psi = \frac{\chi^2}{2m} \left[- \left(\nabla - \frac{iq}{\chi c} \mathbf{A}_{\text{ex}} \right)^2 \psi + G' (\psi^* \psi) \psi \right]. \quad (16.1)$$

We define the wave-corpuscle $\psi = \psi_w$ by a formula similar to (15.1), namely

$$\psi(t, \mathbf{x}) = \psi_w(t, \mathbf{x}) = e^{i\mathfrak{S}} \hat{\psi}(\mathbf{y}), \quad (16.2)$$

$$\hat{\psi}(\mathbf{y}) = \hat{\psi}(|\mathbf{y}|), \quad \mathfrak{S} = \mathfrak{S}(t, \mathbf{y}), \quad \mathbf{y} = \mathbf{x} - \mathbf{r}(t), \quad (16.3)$$

where $\hat{\psi} = \hat{\psi}_a$ is the form factor which relates to the nonlinearity G' according to formula (14.19) with a being the size of the wave-corpuscle, $\mathfrak{S}$ is a real dimensionless phase function, and $\mathbf{r}(t)$ is the center of the wave-corpuscle. Note that the

¹The Einstein quotation is from his Autobiographical notes [295, Autobiographical notes].

WKB method used in quantum mechanics assumes an ansatz similar to (16.2), see Sect. 3.5.1, and the dimensionless phase function is usually written in the form

$$\mathfrak{S} = \frac{S}{\chi}, \quad (16.4)$$

where the phase S is interpreted as the action and the constant χ is equal to the Planck constant $\hbar$. We find it more convenient here to use the dimensionless phase $\mathfrak{S}$.

The defining property of the wave corpuscle is that the shape $|\psi(t, \mathbf{x})|$ of the function ψ does not depend on time, and its time dependence comes only through spatial translation. It turns out that a wave-corpuscle exactly preserves its shape $|\psi|$ in an accelerated motion only for a special class of external EM fields, but this class happens to be sufficiently wide, see Sect. 16.1. In particular, any given trajectory of a charge can be generated by such external EM fields. *Newton's law of motion emerges as the necessary condition for a motion with a preserved shape.* This necessary condition relates the existence of a phase function in the oscillatory factor $e^{i\mathfrak{S}}$ and Newton's law for the spatial translation by $\mathbf{r}(t)$. Such a relation manifests a unification of particle and wave properties in the concept of a wave-corpuscle. We would like to stress that Newton's equations are not postulated as in (6.38) or (6.47), but rather they are derived from the field equations. We also point out that in contrast to the Abraham and Lorentz models and the quantum Abraham models with de Broglie–Bohm laws of quantum motion introduced by M. Kiessling in [196], we do not define a charge as an object with a prescribed geometry. The dynamics and shape of a wave corpuscle is governed by a nonlinear Klein–Gordon or a nonlinear Schrödinger equation in relativistic and non-relativistic cases respectively, and the wave-corpuscle is defined as a special type of solution to these equations.

The specific form of the wave-corpuscle integrates into it both particle attributes such as its center $\mathbf{r}(t)$ as well its wave characteristics. For example, for the wave-corpuscle localized around $\mathbf{y} = \mathbf{0}$, we can naturally define the instantaneous frequency as the time derivative of the phase at the center, that is,

$$\omega = \omega(t) = - \left. \frac{d}{dt} \mathfrak{S}(t, \mathbf{x} - \mathbf{r}(t)) \right|_{\mathbf{x}=\mathbf{r}(t)}, \quad (16.5)$$

with the minus sign chosen based on the standard form of the time oscillation as in $e^{-i\omega t}$. We can also define the de Broglie wavevector by the following formula:

$$\bar{\mathbf{k}} = \bar{\mathbf{k}}(t) = \nabla \mathfrak{S}(t, \mathbf{x} - \mathbf{r}(t))|_{\mathbf{x}=\mathbf{r}(t)}. \quad (16.6)$$

These wave concepts allow us to introduce the dispersion relation and the group velocity for the accelerating wave-corpuscle. The treatment in the following sections includes an analysis of the wave concepts and their relation with particle concepts in the spirit of the de Broglie theory.

We show in Sects. 16.3, 16.4 that the wave-corpuscle provides an approximate solution to the field equations with general external fields with accuracy $\frac{a^2}{R_{\text{ex}}^2}$ where R_{ex} is the scale of variation of the EM potentials, and a is the free charge size parameter. Relativistic accelerating wave corpuscles which are governed by the NKG equation and exactly preserve their shape are treated in Sect. 34.6. In the sections on balanced charge interactions and dressed charge interactions for many charges, we show that the wave corpuscles in general regimes provide approximate solutions to the field equations. Notice that wave-corpuscles describe well localized point-like states in regimes when the charges are spatially separated and the EM fields which act upon them vary significantly only on spatial scales much larger than the typical size of the charge. When charges get closer to each other, they can form a system of bound charges as in the case of the Hydrogen atom. Such states for balanced or dressed charges are not particle-like and are not described by wave-corpuscles, but rather they are similar to the quantum mechanical wave functions, see Sect. 17.3.

We would like to mention that the idea to use the concept of a solitary wave in nonlinear dispersive media to unify particle and wave properties was quite popular. Louis de Broglie tried to use it in his pursuit of material wave mechanics. G. Lochak wrote in his preface to de Broglie's monograph, [77, p. XXXIX]:

"...The first idea concerns the solitons, which we would call *ondes à bosses* (humped waves) at the *Institut Henri Poincaré*. This idea of de Broglie's used to be considered as obsolete and too classical, but it is now quite well known, as I mentioned above, and is likely to be developed in the future, but only provided we realize what the obstacle is and has been for twenty-five years: It resides in the lack of a general principle in the name of which we would be able to choose one nonlinear wave equation from among the infinity of possible equations. If we succeed one day in finding such an equation, a new microphysics will arise."

G. Lochak raised the interesting point of the necessity of having a general principle that would allow the choice of one nonlinearity among infinitely many. In our approach the exact balance of all forces for the resting balanced charge in the equilibrium equation (14.9) is possible for many nonlinearities. We agree with G. Lochak to the extent that there has to be an important physical principle that would allow the choice of the nonlinearity. *Our choice of the nonlinearity to be logarithmic as in (3.27) is justified by its feature of exactly preserving the Planck–Einstein frequency-energy relation*, see Sect. 17.4.1. We have to note though that *the macroscopic dynamics of wave corpuscles which we study in this section is universal, namely it does not depend on a particular nonlinearity*.

16.1 Wave-Corpuscle Preservation in Accelerated Motion

Here we find conditions on the external EM fields $\varphi_{\text{ex}}, \mathbf{A}_{\text{ex}}$ which allow the wave-corpuscle of the form (16.2) to preserve exactly its shape $|\psi|$ during the accelerated motion governed by the NLS equation along a trajectory $\mathbf{r}(t)$. In particular, Newton's law of motion emerges as the necessary condition for such a motion.

16.1.1 A Criterion for Shape Preservation

To trace the shape of a wave-corpuscle, it is convenient to recast Eq. (16.1) for the frame of moving charge using the change of variables $\mathbf{x} - \mathbf{r}(t) = \mathbf{y}$. In $\mathbf{y}$ -coordinates the NLS equation (16.1) takes the form

$$\begin{aligned} & \chi i \partial_t \psi - \chi i \mathbf{v} \cdot \nabla \psi - q \varphi_{\text{ex}} \psi \\ &= \frac{\chi^2}{2m} \left[- \left(\nabla - \frac{i q}{\chi c} \mathbf{A}_{\text{ex}} \right)^2 \psi + G' \psi \right], \end{aligned} \quad (16.7)$$

where $\nabla = \nabla_{\mathbf{y}}$ and $\mathbf{v}$ is the center velocity:

$$\mathbf{v}(t) = \partial_t \mathbf{r}(t). \quad (16.8)$$

Substituting (16.2) in (16.7) and canceling the factor $e^{i\mathfrak{S}}$, we arrive at the following equivalent equation for the phase $\mathfrak{S}$:

$$\begin{aligned} & -\chi \hat{\psi} \partial_t \mathfrak{S} - \chi i \mathbf{v} \cdot \nabla \hat{\psi} + \chi \mathbf{v} \cdot \nabla \mathfrak{S} \hat{\psi} - q \varphi_{\text{ex}} \hat{\psi} \\ &= \frac{\chi^2}{2m} \left[- \left(\nabla - \frac{i q}{\chi c} \mathbf{A}_{\text{ex}} + i \nabla \mathfrak{S} \right)^2 \hat{\psi} + G' \left(\hat{\psi}^* \hat{\psi} \right) \hat{\psi} \right]. \end{aligned} \quad (16.9)$$

Expanding $\left(\nabla - \frac{i q}{\chi c} \mathbf{A}_{\text{ex}} + i \nabla \mathfrak{S} \right)^2 \hat{\psi}$, using the relation

$$-\nabla^2 \hat{\psi} + G' \left(\hat{\psi}^* \hat{\psi} \right) \hat{\psi} = 0, \quad (16.10)$$

which follows from (16.3) and (14.19), we eliminate the term $G' \hat{\psi}$ from Eq. (16.9) and arrive at the equation

$$\begin{aligned} & -\hat{\psi} \chi \partial_t \mathfrak{S} - \chi i \mathbf{v} \cdot \nabla \hat{\psi} + \chi \mathbf{v} \cdot \nabla \mathfrak{S} \hat{\psi} - q \varphi_{\text{ex}} \hat{\psi} \\ &= \frac{\chi^2}{m} \left(\frac{i q}{\chi c} \mathbf{A}_{\text{ex}} - i \nabla \mathfrak{S} \right) \nabla \hat{\psi} \\ &+ \frac{\chi^2}{2m} \left[\left(\nabla \mathfrak{S} - \frac{q}{\chi c} \mathbf{A}_{\text{ex}} \right)^2 - \nabla \cdot \left(i \nabla \mathfrak{S} - \frac{i q}{\chi c} \mathbf{A}_{\text{ex}} \right) \right] \hat{\psi}. \end{aligned} \quad (16.11)$$

Observe that every term in the above equation has either factor $\hat{\psi}$ or $\nabla \hat{\psi}$. Collecting like terms containing $\nabla \hat{\psi}$ and $\hat{\psi}$, we note that Eq. (16.11) holds if the following two equations hold:

$$\left(\chi \nabla \mathfrak{S} - \frac{q}{c} \mathbf{A}_{\text{ex}} - m \mathbf{v} \right) \cdot \nabla \hat{\psi} = 0, \quad (16.12)$$

$$\begin{aligned}
& -\chi \partial_t \mathfrak{S} + \chi \mathbf{v} \cdot \nabla \mathfrak{S} - q \varphi_{\text{ex}} \\
& = -\frac{i\chi}{2m} \nabla \cdot \left(\chi \nabla \mathfrak{S} - \frac{q}{c} \mathbf{A}_{\text{ex}} \right) + \frac{1}{2m} \left(\chi \nabla \mathfrak{S} - \frac{q}{c} \mathbf{A}_{\text{ex}} \right)^2.
\end{aligned} \tag{16.13}$$

Now let us find conditions on the potentials $\varphi_{\text{ex}}, \mathbf{A}_{\text{ex}}$ under which there exists a phase $\mathfrak{S}$ satisfying the above system of Eqs. (16.12), (16.13). Such a phase $\mathfrak{S}$ would provide a wave-corpuscle solution ψ to the NLS equation (16.7) represented by formulas (16.2), (16.3).

Let us consider first Eq. (16.12). Since $\nabla \hat{\psi} = \hat{\psi}'(|\mathbf{y}|) \mathbf{y}/|\mathbf{y}|$, the relation (16.12) is equivalent to the following orthogonality condition:

$$\left(\frac{q}{c} \mathbf{A}_{\text{ex}} - \chi \nabla \mathfrak{S} + m \mathbf{v} \right) \cdot \mathbf{y} = 0. \tag{16.14}$$

To treat the above equation, we use the concept of a *sphere-tangent field*, which is defined as such a vector field $\check{\mathbf{V}}(\mathbf{y})$ that satisfies the orthogonality condition

$$\check{\mathbf{V}}(\mathbf{y}) \cdot \mathbf{y} = 0 \text{ for all } \mathbf{y} \in \mathbb{R}^3. \tag{16.15}$$

Evidently, a sphere-tangent vector field is tangent to spheres centered at the origin. Any vector field $\mathbf{V}(\mathbf{y})$ can be *uniquely* decomposed into a gradient field ∇P and a sphere-tangent field $\check{\mathbf{V}}$ which satisfies the orthogonality condition (16.15), namely

$$\mathbf{V}(\mathbf{y}) = \nabla P(\mathbf{y}) + \check{\mathbf{V}}(\mathbf{y}), \tag{16.16}$$

with details of the decomposition provided in Chap. 38.

We proceed by decomposing the vector potential $\mathbf{A}_{\text{ex}}$ into its sphere-tangent and gradient components, namely

$$\mathbf{A}_{\text{ex}}(t, \mathbf{y}) = \check{\mathbf{A}}_{\text{ex}}(t, \mathbf{y}) + \mathbf{A}_{\text{ex}\nabla}(t, \mathbf{y}), \tag{16.17}$$

with

$$\check{\mathbf{A}}_{\text{ex}}(t, \mathbf{y}) \cdot \mathbf{y} = 0, \tag{16.18}$$

$$\mathbf{A}_{\text{ex}\nabla}(t, \mathbf{y}) = \nabla P(t, \mathbf{y}). \tag{16.19}$$

Equation (16.14) implies that the field $\frac{q}{c} \mathbf{A}_{\text{ex}} - \nabla \mathfrak{S} + m \mathbf{v}$ is purely sphere-tangent. Consequently, its gradient part is zero and we conclude that the following equation is equivalent to (16.14):

$$\chi \nabla \mathfrak{S} = m \mathbf{v} + \frac{q}{c} \mathbf{A}_{\text{ex}\nabla}. \tag{16.20}$$

Now let us turn to Eq. (16.13). Expressing $\nabla \mathfrak{S}$ from (16.20), we rewrite (16.13) in the form

$$\begin{aligned}
& -\chi \partial_t \mathfrak{S} + \mathbf{v} \cdot \left(m\mathbf{v} + \frac{q}{c} \mathbf{A}_{\text{ex}\nabla} \right) - q\varphi_{\text{ex}} \\
& = i \frac{\chi}{2m} \frac{q}{c} \nabla \cdot \check{\mathbf{A}}_{\text{ex}} + \frac{1}{2m} \left(m\mathbf{v} - \frac{q}{c} \check{\mathbf{A}}_{\text{ex}} \right)^2.
\end{aligned} \tag{16.21}$$

Taking the imaginary part of (16.21), we see that the sphere-tangent part of the vector potential must satisfy the following zero divergency condition:

$$\nabla \cdot \check{\mathbf{A}}_{\text{ex}} = 0. \tag{16.22}$$

The real part of (16.21) gives the following equation:

$$\chi \partial_t \mathfrak{S} = m\mathbf{v}^2 + \frac{q}{c} \mathbf{v} \cdot \mathbf{A}_{\text{ex}\nabla} - \frac{1}{2m} \left(m\mathbf{v} - \frac{q}{c} \check{\mathbf{A}}_{\text{ex}} \right)^2 - q\varphi_{\text{ex}}. \tag{16.23}$$

Hence, (16.23) and (16.20) take the form

$$\chi \partial_t \mathfrak{S} = \frac{1}{2} m\mathbf{v}^2 + \frac{q}{c} \mathbf{v} \cdot \mathbf{A}_{\text{ex}} - \frac{q^2}{2mc^2} \check{\mathbf{A}}_{\text{ex}}^2 - q\varphi_{\text{ex}}, \tag{16.24}$$

$$\chi \nabla \mathfrak{S} = m\mathbf{v} + \frac{q}{c} \mathbf{A}_{\text{ex}\nabla}. \tag{16.25}$$

The above relations give an expression for the 4-gradient of $\chi \mathfrak{S}$. The right-hand sides of (16.24), (16.25) can be considered as the coefficients of a differential 1-form. Therefore, if there exists a phase $\mathfrak{S}$ which solves (16.11), the form must be exact, and, as a consequence, must be closed. Conversely, according to the Poincaré's lemma, the form on $\mathbb{R}^4$ is exact if it is closed on $\mathbb{R}^4$, and then the phase $\mathfrak{S}$ can be found by integration of the differential form. Since the right-hand side of (16.25) is always the gradient of a function, the form is closed if the following condition is satisfied:

$$\nabla \left(\frac{1}{2} m\mathbf{v}^2 + \frac{q}{c} \mathbf{v} \cdot \mathbf{A}_{\text{ex}} - \frac{q^2}{2mc^2} \check{\mathbf{A}}_{\text{ex}}^2 - q\varphi_{\text{ex}} \right) = \partial_t \left(m\mathbf{v} + \frac{q}{c} \mathbf{A}_{\text{ex}\nabla} \right). \tag{16.26}$$

This equation can be interpreted as a balance of forces that allows us to exactly preserve the shape of the wave-corpuscle. *Equation (16.26) together with condition (16.22) constitutes a criterion for the preservation of the shape $|\psi|$.*

16.1.2 Trajectory and Phase of a Wave-Corpuscle

If condition (16.26) is fulfilled, the form

$$\left(\frac{1}{2} m\mathbf{v}^2 + \frac{q}{c} \mathbf{v} \cdot \mathbf{A}_{\text{ex}} - \frac{q^2}{2mc^2} \check{\mathbf{A}}_{\text{ex}}^2 - q\varphi_{\text{ex}} \right) dt + \left(m\mathbf{v} + \frac{q}{c} \mathbf{A}_{\text{ex}\nabla} \right) \cdot d\mathbf{y}$$

is exact, and we can find the phase function $\mathfrak{S}$ by the integration:

$$\begin{aligned} \mathfrak{S}(t, \mathbf{y}) = & \frac{1}{\chi} \int_{\Gamma} \left(\frac{1}{2} m \mathbf{v}^2 + \frac{q}{c} \mathbf{v} \cdot \mathbf{A}_{\text{ex}} - \frac{q^2}{2mc^2} \check{\mathbf{A}}_{\text{ex}}^2 - q\varphi_{\text{ex}} \right) dt \\ & + \frac{1}{\chi} \int_{\Gamma} \left(m \mathbf{v} + \frac{q}{c} \mathbf{A}_{\text{ex}\nabla} \right) \cdot d\mathbf{y} \end{aligned} \quad (16.27)$$

where Γ is a curve in the space-time connecting point $(0, \mathbf{0})$ with point $(t, \mathbf{y})$. Since (16.26) is fulfilled, the integral does not depend on the curve. In particular, we take $\Gamma = \Gamma_0$ to be the curve formed by two straight-line segments: the first segment from $(0, \mathbf{0})$ to $(t, \mathbf{0})$ and the second from $(t, \mathbf{0})$ to $(t, \mathbf{y})$:

$$\mathfrak{S}(t, \mathbf{y}) = m \mathbf{v} \cdot \mathbf{y} + \frac{q}{c} \mathbf{y} \cdot \mathbf{A}_{\text{ex}}(t, \mathbf{0}) + s_p(t) + s_{p2}(t, \mathbf{y}) \quad (16.28)$$

where

$$s_p(t) = \frac{1}{\chi} \int_0^t \left(\frac{1}{2} m \mathbf{v}^2(t) + \frac{q}{c} \mathbf{v} \cdot \mathbf{A}_{\text{ex}}(t, \mathbf{0}) - q\varphi_{\text{ex}}(t, \mathbf{0}) \right) dt, \quad (16.29)$$

$$s_{p2}(t, \mathbf{y}) = \frac{q}{\chi c} \int_0^1 \mathbf{y} \cdot (\mathbf{A}_{\text{ex}}(t, s\mathbf{y}) - \mathbf{A}_{\text{ex}}(t, \mathbf{0})) ds, \quad (16.30)$$

and $s_{p2}(t, \mathbf{y})$ is at least quadratic in $\mathbf{y}$. In the above computation we use the following relations:

$$\check{\mathbf{A}}_{\text{ex}}(t, \mathbf{0}) = 0, \quad \mathbf{y} \cdot \check{\mathbf{A}}_{\text{ex}}(t, \mathbf{y}) = 0, \quad \mathbf{A}_{\text{ex}\nabla}(t, \mathbf{0}) = \mathbf{A}_{\text{ex}}(t, \mathbf{0}). \quad (16.31)$$

Singling out the linear part of the phase, we can write the above formulas in the form

$$\mathfrak{S}(t, \mathbf{y}) = \frac{1}{\chi} m \hat{\mathbf{v}} \cdot \mathbf{y} + s_p(t) + s_{p2}(t, \mathbf{y}), \quad (16.32)$$

where

$$\hat{\mathbf{v}} = \mathbf{v} + \frac{q}{mc} \mathbf{A}_{\text{ex}}(t, \mathbf{0}), \quad (16.33)$$

and (16.29) is written in the form

$$s_p(t) = \frac{1}{\chi} \int_0^t \left(\frac{1}{2} m \hat{\mathbf{v}}^2(t) - \frac{q^2}{2mc^2} (\mathbf{A}_{\text{ex}}(t, \mathbf{0}))^2 - q\varphi_{\text{ex}}(t, \mathbf{0}) \right) dt. \quad (16.34)$$

Notice that the integrability condition (16.26) can be rewritten in the form

$$m \partial_t^2 \mathbf{r} = \frac{q}{c} \nabla (\mathbf{v} \cdot \mathbf{A}_{\text{ex}}) - \frac{q^2}{2mc^2} \nabla \check{\mathbf{A}}_{\text{ex}}^2 - q \nabla \varphi_{\text{ex}} - \frac{q}{c} \partial_t \mathbf{A}_{\text{ex}\nabla}. \quad (16.35)$$

The above equation together with (16.22) form a system of equations which involves the EM potentials $\varphi_{\text{ex}}, \mathbf{A}_{\text{ex}}$ and the corpuscle center trajectory $\mathbf{r}(t)$; its fulfillment guarantees that the wave-corpuscle preserves its shape in the course of evolution described by the NLS equation. We refer to (16.35), (16.22) as *non-relativistic wave-corpuscle dynamic balance conditions*. Evidently these conditions do not depend on the nonlinearity G .

By setting $\mathbf{y} = 0$ in (16.35) and taking into account that $\check{\mathbf{A}}_{\text{ex}}(t, 0) = 0$, we obtain the *point balance condition*:

$$m\partial_t^2 \mathbf{r} = \frac{q}{c} \nabla (\mathbf{v} \cdot \mathbf{A}_{\text{ex}})(t, 0) - q \nabla \varphi_{\text{ex}}(t, 0) - \frac{q}{c} \partial_t \mathbf{A}_{\text{ex}}(t, 0). \quad (16.36)$$

To interpret this condition, we note that point $\mathbf{y} = 0$ in the moving frame in the above equation corresponds to point $\mathbf{x} = \mathbf{r}$ in the resting frame and

$$\partial_t \mathbf{A}_{\text{ex}}(t, \mathbf{x})|_{\mathbf{x}=\mathbf{r}} = [\partial_t \mathbf{A}_{\text{ex}}(t, \mathbf{y}) - \mathbf{v} \cdot \nabla \mathbf{A}_{\text{ex}}(t, \mathbf{y})]|_{\mathbf{y}=0}, \quad (16.37)$$

hence the point balance condition (16.36) takes in the rest frame the form

$$\begin{aligned} m\partial_t^2 \mathbf{r} = & \frac{q}{c} \nabla (\mathbf{v} \cdot \mathbf{A}_{\text{ex}})(t, \mathbf{r}) - q \nabla \varphi_{\text{ex}}(t, \mathbf{r}) \\ & - \frac{q}{c} \partial_t \mathbf{A}_{\text{ex}}(t, \mathbf{r}) - \frac{q}{c} \mathbf{v} \cdot \nabla \mathbf{A}_{\text{ex}}(t, \mathbf{r}). \end{aligned} \quad (16.38)$$

Recall that the Lorentz force in terms of EM potentials is given by formula (6.41):

$$\mathbf{f}_{\text{Lor}} = -q \nabla \varphi - \frac{q}{c} \partial_t \mathbf{A} + \frac{q}{c} \nabla (\mathbf{v} \cdot \mathbf{A}) - \frac{q}{c} (\mathbf{v} \cdot \nabla) \mathbf{A},$$

therefore the right-hand side of (16.38) coincides with the Lorentz force. Hence the point balance condition (16.36) in $\mathbf{x}$ -coordinates has the form

$$m\partial_t^2 \mathbf{r} = \mathbf{f}_{\text{Lor}}(t, \mathbf{r}), \quad (16.39)$$

$$\mathbf{f}_{\text{Lor}}(t, \mathbf{r}) = q \mathbf{E}_{\text{ex}}(t, \mathbf{r}) + \frac{q}{c} \mathbf{v} \times \mathbf{B}_{\text{ex}}(t, \mathbf{r}), \quad (16.40)$$

where the EM fields $\mathbf{E}, \mathbf{B}$ are given by (4.6). Therefore the point balance condition coincides with Newton's law of motion (6.38) of a charged point subjected to the Lorentz force $\mathbf{f}_{\text{Lor}}$.

We always assume that the trajectory $\mathbf{r}(t)$ satisfies the point balance condition (16.39), therefore the integrability condition (16.35) after taking into account (16.36) can be written in the form

$$\begin{aligned}
& \frac{q}{c} (\nabla (\mathbf{v} \cdot \mathbf{A}_{\text{ex}}(t, \mathbf{y}) - \nabla (\mathbf{v} \cdot \mathbf{A}_{\text{ex}})(t, 0))) \\
& - \frac{q^2}{c^2} \frac{1}{2m} \nabla \check{\mathbf{A}}_{\text{ex}}^2 - q (\nabla \varphi_{\text{ex}}(t, \mathbf{y}) - \nabla \varphi_{\text{ex}}(t, 0)) \\
& - \frac{q}{c} \partial_t (\mathbf{A}_{\text{ex}\nabla}(t, \mathbf{y}) - \mathbf{A}_{\text{ex}\nabla}(t, \mathbf{0})) = \mathbf{0}.
\end{aligned} \tag{16.41}$$

Notice that the above equation evidently holds for spatially constant fields $\mathbf{A}_{\text{ex}}$ and constant $\nabla \varphi_{\text{ex}}$, that is, in the case when the external EM field is purely electric and homogeneous.

16.1.3 Universality of Dynamic Balance Conditions

The conditions (16.22), (16.41) for preservation of the shape $|\psi|$ are *universal* in the sense that they do not depend on the nonlinearity G or on the form factor $\check{\psi}$. Recall that to derive the conditions we set the coefficients before $\nabla \hat{\psi}$ and $\hat{\psi}$ in (16.11) to be zero. Now we would like to show that this is a necessary requirement if we are looking for universal conditions.

Notice that Eq. (16.11) has the form

$$Q_0 \check{\psi}(r) + \check{\psi}'(r) Q_1 \cdot \mathbf{y}/r = 0,$$

where $\check{\psi}(r)$ is a ground state, and coefficients Q_0, Q_1 obviously do not depend on $\check{\psi}$. If $Q_1 \cdot \mathbf{y}$ is not identically zero, we can separate the variables:

$$\frac{Q_0}{r Q_1 \cdot \mathbf{y}} = -\partial_r \ln \check{\psi}(r),$$

where the left-hand side is $\check{\psi}$ -independent and the right-hand side depends on $\check{\psi}$. This is impossible, hence $Q_1 \cdot \mathbf{y}$ and Q_0 must be zero, leading to Eqs. (16.12), (16.13).

16.1.4 Wave-Corpuscle Motion in the Electric Field

As a simpler but still nontrivial example, we consider here the case of a purely electric homogeneous field with the linear potential φ_{ex} . In this case written in $\mathbf{y}$ -coordinates we have

$$\check{\mathbf{A}}_{\text{ex}} = 0, \quad \varphi_{\text{ex}} = \varphi_{\text{ex}}(t, \mathbf{0}) + \nabla \varphi_{\text{ex}}(t, \mathbf{0}) \cdot \mathbf{y}, \tag{16.42}$$

and the NLS equation (16.1) takes the form

$$i\chi\partial_t\psi = -\frac{\chi^2}{2m}\nabla^2\psi + q\varphi_{\text{ex}}\psi + \frac{\chi^2}{2m}G'\psi. \quad (16.43)$$

The point balance condition (16.39) for the center of the wave-corpuscle then takes the form (in $\mathbf{x}$ -coordinates) of Newton's law of motion in the electric field:

$$m\partial_t^2\mathbf{r} = -q\nabla\varphi_{\text{ex}}(t, \mathbf{r}). \quad (16.44)$$

The phase function can be found according to (16.28):

$$\mathfrak{S}(t, \mathbf{y}) = s_p(t) + \frac{1}{\chi}m\mathbf{v}(t) \cdot \mathbf{y}, \text{ where} \quad (16.45)$$

$$s_p(t) = \frac{m}{2\chi} \int_0^t \mathbf{v}^2(t') dt' - \frac{q}{\chi} \int_0^t \varphi_{\text{ex}}(t', \mathbf{0}) dt', \quad (16.46)$$

$\mathbf{y} = \mathbf{x} - \mathbf{r}$, and $\varphi_{\text{ex}}(t, \mathbf{0})$ when written in $\mathbf{y}$ -coordinates equals $\varphi_{\text{ex}}(t, \mathbf{r})$ in $\mathbf{x}$ -coordinates. Hence we can write in $\mathbf{x}$ -coordinates (slightly abusing the notation)

$$\mathfrak{S}(t, \mathbf{x}) = s_p(t) + \frac{1}{\chi}m\mathbf{v}(t) \cdot (\mathbf{x} - \mathbf{r}(t)), \text{ where} \quad (16.47)$$

$$s_p(t) = \frac{m}{2\chi} \int_0^t \mathbf{v}^2(t') dt' - \frac{q}{\chi} \int_0^t \varphi_{\text{ex}}(t', \mathbf{r}(t')) dt'. \quad (16.48)$$

Note that formula (16.2) is similar to the quasiclassical ansatz (3.60) involving phase $S = \chi\mathfrak{S}$, and we discuss the ansatz in Sect. 3.5.1.

16.1.5 Wave-Corpuscles in the EM Field

Now we give a more general example of fulfillment of the condition (16.41). In this example we prescribe arbitrary EM potentials which are linear with respect to the spatial variable $\mathbf{x}$. First, we take a trajectory $\mathbf{r}(t)$ satisfying Newton's law (16.39) with this potential. Newton's law involves only values of the first derivatives of the EM potential at the point of the trajectory, therefore if we add to the potentials quadratic and higher order terms which vanish on the trajectory, the trajectory still would satisfy Newton's law with the Lorentz force generated by the new potential. Taking that into account, we assume that the second and higher order components of the EM potential's expansion at $\mathbf{r}(t)$ must satisfy certain restrictions. Namely, we assume that $\mathbf{A}_{\text{ex}}$ involves a linear in $\mathbf{y}$ part which is determined by an arbitrary 3×3 matrix $\mathbf{A}_{\text{ex}1}(t)$ with time-dependent elements:

$$\mathbf{A}_{\text{ex}1}(t, \mathbf{y}) = \mathbf{A}_{\text{ex}1}(t) \mathbf{y} = \mathbf{y} \cdot \nabla \mathbf{A}_{\text{ex}}(t, \mathbf{0}). \quad (16.49)$$

We assume that the quadratic and the higher order components of the sphere-tangent component of $\mathbf{A}_{\text{ex}}$ are set to zero, but allow an arbitrary gradient component

$$\mathbf{A}_{\text{ex}}(t, \mathbf{y}) = \mathbf{A}_{\text{ex}}(t, \mathbf{0}) + \mathbf{A}_{\text{ex}1}(t, \mathbf{y}) + \mathbf{A}_{\text{ex}\nabla 2}(t, \mathbf{y}), \quad (16.50)$$

where $\mathbf{A}_{\text{ex}\nabla 2}$ has at least a second order zero at the origin, and its gradient:

$$\mathbf{A}_{\text{ex}\nabla 2}(t, \mathbf{y}) = \nabla P_3(t, \mathbf{y}),$$

with an arbitrary scalar function $P_3(t, \mathbf{y})$ of third order at $\mathbf{0}$. For such a field, its gradient and the sphere-tangent components are given, respectively, by formulas

$$\mathbf{A}_{\text{ex}\nabla} = \nabla P_1 + \nabla P_2 + \nabla P_3, \quad (16.51)$$

$$\check{\mathbf{A}}(t, \mathbf{y}) = \frac{1}{2} [\mathbf{A}_{\text{ex}1}(t) - \mathbf{A}_{\text{ex}1}^T(t)] \mathbf{y} \quad (16.52)$$

where the matrix $\mathbf{A}_{\text{ex}1}^T$ is the matrix $\mathbf{A}_{\text{ex}1}$ transposed,

$$P_1(t, \mathbf{y}) = \mathbf{y} \cdot \mathbf{A}_{\text{ex}}(t, \mathbf{0}), \quad P_2(t, \mathbf{y}) = \frac{1}{2} (\mathbf{y} \cdot \mathbf{A}_{\text{ex}1}(t) \mathbf{y}), \quad (16.53)$$

and the potential $P_3(t, \mathbf{y})$ has a third degree zero at the origin. Note that the action of an anti-symmetric matrix can be written using the cross product, therefore (16.52) can be written as follows:

$$\check{\mathbf{A}}(t, \mathbf{y}) = \frac{1}{2} \check{\mathbf{B}}(t) \times \mathbf{y} \quad (16.54)$$

where $\check{\mathbf{B}} = \nabla \times \check{\mathbf{A}}$. The electric potential φ_{ex} has the form

$$\varphi_{\text{ex}} = \varphi_{\text{ex}}(t, \mathbf{0}) + \mathbf{y} \cdot \nabla \varphi_{\text{ex}}(t, \mathbf{0}) + \varphi_{\text{ex}2}(t, \mathbf{y}), \quad (16.55)$$

where the linear part is arbitrary and $\varphi_{\text{ex}2}(t, \mathbf{y})$ has order two or higher in $\mathbf{y}$, and it is subject to the condition (16.57) which we formulate below.

Now let us check the conditions which guarantee that the wave-corpuscle is an exact solution to the NLS equation. Condition (16.22) is fulfilled for $\mathbf{A}$ defined by (16.52). Hence it is sufficient to satisfy (16.41), which takes the form

$$\frac{q}{c} \nabla (\mathbf{v} \cdot \nabla P_3) - \frac{q^2}{c^2} \frac{1}{2m} \nabla \check{\mathbf{A}}_{\text{ex}}^2 - q \nabla \varphi_{\text{ex}2}(t, \mathbf{y}) - \frac{q}{c} \partial_t (\nabla P_2 + \nabla P_3) = \mathbf{0}. \quad (16.56)$$

To satisfy (16.56), we set

$$\varphi_{\text{ex}2}(t, \mathbf{y}) = -\frac{q}{c^2} \frac{1}{2m} \check{\mathbf{A}}_{\text{ex}}^2 - \frac{1}{c} \partial_t (P_2(t, \mathbf{y}) + P_3(t, \mathbf{y})) + \frac{1}{c} \mathbf{v} \cdot \nabla P_3. \quad (16.57)$$

We then determine the phase $\mathfrak{S}$ of the wave-corpuscle using (16.29):

$$s_p(t) = \frac{1}{\chi} \int_0^t \left(\frac{1}{2} m \mathbf{v}^2(t') + \frac{q}{c} \mathbf{v} \cdot \mathbf{A}_{\text{ex}}(t', 0) - q \varphi_{\text{ex}}(t', 0) \right) dt', \quad (16.58)$$

and, if P_3 is homogeneous of third degree,

$$\begin{aligned} s_{p2}(t, \mathbf{y}) &= \frac{q}{c} \frac{1}{\chi} \int_0^1 \mathbf{y} \cdot \mathbf{A}_{\text{ex}1}(t, \mathbf{y}) s \, ds + \frac{q}{c} \frac{1}{\chi} \int_0^1 \mathbf{y} \cdot \mathbf{A}_{\text{ex}\nabla 2}(t, \mathbf{y}) s^2 \, ds \\ &= \frac{q}{2c} \frac{1}{\chi} \mathbf{y} \cdot \mathbf{A}_{\text{ex}1}(t, \mathbf{y}) + \frac{q}{3c} \frac{1}{\chi} \mathbf{y} \cdot \mathbf{A}_{\text{ex}\nabla 2}(t, \mathbf{y}). \end{aligned} \quad (16.59)$$

Formula (16.2) takes the form

$$\psi(t, \mathbf{x}) = e^{i\mathfrak{S}} \hat{\psi}, \quad \hat{\psi} = \hat{\psi}(|\mathbf{x} - \mathbf{r}(t)|), \quad (16.60)$$

$$\mathfrak{S}(t, \mathbf{y}) = m \frac{1}{\chi} \mathbf{v} \cdot \mathbf{y} + \frac{q}{c} \frac{1}{\chi} \mathbf{y} \cdot \mathbf{A}_{\text{ex}}(t, \mathbf{0}) + s_p(t) + s_{p2}(t, \mathbf{y}) \quad (16.61)$$

where

$$\mathbf{y} = \mathbf{x} - \mathbf{r}(t), \quad \mathbf{v} = \partial_t \mathbf{r}.$$

Note that if the external magnetic field satisfies the following conditions

$$\begin{aligned} \mathbf{y} \cdot \mathbf{A}_{\text{ex}1}(t, \mathbf{y}) &= 0, \\ \mathbf{y} \cdot \mathbf{A}_{\text{ex}\nabla 2}(t, \mathbf{y}) &= 0 \text{ for all } \mathbf{y}, \end{aligned} \quad (16.62)$$

then $s_{p2} = 0$, and the phase function is linear in $\mathbf{y}$, namely

$$\begin{aligned} \psi(t, \mathbf{x}) &= e^{i\mathfrak{S}(t, \mathbf{x})} \hat{\psi}(t, \mathbf{x}), \quad \mathfrak{S}(t, \mathbf{x}) = \frac{1}{\chi} m \mathbf{v}(t) \cdot (\mathbf{x} - \mathbf{r}(t)) + s_p(t), \\ \hat{\psi} &= \hat{\psi}(|\mathbf{x} - \mathbf{r}(t)|), \quad \varphi = \hat{\varphi}(|\mathbf{x} - \mathbf{r}(t)|). \end{aligned} \quad (16.63)$$

The above calculations can be summarized in the following statement

Theorem 16.1.1 *Let the potentials $\varphi, \mathbf{A}$ have the form (16.55), (16.49), (16.50). Assume that $\varphi(t, \mathbf{0})$ and $\nabla \varphi(t, \mathbf{0})$ are continuously differentiable functions of t , $\mathbf{A}(t, \mathbf{0})$ is a continuously differentiable function of t , $\mathbf{A}_1(t)$ is an arbitrary 3×3 matrix which continuously differentially depends on t , and ∇P_3 is a continuously differentiable function of t and $\mathbf{y}$. Let the quadratic part φ_2 of the potential φ satisfy (16.57). Let the trajectory $\mathbf{r}(t)$ satisfy Newton's equation (16.39)–(16.40). Then the wave-corpuscle defined by formula (16.60)–(16.61) provides a solution to the NLS equation (16.1).*

Remark 16.1.1 The above construction does not depend on the nonlinearity $G' = G'_a$ as long as (14.19) is satisfied. It is also uniform with respect to $a > 0$, and

the dependence on a in (16.60) is only through the form factor $\psi(|\mathbf{x} - \mathbf{r}|) = a^{-3/2} \psi_1(a^{-1}|\mathbf{x} - \mathbf{r}|)$. Obviously, if $\psi(t, \mathbf{x})$ is defined by (16.2) then $|\psi(t, \mathbf{x})|^2 \rightarrow \delta(\mathbf{x} - \mathbf{r})$ as $a \rightarrow 0$.

Remark 16.1.2 The form (16.2) for an exact wave-corpuscle solution to the NLS is the same as the WKB ansatz in the quasi-classical approach, see Sect. 3.5.1. The trajectories of the charge centers coincide with the trajectories that can be found by applying well-known quasiclassical asymptotics to solutions of (16.1) if one neglects the nonlinearity. Note though that there are two important effects of the nonlinearity not presented in the standard quasiclassical approach. First of all, due to the nonlinearity, the charge preserves its shape in the course of evolution, whereas in the linear model, any wave packet disperses over time. Second, the quasiclassical asymptotic expansions produce infinite asymptotic series which provide for a formal solution, whereas the properly introduced nonlinearity as in (14.19), (14.21) allows one to obtain an exact solution. For a treatment of mathematical aspects of the approach to nonlinear wave mechanics based on the WKB asymptotic expansions we refer the reader to [201, 5.2], [204, 3.2] and references therein.

Note that an arbitrary vector-function $\mathbf{r}(t)$ can be obtained as a solution of (16.39), (16.40) with an appropriate choice of $\mathbf{E}_{\text{ex}}(t, \mathbf{r})$, $\mathbf{B}_{\text{ex}}(t, \mathbf{r})$. Indeed, let $\mathbf{A}_{\text{ex}} = 0$,

$$\varphi_{\text{ex}} = -m\ddot{\mathbf{r}}(t) \cdot (\mathbf{x} - \mathbf{r}(t)) / q$$

and (16.39), (16.40) is obviously fulfilled with $\mathbf{f}_{\text{Lor}}(t, \mathbf{r}) = -\nabla\varphi_{\text{ex}}$. Thus, we can conclude that the wave-corpuscle (16.60) as an exact solution to the NLS equation (16.1) for a balanced (or dressed) charge with an appropriate choice of the external EM field can model any motion of a point charge.

Remark 16.1.3 We can use in the definition of a wave-corpuscle (16.2) a radial form factor ψ which instead of (14.19) satisfies the eigenvalue problem

$$-\nabla^2 \psi + G'(|\psi|^2) \psi = \lambda \psi. \quad (16.64)$$

Obviously, the above Eq. (16.64) can be considered as the steady state Eq. (14.9) with a modified nonlinearity $G' - \lambda$. Note that ψ is a solution of the NLS equation (16.1) with the nonlinearity G' if and only if the function $e^{-i\frac{\lambda}{2m}\lambda t} \psi$ is a solution of the NLS equation with the nonlinearity $G' - \lambda$. Therefore the wave-corpuscle solutions constructed in Theorem 16.1.1 based on ψ with the nonlinearity $G' - \lambda$ provide a solution to the NLS equation with the original nonlinearity, one has only to add to the phase function $\mathfrak{S}$ an additional term $\frac{\lambda}{2m}\lambda t$. The existence of many solutions of the nonlinear eigenvalue problems of the form (16.64) has been proved in many papers, see [38], [39], [41], [157]. Therefore, there are many wave-corpuscle solutions of a given NLS equation with the same trajectory of motion $\mathbf{r}(t)$ and the same EM fields but with different form factors.

16.2 Particle and Wave Features in Accelerated Motion of a Wave-Corpuscle

The wave corpuscles described by (16.2) have both particle and wave properties. The first important particle-like property of a wave-corpuscle is its localization captured by the form factor $\check{\psi}(|\mathbf{y}|)$, $\mathbf{y} = \mathbf{x} - \mathbf{r}$ where $\mathbf{r}$ is the center of the charge. Obviously, $\check{\psi}(|\mathbf{x} - \mathbf{r}|)$ is obtained from $\check{\psi}(|\mathbf{x}|)$ by a spatial translation, and the wave corpuscle has a well-defined *translational velocity* $\partial_t \mathbf{r}$. Another particle-like property concerns the dynamics of the charge center. It is determined by Newton's equation (16.39) with the Lorentz force (16.40), and the Newtonian dynamics is the fundamental property of particles. Below we discuss the wave properties of wave-corpuscles such as the dispersion relation, de Broglie wavevector and group velocity.

16.2.1 The de Broglie Wavevector

First of all, we define the de Broglie wavevector for a localized charge distribution which satisfies the charge normalization condition (14.17).

Let $\check{\psi}(\mathbf{k}) = [\mathcal{F}\psi](\mathbf{k})$ be the Fourier transform of the wave function $\psi(\mathbf{x})$:

$$\check{\psi}(\mathbf{k}) = [\mathcal{F}\psi](\mathbf{k}) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-i\mathbf{k} \cdot \mathbf{x}} \psi(\mathbf{x}) d\mathbf{x}. \quad (16.65)$$

According to Parseval's theorem, Eq. (14.17) implies a similar normalization condition on the Fourier transform $\check{\psi}(\mathbf{k})$, namely

$$\int_{\mathbb{R}^3} |\check{\psi}(\mathbf{k})|^2 d\mathbf{k} = 1. \quad (16.66)$$

Hence, we can introduce the de Broglie wavevector as the center $\bar{\mathbf{k}}(\psi)$ for $\check{\psi}(\mathbf{k})$ defined by

$$\bar{\mathbf{k}} = \bar{\mathbf{k}}(\psi) = \int_{\mathbb{R}^3} \mathbf{k} |\check{\psi}(\mathbf{k})|^2 d\mathbf{k}. \quad (16.67)$$

Notice that the following elementary identity

$$\mathbf{k} |\check{\psi}(\mathbf{k})|^2 = \mathbf{k} \check{\psi}(\mathbf{k}) \check{\psi}^*(\mathbf{k}) = \frac{1}{2i} \left\{ i\mathbf{k} \check{\psi}(\mathbf{k}) \check{\psi}^*(\mathbf{k}) - \check{\psi}(\mathbf{k}) [i\mathbf{k} \check{\psi}(\mathbf{k})]^* \right\}$$

together with Eq. (16.67) imply the following representation for the de Broglie wavevector:

$$\bar{\mathbf{k}} = \int_{\mathbb{R}^3} \text{Im} \frac{\nabla \psi(\mathbf{x})}{\psi(\mathbf{x})} |\psi(\mathbf{x})|^2 d\mathbf{x}. \quad (16.68)$$

The general formula (16.68) applied to the wave-corpuscle of the form (16.2) yields

$$\bar{\mathbf{k}} = \int_{\mathbb{R}^3} \nabla \mathfrak{S}(t, \mathbf{y}) \left| \hat{\psi}(\mathbf{y}) \right|^2 d\mathbf{y}. \quad (16.69)$$

Using formula (16.32) for the phase and the normalization condition (16.66), we obtain that

$$\begin{aligned} \bar{\mathbf{k}} &= \int_{\mathbb{R}^3} \left(\frac{1}{\chi} m \hat{\mathbf{v}} + \nabla s_{p2}(t, \mathbf{y}) \right) \left| \hat{\psi}(\mathbf{y}) \right|^2 d\mathbf{y} \\ &= \frac{1}{\chi} m \hat{\mathbf{v}} + \int_{\mathbb{R}^3} \nabla s_{p2}(t, \mathbf{y}) \left| \hat{\psi}(\mathbf{y}) \right|^2 d\mathbf{y}. \end{aligned} \quad (16.70)$$

If $s_{p2}(t, \mathbf{y})$ is even, we recollect that $\left| \hat{\psi}(\mathbf{y}) \right|^2$ is even too and conclude that the *de Broglie wavevector* is given by the formula

$$\bar{\mathbf{k}} = \frac{1}{\chi} m \hat{\mathbf{v}}, \quad (16.71)$$

where

$$\hat{\mathbf{v}}(t) = \partial_t \mathbf{r}(t) + \frac{q}{mc} \mathbf{A}_{\text{ex},0},$$

$$\mathbf{A}_{\text{ex},0} = \mathbf{A}_{\text{ex}}(t, \mathbf{y} = \mathbf{0}) = \mathbf{A}_{\text{ex}}(t, \mathbf{x} = \mathbf{r}) = \mathbf{A}_{\text{ex}}(t, \mathbf{r}).$$

Another definition of the wavevector of a wave-corpuscle is given by (16.6). Using (16.32), we observe that formula (16.6) yields the same result (16.71), and the definitions agree.

Remark 16.2.1 Formula (16.71) holds as an exact identity under the assumption that $s_{p2}(t, \mathbf{y})$ is an even function of $\mathbf{y}$. In the general case, it holds asymptotically. Note that the odd terms in $s_{p2}(t, \mathbf{y})$ start with the third order, therefore the non-zero term

$$\int_{\mathbb{R}^3} \nabla s_{p2}(t, \mathbf{y}) \left| \hat{\psi}(\mathbf{y}) \right|^2 d\mathbf{y}$$

is of order

$$\int_{\mathbb{R}^3} |\mathbf{y}|^2 \left| \hat{\psi}(\mathbf{y}) \right|^2 d\mathbf{y} = O(a^2),$$

and provides a vanishing contribution as $a \rightarrow 0$.

16.2.2 The Dispersion Relation and Group Velocity

In this subsection we consider two alternative ways to define dispersion relations and group velocities for a wave-corpuscle. Both ways lead to the same result: *the group velocity defined at the de Broglie wavevector coincides with the translational velocity of the wave-corpuscle*. This identity indicates the wave origin of the charge translational motion.

The first way to define the dispersion relation for the wave-corpuscle is based on the wave-corpuscle representation (16.2). We consider the special case of the wave-corpuscle defined by (16.60) with an even s_{p2} as in Sect. 16.2.1. We define the frequency ω as in (16.5) and, using (16.29), obtain:

$$\begin{aligned} -\omega &= \partial_t \mathfrak{S}(t, \mathbf{x} - \mathbf{r})_{\mathbf{x}=\mathbf{r}} = \partial_t \mathfrak{S}(t, \mathbf{y})_{\mathbf{y}=\mathbf{0}} - \mathbf{v} \cdot \nabla \mathfrak{S}(t, \mathbf{y})_{\mathbf{y}=\mathbf{0}} \\ &= \partial_t s_p - \frac{m}{\chi} \mathbf{v} \cdot \mathbf{v} - \frac{q}{c\chi} \mathbf{v} \cdot \mathbf{A}_{\text{ex},0} = -\frac{1}{2\chi} m \mathbf{v}^2(t) - \frac{q}{\chi} \varphi_{\text{ex}}(t, \mathbf{r}). \end{aligned}$$

Taking into account (16.71), we obtain the following expression for the wave-corpuscle frequency:

$$\omega = \frac{1}{2} \frac{\chi}{m} \left(\bar{\mathbf{k}} - \frac{q}{\chi c} \mathbf{A}_{\text{ex},0} \right)^2 + \frac{q}{\chi} \varphi_{\text{ex}}(t, \mathbf{r}). \quad (16.72)$$

The frequency depends on the wavevector $\mathbf{k} = \bar{\mathbf{k}}$, and such a dependence is called the *dispersion relation*. We find the *group velocity* then by the standard formula

$$\mathbf{v}_{\text{gr}} = \nabla_{\mathbf{k}} \omega(\mathbf{k}) = \frac{\chi}{m} \bar{\mathbf{k}} - \frac{q}{mc} \mathbf{A}_{\text{ex},0}. \quad (16.73)$$

Using (16.71), we observe that

$$\mathbf{v}_{\text{gr}} = \partial_t \mathbf{r}(t), \quad (16.74)$$

that is, *the group velocity exactly coincides with the translational velocity of the wave-corpuscle*.

The second natural way to determine the dispersion relation for the wave-corpuscle is to take the *linear part* of Eq. (16.1) evaluated at the center of the wave-corpuscle $\mathbf{y} = \mathbf{0}$ (or $\mathbf{x} = \mathbf{r}$):

$$\chi i \left(\partial_t + \frac{iq}{\chi} \varphi_{\text{ex}}(\mathbf{r}) \right) \psi = -\frac{\chi^2}{2m} \left(\nabla - \frac{iq}{\chi c} \mathbf{A}_{\text{ex}}(\mathbf{r}) \right)^2 \psi. \quad (16.75)$$

Then we look for plane wave solutions $e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t}$ of this equation and obtain the dispersion relation $\omega(\mathbf{k})$ for the linear part of equation (16.1):

$$\omega(\mathbf{k}) = \frac{\chi}{2m} \left(\mathbf{k} - \frac{q}{\chi c} \mathbf{A}_{\text{ex}}(\mathbf{r}) \right)^2 + \frac{q}{\chi} \varphi_{\text{ex}}(\mathbf{r}). \quad (16.76)$$

This formula coincides with (16.72), and the corresponding group velocity is given by (16.73).

To summarize the above analysis, we may state that even when the charge accelerates, it perfectly combines the properties of a wave and corpuscle, justifying the name wave-corpuscle mechanics for its dynamics. Its wave nature is demonstrated, in particular, in the de Broglie exponential factor and the equality (16.74), indicating the wave origin for the charge motion. The corpuscle properties are manifested in the factor $\psi(|\mathbf{x} - \mathbf{r}(t)|)$ and soliton-like translational propagation with the center $\mathbf{r}(t)$ satisfying the classical Newton's equation (16.39). Importantly, the introduced nonlinearities are stealthy in the sense that they don't show themselves in the point and wave features of the dynamics and kinematics of what looks like soliton-like waves propagating like classical test point charges.

16.3 Wave-Corpuscles in a General Field as an Approximation

Let us consider now the case of a general EM field with potentials $\varphi_{\text{ex}}, \mathbf{A}_{\text{ex}}$. We show that a wave-corpuscle is an approximate solution with a discrepancy of order $\frac{a^2}{R_{\text{ex}}^2}$ where R_{ex}^2 is a spatial scale of variation of the external EM fields. An alternative treatment of wave-corpuscles in a general EM field which is based on methods of classical field theory and uses the concept of a trajectory of concentration can be found in Sect. 34.2.1.

The construction of the wave-corpuscle is accomplished in several steps. As the first step, we take a trajectory $\mathbf{r}(t)$ which satisfies the equation of motion (16.39) with the Lorentz force derived from the potentials $\varphi_{\text{ex}}, \mathbf{A}_{\text{ex}}$. We put $\mathbf{x} - \mathbf{r}(t) = \mathbf{y}$ and write the NLS equation (16.7) in $\mathbf{y}$ -coordinates as follows:

$$\begin{aligned} & \chi i \partial_t \psi - i \chi \mathbf{v} \cdot \nabla \psi - q \varphi_{\text{ex}} \psi \\ &= \frac{\chi^2}{2m} \left[-\nabla^2 \psi + \frac{2iq}{\chi c} \mathbf{A}_{\text{ex}} \cdot \nabla \psi + \frac{iq}{\chi c} (\nabla \cdot \mathbf{A}_{\text{ex}}) \psi + \frac{q^2}{\chi^2 c^2} \mathbf{A}_{\text{ex}}^2 \psi + G' \psi \right]. \end{aligned} \quad (16.77)$$

Then we single out the linear part of the external fields around the trajectory:

$$\mathbf{A}_{\text{ex}}(t, \mathbf{r}(t) + \mathbf{y}) = \mathbf{A}_{\text{ex}}(t, \mathbf{r}(t)) + \mathbf{A}_{\text{ex}1}(t) \mathbf{y} + \mathbf{A}_{\text{ex}2}(t, \mathbf{y}), \quad (16.78)$$

$$\varphi_{\text{ex}}(t, \mathbf{r}(t) + \mathbf{y}) = \varphi_{\text{ex}}(t, \mathbf{r}(t)) + \varphi_{\text{ex}1}(\mathbf{y}) + \varphi_{\text{ex}2}(t, \mathbf{y}), \quad (16.79)$$

where the linear part can be written in the form

$$\varphi_{\text{ex1}}(\mathbf{y}) = \mathbf{y} \cdot \nabla \varphi_{\text{ex}}(t, \mathbf{r}(t)), \quad \mathbf{A}_{\text{ex1}}(t) \mathbf{y} = \mathbf{y} \cdot \nabla \mathbf{A}_{\text{ex}}(t, \mathbf{r}(t)), \quad (16.80)$$

and higher order terms satisfy

$$|\varphi_{\text{ex2}}(s\mathbf{y})| \leq Cs^2, \quad |\mathbf{A}_{\text{ex2}}(t)(s\mathbf{y})| \leq Cs^2.$$

We expand $\mathbf{A}_{\text{ex1}}(t) \mathbf{y}$ into sphere-tangent and potential parts:

$$\mathbf{A}_{\text{ex1}}(t) \mathbf{y} = \check{\mathbf{A}}_{\text{ex1}}(\mathbf{y}) + \nabla P_{\text{ex2}}(\mathbf{y}), \quad (16.81)$$

where $\check{\mathbf{A}}_{\text{ex1}}(\mathbf{y})$ is given by the anti-symmetric part of the matrix $\mathbf{A}_{\text{ex1}}(t)$, and

$$P_{\text{ex2}}(\mathbf{y}) = \frac{1}{2} \mathbf{A}_{\text{ex1}}(t) \mathbf{y} \cdot \mathbf{y}$$

is determined by its symmetric part.

Then we define the auxiliary vector potential corresponding to the trajectory by throwing away higher order terms:

$$\mathbf{A}_{\text{au}}(t, \mathbf{r}(t) + \mathbf{y}) = \mathbf{A}_{\text{ex}}(t, \mathbf{r}(t)) + \mathbf{A}_{\text{ex1}}(t, \mathbf{y}). \quad (16.82)$$

We define the auxiliary electric potential

$$\varphi_{\text{au}}(t, \mathbf{r}(t) + \mathbf{y}) = \varphi_{\text{ex}}(t, \mathbf{r}(t)) + \varphi_{\text{au1}}(t, \mathbf{y}) + \varphi_{\text{au2}}(t, \mathbf{y}), \quad (16.83)$$

where φ_{au1} is the linear part defined by (16.80), $\varphi_{\text{au2}}(t, \mathbf{y})$ is defined so that condition (16.57) holds:

$$\varphi_{\text{au2}}(t, \mathbf{y}) = -\frac{q}{c^2} \frac{1}{2m} \check{\mathbf{A}}_{\text{ex1}}^2 - \frac{1}{c} \partial_t P_{\text{ex2}}(t, \mathbf{y}). \quad (16.84)$$

Note that according to (16.54)

$$\check{\mathbf{A}}_{\text{ex1}} = \frac{1}{2} \check{\mathbf{B}}(t) \times \mathbf{y}, \quad \check{\mathbf{B}}(t) = \nabla \times \mathbf{A}_{\text{ex}}(t, \mathbf{r}(t)).$$

We write the auxiliary NLS equation similar to (16.77):

$$\begin{aligned} & i\chi \partial_t \psi - i\chi \mathbf{v} \cdot \nabla \psi - q\varphi_{\text{au}} \psi \\ &= \frac{\chi^2}{2m} \left[-\nabla^2 \psi + \frac{2iq}{\chi c} \mathbf{A}_{\text{au}} \cdot \nabla \psi + \frac{iq}{\chi c} (\nabla \cdot \mathbf{A}_{\text{au}}) \psi + \frac{q^2}{\chi^2 c^2} \mathbf{A}_{\text{au}}^2 \psi + G' \psi \right]. \end{aligned} \quad (16.85)$$

The wave-corpuscle $\psi = \psi_{\text{au}}$ constructed in Sect. 16.1 is an exact solution to this equation. Now we can rewrite the auxiliary equation (16.85) in the form of a

perturbed NLS equation (16.77):

$$\begin{aligned} & i\chi\partial_t\psi - i\chi\mathbf{v} \cdot \nabla\psi - q\varphi_{\text{ex}}\psi \\ &= \frac{\chi^2}{2m} \left[-\nabla^2\psi + \frac{2iq}{\chi c} \mathbf{A}_{\text{ex}} \cdot \nabla\psi + \frac{iq}{\chi c} (\nabla \cdot \mathbf{A}_{\text{ex}}) \psi + \frac{q^2}{\chi^2 c^2} \mathbf{A}_{\text{ex}}^2 \psi + G'\psi \right] - D, \end{aligned} \quad (16.86)$$

where the discrepancy D has the form

$$\begin{aligned} D = & \frac{iq\chi}{mc} (\mathbf{A}_{\text{ex}} - \mathbf{A}_{\text{au}}) \cdot \nabla\psi + \frac{iq\chi}{2mc} (\nabla \cdot (\mathbf{A}_{\text{ex}} - \mathbf{A}_{\text{au}})) \psi \\ & + q(\varphi_{\text{ex}} - \varphi_{\text{au}}) \psi + \frac{q^2}{2mc^2} (\mathbf{A}_{\text{ex}}^2 - \mathbf{A}_{\text{au}}^2) \psi. \end{aligned} \quad (16.87)$$

In the following subsection we prove that the discrepancy D is of second order with respect to the size parameter.

16.3.1 Estimate of the Discrepancy

Let us assume that the external fields have a spatial scale of variation R_{ex} , namely they can be represented in the form

$$\frac{q}{mc^2} \mathbf{A}_{\text{ex}}(t, \mathbf{y}) = \mathbf{A}'_{\text{ex}}(t, \mathbf{y}/R_{\text{ex}}), \quad \frac{q}{mc^2} \varphi_{\text{ex}}(t, \mathbf{y}) = \varphi'_{\text{ex}}(t, \mathbf{y}/R_{\text{ex}}) \quad (16.88)$$

where $\mathbf{A}'$ and φ' are dimensionless potentials, and the potentials, as functions of dimensionless variables $\mathbf{y}/R_{\text{ex}} = \mathbf{z}$, have bounded first and second derivatives. Assume also that R_{ex} is much larger than the size parameter a , namely

$$\frac{a^2}{R_{\text{ex}}^2} \ll 1.$$

Obviously, large or small values R_{ex} correspond, respectively, to almost homogeneous or highly inhomogeneous EM fields.

Below we show that the discrepancy D in the Eq. (16.86) satisfies the following *discrepancy estimate*:

$$\frac{1}{mc^2} \|D\| \leq C \frac{a^2}{R_{\text{ex}}^2}, \quad (16.89)$$

where a is the charge size parameter and C is a constant. Here the norm $\|\cdot\|$ is the norm of square integrable functions in L^2 space as in (17.7) and (14.25), namely

$$\|\psi\| = \|\psi\|_{L^2} = \left(\int |\psi|^2 d\mathbf{x} \right)^{1/2}. \quad (16.90)$$

The choice of this norm is natural, since the functions ψ are bounded in this norm according to the normalization condition (17.7), and the smallness of D in this norm is equivalent to the smallness of the discrepancy D compared with the principal terms in the equations. Another natural choice for the norm $\|\psi\|$ is the norm in the space C^0 of continuous functions,

$$\|\psi\|_{C^0} = \sup_{\mathbf{x}} |\psi(\mathbf{x})|.$$

If we use this norm, we have to take into account that $\|\psi\|_{C^0} \sim a^{-3/2}$ and to ensure that the terms which enter Eqs. (16.85) and (16.86) are of order 1, we have to multiply the equations by $a^{3/2}$. As a result, the smallness of the rescaled discrepancy $a^{3/2}D$ in this norm is equivalent to the smallness of D relative to the principal terms in the Eqs. (16.85) and (16.86). An analysis similar to the one we present below shows that

$$\frac{1}{mc^2} \|a^{3/2}D\|_{C^0} \leq C \frac{a^2}{R_{\text{ex}}^2}. \quad (16.91)$$

It is convenient to use here the reduced Compton wavelength a_C defined by the formula (14.59), namely

$$a_C = \frac{\chi}{mc},$$

and rewrite (16.87) as follows:

$$\begin{aligned} D = & iqa_C (\mathbf{A}_{\text{ex}} - \mathbf{A}_{\text{au}}) \cdot \nabla \psi \\ & + \frac{iq a_C}{2} (\nabla \cdot (\mathbf{A}_{\text{ex}} - \mathbf{A}_{\text{au}})) \psi + q (\varphi_{\text{ex}} - \varphi_{\text{au}}) \psi + \frac{1}{2m} \frac{q^2}{c^2} (\mathbf{A}_{\text{ex}}^2 - \mathbf{A}_{\text{au}}^2). \end{aligned} \quad (16.92)$$

To estimate (16.87), we can use the identity $\check{\mathbf{A}}_{\text{ex}} \cdot \nabla \psi_{\text{au}} = 0$ and obtain that

$$(\mathbf{A}_{\text{ex}} - \mathbf{A}_{\text{au}}) \cdot \nabla \psi_{\text{au}} = (\mathbf{A}_{\text{ex}\nabla} - \mathbf{A}_{\text{au}\nabla}) \cdot \nabla \psi_{\text{au}}. \quad (16.93)$$

Note that

$$\begin{aligned} |\mathbf{A}_{\text{ex}} - \mathbf{A}_{\text{au}}| & \leq C \frac{1}{R_{\text{ex}}^2} |\mathbf{y}|^2, \quad |\varphi_{\text{ex}} - \varphi_{\text{au}}| \leq C \frac{1}{R_{\text{ex}}^2} |\mathbf{y}|^2, \\ |\nabla \cdot (\mathbf{A}_{\text{ex}} - \mathbf{A}_{\text{au}})| & \leq C \frac{1}{R_{\text{ex}}^2} |\mathbf{y}|. \end{aligned} \quad (16.94)$$

Let us assume here that the ground state $\left| \psi_1(|\mathbf{x}|) \right|^2$ decays fast as $|\mathbf{x}| \rightarrow \infty$ in the sense that for some constant C_0

$$\int \left| \psi_1(|\mathbf{x}|) \right|^2 |\mathbf{x}|^4 d\mathbf{x} \leq C_0, \quad \int \left| \nabla \psi_1(|\mathbf{x}|) \right|^2 |\mathbf{x}|^4 d\mathbf{x} \leq C_0. \quad (16.95)$$

We have then

$$\begin{aligned} \int |\psi|^2 |\mathbf{y}|^2 d\mathbf{y} &= \int a^{-3} \left| \psi_1(a^{-1}\mathbf{y}) \right|^2 |\mathbf{y}|^2 d\mathbf{y} \leq Ca^2, \\ \int |\psi|^2 |\mathbf{y}|^4 d\mathbf{y} &\leq Ca^4, \quad \int |\nabla\psi|^2 |\mathbf{y}|^4 d\mathbf{y} \leq Ca^2. \end{aligned} \quad (16.96)$$

Using the above inequalities to estimate the L^2 norm of (16.92), we conclude that

$$\begin{aligned} \|D\| &\leq C_1 a_C \frac{1}{R_{\text{ex}}^2} \left(\int |\mathbf{y}|^4 |\nabla\psi|^2 d\mathbf{x} \right)^{1/2} + C_1 \frac{a_C}{R_{\text{ex}}^2} \left(\int |\mathbf{y}|^2 |\psi|^2 d\mathbf{x} \right)^{1/2} \\ &\quad + C_1 \frac{1}{R_{\text{ex}}^2} \left(\int |\mathbf{y}|^4 |\psi|^2 d\mathbf{x} \right)^{1/2} \leq C_2 \frac{a^2}{R_{\text{ex}}^2} + C_2 \frac{a_C a}{R_{\text{ex}}^2}. \end{aligned} \quad (16.97)$$

Let us assume also that the size parameter is greater than the Compton wavelength, or not much smaller, namely

$$\frac{a_C}{a} \leq C, \quad (16.98)$$

and obtain (16.89).

Remark 16.3.1 If we replace (16.98) by a weaker assumption

$$\frac{a_C}{R_{\text{ex}}} \leq C,$$

we obtain instead of (16.89) the estimate

$$\|D\| \leq C_3 \frac{a}{R_{\text{ex}}}$$

which again shows that the wave-corpuscle gives the principal term of asymptotic solution to the NLS even when χ and a_C do not tend to zero.

16.3.2 Perturbed Wave-Corpuscles

The above results show that the wave-corpuscle ψ_w is an approximate solution to the NLS equation with general potentials, and the discrepancy of the approximation is small. Now we show how this fact can imply that an exact solution of the NLS equation is close to the wave-corpuscle if its form factor provides a minimum of energy. For simplicity, we assume that the external magnetic field is absent,

$$\mathbf{A}_{\text{ex}} = 0,$$

but the electric field potential can be general and is not assumed to satisfy conditions required for the exact preservation of the wave-corpuscle. We introduce a new unknown function $\tilde{\psi}(t, \mathbf{y})$ (it can be considered as a perturbed form factor $\hat{\psi}(\mathbf{y})$) by a formula analogous to (16.60):

$$\psi = \tilde{\psi}(t, \mathbf{y}) e^{i\mathfrak{S}}, \quad \mathbf{y} = \mathbf{x} - \mathbf{r}(t) \quad (16.99)$$

where $\mathfrak{S}$ is defined by (16.45):

$$\mathfrak{S}(t, \mathbf{y}) = m \frac{1}{\chi} \mathbf{v} \cdot \mathbf{y} + s_p(t), \quad (16.100)$$

$$s_p(t) = \frac{1}{\chi} \int_0^t \left(\frac{1}{2} m \mathbf{v}^2(t) - q \varphi_{\text{ex}}(t, \mathbf{r}(t)) \right) dt, \quad (16.101)$$

and $\mathbf{r}(t)$ satisfies Newton's equation (16.39) which takes the form

$$\partial_t^2 \mathbf{r} = -q \nabla \varphi_{\text{ex}}(t, \mathbf{r}(t)). \quad (16.102)$$

Hence $\mathfrak{S}$ and $\mathbf{r}$ satisfy (16.20) and (16.23), but now $\tilde{\psi}(\mathbf{y})$ is *not assumed to have a fixed shape or to be real-valued*. Consequently, the terms $\partial_t \tilde{\psi}$, $\nabla^2 \tilde{\psi}$, $G'(\tilde{\psi}^* \tilde{\psi}) \tilde{\psi}$ in (16.7) do not drop out.

Substituting representation (16.99) into (16.7), we find that the function $\tilde{\psi}$ satisfies the following equation:

$$i\chi \partial_t \tilde{\psi} - q(\varphi_{\text{ex}} - \varphi_{\text{au}}) \tilde{\psi} = \frac{\chi^2}{2m} \left[-\nabla^2 \tilde{\psi} + G'(\tilde{\psi}^* \tilde{\psi}) \tilde{\psi} \right]. \quad (16.103)$$

Since the difference $\varphi_{\text{ex}} - \varphi_{\text{au}}$ is small, this equation is a small perturbation of the following NLS equation without external fields:

$$i\chi \partial_t \tilde{\psi} = \frac{\chi^2}{2m} \left[-\nabla^2 \tilde{\psi} + G'(\tilde{\psi}^* \tilde{\psi}) \tilde{\psi} \right]. \quad (16.104)$$

Note that since the Eq.(16.103) is written in the moving frame, we look for its solutions which are close to the *rest* solution ψ . To estimate the influence of the field $\varphi_{\text{ex}} - \varphi_{\text{au}}$, we assume that the wave-corpuscle was initially an exact solution and

$$\varphi_{\text{ex}} = \varphi_{\text{au}} \text{ for } t \leq 0; \quad (16.105)$$

therefore

$$\tilde{\psi} = \psi \text{ for } t \leq 0. \quad (16.106)$$

16.3.2.1 Energy of a Perturbed Form Factor

According to (11.160), the canonical energy of a solution $\tilde{\psi}$ to the NLS (16.103) in an electric field with potential $\varphi_{\text{ex}} - \varphi_{\text{au}}$ is given by the formula

$$\begin{aligned} \mathcal{E}(\psi, \varphi_{\text{ex}} - \varphi_{\text{au}}) &= \int_{\mathbb{R}^3} \dot{u} \, d\mathbf{x} \\ &= \frac{\chi^2}{2m} \int_{\mathbb{R}^3} [|\nabla \psi|^2 + G(|\psi|^2)] \, d\mathbf{x} + q \int_{\mathbb{R}^3} \psi^2 (\varphi_{\text{ex}} - \varphi_{\text{au}}) \, d\mathbf{x}. \end{aligned} \quad (16.107)$$

We can apply the formalism of Sect. 11.8 to the form factor $\tilde{\psi}$. In particular, we obtain from the energy equation (11.170) the total energy equation

$$\partial_t \mathcal{E}(\tilde{\psi}, \varphi_{\text{ex}} - \varphi_{\text{au}}) = q \int_{\mathbb{R}^3} |\tilde{\psi}|^2 \partial_t (\varphi_{\text{ex}} - \varphi_{\text{au}}) \, d\mathbf{x}. \quad (16.108)$$

This equation shows that the energy of the perturbed form factor $\tilde{\psi}$ varies slowly. In particular, using (16.105) and (16.106) we conclude that

$$\mathcal{E}(\tilde{\psi}, \varphi_{\text{ex}} - \varphi_{\text{au}}) = \mathcal{E}(\psi, 0) \text{ for } t \leq 0.$$

16.3.3 On Stability of the Perturbed Form Factor

To provide for stability of the wave-corpuscle, we assume the following natural requirement on the ground state which is used in the construction of the nonlinearity G in Sect. 14.3. Namely, we assume that *the ground state is the global minimum of the energy functional*

$$\mathcal{E}(\psi) = \mathcal{E}(\psi, 0) = \int_{\mathbb{R}^3} \frac{\chi^2}{2m} \left[|\nabla \tilde{\psi}|^2 + G(|\tilde{\psi}|^2) \right] d\mathbf{x} \quad (16.109)$$

restricted to the functions that satisfy the normalization condition (14.17):

$$\mathcal{E}(\psi) = \min_{\text{normalized } \tilde{\psi}} \mathcal{E}(\tilde{\psi}). \quad (16.110)$$

In particular, we show in Sect. 35.1.1 that the Gaussian ψ (14.38), which corresponds to the logarithmic nonlinearity G as in (14.41), provides the global minimum of the energy according to the Gross inequality (sometimes called the logarithmic Sobolev inequality). The set of normalized distributions ψ with the minimal energy coincides in non-degenerate cases with the spatial translations of ψ :

$$\left\{ \text{normalized } \tilde{\psi} : \mathcal{E}(\tilde{\psi}) = \mathcal{E}(\dot{\psi}) \right\} = \text{Transl}(\dot{\psi}),$$

$$\text{Transl}(\dot{\psi}) = \left\{ \tilde{\psi} : \dot{\psi}(\mathbf{x} - \mathbf{r}), \mathbf{r} \in \mathbb{R}^3 \right\}.$$

This is true in particular for the logarithmic nonlinearity.

First, to clarify our approach, we consider solutions of Eq. (16.104) corresponding to $\varphi_{\text{ex}} = \varphi_{\text{au}}$. If we consider normalized distributions $\tilde{\psi}$ with a slightly higher than minimal energy,

$$\mathcal{E}(\tilde{\psi}) \leq \mathcal{E}(\dot{\psi}) + \varepsilon, \quad 0 < \varepsilon \ll 1, \quad (16.111)$$

they should converge to $\text{Transl}(\dot{\psi})$ as $\varepsilon \rightarrow 0$. Since the energy $\mathcal{E}(\tilde{\psi}, 0)$ of the solution of the NLS (16.104) is preserved, $\tilde{\psi}(t)$ should converge to $\text{Transl}(\dot{\psi})$ if $\tilde{\psi}(0) \rightarrow \dot{\psi}$; in other words, $\tilde{\psi}(t) \rightarrow \dot{\psi}(\mathbf{x} - \tilde{\mathbf{r}})$ with some $\tilde{\mathbf{r}} = \tilde{\mathbf{r}}(t)$; such a property is called *orbital stability* and is mathematically rigorously proven in [61] for the NLS (16.104) with the logarithmic nonlinearity. Note that, since $\dot{\psi}$ and therefore $\tilde{\psi}(t)$ are spatially localized, we can apply to $\tilde{\mathbf{r}}(t)$ the results of Sect. 17.2.3 and conclude that the limit $\tilde{\mathbf{r}}(t)$ must satisfy Newton's law (16.102). Therefore, the two solutions of the Newton's equation $\tilde{\mathbf{r}}(t)$ and $\mathbf{r}(t)$ coincide not only for $t \leq 0$ but also for $t > 0$.

Now let us consider (16.103) with a small $\varphi_{\text{ex}} - \varphi_{\text{au}}$. The conservation law (11.58) holds for $\tilde{\psi}$, therefore it is normalized for all t . Note that according to (16.106) and (16.105)

$$\dot{\mathcal{E}}(\tilde{\psi}, \varphi_{\text{ex}} - \varphi_{\text{au}}) = \dot{\mathcal{E}}(\dot{\psi}, 0) \text{ for } t \leq 0,$$

and, according to (16.108)

$$\dot{\mathcal{E}}(\tilde{\psi}(t), \varphi_{\text{ex}} - \varphi_{\text{au}}) = \dot{\mathcal{E}}(\dot{\psi}, 0) + q \int_0^t \int |\tilde{\psi}|^2 \partial_t (\varphi_{\text{ex}} - \varphi_{\text{au}}) \, d\mathbf{x} dt. \quad (16.112)$$

Taking into account that

$$\dot{\mathcal{E}}(\tilde{\psi}(t), \varphi_{\text{ex}} - \varphi_{\text{au}}) = \dot{\mathcal{E}}(\tilde{\psi}(t), 0) + q \int |\tilde{\psi}|^2 (\varphi_{\text{ex}} - \varphi_{\text{au}}) \, d\mathbf{x},$$

we conclude that (16.111) holds, namely

$$\dot{\mathcal{E}}(\tilde{\psi}(t), 0) \leq \mathcal{E}(\dot{\psi}) + \varepsilon, \quad (16.113)$$

if the integrals

$$q \int_0^t \int |\tilde{\psi}|^2 \partial_t (\varphi_{\text{ex}} - \varphi_{\text{au}}) \, d\mathbf{x} dt, \quad q \int |\tilde{\psi}|^2 (\varphi_{\text{ex}} - \varphi_{\text{au}}) \, d\mathbf{x} \quad (16.114)$$

are small. As we already pointed out, the inequality (16.111) implies that $\tilde{\psi}(t)$ converges to $\dot{\psi}(\mathbf{x} - \tilde{\mathbf{r}})$ as $\varepsilon \rightarrow 0$, and $\tilde{\mathbf{r}} = \mathbf{r}$ satisfies Newton's equation (16.102). We observe now that, according to (16.79), the difference

$$\varphi_{\text{ex}} - \varphi_{\text{au}} = \varphi_{\text{ex}}(t, \mathbf{r}(t) + \mathbf{y}) - \varphi_{\text{ex}}(t, \mathbf{r}(t)) - \mathbf{y} \cdot \nabla \varphi_{\text{ex}}(t, \mathbf{r}(t))$$

has second order zero at $\mathbf{r}(t)$, and the radius of localization of $\dot{\psi}$ and $|\tilde{\psi}|$ is of order a , therefore the integrals (16.114) are of order $\frac{a^2}{R_{\text{ex}}^2}$ where R_{ex}^2 is the same as in (16.88). Therefore, the ε in (16.113) is small if $\frac{a}{R_{\text{ex}}}$ is small, and we can expect that the perturbed form factor $\tilde{\psi}(t)$ converges to the translated ground state $\dot{\psi}(\mathbf{x} - \mathbf{r}(t))$ as $\frac{a}{R_{\text{ex}}} \rightarrow 0$.

16.4 Wave-Corpuscle for an Accelerating Balanced Charge

Field equations for a non-relativistic balanced charge form a system of NLS equations (12.14) coupled with the Poisson equations (12.16)–(12.17). The NLS equation for a single charge can be written in the form (16.1), and Eq. (12.16) for the electric potentials and (12.17) for the magnetic potentials take, respectively, the form

$$\nabla^2 \varphi = -4\pi q |\psi|^2, \quad (16.115)$$

$$\nabla^2 \mathbf{A} = -4\pi \frac{1}{c} \mathbf{v} q |\psi|^2. \quad (16.116)$$

Evidently, the NLS equation (16.1) for a single charge does not involve $\varphi, \mathbf{A}$. Consequently, it can be solved independently of (16.115), (16.116), and we can apply arguments of Sect. 16.1. We still consider below the EM potentials defined by (16.115), (16.116), since they are useful when we consider interacting charges.

In a particular case when the external EM potentials have the form of linear functions of spatial variables as in Sect. 16.1.5, the wave-corpuscle is an exact solution. Recall that we obtain wave-corpuscles by solving exactly the original Eqs. (13.27)–(13.28) for a certain class of external EM fields. As we show in Sect. 16.1.5, if the charge distribution has the form of a wave-corpuscle, and its shape is preserved, then the center $\mathbf{r}(t)$ of the wave corpuscle must obey the following *Newton's law with the Lorentz force* (16.39):

$$m \partial_t^2 \mathbf{r} = \mathbf{f}_{\text{Lor}}(t, \mathbf{r}). \quad (16.117)$$

Since for a wave-corpuscle $|\psi|^2 = \left| \dot{\psi} \right|^2(\mathbf{y} - \mathbf{r}(t))$, the scalar potential is obtained as a solution of (16.115):

$$\varphi = \dot{\varphi}(\mathbf{x} - \mathbf{r}) \quad (16.118)$$

where $\dot{\varphi}$ is given in terms of $\left|\dot{\psi}\right|^2$ by (14.11), (14.2).

The vector potential is described in Sect. 16.4.3 below.

16.4.1 Current, Charge, Energy and Momentum for a Wave-Corpuscle

The vector potential for a non-relativistic balanced charge is given by formula (12.20) in terms of the current $\mathbf{J}_\ell$, and for a single charge we omit index ℓ . The current densities related to the wave-corpuscle are given by the expressions (12.15), namely

$$\rho = q |\psi|^2 = q \hat{\psi}^2, \quad (16.119)$$

$$\mathbf{J} = \frac{\chi q}{m} \psi \psi^* \operatorname{Im} \frac{\tilde{\nabla} \psi}{\psi} = \frac{q}{m} \left(-\frac{q}{c} \mathbf{A}_{\text{ex}} + \chi \nabla \mathfrak{S} \right) \hat{\psi}^2.$$

Expressing $\chi \nabla \mathfrak{S}$ from (16.25), we obtain

$$\mathbf{J} = \frac{q}{m} \hat{\psi}^2 \left(m \mathbf{v} - \frac{q}{c} \check{\mathbf{A}}_{\text{ex}} \right). \quad (16.120)$$

The current is evidently partitioned into the part generated by the charge motion which is proportional to velocity $\mathbf{v}$, and the part which is related to the external magnetic field. According to (13.34), the momentum density is proportional to the current density: $\mathbf{P} = \frac{m}{q} \mathbf{J}$.

Now let us consider the case when $\check{\mathbf{A}}_{\text{ex}}$ involves only a linear term as in (16.52), (16.54), or, more generally, it is an odd function of $\mathbf{y}$. In this case the integral of $\check{\mathbf{A}}_{\text{ex}} \hat{\psi}^2$ vanishes, and the total current and the total momentum are given by formulas

$$\bar{\mathbf{J}} = \int \mathbf{J} d\mathbf{x} = q \mathbf{v}, \quad \bar{\mathbf{P}} = \frac{m}{q} \bar{\mathbf{J}} = m \mathbf{v}. \quad (16.121)$$

Obviously, the above formula coincides with the expression for the current and the momentum of a point charge. According to (12.29), the *canonical energy density* for solutions of the NLS equations can be written as follows:

$$\tilde{u} = \dot{T}^{00} = q_\ell \varphi_{\text{ex}} \psi_\ell \psi_\ell^* + \frac{\chi^2}{2m} \left[\tilde{\nabla} \psi \cdot \tilde{\nabla}^* \psi^* + G(\psi^* \psi) \right], \quad (16.122)$$

and for the wave-corpuscle

$$\tilde{u} = q_\ell \varphi_{\text{ex}} \dot{\psi}^2 + \frac{\chi^2}{2m} \left[\left(\nabla \mathfrak{S} - \frac{q}{\chi c} \mathbf{A}_{\text{ex}} \right)^2 \dot{\psi}^2 + \left| \nabla \dot{\psi} \right|^2 + G(\dot{\psi}^2) \right]. \quad (16.123)$$

Since $\dot{\psi}$ is a rest solution satisfying (14.9), we can use (14.182) with $\varphi = 0$ and obtain that

$$\int \left[\left| \nabla \dot{\psi} \right|^2 + G \left(\dot{\psi}^2 \right) \right] d\mathbf{x} = \frac{2}{3} \int \left| \nabla \dot{\psi} \right|^2 d\mathbf{x}. \quad (16.124)$$

Expressing $\chi \nabla \mathfrak{S}$ from (16.25), we find that

$$\begin{aligned} \mathring{\mathcal{E}} &= \int \tilde{u} d\mathbf{x} = \int q \varphi_{\text{ex}} \dot{\psi}^2 d\mathbf{x} + \frac{\chi^2}{3m} \int \left| \nabla \dot{\psi} \right|^2 d\mathbf{x} \\ &\quad + \frac{1}{2m} \int \left(m\mathbf{v} - \frac{q}{c} \check{\mathbf{A}}_{\text{ex}} \right)^2 \dot{\psi}^2 d\mathbf{x} \\ &= \frac{1}{2} m \mathbf{v}^2 + \int q \varphi_{\text{ex}} \dot{\psi}^2 d\mathbf{x} + \frac{q^2}{2mc^2} \int \check{\mathbf{A}}_{\text{ex}}^2 \dot{\psi}^2 d\mathbf{x} + \frac{\chi^2}{3m} \int \left| \nabla \dot{\psi} \right|^2 d\mathbf{x}. \end{aligned} \quad (16.125)$$

Taking into account the dependence of $\dot{\psi} = \dot{\psi}_a$ on the size parameter a and (16.54), we conclude that

$$\mathring{\mathcal{E}} = \frac{1}{2} m \mathbf{v}^2 + \int q \varphi_{\text{ex}} \dot{\psi}^2 d\mathbf{x} + mc^2 \frac{a_C^2}{a^2} \Theta_0 + \frac{a^2 q^2}{8mc^2} \int \left(\check{\mathbf{B}}_1 \times \mathbf{y} \right)^2 \dot{\psi}_1^2 d\mathbf{y}, \quad (16.126)$$

where $a_C = \frac{\chi}{mc}$, Θ_0 is defined in (14.58), and $\Theta_0 = 1/2$ for the logarithmic nonlinearity. The last term is $O(a^2/R^2)$ with a macroscopic length scale R related to $|\check{\mathbf{B}}_1|$. In the limit where

$$\frac{a_C}{a} \rightarrow 0, \quad \frac{a^2}{R^2} \rightarrow 0,$$

and $\dot{\psi}^2$ converges to the delta function, we obtain from (16.126) the classical formula

$$\mathring{\mathcal{E}} = \frac{1}{2} m \mathbf{v}^2 + q \varphi_{\text{ex}}(\mathbf{r}) \quad (16.127)$$

for the canonical energy of a point charge.

16.4.2 The Planck–Einstein Relation for a Wave-Corpuscle

Let us consider in more detail the case of a linear magnetic potential with a linear sphere-tangent and constant potential part, namely with $P_2 = 0$, $P_3 = 0$ in (16.51). In this case, formula (16.126) can be simplified. Namely, according to (16.55) and (16.57)

$$\varphi_{\text{ex}} = \varphi_{\text{ex}}(t, \mathbf{0}) + \mathbf{y} \cdot \nabla \varphi_{\text{ex}}(t, \mathbf{0}) - \frac{q}{c^2} \frac{1}{2m} \check{\mathbf{A}}_{\text{ex}}^2, \quad (16.128)$$

and, taking into account that $\varphi_{\text{ex}}(t, \mathbf{y} = \mathbf{0}) = \varphi_{\text{ex}}(t, \mathbf{x} = \mathbf{r})$, we obtain from (16.125) that

$$\mathring{\mathcal{E}} = \frac{1}{2} m \mathbf{v}^2 + q \varphi_{\text{ex}}(t, \mathbf{r}) + m c^2 \frac{a_C^2}{a^2} \Theta_0. \quad (16.129)$$

Now let us compare the above formula with the instantaneous frequency defined by (16.5) for the phase (16.28), (16.29):

$$\omega = \frac{1}{2} \frac{m}{\chi} \mathbf{v}^2 + q \frac{1}{\chi} \varphi_{\text{ex}}(t, \mathbf{r}). \quad (16.130)$$

We see that the *Planck–Einstein frequency-energy relation* for energies and frequencies of two states of the charge is fulfilled:

$$\mathring{\mathcal{E}}(t_2) - \mathring{\mathcal{E}}(t_1) = \chi (\omega(t_2) - \omega(t_1)). \quad (16.131)$$

Note that the above formula holds for any nonlinearity G' , and the Planck–Einstein formula for multiharmonic solutions is obtained in Sect. 17.4.1 for the logarithmic nonlinearity. Notice also that according to (16.129)

$$\mathring{\mathcal{E}} = \chi \omega + \chi m c^2 \frac{a_C^2}{a^2} \Theta_0, \quad (16.132)$$

hence the Planck–Einstein relation in our theory cannot be written in the form $\mathring{\mathcal{E}} = \chi \omega$ but rather it is written as in (16.131) for the differences of energies and frequencies, that is, $\Delta \mathring{\mathcal{E}} = \chi \Delta \omega$.

16.4.3 The Vector Potential for a Non-relativistic Wave Corpuscle

A moving balanced charge generates the magnetic field, and here we describe its potential. The principal part $\mathbf{A}_v$ of the vector potential $\mathbf{A}$ has for small a a simple form:

$$\mathbf{A}_v = \frac{1}{c} q \mathbf{v} \mathring{\varphi}(\mathbf{x} - \mathbf{r}). \quad (16.133)$$

Now let us take a look at the details. The vector potential is given by (12.20) and (16.120), namely

$$\begin{aligned}
\mathbf{A} &= \frac{1}{c} \int \frac{1}{|\mathbf{x} - \mathbf{x}'|} \mathbf{J}(t, \mathbf{x}') d\mathbf{x}' \\
&= \frac{1}{c} \int \frac{1}{|\mathbf{x} - \mathbf{x}'|} \frac{q}{m} \hat{\psi}^2 \left(m\mathbf{v} - \frac{q}{c} \check{\mathbf{A}}_{\text{ex}} \right) d\mathbf{x}' \\
&= \frac{1}{c} q\mathbf{v} \int \frac{1}{|\mathbf{x} - \mathbf{x}'|} \hat{\psi}^2(|\mathbf{y}|) d\mathbf{x}' - \frac{q^2}{mc^2} \int \hat{\psi}^2(|\mathbf{y}|) \frac{1}{|\mathbf{x} - \mathbf{r} - \mathbf{y}|} \check{\mathbf{A}}_{\text{ex}} d\mathbf{y}.
\end{aligned}$$

Hence we obtain a decomposition of the vector potential for a wave corpuscle into two parts: the part generated by the charge motion and proportional to velocity, and the part $\mathbf{A}_b$ which can be considered as induced by the external magnetic field, namely

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c} q\mathbf{v} \hat{\varphi}(\mathbf{x} - \mathbf{r}) + \mathbf{A}_b \quad (16.134)$$

where

$$\mathbf{A}_b = -\frac{q^2}{mc^2} \int \hat{\psi}^2 \frac{1}{|\mathbf{x} - \mathbf{r} - \mathbf{y}|} \check{\mathbf{A}}_{\text{ex}} d\mathbf{y}.$$

Now we would like to show that $\mathbf{A}_b$ is small at macroscopic distances. To see that, we first take a look at the case where $\check{\mathbf{A}}_{\text{ex}}$ is replaced by its linear part:

$$\check{\mathbf{A}}_{\text{ex}}(t, \mathbf{r} + \mathbf{y}) = \frac{1}{2} \check{\mathbf{B}}(t) \times \mathbf{y}, \quad \check{\mathbf{B}}(t) = \nabla \times \mathbf{A}_{\text{ex}}(t, \mathbf{r}). \quad (16.135)$$

In this case the part of the charge of the vector potential generated by the wave-corpuscle has the form

$$\mathbf{A}_b = -\frac{1}{2} \frac{q^2}{mc^2} \check{\mathbf{B}}(t) \times \int \mathbf{y} \hat{\psi}^2(\mathbf{y}) \frac{1}{|\mathbf{x} - \mathbf{r} - \mathbf{y}|} d\mathbf{y}. \quad (16.136)$$

The dependence of $\mathbf{A}_b$ on the charge size a is through the integral

$$\int \mathbf{y} \hat{\psi}_a^2(\mathbf{y}) \frac{1}{|\mathbf{x} - \mathbf{r} - \mathbf{y}|} d\mathbf{y} = a \int \mathbf{y} \hat{\psi}_1^2(\mathbf{y}) \frac{1}{|\mathbf{x} - \mathbf{r} - a\mathbf{y}|} d\mathbf{y}.$$

We consider the behavior of the potential at macroscopic distances, and since a is vanishingly small compared with macroscopic scales, we assume that

$$|\mathbf{x} - \mathbf{r}| \geq R_0 > 0. \quad (16.137)$$

Since $\hat{\psi}^2(\mathbf{y}) = \hat{\psi}^2(-\mathbf{y})$

$$\int \mathbf{y} \hat{\psi}_1^2(\mathbf{y}) \frac{1}{|\mathbf{x} - \mathbf{r}|} d\mathbf{y} = \mathbf{0},$$

and we have:

$$\begin{aligned}
 \int \mathbf{y} \psi_a^2(\mathbf{y}) \frac{1}{|\mathbf{x} - \mathbf{r} - \mathbf{y}|} d\mathbf{y} &= a \int \mathbf{y} \psi_1^2(\mathbf{y}) \left(\frac{1}{|\mathbf{x} - \mathbf{r} - a\mathbf{y}|} - \frac{1}{|\mathbf{x} - \mathbf{r}|} \right) d\mathbf{y} \\
 &= a \int \mathbf{y} \psi_1^2(\mathbf{y}) \frac{|\mathbf{x} - \mathbf{r} - a\mathbf{y}|^2 - |\mathbf{x} - \mathbf{r}|^2}{|\mathbf{x} - \mathbf{r} - a\mathbf{y}| |\mathbf{x} - \mathbf{r}| (|\mathbf{x} - \mathbf{r} - a\mathbf{y}| + |\mathbf{x} - \mathbf{r}|)} d\mathbf{y} \\
 &= -a^2 \int \mathbf{y} \psi_1^2(\mathbf{y}) \frac{2(\mathbf{x} - \mathbf{r}) \cdot \mathbf{y}}{|\mathbf{x} - \mathbf{r} - a\mathbf{y}| |\mathbf{x} - \mathbf{r}| (|\mathbf{x} - \mathbf{r} - a\mathbf{y}| + |\mathbf{x} - \mathbf{r}|)} d\mathbf{y} + O(a^3), \\
 O(a^3) &= a^3 \int \mathbf{y} \psi_1^2(\mathbf{y}) \frac{\mathbf{y}^2}{|\mathbf{x} - \mathbf{r} - a\mathbf{y}| |\mathbf{x} - \mathbf{r}| (|\mathbf{x} - \mathbf{r} - a\mathbf{y}| + |\mathbf{x} - \mathbf{r}|)} d\mathbf{y}.
 \end{aligned}$$

Hence

$$|\mathbf{A}_b| = O\left(\frac{a^2}{R_0^2}\right), \quad \frac{a}{R_0} \ll 1. \quad (16.138)$$

Similarly, we can obtain under assumption (16.137) the above estimate if the second order terms of $\check{\mathbf{A}}_{\text{ex}}(t, \mathbf{r} + \mathbf{y})$ are present. Hence, the principal part of the vector potential is given for small a by the velocity dependent term of (16.134) presented by (16.133).

In a similar way, we can estimate $\mathbf{B}_b = \nabla \times \mathbf{A}_b$. Indeed,

$$\begin{aligned}
 \mathbf{B}_b &= \nabla \times \mathbf{A} - \frac{1}{c} q \nabla \check{\varphi}(\mathbf{x} - \mathbf{r}) \times \mathbf{v} = -\frac{1}{2} \frac{q^2}{mc^2} \nabla_x \times \int \check{\mathbf{B}}(t) \times \frac{\mathbf{y} \psi^2}{|\mathbf{x} - \mathbf{r} - \mathbf{y}|} d\mathbf{y} \\
 &= \frac{1}{2} \frac{q^2}{mc^2} \int \psi^2 \frac{1}{|\mathbf{x} - \mathbf{r} - \mathbf{y}|^3} (\mathbf{x} - \mathbf{r} - \mathbf{y}) \times (\check{\mathbf{B}} \times \mathbf{y}) d\mathbf{y}.
 \end{aligned} \quad (16.139)$$

Similarly to (16.138),

$$\begin{aligned}
 &\int \psi_a^2 \frac{1}{|\mathbf{x} - \mathbf{r} - \mathbf{y}|^3} (\mathbf{x} - \mathbf{r} - \mathbf{y}) \times (\check{\mathbf{B}} \times \mathbf{y}) d\mathbf{y} \\
 &= \int \psi_a^2 \left(\frac{1}{|\mathbf{x} - \mathbf{r} - \mathbf{y}|^3} (\mathbf{x} - \mathbf{r} - \mathbf{y}) - \frac{1}{|\mathbf{x} - \mathbf{r}|^3} (\mathbf{x} - \mathbf{r}) \right) \times (\check{\mathbf{B}} \times \mathbf{y}) d\mathbf{y} \\
 &= a \int \psi_1^2 \left(\frac{1}{|\mathbf{x} - \mathbf{r} - a\mathbf{y}|^3} (\mathbf{x} - \mathbf{r} - a\mathbf{y}) - \frac{1}{|\mathbf{x} - \mathbf{r}|^3} (\mathbf{x} - \mathbf{r}) \right) \times (\check{\mathbf{B}} \times \mathbf{y}) d\mathbf{y}.
 \end{aligned} \quad (16.140)$$

Hence

$$\mathbf{B}_b = O\left(\frac{a^2}{R_0^2}\right) \quad (16.141)$$

if (16.137) holds.

16.4.4 Wave-Corpuscle for an Accelerating Dressed Charge

In Sect. 15.3 we showed that a wave-corpuscle defined by the relations (15.1) for a free moving balanced charge also provides an exact solution to the field equations (13.27)–(13.28) which describe a dressed charge. Now we show that the wave-corpuscle (16.2) which generalizes (15.1) describes a dressed charge accelerating in an external EM field. We use the results of Sects. 16.1 and 16.3.

A dressed non-relativistic charge is described by the NLS equation (13.27) coupled with the Poisson equation (13.28), and the equations can be written using (14.87) in the form

$$\begin{aligned} & \chi i \left(\partial_t + \frac{iq}{\chi} (\varphi_{\text{ex}} + \varphi) \right) \psi \\ &= \frac{\chi^2}{2m} \left[- \left(\nabla - \frac{iq \mathbf{A}_{\text{ex}}}{\chi c} \right)^2 \psi + G'_{\nabla} \psi \right] - q G'_{\varphi} \psi, \end{aligned} \quad (16.142)$$

$$\nabla^2 \varphi = -4\pi q |\psi|^2. \quad (16.143)$$

We define the wave-corpuscle ψ by the formula (16.2), and the corresponding charge electric potential is then defined by

$$\varphi = \dot{\varphi} (|\mathbf{x} - \mathbf{r}(t)|), \quad (16.144)$$

where formula $\dot{\varphi}$ is the potential satisfying (14.77).

Now let us show that the description of wave-corpuscle motion for a dressed charge can be reduced to the same for a balanced charge. For the wave-corpuscle of the form (16.2), $|\psi|^2 = \left| \dot{\psi} \right|^2$, and the dressed charge potential is determined from the Poisson equation:

$$\nabla^2 \varphi = -4\pi q |\psi|^2, \quad \varphi = \dot{\varphi} (\mathbf{x} - \mathbf{r}(t)). \quad (16.145)$$

Electrostatic balance (15.39) holds according to (14.89), allowing to eliminate G'_{φ} and φ . Consequently, equation (16.142) is equivalent to the NLS equation

$$\chi i \left(\partial_t + \frac{iq}{\chi} \varphi_{\text{ex}} \right) \psi = -\frac{\chi^2}{2m} \left(\nabla - \frac{iq \mathbf{A}_{\text{ex}}}{\chi c} \right)^2 \psi + \frac{\chi^2}{2m} G'_{\nabla} \psi, \quad (16.146)$$

coupled with the Poisson equation (16.143) for φ . Evidently φ does not enter (16.146), and the Poisson equation (16.143) can be solved independently. It is also evident that equation (16.146) coincides with NLS equation (16.1) for the balanced charge distribution.

Therefore, the equation (16.142) restricted to wave-corpuscles coincides with equation (16.1) where $G' = G'_{\nabla}$. If the external fields satisfy conditions imposed

in Sect. 16.1, then the dressed charge described by the wave-corpuscle is an exact solution of (16.142). In particular, its center moves according to Newton's law (16.39) with the Lorentz force. Direct computation shows that the total dressed charge field momentum $\mathbf{P}(t)$ and the total current $\mathbf{J}(t)$ for the solution (16.2) are expressed exactly in terms of point charge quantities, namely

$$\mathbf{P}(t) = \frac{m}{q} \mathbf{J}(t) = \int_{\mathbb{R}^3} \frac{\chi q}{m} \operatorname{Im} \frac{\nabla \psi}{\psi} |\psi|^2 \, d\mathbf{x} = \mathbf{p}(t) = m \mathbf{v}(t). \quad (16.147)$$

If the EM field is of general form, we can define the trajectory according to Newton's law (16.39), and the wave-corpuscle ψ_{au} together with the potential φ_{au} provide an approximate solution to (16.142). The discrepancy for this solution arises only in the NLS equation, and it is described in Sect. 16.3. The discrepancy, according to (16.89), is of order $\frac{a^2}{R_{\text{ex}}^2}$ where R_{ex} is the scale of variation of the EM potentials and a is the size parameter of the charge.

Chapter 17

Interaction Theory of Balanced Charges

“I feel that it is a delusion to think of the electrons and the fields as two physically different, independent entities.

Since neither can exist without the other, there is only one reality to be described, which happens to have two different aspects; and the theory ought to recognize this from the start instead of doing things twice.”

*A. Einstein*¹

“Is it after all essential in classical field theory to require that a particle acts upon itself?”

*J. Wheeler and R. Feynman.*²

In the previous chapter we considered a single localized distributed charge in regimes when it can be described by a wave-corpuscle. We did that in the framework of two different models: balanced charges and dressed charges which are described, respectively, by the balanced charges theory (BCT) and the dressed charge theory (DCT). In this chapter we consider interaction of balanced charges. When studying macroscopic aspects, we use two approaches. The first one is similar to the Ehrenfest theorem. It allows us to derive the Newtonian dynamics of localized charges based on conservation laws for the NLS equations or the NKG equations in the non-relativistic and relativistic cases, respectively. The second approach uses wave-corpuscles which were studied in the previous chapter. Wave-corpuscles in the non-relativistic case can be used as approximate solutions to the system of NLS equations to show that the limit dynamics of the centers of the wave-corpuscles is given by Newton’s equations, and that the wave-corpuscles combine de Broglie waves properties with particle properties. We also study balanced charges in the regime of close interaction as in the case of the Hydrogen atom. In this case we study the spectrum of frequencies of

¹The Einstein quotation is from a paper by E. Jaynes [186].

²The Wheeler–Feynman quotation is from their 1949 paper [340].

the corresponding time-harmonic solutions and show that this spectrum converges to the spectrum of the Hydrogen atom as described by the Schrödinger equation in the non-relativistic case. In the relativistic case, the spectrum of frequencies involves Sommerfeld's fine structure. In this chapter we consider balanced charges, leaving the theory of dressed charges to the next chapter.

17.1 Theory of Non-relativistic Balanced Charges

The *non-relativistic equations for balanced charge dynamics* take the form of a system of the NLS-Poisson equations (12.14)–(12.17). In particular, the ℓ -th charge density satisfies the NLS equation:

$$i\chi\partial_t\psi_\ell = -\frac{\chi^2}{2m_\ell}\tilde{\nabla}_\ell^2\psi_\ell + q_\ell\varphi_{\neq\ell}\psi_\ell + \frac{\chi^2}{2m_\ell}G'_\ell(|\psi_\ell|^2)\psi_\ell, \quad (17.1)$$

where $\ell = 1, \dots, N$, the potential $\varphi_{\neq\ell}$ is defined by (12.6), and $\tilde{\nabla}_\ell$ is defined by (12.5). The EM field potentials $(\varphi_\ell, \mathbf{A}_\ell)$ associated with the charges satisfy the Poisson equations (12.16), (12.17), namely

$$\nabla^2\varphi_\ell = -4\pi\rho_\ell, \quad (17.2)$$

$$\nabla^2\mathbf{A}_\ell = -\frac{4\pi}{c}\mathbf{J}_\ell, \quad \ell = 1, \dots, N, \quad (17.3)$$

where the charge density ρ_ℓ and the current density $\mathbf{J}_\ell$ are defined by the following expressions

$$\rho_\ell = q_\ell|\psi_\ell|^2, \quad \mathbf{J}_\ell = \left(\frac{\chi q_\ell}{m_\ell} \operatorname{Im} \frac{\nabla\psi_\ell}{\psi_\ell} - \frac{q_\ell^2}{m_\ell c} \mathbf{A}_{\neq\ell} \right) |\psi_\ell|^2. \quad (17.4)$$

The solutions of the Poisson equations (17.2), (17.3) are defined by Eqs. (12.19), (12.20).

Equations (17.1) for ψ_ℓ and the Poisson equations (12.16), (12.17) for the EM potentials $(\varphi_\ell, \mathbf{A}_\ell)$ can be derived from the non-relativistic Lagrangian $\hat{\mathcal{L}}$ defined by (12.11). This Lagrangian is gauge invariant, and consequently the ℓ -th charge has a 4-current $(\rho_\ell, \mathbf{J}_\ell)$ defined by (11.156)–(11.157) yielding expressions (17.4). The 4-current $(\rho_\ell, \mathbf{J}_\ell)$ defined by (17.4) satisfies for every ℓ the continuity equations (11.54), (11.55), that is,

$$\partial_t\rho_\ell + \nabla \cdot \mathbf{J}_\ell = 0. \quad (17.5)$$

The charge conservation equation (17.5) can also be derived directly from the NLS equation (17.1) by multiplying it and its complex conjugate by ψ_ℓ^* and ψ_ℓ , respectively, and subtracting from one another. Integrating the above equation, we observe

that the ℓ -th total charge is preserved, namely

$$\bar{\rho}_\ell = \int_{\mathbb{R}^3} |\psi_\ell|^2 d\mathbf{x} = \text{const}, \quad (17.6)$$

and the charge normalization condition takes the form (14.17):

$$\bar{\rho}_\ell / q_\ell = \|\psi_\ell\|^2 = \int_{\mathbb{R}^3} |\psi_\ell|^2 d\mathbf{x} = 1, \quad \ell = 1, \dots, N. \quad (17.7)$$

17.1.1 A Charge Singled Out from the Non-relativistic System

The system (17.1)–(17.3) describes N interacting distributed charges and their EM fields. Let us assume that we have a solution of the entire system and single out the ℓ -th charge. It is described by the NLS equation which can be written in the form (17.1); skipping the charge label ℓ , we obtain the NLS equation for a charge in the system:

$$i\chi \partial_t \psi = -\frac{\chi^2}{2m} \tilde{\nabla}^2 \psi + q\varphi_{\text{ex}} \psi + \frac{\chi^2}{2m} G'(|\psi|^2) \psi, \quad (17.8)$$

where $\psi = \psi_\ell$, $G' = G'_\ell$, and the potentials $\varphi_{\neq \ell} + \varphi_{\text{ex}}$, $\mathbf{A}_{\neq \ell} + \mathbf{A}_{\text{ex}}$ are replaced by effective external potentials φ_{ex} , $\mathbf{A}_{\text{ex}}$ (the effective external potentials for a single charge are different from the original external potentials for the entire system, but we use the same notation for them) so that

$$\tilde{\nabla} = \nabla - \frac{iq}{\chi c} \mathbf{A}_{\text{ex}}, \quad (17.9)$$

$$q = q_\ell, \quad m = m_\ell, \quad a = a_\ell.$$

To study the dynamics of such a singled out charge, we consider the external EM fields φ_{ex} and $\mathbf{A}_{\text{ex}}$ as given functions of time and spatial variables.

17.1.2 Exact Wave-Corpuscle Solutions: Accelerating Solitons

The field equation (17.8) has exact solutions in the form of accelerating solitons (wave-corpuscles) which are described in Sect. 16.4. If the external EM field is purely electric, i.e. when $\mathbf{A}_{\text{ex}} = 0$, $\mathbf{E}_{\text{ex}}(t, \mathbf{x}) = -\nabla \varphi_{\text{ex}}(t, \mathbf{x})$, the expression for the wave-corpuscle takes the form (16.63) and the electric potential generated by this charge is given by (16.118). In a simpler case of a free charge where the external fields φ_{ex} and

$\mathbf{A}_{\text{ex}}$ vanish, a simpler solution of (17.8) is provided by (16.63) with $\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}t$ with a constant velocity $\mathbf{v}$. In this case the wave-corpuscle solution (16.63) of the field equation (16.43) can be obtained from the rest solution ψ by the Galilean-gauge transformation. Solutions of a similar form are well-known in the theory of nonlinear Schrödinger equations, see [317] and references therein. For the particular case of the logarithmic nonlinearity solutions of the form (16.63) were found in [40] in the form of accelerating gausssons.

17.2 Non-relativistic Macroscopic Dynamics of Balanced Charges

In this section we consider macroscopic aspects of charge dynamics. Namely, we assume that the spatial scale R of variation of electromagnetic fields is much larger than the size parameter a of a charge, that is,

$$a \ll R.$$

The primary focus of this section is to show that if the size parameter $a \rightarrow 0$, then the dynamics of the centers of localized solutions is approximated by Newton's equations with the Lorentz forces. This is done in the spirit of the well known Ehrenfest Theorem from quantum mechanics, [291, Sections 7, 23]. The explicit wave-corpuscle solutions provide an example of such a dynamics, their wave and particle properties are described in Chap. 16.

17.2.1 Individual Momenta System

The individual momentum density $\mathbf{P}_\ell$ in (11.90) for the Lagrangian $\hat{\mathcal{L}}_0$ in (12.11) is defined by formula (11.163) which takes the form

$$\mathbf{P}_\ell = \frac{i\chi}{2} \left(\psi_\ell \tilde{\nabla}_\ell^* \psi_\ell^* - \psi_\ell^* \tilde{\nabla}_\ell \psi_\ell \right).$$

Note that the so defined momentum density $\mathbf{P}_\ell$ is related to the current $\mathbf{J}_\ell$ in (17.4) by formula (11.166):

$$\mathbf{P}_\ell(t, \mathbf{x}) = \frac{m_\ell}{q_\ell} \mathbf{J}_\ell(t, \mathbf{x}). \quad (17.10)$$

This relation allows us to introduce the velocity distribution for every charge by formula (11.167), namely

$$\mathbf{v}_\ell(t, \mathbf{x}) = \frac{1}{m_\ell} \mathbf{P}_\ell(t, \mathbf{x}) = \frac{1}{q_\ell} \mathbf{J}_\ell(t, \mathbf{x}). \quad (17.11)$$

Let us introduce the total individual momentum $\mathbf{P}_\ell$ for the the ℓ -th charge by

$$\mathbf{P}_\ell = \int_{\mathbb{R}^3} \mathbf{P}_\ell \, d\mathbf{x}. \quad (17.12)$$

From the conservation law (11.92), taking into account (11.141), we obtain the following equations for the total individual momenta:

$$\partial_t \mathbf{P}_\ell = \int_{\mathbb{R}^3} q_\ell \left[\mathbf{E}_{\ell\text{ex}} |\psi_\ell|^2 + \frac{1}{c} \mathbf{v}_\ell \times \mathbf{B}_{\ell\text{ex}} \right] d\mathbf{x} \quad (17.13)$$

where

$$\mathbf{E}_{\ell\text{ex}} = \mathbf{E}_{\neq\ell} + \mathbf{E}_{\text{ex}}, \quad \mathbf{B}_{\ell\text{ex}} = \mathbf{B}_{\neq\ell} + \mathbf{B}_{\text{ex}}, \quad (17.14)$$

and in accordance with (12.6)

$$\mathbf{E}_{\neq\ell} = \sum_{\ell' \neq \ell} \mathbf{E}_{\ell'}, \quad \mathbf{B}_{\neq\ell} = \sum_{\ell' \neq \ell} \mathbf{B}_{\ell'}. \quad (17.15)$$

The integrand of the right-hand side of (17.13) is the Lorentz force density. The EM fields $\mathbf{E}$, $\mathbf{B}$ in the above expressions are determined in terms of the corresponding potentials φ , $\mathbf{A}$ from (17.113) by standard formulas (4.6). There is an alternative rather elementary way to derive Eq. (17.13). Namely, in the simplest case where $\mathbf{A}_{\text{ex}} = 0$, we multiply (12.14) by $\nabla \psi_\ell^*$, take the real part and integrate the result over the entire space using integration by parts. To directly obtain (17.13) in a more involved general case, we similarly multiply (12.14) by $\tilde{\nabla}_\ell^* \psi_\ell^*$ and then integrate the result by parts using some vector algebra manipulations.

17.2.2 The Ehrenfest Theorem for Interacting Balanced Charges

Let us show now that if the size parameter a is small compared to the macroscopic scale of variation of EM fields, the charge evolution can be described approximately by Newton's equations with the Lorentz forces similar to (6.38). Conditions justifying relevant arguments are given in the next section. Mathematical details of handling the approximations are provided in Sect. 35.1.

Using the normalization condition (17.7), we introduce the ℓ -th charge position $\mathbf{r}_\ell(t)$ as the following spatial average:

$$\mathbf{r}_\ell(t) = \mathbf{r}_{\ell a}(t) = \int_{\mathbb{R}^3} \mathbf{x} |\psi_{\ell a}(t, \mathbf{x})|^2 d\mathbf{x}. \quad (17.16)$$

The average velocity $\mathbf{v}_\ell(t)$ is defined based on (17.11) by the formula

$$\mathbf{v}_\ell(t) = \frac{1}{q_\ell} \int_{\mathbb{R}^3} \mathbf{J}_\ell(t, \mathbf{x}) d\mathbf{x}, \quad (17.17)$$

where the current density $\mathbf{J}_\ell$ is defined by (17.4). We demonstrate below that a combination of the continuity equations (17.5) with the momentum equations (17.13) implies the following remarkable property: the positions $\mathbf{r}_\ell(t)$ satisfy with a high accuracy Newton's equations of motion for a system of N point charges if the size parameter a is small enough.

Multiplying the continuity equation (17.5) by $\mathbf{x}$ and integrating, we obtain the identities

$$\begin{aligned} \partial_t \mathbf{r}_\ell(t) &= \int_{\mathbb{R}^3} \mathbf{x} \partial_t |\psi_\ell|^2 d\mathbf{x} \\ &= -\frac{1}{q_\ell} \int_{\mathbb{R}^3} \mathbf{x} \nabla \cdot \mathbf{J}_\ell d\mathbf{x} = \frac{1}{q_\ell} \int_{\mathbb{R}^3} \mathbf{J}_\ell d\mathbf{x}, \end{aligned} \quad (17.18)$$

implying that the velocity of the charge center coincides with the average velocity:

$$\partial_t \mathbf{r}_\ell(t) = \mathbf{v}_\ell(t). \quad (17.19)$$

Then integrating (17.10), we obtain the following kinematic representation for the total momentum:

$$\mathbf{P}_\ell(t) = \frac{m_\ell}{q_\ell} \int_{\mathbb{R}^3} \mathbf{J}_\ell(t, \mathbf{x}) d\mathbf{x} = m_\ell \mathbf{v}_\ell(t), \quad (17.20)$$

which is also exactly the same as for the point charges mechanics. Relations (17.19) and (17.20) yield that

$$m_\ell \partial_t^2 \mathbf{r}_\ell(t) = m_\ell \partial_t \mathbf{v}_\ell(t) = \partial_t \mathbf{P}_\ell, \quad (17.21)$$

and we obtain from (17.13) the following equations of motion for N charges:

$$m_\ell \partial_t^2 \mathbf{r}_\ell(t) = q_\ell \int_{\mathbb{R}^3} \left(\mathbf{E}_{\ell\text{ex}} |\psi_\ell|^2 + \frac{1}{c} \mathbf{v}_\ell \times \mathbf{B}_{\ell\text{ex}} \right) d\mathbf{x}, \quad \ell = 1, \dots, N, \quad (17.22)$$

where $\mathbf{E}_{\ell\text{ex}}$, $\mathbf{B}_{\ell\text{ex}}$ defined by (17.14), (17.15) are the fields which act on the ℓ -th charge, and the EM fields $\mathbf{E}$ and $\mathbf{B}$ are defined in terms of the EM potentials φ and $\mathbf{A}$ by (4.6).

The derivation of the above system is analogous to the well known *Ehrenfest theorem* from quantum mechanics, [40], [144, 8.2], [291, 7, 23], [332, 3.2.5], [331, 1.3.2]. Notice that the system of the equations of motion (17.22) departs from the corresponding system for point charges by having the averaged Lorentz force density instead of the Lorentz force at the exact position $\mathbf{r}_\ell(t)$.

Now we present a derivation of Newton's law of motion for localized charges.

17.2.3 Newtonian Mechanics as an Approximation

Let us suppose that the ℓ -th charge density $q_\ell |\psi_\ell|^2$ and the corresponding current density $\mathbf{J}_\ell$ are localized in an a -vicinity of the position $\mathbf{r}_\ell(t)$ for every ℓ , and that $|\mathbf{r}_\ell(t) - \mathbf{r}_{\ell'}(t)| \geq R > 0$ for $\ell' \neq \ell$ on a time interval $[-T, T]$ with R independent of a . We assume that as $a \rightarrow 0$

$$|\psi_\ell|^2(t, \mathbf{x}) \rightarrow \delta(\mathbf{x} - \mathbf{r}_\ell(t)), \quad \mathbf{v}_\ell(t, x) = \mathbf{J}_\ell/q_\ell \rightarrow \mathbf{v}_\ell(t) \delta(\mathbf{x} - \mathbf{r}_\ell(t)), \quad (17.23)$$

where the coefficients before the Dirac delta-functions are determined by the charge normalization conditions (17.7) and relations (17.17). Using potential representations (12.19), we infer from (17.23) the convergence of the potentials φ_ℓ to the corresponding Coulomb's potentials, namely

$$\begin{aligned} \varphi_\ell(t, \mathbf{x}) &= \frac{1}{c} \int \frac{1}{|\mathbf{x} - \mathbf{y}|} q_\ell |\psi_\ell|^2(t, \mathbf{y}) d\mathbf{y} \rightarrow \frac{q_\ell}{|\mathbf{x} - \mathbf{r}_\ell|}, \\ \nabla_{\mathbf{x}} \varphi_\ell(t, \mathbf{x}) &\rightarrow -\frac{q_\ell (\mathbf{x} - \mathbf{r}_\ell)}{|\mathbf{x} - \mathbf{r}_\ell|^3} \text{ as } a \rightarrow 0. \end{aligned}$$

Similarly (12.20) implies that

$$\begin{aligned} \mathbf{A}_\ell(t, \mathbf{x}) &= \frac{1}{c} \int \frac{1}{|\mathbf{x} - \mathbf{y}|} \mathbf{J}_\ell(t, \mathbf{y}) d\mathbf{y} \rightarrow \mathbf{v}_\ell(t) \frac{q_\ell}{|\mathbf{x} - \mathbf{r}_\ell|}, \\ \mathbf{B}_\ell(t, \mathbf{x}) &= \nabla_{\mathbf{x}} \times \mathbf{A}_\ell(t, \mathbf{x}) \rightarrow q_\ell \nabla_{\mathbf{x}} \times \frac{\mathbf{v}_\ell(t)}{|\mathbf{x} - \mathbf{r}_\ell|} = -\frac{q_\ell}{|\mathbf{x} - \mathbf{r}_\ell|^3} (\mathbf{x} - \mathbf{r}_\ell) \times \mathbf{v}_\ell(t). \end{aligned}$$

Hence, passing to the limit as $a \rightarrow 0$, we can recast the equations of motion (17.22) as the system

$$m_\ell \partial_t^2 \mathbf{r}_\ell = \mathbf{f}_{\ell\text{Lor}} + \epsilon_0, \quad (17.24)$$

where $\epsilon_0 \rightarrow 0$ as $a \rightarrow 0$,

$$\mathbf{f}_{\ell\text{Lor}} = q_\ell \mathbf{E}_{\ell\text{ex}}(\mathbf{r}_\ell) + q_\ell \frac{1}{c} \mathbf{v}_\ell \times \mathbf{B}_{\ell\text{ex}}(\mathbf{r}_\ell), \quad \ell = 1, \dots, N, \quad (17.25)$$

and $\mathbf{E}_{\ell\text{ex}}(\mathbf{x})$, $\mathbf{B}_{\ell\text{ex}}(\mathbf{x})$ are given by (17.14). Notice that the terms $\mathbf{f}_{\ell\text{Lor}}$ in Eqs. (17.24) coincide with the Lorentz forces, and we see that the limit equations of motion obtained from (17.24) coincide with *Newton's equations of motion* for point charges interacting via the Lorentz forces, namely

$$m_\ell \partial_t^2 \mathbf{r}_\ell = \mathbf{f}_{\ell\text{Lor}}, \quad \ell = 1, \dots, N. \quad (17.26)$$

Here the Lorentz force $\mathbf{f}_{\ell\text{Lor}}$ generated by the remaining charges with $\ell' \neq \ell$ has the form

$$\begin{aligned} \mathbf{f}_{\ell\text{Lor}} = & q_\ell \mathbf{E}_{\text{ex}}(\mathbf{r}_\ell) + \frac{1}{c} q_\ell \mathbf{v}_\ell \times \mathbf{B}_{\text{ex}}(\mathbf{r}_\ell) \\ & - q_\ell \sum_{\ell' \neq \ell} \frac{q_{\ell'} (\mathbf{x} - \mathbf{r}_{\ell'})}{|\mathbf{x} - \mathbf{r}_{\ell'}|^3} - q_\ell \mathbf{v}_\ell \times \sum_{\ell' \neq \ell} \frac{q_{\ell'}}{|\mathbf{x} - \mathbf{r}_{\ell'}|^3} (\mathbf{x} - \mathbf{r}_{\ell'}) \times \mathbf{v}_{\ell'}. \end{aligned} \quad (17.27)$$

Note that the nonlinearity G'_a in (17.8) singularly depends on a as $a \rightarrow 0$ according to (14.27), but we still can pass to the limit since the nonlinearity does not enter the system (17.22).

17.2.4 Point Mechanics of Balanced Charges via Wave-Corpuscles

In the previous section we have shown that the dynamics of localized balanced charges can be described by the Newtonian mechanics of their centers. Here we derive the Newtonian dynamics using a different approach. In Sects. 16.4 and 16.1 we proved that a single balanced charge described by a wave corpuscle in an external EM field from a certain class provides an exact solution to the NLS field equations, and the wave-corpuscle center must obey Newton's law of motion with the Lorentz force. In this section we consider N interacting wave-corpuscles. Every charge obeys the relevant NLS equation, where EM potentials are generated by the remaining charges of the system as the solutions of the Poisson equations. Singling out a charge from the system as in Sect. 17.6.1.2, we can consider the sum of the EM potentials of the remaining charges as the potential of an external EM field, and we may use the results of Sects. 16.4 and 16.1. But there is a difference in the case of many charges. Namely, interacting wave-corpuscles with a fixed shape provide only an approximate solution to the field equations, because the EM potentials which they generate generally do not satisfy global conditions which are required for preserving exactly the shape of the wave corpuscles. At the same time, the conditions are satisfied locally at the centers of the wave corpuscles in the linear approximation, allowing therefore for approximate solutions with the discrepancy controlled by the squared ratio $\frac{a^2}{R^2}$ of the corpuscle size a and a macroscopic length scale R .

In this section we construct an approximate solution of the field equations for N interacting charges, where every charge is represented by a wave-corpuscle defined by (16.60), with centers satisfying equations of motion for point charges. We assume here that every shape factor $\left| \psi_\ell(|\mathbf{x}|) \right|^2$ decays fast as $|\mathbf{x}| \rightarrow \infty$, in particular (16.95) holds. The construction begins with determination of all center trajectories from Newton's equation and with an introduction of an auxiliary system of equations for the wave-corpuscles. This system has the following property. If the ℓ -th charge is

singled out, and the sum of EM potentials generated by all the remaining charges is replaced by the linear approximation $\varphi_{\text{aul}}, \mathbf{A}_{\text{aul}}$ of this sum at the position of the ℓ -th charge, then the constructed wave-corpuscle solution for the ℓ -th charge in the so modified field becomes exact. The source of the discrepancy in the original equations is the error of the field linearization at the charge center location $\mathbf{r}_\ell$. To estimate these discrepancies we use the results of Sect. 16.3.

The construction of an approximate solution to the system (17.1)–(17.3) consists of several steps. As the first step, we define the ℓ -th wave-corpuscle by formula (16.60) which takes the form

$$\psi_\ell(t, \mathbf{x}) = e^{i\mathfrak{S}_\ell} \hat{\psi}(|\mathbf{y}_\ell|), \quad \mathbf{y}_\ell = \mathbf{x} - \mathbf{r}_\ell(t). \quad (17.28)$$

Here $\mathbf{r}_\ell(t)$ is the ℓ -th charge trajectory, which we consider at this step to be a given function. The phase function is defined below. As the second step, we define based on (16.118) and (16.134) the *effective potential generated by the ℓ -th wave-corpuscle* as follows:

$$\varphi_\ell = \hat{\varphi}(\mathbf{x} - \mathbf{r}_\ell), \quad \mathbf{A}_{\mathbf{v}\ell} = \frac{1}{c} q \mathbf{v}_\ell \hat{\varphi}(\mathbf{x} - \mathbf{r}_\ell), \quad \mathbf{v}_\ell = \partial_t \mathbf{r}_\ell. \quad (17.29)$$

We also define the *effective potential acting on the ℓ -th wave-corpuscle* by the expressions

$$\varphi_{\neq \ell \text{ex}}(t, \mathbf{x}) = \varphi_{\text{ex}}(t, \mathbf{x}) + \sum_{\ell' \neq \ell} \hat{\varphi}(\mathbf{x} - \mathbf{r}_{\ell'}), \quad (17.30)$$

$$\mathbf{A}_{\neq \ell \text{ex}}(t, \mathbf{x}) = \mathbf{A}_{\text{ex}}(t, \mathbf{x}) + \sum_{\ell' \neq \ell} \mathbf{A}_{\mathbf{v}\ell'}(t, \mathbf{x}). \quad (17.31)$$

As the third step, we write equations defining trajectories $\mathbf{r}_\ell(t)$ and pick their solution (it can be determined uniquely by corresponding initial data). These equations are point charge Newton's laws with the Lorentz forces which are based on the point balance condition (16.39):

$$m_\ell \partial_t^2 \mathbf{r}_\ell = \mathbf{f}_{\ell \text{Lor}}(t, \mathbf{r}_\ell), \quad \ell = 1, \dots, N; \quad (17.32)$$

$$\begin{aligned} \mathbf{f}_{\ell \text{Lor}}(t, \mathbf{x}) = & -q_\ell \nabla \varphi_{\neq \ell \text{ex}}(t, \mathbf{x}) - \frac{q_\ell}{c} \partial_t \mathbf{A}_{\neq \ell \text{ex}}(t, \mathbf{x}) \\ & + \nabla (\mathbf{v}_\ell \cdot \mathbf{A}_{\neq \ell \text{ex}})(t, \mathbf{x}) - \mathbf{v}_\ell \cdot \nabla \mathbf{A}_{\neq \ell \text{ex}}(t, \mathbf{x}). \end{aligned} \quad (17.33)$$

The fourth step of the construction, after the trajectories satisfying the above equations are found, is to define the phase function $\mathfrak{S}_\ell(t, \mathbf{y}_\ell)$ by the formula

$$\begin{aligned} & \mathfrak{S}_\ell(t, \mathbf{y}_\ell) \\ = & m \frac{1}{\chi} \mathbf{v}_\ell \cdot \mathbf{y}_\ell + \frac{q}{c} \frac{1}{\chi} \mathbf{y}_\ell \cdot \mathbf{A}_{\neq \ell \text{ex}}(t, \mathbf{r}_\ell) + s_{\text{lp}}(t) + s_{\text{p2}}(t, \mathbf{y}_\ell), \end{aligned} \quad (17.34)$$

where $\mathbf{y}_\ell = \mathbf{x} - \mathbf{r}_\ell(t)$, $\mathbf{v}_\ell = \partial_t \mathbf{r}_\ell$, and $s_{\ell p}$ and s_{p2} are as in (16.58), (16.59):

$$s_{\ell p}(t) = \frac{1}{\chi} \int_0^t \left(\frac{1}{2} m \mathbf{v}^2(t) + \frac{q}{c} \mathbf{v}_\ell \cdot \mathbf{A}_{\neq \ell \text{ex}}(t, 0) - q \varphi_{\ell \text{ex}}(t, 0) \right) dt, \quad (17.35)$$

$$s_{p2}(t, \mathbf{y}_\ell) = \frac{q}{2c} \frac{1}{\chi} \mathbf{y}_\ell \cdot (\mathbf{y}_\ell \cdot \nabla \mathbf{A}_{\neq \ell \text{ex}}(t, \mathbf{r}_\ell)). \quad (17.36)$$

In the following subsection we show that the wave-corpuscles (17.28) constructed above provide an approximate solution to the NLS-Poisson system (12.14), (13.24).

17.2.5 A Discrepancy Estimate for the Construction

Let us verify now that the above constructed wave-corpuscles provide an approximate solution and show that the discrepancy in the equations is of order $\frac{a^2}{R^2}$, where R is the macroscopic scale of variation of EM fields, and a is the size parameter of the free wave-corpuscle determined by the nonlinearity $G_{\ell a}$.

Let us define for every ℓ -th charge auxiliary electromagnetic potentials $\varphi_{\ell \text{au}}$, $\mathbf{A}_{\ell \text{au}}$ by formulas (16.82)–(16.83) where φ_{ex} and $\mathbf{A}_{\text{ex}}$ are replaced, respectively, with $\varphi_{\neq \ell \text{ex}}$, $\mathbf{A}_{\neq \ell \text{ex}}$:

$$\mathbf{A}_{\ell \text{au}}(t, \mathbf{r}_\ell(t) + \mathbf{y}_\ell) = \mathbf{A}_{\neq \ell \text{ex}}(t, \mathbf{r}_\ell(t)) + \mathbf{y}_\ell \cdot \nabla \mathbf{A}_{\neq \ell \text{ex}}(t, \mathbf{r}_\ell(t)), \quad (17.37)$$

$$\varphi_{\ell \text{au}}(t, \mathbf{r}_\ell(t) + \mathbf{y}_\ell) = \varphi_{\neq \ell \text{ex}} + \mathbf{y}_\ell \cdot \nabla \varphi_{\neq \ell \text{ex}}(t, \mathbf{r}_\ell(t)) + \varphi_{\ell \text{au}2}(t, \mathbf{y}_\ell), \quad (17.38)$$

where $\varphi_{\ell \text{au}2}(t, \mathbf{y})$ is defined by (16.84), that is,

$$\begin{aligned} \varphi_{\ell \text{au}2}(t, \mathbf{y}) &= -\frac{q_\ell}{c^2} \frac{1}{2m} \check{\mathbf{A}}_{\neq \ell \text{ex}1}^2 - \frac{1}{c} \partial_t P_{\neq \ell \text{ex}2}(t, \mathbf{y}) \\ &= -\frac{q_\ell}{2c^2} \frac{1}{8m} [(\nabla \times \mathbf{A}_{\neq \ell \text{ex}}(t, \mathbf{r}_\ell(t))) \times \mathbf{y}]^2 \\ &\quad - \frac{1}{2c} \partial_t [\mathbf{y} \cdot ((\mathbf{y} \cdot \nabla) \mathbf{A}_{\neq \ell \text{ex}}(t, \mathbf{r}_\ell(t)))] . \end{aligned} \quad (17.39)$$

We would like to show now that, if the distances between charges are macroscopic, then the constructed wave corpuscles provide an approximate solution to the system (17.1), (17.2), (17.3) with an accuracy of order $\frac{a^2}{R^2}$.

To estimate the discrepancy in the equations, we write an auxiliary NLS equation similar to (16.77):

$$\begin{aligned}
& \chi i \partial_t \psi_\ell - \chi i \mathbf{v}_\ell \cdot \nabla \psi_\ell - q \varphi_{\ell \text{au}} \psi_\ell \\
&= \frac{\chi^2}{2m_\ell} \left[-\nabla^2 \psi_\ell + G'(\psi_\ell^* \psi_\ell) \psi_\ell \right] \\
&+ \frac{\chi^2}{2m_\ell} \left[2 \frac{i q_\ell}{\chi c} \mathbf{A}_{\ell \text{au}} \cdot \nabla \psi_\ell + \frac{i q}{\chi c} (\nabla \cdot \mathbf{A}_{\ell \text{au}}) \psi_\ell + \frac{q^2}{\chi^2 c^2} \mathbf{A}_{\ell \text{au}}^2 \psi_\ell \right].
\end{aligned} \tag{17.40}$$

The wave-corpucle ψ_ℓ is an exact solution to the above auxiliary equation for every ℓ . The potentials φ_ℓ , $\mathbf{A}_\ell$ are found as solutions to the Poisson equations (17.2), (17.3). The difference between the auxiliary equation (17.40) and the original equation (17.1) comes from the differences $\varphi_{\neq \ell} + \varphi_{\text{ex}} - \varphi_{\ell \text{au}}$ and $\mathbf{A}_{\neq \ell} + \mathbf{A}_{\text{ex}} - \mathbf{A}_{\ell \text{au}}$ between the potentials. As in Sect. 16.3, we can write the auxiliary equation in the form

$$i \chi \partial_t \psi_\ell = -\frac{\chi^2}{2m_\ell} \tilde{\nabla}_\ell^2 \psi_\ell + q_\ell \varphi_{\neq \ell} \psi_\ell + \frac{\chi^2}{2m_\ell} G'_\ell(|\psi_\ell|^2) \psi_\ell - D_\ell. \tag{17.41}$$

According to (16.134) we have

$$\varphi_\ell = \hat{\varphi}(\mathbf{x} - \mathbf{r}_\ell), \quad A_\ell = \frac{1}{c} q_\ell \mathbf{v}_\ell \hat{\varphi}(\mathbf{x} - \mathbf{r}_\ell) + \mathbf{A}_{\ell \text{b}}.$$

Consequently the potential $\mathbf{A}_\ell$ is a sum of the effective potential and the induced potential $\mathbf{A}_{\ell \text{b}}$, and the induced potential contributes to the discrepancy D_ℓ . Note that the differences between the effective potentials and auxiliary potentials and their gradients vanish at the center of the ℓ -th charge. The estimate of the discrepancy D_ℓ is done as in Sect. 16.3.1, but now we have to estimate also the contribution of $\mathbf{A}_{\ell \text{b}}$. To take into account the macroscopic separation of point charges, we introduce the minimal distance between the trajectories:

$$\min_{\ell \neq \ell', 0 \leq t \leq T} |\mathbf{r}_\ell(t) - \mathbf{r}_{\ell'}(t)| = R_{\min} > 0, \tag{17.42}$$

and we assume that $R_{\min}$ does not depend on a . Recall that the dependence on the size parameter a of the form factor $\hat{\psi}_\ell = \hat{\psi}_{\ell,a}$ and corresponding potential $\hat{\varphi}_\ell = \hat{\varphi}_{\ell,a}$ is given by (14.24) and (14.30). Note that $\varphi_{\neq \ell}$ and $\mathbf{A}_{\neq \ell}$ are composed of $\varphi_{\ell'}$ and $\mathbf{A}_{\ell'}$ with $\ell' \neq \ell$. We assume that $|\psi_\ell|$ decays very fast, (for example, in the exponential case as $e^{-\kappa|\mathbf{x} - \mathbf{r}_\ell|/a}$). Though the potentials $\varphi_{\ell'}$ and $\mathbf{A}_{\ell'}$ are of order a^{-1} near $\mathbf{r}_{\ell'}$, under the condition (17.42) the products of the form $(|\mathbf{A}_{\ell'}| + |\varphi_{\ell'}|) \psi_\ell$ become vanishingly small as $a \rightarrow 0$. Therefore it is sufficient to estimate the discrepancy D_ℓ in the domain $|\mathbf{x} - \mathbf{r}_\ell| \leq R_{\min}/2$. In this domain, the EM potentials generated by a charge $\psi_{\ell'}$, $\ell' \neq \ell$, vary at macroscopic scales, and we can also use (16.138), namely $|\mathbf{A}_{\ell \text{b}}| = O(a^2)$. Hence the estimate made in Sect. 16.3.1 differs only by the terms of order $O(a^2)$. According to inequality (16.89) the discrepancy without $\mathbf{A}_{\ell \text{b}}$ found in Sect. 16.3.1 is also of order $O(a^2)$. Consequently, the discrepancy D_ℓ we consider here is also of order $O(a^2)$ for every ℓ .

Note that solutions to the field equations (13.25) depend on the size parameter a through the nonlinearity $G'_{\ell a}$, but the integral equations (17.22) do not involve explicit dependence on a . Equation (16.2), which describes the structure of the wave-corpuscle, involves a only through the radial shape factors $\psi_{\ell} = \psi_{\ell a}$. The dependence of $\psi_{\ell a}$ on a is explicitly singular at zero, as is expected since in the singular limit $a \rightarrow 0$ the wave-corpuscle should turn into a point charge, with the square of amplitude described by a delta function as in (6.55). Nevertheless for *arbitrary small* $a > 0$, the wave-corpuscle structure of every charge is preserved, including its principal wavevector.

17.2.6 Stability Issues

A wave-corpuscle is an exact solution of the NLS equation for a special class of external field potentials, and it is an approximate solution in the case of general field potentials. We explain in this section the reasons why an exact solution to the NLS equation with general potentials should be close to the wave-corpuscle solution. Such a property can be called stability of the wave-corpuscles.

A comprehensive analysis of stability is too complex and is beyond the scope of our analysis. Nevertheless, we would like to give a concise consideration of two aspects of stability for well separated charges in the non-relativistic regime: (i) no “blow-up” or “collapse”; (ii) preservation with high accuracy of the form of a wave-corpuscle solution during the dynamics.

We can expect a wave-corpuscle to be stable if the ground state ψ which is used in the construction of the wave-corpuscles provides a global minimum of energy, namely (16.110) holds. The assumption (16.110) is fulfilled in the important case of the logarithmic nonlinearity G and Gaussian rest state ψ . Another assumption which was made in the previous section is the absence of charge collisions, namely assumption (17.42). Under the above two assumptions, we can apply to every charge the analysis of Sect. 16.3.3 and conclude that a charge distribution can be described by a wave-corpuscle with high accuracy if the ratio $\frac{a}{R}$ is small, where R is a macroscopic scale which describes the interacting charges. An analysis of orbital stability for the logarithmic nonlinearity is given in Sect. 35.1.2.

The absence of a “blow up” can be expected in rather general situations. Here is an argument why there cannot be a “blow-up” in finite time for a class of the nonlinearities we introduced in Sect. 14.3. A “blow-up” could be an issue since the nonlinearity $G'(s)$ provides focusing properties with consequent soliton-like solutions ψ_{ℓ} . The nonlinearity G' does not involve dependence on the derivatives, and only its behavior for large $|\psi|$ can affect the “blow up”. The nonlinearities which we consider in our model grow slowly at infinity, as for example the logarithmic nonlinearity, hence the nonlinearity is subordinated to the linear part as one can see in Sect. 35.1.1 or in [59], and the possibility of the “blow up” is excluded. In particular, it can easily be seen if we define $G'_{\ell a}$ to be a constant for large amplitudes

of the fields, namely for $s \geq \left(\overset{\circ}{\psi}_\ell\right)^2(0)$ as in (14.86). Indeed, the energy of a free charge can be written in terms of the density (11.161). In view of relations (14.86), the nonlinearity $G'(s)$ is bounded, implying $G(|\psi|^2) \geq -C|\psi|^2$ for a constant C . That combined with the charge normalization condition (14.17) implies boundedness of the energy from below, namely

$$\mathcal{E}(\psi) \geq -C \text{ for all } \psi, \quad \|\psi\|^2 = \int_{\mathbb{R}^3} |\psi|^2 \, d\mathbf{x} = 1. \quad (17.43)$$

Note that the boundedness of the energy from below can be proven for slowly growing nonlinearities $G'(s)$ too, see for example Sects. 35.1.1 and 35.1.3. A similar argument in the case of many interacting charges also shows that the energy is bounded from below. Since the energy is a conserved quantity, using the boundedness of the energy from below one can prove along lines of [194] the global existence of a unique solution $\psi_\ell(t, \mathbf{x})$ to (13.25), (12.19), (12.20) for all times $0 \leq t < \infty$ for given initial data $\psi_\ell(0, \mathbf{x})$.

The approximate preservation of the shape is a more delicate issue, and is addressed in Sects. 16.3.3 and 35.1.2.

17.3 Close Interaction of Balanced Charges

In previous sections we considered interacting charges in the case when the EM field generated by every charge varies mildly at locations of all remaining charges, such a case corresponds to charges well separated in space. Now we would like to consider the case when charges are in a close proximity, and the variation of EM fields across the charge may be large. When considering this case, we pursue two goals. The first goal is to find for a system of interacting charges a relation between the frequencies and the energies of the components of the system and to show that it coincides with the Planck–Einstein energy-frequency relation. The second goal is to study in detail the simplest system of interacting charges, namely a system of two charges with opposite signs and a large mass ratio, and to prove, in particular, that its frequency spectrum is the same as for the quantum-mechanical model of a Hydrogen atom.

17.4 Multiharmonic Solutions for a System of Many Charges

In this section we consider a regime of close interaction which models a system of bound balanced charges. This regime differs significantly from the regime of remote interaction considered in Sect. 17.2.

Let us take a closer look at the rest states of the system. Namely, we are interested in special solutions to the field equations (17.1), (17.2), (17.3) for which: (i) the potentials φ_ℓ , $\mathbf{A}_\ell$ are time-independent, namely

$$\varphi_\ell = \varphi_\ell(\mathbf{x}), \quad \mathbf{A}_\ell = \mathbf{A}_\ell(\mathbf{x}), \quad \ell = 1, \dots, N; \quad (17.44)$$

(ii) every wave function ψ_ℓ depends on time harmonically through the gauge factor $e^{-i\omega_\ell t}$ with a possibility of different values of the frequencies ω_ℓ for different ℓ . We refer to such solutions as multiharmonic. Note that multiharmonic solutions naturally arise in the analysis of nonlinear dispersive equations and systems, see [11]. Possible transitions between the rest states are considered in Sect. 17.4.1.

The field equations (17.1) governing the system have the form of the NLS equations

$$i\chi\partial_t\psi_\ell + \frac{\chi^2}{2m_\ell}\tilde{\nabla}_\ell^2\psi_\ell - q_\ell\varphi_{\neq\ell}\psi_\ell = \frac{\chi^2}{2m_\ell}G'_\ell(|\psi_\ell|^2)\psi_\ell, \quad \ell = 1, \dots, N, \quad (17.45)$$

with $\tilde{\nabla}_\ell$ being defined by (12.5), and $\varphi_{\neq\ell}$, $\mathbf{A}_{\neq\ell}$ being defined by (12.7) and (12.6), namely:

$$\varphi_{\neq\ell} = \sum_{\ell' \neq \ell} \varphi_{\ell'}, \quad \mathbf{A}_{\neq\ell} = \sum_{\ell' \neq \ell} \mathbf{A}_{\ell'}. \quad (17.46)$$

The EM potentials φ_ℓ are determined by the Poisson equations (17.2), (17.3):

$$\nabla^2\varphi_\ell = -4\pi q_\ell|\psi_\ell|^2, \quad (17.47)$$

$$\nabla^2\mathbf{A}_\ell = -\frac{4\pi}{c}\mathbf{J}_\ell. \quad (17.48)$$

Let us consider now multiharmonic solutions to the system (17.45), (17.47), that is, solutions of the form

$$\psi_\ell(t, \mathbf{x}) = e^{-i\omega_\ell t}\hat{\psi}_\ell(\mathbf{x}). \quad (17.49)$$

As a consequence of (17.45), the time-independent functions $\hat{\psi}_\ell(\mathbf{x})$ satisfy the following *nonlinear eigenvalue problem*:

$$\chi\omega_\ell\hat{\psi}_\ell + \frac{\chi^2}{2m_\ell}\tilde{\nabla}_\ell^2\hat{\psi}_\ell - q_\ell\varphi_{\neq\ell}\hat{\psi}_\ell - \frac{\chi^2}{2m_\ell}G'_\ell(|\hat{\psi}_\ell|^2)\hat{\psi}_\ell = 0 \quad (17.50)$$

coupled with the Poisson equations (17.47), (17.48). Observe that the Eqs. (17.50) with different ℓ are coupled via the potentials $\varphi_{\neq\ell}$.

Energy of a bounded system. According to the energy-momentum representation (11.65) with $\theta = -1$, the canonical energy of the system with the Lagrangian (12.11) is given by the formula

$$\mathring{\mathcal{E}} = \sum_{\ell} E_{0\ell} + \mathring{\mathcal{E}}_{\text{BEM}}. \quad (17.51)$$

In the above formula the EM field energy $\mathring{\mathcal{E}}_{\text{BEM}}$ for the balanced charges is derived from the Lagrangian (12.11) and has the form

$$\begin{aligned} \mathring{\mathcal{E}}_{\text{BEM}} = & -\frac{1}{8\pi} \int [(\nabla\varphi)^2 - (\nabla\mathbf{A})^2] \, d\mathbf{x} \\ & + \sum_{\ell} \frac{1}{8\pi} \int [(\nabla\varphi_{\ell})^2 - (\nabla\mathbf{A}_{\ell})^2] \, d\mathbf{x}; \end{aligned} \quad (17.52)$$

according to formula (12.29)

$$E_{0\ell} = \int q_{\ell} |\psi_{\ell}|^2 \varphi_{\neq\ell} \, d\mathbf{x} + \int \frac{\chi^2}{2m_{\ell}} \left\{ |\tilde{\nabla}\psi_{\ell}|^2 + G_{\ell} (|\psi_{\ell}|^2) \right\} \, d\mathbf{x}, \quad (17.53)$$

where $\varphi, \mathbf{A}$ are the total potentials (12.7), φ_{ℓ} and $\mathbf{A}_{\ell}$ in (17.46) are expressed by Green's formula (12.19), (12.20) in terms of ψ_{ℓ} . Here and below all integrals with respect to spatial variables are over the entire three-dimensional space $\mathbb{R}^3$. Since the external fields are time-independent, the Lagrangian of the system does not depend explicitly on time. Consequently, the energy $\mathring{\mathcal{E}}$ is a conserved quantity for the system (17.45), (17.47), according to (10.55) with $\nu = 0$.

Comparing $E_{0\ell}$ with the energy density (11.160) for the NLS equation (11.155) with an external field, we see that the energy $E_{0\ell}$ coincides with the energy of the ℓ -th charge in the field with the potential $\varphi_{\neq\ell}$ which is considered as being given. We refer to the so defined energy $E_{0\ell}$ as the ℓ -th particle energy in the system field. Note that for a single particle $\mathring{\mathcal{E}}_{\text{BEM}} = 0$, and $E_{0\ell} = \mathcal{E}_{\ell} = \mathcal{E}$. In the general case with $N \geq 2$, the total energy $\mathring{\mathcal{E}}$, according to (17.51), does not equal the sum of $E_{0\ell}$ because the EM field may have non-zero energy.

Note that the energy $\mathring{\mathcal{E}}$ can be expanded using the Poisson equations into the sum of reduced energies which exclude the gradients of the EM potentials. We show that in the case where $\mathbf{A}_{\ell} = 0$. Multiplying the Poisson equation (17.47) by φ_{ℓ} , and the Poisson equation (11.35) for the total field by φ , we obtain, respectively,

$$\begin{aligned} \frac{1}{8\pi} \int (\nabla\varphi_{\ell})^2 \, d\mathbf{x} &= \frac{1}{2} q_{\ell} \int \varphi_{\ell} |\psi_{\ell}|^2 \, d\mathbf{x}, \\ \frac{1}{8\pi} \int (\nabla\varphi)^2 \, d\mathbf{x} &= \frac{1}{2} \sum_{\ell} q_{\ell} \int \varphi_{\neq\ell} |\psi_{\ell}|^2 \, d\mathbf{x} + \frac{1}{2} \sum_{\ell} q_{\ell} \int \varphi_{\ell} |\psi_{\ell}|^2 \, d\mathbf{x}. \end{aligned}$$

Hence the EM field energy can be written in the form

$$\mathring{\mathcal{E}}_{\text{BEM}} = -\frac{1}{2} \sum_{\ell} q_{\ell} \int \varphi_{\neq\ell} |\psi_{\ell}|^2 \, d\mathbf{x}. \quad (17.54)$$

Substituting the above expression into (17.51), we obtain the following expression for the canonical energy, which does not involve $\nabla\varphi_\ell$:

$$\mathcal{E} = \sum_{\ell} \frac{1}{2} \int q_{\ell} |\psi_{\ell}|^2 \varphi_{\neq \ell} \, d\mathbf{x} + \sum_{\ell} \int \frac{\chi^2}{2m_{\ell}} \{ |\nabla \psi_{\ell}|^2 + G_{\ell} (|\psi_{\ell}|^2) \} \, d\mathbf{x}. \quad (17.55)$$

17.4.1 The Planck–Einstein Frequency–Energy Relation and the Logarithmic Nonlinearity

The nonlinear eigenvalue problem (17.50), (17.47) may have many solutions which represent rest states for a system of charges. Every solution $\left\{ \hat{\psi}_{\ell} \right\}_{\ell=1}^N$ determines the corresponding sets of frequencies $\{\omega_{\ell}\}_{\ell=1}^N$ and energies $\{E_{0\ell}\}_{\ell=1}^N$. We would like to consider a transition from one state $\left\{ \hat{\psi}_{\ell} \right\}_{\ell=1}^N$ to another state $\left\{ \hat{\psi}'_{\ell} \right\}_{\ell=1}^N$ in the sense of relating their energies and frequencies. Notice that we don't study here the process of transition in time.

Recall that the fundamental Planck–Einstein frequency–energy relation reads

$$E = \hbar\omega. \quad (17.56)$$

We would like to find a nonlinearity which has the following property: for any two solutions $\left\{ \hat{\psi}_{\ell} \right\}_{\ell=1}^N, \left\{ \hat{\psi}'_{\ell} \right\}_{\ell=1}^N$ with the corresponding frequencies $\{\omega_{\ell}\}_{\ell=1}^N, \{\omega'_{\ell}\}_{\ell=1}^N$ and energies $\{E_{0\ell}\}_{\ell=1}^N, \{E'_{0\ell}\}_{\ell=1}^N$, the following *Planck–Einstein frequency–energy relation* holds:

$$\chi (\omega_{\ell} - \omega'_{\ell}) = E_{0\ell} - E'_{0\ell}, \quad \ell = 1, \dots, N. \quad (17.57)$$

This relation involves frequencies ω_{ℓ} and energy levels $E_{0\ell}$ for the ℓ -th charge. We want the above relation (17.57) to hold for any regular multiharmonic solution of a system of the form (17.50), (17.47).

Let us consider first the frequencies ω_{ℓ} . They can be determined from the equations as follows. Multiplying the ℓ -th equation (17.50) by $\hat{\psi}_{\ell}^*$, integrating the result with respect to the space variable and using the charge normalization condition, we obtain that

$$\begin{aligned} \chi\omega_{\ell} = & \int \left[\frac{\chi^2}{2m_{\ell}} |\tilde{\nabla} \hat{\psi}_{\ell}|^2 \, d\mathbf{x} + q_{\ell} \varphi_{\neq \ell} |\hat{\psi}_{\ell}|^2 \right] \, d\mathbf{x} \\ & + \int \frac{\chi^2}{2m_{\ell}} G'_{\ell} (|\hat{\psi}_{\ell}|^2) |\hat{\psi}_{\ell}|^2 \, d\mathbf{x}. \end{aligned} \quad (17.58)$$

Comparing the above with Eq. (17.53), we see that

$$\chi\omega_\ell - E_{0\ell} = \frac{\chi^2}{2m_\ell} \int \left[G'_\ell \left(|\hat{\psi}_\ell|^2 \right) |\hat{\psi}_\ell|^2 - G_\ell \left(|\hat{\psi}_\ell|^2 \right) \right] d\mathbf{x}. \quad (17.59)$$

For two different solutions $\left\{ \hat{\psi}_\ell \right\}_{\ell=1}^N$, $\left\{ \hat{\psi}'_\ell \right\}_{\ell=1}^N$, we obtain then the following relations linking their frequencies and energies to the nonlinearities:

$$\begin{aligned} & \chi (\omega_\ell - \omega'_\ell) - (E_{0\ell} - E'_{0\ell}) \\ &= \frac{\chi^2}{2m_\ell} \int G_\ell \left(|\hat{\psi}'_\ell|^2 \right) - G'_\ell \left(|\hat{\psi}'_\ell|^2 \right) |\hat{\psi}'_\ell|^2 d\mathbf{x} \\ & - \frac{\chi^2}{2m_\ell} \int G_\ell \left(|\hat{\psi}_\ell|^2 \right) - G'_\ell \left(|\hat{\psi}_\ell|^2 \right) |\hat{\psi}_\ell|^2 d\mathbf{x}. \end{aligned} \quad (17.60)$$

To find a sufficient condition for the Planck–Einstein frequency-energy relation (17.57) to hold, we take into account the charge normalization condition (17.7), namely $\|\hat{\psi}_\ell\| = 1$. Then (17.60) implies that the desired sufficient condition is as follows:

$$\int \left[G_\ell \left(|\hat{\psi}_\ell|^2 \right) - G'_\ell \left(|\hat{\psi}_\ell|^2 \right) |\hat{\psi}_\ell|^2 \right] d\mathbf{x} = \text{const} \quad (17.61)$$

for every $\hat{\psi}_\ell$ with $\|\hat{\psi}_\ell\| = 1$, where the constant may depend on ℓ . This condition, in turn, is fulfilled if the following differential equation with an independent variable $s = |\hat{\psi}|^2$ holds:

$$s \frac{d}{ds} G_\ell(s) - G_\ell(s) = K_G s, \quad (17.62)$$

with a constant K_G that may depend only on ℓ . Then Eq. (17.59) and the normalization condition $\|\hat{\psi}_\ell\| = 1$ imply that

$$\begin{aligned} \chi\omega_\ell - E_{0\ell} &= \frac{\chi^2}{2m_\ell} \int \left[G'_\ell \left(|\hat{\psi}_\ell|^2 \right) |\hat{\psi}_\ell|^2 - G_\ell \left(|\hat{\psi}_\ell|^2 \right) \right] d\mathbf{x} \\ &= -\frac{\chi^2}{2m_\ell} K_G, \end{aligned} \quad (17.63)$$

where K_G does not depend on ω_ℓ and $E_{0\ell}$, hence (17.57) is fulfilled.

Solving the differential equation (17.62), we obtain the following explicit formula for the *logarithmic nonlinearity*:

$$G_\ell(s) = K_G s \ln s + C s. \quad (17.64)$$

Comparing the representation (17.64) with Eqs. (14.40) and (14.41), we find that if $K_G < 0$ this is exactly the nonlinearity (14.41) which corresponds to the Gaussian

factor. According to Eq. (14.40), formula (17.64) turns into

$$G_\ell(s) = G_{\ell,a}(s) = -\frac{1}{(a_\ell)^2} s \ln s + \frac{1}{(a_\ell)^2} s \left(\ln \frac{1}{\pi^{3/2}} - 2 - 3 \ln a_\ell \right)$$

where $a = a_\ell$ is the size parameter for the ℓ -th charge. If $K_G = 0$, then $G_\ell(|\hat{\psi}_\ell|^2)$ is quadratic, and Eq. (17.50) turns into the linear Schrödinger equation for which the fulfillment of the Planck–Einstein relation is a well-known fundamental property. Note that the above argument can be literally repeated if Eq. (17.50) involves an external electric field as in (12.14); $\varphi_{\neq \ell}$ in the definition (17.53) of $E_{0\ell}$ has to be replaced in this case by $\varphi_{\neq \ell} + \varphi_{\text{ex}}$.

It is absolutely remarkable that the logarithmic nonlinearity, which is singled out by the fulfillment of the Planck–Einstein relation, also has the second crucial property of providing for a localized soliton solution, namely the Gaussian one defined by Eq. (14.38).

Let us study in more detail the situation with the logarithmic nonlinearity singled out by the Planck–Einstein relation. Formula (17.63) for the Gaussian wave function takes the form

$$E_{0\ell} = \chi \omega_\ell + \frac{\chi^2}{2a_\ell^2 m_\ell}. \quad (17.65)$$

Hence, we make the following conclusion

Claim Let $\{\hat{\psi}_\ell^\sigma\}_{\ell=1}^N \in \mathcal{E}^N$ be a set of multiharmonic solutions of the nonlinear eigenvalue problem (17.50), (17.47) with the logarithmic nonlinearity, the index σ labeling the solutions. Let $\{\omega_\ell^\sigma\}_{\ell=1}^N$ and $\{E_{0\ell}^\sigma\}_{\ell=1}^N$ be the frequencies and energies of the corresponding solutions. Then the ℓ -th components of any two solutions $\hat{\psi}_\ell^\sigma = \hat{\psi}_\ell$ and $\hat{\psi}_\ell^{\sigma_1} = \hat{\psi}_\ell'$ satisfy the Planck–Einstein relation (17.57).

More mathematical details related to the above arguments are provided in Sect. 35.1.3.

The Planck–Einstein relation (17.57) implies that the difference of energies after any transition from one multi-frequency state to another is an integer factor times $\chi\omega$. *We would like to stress the remarkable universality of the Planck–Einstein relation (17.57) proven above for multiharmonic solutions and the fact that it holds for every individual particle in a system of many interacting particles.* Namely, the Planck–Einstein relation holds with the same coefficient χ for an arbitrary pair of solutions of (17.50), (17.47) and for an arbitrary component of those solutions, and the validity of this relation does not depend on the number N which equals the number of interacting particles. Observe also that the derivation of (17.63) and (17.65) is based only on properties of the nonlinear terms as in (17.60) and can be extended to more general systems than (17.45), (17.47).

Remark 17.4.1 Note that the above derivation of the Planck–Einstein frequency–energy relation (17.57) uses only the NLS equations (17.50) and the charge normalization condition. The same derivation is valid if the fields φ_ℓ , $\mathbf{A}_\ell$ are time dependent and satisfy not the Poisson equations (17.47), (17.48), but the wave equations (12.9), (12.10). The only essential assumption is that the multiharmonic form (17.49) of the wave functions ψ_ℓ describes an equilibrium of the system. The system of NLS equations (17.45) coupled with wave equations (12.9), (12.10) can be considered as intermediate between relativistic and non-relativistic models of balanced charges.

Though we do not consider the transition in time by itself, the transitions between time-harmonic states are often caused by resonance interactions, [11], [12]. The importance of resonant interactions at microscopic scales in physics is well-known, [297, p. 115]:

“...the interaction between two microscopic physical systems is controlled by a peculiar law of resonance. This law requires that the difference of two proper frequencies of the one system be equal to the difference of two proper frequencies of the other”.

The resonances in the system of balanced charges may occur when the EM field, which interacts with the system, involves a component with the resonant frequency $\omega = \omega_\ell - \omega'_\ell$. The total energy (17.51) in the transition is preserved, therefore there is an energy transfer between the energies of charges $E_{0\ell}$ and the EM energy $\mathcal{E}_{\text{BEM}}$.

If the transition is resonant and local, namely if the ℓ -th charge interacts only with a component of the EM field with the resonant frequency $\omega = \omega_\ell - \omega'_\ell$, the energy of the component of the EM field with the frequency ω , according to (17.57), will change by one or several quanta of energy $E = \chi\omega$. *This property can be interpreted as an interchange of quanta of energy $\chi\omega$ between the atoms and the EM field.*

The connection between the logarithmic nonlinearity and the Planck–Einstein frequency–energy relation was discovered in a different setting by Bialynicki-Birula and Mycielski in [40]. Note though that in [40] a system of N particles is described as in quantum mechanics by a single wave function ψ over $3N$ -dimensional configuration space, whereas in our model every one of N particles has its own wave function ψ_ℓ depending on 3 spatial variables. This is why our approach naturally leads to the study of *multi-harmonic solutions* for interacting particles, for which every individual wave function ψ_ℓ can be associated with its individual frequency ω_ℓ . Another significant difference between our approach and the nonlinear quantum mechanics in [40] is the way the nonlinearity enters the Lagrangian and the field equations for $N \geq 2$ particles. Namely, in our approach every individual particle has its own nonlinearity, whereas in [40] there is a single nonlinearity for the entire system of N particles.

17.5 A Two Particle Hydrogen-Like System

In this section we treat a particular case of multiharmonic solutions of the system (17.50) with $N = 2$ which models a bound proton-electron system. We consider general properties of this system and provide an analysis of the coupling between a proton and an electron. Using this analysis as a motivation, we then write the equation and the energy functional for one charge (electron) in the Coulomb field of the proton. We show that this system has discrete energy levels E_n^0 which obey the Rydberg formula

$$E_n^0 - E_{n'}^0 \simeq -\frac{q^4 m}{2\chi^2} \left(\frac{1}{n^2} - \frac{1}{n'^2} \right), \quad (17.66)$$

if the ratio of the Bohr radius $\hat{a}_B$ to the free charge size parameter a is vanishingly small. We study this problem in more detail along the lines of [38], [39] in Sect. 35.2.

17.5.1 The Electron-Proton System as a Hydrogen Atom Model

We model the Hydrogen atom as a system of two closely interacting balanced charges: an electron and a proton. If we have two balanced charges at large distances, the results of Sect. 17.2 show that their motion can be described with high accuracy by a Keplerian model. In this section we consider a completely different regime of a close interaction. Evidently, the results on interaction of many charges in the regime of remote interaction do not apply to the case of two charges in the Hydrogen atom, since they are in close proximity, and the potentials can vary significantly over their spatial extent. We will see that in this case our model is similar to Schrödinger's Hydrogen atom rather than the Keplerian model. The charges in our analysis are assumed to be spinless. A generalization which would include the spin effect is possible, based on results of Part IV. In this section we provide a detailed sketch of our Hydrogen atom model. More mathematical details are provided in Sect. 35.2.

To model states of a bound proton-electron system, we consider the multiharmonic solutions of the system (17.45), (17.47) with $N = 2$, where the indices ℓ take two values $\ell = 1$ and $\ell = 2$ for electron and proton respectively; the corresponding charges have opposite values

$$q_1 = -q, \quad q_2 = q > 0. \quad (17.67)$$

We neglect magnetic effects and set

$$\mathbf{A}_\ell = 0, \quad \ell = 1, 2. \quad (17.68)$$

The electric fields in the resting Hydrogen atom have to be time-independent, hence $|\psi_\ell|^2$ in (17.47) must be time-independent too. Therefore we assume that only the phase factors depend on time and consider the multi-harmonic solutions of this system, namely solutions of the form (17.49).

Plugging expressions (17.49) into Eqs. (17.45), (17.47), we find that the functions $\hat{\psi}_\ell(\mathbf{x})$ satisfy the nonlinear eigenvalue problem (17.50) where, in accordance with (12.6),

$$\varphi_{\neq 1} = \varphi_2, \quad \varphi_{\neq 2} = \varphi_1, \quad \frac{1}{4\pi} \nabla^2 \varphi_\ell = -q_\ell \left| \hat{\psi}_\ell \right|^2. \quad (17.69)$$

It is convenient to introduce the notation

$$\Phi_1 = \frac{\varphi_1}{q_1}, \quad \Phi_2 = \frac{\varphi_2}{q_2}, \quad (17.70)$$

$$\hat{a}_1 = \frac{\chi^2}{q^2 m_1}, \quad \hat{a}_2 = \frac{\chi^2}{q^2 m_2}. \quad (17.71)$$

The parameter $\hat{a}_1$ (which should not be confused with the size parameter a_1 of the free electron which enters the nonlinearity) turns into *the Bohr radius* if χ coincides with the Planck constant $\hbar$, and m_1, q are the electron mass and charge respectively. Using the above notation, we rewrite the two-particle system (17.50), (17.47) in the form of the following *nonlinear eigenvalue problem*:

$$\frac{\chi}{q^2} \omega_1 \hat{\psi}_1 + \frac{\hat{a}_1}{2} \nabla^2 \hat{\psi}_1 + \Phi_2 \hat{\psi}_1 = \frac{\hat{a}_1}{2} G'_1 \left(|\hat{\psi}_1|^2 \right) \hat{\psi}_1, \quad (17.72)$$

$$\frac{\chi}{q^2} \omega_2 \hat{\psi}_2 + \frac{\hat{a}_2}{2} \nabla^2 \hat{\psi}_2 + \Phi_1 \hat{\psi}_2 = \frac{\hat{a}_2}{2} G'_2 \left(|\hat{\psi}_2|^2 \right) \hat{\psi}_2, \quad (17.73)$$

$$\nabla^2 \Phi_1 = -4\pi |\hat{\psi}_1|^2, \quad \nabla^2 \Phi_2 = -4\pi |\hat{\psi}_2|^2. \quad (17.74)$$

The functions $\hat{\psi}_1$ and $\hat{\psi}_2$ are, respectively, the wave functions for the electron and the proton. As always, we look for solutions which satisfy the charge normalization condition (17.7), namely

$$\left\| \hat{\psi}_1 \right\| = 1, \quad \left\| \hat{\psi}_2 \right\| = 1, \quad (17.75)$$

where, as usual, we use the notation (14.25):

$$\left\| \psi \right\|^2 = \int |\psi|^2 d\mathbf{x}.$$

We denote the set of functions which satisfy the constraint $\left\| \psi \right\|^2 = 1$ by $\mathcal{E}$, see (35.2); if the pair $\{\hat{\psi}_1, \hat{\psi}_2\} \in \mathcal{E}^2$ then (17.75) holds. The nonlinearities G'_1, G'_2

are assumed to be logarithmic as defined by Eq. (14.41) where the size parameter $a = a_\ell$ is different for the electron and proton. One can similarly consider other nonlinearities, but here we stay with the logarithmic ones.

We define solutions Φ_i of (17.74) by Green's formula (4.45) as follows:

$$\Phi_i = \int_{\mathbb{R}^3} \frac{|\hat{\psi}_i|^2(\mathbf{y})}{|\mathbf{y} - \mathbf{x}|} d\mathbf{y}, \quad i = 1, 2. \quad (17.76)$$

In accordance with Eq. (17.51), we introduce the energy functional by the following formula:

$$\begin{aligned} \mathcal{E}(\hat{\psi}_1, \hat{\psi}_2) &= E_{01} + \mathcal{E}_2(\hat{\psi}_2), \\ E_{01} &= q^2 \int \left[\frac{\hat{a}_1}{2} |\nabla \hat{\psi}_1|^2 + \frac{\hat{a}_1}{2} G_1(|\hat{\psi}_1|^2) - |\hat{\psi}_1|^2 \Phi_2 \right] d\mathbf{x}, \end{aligned} \quad (17.77)$$

$$\mathcal{E}_2(\hat{\psi}_2) = q^2 \int \left[\frac{\hat{a}_2}{2} |\nabla \hat{\psi}_2|^2 + \frac{\hat{a}_2}{2} G_2(|\hat{\psi}_2|^2) \right] d\mathbf{x}. \quad (17.78)$$

Equations (17.72), (17.73) can be derived from the Euler equations for a critical point of $\mathcal{E}(\hat{\psi}_1, \hat{\psi}_2)$ with the constraint $\{\hat{\psi}_1, \hat{\psi}_2\} \in \mathcal{E}^2$ if we determine in (17.77) Φ_2, Φ_1 by formula (17.76). The frequencies ω_1, ω_2 are proportional to the Lagrange multipliers for the problem with constraints. According to (17.76),

$$\int |\hat{\psi}_1|^2 \Phi_2 d\mathbf{x} = \int |\hat{\psi}_2|^2 \Phi_1 d\mathbf{x},$$

hence the coupling term in (17.77) is symmetric with respect to $\hat{\psi}_1, \hat{\psi}_2$. Note also that the first term in formula (17.77) coincides with the energy E_{01} of the first charge in the system field as given by (17.53) with $\ell = 1$, and the second term with the expression for the energy $\mathcal{E}_2(\hat{\psi}_2)$ of a free second charge given by (35.1). Obviously, $\mathcal{E}(\hat{\psi}_1, \hat{\psi}_2)$ can also be written as a sum of the energy E_{02} of the second charge in the system field plus the free energy of the first charge.

We can also rewrite the energy functional $\mathcal{E}(\hat{\psi}_1, \hat{\psi}_2)$ in the symmetric form

$$\mathcal{E}(\hat{\psi}_1, \hat{\psi}_2) = \mathcal{E}_1(\hat{\psi}_1, \hat{\psi}_2) + \mathcal{E}_2(\hat{\psi}_1, \hat{\psi}_2), \quad (17.79)$$

where

$$\begin{aligned} \mathcal{E}_1 &= \frac{q^2 \hat{a}_1}{2} \int \left[|\nabla \hat{\psi}_1|^2 + G_1(|\hat{\psi}_1|^2) \right] d\mathbf{x} - \frac{1}{2} q^2 \int \Phi_2 |\hat{\psi}_1|^2 d\mathbf{x}, \\ \mathcal{E}_2 &= \frac{q^2 \hat{a}_2}{2} \int \left[|\nabla \hat{\psi}_2|^2 + G_2(|\hat{\psi}_2|^2) \right] d\mathbf{x} - \frac{1}{2} q^2 \int \Phi_1 |\hat{\psi}_2|^2 d\mathbf{x}. \end{aligned}$$

Remark 17.5.1 Note that system (17.72)–(17.74) is similar to the Hartree equations studied in [225], though it differs from it because of the presence of the nonlinearities G_ℓ .

17.5.1.1 Principal Steps of Analysis

Observe that Eqs. (17.72), (17.73), (17.74) are the Euler equations for critical points of the functional $\mathcal{E}$ under the normalization constraint (17.75), and the frequencies ω_1, ω_2 are the corresponding Lagrange multipliers. Therefore, variational methods can be applied to study the nonlinear eigenvalue problem (17.72)–(17.75). Our analysis is based on two observations. The first one is that the linear Schrödinger operator negative eigenvalues, which describe the Hydrogen atom spectrum, coincide with the critical points of the corresponding quadratic functional $\mathcal{E}_{\text{lin}}(\hat{\psi}_1)$ under the first constraint of (17.75), and then the eigenvalues can be determined by a min-max method. The second observation is that under certain conditions, when applying the min-max method to the nonlinear functional $\mathcal{E}(\hat{\psi}_1, \hat{\psi}_2)$ subjected to the constraints, we can approximate it by $\mathcal{E}_{\text{lin}}(\hat{\psi}_1) + C$ with a certain constant C . The smallness of the ratio $m_1/m_2 \cong 1/1836$ of electron to proton masses plays an important role in the analysis. This smallness implies that, for the critical points with low energies, the potential Φ_2 of the proton is close to Coulomb's potential $1/|\mathbf{x}|$ at spatial scales of the order a_1 . If the characteristic size a_1 of a free electron is much larger than the Bohr radius $\hat{a}_1$, then our analysis shows that the energy levels of our model for the Hydrogen atom are in a good agreement with the Schrödinger theory. As to the assumption that a free electron significantly contracts in size when it is bound to a proton, it seems to be quite reasonable. Note also that the existence of the discrete energy levels of nonlinear functionals under certain general restrictions is well known. Since the energy functional and the constraints are invariant with respect to multiplication by -1 , one can apply the Lusternik–Schnirelman theory, which guarantees the existence of an infinite set of critical points under appropriate conditions, see, for example, [157], [225] and references therein, see also Sect. 35.2.

We already showed in Sect. 17.4.1 that, for the logarithmic nonlinearities G_1, G_2 , the frequencies ω_1, ω_2 and critical values of $\mathcal{E}_{01}, \mathcal{E}_{02}$ satisfy the Planck–Einstein formula $E = \hbar\omega$ (this relation was studied in [16], [17]; in [40] the relation between the Planck–Einstein formula and the logarithmic nonlinearity was discovered in a different setting).

The problem of finding critical values for the energy functional $\mathcal{E}$ defined by (17.77) can be approximated by a simpler problem for a single wave function in a way similar to the Born–Oppenheimer approximation in quantum mechanics, and we discuss the reduction in the next section.

17.5.2 Reduction to One Charge in the Coulomb Field

In this subsection we give a motivation for replacing the problem of determination of critical values of the energy functional $\mathcal{E}$ given by (17.77) for a system of two charges by a simpler problem for a single charge similarly to the Born–Oppenheimer approximation in quantum mechanics. To this end, we use two changes of variables in the two equations, $\mathbf{x} = \hat{a}_1 \mathbf{y}_1$ in (17.72) and $\mathbf{x} = \hat{a}_2 \mathbf{y}_2$ in (17.73), with a_ℓ defined by (17.71), namely

$$\mathbf{x} = \hat{a}_\ell \mathbf{y}_\ell, \quad \ell = 1, 2. \quad (17.80)$$

We introduce the rescaled fields $\phi_\ell, \hat{\psi}_\ell$ as follows:

$$\Phi_\ell(\mathbf{x}) = \frac{\phi_\ell(\mathbf{y}_\ell)}{\hat{a}_\ell}, \quad \hat{\psi}_\ell(\mathbf{x}) = \frac{1}{\hat{a}_\ell^{3/2}} \Psi_\ell(\mathbf{y}_\ell), \quad \ell = 1, 2. \quad (17.81)$$

Hence (17.72), (17.73) takes the form of the following *nonlinear 2-charge system*:

$$\frac{\chi}{q^2} \omega_1 \Psi_1 + \frac{1}{2\hat{a}_1} \nabla_{\mathbf{y}_1}^2 \Psi_1 + \frac{1}{\hat{a}_2} \phi_2 \left(\frac{\hat{a}_1}{\hat{a}_2} \mathbf{y}_1 \right) \Psi_1 = \frac{1}{2\hat{a}_1} G'_1(|\Psi_1|^2) \Psi_1, \quad (17.82)$$

$$\frac{\chi}{q^2} \omega_2 \Psi_2 + \frac{1}{2\hat{a}_2} \nabla_{\mathbf{y}_2}^2 \Psi_2 + \frac{1}{\hat{a}_1} \phi_1 \left(\frac{\hat{a}_2}{\hat{a}_1} \mathbf{y}_2 \right) \Psi_2 = \frac{1}{2\hat{a}_2} G'_2(|\Psi_2|^2) \Psi_2, \quad (17.83)$$

$$\nabla_{\mathbf{y}_1}^2 \phi_1 = -4\pi |\Psi_1|^2, \quad \nabla_{\mathbf{y}_2}^2 \phi_2 = -4\pi |\Psi_2|^2. \quad (17.84)$$

To avoid complicated notation, we use the same letter G to denote the nonlinearity in rescaled variables, one has to note that after substitution (17.81) the nonlinearity G_ℓ involves dependence on the dimensionless ratio $\frac{a_\ell}{\hat{a}_\ell}$ and not on a_ℓ . Equations (17.82), (17.83) involve the ratio

$$b = \frac{\hat{a}_2}{\hat{a}_1}. \quad (17.85)$$

Recall now that the electron/proton mass ratio $\frac{m_1}{m_2}$ is small, and

$$b = \frac{\hat{a}_2}{\hat{a}_1} = \frac{m_1}{m_2} \simeq \frac{1}{1800} \ll 1, \quad (17.86)$$

therefore the ratio b is small too. We rewrite Eqs. (17.82), (17.83) as follows:

$$\frac{\chi \hat{a}_1}{q^2} \omega_1 \Psi_1 + \frac{1}{2} \nabla^2 \Psi_1 + \frac{1}{b} \phi_2 \left(\frac{\mathbf{y}}{b} \right) \Psi_1 = \frac{1}{2} G'_1 \Psi_1, \quad (17.87)$$

$$\frac{\chi \hat{a}_2}{q^2} \omega_2 \Psi_2 + \frac{1}{2} \nabla^2 \Psi_2 + b \phi_1(b\mathbf{y}) \Psi_2 = \frac{1}{2} G'_2 \Psi_2. \quad (17.88)$$

Notice that Eq. (17.88) for the proton wave function Ψ_2 depends on Ψ_1 through the potential $b\phi_1(b\mathbf{y})$ which is small since b is small. A natural approximation is provided by setting $b\phi_1(b\mathbf{y})$ to 0. Then a solution of the so modified Eq. (17.88) is the Gaussian ground state which produces the global minimum of the energy $\mathcal{E}_2(\hat{\psi}_2)$, and this is what we want since we are interested in lower energy levels (see Remark 35.2.5 for more details). We show below that in this case the potential $\frac{1}{b}\phi_2(\frac{\mathbf{y}}{b})$ in (17.87) can be replaced by $\frac{1}{|\mathbf{y}|}$.

To find lower energy levels of the electron, we consider a radial solution (Ψ_1, Ψ_2) of (17.87), (17.88). The equation for the electron frequency ω_1 given by (17.58) with $\ell = 1$ takes the form

$$\frac{\chi \hat{a}_1}{q^2} \omega_1 = \int \frac{1}{2} G'_1 |\Psi_1|^2 + \frac{1}{2} |\nabla \Psi_1|^2 - \frac{1}{b} \phi_2\left(\frac{\mathbf{y}}{b}\right) |\Psi_1|^2 d\mathbf{y} \quad (17.89)$$

with the proton potential ϕ_2 . We would like to replace the potential ϕ_2 in (17.89) by the Coulomb potential $\frac{1}{|\mathbf{y}|}$ as an approximation. To show that for small b one may expect the resulting perturbation of eigenvalues to be small for radial solutions, we can use formula (14.70), namely

$$\frac{1}{b} \phi_2\left(\frac{r}{b}\right) = \frac{1}{r} - \frac{4\pi}{r} \int_{r/b}^{\infty} \left(r_1 - \frac{r}{b}\right) r_1 |\Psi_2(r_1)|^2 dr_1. \quad (17.90)$$

Obviously,

$$\int \frac{1}{b} \phi_2\left(\frac{\mathbf{y}}{b}\right) |\Psi_1|^2 d\mathbf{y} = \int \left(\frac{1}{b} \phi_2\left(\frac{\mathbf{y}}{b}\right) - \frac{1}{|\mathbf{y}|} \right) |\Psi_1|^2 + \frac{1}{|\mathbf{y}|} |\Psi_1|^2 d\mathbf{y}.$$

Restricting ourselves to lower energy levels for $\mathcal{E}(\hat{\psi}_1, \hat{\psi}_2)$, we can conclude using (17.90) that $\frac{1}{b} \phi_2\left(\frac{r}{b}\right)$ can be replaced in the energy functional for (17.87) by the Coulomb potential $\frac{1}{r}$ with an error of order b^2 . Indeed, the term which we expect to be small is

$$\begin{aligned} D_{\text{prot}} &= - \int \left(\frac{1}{b} \phi_2\left(\frac{1}{b} \mathbf{y}_1\right) - \frac{1}{|\mathbf{y}_1|} \right) |\Psi_1|^2 d\mathbf{y}_1 \\ &= 4\pi \int_0^{\infty} \frac{4\pi}{r} |\Psi_2(r)|^2 r^2 \int_{r/b}^{\infty} (r_1 - r/b) r_1 |\Psi_2(r_1)|^2 dr_1 dr. \end{aligned} \quad (17.91)$$

Changing the order of integration and using substitution $r/b = r_2$, we obtain that

$$\begin{aligned} D_{\text{prot}} &= (4\pi)^2 b^2 \int_0^{\infty} \int_{r_2}^{\infty} |\Psi_1(br_2)|^2 r_2 (r_1 - r_2) r_1 |\Psi_2(r_1)|^2 dr_1 dr_2 \\ &= (4\pi)^2 b^2 \int_0^{\infty} |\Psi_2(r_1)|^2 r_1 \int_0^{r_1} |\Psi_1(br_2)|^2 r_2 (r_1 - r_2) dr_2 dr_1. \end{aligned}$$

Hence

$$|D_{\text{prot}}| \leq \frac{(4\pi)^2}{6} b^2 \max_r |\psi_1(r)|^2 \int_0^\infty |\psi_2(r_1)|^2 r_1^4 dr_1 \quad (17.92)$$

where $b \ll 1$. More mathematical details concerning boundedness of the coefficient at b^2 in (17.92) can be found in [16].

Neglecting D_{prot} , we replace ϕ_2 by $\frac{1}{|y|}$ in (17.89) with a small error of order b^2 . Therefore, the frequencies of the electron which correspond to lower energy levels of the Hydrogen atom are determined by replacing the right-hand side of (17.89) by the following energy functional with the Coulomb potential for a single wave function ψ_1 :

$$\mathcal{E}_{\text{Cb}}(\psi_1) = \frac{q^2}{\hat{a}_1} \int_{\mathbb{R}^3} \left[\frac{1}{2} |\nabla \psi_1|^2 + \frac{1}{2} G_1(|\psi_1|^2) - \frac{1}{|y|} |\psi_1|^2 \right] dy. \quad (17.93)$$

Therefore, when studying lower energy levels and corresponding frequencies in the case $b \rightarrow 0$, we can reasonably replace the system (17.87), (17.88) by one equation with the Coulomb potential obtained by variation of $\mathcal{E}_{\text{Cb}}(\psi_1)$. We consider this problem in the next section and in Sect. 35.2. Note that the discrete spectrum of the classical linear non-relativistic Schrödinger operator for the Hydrogen atom with the Coulomb potential can be recovered (if multiplicities are not taken into account) from radial eigenfunctions. Therefore, we can restrict ourselves in the analysis to the radial solutions of (17.87), (17.88) and restrict (17.77) and (17.93) to radial functions.

17.5.2.1 The Charge in the Coulomb Field

As explained above, we replace the original problem of finding frequencies ω_1 based on critical points of $\mathcal{E}(\hat{\psi}_1, \hat{\psi}_2)$ by a simpler problem for the energy functional $\mathcal{E}_{\text{Cb}}(\psi_1)$ subjected to the constraint $\|\psi_1\| = 1$. Note that we look only for lower critical energy levels and corresponding frequencies for both problems. We obtain from (17.87) the corresponding nonlinear eigenvalue problem for the electron wave function ψ_1 :

$$\omega \psi_1 + \frac{1}{2} \nabla^2 \psi_1 + \frac{1}{|y|} \psi_1 = \frac{1}{2} G'_1(|\psi_1|^2) \psi_1 \quad (17.94)$$

where

$$\omega = \frac{\chi \hat{a}_1}{q^2} \omega_1$$

is the dimensionless spectral parameter. As we have already mentioned, a similar reduction to a single Schrödinger equation with the Coulomb potential is made in quantum mechanics via the Born–Oppenheimer approximation.

We essentially exploit the dependence of the nonlinearity $G'_1 = G'_{1a}$ on the small parameter $\kappa = \frac{\hat{a}_1}{a}$ which equals the ratio of the electron Bohr radius $\hat{a}_1$ to the size

parameter $a_1 = a$. If κ is small, then the nonlinearity $G'_{1a}(s) = \kappa^2 G'_1(\kappa^{-3}s)$ is small and plays the role of a small perturbation in the eigenvalue problem (17.94). Recall that the classical energy levels of the Schrödinger operator for the Hydrogen atom are given by the formula

$$E_{0,n} = \hbar\omega_{0,n} = -\frac{q^2}{\hbar\hat{a}_1} \frac{1}{2n^2} = -hcR_\infty \frac{1}{n^2}, \quad n = 1, 2, \dots, \quad (17.95)$$

where,

$$R_\infty = \frac{q^4 m}{4\pi\hbar^3 c}$$

is the Rydberg constant, $\hat{a}_1 = a_B$ is the Bohr radius and we assume that $\chi = \hbar = \frac{\hbar}{2\pi}$.

A detailed mathematical analysis carried out in [16] and Sect. 35.2 shows that the lower energy levels of the functional $\frac{\hat{a}_1}{q^2} \mathcal{E}_{Cb}(\Psi_1)$ are arbitrarily close to $-\frac{1}{2n^2}$ provided that $\kappa = \frac{\hat{a}_1}{a}$ is sufficiently small. Note that the electron energy levels E_n^0 and $E_{n'}^0$ for two harmonic states of the electron satisfy the Planck-Einstein relation

$$\chi(\omega_{1n} - \omega_{1n'}) = E_n^0 - E_{n'}^0, \quad (17.96)$$

where $\omega_{1n}, \omega_{1n'}$ are the electron frequencies. Based on the estimates obtained in Sect. 35.2, we can conclude that, if $\chi = \hbar$ and q, m_1 equal the electron's charge and mass respectively, then the n -th lower frequency ω_{1n} for the solution of (17.72), (17.73), (17.74) is given by the following approximate formula:

$$\chi\omega_{1n} = -\frac{1}{n^2} \frac{q^2}{2\hat{a}_1} \left[1 + O\left(b^2 + \left(\frac{\hat{a}_1}{a}\right)^2 \left| \ln\left(\frac{\hat{a}_1}{a}\right) \right| \right) \right], \quad n = 1, 2, \dots \quad (17.97)$$

We see that this formula is close to the classical expression (17.95). The correction term $O\left(b^2 + \left(\frac{\hat{a}_1}{a}\right)^2 \left| \ln\left(\frac{\hat{a}_1}{a}\right) \right| \right)$ in (17.97) is small if b given by (17.85) and $\kappa = \frac{\hat{a}_1}{a}$ are small. Observe also that, according to Eqs. (17.96), (17.97) and (17.95), the differences of energy levels of the nonlinear eigenvalue problem are close to the same in the Rydberg formula with the relative error of order 10^{-4} if $\frac{\hat{a}_1}{a}$ is of order 10^{-2} . Hence, if we assume that the size a of a free electron is at least 100 times larger than the Bohr radius, then the Hydrogen atom model introduced here is in good agreement with the Hydrogen spectroscopic data. We think that it is quite reasonable to assume that a free electron has much larger size than an electron bound in a Hydrogen atom where it is naturally contracted by the electric force of the positively charged proton.

Remark 17.5.2 If we use the Hydrogen atom model based on dressed charges as in Sect. 19.5, we still obtain discrete energy levels. But the linear eigenvalue problem in the limit $\frac{\hat{a}_1}{a} \rightarrow 0$ involves a potential $\frac{-q^2}{|y|} + q\phi(y)$ where in addition to the Coulomb potential there is a term $q\phi(y)$ due to the electron EM self-interaction. So, the eigenvalue problem for dressed charges does not turn in the limit $\frac{\hat{a}_1}{a} \rightarrow 0$ into

the one for the linear Hydrogen atom Schrödinger operator, and consequently, the energy levels do not converge to the known expressions for the Hydrogen atom.

17.6 Relativistic Balanced Charge Theory

In this section we derive in the localization limit the dynamics for point charges from the energy and momentum conservation laws for relativistic balanced charges. The point dynamics is described by the relativistic version of Newton's law and by Einstein's formula. Einstein's formula is not postulated, but is derived in the regimes with acceleration. The rest mass is not a prescribed quantity but emerges as an integral of motion. The method of derivation of the equations of motion is a relativistic version of the Ehrenfest theorem. We also describe the relativistic version of the Hydrogen atom model which includes Sommerfeld's fine structure.

17.6.1 Relativistic Field Equations for Balanced Charges

Let us consider a system of N charges interacting via their EM fields described by their 4-vector potentials $(\varphi_\ell, \mathbf{A}_\ell)$. The charges are described by their wave functions ψ_ℓ with the subscript index $\ell = 1, \dots, N$ labeling them. The system Lagrangian $\mathcal{L}$ is defined by Eq. (12.2) in Sect. 12.1, and the charge interaction is described by the system of NKG equations for the charge distributions ψ_ℓ and the Maxwell equations for the fields $(\varphi_\ell, \mathbf{A}_\ell)$.

The system equations are the Euler–Lagrange field equations obtained from the Lagrangian $\mathcal{L}$. Namely, the field equations for the charge distributions ψ_ℓ are the following nonlinear Klein–Gordon (NKG) equations (12.8):

$$-\frac{1}{c^2} \tilde{\partial}_\ell \tilde{\partial}_\ell \psi_\ell + \tilde{\nabla}^{\ell 2} \psi_\ell - G'_\ell (\psi_\ell^* \psi_\ell) \psi_\ell - \kappa_{0\ell}^2 \psi_\ell = 0, \quad \ell = 1, \dots, N, \quad (17.98)$$

and the EM fields satisfy the Maxwell equations (11.34), that is,

$$\partial_\mu F_\ell^{\mu\nu} = \frac{4\pi}{c} J_\ell^\nu, \quad \ell = 1, \dots, N. \quad (17.99)$$

The charge density ρ_ℓ and currents $\mathbf{J}_\ell$ generated by the ℓ -th charge which enter the 4-current $J_\ell^\nu = (\rho_\ell, \mathbf{J}_\ell)$ are given by formulas (11.132), (11.133):

$$\rho_\ell = -\frac{\chi q_\ell}{m_\ell c^2} \mathcal{J}_0(\psi_\ell, \varphi_{\neq \ell} + \varphi_{\text{ex}}), \quad (17.100)$$

$$\mathbf{J}_\ell = \frac{\chi q_\ell}{m_\ell} \mathcal{J}_\nabla(\psi_\ell, \mathbf{A}_{\neq \ell} + \mathbf{A}_{\text{ex}}), \quad (17.101)$$

where we use the notation (11.128), (11.129). The potentials $\mathbf{A}_{\neq\ell}$, $\varphi_{\neq\ell}$ are defined by (12.6), and φ_{ex} , $\mathbf{A}_{\text{ex}}$ are the potentials of the external EM field. Note that equations (17.98) for ψ_ℓ are coupled with the equations for the EM potentials via the covariant derivatives (11.13):

$$\tilde{\partial}_t \ell = \partial_t + \frac{iq_\ell}{\chi} (\varphi_{\neq\ell} + \varphi_{\text{ex}}), \quad \tilde{\nabla}_\ell = \nabla - \frac{iq_\ell}{\chi c} (\mathbf{A}_{\neq\ell} + \mathbf{A}_{\text{ex}}). \quad (17.102)$$

The charge density ρ_ℓ and currents $\mathbf{J}_\ell$ for every ℓ satisfy the continuity equations (11.54) as was shown in Sect. 11.2.2.

The total EM field $F^{\mu\nu}$ (12.3) satisfies the Maxwell equation (11.33), namely

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu, \quad J^\nu = (c\rho, \mathbf{J}), \quad (17.103)$$

where, according to (11.31),

$$\rho = \sum_\ell \rho_\ell, \quad (17.104)$$

$$\mathbf{J} = \sum_\ell \mathbf{J}_\ell. \quad (17.105)$$

We assume that the EM fields satisfy the Lorentz gauge, namely

$$\frac{1}{c} \partial_t \varphi_\ell + \nabla \cdot \mathbf{A}_\ell = 0. \quad (17.106)$$

Under this assumption, Eqs. (17.99) take the form of the wave equations:

$$\nabla^2 \varphi_\ell - \frac{1}{c^2} \partial_t^2 \varphi_\ell = -4\pi \rho_\ell, \quad (17.107)$$

$$\frac{1}{c^2} \partial_t^2 \mathbf{A}_\ell - \nabla^2 \mathbf{A}_\ell = \frac{4\pi}{c} \mathbf{J}_\ell, \quad \ell = 1, \dots, N, \quad (17.108)$$

where ρ_ℓ , $\mathbf{J}_\ell$ are defined by (17.100), (13.11). Note that the Lorentz gauge is preserved by the wave equations thanks to fulfillment of the continuity equations (11.55).

The energy-momentum tensors of balanced charges are reduced in Sect. 11.4 to the EnMT of the NKG equations and the Maxwell equations. In particular, the expression for the energy and momentum densities for the ℓ -th charge are given, respectively, by (11.142), (11.143).

17.6.1.1 Charge Conservation and the Charge Normalization Condition

The NKG equations (17.98) are gauge invariant; therefore ρ_ℓ and $\mathbf{J}_\ell$ defined by (17.100), (17.101) satisfy the conservation/continuity equation (11.55):

$$\partial_t \rho_\ell + \nabla \cdot \mathbf{J}_\ell = 0, \quad \ell = 1, \dots, N. \quad (17.109)$$

A simple and straightforward direct derivation of the continuity equation is as follows. We multiply Eq. (17.98) by ψ_ℓ^* , take the imaginary part and obtain the desired conservation Eq. (17.109). Integrating (17.109) we observe that

$$\partial_t \int_{\mathbb{R}^3} \rho_\ell(t, \mathbf{x}) \, d\mathbf{x} = 0,$$

that is, the continuity equation implies preservation of the charge value for every charge. We denote the total conserved value of the charge by $\bar{\rho}_\ell$, that is,

$$\bar{\rho}_\ell = \int_{\mathbb{R}^3} \rho_\ell(t, \mathbf{x}) \, d\mathbf{x} = \text{const}. \quad (17.110)$$

Let us consider now the balanced charge normalization. As in Sect. 14.2, we choose the normalization by requiring that Coulomb's law for a resting charge holds with the value q_ℓ of the charge. The charge density is given by (17.100) and the Eqs. (17.107) for a resting charge take the form

$$\nabla \varphi_\ell = -4\pi \rho_\ell,$$

and their solution is given by (12.19). The analysis of Sect. 14.17 shows that the charge normalization condition has the form

$$\int_{\mathbb{R}^3} \rho_\ell(t, \mathbf{x}) \, d\mathbf{x} = \bar{\rho}_\ell = q_\ell. \quad (17.111)$$

17.6.1.2 Singling Out a Charge from the System and the Equation for a Test Charge

The system of field equations (17.98), (17.107), (17.108), (17.100), (13.11) in the absence of external fields describes N interacting distributed charges and their EM fields. Let us assume that we have a solution of the entire system and single out the ℓ -th charge. It is described by the NKG equation (11.125), which can be written in the form

$$-\frac{1}{c^2} \tilde{\partial}_t \tilde{\partial}_t \psi + \tilde{\nabla}^2 \psi - G'(\psi^* \psi) \psi - \kappa_0^2 \psi_\ell = 0, \quad (17.112)$$

where $\psi = \psi_\ell$, $G' = G'_\ell$,

$$q = q_\ell, \quad \varphi_{\text{ex}} = \varphi_{\neq \ell}, \quad \mathbf{A}_{\text{ex}} = \mathbf{A}_{\neq \ell}, \quad \kappa_0 = \frac{m_\ell c}{\chi}, \quad m = m_\ell$$

and

$$\tilde{\partial}_t = \partial_t + \frac{iq}{\chi} \varphi_{\neq \ell}, \quad \tilde{\nabla} = \nabla - \frac{iq}{\chi c} \mathbf{A}_{\neq \ell}. \quad (17.113)$$

Therefore the equation of motion of one of the charges of the system is reduced to an equation of motion of a single charge in an external EM field with potentials $\varphi_{\neq \ell}$, $\mathbf{A}_{\neq \ell}$.

To study the dynamics of a singled out charge, we may consider $\varphi_{\neq \ell} = \varphi_{\text{ex}}$ and $\mathbf{A}_{\neq \ell} = \mathbf{A}_{\text{ex}}$ as given functions of time and spatial variables, and they can be interpreted as external potentials. This approach is sufficient to describe charge dynamics at macroscopic scales, as we will see in the following sections. The Lagrangian theory for a single equation (17.112), in particular the energy, momentum and corresponding conservation laws, is described in Sect. 11.6.

Equation (17.112) can be used to describe the dynamics of a test charge. By a test charge we mean a charge which is so small that its influence on the remaining charges and their fields can be neglected. If we add a test charge to a system of N interacting charges, we can consider the resulting system of $N + 1$ charges, and we set $\ell = N + 1$ for the test charge. Then in the Eqs. (17.112)–(17.113) the external potentials are given by the formula

$$\begin{aligned} \varphi_{\text{ex}} = \varphi_{\neq N+1} &= \sum_{\ell'=1}^N \varphi_{\ell'} = \varphi, \\ \mathbf{A}_{\text{ex}} = \mathbf{A}_{\neq N+1} &= \sum_{\ell'=1}^N \mathbf{A}_{\ell'} = \mathbf{A}, \end{aligned}$$

that is the “external” potentials for a test charge coincide with the potentials φ , $\mathbf{A}$ of the total electromagnetic field in (17.103).

17.6.2 A Relativistic Localized Distributed Charge as a Particle

“What impresses our senses as matter is really a great concentration of energy into a comparatively small space”.

A. Einstein and L. Infeld³

³The quotation is from Einstein and Infeld’s book [105] p. 257.

In the non-relativistic case we have derived the Newtonian dynamics of localized charges in Sects. 17.2.3 and 17.2.4. In this section we develop a relativistic version of the Ehrenfest theorem. It is used to treat the dynamics of localized relativistic balanced charges and to derive the relativistic point mechanics including the derivation of Einstein's celebrated mass energy formula from the field equations.

Let us begin by recalling the fundamentals of the relativistic dynamics of a mass point. The relativistic dynamics of an accelerating mass point driven by the external electromagnetic (EM) field is described by the following equation, [265, Section 29]:

$$\partial_t (M\mathbf{v}) = \mathbf{f}_{\text{Lor}}(t, \mathbf{r}), \quad M = m_0\gamma, \quad \gamma = \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-1/2}, \quad (17.114)$$

where $\mathbf{v} = \partial_t \mathbf{r}$ is its velocity, m_0 is the rest mass of the mass point, $\mathbf{f}_{\text{Lor}}$ is the Lorentz force (6.39), and γ is the Lorentz factor. Equations (17.114) for the space components of the 4-vector are usually complemented with the following equation for the time (energy) component:

$$\partial_t (Mc^2) = \mathbf{f}_{\text{Lor}} \cdot \mathbf{v}. \quad (17.115)$$

In particular, for small velocities when $|\mathbf{v}|/c \ll 1$, we readily recover Newton's equation with the Lorentz force as the non-relativistic approximation of Eq. (17.114) by setting $\gamma = 1$. Note that the rest mass m_0 of a point in (17.114) is an intrinsic property of a point and is prescribed.

In a relativistic field theory, the relativistic field dynamics is derived from a relativistic covariant Lagrangian. The field equations, the total energy and momentum, forces and their densities are naturally defined in terms of the Lagrangian both in the cases of closed and non-closed (with external forces) systems (see for instance, [4, 7.1-7.5], [248, 3.1-3.3, 3.5], [265, 37], [304, 4.1], see also Sect. 11.6.1). For a closed system, the total energy-momentum is a 4-vector, and the total momentum has a simple form $\mathbf{P} = M\mathbf{v}$ where the constant velocity $\mathbf{v}$ originates from the corresponding parameter of the Lorentz group. Therefore, one can naturally define and interpret the mass M for a closed relativistic system as the coefficient of proportionality between the momentum $\mathbf{P}$ and the velocity $\mathbf{v}$. Since the energy of a closed system is the 4-th component of the 4-vector, the celebrated Einstein energy-mass relation follows from its relativistic invariance, [4, Sect. 7.1-7.5], [248, Sect. 3.1-3.3, 3.5], [265, Sect. 37], [304, Sect. 4.1]:

$$E = Mc^2, \quad M = m_0\gamma, \quad (17.116)$$

where m_0 is the rest mass and γ is the Lorentz factor. Observe that the above definition of mass is based on the relativistic argument for uniform motion with constant velocity $\mathbf{v}$ without acceleration. An immediate implication of Einstein's mass-energy relation (17.116) is that the rest mass of a closed system is essentially equivalent to the internal energy of the system. As stated by Pauli, [265, p. 123]:

“We can thus consider it as proved that the relativity principle, in conjunction with the momentum and energy conservation laws, leads to the fundamental principle of the

equivalence of mass and (any kind of) energy. We may consider this principle (as was done by Einstein) as the most important of the results of the theory of special relativity.”

Now let us take a brief look at the difference in determination of the mass in the dynamics of a single point and in a general Lagrangian Lorentz invariant field theory. Since Eq. (17.114) for a single mass point has the form of Newton’s law, one can determine as in Newtonian mechanics the mass M as a measure of inertia from the known force $\mathbf{f}_{\text{Lor}}$ and the acceleration $d^2\mathbf{r}/dt^2$ (the variability of γ can be ignored for mild accelerations). Note that an accelerating particle in an external field is not a closed system, and there are principal differences between closed and non-closed systems. In a relativistic field theory for a closed system, the energy-momentum is a four-vector, and the energy is given by Einstein’s formula (17.116); this allows us to define the total mass and rest mass of the system in the case of uniform motion. However, in the case of a general non-closed system (which is the subject of our primary interest since we study field regimes with acceleration), there is no canonical way to determine mass, position, velocity and acceleration, in particular one cannot identify the velocity parameter $\mathbf{v}$ of the Lorentz group with velocity of the system. For a non-closed system, there is a problem even with a sound definition of the center of mass: “the centre of mass loses its physical importance”, [248, p. 203]; the center of energy (also known as the center of mass or centroid or ergocenter) and the total energy-momentum are frame dependent, hence they are not 4-vectors, [248, Sect. 7.1, 7.2], [212, Sect. 24]. To summarize, in a general relativistic field theory the rest mass is defined for a uniform motion, whereas in Newtonian mechanics the concept of inertial mass is introduced through an accelerated motion.

Our manifestly relativistic Lagrangian field theory describes a single charge by a complex-valued scalar field (wave function or charge distribution) $\psi(t, \mathbf{x})$ satisfying the nonlinear Klein–Gordon (NKG) equation (34.128):

$$-\frac{1}{c^2}\tilde{\partial}_t^2\psi + \tilde{\nabla}^2\psi - G'(|\psi|^2)\psi - \frac{m^2c^2}{\chi^2}\psi = 0,$$

where m is a positive mass parameter, and χ is a constant which coincides with (or is close to) the Planck constant $\hbar$. The expressions for the covariant derivatives in (34.128) are given by (11.123), the Lagrangian for the nonlinear Klein–Gordon (NKG) equation (34.128) is given by (11.121). Notice that $|\psi|^2$ is interpreted as a charge density and not as a probability density. We prove below that for certain regimes of accelerated motion the charge dynamics can be described by the relativistic point mass equations (17.114)–(17.115). Since the theory is self-contained, such equations cannot be postulated. In particular, the Einstein energy-mass formula (17.116) or any asymptotic law of motion must be derived within the framework of the theory.

We show that the relativistic point mass equation (17.114), (17.115) is an approximation which describes the behavior of the field ψ when it is well localized. The charge localization can be described by the ratio a/R_{ex} where R_{ex} is a typical length scale for the spatial variation of the external EM forces. Consequently, the charge is

well localized when the ratio a/R_{ex} is very small. The charge localization is facilitated by the cohesive forces associated with the nonlinearity $G'(s)$ which can be interpreted as the Poincaré forces. In the particular case of the logarithmic nonlinearity G defined by Eq. (14.42), a free resting charge with the minimal energy has a Gaussian shape described by Eq. (14.43), namely $|\psi| = \pi^{-3/4} a^{-3/2} \exp(-|x|^2/2a^2)$. *We identify the charge location with the point where the energy of the field ψ is concentrated*, namely with the location of the ergocenter. If a field ψ satisfies the field equation (34.128), its ergocenter $\mathbf{r}$ and energy E satisfy equations which can be derived from the conservation laws for the NKG equation (34.128). We prove that, under the assumption of localization in the asymptotic limit $a/R_{\text{ex}} \rightarrow 0$, these equations turn into the relativistic point mass equations (17.114), (17.115). Remarkably, the value of the inertial mass determined from Newton's equations in the Newtonian approach coincides exactly with the mass given by Einstein's formula (17.116) if initially the charge is in a rest state or in a uniformly moving state. Of course, a convincing argument for the equivalence of the inertial mass and the energy based on the analysis of the charge momentum when it interacts with the electromagnetic field has been made by Einstein [103], but here the same is obtained through a thorough mathematical analysis of a concrete Lagrangian field model. *Importantly, since the limit mass point equations are derived and not postulated, the resulting rest mass is shown to be an integral of motion rather than a prescribed constant. Consequently, the rest mass may take different values depending on the state of the field. In particular, in addition to the primary Gaussian ground state, there is a sequence of rest states with higher rest energies and rest masses.* The possibility of different rest masses comes from the fact that in our theory an elementary charge is not a point but is a distribution described by a wave function ψ . In other words, the charge though elementary has infinitely many degrees of freedom with internal interactions of non-electromagnetic origin which contribute to its internal energy.

Below we derive from the NKG field equation (34.128) the relativistic law of motion for the ergocenter $\mathbf{r}(t)$. We show that, if the charge is localized, the trajectory $\mathbf{r}(t)$ converges to a solution of the relativistic equation (17.114) as $a/R_{\text{ex}} \rightarrow 0$. The derivation is based entirely on the analysis of the NKG equation (34.128) and corresponding conservation laws. The mathematically rigorous (but more technical) treatment is given in Sect. 34.3.

17.6.2.1 Relativistic Dynamics of the Energy Center of a Localized Charge

In this section we use the conservation laws for the NKG equation to derive equations for the energy center $\mathbf{r}$. Based on the obtained equations and under asymptotic localization assumptions, we then derive the relativistic point mass equations (17.114), (17.115) for the energy center $\mathbf{r}$.

We begin by introducing expressions for the total energy, momentum and force:

$$E(t) = \int u(t, \mathbf{x}) d\mathbf{x}, \quad (17.117)$$

$$P^i(t) = \int p^i(t, \mathbf{x}) d\mathbf{x}, \quad (17.118)$$

$$F^i(t) = \int f^i(t, \mathbf{x}) d\mathbf{x}, \quad i = 1, 2, 3, \quad (17.119)$$

where u , p^i , f^i are, respectively, the energy, momentum and Lorentz force densities for the NKG equation introduced in Sect. 11.6.1. The coordinates r^i of the energy center $\mathbf{r}$ (ergocenter) are given by the following expressions:

$$r^i = \frac{1}{E(t)} \int x^i u(t, \mathbf{x}) d\mathbf{x}. \quad (17.120)$$

The continuity equation (11.127) after multiplication by $(x^i - r^i)$ readily implies the following expression for the current density $\mathbf{J}$:

$$\partial_t((x^i - r^i)\rho) + \rho \partial_t r^i + \nabla \cdot ((x^i - r^i)\mathbf{J}) = J^i, \quad i = 1, 2, 3. \quad (17.121)$$

Integrating over the entire space the energy-momentum conservation laws (11.139) and (11.144), we obtain equations:

$$\partial_t P^i = F^i, \quad i = 1, 2, 3; \quad (17.122)$$

$$\frac{1}{c} \partial_t E = F^0. \quad (17.123)$$

Using Eq. (17.121), we obtain the following representation for the energy component F^0 in (17.122) with f^0 given by (11.141):

$$F^0 = \int \frac{1}{c} \mathbf{J} \cdot \mathbf{E}_{\text{ex}} d\mathbf{x} = \bar{\rho} \frac{1}{c} \partial_t \mathbf{r} \cdot \mathbf{E}_{\text{ex}}(t, \mathbf{r}) + \delta_F^0, \quad (17.124)$$

$$\begin{aligned} \delta_F^0 = \sum_i \int \frac{1}{c} (\mathbf{E}_{\text{ex}}^i \partial_t((x^i - r^i)\rho) + \mathbf{E}_{\text{ex}}^i \nabla \cdot ((x^i - r^i)\mathbf{J})) d\mathbf{x} \\ + \frac{1}{c} \partial_t \mathbf{r} \cdot \int \rho (\mathbf{E}_{\text{ex}}(t, \mathbf{x}) - \mathbf{E}_{\text{ex}}(t, \mathbf{r})) d\mathbf{x}. \end{aligned} \quad (17.125)$$

Integrating the angular momentum conservation law (10.103) with $\nu = 0$ and $\gamma = i$ over the entire space, we obtain

$$\frac{1}{c} \partial_t (c^2 t P^i - r^i \mathbf{E}) = ct F^i - F^0 r^i - \delta_f^i, \quad (17.126)$$

where the *particle discrepancy terms* δ_f^i are of the form

$$\delta_f^i = \int (x^i - r^i) f^0 d\mathbf{x}, \quad i = 1, 2, 3. \quad (17.127)$$

The identities (17.122), (17.123) and (17.126) imply that

$$P^i = \frac{\mathbf{E}}{c^2} \partial_t r^i - \delta_f^i. \quad (17.128)$$

Now, combining (17.128) with (17.122), we obtain the relation

$$\partial_t \left(\frac{1}{c^2} \mathbf{E} \partial_t \mathbf{r} - \mathbf{f} \right) = \mathbf{F}, \quad (17.129)$$

with $\mathbf{F} = (F^1, F^2, F^3)$ defined by (17.119), (11.141). The expressions for $\mathbf{F}$ can then be written in the form

$$\mathbf{F} = \mathbf{f}_{\text{Lor}}(t, \mathbf{r}) + \mathbf{F}, \quad (17.130)$$

where $\mathbf{f}_{\text{Lor}}(t, \mathbf{r})$ is the Lorentz force (6.39) with the remainder term

$$\begin{aligned} \delta_F = & \frac{1}{q} \int (\mathbf{f}_{\text{Lor}}(t, \mathbf{x}) - \mathbf{f}_{\text{Lor}}(t, \mathbf{r})) \rho d\mathbf{x} \\ & + \int \frac{1}{c} \left(\partial_t ((\mathbf{x} - \mathbf{r}) \rho) + \sum_l \partial_l ((\mathbf{x} - \mathbf{r}) \mathbf{J}^l) \right) \times \mathbf{B}_{\text{ex}} d\mathbf{x}. \end{aligned} \quad (17.131)$$

Equations (17.124) and (17.129) result in the following system of two ergocenter equations: the spatial part

$$\partial_t \left(\frac{\mathbf{E}}{c^2} \partial_t \mathbf{r} - \mathbf{f} \right) = \mathbf{f}_{\text{Lor}}(t, \mathbf{r}) + \delta_F, \quad (17.132)$$

and the time part

$$\frac{1}{c} \partial_t \mathbf{E} = \frac{1}{c} \partial_t \mathbf{r} \cdot \mathbf{f}_{\text{Lor}}(t, \mathbf{r}) + \delta_F^0. \quad (17.133)$$

Observe now that the particle discrepancy terms δ_f , δ_F , δ_F^0 involve in their integrands the factors $(\mathbf{x} - \mathbf{r})$, $\mathbf{B}_{\text{ex}}(t, \mathbf{x}) - \mathbf{B}_{\text{ex}}(t, \mathbf{r})$, $\mathbf{E}_{\text{ex}}(t, \mathbf{x}) - \mathbf{E}_{\text{ex}}(t, \mathbf{r})$ which vanish at the energy center $\mathbf{r}$, and they are small in a small neighborhood of $\mathbf{r}$. In addition to that, the integrands involve the factors $\mathbf{J}$, ρ which cannot be large outside a small neighborhood of the energy center if the solution ψ is localized.

Under the assumption that the charge is localized, we can neglect the particle discrepancy terms $\delta_f, \delta_F, \delta_F^0$ in Eqs. (17.132) and (17.133), obtaining the following limit system for $\mathbf{r}$ and $\mathbf{E}$:

$$\partial_t \mathbf{E} = \partial_t \mathbf{r} \cdot \mathbf{f}_{\text{Lor}}(t, \mathbf{r}), \quad (17.134)$$

$$\partial_t \left(\frac{\mathbf{E}}{c^2} \partial_t \mathbf{r} \right) = \mathbf{f}_{\text{Lor}}(t, \mathbf{r}). \quad (17.135)$$

In the above derivation we have omitted mathematical details, some of which are discussed in Sect. 17.6.2.3, and a more detailed, mathematically rigorous treatment is provided in Sect. 34.3.

Let us take a closer look at Eqs. (17.134), (17.135). Equation (17.135) evidently has the form of the relativistic version of Newton's law of motion with the Lorentz force, namely

$$\partial_t (M \partial_t \mathbf{r}) = \mathbf{f}_{\text{Lor}}(t, \mathbf{r}), \quad (17.136)$$

provided the mass M is given by Einstein's formula

$$M = \frac{E}{c^2}. \quad (17.137)$$

Equation (17.134) has the form of the time-component for the relativistic point dynamics, see [265, Section 29, 37], [27, Section II.1]. Let us derive now a relation between the inertial mass M of a moving charge and the rest mass. Using (17.134), we readily obtain that

$$M \partial_t \mathbf{r} \cdot \partial_t (M \partial_t \mathbf{r}) = M \partial_t \mathbf{r} \cdot q \mathbf{E}_{\text{ex}}(t, \mathbf{r}) = M c^2 \partial_t M, \quad (17.138)$$

which implies the relation

$$M^2 - \frac{1}{c^2} M^2 (\partial_t \mathbf{r})^2 = M_0^2, \quad (17.139)$$

where M_0^2 is the constant of integration which can be naturally interpreted as the square of the rest mass. Consequently, we recover from Eq. (17.139) the well-known formula

$$M = \gamma M_0, \quad \gamma = (1 - (\partial_t \mathbf{r})^2 / c^2)^{-1/2}. \quad (17.140)$$

The relations (17.136) and (17.140) readily imply the accelerated motion equation (17.114). As to the Eqs. (17.134), (17.135), they describe the asymptotic behavior of the energy and the ergocenter of the charge when its wave function ψ remains localized in the course of motion.

Remark 17.6.1 The constant M_0 in Eq. (17.139) is evidently equal to the mass M if $\partial_t \mathbf{r} = \mathbf{0}$. This allows us to interpret M_0 as the rest mass as in (17.114). We would like to stress again that the rest mass M_0 in our treatment is not a prescribed quantity, but

it is derived in (17.139) as an integral of motion (or, more precisely, an approximate integral of the field equation which becomes precise in an asymptotic limit). As for any integral of motion, it can take different values for different “trajectories” of the field. This is demonstrated by the different values of the rest mass M_0 for different rest states described in Sects. 14.4, 14.4.1. The integral of motion M_0 can be related to the mass m_0 of one of the resting charges considered in Sect. 14.4 by the identity

$$M_0 = m_0 \quad (17.141)$$

if either velocity vanishes on a time interval or asymptotically as $t \rightarrow -\infty$ or $t \rightarrow \infty$. If velocity $\partial_t \mathbf{r}$ vanishes just at a time instant t_0 , it is possible to express the value of M_0 in terms of $\mathbf{E} = \mathbf{E}(\psi)$ by formulas (34.161), (17.139), but the corresponding $\psi = \psi(t_0)$ may have no relation to the rest solutions of the field equation with a time-independent profile $|\psi|^2$. It is also possible that $\partial_t \mathbf{r}$ never equals zero, and in fact this is a general case, since all three components of velocity may vanish simultaneously only in very special situations. Hence, there is a possibility of localized regimes where the value of the “rest mass” M_0 may differ from the rest mass of the free charge. In such regimes, the value of the rest mass cannot be derived based on the analysis of the uniform motion and resting charges as in Sects. 15.2 and 14.4. This wide variety of possibilities makes even more remarkable the fact that the inertial mass is well-defined, and that Einstein’s formula (17.116) holds even in such general regimes where the standard analysis based on the Lorentz invariance of the uniform motion as in Sect. 15.2 does not apply. In the general case where the localization is not assumed, the functional

$$\hat{M}_0^2 = \frac{\mathbf{E}^2}{c^4} \left(1 - \frac{1}{c^2} (\partial_t \mathbf{r})^2 \right) \quad (17.142)$$

extends formula (17.139) to general fields and produces a “generalized rest mass” M_0 defined in terms of the energy and the ergocenter as follows:

$$M_0^2 = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \hat{M}_0^2(t) dt. \quad (17.143)$$

The above formula obviously defines the value of the rest mass for a more general class of field trajectories, and according to (34.161) and (17.139) produces the value of the integral of motion in the case of asymptotic localization.

Remark 17.6.2 Equation (17.114) does not involve radiation reaction effects since in the balanced charges theory the EM self-interaction is completely eliminated. The radiation effects acting on a point charge are described by the Lorentz–Abraham–Dirac equation [85], [345]. We derive the Lorentz–Abraham–Dirac equation in [10] using a modification of the balanced charges theory which allows some EM self-interaction of the charge. The method of derivation can be considered as a refinement of Dirac’s [85] mass renormalization approach in the spirit of the Ehrenfest theorem.

17.6.2.2 The Spectroscopic and Inertial Masses

A model for the Hydrogen atom in the framework of balanced charges theory has been introduced and described in [16], [17] and Sects. 17.5, 17.6.4, 35.2. In the non-relativistic case, it provides the asymptotic formula $E_n = -hcR_\infty/n^2$ for the Hydrogen energy levels where the factor R_∞ is given by the formula

$$R_\infty = \frac{q^4 m}{4\pi\hbar^3 c}, \quad (17.144)$$

assuming that $\chi = \hbar = \frac{h}{2\pi}$.

In the relativistic case, a more complex formula which involves an equivalent of the Sommerfeld fine structure with the same factor hcR_∞ can be derived from an analysis of (17.183). The constant R_∞ in Eq. (17.144) coincides with the expression for the Rydberg constant if m is the electron mass, $m = m_e$, and $q = e$ equals the electron charge. It seems natural, therefore, to refer to the mass parameter m as the *spectroscopic mass*. We make a distinction between the spectroscopic mass and the inertial mass, since in our theory the mass parameter m of a charge is somewhat smaller than the inertial mass m_0 defined by the formula (15.34) with $\omega = \omega_0$, namely

$$m_0 = m \left(1 + \frac{1}{2} \frac{a_C^2}{a^2} \right), \quad a_C = \frac{\hbar}{mc}. \quad (17.145)$$

The difference $m_0 - m$ depends evidently on the size parameter a . The question stands now: is there any experimental evidence which shows that the inertial mass m_0 and the spectroscopic mass m may be different as in our theory?

Quantum mechanics and quantum electrodynamics allow us to interpret the spectroscopic data and extract from it the mass of the electron known as the *recommended value of the electron mass* $m = m_e = A_r(e)$. The recommended value of the electron mass $A_r(e)$ when expressed in units u can be found in [247, Table XLIX, p. 710]:

$$A_r(e) = 5.4857990943(23) \times 10^{-4} [4.2 \times 10^{-10}] \text{ (recommended value)}, \quad (17.146)$$

where, we recall the common convention, 5.4857990943 represents the mean value of the experimental data, 0.0000000023 represents its standard uncertainty (deviation) and 4.2×10^{-10} represents its relative uncertainty. The value $A_r(e)$ is found based on very extensive spectroscopic data.

But there is also another class of measurements, namely the Penning trap measurements, which can be considered as the most direct measurement of the electron mass as the inertial one, and it gives the following mass value, [117]:

$$A_r(e) = 5.485799111(12) \times 10^{-4} [2.1 \times 10^{-9}] \text{ (Penning trap)}. \quad (17.147)$$

Observe now that the value of the electron inertial mass coming from the Penning trap measurement (17.147) is larger than the spectroscopic mass (17.146), and an

elementary statistical calculation shows definitely that the difference is statistically significant. Indeed, using standard statistical analysis, we obtain for the difference $\delta A_r(e) = m_0 - m$ the following:

$$m_0 - m = \delta A_r(e) = 0.17(12) \times 10^{-11}, [3.1 \times 10^{-9}]. \quad (17.148)$$

If we take the *recommended value* $A_r(e)$ from (17.146) as our “golden standard”, then the mean value of the Penning trap measurements from (17.147) is more than 7 standard deviations above the mean recommended value $A_r(e) = 5.4857990943$ and the same is true for approximately one half of all Penning trap measurements which are greater than the mean value. *Hence, there is a strong indication that the inertial mass m_0 as represented by the Penning trap measurements (17.147) is larger than the recommend value m from (17.146).*

We have not succeeded in finding in the existing literature an explanation of the above mass difference, but in our theory such a difference, including its positive sign, can easily be explained by the finite value of the electron size parameter a . The resulting relation between the two masses is given by the formula (17.145), which can be recast as

$$\frac{a_C^2}{a^2} = 2 \frac{m_0 - m}{m}. \quad (17.149)$$

The formula (17.149) relates the size parameter a to the relative difference of the two mass values m_0 and m . We take the above mentioned measurement in the Penning trap device as in (17.147) as the value m_0 of the inertial mass, and using relation (17.148), we get the following approximate inequalities:

$$0.87 \times 10^{-9} \leq \frac{m_0 - m}{m} \leq 0.53 \times 10^{-8}. \quad (17.150)$$

If we assign the effect to the non-zero value of a , we obtain using the formula (17.149) approximate inequalities for the size parameter

$$0.97 \times 10^4 \leq \frac{a}{a_C} \leq 2.4 \times 10^4. \quad (17.151)$$

These inequalities are consistent with our prior assessments for a (made in [15, p. 948], [16, p. 948], and Sects. 17.5, 35.2 on the basis of the analysis of the frequency spectrum of the Hydrogen atom) to be at least of order $10^2 \alpha^{-1} a_C$ where $\alpha^{-1} \simeq 137$ is the inverse of the Sommerfeld fine structure constant. Note that deriving inequalities (17.151) we used an asymptotic formula for the spectrum from our theory of the Hydrogen atom, and the terms which were neglected may result in a much larger interval for a/a_C . Consequently inequalities (17.151) are somewhat hypothetical rather than rigorous estimates of the ratio a/a_C .

17.6.2.3 Charge Localization Assumptions

A mathematically rigorous derivation of the relativistic charge dynamics is based on the concept of trajectory of concentration and is provided in Sect. 34.3. Here we provide a heuristic discussion of related localization assumptions.

The ergocenter obeys the relativistic version of Newton's equation if the particle discrepancy terms $\delta_f, \delta_F, \delta_F^0$ in the ergocenter equations (17.133), (17.132) can be neglected. These terms would vanish exactly if the charge and corresponding currents are localized exactly at the center $\mathbf{r}$, or if $\mathbf{r}$ is a center of symmetry, and the external EM potentials are constant as in the case of a uniformly moving charge. In the general case, we may only expect that these terms vanish asymptotically in the localization limit:

$$\delta_f \rightarrow 0, \quad \delta_F \rightarrow 0, \quad \delta_F^0 \rightarrow 0. \quad (17.152)$$

Notice that there are two kinds of quantities which enter the discrepancy terms. The first kind of quantities are presented by $\mathbf{E}_{\text{ex}}(t, \mathbf{x}) - \mathbf{E}_{\text{ex}}(t, \mathbf{r})$ and $\mathbf{B}_{\text{ex}}(t, \mathbf{x}) - \mathbf{B}_{\text{ex}}(t, \mathbf{r})$. They vanish if the fields are constant, and they are small if the fields are almost constant. The magnitude of inhomogeneity of the EM fields can be described by the typical length R_{ex} at which they vary significantly. Hence, to ensure that the fields are almost constant near the charge, we assume that R_{ex} is much larger than the charge size a and impose on the external field the following *asymptotic local homogeneity* condition:

$$a/R_{\text{ex}} \rightarrow 0. \quad (17.153)$$

Another kind of quantity which enters the discrepancy terms involves factors $(\mathbf{x} - \mathbf{r})$ as in δ_F in (17.131):

$$\begin{aligned} & \int \frac{1}{c} \left(\partial_t ((\mathbf{x} - \mathbf{r}) \rho) + \sum_l \partial_l \cdot ((\mathbf{x} - \mathbf{r}) \mathbf{J}^l) \right) \times \mathbf{B}_{\text{ex}} d\mathbf{x} \\ &= \frac{1}{c} \partial_t \int \rho (\mathbf{x} - \mathbf{r}) \times \mathbf{B}_{\text{ex}} d\mathbf{x} - \frac{1}{c} \int \rho (\mathbf{x} - \mathbf{r}) \times \partial_t \mathbf{B}_{\text{ex}} d\mathbf{x} \\ & \quad + \int \frac{1}{c} \int \sum_l \mathbf{J}^l (\mathbf{x} - \mathbf{r}) \times \partial_l \mathbf{B}_{\text{ex}} d\mathbf{x}. \end{aligned}$$

These quantities vanish for spatially constant $\mathbf{B}_{\text{ex}}, \mathbf{E}_{\text{ex}}$ if ρ and every component of $\mathbf{J}$ are center-symmetric with respect to $\mathbf{r}$. Similar quantities are present in δ_F^0 and δ_f defined by (17.125), (17.127). For example

$$\delta_f^i = \int (x^i - r^i) \left(\rho \mathbf{E}_{\text{ex}} + \frac{1}{c} \mathbf{J} \times \mathbf{B}_{\text{ex}} \right)^i d\mathbf{x} \quad (17.154)$$

vanish under the same central symmetry assumption. Hence, to satisfy the charge localization condition (17.152), it is sufficient to impose two separate requirements:

(i) the asymptotic condition (17.153) and (ii) asymptotic central symmetry of ρ and components of $\mathbf{J}$.

To clarify the meaning of the above localization conditions, let us look at the simplest case where ρ and $\mathbf{J}$ are derived for the uniform motion of a free particle described by the solution $\psi(t, \mathbf{x}) = \psi_{\text{free}}(t, \mathbf{x})$ considered in Sect. 15.2. The solution has the following properties:

(i) the energy density u is center-symmetric with respect to $\mathbf{r}(t) = \mathbf{v}t$, hence the ergocenter coincides with $\mathbf{r}(t)$;

(ii) the charge density ρ is given by (15.27), and according to (15.24) it converges to $q\delta(\mathbf{x} - \mathbf{r})$ as $a \rightarrow 0$;

(iii) the current $\mathbf{J}$ is given by (15.30), its components are center-symmetric and converge to the corresponding components of $q\mathbf{v}\delta(\mathbf{x} - \mathbf{r})$.

Hence, the localization assumptions (17.152) are fulfilled for ρ and $\mathbf{P}$ derived for $\psi_{\text{free}}(t, \mathbf{x})$ for general fields $\mathbf{E}_{\text{ex}}, \mathbf{B}_{\text{ex}}$ which are regular near $\mathbf{r}$ when $a/R_{\text{ex}} \rightarrow 0$. If the motion is almost uniform, namely if the external fields are not too strong and the solution $\psi(t, \mathbf{x})$ of (34.128) is close to $\psi_{\text{free}}(t, \mathbf{x})$, we may expect that such a solution also satisfies the localization conditions. We present an example in Sect. 17.6.2.4.

Now let us briefly discuss condition (17.153). The charge velocity may have a magnitude comparable with the speed of light. The external fields cause the charge acceleration with a change of velocity $\mathbf{v}$ of the order $\bar{v}_1$. Suppose that accelerations are not violent, namely $\bar{v}_1 \ll c$, and that the corresponding forces according to (17.135) are of the order $\hat{f}_{\text{Lor}}$ where $\hat{f}_{\text{Lor}}$ is a typical magnitude of the Lorentz force $\mathbf{f}_{\text{Lor}}(t, \mathbf{r})$. The spatial scale R_{ex} at which the forces associated with the electromagnetic fields $\mathbf{E}_{\text{ex}}, \mathbf{B}_{\text{ex}}$ vary by the same order of magnitude as $\hat{f}_{\text{Lor}}$ can be defined as follows:

$$\frac{1}{R_{\text{ex}}} = \frac{q}{\hat{f}_{\text{Lor}}} \max_{|\mathbf{x} - \mathbf{r}| \leq \theta a} \left(\frac{1}{c} |\nabla \mathbf{B}_{\text{ex}}| + |\nabla \mathbf{E}_{\text{ex}}| \right), \quad (17.155)$$

with $\theta \gg 1$. In view of this definition, condition (17.153) ensures that the variability of EM fields causes a vanishing perturbation to the Lorentz force in (17.135).

The estimates of smallness of the discrepancy terms can be made in certain asymptotic regimes, but they are laborious, and we would like to make some guiding comments. The NKG equation (34.210) evidently involves five parameters, namely $c, \hbar, m, q, a$, and the external fields can also be considered as functional parameters. The limits (17.152) can be considered simultaneously with certain combinations of the parameters tending to their limits, and the limits in (17.152) have to be taken together with these parameter limits. Note that all the quantities which enter Eqs. (17.132), (17.133) are given by integrals over the entire space of certain densities. The integrals over the entire space are obviously the limits of integrals over domains $|\mathbf{x} - \mathbf{r}| < \theta a$ with $\theta \rightarrow \infty$ and the value of θ in the asymptotic regimes can be related to the values of the other parameters mentioned. An example of such an asymptotic situation with nontrivial relations between parameters is briefly discussed in the following Sect. 17.6.2.4, for a detailed treatment see Sect. 34.6. A detailed, mathematically

rigorous treatment of the derivation of the relativistic point dynamics is given in Sect. 34.3.

17.6.2.4 Example of Relativistic Accelerating Regime for a Localized Charge

Up to now, we were considering relativistic features of an accelerating charge assuming its localization. The localization assumptions, though natural, are rather technical when it comes to rigorous treatment. With that in mind, we study in this section a particular case where the dynamics demonstrates a wide variety of accelerated relativistic motions of the charge. This case is simple enough for a detailed analysis and the verification of the localization assumptions. The analysis is still rather involved, and that comes as no surprise since for general external EM fields the NKG equation (34.128) has no closed form solutions. We succeeded though in finding a large family of relevant regimes for which we can obtain almost explicit representations of solutions allowing for a detailed study of the relativistic features of the charge accelerating in an external EM field. Here we follow [18] and describe the principal steps of the construction; a mathematically rigorous detailed treatment is given in [19] and in Sect. 34.6 where properties of the solutions are described. In particular, our construction in Sect. 34.6 shows that in certain accelerating regimes Einstein's formula is a more robust phenomenon than the Lorentz contraction.

Here we present an example of an accelerating charge, where its localization, required in Sect. 17.6.2.1, is maintained in the strongest possible form. Namely, we consider a regime where the accelerating charge *exactly preserves its Gaussian shape*. The charge wave function is similar to $\psi_{\text{free}}(t, \mathbf{x})$ in (15.23), and it is localized about $\mathbf{r}(t)$, but its velocity $\mathbf{v} = \partial_t \mathbf{r}$ is not constant, and the charge has a non-zero acceleration. *The shape of the wave in an accelerated regime is exactly the same as for a free particle, and it is only the phase factor that is affected by the acceleration caused by the external force.* Such an almost rigid accelerated motion is possible only for a properly chosen potential φ_{ex} . We consider here the simplest but still relevant case where the charge moves and accelerates in the direction of the axis x_3 with the potential φ_{ex} being a function of only the variable x_3 and the time t , and where no external magnetic field is present, that is, $\mathbf{A}_{\text{ex}} = 0$. The NKG equation (17.158) involves five parameters, namely the speed of light c , the Planck constant χ , the mass parameter m , the charge value q and the charge size a . A combination of the parameters which is important in our analysis is the reduced Compton wavelength $a_c = \frac{\chi}{mc}$. Another intrinsic length scale for the NKG equation (34.128) is the size parameter a . In the asymptotic localization regime the parameters form a sequence: $a = a_n$, $\chi = \chi_n$, $n = 1, 2, \dots$. We make the following assumption: the Compton wavelength is much smaller than the size parameter a , namely

$$\zeta = \zeta_n = \frac{a_c}{a} = \frac{\chi_n}{a_n mc} \ll 1. \quad (17.156)$$

This assumption makes possible a detailed enough analysis of the problem. Here we present only a sketch, a detailed exposition of the construction is given in Sect. 34.6.

When the external potential φ_{ex} depends only on t and x_3 , the Eq. (34.128) in three-dimensional space with a logarithmic nonlinearity (14.42) can be exactly reduced to a problem in one-dimensional space by the following substitution:

$$\psi = \pi^{-1/2} a^{-1} \exp\left(-\frac{1}{2a^2} (x_1^2 + x_2^2)\right) \psi_{1D}(t, x_3) \quad (17.157)$$

with $\psi_{1D}(t, x_3)$ being dependent only on time and x_3 . The corresponding reduced 1D (one-dimensional) NKG equation for $\psi = \psi_{1D}(t, x_3)$ with *one spatial variable* is as follows:

$$-\frac{1}{c^2} \tilde{\partial}_t \tilde{\partial}_t \psi + \partial_3^2 \psi - G'_{a1D}(\psi^* \psi) \psi - \kappa_0^2 \psi = 0, \quad (17.158)$$

where the 1D logarithmic nonlinearity has the form

$$G'_{a1D}(|\psi|^2) = -a^{-2} [\ln(\pi^{1/2} |\psi|^2) + 1] - a^{-2} \ln a, \quad (17.159)$$

and

$$\kappa_0 = \frac{mc}{\hbar}, \quad \tilde{\partial}_t = \partial_t + \frac{iq}{\hbar} \varphi, \quad \varphi = \varphi_{\text{ex}}(t, x_3). \quad (17.160)$$

If $\varphi = 0$, this equation has a Gaussian as a rest solution, namely

$$\check{\psi}(x_3) = \pi^{-1/4} a^{-1/2} e^{-x_3^2/2a^2}. \quad (17.161)$$

We assume that the solution of (17.158) has the form of a wave-corpuscle, namely the Gaussian shape of ψ is preserved, but we allow its phase to vary:

$$\psi(t, x) = e^{i\frac{\omega_0}{c} \gamma(x-r)\beta - i\mathfrak{S}_0} \check{\psi}(x-r), \quad \mathfrak{S}_0 = \mathfrak{S}_0(t, x-r), \quad (17.162)$$

where β is the *normalized charge velocity*, namely

$$\beta = \frac{v}{c}, \quad v = \partial_t r, \quad (17.163)$$

and we write x instead of x_3 for notational simplicity. We consider the center location $r(t)$ to be a given function, and the ansatz (17.162) is written under the assumption that the charge is at rest for $t \leq 0$, the case of general uniform movement for $t \leq 0$ is discussed in Remark 34.6.2. Importantly, we treat the potential φ_{ex} as an unknown which should be determined. We look for the external potential φ in the form

$$\varphi(t, x) = \varphi_{\text{ac}}(t, x) + \varphi_{\text{b}}(t, x; \zeta) \quad (17.164)$$

where the *accelerating potential* φ_{ac} is linear in y , namely

$$\varphi_{ac} = \varphi_0(t) + \varphi'_{ac}y, \quad y = x - r, \quad (17.165)$$

and $\varphi_b(t, x; \zeta)$ is a small *balancing potential* which vanishes as $\zeta \rightarrow 0$. According to our analysis, the coefficient $\varphi'_{ac} = \partial_x \varphi_{ac}$ is determined by the trajectory $r(t)$ according to the *relativistic law of motion* (17.114), which has the form

$$\partial_t(m\gamma v) + q\partial_x \varphi_{ac}(t, r) = 0. \quad (17.166)$$

The coefficient $\varphi_0(t)$ can be prescribed as an arbitrary function of time which has bounded derivatives. We are able to find for a general trajectory $r(t)$ a *small* balancing potential φ_b such that the wave-corpuscle (17.162) is an exact solution to the NKG equation (17.158) in a wide strip $\mathcal{E}(\theta) = \{|x_3 - r(t)| \leq \theta a\}$ around the trajectory, and θ grows to infinity as $\zeta \rightarrow 0$. In the indicated case we can describe the potential, in particular its dependence on ζ , in sufficient detail.

Here is a summary of the properties of the relativistic wave-corpuscle regime. For a given trajectory $r(t)$, we construct a potential φ_{ex} which makes the Gaussian wave with center $r(t)$ an exact solution of the field equation in the wide strip $\mathcal{E}(\theta)$. Since the shape of $|\psi|$ is preserved, and $\theta \rightarrow \infty$, such a function is localized around $r(t)$. The possibility to preserve localization which we discuss here concerns microscopic details of the charge evolution, and naturally the ratio $\zeta = a_C/a$ of the charge size to the Compton wavelength a_C plays an important role. For the details of the construction see Sect. 34.6.

The possibility of a uniform global motion without acceleration is well-known, see Sect. 15.2 and [37]. The fact that relativistic acceleration imposes restrictions on the spatial extension of rigid bodies was noted in a different setting in [112].

17.6.3 The Relativistic Interaction of Balanced Charges

Relativistic theory of many interacting point particles is known to have fundamental difficulties. A. Barut pointed out, [27, Section II.1, System of colliding particles]:

“The invariant formulation of the motion of two or more interacting particles is complicated by the fact that each particle will have a different proper time. ... No exact general theory seems to be available.”

Some of these difficulties are analyzed by H. Goldstein in his classical monograph, [137, Section 7.10]:

“The great stumbling block however is the treatment of the type of interaction that is so natural and common in non-relativistic mechanics – direct interaction between particles. ... To say that the force on a particle depends upon the positions or velocities of other particles at the same time implies propagation of effects with infinite velocity from one particle to another – ‘action at a distance.’ In special relativity, where signals cannot travel faster than

the speed of light, action-at-a-distance seems outlawed. And in a certain sense this seems to be the correct picture. It has been proven that if we require certain properties of the system to behave in the normal way (such as conservation of total linear momentum), then there can be no covariant direct interaction between particles except through contact forces.”

The Lagrangian and Hamiltonian formalism for interacting point charges was studied in a number of papers, see Marnelius [232] and references therein; still the construction of a satisfactory theory meets certain difficulties because of the non-locality in time. Meanwhile, Lagrangian relativistic field theories, for example Maxwell’s theory, or the theory of the Klein–Gordon equations, or the theory of balanced charges are local and do not face those principal difficulties on interacting particles.

We show below a consistent derivation of the relativistic theory of interacting point charges as an approximation of the theory of balanced charges. The derivation is based on the results of the previous section on the NKG relativistic dynamics. In the localization limit, the charges are localized at their ergocenters, and their EM interaction is realized through the Green functions of the Maxwell equations. Therefore, the interactions inevitably involve time retardations which are present in the Green functions. The system of relativistic balanced charges involves the NKG equations (12.8) and the Maxwell equations which under the Lorentz gauge can be written in the form (12.9)–(12.10).

Assuming that the charges are localized, and their trajectories are separated in space, we can apply to the NKG equation results of the previous section and obtain Eqs. (17.136), (17.140) for the ℓ -th charge, which takes the following form:

$$\partial_t (M_\ell \partial_t \mathbf{r}_\ell) = \mathbf{f}_{\ell\text{Lor}}(t, \mathbf{r}_\ell), \quad \ell = 1, \dots, N, \quad (17.167)$$

$$M_\ell = \gamma M_{0\ell}, \quad \gamma = (1 - (\partial_t \mathbf{r}_\ell)^2 / c^2)^{-1/2}, \quad (17.168)$$

where $\mathbf{f}_{\ell\text{Lor}}$ is the Lorentz force associated with the remaining charges of the system. The Lorentz force $\mathbf{f}_{\ell\text{Lor}}$ is represented by the expression (17.179) involving retardation effects.

To find an expression for the Lorentz force, we have to consider Maxwell equations (17.107)–(17.108) in the case of charges localized at their trajectories $\mathbf{r}_\ell$. We assume that the solution of (12.9)–(12.10) is given in terms of retarded potentials by (4.42):

$$\varphi(t, \mathbf{x}) = \int \frac{\rho(t', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}', \quad \mathbf{A}(t, \mathbf{x}) = \frac{1}{c} \int \frac{\mathbf{J}(t', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}', \quad (17.169)$$

with the retarded time value

$$t' = t - \frac{1}{c} |\mathbf{x} - \mathbf{x}'|. \quad (17.170)$$

For given $(t, \mathbf{x})$, the above Eq. (17.170) determines the light half-cone with the vertex at $(t, \mathbf{x})$. Since the solution $\mathbf{r}_\ell(t)$ of (17.167), (17.168) has a velocity smaller

than the speed of light, every trajectory

$$\mathbf{x}' = \mathbf{r}_\ell(t')$$

intersects the half-cone at a single point at $t' = t'_\ell$, $\mathbf{x}' = \mathbf{r}_\ell(t'_\ell)$. The value of t'_ℓ depends on the vertex coordinates $(t, \mathbf{x})$ and on the trajectory $\mathbf{r}_\ell$:

$$t'_\ell = t'_\ell(t, \mathbf{x}). \quad (17.171)$$

Assuming that the solutions ψ_ℓ concentrate at the trajectory $\mathbf{r}_\ell(t)$ as $a \rightarrow 0$, we can replace $\rho_\ell(t', \mathbf{x}')$ and $\mathbf{J}_\ell(t', \mathbf{x}')$, respectively, by delta-functions

$$\hat{\rho}_\ell(t') \delta(\mathbf{x}' - \mathbf{r}_\ell(t')), \quad (17.172)$$

$$\hat{\mathbf{J}}_\ell(t') \delta(\mathbf{x}' - \mathbf{r}_\ell(t')). \quad (17.173)$$

The coefficient $\hat{\rho}_\ell(t')$ can easily be determined from the normalization condition (17.111), namely

$$\int \rho_\ell(t', \mathbf{x}') d\mathbf{x}' = \hat{\rho}_\ell(t') = q_\ell.$$

Now we derive the expression for $\hat{\mathbf{J}}_\ell(t')$. Multiplying the continuity equation (17.109) by $(\mathbf{x} - \mathbf{r}_\ell)$, we obtain that

$$\int (\mathbf{x} - \mathbf{r}_\ell) \partial_t \rho_\ell d\mathbf{x} + \int (\mathbf{x} - \mathbf{r}_\ell) \nabla \cdot \mathbf{J}_\ell d\mathbf{x} = 0. \quad (17.174)$$

Using the commutation relation between coordinate multiplication and differentiation operators

$$\partial_j x_i - x_i \partial_j = \delta_{ij},$$

we conclude that

$$\partial_t \int (\mathbf{x} - \mathbf{r}_\ell) \rho_\ell d\mathbf{x} + \partial_t \mathbf{r}_\ell \int \rho_\ell d\mathbf{x} - \int \mathbf{J}_\ell d\mathbf{x} = 0. \quad (17.175)$$

Taking into account Eq. (17.175) and relations (17.172), (17.173), we obtain that asymptotically

$$\int (\mathbf{x} - \mathbf{r}_\ell) \rho_\ell d\mathbf{x} \sim 0, \quad \int \mathbf{J}_\ell d\mathbf{x} \sim \hat{\mathbf{J}}_\ell,$$

and conclude that

$$\hat{\mathbf{J}}_\ell = q_\ell \partial_t \mathbf{r}_\ell. \quad (17.176)$$

Note that the above expression coincides with the one for the uniformly moving charge in (15.32). Substituting (17.172), (17.173), we obtain from (17.169) the limit

potentials

$$\varphi_{\neq\ell}(t, \mathbf{x}) = \sum_{\ell' \neq \ell} \frac{q_{\ell'}}{|\mathbf{x} - \mathbf{r}_{\ell'}(t'_{\ell'})|}, \quad (17.177)$$

$$\mathbf{A}_{\neq\ell}(t, \mathbf{x}) = \frac{1}{c} \sum_{\ell' \neq \ell} \frac{q_{\ell'}}{|\mathbf{x} - \mathbf{r}_{\ell'}(t'_{\ell'})|} \partial_t \mathbf{r}_{\ell'}(t'_{\ell'}), \quad (17.178)$$

and the Lorentz force in the Eq. (17.167) takes a form similar to (17.27):

$$\begin{aligned} \mathbf{f}_{\ell\text{Lor}}(t, \mathbf{x}) = & -q_{\ell} \sum_{\ell' \neq \ell} \frac{q_{\ell'} (\mathbf{x} - \mathbf{r}_{\ell'})}{|\mathbf{x} - \mathbf{r}_{\ell'}(t'_{\ell'})|^3} \\ & - q_{\ell} \partial_t \mathbf{r}_{\ell}(t) \sum_{\ell' \neq \ell} \frac{q_{\ell'}}{|\mathbf{x} - \mathbf{r}_{\ell'}(t'_{\ell'})|^3} (\mathbf{x} - \mathbf{r}_{\ell'}) \times \partial_t \mathbf{r}_{\ell'}(t'_{\ell'}), \end{aligned} \quad (17.179)$$

but now the charge interaction is not instantaneous but involves relevant retardation since the time moment $t'_{\ell'}$ defined by (17.171) satisfies the inequality $t'_{\ell'} < t$.

We can consider quite similarly the case where the Green function equals the half sum of retarded and advanced potentials. In this case we have two half-cones (retarded and advanced) and two points of intersection $t'_{-, \ell'}$ and $t'_{+, \ell'}$ with the half-cones. In this case the Lorentz force has the form

$$\mathbf{f}_{\ell\text{Lor}} = \frac{1}{2} \mathbf{f}_{-, \ell\text{Lor}} + \frac{1}{2} \mathbf{f}_{+, \ell\text{Lor}}$$

where $\mathbf{f}_{\pm, \ell\text{Lor}}$ is given by formula (17.179) with $t'_{\ell'}$ replaced by $t'_{\pm, \ell'}$. This type of interaction is considered in the Wheeler–Feynman theory. A relativistic interaction of this form of two point charges was considered in detail in [292].

17.6.4 A Relativistic Hydrogen Atom Model

In this section we provide a concise comparison of the non-relativistic treatment of the Hydrogen atom in Sect. 17.5 with the treatment in the framework of the relativistic version of our model. We start directly from the relativistic system (17.98)–(17.102) and look for time-harmonic solutions with $\mathbf{A}_{\ell} = 0$ and time-independent φ_{ℓ} using substitutions (14.49) and (17.49) as follows:

$$\psi_{\ell}(t, \mathbf{x}) = e^{-i(\omega_{\ell} + \omega_0)t} \hat{\psi}_{\ell}(\mathbf{x}), \quad \varphi_{\ell}(t, \mathbf{x}) = \varphi_{\ell}(\mathbf{x}), \quad (17.180)$$

where $\ell = 1, 2$ and $\omega_0 = \frac{m_{\ell} c^2}{\chi} = c \kappa_{0\ell}$. We arrive then at the following system of equations similar to the system (17.50), (17.47):

$$\frac{1}{c^2} \left[\left(\omega_\ell + \frac{m_\ell c^2 - q_\ell \varphi_{\neq \ell}}{\chi} \right)^2 - \left(\frac{m_\ell c^2}{\chi} \right)^2 \right] \psi_\ell + \nabla^2 \psi_\ell - G'_\ell (\psi_\ell^* \psi_\ell) \psi_\ell = 0, \quad (17.181)$$

where $\varphi_{\neq 1} = \varphi_2$, $\varphi_{\neq 2} = \varphi_1$ and the time-independent $\varphi_{\neq \ell}$ satisfy the Poisson equation (17.47). Based on the smallness of the electron/proton mass ratio, as in the non-relativistic case, we arrive at an eigenvalue problem for the electron density which is a relativistic version of Eq. (17.87), namely

$$\left[\left(m_1 c^2 + \chi \omega_1 + \frac{q^2}{|\mathbf{x}|} \right)^2 - m_1^2 c^4 \right] \hat{\psi}_1 + c^2 \chi^2 \left[\nabla^2 - G' \left(|\hat{\psi}_1|^2 \right) \right] \hat{\psi}_1 = 0. \quad (17.182)$$

After the same change of variables (17.80)–(17.81) which transformed (17.72) into (17.94), using the relations between the reduced Compton wavelength $a_C = \frac{\chi}{mc} = \alpha \hat{a}_1$, the Bohr radius $\hat{a}_1 = \frac{\chi^2}{mq^2}$ and the Sommerfeld fine structure constant $\alpha = \frac{q^2}{\chi c}$, we obtain from (17.182) the following dimensionless equation:

$$\left(\frac{1}{\alpha} + \frac{a_1}{c} \omega_1 + \frac{\alpha}{|\mathbf{y}|} \right)^2 \Psi_1 + \nabla^2 \Psi_1 - \hat{G}' (|\Psi_1|^2) \Psi_1 - \frac{1}{\alpha^2} \Psi_1 = 0. \quad (17.183)$$

This equation depends on the parameter a only through the rescaled nonlinearity $\hat{G}'(s) = \kappa^2 G'(\kappa^{-3}s)$. If the ratio $\kappa = \frac{a_1}{a}$ is small, the nonlinearity in (17.183) (and in (17.182)) can be treated as a small perturbation, and the linear part of (17.182) essentially determines the lower energy levels. Note that if we set $\chi = \hbar$, the linear part of (17.182) coincides with the *following relativistic version of the Schrödinger equation* (see [291, 42], [331, 1.5.2, 1.5.4], [143, 1.9])

$$(-\hbar^2 c^2 \nabla^2 + m^2 c^4) \hat{\psi} = \left(E + \frac{q^2}{|\mathbf{x}|} \right)^2 \hat{\psi} \quad (17.184)$$

with the energy levels $E = mc^2 + \chi \omega_1$. The energy levels of the linear relativistic Schrödinger equation (17.184), in contrast to the non-relativistic Hydrogen Schrödinger equation, include the so-called fine structure, [291, 42], [138, 5.3]. The fine structure energy levels are given by Sommerfeld's formula, and the relative scale of the fine structure is controlled by α^2 where Sommerfeld's fine structure constant $\alpha = \frac{q^2}{\hbar c} \simeq \frac{1}{137}$. Hence relativistic effects at atomic scales in our relativistic model are present even in the case of zero velocities if the square of Sommerfeld's fine structure constant is not assumed to be negligible.

Remark 17.6.3 In our treatment of charges in Sect. 19.2 at macroscopic scales, we assume the electron size a to be very small, and in this section we assume $\kappa = \hat{a}_1/a$

to be very small. Since there is a huge gap of scales between the macroscopic and atomic scales, there is no contradiction if we take into account the small value of the Bohr radius $\hat{a}_1 \sim 5.3 \times 10^{-11}$ m compared with the scale of variation of EM fields. Note that the error ϵ_0 of approximation by Newtonian trajectory in (17.24) is of order $a^2/R_{\text{macr}}^2 \ll 1$ where R_{macr} is the scale of spatial variation of EM fields which act on a charge. In the treatment of the Hydrogen atom in this section we assume $\kappa^2 = \hat{a}_1^2/a^2 \ll 1$. Taking $a \sim 10^2 \hat{a}_1$ we arrive at the restriction $R_{\text{macr}} \gg 5.3 \times 10^{-9}$ m which is an estimate of the scale of spatial variation of EM fields for which Newton's equations with the Lorentz force hold with a good accuracy.

Chapter 18

Relation to Quantum Mechanical Models and Phenomena

Some of important differences and similarities between our neoclassical theory and quantum mechanics have already been discussed in the concise treatment in Chap. 3, particularly in Sects. 3.5.1, 3.5.4, 3.8. In this chapter we elaborate further on differences and similarities between the two theories.

18.1 Comparison with the Schrödinger Wave Theory

Let us recall that the Schrödinger wave mechanics is constructed based on the classical point particle Hamiltonian

$$\mathcal{E} = H(\mathbf{p}, \mathbf{x}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}) \quad (18.1)$$

which is subjected to the so-called quantization procedure, [268, 2, 11], [267, 3, 4]. Namely we set into correspondence with the quantities $\mathcal{E}$ and $\mathbf{p}$ the following operators

$$\mathcal{E} \rightarrow i\hbar \frac{\partial}{\partial t}, \quad \mathbf{p} \rightarrow -i\hbar \nabla, \quad (18.2)$$

and substitute them into Eq. (18.1). As a result, we obtain the celebrated Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H(-i\hbar \nabla, \mathbf{x}) \psi = -\frac{\hbar^2 \nabla^2 \psi}{2m} + V(\mathbf{x}) \psi. \quad (18.3)$$

In other words, the quantization procedure represented by the substitution (18.2) produces from the classical point Hamiltonian $H(\mathbf{p}, \mathbf{x})$ in (18.1) the quantum mechanical Hamiltonian operator $H(-i\hbar \nabla, \mathbf{x})$, consequently leading to the Schrödinger wave equation (18.3).

The quantization agrees with *Bohr's correspondence principle*, which requires a correspondence between quantum and classical mechanics in the limit of large quantum numbers, [183, 3.2], [138, 1.1a], [299, 5.2]. If we introduce the polar representation

$$\psi = e^{i\frac{S}{\hbar}} R, \quad R \geq 0, \quad (18.4)$$

then the Schrödinger equation (18.3) can be recast as the following system of two coupled equations for the real-valued phase function S and the amplitude R , [176, 3.2.1] (see also Chap. 41):

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2 \nabla^2 R}{2mR} = 0, \quad (18.5)$$

$$\frac{\partial R^2}{\partial t} + \nabla \cdot \frac{R^2 \nabla S}{m} = 0. \quad (18.6)$$

Then expanding the phase S into a power series with respect to $\hbar$, that is,

$$S = S_0 + \hbar S_1 + \hbar^2 S_2 + \dots, \quad (18.7)$$

we obtain from Eq. (18.5) the so called WKB approximation, see [267, Section 12] and Sects. 3.9, 3.5.1. In particular, the function S_0 satisfies the Hamilton–Jacobi equation

$$\frac{\partial S_0}{\partial t} + \frac{(\nabla S_0)^2}{2m} + V = 0 \text{ or } \frac{\partial S_0}{\partial t} + H(\nabla S_0, \mathbf{x}) = 0 \quad (18.8)$$

suggesting that the Schrödinger wave equation (18.3) “remembers” how it was constructed from the original point particle Hamiltonian H (18.1).

Summarizing the above considerations, we can state that the Schrödinger wave theory starts with classical point mechanics and modifies it by the quantization procedure into the wave theory. The neoclassical theory advanced here is fundamentally different, and in some ways it goes in the opposite direction. Namely, it starts with the field (wave) equations derived from the system field Lagrangian (12.11), and the classical point particle mechanics arises then as an approximation.

The fundamental contrast between the two approaches shows clearly in the case of many interacting charges. Indeed, for a system of N charges the neoclassical theory involves N elementary wave functions (with the corresponding elementary EM fields) defined over a 4-dimensional space-time continuum, whereas the same system of N charges in the Schrödinger wave mechanics has a single wave function defined over a $3N$ -dimensional “configuration space”. Interestingly, in spite of those fundamental differences, there is solid common ground between the neoclassical theory and quantum mechanics. This common ground rests on similarities between the neoclassical field equations and the Schrödinger and Klein–Gordon equations

in the non-relativistic and relativistic case respectively. As to the classical particle mechanics, we have shown that in relevant regimes classical Newtonian mechanics can be deduced from the field equations as an approximation (see Sects. 17.2.4, 17.2.3, 34.1).

18.1.1 *Uncertainty Relations*

There is a marked difference between the essence of the uncertainty relations in quantum mechanics and in the neoclassical theory advanced here. For simplicity's sake, let us consider the non-relativistic versions of both theories.

The difference in the two theories shows already in the origin of the uncertainty. Let us start with quantum mechanics and recall that the uncertainty there is bound to the probabilistic interpretation of the wave function. Just as an example, suppose that the physical variable of interest is the charge position described by a point $\mathbf{x}$ in the three-dimensional space. According to quantum mechanics, an elementary charge is a point-like object (see Sect. 3.8), and the position vector $\mathbf{x}$ describes unambiguously its spatial location. The uncertainty of the charge location comes entirely through its complex-valued wave function $\psi(\mathbf{x})$. Namely, if the charge state is described by a wave function $\psi(\mathbf{x})$, the probability to locate it in an infinitesimally small spatial domain of volume $d\mathbf{x}$ is $|\psi(\mathbf{x})|^2 d\mathbf{x}$. Since in quantum mechanics the wave function describes completely a quantum state of a charge, its location is fundamentally uncertain and described in probabilistic terms from the outset. In other words, [176, p. xvii]:

“The uncertainty is postulated to be intrinsic to the system.”

In the neoclassical theory advanced here, an elementary charge is never exactly a point object, but it can be localized and treated then as a point-like object. Its state just as in quantum mechanics is described by a complex-valued wave function $\psi(\mathbf{x})$, but its physical interpretation is very different from quantum mechanical one. The wave function $\psi(\mathbf{x})$ of the balanced charge is interpreted as a "material" wave of the charged matter distributed in space, and $q |\psi(\mathbf{x})|^2 d\mathbf{x}$ is interpreted as a fraction of the entire charge residing in an infinitesimally small spatial domain of volume $d\mathbf{x}$. Hence in the neoclassical theory the uncertainty in the charge location originates in its actual spatial distribution over the space. Consequently, the uncertainty here is simply about a natural ambiguity in assigning a single geometric point to a spatially distributed object. Such an uncertainty differs markedly from quantum mechanical probabilistic uncertainty.

The differences in the interpretation of uncertainty discussed above are of physical significance, and they are manifest when it comes to relating theoretical predictions to experimental observations. To see why, let us take a look at general aspects of the uncertainty relations associated with waves of whatever nature. Observe first that the uncertainty relations point to the fact that some physical quantities associated with the underlying wave motion cannot be defined unambiguously with absolute

certainty. For instance, if the wave describes a perturbation such as a wave packet of finite spatial extension, then its location, understood as a single geometric point, cannot be identified unambiguously with an accuracy exceeding its spatial extension. One can argue though that if the perturbation has a fixed shape, say it is a ball, it is reasonable to identify its location with the ball center. That being accepted, still a general object such as a disturbance on the surface of a water pool has a natural "location" as a particular area on the surface rather than a single point. This kind of uncertainty is natural to any object which is not an ideal geometric point. To raise the uncertainty analysis to the level of quantitative physics, one has to relate it to the system configurations.

W. Pauli writes in the section "The uncertainty principle" in [268, 3]:

"The kinematics of waves does not allow the simultaneous specification of the location and the exact wavelength of a wave. Indeed, one can only speak of the location of a wave in the case of a spatially localized wave packet. The number of different wavelengths contained in the Fourier spectrum increases as the wave packet becomes more localized. A relation of the form $\Delta k_i \Delta x_i \geq \text{constant}$ seems reasonable, and we now want to derive this relation quantitatively."

Then in the same section he derives the well known Heisenberg uncertainty principle for a wave packet in the form

$$\Delta k \Delta x \geq \frac{1}{2}, \quad \Delta p \Delta x \geq \frac{\hbar}{2}, \quad (18.9)$$

where Δx , Δk and $\Delta p = \hbar \Delta k$ are, respectively the spatial range, the wave number range and the momentum range for the wave packet $\psi(\mathbf{x})$. These quantities are defined by

$$\Delta x^2 = \int_{\mathbb{R}^3} (\mathbf{x} - \bar{\mathbf{x}})^2 |\psi(\mathbf{x})|^2 d\mathbf{x}, \quad (18.10)$$

$$\Delta k^2 = \int_{\mathbb{R}^3} (\mathbf{k} - \bar{\mathbf{k}})^2 |\hat{\psi}(\mathbf{k})|^2 d\mathbf{k} \quad (18.11)$$

with average values $\bar{\mathbf{x}}$ and $\bar{\mathbf{k}}$ being given by

$$\bar{\mathbf{x}} = \int_{\mathbb{R}^3} \mathbf{x} |\psi(\mathbf{x})|^2 d\mathbf{x}, \quad (18.12)$$

$$\bar{\mathbf{k}} = \int_{\mathbb{R}^3} \psi^*(\mathbf{x}) i \nabla \psi(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^3} \mathbf{k} |\check{\psi}(\mathbf{k})|^2 d\mathbf{k}, \quad (18.13)$$

where the Fourier transform $\check{\psi}(\mathbf{k})$ of $\psi(\mathbf{x})$ is defined by equality (16.65), that is,

$$\check{\psi}(\mathbf{k}) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-i\mathbf{k} \cdot \mathbf{x}} \psi(\mathbf{x}) d\mathbf{x}. \quad (18.14)$$

Quantities Δx and Δk defined by equalities (18.10), (18.11) measure the uncertainty in the location and the wave number respectively. *Hence the exact form of the Heisenberg uncertainty relation (18.9) is a direct consequence of the fundamental definition of the momentum as $\mathbf{p} = \hbar \mathbf{k}$ (18.2) and the definition of the uncertainty as in (18.10) based on the probabilistic interpretation of the wave function.*

An important feature of the uncertainty relations in the linear theory is that any freely propagating wave packet spreads out with time. In particular, according to formulas (45.14)–(45.17) (see also [268, 3]) for the Gaussian wave packet,

$$(\Delta x)^2 = \frac{1}{4(\Delta k)^2} + \frac{\hbar^2 (\Delta k)^2}{m^2} t^2 \quad (18.15)$$

(for details, see Chap. 45). Note then that if the initial uncertainty at $t = 0$ is Δx , then the corresponding $\Delta k = \frac{1}{2\Delta x}$, and the wave packet spread Δx doubles for the time

$$t = 2\sqrt{3} \frac{m}{\hbar} (\Delta x)^2. \quad (18.16)$$

We would also like to point out that the very concept of a wave packet is based on the medium linearity, and the same is true for the uncertainty relations (18.9), (18.15) as general wave phenomena.

Let us turn now to the neoclassical theory and consider a freely moving elementary charge just as we did above for quantum mechanics case. The neoclassical freely moving charge is described by a localized solution called a wave-corpuscle that has been studied extensively in Sect. 15.1. *The wave-corpuscle naturally maintains its shape and does not spread out in the course of motion, all of which makes it markedly different from the wave-packet in quantum mechanics.* In particular, when it comes to uncertainty in location, the wave-corpuscle is a distributed function and not a delta-function as in the case of a point charge. Consequently, the wave-corpuscle “location” is not sharply defined. Notice also that, although formula (18.12) can be used to define the charge location, one has to admit that it is just a justifiable convention, for the charge is not an ideal point but an extended object. With that in mind, let us denote the uncertainty in the position x of the wave-corpuscle by $\tilde{\Delta}x$. We deliberately use a different symbol $\tilde{\Delta}$ to distinguish the wave-corpuscle uncertainty from similar uncertainty in quantum mechanics. The wave-corpuscle described by Eqs. (15.1), (15.2) is a material wave for which we can reasonably assign a spatial extension Δx based on the charge density $\rho(t, \mathbf{x}) = q |\psi(\mathbf{x})|^2$, namely

$$(\tilde{\Delta}x)^2 = \int_{\mathbb{R}^3} (\mathbf{x} - \bar{\mathbf{x}})^2 \frac{\rho(t, \mathbf{x})}{q} d\mathbf{x} = \int_{\mathbb{R}^3} (\mathbf{x} - \bar{\mathbf{x}})^2 |\psi(\mathbf{x})|^2 d\mathbf{x}, \quad (18.17)$$

where

$$\bar{\mathbf{x}} = \int_{\mathbb{R}^3} \mathbf{x} \frac{\rho(t, \mathbf{x})}{q} d\mathbf{x} = \int_{\mathbb{R}^3} \mathbf{x} |\psi(\mathbf{x})|^2 d\mathbf{x}. \quad (18.18)$$

Evidently the above formulas are similar to relations (18.10), (18.12). Notice that $\bar{\mathbf{x}}$ defined by (18.18) coincides with the center $\mathbf{r}(t)$ of the wave-corpuscle.

Let us compare now the neoclassical wave-corpuscle with a quantum mechanical wave packet. Notice that a common feature of the two is the wave origin of their kinematics as manifested by the equality of the velocity to the group velocity, see Sects. 15.1.2 and 16.2.2. But in sharp contrast to the wave packet, the wave-corpuscle maintains its shape and does not disperse. Indeed, we show in Sect. 16.1.5 that the wave-corpuscle as defined by relation (16.2) is an exact solution to the field equations, and it propagates in space without dispersion even when it accelerates. Note that the free relativistic balanced charge described by the relations (15.23)–(15.25) also does not disperse as it moves and preserves its shape up to the Lorentz contraction. *Therefore, in both the cases of a relativistic or non-relativistic wave-corpuscles, the uncertainty $\tilde{\Delta}x$ does not depend on t .* Hence, the uncertainty $\tilde{\Delta}x$, unlike the uncertainty Δx for the linear Schrödinger dynamics, does not grow without bound for large times.

For non-relativistic wave-corpuscles, the normalized charge density depends on time in a special way:

$$\frac{\rho(t, \mathbf{x})}{q} = \left| \dot{\psi} \right|^2 (\mathbf{x} - \mathbf{r}(t)) \quad (18.19)$$

indicating that the charge motion is reduced to spatial translations of a fixed rest charge distribution. This readily implies that its velocity $\mathbf{v} = \partial_t \mathbf{r}(t)$ is well-defined without any uncertainty, for every point of the charge distribution moves with that velocity. The total momentum $\mathbf{P}$ is canonically defined as the integral of the momentum density, and formula (15.10) shows that $\mathbf{P} = m\mathbf{v}$ where $\mathbf{v} = \partial_t \mathbf{r}(t)$, and we can argue that the momentum uncertainty $\tilde{\Delta}p = 0$ for the wave-corpuscle solutions.

The de Broglie wavevector $\bar{\mathbf{k}}$ for a charge distribution given by a wave-corpuscle in the non-relativistic case is defined by formula (16.67), which is quite similar to (18.13). We can then define the corresponding wave number range $\left(\tilde{\Delta}k \right)^2$ by (18.11). Since the uncertainty principle is based on the commutation relation between x_i and ∂_j , we observe that, similarly to (18.9),

$$\tilde{\Delta}k \tilde{\Delta}x \geq \frac{1}{2}. \quad (18.20)$$

Note now that for a general charge distribution the above relation formally is identical to relation (18.9), but the interpretation is quite different since $|\psi|^2$ is not a probability density. Indeed, the wave-corpuscles are very special charge distributions for which velocity $\mathbf{v} = \partial_t \mathbf{r}$ and momentum $\mathbf{P}$ are in unique correspondence with the trajectory $\mathbf{r}(t)$, and though $\tilde{\Delta}k$ is non-zero, the momentum $\mathbf{P}$ is uniquely defined without uncertainty. Note that this does not contradict (18.20), since the relation $\mathbf{P} = \chi \bar{\mathbf{k}}$ which follows from (16.71) holds for the total momentum and not for the momentum density.

Summarizing, we can conclude that *in the neoclassical theory advanced here the Heisenberg uncertainty principle does not hold as a universal principle, though it might well hold in many situations.*

18.1.2 Quantum Statistics and Non-locality

The neoclassical theory of distributed charges treated in this book covers macroscopic regimes described by the wave-corpuscles studied in Chap. 16. Although wave-corpuscle solutions don't describe all regimes of interest per se, we will use in this section the term wave-corpuscle mechanics (WCM) to describe the mechanics of charges in all regimes for both balanced and dressed charges. Since the wave-corpuscle mechanics naturally covers all spatial scales, one can wonder how it relates to quantum mechanics (QM), including the probabilistic interpretation of the wave function, the Hydrogen atom frequencies, the Davisson–Germer experiment, scattering of a charge by a potential created by another charge and more. In Sects. 17.5, 18.1 we have already provided comparisons of some features of WCM and QM. But a systematic comparison of all fundamental features of the two mechanics requires more extensive studies of WCM, and at this point we can only formulate hypotheses on some significant elements of WCM and their relations to the fundamentals of QM.

An important issue that has not been studied yet in WCM is a regime of time limited close interaction between a single elementary charge and the external electromagnetic field. More precisely, it is a regime when an initially free moving charge undergoes for a limited time a close interaction with an external EM field, and after that the charge continues to move freely again but with altered location and velocity. A typical example of such an interaction is when a moving charge passes through a bounded domain in space with a strong external EM field inside of it. Let us try to imagine what can happen, according to WCM, to an elementary charge in the course of such a time limited close interaction. Recall that in WCM, when charges are far apart, every charge is represented by a particle-like, well localized wave-corpuscle as in (16.2), and the localization is a result of a fine balance of forces including the nonlinear self-interaction. Importantly, the cohesive action of the nonlinear self-interaction is subtle, and by no means is it a brute attractive force, for there is no action at a distance in WCM. Now, when one charge comes close to another, or if it enters a domain with an external EM field strongly varying at a sufficiently small spatial scale, the fine balance of forces holding the charge together is disrupted. We can already see consequences of such a disruption in WCM Hydrogen atom model in Sect. 17.5 where the bounded electron size reduces by a factor of order 100 compared to that of the free electron under the attractive action of a single proton. A disruption of the subtle cohesive action of the nonlinear self-interaction by external fields can also cause the charge wave function to spread out substantially and become wave-like. We can imagine further that during the time of interaction the evolution of the extended wave function is determined by an interplay of two factors: (i) the linear Schrödinger

component of the field equation; (ii) its nonlinear component due to the nonlinear self-interaction. Shortly after the interaction ends, the wave-function of the charge contracts restoring the particle-like form, but the charge position and velocity after the contraction will depend sensitively on details of the interaction. So, effectively, a limited time interaction switches one particle-like state of the charge to another.

Based on the above hypothetical scenario of a time limited close interaction, one can explain how the entirely deterministic WCM can conceivably lead to some of the probabilistic aspects of QM. Indeed, suppose for the sake of argument that the time scale of the interaction process is smaller than an observer can resolve, and, consequently, he sees the interaction result as effectively instantaneous transition from one particle-like state to another. The interaction process can alter the total momentum of the charge quite considerably. This momentum alteration combined with effects of the nonlinear self-interaction can cause an extreme sensitivity to the initial data, and that, in turn, can make the transition look like it is random and subjected to a statistical theory. An interesting feature of the nonlinear self-interaction in WCM that might be relevant to the extreme sensitivity is that it is not analytic and singular for small wave-amplitudes (see examples of WCM nonlinearity in Sects. 14.3.2 and 14.5.2). Consequently, small wave amplitudes can play a far more important role in WCM than in the case of conventional nonlinearities which are analytic for small amplitudes. Going further, we observe that WCM field equations are similar to the Schrödinger equation. Hence, it is conceivable that the statistics of the transition will be determined with a certain degree of accuracy by a wave function satisfying an effective linear Schrödinger equation. Some general ideas on the “determinism beneath quantum mechanics” at the Planck scale were put forward recently by ‘t Hooft (see [319] and references therein). A hypothetical scenario of the Davisson–Germer experiment based on the above considerations is given in Sect. 3.5.4.

Another qualitatively important regime is a regime of close interaction for an extended or even infinite period of time. This regime can occur, in particular, in complex systems involving many charges such as atoms, molecules or solids. As we have already indicated in Sect. 17.5, WCM Hydrogen atom has features which are very similar to the Schrödinger atom. In particular, the primary binding force in that model between the electron and the nucleus is the EM attraction. As to solids, let us briefly recall the basics of their treatment in QM. As in any theory of many particles, the fundamental QM theory of a solid is of enormous complexity, but the standard simplified QM treatment of charges in crystalline solids is based on a free-electron model with the following basic assumptions, [9, 1]: (i) positively charged ionized atoms, consisting of nuclei and “core electrons” tightly bound to them, form an immobile periodic lattice structure; (ii) “valence electrons”, also called conductance electrons, are “allowed to wander far away from their parent atoms”; (iii) the conductance electrons are non-interacting and independent and the interaction between a conductance electron and the periodic lattice is modeled via a periodic potential. Such a simplified QM theory is effectively reduced to the one-electron theory for the Schrödinger operator with the periodic potential. Consequently, the eigenfunctions of such an operator are of the Bloch form and are extended over the

entire crystal. The fundamental WCM theory of a solid is of an enormous complexity as well, but similarly to QM theory we can introduce a simplified WCM model for a solid based on the same assumptions as in QM theory. Hence, as in QM model, there is an immobile periodic lattice of ionized atoms described by a periodic potential corresponding to an external electric field. The one-electron WCM model is similar to QM one, but it differs from it by the presence of the nonlinear self-interaction. In this non-relativistic WCM model, a mobile conductance/valence electron is subjected to attractive forces of the periodic lattice and nonlinear self-interaction forces. In a solid, the distance between atoms is pretty small, it is of the order of the Bohr radius and is much smaller than the spatial extent of the free electron. Since the cohesive forces are relatively weak compared with attraction by many different ionized atoms, the action of the nonlinear self-interaction can be disrupted, and the wave function can spread out significantly, and it might even resemble a Bloch eigenmode, in which case the electron would occupy the entire crystal sample.

The above considerations bring us naturally to issues of the charge size and locality in WCM theory. As the above considerations suggest, the electron size in WCM can vary significantly depending on whether it is free or if it is bound in an atom, or if it is a conductance electron in a crystalline solid. In particular, the size of the electron can decrease or increase dramatically when it undergoes a strong close interaction with an external EM field or a system of other charges. As to the locality of WCM, it is perfectly local in one sense but can be non-local in another sense. Namely, WCM theory is perfectly local in the sense that there is no action-at-a-distance. But the charge evidently is not perfectly local, since in WCM it is not a point but at best a localized wave, which, under the conditions indicated above, can spread out significantly. Consequently, it is conceivable within WCM that an elementary charge being a spatially distributed quantity can be simultaneously at two distant spatial locations, and in this sense WCM might deviate significantly from being a local theory.

18.1.3 Relation to Hidden Variables Theories

One can also wonder what the relation is between WCM and hidden variables theories, see [138, 12.2], [176, 1.5, 3.7.2], and a review article [131] with references therein. Particularly, it is interesting to look at how WCM compares with Bohmian Mechanics (BM) (the de Broglie–Bohm theory), [45], [176, 3.1, 3.2], [97], [98, 8], a well known example of a hidden variables theory.

Even a brief look at WCM and BM shows their significant differences: (i) in BM an elementary charge is a point, whereas in WCM it is a distributed quantity, a wave; (ii) WCM theory is local in the sense that there is no action at a distance, but this is not so for BM; (iii) WCM is a genuine Lagrangian Mechanics and, consequently, Newton's Third Law is always satisfied, but this is not so for the BM, [176, 3.3.2]. In addition to that, as we have already indicated above, WCM might approximately account for QM statistics via the dynamic instability, and the verification of that, including the

accuracy of approximation, is a subject of future studies. But it is absolutely clear already that the statistical predictions of WCM cannot be precisely the same as those of QM, since WCM field equations might only approximately and under certain conditions be reduced to QM evolution equation. The latter factor evidently shows that WCM differs from BM, where the Schrödinger equation remains to be an exact equation for the wave function as a part of the BM variables.

An interesting connection between WCM and the de Broglie–Bohm theory has already been pointed out in Sect. 3.3. Namely, the right-hand side of the equality (3.29) that defines the nonlinearity G' – a key component of WCM, equals up to a factor $-\frac{\lambda^2}{2m}$ the quantum potential $Q(\psi(r))$ in (2.28) – a key ingredient of the de Broglie–Bohm theory discussed in Sect. 2.11.

18.1.4 Comparative Summary of the Neoclassical Theory and the Schrödinger Wave Mechanics

As we already mentioned, the non-relativistic version of our wave mechanics has many features in common with the Schrödinger wave mechanics. In particular, the charges wave functions are complex-valued, they also satisfy the Schrödinger equation (but with a nonlinearity), and the charge normalization condition is the same as in the Schrödinger wave mechanics. Our theory provides for a Hydrogen atom model which has a lot in common with that of the Schrödinger model as is shown in Sect. 17.5. Nevertheless, there are features of our wave theory that distinguish it significantly from the Schrödinger wave mechanics and we provide below a comparative summary.

- The single-particle wave function is interpreted as a material wave, as in Schrödinger’s original thinking, and not as a probability amplitude in the sense of Born.
- Every single particle Lagrangian has a nonlinear self-interaction term providing for a cohesive force which holds it together as it moves freely or accelerates.
- A single charge which is either free or in an external mildly varying EM field is described by a soliton-like wave function parametrized by the position and the momentum related to the corresponding point mechanics. It propagates in space without dispersion even when it accelerates, and this addresses one of the “grave difficulties” with Schrödinger’s interpretation of the wave function expressed by M. Born mentioned in Sect. 2.10.
- Charges are always coupled with and inseparable from the EM field.
- When balanced charges are separated by distances considerably larger than their sizes, their wave functions and the corresponding EM fields maintain a soliton-like (wave-corpuscle) representation.

- The correspondence between wave mechanics and point mechanics is obtained either through the Ehrenfest theorem or through the wave-corpuscle representations; both ways allow us to define point momenta and positions. The wave-corpuscle representation allows us to define the de Broglie wavevector. In addition to that, the corresponding group velocity matches exactly the velocity of a soliton-like solution and the point mechanics velocity.
- The orthodox Schrödinger wave equation is linear, and any freely propagating wave packet spreads out with time, whereas our wave-corpuscle preserves its shape. Though the spatial spread of a wave corpuscle is non-zero, the momentum of a free wave corpuscle is precisely defined, indicating that the Heisenberg uncertainty principle between the position and momentum of a particle cannot be a universal principle in the neoclassical theory advanced here.
- In the case of many interacting charges, every charge is described by its own wave function over the same three-dimensional space (four-dimensional space-time continuum), in contrast to the Schrödinger wave mechanics for many charges requiring a multi-dimensional configuration space.
- Our theory has a relativistic version based on a local, gauge and Lorentz invariant Lagrangian with most of the above listed features.

18.2 The Size of a Free Electron as a New Fundamental Scale

The neoclassical theory of electron involves the size parameter $a = a_e$ which describes the size of a free electron. This scale is special to our theory and it is not present in either quantum mechanics or classical mechanics.

We present in this section experimental and theoretical evidence supporting the idea of the existence of a specific nanometer scale such as the size of the free electron a_e which we currently estimate as $a_e \sim 100a_B \approx 5$ nm. When entertaining the idea of the size of a free electron a_e of the order of several nanometers to be a new fundamental spatial scale, we expect it to manifest itself in electron transitions to free states and, maybe, when the electron is confined to a slab of solid-state material of dimensions comparable with 5 nm. With that in mind, we have identified three areas where the size of the free electron can be of critical importance. The first area is field emission physics dealing with an electron that leaves the surface of a cathode and becomes free when subjected to a sufficiently strong external electric field. The second area is a relatively new one—plasmonics. Recent remarkable advances in nano-technology have probed electromagnetic interactions at nano-scales and provided new challenges to the electromagnetic theory. The third area is plasma physics or more specifically the concept of a “*charge cloud*” related to the Particle-in-Cell (PIC) method. Some plasma experiments

“demonstrate that the physical interpretation of the numerical spatial charge and force sharing goes beyond simple smoothing of point particles and clearly supports the concept of finite-size particles”, [258].

For more detailed references, see Sects. 18.2.1, 18.2.2, 18.2.3.

18.2.1 *Electron Field Emission Physics*

Electron emission physics is a well established area of research, see [315], [261], [189]. We are interested in electron emission from a nanostructured surface. As a concise summary of experimental evidence of field emission localized to nanometer-scale emitting areas at extremely high current densities, we quote G. Furse, [127, p. X-XI]:

“M.I. Elinson and co-workers found that the maximum current values were dependent on emitter geometry and showed that by increasing the tip cone angle, the current density could be increased by about an order of magnitude without emitter tip damage. Researchers in this group outlined a program to search for suitable materials for the field emission cathodes and studied materials based on metal-like and semiconductor compounds, such as LaB_6 and ZrC . A number of investigations in the high current density region have also been performed by G.N. Shuppe’s group.

... In studies by G.A. Mesyats and G.N. Furse, current densities of 10^9 A/cm^2 were observed with nanosecond range pulse lengths. Current densities up to $5 \times 10^9 \text{ A/cm}^2$ were demonstrated for field emission localized to nanometer-scale emitting areas. In experiments by V. N. Shredniket al., current densities up to $(10^9\text{--}10^{10}) \text{ A/cm}^2$ are recorded from nanometer-sized tips under steady-state conditions. Recently, G.N. Furse and D.V. Galazanov, using tips with an apex radius of $\sim 10 \text{ \AA}$ were able to reach current densities of $10^9\text{--}10^{10} \text{ A/cm}^2$. These current densities are close to the theoretical supply limit of a metal’s conduction band when the electron tunneling probability is unity.”

18.2.2 *Nanoplasmonics*

Plasmonics has grown in recent years into a well established area of research with great potential. Our interest to this area is rooted in mechanisms involved in plasmonic resonance responses and an implied narrow spatial dimension range between 1 nm and 25 nm. In our concise presentation of essential features of plasmonics, we rely mainly on “Nanoplasmonics: From Present into Future” by Mark I. Stockman from [305, Sect. 1], [313] and “Modelling at Nanoscale” by Paolo Di Sia from [198, Sect.1.6] and references therein.

According to [305, Sect. 1]:

“Nanoplasmonics is a branch of optical condensed matter science devoted to optical phenomena on the nanoscale in nanostructured metal systems. A remarkable property of such systems is their ability to keep the optical energy concentrated on the nanoscale due to modes called surface plasmons (SPs).”

The resonant frequencies ω associated with SPs satisfy the following two conditions:

$$\mathbf{Re}\varepsilon_m(\omega) < 0, \quad \mathbf{Im}\varepsilon_m(\omega) \ll -\mathbf{Re}\varepsilon_m(\omega), \quad (18.21)$$

where $\varepsilon_m(\omega)$ is the frequency dependent dielectric constant of the nanostructured metal system. The above relations hold for good plasmonic metals, for instance silver, gold, copper, alkaline metals. *A defining property of nanoplasmonics is that the energy stored at optical frequencies on the nanoscale is in the form of electromechanical rather than electromagnetic energy.* This and other significant features of the nanophotonics are sketched below.

The metallic nanosystem size R is bound by two fundamental spatial scales: the skin depth l_s and the so-called nonlocality length l_{nl} that bound the nanosystem size R ,

$$l_{nl} \ll R \ll l_s \quad (18.22)$$

where l_s is the *skin depth* and l_{nl} is the so-called *nonlocality length*. We refer to the bounds (18.22) as *nanosystem bounds*.

The *skin depth* l_s is determined by the following expression

$$l_s = \frac{\lambda}{2\pi} \left[\mathbf{Re} \left\{ \frac{-\varepsilon_m^2}{\varepsilon_m + \varepsilon_d} \right\}^{1/2} \right]^{-1}, \quad \frac{\lambda}{2\pi} = \frac{\omega}{c}, \quad (18.23)$$

where λ is the vacuum wavelength and ε_d is the dielectric constant of the surrounding nanosystem dielectric medium. Observe that the above expression for l_s is determined entirely by macroscopic frequency dependent $\varepsilon_m(\omega)$ and ε_d , and that the geometry of the metal nanosystem enters l_s only through the frequency ω . The relevance and significance of the skin depth l_s to the physics of the nanosystem is that, in view of $R \ll l_s$, its interior is fully accessible to the external electromagnetic field which can effectively excite it.

The *nonlocality length* l_{nl} is defined as the distance that an electron passes in space moving with the Fermi velocity v_F during a characteristic period of the optical field

$$l_{nl} = \frac{v_F}{\omega}. \quad (18.24)$$

The nonlocality length l_{nl} is evidently of a quantum nature. It is relevant to nanoplasmonics since if the system size R is smaller than l_{nl} , the related Landau damping causes broadening and disappearance of SP resonances.

The following estimates hold for single-valence plasmonic metals (silver, gold, copper, alkaline metals)

$$l_{nl} \approx 1 \text{ nm}, \quad l_s \approx 25 \text{ nm in the entire optical region.} \quad (18.25)$$

We would like to stress once again that it is absolutely critical to nanoplasmonics that the total energy associated with SPs is mostly of an electromechanical and not electromagnetic nature. Consequently, the electromagnetic wavelength does not define the limit of the spatial localization of energy, but it is the nanosystem size $R \ll l_s$ that defines the spatial scale of the optical energy localization. Notice that l_s is the smallest electromagnetic scale.

The mechanism of SP resonances is as follows. Recall that the nanosystem's size $R \ll l_s$, and consequently the external electromagnetic field penetrates it entirely. This field effectively displaces conductance electrons with respect to the lattice, causing charge polarization on opposite sides of the nanosystem surface. In other words, the nanosystem excited by an external EM field can be viewed as a resonator, and the oscillating electron density forms a localized surface plasmon. The geometry of a nanosystem as resonator determines the resonant frequencies ω_{sp} of the surface plasmon modes. A typical example of a nanosystem is a 10 nm silver sphere with about 10^5 conduction electrons.

18.2.3 Finite-Size Particles or Clouds in Plasma Physics

Plasma physics is a field of enormous complexity, and different approaches have been used to model and understand plasma properties. Many characteristics of plasmas have been successfully studied by computer simulations. The concept of a “charge cloud” emerged in these studies as one of the key components in the physics of plasmas. We collect in this section evidence suggesting that the charge cloud concept has real physical significance and goes far beyond being just a convenient tool in computer simulations. When presenting this evidence, we rely on the original work and results by a number of authors, see [42], [75, p. 406], [259], [258] and references therein.

The motivations and rationale for introducing the charge cloud concept were as follows, [259]:

“A model consisting of finite-size particles, which we call clouds, was introduced in order to reduce the collisional effects between particles and the noise enhanced by particle crossing, but keeping the long wavelength behavior unchanged. We assume that the particles have finite size, are tenuous, and may pass through each other without rotation and internal motion as sketched in Our clouds-in-cell method uses the same density and force averaging as the particle-in-cell method. ... The concept of finite-size charges in physics is old and not new in plasmas; two decades ago Vlasov allowed for arbitrary charge shape factors and called his particles clouds.”

With our goals in mind, the most significant conclusion of the analysis of “charge clouds” and “spatial charge” carried out in [258] is:

“The experiments also demonstrate that the physical interpretation of the numerical spatial charge and force sharing goes beyond simple smoothing of point particles and clearly supports the concept of finite-size particles or clouds.”

A detailed analysis of numerical simulations involving finite-size particles or “clouds” and their physical effects including the reduction in the collision rate are provided in [75, p. 406]:

“If we could replace the Coulombic force between particles by one which is Coulombic at large distances but which goes to zero for short distances, then we would retain the collective behavior while reducing the collision rate. ... This is just the type of force which will exist between two circular (spherical) charge clouds which are free to pass through each other. ... When the charges are far apart, the force is just Coulombic; but when they start to overlap, it starts to drop off—and it will go to zero when they lie exactly on top of each other.”

In particular, J. Dawson’s analysis [75, p. 406] includes studies of significant properties of clouds of Gaussian shape.

Chapter 19

The Theory of Electromagnetic Interaction of Dressed Charges

In this chapter we develop the theory of electromagnetic interaction of dressed charges. The Lagrangian formalism for relativistic and non-relativistic dressed charges is described in Sects. 13.1 and 13.2. The theory of dressed charges in the relativistic case is based on the NKG equations (13.5) for charge distributions coupled with the Maxwell equations (11.33) for the EM fields. In the non-relativistic case, the field equations for dressed charges are the NLS equations (13.20) coupled with the Poisson equations (13.21). Both cases are described, respectively, by Lagrangians (13.1) and (13.17).

We have already studied dressed charges resting states and charges that move uniformly in Chaps. 14 and 15 respectively. In both cases the external forces and interactions between charges are absent. We would like to consider now dressed charges interacting with each other electromagnetically. The difference with the balanced charges theory treated in the previous chapter is that the electrostatic self-interaction is present, and an additional nonlinear term is introduced to compensate it. The construction of the corresponding nonlinearity is described in Sects. 14.6 and 14.5 respectively. The nonlinearity involves two components, G_{∇} and G_{φ} , see (14.87). The first component G_{∇} is required to compensate for elastic forces which originate from the term $|\nabla\psi|^2$ in the Lagrangian. The second component G_{φ} is required to compensate the electrostatic self-interaction of the charge which originates from its electrostatic potential φ .

The interaction of dressed charges is possible only through the EM field which is generated by all charges of the system. Thanks to the field additivity property, see Sect. 11.2.4, the field can be partitioned into the field of the charge itself and the field generated by all remaining charges of the system; the latter in many cases can be understood as a given external field for the charge. This makes it important to study a single charge in an external field. The EM field of the charge is represented in the non-relativistic Lagrangian (13.17) only by its scalar potential φ and the corresponding electric field $\mathbf{E} = -\nabla\varphi$; any radiation phenomena are excluded in this model.

The macroscopic dynamics of dressed charges in the regime of remote interaction is described by Newton's equation of motion. In the regime of the remote interaction, the spatial variation of the EM field generated by every charge is small at the location of another charge, and the smallness is guaranteed by a non-zero minimal macroscopic distance $R_{\min} > 0$ between any two charges. Another relevant spatial scale R_{ex} measures a typical scale of the spatial inhomogeneity of the external fields near charges. Consequently, *the regime of remote interaction* is a regime where the size parameter a is much smaller than the macroscopic spatial scale $R = \min(R_{\min}, R_{\text{ex}})$ and can be characterized by the condition $a/R \ll 1$.

As in the case of balanced charges, there are two distinct ways to correspond our field theory to the classical point charge mechanics when all charges are well separated, and the condition $a/R \ll 1$ is satisfied. The first way is via averaged quantities in the spirit of the *Ehrenfest Theorem*, well known in quantum mechanics [291, Sections 7, 23]. The second one is via a construction of approximate solutions to the field equations (13.25) when every charge is represented by a wave-corpuscle (16.2).

Now we briefly describe similarities and differences between the theories of dressed charges (DCT) and balanced charges (BCT). When we consider dressed charges, we introduce an additional nonlinearity to balance the electrostatic self-interaction and to obtain a localized resting charge ψ . For balanced charges we exclude the EM self-interactions from the very beginning by eliminating them in the field Lagrangian. As for the analysis of a single resting charge or uniformly moving charge, both the dressed charge and balanced charge models can be treated similarly; the accelerating regimes in EM fields with a mild variation are also described by wave corpuscles which obey the same Newton's equations in both cases. But the difference between the two theories reveals itself when we consider closely interacting charges with the fields varying significantly at the charge location. Namely, an analysis of the Hydrogen atom model shows that the electrostatic self-interaction of a dressed charge leads to a non-vanishing perturbation of the frequency spectrum of Hydrogen. In contrast, the balanced charge model allows us to obtain asymptotically classical Hydrogen spectrum with any desired precision.

19.1 The Ehrenfest Theorem for Non-relativistic Dynamics of the Charge Center

Taking into account the normalization condition (17.7), we introduce the *charge position (or average position)* $\mathbf{r}(t)$ by the formula

$$\mathbf{r}(t) = \int_{\mathbb{R}^3} \mathbf{x} |\psi(t, \mathbf{x})|^2 d\mathbf{x}. \quad (19.1)$$

Following (11.167) and (11.168), we also introduce the *total velocity* $\mathbf{v}(t)$

$$\mathbf{v}(t) = \frac{1}{q} \int_{\mathbb{R}^3} \mathbf{J}(t, \mathbf{x}) d\mathbf{x} = \frac{1}{m} \int_{\mathbb{R}^3} \mathbf{P}(t, \mathbf{x}) d\mathbf{x}. \quad (19.2)$$

Using representations (11.103), we obtain from the conservation law (13.37) the total momentum conservation law. According to (11.103), $\Theta^{0i} = 0$ and $T^{0i} = T^{0i} = p^i$, the equation for the total momentum of the dressed charge is reduced to the equation for the NLS momentum density $\mathbf{P} = (p^1, p^2, p^3)$ defined by equation (11.162), namely

$$\partial_t p^i = c \partial_0 p^i = - \sum_{j=1,2,3} c \partial_j T^{ji} + \left(\rho \mathbf{E}_{\text{ex}} + \frac{1}{c} \mathbf{J} \times \mathbf{B}_{\text{ex}} \right)^i, \quad i = 1, 2, 3. \quad (19.3)$$

Integrating the momentum conservation law (19.3) over the entire space $\mathbb{R}^3$, we obtain the following equation for the total momentum $\mathbf{P}$:

$$\partial_t \mathbf{P} = \int_{\mathbb{R}^3} \left[\rho \mathbf{E}_{\text{ex}} + \frac{1}{c} \mathbf{J} \times \mathbf{B}_{\text{ex}} \right] (t, \mathbf{x}) d\mathbf{x}. \quad (19.4)$$

Importantly, the dressed charge EM self-interaction does not enter this equation, which has exactly the same form as Eq. (17.13) derived for balanced charges where the self-interaction was absent from the very start. Hence, the treatment is the same as in Sect. 17.2.2. The charge conservation law (11.158) multiplied by $(\mathbf{x} - \mathbf{r})$ implies that

$$\partial_t ((\mathbf{x} - \mathbf{r}) \rho) + \rho \partial_t \mathbf{r} + (\mathbf{x} - \mathbf{r}) \nabla \cdot \mathbf{J} = 0. \quad (19.5)$$

We integrate the above equation and take into account that the integral of the first term vanishes thanks to (17.7) and (19.1):

$$\partial_t \mathbf{r} \int_{\mathbb{R}^3} \rho d\mathbf{x} - \int_{\mathbb{R}^3} (\mathbf{x} - \mathbf{r}) \nabla \cdot \mathbf{J} d\mathbf{x} = \mathbf{0}. \quad (19.6)$$

Using (19.2), we find that velocity of the charge position coincides with the total velocity $\mathbf{v}$:

$$\partial_t \mathbf{r} = -\frac{1}{q} \int_{\mathbb{R}^3} (\mathbf{x} - \mathbf{r}) \nabla \cdot \mathbf{J} d\mathbf{x} = \frac{1}{q} \int_{\mathbb{R}^3} \mathbf{J} d\mathbf{x} = \frac{1}{m} \mathbf{P} = \mathbf{v}. \quad (19.7)$$

The above formula and (19.4) gives an equation which relates the acceleration $\partial_t^2 \mathbf{r}$ to the total force:

$$m \partial_t^2 \mathbf{r} = \int_{\mathbb{R}^3} \left[\rho \mathbf{E}_{\text{ex}} + \frac{1}{c} \mathbf{J} \times \mathbf{B}_{\text{ex}} \right] (t, \mathbf{x}) d\mathbf{x}. \quad (19.8)$$

The above formula has the form of Newton's law, where the force equals the integral of the Lorentz force density. In particular, for the spatially homogeneous EM fields

$\mathbf{E}_{\text{ex}}(t)$ and $\mathbf{B}_{\text{ex}}(t)$, Eq. (19.8) is reduced to

$$m\partial_t^2 \mathbf{r}(t) = q\mathbf{E}_{\text{ex}}(t) + \frac{q}{c}\partial_t \mathbf{r}(t) \times \mathbf{B}_{\text{ex}}(t), \quad (19.9)$$

and we recognize the equation of motion of a point charge in a homogeneous EM field with the familiar expression for the Lorentz force.

Now we derive from (13.37) the energy conservation law:

$$\partial_t u = c\partial_0 u = - \sum_{j=1,2,3} \partial_j s_j + \mathbf{J} \cdot \mathbf{E}_{\text{ex}}. \quad (19.10)$$

Integrating energy conservation law (19.10), we obtain an equation for the total energy

$$\mathbf{E} = \int_{\mathbb{R}^3} (u(t, \mathbf{x}) + u_{\text{em}}(t, \mathbf{x})) d\mathbf{x}, \quad (19.11)$$

namely

$$\partial_t \mathbf{E} = \int_{\mathbb{R}^3} \mathbf{J} \cdot \mathbf{E}_{\text{ex}}(t, \mathbf{x}) d\mathbf{x}. \quad (19.12)$$

For the spatially homogeneous EM fields $\mathbf{E}_{\text{ex}}(t)$ and $\mathbf{B}_{\text{ex}}(t)$, the above equation has the form

$$\partial_t \mathbf{E} = q\mathbf{v}(t) \cdot \mathbf{E}_{\text{ex}}(t), \quad \mathbf{v}(t) = \partial_t \mathbf{r}(t). \quad (19.13)$$

In addition to that, in this case (19.9) implies that

$$\frac{m}{2}\partial_t (\mathbf{v} \cdot \mathbf{v}) = q\mathbf{v} \cdot \mathbf{E}_{\text{ex}}(t). \quad (19.14)$$

Combining this expression with (19.13), we obtain the following energy representation:

$$\mathbf{E} = \frac{m}{2}\mathbf{v} \cdot \mathbf{v} + \text{constant}. \quad (19.15)$$

Note that the expression for the energy obtained from the symmetric energy-momentum tensor is different from the expression for the canonical energy-momentum tensor, the latter one involves an electric potential as in (16.127). Notice that the correspondence found above between field quantities and point mechanics quantities via the charge position and velocity defined as average values (19.1), (19.2) is similar to the well known *Ehrenfest Theorem* in quantum mechanics, [291, Sections 7, 23]. *The key argument for the Ehrenfest theorem is the momentum density kinematic representation (11.166)–(11.168).*

19.2 Many Interacting Dressed Charges

The case of many interacting dressed charges can be considered similarly to many balanced charges, the difference is the presence of the electric self-interaction and the absence of magnetic interaction between the charges. As in the case of balanced charges, we use two approaches. First, the analysis of dynamics of averaged quantities similar to the Ehrenfest theorem. The second approach uses wave-corpuscule solutions studied in Chap. 16.

19.2.1 The Ehrenfest Theorem for Dynamics of Many Interacting Dressed Charges

Now, similarly to the case of a single charge in Sect. 19.1, we consider a system of interacting charges and derive the Ehrenfest theorem for the dynamics of the charge centers. A qualitatively new physical phenomenon in the theory of two or more charges compared with the theory of a single charge is obviously the interaction between them. It is important to note that *any individual “bare” charge interacts directly only with the EM field* and consequently different charges interact with each other only indirectly through the EM field. The case of many interacting dressed charges can be considered similarly to balanced charges, but now the electric self-interaction is present and the magnetic interaction between the charges is absent. We will see below that the self-interaction does not affect the dynamics of well-separated charges.

Integration of the continuity equations (13.26) over space implies the conservation of the total ℓ -th conserved charge. As always, we require every total ℓ -th conserved charge to be exactly q_ℓ and, hence, to satisfy the charge normalization condition (17.7). We introduce the ℓ -th charge center $\mathbf{r}_\ell(t)$ and the average velocity $\mathbf{v}_\ell(t)$ as in (19.1):

$$\mathbf{r}_\ell(t) = \mathbf{r}_{\ell a}(t) = \int_{\mathbb{R}^3} \mathbf{x} |\psi_\ell(t, \mathbf{x})|^2 d\mathbf{x}, \quad (19.16)$$

$$\mathbf{v}_\ell(t) = \mathbf{v}_{\ell a}(t) = \frac{1}{q} \int_{\mathbb{R}^3} \mathbf{J}_\ell(t, \mathbf{x}) d\mathbf{x}. \quad (19.17)$$

Sometimes we explicitly show the dependence of $\psi_\ell = \psi_{\ell a}$, $\mathbf{v}_\ell = \mathbf{v}_{\ell a}$ etc. on the size parameter a which enters the nonlinearity G . We derive from the charge conservation law (13.26) as in Sect. 17.2.2 the identity (17.19), namely

$$\partial_t \mathbf{r}_\ell(t) = \mathbf{v}_\ell(t).$$

Hence the average velocity of a charge defined by formulas (19.17) equals the velocity of the charge position. The total individual momenta $\mathbf{P}_\ell$ and energies $\mathbf{E}_\ell$ for the ℓ -th

dressed charge are, as always, obtained by integration:

$$\mathbf{P}_\ell = \int_{\mathbb{R}^3} \mathbf{P}_\ell \, d\mathbf{x}, \quad \mathbf{E}_\ell = \int_{\mathbb{R}^3} \tilde{u}_\ell \, d\mathbf{x}. \quad (19.18)$$

Based on the equation (13.21), it is natural to introduce for every ℓ -th charge the corresponding potential φ_ℓ using the Green function (37.2) as in (12.19), namely

$$\varphi_\ell(t, \mathbf{x}) = q_\ell \int_{\mathbb{R}^3} \frac{|\psi_\ell|^2(t, \mathbf{y})}{|\mathbf{y} - \mathbf{x}|} \, d\mathbf{y}, \quad \varphi = \sum_\ell \varphi_\ell. \quad (19.19)$$

We derive from the conservation law (13.37) similarly to (19.3) the momentum conservation law

$$\partial_t p_\ell^i + c \partial_i \tilde{\sigma}_{\ell ij} = \sum_{\ell' \neq \ell} \rho_\ell \mathbf{E}_{\ell'}^i + \left(\rho \mathbf{E}_{\text{ex}} + \frac{1}{c} \mathbf{J}_\ell \times \mathbf{B}_{\text{ex}} \right)^i. \quad (19.20)$$

The EM momentum does not contribute to the above equation since $\Theta_\ell^{0j} = \Theta_\ell^{j0} = 0$ according to (11.103). Integrating, we obtain the following equations:

$$\partial_t \mathbf{P}_\ell = q_\ell \int_{\mathbb{R}^3} \left[\left(\sum_{\ell' \neq \ell} \mathbf{E}_{\ell'} + \mathbf{E}_{\text{ex}} \right) |\psi_\ell|^2 + \frac{1}{c} \mathbf{v}_\ell \times \mathbf{B}_{\text{ex}} \right] d\mathbf{x} \quad (19.21)$$

where, according to (17.11), $\mathbf{v}_\ell = \mathbf{J}_\ell / q_\ell$. Since the dressed charge self-interaction does not enter this equation, we can treat the equations the same way as (17.22) in Sect. 17.2.2.

Then we readily obtain from the momentum density kinematic representation (11.168) the kinematic representation (17.20) for the ℓ -th charge total momentum. Combining relations (19.21), (17.19) and (17.20), we obtain the following system of equations of motion for N charges:

$$\begin{aligned} m_\ell \partial_t^2 \mathbf{r}_\ell &= \partial_t \mathbf{P}_\ell \\ &= q_\ell \int_{\mathbb{R}^3} \left[\left(\sum_{\ell' \neq \ell} \mathbf{E}_{\ell'} + \mathbf{E}_{\text{ex}} \right) |\psi_\ell|^2 + \frac{1}{c} \mathbf{v}_\ell \times \mathbf{B}_{\text{ex}} \right] d\mathbf{x}, \end{aligned} \quad (19.22)$$

where

$$\mathbf{E}_\ell(t, \mathbf{x}) = -\nabla \varphi_\ell(t, \mathbf{x}), \quad \mathbf{E}_{\text{ex}} = -\nabla \varphi_{\text{ex}}(t, \mathbf{x}). \quad (19.23)$$

The above system is analogous to the *Ehrenfest Theorem*, well known in quantum mechanics [291, Sections 7, 23]. Observe also that the system of equations of motion (19.22) is consistent with Newton's third law of motion "action equals reaction" as

it follows from the relations (13.31). The interaction integral in the above formula can be written according to relations (19.19) and (11.102) as follows:

$$\begin{aligned} & \int_{\mathbb{R}^3} \sum_{\ell' \neq \ell} \mathbf{E}^{\ell'} |\psi_\ell|^2 \, d\mathbf{x} \\ &= - \sum_{\ell' \neq \ell} q_{\ell'} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{(\mathbf{y} - \mathbf{x}) |\psi_{\ell'}|^2(t, \mathbf{y}) |\psi_\ell(t, \mathbf{x})|^2}{|\mathbf{y} - \mathbf{x}|^3} \, d\mathbf{y} d\mathbf{x}. \end{aligned} \quad (19.24)$$

19.3 Mechanics of Localized Charge Centers as an Approximation

Based on equation (19.22), one can derive Newton's equations of motion from (19.22) in the limit of charge localization. Indeed, let us suppose that for every ℓ the charge density $|\psi_\ell|^2$ and the corresponding current density $\mathbf{J}_\ell = q_\ell \mathbf{v}_\ell$ are localized at the center $\mathbf{r}_\ell(t)$ having the localization scale a_ℓ which is small compared with the typical variation scale R_{EM} of the EM field. Namely, for a bounded R_{EM} and a_ℓ converging to 0, we assume that $a_\ell/R_{\text{EM}} \rightarrow 0$ and

$$|\psi_{\ell a}|^2(t, \mathbf{x}) \rightarrow \delta(\mathbf{x} - \mathbf{r}_\ell(t)), \quad \mathbf{v}_{\ell a}(t, \mathbf{x}) \rightarrow \mathbf{v}_\ell(t) \delta(\mathbf{x} - \mathbf{r}_\ell(t)), \quad (19.25)$$

where the coefficients at the delta-functions are determined from the charge normalization conditions (17.7) and relations (19.17). Using potential representations (19.19), we infer from (19.25) the convergence of the potentials φ_ℓ to the corresponding Coulomb potentials, namely

$$\varphi_\ell(t, \mathbf{x}) \rightarrow \varphi_{\ell 0}(t, \mathbf{x}) = \frac{q_\ell}{|\mathbf{x} - \mathbf{r}_\ell(t)|} \text{ as } a_\ell/R_{\text{EM}} \rightarrow 0. \quad (19.26)$$

Hence, we can recast the equations of motion (19.22) as

$$m_\ell \partial_t^2 \mathbf{r}_\ell = \mathbf{f}_{\ell \text{Lor}} + \mathbf{p}_\ell, \quad (19.27)$$

where the Lorentz force $\mathbf{f}_{\ell \text{Lor}}$ is given by formulas (17.25), (17.14) in the particular case $\mathbf{B}_{\ell \text{ex}} = \mathbf{B}_{\text{ex}}$:

$$\mathbf{f}_{\ell \text{Lor}} = q_\ell \mathbf{E}_{\ell \text{ex}}(\mathbf{r}_\ell) + \frac{1}{c} \mathbf{v}_\ell \times \mathbf{B}_{\text{ex}}(\mathbf{r}_\ell), \quad \ell = 1, \dots, N, \quad (19.28)$$

with small discrepancies $\epsilon_{\mathbf{p}_\ell} \rightarrow 0$ as $a_\ell/R_{\text{EM}} \rightarrow 0$. In the localization limit $a_\ell \rightarrow 0$, we obtain Newton's law

$$m_\ell \partial_t^2 \mathbf{r}_\ell = \mathbf{f}_{\ell \text{Lor}}. \quad (19.29)$$

In particular, in the case when there is no external EM field, we obtain the equations (“static limit, zeroth order in (v/c) ”, [179, Section 12.6])

$$m_\ell \partial_t^2 \mathbf{r}_\ell = - \sum_{\ell' \neq \ell} \frac{q_\ell q_{\ell'} (\mathbf{r}_{\ell'} - \mathbf{r}_\ell)}{|\mathbf{r}_{\ell'} - \mathbf{r}_\ell|^3}, \quad \ell = 1, \dots, N. \quad (19.30)$$

We can also show that the total energy of every charge given by the integral (19.11) converges to the kinetic energy of a point charge. Indeed, similarly to (19.12), we obtain using (13.37):

$$\partial_t \mathbf{E}_\ell = \int_{\mathbb{R}^3} \mathbf{J}_\ell \cdot \mathbf{E}_{\text{ex}}^\ell(t, \mathbf{x}) \, d\mathbf{x} = q_\ell \int_{\mathbb{R}^3} \mathbf{v}_\ell \cdot \left(\sum_{\ell' \neq \ell} \mathbf{E}^{\ell'} + \mathbf{E}_{\text{ex}} \right) d\mathbf{x}. \quad (19.31)$$

It then follows from (19.31) that

$$\partial_t \mathbf{E}_\ell = \mathbf{v}_\ell \cdot \mathbf{f}_{\ell \text{Lor}} + \epsilon_{\mathbf{E}_\ell} \quad (19.32)$$

with small discrepancies $\epsilon_{\mathbf{E}_\ell} \rightarrow 0$ as $a_\ell/R_{\text{EM}} \rightarrow 0$. Combining equalities (17.20), (19.27) and (19.32), we get

$$\begin{aligned} \frac{1}{2} \partial_t (m_\ell \mathbf{v}_\ell \cdot \mathbf{v}_\ell) &= \mathbf{v}_\ell \cdot \partial_t (m \mathbf{v}_\ell) \\ &= \mathbf{v}_\ell \cdot \mathbf{f}_\ell + \mathbf{v}_\ell \cdot \mathbf{p}_\ell = \frac{d\mathbf{E}_\ell}{dt} + \mathbf{v}_\ell \cdot \mathbf{p}_\ell - \epsilon_{\mathbf{E}_\ell}, \end{aligned} \quad (19.33)$$

implying

$$\partial_t \left(\mathbf{E}_\ell - \frac{m_\ell}{2} \mathbf{v}_\ell \cdot \mathbf{v}_\ell \right) = \epsilon_{\mathbf{E}_\ell} - \mathbf{v}_\ell \cdot \mathbf{p}_\ell,$$

which, up to small errors, are kinematic representations for the energies of individual charges well known from point charge mechanics.

19.4 Point Mechanics of Dressed Charges Via Wave-Corpuscles

Point mechanics can be alternatively derived as a property of approximate wave-corpuscle solutions to the field equations (13.20), (13.21) (or, equivalently, (13.25)) for N interacting charges. The construction for dressed charges is similar to the one for balanced charges in Sect. 17.2.4. As before, wave-corpuscles are defined by equality (17.28), where the trajectories satisfy Newton’s equations (17.32) with the Lorentz forces, and the phase functions are defined by equations (17.35), (17.36). When exploiting similarities between wave-corpuscles for dressed and balanced charges, one has to keep in mind the following differences. First, the non-relativistic model

of dressed charges does not involve the magnetic field generated by the charges, and only involves the external magnetic field as in (13.19). Consequently, the effective vector potential (17.31) acting on the ℓ -th charge does not depend on the charges of the system and has the form

$$\mathbf{A}_{\neq \ell \text{ex}}(t, \mathbf{x}) = \mathbf{A}_{\text{ex}}(t, \mathbf{x}). \quad (19.34)$$

Hence the dynamics of the centers of wave-corpuscles for dressed charges is described by system (17.32) where the Lorentz force involves only the external magnetic field and the charges interact only through their Coulomb forces:

$$\begin{aligned} m \partial_t^2 \mathbf{r}_\ell &= \mathbf{f}_{\ell \text{Lor}}(t, \mathbf{r}_\ell), \quad \ell = 1, \dots, N, \\ \mathbf{f}_{\ell \text{Lor}}(t, \mathbf{x}) &= -q \nabla \varphi_{\neq \ell \text{ex}}(t, \mathbf{x}) - \frac{q}{c} \partial_t \mathbf{A}_{\text{ex}}(t, \mathbf{x}) \\ &\quad + \nabla (\mathbf{v} \cdot \mathbf{A}_{\text{ex}})(t, \mathbf{x}) - \mathbf{v} \cdot \nabla \mathbf{A}_{\text{ex}}(t, \mathbf{x}). \end{aligned} \quad (19.35)$$

In view of equality (15.39), the scalar potential $\varphi_{\neq \ell \text{ex}}(t, \mathbf{x})$ has exactly the same form as in equality (17.30) for balanced charges, that is,

$$\varphi_{\neq \ell \text{ex}}(t, \mathbf{x}) = \varphi_{\text{ex}}(t, \mathbf{x}) + \sum_{\ell' \neq \ell} \hat{\varphi}(\mathbf{x} - \mathbf{r}_{\ell'}).$$

The estimates of the discrepancy are quite similar to those for balanced charges.

19.5 A Hydrogen Atom Model

We model a Hydrogen atom as a system of two bound dressed charges: an electron and a proton. The analysis is similar to the one in Sect. 17.5.1 for balanced charges, but there is a difference which has its origin in the EM self-interaction of the charge.

Indeed, to model interaction of two charges *at a short distance* we consider the original system (13.25) for two charges with $-q_1 = q_2 = q > 0$, that is,

$$\begin{aligned} i\chi \partial_t \psi_1 &= -\frac{\chi^2 \nabla^2 \psi_1}{2m_1} - q(\varphi_1 + \varphi_2) \psi_1 + \frac{\chi^2}{2m_1} G'_1(|\psi_1|^2) \psi_1, \\ -\nabla^2 \varphi_1 &= -4\pi q |\psi_1|^2, \\ i\chi \partial_t \psi_2 &= -\frac{\chi^2 \nabla^2 \psi_2}{2m_2} + q(\varphi_1 + \varphi_2) \psi_2 + \frac{\chi^2}{2m_2} G'_2(|\psi_2|^2) \psi_2, \\ \nabla^2 \varphi_2 &= -4\pi q |\psi_2|^2. \end{aligned} \quad (19.36)$$

Note that this model describes proton-electron interaction if $q = e$ is the electron charge, $\chi = \hbar$ is the Planck constant, the first charge is the electron, the second is

the proton; m_1 and m_2 are the electron and the proton masses, respectively. Let us take a look now at time-harmonic solutions to the system (19.36) in the form

$$\begin{aligned}\psi_1(t, \mathbf{x}) &= e^{-i\omega_1 t} \hat{\psi}_1(\mathbf{x}), & \psi_2 &= e^{-i\omega_2 t} \hat{\psi}_2(\mathbf{x}), \\ \Phi_1 &= -\frac{\varphi_1}{q}, & \Phi_2 &= \frac{\varphi_2}{q}.\end{aligned}\quad (19.37)$$

The system (19.36) for such solutions turns into the following nonlinear eigenvalue problem:

$$\begin{aligned}-\frac{\hat{a}_1}{2} \nabla^2 \hat{\psi}_1 + (\Phi_1 - \Phi_2) \hat{\psi}_1 + \frac{\hat{a}_1}{2} G'_1 \left(|\hat{\psi}_1|^2 \right) \hat{\psi}_1 &= \frac{\chi}{q^2} \omega_1 \hat{\psi}_1, \\ -\frac{\hat{a}_2}{2} \nabla^2 \hat{\psi}_2 + (\Phi_2 - \Phi_1) \hat{\psi}_2 + \frac{\hat{a}_2}{2} G'_2 \left(|\hat{\psi}_2|^2 \right) \hat{\psi}_2 &= \frac{\chi}{q^2} \omega_2 \hat{\psi}_2,\end{aligned}\quad (19.38)$$

where $\hat{a}_1, \hat{a}_2$ are given by (17.71):

$$\hat{a}_1 = \frac{\chi^2}{q^2 m_1}, \quad \hat{a}_2 = \frac{\chi^2}{q^2 m_2}.$$

In particular, $\hat{a}_1$ coincides with the Bohr radius. We seek solutions of (19.38) satisfying the charge normalization conditions

$$\int_{\mathbb{R}^3} |\hat{\psi}_1|^2 d\mathbf{x} = 1, \quad \int_{\mathbb{R}^3} |\hat{\psi}_2|^2 d\mathbf{x} = 1. \quad (19.39)$$

The potentials Φ_i are presented by (17.76). Let us introduce the following energy functional

$$\begin{aligned}\mathcal{E}(\hat{\psi}_1, \hat{\psi}_2) &= q^2 \int_{\mathbb{R}^3} \left[\frac{\hat{a}_1}{2} |\nabla \hat{\psi}_1|^2 + \hat{a}_1 G_1 \left(|\hat{\psi}_1|^2 \right) \right] d\mathbf{x} \\ &+ q^2 \int_{\mathbb{R}^3} \left[\frac{\hat{a}_2}{2} |\nabla \hat{\psi}_2|^2 + \hat{a}_2 G_2 \left(|\hat{\psi}_2|^2 \right) \right] d\mathbf{x} \\ &- q^2 \int_{\mathbb{R}^3} (\Phi_1 - \Phi_2) \left(|\hat{\psi}_2|^2 - |\hat{\psi}_1|^2 \right) + \frac{1}{8\pi} |\nabla (\Phi_1 - \Phi_2)|^2 d\mathbf{x}.\end{aligned}\quad (19.40)$$

Taking into account the smallness of the mass ratio m_1/m_2 , we can use, as it is done in quantum mechanics, the *Born–Oppenheimer approximation*, [138, Section 7.5d], that is, an approximation when the proton is treated as a point charge with fixed position at the origin, and its EM field is modeled by the Coulomb potential $\frac{1}{|\mathbf{x}|}$. In other words, effectively we have an electron in an external EM field with Coulomb

potential $\frac{1}{|\mathbf{x}|}$. In this approximation the energy $\mathcal{E}(\hat{\psi}_1, \hat{\psi}_2)$ of the proton-electron system is described by the following expression

$$\begin{aligned} \mathcal{E}_1(\hat{\psi}_1) &= q^2 \int_{\mathbb{R}^3} \frac{\hat{a}_1}{2} \left[|\nabla \hat{\psi}_1|^2 + G_1 \left(|\hat{\psi}_1|^2 \right) \right] d\mathbf{x} \\ &+ q^2 \int_{\mathbb{R}^3} \left(\Phi_1 - \frac{1}{|\mathbf{x}|} \right) |\hat{\psi}_1|^2 - \frac{1}{8\pi} |\nabla \Phi_1|^2 d\mathbf{x}, \end{aligned} \quad (19.41)$$

and the nonlinear eigenvalue problem (19.38)–(19.39) is approximated by the following equation

$$\begin{aligned} -\frac{\hat{a}_1}{2} \nabla^2 \hat{\psi}_1 + \left(\Phi_1 - \frac{1}{|\mathbf{x}|} \right) \hat{\psi}_1 + \frac{\hat{a}_1}{2} G'_1 \left(|\hat{\psi}_1|^2 \right) \hat{\psi}_1 &= \frac{\chi}{q^2} \omega_1 \hat{\psi}_1, \\ \Phi_1 &= \int_{\mathbb{R}^3} \frac{|\hat{\psi}_1|^2(\mathbf{y})}{|\mathbf{y} - \mathbf{x}|} d\mathbf{y}, \quad \int_{\mathbb{R}^3} |\hat{\psi}_1|^2 d\mathbf{x} = 1. \end{aligned} \quad (19.42)$$

The energy expression (17.89) and the Eq. (19.42) resemble the corresponding expressions for the Schrödinger Hydrogen atom model with two important alterations: the nonlinear term $G_1 \left(|\hat{\psi}_1|^2 \right)$ and the electrostatic self-interaction term $\Phi_1 |\hat{\psi}_1|^2$. Notice that, for the resting free electron, these two terms exactly compensate each other, but for the Hydrogen states they do not, since the nonlinear self-interaction and the electrostatic self-interaction depend on the charge distribution differently.

Comparison with the Rydberg formula. Let us compare now the electron frequencies associated with equations (19.38) with the classical Rydberg formula. We follow here the line of argument of Sects. 17.5.1–17.5.2. As the first step, we make two changes of variables in two equations (19.38):

$$\mathbf{x}_\ell = \hat{a}_\ell \mathbf{y}_\ell, \text{ namely } \mathbf{x}_1 = \hat{a}_1 \mathbf{y}_1, \quad \mathbf{x}_2 = \hat{a}_2 \mathbf{y}_2,$$

and introduce rescaled fields

$$\Phi_\ell(\mathbf{x}) = \frac{\phi_\ell(\mathbf{y}_\ell)}{\hat{a}_\ell}, \quad \hat{\psi}_\ell(\mathbf{x}) = \frac{1}{\hat{a}_\ell^{3/2}} \Psi_\ell(\mathbf{y}_\ell), \quad \ell = 1, 2. \quad (19.43)$$

Then we recast nonlinearity $G'_1 = G'_{a,1}$ into the form

$$\frac{\hat{a}_1}{2} G'_{a,1} \left(|\hat{\psi}_1|^2 \right) = \frac{\hat{a}_1}{2} a^{-2} G'_{11} \left(a^3 \hat{a}_1^{-3} |\Psi_1|^2 \right) = \frac{1}{2\hat{a}_1} \bar{G}'_{1,a/\hat{a}_1} (|\Psi_1|^2).$$

Then equations for Ψ_1 and Ψ_2 in the rescaled variables take the form

$$\begin{aligned} -\frac{1}{2}\nabla^2\Psi_1 + \left(\phi_1 - \frac{1}{b}\phi_2\left(\frac{1}{b}y_2\right)\right)\Psi_1 + \frac{1}{2}\bar{G}'_1(|\Psi_1|^2)\Psi_1 &= \frac{\chi\hat{a}_1}{q^2}\omega_1\Psi_1, \\ -\frac{1}{2}\nabla^2\Psi_2 + (\phi_2 - b\phi_1(by_1))\hat{\psi}_2 + \frac{1}{2}\bar{G}'_2(|\Psi_2|^2)\hat{\psi}_2 &= \frac{\chi\hat{a}_2}{q^2}\omega_2\Psi_2, \end{aligned} \quad (19.44)$$

where

$$b = \frac{\hat{a}_2}{\hat{a}_1} = \frac{m_1}{m_2} \simeq \frac{1}{1800}. \quad (19.45)$$

Let us introduce normalized frequencies

$$\Omega_1 = \frac{\chi\hat{a}_1}{q^2}\omega_1, \quad \Omega_2 = \frac{\chi\hat{a}_2}{q^2}\omega_2. \quad (19.46)$$

Notice that according to (17.90), the potential $\frac{1}{b}\phi_2(\frac{1}{b}r)$ approaches $\frac{1}{r}$ as $b \rightarrow 0$, and the contribution of the nonlinearity $\bar{G}'_1 = G'_{1,a/a_1}$ tends to zero as $a/a_1 \rightarrow \infty$. Consequently, the normalized electron frequencies $\Omega_1 = \frac{\chi\hat{a}_1}{q^2}\omega_1$ which correspond to lower energies converge to frequencies associated with the following nonlinear eigenvalue problem:

$$\begin{aligned} -\frac{1}{2}\nabla^2\Psi_1 + \left(\phi_1 - \frac{1}{|y|}\right)\Psi_1 &= \Omega_1\Psi_1, \\ \nabla^2\phi_1 &= -4\pi|\Psi_1|^2. \end{aligned} \quad (19.47)$$

If the potential ϕ_1 in the first equation (19.47) is neglected, it turns into the linear Schrödinger equation with Coulomb potential which has the frequencies

$$\Omega_{1,n} = -\frac{1}{2n^2}, \quad n = 1, 2, \dots \quad (19.48)$$

Combining equations (19.46) and (19.48) and setting $\chi = \hbar = h/2\pi$, we obtain the classical Rydberg formula:

$$\omega_{1,n} = \frac{-1}{2n^2} \frac{q^2}{\hbar\hat{a}_1} = -\frac{hc}{n^2}R_\infty, \quad R_\infty = \frac{q^4m}{4\pi\hbar^3c}.$$

Observe that the eigenvalue problem (19.47) involves the potential ϕ_1 which can be treated as a perturbation of the linear Schrödinger equation. The problem (19.47), though, does not involve small parameters, and, consequently, the perturbation cannot be made arbitrarily small. Based on that, we can conclude that though the problem (19.42) has discrete energy levels which can be related to the energy levels of the Schrödinger Hydrogen atom model, a good quantitative agreement between the two

would require a different method of compensating the electrostatic self-interaction. For such an approach based on balanced charges, see Chap. 17.

19.6 The Relativistic Theory of Interacting Dressed Charges

We consider here general properties of systems of interacting charges described by the relativistic Lagrangian introduced in Sect. 13.1. The system Lagrangian $\mathcal{L}$ is of the general form as in Sect. 11.1, namely it is given by equality (13.1).

19.6.1 Single Charge

The state of a single elementary charge without spin is represented by its wave function $\psi = \psi(t, \mathbf{x})$, $(t, \mathbf{x}) \in \mathbb{R}^4$, which is a complex-valued scalar field on the space-time continuum $\mathbb{R}^4$. The charge is coupled to classical EM field through its potentials $A^\mu = (\varphi, \mathbf{A})$ which are related to the EM field by the standard formulas (4.6). The system of a single charge and the EM field has the Lagrangian defined by equality (13.13), which can also be written in the form

$$\begin{aligned} \mathcal{L} = \frac{\chi^2}{2m} \left\{ \frac{1}{c^2} \left| \tilde{\partial}_t \psi \right|^2 - \left| \tilde{\nabla} \psi \right|^2 - \kappa_0^2 |\psi|^2 - G(\psi^* \psi) \right\} \\ + \frac{1}{8\pi} \left[\left(\nabla \varphi + \frac{1}{c} \partial_t \mathbf{A} \right)^2 - (\nabla \times \mathbf{A})^2 \right], \end{aligned} \quad (19.49)$$

where $\tilde{\partial}_t$ and $\tilde{\nabla}$ are the covariant differentiation operators defined by (13.8). The field Euler–Lagrange equations for the above Lagrangian are the nonlinear Klein–Gordon equation (13.14) coupled with the Maxwell equations (13.15).

A distinct component of the Lagrangian in (13.13) is the term $G(\psi^* \psi)$ associated with a real-valued *self-interaction nonlinearity* $G(s)$, $s \geq 0$. This nonlinearity models the charge intrinsic forces of a non-electromagnetic origin that, in particular, provide for charge being particle-like under proper conditions. The remaining part of the Lagrangian expression (13.13) is the standard Lagrangian of the EM field coupled to the charge via the covariant derivatives. Observe that the Lagrangian $\mathcal{L}$ defined by (13.13) is manifestly (i) local; (ii) Lorentz and gauge invariant, and (iii) it has a local nonlinear term providing for a cohesive self-force similar to the Poincaré force for the Lorentz–Poincaré model of an extended charge.

The gauge invariance of the Lagrangian, namely its NKG part, allows us to introduce in a standard fashion as in Sect. 11.2.2 the *charge density* ρ and the *current density* $\mathbf{J}$ by (11.132)–(11.133) satisfying the continuity equation (11.54), that is,

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0. \quad (19.50)$$

Consequently, the total charge is conserved in the course of evolution:

$$\int_{\mathbb{R}^3} \rho(t, \mathbf{x}) \, d\mathbf{x} = \text{const.} \quad (19.51)$$

Using the same argument as for condition (14.123), we impose a *charge normalization* condition, namely

$$\int_{\mathbb{R}^3} \frac{\rho(t, \mathbf{x})}{q} \, d\mathbf{x} = - \int_{\mathbb{R}^3} \left(\frac{\chi}{mc^2} \text{Im} \frac{\partial_t \psi}{\psi} + \frac{q\varphi}{mc^2} \right) |\psi|^2 \, d\mathbf{x} = 1. \quad (19.52)$$

We would like to stress that the equation (19.52) *constitutes an independent and physically significant constraint for the total charge to be exactly q as in the Coulomb potential, rather than an arbitrary constant; such a normalization provides an interpretation for the parameter q* . For the fundamental pair $\{\hat{\psi}, \hat{\varphi}\}$ the corresponding charge density defined by (11.132)–(11.133) turns into (14.119).

19.7 Dressed Charge Equations in Dimensionless Form and the Non-relativistic Limit

We introduce here changes of variables allowing to recast the original field equations into a dimensionless form. The equations in dimensionless form allow us to clarify the theory for a single charge: out of all the constants involved, there is only one parameter of significance, denoted by α , and it coincides with the Sommerfeld fine structure constant $\alpha = \frac{q^2}{\chi c} \simeq \frac{1}{137}$ if $\chi = \hbar$ and q, m are the electron charge and mass respectively.

Recall that a derivation of the non-relativistic equations from the relativistic is given in Sect. 12.2.1 and uses the frequency shifted Lagrangian L_{ω_0} defined by (11.151)–(11.152). The Euler–Lagrange field equations corresponding to L_{ω_0} under the Lorentz gauge are the following NKG–Maxwell equations:

$$\begin{aligned} \frac{1}{c^2} \tilde{\partial}_t^2 \psi - \frac{im}{\chi} \left(2\partial_t \psi + 2\frac{iq\bar{\varphi}}{\chi} \hat{\psi} \right) - \tilde{\nabla}^2 \psi + G'(|\psi|^2) \psi &= 0, \\ \frac{1}{4\pi} \nabla \cdot \left(\frac{1}{c} \partial_t \mathbf{A} + \nabla \varphi \right) &= \left(\frac{\chi q}{mc^2} \text{Im} \frac{\partial_t \hat{\psi}}{\hat{\psi}} + \frac{q^2}{mc^2} \bar{\varphi} \right) |\psi|^2 - q |\psi|^2, \\ - \left(\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c} \partial_t \left(\frac{1}{c} \partial_t \mathbf{A} + \nabla \varphi \right) \right) &= \frac{4\pi}{c} \left(-\frac{\chi q}{m} \text{Im} \frac{\nabla \psi}{\psi} - \frac{q^2}{mc} \bar{\mathbf{A}} \right) |\psi|^2, \end{aligned}$$

where the covariant derivatives $\tilde{\partial}_t$, $\tilde{\nabla}^2$ are given by (11.138). It is convenient to introduce the notations as in (14.134):

$$a_B = \frac{\chi^2}{mq^2}, \quad \alpha = \frac{q^2}{\chi c}, \quad \omega_0 = \frac{mc^2}{\chi} = \frac{c}{\alpha a_B}, \quad (19.53)$$

and new variables $\tau, \mathbf{y}, \Psi, \Phi, \mathbf{A}$ as in (14.133):

$$\alpha^2 \omega_0 t = \tau, \quad \mathbf{x} = a_B \mathbf{y}, \quad (19.54)$$

$$\psi(\mathbf{x}) = \frac{1}{a_B^{3/2}} \Psi\left(\frac{\mathbf{x}}{a_B}\right), \quad \varphi(\mathbf{x}) = \frac{q}{a_B} \Phi\left(\frac{\mathbf{x}}{a_B}\right), \quad \mathbf{A}(\mathbf{x}) = \frac{q}{a_B} \mathbf{A}\left(\frac{\mathbf{x}}{a_B}\right).$$

In the new variables the NKG-Maxwell equations turn into the following dimensionless form:

$$\alpha^2 (\partial_\tau + i\bar{\Phi})^2 \Psi - 2i (\partial_\tau + i\bar{\Phi}) \Psi - (\nabla_y - i\alpha \mathbf{A})^2 \Psi + \mathbf{G}'(|\Psi|^2) \Psi = 0, \quad (19.55)$$

$$\frac{1}{4\pi} \nabla_y \cdot (\alpha \partial_\tau \mathbf{A} + \nabla_y \Phi) = \left(\alpha^2 \operatorname{Im} \frac{\partial_\tau \Psi}{\Psi} + \alpha^2 \Phi \right) |\Psi|^2 - |\Psi|^2, \quad (19.56)$$

$$- (\nabla_y \times (\nabla_y \times \mathbf{A}) + \alpha \partial_\tau (\alpha \partial_\tau \mathbf{A} + \nabla_y \Phi)) = -4\pi \alpha \left(\operatorname{Im} \frac{\nabla_y \Psi}{\Psi} + \alpha \mathbf{A} \right) |\Psi|^2. \quad (19.57)$$

Note also that the Lorentz gauge takes the form

$$\alpha \partial_\tau \Phi + \nabla_y \cdot \mathbf{A} = 0. \quad (19.58)$$

We would like to show that the dimensionless form of the non-relativistic equations field equations (13.27)–(13.28) can be obtained from the field equations (19.55)–(19.57) in the limit $\alpha \rightarrow 0$. To have a nonvanishing external magnetic field in the limit $\alpha \rightarrow 0$, we set

$$\mathbf{A}_{\text{ex}} = \alpha^{-1} \mathbf{A}_{\text{ex}}^0. \quad (19.59)$$

Plugging the expression (19.59) into the Eqs. (19.55)–(19.57), we obtain in the limit $\alpha \rightarrow 0$ the following dimensionless version of the field equations (13.27)–(13.28):

$$\begin{aligned}
 i\partial_\tau \Psi = & -\frac{1}{2} (\nabla_y - i\mathbf{A}_{\text{ex}}^0)^2 \Psi + (\Phi + \Phi_{\text{ex}}) \Psi + \frac{1}{2} \mathbf{G}'(|\Psi|^2) \Psi, \\
 -\nabla_y^2 \cdot \Phi = & 4\pi |\Psi|^2, \quad \nabla_y \times (\nabla_y \times \mathbf{A}) = 0.
 \end{aligned} \tag{19.60}$$

On taking the limit of (19.58), we obtain the condition

$$\nabla_y \cdot \mathbf{A} = 0.$$

Since $\nabla_y \times (\nabla_y \times \mathbf{A}) = 0$ and is divergent-free, the vector potential $\mathbf{A}$ which is bounded in the entire space must be a constant, hence our assumption on eliminating the magnetic field of non-relativistic dressed charge is substantiated.

We would like to make a few comments on the relative magnitude of terms that have to be omitted in Eqs. (19.55)–(19.57) in order to obtain Eq. (19.60). The non-relativistic case is defined as the one where the charge velocity v is much smaller than the speed of light c , and a careful look at the omitted terms in (19.55)–(19.57) that have factors α and α^2 shows that they can be small not only because of the smallness of α , but also because of the smallness of typical values of velocities compared to the speed of light. Indeed, every term that has factor α involves time derivatives with only one exception: the term $\alpha^2 \Phi^2 \Psi$. In fact, the term $\alpha^2 \Phi^2 \Psi$ could be preserved in the non-relativistic system which would be similar to (19.60) with properties analogous to (13.27). The analysis of that system is more complicated, and the treatment is similar to the one in Sect. 14.6.1 for the rest solution of the relativistic equation which involves such a term.

Chapter 20

Comparison of EM Aspects of Dressed and Balanced Charges Theories

In the balanced charges theory (BCT) as well as in the dressed charges theory (DCT), the charges generate EM fields which, in turn, act upon the charges. In the DCT the EM field action on any of the charges from the system of all charges is reduced to the action of the single total EM field with the potential A^μ , and according to (20.2) *this field is the same for all charges*.

In the BCT, just as in the DCT, a single elementary ℓ -th charge is affected by other charges through a single potential external to it as in (12.6), namely

$$A_{\neq \ell}^\mu(x) = \sum_{\ell' \neq \ell} A_{\ell'}^\mu(x) = A^\mu(x) - A_\ell^\mu(x).$$

An evident difference between the BCT and DCT is that this single external field $A_{\neq \ell}^\mu(x)$ *is not the same for different charges*. But in both cases it is the EM field A^μ in (20.2) which can be measured by observation of the trajectory of a small well localized test charge placed into the system, because this field enters Newton's equations of motion for the test charge; the self-interaction does not affect its trajectory according to the results of Sects. 17.2.3 and 19.2. In the following Sect. 20.1 we compare the Lagrangians for the balanced charges and dressed charges theories.

20.1 Lagrangian Formalism for Dressed Charges Versus Balanced Charges

The Lagrangians for balanced charges and for dressed charges (with self-interactions) have the form (11.14). The difference between the Lagrangian for balanced charges and the Lagrangian for dressed charges is the value of the coefficient θ in (11.14), namely

$$\theta = 0 \quad \text{for dressed charges, } \theta = -1 \quad \text{for balanced charges.} \quad (20.1)$$

The analysis of Sect. 11.1 equally applies to both cases, in particular the field partition (11.58) and (11.60) holds both for dressed and balanced charges, namely

$$A^\mu = \sum_{\ell} A_{\ell}^{\mu}, \quad F^{\mu\nu} = \sum_{\ell} F_{\ell}^{\mu\nu}. \quad (20.2)$$

One has to take into account, though, that the above formula in the case of dressed charges is obtained under the assumption that the solution operator of Maxwell's equation is fixed, whereas for balanced charges it is postulated in (11.7). Another difference can be seen when comparing covariant derivatives (12.5) for balanced charges with (13.8) for dressed charges. The latter involve the same EM field potential $(\varphi, \mathbf{A})$ for all charges.

In both cases, the field equations derived from the Lagrangian are the NKG equations (12.8) for the current ψ_{ℓ} and the Maxwell equations (11.59) for the elementary EM fields $F_{\ell}^{\mu\nu}$ with the vector form (4.1)–(4.4) for every field. The Maxwell equations for the elementary EM fields (11.59) readily imply that the total field also satisfies the Maxwell equations (11.33). Observe that every NKG equation for a balanced charge (12.8), (11.59) indicates that the ℓ -th charge is driven by the potential $A_{\neq\ell}^{\nu}$ indicating that there is no EM self-interaction. At the same time, dressed charges involve the EM self-interaction, which constitutes the main difference between dressed and balanced charges.

Another difference comes from the EM part of the Lagrangian (11.14), which takes the form

$$\mathcal{L}_{\text{em}} = L_{\text{em}}(A) + \theta \sum_{\ell} L_{\text{em}}(A_{\ell}) \quad (20.3)$$

and is obviously different for $\theta = 0$ and $\theta = -1$. When $\theta = -1$ we obtain the Lagrangian $\mathcal{L}_{\text{BEM}}$ defined by (12.46) where (20.2) is assumed. Below we discuss this Lagrangian in more detail.

20.2 BEM Theory (Reduced Balanced Charge Theory)

Since the electromagnetic part of the Lagrangian for balanced charge theory differs from the electromagnetic part of the Lagrangian for dressed charges and from the classical EM Lagrangian $L_{\text{em}}(A)$, it is instructive to introduce an intermediate theory which involves individual EM fields A_{ℓ} , does not involve charge distributions ψ_{ℓ} , and the EM fields are not generated by the charges ψ_{ℓ} , but rather by prescribed currents J_{ℓ} . This simpler theory (BEM theory) can help to better understand the basic properties of EM interactions for balanced changes and the corresponding energy-momentum transfer in the space-time.

To introduce the BEM theory, we single out the Maxwell equations (11.59) for the EM fields generated by the ℓ -th charge from the NKG-Maxwell system (17.98)–(17.103). Now we consider the currents J_ℓ^ν in the Maxwell equations (11.59) as given functions of time and spatial variables. As a result, we obtain a linear system of Maxwell equations with the currents J_ℓ^ν as the sources. To introduce a Lagrangian formalism for the singled out equations, we introduce the following Lagrangian:

$$\mathcal{L}_{\text{BEMJ}} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{1}{16\pi} \sum_{1 \leq \ell \leq N} F_\ell^{\mu\nu} F_{\ell\mu\nu} - \frac{1}{c} \sum_{1 \leq \ell \leq N} J_{\ell\mu} A_{\neq\ell}^\mu. \quad (20.4)$$

Based on the representation (12.48), it can be alternatively written as

$$\begin{aligned} \mathcal{L}_{\text{BEMJ}} &= -\frac{1}{16\pi} \sum_{\{\ell, \ell'\}: \ell' \neq \ell} F_\ell^{\mu\nu} F_{\ell'} - \frac{1}{c} \sum_{1 \leq \ell \leq N} J_{\ell\mu} A_{\neq\ell}^\mu \\ &= -\frac{1}{16\pi} \sum_{1 \leq \ell \leq N} F_\ell^{\mu\nu} F_{\neq\ell\mu\nu} - \frac{1}{c} \sum_{1 \leq \ell \leq N} J_\ell^\nu A_{\neq\ell\nu}. \end{aligned} \quad (20.5)$$

Obviously, the Maxwell equations (11.59) can be derived as the Euler–Lagrange equations from the above Lagrangians. *The BEM theory does not involve balanced charges*, but we sometimes refer to the currents J_ℓ as currents generated by b-charges or simply as b-charges.

We would like to compare BEM theory with the Maxwell equations (11.33) where the current J^ν and the field $F^{\mu\nu}$ is expanded according to (11.31) and (11.58). The corresponding classical Lagrangian, similar to (12.47), is given by

$$\mathcal{L}_{\text{CEMJ}} = -\frac{1}{16\pi} \sum_{\ell, \ell'} F_\ell^{\mu\nu} F_{\ell'\mu\nu} - \frac{1}{c} \sum_{\ell, \ell'} J_{\ell\mu} A_{\ell'}^\mu, \quad (20.6)$$

and we call the corresponding EM theory classical EM (CEM) theory. The field equations for both theories are the same, but since the Lagrangians are different, the EM energy is defined differently.

20.2.1 BEM and CEM Theories

As we mentioned, the difference between the dressed charges and balanced charges is not only in the elimination of the self-interaction of the charge, but also in a different structure of the EM part of the Lagrangian (see (20.6) and (20.4)), and this difference affects the definitions of the energy of the EM field. For example in BEM a nonzero EM energy can only be in a system of more than one charge similarly to Wheeler–Feynman theory. Below we discuss only the differences in the EM aspects of the theories and do not consider the balanced charges themselves, but rather the

BEM theory with prescribed currents which has the same structure as the EM part of the balanced charges theory (BCT). We hope that a better understanding of such an intermediate EM theory can help to understand the BCT. *We stress again that the intermediate BEM theory does not involve balanced charges, they are replaced by classical prescribed charge distributions*; for brevity we still call them *b-charges*.

We show in Sect. 20.2.6 that in a certain sense the CEM theory (classical EM theory) can be interpreted as a limit of the BEM theory. This can already be seen by comparing formulas (12.7) and (12.6) for a large N , since these large sums differ just by one term. But the differences between the CEM and BEM theories become more pronounced for smaller systems with fewer b-charges. The BEM theory predictions concerning energy transfer can significantly deviate from those of the CEM theory in the following situations: (i) there are just a few b-charges; (ii) there is a large but highly coherent system of b-charges similar to those *collective, coherent systems* (superconducting ring, laser and more) described by C. Mead [242, p.5] in his “collective electrodynamics”. Based on our analysis, one may expect noticeable differences between CEM theory and BEM theory, for example when the ratio $1/n_\ell$ in (20.50) is not small. These differences can become more pronounced when $\mathcal{L}_{\text{em}}$ in (12.46) is comparable with the classical EM Lagrangian $\mathcal{L}_{\text{CEM}}$.

An important signature of the BEM theory differentiating it from the CEM theory is a *mechanism of negative radiation for certain prescribed currents*, i.e. a situation when the EM energy propagates at the speed of light toward the current source rather than away from it as we show in Sect. 20.2. This mechanism can conceivably work for a limited time in a system of several bound charges resulting in effective energy gain coming from matching energy loss of b-charges outside of this system. Such energy transfer is completely accounted for by “interacting” elementary EM fields generated by the involved b-charges.

20.2.2 Individual EM Energy-Momentum Tensors

The EnMT is a quadratic function of the fields and the currents which generate them. Therefore it is natural to take explicitly into account contributions to the electromagnetic EnMT from any given pair of currents which we call in this section “interacting charges” (though, of course, one current does not affect another).

The symmetric EnMT $T^{\mu\nu}$ of the system Lagrangian $\mathcal{L}$ is defined by formula (11.110) where $T^{\mu\nu}(L_\ell)$ are absent, $\theta = -1$, and A is defined by (11.7). As a result, we obtain the following expression for the total EM tensor:

$$\begin{aligned}\mathcal{E}^{\mu\nu} &= T^{\mu\nu}(L_{\text{em}}(A)) - \sum_{\ell} T^{\mu\nu}(L_{\text{em}}(A_\ell)) \\ &= \Theta^{\mu\nu}(F^{\mu\nu}) - \sum_{1 \leq \ell \leq N} \Theta^{\mu\nu}(F_\ell^{\mu\nu}),\end{aligned}\tag{20.7}$$

where the individual EnMT are given by (11.100), and individual fields $F_\ell^{\mu\nu}$ satisfy (11.59) with prescribed currents. Since the expression (9.13) for $T^{\mu\nu}(L_{\text{em}}(A)) = \Theta^{\mu\nu}(F)$ is quadratic with respect to A , we can expand $T^{\mu\nu}(L_{\text{em}}(A))$ and $\mathcal{E}^{\mu\nu}$ as follows:

$$T^{\mu\nu}(L_{\text{em}}(A)) = \Theta^{\mu\nu}(F^{\mu\nu}) = \sum_{\ell', \ell} \mathcal{E}_{\ell\ell'}^{\mu\nu}, \quad (20.8)$$

$$\mathcal{E}^{\mu\nu} = \sum_{\ell', \ell: \ell' \neq \ell} \mathcal{E}_{\ell\ell'}^{\mu\nu}, \quad (20.9)$$

with the individual EnMT components $\mathcal{E}_{\ell\ell'}^{\mu\nu}$ given by

$$\mathcal{E}_{\ell\ell'}^{\mu\nu} = \frac{1}{4\pi} \left(g^{\mu\gamma} F_{\ell\gamma\xi} F_{\ell'}^{\xi\nu} + \frac{1}{4} g^{\mu\nu} F_{\ell\gamma\xi} F_{\ell'}^{\gamma\xi} \right). \quad (20.10)$$

Obviously, we can rewrite (20.9) as

$$\mathcal{E}^{\mu\nu} = \sum_{1 \leq \ell \leq N} \mathcal{E}_{\neq \ell}^{\mu\nu}$$

where

$$\mathcal{E}_{\neq \ell}^{\mu\nu} = \sum_{\ell' \neq \ell} \mathcal{E}_{\ell\ell'}^{\mu\nu} = \frac{1}{4\pi} \left(g^{\mu\gamma} F_{\ell\gamma\xi} F_{\neq \ell}^{\xi\nu} + \frac{1}{4} g^{\mu\nu} F_{\ell\gamma\xi} F_{\neq \ell}^{\gamma\xi} \right). \quad (20.11)$$

The relation between the classical $\Theta^{\mu\nu}$ and the reduced EnMTs $\mathcal{E}^{\mu\nu}$ is given by (20.7).

Based on the formula (20.10) for the components $\mathcal{E}_{\ell\ell'}^{\mu\nu}$ of the symmetric EnMT, we can represent their entries in terms of the fields $\mathbf{E}_\ell$ and $\mathbf{B}_\ell$, namely

$$u_{\text{em}\ell\ell'} = \mathcal{E}_{\ell\ell'}^{00} = \frac{1}{8\pi} (\mathbf{E}_\ell \cdot \mathbf{E}_{\ell'} + \mathbf{B}_\ell \cdot \mathbf{B}_{\ell'}), \quad (20.12)$$

$$cP_{i\ell\ell'} = \mathcal{E}_{\ell\ell'}^{0i} = \mathcal{E}_{\ell\ell'}^{i0} = \frac{1}{4\pi} \mathbf{E}_\ell \times \mathbf{B}_{\ell'},$$

$$\mathcal{E}_{\ell\ell'}^{ij} = -\frac{1}{8\pi} \left[E_{\ell i} E_{\ell' j} + B_{\ell i} B_{\ell' j} - \frac{1}{2} \delta_{ij} (\mathbf{E}_\ell \cdot \mathbf{E}_{\ell'} + \mathbf{B}_\ell \cdot \mathbf{B}_{\ell'}) \right], \quad (20.13)$$

$$\mathcal{E}_{\ell\ell'}^{\alpha\beta} = \begin{bmatrix} u_{\text{em}\ell\ell'} & c\mathbf{P}_{\text{em}\ell\ell'} \\ c\mathbf{P}_{\text{em}\ell\ell'} & -\tau_{\ell\ell'ij} \end{bmatrix}, \quad (20.14)$$

which are evidently similar to (9.14)–(9.16). In particular, we have the following expression for the Poynting vectors similar to (9.23) for the pair of two individual fields:

$$\mathbf{S}_{\ell\ell'} = c^2 \mathbf{P}_{\text{em}\ell\ell'} = \frac{c}{4\pi} \mathbf{E}_\ell \times \mathbf{B}_{\ell'}. \quad (20.15)$$

It readily follows from relations (20.20) and (20.11) that the interaction EnMT $\mathfrak{E}_{\{\ell,\ell'\}}^{\mu\nu}$ has the following representation in terms of the classical EnMT $\Theta^{\mu\nu}$:

$$\mathfrak{E}_{\{\ell,\ell'\}}^{\mu\nu} = \Theta^{\mu\nu} (F_\ell^{\mu\nu} + F_{\ell'}^{\mu\nu}) - \Theta^{\mu\nu} (F_\ell^{\mu\nu}) - \Theta^{\mu\nu} (F_{\ell'}^{\mu\nu}). \quad (20.16)$$

The representation (20.16) is evidently a particular case of (20.7) for two individual fields.

20.2.3 Individual EnMT Conservation Laws

Now we introduce conservation laws for individual EnMT $\mathfrak{E}_{\ell\ell'}^{\mu\nu}$ defined by (20.10). Combining the Maxwell field equations (11.59) for $F^{\mu\xi} = \partial^\mu A^\xi - \partial^\xi A^\mu$, an elementary identity

$$\partial^\mu F^{\xi\nu} - \partial^\xi F^{\mu\nu} = \partial^\nu F^{\mu\xi} \quad (20.17)$$

and the antisymmetry of the EM field tensors $F_\ell^{\mu\nu}$, we obtain

$$\partial_\mu \mathfrak{E}_{\ell\ell'}^{\mu\nu} = \frac{1}{c} J_{\ell\xi} F_{\ell'}^{\xi\nu} + \frac{1}{4\pi} \left[-\frac{1}{2} F_{\ell\mu\xi} \partial^\nu F_{\ell'}^{\mu\xi} + \frac{1}{4} \partial^\nu (F_{\ell\gamma\xi} F_{\ell'}^{\gamma\xi}) \right]. \quad (20.18)$$

The relations (20.8) readily imply the total energy-momentum conservation law

$$\partial_\mu \mathfrak{E}^{\mu\nu} = \frac{1}{2} \sum_{\ell' \neq \ell} \partial_\mu (\mathfrak{E}_{\ell\ell'}^{\mu\nu} + \mathfrak{E}_{\ell'\ell}^{\mu\nu}) = -\frac{1}{c} \sum_{1 \leq \ell \leq N} J_\xi^\ell F_{\neq \ell}^{\nu\xi}, \quad (20.19)$$

with the sum of the negative of the Lorentz forces in the right-hand side.

Let us introduce an *elementary EM field interaction energy for the pair* $\{\ell, \ell'\}$

$$\mathfrak{E}_{\{\ell,\ell'\}}^{\mu\nu} = \mathfrak{E}_{\ell\ell'}^{\mu\nu} + \mathfrak{E}_{\ell'\ell}^{\mu\nu}, \quad (20.20)$$

which in view of (20.18) satisfies the following *elementary energy-momentum conservation laws*:

$$\partial_\mu \mathfrak{E}_{\{\ell,\ell'\}}^{\mu\nu} = -\frac{1}{c} (J_{\ell\xi} F_{\ell'}^{\nu\xi} + J_{\ell'\xi} F_\ell^{\nu\xi}) \quad (20.21)$$

with the right-hand side being the negative of the sum of the corresponding Lorentz force density.

In view of (20.12), (20.13) and (20.15), the entries of the tensor $\mathfrak{E}_{\{\ell,\ell'\}}^{\mu\nu}$ are as follows:

$$\begin{aligned}
u_{\text{em}\{\ell,\ell'\}} &= \mathcal{E}_{\{\ell,\ell'\}}^{00} = \frac{1}{4\pi} (\mathbf{E}_\ell \cdot \mathbf{E}_{\ell'} + \mathbf{B}_\ell \cdot \mathbf{B}_{\ell'}), \\
c\mathbf{P}_{\text{em}\{\ell,\ell'\}i} &= \mathcal{E}_{\{\ell,\ell'\}}^{0i} = \mathcal{E}_{\{\ell,\ell'\}}^{i0} = \frac{1}{4\pi} (\mathbf{E}_\ell \times \mathbf{B}_{\ell'} + \mathbf{E}_{\ell'} \times \mathbf{B}_\ell), \\
\mathbf{S}_{\{\ell,\ell'\}} &= \frac{c}{4\pi} (\mathbf{E}_\ell \times \mathbf{B}_{\ell'} + \mathbf{E}_{\ell'} \times \mathbf{B}_\ell),
\end{aligned} \tag{20.22}$$

$$\begin{aligned}
\tau_{\{\ell,\ell'\}ij} &= \mathcal{E}_{\{\ell,\ell'\}}^{ij} \\
&= -\frac{1}{4\pi} \left[E_{\ell i} E_{\ell' j} + B_{\ell i} B_{\ell' j} - \frac{1}{2} \delta_{ij} (\mathbf{E}_\ell \cdot \mathbf{E}_{\ell'} + \mathbf{B}_\ell \cdot \mathbf{B}_{\ell'}) \right].
\end{aligned} \tag{20.23}$$

The expressions (20.22) and (20.23) can be alternatively derived from their classical counterparts (9.14), (9.15) and (9.23) based on relation (20.16).

Notice also that, as we can see from the expression (20.22) for the interaction energy density for a pair of fields, the interaction energy density can be negative. The latter is analogous to the negative sign of the electrostatic energy for two classical point charges when one of them is positive and the other one is negative, and the interaction energy is defined to be zero when charges are separated by infinite distance. This way to calibrate the interaction energy is in line with defining the interaction energy as work done to assemble a system of charges from the state when they don't interact, that is, when they are separated by infinite distance.

The vector form of the elementary conservation law (20.21) is similar to the conservation law (9.20), [180, Section 6.8], namely

$$\frac{1}{c} [\partial_t u_{\text{em}\{\ell,\ell'\}} + \nabla \cdot \mathbf{S}_{\{\ell,\ell'\}}] = -\frac{1}{c} [\mathbf{J}_\ell \cdot \mathbf{E}_{\ell'} + \mathbf{J}_{\ell'} \cdot \mathbf{E}_\ell], \tag{20.24}$$

$$\partial_t \mathbf{P}_{\text{em}\{\ell,\ell'\}i} - \partial_j \tau_{\{\ell,\ell'\}ij} = f_{\{\ell,\ell'\}i}, \quad i = 1, 2, 3, \tag{20.25}$$

where $f_{\{\ell,\ell'\}i}$ is the Lorentz force density satisfying

$$f_{\{\ell,\ell'\}i} = f_i^{\ell\ell'} + f_i^{\ell'\ell}, \quad f_i^{\ell\ell'} = \rho^\ell E_i^{\ell'} + \frac{1}{c} (\mathbf{J}_\ell \times \mathbf{B}_{\ell'})_i. \tag{20.26}$$

The system of individual EM fields generated by prescribed currents satisfies the elementary conservation laws (20.24), (20.25), which can also be directly derived from the Maxwell equations (4.3), (4.4) for $\mathbf{E}_\ell$, $\mathbf{B}_\ell$. Indeed, using the general vector identity (43.8) and the Maxwell equations for the indices ℓ and ℓ' , we obtain

$$\begin{aligned}
\nabla \cdot (\mathbf{E}_\ell \times \mathbf{B}_{\ell'}) &= \mathbf{B}_{\ell'} \cdot (\nabla \times \mathbf{E}_\ell) - \mathbf{E}_\ell \cdot (\nabla \times \mathbf{B}_{\ell'}) \\
&= -\mathbf{B}_{\ell'} \cdot \frac{1}{c} \partial_t \mathbf{B}_\ell - \mathbf{E}_\ell \cdot \left(\frac{1}{c} \partial_t \mathbf{E}_{\ell'} + \frac{4\pi}{c} \mathbf{J}_{\ell'} \right).
\end{aligned} \tag{20.27}$$

Adding then to the identity (20.27) a similar one obtained from it by swapping indices ℓ and ℓ' , we obtain the elementary energy conservation law (20.24). A similar direct derivation is possible for the elementary momentum conservation law (20.25).

Observe also that according to the individual conservation laws (20.21), the energy and the momentum are assigned not only to the elementary EM fields by themselves but also to their interacting pairs.

Note that the conservation law (20.24) can easily be derived from the classical energy conservation law (the Poynting theorem) as follows. Let us consider the classical energy conservation law associated with three Maxwell equations (11.59): (i) for the ℓ -th EM field, (ii) for the ℓ' -th EM field and (iii) for their sum. Then, subtracting from the conservation law for the sum of the fields the sum of the conservation laws for the ℓ -th and ℓ' -th fields, we obtain exactly the conservation law (20.24).

20.2.4 Elementary Currents for Point Charges

The BEM theory describes fields and radiation phenomena with prescribed elementary currents. Now we look at the case where they are caused by a system of prescribed point charges. In the case of point charges, we use the acceleration fields $\mathbf{E}_a(t, \mathbf{x})$ and $\mathbf{B}_a(t, \mathbf{x})$ defined by formulas (4.66), (4.67) to find the time-averaged radiation power. Consequently, for every pair $\{\ell, \ell'\}$ of b-charges we have

$$\mathbf{S}_{\ell\ell'} = \frac{c}{4\pi} \mathbf{E}_{\ell a} \times \mathbf{B}_{\ell' a} \text{ and } \mathbf{S}_{\ell'\ell} = \frac{c}{4\pi} \mathbf{E}_{\ell' a} \times \mathbf{B}_{\ell a}. \quad (20.28)$$

Suppose that one of the two b-charges, say the ℓ -th charge, is at rest or moves uniformly implying that $\dot{\beta}_\ell = 0$. Then it follows from formulas (4.66), (4.67) that $\mathbf{E}_{\ell a} = 0$ and $\mathbf{B}_{\ell a} = 0$ implying that $\mathbf{S}_{\ell\ell'}$ and $\mathbf{S}_{\ell'\ell}$ vanish, namely

$$\mathbf{S}_{\ell\ell'} = \mathbf{S}_{\ell'\ell} = 0 \text{ if } \dot{\beta}_\ell = 0 \text{ or } \dot{\beta}_{\ell'} = 0. \quad (20.29)$$

Suppose that the position functions $\mathbf{r}_\ell(t)$ and $\mathbf{r}_{\ell'}(t)$ of the corresponding b-charges are almost periodic functions as described in Chap. 42, and that $\text{mod}(\mathbf{r}_\ell(t))$ and $\text{mod}(\mathbf{r}_{\ell'}(t))$ have no common frequencies. Then the time-averaged flux $\langle \mathbf{S}_{\ell\ell'} \rangle$ is exactly zero, i.e.

$$\langle \mathbf{S}_{\ell\ell'} \rangle = 0 \text{ if } \text{mod}(\mathbf{r}_\ell(t)) \cap \text{mod}(\mathbf{r}_{\ell'}(t)) = \emptyset. \quad (20.30)$$

Now let us compare this estimate with a similar computation in the framework of the CEM theory with the total field $\mathbf{E} = \mathbf{E}_{\ell' a} + \mathbf{E}_{\ell a}$, $\mathbf{B} = \mathbf{B}_{\ell' a} + \mathbf{B}_{\ell a}$ or in the framework of the BEM theory with two prescribed charges and a test charge. In the framework of the BEM theory, if we consider the action of the system of two charges on a test charge or on a distant system of test charges which does not noticeably affect the radiation of the two charges, the field of two charges turns into an external field given

by the same formula $\mathbf{E} = \mathbf{E}_{\ell'a} + \mathbf{E}_{\ell a}$, $\mathbf{B} = \mathbf{B}_{\ell'a} + \mathbf{B}_{\ell a}$. The radiation power at large distances generated by two charges in accordance with (4.65) is expressed by the formula

$$\mathbf{S} = \frac{c}{4\pi} (\mathbf{E}_{\ell'a} + \mathbf{E}_{\ell a}) \times (\mathbf{B}_{\ell a} + \mathbf{B}_{\ell'a}). \quad (20.31)$$

If the ℓ -th charge is at rest, $\mathbf{B}_{\ell a} = 0$ and $\mathbf{E}_{\ell a} = 0$, and we obtain the leading part and its time average

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E}_{\ell'a} \times \mathbf{B}_{\ell'a}, \quad \langle \mathbf{S} \rangle = \frac{c}{4\pi} \langle \mathbf{E}_{\ell'a} \times \mathbf{B}_{\ell'a} \rangle, \quad (20.32)$$

which differs from (20.30) and (20.29).

Note that formulas (20.30) and (20.29) describe the radiation of EM interaction energy between the two charges, and it is not the same as the EM radiation of the fields which act on test charges. One may ask why we are interested in the interaction energy. The answer is that this is the energy which is derived from the Lagrangian and the interaction energy is conserved when the system is closed. Of course the system where the motion of sources is prescribed is not closed, and there must be an external source of energy to provide for the prescribed currents and for the radiation which acts on test charges.

Let us consider in conclusion the case where both the charges move slowly, hence $|\beta_\ell| \ll 1$, $|\beta_{\ell'}| \ll 1$. Notice that the relations (4.68) defining the point charge EM fields are similar to the dipole expressions (5.26) and can be obtained from them by a substitution $\ddot{\mathbf{p}}_\ell(t) \rightarrow cq\dot{\mathbf{v}}_\ell(t)$. Consequently, expressions for the energy flux and the radiated power for the pair of slowly moving charges can be readily obtained from the dipole expressions (20.33) by a substitution $\ddot{\mathbf{p}}_\ell(t) \rightarrow cq\dot{\mathbf{v}}_\ell(t)$ with an additional correcting asymptotic factor $\sigma_\ell = (1 + O(|\beta_\ell|))$. In particular,

$$\begin{aligned} S_{\ell\ell'} &= \frac{q^2}{4\pi c} \frac{\dot{\mathbf{v}}_\ell(t) \cdot \dot{\mathbf{v}}_{\ell'}(t) - (\dot{\mathbf{v}}_\ell(t) \cdot \hat{\mathbf{R}})(\hat{\mathbf{R}} \cdot \dot{\mathbf{v}}_{\ell'}(t))}{|\mathbf{R}|^2} \hat{\mathbf{R}} \sigma_{\ell\ell'} \\ &= \frac{q^2}{4\pi c} \frac{[(\dot{\mathbf{v}}_\ell(t) \times \hat{\mathbf{R}}) \cdot (\dot{\mathbf{v}}_{\ell'} \times \hat{\mathbf{R}})]}{4\pi c^3 |\mathbf{R}|^2} \hat{\mathbf{R}} \sigma_{\ell\ell'} \end{aligned}$$

where

$$\sigma_{\ell\ell'} = (1 + O(|\beta_\ell|)) (1 + O(|\beta_{\ell'}|)).$$

20.2.5 Energy Flux for a Pair of Elementary Dipoles

Now we consider b-charges described by elementary dipoles. Using expression (5.26) for the EM fields and the vector identities (43.15) and (43.16), we consequently obtain the following formulas for the energy flux $\mathbf{S}_{\ell\ell'}$ (20.15) of the pair of “interacting

b-charges” and the corresponding total powers $P_{\ell\ell'}$ and $P_{\{\ell,\ell'\}}$ radiated through a sphere centered at the dipole location:

$$\begin{aligned} \mathbf{S}_{\ell\ell'} &= \frac{c}{4\pi} \mathbf{E}_{\ell} \times \mathbf{B}_{\ell'} \\ &= \left[\ddot{\mathbf{p}}_{\ell}(t_0) \cdot \ddot{\mathbf{p}}_{\ell'}(t_0) - \left(\ddot{\mathbf{p}}_{\ell}(t_0) \cdot \hat{\mathbf{R}} \right) \left(\hat{\mathbf{R}} \cdot \ddot{\mathbf{p}}_{\ell'}(t_0) \right) \right] \frac{1}{4\pi c^3 |\mathbf{R}|^2} \hat{\mathbf{R}} \\ &= \left[\left(\ddot{\mathbf{p}}_{\ell}(t_0) \times \hat{\mathbf{R}} \right) \cdot \left(\ddot{\mathbf{p}}_{\ell'}(t_0) \times \hat{\mathbf{R}} \right) \right] \frac{1}{4\pi c^3 |\mathbf{R}|^2} \hat{\mathbf{R}}, \end{aligned} \quad (20.33)$$

$$\begin{aligned} P_{\ell\ell'} &= \int_{|\mathbf{x}|=|\mathbf{R}|} \mathbf{S}_{\ell\ell'} \cdot d\boldsymbol{\sigma} = \frac{2}{3c^3} \ddot{\mathbf{p}}_{\ell}(t_0) \cdot \ddot{\mathbf{p}}_{\ell'}(t_0), \\ P_{\{\ell,\ell'\}} &= P^{\ell\ell'} + P^{\ell'\ell} = \frac{4}{3c^3} \ddot{\mathbf{p}}_{\ell}(t_0) \cdot \ddot{\mathbf{p}}_{\ell'}(t_0). \end{aligned} \quad (20.34)$$

Let us assume now that the dipole functions $\mathbf{p}_{\ell}(t)$ depend on time t almost periodically as in Chap. 42. Then the representation (20.34) together with relations (20.22), (5.17) and (42.10) consequently imply the following formulas for the time-averaged radiated powers

$$\begin{aligned} \langle \mathbf{S}_{\ell\ell'} \rangle &= \frac{\left[\left\langle \left(\ddot{\mathbf{p}}_{\ell} \times \hat{\mathbf{R}} \right) \cdot \left(\ddot{\mathbf{p}}_{\ell'} \times \hat{\mathbf{R}} \right) \right\rangle \right]}{4\pi c^3 |\mathbf{R}|^2} \hat{\mathbf{R}} \\ &= \frac{\hat{\mathbf{R}}}{8\pi c^3 |\mathbf{R}|^2} \sum_{\omega \in \Lambda_{\mathbf{p}_{\ell}} \cap \Lambda_{\mathbf{p}_{\ell'}}} \omega^4 \operatorname{Re} \left\{ \left(\mathbf{p}_{\ell\omega} \times \hat{\mathbf{R}} \right) \left(\mathbf{p}_{\ell'\omega}^* \times \hat{\mathbf{R}} \right) \right\}, \end{aligned} \quad (20.35)$$

$$\langle P_{\ell\ell'} \rangle = \frac{2 \langle \ddot{\mathbf{p}}_{\ell} \cdot \ddot{\mathbf{p}}_{\ell'} \rangle}{3c^3} = \sum_{\omega \in \Lambda_{\mathbf{p}_{\ell}} \cap \Lambda_{\mathbf{p}_{\ell'}}} \frac{\omega^4}{3c^3} \operatorname{Re} \left\{ \mathbf{p}_{\ell\omega} \cdot \mathbf{p}_{\ell'\omega}^* \right\}, \quad (20.36)$$

$$\langle P_{\{\ell,\ell'\}} \rangle = \frac{4 \langle \ddot{\mathbf{p}}_{\ell} \cdot \ddot{\mathbf{p}}_{\ell'} \rangle}{3c^3} = \sum_{\omega \in \Lambda_{\mathbf{p}_{\ell}} \cap \Lambda_{\mathbf{p}_{\ell'}}} \frac{2\omega^4}{3c^3} \operatorname{Re} \left\{ \mathbf{p}_{\ell\omega} \cdot \mathbf{p}_{\ell'\omega}^* \right\}, \quad (20.37)$$

where $\Lambda_{\mathbf{p}_{\ell}}$ and $\Lambda_{\mathbf{p}_{\ell'}}$ are, respectively, the frequency spectra of $\mathbf{p}_{\ell}(t)$ and $\mathbf{p}_{\ell'}(t)$. It readily follows from the formula (20.37) that *the time-averaged radiated power $\langle P_{\{\ell,\ell'\}} \rangle$ can take any real value: negative, zero or positive*. Indeed, according to the formulas (20.37), (42.10), $\langle P_{\{\ell,\ell'\}} \rangle$ vanishes if the frequency spectra $\Lambda_{\mathbf{p}_{\ell}}$ and $\Lambda_{\mathbf{p}_{\ell'}}$ don't have any common frequencies, i.e.

$$\langle P_{\{\ell,\ell'\}} \rangle = \frac{4 \langle \ddot{\mathbf{p}}_{\ell}(t) \cdot \ddot{\mathbf{p}}_{\ell'}(t) \rangle}{3c^3} = 0 \text{ if } \Lambda_{\mathbf{p}_{\ell}} \cap \Lambda_{\mathbf{p}_{\ell'}} = \emptyset. \quad (20.38)$$

In particular, the relation (20.38) shows that if both the ℓ -th and the ℓ' -th b-charges are monochromatic with different frequencies, then the time-averaged radiated power is exactly zero or, in other words, there is no radiation. The formulas (20.37), (42.10) readily imply

$$\langle P_{\{\ell, \ell'\}} \rangle = \frac{2}{3c^3} \sum_{\omega \in \Lambda_{\mathbf{p}_\ell}} \omega^4 |\mathbf{p}_{\ell\omega}|^2 > 0 \text{ if } \mathbf{p}_{\ell'}(t) = \mathbf{p}_\ell(t), \quad (20.39)$$

$$\langle P_{\{\ell, \ell'\}} \rangle = -\frac{2}{3c^3} \sum_{\omega \in \Lambda_{\mathbf{p}_\ell}} \omega^4 |\mathbf{p}_{\ell\omega}|^2 < 0 \text{ if } \mathbf{p}_{\ell'}(t) = -\mathbf{p}_\ell(t). \quad (20.40)$$

Evidently the relation (20.40) describes a situation when for a given pair $\{\ell, \ell'\}$ the time-averaged radiated power $\langle P_{\{\ell, \ell'\}} \rangle$ is negative, that is, the radiated energy propagates with the speed of light toward the source rather than away from it.

20.2.5.1 Radiated Power of a System of Identical Dipoles

Now we want to see how classical EM phenomena including radiation are present within the BEM theory.

As a basis for a simple comparison of radiative properties in the BEM and CEM theories let us consider a system of $N > 1$ b-charges with every charge having the same dipole moment $\mathbf{p}(t)$. Then, in view of (20.39), we have for the system the following time-average radiated power:

$$\begin{aligned} \langle P_{\text{BEM}} \rangle &= \sum_{\ell' \neq \ell} \langle P^{\{\ell, \ell'\}} \rangle \\ N(N-1) \frac{2 \langle \ddot{\mathbf{p}}_\ell(t) \cdot \ddot{\mathbf{p}}_\ell(t) \rangle}{3c^3} &= N(N-1) \frac{1}{3c^3} \sum_{\omega \in \Lambda_{\mathbf{p}}} \omega^4 |\mathbf{p}_\omega|^2. \end{aligned} \quad (20.41)$$

In the CEM theory, a similar system of N identical dipoles $\mathbf{p}$ has an effective dipole moment $N\mathbf{p}$. Substituting this number for $\mathbf{p}$ in the expression (20.36) we get

$$\langle P_{\text{CEM}} \rangle = N^2 \frac{2 \langle \ddot{\mathbf{p}}_\ell(t) \cdot \ddot{\mathbf{p}}_\ell(t) \rangle}{3c^3} = N^2 \frac{1}{3c^3} \sum_{\omega \in \Lambda_{\mathbf{p}}} \omega^4 |\mathbf{p}_\omega|^2. \quad (20.42)$$

Relating representations (20.41) and (20.42), we readily obtain that

$$\langle P_{\text{BEM}} \rangle = \left(1 - \frac{1}{N}\right) \langle P_{\text{CEM}} \rangle. \quad (20.43)$$

20.2.6 The Lagrangian for Clusters of Charges

We would like to show here that EM theory of dressed charges can be viewed as a limit case of the BEM-type theory. Particularly, we are interested in the following: how the single classical EM field is modeled by elementary EM fields as in the theory of balanced charges.

Here we introduce the Lagrangian for a cluster of identical charges. We make a comparison of two Lagrangians for N clusters of charges with n_ℓ particles within every cluster which are formally derived from the dressed charges and balanced charges theories respectively. To this end, we consider first a Lagrangian as in the dressed charges theory with $n_1 + \dots + n_N$ charges and fixed nonlinearities $G_{\ell w}$. As an example of a relation between the classical EM field and elementary EM fields, we consider clusters of many tightly bound identical balanced charges, every cluster is labeled by index ℓ and involves n_ℓ charges with charge densities $\psi_{\bar{\ell}}$ where

$$\bar{\ell} = (\ell, s), \quad s = 1, \dots, N_\ell.$$

The cluster of b-charges is defined as follows: for every ℓ we introduce n_ℓ identical b-charges distributions and elementary EM potentials:

$$\psi_{(\ell,s)} = \psi_\ell, \quad A_{(\ell,s)}^\mu = A_\ell^\mu, \quad F_{(\ell,s)}^{\mu\nu} = F_\ell^{\mu\nu}, \quad (20.44)$$

where

$$1 \leq \ell \leq N, \quad 1 \leq s \leq n_\ell,$$

and

$$m_{\ell,s} = m_\ell, \quad q_{\ell,s} = q_\ell, \quad s = 1, \dots, N_\ell. \quad (20.45)$$

The system Lagrangian (11.14) which describes $n_1 + \dots + n_N$ charges can be written in the form

$$\mathcal{L} = L_{\text{ch}} + \mathcal{L}_{\text{em}} \quad (20.46)$$

where L_{ch} is the part which describes the charges,

$$L_{\text{ch}} = \sum_{\bar{\ell}} L_\ell (\{\psi_{\bar{\ell}}, \{\psi_{\bar{\ell};\mu}\}\}) = \sum_{1 \leq \ell \leq N} \sum_{1 \leq s_\ell \leq N_\ell} L_\ell \left(\psi^{(\ell,s_\ell)}, \psi_{;\mu}^{(\ell,s_\ell)} \right). \quad (20.47)$$

The remaining EM part $\mathcal{L}_{\text{em}}$ of the Lagrangian, which describes the EM fields, may take a different form in two subcases:

$$\mathcal{L}_{\text{em}} = \mathcal{L}_{\text{CEM}} \quad \text{or} \quad \mathcal{L}_{\text{em}} = \mathcal{L}_{\text{BEM}}$$

where

$$\begin{aligned}\mathcal{L}_{\text{CEM}} &= -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} = -\frac{1}{16\pi} \sum_{\ell', \ell} n_{\ell'} n_{\ell} F_{\ell}^{\mu\nu} F_{\ell'}^{\mu\nu} \\ &= -\frac{1}{16\pi} \sum_{\ell' \neq \ell} n_{\ell'} n_{\ell} F_{\ell}^{\mu\nu} F_{\mu\nu}^{\ell'} - \frac{1}{16\pi} \sum_{\ell=1}^N n_{\ell}^2 F_{\ell}^{\mu\nu} F_{\ell\mu\nu}.\end{aligned}\quad (20.48)$$

We use (20.2) to obtain the above formula from (11.17).

Now we compare the Lagrangians in the BEM and CEM theories. For the BEM Lagrangian (20.3) takes the form (12.48), namely

$$\begin{aligned}\mathcal{L}_{\text{BEM}} &= -\frac{1}{16\pi} \sum_{\{(\ell, s), (\ell', s') : (\ell', s') \neq (\ell, s)\}} F_{(\ell, s)}^{\mu\nu} F_{(\ell', s')\mu\nu} \\ &= -\frac{1}{16\pi} \sum_{\ell' \neq \ell} n_{\ell'} n_{\ell} F_{\ell}^{\mu\nu} F_{\mu\nu}^{\ell'} - \frac{1}{16\pi} \sum_{\ell} n_{\ell} (n_{\ell} - 1) F_{\ell}^{\mu\nu} F_{\mu\nu}^{\ell}.\end{aligned}\quad (20.49)$$

The difference between $\mathcal{L}_{\text{CEM}}$ and $\mathcal{L}_{\text{BEM}}$ can be attributed to interactions inside every cluster. In particular, we have equalities

$$\mathcal{L}_{\text{CEM}\ell} = n_{\ell}^2 F_{\ell}^{\mu\nu} F_{\mu\nu}^{\ell}, \quad \mathcal{L}_{\text{BEM}\ell} = n_{\ell} (n_{\ell} - 1) F_{\ell}^{\mu\nu} F_{\mu\nu}^{\ell},$$

which readily imply the following expression for the relative difference

$$\mathcal{L}_{\text{BEM}\ell} / \mathcal{L}_{\text{CEM}\ell} - 1 = 1/n_{\ell}. \quad (20.50)$$

The difference (20.50) evidently becomes small as the number n_{ℓ} of particles in the cluster becomes large.

To summarize, EM theory with a single EM field (as in dressed charges theory) can be formally derived from the theory of clusters of balanced charges as a limit obtained by binding together n_{ℓ} identical particles with $n_{\ell} \rightarrow \infty$.

20.2.6.1 A Cluster of Dressed Charges in Equilibrium

A cluster formed by dressed charges described by the Lagrangian (20.46), (20.48) in an equilibrium is described by the real-valued functions ψ_{ℓ} and $\varphi_{\ell, s}$ which satisfy the following system of equations:

$$-\Delta\varphi_{\ell w} = 4\pi \sum_{s=1}^{N_{\ell}} q_{\ell, s} \left(1 - \frac{q_{\ell, s} \varphi_{\ell w}}{m_{\ell, s} c^2} \right) (\psi_{\ell, s})^2, \quad (20.51)$$

$$-\Delta\psi_{\ell, s} + \frac{m_{\ell, s} \varphi_{\ell w}^{\ell}}{\chi^2} q^{\ell, s} \left(2 - \frac{q \varphi_{\ell w}}{m_{\ell, s} c^2} \right) \psi_{\ell, s} + G'_{\text{w}} (|\psi_{\ell, s}|^2) \psi_{\ell, s} = 0. \quad (20.52)$$

Note that the dressed charge theory developed in [15] and described in Sect. 13.1, Chap. 19 coincides with the case where every cluster contains exactly one charge. In this case the charge equilibrium equation has the form

$$-\Delta\dot{\varphi}_\ell = 4\pi q_\ell \left(1 - \frac{q_\ell \dot{\varphi}_\ell}{mc^2}\right) \dot{\psi}_\ell^2, \quad (20.53)$$

$$-\Delta\dot{\psi}_\ell + \frac{m_\ell \dot{\varphi}_\ell}{\chi^2} q_\ell \left(2 - \frac{q_\ell \dot{\varphi}_\ell}{m_\ell c^2}\right) \dot{\psi}_\ell + G'_w \left(|\dot{\psi}_\ell|^2\right) \dot{\psi}_\ell = 0. \quad (20.54)$$

Comparing with (14.120) and (14.121) and setting

$$\begin{aligned} \varphi_{\ell w} &= n_\ell \dot{\varphi}_\ell, & q_{\ell w} &= n_\ell q_\ell, & m_{\ell w} &= n_\ell^2 m_\ell, \\ \psi_{\ell, s} &= \dot{\psi}_\ell = \psi_{\ell w}, & s &= 1, \dots, n_\ell, \end{aligned} \quad (20.55)$$

we then find that φ_w^ℓ satisfies the following equation

$$-\Delta\varphi_w^\ell = 4\pi q_{\ell w} \left(1 - \frac{q_{\ell w} \varphi_{\ell w}}{m_{\ell w} c^2}\right) \dot{\psi}_\ell^2, \quad (20.56)$$

which has exactly the same form as (20.53) in the dressed charge theory. The charge normalization condition is also fulfilled. Equations (14.120) and (20.54) can be rewritten in the form

$$-\Delta\dot{\psi}_\ell + \frac{m_{\ell w} \varphi_{\ell w}}{N_\ell^4 \chi^2} q_{\ell w} \left(2 - \frac{q_{\ell w} \varphi_{\ell w}}{m_{\ell w} c^2}\right) \dot{\psi}_\ell + G'_w \left(|\dot{\psi}_\ell|^2\right) \dot{\psi}_\ell = 0. \quad (20.57)$$

This equation has the form of (14.121) for $\dot{\psi}_\ell = \dot{\psi}_w^\ell$ if we set

$$\chi_w = n_\ell^2 \chi. \quad (20.58)$$

Observe that relations (20.55)–(20.60) readily imply the following identities for the model constants:

$$\kappa_w^\ell = \frac{m_w^\ell c}{\chi_w} = \frac{m_\ell c}{\chi} = \kappa_\ell. \quad (20.59)$$

If we introduce the Lagrangian $\mathcal{L}_w \left(\{\psi_w^\ell\}, \{\psi_{w;\mu}^\ell\}\right)$ with (i) constants defined by (20.55), (20.58) and (ii) EM potential defined by

$$A_w^\mu = \sum_{\ell, s} A_{\ell, s}^\mu = \sum_{\ell} A_{\ell w}^\mu, \quad A_{\ell w}^\mu = n_\ell A_\ell^\mu, \quad 1 \leq \ell \leq N, \quad (20.60)$$

then the Euler–Lagrange equations for the Lagrangian $\mathcal{L} \left(\{\psi_{(\ell, s_\ell)}\}, \{\psi_{(\ell, s_\ell); \mu}\}\right)$ defined by (20.47) with restrictions (20.45) are equivalent to the Euler–Lagrange equations

for $\mathcal{L}_w$. Hence, we can conclude that the introduction of clusters of charges can be described by a proper rescaling of the constants of the dressed charge theory. Note that the quadratic dependence of m_w^ℓ on the number n_ℓ of charges in the cluster is natural in the relativistic theory since the energy of interactions depends on the number of particles quadratically, and the relativistic mass of the cluster is proportional to its energy.

20.2.6.2 Comparison with a Cluster of Balanced Charges

Now let us relate the above treatment to a similar treatment of a system of balanced charges in equilibrium, where $\mathcal{L}_{\text{em}} = \mathcal{L}_{\text{BEM}}$. To make the comparison more transparent, we consider a generalized version of the theory of balanced charges, namely the theory where the basic equilibrium object is a cluster of n_ℓ b-charges, $n_\ell \geq 1$. The Lagrangian $\mathcal{L}_{\text{BEMC}}(\{\psi^{(\ell,s\ell)}\}, \{\psi_{;\mu}^{(\ell,s\ell)}\})$ for clusters is given by (20.46) with $\mathcal{L}_{\text{em}} = \mathcal{L}_{\text{BEM}}$, and the variables are subjected to additional constraints as in (20.45). The equilibrium state for a generalized cluster of $n = n_\ell$ identical b-charges is denoted by $\psi_{\ell,s;b,n}, \dot{\varphi}_{\ell,s;b,n}$, $s = 1, \dots, n$. The equilibrium condition takes the following form similar to (14.120) and (14.121):

$$-\Delta \dot{\varphi}_{\ell b,n} = 4\pi \sum_{s=2}^{N_\ell} q_{\ell,s;b,n} \left(1 - \frac{q_{\ell,s;b,n} \dot{\varphi}_{\ell b,n}}{m_{b,n}^{\ell,s} c^2} \right) (\psi_{\ell,s;b,n})^2, \quad (20.61)$$

$$\begin{aligned} -\Delta \psi_{\ell,s;b,n} + \frac{m_{\ell,s} \dot{\varphi}_{\ell b,n}}{\chi_{b,n}^2} q_{\ell,s;b,n} \left(2 - \frac{q \dot{\varphi}_{\ell b,n}}{m_{\ell,s;b,n} c^2} \right) \psi_{\ell,s;b,n} \\ + G'_{b,n} (|\psi_{\ell,s;b,n}|^2) \psi_{\ell,s;b,n} = 0. \end{aligned} \quad (20.62)$$

Note that the only difference between (20.61) and (20.51) is that the first term with $s = 1$ in the sum in (20.61) is omitted. Now $\psi_{\ell b,2}, \dot{\varphi}_{\ell b,2}$ is the basis for comparison, and once again we fix the nonlinearity $G'_{b,n} = G'_{b,2} = G'_b$ and set, similarly to (20.55), (20.58),

$$\begin{aligned} \dot{\varphi}_{\ell b,n} &= (n-1) \dot{\varphi}_{\ell b,2}, & q_{\ell b,n} &= (n-1) q_{\ell b,2}, \\ m_{\ell b,n} &= (n-1)^2 m_{\ell b,2}, & \chi_{\ell b,n} &= (n-1)^2 \chi_{\ell b,2}. \end{aligned} \quad (20.63)$$

Let us introduce the Lagrangian $\mathcal{L}_{\text{BEM},n}(\{\psi_{\ell b,n}\}, \{\varphi_{\ell b,n}\})$ with so defined constants. Observe that the Euler–Lagrange equations for $\mathcal{L}_{\text{BEMC}}(\{\psi_{\ell,s\ell}\}, \{\psi_{\ell,s\ell;\mu}\})$ with restrictions (20.45) are equivalent to the Euler–Lagrange equations for $\mathcal{L}_{\text{BEM},n}(\{\psi_{\ell b,n}\}, \{\varphi_{\ell b,n}\})$. Obviously if all $n = n_\ell \geq 2$, then $\mathcal{L}_{\text{BEM},n}$ has the same form as $\mathcal{L}_w$. In addition, if

$$G'_{b,2} = G'_w, \quad q_{\ell b,2} = q_\ell, \quad m_{\ell b,2} = m_\ell, \quad \chi_{\ell b,2} = \chi,$$

then the relative difference of the coefficients of the Lagrangians $\mathcal{L}_{\text{BEM},n}$ and $\mathcal{L}_{\text{w}}$ with the same $n \geq 2$ is of order $\frac{1}{n}$. For example, $(q_{\ell\text{w}} - q_{\ell\text{b},n})/q_{\ell\text{w}} = 1/n$. Notice that the most important case $n = 1$, which directly corresponds to the balanced charges studied in Chap. 17, is special. There is a non-vanishing difference between the fundamental equilibrium $(\psi_{\ell\text{b},1}^\circ, \varphi_{\ell\text{b},1}^\circ) = (\psi_\ell^\circ, \varphi_\ell^\circ)$ determined by (14.77), (14.78) for this special case and the smallest cluster $(\psi_{\ell\text{b},2}^\circ, \varphi_{\ell\text{b},2}^\circ)$.

Based on the above analysis, we may conclude that the dressed charge theory can be considered as an approximation for the generalized theory of clusters of balanced charges if the dressed charge is identified with a cluster of a large number of balanced charges.

Part IV

The Neoclassical Theory of Charges with Spin

In this part we construct a spinorial version of our neoclassical theory of elementary charges developed in Part III, integrating into it the concept of spin $1/2$; we mainly follow our paper [21]. The spinorial version of our theory has many features identical to those of the Dirac theory, including the gyromagnetic ratio, expressions for currents and the antimatter states. The treatment of spinorial and relativistic aspects of our theory exploits extensively the spacetime algebra (STA), that is, the Clifford algebra of the Minkowski vector space. We provide here a sufficiently detailed exposition of all significant features of the STA needed for the construction of the relativistic Lagrangian framework and electromagnetic interactions.

This part is organized as follows. In Chap. 22 we provide the basic facts of the Dirac equation. In Chap. 23 we provide basic information on the STA needed to carry out calculations. In Chap. 24 we discuss important properties of the STA version of the Dirac theory. In Chap. 25 we develop the spinorial version of our neoclassical theory, and in Chap. 26 we study properties of a free charge. In Chap. 27 we consider the interpretation of the neoclassical solutions and compare the main features of the theory developed here with those of the Dirac theory.

Chapter 21

Introduction

When developing the spinorial version of our theory, we kept in mind that it has to incorporate in one form or another some important features of the Dirac theory of spin 1/2 particles that are verified experimentally. To integrate spin into our Lagrangian relativistic field theory, we used methods developed by D. Hestenes and other authors, see [169], [170], [82], [93], [307] and references therein. In particular, we used Hestenes's "real" form of the Dirac equation based on the spacetime algebra (STA), that is, the Clifford algebra of the Minkowski vector space. The geometric transparency of the STA combined with a rich multivector algebraic structure was a decisive incentive for using it instead of Dirac's γ -matrices.

Since the spinorial version of our neoclassical theory is obtained by a modification of its original spinless version, it is useful to take a look once again at its basic features. Recall that according to our spinless theory developed in Part III, a balanced charge is described by a pair (ψ, A^μ) where ψ is the charge's complex-valued scalar wave function, and $A^\mu = (\varphi, \mathbf{A})$ is the 4-vector potential of the EM field generated by the charge. A balanced charge does not interact with itself electromagnetically. Its wave function ψ describes the charge distribution, and the elementary potential A^μ mediates its EM interactions with all other elementary charges. Importantly, (i) all internal forces of an elementary charge are exclusively of non-electromagnetic origin; (ii) every elementary charge is a source of its elementary EM field which represents force exerted by this charge on any other charge but not upon itself. A system of any number of elementary charges is furnished with a relativistic Lagrangian that yields EM interactions with the following features: (i) elementary charges interact only through their elementary EM potentials and fields; (ii) the field equations for the elementary EM fields are exactly the Maxwell equations with proper conserved currents; (iii) the wave function's evolution is governed by the nonlinear Klein–Gordon equations; (iv) the EM force density is described exactly by its well-known Lorentz expression; (v) the Newton equations with the Lorentz forces are not postulated but are derived as an approximation when charges are well separated; (vi) a free charge moves uniformly preserving its shape up to the Lorentz contraction. Since an overwhelming number of EM phenomena are explained within classical EM theory by

the Maxwell equations and the Lorentz forces, our neoclassical EM theory is equally successful in explaining the same phenomena.

Particle-like states of elementary charges are recovered in our theory from the original field concepts as localized states. A possibility for an elementary charge to localize is facilitated by internal forces of non-electromagnetic origin which are represented by the nonlinearity. In this part of the book we choose the nonlinearity to be logarithmic as in (14.42), namely

$$G(s) = G_a(s) = -a^{-2}s [\ln(a^3|s|) + \ln\pi^{3/2} + 2], \quad -\infty < s < \infty, \quad (21.1)$$

where $a > 0$ is the size parameter. Note though that most of our analysis is valid for any nonlinearity determined by (14.21), see the examples in Sect. 14.3.2.

To give a flavor of the spinorial version of our neoclassical theory proposed here, we can state that the Euler–Lagrange field equation governing the motion of a free charge is the following spinorial version of the nonlinear Klein–Gordon equation:

$$\mathcal{P}^2\psi - \left[\kappa_0^2 + G' \left(\langle \psi \tilde{\psi} \rangle \right) \right] \psi = 0, \quad \kappa_0 = \frac{mc}{\chi}, \quad (21.2)$$

where $\mathcal{P}$ is a spinorial version of the momentum operator similar to the same in the Dirac theory, ψ is the spinor field and G is the nonlinearity. We show that *the spinorial version of our theory developed here has many important features identical to those of the Dirac theory such as the gyromagnetic ratio, expressions for currents including the spin current, and antimatter states*. We focus mostly on the case of a free charge, since the difference with the spinless scalar case shows itself already in this case. A detailed analysis of more complex cases, including a charge in an external electromagnetic field or systems of interacting charges, is left for future work.

While our theory has many features identical to the Dirac theory as pointed out above, it differs significantly from the Dirac theory. The first significant difference is that from the outset our neoclassical theory is a consistent relativistic Lagrangian field theory. This difference is manifest in the treatment of the energy. The Dirac theory using quantum mechanics (QM) framework essentially identifies the energy with the frequency through the Planck–Einstein relation $E = \hbar\omega$ considered to be fundamentally exact. In our neoclassical Lagrangian field theory, the energy-momentum density is constructed based on the system Lagrangian and the Noether theorem, and its relation to frequencies in relevant regimes is non-trivial. In particular, in our theory the Planck–Einstein relation holds only approximately as a non-relativistic approximation for time-harmonic states taking the form $E \approx \hbar|\omega|$. Observe that it is the compliance of the Dirac theory with the foundations of QM that requires the identity $E = \hbar\omega$ to hold, and that results in the well known unbounded negative energy problem. Indeed, QM evolution equation $\hbar i \partial_t \psi = H\psi$ with H being the energy operator is just an operator form of the Planck–Einstein relation $E = \hbar\omega$. This evolution equation requires naturally negative frequencies which then, according to the Planck–Einstein relation, have to be interpreted as negative energies.

The second significant difference is that our Lagrangian theory is free from “infinities” which constitute a known problem for QM and quantum electrodynamics.

The third difference is that an elementary charge in our theory is an extended object which can be localized in certain situations, whereas in QM it is always a point-like object. As R. Feynman put it, [122, p. 21–6]: “The wave function $\psi(\mathbf{r})$ for an electron in an atom does not, then, describe a smeared-out electron with a smooth charge density. The electron is either here, or there, or somewhere else, but wherever it is, it is a point charge.” In our theory the ability of an elementary charge to localize as a small point-like object in certain situations is provided by a nonlinear non-electromagnetic self-interaction G defined by (21.1). In particular, the free charge spinor wave function in our theory is a plane wave modulated by a Gaussian amplitude factor compared to a plane wave free charge solution in the Dirac theory. Such a localization property of our neoclassical free charge solutions allows us to evaluate the conserved quantities by integration, whereas this is not possible in the case of plane waves. The presence of the nonlinearity in our theory also invalidates to some degree the linear superposition principle, whereas it is of fundamental importance in QM.

The fourth important difference is that in our theory there is no electromagnetic self-interaction for an elementary charge.

An important insightful point on the nature of spin is made by D. Hestenes [173, p. 1029]:

“It is significant that in Schrödinger theory i and $\hbar$ always appear together in the product $i\hbar$. The significance is made manifest in the study of the Pauli theory to follow. There it is shown that Planck’s constant $\hbar$ enters the theory only as twice the magnitude of the electron’s spin. This strongly suggests that there is a general connection of spin to the appearance both of Planck’s constant and of “complex numbers” in quantum theory. It should especially be noted that this idea has arisen only from insistence on the internal consistency of quantum theory as it exists today; it has not been imposed on the theory by external considerations.”

Chapter 22

The Dirac Equation

“I consider that I understand an equation when
I can predict the properties of its solutions,
without actually solving it.”
*P. Dirac.*¹

The *Dirac equation* for a charge e in an external electromagnetic field with 4-potential A_μ is of the form, [299, 5.3.5.4], [143, 3], [307, 8.1], [331, 2.1]

$$\hbar\gamma^\mu i\partial_\mu\Psi - \frac{e}{c}\gamma^\mu A_\mu\Psi = mc\Psi, \quad (22.1)$$

or

$$\hbar i\partial\Psi - \frac{e}{c}A\Psi = mc\Psi \quad (22.2)$$

where

$$\partial = \gamma^\mu\partial_\mu, \quad A = \gamma^\mu A_\mu, \quad (22.3)$$

and γ^μ are the *Dirac gamma matrices* satisfying the fundamental relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{I}_4.$$

Notice that the left-hand side of the equation (22.2) is an operator form for the classical canonical momentum $p - \frac{e}{c}A$ of the charge in an external electromagnetic field.

A particularly important representation of Dirac’s gamma matrices is as follows, [299, 5.3.5.4, 6.2.5], [143, 3, p. 129]

¹[128, p. 664].

$$\gamma^0 = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}, \quad i = 1, 2, 3, \quad (22.4)$$

where σ^i are the Hermitian *Pauli matrices* defined by

$$\sigma^0 = \mathbb{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (22.5)$$

Another Dirac matrix that appears in the analysis is, [299, 6.2.5]

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{bmatrix}, \quad (\gamma^5)^2 = 1, \quad \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5. \quad (22.6)$$

Observe also that the Dirac equation (22.2) can be written in terms of the *momentum operator* $\hat{P}_\mu$ and the *covariant derivative* D_μ , namely, [146, 3, p. 130], [299, 5.3.5.4], [331, 2.1.2]

$$\hat{P}\Psi = mc\Psi, \quad \hat{P} = \gamma^\mu \hat{P}_\mu, \quad D = \gamma^\mu D_\mu, \quad (22.7)$$

where

$$\hat{P}_\mu = i\hbar D_\mu = i\hbar\partial_\mu - \frac{e}{c}A_\mu, \quad D_\mu = \partial_\mu + i\frac{e}{\hbar c}A_\mu. \quad (22.8)$$

The conserved *charge current density* J^μ in the Dirac theory is commonly written in the following form, [143, 3.5], [331, 2.1.2]:

$$J^\mu(x) = ec\Psi^\dagger(x)\gamma^0\gamma^\mu\Psi(x) = ec\bar{\Psi}(x)\gamma^\mu\Psi(x), \quad (22.9)$$

where $\Psi^\dagger$ stands for the Hermitian conjugate to Ψ , and $\bar{\Psi}$ is the so-called *spinor adjoint* defined by

$$\bar{\Psi} = \Psi^\dagger\gamma^0. \quad (22.10)$$

Under the assumption that the wave function Ψ satisfies the Dirac equation (22.2), the Dirac current J^μ in (22.9) can also be represented in a physically appealing way known as the *Gordon decomposition*, [143, 8.1], [307, 8.1], [331, p. 148]:

$$\begin{aligned} J^\mu &= ec\bar{\Psi}\gamma^\mu\Psi = J_c^\mu + J_s^\mu \\ &= \frac{e}{2m} \left[\bar{\Psi}\hat{P}^\mu\Psi - \overline{(\hat{P}^\mu\Psi)}\Psi \right] - \frac{ie}{2m}\partial^\nu(\bar{\Psi}\sigma_\nu^\mu\Psi), \end{aligned} \quad (22.11)$$

where σ_ν^μ is defined by (6.28),

$$J_c^\mu = \frac{e}{2m} \left[\bar{\Psi}\hat{P}^\mu\Psi - \overline{(\hat{P}^\mu\Psi)}\Psi \right] \text{ is the convection current density,} \quad (22.12)$$

$$J_s^\mu = -\frac{ie}{2m}\hat{P}^\nu(\bar{\Psi}\sigma_\nu^\mu\Psi) \text{ is the spin current density.}$$

Importantly, in the Dirac theory both the convection and spin currents are independently conserved. In contrast, in the Pauli theory it is only the convection current which is conserved according the Pauli–Schrödinger equation, whereas the spin current must be added “ad hoc”, [144, 12.5], [149].

The Lagrangian density for the free electron is as follows, [143, 2.2], [51, 6.8], [331, 2.1]

$$L = \bar{\Psi} (\text{ci}\hbar\partial\Psi - mc^2\Psi), \quad (22.13)$$

and the corresponding Dirac equation for the free electron is

$$\hbar i\partial\Psi = mc\Psi. \quad (22.14)$$

Evidently the above equation is a particular case of the Dirac equation (22.7) when $A = 0$.

The EnMT $T^{\mu\nu}$ for a free electron and the corresponding energy T^{00} are, [143, 2.2], [331, 2.1],

$$T^{\mu\nu} = \bar{\Psi} \text{ci}\hbar\gamma^\mu \partial^\nu \Psi - \delta^{\mu\nu} \bar{\Psi} \text{ci}\hbar\partial\Psi + \delta^{\mu\nu} mc^2 \bar{\Psi}\Psi, \quad (22.15)$$

$$T^{00} = \bar{\Psi} \text{ci}\hbar\gamma^0 \partial_0 \Psi = -\bar{\Psi} \text{ci}\hbar\gamma^i \partial_i \Psi + mc^2 \bar{\Psi}\Psi, \quad (22.16)$$

$$T^{00} = \bar{\Psi} \text{ci}\hbar\Psi - \bar{\Psi} \text{ci}\hbar\partial\Psi + mc^2 \bar{\Psi}\Psi.$$

The Dirac Lagrangian for the electron in the electromagnetic field has the form

$$L = \text{c}\bar{\Psi} \left(\hat{P}\Psi - mc\Psi \right), \quad \hat{P} = i\hbar\partial - \frac{e}{c}A = i\hbar D.$$

Chapter 23

Basics of Spacetime Algebra (STA)

The *Spacetime Algebra* (STA) is perfectly suited for our conceptual purposes as well as the computation, and in this section we formulate its basic properties. The spacetime algebra is a particular case of the *Clifford algebra*. A general *Clifford algebra*, also called *Geometric Algebra* (GA), is an associative algebra generated by an n -dimensional vector space V over the set of real scalars $\mathbb{R}$ furnished with a symmetric quadratic form g . We denote such a Clifford algebra by $\text{Cl}(V, g)$ and call its elements *multivectors*, referring to elements of the generating linear space V as vectors. The *Clifford product*, also called *geometric product*, of any two multivectors A and B is denoted by juxtaposition, that is, AB . The Clifford product is fundamentally determined by the requirement to satisfy the following identity for any two vectors a and b from the generating vector space V :

$$ab + ba = 2g(a, b)1, \quad (23.1)$$

where 1 is the multiplicative identity which we often skip in the notation. The Clifford algebra is naturally furnished with an *inner “.” (interior, dot) product and outer (exterior, Grassmann) product “^”* so that for any two vectors a and b in V

$$a \cdot b = \frac{ab + ba}{2} = g(a, b), \quad a \wedge b = -b \wedge a = \frac{ab - ba}{2}, \quad (23.2)$$

implying

$$ab = a \cdot b + a \wedge b. \quad (23.3)$$

According to (23.2), the orthogonality of two vectors a and b in the Clifford algebra, that is, $a \cdot b = 0$, has an equivalent algebraic representation as the anticommutativity of a and b .

The *Spacetime Algebra* is the Clifford algebra based on the real 4-dimensional Minkowski space $\mathbb{M}^4$, and it is denoted by $\text{Cl}(1, 3)$ where $(1, 3)$ is the signature

of the Minkowski space metric. The 3-dimensional Euclidean space is denoted by $\mathbb{R}^3$, and the Clifford Algebra corresponding to it is $\text{Cl}(3, 0)$. In setting up the STA we follow [169], [93] and [307]. Though many properties of Clifford Algebras hold universally across different dimensions and signatures, we formulate them mostly for the case of our primary interest which is the Spacetime Algebra $\text{Cl}(1, 3)$. We want to stress that the concise review of the STA presented here is not meant to be a complete and/or systematic presentation of the Clifford Algebras theory, but rather it is a selection of properties of the STA which are important for our purposes. We want to acknowledge the work done by D. Hestenes who pioneered and developed many aspects of the STA and its applications to physics. There is a number of excellent presentations of Clifford Algebras and their applications to physics written by D. Hestenes and his followers, see, for instance, [171], [170], [169], [93], [307], [95].

The standard model for spacetime is the real Minkowski vector space $\mathbb{M}^4$ with the standard metric $g_{\mu\nu}$ defined by

$$\{g_{\mu\nu}\} = \{g^{\mu\nu}\}, \quad g_{00} = 1, \quad g_{jj} = -1, \quad j = 1, 2, 3, \quad (23.4)$$

$$g_{\mu\nu} = 0, \quad \mu \neq \nu.$$

A basis for the STA can be generated by a standard frame $\{\gamma_\mu : \mu = 0, 1, 2, 3\}$ of orthonormal vectors, with a *timelike vector* γ_0 in the forward light cone, and γ_μ are assumed to satisfy the following relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\gamma_\mu \cdot \gamma_\nu = 2g_{\mu\nu}, \quad (23.5)$$

namely

$$\gamma_0^2 = 1, \quad \gamma_i^2 = -1, \quad \gamma_0 \cdot \gamma_i = 0, \quad \gamma_i \cdot \gamma_j = -\delta_{ij}, \quad i, j = 1, 2, 3. \quad (23.6)$$

Notice that (23.5)–(23.6) are the defining relations of the *Dirac matrix algebra*. That explains our choice to denote an orthonormal frame by $\{\gamma_\mu\}$, but it must be remembered that the $\{\gamma_\mu\}$ are basis vectors and not a set of matrices in “isospace”. Now we show in brief how relations (23.5) determine the structure of the algebra $\text{Cl}(1, 3)$. The algebra consists of linear combinations of products of basis elements γ_μ . Every such monomial can be reduced using the anticommutation relation (23.5) to linear combinations of monomials of the form

$$\gamma_0^{s_0} \gamma_1^{s_1} \gamma_2^{s_2} \gamma_3^{s_3}. \quad (23.7)$$

Then, since $\gamma_\mu^2 = g_{\mu\mu}$, it can further be reduced to linear combinations of monomials where either $s_\mu = 1$ or $s_\mu = 0$. We call $k = s_0 + s_1 + s_2 + s_3$ the degree of the reduced monomial, obviously $0 \leq k \leq 4$. The reduced monomials with a given degree k form a basis of a linear subspace in the algebra. Obviously, the dimension of the subspace corresponding to degree k equals the binomial coefficient $C(4, k)$, and

the dimension of the algebra $\text{Cl}(1, 3)$ considered as a linear space equals $2^4 = 16$. Below the structure of the algebra is described in more detail.

To facilitate algebraic manipulations, it is convenient to introduce the reciprocal frame $\{\gamma^\mu\}$ defined by the equations

$$\gamma^\mu = g^{\mu\nu}\gamma_\nu, \quad \gamma_\mu \cdot \gamma^\nu = \delta_\mu^\nu, \quad (23.8)$$

with the summation convention understood. Observe that two different vectors γ_μ anticommute. Since we are in a space of mixed signature, we distinguish between a *frame* $\{\gamma_\mu\}$ and its *reciprocal* $\{\gamma^\mu\}$, namely

$$\gamma^0 = \gamma_0, \quad \gamma^i = -\gamma_i, \quad i = 1, 2, 3. \quad (23.9)$$

Notice that, following the common practice, we use Greek letters $\mu, \nu, \dots$ for indices taking values 0, 1, 2, 3 and Latin letters $i, j, \dots$ for indices taking values 1, 2, 3. The γ_μ determine a unique *right-handed unit pseudoscalar*

$$\mathbf{I} = \gamma_0\gamma_1\gamma_2\gamma_3, \quad \mathbf{I}^2 = -1, \quad (23.10)$$

that anticommutes with vectors γ_μ

$$\mathbf{I}\gamma_\mu = -\gamma_\mu\mathbf{I}, \quad \mu = 0, 1, 2, 3. \quad (23.11)$$

For any vector a , a frame $\{\gamma_\mu\}$ determines a set of *rectangular coordinates*

$$a = a^\mu\gamma_\mu = a^0\gamma_0 + a^i\gamma_i, \quad \{a^\mu\} = \{a^0, \mathbf{a}\}. \quad (23.12)$$

In particular, for any spacetime point x ,

$$x = x^\mu\gamma_\mu = ct\gamma_0 + x^i\gamma_i, \quad \{x^\mu\} = \{x^0, \mathbf{x}\} = \{ct, \mathbf{x}\}. \quad (23.13)$$

The frame $\{\gamma_\mu\}$ also defines the following basis for the algebra $\text{Cl}(1, 3)$:

$$\begin{array}{ccccc} 1 & \{\gamma_\mu\} & \{\gamma_\mu \wedge \gamma_\nu\} & \{\mathbf{I}\gamma_\mu\} & \{\mathbf{I}\} \\ \text{1 scalar} & \text{4 vectors} & \text{6 bivectors} & \text{4 trivectors} & \text{1 pseudoscalar} \end{array}, \quad (23.14)$$

where “ $\wedge$ ” is the external (Grassman) product, and $\mu < \nu$ in $\gamma_\mu \wedge \gamma_\nu$. This choice of the basis subspaces is more convenient and better describes the structure of the algebra than the basis formed by the monomials (23.7). A general element M of the spacetime algebra is called a *multivector* and can be written as

$$M = \alpha + a + B + \mathbf{I}b + \mathbf{I}\beta, \quad (23.15)$$

where α and β are scalars, a and b are vectors and B is a bivector. The representation (23.15) is a decomposition of M into its k -vector parts (grades), which can be expressed more explicitly by putting it in the form

$$M = \sum_{0 \leq k \leq 4} \langle M \rangle_k, \quad \text{where } \langle M \rangle_0 = \langle M \rangle = \alpha, \quad (23.16)$$

$$\langle M \rangle_1 = a, \quad \langle M \rangle_2 = B, \quad \langle M \rangle_3 = Ib, \quad \langle M \rangle_4 = I\beta,$$

where the subscript (k) means “ k -vector part”. Notice the special notation $\langle M \rangle = \langle M \rangle_0$ for the scalar part for the multivector M . The space of k -vectors, that is, multivectors of grade k in (23.14) is denoted by Λ^k .

It is instructive to see grade decomposition for the geometric product of two multivectors $A_r \in \Lambda^r$ and $B_s \in \Lambda^s$, [171, 1.1], [167, 2.2], [283, 2.4.2]

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{r+s} \quad (23.17)$$

$$= \sum_{k=0}^m \langle A_r B_s \rangle_{|r-s|+2k}, \quad \text{where } m = \frac{1}{2}(r+s-|r-s|),$$

where it is understood that for any multivector M

$$\langle M \rangle_k \equiv 0 \text{ for any } k > 4. \quad (23.18)$$

The inner (dot) “ $\cdot$ ” and outer (Grassman) “ $\wedge$ ” products are defined first for homogeneous multivectors $A_r \in \Lambda^r$ and $B_s \in \Lambda^s$ by, [171, 1.1],

$$A_r \cdot B_s = \langle A_r B_s \rangle_{|r-s|}, \quad \text{if } r, s > 0; \quad (23.19)$$

$$A_r \cdot A_s = 0, \quad \text{if } r = 0 \text{ or } s = 0;$$

$$A_r \cdot B_s = (-1)^{r(s-r)} B_s \cdot A_r \text{ for } s \geq r; \quad (23.20)$$

$$A_r \wedge B_s = \langle A_r B_s \rangle_{r+s} = (-1)^{rs} B_s \wedge A_r. \quad (23.21)$$

Then the products are extended by linearity to arbitrary multivectors A and B . In particular, if a is a vector and A_r is a multivector of the grade r , we have, [171, 1-1], [93, 4.1.2]

$$a A_r = a \cdot A_r + a \wedge A_r, \quad A_r a = A_r \cdot a + A_r \wedge a, \quad (23.22)$$

where

$$a \cdot A_r = (-1)^{r-1} A_r \cdot a = \frac{1}{2} (a A_r - (-1)^r A_r a), \quad (23.23)$$

$$a \wedge A_r = (-1)^r A_r \wedge a = \frac{1}{2} (a A_r + (-1)^r A_r a).$$

The STA has a much richer structure than the algebra of complex numbers, and it can be furnished with several natural conjugations (involutions), [82, 5.3], [30, 1.4.8], [270, 3.1, 3.2]. The most important of those is called *reversion* (principal anti-automorphism), and the *reverse* $\tilde{M}$ of a general multivector M is defined by

$$\tilde{M} = \alpha + a - B - Ib + I\beta, \quad (23.24)$$

$$\langle \tilde{M} \rangle_k = \widetilde{\langle M \rangle_k} = (-1)^{\frac{k(k-1)}{2}} \langle M \rangle_k, \quad 0 \leq k \leq 4. \quad (23.25)$$

The reversion operation justifies its name since it reverses the order of the multipliers:

$$(MN)^\sim = \tilde{N}\tilde{M}. \quad (23.26)$$

Grade involution is a conjugation defined by

$$\langle \hat{M} \rangle_k = (-1)^k \langle M \rangle_k, \quad 0 \leq k \leq 4. \quad (23.27)$$

There is yet another *Hermitian conjugation* also called the *relative reversion* $M^\dagger$ of a multivector M defined by

$$M^\dagger = \gamma_0 \tilde{M} \gamma_0, \quad (23.28)$$

and it corresponds to the Hermitian conjugation in the Dirac Algebra. Every multivector can then be decomposed into γ_0 -even and γ_0 -odd components

$$M = M_e + M_o, \text{ where } M_e = \frac{1}{2} (M^\dagger + M), \quad M_o = \frac{1}{2} (M^\dagger - M), \quad (23.29)$$

and evidently

$$M_e^\dagger = M_e, \quad M_o^\dagger = -M_o, \quad \tilde{M}^\dagger = \widetilde{M^\dagger}. \quad (23.30)$$

The grade structure (23.15) of the STA and the grade involution operator defined by (23.27) provide for a natural decomposition of any multivector M into the sum of an *even part* M_+ and an *odd part* M_- as follows:

$$\begin{aligned} M_+ &= \alpha + B + I\beta, \quad M_- = a + Ib, \\ M_\pm &= \frac{1}{2} (M \pm \hat{M}) = \frac{1}{2} (M \mp IMI). \end{aligned} \quad (23.31)$$

Notice that *the even and odd parts, respectively, commute and anticommute with I*, that is,

$$M_+ I = I M_+, \quad M_- I = -M_- I. \quad (23.32)$$

Importantly, the set of all even elements M_+ of the STA $\text{Cl}(1, 3)$ forms a Clifford algebra on its own, we denote it by $\text{Cl}_+(1, 3)$. This even subalgebra $\text{Cl}_+(1, 3)$ is

isomorphic to the geometric algebra (GA) $\text{Cl}(3, 0)$ of three-dimensional Euclidean space with multivectors of the form, [166, I], [168, VI],

$$N = \alpha + I\beta + a + Ib \in \text{Cl}(3, 0), \quad (23.33)$$

where α and β are scalars, a and b are vectors and I is the unit pseudoscalar in $\text{Cl}(3, 0)$. The even subalgebra $\text{Cl}_+(1, 3)$ is very important to the STA version of the Dirac electron theory where it is the space of values of the Dirac spinorial wave function, [93, 5.2.4].

Notice that the scalar part of $\langle M \rangle$ has the following properties

$$\langle M \rangle = \langle \tilde{M} \rangle, \quad \langle MN \rangle = \langle NM \rangle, \quad \langle \langle M \rangle_k \langle N \rangle_s \rangle = 0, \quad \text{if } k \neq s, \quad (23.34)$$

where M and N are multivectors. The above equalities imply the following identities involving Hermitian conjugation

$$\langle MN^\dagger \rangle = \langle N^\dagger M \rangle = \langle NM^\dagger \rangle. \quad (23.35)$$

Based on the above we define first a *scalar-valued *-product* for any two arbitrary multivectors A and B by, [171, 1.1], [93, 4.1.3], [94], [95, 3.1.2], [270, 3.2.3]

$$A * B = \langle AB \rangle = \sum_{0 \leq k \leq 4} \langle A_{(k)} B_{(k)} \rangle. \quad (23.36)$$

The above scalar *-product is symmetrical and reversible

$$A * B = B * A = \tilde{A} * \tilde{B} = \tilde{B} * \tilde{A}. \quad (23.37)$$

Another (fiducial) *scalar product* $A \cdot B = \langle A, B \rangle$ of two arbitrary multivectors A and B is defined by, [307, 3.4], [252, 4.2.4]

$$\begin{aligned} A \cdot B &= \langle A, B \rangle = \tilde{A} * B = \langle \tilde{A} B \rangle = \langle A \tilde{B} \rangle \\ &= \sum_{0 \leq k \leq 4} \langle \widetilde{\langle A \rangle_k} \langle B \rangle_k \rangle = \sum_{0 \leq k \leq 4} (-1)^{\frac{k(k-1)}{2}} \langle A_{(k)} B_{(k)} \rangle. \end{aligned} \quad (23.38)$$

Notice that we use the symbol “ $\cdot$ ” for the scalar product since the “normal” dot symbol “ $\cdot$ ” is already taken for the inner product. Unfortunately, the symbols “ $*$ ” and “ $\cdot$ ” are used differently in different texts and one has to pay attention when using those symbols. For a detailed and insightful analysis of the relations between different products and their geometric meaning see [94]. The relations (23.34)–(23.41) readily imply the following useful properties of the scalar products:

$$(AB) * C = A * (BC) = \langle ABC \rangle, \quad (23.39)$$

$$\begin{aligned}
 (AB) \cdot C &= B \cdot (\tilde{A}C) = A \cdot (C\tilde{B}), \\
 A \cdot (BC) &= (\tilde{B}A) \cdot C = (A\tilde{C}) \cdot B.
 \end{aligned}
 \tag{23.40}$$

A grade- r multivector A is called *simple* or a *blade* if it is a product of r anticommuting vectors, that is,

$$A = a_1 \wedge a_2 \cdots \wedge a_r, \text{ where } a_k a_j = -a_k a_j \text{ for } k \neq j. \tag{23.41}$$

Blades naturally correspond to subspaces, and they are instrumental to establishing relations between geometric and algebraic properties. An important property of every grade- r blade A_r is that it has the inverse, [171, 1-1], [95, 3.5.2]

$$A_r^{-1} = \frac{\tilde{A}_r}{A_r * \tilde{A}_r} = (-1)^{r(r-1)/2} \frac{A_r}{A_r * \tilde{A}_r}. \tag{23.42}$$

In the case where A_r and B_r are simple r -vectors, the scalar products (23.36), (23.38) have the following representations via the determinant, [95, 3.1.2]

$$A_r * B_r = \langle A_r B_r \rangle = A_r \cdot B_r = \det \begin{bmatrix} \langle a_1, b_r \rangle & \cdots & \langle a_1, b_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle a_r, b_r \rangle & \cdots & \langle a_r, b_1 \rangle \end{bmatrix}, \quad r > 0, \tag{23.43}$$

$$A_r \cdot B_r = A_r * \tilde{B}_r = \det \begin{bmatrix} \langle a_1, b_1 \rangle & \cdots & \langle a_1, b_r \rangle \\ \vdots & \ddots & \vdots \\ \langle a_r, b_1 \rangle & \cdots & \langle a_r, b_r \rangle \end{bmatrix}, \quad r > 0, \tag{23.44}$$

and

$$A_r * B_s = 0, \quad r \neq s; \quad a * b = ab, \text{ if } a \text{ and } b \text{ are scalars.} \tag{23.45}$$

Observe that in the case of the Clifford Algebra $\text{Cl}(3, 0)$ of 3-dimensional Euclidean space, the scalar product is positive, $A \cdot A = \langle \tilde{A}A \rangle \geq 0$, for any multivector $A \in \text{Cl}(3, 0)$, and that is the primary motivation to define the scalar product by the formula (23.38). The scalar product also allows us to define a *positive definite magnitude* $|M|$ for any multivector M by

$$|M|^2 = \left| \langle \tilde{M}M \rangle \right| = \left| \langle M\tilde{M} \rangle \right|. \tag{23.46}$$

Notice that in the case of vectors we always have

$$A \cdot B = A \cdot B = A * B \text{ if } A, B \in \Lambda^1. \tag{23.47}$$

Being given a basis $\{\gamma_\mu\}$ for $\mathcal{M}^4$, we define a basis $\{\sigma_k\}$ for the 3-dimensional Euclidean space $\mathcal{P}^3$ by

$$\sigma_k = \gamma_k \wedge \gamma_0 = \gamma_k \gamma_0 = -\tilde{\sigma}_k = -\sigma^k, \quad \sigma_k^2 = 1, \quad k = 1, 2, 3, \quad (23.48)$$

$$\begin{aligned} \sigma_i \sigma_j &= -\gamma_i \gamma_j = -\gamma_i \wedge \gamma_j = \epsilon_{ijk} \mathbf{I} \sigma_k, \quad i \neq j, \\ \sigma_1 \sigma_2 \sigma_3 &= \mathbf{I}, \quad (\mathbf{I} \sigma_k)^2 = -1, \end{aligned} \quad (23.49)$$

where ϵ_{ijk} is the *alternating tensor*, also called *Levi-Civita symbol*, defined by

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is a cyclic permutation of } 123, \\ -1 & \text{if } ijk \text{ is an anticyclic permutation of } 123, \\ 0 & \text{if otherwise.} \end{cases} \quad (23.50)$$

Notice also that the following identities hold

$$\begin{aligned} \gamma_0 \sigma_k &= -\sigma_k \gamma_0, \quad \gamma_0 \mathbf{I} = -\mathbf{I} \gamma_0, \quad \gamma_0 \mathbf{I} \sigma_k = \mathbf{I} \sigma_k \gamma_0, \\ \sigma_k \mathbf{I} &= \mathbf{I} \sigma_k, \quad k = 1, 2, 3; \end{aligned} \quad (23.51)$$

$$\begin{aligned} \sigma_i \cdot \sigma_j &= \delta_{ij}, \quad \frac{1}{2} (\sigma_i \sigma_j - \sigma_j \sigma_i) = \epsilon_{ijk} \mathbf{I} \sigma_k, \\ \frac{1}{2} (\mathbf{I} \sigma_i \mathbf{I} \sigma_j - \mathbf{I} \sigma_j \mathbf{I} \sigma_i) &= \epsilon_{ijk} \mathbf{I} \sigma_k. \end{aligned} \quad (23.52)$$

The bivectors σ_k are called *relative vectors* and they correspond to timelike planes. The *relative vectors* σ_k generate the even subalgebra $\text{Cl}_+(1, 3)$ which is isomorphic to the geometric algebra (GA) $\text{Cl}(3, 0)$ of three-dimensional Euclidean space, [165, 3], [166, 1]. Relative bivectors $\mathbf{I} \sigma_k$ according to (23.49) are spacelike bivectors.

Observe that, using $\mathbf{I}^2 = -1$, we can recast the relations (23.49) as

$$\mathbf{I} \sigma_i \mathbf{I} \sigma_j = -\sigma_i \sigma_j = \gamma_i \gamma_j = \gamma_i \wedge \gamma_j = -\epsilon_{ijk} \mathbf{I} \sigma_k, \quad i \neq j, \quad (23.53)$$

implying that the span $\langle 1, \mathbf{I} \sigma_1, \mathbf{I} \sigma_2, \mathbf{I} \sigma_3 \rangle$ is a subalgebra $\mathbf{Q}$ which is isomorphic to the even subalgebra $\text{Cl}_+(3, 0)$ of the geometric algebra $\text{Cl}(3, 0)$ of three-dimensional Euclidean space. Since $\text{Cl}_+(3, 0)$ is isomorphic to the *quaternion algebra*, [170, 2.3], [93, 2.4.2], [82, 6.1], the subalgebra $\mathbf{Q}$ is also isomorphic to the quaternion algebra, and we refer to it by that name, that is,

$$\mathbf{Q} = \langle 1, \mathbf{I} \sigma_1, \mathbf{I} \sigma_2, \mathbf{I} \sigma_3 \rangle \text{ is the quaternion subalgebra.} \quad (23.54)$$

Notice that the quaternion subalgebra $\mathbf{Q}$ can also be characterized as the one consisting of even multivectors which are also γ_0 -even, that is,

$$\mathbf{Q} = \left\{ M \in \text{Cl}_+(3, 0) : M^\dagger = \gamma_0 \tilde{M} \gamma_0 = M \right\}. \quad (23.55)$$

The quaternion subalgebra $\mathbf{Q}$ is very important to the STA version of the Pauli electron theory, where it is the space of values of the Pauli spinorial wave function.

Chapter 24

The Dirac Equation in the STA

“Equations are more important to me,
because politics is for the present,
but an equation is something for eternity.”
A. Einstein.¹

Since the Dirac theory has been very thoroughly analyzed and tested experimentally, we would like to consider its STA version in sufficient detail and compare it with the neoclassical theory developed here. In addition to that, the Dirac equation in the STA and its analysis provides us with a number of valuable tools useful for our own constructions, and we consider its important features in this section.

The STA version of the Dirac spinor Ψ is the wave function ψ taking values in the even subalgebra $\text{Cl}_+(1, 3)$ of the Clifford algebra $\text{Cl}(1, 3)$, and we refer to it as to the *Dirac spinor* or just *spinor*. Notice that for any ψ from $\text{Cl}_+(1, 3)$ we have $\psi\tilde{\psi} = \widetilde{\psi\psi}$, hence this product is a linear combination of the scalar and the pseudoscalar I , that is,

$$\psi\tilde{\psi} = \tilde{\psi}\psi = \varrho e^{I\beta} = \varrho (\cos \beta + I \sin \beta), \text{ where} \quad (24.1)$$

$$\varrho \geq 0 \text{ and } \beta \text{ are scalars.}$$

This leads to the following *canonical Lorentz invariant decomposition* which holds for every even multivector ψ , [162], [169, VII.D], [93, 8.2], [82, 9.3],

$$\psi = \varrho^{\frac{1}{2}} e^{\frac{I\beta}{2}} R = R \varrho^{\frac{1}{2}} e^{\frac{I\beta}{2}}, \quad R\tilde{R} = R\tilde{R} = 1, \quad (24.2)$$

where $\varrho > 0$ and β are scalars, and R is the Lorentz rotor, that is, $x' = Rx\tilde{R}$ is the Lorentz transformation. According to D. Hestenes, the canonical decomposition

¹[128, p.664].

(24.1) can be regarded as an invariant decomposition of the Dirac wave function into a 2-parameter *statistical factor* $\varrho^{\frac{1}{2}} e^{\frac{i\beta}{2}}$ and a 6-parameter *kinematical factor* R .

It is worth pointing out that the identity (24.1) clearly shows that, though the reversion operation $\tilde{\psi}$ is analogous to complex conjugation for complex numbers, the even subalgebra $\text{Cl}_+(1, 3)$ is a richer entity than the set of complex numbers allowing $\psi\tilde{\psi}$ to be negative and not scalar-valued.

To introduce an STA form of the Dirac equation, we define first an STA version of the *Dirac operator* denoted sometimes by nabla dagger, [178, 2-1-2], [143, 3]. We denote this STA version of the Dirac operator by $\partial = \partial_x$. It is often called the *vector derivative* with respect to the vector x and defined by, [168], [169, II], [171], [93],

$$\partial = \partial_x = \gamma^\mu \partial_\mu, \text{ where } \partial_\mu = \frac{\partial}{\partial x^\mu}. \quad (24.3)$$

Notice that since ∂ is a vector, it may not commute with other multivectors.

In the case of the Clifford algebra $\text{Cl}(3, 0)$ of the 3-dimensional Euclidean space, the *vector derivative* ∇ is defined by

$$\nabla = \sum_{j=1}^3 \sigma_j \partial_j, \text{ where } \partial_j = \frac{\partial}{\partial x^j}, \text{ and } \sigma_j \text{ is a basis of } \text{Cl}(3, 0). \quad (24.4)$$

The *covariant Dirac equation in STA*, known also as the real Dirac equation, was obtained by D. Hestenes [169, VII], [93, 13.3.3, 13.3.3.4], [283, 6.7, 6.8] and it is

$$\hbar \partial \psi \mathbf{I} \sigma_3 - \frac{e}{c} A \psi = mc \psi \gamma_0, \text{ where } \mathbf{I} \sigma_3 = \gamma_1 \gamma_2. \quad (24.5)$$

The *real Dirac equation* (24.5) is equivalent to the original Dirac equation. Equation (24.5) can also be recast as

$$(\mathcal{P} - mc \overleftarrow{\gamma}_0) \psi = 0, \text{ or } \mathcal{P} \psi = mc \psi \gamma_0, \quad (24.6)$$

where the momentum operator $\mathcal{P}$ and the operator $\overleftarrow{\gamma}_0$ are defined by

$$\mathcal{P} \psi = \hbar \partial \psi \mathbf{I} \sigma_3 - \frac{e}{c} A \psi, \quad \overleftarrow{\gamma}_0 \psi = \psi \gamma_0. \quad (24.7)$$

The momentum operator $\mathcal{P}$ can be alternatively represented by

$$\mathcal{P} \psi = \gamma^\mu \mathcal{P}_\mu \psi, \text{ where } \mathcal{P}_\mu \psi = \hbar \partial_\mu \psi \mathbf{I} \sigma_3 - \frac{e}{c} A_\mu \psi. \quad (24.8)$$

We refer to the Eqs. (24.5), (24.6) as the *Dirac–Hestenes equations*. Observe that $\mathcal{P}$ and $\overleftarrow{\gamma}_0$ commute since the multivectors $\mathbf{I} \sigma_3 = \gamma_1 \gamma_2$ and γ_0 commute, that is,

$$(\mathbf{I} \sigma_3) \gamma_0 = \gamma_0 (\mathbf{I} \sigma_3), \quad \overleftarrow{\gamma}_0 \mathcal{P} = \mathcal{P} \overleftarrow{\gamma}_0. \quad (24.9)$$

The free electron canonical momentum operator $\mathring{\mathcal{P}}$ is obtained as a particular case of $\mathcal{P}$ in (24.7) when $A = 0$, that is,

$$\mathring{\mathcal{P}}\psi = \hbar\partial\psi\mathbf{I}\sigma_3, \text{ and } \mathcal{P}\psi = \mathring{\mathcal{P}}\psi - \frac{e}{c}A\psi = \hbar\partial\psi\mathbf{I}\sigma_3 - \frac{e}{c}A\psi. \quad (24.10)$$

One can also introduce for spinor-valued ψ the *covariant derivative operator* $\mathcal{D}$:

$$\mathcal{D}\psi = \partial\psi + \frac{e}{\hbar c}A\psi\mathbf{I}\sigma_3, \text{ implying } \mathcal{P} = \hbar\partial\psi\mathbf{I}\sigma_3 - \frac{e}{c}A\psi\hbar = \mathcal{D}\psi\mathbf{I}\sigma_3. \quad (24.11)$$

Conserved quantities of interest, including the electric current and the energy-momentum tensor (EnMT), can be obtained from the following *real Dirac–Hestenes Lagrangian* density for an electron in an external electromagnetic field, [219, 4.4], [166, Ap. B], [172, App. B], [283, 7.8],

$$L = c \left\langle \hbar\partial\psi\mathbf{I}\gamma_3\tilde{\psi} - \frac{e}{c}A\psi\gamma_0\tilde{\psi} - mc\psi\tilde{\psi} \right\rangle. \quad (24.12)$$

Using expressions (24.11) for the canonical momentum $\mathcal{P}$ and the covariant derivative $\mathcal{D}$ we can transform the Dirac–Hestenes Lagrangian into the following form

$$L = c \left\langle [(\mathcal{P} - mc\overleftarrow{\gamma}_0)\psi] \gamma_0\tilde{\psi} \right\rangle = c \left\langle [(\hbar\mathcal{D}\psi\mathbf{I}\sigma_3 - mc\overleftarrow{\gamma}_0)\psi] \gamma_0\tilde{\psi} \right\rangle. \quad (24.13)$$

The Lagrangian representation (24.13) implies

$$L = 0 \text{ for any } \psi \text{ satisfying the Dirac equation (24.6),} \quad (24.14)$$

and that is typical for the first order systems, [93, 13.3]. One can also verify that the corresponding Euler–Lagrange field equation is equivalent to the Dirac–Hestenes equation (24.5).

The *free electron Dirac–Hestenes Lagrangian* $\mathring{L}$ (when $A = 0$) equals

$$\begin{aligned} \mathring{L} &= c \left\langle \hbar\partial\psi\mathbf{I}\gamma_3\tilde{\psi} - mc\psi\tilde{\psi} \right\rangle = c \left\langle [(\mathring{\mathcal{P}} - mc\overleftarrow{\gamma}_0)\psi] \gamma_0\tilde{\psi} \right\rangle \\ &= c \left\langle [(\hbar\partial\psi\mathbf{I}\sigma_3 - mc\overleftarrow{\gamma}_0)\psi] \gamma_0\tilde{\psi} \right\rangle, \text{ where } \mathring{\mathcal{P}}_\mu\psi = \hbar\partial_\mu\psi\mathbf{I}\sigma_3. \end{aligned} \quad (24.15)$$

24.1 Conservation Laws

Our treatment of the charge and energy-momentum conservation laws is based on the Dirac–Hestenes Lagrangian and the Noether theorem.

24.1.1 Electric Charge Conservation

We introduce the so-called *global electromagnetic gauge transformation* as follows, [159, 3], [219, 3.2]:

$$x' = x, \quad \psi'(x') = \psi(x) e^{I\sigma_3 \epsilon}, \quad \epsilon \text{ is any real number.} \quad (24.16)$$

Consequently, the global electromagnetic gauge transformation preserves the vector derivative $\partial\psi$, that is,

$$\partial' = \partial, \quad \partial'\psi'(x') = \partial\psi(x) e^{I\sigma_3 \epsilon}. \quad (24.17)$$

The infinitesimal form of (24.16) for small ϵ is

$$\delta x' = 0, \quad \bar{\delta}\psi = \psi I\sigma_3 \epsilon. \quad (24.18)$$

The *local electromagnetic gauge transformation* is conceived to keep the covariant derivative $\mathcal{D}\psi$ defined by (24.11) invariant. It involves both ψ and A and is of the form

$$x' = x, \quad \psi'(x) = \psi(x) e^{I\sigma_3 \frac{e}{\hbar c} \epsilon(x)}, \quad A'(x) = A(x) - \partial\epsilon, \quad (24.19)$$

where $\epsilon(x)$ is a real-valued function of x . Then, since $\partial = \gamma^\mu \partial_\mu$, we consequently obtain

$$\partial' = \partial, \quad \partial'\psi'(x') = \left[\partial\psi(x) + \frac{e}{\hbar c} \partial\epsilon \psi(x) I\sigma_3 \right] e^{I\sigma_3 \frac{e}{\hbar c} \epsilon(x)}, \quad (24.20)$$

and

$$\mathcal{D}'\psi'(x') = \mathcal{D}\psi(x). \quad (24.21)$$

The infinitesimal form of (24.19) for small $\epsilon(x)$ is

$$\delta x' = 0, \quad \bar{\delta}\psi = \psi I\sigma_3 \frac{e}{\hbar c} \epsilon, \quad \bar{\delta}\partial\psi = \partial\psi + \frac{e}{\hbar c} \partial\epsilon \psi I\sigma_3, \quad \bar{\delta}A = -\partial\epsilon. \quad (24.22)$$

The last equality in (24.22) indicates that, to have the local gauge invariance, we have to couple the spinor field ψ with a vector field to “compensate” for the term $\frac{e}{\hbar c} \partial\epsilon \psi I\sigma_3$. And this is exactly what the electromagnetic potential A does, yielding the well known *minimal coupling*.

One readily verifies that the Dirac–Hestenes Lagrangian (24.12) is invariant with respect to the electromagnetic gauge transformation (24.16). Then, according to Noether’s theorem, there is a conserved electric current J^μ defined by

$$\pi^\mu = \frac{\partial L}{\partial \psi_{,\mu}} = c\hbar \mathbf{I} \gamma_3 \tilde{\psi} \gamma^\mu, \quad (24.23)$$

$$J^\mu = \pi^\mu * \bar{\delta}\psi = c \left\langle \hbar \mathbf{I} \gamma_3 \tilde{\psi} \gamma^\mu \psi \mathbf{I} \sigma_3 \epsilon \right\rangle = c \left\langle \hbar \tilde{\psi} \gamma^\mu \psi \gamma_0 \epsilon \right\rangle.$$

Multiplying the current expression in (24.23) by a proper constant and using the STA properties (23.26), (23.34)–(23.37) together with the momentum $\mathcal{P}$ representation (24.11), we obtain the following expression for the electric current

$$J^\mu = ec \left\langle \gamma^\mu \psi \gamma_0 \tilde{\psi} \right\rangle, \quad J = ec \psi \gamma_0 \tilde{\psi}, \quad J^\mu = (c\rho, \mathbf{J}). \quad (24.24)$$

Assuming that ψ satisfies the Dirac equation (24.6), (24.7), we can recast the above expression into

$$J^\mu = \frac{e}{m} \left\langle \gamma^\mu (\mathcal{P}\psi) \tilde{\psi} \right\rangle, \quad (24.25)$$

$$J = J^\mu \gamma_\mu = \frac{e}{m} (\mathcal{P}\psi) \tilde{\psi} = \frac{e}{m} \left(\chi \partial \psi \mathbf{I} \sigma_3 - \frac{e}{c} A \psi \right) \tilde{\psi}.$$

The expression $c\psi\gamma_0\tilde{\psi}$ is known as the *Dirac probability current* in QM, [169, VII.D], [172, 10], whereas the expression $\frac{e}{m} (\mathcal{P}\psi) \tilde{\psi}$ is known as the *Gordon current* [162, 5], [169, VII.H]. We want to stress that the current J expressions (24.24) and (24.25) are evidently two very different expressions which are equal only because ψ satisfies the Dirac equation (24.6), (24.7). Consequently, one may interpret the Dirac equation as a requirement that two generally different currents defined by (24.24) and (24.25) must be the same. In addition to that, notice that the expression $\frac{e}{m} (\chi \partial \psi \mathbf{I} \sigma_3 - \frac{e}{c} A \psi) \tilde{\psi}$ in (24.25) for a general even ψ , that is, ψ not necessarily satisfying the Dirac equation, can take multivector values. So, in the case of a general even ψ , the proper Gordon current expression based on its components J^μ is

$$J = J^\mu \gamma_\mu = \left\langle \frac{e}{m} (\mathcal{P}\psi) \tilde{\psi} \right\rangle_1 = \frac{e}{m} \left\langle \left(\chi \partial \psi \mathbf{I} \sigma_3 - \frac{e}{c} A \psi \right) \tilde{\psi} \right\rangle_1, \quad (24.26)$$

and in the special case when ψ satisfies the Dirac equation the projection operation $\langle \rangle_1$ on the vector space can be naturally omitted since $(\chi \partial \psi \mathbf{I} \sigma_3 - \frac{e}{c} A \psi) \tilde{\psi}$ has to be a vector in this case. The current J satisfies the conservation law

$$\partial \cdot J = 0 \text{ or } \partial_\mu J^\mu = 0, \quad J^\mu = (c\rho, \mathbf{J}), \quad (24.27)$$

where ρ is the charge density and $\mathbf{J}$ is the charge current.

The Gordon current expression (24.25) satisfies the following *Gordon decomposition* law, [166, 3]

$$J^\mu = J_c^\mu + J_s^\mu, \quad (24.28)$$

$$J_c^\mu = \frac{e}{m} \left\langle (\mathcal{P}^\mu \psi) \tilde{\psi} \right\rangle, \quad J_s^\mu = \frac{\hbar e}{m} \left\langle [\gamma^\mu, \gamma^\nu] \partial_\nu \psi \mathbf{I} \sigma_3 \tilde{\psi} \right\rangle,$$

where J_c^μ and J_s^μ are, respectively, the convection and magnetization (spin) currents. To justify the use of magnetization and spin terms, let us recall that the magnetization bivector M and the intimately related to it spin angular momentum bivector S are defined in the STA by the following expressions, [162, 4], [166, 2, 3], [169, VII.C]:

$$S = \frac{\hbar}{2} R \mathbf{I} \sigma_3 \tilde{R} = \frac{\hbar}{2} R \gamma_2 \gamma_1 \tilde{R}, \quad \psi = \varrho^{\frac{1}{2}} e^{\frac{i\beta}{2}} R = R \varrho^{\frac{1}{2}} e^{\frac{i\beta}{2}}, \quad (24.29)$$

$$M = \frac{\hbar e}{2mc} \psi \mathbf{I} \sigma_3 \tilde{\psi} = \frac{\hbar e}{2mc} \psi \gamma_2 \gamma_1 \tilde{\psi} = \frac{e}{mc} \varrho e^{i\beta} S, \quad (24.30)$$

where ψ satisfies the canonical relations (24.1), (24.2). Then the following relations between the components J_s^μ (24.25) and the magnetization bivector M hold:

$$J_s = J_s^\mu \gamma_\mu = c \partial \cdot M, \quad J_s^\mu = c \gamma^\mu \cdot (\partial \cdot M) = c \gamma^\mu \cdot (\gamma^\nu \cdot \partial_\nu M) \quad (24.31)$$

$$= c (\gamma^\mu \wedge \gamma^\nu) \cdot \partial_\nu M = \frac{\hbar e}{m} \left\langle (\gamma^\mu \wedge \gamma^\nu) \partial_\nu \psi \mathbf{I} \sigma_3 \tilde{\psi} \right\rangle.$$

It is instructive to see that the STA Gordon current decomposition representation (24.28) perfectly matches a similar formula in the conventional Dirac theory, [143, 8.1], [307, 8.1], [331, p. 148]:

$$J^\mu = e c \bar{\Psi} \gamma^\mu \Psi = J_c^\mu + J_s^\mu \quad (24.32)$$

$$= \frac{e}{2m} \left[\bar{\Psi} \hat{P}^\mu \Psi - \overline{(\hat{P}^\mu \Psi)} \Psi \right] - \frac{ie}{2m} \partial^\nu (\bar{\Psi} \sigma_\nu^\mu \Psi),$$

where σ_ν^μ is defined by

$$\sigma_{\mu\nu} = \frac{i}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu), \quad (24.33)$$

and

$$J_c^\mu = \frac{e}{2m} \left[\bar{\Psi} \hat{P}^\mu \Psi - \overline{(\hat{P}^\mu \Psi)} \Psi \right] \text{ is the convection current density,} \quad (24.34)$$

$$J_s^\mu = -\frac{ie}{2m} \hat{P}^\nu (\bar{\Psi} \sigma_\nu^\mu \Psi) \text{ is the spin current density.}$$

Notice that the spin (magnetization) current J_s in view of the representation $J_s = \partial \cdot M$ in (24.31) is conserved since

$$\partial \cdot J_s = c \partial \cdot (\partial \cdot M) = c (\partial \wedge \partial) \cdot M = 0. \quad (24.35)$$

Combining (24.27) and (24.28), we obtain also the conservation law for the convection current J_c^μ :

$$\partial_\mu J_c^\mu = 0. \quad (24.36)$$

Notice also that, assuming that J is the Dirac probability current defined by (24.24), we can recast the Lagrangian in (24.13)

$$L = \dot{L} - \frac{1}{c} \langle AJ \rangle = c \left\langle \left[(\hbar \partial \psi \mathbf{I} \sigma_3 - mc \overleftarrow{\gamma}_0 \psi) \right] \gamma_0 \tilde{\psi} \right\rangle - \frac{1}{c} \langle AJ \rangle, \quad (24.37)$$

indicating that the current J defined by (24.24) is in accord with the classical theory. Indeed, in the classical theory the EM interaction between the EM field four-potential A and the current J is described by the expression $\frac{1}{c} \langle AJ \rangle$.

24.1.2 Energy-Momentum Conservation

Applying the Noether theorem to the Dirac Lagrangian (24.12), (24.13) and using relations (24.10), (24.14), we consequently obtain for the canonical EnMT $\dot{T}^{\mu\nu}$ the following expression:

$$\pi^\mu = \frac{\partial L}{\partial \psi_{,\mu}} = c \hbar \mathbf{I} \gamma_3 \tilde{\psi} \gamma^\mu, \quad \dot{T}^{\mu\nu} = \pi^\mu * \partial^\nu \psi = c \left\langle \hbar \mathbf{I} \gamma_3 \tilde{\psi} \gamma^\mu \partial^\nu \psi \right\rangle. \quad (24.38)$$

The above expression for EnMT $\dot{T}^{\mu\nu}$ after an elementary transformation turns into

$$\begin{aligned} \dot{T}^{\mu\nu} &= c \left\langle \hbar \mathbf{I} \gamma_3 \tilde{\psi} \gamma^\mu \partial^\nu \psi \right\rangle = c \left\langle \gamma_0 \tilde{\psi} \gamma^\mu \hbar \partial^\nu \psi \mathbf{I} \sigma_3 \right\rangle \\ &= c \left\langle \gamma^\mu \left(\hat{\mathcal{P}}^\nu \psi \right) \gamma_0 \tilde{\psi} \right\rangle. \end{aligned} \quad (24.39)$$

Then the EnMT conservation law takes the form

$$\partial_\mu \dot{T}^{\mu\nu} = -\partial^\nu L = \frac{1}{c} (\partial^\nu A^\mu) J_\mu. \quad (24.40)$$

Observe that the canonical EnMT $\dot{T}^{\mu\nu}$ involves $\hat{\mathcal{P}}^\nu$ and evidently is not gauge invariant. To find its gauge invariant modification $T^{\mu\nu}$, we use the charge conservation law $\partial^\mu J_\mu = 0$, and obtain the following identity

$$\begin{aligned} (\partial^\nu A^\mu) J_\mu &= (\partial^\nu A^\mu - \partial^\mu A^\nu) J_\mu + (\partial^\mu A^\nu) J_\mu \\ &= F^{\nu\mu} J_\mu + \partial^\mu (A^\nu J_\mu) = F^{\nu\mu} J_\mu + \partial_\mu (A^\nu J^\mu) \end{aligned} \quad (24.41)$$

where $F^{\nu\mu} = \partial^\nu A^\mu - \partial^\mu A^\nu$ are components of the EM field bivector $F = \frac{1}{2} F_{\nu\mu} \gamma^\nu \wedge \gamma^\mu$. The above identity allows us to recast the conservation law (24.40) as

$$\partial_\mu \left(\dot{T}^{\mu\nu} - \frac{1}{c} A^\nu J^\mu \right) = \frac{1}{c} F^{\nu\mu} J_\mu. \quad (24.42)$$

The equality (24.42) in turn suggests to introduce the following gauge invariant modification $T^{\mu\nu}$ of the canonical EnMT $\dot{T}^{\mu\nu}$:

$$T^{\mu\nu} = \dot{T}^{\mu\nu} - \frac{1}{c} A^\nu J^\mu = c \left\langle \gamma^\mu (\mathcal{P}^\nu \psi) \gamma_0 \tilde{\psi} \right\rangle. \quad (24.43)$$

Then (24.42) can be recast into the conservation law

$$\partial_\mu T^{\mu\nu} = \frac{1}{c} F^{\nu\mu} J_\mu, \quad (24.44)$$

where $\frac{1}{c} F^{\nu\mu} J_\mu$ are the components of the Lorentz force density. Using the identity

$$F^{\nu\mu} J_\mu = (\gamma^\nu \wedge \gamma^\mu) \cdot F J_\mu = \gamma^\nu \cdot (\gamma^\mu \cdot F) J_\mu = \gamma^\nu \cdot (J \cdot F), \quad (24.45)$$

and introducing the vectors

$$T^\mu = T^{\mu\nu} \gamma_\nu, \quad (24.46)$$

we can recast the EnMT conservation (24.43) into a concise vector form

$$\partial_\mu T^\mu = \frac{1}{c} J \cdot F, \text{ where } J \cdot F \text{ is the Lorentz force vector.} \quad (24.47)$$

The properties of the gauge invariant EnMT $T^{\mu\nu}$ and T^μ are thoroughly studied in [166, 3].

24.2 Free Electron Solutions to the Dirac Equation

This section provides basic information on the plane wave solutions to the Dirac–Hestenes equations following [164, 6], [166, 4], [169, VIII.B], [93, 8.3.2]. The free electron satisfies the Dirac equation (24.5) with $A = 0$, that is,

$$\hbar \partial \psi \mathbf{I} \sigma_3 = mc \psi \gamma_0, \text{ where } \mathbf{I} \sigma_3 = \gamma_1 \gamma_2. \quad (24.48)$$

A *positive energy plane-wave solution* ψ_- to the Dirac equation (24.48) for the electron is defined as follows:

$$\text{positive energy solution: } \psi_- = \psi_0 e^{-\mathbf{I} \sigma_3 k \cdot x}, \text{ where } \gamma_0 \cdot k > 0, \quad (24.49)$$

and ψ_0 is a constant spinor. Notice that in ψ_- the subindex “-” signifies the sign of the electron charge. Recall that the *wavevector* k is related to the *momentum vector* p by $p = \hbar k$, and we obtain the following spacetime split representations in terms of relative vectors:

$$k\gamma_0 = \frac{\omega}{c} + \mathbf{k}, \quad p\gamma_0 = \hbar k\gamma_0 = \frac{\hbar\omega}{c} + \hbar\mathbf{k} = \frac{E}{c} + \mathbf{p}. \quad (24.50)$$

If the charge is at rest in the γ_0 -frame interpreted as $\mathbf{p} = \mathbf{0}$, then according to the above formula

$$p = p \cdot \gamma_0 = p_0 = \frac{\hbar\omega_0}{c} = mc. \quad (24.51)$$

Since $\partial = \gamma^\mu \partial_\mu$, we have

$$\partial\psi = \partial\psi_0 e^{-I\sigma_3 k \cdot x} = -k\psi_0 e^{-I\sigma_3 k \cdot x} I\sigma_3 = -k\psi I\sigma_3, \quad (24.52)$$

implying that $\psi = \psi_0 e^{-I\sigma_3 k \cdot x}$ is a solution to the Dirac equation (24.48) if and only if ψ_0 satisfies

$$p\psi_0 = mc\psi_0\gamma_0. \quad (24.53)$$

Multiplying the above equation on the right by $\tilde{\psi}_0$, we obtain

$$p\psi_0\tilde{\psi}_0 = mc\psi_0\gamma_0\tilde{\psi}_0. \quad (24.54)$$

We assume the constant spinor ψ_0 to be normalized with the following canonical representation (24.2):

$$\psi_0 = e^{\frac{I\beta_0}{2}} R_0, \quad \psi_0\tilde{\psi}_0 = e^{\beta_0 I}, \quad \text{where } \beta_0 \text{ is real,} \quad (24.55)$$

R_0 is the Lorentz rotor: $R_0\tilde{R}_0 = \tilde{R}_0R_0 = 1$.

Then it follows from (24.54) and (24.55) that

$$pe^{\beta_0 I} = mcR_0\gamma_0\tilde{R}_0, \quad (24.56)$$

and since both the p and $R_0\gamma_0\tilde{R}_0$ are vectors, we must have

$$\psi_0\tilde{\psi}_0 = e^{\beta_0 I} = \pm 1, \quad \text{that is, } \beta_0 = 0, \pi. \quad (24.57)$$

Since $\gamma_0 \cdot p > 0$ and $\gamma_0 \cdot R_0\gamma_0\tilde{R}_0 > 0$, as follows from (24.49), we must have $e^{\beta_0 I} = 1$ in (24.56), that is,

$$p = mcR_0\gamma_0\tilde{R}_0. \quad (24.58)$$

The rotor R_0 solving the problem (24.58) is the product

$$R_0 = LU, \quad (24.59)$$

where the boost L is defined by

$$L = \frac{1 + v\gamma_0}{[2(1 + v \cdot \gamma_0)]^{1/2}}, \quad (24.60)$$

$$v = \frac{p}{mc} = \gamma \left(1 + \frac{\mathbf{v}}{c}\right) \gamma_0 = \frac{1}{mc} \left(\frac{E}{c} + \mathbf{p}\right) \gamma_0,$$

or in view of (24.49)

$$L = L(\mathbf{p}) = \frac{E_0 + E(\mathbf{p}) + \mathbf{cp}}{[2E_0(E_0 + E(\mathbf{p}))]^{1/2}}, \quad (24.61)$$

$$\text{where } E(\mathbf{p}) = E_0 \sqrt{\frac{\mathbf{p}^2}{c^2} + 1}, \quad E_0 = mc^2 = p_0 c = \hbar k_0 c = \hbar \omega, \quad (24.62)$$

and the rotor U is a pure rotation in the γ_0 -frame, that is, $U\gamma_0 = \gamma_0 U$.

A *negative energy plane-wave solution* ψ_+ to the Dirac equation (24.48) is defined by a formula similar to (24.48) but with the phase factor $e^{+I\sigma_3 k \cdot x}$, namely

$$\text{negative energy solution: } \psi_+ = \psi_0 e^{I\sigma_3 k \cdot x}, \quad \text{where } k \cdot \gamma_0 > 0. \quad (24.63)$$

Notice that in ψ_+ the subindex “+” signifies that the sign of the positron charge is opposite to the negative sign of the electron charge. For negative energy solutions in place of (24.56) we have

$$-p e^{\beta_0 I} = mc R_0 \gamma_0 \tilde{R}_0, \quad (24.64)$$

and, consequently, $e^{\beta_0 I} = -1$, implying

$$\text{for negative energy: } p = mc R_0 \gamma_0 \tilde{R}_0, \quad \psi_0 \tilde{\psi}_0 = e^{\beta_0 I} = -1. \quad (24.65)$$

Positive and negative energy plane wave states are commonly interpreted as, respectively, an electron state and a positron (antiparticle) state with positive energy, [331, 2.1.6]. Their representations for a positive energy electron and a negative energy positron can be summarized, respectively, by

$$\text{pos. en. : } \psi_0 \tilde{\psi}_0 = e^{\beta_0 I} = 1: \quad \psi_- = L(\mathbf{p}) U_r e^{-I\sigma_3 k \cdot x}, \quad (24.66)$$

$$\text{neg. en. : } \psi_0 \tilde{\psi}_0 = e^{\beta_0 I} = -1: \quad \psi_+ = L(\mathbf{p}) U_r I e^{I\sigma_3 k \cdot x}, \quad (24.67)$$

where $p = \hbar k$, and the subscript r at the spatial rotor U_r labels the spin state with

$$U_0 = 1, \quad U_1 = -\mathbf{I}\sigma_2 = \gamma_1\gamma_3, \quad U_1\gamma_3\tilde{U}_1 = -\gamma_3. \quad (24.68)$$

Electron and positron states in (24.66)–(24.67) can be related to each other by the so-called *charge conjugation* transformation, [169, VII.C, VIII.B], [331, 2.1.6], defined by

$$\psi^C = \psi\sigma_2, \text{ where } \sigma_2 = \gamma_2\gamma_0. \quad (24.69)$$

Namely, σ_2 anticommutes with γ_0 and $\mathbf{I}\sigma_3$, therefore if ψ solves the Dirac equation (24.5) with charge e , its conjugate ψ^C solves the Dirac equation with the charge $-e$, that is,

$$\hbar\partial\psi^C\mathbf{I}\sigma_3 + \frac{e}{c}A\psi^C = mc\psi^C\gamma_0, \text{ where } \mathbf{I}\sigma_3 = \gamma_1\gamma_2. \quad (24.70)$$

Notice also that the following identity holds for any real α

$$e^{-\mathbf{I}\sigma_3\alpha}\sigma_2 = \sigma_2e^{\mathbf{I}\sigma_3\alpha}, \quad (24.71)$$

implying together with (23.10), (23.51) and (24.66)–(24.67) that

$$\psi_-^C = L(\mathbf{p})U'_r\mathbf{I}e^{-\mathbf{I}\sigma_3k\cdot x}, \quad U'_r = U_r(-\mathbf{I}\sigma_2). \quad (24.72)$$

Observe that ψ_-^C in the above equation is a state similar to ψ_+ in (24.67) indicating that the charge conjugation transforms an electron state into an antiparticle (positron) state with positive energy. Note that in view of the last equality in (24.68), the factor $-\mathbf{I}\sigma_2 = U_1$ represents a spatial rotation that “flips” the direction of the spin vector, [169, VIII.B]. In fact, the charge conjugation $\psi \rightarrow \psi^C$ reverses the charge, energy, momentum, and spin of an electron state, transferring it into a positron state describing the antiparticle with opposite charge $-e$ in the same potential A^μ , [331, 2.1.6].

Chapter 25

The Basics of the Neoclassical Theory of Charges with Spin 1/2

We develop in this section a spinorial version of our neoclassical Lagrangian field theory of elementary charges. The initial step in this development is to assume that the wave function ψ of a single charge such as an electron takes values in the even algebra $\text{Cl}_+(1, 3)$ just as in the Dirac theory. We focus here on the theory of a single charge in an external electromagnetic field. Extension of this theory to the case of many elementary charges is similar to the same for spinless charges constructed and studied in [17]–[18] and in Part III of this book.

The Lagrangian of a *single elementary charge in an external electromagnetic field described by the 4-potential $\check{A}$* is

$$L = \frac{1}{2m} \left\{ \left\langle \mathcal{P}\psi (\mathcal{P}\psi)^\sim \right\rangle - \chi^2 \left[\kappa_0^2 \left\langle \psi \check{\psi} \right\rangle + G \left(\left\langle \psi \check{\psi} \right\rangle \right) \right] \right\}, \quad \kappa_0 = \frac{mc}{\chi}, \quad (25.1)$$

where (i) m is the electron mass; (ii) χ is a constant approximately equal to the Planck constant $\hbar$; (iii) G is a *nonlinear self-interaction term of non-electromagnetic origin*, and (iv)

$$\mathcal{P}\psi = \chi \partial_t \psi \mathbf{I} \sigma_3 - \frac{e}{c} \check{A} \psi \quad (25.2)$$

is the *momentum operator* which is identical to the same in the Dirac–Hestenes equation (24.6). Notice that we have somewhat departed from the common notation of the Dirac theory denoting the external EM 4-potential by $\check{A}$ instead of A . The reason for such an alteration is that *there is no electromagnetic self-interaction for an elementary charge in our theory*, and every charge is associated with its individual wave function ψ and elementary EM four-potential A . So, to avoid any confusion and to distinguish the external 4-potential from the elementary 4-potential A , we use $\check{A}$ for the external one.

The nonlinearity $G(s)$ in (25.1) is defined by the formula (21.1), and its derivative is given by formula (14.41), namely

$$G'(s) = G'_a(s) = -a^{-2} \left[\ln(a^3 |s|) + \ln \pi^{3/2} + 3 \right], \quad -\infty < s < \infty. \quad (25.3)$$

Notice that (21.1) and (25.3) imply the following identity

$$sG'_a(s) - G_a(s) = -a^{-2}s. \quad (25.4)$$

As has already been explained, the role of the nonlinear self-interaction term G is to provide for the localization of the elementary charge in relevant situations.

Just as in the Dirac theory, it is useful to single out the “free” part $\mathring{P}$ of $\mathcal{P}$, namely

$$\mathcal{P}\psi = \mathring{P}\psi - \frac{e}{c}\check{A}\psi, \text{ where } \mathring{P}\psi = \chi\partial\psi\mathbf{I}\sigma_3. \quad (25.5)$$

The coordinate forms $\mathcal{P}_\mu$ and $\mathring{P}_\mu$ of the above momenta operators are

$$\begin{aligned} \mathcal{P}_\mu\psi &= \chi\partial_\mu\psi\mathbf{I}\sigma_3 - \frac{e}{c}\check{A}_\mu\psi, & \mathcal{P}\psi &= (\gamma^\mu\mathcal{P}_\mu)\psi, \\ \mathring{P}_\mu\psi &= \chi\partial_\mu\psi\mathbf{I}\sigma_3, & \mathring{P}\psi &= (\gamma^\mu\mathring{P}_\mu)\psi. \end{aligned} \quad (25.6)$$

When transforming expressions involving the reversion operation, we often use the following elementary identities:

$$\widetilde{\sigma_3} = -\sigma_3, \quad \widetilde{\mathbf{I}} = \mathbf{I}, \quad \widetilde{\mathbf{I}\sigma_3} = -\mathbf{I}\sigma_3 = -\sigma_3\mathbf{I}. \quad (25.7)$$

A Lagrangian treatment of the conservation laws based on a multivector Noether theorem has been developed in [219, 4–6], [93, 12.4, 13], and we adopt most of that approach here. For more details of mathematical aspects of the Lagrangian field theory for multivector-valued fields, we refer the reader to [283, 7]. To obtain the Euler–Lagrange equations for the Lagrangian L defined by (25.1), we find first its derivatives

$$\frac{\partial L}{\partial\psi} = -\frac{1}{m} \left\{ (\mathcal{P}\psi)^\sim \frac{e}{c}\check{A} + \chi^2 \left[\kappa_0^2 + G' \left(\left\langle \psi\tilde{\psi} \right\rangle \right) \right] \tilde{\psi} \right\}, \quad (25.8)$$

$$\pi^\mu = \frac{\partial L}{\partial\psi_{,\mu}} = \frac{\chi}{m} \mathbf{I}\sigma_3 (\mathcal{P}\psi)^\sim \gamma^\mu. \quad (25.9)$$

Notice that we have dropped the projection operation $\langle * \rangle_X$ in the right-hand sides of (25.8), (25.9) since their expressions take values in the even subalgebra. Using expressions (25.8), (25.9), we obtain the Euler–Lagrange equation

$$- (\mathcal{P}\psi)^\sim \frac{e}{c}\check{A} - \chi^2 \left[\kappa_0^2 + G' \left(\left\langle \psi\tilde{\psi} \right\rangle \right) \right] \tilde{\psi} - \partial_\mu \chi \mathbf{I}\sigma_3 (\mathcal{P}\psi)^\sim \gamma^\mu = 0. \quad (25.10)$$

Application of the reversion operation to the above equation yields

$$- \frac{e}{c}\check{A}\mathcal{P}\psi - \left[\kappa_0^2 + G' \left(\left\langle \psi\tilde{\psi} \right\rangle \right) \right] \psi + \chi\partial (\mathcal{P}\psi) \mathbf{I}\sigma_3 = 0, \quad (25.11)$$

which, in turn, in view of the expression (25.2) for $\mathcal{P}$, can be transformed into a more concise form of the *field equation*

$$\mathcal{P}^2\psi - \left[\kappa_0^2 + G' \left(\langle \psi \tilde{\psi} \rangle \right) \right] \psi = 0. \quad (25.12)$$

Hence, the field equation (25.12) is the master evolution equation for the wave function in our theory based on the Lagrangian (25.1). The expression $\mathcal{P}^2\psi$ in Eq. (25.12) can be transformed into the following form showing the external EM field

$$\mathcal{P}^2\psi = \mathring{\mathcal{P}}^2\psi - \frac{\chi e}{c} \left[F + 2\check{A} \cdot \partial \right] \psi \mathbf{I}\sigma_3 + \frac{e^2}{c^2} \check{A}^2\psi, \quad (25.13)$$

where $F = \partial \wedge \check{A}$ is the bivector of the electromagnetic field.

Using the commutativity (24.9) of the operator $\check{\gamma}_0$ and the momentum operator $\mathcal{P}$, one can factorize the expression $\mathcal{P}^2\psi - \kappa_0^2\psi$ in the Eq. (25.12) yielding

$$(\mathcal{P} + mc\check{\gamma}_0) (\mathcal{P} - mc\check{\gamma}_0) \psi - \chi^2 G' \left(\langle \psi \tilde{\psi} \rangle \right) \psi = 0. \quad (25.14)$$

It is instructive to compare the above field equation (25.12) with the Dirac–Hestenes equations (24.6), (24.7). Just by looking at the two equations, one can see two significant differences. First of all, the field equation (25.12) contains a nonlinear self-interaction term $G' \left(\langle \psi \tilde{\psi} \rangle \right)$, that is, a concept not present in the Dirac theory. For comparison purposes it is instructive to eliminate this nonlinear term from the field equation (25.12) resulting in

$$\text{the truncated field equation: } (\mathcal{P}^2 - m^2c^2) \psi = 0. \quad (25.15)$$

Now one can see another significant difference between the truncated field equation (25.15) and the Dirac–Hestenes equation (24.6). Indeed, the Dirac–Hestenes equation (24.6) is linear in $\mathcal{P}$ whereas the truncated field equation (25.15) is quadratic in $\mathcal{P}$. In spite of this difference it is possible to establish an intimate relation between the two equations by factorizing the truncated field equation (25.15). To do that we use the commutativity (24.9) of the operator $\check{\gamma}_0$ and the momentum operator $\mathcal{P}$ and factorize Eq. (25.12) into the following form

$$(\mathcal{P} + mc\check{\gamma}_0) (\mathcal{P} - mc\check{\gamma}_0) \psi = 0, \text{ truncated equation factorized.} \quad (25.16)$$

The above factorization of the truncated field equation is not unique. In fact, one can drop the operator $\check{\gamma}_0$ from it, and what is left is still a correct representation of the original field equation (25.12). An important justification for the factorization (25.16) with the operator $\check{\gamma}_0$ is as follows. For even ψ both the vectors $\mathcal{P}\psi$ and $mc\check{\gamma}_0\psi$ are odd, hence each of the equation

$$(\mathcal{P} - mc \overleftarrow{\gamma}_0) \psi = 0, \quad (\mathcal{P} + mc \overleftarrow{\gamma}_0) \psi = 0 \quad (25.17)$$

can have even solutions. On the other hand, the equations

$$(\mathcal{P} - mc) \psi = 0, \quad (\mathcal{P} + mc) \psi = 0 \quad (25.18)$$

cannot have a nontrivial even solution since for even ψ the multivector $\mathcal{P}\psi$ is always odd.

Observe now that any linear combination of solutions to Eq. (25.17) is a solution to the truncated field equations (25.16). Hence any solution to the Dirac–Hestenes equation (24.5)–(24.7) also solves the truncated form (25.16) of the neoclassical field equation. In particular, let us take the external potential $\hat{A}$ to be the Coulomb potential, that is, $\hat{A} = \hat{A}_c^\mu = \left(-\frac{Ze^2}{|\mathbf{x}|}, 0\right)$ where Z is the nucleus charge. Then solutions to the Dirac–Hestenes equation for the Coulomb potential

$$\hbar \partial \psi \mathbf{I} \sigma_3 - \frac{e}{c} \hat{A}_c \psi = mc \psi \gamma_0, \quad \hat{A}_c^\mu = \left(-\frac{Ze^2}{|\mathbf{x}|}, 0\right), \quad (25.19)$$

are solutions to the truncated field equation (25.16) and, consequently, are approximate solutions to the neoclassical field equation (25.12) with neglected nonlinearity G . Notice that the typical spatial scale of electron states in the Coulomb potential is the Bohr radius, and if the size parameter a is much larger than the Bohr radius, then the nonlinearity can be neglected, see [17] and Sects. 17.5.1 and 35.2. Since the Dirac–Hestenes equation for the Coulomb potential (25.19) is exactly equivalent to the original Dirac equation for the same potential, [169, VII], we can claim that the frequency spectrum of the neoclassical field equation (25.12) includes as an approximation the well known frequency spectrum of the Dirac equation, [299, 8.2].

Interestingly, in [289] the Eq. (25.15) (called the “square of the Dirac equation”) is derived by conformal differential geometry. The general setup in [289], though very different from our neoclassical approach, has some common features including the underlying continuum and that QM is not a starting point but rather an approximation.

25.1 Conservation Laws

Our treatment of the charge and energy-momentum conservation is based on the Noether theorem and consequently requires the knowledge of relevant groups of transformations which leave the Lagrangian invariant.

25.1.1 Charge and Current Densities

The neoclassical Lagrangian (25.8) is invariant with respect to the global charge gauge transformation (24.18) as in the case of the Dirac theory. Consequently,

Noether's current reduces in this case to the following expression for the electric current

$$\begin{aligned} J^\mu &= \pi^\mu * \bar{\delta}\psi = \frac{\partial \mathcal{L}}{\partial \psi_{,\mu}} * \bar{\delta}\psi = \frac{\chi}{m} \left\langle \mathbf{I}\sigma_3 (\mathcal{P}\psi)^\sim \gamma^\mu \psi \mathbf{I}\sigma_3 \epsilon \right\rangle \\ &= -\frac{\chi}{m} \left\langle (\mathcal{P}\psi)^\sim \gamma^\mu \psi \right\rangle \epsilon = -\frac{\chi}{m} \left\langle \gamma^\mu (\mathcal{P}\psi) \tilde{\psi} \right\rangle \epsilon, \end{aligned} \quad (25.20)$$

where we have used expressions (25.9) and (24.18) for π^μ and $\bar{\delta}\psi$ respectively. Multiplication of the above expression for J^μ by a suitable constant and consequent transformations yield the following expressions for the current components

$$\begin{aligned} J^\mu &= \frac{e}{m} \left\langle \gamma^\mu (\mathcal{P}\psi) \tilde{\psi} \right\rangle = \frac{e}{m} \gamma^\mu \cdot \left\langle (\mathcal{P}\psi) \tilde{\psi} \right\rangle_1 \\ &= \frac{e}{m} \left\langle \gamma^\mu \left(\chi \partial \psi \mathbf{I}\sigma_3 - \frac{e}{c} \check{A} \psi \right) \tilde{\psi} \right\rangle = \frac{e}{m} \gamma^\mu \cdot \left\langle \left(\chi \partial \psi \mathbf{I}\sigma_3 - \frac{e}{c} \check{A} \psi \right) \tilde{\psi} \right\rangle_1, \end{aligned} \quad (25.21)$$

implying the following concise form for the current vector

$$J = \frac{e}{m} \gamma_\mu J^\mu = \frac{e}{m} \left\langle (\mathcal{P}\psi) \tilde{\psi} \right\rangle_1 = \frac{e}{m} \left\langle \left(\chi \partial \psi \mathbf{I}\sigma_3 - \frac{e}{c} \check{A} \psi \right) \tilde{\psi} \right\rangle_1. \quad (25.22)$$

Observe that our expressions (25.21) and (25.22) for the current J are exactly the same as the current expressions (24.5) and (24.6) in the Dirac theory.

Notice that any even ψ according to (24.1) satisfies $\psi \tilde{\psi} = \tilde{\psi} \psi = \rho e^{\beta 1}$. Consequently, for any vector $\check{A}$ we have

$$\begin{aligned} \left\langle \tilde{\psi} \gamma^\mu \check{A} \psi \right\rangle &= \left\langle \gamma^\mu \check{A} \psi \tilde{\psi} \right\rangle = \left\langle \left(\gamma^\mu \cdot \check{A} + \gamma^\mu \wedge \check{A} \right) \rho e^{\beta 1} \right\rangle \\ &= \left\langle \gamma^\mu \cdot \check{A} \rho e^{\beta 1} \right\rangle = \check{A}^\mu \left\langle \rho e^{\beta 1} \right\rangle = \check{A}^\mu \left\langle \tilde{\psi} \psi \right\rangle. \end{aligned} \quad (25.23)$$

Using the above identity we can transform the current components J^μ in (25.21) as follows:

$$\begin{aligned} \left\langle \gamma^\mu (\mathcal{P}\psi) \tilde{\psi} \right\rangle &= \left\langle \gamma^\mu \left(\chi \partial \psi \mathbf{I}\sigma_3 - \frac{e}{c} \check{A} \psi \right) \tilde{\psi} \right\rangle \\ &= \chi \left\langle \gamma^\mu \gamma^\nu \partial_\nu \psi \mathbf{I}\sigma_3 \tilde{\psi} \right\rangle - \left\langle \frac{e}{c} \check{A}^\mu \psi \tilde{\psi} \right\rangle \\ &= \frac{\chi}{2} \left\langle (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \partial_\nu \psi \mathbf{I}\sigma_3 \tilde{\psi} \right\rangle \\ &\quad + \frac{\chi}{2} \left\langle (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \partial_\nu \psi \mathbf{I}\sigma_3 \tilde{\psi} \right\rangle - \left\langle \tilde{\psi} \frac{e}{c} \check{A}^\mu \psi \right\rangle \\ &= \frac{\chi}{2} \left\langle 2g^{\mu\nu} \partial_\nu \psi \mathbf{I}\sigma_3 \tilde{\psi} \right\rangle + \chi \left\langle (\gamma^\mu \wedge \gamma^\nu) \partial_\nu \psi \mathbf{I}\sigma_3 \tilde{\psi} \right\rangle - \left\langle \frac{e}{c} \check{A}^\mu \psi \tilde{\psi} \right\rangle \\ &= \left\langle (\mathcal{P}^\mu \psi) \tilde{\psi} \right\rangle + \chi \left\langle (\gamma^\mu \wedge \gamma^\nu) \partial_\nu \psi \mathbf{I}\sigma_3 \tilde{\psi} \right\rangle, \end{aligned} \quad (25.24)$$

yielding

$$J^\mu = \frac{e}{m} \left\langle (\mathcal{P}^\mu \psi) \tilde{\psi} \right\rangle + \frac{\chi e}{m} \left\langle [\gamma^\mu, \gamma^\nu] \partial_\nu \psi \mathbf{I} \sigma_3 \tilde{\psi} \right\rangle, \quad (25.25)$$

where $\mathcal{P}^\mu \psi = \chi \partial^\mu \psi \mathbf{I} \sigma_3 - \frac{e}{c} \check{A}^\mu \psi$,

where we used the commutator product $[\gamma^\mu, \gamma^\nu]$ notation (see (28.51)). Observe that the current expression (25.25) is exactly the same as the Gordon current decomposition (24.28) for the current in the Dirac theory if we substitute $\hbar$ with χ , namely

$$J^\mu = J_c^\mu + J_s^\mu, \text{ where} \quad (25.26)$$

$$J_c^\mu = \frac{e}{m} \left\langle (\mathcal{P}^\mu \psi) \tilde{\psi} \right\rangle, \quad J_s^\mu = \frac{\chi e}{m} \left\langle [\gamma^\mu, \gamma^\nu] \partial_\nu \psi \mathbf{I} \sigma_3 \tilde{\psi} \right\rangle,$$

where J_c^μ and J_s^μ are, respectively, the convection and magnetization (spin) currents. Consequently, just as in the case of the Dirac theory as indicated by conservation laws (24.35), (24.36), the currents are conserved individually:

$$\partial_\mu J_c^\mu = 0, \quad \partial \cdot J_s = 0. \quad (25.27)$$

Notice that the representation (24.29)–(24.31) for the magnetization/spin current in the Dirac theory holds in the neoclassical case as well, namely

$$J_s = J_s^\mu \gamma_\mu = c \partial \cdot M, \quad J_s^\mu = c \gamma^\mu \cdot (\partial \cdot M) = c \gamma^\mu \cdot (\gamma^\nu \cdot \partial_\nu M) \quad (25.28)$$

$$= c (\gamma^\mu \wedge \gamma^\nu) \cdot \partial_\nu M = \frac{\hbar e}{m} \left\langle (\gamma^\mu \wedge \gamma^\nu) \partial_\nu \psi \mathbf{I} \sigma_3 \tilde{\psi} \right\rangle.$$

Observe also that the magnetization bivector M defined in (24.30) can be related to the spin angular momentum bivector S as follows, [162, 4], [166, 2, 3], [169, VII.C]

$$S = \frac{\hbar}{2} R \mathbf{I} \sigma_3 \tilde{R} = \frac{\hbar}{2} R \gamma_2 \gamma_1 \tilde{R}, \quad \psi = \varrho^{\frac{1}{2}} e^{\frac{i\beta}{2}} R = R \varrho^{\frac{1}{2}} e^{\frac{i\beta}{2}}, \quad (25.29)$$

$$M = \frac{\hbar e}{2mc} \psi \mathbf{I} \sigma_3 \tilde{\psi} = \frac{\hbar e}{2mc} \psi \gamma_2 \gamma_1 \tilde{\psi} = \frac{e}{mc} e^{i\beta} \varrho S. \quad (25.30)$$

25.1.2 Gauge Invariant Energy-Momentum Tensor

In the case of the neoclassical Lagrangian (25.8), the general expression for the canonical EnMT $\hat{T}^{\mu\nu}$ with the help of (25.9) reduces to

$$\hat{T}^{\mu\nu} = \pi^\mu * \partial^\nu \psi - \delta^{\mu\nu} L, \quad \text{where } \pi^\mu = \frac{\partial L}{\partial \psi_{,\mu}} = \frac{\chi}{m} \mathbf{I} \sigma_3 (\mathcal{P} \psi) \tilde{\gamma}^\mu, \quad (25.31)$$

implying

$$\dot{T}^{\mu\nu} = \frac{\chi}{m} \left\langle \gamma^\mu \partial^\nu \psi \mathbf{I} \sigma_3 (\mathcal{P}\psi)^\sim \right\rangle - \delta_\nu^\mu L, \quad \text{where } \mathcal{P}\psi = \chi \partial \psi \mathbf{I} \sigma_3 - \frac{e}{c} \check{A} \psi. \quad (25.32)$$

The corresponding conservation law is of the form (10.55):

$$\partial_\mu \dot{T}^{\mu\nu} = -\partial^\nu L, \quad \text{where } \partial^\nu L = \frac{\partial L}{\partial x_\nu}. \quad (25.33)$$

Since the explicit dependence on x_ν in L comes only through the EM potential $\check{A}$, we find that

$$\begin{aligned} \partial^\nu L &= \partial^\nu \frac{1}{2m} \left\langle (\mathcal{P}\psi) (\mathcal{P}\psi)^\sim \right\rangle = \frac{1}{m} \left\langle [\partial^\nu (\mathcal{P}\psi)] (\mathcal{P}\psi)^\sim \right\rangle \\ &= -\frac{1}{m} \frac{e}{c} \left\langle \left(\partial^\nu \check{A} \right) \psi (\mathcal{P}\psi)^\sim \right\rangle = -\frac{1}{m} \frac{e}{c} \left(\partial^\nu \check{A}^\mu \right) \left\langle \gamma_\mu \psi (\mathcal{P}\psi)^\sim \right\rangle \\ &= -\frac{1}{c} \frac{e}{m} \left(\partial^\nu \check{A}^\mu \right) \left\langle \gamma_\mu (\mathcal{P}\psi) \tilde{\psi} \right\rangle = -\frac{1}{c} \left(\partial^\nu \check{A}^\mu \right) J_\mu. \end{aligned} \quad (25.34)$$

The canonical EnMT $\dot{T}^{\mu\nu}$ defined by (25.32) evidently is not gauge invariant. To modify it into a gauge invariant form, we use the expression (25.21) for the current components J^μ and transform EnMT $\dot{T}^{\mu\nu}$ as follows:

$$\begin{aligned} \dot{T}^{\mu\nu} + \delta_\nu^\mu L &= \frac{\chi}{m} \left\langle \gamma^\mu \partial^\nu \psi \mathbf{I} \sigma_3 (\mathcal{P}\psi)^\sim \right\rangle = \frac{1}{m} \left\langle \gamma^\mu \chi \partial^\nu \psi \mathbf{I} \sigma_3 (\mathcal{P}\psi)^\sim \right\rangle \\ &= \frac{1}{m} \left\langle \gamma^\mu (\mathcal{P}^\nu \psi) (\mathcal{P}\psi)^\sim \right\rangle + \frac{1}{m} \left\langle \gamma^\mu \frac{e}{c} \check{A}^\nu \psi (\mathcal{P}\psi)^\sim \right\rangle \\ &= \frac{1}{m} \left\langle \gamma^\mu (\mathcal{P}^\nu \psi) (\mathcal{P}\psi)^\sim \right\rangle + \frac{1}{c} \check{A}^\nu \frac{e}{m} \left\langle \gamma^\mu (\mathcal{P}\psi) \tilde{\psi} \right\rangle \\ &= \frac{1}{m} \left\langle \gamma^\mu (\mathcal{P}^\nu \psi) (\mathcal{P}\psi)^\sim \right\rangle + \frac{1}{c} \check{A}^\nu J^\mu. \end{aligned} \quad (25.35)$$

The above equality suggests to introduce the following expression for the gauge invariant EnMT $T^{\mu\nu}$:

$$T^{\mu\nu} = \dot{T}^{\mu\nu} - \frac{1}{c} \check{A}^\nu J^\mu = \frac{1}{m} \left\langle \gamma^\mu (\mathcal{P}^\nu \psi) (\mathcal{P}\psi)^\sim \right\rangle - \delta_\nu^\mu L. \quad (25.36)$$

Using the conservation law $\partial_\mu J^\mu = 0$ and the canonical EnMT $\hat{T}^{\mu\nu}$ conservation law (25.33), we obtain

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= \partial_\mu \hat{T}^{\mu\nu} - \frac{1}{c} \left(\partial_\mu \check{A}^\nu \right) J^\mu = -\partial^\nu L - \frac{1}{c} \left(\partial_\mu \check{A}^\nu \right) J^\mu \\ &= \frac{1}{c} \left(\partial^\nu \check{A}^\mu \right) J_\mu - \frac{1}{c} \left(\partial^\mu \check{A}^\nu \right) J_\mu = \frac{1}{c} F^{\nu\mu} J_\mu,\end{aligned}\quad (25.37)$$

where $\frac{1}{c} F^{\nu\mu} J_\mu$ is the Lorentz force density.

Chapter 26

Neoclassical Free Charge with Spin

In this section we carry out a rather detailed analysis of the basic case of the free charge when $\check{A} = 0$. The charge Lagrangian (25.1) in the case of the free charge takes the form

$$L = \frac{\chi^2}{2m} \left\{ \left\langle \gamma^\beta \gamma^\alpha (\partial_\alpha \psi) (\partial_\beta \tilde{\psi}) \right\rangle - \left[\kappa_0^2 \langle \psi \tilde{\psi} \rangle + G \left(\langle \psi \tilde{\psi} \rangle \right) \right] \right\}, \quad (26.1)$$

where

$$\kappa_0 = \frac{mc}{\chi}. \quad (26.2)$$

The field equation (25.12) with $\check{A} = 0$ after straightforward transformations turns into the *free charge spinor field equation*

$$-\chi^2 \partial_\mu \partial^\mu \psi - \left[\kappa_0^2 + G' \left(\langle \psi \tilde{\psi} \rangle \right) \right] \psi = 0. \quad (26.3)$$

The above equation is similar to a *scalar nonlinear Klein–Gordon* (NKG) Eq. (11.125), namely

$$-\frac{1}{c^2} \partial_t^2 \psi + \nabla^2 \psi - \kappa_0^2 \psi - G'(|\psi|^2) \psi = 0, \quad (26.4)$$

where ψ is a complex-valued scalar. The scalar equation (26.4) and its solutions are relevant to the analysis of the spinor field equation (26.3) and its basic properties are considered in the following section.

26.1 Scalar Equation

The fundamental rest solution to the scalar NKG equation (26.4) is of the form (14.49):

$$\psi_{\pm}(t, \mathbf{x}) = e^{\mp i\omega_0 t} \psi^{\circ}(|\mathbf{x}|), \text{ where } \omega_0 = \frac{mc^2}{\chi} = \kappa_0 c, \quad (26.5)$$

where

$$\psi^{\circ}(s) = \psi_a^{\circ}(s) = a^{-3/2} \pi^{-3/4} \exp\left(-\frac{s^2}{2a^2}\right), \quad s \geq 0. \quad (26.6)$$

As was shown in Example 14.3.5, $\psi^{\circ}(|\mathbf{x}|)$ satisfies Eq. (14.19), namely

$$\nabla^2 \psi^{\circ} - G'(\psi^{\circ 2}) \psi^{\circ} = 0. \quad (26.7)$$

Observe that the solution $\psi_{\pm}$ is the product of the time-harmonic factor $e^{\mp i\omega_0 t}$ and the Gaussian factor $\psi^{\circ}(|\mathbf{x}|)$ describing the shape of the wave. An STA representation of the above solution which is manifestly coordinate free is as follows:

$$\psi(x) = \psi_{\mp}(v, x) = e^{\mp i\kappa_0 x \cdot v} \psi^{\circ}\left(\sqrt{(x \cdot v)^2 - x^2}\right), \quad (26.8)$$

where v is the proper velocity of the free electron, and one can think of v as describing the rest frame of the electron.

It is instructive to find a representation of the solution $\psi(v, x)$ in (26.8) in the frame of an arbitrary inertial observer γ_0 by relating it to the inertial observer $v = \gamma'_0$. Such a representation can be effectively obtained by introducing a subspace of the vector space $\text{Span}\{v, \gamma_0\}$ and the corresponding orthogonal decomposition as in [265, p. 10], [160, 1]

$$x = x_{\parallel} + x_{\perp}, \quad (26.9)$$

where

$$x_{\parallel} \in \text{Span}\{v, \gamma_0\} \text{ and } x_{\perp} \text{ is orthogonal to } \text{Span}\{v, \gamma_0\}.$$

We will also need the corresponding relative velocity $\mathbf{v}$ which is defined by

$$\frac{\mathbf{v}}{c} = \frac{v \wedge \gamma_0}{v \cdot \gamma_0}, \quad v = \gamma \left(1 + \frac{\mathbf{v}}{c}\right) \gamma_0 = \gamma \gamma_0 \left(1 - \frac{\mathbf{v}}{c}\right) \quad (26.10)$$

where

$$\gamma = v \cdot \gamma_0 = \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-1/2} \text{ is the Lorentz factor.} \quad (26.11)$$

Then we obtain the following identities:

$$x \cdot v = \gamma \left[x_0 - \mathbf{x}_{\parallel} \cdot \frac{\mathbf{v}}{c} \right] \text{ where } \mathbf{x}_{\parallel} = x_{\parallel} \wedge \gamma_0, \quad (26.12)$$

$$\sqrt{(x \cdot v)^2 - x^2} = \left| \gamma \left(\mathbf{x}_{\parallel} - x_0 \frac{\mathbf{v}}{c} \right) + \mathbf{x}_{\perp} \right|, \text{ where } \mathbf{x}_{\perp} = x_{\perp} \wedge \gamma_0. \quad (26.13)$$

Observe that the right-hand sides correspond to standard Lorentz boost transformations for, respectively, time and space components of the vector x , [265, p.10].

Consequently, we get the following representation of the scalar solution (26.8) in the frame of an arbitrary observer γ_0 :

$$\psi(x) = \exp \left\{ \mp i \kappa_0 \gamma \left[x_0 - \mathbf{x}_{\parallel} \cdot \frac{\mathbf{v}}{c} \right] \right\} \psi^{\circ} \left(\left| \gamma \left(\mathbf{x}_{\parallel} - x_0 \frac{\mathbf{v}}{c} \right) + \mathbf{x}_{\perp} \right| \right). \quad (26.14)$$

The charge and current densities in the scalar case are given by the expressions (11.130), (11.131):

$$\rho = -\frac{\chi q}{mc^2} \operatorname{Im} \frac{\partial_t \psi}{\psi} |\psi|^2, \quad \mathbf{J} = \frac{\chi q}{m} \operatorname{Im} \frac{\nabla \psi}{\psi} |\psi|^2,$$

implying for the solutions $\psi_{\pm}$ in (26.5) the following representation for the total conserved charge

$$q_{\pm} = \int_{\mathbb{R}^3} \rho_{\pm}(t, \mathbf{x}) \, d\mathbf{x} = \pm \frac{\chi q \omega_0}{mc^2} = \pm q.$$

The conserved energy $\mathcal{E}$ and momentum $\mathbf{p}$ densities are given by (11.142) and (11.143). In particular, the expression (14.60) for the total conserved energy $E_{\pm}$ for the wave function $\psi_{\pm}$ defined by (26.8) takes the form

$$E_{\pm} = \chi \omega_0 \left(1 + \frac{a_c^2}{2a^2} \right) > 0. \quad (26.15)$$

So we observe that *the energy of rest solutions for both the positive and negative frequencies is always positive*. A very detailed theory of the scalar Klein–Gordon equation including the Lagrangian treatment can be found in [143, 1.5], [331, 1.1, p.20].

26.2 Solutions to the Spinor Field Equation

We seek a solution to the free charge spinor equation (26.3) which is expected to incorporate the features of the scalar solution as in (26.8) and the plane-wave solution (24.49) to the Dirac equation. We find that such a spinor rest solution does exist and is of the form $\psi(x) = \psi_{\mp}(v, x)$

$$\psi_{\mp}(v, x) = \psi_0 e^{\mp i \sigma_3 \kappa_0 x \cdot v} \psi^{\circ} \left(\sqrt{(x \cdot v)^2 - x^2} \right), \quad \kappa_0 = \frac{\omega_0}{c}, \quad (26.16)$$

where the Gaussian factor $\mathring{\psi}$ is defined by (26.6) and ψ_0 is a normalized constant spinor from the even subalgebra $\text{Cl}_+(1, 3)$ satisfying the canonical representation (24.55) and the following special conditions

$$\langle \psi_0 \tilde{\psi}_0 \rangle = 1 \text{ or } \langle \psi_0 \tilde{\psi}_0 \rangle = -1, \quad (26.17)$$

that is, $\beta_0 = 0$ or $\beta_0 = \pi$. One can readily see a distinct feature of the spinor solution (26.16) compared to the plane-wave solution (24.49) to the Dirac equation. It is the amplitude factor $\mathring{\psi}$ which can be attributed to the nonlinearity $G(\langle \psi \tilde{\psi} \rangle)$ in the spinor equation (26.3). The origin of the special constraints (26.17) can be traced to the particular way ψ enters the nonlinearity, namely as $G(\langle \psi \tilde{\psi} \rangle)$. For ψ of the form (26.16) to be a solution to the free charge spinor equation (26.3), there has to be an effective reduction to the scalar equation (26.4) with the nonlinearity $G'(\mathring{\psi}^2)$. The constraint (26.17) is essential for such a reduction. Indeed, if the spinor wave function ψ is defined by (26.16) and satisfies the condition (26.17), then

$$\langle \psi \tilde{\psi} \rangle = \langle \psi_0 \tilde{\psi}_0 \rangle \mathring{\psi}^2, \quad (26.18)$$

implying that

$$G'(\langle \psi \tilde{\psi} \rangle) = G'(\langle \psi_0 \tilde{\psi}_0 \rangle \mathring{\psi}^2) = G'(\mathring{\psi}^2). \quad (26.19)$$

When establishing identities (26.18), (26.19) we used the identity (25.7) and that $G'(s)$ defined by (25.3) is an even function. The identities (26.19) allow us to reduce the spinor equation (26.3) to the scalar equation (26.4).

26.3 Charge and Current Densities

Let us consider the solution $\psi_{\mp}(x)$ as in (26.16) with $v = \gamma_0$, that is,

$$\psi_{\mp}(x) = \psi_{\mp}(\gamma_0, x) = \psi_0 e^{\mp i \sigma_3 \kappa_0 x_0} \mathring{\psi} \left(((x \cdot \gamma_0)^2 - x^2)^{1/2} \right) \quad (26.20)$$

where $x_0 = x \cdot \gamma_0$, the Gaussian factor $\mathring{\psi}$ is defined by (26.6) and ψ_0 satisfies the condition (26.17), that is, $\langle \psi_0 \tilde{\psi}_0 \rangle = \pm 1$. Then the current component J^μ defined by (25.21) for the free charge with $\dot{A} = 0$ takes the following form:

$$J^\mu = \frac{q}{m} \langle \gamma^\mu \chi \partial \psi \mathbf{I} \sigma_3 \tilde{\psi} \rangle = \frac{q}{m} \langle \gamma^\mu \chi \partial \psi \mathbf{I} \sigma_3 \tilde{\psi} \rangle. \quad (26.21)$$

Since for the rest solution $\psi_{\mp}$ defined by (26.20) $\partial_0 \psi = 0$, the following relations hold:

$$\begin{aligned} \partial_0 \psi_{\mp} &= \psi_0 (\mp \mathbf{I} \sigma_3 \kappa_0) e^{\mp \mathbf{I} \sigma_3 \kappa_0 x_0} \dot{\psi} \\ &= \psi_0 e^{\mp \mathbf{I} \sigma_3 \kappa_0 x_0} (\mp \mathbf{I} \sigma_3 \kappa_0) \dot{\psi} = \psi_{\mp} (\mp \mathbf{I} \sigma_3 \kappa_0) \dot{\psi}, \\ \partial_j \psi_{\mp} &= \psi_0 e^{\mp \mathbf{I} \sigma_3 \kappa_0 x_0} \partial_j \dot{\psi} = \psi_{\mp} \partial_j \ln \dot{\psi}, \end{aligned} \quad (26.22)$$

$$\begin{aligned} \partial \psi_{\mp} &= \gamma^{\mu} \partial_{\mu} \psi_0 e^{\mp \mathbf{I} \sigma_3 \kappa_0 x_0} \dot{\psi} \\ &= \left[\gamma^0 \psi_0 (\mp \mathbf{I} \sigma_3 \kappa_0) \dot{\psi} + \sum_{1 \leq j \leq 3} \gamma^j \psi_0 \partial_j \dot{\psi} \right] e^{\mp \mathbf{I} \sigma_3 \kappa_0 x_0}. \end{aligned} \quad (26.23)$$

Notice that, in view of the identity (25.7), we have

$$\tilde{\psi}_{\mp} = e^{\pm \mathbf{I} \sigma_3 \kappa_0 x_0} \tilde{\psi}_0 \dot{\psi}. \quad (26.24)$$

The above relation combined with the equality (26.23) yields

$$\begin{aligned} \left\langle \gamma^0 \partial \psi_{\mp} \mathbf{I} \sigma_3 \tilde{\psi}_{\mp} \right\rangle &= \left\langle \psi_0 (\mp \mathbf{I} \sigma_3 \kappa_0) e^{\mp \mathbf{I} \sigma_3 \kappa_0 x_0} \dot{\psi} \mathbf{I} \sigma_3 e^{\pm \mathbf{I} \sigma_3 \kappa_0 x_0} \tilde{\psi}_0 \dot{\psi} \right\rangle \\ &\quad + \sum_{1 \leq j \leq 3} \dot{\psi} \partial_j \dot{\psi} \left\langle \gamma^0 \gamma^j \psi_0 \mathbf{I} \sigma_3 \tilde{\psi}_0 \right\rangle \\ &= \pm \dot{\psi}^2 \kappa_0 \left\langle \psi_0 \tilde{\psi}_0 \right\rangle + \frac{1}{2} \sum_{1 \leq j \leq 3} \left(\partial_j \dot{\psi}^2 \right) \left\langle \gamma^0 \gamma^j \psi_0 \mathbf{I} \sigma_3 \tilde{\psi}_0 \right\rangle. \end{aligned} \quad (26.25)$$

Hence, based on (26.23) and the above equality, we obtain

$$\begin{aligned} J_{\mp}^0 &= c \rho_{\mp} = \frac{\chi q}{m} \left\langle \gamma^0 (\partial \psi_{\mp}) \mathbf{I} \sigma_3 \tilde{\psi}_{\mp} \right\rangle \\ &= \frac{\chi q}{m} \left[\pm \dot{\psi}^2 \kappa_0 \left\langle \psi_0 \tilde{\psi}_0 \right\rangle + \sum_{1 \leq j \leq 3} \frac{1}{2} \left(\partial_j \dot{\psi}^2 \right) \left\langle \gamma^0 \gamma^j \psi_0 \mathbf{I} \sigma_3 \tilde{\psi}_0 \right\rangle \right], \end{aligned} \quad (26.26)$$

implying the following expressions for the total charge

$$\mathbf{q}_{\mp} = \int \rho_{\mp} \, \mathbf{d}\mathbf{x} = \pm \frac{\chi \kappa_0}{mc} q \left\langle \psi_0 \tilde{\psi}_0 \right\rangle = \pm q \left\langle \psi_0 \tilde{\psi}_0 \right\rangle. \quad (26.27)$$

Observe that the different signs $\pm$ of the charge $\mathbf{q}_{\mp}$ above can be traced to the different signs of the frequencies in expressions (26.20) for $\psi_{\mp}(x)$.

26.4 Energy-Momentum Density

Let us find the energy-momentum density for the solutions of the spinorial field equations $\psi_{\mp}(x)$ defined by (26.20). To facilitate efficient computation, we use the following identities. Suppose p_{α} , c_{α} and φ are multivectors satisfying

$$p_{\alpha} = \varphi c_{\alpha}, \quad c_{\alpha} c_{\beta} = c_{\beta} c_{\alpha}, \quad \tilde{c}_{\alpha} = c_{\alpha}. \quad (26.28)$$

Then

$$p_{\alpha} \tilde{p}_{\beta} = p_{\beta} \tilde{p}_{\alpha} = \varphi c_{\alpha} c_{\beta} \tilde{\varphi}. \quad (26.29)$$

Observe now that if we consider the derivatives $\partial_{\alpha} \psi_{\mp}$ defined by (26.22) and set

$$p_{\alpha} = \partial_{\alpha} \psi_{\mp}, \quad c_0 = \mp \mathbf{I} \sigma_3 \kappa_0 \dot{\psi}, \quad c_j = \partial_j \ln \dot{\psi}, \quad \varphi = \psi_{\mp}, \quad (26.30)$$

then the relations (26.29) are satisfied, that is,

$$(\partial_{\alpha} \psi_{\mp}) (\partial_{\beta} \tilde{\psi}_{\mp}) = (\partial_{\beta} \psi_{\mp}) (\partial_{\alpha} \tilde{\psi}_{\mp}). \quad (26.31)$$

Notice that the following relations hold for the solutions $\psi_{\mp}$ defined by (26.20):

$$\langle \psi_{\mp} \tilde{\psi}_{\mp} \rangle = \langle \psi_0 \tilde{\psi}_0 \rangle \dot{\psi}^2, \quad (26.32)$$

$$\partial_0 \psi_{\mp} = \mp \psi_0 \mathbf{I} \sigma_3 \kappa_0 e^{\mp \mathbf{I} \sigma_3 \kappa_0 x_0} \dot{\psi}, \quad \partial_0 \tilde{\psi}_{\mp} = \pm \dot{\psi} \mathbf{I} \sigma_3 \kappa_0 e^{\pm \mathbf{I} \sigma_3 \kappa_0 x_0} \tilde{\psi}_0, \quad (26.33)$$

$$\begin{aligned} (\partial_0 \psi_{\mp}) (\partial_0 \tilde{\psi}_{\mp}) &= -\dot{\psi}^2 \psi_0 \mathbf{I} \sigma_3 \kappa_0 e^{\mp \mathbf{I} \sigma_3 \kappa_0 x_0} \mathbf{I} \sigma_3 \kappa_0 e^{\pm \mathbf{I} \sigma_3 \kappa_0 x_0} \tilde{\psi}_0 \\ &= \dot{\psi}^2 \kappa_0^2 \psi_0 \tilde{\psi}_0, \end{aligned} \quad (26.34)$$

$$\partial_j \psi_{\mp} = \psi_0 e^{\mp \mathbf{I} \sigma_3 \kappa_0 x_0} \partial_j \dot{\psi}, \quad \partial_j \tilde{\psi}_{\mp} = e^{\pm \mathbf{I} \sigma_3 \kappa_0 x_0} \tilde{\psi}_0 \partial_j \dot{\psi}, \quad (26.35)$$

$$(\partial_j \psi_{\mp}) (\partial_j \tilde{\psi}_{\mp}) = \psi_0 \tilde{\psi}_0 (\partial_j \dot{\psi})^2. \quad (26.36)$$

Using then the expression (26.1) for the Lagrangian L and the identities (26.31), we find its value on the fields $\psi_{\mp}$ to be

$$\begin{aligned} L &= \frac{\chi^2}{2m} \langle (\partial^{\alpha} \psi_{\mp}) (\partial_{\alpha} \tilde{\psi}_{\mp}) \rangle \\ &\quad - \frac{\chi^2}{2m} \left[\kappa_0^2 \langle \psi_{\mp} \tilde{\psi}_{\mp} \rangle + G \left(\langle \psi_{\mp} \tilde{\psi}_{\mp} \rangle \right) \right]. \end{aligned} \quad (26.37)$$

The canonical EnMT $\hat{T}^{\mu\nu}$ defined by (25.32) takes the following form for the free charge with $\check{A} = 0$

$$\begin{aligned}\hat{T}^{\mu\nu} &= \frac{\chi}{m} \left\langle \gamma^\mu \partial^\nu \psi \mathbf{I} \sigma_3 (\chi \partial \psi \mathbf{I} \sigma_3)^\sim \right\rangle - \delta_\nu^\mu L \\ &= \frac{\chi^2}{m} \left\langle \gamma^\mu \partial^\nu \psi (\partial \psi)^\sim \right\rangle - \delta_\nu^\mu L = \frac{\chi^2}{m} \left\langle \gamma^\mu \partial^\nu \psi \partial_\alpha \tilde{\psi} \gamma^\alpha \right\rangle - \delta_\nu^\mu L \\ &= \frac{\chi^2}{m} \left\langle \gamma^\alpha \gamma^\mu \partial^\nu \psi \partial_\alpha \tilde{\psi} \right\rangle - \delta_\nu^\mu L,\end{aligned}\quad (26.38)$$

where we used the identity (25.7). The above formula yields the following representation for the energy density $\mathcal{E}$:

$$\mathcal{E} = \hat{T}^{00} = \frac{\chi^2}{m} \left\langle \gamma^\alpha \gamma^0 \partial^0 \psi \partial_\alpha \tilde{\psi} \right\rangle - L. \quad (26.39)$$

In particular, for $\psi = \psi_\mp$, we use (26.39) and (26.37) to obtain

$$\begin{aligned}\mathcal{E}_\mp &= \frac{\chi^2}{m} \left\langle \gamma^\alpha \gamma^0 \partial^0 \psi_\mp \partial_\alpha \tilde{\psi}_\mp \right\rangle - L \\ &= \frac{\chi^2}{m} \left\langle \gamma^\alpha \gamma^0 \partial^0 \psi_\mp \partial_\alpha \tilde{\psi}_\mp \right\rangle - \frac{\chi^2}{2m} \left\langle (\partial^\alpha \psi_\mp) \left(\partial_\alpha \tilde{\psi}_\mp \right) \right\rangle \\ &\quad + \frac{\chi^2}{2m} \left[\kappa_0^2 \left\langle \psi_\mp \tilde{\psi}_\mp \right\rangle + G \left(\left\langle \psi_\mp \tilde{\psi}_\mp \right\rangle \right) \right].\end{aligned}\quad (26.40)$$

The expression above can be transformed into

$$\begin{aligned}\mathcal{E}_\mp &= \frac{\chi^2}{2m} \left\{ \left\langle (\partial_0 \psi_\mp) \left(\partial_0 \tilde{\psi}_\mp \right) \right\rangle + \sum_{1 \leq j \leq 3} \left\langle (\partial_j \psi_\mp) \left(\partial_j \tilde{\psi}_\mp \right) \right\rangle \right\} \\ &\quad + \frac{\chi^2}{2m} \left\{ \left[\kappa_0^2 \left\langle \psi \tilde{\psi} \right\rangle + G \left(\left\langle \psi \tilde{\psi} \right\rangle \right) \right] \right\} - \sum_{1 \leq j \leq 3} \frac{\chi^2}{m} \left\langle \gamma^j \gamma^0 \partial^0 \psi_\mp \partial_j \tilde{\psi}_\mp \right\rangle.\end{aligned}\quad (26.41)$$

Using the identities (26.32)–(26.36), we transform the above representation further into

$$\begin{aligned}\mathcal{E}_\mp &= \frac{\chi^2 \left\langle \psi_0 \tilde{\psi}_0 \right\rangle}{2m} \left[2\psi^2 \kappa_0^2 + \sum_{1 \leq j \leq 3} \left(\partial_j \psi \right)^2 + G \left(\psi^2 \right) \right] \\ &\quad \pm \frac{\chi^2}{2m} \kappa_0 \sum_{1 \leq j \leq 3} \left\langle \gamma^j \gamma^0 \psi_0 \mathbf{I} \sigma_3 \tilde{\psi}_0 \right\rangle \partial_j \psi^2.\end{aligned}\quad (26.42)$$

Then, using the above formula and relations (26.7), (25.4), we obtain the following representation for the total energy $E_{\mp}$ of the free charge solutions $\psi_{\mp}$:

$$\begin{aligned}
 E_{\mp} &= \int \mathcal{E}_{\mp} d\mathbf{x} \\
 &= \frac{\chi^2 \langle \psi_0 \tilde{\psi}_0 \rangle}{2m} \int \left[2\dot{\psi}^2 \kappa_0^2 - \sum_{1 \leq j \leq 3} \left(\partial_j^2 \dot{\psi} \right) \dot{\psi} + G \left(\dot{\psi}^2 \right) \right] d\mathbf{x} \\
 &= \frac{\chi^2 \langle \psi_0 \tilde{\psi}_0 \rangle}{2m} \int \left[2\dot{\psi}^2 \kappa_0^2 - G' \left(\dot{\psi}^2 \right) \dot{\psi}^2 + G \left(\dot{\psi}^2 \right) \right] d\mathbf{x} \\
 &= \frac{\chi^2 \langle \psi_0 \tilde{\psi}_0 \rangle}{2m} \int \left(2\kappa_0^2 + \frac{1}{a^2} \right) \dot{\psi}^2 d\mathbf{x} = \frac{\chi^2 \langle \psi_0 \tilde{\psi}_0 \rangle}{2m} \left(2\kappa_0^2 + \frac{1}{a^2} \right) \\
 &= \langle \psi_0 \tilde{\psi}_0 \rangle \chi \omega_0 \left(1 + \frac{a_C^2}{2a^2} \right),
 \end{aligned} \tag{26.43}$$

where

$$a_C = \kappa_0^{-1} = \frac{\chi}{mc} \tag{26.44}$$

is the reduced Compton wavelength. Since we want the energy $E_{\pm}$ defined by (26.43) to be positive, we require in accordance with the canonical spinor representation (24.55) for ψ_0 and constraints (26.17) that $\langle \psi_0 \tilde{\psi}_0 \rangle = \cos \beta_0 = 1$, that is, $\beta_0 = 0$. This requirement in view of (24.55) is equivalent to the following constraint for free charge solutions

$$\psi_0 \tilde{\psi}_0 = 1, \text{ that is, } \psi_0 \text{ is the Lorentz rotor.} \tag{26.45}$$

Under the above constraint the formula (26.43) turns into

$$E_{\mp} = \chi \omega_0 \left(1 + \frac{a_C^2}{2a^2} \right), \quad a_C = \kappa_0^{-1} = \frac{\chi}{mc}. \tag{26.46}$$

Let us take a closer look at the origin of the factor $\langle \psi_0 \tilde{\psi}_0 \rangle$ in expressions (26.43) and (26.27) for $E_{\mp}$ and $q_{\mp}$. The similar dependence on this factor of evidently spinorial nature occurs in the quadratic part of the charge Lagrangian L (without the nonlinear term G) defined by (25.1). Observe that multiplication of the Lagrangian by any constant, positive or negative, does not change the Euler–Lagrange equation, but it does alter the energy and the charge densities defined canonically by the Lagrangian.

To summarize, the factor $\langle \psi_0 \tilde{\psi}_0 \rangle$ can alter the sign of the Lagrangian, the energy and the charge since it can be positive or negative. It is special for the spinorial wave functions, whereas the similar factor $\psi_0 \psi_0^*$ for complex-valued ones is always positive.

Chapter 27

Neoclassical Solutions: Interpretation and Comparison with the Dirac Theory

As to general aspects of the interpretation of the wave function and observables in the STA settings, we rely mostly on works of D. Hestenes, see [162, 4], [166, 2], [169, VII.D] and references therein. The key points are as follows. First of all, based on the general canonical representation (24.1) for the wave function $\psi = \psi(x)$, we assign at each spacetime point x the *local rotor* $R = R(x)$. This rotor determines the Lorentz rotation of a given fixed frame of vectors $\{\gamma_\mu\}$ into the *local rest frame* of vectors $\{e_\mu = e_\mu(x)\}$ given by

$$e_\mu = e_\mu(x) = R\gamma_\mu\tilde{R}, \quad R = R(x). \quad (27.1)$$

Importantly, in view of the canonical representation (24.1), we have

$$\psi\gamma_\mu\tilde{\psi} = \varrho R\gamma_\mu\tilde{R} = \varrho e_\mu. \quad (27.2)$$

The interpretation of the above fields in the Dirac theory is as follows. The vector field

$$\psi\gamma_0\tilde{\psi} = \varrho e_0 = \varrho v \quad (27.3)$$

is the *Dirac current* (*probability current* in the standard Born interpretation) that determines the local rest frame v . The *local spin vector density* is defined by

$$s = \frac{1}{2}\hbar\psi\gamma_3\tilde{\psi} = \frac{1}{2}\hbar\varrho e_3. \quad (27.4)$$

The *spin angular momentum* $S = S(x)$ (*proper spin*) is a bivector field related to the spin vector field $s = s(x)$ as follows, [162, 4], [166, 2]:

$$\begin{aligned}
 S &= \frac{1}{2} \hbar e_2 e_1 = \frac{1}{2} \hbar R I \sigma_3 \tilde{R} = \frac{1}{2} \hbar R \gamma_2 \gamma_1 \tilde{R} \\
 &= \frac{1}{2} \hbar R I \sigma_3 \tilde{R} = I s e_0 = I (s \wedge e_0).
 \end{aligned} \tag{27.5}$$

Notice that, according to (25.29), (25.30), the *proper spin density* is ϱS and the *magnetization* or *magnetic moment density* M of the charge is defined by the following expression:

$$M = \frac{\hbar e}{2mc} \psi \gamma_2 \gamma_1 \tilde{\psi} = e^{I\beta} \frac{q}{mc} \varrho S. \tag{27.6}$$

Let us turn now to our neoclassical free charge solutions (26.20) satisfying energy positivity constraint (26.45), that is,

$$\psi_{\mp}(x) = \psi_0 e^{\mp I \sigma_3 \kappa_0 x_0} \tilde{\psi}, \quad \text{where} \quad \psi_0 \tilde{\psi}_0 = 1, \tag{27.7}$$

where the Gaussian form factor $\tilde{\psi}(s)$ is defined by (26.6), and the time-like vector unit vector γ_0 describes the constant rest frame $v = \gamma_0$ of the charge for every x . Then formulas (26.27) and (26.46) yield the following ultimate expressions for the total charge $q_{\mp}$ and the total energy $E_{\mp}$ of the solutions $\psi_{\mp}$:

$$q_{\mp} = \int \rho_{\mp} dx = \pm q, \quad \text{for the total charge,} \tag{27.8}$$

$$E_{\mp} = \int \mathcal{E}_{\mp} dx = \chi \omega_0 \left(1 + \frac{a_C^2}{2a^2} \right), \quad \text{for the total energy,} \tag{27.9}$$

where

$$\kappa_0 = \frac{mc}{\chi}, \quad \omega_0 = \kappa_0 c = \frac{mc^2}{\chi}, \quad a_C = \kappa_0^{-1} = \frac{\chi}{mc}, \tag{27.10}$$

and χ is a constant equal to the Planck constant $\hbar$. Notice that subindices $\mp$ in $q_{\mp}$ and $E_{\mp}$ are picked so that, if the charge is the electron, then its index is “−” to match the sign of the charge.

Observe that formula (27.8) demands the total charges $q_{\mp}$ associated with the wave functions ψ_{-} and ψ_{+} to have opposite signs. Following the established tradition, we call them *charge* and *anticharge* (for instance, electron and positron). We introduce the following representation of $\psi_{\mp}(x)$ based on (27.7) and (26.45):

$$\psi_{\mp}(x) = R_{\mp}(x) \tilde{\psi}(x), \tag{27.11}$$

where

$$R_{\mp} = \psi_0 e^{\mp I \sigma_3 \kappa_0 x_0}, \quad \psi_0 \tilde{\psi}_0 = 1 \quad R_{\mp} \tilde{R}_{\mp} = 1.$$

Then, following D. Hestenes, [164, 6, 9] and using the above representation, we obtain a formula involving the rotational velocities $\Omega_{\mp}$:

$$\frac{d\psi_{\mp}}{dt} = \frac{cd\psi_{\mp}}{dx_0} = c\frac{1}{2}\Omega_{\mp}\psi_{\mp}, \quad \Omega_{\mp} = \mp 2\kappa_0 c R_{\mp} (\mathbf{I}\sigma_3) \tilde{R}_{\mp}. \quad (27.12)$$

In other words, *the local rest frame rotates “about the spin axis” $s = R_{\mp}(\gamma_3) \tilde{R}_{\mp}$ with the angular speed $|\Omega_{\mp}| = 2\kappa_0 c = 2\omega_0$ which equals twice the frequency ω_0 corresponding to the rest energy $mc^2 = \chi\omega_0$.* Observe that, according to relations (27.12), the charge and “anticharge” (for instance, electron and positron) wave functions ψ_- and ψ_+ differ only by the opposite sign in the rotation described by the factor $e^{\pm \mathbf{I}\sigma_3 \kappa_0 x_0}$ in (27.11).

Let us consider now the magnetization M and the proper spin ϱS bivector densities defined by (27.5), (27.6) for the neoclassical free charge solutions $\psi_{\mp}$ described by formulas (27.7) and (27.11). Notice that for the free charge at rest we have $\varrho = \dot{\psi}^2(x)$, and the following relation holds for the magnetization and proper spin densities:

$$M_{\mp} = \frac{\hbar e}{2mc} \psi_{\mp} \gamma_2 \gamma_1 \tilde{\psi}_{\mp} = \frac{q}{mc} \dot{\psi}^2(x) S_{\mp}, \text{ where } S_{\mp} = \frac{1}{2} \hbar \psi_0 \mathbf{I} \sigma_3 \tilde{\psi}_0, \quad (27.13)$$

where we took into account that $\beta = 0$ since $\psi_0 \tilde{\psi}_0 = 1$. Observe also that the above formula shows that both the states $\psi_{\mp}$ have the same and constant $M_{\mp}$ and $S_{\mp}$. Integrating the magnetization density of the free charge at rest $M_{\mp}$ in (27.13), we obtain the total generalized magnetic momentum bivector

$$\mathcal{M}_{\mp} = \int_{\mathbb{R}^3} M_{\mp} d\mathbf{x} = \frac{q}{mc} S_{\mp}. \quad (27.14)$$

For general issues of the STA treatment of localized charge distributions and the proper momentum bivector M , see [161, 1].

Let us compare now the neoclassical free charge solutions (27.7) with the Dirac free charge solutions (24.48). First of all, the spinorial aspect of the neoclassical theory proposed here is identical to that in the Dirac theory since in both cases the wave function ψ takes values in the even algebra $\text{Cl}_+(1, 3)$. The governing field equation for the neoclassical spinor field is (26.3), and the Dirac spinor field satisfies the Dirac equation (24.5). The neoclassical field equation (26.3) can be viewed structurally as a spinorial version of the Klein–Gordon equation with added nonlinearity, and it is also related to the Dirac equation. In particular, *solutions to the Dirac equations are also solutions to our field equations if the nonlinearity there is neglected.* But when it comes to the structure of solutions, the first significant difference of the neoclassical free charge solution (27.7) compared with the Dirac free charge plane wave solution is the Gaussian factor $\dot{\psi}(x)$. In other words, *the neoclassical free charge solution is a localized soliton-like wave, whereas the Dirac free charge solution is a plane wave.*

Below we compare other features of the neoclassical free charge wave function and the Dirac free charge plane wave function.

27.1 The Gyromagnetic Ratio and Currents

Recall that the gyromagnetic ratio g is defined as a coefficient that relates the magnetic dipole moment $\mathbf{m}$ and the angular momentum $\mathbf{L}$ for a system of localized currents, namely

$$\mathbf{m} = \frac{gq}{2mc} \mathbf{L}. \quad (27.15)$$

Comparing the above relation (27.15) with (27.14), we conclude that in our theory the gyromagnetic ratio $g = 2$, just as in the Dirac theory. *The expressions (25.26)–(25.30) for the current and its Gordon decomposition, which includes the magnetization (spin) current in the neoclassical theory, are identical to the same in the Dirac theory (24.28)–(24.30).* The value of the gyromagnetic ratio $g = 2$ in our theory is in fact not so surprising since the minimal coupling as in (25.2) implies that $g = 2$, [178, 2-2-3]. Interestingly, there is an example of a classical particle with the gyromagnetic ratio $g = 2$, [169, V].

27.2 The Energies and Frequencies

The issue of negative energies in the Dirac theory constitutes a well known serious problem discussed extensively in the literature, see for instance [143, 12], [178, 2.4.2], [331, 2.1.6] and references therein. One of the proposed ways to deal with it is to *reinterpret an electron state of negative energy/frequency as a positron state of positive energy/frequency using the charge conjugation transformation (24.69)*. This operation in the conventional setting involves complex conjugation of the Dirac wave function and reverses the sign of the charge, its frequency, energy, momentum, and spin, [152, 5.4], [331, 2.1.6, p. 109]. Effectively, the charge conjugation operation changes the sign of the frequency of the wave function which then satisfies a complex conjugate version of the original Dirac equation with the opposite sign of the charge there.

In quantum mechanics and in the Dirac theory in particular, the energy is identified with the frequency via the Planck–Einstein relation $\mathcal{E} = \hbar\omega$. In contrast, in our neoclassical theory the energy and the frequency are two distinct though closely related concepts, see [17]. The relation between them in the relativistic case is by no means as explicit as the Planck–Einstein relation $\mathcal{E} = \hbar\omega$. Importantly, in our theory the frequencies may be positive or negative when the energy is positive. The positivity of the energies of the free charge in our theory was obtained by simply limiting the values of the spinor constant ψ_0 for the free charge solutions $\psi_{\mp}$ in (27.7) to be a Lorentz rotor, that is, to satisfy the energy positivity constraint (26.45), i.e. $\psi_0 \tilde{\psi}_0 = 1$. Consequently, in our theory a positron state differs from the quantum mechanical positron state and, importantly, it is not obtained by applying the charge conjugation (24.69) to an electron state. Also there were no changes of frequencies or any transformation of the evolution equation. If the energy positivity constraint

(26.45) is satisfied, then according to (27.9) the energies $E_{\mp}$ of the free charge at rest satisfy approximately $E_{\mp} \approx \hbar |\omega_0|$, and they stay positive, whereas the corresponding frequencies $\mp \kappa_0 c = \mp \omega_0$ of the solutions $\psi_{\mp}$ in (27.7) can be positive or negative. The above analysis indicates a significant difference in the treatment of negative energies in our theory compared to the Dirac theory or QM. One may notice though that so far we have analyzed only energies of free charges. We expect the treatment of a charge in an external field to be more complex, but this study is left for the future.

27.3 Antimatter States

Similarly to the Dirac theory, our theory naturally integrates into it the concept of an antiparticle. According to (27.11), there are two directions of “spinning” in the rest frame, and that naturally leads to the concepts of charge and anticharge with the frequencies of the opposite signs. Note that since the value of a charge is preserved even in the external EM field, the charge cannot turn into the anticharge as a result of electromagnetic interactions. All the properties of the charge and anticharge are exactly the same except for the difference in sign. We would like to stress once again a noticeable difference between the antimatter states in our theory and the same in the Dirac theory. In our theory the matter and antimatter states correspond to $\langle \psi_0 \tilde{\psi}_0 \rangle = 1$, whereas in the Dirac theory the usual way to introduce the antimatter state (positron) is by applying charge conjugation (24.69), which requires $\langle \psi_0 \tilde{\psi}_0 \rangle = \langle e^{\beta_0 I} \rangle = \cos \beta_0 = -1$ as in (24.55), (24.67). Consequently, in order to introduce the antimatter state in the Dirac theory one has to invoke the parameter β of the canonical spinor representation (24.2), and the interpretation of the parameter β has known difficulties, [166, 3], [169, VII.D, G].

Chapter 28

Clifford and Spacetime Algebras

Though we are interested primarily in Spacetime algebra (STA), it is instructive to consider it within the general framework of Clifford algebras. In fact, this does not complicate the treatment of the subject at all, but rather underlines its geometric and algebraic essence.

Let p, q be nonnegative integers and $V(p, q)$ be a $(p + q)$ -dimensional linear space over the field of real numbers. We denote $p + q$ by n . Suppose that g is a non-degenerate symmetric metric tensor (quadratic form) on $V(p, q)$ with the signature (p, q) , that is, it has p positive and q negative eigenvalues. The vector space $V(p, q)$ and the metric g define, based on the identity (23.1), a (pseudo-Euclidean) geometric algebra (GA) $\text{Cl}(p, q) = \text{Cl}(V(p, q), g)$. We denote by γ_j an orthonormal basis of $V(p, q)$, and assume it to satisfy the following fundamental identities:

$$\begin{aligned}\gamma_j^2 &= 1, \quad 1 \leq j \leq p; & \gamma_j^2 &= -1, \quad p + 1 \leq j \leq n, \\ \gamma_j \gamma_i &= -\gamma_i \gamma_j, \quad 1 \leq i \neq j \leq n, & p + q &= n.\end{aligned}\tag{28.1}$$

The Spacetime Algebra is the Clifford algebra $\text{Cl}(1, 3)$ based on the real 4-dimensional *Minkowski space* $\mathbb{M}^4$ ($p = 1, q = 3$). The common 3-dimensional Euclidean space is denoted by $\mathbb{R}^3$, and the Clifford Algebra corresponding to it is $\text{Cl}(3, 0)$.

When selecting subjects and their exposition, we relied mostly on [171, 3-8], [93], [95], [270]. This section is only a concise exposition of selected subjects from an extensive theory of Clifford algebras. For a systematic and detailed theory we refer the reader to the monographs cited above.

28.1 Isometries, Reflections, Versors and Rotors

Following [171, 3-8], [95, 7.6.1], we provide here some basic facts on the representation of isometries for the metric quadratic form g in terms of its associated (pseudo-Euclidean) geometric algebra (GA) $\text{Cl}(p, q) = \text{Cl}(V(p, q), g)$.

A linear transformation $f(x)$ of $V(p, q)$ into itself is said to be its *orthogonal transformation* if

$$f(x) \cdot f(y) = x \cdot y, \quad x, y \in V(p, q). \quad (28.2)$$

The simplest kind of isometry is a *simple reflection* of x in the dual hyperplane u , that is, the hyperplane orthogonal to u , [95, 7.1], [270, 4.1.4], [171, 3-8]:

$$\text{reflection of } x \text{ in the dual hyperplane } u : x \mapsto -uxu^{-1}, \quad (28.3)$$

where u is assumed to be a non-null vector, that is, $u^2 \neq 0$. This transformation reverses the direction of all vectors collinear with u and leaves the hyperplane of vectors orthogonal to u invariant. Another reflection is defined by the above but without the minus sign, and it is called the reflection of x in the u -line:

$$\text{reflection of } x \text{ in the } u\text{-line} : x \mapsto uxu^{-1}. \quad (28.4)$$

It turns out that every isometry $f = f(x)$ of $V(p, q)$ can be expressed as a composite of at most $n = p + q$ simple reflections, that is, there exist vectors $u_1, u_2, \dots, u_k$, where $k \leq n$, such that, [171, 3-8], [95, 7.6.1], [270, 3.3]

$$f(x) = (-1)^k u_k \cdots u_2 u_1 x u_1^{-1} u_2^{-1} \cdots u_k^{-1} \quad (28.5)$$

implying that every isometry $f = f(x)$ of $V(p, q)$ can be expressed in the form

$$f(x) = (-1)^k UxU^{-1}, \quad U = u_k \cdots u_2 u_1. \quad (28.6)$$

Multivectors which can be factored into a product of non-null vectors are so important that they have a class name: versors. A versor which can be factored into a product of k non-null vectors will be called a k -versor, and a 0-versor is a scalar. Naturally, a versor will be called an even (odd) versor if it is an even (odd) multivector. The importance of the versors is indicated by the representation (28.6) where by construction the multivector U must be an invertible versor. Obviously U in equation (28.6) can be replaced by a nonzero scalar multiple of U without altering the equation but, *except for this ambiguity in scale and sign, U is uniquely determined by the isometry f .*

For a versor U one can define the *versor product operation*

$$x \mapsto UxU^{-1}, \quad x \in V(p, q) \quad (28.7)$$

which maps the vector space $V(p, q)$ into itself. This operation is evidently based on the geometric product, hence it can be extended to multivectors:

$$M \longmapsto UMU^{-1}, \quad M \in \text{Cl}(p, q). \quad (28.8)$$

One can verify that the *versor product operation preserves the grade*, [270, 3.3.2], namely if a_j , $1 \leq j \leq k$, are vectors then

$$U \left(\bigwedge_{j=1}^k a_j \right) U^{-1} = \bigwedge_{j=1}^k (Ua_j U^{-1}). \quad (28.9)$$

Yet another important concept related to isometries is the concept of a *rotor defined as an even versor S satisfying*

$$\tilde{S}S = 1. \quad (28.10)$$

It turns out that every rotor S can be represented as a product of simple rotors S_j , [171, 3-8], [95, 7.4.3]

$$S = S_m S_{m-1} \dots S_1, \quad m \leq \frac{1}{2}n, \quad (28.11)$$

$$S_j = a_j b_j = e^{i_j \alpha_j / 2}, \quad \text{where } a_j^2 = b_j^2 = \pm 1, \quad 1 \leq j \leq m, \quad (28.12)$$

where α_j is a scalar and i_j is the pseudoscalar of the plane generated by vectors a_j and b_j . In addition to that, in the case of Euclidean and Minkowski spaces, that is, when $q = 0, 1$, the simple rotors S_j can be chosen to be orthogonal and commuting, that is,

$$S_j S_s = S_s S_j \text{ for Euclidean and Minkowski spaces.} \quad (28.13)$$

It follows then from (28.10)–(28.13) that every rotor S can be represented as follows:

$$S = e^{B/2}, \quad B = \alpha_m i_m + \alpha_{m-1} i_{m-1} + \dots + \alpha_1 i_1, \quad (28.14)$$

showing a decomposition of a bivector B into commuting orthogonal blades.

Notice that since every rotor S by its very definition is an even versor, and it always satisfies the relation (28.10), the rotor product $SM\tilde{S}$ preserves the grade, that is, (28.9) turns into

$$S \left(\bigwedge_{j=1}^k a_j \right) \tilde{S} = \bigwedge_{j=1}^k (S a_j \tilde{S}). \quad (28.15)$$

28.2 Clifford Algebra Bases

In this section we follow [171, 1-3.], [270, 3.5].

It is often convenient to use a multiindex J to describe concisely the linear basis γ_J of the STA in (23.14). Namely, $J = \{j_1, \dots, j_s\}$ with $j_1, \dots, j_s$ taking values from the set $\{1, \dots, n\}$. Sometimes we assume J to satisfy the following “order” condition

$$J = \{j_1, \dots, j_s\}, \text{ where } 0 \leq j_1 < \dots < j_s \leq 3, \quad s = |J| \leq n, \quad (28.16)$$

where $|J|$ stands for the number of indices in J . Evidently, the number of different indices J satisfying (28.16) is 2^n , and it is 16 for the STA. Let us introduce

$$\gamma_J = \gamma_{j_1} \wedge \dots \wedge \gamma_{j_s} \text{ for } J = \{j_1, \dots, j_s\}, \text{ and } \gamma_J = 1 \text{ for } J = \emptyset. \quad (28.17)$$

Notice that in view of the alternating property (23.2) of the outer product $\wedge$, a permutation σ of indices $j_1, \dots, j_s$ changes the sign γ_J according to the formula

$$\gamma_{\sigma(J)} = s(\sigma) \gamma_J, \text{ where } s(\sigma) = 1 \text{ if } \sigma \text{ is even and } s(\sigma) = -1 \text{ if } \sigma \text{ is odd.} \quad (28.18)$$

Evidently, the set of all multivectors γ_J with a multiindex J satisfying (28.16) forms a complete basis of the STA as a linear space. It is also convenient to introduce a *reciprocal (dual)* to γ_J basis γ^J by the formula

$$\gamma^J = \varepsilon(J) \tilde{\gamma}_J, \text{ where } \varepsilon(J) = g_{j_1 j_1} \dots g_{j_s j_s} = \gamma_J \tilde{\gamma}_J, \quad (28.19)$$

for every J satisfying (28.16). Notice that $\varepsilon(J) = \pm 1$ and, in view of (28.19) and (23.38), the following relation holds for the reciprocal bases:

$$\gamma_J * \gamma^{J'} = \varepsilon(J) \langle \gamma_J \tilde{\gamma}_{J'} \rangle = \delta_J^{J'}; \quad (28.20)$$

it is a generalization of the reciprocal bases relations (23.8) for vectors. Observe that, according to (28.19) and (28.20), the “lifting” of indices involves the reversion operation.

An alternative way to define the reciprocal (dual) basis in the Clifford algebra $\text{Cl}(p, q)$ is to define it first for the basis vectors γ_j by, [270, 3.5]

$$\gamma^j = (-1)^j \left[\bigwedge_{r=1}^{j-1} \gamma_r \wedge \bigwedge_{r=j+1}^n \gamma_r \right] \text{I}^{-1}, \quad \text{I} = \bigwedge_{r=1}^n \gamma_r. \quad (28.21)$$

The multivector I above is the so-called *pseudoscalar* of the Clifford algebra $\text{Cl}(p, q)$. The reciprocal basis γ^j defined by (28.21) satisfies

$$\gamma_j \cdot \gamma^{j'} = \delta_j^{j'}. \quad (28.22)$$

Then we define all basis multivectors γ^J by the formula

$$\gamma^J = \left(\bigwedge_{r=1}^s \gamma^{j_r} \right), \quad J = \{j_1, \dots, j_s\}, \quad (28.23)$$

and they satisfy the relation (28.20).

Every multivector X can then be expanded as follows:

$$X = \sum_J \frac{1}{|J|!} X_J \gamma^J = \sum_J \frac{1}{|J|!} X^J \gamma_J, \quad \text{where} \quad (28.24)$$

$$X_J = X * \gamma_J, \quad X^J = X * \gamma^J.$$

Notice that in the above sum we allow $J = \{j_1, \dots, j_s\}$ to take any values with different $j_1, \dots, j_s$, and the $1/|J|!$ factor is due to the identity (28.18). Observe also that the grade projection operator $\langle \cdot \rangle_k$ satisfies

$$\langle X \rangle_k = \frac{1}{k!} \sum_{|J|=k} X_J \gamma^J = \frac{1}{k!} \sum_{|J|=k} X^J \gamma_J. \quad (28.25)$$

The grade projection operator $\langle \cdot \rangle_k$ can be generalized to a projection on any subspace $\mathcal{X}$ of the spacetime algebra $\text{Cl}(1, 3)$ which is the direct sum of a set of grade subspaces Λ^k . Namely, the projection operator $\langle \cdot \rangle_{\mathcal{X}}$ is defined by

$$\langle Y \rangle_{\mathcal{X}} = \sum_{\gamma_J \in \mathcal{X}} \frac{1}{|J|!} Y_J \gamma^J. \quad (28.26)$$

28.3 Inner and Outer Product Properties

The inner and outer products defined by (23.19), (23.19) for homogeneous multivectors are instrumental in utilizing the potential of geometric algebra. Their general algebraic properties follow straightforwardly from the axioms of geometric algebra, [171, Sect. 1.1], [167, 2]. Similarly to the geometric product, the inner and outer products are not commutative. In particular, it follows from (23.25) that they satisfy the following reordering rules for homogeneous multivectors

$$A_r \cdot B_s = \langle A_r B_s \rangle_{s-r} = (-1)^{r(s-r)} B_s \cdot A_r, \quad 0 < r \leq s, \quad (28.27)$$

$$A_r \wedge B_s = \langle A_r B_s \rangle_{s+r} = (-1)^{rs} B_s \wedge A_r. \quad (28.28)$$

The inner product is not associative, but homogeneous multivectors obey the following rules, [171, 1.1]:

$$A_r \cdot (B_s \cdot C_t) = (A_r \wedge B_s) \cdot C_t, \text{ for } r + s \leq t, \quad r, s > 0, \quad (28.29)$$

$$A_r \cdot (B_s \cdot C_t) = (A_r \cdot B_s) \cdot C_t, \text{ for } r + t \leq s. \quad (28.30)$$

Repeated application of the fundamental inner product vector identity (23.2) in the form $ab = 2a \cdot b - ba$ yields the following valuable vector identity:

$$a \cdot (a_1 a_2 \cdots a_n) = \sum_{k=1}^n (-1)^k a \cdot a_k (a_1 a_2 \cdots \check{a}_k \cdots a_n), \quad (28.31)$$

where a_k are vectors and the inverted circumflex means the k -th vector is omitted from the product. The vector identity (28.31) can be generalized to multivectors. Namely, if B_r is an r -vector with $1 \leq r \leq n$, and a_k with $1 \leq k \leq n$ are vectors, then

$$\begin{aligned} & B_r \cdot a_1 \wedge \cdots \wedge a_n \\ &= \sum_{j_1 < j_2 < \cdots < j_r} \sigma(j_1, \dots, j_r) (B_r \cdot a_{j_1} \wedge \cdots \wedge a_{j_r}) a_{j_{r+1}} \wedge \cdots \wedge a_{j_n} \end{aligned} \quad (28.32)$$

where $j_1, \dots, j_n$ is a permutation of $1, \dots, n$, and $\sigma(j_1, \dots, j_n)$ equals 1 if the permutation is even and -1 if it is odd.

Notice that, according to (23.19), the form of the dot product is not universal and depends upon the grades. For example, the relations (28.27)–(28.30) hold only under the indicated grade constraints. There are certain advantages in introducing the inner product slightly differently based on the so-called contraction (right and left) operators “ $\lrcorner$ ” and “ $\lrcorner$ ” for which many important identities hold universally and unconditionally, [95, 3]. Namely, the *left contraction* $\lrcorner$ is defined implicitly by

$$\text{left contraction: } X * (A \lrcorner B) = (X \wedge A) * B. \quad (28.33)$$

Consequently, the *left contraction* $\lrcorner$ is a product producing a $(k - s)$ -blade from a k -blade and s -blade with the following defining properties, [95, 3.2.2]

$$\alpha \lrcorner B = \alpha B, \text{ if } \text{grade}(B) > 0, \quad (28.34)$$

$$a \lrcorner b = a \cdot b, \quad (28.35)$$

$$a \lrcorner (B \wedge C) = (a \lrcorner B) + (-1)^{\text{grade}(B)} B \wedge (a \lrcorner C), \quad (28.36)$$

$$A \lrcorner (B \lrcorner C) = (A \wedge B) \lrcorner C, \quad (28.37)$$

where α is a scalar, a and b are vectors, and A , B and C are blades (which could be scalars or vectors as well as higher-dimensional blades). Notice that in contrast to the important identity (28.29) which holds under the indicated grade constraints, the similar identity (28.37) holds unconditionally.

The *right contraction operator* $\rfloor$ is defined implicitly by the relation

$$\text{right contraction: } (B \rfloor A) * X = B * (A \wedge X). \quad (28.38)$$

The relationship between the two contractions follows from the reversion symmetry of the scalar product (23.37), namely

$$B \rfloor A = (\tilde{A} \rfloor \tilde{B})^\sim = (-1)^{\text{grade}(A)(\text{grade}(B)+1)} A \rfloor B. \quad (28.39)$$

For example, we have for a vector a and a multivector B the following identities for the right and left contractions:

$$\begin{aligned} a \rfloor B &= \frac{1}{2} (aB - \hat{B}a), & B \rfloor a &= \frac{1}{2} (Ba - a\hat{B}), \\ a \wedge B &= \frac{1}{2} (aB + \hat{B}a), & B \wedge a &= \frac{1}{2} (Ba + a\hat{B}), \end{aligned} \quad (28.40)$$

$$\begin{aligned} aB &= a \rfloor B + a \wedge B, \\ \hat{B}a &= \hat{B} \rfloor a + \hat{B} \wedge a = -a \rfloor B + a \wedge B, \\ aB &= \hat{B}a + 2a \rfloor B = -\hat{B}a + 2a \wedge B, \end{aligned} \quad (28.41)$$

where $\hat{B}$ is the grade involution of B defined below in (23.27).

When dealing with expressions involving different products, we often use parentheses to avoid ambiguity, and that unfortunately clutters expressions. To minimize the use of parentheses, we follow to the *precedence convention*, [171, 1.1]: *in ambiguous algebraic expressions outer products are to be formed before inner products, and both of those before geometric products*. For example,

$$\begin{aligned} (A \cdot B) C &= A \cdot BC \neq A \cdot (BC), \\ (A \wedge B) C &= A \wedge BC \neq A \wedge (BC), \\ A \cdot (B \wedge C) &= A \cdot B \wedge C \neq (A \cdot B) \wedge C, \\ (A \cdot B) (C \wedge D) &= A \cdot BC \wedge D. \end{aligned} \quad (28.42)$$

28.4 Bivectors

A *bivector* is often characterized as an “oriented area” or “directed plane segment” and it plays a very important role in the theory of Clifford algebras and its applications to geometry and physics, [168, VI.A], [93, 2.4, 2.7], [95, 2.3, 3.7, 7.2], [185, 0.1-2]. The standard vector algebra counterpart of bivector is “axial vector”, and the

relation between the two is characterized well by J. Stratton, [314, p. 67]: “in vector analysis it is customary to distinguish polar vectors, such as are employed to represent translations and mechanical forces, from axial vectors with which there are associated directions of rotation. Geometrically, a polar vector is represented by a displacement or line, whereas an axial vector corresponds to an area. A typical axial vector is that which results from the vector or cross product of two polar vectors, and we must conclude from the above that an axial vector is in fact an antisymmetric tensor and its components should properly be denoted by two indices rather than one.”

Specific examples of bivector applications in geometry and EM theory are: (i) bivectors give algebraic expression for rotations and spins and the angle between two vectors can be represented by a bivector; (ii) quantities represented conventionally by “axial vectors” such as torque, angular momentum and magnetism related fields **B**, **H**, and **M** are bivectors; (iii) the relativistic representation of an EM field is a bivector quantity.

In this section we provide a list of selected properties of bivectors, and for a detailed exposition of the subject we refer the reader to [171, 1.5], [93, 2.4], [95, 2.3, 3.7, 7.2], [185, 0.1-2]. Bivectors can be represented in the following form, [82, 5.2],

$$B = \frac{1}{2} B^{\mu\nu} \gamma_\mu \wedge \gamma_\nu, \text{ where} \quad (28.43)$$

$$B^{\mu\nu} = -B^{\nu\mu} = (\gamma_\mu \wedge \gamma_\nu) \cdot B = \gamma_\mu \cdot (\gamma_\nu \cdot B).$$

If a is a vector and B is a bivector, general formulas (23.22) and (23.23) imply that

$$aB = a \cdot B + a \wedge B, \quad Ba = B \cdot a + B \wedge a, \quad (28.44)$$

$$a \cdot B = -B \cdot a = \frac{1}{2} (aB - Ba),$$

$$a \wedge B = a \wedge B = \frac{1}{2} (aB + Ba).$$

If a , b and c are vectors, and B is a bivector, then, using the above formulas together with defining relations (23.2) and (23.3) for vectors, one can obtain the following useful identities, [160, 1]:

$$a \cdot (b \wedge c) = (a \cdot b) c - (a \cdot c) b = -(b \wedge c) \cdot a, \quad (28.45)$$

$$a \cdot (b \wedge B) = (a \cdot b) B - b \wedge (a \cdot B). \quad (28.46)$$

An important and nontrivial fact about bivectors is that for Euclidean and Minkowski spaces *every bivector B can be expressed as a sum of distinct commuting blades*, [171, 3-4, 3-8], [95, 7.4.3], namely

$$B = \sum_{s=1}^m B_s, \text{ where } m \leq \frac{p+q}{2}, \quad B_s B_q = B_q B_s. \quad (28.47)$$

For Euclidean space we have more:

$$B = \sum_{s=1}^m B_s, \text{ where } m \leq \frac{p+q}{2}, \text{ and} \quad (28.48)$$

$$B_s B_q = B_q B_s = B_q \wedge B_s, \text{ for } s \neq q, \quad B_s^2 \neq 0.$$

In the case of 4-dimensional Minkowski space and the STA, a very detailed form of the representation (28.47) which includes null-bivectors and null-blades is provided in [307, A.1, A.2]. In particular, any bivector F such that $F^2 \neq 0$ can be put in the canonical form, [165, 3], [307, A.1, A.2]

$$F = f e^{I\varphi}, \quad (28.49)$$

where a timelike bivector f and a scalar φ satisfy

$$f^2 = [(F \cdot F) + |F \wedge F|^2]^{1/2}, \quad (28.50)$$

$$e^{I\varphi} = \left[\frac{F^2}{f^2} \right]^{1/2} = \frac{[(F \cdot F) + F \wedge F]^{1/2}}{[(F \cdot F)^2 + |F \wedge F|^2]^{1/2}}.$$

More properties of bivectors related to the commutator product are provided in the following section. For important representation of bivectors related to the Lorentz transformations and their rotors, see Sect. 30.4.

28.5 The Commutator Product and Bivectors

Another product of importance in geometric algebra $\text{Cl}(p, q)$ is the commutator product $A \diamond B$ of two multivectors A and B defined [93, 4.1.3], [171, 1.1], [172, I.1], [167, 2.4] by

$$A \diamond B = [A, B] = \frac{1}{2} (AB - BA). \quad (28.51)$$

The commutator product arises naturally in infinitesimal treatment of rotations. Sometimes the commutator product is denoted by $A \times B$, but we want to reserve symbol “ $\times$ ” for the conventional vector product to avoid any confusion, and for that reason we pick symbol “ $\diamond$ ” for the commutator product.

The commutator product is evidently anticommutative

$$A \diamond B = -A \diamond B, \quad (28.52)$$

and, instead of being associative, it satisfies the so-called *Jacobi identity*

$$A \diamond (B \diamond C) + C \diamond (A \diamond B) + B \diamond (C \diamond A) = 0. \quad (28.53)$$

The commutator product is a “*derivation*” with respect to the geometric product, that is,

$$A \diamond (BC) = (A \diamond B)C + B(A \diamond C). \quad (28.54)$$

The relation of the commutator product to the inner and outer products is grade dependent. For a vector a

$$a \diamond \langle M \rangle_k = a \wedge \langle M \rangle_k \text{ if } k \text{ is odd, } \quad a \diamond \langle M \rangle_k = a \cdot \langle M \rangle_k \text{ if } k \text{ is even.} \quad (28.55)$$

The commutator product is especially useful in computations with bivectors. For any bivector $B = \langle B \rangle_2$ and vector a

$$B \diamond a = \frac{1}{2} (Ba - aB) = B \cdot a. \quad (28.56)$$

If $e_1, \dots, e_r$ is a set of orthogonal and, therefore, anticommuting vectors, then

$$B \diamond (e_1 \cdots e_r) = \sum_{s=1}^r e_1 \cdots (B \cdot e_s) \cdots e_r, \text{ if } B = \langle B \rangle_2 \quad (28.57)$$

for any bivector B . The commutator of a bivector B and a multivector A_r with the grade r preserves its grade, namely

$$B \diamond A_r = \langle B \diamond A_r \rangle_r, \quad B = \langle B \rangle_2, \quad (28.58)$$

and this is the basis for incorporating the theory of Lie groups into geometric algebra. In particular, the grade preservation property (28.58) of the commutator product implies that the space of bivectors is closed under the commutator product. Therefore, it forms a Lie algebra which is, in fact, the Lie algebra of the Lorentz group.

For a bivector $B = \langle B \rangle_2$ and A_r with the grade $r > 1$ we have

$$BA_r = B \cdot A_r + B \diamond A_r + B \wedge A_r, \quad r > 1, \quad (28.59)$$

where

$$\frac{1}{2} (BA_r - A_r B) = B \diamond A_r, \quad \frac{1}{2} (BA_r + A_r B) = B \cdot A_r + B \wedge A_r. \quad (28.60)$$

Notice that the derivation property combined with grade preservation property implies that for any bivector $B = \langle B \rangle_2$, [171, 1.1]

$$B \diamond (A \cdot C) = (B \diamond A) \cdot C + A \cdot (B \diamond C), \quad (28.61)$$

$$B \diamond (A \wedge C) = (B \diamond A) \wedge C + A \wedge (B \diamond C). \quad (28.62)$$

Three bivectors $\sigma_i = \gamma_i \gamma_0$ satisfy the following algebraic relations involving the commutator product, [93, 5.1.2]:

$$\begin{aligned} \sigma_i \diamond \sigma_j &= \epsilon_{ijk} \mathbf{I} \sigma_k, \\ (\mathbf{I} \sigma_i) \diamond (\mathbf{I} \sigma_j) &= -\epsilon_{ijk} \mathbf{I} \sigma_k, \\ (\mathbf{I} \sigma_i) \diamond \sigma_j &= -\epsilon_{ijk} \sigma_k, \end{aligned} \quad (28.63)$$

where ϵ_{ijk} is the alternating tensor (*Levi-Civita symbol*) (23.50).

28.6 Pseudoscalar, Duality and the Cross Product

The concept of a dual to a blade is an algebraic counterpart of an orthogonal complement to a subspace, [171, 1.2], [93, 4.1.4], [95, 3.5], [167]. Consequently the concept of duality is as useful for algebraic transformation as the concept of orthogonality in geometry. Let us consider the geometric algebra $\text{Cl}(p, q)$, $p + q = n$. The (unit) pseudoscalar $\mathbf{I}$ in this algebra is defined as the product of unit basis vectors γ_j

$$\begin{aligned} \mathbf{I} &= \gamma_1 \gamma_2 \cdots \gamma_n, \quad \text{where} \\ \gamma_j^2 &= -1 \text{ for } 1 \leq j \leq q, \quad \gamma_j^2 = 1 \text{ for } q+1 \leq j \leq n = p+1. \end{aligned} \quad (28.64)$$

Then evidently

$$\begin{aligned} \tilde{\mathbf{I}} &= (-1)^q, \quad \tilde{\mathbf{I}} = \mathbf{I} \text{ for the STA,} \\ \tilde{\mathbf{I}} &= -\mathbf{I} \text{ for GA of 3-dimensional space} \end{aligned} \quad (28.65)$$

and, in view of the formula (23.42), we have

$$\mathbf{I}^{-1} = (-1)^{q+n(n-1)/2} \mathbf{I}, \quad \text{or } \mathbf{I}^2 = (-1)^{q+n(n-1)/2}. \quad (28.66)$$

We define a *dual* $A^\sharp$ to a multivector A by

$$A^\sharp = \mathbf{A} \mathbf{I}^{-1} = A \cdot \mathbf{I}^{-1} \text{ so } A = A^\sharp \mathbf{I}, \quad (28.67)$$

[167, 2.3], [95, 3.5.3], and the reason for using the inverse I^{-1} in the definition of $A^\sharp$ is to have $I^\sharp = 1$. Consequently,

$$A^{\sharp\sharp} = AI^{-2} = (-1)^{q+n(n-1)/2} A, \quad (28.68)$$

that is if the duality operation is applied twice, it returns the same multivector up to a factor $(-1)^{q+n(n-1)/2}$. In addition to that, if A_r is a grade r blade, then

$$IA_r = (-1)^{r(n-r)} A_r I, \quad (28.69)$$

as it can be verified using a basis. In particular, the relations (28.69) imply that in all cases I commutes with even-grade multivectors. Combining equalities (28.67) and (28.69), we obtain

$$A_r^\sharp = A_r I^{-1} = A_r \cdot I^{-1} = (-1)^{r(n-r)} I^{-1} A_r. \quad (28.70)$$

Note also that the dual $A_r^\sharp = A_r I^{-1}$ is a grade- $(n-r)$ blade which is orthogonal to A_r , namely

$$A_r^\sharp = \langle A_r^\sharp \rangle_{n-r}. \quad (28.71)$$

Combining relations (23.42), (28.66) and (28.70), we obtain

$$\frac{A_r^\sharp A_r}{A_r * \tilde{A}_r} = (-1)^{r(n-r)+r(r-1)/2+q+n(n-1)/2} I. \quad (28.72)$$

Observe that

$$I^2 = -1, \text{ and } A^{\sharp\sharp} = -A \text{ for the STA and GA of 3-dimensional space.} \quad (28.73)$$

An important feature of the duality transformation is that it relates the *inner and outer product*. Namely, if A_r and B_s are multivectors of respective grades r and s , then the following identity holds:

$$A_r \cdot (B_s I) = (A_r \wedge B_s) I = (-1)^{s(n-s)} (A_r I) \cdot B_s. \quad (28.74)$$

Or, using the fact that $I^{-2} = (-1)^q (-1)^{n(n-1)/2}$ is a scalar, we can recast the above as

$$(A_r \wedge B_s)^\sharp = A_r \cdot B_s^\sharp = (-1)^{s(n-s)} A_r^\sharp \cdot B_s. \quad (28.75)$$

Hence, according to (28.75), the duality operation interchanges inner and outer products, and that makes it particularly useful in computation.

The duality relationships (28.75) for Euclidean spaces can also be expressed equivalently in the form which uses the contraction, [95, 3.5.4]

$$(A \wedge B)^\sharp = A \rfloor (B^\sharp), \quad (A \rfloor B)^\sharp = A \wedge (B^\sharp). \quad (28.76)$$

In the case of 3-dimensional Euclidean space, the corresponding duality transformation allows us to relate the conventional “cross product” $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ to GA products, namely

$$\mathbf{a} \wedge \mathbf{b} = \frac{1}{2} (\mathbf{a}\mathbf{b} - \mathbf{b}\mathbf{a}) = I (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{b}) I, \quad (28.77)$$

$$\mathbf{a} \times \mathbf{b} = (\mathbf{a} \wedge \mathbf{b}) I^{-1} = -\frac{1}{2} (\mathbf{a}\mathbf{b} - \mathbf{b}\mathbf{a}) I = -\mathbf{a} \cdot (I\mathbf{b}) = (\mathbf{a} \wedge \mathbf{b})^\sharp. \quad (28.78)$$

Here is more useful identities involving vectors and the pseudoscalar I , [170]:

$$\mathbf{a} \wedge (I\mathbf{b}) = (\mathbf{a} \cdot \mathbf{b}) I, \quad \mathbf{a} \cdot (I\mathbf{b}) = (\mathbf{a} \wedge \mathbf{b}) I = -\mathbf{a} \times \mathbf{b}. \quad (28.79)$$

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} - (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} = -\mathbf{a} \times (\mathbf{b} \times \mathbf{c}), \quad (28.80)$$

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = I [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})], \quad (28.81)$$

$$(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \cdot \mathbf{d} = (\mathbf{a} \wedge \mathbf{b}) (\mathbf{c} \cdot \mathbf{d}) - (\mathbf{a} \wedge \mathbf{c}) (\mathbf{b} \cdot \mathbf{d}) + (\mathbf{b} \wedge \mathbf{c}) (\mathbf{a} \cdot \mathbf{d}). \quad (28.82)$$

The *Hodge map* “ $*$ ” can also be expressed in terms of the dual transformation [185, 0.4]

$$* \mathbf{b} = I\mathbf{b} = \mathbf{b}I = -\mathbf{b}^\sharp \text{ for any vector } \mathbf{b}. \quad (28.83)$$

Chapter 29

Multivector Calculus

Multivectors are very effective in describing geometric properties of physical quantities. In particular, they can represent physical fields such as EM fields and spinorial charge fields in coordinate-free and manifestly Lorentz-covariant form. Since partial derivatives of the fields are commonly involved, it is important to extend their use to multivector variables. Specifically we want to have a multivector calculus which would allow us to produce the Euler–Lagrange field equations in the most concise and geometrically transparent multivector form. Such a multivector calculus and its applications to the Lagrangian field theories has been developed, see [93, 11, 12], [171, 2.2], [252, 6.2.3], [219], [307, 7.1], [283, 2.11] and references therein. We provide in this section basic features of this calculus. The authors of the above monographs use somewhat different notation and conventions. That, of course, does not affect anything of significance but rather requires you to pay attention when using formulas.

29.1 Definition and Basic Properties of Multivector Derivatives

A multivector variable X is assumed to take values in a subspace $\mathcal{X}$ of the spacetime algebra $\text{Cl}(1, 3)$ which is a direct sum of a set of grade subspaces $\bigwedge^k$. For instance, $\mathcal{X}$ can be a space of vectors as in the case of the EM potential A , or it can be a space of bivectors as in the case of an EM field. $\mathcal{X}$ can also be a space of all even multivectors, which forms an important even algebra $\text{Cl}_+(1, 3)$, as in the case of a spin- $\frac{1}{2}$ spinorial variable ψ , or it can be a space of odd multivectors which is the case of the geometric derivative $\partial\psi$ of ψ . We associate with a multivector variable X with values in $\mathcal{X}$ a projection operator $\langle Y \rangle_{\mathcal{X}}$ defined by (28.26), that is,

$$\langle Y \rangle_X = \langle Y \rangle_{\mathcal{X}} \text{ is the projection of } Y \text{ on the space } \mathcal{X}. \quad (29.1)$$

The *multivector differentiation* operation is fundamentally based on the concept of the *directional derivative* $F'_A(X_0)$ of a multivector-valued $F(X)$ in the direction of a multivector $A \in \mathcal{X}$ at a given point X_0 . This directional derivative $F'_A(X_0) = (D_A F)(X_0)$ is defined as the following limit

$$F'_A(X_0) = (D_A F)(X_0) = \lim_{\varepsilon \rightarrow 0} \frac{F(X_0 + \varepsilon A) - F(X_0)}{\varepsilon}, \quad (29.2)$$

with an assumption of the existence of the limit. Importantly, the *directional derivative operator* D_A is *grade preserving* and it is *linear in A* , [283, 2.11], [252, 6.2.3.1]

$$F'_{\alpha A + \beta B}(X_0) = \alpha F'_A(X_0) + \beta F'_B(X_0) \text{ or } D_{\alpha A + \beta B} = \alpha D_A + \beta D_B. \quad (29.3)$$

As a consequence of the linearity of D_A with respect to A we obtain expression

$$D_A = \sum_{\gamma^J \in \mathcal{X}} \frac{1}{|J|!} (A * \gamma^J) D_{\gamma^J}. \quad (29.4)$$

The above representation for D_A suggests to introduce an important geometric multivector differentiation operator ∂_X called the *vector derivative* and defined by, [252, 6.2.3], [283, 2.11.2]

$$\partial_X = \sum_{\gamma^J \in \mathcal{X}} \frac{1}{|J|!} \gamma^J D_{\gamma^J}. \quad (29.5)$$

Observe that relations (29.4) and (29.5) imply the following relations between the directional derivative operator D_A and the multivector differentiation operator ∂_X :

$$D_A = A * \partial_X, \quad \text{and } F'_A(X_0) = (D_A F)(X_0) = (A * \partial_X) F(X_0). \quad (29.6)$$

Notice that in algebraic manipulations the directional derivative operator D_A and the multivector differentiation operator ∂_X can be treated, respectively, as scalar and multivector quantities.

Often $(A * \partial_X) F$ is written simply as $A * \partial_X F$ with a *presumption that the operation $A * \partial_X$ was carried out first*. Needless to say, “forgetting” the operation order convention can lead to serious pitfalls when interpreting “self-explanatory” expressions such as those in (29.6). For instance,

$$A * \partial_X F \equiv (A * \partial_X) F \neq A * (\partial_X F), \quad (29.7)$$

since for a multivector-valued F the expression in the left-hand side of (29.7) evidently involves all grades of F whereas the expression in the right-hand side is a scalar. To clarify and stress the operation order, we often use parentheses.

Sometimes it is convenient to identify the linear dependence of the directional derivative $F'_A(X_0)$ on A in the following form, [171, Section 2.2]

$$F'_A(X_0) = \underline{F}(A), \quad (29.8)$$

where $\underline{F}(A)$ is a linear multivector-valued function of the multivector A called the *differential* of F at X_0 with the dependence on X_0 being suppressed in the notation. A particularly useful quantity when it comes to a geometrically appealing treatment of multivector-valued differentials is the *adjoint differential* $\overline{F}(B)$ defined by, [171, Section 2-2.]

$$B * \underline{F}(A) = \overline{F}(B) * A, \quad (29.9)$$

where one might view the arbitrary multivector B as a geometric substitute for a “coordinate”.

In the *special case of a scalar-valued function* $F(X)$, which can be a Lagrangian density, a particularly simple representation of the directional derivative F'_A in terms of $\partial_X F$ holds. Namely, using the grade preservation property of the directional derivative and the fact that for a scalar-valued F the directional derivatives F'_{γ_j} are scalar-valued as well, we obtain from (29.4)–(29.6)

$$F'_A(X_0) = \underline{F}(A) = (A * \partial_X) F(X_0) = A * [\partial_X F(X_0)], \text{ where } \partial_X F(X_0) \in \mathcal{X}. \quad (29.10)$$

In other words, in the case of a scalar F according to (29.10) we can think of the derivative $\partial_X F$ as simply a multivector in $\mathcal{X}$.

Yet another useful set of differential operators “ $\partial_X \star$ ”, where “ $\star$ ” stands for one of the four multiplication operations “ $\cdot$ ”, “ $\wedge$ ”, “ $\cdot$ ” or the Clifford product, are defined by, [252, 6.2.3]

$$\partial_X \star F(X_0) = \sum_{\gamma^J \in \mathcal{X}} \frac{1}{|J|!} \gamma^J \star F'_{\gamma^J}(X_0) = \sum_{\gamma^J \in \mathcal{X}} \frac{1}{|J|!} \gamma^J \star (D_{\gamma^J} F)(X_0). \quad (29.11)$$

Using the coordinate decomposition (28.24)

$$F = \sum_S \frac{1}{|S|!} \gamma_S F^S, \quad (29.12)$$

we can recast the formula (29.11) as

$$\partial_X \star F(X_0) = \sum_{\gamma^J \in \mathcal{X}, S} \frac{1}{|J|! |S|!} (\gamma^J \star \gamma_S) (D_{\gamma^J} F^S)(X_0). \quad (29.13)$$

In particular, the above formula suggests that if x is a vector then $\partial_x = \partial$ and $\partial \wedge$, $\partial \cdot$, and ∂ defined by (29.11) can be viewed as multivector versions of the “curl”, “scalar divergence” and “gradient” operators.

Another two important tools to facilitate effective differentiation are the chain and the Leibnitz rules, [252, Section 6.2.3]

$$(F \circ G)'_A (X_0) = F'_{G'_A(X_0)} (G (X_0)) \text{ or } D_A (F \circ G) = D_{D_A G} F, \quad (29.14)$$

$$(G \star F)'_A (X_0) = G'_A (X_0) \star F (X_0) + G (X_0) \star F'_A (X_0), \text{ or} \quad (29.15)$$

$$D_A (G \star F) = D_A G \star F + G \star D_A F.$$

Since for any multivector B the quantity $F \star B$ is evidently scalar-valued, using (29.9) and (29.10) together with (29.15) we obtain

$$(A \star \partial_X) (F \star B) = [(A \star \partial_X) F] \star B \quad (29.16)$$

$$= \underline{F} (A) \star B = \overline{F} (B) \star A = A \star [\partial_X (F \star B)]$$

implying the following representation for the *adjoint differential*

$$\overline{F} (B) = \partial_X (F \star B). \quad (29.17)$$

29.2 The Vector Derivative and Its Basic Properties

We present here basic properties of the vector derivative following [171, 2], [172, 2], [93, 6.1.3].

The vector derivative $\partial = \partial_x$ is defined by (24.3), namely

$$\partial_x = \partial = \gamma^\mu \partial_\mu = \gamma^0 \partial_0 + \gamma^1 \partial_1 + \gamma^2 \partial_2 + \gamma^3 \partial_3. \quad (29.18)$$

It is evidently a special case of the multivector derivative ∂_X defined by relations (29.4) and (29.5) when $X = x$ is a vector. Using the symmetry of the partial derivative $\partial_\mu \partial_\nu$ with respect to μ and ν , and the fundamental relations (23.5)–(23.6) defining the basis vectors γ^μ , we readily obtain

$$\partial^2 = \partial \cdot \partial = \gamma^\mu \gamma^\nu \partial_\mu \partial_\nu = g^{\mu\nu} \partial_\mu \partial_\nu = \partial_\mu \partial^\mu = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2, \quad (29.19)$$

implying in particular that ∂^2 is a scalar operator.

The change of vector variables from x to $x' = f(x)$ induces the adjoint linear transformation of the derivative [171, Section 2-1.]

$$\partial_x = \tilde{f} (\partial_{x'}), \text{ for } x' = f(x), \quad (29.20)$$

and the following *chain rule for the vector derivative* holds [93, 6.1.3]:

$$\partial_x G (f(x)) = \tilde{f} (\partial_{x'}) G (x') \text{ for } x' = f(x); \quad (29.21)$$

$\partial (x \cdot a) = a$ if a is a vector independent of x ;

$$\partial x^\mu = \gamma^\mu.$$

If $a = a(x)$ is a vector field depending on the vector x , then

$$\partial \cdot a = \gamma^\mu \partial_\mu \cdot a_\nu \gamma^\nu = \partial_\mu a_\nu \gamma^\mu \cdot \gamma^\nu = \partial_\mu a_\nu g^{\mu\nu} = \partial_\mu a^\mu.$$

Using the antisymmetry of $\gamma^\mu \wedge \gamma^\nu$ in indices μ and ν , we obtain:

$$\partial \wedge a = \gamma^\mu \partial_\mu \wedge a_\nu \gamma^\nu = \partial_\mu a_\nu \gamma^\mu \wedge \gamma^\nu = \frac{1}{2} (\partial_\mu a_\nu - \partial_\nu a_\mu) \gamma^\mu \wedge \gamma^\nu.$$

Hence,

$$\partial a = \partial \cdot a + \partial \wedge a.$$

Here

$$\partial (ab) = (\partial a) b - a (\partial b) + 2 (a \cdot \partial) b.$$

An important result for the vector derivative is that the exterior derivative of an exterior derivative always vanishes

$$\partial \wedge (\partial \wedge A) = 0, \quad \partial \cdot (\partial \cdot A) = 0,$$

[171, 2].

Here is a table of vector derivatives of some elementary functions [171, 2-1.], [172, 2], [93, 6.1.3]

$$\begin{aligned} \partial (x \cdot a) &= (a \cdot \partial) x = a, & \partial (x \cdot A_r) &= (A_r \cdot \partial) x = r A_r, \\ \partial (|x|^2) &= \partial (x^2) = 2x, & \partial (x \wedge A) &= (A \wedge \partial) x = (4 - r) A_r, \\ \partial \wedge x &= 0, & \partial (A_r x) &= \gamma^\mu A_r \gamma_\mu = (-1)^4 (4 - 2r) A_r, \\ \partial x &= \partial \cdot x = 4, & \partial (\log |x|) &= \frac{x}{|x|^2} = x^{-1}, \\ \partial (|x|^k) &= k |x|^{k-2} x, & \partial \left(\frac{x}{|x|^k} \right) &= \frac{4-k}{|x|^k} x, \end{aligned} \quad (29.22)$$

where $|x|^2 = x \cdot \tilde{x}$, a is a vector function and A_r is a grade r multivector function.

Another identity [171, 2-1.], [172, 2]

$$[a, b] = \partial (a \wedge b) + a (\partial \cdot b) - b (\partial \cdot a),$$

where $[a, b]$ is the commutator of directional derivatives

$$[a, b] = (a \cdot \partial) b - (b \cdot \partial) a,$$

also called the *Lie bracket*. The left-hand side of this identity may be identified as a Lie bracket; a more general concept of the Lie bracket is introduced later on.

Notice that since ∂ is a vector, it may not commute with other multivectors. Consequently, one has to exercise caution in handling expressions involving ∂ . One way to avoid ambiguity and enforce the desired scope of action of the operator ∂ is to use parentheses. A common way to define the scope of the vector derivative ∂ is by using overdots, a convention used to specify which terms in an expression are subject to the differentiation as, for instance

$$\dot{\partial} (C \dot{A} B) \equiv C (\partial A) B, \quad (29.23)$$

where the overdot over $\dot{A}$ is used to identify the only multivector in the expression to be differentiated.

To have expressions less cluttered with parentheses, the following *series of conventions* is used to clarify the scope of ∂ , [93, Section 6.1.3]:

(i) in the absence of parentheses (brackets), ∂ acts on the object to its immediate right;

(ii) when the ∂ is followed by parentheses (brackets), the derivative acts on all of the terms in parentheses (brackets);

(iii) when the ∂ acts on a multivector to which it is not adjacent, we use overdots to describe the scope.

29.3 Examples of Multivector Derivatives

To facilitate multivector differentiation of a Lagrangian, we consider a number of key examples of the multivector derivatives. The first very useful identity for the differentiation ∂_X is

$$\partial_X (Y * X) = \langle Y \rangle_X, \quad (29.24)$$

the projection $\langle Y \rangle_X$ is defined in (29.1). The following identities can be used for integration by parts

$$\begin{aligned} \partial_\mu \langle MN \rangle &= \langle (\partial_\mu M) N \rangle + \langle M (\partial_\mu N) \rangle, \text{ or} \\ \partial_\mu (M * N) &= (\partial_\mu M) * N + M * (\partial_\mu N) \end{aligned} \quad (29.25)$$

where $M = M(x)$ and $N = N(x)$ are multivector-valued functions of the spacetime variable x . Here is another identity involving the vector derivative ∂ following from (29.25):

$$\begin{aligned} \langle (\partial M) N \rangle &= \langle (\partial_\mu M) N \gamma^\mu \rangle = \partial_\mu \langle MN \gamma^\mu \rangle - \langle M (\partial_\mu N) \gamma^\mu \rangle \\ &= \partial \cdot \langle MN \rangle_1 - \left\langle \gamma^\mu \left(\partial_\mu \tilde{N} \right) \tilde{M} \right\rangle = \partial \cdot \langle MN \rangle_1 - \left\langle \tilde{M} \left(\partial \tilde{N} \right) \right\rangle, \end{aligned} \quad (29.26)$$

or

$$(\partial M) * N = \partial \cdot \langle M \tilde{N} \rangle_1 - \tilde{M} * \partial \tilde{N} = \partial \cdot \langle M \tilde{N} \rangle_1 - M * \widetilde{\partial \tilde{N}}. \quad (29.27)$$

To facilitate effective computations of multivector derivatives of Lagrangians, we provide below a formula useful for typical Lagrangian expressions. Suppose that $L(X)$ is a scalar function of a multivector X represented by the formula

$$L(X) = \langle L_1(X) L_2(X) \dots L_n(X) \rangle. \quad (29.28)$$

Suppose also that in the above formula every multiplier $L_s(X)$ is a linear multivector function of the form

$$L_s(X) = \sum_p A_{sp} X B_{sp} + \sum_p M_{sp} \tilde{X} N_{sp} + C_s, \quad (29.29)$$

where A_s, B_s, M_s, N_s, C_s are multivectors independent of the variable X . Then the following formula holds for the derivative L' of L with respect to X :

$$L' = \partial_X L = \sum_{s=1}^n \langle K_s \rangle_X, \quad K_s = \sum_p B_{sp} Q_s A_{sp} + \sum_p \tilde{M}_{sp} \tilde{Q}_s \tilde{N}_{sp}, \quad (29.30)$$

$$Q_s = L_1(X) \dots \check{L}_s(X) \dots L_n(X), \text{ for } n \geq 2,$$

where $\check{L}_s(X)$ indicates that this multiplier is omitted in the product, and the projection $\langle Y \rangle_X$ is defined in (29.1). The formula (29.30) is an elementary consequence of the formula for the derivative of the product (29.15) and basic properties (23.34)–(23.37) of the operation $\langle \cdot \rangle$. In the case of $n = 1$, that is,

$$L(X) = \sum_p \langle A_{sp} X B_{sp} + M_{sp} \tilde{X} N_{sp} + C \rangle, \quad (29.31)$$

the derivative takes the form

$$L' = \partial_X L = \sum_p B_{sp} A_{sp} + \sum_p \tilde{M}_{sp} \tilde{N}_{sp}. \quad (29.32)$$

In the special case of the Lagrangian of the form (29.30) such that

$$L = \langle L_1 \tilde{L}_1 \rangle, \text{ with } L_1 = L_1(X) = \sum_p A_p X B_p + C, \quad (29.33)$$

where A, B and C are multivectors independent of the variable X , the formula (29.30) implies that

$$L' = \partial_X L = 2 \sum_p \langle B_p \tilde{L}_1 A_p \rangle_X. \quad (29.34)$$

In another special case of the Lagrangian of the form (29.30) involving Hermitian conjugation such that

$$L = \left\langle L_1 L_1^\dagger \right\rangle, \text{ with } L_1 = L_1(X) = \sum_p A_p X B_p + C, \quad (29.35)$$

where A , B and C are multivectors independent of the variable X , the formula (29.30) implies the expression

$$L' = \partial_X L = 2 \left\langle \sum_p B_p L_1^\dagger A_p \right\rangle_X. \quad (29.36)$$

Another case is the Lagrangian of the form

$$L(X) = f\left(\langle X \tilde{X} \rangle\right), \text{ where } f \text{ is a scalar-valued function.} \quad (29.37)$$

In this case

$$L' = \partial_X L = 2f'\left(\langle X \tilde{X} \rangle\right) \tilde{X}. \quad (29.38)$$

In the case of

$$L(X) = f\left(\langle X X^\dagger \rangle\right), \text{ where } f \text{ is a scalar-valued function,} \quad (29.39)$$

we have

$$L' = \partial_X L = 2f'\left(\langle X X^\dagger \rangle\right) X^\dagger. \quad (29.40)$$

Here is a table of multivector derivatives of some elementary functions [171, 2.2]:

$$\begin{aligned} (A * \partial_X) X &= \langle A \rangle_X, \\ (A * \partial_X) \tilde{X} &= \left\langle \tilde{A} \right\rangle_X, \\ \partial_X X &= d, \\ \partial_X (|X|^2) &= 2\tilde{X}, \\ (A * \partial_X) X^k &= \left\langle \tilde{A} \right\rangle_X X^{k-1} + X \left\langle \tilde{A} \right\rangle_X X^{k-2} + \dots + X^{k-1} \left\langle \tilde{A} \right\rangle_X, \\ \partial_X (|X|^k) &= k |X|^{k-2} \tilde{X}, \\ \partial_X \log |X| &= \frac{\tilde{X}}{|X|^2}, \\ (A * \partial_X) [|X|^k X] &= |X|^k \left[\left\langle \tilde{A} \right\rangle_X + k \frac{(A * \tilde{X}) X}{|X|^2} \right], \end{aligned}$$

where $|X|^2 = X * \tilde{X}$ and d is the dimension of the subspace associated with the variable X .

Chapter 30

Relativistic Concepts in the STA

The STA makes it possible to formulate and analyze conventional relativistic physics in an invariant form without reference to a coordinate system, [160, 2], [169, III.], [93, 5].

30.1 Inertial Systems and the Spacetime Split

A given inertial system in the STA is completely characterized by a single timelike, future pointing, unit vector γ_0 , [169, III.]. So from now on, we use γ_0 as a name for the observer and refer to the corresponding system as the γ_0 -system. Consequently, the inertial observer γ_0 is at rest in the γ_0 -system, and the vector γ_0 is tangent to the straight world line of this inertial observer. In other words, the inertial observer can be identified with the straight world line of a free particle. Using this idea, D. Hestenes introduced a simple algebraic device called the *spacetime split* that gives a possibility to relate proper (covariant) descriptions of physical properties to relative descriptions with respect to inertial systems parametrized by γ_0 , [169, III]. The spacetime split connects very efficiently the covariant language of the STA for the Minkowski space with the language of GA for the 3-dimensional Euclidean space of classical physics.

The inertial observer γ_0 determines a unique mapping of spacetime into the even subalgebra $Cl_+(1, 3)$ of the STA. Namely, for each spacetime point $x = x^\mu \gamma_\mu$ the mapping is specified by

$$x \rightarrow x\gamma_0 = x \cdot \gamma_0 + x \wedge \gamma_0, \quad (30.1)$$

which can be rewritten as

$$x\gamma_0 = x_0 + \mathbf{x}, \quad \text{where } x_0 = x \cdot \gamma_0 = ct, \quad \mathbf{x} = x \wedge \gamma_0. \quad (30.2)$$

Hence, we have the following *representation of a vector x that relates it to γ_0* :

$$x = x\gamma_0^2 = (x_0 + \mathbf{x})\gamma_0 = x_0\gamma_0 + \mathbf{x} \cdot \gamma_0, \quad (30.3)$$

and following D. Hestenes we call the bivector $\mathbf{x} = x \wedge \gamma_0$ the *relative vector*. In other words, if x is the position vector then the relative vector $\mathbf{x}$ is its position relative to the observer γ_0 . Applying relations (23.2) to the relative vector $x \wedge \gamma_0$, one can verify that it anticommutes with γ_0 , that is,

$$\gamma_0 \mathbf{x} = (x \wedge \gamma_0) = - (x \wedge \gamma_0) \gamma_0 = -\mathbf{x} \gamma_0. \quad (30.4)$$

If one wants to pick coordinates for a relative vector $\mathbf{x}$, the vector γ_0 has to be included into a frame γ_μ with properties described in Chap. 23. In the frame $\{\gamma_\mu\}$ a spacetime point x has coordinates x^μ so that

$$x = x^\mu \gamma_\mu, \quad x^\mu = x \cdot \gamma^\mu. \quad (30.5)$$

We recall that the vectors γ_j form a right-handed set of orthonormal spacelike vectors perpendicular to γ_0 . Then any relative vector $\mathbf{x}$ satisfies the following representation

$$\mathbf{x} = x^j \sigma_j, \quad \text{where } \sigma_j = \gamma_j \gamma_0, \quad j = 1, 2, 3, \quad (30.6)$$

and properties of the basis σ_j of relative vectors are listed in (23.48)–(23.55).

Notice that

$$x = \tilde{x} = \gamma_0 (x_0 - \mathbf{x}), \quad \gamma_0 x = x_0 - \mathbf{x}. \quad (30.7)$$

Consequently, using (30.3), (30.7) and taking into account the normalization $\gamma_0^2 = 1$, we recover the well known relation

$$x^2 = x\gamma_0\gamma_0x = x_0^2 - \mathbf{x}^2 = (ct)^2 - \mathbf{x}^2. \quad (30.8)$$

Justifying their name, relative vectors take values in the even subalgebra $\text{Cl}_+(1, 3)$ which is isomorphic to the geometric algebra $\text{Cl}(3, 0)$ of three-dimensional Euclidean space, [166, 1]. This implies that

$$\mathbf{x}^2 \text{ is a scalar and } \mathbf{x}^2 \geq 0, \quad (30.9)$$

and it can be verified based on (30.6) and (23.48)–(23.52).

The geometric product of two relative vectors $\mathbf{a}$ and $\mathbf{b}$ can be decomposed into an inner product $\mathbf{a} \cdot \mathbf{b}$ and an outer product $\mathbf{a} \wedge \mathbf{b}$, namely, [160, 2],

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}, \quad (30.10)$$

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2} (\mathbf{ab} + \mathbf{ba}), \quad (30.11)$$

$$\mathbf{a} \wedge \mathbf{b} = \frac{1}{2} (\mathbf{ab} - \mathbf{ba}) = \mathbf{I}(\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{b})\mathbf{I}, \quad (30.12)$$

where $\mathbf{a} \times \mathbf{b}$ is the conventional “cross product” of $\mathbf{a}$ and $\mathbf{b}$. Consequently, we can express the cross product $\mathbf{a} \times \mathbf{b}$ in terms of GA products as follows

$$\mathbf{a} \times \mathbf{b} = (\mathbf{a} \wedge \mathbf{b})\mathbf{I}^{-1} = -\frac{1}{2} (\mathbf{ab} - \mathbf{ba})\mathbf{I} = -\mathbf{a} \cdot (\mathbf{Ib}). \quad (30.13)$$

Equalities (30.13) and (28.67) show that the cross product vector $\mathbf{a} \times \mathbf{b}$ is dual to the bivector $\mathbf{a} \wedge \mathbf{b}$. Such a relation can hold only in three dimensions—the only space for which the dual of a bivector is a vector. With GA products available, the cross product is largely redundant and can always be expressed in terms of GA products. For example, [93, 2.4.3]

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}). \quad (30.14)$$

Let $x(\tau)$ be a point particle *world line* in the spacetime where τ is the *proper time* defined by

$$d\tau^2 = (dx)^2 = (dx_0)^2 - (dx_1)^2 - (dx_2)^2 - (dx_3)^2. \quad (30.15)$$

Notice that our definition of the proper time follows conventions in [27, II.1], [263, 17-3], that is, the *proper time* $d\tau$ in the rest frame is identified with $d\tau = dx_0 = cdt$ whereas another common convention is to define the proper time in the rest frame so that $d\tau = dt$.

The *proper velocity* (four-velocity) $v = v^\mu \gamma_\mu$ of a point particle with a *world line* $x(\tau)$ in the spacetime is defined by

$$v = \frac{dx(\tau)}{d\tau} = \dot{x}, \quad (30.16)$$

and, consequently, v always satisfies

$$v^2 = v_0^2 - v_1^2 - v_2^2 - v_3^2 = 1. \quad (30.17)$$

Observe that, under the alternative convention for the proper time, we would have $v^2 = c^2$ instead of $v^2 = 1$.

The *relative velocity* $\mathbf{v}$ of the point particle is defined by

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \frac{d\tau}{dt} \frac{d\mathbf{x}}{d\tau} = c \frac{v \wedge \gamma_0}{v \cdot \gamma_0}. \quad (30.18)$$

Since $v^2 = 1$, the following relation holds between the proper and the relative velocities:

$$v = v\gamma_0^2 = (v \cdot \gamma_0 + v \wedge \gamma_0) \gamma_0 = \gamma \left(1 + \frac{\mathbf{v}}{c}\right) \gamma_0 = \tilde{v} = \gamma \gamma_0 \left(1 - \frac{\mathbf{v}}{c}\right), \quad (30.19)$$

$$\text{where } \gamma = v \cdot \gamma_0 = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \text{ is the Lorentz factor,}$$

implying, in particular, identities

$$v\gamma_0 = \gamma \left(1 + \frac{\mathbf{v}}{c}\right), \quad \gamma_0 v = \gamma \left(1 - \frac{\mathbf{v}}{c}\right). \quad (30.20)$$

Considerations similar to the above yields the following relations for the derivatives, [162, 6]:

$$\gamma_0 \partial = \gamma_0 \cdot \partial + \gamma_0 \wedge \partial = \partial_0 - \sigma_j \partial^j = \partial_0 + \nabla_{\mathbf{x}} = c^{-1} \partial_t + \nabla_{\mathbf{x}}, \quad (30.21)$$

$$d_\tau = \frac{d}{d\tau} = v \cdot \partial = \frac{v_0}{c} d_t \text{ proper time derivative,} \quad (30.22)$$

$$d_t = \partial_t + \mathbf{v} \cdot \nabla_{\mathbf{x}} \text{ hydrodynamic derivative.} \quad (30.23)$$

To describe the motion of a point particle of mass m , we introduce its *proper momentum vector* p defined by

$$p = mc v, \text{ where } v \text{ is the proper velocity.} \quad (30.24)$$

Multiplying the above equality by γ_0 , one obtains from p the energy (or relative mass) E and the relative momentum $\mathbf{p}$:

$$p\gamma_0 = p \cdot \gamma_0 + p \wedge \gamma_0 = \frac{E}{c} + \mathbf{p} = \gamma (mc + m\mathbf{v}), \quad (30.25)$$

where E is the energy and $\mathbf{p}$ is the *relative momentum*. The relation (30.25) accounts for the following familiar energy and momentum formulas:

$$E = \gamma mc^2 = \frac{mc^2}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}, \quad \mathbf{p} = \gamma m\mathbf{v} = \frac{m\mathbf{v}}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}, \quad (30.26)$$

$$E^2 - c^2 \mathbf{p}^2 = m^2 c^4, \quad E = E(\mathbf{p}) = E_0 \sqrt{\frac{\mathbf{p}^2}{c^2} + 1}, \quad E_0 = mc^2. \quad (30.27)$$

30.2 Multivector and Bivector Spacetime Split

There is a straightforward generalization of the vector spacetime split defined by (30.3) to arbitrary multivectors which can be done as follows. Suppose that we have an inertial observer γ_0 , and let A be a homogeneous multivector of a positive grade. Then proceeding similarly to (30.3), we get

$$A = A\gamma_0^2 = (A \cdot \gamma_0 + A \wedge \gamma_0) \gamma_0 = A_+ + A_-, \quad \text{where} \quad (30.28)$$

$$A_+ = (A \cdot \gamma_0) \gamma_0, \quad A_- = (A \wedge \gamma_0) \gamma_0.$$

Using relations (23.22), (23.23), we see that the components $A_{\pm}$ can be transformed into

$$A_+ = (A \cdot \gamma_0) \gamma_0 = \frac{1}{2} (A - (-1)^{\text{grade}(A)} \gamma_0 A \gamma_0), \quad (30.29)$$

$$A_- = (A \wedge \gamma_0) \gamma_0 = \frac{1}{2} (A + (-1)^{\text{grade}(A)} \gamma_0 A \gamma_0),$$

implying

$$\gamma_0 A_+ \gamma_0 = (-1)^{\text{grade}(A)-1} A_+, \quad \gamma_0 A_- \gamma_0 = (-1)^{\text{grade}(A)} A_-, \quad (30.30)$$

or

$$\gamma_0 A_+ = (-1)^{\text{grade}(A)-1} A_+ \gamma_0, \quad \gamma_0 A_- = (-1)^{\text{grade}(A)} A_- \gamma_0. \quad (30.31)$$

According to (30.31), the defining difference between A_+ and A_- is that if A has an odd grade, then A_+ commutes with γ_0 and A_- anticommutes with γ_0 , and if A has an even grade, then A_+ anticommutes with γ_0 and A_- commutes with γ_0 . This principal difference between A_+ and A_- can be reformulated even more concisely in terms of the STA basis associated with the frame γ_μ as defined in Sect. 28.2. Namely, A_+ is a linear combination of those basis blades which contain γ_0 , whereas A_- is a linear combination of those basis blades which do not contain γ_0 .

In the important case of a general bivector Ω , the specetime split defined by (30.28), (30.29) is reduced to, [160, 1], [82, 8.1],

$$\Omega = \Omega \gamma_0^2 = (\Omega \cdot \gamma_0 + \Omega \wedge \gamma_0) \gamma_0 = \Omega_+ + \Omega_- \quad (30.32)$$

with the following alternative representations for Ω_+ and Ω_- :

$$\begin{aligned} \Omega_+ &= (\Omega \cdot \gamma_0) \gamma_0 = (\Omega \cdot \gamma_0) \wedge \gamma_0 \\ &= \frac{1}{2} (\Omega - \gamma_0 \Omega \gamma_0) = \frac{1}{2} (\Omega + \gamma_0 \tilde{\Omega} \gamma_0), \\ \Omega_- &= (\Omega \wedge \gamma_0) \gamma_0 = (\Omega \wedge \gamma_0) \cdot \gamma_0 \\ &= \frac{1}{2} (\Omega + \gamma_0 \Omega \gamma_0) = \frac{1}{2} (\Omega - \gamma_0 \tilde{\Omega} \gamma_0). \end{aligned} \quad (30.33)$$

There is yet another representation for Ω_+ and Ω_- in terms of Hermitian conjugative $\Omega^\dagger$ defined by (23.28), namely

$$\Omega_+ = \frac{1}{2} (\Omega + \Omega^\dagger), \quad \Omega_- = \frac{1}{2} (\Omega - \Omega^\dagger), \quad \Omega^\dagger = \gamma_0 \tilde{\Omega} \gamma_0. \quad (30.34)$$

Notice that the two components Ω_+ and Ω_- of a bivector Ω satisfy the following commutative relations:

$$\Omega_+ \gamma_0 = -\gamma_0 \Omega_+, \quad \Omega_- \gamma_0 = \gamma_0 \Omega_-. \quad (30.35)$$

The coordinate representation of the spacetime split (30.32), (30.33) has the form

$$\begin{aligned} \Omega &= \frac{1}{2} \Omega^{\mu\nu} \gamma_\mu \wedge \gamma_\nu = \Omega_+ + \Omega_-, \quad \text{where } \Omega^{\mu\nu} = -\Omega^{\nu\mu}, \\ \Omega_+ &= \Omega^{j0} \gamma_j \wedge \gamma_0, \quad \Omega_- = \Omega^{ij} \gamma_i \wedge \gamma_j. \end{aligned} \quad (30.36)$$

We see that, as observed above, Ω_+ is a linear combination of the bivector basis blades containing γ_0 , whereas Ω_- is a linear combination of the bivector basis blades not containing γ_0 . We have

$$\Omega = \Omega_+ + \mathbf{I} \Omega_-, \quad \text{where } \Omega_+ = \Omega_+, \quad \mathbf{I} \Omega_- = \Omega_-,$$

and both $\Omega_\pm$ are relative vectors satisfying

$$\Omega_\pm = \Omega_\pm^k \sigma_k, \quad \Omega_+^k = \Omega^{k0}, \quad \Omega_-^k = -\frac{1}{2} \Omega^{ij} \epsilon_{ijk},$$

with σ_k and ϵ_{ijk} being defined by (23.48)–(23.50) and (23.53).

30.3 Electromagnetic Field Spacetime Split

Applying the results of Sect. 30.2 to the electromagnetic bivector field F (see Chap. 31), we can decompose it, respectively, into electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ according to the observer γ_0 , namely

$$\begin{aligned} F &= F_+ + F_- = \mathbf{E} + \mathbf{IB} = \mathbf{E} + \mathbf{BI}, \\ F_+ &= \mathbf{E} = E^j \sigma_j, \quad F_- = \mathbf{B} = B^j \sigma_j, \end{aligned} \quad (30.37)$$

where $\mathbf{E}$ and $\mathbf{B}$ are relative vectors defined by

$$\begin{aligned} \mathbf{E} &= (F \cdot \gamma_0) \gamma_0 = (F \cdot \gamma_0) \wedge \gamma_0, \\ \mathbf{IB} &= \mathbf{BI} = (F \wedge \gamma_0) \gamma_0 = (F \wedge \gamma_0) \cdot \gamma_0, \end{aligned} \quad (30.38)$$

and properties of the basis σ_j of relative vectors are listed in (23.48)–(23.55).

Notice first that the spacetime split (30.37)–(30.38) of the EM field indicates that the electric field $\mathbf{E}$ is a relative vector. It also suggests that the *magnetic field is represented by a relative bivector $\mathbf{IB}$ rather than relative vector $\mathbf{B}$ alone*, [168, VI.A], [185, 0.1-2]. We recall that in the standard vector algebra $\mathbf{E}$ is treated as a “polar vector”, and $\mathbf{B}$ is treated as an “axial vector”, which is essentially an oriented area, [314, p. 67], [263, 18-2].

Note that relations (28.27), (28.28), (23.2), (23.3), (28.44) imply the following alternative representations for $\mathbf{E}$ and $\mathbf{B}$:

$$\mathbf{E} = -\gamma_0 (F \cdot \gamma_0), \quad \mathbf{IB} = \mathbf{BI} = \gamma_0 (F \wedge \gamma_0), \quad (30.39)$$

implying

$$\mathbf{E}\gamma_0 = -\gamma_0\mathbf{E} = F \cdot \gamma_0, \quad \mathbf{IB}\gamma_0 = \gamma_0\mathbf{IB} = F \wedge \gamma_0. \quad (30.40)$$

Using the Hermitian conjugation defined by (23.28) and the representations (30.38)–(30.40), we obtain

$$\mathbf{E}^\dagger = \mathbf{E}, \quad \mathbf{B}^\dagger = \mathbf{B}, \quad \mathbf{I}^\dagger = -\mathbf{I}, \quad (30.41)$$

and consequently

$$F^\dagger = \gamma_0 \tilde{F} \gamma_0 = -\gamma_0 F \gamma_0 = \mathbf{E}^\dagger + \mathbf{I}^\dagger \mathbf{B}^\dagger = \mathbf{E} - \mathbf{IB}. \quad (30.42)$$

Hence, we also have the following representations

$$\begin{aligned}
 \mathbf{E} = F_+ &= \frac{1}{2} (F + F^\dagger) = \frac{1}{2} (F + \gamma_0 \tilde{F} \gamma_0) \\
 &= \frac{1}{2} (F - \gamma_0 F \gamma_0) = E^j \sigma_j, \\
 \mathbf{IB} = F_- &= \frac{1}{2} (F - F^\dagger) = \frac{1}{2} (F - \gamma_0 \tilde{F} \gamma_0) \\
 &= \frac{1}{2} (F + \gamma_0 F \gamma_0) = IB^j \sigma_j.
 \end{aligned} \tag{30.43}$$

Observe that the following useful identities hold:

$$F^2 = (\mathbf{E}^2 - \mathbf{B}^2) + 2I(\mathbf{E} \cdot \mathbf{B}). \tag{30.44}$$

30.4 Lorentz Transformations and Their Rotors

Let us turn now to the basics of the representation of the Lorentz transformation in the STA language, [166, Ap. A], [168, Sect. IV, V], [169, IV, VIIB], [170, 9.1], [93, 5.3, 5.4, 13.1.1], [82, 5.4], [307, App. A1, A2]. Such a representation is an STA form of the so-called bispinor representation of the Lorentz transformations, as briefly introduced in Sect. 6.2.1.

The Lorentz rotation $\Lambda = \Lambda^\nu_\mu$, defined as a proper and orthochronous Lorentz transformation, has the following spinor representation in the STA:

$$\gamma'_\mu = R \gamma_\mu \tilde{R} = \Lambda^\nu_\mu \gamma_\nu, \quad [\Lambda^{-1}]^\mu_\nu = \Lambda^\mu_\nu, \quad \Lambda^\mu_\sigma \Lambda^\sigma_\nu = \delta^\mu_\nu \tag{30.45}$$

where $R = R_\Lambda$ is an *even multivector called the Lorentz rotor* normalized by the condition (30.46),

$$R\tilde{R} = \tilde{R}R = 1, \tag{30.46}$$

and γ'_μ is the basis of the new transformed frame. Notice that the property of the Lorentz transformation to be proper and orthochronous follows from the rotor representation, and the γ_0 component of γ'_0 is positive, [93, 5.4], that is,

$$\gamma_0 \cdot \gamma'_0 = \langle \gamma_0 R \gamma_0 \tilde{R} \rangle > 0. \tag{30.47}$$

Consequently, we have the following *rotor form of the Lorentz transformation* of vectors in the frame γ'_μ :

$$x' = R x \tilde{R}. \tag{30.48}$$

Recall that the defining property of the Lorentz transformation is the preservation of the metric, and the transformation (30.48) satisfies this requirement. Indeed, using (30.46) and the basic properties of the scalar projection $\langle \cdot \rangle$, we obtain for any two vectors x and y that

$$(R\tilde{x}\tilde{R}) \cdot (R\tilde{y}\tilde{R}) = \langle (R\tilde{x}\tilde{R}) (R\tilde{y}\tilde{R}) \rangle = \langle xy \rangle = x \cdot y. \quad (30.49)$$

Using (30.46), (30.49) and the STA defining vector identity $x \wedge y = xy - x \cdot y$, one can verify that the Lorentz transformation (30.48) also preserves the geometric and wedge product, namely for any two vectors x and y

$$(R\tilde{x}\tilde{R}) (R\tilde{y}\tilde{R}) Rxy\tilde{R} = Rxy\tilde{R}, \quad (R\tilde{x}\tilde{R}) \wedge (R\tilde{y}\tilde{R}) = R(x \wedge y). \quad (30.50)$$

Consequently, the Lorentz transformation (30.48) preserves the entire STA algebraic structure and is naturally extended to any multivectors as a linear transformation, that is,

$$\begin{aligned} M' &= RM\tilde{R}, \quad (MN)' = M'N', \\ (M + N)' &= M' + N', \quad \langle M'N' \rangle = \langle MN \rangle. \end{aligned} \quad (30.51)$$

The rotors form a multiplicative group called the *rotor group*, which is a *double-valued representation of the Lorentz rotation group* (also called the *restricted Lorentz group*). Based on relations (30.45), (30.46) one can derive the following explicit representation of the Lorentz rotor $R = R_\Lambda$ in terms of the conventional Lorentz rotation Λ , [169, IV], [160, 1]:

$$R = \pm (A\tilde{A})^{-1/2} A, \quad A = A(\Lambda) = \gamma'_\mu \cdot \gamma^\mu = \Lambda^\nu_\mu \gamma_\nu \gamma^\mu. \quad (30.52)$$

In addition to that, the rotor R can always be represented in the exponential form, [93, 5.4.1], [307, A.1]

$$R = \pm e^{\frac{B}{2}}, \text{ parametrized by bivector } B = \frac{1}{2} \xi^{\mu\nu} \gamma_\mu \wedge \gamma_\nu, \text{ where} \quad (30.53)$$

$$\xi^{\mu\nu} = -\xi^{\nu\mu}.$$

For any Lorentz rotation Λ , its rotor R , that is, *any even multivector satisfying $R\tilde{R} = 1$, can be decomposed into a commuting product of a boost (pure Lorentz rotation) and a spatial rotation*, [93, 5.4.1], [82, 5.4], [307, A.1, A.2]

$$R = e^{\frac{\varphi + \theta \hat{B}}{2}} \hat{B} = e^{\frac{\varphi}{2} \hat{B}} e^{\frac{\theta}{2} \hat{B}} = e^{\frac{\theta}{2} \hat{B}} e^{\frac{\varphi}{2} \hat{B}}, \text{ where} \quad (30.54)$$

$$\hat{B} \text{ is a unit bivector blade, } \hat{B}^2 = 1.$$

The transformation $e^{\frac{\varphi}{2}\hat{B}}$ is a boost, and $e^{\frac{\theta}{2}I\hat{B}}$ is a spatial rotation. The boost associated with a bivector blade $\hat{B}$ can also be represented in the following form

$$R = e^{\frac{\varphi}{2}\hat{B}} = \cosh\left(\frac{\varphi}{2}\right) + \hat{B} \sinh\left(\frac{\varphi}{2}\right), \quad (30.55)$$

where the unit bivector blade $\hat{B}$ has the following coordinate representation

$$\begin{aligned} \hat{B} &= B^1\sigma_1 + B^2\sigma_2 + B^3\sigma_3 = (B^1\gamma_1 + B^2\gamma_2 + B^3\gamma_3) \wedge \gamma_0, \\ \hat{B}^2 &= 1, \end{aligned} \quad (30.56)$$

with the bivectors (relative vectors) $\sigma_k = \gamma_k\gamma_0$ being defined by (23.51). Here is a typical example of the Lorentz boost, [93, 5.3],

$$\begin{aligned} R &= e^{\frac{\varphi}{2}\gamma_3\gamma_0} = \cosh\left(\frac{\varphi}{2}\right) + \gamma_3\gamma_0 \sinh\left(\frac{\varphi}{2}\right), \text{ where} \\ \tan \varphi &= \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{|\mathbf{v}|}{c}. \end{aligned} \quad (30.57)$$

The spatial rotation $e^{\frac{\theta}{2}I\hat{B}}$ with $\hat{B}$ satisfying (30.56) can be recast similarly as

$$R = e^{\frac{\theta}{2}I\hat{B}} = \cos\left(\frac{\theta}{2}\right) + I\hat{B} \sin\left(\frac{\theta}{2}\right). \quad (30.58)$$

30.4.1 Lorentz Rotor Spacetime Split

Let us consider now a Lorentz boost L from one proper velocity u to another proper velocity v , [149, 2], [164, 2], [170, 9.1], [171, 3-8], [93, 5.4.4]. Suppose we are travelling with a proper velocity u and want to boost to a proper velocity v . So we seek a “pure” boost rotor L which contains no additional rotational factors such that

$$v = Lu\tilde{L}, \text{ so that } La\tilde{L} = a \text{ for every } a \text{ such } a \cdot (u \wedge v) = 0. \quad (30.59)$$

In view of (30.54), this implies that $u \wedge v$ is the bivector associated with the rotor L , and since $u \wedge v$ anticommutes with u and v , we consequently obtain that

$$u\tilde{L} = Lu, \quad v = Lu\tilde{L}, \quad v = L^2u, \quad L^2 = vu. \quad (30.60)$$

The last equality in (30.60) determines the rotor L uniquely as a pure boost given by the following formula:

$$L = \frac{1 + vu}{[2(1 + v \cdot u)]^{1/2}} = \exp\left(\frac{\alpha}{2} \frac{v \wedge u}{|v \wedge u|}\right), \quad \cosh(\alpha) = u \cdot v. \quad (30.61)$$

In particular for $u = \gamma_0$, the above formula together with (30.19) yields

$$L = \frac{1 + v\gamma_0}{[2(1 + v \cdot \gamma_0)]^{1/2}}, \quad v\gamma_0 = \gamma \left(1 + \frac{\mathbf{v}}{c}\right) \quad (30.62)$$

where

$$\gamma = v \cdot \gamma_0 = \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-1/2}.$$

The boost L in (30.62) can be written in the exponential form:

$$L = e^{\frac{\mathbf{a}}{2}} = \cosh\left(\frac{\alpha}{2}\right) + \frac{\mathbf{v}}{|\mathbf{v}|} \sinh\left(\frac{\alpha}{2}\right), \quad \text{where } \mathbf{a} = \alpha \frac{\mathbf{v}}{|\mathbf{v}|}, \quad (30.63)$$

$$\mathbf{v} = c \frac{v \wedge \gamma_0}{v \cdot \gamma_0}, \quad \cosh(\alpha) = \gamma = v \cdot \gamma_0 = \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-1/2},$$

and $\mathbf{a}$ is called the *rapidity bivector*, [170, 9.1].

In the case of a point particle, we can recast using (30.26) and (30.27) the pure boost L in terms of the relative momentum $\mathbf{p}$ as follows:

$$L = L(\mathbf{p}) = \frac{E_0 + E(\mathbf{p}) + c\mathbf{p}}{[2E_0(E_0 + E(\mathbf{p}))]^{1/2}}, \quad (30.64)$$

where $E(\mathbf{p}) = E_0 \sqrt{\frac{\mathbf{p}^2}{c^2} + 1}$, $E_0 = mc^2$.

Let us consider now a general Lorentz rotor R and apply it to a standard frame $\{\gamma_\mu\}$. The result is another frame of vectors $\{e_\mu\}$ given by

$$e_\mu = R\gamma_\mu\tilde{R}. \quad (30.65)$$

A *spacetime rotor split of the Lorentz* R is accomplished by splitting it into the product of two rotors, [169, IV], [170, 9.1], [9.3, 5.4.4]

$$R = LU, \quad (30.66)$$

where L and U are, respectively, a *pure boost* and a *pure rotation* satisfying

$$L^\dagger = \gamma_0 \tilde{L} \gamma_0 = L \text{ or, equivalently, } \gamma_0 \tilde{L} = L \gamma_0, \quad (30.67)$$

$$U^\dagger = \gamma_0 \tilde{U} \gamma_0 = \tilde{U} \text{ or, equivalently, } U \gamma_0 \tilde{U} = \gamma_0. \quad (30.68)$$

Using relations (30.66)–(30.68), we obtain that

$$e_0 = R\gamma_0\tilde{R} = L\gamma_0\tilde{L} = L^2\gamma_0, \text{ hence } L^2 = e_0\gamma_0. \quad (30.69)$$

Now, using the last equality in (30.69), we can uniquely recover L from formula (30.62) with $v = e_0$, that is,

$$L = \frac{1 + e_0 \gamma_0}{[2(1 + e_0 \cdot \gamma_0)]^{1/2}}. \quad (30.70)$$

Then we define the rotor U by

$$U = \tilde{L}R, \quad U\tilde{U} = \tilde{L}R\tilde{R}L = 1, \quad (30.71)$$

and verify that it satisfies relations (30.68). It is straightforward to see that U is a pure rotation in the γ_0 frame of the form

$$U = e^{\frac{1}{2}\mathbf{b}}, \quad \text{where } \mathbf{b} \text{ is a rotation angle (relative vector)}. \quad (30.72)$$

Hence, we have a decomposition (30.66) of a Lorentz rotor R into a boost and a rotation. Observe that unlike the invariant decomposition into a boost and a rotation in (30.54), the boost L and the rotation U will not usually commute. Notice also that in view of (30.67)–(30.68) the decomposition (30.66) depends on γ_0 and consequently is frame-dependent.

30.4.2 Lorentz Boosts and Spacetime Splits

One can naturally obtain Lorentz boost transformations by expressing the spacetime split representation of one inertial observer in terms of the same for another one. Indeed, let one inertial observer be associated with the proper velocity $v = \gamma'_0$ which can be thought of as the rest frame of a moving particle, and another one with the proper velocity γ_0 . We recall then that both $v = \gamma'_0$ and γ_0 are time-like vectors, and the corresponding relative velocity $\mathbf{v}$ is defined by

$$\frac{\mathbf{v}}{c} = \frac{v \wedge \gamma_0}{v \cdot \gamma_0}, \quad v = \gamma \left(1 + \frac{\mathbf{v}}{c}\right) \gamma_0 = \gamma \gamma_0 \left(1 - \frac{\mathbf{v}}{c}\right), \quad (30.73)$$

$$\text{where } \gamma = v \cdot \gamma_0 = \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-1/2} \text{ is the Lorentz factor.}$$

When relating two observers, the subspace $\text{Span}\{v, \gamma_0\}$ of the vector space is of importance. We introduce the orthogonal decomposition as in [265, p. 10], [160, 1]

$$x = x_{\parallel} + x_{\perp}, \quad \text{where} \quad (30.74)$$

$$x_{\parallel} \in \text{Span}\{v, \gamma_0\} \text{ and } x_{\perp} \text{ is orthogonal to } \text{Span}\{v, \gamma_0\}.$$

Notice that $\text{Span}\{\nu, \gamma_0\}$ is a two-dimensional Minkowski space for which the corresponding Clifford algebra is equivalent to the set of hyperbolic numbers $\alpha_0 + \alpha_1 h$ where h is the pseudoscalar defined by

$$|\mathbf{v}| h = \mathbf{v}, \quad h^2 = 1. \quad (30.75)$$

The hyperbolic numbers are similar to complex numbers, but the corresponding hyperbolic geometry is the non-Euclidean Minkowski geometry. Hyperbolic numbers and their geometry is a well studied subject, [57, 347].

Now let us introduce for an arbitrary vector y in $\text{Span}\{\nu, \gamma_0\}$ the following hyperbolic representations:

$$\begin{aligned} y\nu &= y \cdot \nu + y \wedge \nu = y'_0 + y'_1 h, \text{ where} \\ y'_0 &= y \cdot \nu, \quad y'_1 = y \wedge \nu, \end{aligned} \quad (30.76)$$

$$\begin{aligned} y\gamma_0 &= y \cdot \gamma_0 + y \wedge \gamma_0 = y_0 + y_1 h, \text{ where} \\ y_0 &= y \cdot \gamma_0, \quad y_1 = y \wedge \gamma_0. \end{aligned} \quad (30.77)$$

Using (30.73), (30.75) and (30.77), we obtain

$$\begin{aligned} y\nu &= (y_0 + y_1 h) \gamma_0 \nu = \gamma (y_0 + y_1 h) \left(1 - \frac{|\mathbf{v}|}{c} h\right) \\ &= \gamma \left[\left(y_0 - y_1 \frac{|\mathbf{v}|}{c}\right) + \left(y_1 - y_0 \frac{|\mathbf{v}|}{c}\right) h \right]. \end{aligned} \quad (30.78)$$

Comparing (30.76) with (30.78), we find that

$$y'_0 = y_0 - y_1 \frac{|\mathbf{v}|}{c}, \quad y'_1 = y_1 - y_0 \frac{|\mathbf{v}|}{c}. \quad (30.79)$$

It is evident that the above relation between coordinates for two inertial observers ν and γ_0 is precisely the Lorentz boost transformation.

Now, using the decomposition (30.74) for x together with relations (30.76)–(30.79), and denoting $y = x_{\parallel}$, we obtain the following identities

$$x_{\parallel} \cdot \nu = x_{\parallel} \cdot \nu = x'_{\parallel 0} = x_{\parallel 0} - x_{\parallel 1} \frac{|\mathbf{v}|}{c}, \quad (30.80)$$

$$\begin{aligned} (x \cdot \nu)^2 - x^2 &= (x_{\parallel} \cdot \nu)^2 - x_{\parallel}^2 + x_{\perp}^2 = x_{\parallel 1}^2 + x_{\perp}^2 \\ &= \left(x_{\parallel 1} - x_{\parallel 0} \frac{|\mathbf{v}|}{c}\right)^2 + x_{\perp}^2. \end{aligned} \quad (30.81)$$

The latter can also be recast as

$$\sqrt{(x \cdot v)^2 - x^2} = \left| \gamma \left(x_{\parallel 1} - x_{\parallel 0} \frac{|\mathbf{v}|}{c} \right) h + \mathbf{x}^\perp \right|. \quad (30.82)$$

30.4.3 Field Transformations

We associate with every Lorentz rotor R the following so-called *active transformations* for vectors, multivectors and 1/2-spinors:

$$\begin{aligned} x &\rightarrow x' = Rx\tilde{R} && \text{for vectors,} \\ M &\rightarrow M' = RM\tilde{R} && \text{for multivectors,} \\ \psi(x) &\rightarrow \psi'(x') = R\psi(x) && \text{for 1/2-spinors.} \end{aligned} \quad (30.83)$$

Active and passive transformations are discussed in more detail in the following subsection.

The *infinitesimal form of the general rotor* R defined by (30.53) is

$$R = e^{\frac{B}{2}} = e^{\frac{1}{4}\xi^{\mu\nu}\gamma_\mu \wedge \gamma_\nu} = 1 + \frac{B}{2} + \cdots = 1 + \frac{1}{4}\xi^{\mu\nu}\gamma_\mu \wedge \gamma_\nu + \cdots. \quad (30.84)$$

Based on the infinitesimal form (30.84) for the rotor R and the commutator product defined in (28.51), we obtain the following infinitesimal forms for, respectively, vectors x , multivectors M and 1/2-spinors ψ :

$$\delta x = \frac{1}{2} (Bx - xB) = \frac{1}{2} B \diamond x = B \cdot x = \xi^{\mu\nu} x_\nu \gamma_\mu, \quad (30.85)$$

$$\delta M = \frac{1}{2} (BM - MB) = \frac{1}{2} B \diamond M, \quad (30.86)$$

$$\bar{\delta}\psi = \frac{B}{2}\psi = \frac{1}{4}\xi^{\mu\nu} (\gamma_\mu \wedge \gamma_\nu) \psi = \frac{1}{4} \frac{\partial \delta x^\mu}{\partial x_\nu} (\gamma_\mu \wedge \gamma_\nu) \psi. \quad (30.87)$$

We recall that $\bar{\delta}\psi$ refers to the so-called total variation associated with the difference $\psi'(x') - \psi(x)$.

30.5 Active and Passive Transformations

There are two ways of representing transformations between two different basis systems or frames known as *active and passive transformations*, [169, IV], [304, 3.3], [331, 1.2]. An active transformation assumes that the physical state, say the

position vector or spin state, is transformed within the same coordinate system or reference frame. In other words, the original and the transformed states are observed from the same reference frame. A passive transformation assumes that the physical state remains the same but the reference frame (basis system) is transformed. The passive transformations are often called coordinate transformations as they always imply a change in the space-time coordinates.

Since we are interested in relativistic theories, we focus primarily on the Lorentz transformations, see Sect. 6.2.1, that is,

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}. \quad (30.88)$$

In the passive case, it is accompanied by a (usually implicit) transformation of the coordinate frame:

$$\gamma'_{\mu} = \Lambda^{\nu}_{\mu} \gamma_{\nu} = R \gamma_{\mu} \tilde{R} \text{ or equivalently } \gamma'^{\mu} = \Lambda^{\mu}_{\nu} \gamma^{\nu} = R \gamma^{\mu} \tilde{R}. \quad (30.89)$$

Using (6.16), (6.17), we obtain the transformation laws for vector quantities

$$\begin{aligned} x' = x'^{\mu} \gamma'_{\mu} &= \Lambda^{\mu}_{\nu} x^{\nu} \Lambda^{\alpha}_{\mu} \gamma_{\alpha} = \delta^{\alpha}_{\nu} x^{\nu} \gamma_{\alpha} = x^{\nu} \gamma_{\nu} = x, & A' &= A, \\ \partial &= \gamma^{\mu} \partial_{\mu} = \Lambda^{\nu}_{\mu} \gamma^{\mu} \partial'_{\nu} = \gamma'^{\nu} \partial'_{\nu} = \partial', \end{aligned} \quad (30.90)$$

where A is the electromagnetic potential, and, since it is a vector, its transformation law is identical to the vector x , and the same argument applies to the vector derivative ∂ . As we can see from (30.90), passive transformations simply assert that two different coordinates x'^{μ} and x^{μ} represent the same point x and likewise for any vector quantities. But in the STA vectors and multivectors are manifestly coordinate free, and that makes the passive transformations essentially redundant, [171, 3.10]. In other words, as suggested by D. Hestenes, [169, IV] “STA enables us to dispense with coordinates entirely” and consider only active transformations.

Since we want the Dirac-Hestenes equation (24.5) to be relativistic invariant, the right choice for the transformation law for the spinor ψ turns out to be

$$\psi'(x') = \psi(x) \tilde{R}. \quad (30.91)$$

Summarizing, *passive transformations reduce to*

$$\begin{aligned} x' &= x, & \partial' &= \partial, & \gamma'^{\mu} &= R \gamma^{\mu} \tilde{R}, \\ \psi'(x') &= \psi(x) \tilde{R}, & A'(x') &= A(x). \end{aligned} \quad (30.92)$$

To verify the relativistic invariance of the Dirac-Hestenes equation (24.5), we take it first in the prime variables, that is,

$$\hbar \partial \psi \mathbf{I} \sigma'_3 - \frac{e}{c} A' \psi' = mc \psi' \gamma'_0. \quad (30.93)$$

Making the passive transformation substitutions (30.92) into (30.93) and using $\tilde{R}R = 1$, we obtain

$$\hbar \partial \psi \mathbf{I} \sigma_3 \tilde{R} - \frac{e}{c} A' \psi \tilde{R} = mc \psi \gamma_0 \tilde{R}, \quad (30.94)$$

which readily turns into the Dirac-Hestenes equation (24.5) after multiplication by R from the right. As one can see from (30.93), the passive transformation effectively is just a change of constants in the Dirac-Hestenes equation (24.5). In the Dirac matrix formulation it corresponds to a mere change of a representation of the matrices γ^μ .

In the *active transformation case*, each spacetime point $x = x^\mu \gamma_\mu$ is mapped to a new spacetime point

$$x' = x'^\mu \gamma_\mu = \Lambda^\mu{}_\nu x^\nu \gamma_\mu = x'^\nu R \gamma_\nu \tilde{R} = R x \tilde{R}, \quad (30.95)$$

where we used equalities (30.45), (30.46) defining the Lorentz rotor R . Notice also that in the active transformation case the basis γ_μ remains unaltered, that is,

$$\gamma_\mu = \gamma'_\mu. \quad (30.96)$$

To keep the Dirac-Hestenes equation (24.5) relativistic invariant, the right choice for the transformation law for the spinor ψ in an active transformation is

$$\begin{aligned} x' &= R x \tilde{R}, \quad \partial' = R \partial \tilde{R}, \quad \gamma'^\mu = \gamma^\mu, \\ \psi' (x') &= R \psi (x), \quad A' (x') = R A \tilde{R} (x). \end{aligned} \quad (30.97)$$

In the active transformation case the Dirac-Hestenes equation (24.5) in the prime variables takes the form

$$\hbar \partial' \psi' \mathbf{I} \sigma_3 - \frac{e}{c} A' \psi' = mc \psi' \gamma_0. \quad (30.98)$$

Making the active transformation substitutions (30.97) into (30.98) and using $\tilde{R}R = 1$ we obtain

$$\hbar R \partial \psi \mathbf{I} \sigma_3 - \frac{e}{c} R A \psi = mc R \psi \gamma_0, \quad (30.99)$$

which readily turns into the Dirac-Hestenes equation (24.5) after multiplication by $\tilde{R}$ from the left.

30.6 The Motion Equation of a Point Charged Particle

Recall that the relativistic covariant motion equation of a point particle of mass m and charge e is, [27, II.1]

$$mc^2 \partial_\tau v^\mu = e F^{\mu\nu} v_\nu, \quad (30.100)$$

where τ is the proper time defined by (30.15). The corresponding STA version of the above equation is, [160, 2]

$$mc^2 \partial_\tau v = eF \cdot v, \text{ or } c \partial_\tau p = eF \cdot v \quad (30.101)$$

where p is the proper momentum defined by (30.24), F is the bivector of the electromagnetic field and v is the proper velocity defined by (30.16). Using relations (30.37)–(30.42) and (30.13) we obtain

$$\begin{aligned} (F \cdot v) \gamma_0 &= \frac{1}{2} (Fv - vF) \gamma_0 = \frac{1}{2} [F(v\gamma_0) + (v\gamma_0)F^\dagger] \\ &= \frac{1}{2} \gamma \left[(\mathbf{E} + \mathbf{IB}) \left(1 + \frac{\mathbf{v}}{c}\right) + \left(1 + \frac{\mathbf{v}}{c}\right) (\mathbf{E} - \mathbf{IB}) \right] \\ &= \frac{1}{2} \gamma \left[2\mathbf{E} + \mathbf{E} \frac{\mathbf{v}}{c} + \frac{\mathbf{v}}{c} \mathbf{E} + (\mathbf{IB}) \frac{\mathbf{v}}{c} - \frac{\mathbf{v}}{c} (\mathbf{IB}) \right] \\ &= \gamma \left[\mathbf{E} + \mathbf{E} \cdot \frac{\mathbf{v}}{c} + \mathbf{I} \left(\mathbf{B} \wedge \frac{\mathbf{v}}{c} \right) \right] = \gamma \left[\mathbf{E} \cdot \frac{\mathbf{v}}{c} + \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \right]. \end{aligned} \quad (30.102)$$

To find the relative vector form of the motion equation (30.101), we multiply it by γ_0 on the right; using (30.24) we get

$$c (\partial_\tau p) \gamma_0 = c (\partial_\tau p \gamma_0) = \gamma \partial_t \left(\frac{E}{c} + \mathbf{p} \right) = e (F \cdot v) \gamma_0. \quad (30.103)$$

Then relations (30.102), (30.103) and (30.13) imply

$$\partial_t E = e \mathbf{E} \cdot \mathbf{v}, \quad \partial_t \mathbf{p} = e \left[\mathbf{E} - \frac{\mathbf{v}}{c} \cdot (\mathbf{IB}) \right] = e \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right), \quad (30.104)$$

where we recognize in the right-hand side of the second equation the usual vector form for the Lorentz force.

30.7 Spinor Point Particle Mechanics

In this section we provide a concise STA representation of relativistic spinor particle mechanics including the Larmor and the Thomas precessions following [160, 4], [161, 1-2], [164, 3], [169, V], [170, 3.9, 5.6, 7.3, 9.3-5].

Being given a fixed frame $\{\gamma_\mu\}$, one can use the rotor equation

$$e_\mu = R \gamma_\mu \tilde{R}, \text{ where } R = R(\tau) \text{ is the Lorentz rotor,} \quad (30.105)$$

to describe the relativistic kinematics of a rigid body of negligible dimensions (a point particle) traversing a world line $x = x(\tau)$ where τ is the proper time defined

by (30.15). The frame $\{e_\mu\} = \{e_\mu(\tau)\}$, called the *comoving frame*, traverses the world line along with the particle. The spacelike vectors $e_k(\tau)$, $k = 1, 2, 3$, can be identified with the principal axes of the body. First, *we identify the vector e_0 with the particle proper velocity v , namely*

$$\partial_\tau x = v = e_0 = R\gamma_0\tilde{R}, \text{ where } R = R(\tau). \quad (30.106)$$

For a particle with an intrinsic angular momentum or spin, *we identify vector e_3 with the spin direction s*

$$s = e_3 = R\gamma_3\tilde{R}. \quad (30.107)$$

To get an equation of motion for R , we recall first that it is always an even multivector normalized by the condition $R\tilde{R} = \tilde{R}R = 1$ implying

$$\partial_\tau (R\tilde{R}) = (\partial_\tau R)\tilde{R} + R(\partial_\tau \tilde{R}) = 0. \quad (30.108)$$

We introduce then the *generalized angular (rotational) velocity Ω of the comoving frame* by

$$\Omega = 2(\partial_\tau R)\tilde{R}, \quad (30.109)$$

and notice that Ω must be an even multivector since R is even. In addition to that, Ω must be a bivector since it also satisfies

$$\tilde{\Omega} = R(\partial_\tau \tilde{R}) = -\Omega, \quad (30.110)$$

as follows from (30.108).

If we now multiply both sides of equality (30.109) by R , we obtain the desired *rotor motion equation*

$$\partial_\tau R = \frac{1}{2}\Omega R, \text{ where } \Omega = \Omega(\tau) \text{ is a bivector.} \quad (30.111)$$

Using the above equation together with definition (30.105) for the comoving frame $\{e_\mu\}$, we obtain its equations of motion

$$\partial_\tau e_\mu = \frac{1}{2}(\Omega e_\mu - e_\mu \Omega) = \Omega \cdot e_\mu, \quad (30.112)$$

confirming that the bivector Ω can be interpreted as the *generalized angular velocity* of the comoving frame. In particular, for $\mu = 0$ we get the following motion equation

$$\partial_\tau v = \Omega \cdot v, \quad v = e_0. \quad (30.113)$$

Comparison of the kinematic motion equation (30.113) with the point charge motion equation (30.101) suggests the following relation between Ω and F :

$$\Omega = \frac{e}{mc^2} F. \quad (30.114)$$

Combining the above equation together with the rotor equation (30.111), we obtain the following *spinor form of the point charge motion equation*

$$\partial_\tau R = \frac{1}{2} \frac{e}{mc^2} FR. \quad (30.115)$$

We would like to stress that the Eqs. (30.114) and (30.115) evidently carry more information than the relativistic point charge motion equation (30.101). Conceivably, they can be derived from a model for a structured charge as an approximation. In fact equation (30.115) is an approximation to the Dirac equation, [159, (6.16), (6.17)].

Chapter 31

Electromagnetic Theory in the STA

The electromagnetic theory in the STA is based on the potential A , the electromagnetic field $F = \partial A$ and the current J which are, respectively, vector-, bivector- and vector- valued STA quantities, [158, 4], [169, VI], [82, 7, 8], [93, 7], [332, 2.8]. Their coordinate representations are

$$\begin{aligned} A &= A_\mu \gamma^\mu, \quad F = \partial \wedge A = \frac{1}{2} F_{\mu\nu} \gamma^\mu \wedge \gamma^\nu, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \quad J = J_\mu \gamma^\mu, \end{aligned} \quad (31.1)$$

with coordinate and bases conventions specified in Sect. 28.2. The scalar components of the bivector F are given by

$$F_{\mu\nu} = \gamma^\mu \cdot F \cdot \gamma^\nu = \gamma^\nu \cdot (\gamma^\mu \cdot F) = (\gamma^\mu \wedge \gamma^\nu) F. \quad (31.2)$$

The Maxwell equations in vacuum take in the STA the form [166, 1], [82, 7.4], [93, 13.2]

$$\partial F = \frac{4\pi}{c} J, \quad (31.3)$$

or, equivalently

$$\partial \cdot F = \frac{4\pi}{c} J, \quad \partial \wedge F = 0. \quad (31.4)$$

The STA version of the electromagnetic Larmor Lagrangian is, [93, 13.2.1], [283, 7.4.1], [172, Ap. B],

$$\begin{aligned}
L &= -\frac{1}{4\pi} F \cdot \tilde{F} - \frac{1}{c} A \cdot J = \frac{1}{4\pi} F \cdot F - \frac{1}{c} A \cdot J = \\
&= \left\langle \frac{1}{4\pi} F^2 - \frac{1}{c} AJ \right\rangle, \quad F = \partial \wedge A,
\end{aligned} \tag{31.5}$$

and based on (31.1) it can readily be transformed into familiar form

$$L = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} A_\mu J^\mu. \tag{31.6}$$

The Maxwell equation in vacuum is a particular case of the general Maxwell equation (31.3) when $J = 0$, that is,

$$\partial F = 0. \tag{31.7}$$

To match the above STA treatment of the electromagnetic field with the conventional theory, one can use the relative vector representation (30.37) associated with a fixed inertial system as in [158, 4], [185, 3, 7] (see also [82, 8], [30, 6, 7]), namely

$$F = \mathbf{E} + \mathbf{I}\mathbf{B} = \mathbf{E} + \mathbf{B}\mathbf{I}, \quad \mathbf{E} = E^j \sigma_j, \quad \mathbf{B} = B^j \sigma_j. \tag{31.8}$$

Observe that the relative representation (31.8) indicates that the *complete EM field* F has two *geometrically distinct components*: (i) a relative vector component related to the electric field $\mathbf{E}$; (ii) a relative bivector component $\mathbf{B}\mathbf{I}$ related to the magnetic field. Hence, the magnetic field is correctly described by the relative bivector $\mathbf{B}\mathbf{I}$ rather than its dual relative vector $\mathbf{B}$, an observation made by D. Hestenes in [158, 4].

31.1 Electromagnetic Fields in Dielectric Media

In a material dielectric media, just as in the case of the conventional macroscopic EM theory, the EM phenomena in the STA are described in terms of two bivector fields: (i) the bivector EM field $F = \mathbf{E} + \mathbf{I}\mathbf{B}$ as in (31.8) representing forces; (ii) the material bivector field G representing at macroscopic scales material charges and currents defined by

$$G = \mathbf{D} + \mathbf{I}\mathbf{H} = \mathbf{D} + \mathbf{H}\mathbf{I}, \quad \mathbf{D} = D^j \sigma_j, \quad \mathbf{H} = H^j \sigma_j. \tag{31.9}$$

The physical content of the material bivector field $G = \mathbf{D} + \mathbf{I}\mathbf{H}$ can be seen in its relation to the *generalized magnetization (polarization)* bivector field Q , that is,

$$Q = -\mathbf{P} + \mathbf{I}\mathbf{M}, \quad \mathbf{P} = P^j \sigma_j, \quad \mathbf{M} = M^j \sigma_j, \tag{31.10}$$

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \quad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}, \tag{31.11}$$

where the fields $\mathbf{P}$ and $\mathbf{M}$ are, respectively, the *dipole and magnetic-dipole moments per unit volume* (also called the *electric and magnetic polarizations*). To avoid confusion we recall that in the STA $\mathbf{P}$ and $\mathbf{M}$ are relative vectors, that is, timelike bivectors.

We would like to stress that the complete relative bivectors $\mathbf{IB}$, $\mathbf{IH}$ and $\mathbf{IM}$ provide for more sound representation of magnetic quantities rather than relative vectors $\mathbf{B}$, $\mathbf{H}$ and $\mathbf{M}$ alone, [168, VI.A], [185, 0.1-2]. This is justified in particular by the fact that in standard vector algebra $\mathbf{B}$, $\mathbf{H}$ and $\mathbf{M}$ are treated as “axial vectors” which are in essence oriented areas, [314, p. 67], [263, 18-2].

Recall also that fields $\mathbf{P}$ and $\mathbf{M}$ in a dielectric medium manifest themselves physically through induced polarization charges ρ_{pol} and currents J_{pol} according to the following relations, [50, 2.2], [282, 3]

$$\rho_{\text{pol}} = -\nabla \cdot \mathbf{P}, \quad \mathbf{J}_{\text{pol}} = \partial_t \mathbf{P} + c \nabla \times \mathbf{M}. \quad (31.12)$$

Observe that the polarization charges ρ_{pol} and currents J_{pol} defined above evidently satisfy the conservation law

$$\partial_t \rho_{\text{pol}} + \nabla \cdot \mathbf{J}_{\text{pol}} = 0. \quad (31.13)$$

A concise STA equivalent of (31.12) is

$$J_{\text{pol}} = c \partial \cdot Q, \quad (31.14)$$

as can be readily verified with the help of GA representation (30.13) for the cross product. The bivector fields F , G and M are related by

$$G = F - Q, \quad (31.15)$$

with an additional assumption that the material fields G and Q are determined by EM field F through *material (constitutive) relations*, that is,

$$Q = Q(F) \quad G = G(F) = F - Q(F). \quad (31.16)$$

For the STA treatment of constitutive relations, see [175].

Then the STA version of the Maxwell equations in a material dielectric medium is, [166, 1], [169, VI.B], [93, 7.1.3],

$$\partial \wedge F = 0, \quad \partial \cdot G = \frac{4\pi}{c} J, \quad G = G(F) = F - Q(F), \quad (31.17)$$

where $J = J_c$ is a *convection (impressed) current* which is a vector-valued quantity in the STA. The polarization currents J_{pol} enter the Maxwell equations implicitly through generalized magnetization Q and relations (31.10)–(31.14). Consequently, the *true, total physical current* J_t is the sum of the *convection (impressed) current*

J_c and the polarization (magnetization) current $J_{\text{pol}} = c\partial \cdot Q$, that is,

$$J_t = J_c + J_{\text{pol}} = J + c\partial \cdot Q. \quad (31.18)$$

Notice that, as a consequence of the general formula (28.29), we have $\partial \cdot (\partial \cdot K) = (\partial \wedge \partial) K = 0$ for any field K . Since $J = \frac{c}{4\pi} \partial \cdot G$, and $J_{\text{pol}} = c\partial \cdot Q$, *each of these currents is individually conserved* implying the conservation of the total current $J_t = J + J_{\text{pol}}$, namely

$$\partial \cdot J = 0, \quad \partial \cdot J_{\text{pol}} = 0, \quad \partial \cdot J_t = 0. \quad (31.19)$$

Such an independent conservation of the convection and polarization currents can be attributed to differences in their microscopic origins. Notice that conservation laws (31.19) are the STA version of the conventional polarization current conservation law (31.13).

In a chosen inertial system with a given γ_0 , the Maxwell equations (31.17) take the following spacetime split form, [185, 1.3]

$$\begin{aligned} \nabla \wedge (\mathbf{IB}) &= 0, & \nabla \wedge \mathbf{E} + \frac{1}{c} \partial_t (\mathbf{IB}) &= 0, \\ \nabla \cdot (\mathbf{IH}) + \frac{1}{c} \partial_t \mathbf{D} &= -\frac{4\pi}{c} \mathbf{J}, & \nabla \cdot \mathbf{D} &= 4\pi \rho, \end{aligned} \quad (31.20)$$

with the current conservation law

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0. \quad (31.21)$$

Observe that the magnetic quantities enter the spacetime split form of the Maxwell equations as bivectors $\mathbf{IB}$ and $\mathbf{IH}$, whereas electric quantities $\mathbf{E}$ and $\mathbf{D}$ enter them as vectors.

Notice also that the STA and the spacetime split forms (31.17), (31.20) of the Maxwell equations is equivalent, of course, to their conventional form, [50, 2.2.1], which we provide here for comparison

$$\begin{aligned} \nabla \times \mathbf{H} &= \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \partial_t \mathbf{D}, & \nabla \times \mathbf{E} &= -\frac{1}{c} \partial_t \mathbf{B}, \\ \nabla \cdot \mathbf{B} &= 0, & \nabla \cdot \mathbf{D} &= 4\pi \rho. \end{aligned}$$

Here are a few useful identities in the STA setting, [161, 1]

$$M = -\mathbf{P} + \mathbf{IM}, \quad -\mathbf{P} = (M \cdot \gamma_0) \gamma_0, \quad \mathbf{IM} = (M \wedge \gamma_0) \gamma_0, \quad (31.22)$$

$$M \cdot F = -\mathbf{P} \cdot \mathbf{E} - \mathbf{M} \cdot \mathbf{B}, \quad (31.23)$$

$$\mathbf{M} \cdot \mathbf{B} = -[(M \wedge \gamma_0) \cdot \gamma_0] \cdot F - (M \wedge \gamma_0) \cdot (\gamma_0 \wedge F),$$

$$\mathbf{f} = e\mathbf{E} + (\mathbf{P} \cdot \partial)\mathbf{E} + \partial(\mathbf{M} \cdot \mathbf{B}),$$

$$\mathbf{f} = f \wedge v = mc^2 (\partial_t v) v = f v,$$

$$\mathbf{P} \cdot \partial = -[(M \cdot v) \wedge v] \cdot (v \wedge \partial) = -M \cdot (v \wedge \partial).$$

31.2 Time-Harmonic Solutions to the Maxwell Equation in Vacuum

In the STA treatment and interpretation of time-harmonic solutions to the Maxwell equations in vacuum, we follow [158, 4] (see also [82, 8], [93, 7]). Similarly to the conventional theory, we introduce a *positive frequency time-harmonic solution* F_+ to the Maxwell equation (31.7) in vacuum of the form

$$F_+ = f e^{I k \cdot x} = f e^{I k_\mu x^\mu}, \quad k_0 = \frac{\omega}{c} > 0, \quad (31.24)$$

where $k = k^\mu \gamma_\mu$ is the STA form of the wavevector, $k\gamma_0 = k_0 + \mathbf{k} = \frac{\omega}{c} - \mathbf{k}$ is its relative form, and f is a constant bivector. Hence, we can recast F_+ as follows:

$$F_+ = f e^{I k \cdot x} = f \exp[I(\omega t - \mathbf{k} \cdot \mathbf{x})], \quad \omega > 0. \quad (31.25)$$

Plugging expression (31.24) into the Maxwell equation (31.7), we obtain

$$kF = 0, \text{ or } \left(\frac{\omega}{c} - \mathbf{k}\right)F = 0. \quad (31.26)$$

Multiplying the above equation on the left by k , we get $k(kF) = (k^2)F = 0$, and since F is not zero, we obtain the well known property of the wavevector to be lightlike or null, that is,

$$k^2 = \left(\frac{\omega}{c}\right)^2 - \mathbf{k}^2 = 0. \quad (31.27)$$

Let us rewrite the second equation in (31.26) as

$$(1 - \hat{\mathbf{k}})F = 0, \text{ where } \hat{\mathbf{k}} = \frac{c\mathbf{k}}{\omega}, \quad (31.28)$$

and then, using the relative representation (31.8) for F , transform the above equation into

$$\hat{\mathbf{k}}F = F \text{ or } \hat{\mathbf{k}}\mathbf{E} = \mathbf{I}\mathbf{B}. \quad (31.29)$$

Observe now that $\mathbf{I}\mathbf{B}$ is evidently a pure relative bivector, implying the following relations between $\hat{\mathbf{k}}$, $\mathbf{E}$ and $\mathbf{B}$, including their orthogonality:

$$\hat{\mathbf{k}} \cdot \mathbf{E} = 0, \quad \hat{\mathbf{k}}\mathbf{E} = \hat{\mathbf{k}} \wedge \mathbf{E} = -\mathbf{E}\hat{\mathbf{k}} = \mathbf{I}\mathbf{B}, \quad (31.30)$$

$$\mathbf{E} = \mathbf{I}\hat{\mathbf{k}}\mathbf{B} = -\mathbf{I}\mathbf{B}\hat{\mathbf{k}}, \quad \hat{\mathbf{k}}\mathbf{B} = -\mathbf{B}\hat{\mathbf{k}} = \hat{\mathbf{k}} \wedge \mathbf{B}, \quad \hat{\mathbf{k}} \cdot \mathbf{B} = \mathbf{E} \cdot \mathbf{B} = 0. \quad (31.31)$$

Using the above equalities, we obtain

$$\hat{\mathbf{E}}\hat{\mathbf{B}}\hat{\mathbf{k}} = \mathbf{I}, \quad \text{where } \hat{\mathbf{E}} = \frac{\mathbf{E}}{\sqrt{\mathbf{E} \cdot \mathbf{E}}}, \quad \hat{\mathbf{B}} = \frac{\mathbf{B}}{\sqrt{\mathbf{B} \cdot \mathbf{B}}}. \quad (31.32)$$

The relation (31.32) shows that the triple $\{\hat{\mathbf{E}}, \hat{\mathbf{B}}, \hat{\mathbf{k}}\}$, in that order, forms a right-handed orthonormal frame of relative vectors. *It is interesting that the constructed positive frequency solution F_+ is right circularly polarized.*

Using relations (31.29)–(31.31), we can express the complete EM field F in terms of the electric field $\mathbf{E}$, namely

$$F = (1 + \hat{\mathbf{k}}) \mathbf{E} = \mathbf{E} (1 - \hat{\mathbf{k}}). \quad (31.33)$$

Let us look now at the time dependence of the EM field F defined by (31.24). At any point of the plane $\mathbf{k} \cdot \mathbf{x} = 0$, the field is given by the formula

$$F = \mathbf{E}(t) + \mathbf{I}\mathbf{B}(t) = f \exp(\mathbf{I}\omega t), \quad (31.34)$$

where

$$f = F|_{x=0} = \mathbf{E}_0 + \mathbf{I}\mathbf{B}_0 = \mathbf{E}_0 (1 - \hat{\mathbf{k}}), \quad \text{where} \quad (31.35)$$

$$\hat{\mathbf{k}} \cdot \mathbf{B}_0 = \hat{\mathbf{k}} \cdot \mathbf{E}_0 = \mathbf{B}_0 \cdot \mathbf{E}_0 = 0.$$

Notice that $(1 - \hat{\mathbf{k}}) \hat{\mathbf{k}} = \hat{\mathbf{k}} - 1$ implying the following identity

$$(1 - \hat{\mathbf{k}}) e^{\mathbf{I}\alpha} = (1 - \hat{\mathbf{k}}) e^{-\mathbf{I}\hat{\mathbf{k}}\alpha}, \quad \text{where } \alpha \text{ is any scalar.} \quad (31.36)$$

It follows then from (31.33)–(31.36) that

$$F = \mathbf{E}(t) + \mathbf{I}\mathbf{B}(t) = \mathbf{E}_0 (1 - \hat{\mathbf{k}}) e^{-\mathbf{I}\hat{\mathbf{k}}\omega t}, \quad \text{for } \mathbf{k} \cdot \mathbf{x} = 0. \quad (31.37)$$

Taking the relative vector and bivector parts of the above representation, we obtain

$$\mathbf{E}(t) = \mathbf{E}_0 e^{-\mathbf{I}\hat{\mathbf{k}}\omega t} = \mathbf{E}_0 (\cos \omega t - \mathbf{I}\hat{\mathbf{k}} \sin \omega t), \quad (31.38)$$

$$\mathbf{B}(t) = \mathbf{E}_0 \mathbf{I}\hat{\mathbf{k}} (\cos \omega t - \mathbf{I}\hat{\mathbf{k}} \sin \omega t) = \mathbf{E}_0 \exp \left[-\mathbf{I}\hat{\mathbf{k}} \left(\omega t - \frac{\pi}{2} \right) \right]. \quad (31.39)$$

The relation (31.38) shows explicitly that, as t increases, the electric vector $\mathbf{E}(t)$ rotates clockwise in the plane as viewed by an observer facing the oncoming wave train. The relation (31.39) shows that magnetic vector $\mathbf{B}(t)$ follows 90° behind. Thus the usual picture of a circularly polarized wave arises with $\hat{\mathbf{I}}\hat{\mathbf{k}}$ generating the rotation.

To obtain the *left circularly polarized solution* F_- we simply change the orientation of the generator $\mathbf{I}$ for $-\mathbf{I}$ in expression (31.24), that is,

$$F_- = f e^{-\mathbf{I}\mathbf{k}\cdot\mathbf{x}} = f e^{-\mathbf{I}k_\mu x^\mu} = f \exp[-\mathbf{I}(\omega t - \mathbf{k} \cdot \mathbf{x})], \quad k_0 = \frac{\omega}{c} > 0. \quad (31.40)$$

Notice that the solution F_- can also be viewed effectively as a *negative frequency solution*. In fact, it is convenient to allow the frequency to take both positive and negative values. *The sign of the frequency is then associated with the polarization of the wave, F_+ being the positive frequency solution, and F_- being the negative frequency solution.*

Solutions of general polarization are linear combinations of F_+ and F_- and are of the form, [93, 7.4.1]

$$F = \hat{\mathbf{n}} \left(1 - \hat{\mathbf{k}}\right) (a_+ e^{\mathbf{I}\mathbf{k}\cdot\mathbf{x}} + a_- e^{-\mathbf{I}\mathbf{k}\cdot\mathbf{x}}), \quad \text{where} \quad (31.41)$$

$$a_\pm = \eta_\pm + \xi_\pm \mathbf{I}, \quad \text{with real } \eta_\pm, \xi_\pm,$$

$\hat{\mathbf{n}}$ is a unit relative vector orthogonal to $\hat{\mathbf{n}}$.

The circularly-polarised solutions can also be written in a more covariant notation as, [93, 7.4.1]

$$F = kn (a_+ e^{\mathbf{I}\mathbf{k}\cdot\mathbf{x}} + a_- e^{-\mathbf{I}\mathbf{k}\cdot\mathbf{x}}), \quad k \cdot n = 0, \quad (31.42)$$

$$k \cdot n = 0, \quad \text{and } a_\pm = \eta_\pm + \xi_\pm \mathbf{I}, \quad \text{with real } \eta_\pm, \xi_\pm.$$

Chapter 32

The Wave Function and Local Observables in the STA

We have already provided in Chap. 24 a concise overview of the STA version of the Dirac theory, [162, 4–6], [169, VII.D], [93, 8.2], [82, 9.3]. In this section we consider Hestenes's construction and interpretation of the wave function as well as observables in more detail with an intension to use them in our neoclassical theory.

The Dirac wave function $\psi(x)$ is represented in the STA by the *real Dirac wave function* $\psi = \psi(x)$ which always takes values in the even subalgebra $\text{Cl}_+(1, 3)$. The significance of the even subalgebra $\text{Cl}_+(1, 3)$ rests on the fact that it is isomorphic to the GA $\text{Cl}(3, 0)$ of three-dimensional Euclidean space, [166, 1]. So the wave function in the STA is always assumed to take values in the even subalgebra $\text{Cl}_+(1, 3)$.

Notice that $\psi\tilde{\psi} = \tilde{\psi}\psi$ for any ψ from $\text{Cl}_+(1, 3)$ implying that this product is a linear combination of the scalar and the pseudoscalar I , hence

$$\psi\tilde{\psi} = \tilde{\psi}\psi = \varrho e^{I\beta} = \varrho (\cos \beta + I \sin \beta), \quad \text{where } \varrho \geq 0 \text{ and } \beta \text{ are scalars.} \quad (32.1)$$

This leads to the following *canonical Lorentz invariant decomposition* which holds for every even multivector ψ , [162], [169, VII.D], [93, 8.2], [82, 9.3]

$$\psi = \varrho^{\frac{1}{2}} e^{\frac{I\beta}{2}} R = R \varrho^{\frac{1}{2}} e^{\frac{I\beta}{2}}, \quad R\tilde{R} = \tilde{R}R = 1, \quad (32.2)$$

where $\varrho > 0$ and β are scalars, and R is the Lorentz rotor described in Chap. 30. Hence, according to Hestenes, the canonical decomposition (32.2) can be regarded as an invariant decomposition of the Dirac wave function into a 2-parameter *statistical factor* $\varrho^{\frac{1}{2}} e^{\frac{I\beta}{2}}$ and a 6-parameter *kinematical factor* R . The rotor R can always be decomposed into a commuting product (30.54) of a boost (pure Lorentz rotation) and a spatial rotation. D. Hestenes refers to the factor $e^{I\beta}$ in (32.1) as the “*duality rotation*”, [162, 4], [169, VII.D]. A variation of the canonical representation (32.2) and its analysis are provided in [307, A.1, A.2].

Importantly, the parameter β must be small in most of the cases since it is a prerequisite for the Pauli equation to be obtained as the non-relativistic limit of the Dirac equation, [149, 2]. More insights on the role of the parameter β can be found

in [51, 3.1, 7.2]. In particular, the parameter β rotates bivectors, but not vectors, and its change to $\beta + \pi$ results in the reversal of a bivector and can be associated with the transformation of an electron into a positron with the opposite orientation of the bivector spin. The parameter β also enters the mass term of the Lagrangian of the positron in the form $-mc^2 \cos \beta$, where $0 \leq \beta \leq \pi$, and it is possible that the transformation of β into $\beta + \pi$ changes the mass m of the electron into $-m$, that is, a negative mass for the positron, and that might be related to the disappearance of the mass in the process of electron–positron annihilation.

When deriving the canonical decomposition (32.2), one uses in particular the following commutativity properties of any even multivector ψ :

$$\psi \mathbf{I} = \mathbf{I} \psi, \quad \psi \tilde{\psi} = \tilde{\psi} \psi. \quad (32.3)$$

Being given a wave function $\psi = \psi(x)$, one can use the canonical decomposition (32.2) to assign at each spacetime point x the rotor $R = R(x)$ of $\psi(x)$. The rotor $R(x)$ determines a Lorentz rotation of a given fixed frame of vectors $\{\gamma_\mu\}$ into a frame of vectors $\{e_\mu = e_\mu(x)\}$ given by

$$e_\mu = e_\mu(x) = R \gamma_\mu \tilde{R}, \quad R = R(x). \quad (32.4)$$

D. Hestenes stresses in [169, VII.D] that “the physical interpretation given to the frame field $\{e_\mu\}$ is a key to the interpretation of the entire Dirac theory. Specifically, the e_μ can be interpreted directly as descriptors of the kinematics of electron motion.” Notice that e_μ are vectors since every rotor is a versor and the versor multiplication operation $M \mapsto RM\tilde{R}$ is grade preserving, see Sect. 28.1.

Importantly, since the pseudoscalar commutes with a vector, it follows from (32.2) and (32.4) that we have equation

$$\psi \gamma_\mu \tilde{\psi} = \varrho R \gamma_\mu \tilde{R} = \varrho e_\mu, \quad (32.5)$$

that is, the expression on the left in fact depends only on the rotor R and the scalar ϱ , and it does not depend on the parameter β as in (32.2).

The interpretation of the above fields in the Dirac theory is as follows. The vector field

$$\psi \gamma_0 \tilde{\psi} = \varrho e_0 = \varrho v \quad (32.6)$$

is the *Dirac current (probability current)* in the standard Born interpretation). The timelike vector $v = v(x) = e_0(x)$ at each spacetime point x is interpreted as the probable (proper) velocity of the electron, and $\varrho = \varrho(x)$ is the relative probability (i.e. proper probability density) that the electron actually is at x . *Notice that the very possibility of a probabilistic interpretation rests on the fact that $\gamma_0 \cdot v = \langle \gamma_0 \tilde{\psi} \gamma_0 \psi \rangle$ is always positive, that is,*

$$\langle \gamma_0 \tilde{\psi} \gamma_0 \psi \rangle = \langle \psi^\dagger \psi \rangle \geq 0, \text{ for any } \psi \in \text{Cl}_+(1, 3), \quad (32.7)$$

where $\psi^\dagger = \gamma_0 \tilde{\psi} \gamma_0$ is the relative reversion (23.28) of ψ . Indeed, any even multivector ψ can be represented as

$$\psi = \alpha_1 + \mathbf{b}_1 + \mathbf{b}_2 \mathbf{I} + \alpha_2 \mathbf{I}, \quad (32.8)$$

where α_1, α_2 are scalars and $\mathbf{b}_1, \mathbf{b}_2$ are relative vectors, see Sect. 30.1. Then it follows that

$$\psi^\dagger = \gamma_0 \tilde{\psi} \gamma_0 = \alpha_1 + \mathbf{b}_1 - \mathbf{b}_2 \mathbf{I} - \alpha_2 \mathbf{I}, \quad (32.9)$$

and, consequently,

$$\langle \psi^\dagger \psi \rangle = \alpha_1^2 + \mathbf{b}_1^2 + \mathbf{b}_2^2 + \alpha_2^2 \geq 0. \quad (32.10)$$

Another important property of the wave function ψ is that for every vector a the multivector $\psi a \tilde{\psi}$ is in fact a vector. Indeed, as a consequence of the general commuting property (28.69), we have

$$a \mathbf{I} = -\mathbf{I} a, \quad a e^{-\frac{1\beta}{2}} = e^{\frac{1\beta}{2}} a, \quad \text{for any vector } a, \text{ and scalar } \beta. \quad (32.11)$$

Then using the canonical form (32.2) of the wave function ψ and the grade preserving property (28.15) of the rotor product we obtain

$$\psi a \tilde{\psi} = \varrho R a \tilde{R} \text{ is a vector for any vector } a. \quad (32.12)$$

Local spin vector density is defined by

$$s = \frac{1}{2} \hbar \psi \gamma_3 \tilde{\psi} = \frac{1}{2} \hbar \varrho e_3. \quad (32.13)$$

The *spin angular momentum* $S = S(x)$ (proper spin) is a bivector field related to the spin vector field $s = s(x)$ by, [162, 4], [166, 2]

$$S = \frac{1}{2} \hbar e_2 e_1 = \frac{1}{2} \hbar \psi \mathbf{I} \sigma_3 = \frac{1}{2} \hbar R \gamma_2 \gamma_1 \tilde{R} = \frac{1}{2} \hbar R \mathbf{I} \sigma_3 \tilde{R} = \mathbf{I} s v = \mathbf{I} s \wedge v. \quad (32.14)$$

The tensor components of S are

$$S^{\mu\nu} = S \cdot \gamma^\mu \wedge \gamma^\nu = \langle S \gamma^\mu \wedge \gamma^\nu \rangle = \langle S \gamma^\mu \gamma^\nu \rangle = \mathbf{I} s \wedge v \wedge \gamma^\mu \wedge \gamma^\nu.$$

The proper spin density is ϱS .

The magnetization or magnetic moment density of the electron, [162, 4], [166, 2], [169, VII.D] is defined by the following expression

$$M = \frac{\hbar e}{2mc} \psi \gamma_2 \gamma_1 \tilde{\psi} = \frac{e}{mc} e^{1\beta} \varrho S,$$

showing that the ratio of magnetic moment M to spin density ρS differs from the usual ferromagnetic ratio $\frac{e}{mc}$ attributed to the electron by the factor $e^{I\beta}$. Based on this fact it was suggested by D. Hestenes to interpret β geometrically as the angle of a “*duality rotation*” of S to M .

Differentiating $R\tilde{R} = 1$, one obtains

$$\partial_\mu (R\tilde{R}) = (\partial_\mu R) \tilde{R} + R (\partial_\mu \tilde{R}) = 0 \quad (32.15)$$

or

$$(\partial_\mu R) \tilde{R} = - \left[(\partial_\mu R) \tilde{R} \right]^\sim = \Omega_\mu. \quad (32.16)$$

If we introduce $\Omega_\mu = 2 (\partial_\mu R) \tilde{R}$, then, since R is an even multivector, Ω_μ is an even vector as well. Since according to (32.16) Ω_μ changes sign under reversion, it must be a bivector and consequently we have

$$\partial_\mu R = \frac{\Omega_\mu}{2} R, \text{ where } \Omega_\mu \text{ is a bivector.} \quad (32.17)$$

Chapter 33

Multivector Field Theory

Multivector field theory allows us to preserve rich multivector algebraic features within the Lagrangian framework.

33.1 Transformations Laws

When considering coordinate transformations based on the STA, it is useful to keep in mind the following points made by D. Hestenes in [169, III] and [172, Ap. A]:

“STA makes it possible to formulate and analyze conventional relativistic physics in invariant form without reference to a coordinate system ... The “proper formulation” given here takes another step to move from covariance to invariance by relating particle motion directly to Minkowski’s “absolute spacetime” without reference to any coordinate system.” “...the Poincaré invariance should be interpreted as an equivalence of spacetime points rather than an equivalence of observers. It describes a physical property of the Minkowski model of spacetime.”

We provide in this section a concise account of transformation laws based on the STA primarily following [172, App. A].

A general transformation law of the position vector is assumed to be of the form

$$x \rightarrow x' = h(x). \quad (33.1)$$

The corresponding transformation of a vector field $a = a(x)$ into a vector field $a' = a'(x')$ is given by the *differential* $\underline{h}$ of h , namely

$$\underline{h} : a \rightarrow a' = \underline{h}a = a \cdot \partial h|_{x=h^{-1}(x')}. \quad (33.2)$$

The linear mapping $\underline{h}$ has a natural extension to multivectors, called an *automorphism*, by linearity and

$$\underline{h}(a_1 \wedge a_2 \wedge \dots \wedge a_k) = (\underline{h}a_1) \wedge (\underline{h}a_2) \wedge \dots \wedge (\underline{h}a_k). \quad (33.3)$$

Consequently, we have for any multivector M

$$\underline{h}M = \sum_{0 \leq k \leq 4} \underline{h} \langle M \rangle_k, \quad \text{where } \underline{h}(\langle M \rangle_0) = \langle M \rangle_0. \quad (33.4)$$

Application of $\underline{h}$ to the pseudounit I yields the determinant of the $\underline{h}$, that is,

$$\underline{h}I = \det(\underline{h}) I. \quad (33.5)$$

Let us consider now a special form of the transformation (33.1), suitable for infinitesimal treatment:

$$x' = h(x) = x + \xi(x), \quad (33.6)$$

where for any unit vector $\hat{a}$ the quantity $\hat{a} \cdot \xi$ is assumed to be small. Using relations (33.2)–(33.4), we get the following representation of the action of $\underline{h}$:

$$\underline{h}a = a + a \cdot \partial \xi, \quad \underline{h}a = M + (M \cdot \partial) \wedge \xi. \quad (33.7)$$

Suppressing higher order terms, we get

$$M'(x') = M(x + \xi) = M(x) + [M(x) \cdot \partial] \wedge \xi + \dots, \quad (33.8)$$

implying

$$M'(x) = M(x) - (\xi \cdot \partial) M(x) + [M(x) \cdot \partial] \wedge \xi + \dots \quad (33.9)$$

In particular, in the case of the Lorentz field transformations and for the Poincaré group of transformations since $R\tilde{R} = 1$, we have

$$h(x) = Rx\tilde{R} + c, \quad \underline{h}a = Ra\tilde{R}, \quad \underline{h}M = RM\tilde{R}. \quad (33.10)$$

For a spinorial field, the transformation law is

$$\psi' = R\psi. \quad (33.11)$$

33.2 Lagrangian Treatment and Conservation Laws for Multivector Fields

We are motivated to incorporate multivectors into the Lagrangian treatment, for this allows us to obtain the field equations and conservation laws in the simplest and geometrically sound form available due to the multivector algebraic structure.

Such a Lagrangian treatment, including the conservation laws based on a multivector Noether's theorem, has been developed in [219, 4-6], [93, 12.4, 13], and we adopt most of that approach here. For more details of mathematical aspects of the Lagrangian field theory for multivector-valued fields we refer the reader to [283, 7].

The Lagrangian densities involving multivector fields ψ are often defined as manifestly real scalar functions of the form $\langle \hat{L} \rangle$ where $\hat{L}$ is a multivector expression involving fields ψ together with their reverses $\tilde{\psi}$. Interestingly, the variational treatment of such Lagrangian densities does not require us to assume $\tilde{\psi}$ to be independent, [93, 13.3], whereas the conventional treatment of complex-valued fields ψ does require their complex conjugates ψ^* to be treated as an independent variable. This factor cuts in half the number of field equations and simplifies considerably expressions of conserved quantities for the multivector case compared to the conventional treatment for complex-valued fields. Apart from the point made above, the variational aspects of the Lagrangian treatment involving multivector fields are essentially identical to those for classical fields described in Chap. 10. So we address here only the new primarily algebraic aspects that are special to the multivector fields.

First, we assume that every field ψ^ℓ takes values in a subspace $\mathcal{X}_\ell$ of the spacetime algebra $\text{Cl}(1, 3)$ which is a direct sum of any set of grade subspaces $\bigwedge^k$. For instance, a particularly important subspace is the even algebra $\text{Cl}_+(1, 3)$, as in the case of a spin- $\frac{1}{2}$ spinorial variable ψ , or it can be a space of odd multivectors which is the case for the vector derivative $\partial\psi$ of ψ . In the case of the electromagnetic potential A , the subspace $\mathcal{X}_A$ is the vector subspace $\bigwedge$. The subspace $\mathcal{X}_\ell$ can also be any grade space $\bigwedge^k$.

As to the symmetry transformations, we assume settings similar to Chap. 10 with the only difference being that the fields take values in subspaces $\mathcal{X}^\ell$, namely

$$x^{\mu'} = x^\mu + \delta x^\mu, \quad \psi'_\ell(x) = \psi_\ell(x) + \delta\psi_\ell(x) \in \mathcal{X}_\ell, \quad (33.12)$$

with the *total field variation* $\bar{\delta}$ defined by

$$\bar{\delta}\psi_\ell(x) = \psi'_\ell(x') - \psi_\ell(x) = \delta\psi_\ell(x) + \psi_{\ell,\mu}\delta x^\mu, \quad (33.13)$$

where $\psi_{\ell,\mu} = \partial_\mu\psi_\ell \in \mathcal{X}_\ell$. More precisely, the symmetry transformations form a Lie group of transformations parametrized by parameters $\{\epsilon^k\}$ of the form

$$x^{\mu'} = X(x^\mu, \{\epsilon^k\}), \quad \psi'_\ell(x') = \Psi_\ell(x^\mu, \psi_\ell(x), \{\epsilon^k\}) \in \mathcal{X}_\ell. \quad (33.14)$$

The infinitesimal form of (33.14) for small ϵ^k is

$$\delta x^\mu = \Gamma_k^\mu(x^\mu)\epsilon^k, \quad \bar{\delta}\psi_\ell = \Phi_{\ell k}(x^\mu, \psi_\ell)\epsilon^k \in \mathcal{X}_\ell, \quad (33.15)$$

where Γ_k^μ and G_k^μ are the *group generators*.

The *Euler–Lagrange field equations* here are of the same familiar form

$$\Lambda_\ell = \frac{\partial \mathcal{L}}{\partial \psi_\ell} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \psi_{\ell,\mu}} \right) = 0 \in \mathcal{X}_\ell, \quad (33.16)$$

with the only difference being that the Eulerian Λ_ℓ and the corresponding derivatives now take values in the subspaces $\mathcal{X}_\ell$.

The total Lagrangian density variation and corresponding action integral variation are

$$\bar{\delta} \mathcal{L} = \partial_\mu \left(\sum_\ell \pi_\ell^\mu * \bar{\delta} \psi_\ell - \dot{T}^{\mu\nu} \delta x^\nu \right), \quad \text{for } \Lambda_\ell = 0. \quad (33.17)$$

The *conjugate momentum* π_ℓ^μ and the *canonical energy-momentum tensor* $\dot{T}_\nu^\mu$ are defined by (33.18):

$$\begin{aligned} \pi_\ell^\mu &= \frac{\partial \mathcal{L}}{\partial \psi_{\ell,\mu}} \in \mathcal{X}_\ell, & \dot{T}^{\mu\nu} &= \sum_\ell \pi_\ell^\mu * \psi_{\ell,\nu}^\cdot - \delta_\nu^\mu \mathcal{L}, \\ \psi_{\ell,\mu} &= \frac{\partial \psi_\ell}{\partial x^\mu} \in \mathcal{X}_\ell. \end{aligned} \quad (33.18)$$

The corresponding action integral variation is

$$\bar{\delta} W = \int_D \bar{\delta} \mathcal{L} \, dx = \dot{G}(\Sigma_2) - \dot{G}(\Sigma_1), \quad \text{for } \Lambda_\ell = 0, \quad (33.19)$$

where the *canonical symmetry transformation generator* $\dot{G}(\Sigma)$ is defined now by the surface integral

$$\dot{G}(\Sigma) = \int_\Sigma \left[\sum_\ell \pi_\ell^\mu * \bar{\delta} \psi_\ell - \dot{T}^{\mu\nu} \delta x_\nu \right] d\sigma_\mu. \quad (33.20)$$

The system invariance with respect to the symmetry transformation (33.12)–(33.15) takes the form

$$\bar{\delta} \mathcal{L} = \partial_\mu \left(\sum_\ell \pi_\ell^\mu * \bar{\delta} \psi_\ell - \dot{T}^{\mu\nu} \delta x^\nu \right) = 0, \quad \text{for } \Lambda_\ell = 0, \quad (33.21)$$

$$\bar{\delta} W = \dot{G}(\Sigma_2) - \dot{G}(\Sigma_1) = 0, \quad \text{for } \Lambda_\ell = 0. \quad (33.22)$$

The relation (33.22) implies that $\dot{G}(\Sigma)$ is a constant for any space-like surface Σ extending to infinity.

The invariance condition (33.21) can be recast as the continuity equation

$$\partial_\mu J^\mu = 0, \quad J^\mu = \sum_\ell \pi_\ell^\mu * \bar{\delta} \psi_\ell - \dot{T}^{\mu\nu} \delta x_\nu, \quad \text{for } \Lambda_\ell = 0, \quad (33.23)$$

where J^μ is the *multivector representation for Noether's current*. A more precise form of the conserved Noether's current based on (33.15) is

$$J_k^\mu = \sum_\ell \pi_\ell^\mu * \Phi_k^\ell - \dot{T}^{\mu\nu} \Gamma_{k\nu}, \quad \partial_\mu J_k^\mu = 0, \quad (33.24)$$

where J_k^μ is Noether's current corresponding to the parameter ϵ^k .

In the important case where the Lagrangian density is invariant under space-time translations, which is a particular case of the Poincaré transformation (33.12) with $\xi^{\mu\nu} = 0$ and $\delta x^\mu = a^\mu$, we have $\bar{\delta}\psi_\ell = 0$ in (33.13), and the canonical action generator $\dot{G}(\Sigma)$ takes the form

$$\dot{G}(\Sigma) = -P^\nu(\Sigma) a_\nu, \quad P^\nu(\Sigma) = \int_\Sigma \dot{T}^{\mu\nu} d\sigma_\mu. \quad (33.25)$$

The quantity $P^\mu(\Sigma)$ is the energy-momentum 4-vector which depends on the surface Σ and in view of (33.22) is a conserved quantity, that is,

$$P^\nu(\Sigma) \text{ is a constant for any space-like } \Sigma \text{ extending to } \infty. \quad (33.26)$$

33.3 The Symmetric Energy-Momentum Tensor

When constructing the symmetric EnMT for multivector-valued fields, we follow Barut's approach described in Sect. 10.7.2. We assume the system to be invariant with respect to the Lorentz–Poincaré transformations, hence the energy-momentum is conserved. We would like to modify the canonical EnMT $\dot{T}^{\mu\nu}$ and the corresponding canonical symmetry generator $\dot{G}(\Sigma)$ defined by (33.18), (33.20) to another EnMT $T^{\mu\nu}$ with its symmetry generator $G(\Sigma)$ so that the key relation (33.22) is preserved, namely

$$\bar{\delta}W = \bar{\delta} \int_D \mathcal{L} dx = G(\Sigma_2) - G(\Sigma_1) = 0. \quad (33.27)$$

To get such a modification, we pick a field C^μ satisfying

$$\partial_\mu C^\mu = 0, \quad (33.28)$$

and introduce the modified generator

$$\begin{aligned} G(\Sigma) &= \int_\Sigma \left[\sum_\ell \pi_\ell^\mu \cdot \bar{\delta}\psi_\ell - \dot{T}^{\mu\nu} \delta x_\nu + C^\mu \right] d\sigma_\mu \\ &= \dot{G}(\Sigma) + \int_\Sigma C^\mu d\sigma_\mu. \end{aligned} \quad (33.29)$$

Then the following relation holds

$$G(\Sigma_2) - G(\Sigma_1) = \mathring{G}(\Sigma_2) - \mathring{G}(\Sigma_1) + \int_D \partial_\mu C^\mu dx = \mathring{G}(\Sigma_2) - \mathring{G}(\Sigma_1), \quad (33.30)$$

implying that the modified generator $G(\Sigma)$ defined by (33.29) satisfies the desired relation (33.27). Our particular choice of a suitable field C^μ is based on the requirement that the generator $G(\Sigma)$ must satisfy the following special representation

$$G(\Sigma) = - \int_\Sigma T^{\mu\nu} \delta x_\nu d\sigma_\mu = - \int_\Sigma T^{\mu\nu} (\xi_{\nu\rho} x^\rho + a_\nu) d\sigma_\mu. \quad (33.31)$$

If that is accomplished, we obtain

$$G(\Sigma) = -P^\nu(\Sigma) a_\nu - \frac{1}{2} J^{\nu\rho}(\Sigma) \xi_{\nu\rho}, \quad (33.32)$$

where the 4-vector *energy-momentum* $P^\mu(\Sigma)$ and the *angular momentum* $J^{\nu\rho}(\Sigma)$ are defined by

$$P^\nu(\Sigma) = \int_\Sigma T^{\mu\nu} d\sigma_\mu, \quad (33.33)$$

$$J^{\nu\rho}(\Sigma) = \int_\Sigma J^{\mu\nu\rho} d\sigma_\mu, \quad J^{\mu\nu\rho} = T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu.$$

Consequently, in view of (33.27), the following conservation law holds

$$P^\nu(\Sigma) \text{ and } J^{\nu\rho}(\Sigma) \text{ are constant for any space-like } \Sigma \text{ extending } \infty. \quad (33.34)$$

The *differential form of the above conservation laws* is

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu J^{\mu\nu\rho} = \partial_\mu (T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu) = T^{\rho\nu} - T^{\nu\rho} = 0. \quad (33.35)$$

Notice the *particle-like form* $J^{\mu\nu\rho} = T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu$ of the *density of the angular momentum* $J^{\mu\nu\rho}$ in (10.91).

Similarly to classical field theory, the corresponding coordinate and field variations δx^μ and $\delta\psi_\ell$ are defined by

$$x^{\mu'} = x^\mu + \delta x^\mu, \quad \psi'_\ell(x) = \psi_\ell(x) + \delta\psi_\ell(x), \quad (33.36)$$

and the total field variation $\bar{\delta}\psi_\ell^\ell$ is defined by

$$\bar{\delta}\psi_\ell(x) = \psi'_\ell(x') - \psi_\ell(x) = \delta\psi_\ell(x) + \partial_\mu \psi_\ell \delta x^\mu. \quad (33.37)$$

We have assumed the system to be invariant with respect to the infinitesimal inhomogeneous Lorentz (Poincaré) transformations (6.22), that is,

$$x'^{\mu} = x^{\mu} + \delta x^{\mu}, \quad \delta x^{\mu} = \xi^{\mu\nu} x_{\nu} + a^{\mu}, \quad \xi^{\mu\nu} = -\xi^{\nu\mu}, \quad (33.38)$$

and the field transformations associated with them

$$\bar{\delta}\psi_{\ell} = \frac{1}{2}\xi^{\mu\nu}\underline{S}_{\ell\mu\nu}\psi_{\ell}, \quad \underline{S}_{\ell\mu\nu} = -\underline{S}_{\ell\nu\mu}, \quad (33.39)$$

where $\underline{S}_{\ell}^{\mu\nu}$ in a general setting are linear infinitesimal operators acting on the subspace $\mathcal{X}_{\psi_{\ell}}$ of values of ψ_{ℓ} . In the case of most interest to us, of vectors, multivectors and 1/2-spinors, the corresponding field transformations in the STA are defined by (30.53), (30.85)–(30.87), and they are parametrized by the rotor R and the corresponding bivector

$$B = \frac{1}{2}\xi^{\mu\nu}\gamma_{\mu} \wedge \gamma_{\nu}. \quad (33.40)$$

Consequently, the infinitesimal Lorentz transformation in the STA corresponding to (33.38), (33.39) for the position vector is

$$\delta x = \frac{1}{2}(Bx - xB) = \frac{1}{2}B \diamond x = B \cdot x = \xi^{\mu\nu}x_{\nu}\gamma_{\mu}. \quad (33.41)$$

The infinitesimal Lorentz transformation for the fields in the STA are:

$$\begin{aligned} \bar{\delta}\psi &= \frac{1}{2}(B\psi - \psi B) = \frac{1}{2}B \diamond \psi = B \cdot \psi = \xi^{\mu\nu}\psi_{\nu}\gamma_{\mu}, \\ \psi &= (\gamma_{\mu} \cdot \psi) \gamma_{\nu} \text{ if } \psi^{\ell} = \psi \text{ is a vector field;} \end{aligned} \quad (33.42)$$

$$\begin{aligned} \bar{\delta}\psi &= \frac{1}{2}(B\psi - \psi B) = \frac{1}{2}B \diamond \psi = \frac{1}{2}\xi^{\mu\nu}(\gamma_{\mu} \wedge \gamma_{\nu}) \diamond \psi, \\ \underline{S}_{\ell\mu\nu}\psi &= (\gamma_{\mu} \wedge \gamma_{\nu}) \diamond \psi \text{ if } \psi_{\ell} = \psi \text{ is a multivector field;} \end{aligned} \quad (33.43)$$

$$\begin{aligned} \bar{\delta}\psi &= \frac{B}{2}\psi = \frac{1}{4}\xi^{\mu\nu}(\gamma_{\mu} \wedge \gamma_{\nu})\psi = \frac{1}{4}\frac{\partial\delta x^{\mu}}{\partial x_{\nu}}\psi(\gamma_{\mu} \wedge \gamma_{\nu}), \\ \underline{S}_{\ell\mu\nu}\psi &= \frac{1}{2}(\gamma_{\mu} \wedge \gamma_{\nu})\psi \text{ if } \psi_{\ell} = \psi \text{ is a 1/2-spinor field.} \end{aligned} \quad (33.44)$$

The rest of the argument is identical to that of Sect. 10.7.2 with the only difference being that we use the STA form (33.42)–(33.44) for the infinitesimal Lorentz transformation for the fields. Consequently, we arrive at the following *STA representation of the symmetric EnMT and its generator*:

$$T^{\mu\nu} = \dot{T}^{\mu\nu} - \sum_{\ell} \frac{\partial}{\partial x^{\rho}} \left(\sum_{\ell} f_{\ell}^{\mu\rho\nu} \right), \quad G(\Sigma) = - \int_{\Sigma} T^{\mu\nu} \delta x_{\nu} d\sigma_{\mu}, \quad (33.45)$$

where

$$f_{\ell}^{\mu\rho\nu} = \frac{1}{2} \left\langle \pi^{\mu} (\underline{S}_{\ell}^{\rho\nu} \psi_{\ell}) + \pi^{\rho} (\underline{S}_{\ell}^{\nu\mu} \psi_{\ell}) - \pi^{\nu} (\underline{S}_{\ell}^{\mu\rho} \psi_{\ell}) \right\rangle. \quad (33.46)$$

A straightforward examination based on the antisymmetry of $\underline{S}_{\ell}^{\mu\nu}$ shows that $f^{\ell\mu\rho\nu}$ is antisymmetric in μ and ρ , that is,

$$f_{\ell}^{\mu\rho\nu} = -f_{\ell}^{\rho\mu\nu}. \quad (33.47)$$

Notice that in formulas (33.45), (33.46) involving STA operations we have incorporated fields ψ_{ℓ} in the expression of $f_{\ell}^{\mu\rho\nu}$.

In particular, under an assumption that fields decay at infinity, the system total conserved energy $\mathcal{E}(t)$ is

$$\mathcal{E}(t) = \int_{\mathbb{R}^3} T^{00} d\mathbf{x} = \int_{\mathbb{R}^3} \dot{T}^{00} d\mathbf{x}. \quad (33.48)$$

In the case when the Lagrangian density $\mathcal{L}$ depends explicitly on x and the system is not translation invariant the conservation law (33.35) can be generalized as follows:

$$\partial_{\mu} T^{\mu\nu} = \partial_{\mu} \dot{T}^{\mu\nu} = -\partial^{\nu} \mathcal{L}. \quad (33.49)$$

The proof of (33.49) is as in the case of the similar relation (10.84).

Part V

Mathematical Aspects of the Theory of Distributed Elementary Charges

We present in this part of the book rigorous mathematical formulations and proofs of the properties of balanced charges which have been introduced in previous parts of the book. We treat here several important and mathematically nontrivial problems which arise in our study of balanced charges. The problems belong to two general areas.

The first area, to which we pay most of our attention, concerns the macroscopic dynamics of localized charges. Reflecting on a similar subject Albert Einstein remarks in his letter to Ernst Cassirer of March 16, 1937, [312, pp. 393–394]:

“One must always bear in mind that up to now we know absolutely nothing about the laws of motion of material points from the standpoint of “classical field theory.” For the mastery of this problem, however, no special physical hypothesis is needed, but “only” the solution of certain mathematical problems.”

So the first area of our mathematical analysis concerns this kind of problems. In particular, we study the problem of derivation of the macroscopic Newtonian point mechanics described by ordinary differential equations from the field dynamics described by partial differential equations (PDE), i.e., by the NLS equation in the non-relativistic case and the NKG equation in the relativistic case. A key ingredient of the problem of relating ODEs and PDEs is the concept of the “trajectory of concentration,” which assigns an exact mathematical meaning to the concept of localization which was used in our derivation of Newtonian mechanics for centers of localized distributed charges.

We present two variants of the theory of trajectories of concentration for the non-relativistic and the relativistic cases. In particular, in the relativistic case we give a mathematically rigorous derivation of Einstein’s formula $E = mc^2$ for the elementary charge motion in accelerating regimes, which is fundamentally important since the mass of a particle in Newtonian mechanics is defined via its acceleration.

The second area of our mathematical studies focuses on the energy functionals and their critical points. Based on the properties of the energy functional, we derive the stability properties of the NLS equation with the logarithmic nonlinearity. A relation between the critical energy levels of the energy and the frequencies of multi-harmonic solutions of the field equations leads to the derivation of the Planck–Einstein formula $\Delta E = \hbar \Delta \omega$. A rigorous study of the Hydrogen atom model and a derivation of the Hydrogen atom spectrum are based on a mathematical analysis of the nonlinear eigenvalue problem for a balanced charge in the Coulomb field.

Chapter 34

Trajectories of Concentration

In this chapter we introduce the concept of *concentration at a trajectory*. This concept allows us to mathematically rigorously derive Newton's law of motion for point charges; in particular, we derive Einstein's formula in the accelerating relativistic regimes.

The first step in introducing the concept of concentration at a trajectory is to describe the class of trajectories.

Definition 34.0.1 (*trajectory*) A trajectory $\hat{\mathbf{r}}(t)$, $T_- \leq t \leq T_+$, is a twice continuously differentiable function of time t with values in $\mathbb{R}^3$ satisfying

$$|\hat{\mathbf{v}}|, |\partial_t \hat{\mathbf{v}}| \leq C \text{ where } \hat{\mathbf{v}}(t) = \partial_t \hat{\mathbf{r}}(t), \quad T_- \leq t \leq T_+. \quad (34.1)$$

Note that we consider problems involving a set of parameters, and the constant C in the above inequality is assumed to be uniform for the set of parameters.

Being given a trajectory $\hat{\mathbf{r}}(t)$, we consider the following family of neighborhoods contracting to it. With that in mind, we introduce a ball $\Omega(\hat{\mathbf{r}}(t), R)$ of radius R centered at $\hat{\mathbf{r}}(t)$, that is,

$$\Omega(\hat{\mathbf{r}}(t), R) = \left\{ \mathbf{x} : |\mathbf{x} - \hat{\mathbf{r}}(t)|^2 \leq R^2 \right\} \subset \mathbb{R}^3, \quad R > 0. \quad (34.2)$$

Definition 34.0.2 (*concentrating neighborhoods*) Concentrating neighborhoods $\hat{\Omega}(\hat{\mathbf{r}}(t), R_n) \subset [T_-, T_+] \times \mathbb{R}^3$ of a trajectory $\hat{\mathbf{r}}(t)$ are defined as a family of tubular domains

$$\hat{\Omega} = \hat{\Omega}(\hat{\mathbf{r}}(t), R_n) = \left\{ (t, \mathbf{x}) : T_- \leq t \leq T_+, \quad |\mathbf{x} - \hat{\mathbf{r}}(t)|^2 \leq R_n^2 \right\}, \quad (34.3)$$

where $T_- < T_+$ are some constants, and R_n satisfy the contraction condition:

$$R_n \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (34.4)$$

The section of the tubular domain $\hat{\Omega}$ at a fixed time t is given by relation (34.2) and is denoted by Ω_n , namely

$$\Omega_n = \Omega_n(t) = \Omega(\hat{\mathbf{r}}(t), R_n) \subset \mathbb{R}^3. \quad (34.5)$$

When we consider the concentration at a trajectory, we treat two different cases: relativistic and non-relativistic. In the relativistic case, we study in detail the concentration of solutions of the NKG equation. In particular, we derive Einstein's formula $E = mc^2$ and construct examples of concentrating solutions for accelerating regimes. These examples show in particular that Einstein's formula in accelerating regimes is a more robust phenomenon than the Lorentz contraction. In the non-relativistic case, we study the concentration of solutions of the NLS equation.

In the following sections we impose certain regularity conditions on the charge distribution and external EM fields in the tubular domain $\hat{\Omega}$ of the trajectory. To formulate the conditions, we make use of the following notation:

$$\partial_0 = c^{-1} \partial_t, \quad (34.6)$$

$$\begin{aligned} \nabla_{\mathbf{x}} \varphi &= \nabla \varphi = (\partial_1 \varphi, \partial_2 \varphi, \partial_3 \varphi), \\ |\nabla_{\mathbf{x}} \varphi|^2 &= |\nabla \varphi|^2 = |\partial_1 \varphi|^2 + |\partial_2 \varphi|^2 + |\partial_3 \varphi|^2, \\ \nabla_{0,\mathbf{x}} \varphi &= (\partial_0 \varphi, \partial_1 \varphi, \partial_2 \varphi, \partial_3 \varphi), \\ |\nabla_{0,\mathbf{x}} \varphi|^2 &= |\partial_0 \varphi|^2 + |\partial_1 \varphi|^2 + |\partial_2 \varphi|^2 + |\partial_3 \varphi|^2. \end{aligned} \quad (34.7)$$

34.1 Derivation of Non-relativistic Point Dynamics for Localized Solutions of the NLS

We consider solutions to the NLS equation (11.153) which are localized around a trajectory in 3D space, and we will prove that the necessary condition for such a trajectory is the fulfillment of Newton's equations of motion. The NLS equation (11.153) is of the form

$$i \tilde{\partial}_t \psi = \frac{\chi}{2m} \left[-\tilde{\nabla}^2 \psi + G'_a(\psi^* \psi) \psi \right],$$

where the *covariant differentiation operators* $\tilde{\partial}_t$ and $\tilde{\nabla}$ are defined by formulas (11.154). The covariant operators involve potentials $\varphi_{\text{ex}}(t, \mathbf{x})$, $\mathbf{A}_{\text{ex}}(t, \mathbf{x})$ of external electric and magnetic fields, the potentials are twice differentiable functions of time t and spatial variables $\mathbf{x} \in \mathbb{R}^3$ and are assumed to be given.

In the case when the potentials φ and $\mathbf{A}$ are zero, the charge can be considered as “free”, and the NLS equation (11.153) has localized solutions corresponding to resting and uniformly moving charges, see Sects. 14.1 and 15.1. Newton's law

involves an acceleration, and, to produce an accelerated motion, non-zero potentials are required. One may expect point-like dynamics for localized solutions, and, as we show in Sect. 14.3, the nonlinearity G' provides the existence of localized solutions to the NLS equation. Now we consider in detail the localization in accelerating regimes. The only condition which we impose here on the nonlinearity (in addition to natural continuity assumptions, see Condition 34.1.1), is the existence of a positive radial solution $\psi_1 = \psi_1(|\mathbf{x}|)$ of the steady-state equation (14.19) for a free charge:

$$-\nabla^2 \psi_1 + G'_1(|\psi_1|^2) \psi_1 = 0. \quad (34.8)$$

We assume that $\psi(|\mathbf{x}|)$ is twice continuously differentiable for all $\mathbf{x} \in \mathbb{R}^3$ and is square integrable, that is,

$$\int_{\mathbb{R}^3} |\psi|^2 d\mathbf{x} = v_0 < \infty. \quad (34.9)$$

Importantly, we assume that the nonlinearity depends explicitly on the size parameter $a > 0$ as in (14.27). A typical example of the ground state ψ_a is the Gaussian $\psi_a = a^{-3/2} e^{-r^2/a^2}$ corresponding to the logarithmic nonlinearity G'_a discussed in Example 14.3.5. Evidently, $v_0^{-1} \psi_a^2$ converges to Dirac's delta-function $\delta(\mathbf{x})$ as $a \rightarrow 0$. We study the behavior of localized solutions of the NLS equation in the asymptotic regime $a \rightarrow 0$.

A crucial role in our analysis is played by the key concept of concentrating solutions. In simple terms, we say that solutions $\psi = \psi_n$ concentrate at a given trajectory $\hat{\mathbf{r}}(t)$ if their charge densities $q|\psi|^2(\mathbf{x}, t)$ restricted to R_n -neighborhoods Ω_n of $\hat{\mathbf{r}}(t)$ locally converge to the delta-function $q_\infty(t) \delta(\mathbf{x} - \hat{\mathbf{r}}(t))$ as

$$a_n \rightarrow 0, \quad R_n \rightarrow 0, \quad a_n/R_n \rightarrow 0, \quad n \rightarrow \infty. \quad (34.10)$$

A precise definition of the concept of *concentrating* solutions requires somewhat more detail. Namely, there are two relevant spatial scales for the NLS equation: (i) microscopic size parameter a which determines the size of a free elementary charge, and (ii) macroscopic length scale R_{ex} of order 1 at which the potentials φ and $\mathbf{A}$ vary significantly. We also introduce a third intermediate spatial scale R_n which can be called the *confinement scale*, and make an assumption that ψ_n *asymptotically vanish at the boundary* $\partial\Omega_n = \{|\mathbf{x} - \hat{\mathbf{r}}(t)| = R_n\}$ of the neighborhood Ω_n of the trajectory $\hat{\mathbf{r}}(t)$. Since, according to relations (34.10) $R_n/a \rightarrow \infty$, the assumption of asymptotic vanishing is quite natural for solutions with the typical spatial scale a which are localized at $\hat{\mathbf{r}}$. Since at the same time $R_n/R_{\text{ex}} \rightarrow 0$, general EM potentials $\varphi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$ can be well approximated in Ω_n by their linearizations at $\mathbf{x} = \hat{\mathbf{r}}(t)$. A precise definition of the concentrating solutions (or concentrating asymptotic solutions) involves two major assumptions:

- (1) volume integrals of the charge density $q |\psi|^2$ and the momentum density $\mathbf{P}$ over the balls Ω_n should be bounded;
- (2) surface integrals over the spheres $\partial\Omega_n$ of certain quadratic expressions, which involve ψ and its first order derivatives and originate from the elements of the energy-momentum tensor of the NLS equation, tend to zero as $R_n \rightarrow 0$, $a_n/R_n \rightarrow 0$.

Detailed presentations of the above assumptions are given in Definitions 34.1.3 and 34.2.1.

A concise formulation of our main result, Theorem 34.2.1, is as follows. We prove that *if a sequence of asymptotic solutions of the NLS equation (11.153) concentrates at a trajectory $\hat{\mathbf{r}}(t)$, then this trajectory must satisfy Newton's equation*

$$m\partial_t^2 \hat{\mathbf{r}} = \mathbf{f}_{\text{Lor}}(t, \hat{\mathbf{r}}), \quad (34.11)$$

where $\mathbf{f}_{\text{Lor}}$ is the Lorentz force which is defined by the classical formula

$$\mathbf{f}_{\text{Lor}}(t, \hat{\mathbf{r}}) = q\mathbf{E}(t, \hat{\mathbf{r}}) + \frac{q}{c}\partial_t \hat{\mathbf{r}} \times \mathbf{B}(t, \hat{\mathbf{r}}), \quad (34.12)$$

with the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ defined in terms of the potentials $\varphi, \mathbf{A}$ in (11.154) according to the standard formula (4.6). Notice that, according to Newton's equation (34.11), the parameter m entering the NLS equation (11.153) can be interpreted as the mass of the charge.

Now we would like to comment on certain subjects related to our main Theorem 34.2.1. First of all, the assumptions imposed on the concentrating solutions are not restrictive. Indeed, the solutions are local and have only to be defined in a tubular neighborhood of the trajectory, and no initial or boundary conditions are imposed on them. Second, only the charge and the momentum conservation laws are used in the derivation of the theorem. Since only the conservation laws are used for the argument, we are able to introduce the asymptotic solutions for the NLS equation as functions ψ for which the charge and the momentum conservation laws hold approximately. Namely, we assume that certain integrals which involve quantities that enter the conservation laws vanish in an asymptotic limit, see Definition 34.2.1 for details. Remarkably, such mild restrictions still completely determine all possible trajectories of concentration, that is, the trajectory $\hat{\mathbf{r}}(t)$ is uniquely defined by the initial data $\hat{\mathbf{r}}(t_0)$ and $\partial_t \hat{\mathbf{r}}(t_0)$ as a solution of (34.11).

Equation (34.11) yields a *necessary condition* for solutions of the NLS equation to concentrate at a trajectory. To obtain this necessary condition, we have to derive the point dynamics governed by an ordinary differential equation (34.11) from the dynamics of waves governed by a partial differential equation. The concept of “*concentration of functions at a given trajectory $\hat{\mathbf{r}}(t)$* ”, see Definition 34.1.3, is the first step in relating spatially localized fields ψ to point trajectories. The definition of concentration of functions has a sufficient flexibility to allow for general regular trajectories $\hat{\mathbf{r}}(t)$, and plenty of functions are localized about the trajectory.

But if functions from a sequence concentrating at a given trajectory *also satisfy or asymptotically satisfy the conservation laws for the NLS equation*, then, according to Theorems 34.1.3 and 34.2.1, the trajectory and the limit energy must satisfy Newton's equation.

To derive Newton's equation (34.11), we consider $\psi(\mathbf{x}, t)$ restricted to a narrow tubular neighborhood of the trajectory of radius R_n and consider adjacent charge centers

$$\mathbf{r}_n(t) = \frac{1}{\bar{\rho}_n} \int_{|\mathbf{x}-\hat{\mathbf{r}}| \leq R_n} \mathbf{x} q |\psi_n|^2 d\mathbf{x}, \quad \bar{\rho}_n = \int_{|\mathbf{x}-\hat{\mathbf{r}}| \leq R_n} q |\psi_n|^2 d\mathbf{x},$$

of the concentrating solutions. Then we infer integral equations for the restricted momentum and for the adjacent centers from the momentum conservation law and the continuity equation, and pass to the limit as $R_n \rightarrow 0$, $a_n \rightarrow 0$ as is shown in the proofs of Theorems 34.1.1–34.1.3. Note that the determination of the restricted charge density $\bar{\rho}$ and a similar momentum density involves integration over a large, relative to a , spatial domain of radius R_n , $R_n \gg a$. Therefore it is natural to call our method of determination of the point trajectories *semi-local*. This semi-local feature applied to the nonlinear Klein–Gordon equation in [18], [19] and in Sects. 17.6.2, 34.3 allows us to derive Einstein's relation between mass and energy with the energy being an integral quantity.

If the form factor $\psi_1(\theta)$ from Eq.(34.8) decays fast enough as $\theta \rightarrow \infty$ (faster than θ^{-2}), we also prove the converse statement (Theorem 34.2.3) to Theorem 34.2.1. Namely, if $\hat{\mathbf{r}}(t)$ is a solution to Eq. (34.11), then it is a trajectory of concentration for a sequence of asymptotic solutions to the NLS. The proof of this theorem is based on an explicit construction of *wave-corpucle solutions* of the NLS of the form $\psi(t, \mathbf{x}) = e^{i\mathfrak{S}} \hat{\psi}(\mathbf{x} - \hat{\mathbf{r}})$ for a general enough class of potentials φ and $\mathbf{A}$. We choose auxiliary potentials φ_{aux} and $\mathbf{A}_{\text{aux}}$ from this class to approximate φ and $\mathbf{A}$ at $\hat{\mathbf{r}}(t)$. The wave-corpucle solutions constructed for the auxiliary potentials φ_{aux} and $\mathbf{A}_{\text{aux}}$ turn out to form a sequence of concentrating asymptotic solutions of the NLS. Therefore, for given twice continuously differentiable potentials φ and $\mathbf{A}$, a given trajectory coincides with a trajectory of concentration of asymptotic solutions of the NLS if and only if this trajectory satisfies Newton's law (34.11). The phase function $\mathfrak{S}$ of the wave-corpucle can be interpreted as the phase of the de Broglie wave (see Sect. 16.2 for details). Hence, the described relation between Newtonian trajectories and concentrating asymptotic solutions can be interpreted as the wave-particle duality.

The derivation of Newton's equation is based on the conservation laws for the NLS equation which are considered in the following section.

Continuity and momentum equations. Solutions of the NLS equation (11.153) satisfy the continuity (charge conservation) equation (11.158), namely

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0,$$

where the charge density ρ and current density $\mathbf{J}$ are given by (11.156), (11.157). They also satisfy the momentum equation (11.169), namely

$$\partial_t \mathbf{P} + \partial_i T^{ij} = \mathbf{f}, \quad (34.13)$$

where the momentum density $\mathbf{P}$ is given by (11.163), and the Lorentz force density $\mathbf{f}$ is given by the formula

$$\mathbf{f} = \rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B}. \quad (34.14)$$

The electromagnetic fields $\mathbf{E}$, $\mathbf{B}$ in the above formula are defined by formula (4.6) in terms of the potentials φ_{ex} , $\mathbf{A}_{\text{ex}}$ which enter the covariant derivatives in (11.153). In what follows, we often make use of the following elementary identity

$$\int_{\Omega_n} \partial_t f(t, \mathbf{x}) \, d\mathbf{x} = \partial_t \int_{\Omega_n} f(t, \mathbf{x}) \, d\mathbf{x} - \int_{\partial\Omega_n} f(t, \mathbf{x}) \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \, d\sigma, \quad (34.15)$$

where $\bar{\mathbf{n}}$ is the external normal to $\partial\Omega_n$, $\hat{\mathbf{v}} = \partial_t \hat{\mathbf{r}}$, and f is a function of $t, \mathbf{x}$.

34.1.1 Localized NLS Equations

Let us consider the NLS equation (11.153) in a neighborhood of the trajectory $\hat{\mathbf{r}}(t)$, and we assume that the mass parameter m , the charge value q , the Planck constant χ and the speed of light c are fixed. The size parameter a and potentials $\varphi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$ form a sequence.

We formulate first the continuity assumptions imposed on the nonlinearity G'_1 which enters NLS equation (11.153) (the corresponding properties of G'_a with $a \neq 1$ are determined by (14.27)).

Definition 34.1.1 We say that the nonlinearity $G'_1(s)$ is regular if it satisfies the following conditions. It is a real-valued function which is continuous for $s > 0$. It coincides with the derivative of the potential $G_1(s)$ which is differentiable for $s > 0$ and continuous for $s \geq 0$. We assume that the function $G_1(\psi^* \psi)$ of the complex variable $\psi \in \mathbb{C}$ is differentiable for all ψ and its differential has the form

$$dG_1(\psi^* \psi) = g(\psi) d\psi^* + g^*(\psi) d\psi,$$

where

$$g(\psi) = \begin{cases} G'_1(\psi^* \psi) \psi & \text{for } \psi \in \mathbb{C}, \psi \neq 0, \\ 0 & \text{for } \psi = 0, \end{cases}$$

and $g(\psi)$ is continuous for $\psi \in \mathbb{C}$.

Note that the above condition allows a mild singularity of G'_1 at zero, for example the logarithmic nonlinearity (14.39) satisfies this condition.

Definition 34.1.2 (*localized NLS equations*) Let $\hat{\mathbf{r}}(t)$ be a trajectory with its concentrating neighborhoods $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$, and let $a = a_n$, $\varphi = \varphi_n$, $\mathbf{A} = \mathbf{A}_n$ be a sequence of parameters and potentials entering the NLS equation (11.153). We say that the NLS equations are localized in $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$ if the following conditions are satisfied. We require that R_n and a are vanishingly small:

$$a = a_n \rightarrow 0, \quad R_n \rightarrow 0, \quad (34.16)$$

and the ratio $\theta = R/a$ grows to infinity:

$$\theta_n = \frac{R_n}{a_n} \rightarrow \infty \text{ as } n \rightarrow \infty. \quad (34.17)$$

We also assume that at the trajectory $\hat{\mathbf{r}}(t)$ the limit linear potentials $\varphi_\infty(t, \mathbf{x})$, $\mathbf{A}_\infty(t, \mathbf{x})$ are defined by the following formulas:

$$\varphi_\infty(t, \mathbf{x}) = \varphi_\infty(t) + (\mathbf{x} - \hat{\mathbf{r}}) \cdot \nabla \varphi_\infty(t), \quad (34.18)$$

$$\mathbf{A}_\infty(t, \mathbf{x}) = \mathbf{A}_\infty(t) + (\mathbf{x} - \hat{\mathbf{r}}) \cdot \nabla \mathbf{A}_\infty(t), \quad (34.19)$$

where the coefficients φ_∞ , $\nabla \varphi_\infty$, $\mathbf{A}_\infty$, $\nabla \mathbf{A}_\infty$ satisfy the following boundedness conditions:

$$|\varphi_\infty(t)| \leq C, \quad |\nabla_{0,\mathbf{x}} \varphi_\infty| \leq C \text{ for } T_- \leq t \leq T_+, \quad (34.20)$$

$$|\mathbf{A}_\infty(t)| \leq C, \quad |\nabla_{0,\mathbf{x}} \mathbf{A}_\infty| \leq C \text{ for } T_- \leq t \leq T_+. \quad (34.21)$$

We assume that the EM potentials $\varphi_n(t, \mathbf{x})$, $\mathbf{A}_n(t, \mathbf{x})$ are twice continuously differentiable in $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$. The potentials $\varphi_n(t, \mathbf{x})$, $\mathbf{A}_n(t, \mathbf{x})$ locally converge to the limit potentials $\varphi_\infty(t, \mathbf{x})$, $\mathbf{A}_\infty(t, \mathbf{x})$, namely they satisfy the following relations:

(i) convergence as $n \rightarrow \infty$:

$$\begin{aligned} & \max_{T_- \leq t \leq T_+, \mathbf{x} \in \Omega_n} |\varphi_n(t, \mathbf{x}) - \varphi_\infty(t, \mathbf{x})| \\ & + \max_{T_- \leq t \leq T_+, \mathbf{x} \in \Omega_n} |\nabla_{0,\mathbf{x}} \varphi_n(t, \mathbf{x}) - \nabla_{0,\mathbf{x}} \varphi_\infty(t, \mathbf{x})| \rightarrow 0, \end{aligned} \quad (34.22)$$

$$\begin{aligned} & \max_{T_- \leq t \leq T_+, \mathbf{x} \in \Omega_n} |\mathbf{A}_n(t, \mathbf{x}) - \mathbf{A}_\infty(t, \mathbf{x})| \\ & + \max_{T_- \leq t \leq T_+, \mathbf{x} \in \Omega_n} |\nabla_{0,\mathbf{x}} \mathbf{A}_n(t, \mathbf{x}) - \nabla_{0,\mathbf{x}} \mathbf{A}_\infty(t, \mathbf{x})| \rightarrow 0; \end{aligned} \quad (34.23)$$

(ii) uniform in n estimates:

$$|\varphi_n(t, \mathbf{x})| \leq C, \quad |\nabla_{0,\mathbf{x}} \varphi_n(t, \mathbf{x})| \leq C \quad \text{for } (t, \mathbf{x}) \in \hat{\Omega}(\hat{\mathbf{r}}, R_n), \quad (34.24)$$

$$|\mathbf{A}_n(t, \mathbf{x})| \leq C, \quad |\nabla_{0,\mathbf{x}} \mathbf{A}_n(t, \mathbf{x})| \leq C \quad \text{for} \quad (t, \mathbf{x}) \in \hat{\Omega}(\hat{\mathbf{r}}, R_n). \quad (34.25)$$

The limit EM fields $\mathbf{E}_\infty, \mathbf{B}_\infty$ at the trajectory are defined in terms of the limit linear potentials by (4.6), namely

$$\mathbf{E}_\infty = -\nabla \varphi_\infty(t, \hat{\mathbf{r}}) - \frac{1}{c} \partial_t \mathbf{A}_\infty(t, \hat{\mathbf{r}}), \quad \mathbf{B}_\infty = \nabla \times \mathbf{A}_\infty(t, \hat{\mathbf{r}}). \quad (34.26)$$

Note that according to (34.22), (34.23)

$$\mathbf{E} = \mathbf{E}_n \rightarrow \mathbf{E}_\infty, \quad \mathbf{B} = \mathbf{B}_n \rightarrow \mathbf{B}_\infty \text{ in } \hat{\Omega}(\hat{\mathbf{r}}(t), R_n). \quad (34.27)$$

Throughout this chapter we denote constants which do not depend on n by the letter C with different indices. Sometimes the same letter C with the same index may denote in different formulas different constants. Below we often omit the index n in a_n, φ_n etc.

The most important case where we apply the above definition is described in the following example.

Example 34.1.1 If the potentials $\varphi_n, \mathbf{A}_n$ are the restrictions of fixed twice continuously differentiable potentials $\varphi, \mathbf{A}$ to the domain $\hat{\Omega}(\hat{\mathbf{r}}(t), R_n)$, then $\varphi_\infty, \mathbf{A}_\infty$ are the linear parts of $\varphi, \mathbf{A}$ at $\hat{\mathbf{r}} = \hat{\mathbf{r}}(t)$, and conditions (34.21)–(34.24) are satisfied with

$$\begin{aligned} \varphi_\infty(t, \mathbf{x}) &= \varphi(t, \hat{\mathbf{r}}) + (\mathbf{x} - \hat{\mathbf{r}}) \nabla \varphi(t, \hat{\mathbf{r}}), \\ \mathbf{A}_\infty(t, \mathbf{x}) &= \mathbf{A}(t, \hat{\mathbf{r}}) + (\mathbf{x} - \hat{\mathbf{r}}) \nabla \mathbf{A}(t, \hat{\mathbf{r}}). \end{aligned} \quad (34.28)$$

Hence the coefficients in (34.18), (34.19) are defined as follows:

$$\varphi_\infty(t) = \varphi(t, \hat{\mathbf{r}}), \quad \nabla \mathbf{A}_\infty(t) = \nabla \mathbf{A}(t, \hat{\mathbf{r}}), \quad (34.29)$$

where the EM fields $\mathbf{E}_\infty, \mathbf{B}_\infty$ are directly expressed in terms of $\varphi, \mathbf{A}$ by (4.6), namely

$$\mathbf{E}_\infty = -\nabla \varphi(t, \hat{\mathbf{r}}) - \frac{1}{c} \partial_t \mathbf{A}(t, \hat{\mathbf{r}}), \quad \mathbf{B}_\infty = \nabla \times \mathbf{A}(t, \hat{\mathbf{r}}). \quad (34.30)$$

We introduce the local value of the charge restricted to $\Omega(\hat{\mathbf{r}}(t), R_n)$ by the formula

$$\bar{\rho}_n(t) = \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \rho_n \, d\mathbf{x} = \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} q |\psi_n|^2 \, d\mathbf{x} \quad (34.31)$$

with $\rho_n(t, \mathbf{x})$ being the charge density defined by (11.156), and we call $\bar{\rho}_n$ the adjacent charge value.

Now we give the definition of concentrating solutions and trajectory of concentration in the non-relativistic case.

Definition 34.1.3 (*concentrating solutions*) Let $\hat{\mathbf{r}}(t)$ be a trajectory. We say that solutions ψ of the NLS equations concentrate at the trajectory $\hat{\mathbf{r}}(t)$ if the following conditions are fulfilled. First, a sequence of concentrating neighborhoods $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$, parameters $a = a_n$ and potentials $\varphi = \varphi_n$, $\mathbf{A} = \mathbf{A}_n$ are selected to form a sequence of the NLS equations localized in $\hat{\Omega}_n(\hat{\mathbf{r}}, R_n)$ as in Definition 34.1.2, and the non-linearity in the equations is regular as in Definition 34.1.1. Second, there exists a sequence of functions $\psi = \psi_n$ which are twice continuously differentiable, namely $\psi_n \in C^2(\hat{\Omega}(\hat{\mathbf{r}}, R_n))$, such that the charge density $\rho = q |\psi_n|^2$ and the momentum density $\mathbf{P}$ defined by (11.163) for this sequence satisfy the following conditions (i)–(vi):

- (i) Every function ψ_n is a solution to the NLS equation (11.153) in $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$.
- (ii) The momentum density $\mathbf{P}$ defined by (11.163) for this sequence is such that the following integrals are bounded:

$$\left| \int_{\Omega_n} \mathbf{P}_n(t) \, d\mathbf{x} \right| \leq C. \quad (34.32)$$

- (iii) The local charge value defined by (34.31) is bounded from above and below for sufficiently large n :

$$C \geq \bar{\rho}_n(t) \geq c_0 > 0 \text{ for } n \geq n_0, \quad T_- \leq t \leq T_+. \quad (34.33)$$

- (iv) There exists a $t_0 \in [T_-, T_+]$ such that the sequence of local charge values converges:

$$\lim_{n \rightarrow \infty} \bar{\rho}_n(t_0) = \bar{\rho}_\infty. \quad (34.34)$$

- (v) We impose conditions on the following surface integrals over the spheres $\partial\Omega_n = \{|\mathbf{x} - \hat{\mathbf{r}}| = R_n\}$:

$$Q_0 = Q_0^{(j)} = \int_{t_0}^t \int_{\partial\Omega_n} \bar{\mathbf{n}}_i T^{ij} \, d\sigma dt' \quad (34.35)$$

where T^{ij} are given by (11.164), (11.165);

$$Q_{01} = \int_{t_0}^t \int_{\partial\Omega_n} \mathbf{P} \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} d\sigma dt' \quad (34.36)$$

where $\mathbf{P}$ is defined by (11.163);

$$Q_{20} = - \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho \, d\sigma \quad (34.37)$$

where ρ is defined by (11.156);

$$Q_{22} = \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \cdot \bar{\mathbf{n}} \cdot \mathbf{J} d\mathbf{x} \quad (34.38)$$

where $\mathbf{J}$ is given by (11.157); and

$$Q_{23} = \int_{t_0}^t \int_{\partial\Omega_n} \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho d\mathbf{x} dt' - \int_{t_0}^t \int_{\partial\Omega_n} \bar{\mathbf{n}} \cdot \mathbf{J} d\mathbf{x} dt'. \quad (34.39)$$

We assume that

$$Q_0 \rightarrow 0, \quad (34.40)$$

$$Q_{01} \rightarrow 0, \quad (34.41)$$

$$Q_{20} \rightarrow 0, \quad (34.42)$$

$$Q_{22} \rightarrow 0, \quad (34.43)$$

$$Q_{23} \rightarrow 0 \quad (34.44)$$

uniformly on the time interval $[T_-, T_+]$.

(vi) We also consider the following integrals over Ω_n with vanishing at $\hat{\mathbf{r}}$ weights:

$$Q_{30} = \int_{\Omega_n} (\mathbf{E} - \mathbf{E}_\infty) \rho d\mathbf{x}, \quad (34.45)$$

$$Q_{31} = \int_{\Omega_n} \frac{1}{c} \mathbf{J} \times (\mathbf{B} - \mathbf{B}_\infty) d\mathbf{x}; \quad (34.46)$$

we assume that

$$\int_{t_0}^t (Q_{30} + Q_{31}) dt' \rightarrow 0 \quad (34.47)$$

uniformly on the time interval $[T_-, T_+]$.

Definition 34.1.4 If all the above conditions are fulfilled, we call $\hat{\mathbf{r}}(t)$ a trajectory of concentration for the NLS equation (11.153).

Obviously conditions (ii)–(iv) and (vi) provide boundedness and convergence of certain volume integrals over Ω_n , and condition (v) provides asymptotic vanishing of surface integrals over the boundary.

Notice that condition (ii) provides for the boundedness of the momentum over the domain Ω_n , condition (v) provides for a proper confinement of ψ to Ω_n , and the estimate from below in condition (iii) ensures that the sequence is non-trivial. According to (34.33), $\bar{\rho}_n(t_0)$ is a bounded sequence, consequently it always contains

a converging subsequence. Hence condition (iv) is not really an additional constraint but rather it assumes that such a subsequence is selected. The choice of a particular subsequence limit $\bar{\rho}_\infty$ is discussed in Remark 34.1.2. This condition describes the amount of charge which concentrates at the trajectory at the time t_0 . Condition (i) can be relaxed and replaced by the assumption that ψ_n is an asymptotic solution, see Definition 34.2.1 for details. We could also allow parameters χ, m, q to form sequences and depend on n , but for simplicity of the mathematical treatment we assume them fixed.

The wave-corpuscles constructed in Chap. 16 provide a non-trivial example of solutions which concentrate at trajectories of accelerating charges.

34.1.2 Properties of Concentrating Solutions of NLS

We define the adjacent charge center $\mathbf{r}_n$ by the formula

$$\mathbf{r}_n(t) = \frac{1}{\bar{\rho}_n} \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \mathbf{x} \rho_n d\mathbf{x} \quad (34.48)$$

with the charge density ρ given by (11.156). Since $\psi_a(t, \mathbf{x})$ is continuously differentiable with respect to $(t, \mathbf{x})$, and $\hat{\mathbf{r}}(t)$ is differentiable, the vector $\mathbf{r}(t)$ is a differentiable function of time, and we denote by $\mathbf{v}$ velocity of the adjacent charge center:

$$\mathbf{v} = \mathbf{v}_n(t) = \partial_t \mathbf{r}.$$

Lemma 34.1.1 *Let charge densities ρ_n satisfy (34.33). Then the adjacent ergocenters $\mathbf{r}_n(t)$ of the solutions converge to $\hat{\mathbf{r}}(t)$ uniformly on the time interval $[T_-, T_+]$.*

Proof By (34.48)

$$\int_{\Omega_n} (\mathbf{x} - \mathbf{r}) \rho_n d\mathbf{x} = 0, \quad (34.49)$$

and, according to (34.33),

$$\left| \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} (\mathbf{x} - \hat{\mathbf{r}}) \rho_n d\mathbf{x} \right| \leq R_n \int_{\Omega(\mathbf{0}, R_n)} \rho_n d\mathbf{y} \rightarrow 0. \quad (34.50)$$

Therefore

$$(\mathbf{r} - \hat{\mathbf{r}}) \bar{\rho} = \int_{\Omega_n} (\mathbf{x} - \hat{\mathbf{r}}) \rho_n d\mathbf{x} - \int_{\Omega_n} (\mathbf{x} - \mathbf{r}) \rho_n d\mathbf{x} \rightarrow 0.$$

Using (34.33), we conclude that

$$|\hat{\mathbf{r}} - \mathbf{r}_n| \rightarrow 0 \quad (34.51)$$

uniformly on $[T_-, T_+]$.

Lemma 34.1.2 *Let the current $\mathbf{J}$ and the charge density ρ in (11.158) satisfy (34.44). Then the adjacent charge values uniformly on $[T_-, T_+]$ converge to a constant:*

$$\bar{\rho}_n(t) \rightarrow \bar{\rho}_\infty \text{ as } n \rightarrow \infty. \quad (34.52)$$

Proof Integrating the continuity equation, we obtain that

$$\bar{\rho}(t) - \bar{\rho}(t_0) - \int_{t_0}^t \int_{\partial\Omega_n} \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho \, d\mathbf{x} \, dt' + \int_{t_0}^t \int_{\partial\Omega_n} \bar{\mathbf{n}} \cdot \mathbf{J} \, d\mathbf{x} \, dt' = 0. \quad (34.53)$$

We use (34.44) and obtain (34.52).

Lemma 34.1.3 *Assume that (34.32) holds. Then there is a subsequence of the concentrating sequence of solutions of the NLS such that*

$$\int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \mathbf{P}_n(t_0) \, d\mathbf{x} \rightarrow \mathbf{p}_\infty \text{ as } n \rightarrow \infty. \quad (34.54)$$

Assume in addition that the boundary integrals (34.35) and (34.36) satisfy assumptions (34.40) and (34.41). Then for this subsequence

$$\int_{\Omega_n} \mathbf{P}_n(t) \, d\mathbf{x} = \int_{t_0}^t \int_{\Omega_n} \mathbf{f} \, d\mathbf{x} \, dt' + \mathbf{p}_\infty + \mathcal{Q}_{00}, \quad (34.55)$$

where

$$\mathcal{Q}_{00} \rightarrow 0 \text{ as } n \rightarrow \infty \quad (34.56)$$

uniformly on $[T_-, T_+]$.

Proof According to (34.32), we can select a subsequence which has a limit $\mathbf{p}_\infty$, and (34.54) holds. We use the momentum equation (11.169):

$$\partial_t \mathbf{P} + \partial_i T^{ij} = \mathbf{f}. \quad (34.57)$$

Integrating it over $\Omega(\hat{\mathbf{r}}(t), R_n) = \Omega_n$ and then with respect to time, we obtain the equation

$$\int_{t_0}^t \int_{\Omega_n} \partial_t \mathbf{P}(t') \, d\mathbf{x} \, dt' - \int_{t_0}^t \int_{\Omega_n} \mathbf{f} \, d\mathbf{x} \, dt' + \mathcal{Q}_0 = 0 \quad (34.58)$$

where Q_0 is defined by (34.35). Using (34.15), we rewrite the equation in the form

$$\int_{\Omega_n} \mathbf{P}(t) \, d\mathbf{x} - \int_{\Omega_n} \mathbf{P}(t_0) \, d\mathbf{x} - Q_{01} - \int_{t_0}^t \int_{\Omega_n} \mathbf{f} d\mathbf{x} dt' + Q_0 = 0 \quad (34.59)$$

where Q_{01} is defined by (34.36), $\Omega_n = \Omega(\mathbf{r}(t), R_n)$. Using (34.40), (34.41) and (34.54), we obtain (34.55) and (34.56) from (34.59).

Lemma 34.1.4 *In addition to the assumptions of Lemmas 34.1.1–34.1.3, assume that the boundary integrals (34.37) and (34.38) asymptotically vanish as $n \rightarrow \infty$, namely (34.42) and (34.43) are fulfilled uniformly on the time interval $[T_-, T_+]$. Then*

$$\int_{\Omega_n} \mathbf{J} \, d\mathbf{x} = \mathbf{v} \bar{\rho}_\infty + Q_{200}, \quad (34.60)$$

$$\int_{\Omega_n} \mathbf{P} \, d\mathbf{x} = m \mathbf{v} \frac{\bar{\rho}_\infty}{q} + \frac{m}{q} Q_{200}, \quad (34.61)$$

where $\mathbf{v} = \partial_t \mathbf{r}$, and

$$Q_{200} \rightarrow 0 \quad (34.62)$$

uniformly on the time interval $[T_-, T_+]$.

Proof According to the continuity equation, we obtain the identity

$$\int_{\Omega_n} (\mathbf{x} - \mathbf{r}) \partial_t \rho \, d\mathbf{x} + \int_{\Omega_n} (\mathbf{x} - \mathbf{r}) \nabla \cdot \mathbf{J} \, d\mathbf{x} = 0.$$

Using the commutation relation

$$\partial_j (x_i f) - x_i \partial_j f = \delta_{ij} f \quad (34.63)$$

to transform the second integral, we obtain the following equation:

$$\begin{aligned} \int_{\Omega_n} \partial_t ((\mathbf{x} - \mathbf{r}) \rho) \, d\mathbf{x} + \partial_t \mathbf{r} \int_{\partial\Omega_n} \rho \, d\mathbf{x} \\ + \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \mathbf{n} \cdot \mathbf{J} \, d\mathbf{x} = \int_{\Omega_n} \mathbf{J} \, d\mathbf{x}. \end{aligned} \quad (34.64)$$

Using the definition of the charge center $\mathbf{r}$ and (34.15), we infer that the first term in the above equation has the following form:

$$\int_{\Omega_n} \partial_t ((\mathbf{x} - \mathbf{r}) \rho) \, d\mathbf{x} = - \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho \, d\sigma, \quad (34.65)$$

where $\hat{\mathbf{v}} = \partial_t \hat{\mathbf{r}}$. We can express $\mathbf{v} = \partial_t \mathbf{r}$ from (34.64), and using (34.32), (34.38), (34.51) and (34.33) to estimate the integrals which enter (34.64), we conclude that

$$|\mathbf{v}| \leq C \quad \text{for} \quad -T \leq t \leq T. \quad (34.66)$$

Notice that we can rewrite (34.64) in the form

$$\int_{\Omega} \mathbf{J} \, d\mathbf{x} = \mathbf{v} \bar{\rho}_{\infty} + Q_{20} + Q_{21} + Q_{22}, \quad (34.67)$$

where $\bar{\rho}_{\infty}$ is the same as in (34.34),

$$Q_{21} = \mathbf{v} \left(\int_{\Omega} \rho_n \, d\mathbf{x} - \bar{\rho}_{\infty} \right).$$

According to (34.34) and (34.66), $Q_{21} \rightarrow 0$. Using (34.37) and (34.38), we conclude that (34.60) and (34.62) hold with $Q_{200} = Q_{20} + Q_{21} + Q_{22}$; using (11.166), we deduce (34.61) from (34.60).

34.1.3 Derivation of Newton's Equation for the Trajectory of Concentration

Theorem 34.1.1 *Given a concentrating sequence of solutions of the NLS equation, the adjacent center velocities $\mathbf{v} = \mathbf{v}_n = \partial_t \mathbf{r}_n$ satisfy the equation*

$$m\mathbf{v} \frac{1}{q} \bar{\rho}_{\infty} = \int_{t_0}^t \left(\bar{\rho}_{\infty} \mathbf{E}_{\infty}(t') + \bar{\rho}_{\infty} \mathbf{v} \times \frac{1}{c} \mathbf{B}_{\infty}(t') \right) dt' + \mathbf{p}_{\infty} + Q_6, \quad (34.68)$$

where $Q_6 \rightarrow 0$ uniformly on $[T_-, T_+]$, $\mathbf{p}_{\infty}$ is the same as in (34.54), and $\mathbf{E}_{\infty}, \mathbf{B}_{\infty}$ are the same as in (34.26).

Proof We substitute (34.61) into (34.55) and obtain the following equation:

$$m\mathbf{v} \frac{1}{q} \bar{\rho}_{\infty} + \frac{m}{q} Q_{200} = \int_{t_0}^t \int_{\Omega_n} \mathbf{f} \, d\mathbf{x} dt' + \mathbf{p}_{\infty} + Q_{00}, \quad (34.69)$$

where the Lorentz force density $\mathbf{f}$ is given by (34.14).

To evaluate the terms involving the Lorentz force density, we use the limit EM fields defined in accordance with (34.18), (34.19) by formula (34.26). We obtain the following expression:

$$\int_{\Omega_n} \mathbf{f} \, d\mathbf{x} = \mathbf{E}_{\infty} \int_{\Omega_n} \rho \, d\mathbf{x} + \int_{\Omega_n} \frac{1}{c} \mathbf{J} \, d\mathbf{x} \times \mathbf{B}_{\infty} + Q_3 \quad (34.70)$$

where $Q_3 = Q_{30} + Q_{31}$ is expressed in terms of (34.45), (34.46). Therefore, using (34.60), we obtain from (34.70) that

$$\int_{\Omega_n} \mathbf{f} \, d\mathbf{x} = \bar{\rho}_\infty \mathbf{E}_\infty + \bar{\rho}_\infty \mathbf{v} \times \frac{1}{c} \mathbf{B}_\infty + Q_3 + Q_4 + Q_5, \quad (34.71)$$

$$Q_4 = \frac{1}{c} Q_{200} \times \mathbf{B}_\infty, \quad Q_5 = \mathbf{E}_\infty (\bar{\rho}_n - \bar{\rho}_\infty). \quad (34.72)$$

Using (34.47), (34.52), (34.62), (34.20) and (34.21), we obtain (34.68) with

$$Q_6 = Q_{00} - \frac{m}{q} Q_{200} + \int_{t_0}^t (Q_3 + Q_4 + Q_5) \, dt'.$$

The following statement provides an explicit necessary condition for a trajectory to be a trajectory of concentration.

Theorem 34.1.2 (Trajectory of concentration criterion) *Let solutions ψ of the NLS equation (11.153) concentrate at $\hat{\mathbf{r}}(t)$. Then the trajectory $\hat{\mathbf{r}}(t)$ satisfies the equation*

$$m \partial_t^2 \hat{\mathbf{r}} = \mathbf{f}_\infty \quad (34.73)$$

with the Lorentz force $\mathbf{f}_\infty(t)$ expressed in terms of the limit potentials (34.18), (34.19) by the following formula

$$\mathbf{f}_\infty(t) = -q \mathbf{E}_\infty + \frac{q}{c} \partial_t \hat{\mathbf{r}} \times \mathbf{B}_\infty. \quad (34.74)$$

Proof The function $\mathbf{v}(t)$ can be considered a solution of the integral equation (34.68) on the interval $[T_-, T_+]$, this equation is obviously linear. Since the term $Q_6 \rightarrow 0$ uniformly on the interval, the sequence $\mathbf{v}_n(t)$ of its solutions converges uniformly to the solution of the equation

$$m \mathbf{v}_\infty \frac{1}{q} \bar{\rho}_\infty = \bar{\rho}_\infty \int_{t_0}^t \left(\mathbf{E}_\infty(t') + \mathbf{v}_\infty \times \frac{1}{c} \mathbf{B}_\infty(t') \right) dt' + \mathbf{p}_\infty. \quad (34.75)$$

Now we prove that $\mathbf{v}_\infty = \hat{\mathbf{v}} = \partial_t \hat{\mathbf{r}}$. Note that, according to Lemma 34.1.1,

$$\int_{t_0}^t \mathbf{v}_n \, dt' = \mathbf{r}_n(t) - \mathbf{r}_n(t_0) \rightarrow \hat{\mathbf{r}}(t) - \hat{\mathbf{r}}(t_0),$$

therefore

$$\int_{t_0}^t \mathbf{v}_\infty \, dt' = \hat{\mathbf{r}}(t) - \hat{\mathbf{r}}(t_0);$$

this identity implies that $\partial_r \hat{\mathbf{r}} = \hat{\mathbf{v}}(t) = \mathbf{v}_\infty(t)$, and therefore $\hat{\mathbf{r}}(t)$ satisfies (34.75):

$$m \partial_t \hat{\mathbf{r}} \frac{1}{q} \bar{\rho}_\infty = \bar{\rho}_\infty \int_{t_0}^t \left(\mathbf{E}_\infty(t') + \partial_t \hat{\mathbf{r}} \times \frac{1}{c} \mathbf{B}_\infty(t') \right) dt' + \mathbf{p}_\infty. \quad (34.76)$$

We take the derivative of this equation and obtain (34.73).

As a corollary of Theorem 34.1.2 we obtain the following theorem, which describes the whole class of trajectories of concentration of the NLS equation (11.153):

Theorem 34.1.3 (Non-relativistic Newton's law) *Assume that potentials $\varphi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$ are defined and regular in a domain $D \subset \mathbb{R} \times \mathbb{R}^3$, the trajectory $(t, \hat{\mathbf{r}}(t))$ lies in this domain and the limit potentials φ_∞ , $\mathbf{A}_\infty$ are the linear parts of the potentials $\varphi, \mathbf{A}$ at $\hat{\mathbf{r}}$ as in (34.28). Let EM fields $\mathbf{E}(t, \mathbf{x})$, $\mathbf{B}(t, \mathbf{x})$ be determined in terms of the potentials by formula (4.6). Let solutions ψ of the NLS equation (11.153) concentrate at $\hat{\mathbf{r}}(t)$. Then Eq. (34.73) for the trajectory $\hat{\mathbf{r}}$ takes the form of Newton's law of motion with the Lorentz force in the external EM field $\mathbf{E}(t, \mathbf{x})$, $\mathbf{B}(t, \mathbf{x})$:*

$$m \partial_t^2 \hat{\mathbf{r}} = q \mathbf{E}(t, \hat{\mathbf{r}}) + \frac{q}{c} \partial_t \hat{\mathbf{r}} \times \mathbf{B}(t, \hat{\mathbf{r}}). \quad (34.77)$$

Hence, for the NLS equation with given potentials $\varphi, \mathbf{A}$, any concentration trajectory must coincide with the solution of the Eq. (34.77). For given potentials $\varphi, \mathbf{A}$, the concentration trajectory through a point $\hat{\mathbf{r}}(t_0)$ is uniquely determined by velocity $\partial_t \hat{\mathbf{r}}(t_0)$; in particular, it does not depend on the parameter χ which enters the NLS equation. All possible concentration trajectories are the solutions of the equation (34.77).

Remark 34.1.1 We can modify the definition of concentration to a trajectory by allowing χ not to be fixed but to form a sequence; the statements of Theorems 34.1.2 and 34.1.3 remain true in this case too.

Remark 34.1.2 Consider the situation of Example 34.1.1 and Theorem 34.1.3. The sequences $a_n, R_n, \theta_n, \varphi_n, \mathbf{A}_n, \psi_n$ enter the definition of a concentrating solution. But if we take two different sequences which fit the definition, we obtain the same Eq. (34.77). More than that, the trajectory $\hat{\mathbf{r}}(t)$ is uniquely defined by the initial data $\hat{\mathbf{r}}(t_0), \partial_t \hat{\mathbf{r}}(t_0)$ and consequently it does not depend on the particular sequence. Remarkably, the equation for the trajectory is independent of the value of $\bar{\rho}_\infty$, and this property, as we will see later, does not hold in the relativistic case.

34.1.4 Wave-Corpuscles as Concentrating Solutions

The wave-corpuscles constructed in Sect. 16.1 in Theorem 16.1.1 using rapidly decaying form factors ψ provide an example of concentrating solutions. To check

this fact, we verify all the requirements of Definition 34.1.3. It is important to note that the dependence on the size parameter a in (16.2) is only through the form factor $\psi(|\mathbf{x} - \mathbf{r}|) = a^{-3/2} \psi_1(a^{-1}|\mathbf{x} - \mathbf{r}|)$ where ψ_1 is a given smooth function. The verification of the conditions imposed in the Definition 34.1.3 is straightforward. For example,

$$\begin{aligned} \bar{\rho}_n &= q \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} a^{-3} \left| \psi_1(a^{-1}|\mathbf{x} - \mathbf{r}|) \right|^2 d\mathbf{x} \\ &= q \int_{\Omega(\mathbf{0}, R_n/a)} \psi_1^2(|\mathbf{y}|) d\mathbf{y} \rightarrow q \int_{\mathbb{R}^3} \psi_1^2(|\mathbf{y}|) d\mathbf{y}, \end{aligned} \quad (34.78)$$

where $R_n/a \rightarrow \infty$ according to (34.17), and we obtain (34.33) and (34.34). To estimate integral (34.32), we note that, according to the definition of the momentum density (11.163), (11.154),

$$\mathbf{P} = i\chi \frac{1}{2} \left[\frac{\nabla \psi^*}{\psi^*} - \frac{\nabla \psi}{\psi} + 2 \frac{iq}{\chi c} \mathbf{A} \right] \psi^* \psi, \quad (34.79)$$

and for the wave-corpuscles

$$\mathbf{P} = \chi \left[\nabla \mathfrak{S} - \frac{q}{\chi c} \mathbf{A} \right] \psi^2. \quad (34.80)$$

Since the phase $\mathfrak{S}$ in (16.61) is a smooth function which does not depend on a , and $\mathbf{A}$ satisfies (34.25), the estimate (34.32) follows from (34.78). Using (34.80) and (34.85), we obtain (34.41):

$$|Q_{01}| \leq |t - t_0| \max_{T_- \leq s \leq T_+} \left| \partial_t \hat{\mathbf{r}} \right| \max_{T_- \leq s \leq T_+} \int_{\partial\Omega_n} |\mathbf{P}| d\sigma \rightarrow 0. \quad (34.81)$$

Similarly, we obtain (34.42), (34.43) and (34.44). Note that (34.30) holds, hence

$$|\mathbf{E}_n(0) - \mathbf{E}_\infty(0)| = |\mathbf{E}(0) - \mathbf{E}_\infty(0)| = 0, \quad (34.82)$$

$$|\mathbf{B}_n(0) - \mathbf{B}_\infty(0)| = |\mathbf{B}(0) - \mathbf{B}_\infty(0)| = 0. \quad (34.83)$$

Using the continuity of $\mathbf{E}(t, \mathbf{y})$, $\mathbf{B}(t, \mathbf{y})$, we conclude that $|\mathbf{E}_n - \mathbf{E}_\infty| \rightarrow 0$ and $|\mathbf{B}_n - \mathbf{B}_\infty| \rightarrow 0$ in Ω_n since $R_n \rightarrow 0$. Therefore fulfillment of relation (34.47) follows from (34.78) and (34.80).

To obtain limit (34.40), we split the tensor T^{ij} into diagonal and non-diagonal parts presented, respectively, by (11.164) and (11.165):

$$T^{ij} = T_{i \neq j}^{ij} + T_{i=j}^{ij}. \quad (34.84)$$

According to (34.35),

$$Q_0 = Q_{0,\neq} + Q_{0,=},$$

where

$$Q_{0,\neq} = \int_{t_0}^t \int_{\partial\Omega_n} \bar{\mathbf{n}}_i T_{i\neq j}^{ij} \, d\sigma dt', \quad Q_{0,=} = \int_{t_0}^t \int_{\partial\Omega_n} \bar{\mathbf{n}}_i T_{i=j}^{ij} \, d\sigma dt'.$$

To estimate $|Q_{0,\neq}|$, it is sufficient, according to (11.165), to prove that

$$\max_{T_- \leq t \leq T_+} \int_{\partial\Omega_n} \left(\frac{\chi^2}{2m} |\nabla \psi(t, \mathbf{x})|^2 + |\psi(t, \mathbf{x})|^2 \right) d\sigma \rightarrow 0. \quad (34.85)$$

Using this estimate and (11.165), we obtain that

$$|Q_{0,\neq}| \leq |t - t_0| \max_{T_- \leq s \leq T_+} \int_{\partial\Omega_n} |T_{i\neq j}^{ij}| \, d\sigma \rightarrow 0. \quad (34.86)$$

To estimate $Q_{0,=}$, we use (11.164) and (11.161):

$$\begin{aligned} \int_{\partial\Omega_n} \bar{\mathbf{n}}_i T^{ii} d\sigma &= - \int_{\partial\Omega_n} \frac{\chi^2}{m} \bar{\mathbf{n}}_i \tilde{\partial}_i \psi \tilde{\partial}_i^* \psi^* d\sigma \\ &+ \int_{\partial\Omega_n} \left[\frac{\chi^2}{2m} \left(G(|\psi|^2) + |\tilde{\nabla} \psi|^2 \right) + i \frac{\chi}{2} \left(\psi \tilde{\partial}_i^* \psi^* - \psi^* \tilde{\partial}_i \psi \right) \right] \sum_i \bar{\mathbf{n}}_i d\sigma. \end{aligned} \quad (34.87)$$

Since $|\psi|^2 = |\tilde{\psi}|^2$ is a radial function, and $\bar{\mathbf{n}}_i = y_i / |\mathbf{y}|$ is odd with respect to y_i -reflection, we see that in the above integral

$$\int_{\partial\Omega_n} G(|\psi|^2) \bar{\mathbf{n}}_i d\sigma = 0. \quad (34.88)$$

We also note that

$$\begin{aligned} i \frac{\chi}{2} \left(\psi \tilde{\partial}_i^* \psi^* - \psi^* \tilde{\partial}_i \psi \right) &= i \frac{\chi}{2} \left(\psi \partial_i \psi^* - \psi^* \partial_i \psi \right) + q \varphi |\psi|^2 \\ &= \chi |\psi|^2 \partial_i \mathfrak{S} + q \varphi |\psi|^2, \end{aligned} \quad (34.89)$$

where $\partial_i \mathfrak{S}$ and φ are bounded functions. Under the assumption (34.85), straightforward estimates produce that

$$i \frac{\chi}{2} \int_{\partial\Omega_n} \left(\psi \tilde{\partial}_i^* \psi^* - \psi^* \tilde{\partial}_i \psi \right) d\sigma \rightarrow 0,$$

$$\int_{\partial\Omega_n} \frac{\chi^2}{m} \bar{\mathbf{n}}_i \tilde{\partial}_i \psi \tilde{\partial}_i^* \psi^* d\sigma \rightarrow 0,$$

$$\int_{\partial\Omega_n} \frac{\chi^2}{2m} |\tilde{\nabla} \psi|^2 d\sigma \rightarrow 0$$

uniformly on $[T_-, T_+]$. Hence, if (34.85) holds, we obtain that

$$Q_{0,=} \rightarrow 0, \quad (34.90)$$

and taking into account (34.86) we conclude that (34.40) holds.

Let us prove now that (34.85) holds under certain decay conditions. Since the phase Θ in (16.61) is bounded and has bounded derivatives in $\Omega(\hat{\mathbf{r}}(t), R_n)$, it is sufficient to estimate the surface integrals of $|\nabla \psi|$ and $|\psi|^2$ to obtain (34.85). Obviously,

$$\begin{aligned} & \int_{\partial\Omega(\hat{\mathbf{r}}(t), R_n)} a^{-3} \left| \nabla \psi_1 (a^{-1}|\mathbf{x} - \mathbf{r}|) \right|^2 d\sigma \\ &= 4\pi R_n^2 a^{-5} \left| \psi_1' (R_n/a) \right|^2 = 4\pi a R_n^{-4} \theta^6 \left| \psi_1' (\theta) \right|^2, \end{aligned} \quad (34.91)$$

$$\begin{aligned} & \int_{\partial\Omega(\hat{\mathbf{r}}(t), R_n)} a^{-3} \left| \psi_1 (a^{-1}|\mathbf{x} - \mathbf{r}|) \right|^2 d\sigma \\ &= 4\pi a^{-3} R_n^2 \left| \psi_1 (R_n/a) \right|^2 = 4\pi a R_n^{-2} \theta^4 \left| \psi_1 (\theta) \right|^2, \end{aligned} \quad (34.92)$$

where $R_n/a = \theta \rightarrow \infty$, $R_n \rightarrow 0$. We assume that the form factor $\psi_1(r)$ and its derivative $\psi_1'(r)$ satisfy the following decay conditions:

$$\theta^2 \left| \psi_1 (\theta) \right| \leq C \quad \text{as } \theta \rightarrow \infty, \quad (34.93)$$

$$\theta^3 \left| \psi_1' (\theta) \right| \leq C \quad \text{as } \theta \rightarrow \infty, \quad (34.94)$$

and we take such a_n, R_n that

$$a_n R_n^{-4} \rightarrow 0, \quad a_n \rightarrow 0, \quad R_n \rightarrow 0. \quad (34.95)$$

Under these assumptions, we conclude that (34.85) is fulfilled.

Combining the above arguments, we obtain the following statement:

Theorem 34.1.4 *Let ψ_1 satisfy (34.93), (34.94) and a_n, R_n satisfy (34.95). Let $\psi = \psi_n$ be wave-corpucle solutions of the NLS equation constructed in Theorem 16.1.1. Then the sequence ψ_n concentrates at the trajectory $\hat{\mathbf{r}}(t) = \mathbf{r}(t)$.*

34.2 Concentration of Asymptotic Solutions

One can see from the proofs of Sect. 34.1 that we use in the derivation of the Newtonian dynamics only the conservation laws (11.158) and (11.169). Since we use only asymptotic properties, it is natural to consider fields which satisfy the NLS equations and the conservation laws not exactly, but approximately. Namely, we assume now that the conservation laws (11.158) and (11.169) are replaced by

$$\partial_t \rho + \partial_t \rho' + \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J}' = 0, \quad (34.96)$$

$$\partial_t \mathbf{P} + \partial_t \mathbf{P}' + \partial_i T^{ij} + \partial_i T^{ij'} = \mathbf{f} + \mathbf{f}', \quad (34.97)$$

where the charge density ρ , current density $\mathbf{J}$ and tensor elements T^{ij} are defined by (11.156), (11.157) and (11.165) in terms of given functions $\psi, \varphi, \mathbf{A}$, and $\rho', \mathbf{J}', \mathbf{P}', T^{ij'}$ and $\mathbf{f}'$ are perturbation terms which vanish as $n \rightarrow \infty$. Now we modify the proofs of the statements in Sect. 34.1 concerning concentrating solutions to obtain similar statements for concentrating asymptotic solutions.

In the proof of Lemma 34.1.3, Eq. (34.57) is replaced by (34.97). This leads to replacement of the Eq. (34.59) by the equation

$$\begin{aligned} \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \mathbf{P}(t) \, d^3x - \int_{\Omega(\hat{\mathbf{r}}(t_0), R_n)} \mathbf{P}(t_0) \, d^3x - Q_{01} \\ - \int_{t_0}^t \int_{\Omega_n} \mathbf{f} \, d\mathbf{x} dt' + Q_0 + Q'_0 = 0 \end{aligned} \quad (34.98)$$

with

$$\begin{aligned} Q'_0 = \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \mathbf{P}'(t) \, d^3x - \int_{\Omega(\hat{\mathbf{r}}(t_0), R_n)} \mathbf{P}'(t_0) \, d^3x \\ - \int_{t_0}^t \int_{\partial\Omega_n} \mathbf{P}' \hat{\mathbf{v}} \cdot \hat{\mathbf{n}} \, d\sigma dt' + \int_{t_0}^t \int_{\Omega_n} \partial_i T^{ij'} \, d\mathbf{x} dt' - \int_{t_0}^t \int_{\Omega_n} \mathbf{f}' \, d\mathbf{x} dt'. \end{aligned} \quad (34.99)$$

If

$$Q'_0 \rightarrow 0, \quad (34.100)$$

the proof of Lemma 34.1.3 remains valid with $Q_{00} = Q_0 + Q'_0 - Q_{01}$, and we obtain the following lemma:

Lemma 34.2.1 *Let the conservation law (11.169) be replaced by (34.97) with condition (34.100) satisfied. Then the statement of Lemma 34.1.3 remains true:*

$$\int_{\Omega_n} \mathbf{P}_n(t) \, d\mathbf{x} = \int_{t_0}^t \int_{\Omega_n} \mathbf{f} \, d\mathbf{x} dt' + \mathbf{p}_\infty + Q_{00}, \quad (34.101)$$

where

$$Q_{00} \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (34.102)$$

uniformly on $[T_-, T_+]$.

Sufficient conditions for fulfillment of (34.100) obviously are the following assumptions:

$$\int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \mathbf{P}'(t) \, d\mathbf{x} \rightarrow 0, \quad (34.103)$$

$$\int_{t_0}^t \int_{\partial\Omega_n} \mathbf{P}' \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \, d\sigma dt' \rightarrow 0, \quad (34.104)$$

$$\int_{t_0}^t \int_{\partial\Omega_n} \bar{\mathbf{n}}_i T^{ij'} \, d\sigma dt' \rightarrow 0, \quad (34.105)$$

$$\int_{t_0}^t \int_{\Omega_n} \mathbf{f}' \, d\mathbf{x} dt' \rightarrow 0 \quad (34.106)$$

uniformly on the time interval $[T_-, T_+]$.

Now let us take a look at the proof of Lemma 34.1.4. Since we have replaced the continuity equation (11.158) by (34.96), the Eq. (34.64) now involves additional terms:

$$\begin{aligned} & \int_{\Omega_n} \partial_t ((\mathbf{x} - \mathbf{r}) \rho) \, d\mathbf{x} + \partial_t \mathbf{r} \int_{\Omega_n} \rho \, d\mathbf{x} \\ & + \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \bar{\mathbf{n}} \cdot \mathbf{J} \, d\mathbf{x} + Q'_2 = \int_{\Omega_n} \mathbf{J} \, d\mathbf{x} \end{aligned}$$

with

$$\begin{aligned} Q'_2 &= \int_{\Omega_n} \partial_t ((\mathbf{x} - \mathbf{r}) \rho') \, d\mathbf{x} + \partial_t \mathbf{r} \int_{\Omega_n} \rho' \, d\mathbf{x} \\ &+ \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \bar{\mathbf{n}} \cdot \mathbf{J}' \, d\mathbf{x} - \int_{\Omega_n} \mathbf{J}' \, d\mathbf{x}. \end{aligned} \quad (34.107)$$

We assume that

$$Q'_2 \rightarrow 0 \quad (34.108)$$

and conclude that the statement of Lemma 34.1.4 holds:

Lemma 34.2.2 *Let the continuity equation (11.158) be replaced by (34.96) with condition (34.108) fulfilled. Then the statement of Lemma 34.1.4 is true.*

Note that, according to (34.65), sufficient conditions for fulfillment of (34.108) are as follows:

$$\int_{\Omega_n} \rho' \, d\mathbf{x} \rightarrow 0, \quad (34.109)$$

$$\int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \, \bar{\mathbf{n}} \cdot \mathbf{J}' \, d\mathbf{x} \rightarrow 0, \quad (34.110)$$

$$\int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \, \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho' \, d\sigma \rightarrow 0. \quad (34.111)$$

Now let us consider the proof of Lemma 34.1.2. Equation (34.53) is replaced by

$$\bar{\rho}(t) - \bar{\rho}(t_0) - \int_{t_0}^t \int_{\partial\Omega_n} \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho \, d\mathbf{x} dt' + \int_{t_0}^t \int_{\partial\Omega_n} \bar{\mathbf{n}} \cdot \mathbf{J} \, d\mathbf{x} dt' + Q'_3 = 0, \quad (34.112)$$

where

$$Q'_3 = \bar{\rho}'(t) - \bar{\rho}'(t_0) - \int_{\partial\Omega_n} \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho' \, d\mathbf{x} + \int_{\partial\Omega_n} \bar{\mathbf{n}} \cdot \mathbf{J}' \, d\mathbf{x}. \quad (34.113)$$

The proof of Lemma 34.1.2 is preserved if we assume that

$$Q'_3 \rightarrow 0. \quad (34.114)$$

Lemma 34.2.3 *Let the continuity equation (11.158) be replaced by (34.96) with condition (34.114) fulfilled. Then the statement of Lemma 34.1.2 is true.*

Now we would like to collect the assumptions required to make the statements, which were formulated for exact solutions in Sect. 34.1, valid for asymptotic solutions. Namely, we introduce the following definition:

Definition 34.2.1 (*Concentrating asymptotic solutions*) We assume all conditions of Definition 34.1.3 except condition (i), which is replaced by the following weaker condition: asymptotic conservation laws (34.97) and (34.96) are fulfilled and conditions (34.108), (34.100), (34.114) hold. Then we say that asymptotic solutions of the NLS equation concentrate at $\hat{\mathbf{r}}(t)$. We call $\hat{\mathbf{r}}(t)$ a concentration trajectory of the NLS equation in the asymptotic sense if there exists a sequence of asymptotic solutions which concentrates at $\hat{\mathbf{r}}(t)$.

Using Lemmas 34.2.1 and 34.2.2 instead of Lemmas 34.1.3 and 34.1.4, we obtain statements of Theorems 34.1.2 and 34.1.3 under the assumption that asymptotic solutions ψ of the NLS equation (11.153) concentrate at $\hat{\mathbf{r}}(t)$. In particular, we obtain:

Theorem 34.2.1 *Assume that potentials $\varphi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$ are twice continuously differentiable in a domain $D \subset \mathbb{R} \times \mathbb{R}^3$, the trajectory $(t, \hat{\mathbf{r}}(t))$ lies in this domain and the limit potentials φ_∞ , $\mathbf{A}_\infty$ are the restriction of fixed potentials φ , $\mathbf{A}$ as in (34.28). Let EM fields $\mathbf{E}(t, \mathbf{x})$, $\mathbf{B}(t, \mathbf{x})$ be determined in terms of the potentials by formula (4.6). Let asymptotic solutions ψ of the NLS equation (11.153) concentrate at $\hat{\mathbf{r}}(t)$. Then the trajectory $\hat{\mathbf{r}}$ satisfies Newton's law of motion (34.77).*

34.2.1 Point Trajectories as Trajectories of Asymptotic Concentration

Let us consider the NLS equation (11.153) in a domain $D \subset \mathbb{R} \times \mathbb{R}^3$, and assume that potentials $\varphi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$ are defined and twice continuously differentiable in the domain D . Let us also consider equations (16.39)–(16.40) which describe Newtonian dynamics of a point charge in an EM field, as well as a solution $\mathbf{r}(t)$ of equations (16.39)–(16.40), and assume that the trajectory $(t, \mathbf{r}(t))$ lies in D during the time interval $T_- \leq t \leq T_+$. Now we can construct asymptotic solutions of the NLS which concentrate at $\mathbf{r}(t)$ as follows.

As a first step, we find the linear part of $\varphi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$ at $\mathbf{r}(t)$ as in (34.28):

$$\begin{aligned}\varphi_\infty(t, \mathbf{x}) &= \varphi(t, \mathbf{r}) + (\mathbf{x} - \mathbf{r}) \nabla \varphi(t, \mathbf{r}), \\ \mathbf{A}_\infty(t, \mathbf{x}) &= \mathbf{A}(t, \mathbf{r}) + (\mathbf{x} - \mathbf{r}) \nabla \mathbf{A}(t, \mathbf{r}).\end{aligned}\quad (34.115)$$

As a second step, we construct the auxiliary potentials

$$\begin{aligned}\varphi_{\text{aux}}(t, \mathbf{x}) &= \varphi(t, \mathbf{r}) + (\mathbf{x} - \mathbf{r}) \nabla \varphi(t, \mathbf{r}) + \varphi_2(t, \mathbf{x} - \mathbf{r}), \\ \mathbf{A}_{\text{aux}}(t, \mathbf{x}) &= \mathbf{A}(t, \mathbf{r}) + (\mathbf{x} - \mathbf{r}) \nabla \mathbf{A}(t, \mathbf{r}),\end{aligned}\quad (34.116)$$

where φ_2 is determined by (16.57) and $\varphi_2(t, \mathbf{x} - \mathbf{r})$ is quadratic with respect to $(\mathbf{x} - \mathbf{r})$.

The wave-corpuscle ψ described in Theorem 16.1.1 is a solution to the NLS equation with the potentials $(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}})$, therefore it exactly satisfies corresponding conservation laws. From fulfillment of (11.158) and (11.169) for $\mathbf{P}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}})$, $\mathbf{J}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}})$, $T^{ij}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}})$, $\mathbf{f}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}})$ we obtain fulfillment of (34.96), (34.97) with

$$\mathbf{P}' = \mathbf{P}(\varphi, \mathbf{A}) - \mathbf{P}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}}), \quad (34.117)$$

$$\mathbf{J}' = \mathbf{J}(\varphi, \mathbf{A}) - \mathbf{J}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}}), \quad (34.118)$$

$$\mathbf{f}' = \mathbf{f}(\varphi, \mathbf{A}) - \mathbf{f}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}}), \quad (34.119)$$

$$T^{ij'} = T^{ij}(\varphi, \mathbf{A}) - T^{ij}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}}). \quad (34.120)$$

The expression for ρ does not depend on the potentials, therefore

$$\rho' = 0. \quad (34.121)$$

Now we need to verify that conditions (34.108), (34.100), (34.114) hold for the wave-corpuscle ψ .

Using (11.163), (34.80), we conclude that

$$\mathbf{P}(\varphi, \mathbf{A}) - \mathbf{P}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}}) = -\frac{q}{c}(\mathbf{A} - \mathbf{A}_{\text{aux}})\psi^*\psi, \quad (34.122)$$

and, according to (11.166),

$$\mathbf{J}(\varphi, \mathbf{A}) - \mathbf{J}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}}) = -\frac{q^2}{mc}(\mathbf{A} - \mathbf{A}_{\text{aux}})\psi^*\psi. \quad (34.123)$$

We can obtain then from expression (34.14) for the Lorentz density the following equality:

$$\begin{aligned} \mathbf{f}(\varphi, \mathbf{A}) - \mathbf{f}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}}) &= \rho(\mathbf{E} - \mathbf{E}_{\text{aux}}) + \frac{1}{c}(\mathbf{J} \times \mathbf{B} - \mathbf{J}_{\text{aux}} \times \mathbf{B}_{\text{aux}}) \\ &= \rho(\mathbf{E} - \mathbf{E}_{\text{aux}}) + \frac{1}{c}((\mathbf{J} - \mathbf{J}_{\text{aux}}) \times \mathbf{B} + \mathbf{J}_{\text{aux}} \times (\mathbf{B} - \mathbf{B}_{\text{aux}})). \end{aligned}$$

Using (34.80) and (11.166), we rewrite expression for $\mathbf{f}'$ in the form

$$\begin{aligned} \mathbf{f}' &= \rho(\mathbf{E} - \mathbf{E}_{\text{aux}}) \\ &\quad - \frac{q^2}{mc^2}\psi^2(\mathbf{A} - \mathbf{A}_{\text{aux}}) \times \mathbf{B} + \frac{\chi q}{mc}\psi^2 \left[\nabla \mathfrak{S} - \frac{q}{\chi c}\mathbf{A}_{\text{aux}} \right] \times (\mathbf{B} - \mathbf{B}_{\text{aux}}). \end{aligned} \quad (34.124)$$

The difference of tensor elements (34.120) can be estimated based on relations (11.165) and (11.164). Note that, according to the construction of φ_{aux} and $\mathbf{A}_{\text{aux}}$, the differences $\varphi - \varphi_{\text{aux}}$ and $\mathbf{A} - \mathbf{A}_{\text{aux}}$ have zeros of second order at $\mathbf{r}(t)$. Consequently, the differences $\mathbf{E} - \mathbf{E}_{\text{aux}}$ and $\mathbf{B} - \mathbf{B}_{\text{aux}}$ have zero of first order at $\mathbf{x} = \mathbf{r}$:

$$|\mathbf{A} - \mathbf{A}_{\text{aux}}| \leq C|\mathbf{x} - \mathbf{r}|^2, \quad |\varphi - \varphi_{\text{aux}}| \leq C|\mathbf{x} - \mathbf{r}|^2, \quad (34.125)$$

$$|\mathbf{E} - \mathbf{E}_{\text{aux}}| \leq C|\mathbf{x} - \mathbf{r}|, \quad |\mathbf{B} - \mathbf{B}_{\text{aux}}| \leq C|\mathbf{x} - \mathbf{r}|, \quad (34.126)$$

and they are vanishingly small in $\Omega(\mathbf{r}(t), R_n)$.

Now let us estimate terms that enter (34.99), (34.107), (34.113).

Lemma 34.2.4 *Let $\mathbf{P}'$, $\mathbf{J}'$, $\mathbf{f}'$, $T^{ij'}$ be defined by relations (34.117)–(34.120), with conditions (34.93), (34.94) and (34.95) being fulfilled. Then relations (34.103)–(34.106) and (34.109)–(34.111) and (34.114) hold.*

Proof The proof is based on inequalities (34.126) and (34.125). To obtain the limit relation (34.103), we use (34.122) and (34.78):

$$\begin{aligned} \left| \int_{\Omega(\mathbf{r}(t), R_n)} \mathbf{P}'(t) \, d\mathbf{x} \right| &\leq C \int_{\Omega(\mathbf{r}(t), R_n)} |\mathbf{x} - \mathbf{r}|^2 \psi_a^2(|\mathbf{x} - \mathbf{r}|) \, d\mathbf{x} \\ &\leq C R_n^2 \int_{\Omega(\mathbf{0}, R_n/a)} \psi_a^2(|\mathbf{y}|) \, d\mathbf{y} \leq C_1 R_n^2, \end{aligned}$$

implying (34.103) for $R_n \rightarrow 0$. To obtain the limit relation (34.104), we observe that, according to (34.122),

$$\begin{aligned} \left| \int_{\partial\Omega(\mathbf{r}(t), R_n)} \mathbf{P}' \hat{\mathbf{v}} \cdot \hat{\mathbf{n}} \, d\sigma \right| &\leq C \int_{\partial\Omega(\mathbf{r}(t), R_n)} |\mathbf{x} - \mathbf{r}|^2 \psi_a^2(|\mathbf{x} - \mathbf{r}|) \, d\sigma \quad (34.127) \\ &= 4\pi C R_n^4 a^{-3} \psi_1^2(R_n/a) = 4\pi C R_n \theta^3 \psi_1^2(\theta) \rightarrow 0. \end{aligned}$$

A straightforward estimate of (34.124) yields (34.106). Note that, according to (11.164) and (11.161), the terms in T^{ij} that involve G do not depend on $(\varphi, \mathbf{A})$, hence G does not enter $T^{ij'}$. According to (11.165), (11.164), (34.89) and (11.154), $T^{ij}(\varphi, \mathbf{A})$ is a quadratic function of the EM potentials, and $T^{ij'}$ equals the sum of terms every one of which involves the factors $\mathbf{A} - \mathbf{A}_{\text{aux}}$ or $\varphi - \varphi_{\text{aux}}$ satisfying (34.125), and also the factors $\psi \nabla \psi^*$ or $\nabla \psi \psi^*$ or $\psi \psi^*$. The factors $\mathbf{A} - \mathbf{A}_{\text{aux}}$ or $\varphi - \varphi_{\text{aux}}$ in $\Omega(\mathbf{r}(t), R_n)$ produce the coefficient $C R_n^2$. All the terms are easily estimated, for example an elementary inequality

$$\int_{\partial\Omega(\mathbf{r}(t), R_n)} \left| \psi_a \right| \left| \nabla \psi_a \right| \, d\sigma \leq \frac{1}{2} \int_{\partial\Omega(\mathbf{r}(t), R_n)} \left| \psi_a \right|^2 + \left| \nabla \psi_a \right|^2 \, d\sigma$$

allows us to use (34.91)–(34.92), and we obtain

$$\int_{\partial\Omega(\mathbf{r}(t), R_n)} \left| \psi_a \right| \left| \nabla \psi_a \right| \, d\sigma \rightarrow 0.$$

Therefore (34.105) is fulfilled. Since $\rho' = 0$, conditions (34.109) and (34.111) are fulfilled, and (34.110) follows from (34.103) and (11.166). To verify condition (34.114), we note that $\rho' = 0$, and using (11.166) we estimate the surface integral in (34.113) involving $\mathbf{J}'$ similarly to (34.127).

The above lemmas and Theorem 34.2.1 imply the following theorem:

Theorem 34.2.2 *Let potentials $\varphi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$ be twice continuously differentiable in a domain D . Let the form factor ψ_1 satisfy conditions (34.93), (34.94). Let*

sequences a_n, R_n satisfy (34.95). Let $\mathbf{r}(t)$ be a solution of equations (16.39)–(16.40) with trajectory $(t, \mathbf{r}(t))$ in D . Then the wave-corpuscles constructed in Theorem 34.1.4 based on $\mathbf{r}(t)$ and potentials $\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}}$ given by (34.116) provide asymptotic solutions in the sense of Definition 34.2.1 that concentrate at $\mathbf{r}(t)$.

As a corollary, we obtain the following theorem:

Theorem 34.2.3 *Let $\varphi(t, \mathbf{x}), \mathbf{A}(t, \mathbf{x})$ be defined and twice continuously differentiable in a domain D . Then a trajectory $\mathbf{r}(t)$ which lies in D is a concentration trajectory of asymptotic solutions of the NLS equation (11.153) if and only if it satisfies Newton's law (16.39) with the Lorentz force (16.40).*

Since asymptotic solutions in the sense of Definition 34.2.1 in Theorem 34.2.2 are constructed as wave-corpuscles, we obtain the following corollary.

Corollary 34.2.1 *If a sequence of concentrating asymptotic solutions concentrates at a trajectory $\mathbf{r}(t)$, then there exists a sequence of concentrating wave-corpuscle asymptotic solutions which concentrate at the same trajectory.*

34.3 Trajectories of Concentration in Relativistic Field Dynamics

In this section we consider trajectories of concentration in the relativistic context. Our goal in this section is to provide a rigorous proof of Einstein's relation $E = Mc^2$ in the framework of classical field theory, namely in the framework of the theory of a balanced charge in an external EM field. We mainly follow our paper [19]; the basics of the relativistic dynamics of a mass point and the relativistic field theory are discussed in Sect. 17.6.2.

34.3.1 Rigorous Derivation of Einstein's Formula for a Balanced Charge

The idea that a particle can be viewed as a field excitation carrying a certain amount of energy is a rather old one. Einstein and Infeld wrote, [105, p. 257]:

“What impresses our senses as matter is really a great concentration of energy into a comparatively small space. We could regard matter as the regions in space where the field is extremely strong.”

But implementation of this idea in a mathematically sound theory is a challenging problem, and it is the subject of this section.

Let us turn first to the relativistic dynamics of a mass point which accelerates under the action of a force $\mathbf{f}(t, \mathbf{r})$. This dynamics is governed by the relativistic version (17.114) of Newton's equation [27], [265], [248]:

$$\partial_t (M \mathbf{v}(t)) = \mathbf{f}(t, \mathbf{r}), \quad \mathbf{v}(t) = \partial_t \mathbf{r}(t), \quad M = m_0 \gamma.$$

Note that in (17.114) the rest mass m_0 is prescribed as an intrinsic property of the mass point, for the mass point does not have internal degrees of freedom. The Eq. (17.114), just as in the case of classical Newtonian mechanics, suggests that the mass M is a measure of inertia, that is, the coefficient that relates the acceleration to the known force $\mathbf{f}$. In contrast, Einstein's relation between mass and energy in a relativistic field theory is usually defined as follows (see for instance, [4, 7.1–7.5], [27], [248, 3.1–3.3, 3.5], [265, 37], [304, 4.1], see also Sect. 11.6.1). A closed relativistic system possesses a total momentum and energy which form a 4-vector. The total momentum has a simple form $\mathbf{P} = M \mathbf{v}$ where the constant velocity $\mathbf{v}$ originates from the corresponding parameter of the Lorentz group. Then one can naturally define and interpret the mass M for a closed relativistic system as the coefficient of proportionality between momentum $\mathbf{P}$ and velocity $\mathbf{v}$. So, we can conclude that, according to the relativity principles, the rest mass is naturally defined for a uniform motion, but in Newtonian mechanics the concept of inertial mass is introduced through an accelerated motion. Note that the relativistic and non-relativistic masses are sometimes considered to be “rival and contradictory”, [111].

The principal problem we want to take on here is as follows. We would like to construct a field model of a charge where the internal energy of a localized wave affects its acceleration in an external field the same way the inertial mass does in Newtonian mechanics. We could not find such a model in the literature, so we introduce and study it here. The model allows us, in particular, to consider in the same framework uniform motion in absence of external forces and accelerated motion in an external field. Hence, within the same framework (but in different regimes), we can determine the mass either from the analysis of the 4-vector of the energy-momentum or using the Newtonian approach. When both the approaches are relevant, they agree, see Remark 17.6.1.

The proposed model is based on the relativistic theory of balanced charges, see Sect. 17.6. Namely, the state of a single charge is described by a complex-valued scalar field (wave function) $\psi(t, \mathbf{x})$ of the time t and the position vector $\mathbf{x} \in \mathbb{R}^3$, and its time evolution is governed by the following nonlinear Klein–Gordon (NKG) equation (11.125):

$$-\frac{1}{c^2} \tilde{\partial}_t^2 \psi + \tilde{\nabla}^2 \psi - G'(|\psi|^2) \psi - \frac{m^2 c^2}{\chi^2} \psi = 0 \quad (34.128)$$

where m is a positive mass parameter, and χ is a constant which in physical applications coincides with (or is close to) the Planck constant $\hbar$. The covariant derivatives in Eq. (34.128) are defined by (11.123), namely

$$\tilde{\partial}_t = \partial_t + \frac{iq}{\chi} \varphi, \quad \tilde{\nabla} = \nabla - \frac{iq}{\chi c} \mathbf{A}$$

where q is the value of the charge, and $\varphi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$ are the potentials of the external EM field. The external EM field acts upon the charge, and we will see that it results ultimately in the Lorentz force when Newton's point mass equation is relevant. In this chapter, for simplicity, we treat the case where only electric forces are present, and the magnetic potential is set to be zero, that is,

$$\mathbf{A} = \mathbf{0}, \quad \tilde{\nabla} = \nabla. \quad (34.129)$$

A less detailed treatment of the case with a non-zero external magnetic field is provided in [18] and Sect. 17.6.2.

The nonlinear term G' in the NKG (34.128) provides for the existence in infinite time intervals of localized solutions for resting or uniformly moving charges. The nonlinearity G' and its properties are considered in Sect. 14.3. Importantly, the nonlinearity G' involves a size parameter a that determines the spatial scale of the charge when at rest.

A sketch of our line of argument is as follows. We derive from the Lagrangian of the NKG equation (34.128) the standard expression (11.142) for the energy density which allows us to define the energy involved in Einstein's formula. To be able to apply the Newtonian approach, it is necessary to relate point trajectories $\mathbf{r}(t)$ to wave solutions ψ and derive point dynamics from the field equations. The latter is accomplished based on the concept of solutions "concentrating at a trajectory".

Roughly speaking (see Sect. 34.5.1 for the exact definitions), solutions of the NKG equation concentrate at a given trajectory $\hat{\mathbf{r}}(t)$ if their energy densities $u(\mathbf{x}, t)$ restricted to R_n -neighborhoods of $\hat{\mathbf{r}}(t)$ locally converge to the delta-function $E(t) \delta(\mathbf{x} - \hat{\mathbf{r}}(t))$ as $R_n \rightarrow 0$. A concise formulation of the main result, Theorem 34.5.1, is as follows. We prove that *if a sequence of solutions of the NKG equation concentrates at a trajectory $\hat{\mathbf{r}}(t)$, then the restricted energy $\tilde{E}_n$ of the solutions converges to a function $\tilde{E}(t)$ so that the following relativistic version of Newton's equation holds:*

$$\partial_t \left(\frac{\tilde{E}}{c^2} \hat{\mathbf{v}} \right) = \mathbf{f}(t, \hat{\mathbf{r}}), \quad \hat{\mathbf{v}} = \partial_t \hat{\mathbf{r}}. \quad (34.130)$$

The electric force $\mathbf{f}$ in (34.130) is defined by the formula

$$\mathbf{f}(t, \hat{\mathbf{r}}) = -\bar{\rho} \nabla \varphi(t, \hat{\mathbf{r}}), \quad (34.131)$$

where φ is the electric potential as in the NKG equation, $\bar{\rho}$ is a constant describing the limit charge. The limit restricted energy $\tilde{E}(t)$ satisfies the relation

$$\frac{\tilde{E}(t)}{c^2} = M_0 \gamma, \quad \gamma = (1 - \hat{\mathbf{v}}^2/c^2)^{-1/2}, \quad (34.132)$$

where M_0 is a constant which plays the role of a generalized rest mass. Observe that Eq. (34.130) takes the form of the relativistic version of Newton's law if the mass is

defined by Einstein's formula $M(t) = \frac{1}{c^2} \bar{\mathcal{E}}(t)$. The relation between the generalized rest mass M_0 and the rest mass m_0 of resting solutions is discussed in Remark 17.6.1. Note that the same NKG equation (34.128) with the same value of the mass parameter m has rest solutions with different energies, and consequently different rest masses, see Sects. 14.1 and 15.2.

Therefore we can make the following conclusion: *in the framework of the NKG field theory, the relativistic material point dynamics is represented by concentrating solutions of the NKG equation with the mass determined by Einstein's formula in terms of the limit restricted energy of the solutions.*

We can add to the above outline a few guiding points concerning the mathematical aspects of our approach. Equation (34.130) derived in Theorem 34.5.1 produces a *necessary condition* for solutions of the NKG equation to concentrate at a trajectory and reveals their asymptotic point-like dynamics. To obtain this necessary condition, we have to derive the point dynamics governed by an ordinary differential equation (34.130) from the dynamics of waves governed by a partial differential equation. The concept of “*concentration of functions at a given trajectory $\hat{\mathbf{r}}(t)$* ”, see Definition 34.5.2 below, is the first step in relating spatially localized fields ψ to point trajectories. The definition of concentration of functions has a sufficient flexibility to allow for general regular trajectories $\hat{\mathbf{r}}(t)$, and plenty of functions are localized about the trajectory, see Example 34.5.1. But if a sequence of functions concentrating at a given trajectory *are also solutions of the NKG equation, then, according to Theorem 34.5.1, the trajectory and the limit energy must satisfy the relativistic Newton's equations together with Einstein's formula.* To derive the Eqs. (34.130) and (34.132), we introduce the energy $\tilde{\mathcal{E}}_n(t)$ restricted to a narrow tubular neighborhood of the trajectory with radius R_n and define adjacent ergocenters $\mathbf{r}_n(t)$ of the concentrating solutions; then we infer integral equations for the restricted energy and adjacent ergocenters from the energy and momentum conservation laws and the continuity equation, and then we pass to the limit as $R_n \rightarrow 0$, see Theorems 34.5.2 and 34.5.3 and the proof of Theorem 34.5.1. The intrinsic length scales a and $a_C = \frac{\lambda}{mc}$ of concentrating solutions are much smaller than the radius R_n , namely $R_n/a \rightarrow \infty$, $R_n/a_C \rightarrow \infty$. The determination of the restricted energy and charge and adjacent ergocenters involves integration over a large (relative to a and a_C) spatial domain of radius R_n . Therefore, Eq. (34.130) inherits integral, non-local characteristics of concentrating solutions which cannot be reduced to their behavior at the trajectory, namely the limit restricted energy $\bar{\mathcal{E}}(t)$ and restricted charge $\bar{\rho}_\infty$. Therefore it is natural to call our method of determination of the point trajectories *semi-local*. This semi-local feature allows us to capture Einstein's relation between mass and energy.

In many problems of physics and mathematics, a common way to establish a relation between point dynamics and wave dynamics is by means of the WKB method, see for instance [235], [254, Sect. 7.1], see also Sect. 3.5.1. We recall that the WKB method is based on the quasiclassical ansatz for solutions to a hyperbolic partial differential equation and their asymptotic expansion. The leading term of the expansion results in the Hamilton–Jacobi equation; wave packets and their energy propagate along its characteristics. The characteristics represent point dynamics and are determined from a system of ODEs which can be interpreted as a law of motion or a law

of propagation. The construction of the characteristics involves only local data. The approach proposed here also relates waves governed by certain PDEs in asymptotic regimes to the point dynamics, but it differs significantly from the WKB method. In particular, our approach is not based on any specific ansatz, and it is not entirely local but rather is semi-local.

An asymptotic derivation of Newtonian dynamics, which does not rely on the fast oscillation of solutions as in the WKB method, is made for soliton-like solutions of Nonlinear Schrödinger and Nonlinear Klein–Gordon equations in a number of papers, see [53], [126], [190], [226]. In the mentioned papers, the derivation of the limit Newtonian dynamics of the soliton center relies on the asymptotic analysis of an ansatz structure of soliton-like solutions. By contrast, our semi-local method is in the spirit of the Ehrenfest theorem (though it is applied here in the relativistic setting) and is not based on any specific ansatz for the sequence of concentrating solutions.

A simple and important example of concentrating solutions is provided by a resting or uniformly moving charge in Example 34.5.2. Examples of *accelerating* charges are constructed in Sect. 17.6.2 where we show that all the assumptions on concentrating solutions imposed in Theorem 34.5.1 are satisfied. For a given accelerating rectilinear translational motion and a fixed Gaussian shape of $|\psi|$, we construct an electric potential φ consisting of (i) an explicitly written principal component yielding the desired acceleration and (ii) an additional vanishingly small “balancing” component allowing for the shape $|\psi|$ to be exactly preserved. The construction of the balancing component is reduced to solving a system of characteristic ODEs which allows for a detailed analysis.

34.4 Basic Properties of the Klein–Gordon Equation

34.4.1 Nonlinearity Properties

In this section we describe assumptions under which the derivation of Newton’s law with the Lorentz forces for localized charges can be rigorously justified. In the case of many well separated charges, every single charge “senses” all other charges through the sum of their EM fields, and, because of this feature of the EM interaction, the description of a single charge in an external EM field is of special importance.

In this section we describe the nonlinearity $G'(|\psi|^2)$ that enters the NKG equation (34.128). We assume that $G'(s)$ is of class $C^1(\mathbb{R} \setminus 0)$, it may have a mild singularity point at $s = 0$. Namely, we assume that the function $G'(|\psi|^2)\psi$, $\psi \in \mathbb{C}$, can be extended to a function which belongs to the Hölder classes $C^\alpha(\mathbb{C})$ with any α such that $1 > \alpha > 0$, and the antiderivative $G(|\psi|^2)$, namely

$$G(s) = \int_0^s G'(s') ds',$$

is a function of class $C^{1+\alpha}(\mathbb{C})$ for any $\alpha < 1$ with respect to the variable $\psi \in \mathbb{C}$. The dependence of $G'(s)$ on the size parameter $a > 0$ is as in (14.27).

The regularity assumption includes the following estimate. We assume that for every C_1 there exists a constant C (which can depend on α) such that the following inequalities hold:

$$|G_1(|\psi|^2)| \leq C |\psi|^{1+\alpha}, \quad |G'_1(|\psi|^2)| |\psi| \leq C |\psi|^\alpha \quad \text{for } |\psi| \leq C_1. \quad (34.133)$$

Examples of functions $G'(|\psi|^2)$, which allow for resting, time-harmonic localized solutions of (34.128), are given in Sect. 14.3.2, see also Sect. 14.4.

34.4.2 Conservation Laws for the Klein–Gordon Equation

Our derivation of Einstein's formula is based on the conservation laws for the NKG equation, namely the continuity equation (11.127), the energy conservation equation (11.144) and the momentum conservation law (11.92); we also use the symmetry of the energy-momentum tensor (EnMT) for the NKG equation. For the reader's convenience we formulate the conservation laws in this section.

The expression (11.142) for the energy density has the form

$$\mathcal{E} = u = \frac{\chi^2}{2m} \left[\frac{1}{c^2} \tilde{\partial}_t \psi \tilde{\partial}_t^* \psi^* + \tilde{\nabla} \psi \tilde{\nabla}^* \psi^* + G(\psi^* \psi) + \kappa_0^2 \psi \psi^* \right] \quad (34.134)$$

with $\kappa_0 = \frac{mc}{\chi}$, and the momentum density is given by (11.143). The charge density $\rho(t, \mathbf{x})$ is given by (11.132), and the current density $\mathbf{J}$ by (11.133). They satisfy the continuity equation (11.127):

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0. \quad (34.135)$$

This is readily verified by multiplying both sides of Eq. (34.128) by ψ^* and taking the imaginary part. The Eq. (34.135), in turn, implies the total charge conservation (17.110). The energy-momentum tensor for the NKG equation is given by (11.135), and from the conservation law (11.92) we derive energy and momentum conservation laws. Using the symmetry of the EnMT for the NKG equation, we write the energy conservation law (11.144) in the form

$$\partial_t u = -c^2 \nabla \cdot \mathbf{P} - \nabla \varphi \cdot \mathbf{J}. \quad (34.136)$$

Alternatively, the above equation can be directly verified by multiplying (34.128) by $\tilde{\partial}_t^* \psi^*$ defined by (11.123), taking the real part and carrying out elementary transformations, see Chap. 36. The momentum conservation law (11.92) can be written in the form

$$\partial_t \mathbf{P} - \mathbf{f} + \nabla \mathcal{L}_{\text{KG}}(\psi) + \frac{\chi^2}{2m} \partial_j (\partial_j \psi \nabla \psi^* + \partial_j \psi^* \nabla \psi) = 0, \quad (34.137)$$

where we use the summation convention. The Lagrangian $\mathcal{L}_{\text{KG}}(\psi)$ is defined by (11.121), and $\mathbf{f}$ is the external Lorentz force density. Since the magnetic field is absent, the Lorentz force density, according to (11.140), (11.141), (4.6) with $\mathbf{A} = \mathbf{0}$, takes the form

$$\mathbf{f} = -\rho \nabla \varphi. \quad (34.138)$$

The conservation law (34.137) can be directly verified by multiplying (34.128) by $\tilde{\nabla}^* \psi^*$, taking the real part and carrying out elementary transformations, see Chap. 36.

34.5 Relativistic Dynamics of Localized Solutions

The simple derivation of the relativistic point dynamics given in [18] and Sect. 17.6.2 was based on a treatment of relevant fields in the entire space. At the same time, Newton's equation (17.114), which describes the point dynamics, involves only the values of the EM fields exactly at the location of the point. Therefore, *it is natural to make assumptions of localization of the NKG equations and their solutions which should involve only the behavior of the fields in a vicinity of the trajectory*. The assumptions are asymptotic. The speed of light c , the charge q and the mass parameter m are assumed to be fixed. There are two intrinsic length scales relevant for the NKG equation: (i) the size parameter a that enters the nonlinearity, and (ii) the quantity $a_C = \frac{\chi}{mc}$ known as the reduced Compton wavelength (if χ equals the Planck constant $\hbar$). We suppose in our analysis in this section that the parameter a_C and the size parameter a are vanishingly small, namely we take a sequence of values $\chi_n \rightarrow 0$, $a_n \rightarrow 0$, $n = 1, 2, \dots$. The corresponding values of the potential φ_n and solutions $\psi_n(t, \mathbf{x})$ of the NKG equation are then defined in contracting neighborhoods of a trajectory $\hat{\mathbf{r}}(t)$ of radius R_n with

$$R_n \rightarrow 0. \quad (34.139)$$

34.5.1 Concentrating Solutions of the NKG Equation

We consider here the NKG equations and their solutions in concentrating neighborhoods of a trajectory $\hat{\mathbf{r}}(t)$, see Definitions 34.0.1 and 34.0.2. We make certain regularity assumptions on the behavior of the potentials and solutions restricted to the domain $\hat{\mathcal{Q}}(\hat{\mathbf{r}}, R_n)$ around the trajectory $\hat{\mathbf{r}}(t)$. The assumptions relate the “microscopic” scales a and a_C to the “macroscopic” length scale of the order 1 relevant to the trajectory $\hat{\mathbf{r}}(t)$ and the electric potential $\varphi(t, \mathbf{x})$. We use the notation (34.6), (34.7).

Definition 34.5.1 (*localized KG equations*) Let $\hat{\mathbf{r}}(t)$ be a trajectory with its concentrating neighborhoods $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$ (as in Definitions 34.0.1 and 34.0.2), and let $\chi = \chi_n$, $a = a_n$, $\varphi = \varphi_n$ be a sequence of parameters defining the NKG equation (34.128). A sequence of the NKG equations is called localized in $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$ if the following conditions are satisfied. First of all, χ and a are vanishingly small:

$$a = a_n \rightarrow 0, \quad \chi = \chi_n \rightarrow 0; \quad (34.140)$$

the ratio $\zeta = a_C/a$ remains bounded,

$$\zeta = \frac{a_C}{a} \leq C \quad \text{where } a_C = \frac{\chi}{mc}, \quad (34.141)$$

and the ratio $\theta = R/a$ grows to infinity,

$$\theta_n = \frac{R_n}{a_n} \rightarrow \infty \text{ as } n \rightarrow \infty. \quad (34.142)$$

Second, the electric potentials $\varphi_n(t, \mathbf{x})$ are twice continuously differentiable in $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$ and satisfy the following constraints:

(i) the potentials φ_n satisfy the following uniform in n estimate:

$$\max_{T_- \leq t \leq T_+, \mathbf{x} \in \Omega_n} (|\varphi_n(t, \mathbf{x})| + |\nabla_{0,\mathbf{x}} \varphi_n(t, \mathbf{x})|) \leq C \quad (34.143)$$

where $\nabla_{0,\mathbf{x}}$ is defined in (34.7);

(ii) the potentials $\varphi_n(t, \mathbf{x})$ locally converge to a linear potential $\varphi_\infty(t, \mathbf{x})$, namely

$$\begin{aligned} & \max_{T_- \leq t \leq T_+, \mathbf{x} \in \Omega_n} |\varphi(t, \mathbf{x}) - \varphi_\infty(t, \mathbf{x})| \\ & + \max_{T_- \leq t \leq T_+, \mathbf{x} \in \Omega_n} |\nabla_{0,\mathbf{x}} \varphi(t, \mathbf{x}) - \nabla_{0,\mathbf{x}} \varphi_\infty(t, \mathbf{x})| \rightarrow 0 \end{aligned} \quad (34.144)$$

as $n \rightarrow \infty$, where

$$\varphi_\infty(t, \mathbf{x}) = \varphi_\infty(t, \hat{\mathbf{r}}) + (\mathbf{x} - \hat{\mathbf{r}}) \nabla \varphi_\infty(t), \quad (34.145)$$

and the coefficients $\varphi_\infty(t, \hat{\mathbf{r}})$ and $\nabla \varphi_\infty(t)$ are bounded:

$$|\varphi_\infty(t, \hat{\mathbf{r}})|, \quad |\nabla \varphi_\infty|, \quad |\partial_t \nabla \varphi_\infty| \leq C \text{ for } T_- \leq t \leq T_+. \quad (34.146)$$

Since the parameters m, c, q are fixed, the following inequalities always hold

$$C_0 \chi \leq a_C \leq C_1 \chi, \quad \chi_n \leq C_2 a_n. \quad (34.147)$$

Based on the above inequalities, we often replace χ with a_C in estimates without special comments. Throughout the chapter we denote constants which do not depend on χ_n , a_n , and θ_n by the letter C with different indices. Sometimes the same letter C with the same indices may denote in different formulas different constants. Below we often omit the index n in χ_n , a_n , φ_n etc.

Obviously, if the potentials φ_n are the restrictions of a fixed twice continuously differentiable function φ to $\hat{\Omega}(\hat{\mathbf{r}}(t), R_n)$, then conditions (34.143) and (34.144) are satisfied with

$$\varphi_\infty(t, \mathbf{x}) = \varphi(t, \hat{\mathbf{r}}) + (\mathbf{x} - \hat{\mathbf{r}}) \nabla \varphi(t, \hat{\mathbf{r}}). \quad (34.148)$$

Now we give the definition of concentrating solutions and trajectory of concentration in the relativistic case:

Definition 34.5.2 (*concentrating solutions*) Let $\hat{\mathbf{r}}(t)$ be a trajectory. We say that solutions ψ of the NKG equation (34.128) concentrate at the trajectory $\hat{\mathbf{r}}(t)$ if the following conditions hold. First of all, a sequence of concentrating neighborhoods $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$ and parameters $a = a_n$, $\chi = \chi_n$ and potentials $\varphi = \varphi_n$ are selected to form a sequence of the NKG equations (34.128) localized in $\hat{\Omega}_n(\hat{\mathbf{r}}, R_n)$ as in Definition 34.5.1. Second, for every domain $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$ there exists a function $\psi = \psi_n \in C^2(\hat{\Omega}(\hat{\mathbf{r}}, R_n))$ which is a solution to the NKG equation (34.128) in $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$, and this solution satisfies the following constraints:

- (i) there exists a constant C which does not depend on n but may depend on the sequence such that

$$\begin{aligned} & \max_{T_- \leq t \leq T_+} \int_{\Omega_n} \left[a_C^2 |\nabla_{0,\mathbf{x}} \psi(t, \mathbf{x})|^2 + |\psi(t, \mathbf{x})|^2 \right] d\mathbf{x} \\ & + \max_{T_- \leq t \leq T_+} \int_{\Omega_n} a_C^2 |G(|\psi(t, \mathbf{x})|^2)| d\mathbf{x} \leq C, \end{aligned} \quad (34.149)$$

where $G = G_a$ is the nonlinearity in (11.121);

- (ii) the restriction of functions ψ to the boundary $\partial\Omega_n = \{|\mathbf{x} - \hat{\mathbf{r}}| = R_n\}$ vanishes asymptotically:

$$\begin{aligned} & \max_{T_- \leq t \leq T_+} \int_{\partial\Omega_n} (a_C^2 |\nabla_{0,\mathbf{x}} \psi(t, \mathbf{x})|^2 + |\psi(t, \mathbf{x})|^2) d\sigma \\ & + \max_{T_- \leq t \leq T_+} \int_{\partial\Omega_n} a_C^2 |G(|\psi(t, \mathbf{x})|^2)| d\sigma \rightarrow 0; \end{aligned} \quad (34.150)$$

- (iii) the restricted to $\Omega(\hat{\mathbf{r}}(t), R_n)$ energy

$$\bar{\mathcal{E}}_n(t) = \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} u_n d\mathbf{x}, \quad (34.151)$$

where u_n is the energy density defined by expression (11.142), is bounded from below for sufficiently large n :

$$\bar{\mathcal{E}}_n(t) \geq c_0 > 0 \text{ for } n \geq n_0, \quad T_- \leq t \leq T_+; \quad (34.152)$$

(iv) there exists a $t_0 \in [T_-, T_+]$ such that the sequence of restricted energies $\bar{\mathcal{E}}_n(t_0)$ converges:

$$\lim_{n\hat{\mathbf{r}}(t) \rightarrow \infty} \bar{\mathcal{E}}_n(t_0) = \bar{\mathcal{E}}_\infty(t_0). \quad (34.153)$$

If the above conditions are fulfilled, we call $\hat{\mathbf{r}}(t)$ a trajectory of concentration.

Notice that condition (i) provides for the boundedness of the restricted energy over the domains Ω_n . Condition (ii) provides for a proper confinement of ψ to Ω_n , and condition (iii) ensures that the sequence is non-trivial. Note that conditions (ii) and (iii) give the precise meaning to our heuristic requirement that local solutions ψ_n should converge to a delta-function which is zero everywhere but at the point $\hat{\mathbf{r}}(t)$. According to (34.166) and (34.149), $\bar{\mathcal{E}}_n(t_0)$ is a bounded sequence, and, consequently, it always contains a converging subsequence. Hence, condition (iv) is not really an additional constraint but rather it assumes that such a subsequence is selected. The choice of a particular subsequence limit $\bar{\mathcal{E}}_\infty(t_0)$ can then be interpreted as a normalization, see Remark 34.5.2. Obviously, this condition describes the amount of energy which concentrates at the trajectory at the time t_0 .

Example 34.5.1 The conditions of Definition 34.5.2 can easily be verified for rather general sequences of functions which are localized around an arbitrary trajectory $\hat{\mathbf{r}}(t)$ (if we do not assume that the functions are solutions of the NKG equation). In particular, let a sequence of functions be defined by the formula

$$\psi(t, \mathbf{x}) = a^{-3/2} \psi_0((\mathbf{x} - \hat{\mathbf{r}}(t))/a), \quad a = a_n, \quad (34.154)$$

where a is the same parameter as in G_a . The function $\psi_0(\mathbf{z})$ is a smooth fixed function which decays together with its derivatives as $|\mathbf{z}| \rightarrow \infty$:

$$\max_{|\mathbf{z}|=\theta} (|\nabla_{\mathbf{z}} \psi_0(\mathbf{z})|^2 + |\psi_0(\mathbf{z})|^2) \leq Y^2(\theta), \quad (34.155)$$

where $Y(\theta)$ is a continuous function which decays fast enough:

$$Y(\theta) \leq C_0 \theta^{-N} \text{ if } \theta \geq 1; \quad Y(\theta) \leq C_0 \text{ if } \theta \leq 1, \quad (34.156)$$

with a sufficiently large $N > 3/2$, and θ is defined in (34.142). We assume that N and the sequences $\theta_n \rightarrow \infty$, $a = a_n \rightarrow 0$, satisfy the condition

$$3 - (1 + \alpha)N < 0, \quad a^{-1} \theta_n^{2-(1+\alpha)N} \rightarrow 0 \quad (34.157)$$

with $1 > \alpha > 0$, and α is close to 1 as in (34.133). To verify (34.149), we change variables $(\mathbf{x} - \hat{\mathbf{r}}(t)) / a = \mathbf{z}$, take into account that according to (34.141) $\zeta^2 = \chi^2 / a^2 \leq C$, and obtain inequalities

$$\begin{aligned} & a_C^2 \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} |\nabla_{\mathbf{x}} \psi(t, \mathbf{x})|^2 + \frac{1}{c^2} |\partial_t \psi(t, \mathbf{x})|^2 d\mathbf{x} \\ & \leq C \int_{\mathbf{z} \in \Omega(0, \theta_n)} |\nabla_{\mathbf{z}} \psi_0(t, \mathbf{z})|^2 \left(1 + |\partial_t \hat{\mathbf{r}}|^2\right) d\mathbf{z} \\ & \leq C_1 + C_1 \int_1^\infty r^2 r^{-2N} dr \leq C'_1, \\ & \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} |\psi(t, \mathbf{x})|^2 d\mathbf{x} = \int_{\mathbf{z} \in \Omega(0, \theta_n)} |\psi_0(t, \mathbf{z})|^2 d\mathbf{z} \leq C_2. \end{aligned}$$

Now we would like to estimate the integral which involves the nonlinearity

$$G_a(|\psi(t, \mathbf{x})|^2) = a^{-5} G_1(|\psi_0((\mathbf{x} - \hat{\mathbf{r}}(t)) / a)|^2) = a^{-5} G_1(|\psi_0(\mathbf{z})|^2).$$

According to (34.133) and (34.157),

$$\begin{aligned} a_C^2 \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} |G_a(|\psi(t, \mathbf{x})|^2)| d\mathbf{x} &= \zeta^2 \int_{\Omega(0, \theta_n)} |G_a(|\psi_0(t, \mathbf{z})|^2)| d\mathbf{z} \quad (34.158) \\ &\leq C \zeta^2 \int_{\Omega(0, \theta_n)} |\psi_0(t, \mathbf{z})|^{1+\alpha} d\mathbf{z} \\ &\leq \zeta^2 C_1 + \zeta^2 C_2 \int_1^{\theta_n} \theta^{-N(1+\alpha)+2} d\theta \leq \zeta^2 C_3. \end{aligned}$$

Hence condition (34.149) is fulfilled. To verify (34.150), we estimate integrals over the boundary using (34.157) and (34.133):

$$\begin{aligned} & \int_{\partial\Omega(\hat{\mathbf{r}}(t), R_n)} a_C^2 |\nabla_{\mathbf{x}} \psi(t, \mathbf{x})|^2 + a_C^2 \frac{1}{c^2} |\partial_t \psi(t, \mathbf{x})|^2 + |\psi(t, \mathbf{x})|^2 d\sigma \\ & \leq C a^{-1} \int_{\partial\Omega(0, \theta_n)} |\nabla_{\mathbf{z}} \psi_0(t, \mathbf{z})|^2 + |\psi_0(t, \mathbf{z})|^2 d\sigma \leq C_1 \theta_n^2 Y^2(\theta_n) a^{-1} \leq C, \end{aligned}$$

and

$$\begin{aligned} & a_C^2 \int_{\partial\Omega(\hat{\mathbf{r}}(t), R_n)} |G_a(|\psi(t, \mathbf{x})|^2)| d\sigma = \zeta^2 a^{-1} \int_{\partial\Omega(0, \theta_n)} |G_1(|\psi_0(t, \mathbf{z})|^2)| d\sigma \\ & \leq C \zeta^2 a^{-1} \int_{\partial\Omega(0, \theta_n)} |\psi_0(t, \mathbf{z})|^{1+\alpha} d\sigma \leq C_1 \zeta^2 a^{-1} \theta_n^2 Y^{1+\alpha}(\theta_n) \\ & \leq C_2 a^{-1} \theta_n^2 \theta_n^{-N(1+\alpha)} \leq C'_2. \end{aligned}$$

Consequently, we obtain the desired (34.150). The energy $\bar{\mathcal{E}}_n$ involves the positive term

$$\begin{aligned} & \frac{\chi^2}{2m} \kappa_0^2 \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} |\psi(t, \mathbf{x})|^2 \\ &= \frac{mc^2}{2} \int_{\Omega(0, \theta_n)} |\psi_0|^2 d\mathbf{z} \rightarrow \frac{mc^2}{2} \int_{\mathbb{R}^3} |\psi_0|^2 d\mathbf{z} \geq \frac{1}{C_4} > 0. \end{aligned}$$

According to (11.142), the only negative contribution to the energy can come from the nonlinearity G , and, according to (34.158), this contribution cannot be greater than $\zeta^2 C_3$, hence condition (34.152) is satisfied if ζ is small enough.

We consider in the following important example the uniform motion of charges from Sect. 15.2. We show that the uniformly moving solutions are an example of concentrating solutions. A general construction of concentrating solutions for an accelerating motion is given in Sect. 34.6.

Example 34.5.2 As a simple example of solutions which concentrate at a trajectory $\hat{\mathbf{r}}(t)$, we take wave-corpuscle solutions defined by (15.23) in Sect. 15.2. Then the trajectory $\hat{\mathbf{r}}(t) = \mathbf{v}t$ is a straight line, $\varphi = 0$. We assume that the function $\check{\psi}_1 = \psi_0$ from (15.23) satisfies (34.155), (34.156) (the ground states from Examples 14.3.1–14.3.5 satisfy this assumption). We take sequences $a_n \rightarrow 0$, $\chi_n \rightarrow 0$, $R_n \rightarrow 0$, $\theta_n \rightarrow \infty$ such that conditions (34.157) are satisfied. Since $\theta_n \rightarrow \infty$, the restricted energy and charge defined as integrals over $\Omega(\hat{\mathbf{r}}(t), R_n)$ converge to the integrals over the entire space, and $\bar{\mathcal{E}}_\infty$, $\bar{\rho}_\infty$ are given by (15.27) and (15.28), namely

$$\bar{\mathcal{E}}_n(t) \rightarrow \gamma mc^2 (1 + \Theta(\omega)), \quad \bar{\rho}_n(t) \rightarrow q.$$

Therefore, (34.152) holds for large n . The solutions concentrate at $\hat{\mathbf{r}}(t) = \mathbf{v}t$ and have the following additional properties: (i) the energy density u is center-symmetric with respect to $\hat{\mathbf{r}}(t) = \mathbf{v}t$, hence the ergocenter $\mathbf{r}(t)$ defined by (34.180) coincides with the center $\hat{\mathbf{r}}(t)$; (ii) the charge density ρ is given by (15.27), and according to (15.24) it converges to the delta-function $q\delta(\mathbf{x} - \mathbf{r})$ as $a \rightarrow 0$; (iii) the current $\mathbf{J}$ is given by (15.30), its components are center-symmetric and converge to the corresponding components of $q\mathbf{v}\delta(\mathbf{x} - \mathbf{r})$.

The following statement is the main result of this section. It describes trajectories relevant to concentrating solutions to the NKG equations.

Theorem 34.5.1 (relevant trajectories of concentration) *Let solutions ψ of the NKG equation (34.128) concentrate at $\hat{\mathbf{r}}(t)$. Then their restricted energies $\bar{\mathcal{E}}_n(t)$ converge to the limit restricted energy $\bar{\mathcal{E}}_\infty(t)$ which satisfies Eq. (34.171). The limit energy $\bar{\mathcal{E}}_\infty(t)$ and the trajectory $\hat{\mathbf{r}}(t)$ satisfy equation*

$$\partial_t \left(\frac{1}{c^2} \bar{\mathcal{E}}_\infty(t) \partial_t \hat{\mathbf{r}} \right) = \mathbf{f}_\infty \quad (34.159)$$

with the electric force $\mathbf{f}_\infty(t)$ given by the formula

$$\mathbf{f}_\infty(t) = -\bar{\rho}_\infty \nabla \varphi_\infty(t), \quad (34.160)$$

where the charge $\bar{\rho}_\infty$ does not depend on t . We interpret the coefficient at $\partial_t \hat{\mathbf{r}}$ in (34.159) as the mass M and obtain Einstein's formula

$$M = \frac{1}{c^2} \bar{\mathcal{E}}_\infty(t). \quad (34.161)$$

The following formula holds:

$$M = \gamma M_0, \quad \gamma = \left(1 - (\partial_t \hat{\mathbf{r}})^2 / c^2\right)^{-1/2}, \quad (34.162)$$

where M_0 is a constant.

The proof of Theorem 34.5.1 is given in Sect. 34.5.3. The constant M_0 can be interpreted as the rest mass of the charge.

Note that the formula (34.161) defines the mass M based on the observation that (34.159) is of the form of the relativistic Newton's equation (17.114). Therefore formula (34.159) provides an alternative Newtonian method of deriving Einstein's formula.

Remark 34.5.1 If the external electric field φ is not zero, the system described by (11.121) is not a closed system. For non-closed systems, the center of energy (also known as the center of mass or centroid) and the total energy-momentum are frame dependent, hence they are not 4-vectors, [248, 7.1, 7.2], [212, 24]. Therefore, in contrast to the case of a closed system, one cannot present velocity parameter $\mathbf{v}$ of the Lorentz group which can be identified with the velocity of the system.

Remark 34.5.2 The sequences $a_n, R_n, \theta_n, \varphi_n, \chi_n, \psi_n$ enter the definition of a concentrating solution. But if we take two different sequences which fit the definition for the same trajectory, we obtain the same Eqs. (34.159)–(34.162) and the limit energy $E_\infty(t)$. More than that, as long as the gradient $\nabla \varphi_\infty(t)$ of the limit potential is given, and at a moment of time t_0 the position and velocity $\hat{\mathbf{r}}(t_0), \partial_t \hat{\mathbf{r}}(t_0)$ and the limit restricted energy and charge $E_\infty(t_0), \bar{\rho}_\infty$ are fixed, then the trajectory $\hat{\mathbf{r}}(t)$ and the energy $E_\infty(t)$ are uniquely defined as solutions of equations (34.159)–(34.162), and consequently they do not depend on particular sequences. Note also that, if we eliminated point (iv) in Definition 34.5.2, we could still pick a subsequence for which (34.153) holds. For any such a subsequence, we obtain (34.159), but $\bar{\mathcal{E}}_\infty(t_0)$ could be different. Taking into account (34.162) and Corollary 34.5.1, we see that the choice of different subsequences leads to multiplication of Eqs. (34.159), (34.162) by a constant and corresponds to simultaneous multiplication of the rest mass M_0 and the charge $\bar{\rho}_\infty$ by the same constant.

In Sect. 17.6.2 we construct a non-trivial example of sequences of the NKG equations and their solutions which concentrate at rectilinear accelerating trajectories.

34.5.2 Properties of Concentrating Solutions

Everywhere in this section, we assume that we have a trajectory $\hat{\mathbf{r}}(t)$ with its concentrating neighborhoods $\hat{\Omega}_n(\hat{\mathbf{r}}, R_n)$. We often use in this section the following elementary inequalities for the densities defined in Sect. 34.4.2:

$$|\mathbf{P}| \leq \frac{\chi^2}{mc^2} \left| \partial_t \psi + \frac{iq}{\chi} \varphi \psi \right| |\nabla \psi| \leq C \chi^2 |\partial_t \psi|^2 + C \chi^2 |\nabla \psi|^2 + C |\psi|^2, \quad (34.163)$$

$$|\mathbf{J}| \leq \frac{\chi q}{m} |\nabla \psi| |\psi| \leq C (\chi |\nabla \psi| + |\psi|) |\psi| \leq C' \chi^2 |\nabla \psi|^2 + C' |\psi|^2, \quad (34.164)$$

$$|\rho| \leq \frac{\chi q}{mc^2} \left| \tilde{\partial}_t \psi \right| |\psi| \leq C \chi^2 |\partial_t \psi|^2 + C |\psi|^2. \quad (34.165)$$

Similarly,

$$|u| + |\mathcal{L}| \leq C \chi^2 |\partial_t \psi|^2 + C \chi^2 |\nabla \psi|^2 + C \chi^2 |G(\psi^* \psi)| + C |\psi|^2. \quad (34.166)$$

We define the restricted to $\Omega(\hat{\mathbf{r}}(t), R_n)$ charge $\bar{\rho}_n$ by the following formula:

$$\bar{\rho}_n(t) = \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \rho_n \, d\mathbf{x}. \quad (34.167)$$

Lemma 34.5.1 *Let solutions ψ of the NKG equation (34.128) concentrate at $\hat{\mathbf{r}}(t)$. Then there exists a number $\bar{\rho}_\infty$ and a subsequence of the solutions ψ_n such that*

$$\bar{\rho}_n(t) \rightarrow \bar{\rho}_\infty \quad \text{uniformly for } T_- \leq t \leq T_+. \quad (34.168)$$

Proof According to (34.165) and (34.149), for any $t_0 \in [T_-, T_+]$, the sequence $\bar{\rho}_n(t_0)$ is bounded, hence it always contains a converging subsequence. We pick such a subsequence, denote its limit by $\bar{\rho}_\infty$ and integrate the continuity equation (34.135):

$$\partial_t \int_{\Omega_n} \rho_n \, d\mathbf{x} - \int_{\partial\Omega_n} \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho_n \, d\sigma + \int_{\partial\Omega_n} \bar{\mathbf{n}} \cdot \mathbf{J} \, d\sigma = 0. \quad (34.169)$$

Integrating with respect to time from t_0 to t and using (34.165), (34.164) and (34.150), we obtain

$$\begin{aligned} |\bar{\rho}_n(t) - \bar{\rho}_\infty| &\leq |\bar{\rho}_n(t) - \bar{\rho}_n(t_0)| + |\bar{\rho}_n(t_0) - \bar{\rho}_\infty| \\ &\leq (T_+ - T_-) \left| \hat{\mathbf{v}} \right| \int_{\partial\Omega_n} |\rho_n| \, d\sigma + (T_+ - T_-) \int_{\partial\Omega_n} |\mathbf{J}| \, d\mathbf{x} \\ &\leq C \int_{\partial\Omega_n} \left(\chi^2 |\nabla_{0,x} \psi|^2 + C'_1 |\psi|^2 \right) d\sigma + |\bar{\rho}_n(t_0) - \bar{\rho}_\infty| \rightarrow 0. \end{aligned}$$

Therefore (34.168) holds.

Theorem 34.5.2 (restricted energy convergence) *Let solutions ψ of the NKG equation (34.128) concentrate at $\hat{\mathbf{r}}(t)$. Let $\bar{\rho}_n(t_0) \rightarrow \bar{\rho}_\infty$. Then the restricted energy $\bar{\mathcal{E}}_n(t)$ converges uniformly to the limit restricted energy $\bar{\mathcal{E}}_\infty$*

$$\bar{\mathcal{E}}_n(t) \rightarrow \bar{\mathcal{E}}_\infty(t) \quad \text{uniformly on } [T_-, T_+], \quad (34.170)$$

where

$$\bar{\mathcal{E}}_\infty(t) = \bar{\mathcal{E}}_\infty(t_0) - \bar{\rho}_\infty \int_{t_0}^t \partial_t \hat{\mathbf{r}} \cdot \nabla \varphi_\infty dt'. \quad (34.171)$$

Proof Integrating (34.136) with respect to x and t , we obtain that

$$\bar{\mathcal{E}}_n(t) - \bar{\mathcal{E}}_\infty(t_0) = - \int_{t_0}^t \nabla \varphi_\infty \cdot \int_{\Omega_n} \mathbf{J} d\mathbf{x} dt' + Q_1, \quad (34.172)$$

where φ_∞ is defined by (34.145), and

$$\begin{aligned} Q_1 &= \int_{t_0}^t \int_{\partial\Omega_n} (\hat{\mathbf{v}} \cdot \bar{n} u_n - c^2 \bar{\mathbf{n}} \cdot \mathbf{P}) d\sigma dt' \\ &\quad - \int_{t_0}^t \int_{\Omega_n} \nabla(\varphi - \varphi_\infty) \cdot \mathbf{J} d\mathbf{x} dt' + \bar{\mathcal{E}}_n(t_0) - \bar{\mathcal{E}}_\infty(t_0). \end{aligned}$$

We put $\varphi - \varphi_\infty = \varphi_b$, and, using the continuity equation (34.135), we obtain that

$$\begin{aligned} \int_{\Omega_n} \nabla \varphi_b \cdot \mathbf{J} d\mathbf{x} &= \partial_t \int_{\Omega_n} \varphi_b \rho d\mathbf{x} \\ &+ \int_{\partial\Omega_n} (\bar{\mathbf{n}} \cdot \mathbf{J} - \hat{\mathbf{v}} \cdot \bar{n} \rho) \varphi_b d\sigma - \int_{\Omega_n} \partial_t \varphi_b \rho d\mathbf{x}. \end{aligned} \quad (34.173)$$

Therefore, according to (34.144), (34.165), (34.164), (34.149) and (34.150),

$$\begin{aligned} \left| \int_{t_0}^t \int_{\Omega_n} \nabla \varphi_b \cdot \mathbf{J} d\mathbf{x} \right| &\leq 2 \max_{T_- \leq t \leq T_+, x \in \Omega_n} |\varphi_b| \max_{T_- \leq t \leq T_+} \int_{\Omega_n} |\rho| d\mathbf{x} \\ &+ C \max_{T_- \leq t \leq T_+} \int_{\partial\Omega_n} (|\mathbf{J}| + |\rho|) \varphi_b d\sigma \\ &+ \max_{T_- \leq t \leq T_+, x \in \Omega_n} |\partial_t \varphi_b| \max_{T_- \leq t \leq T_+} \int_{\Omega_n} |\rho| d\mathbf{x} \rightarrow 0. \end{aligned} \quad (34.174)$$

Similarly, we estimate the first term in Q_1 :

$$\left| \int_{t_0}^t \int_{\partial\Omega_n} \hat{\mathbf{v}} \cdot \bar{n} u_n - c^2 \bar{\mathbf{n}} \cdot \mathbf{P} d\sigma dt' \right| \leq C \max_{T_- \leq t \leq T_+} \int_{\partial\Omega_n} (|u_n| + |\mathbf{P}|) d\sigma dt',$$

and we conclude that $|Q_1| \rightarrow 0$.

Multiplication of the continuity equation (34.135) by the vector $\mathbf{x} - \hat{\mathbf{r}}$ yields the following expression for $\mathbf{J}$:

$$\partial_t (\rho (\mathbf{x} - \hat{\mathbf{r}})) + \partial_t \hat{\mathbf{r}} \rho + \partial_j ((\mathbf{x} - \hat{\mathbf{r}}) \mathbf{J}_j) = \mathbf{J}. \quad (34.175)$$

Integrating over Ω_n , we see that

$$\int_{\Omega_n} \mathbf{J} \, d\mathbf{x} = \hat{\mathbf{v}} \bar{\rho} + \partial_t \int_{\Omega_n} (\rho (\mathbf{x} - \hat{\mathbf{r}})) \, d\mathbf{x} + \int_{\partial\Omega_n} (\mathbf{x} - \hat{\mathbf{r}}) (\bar{\mathbf{n}} \cdot \mathbf{J} - \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho) \, d\sigma,$$

and

$$\int_{t_0}^t \nabla \varphi_\infty \cdot \int_{\Omega_n} \mathbf{J} \, d\mathbf{x} \, dt' = \int_{t_0}^t \nabla \varphi_\infty \cdot \partial_t \hat{\mathbf{r}} \bar{\rho}_\infty \, dt' + Q_{10} \quad (34.176)$$

where

$$\begin{aligned} Q_{10} = & \nabla \varphi_\infty \cdot \int_{\Omega_n} (\rho (\mathbf{x} - \hat{\mathbf{r}})) \, d\mathbf{x} \Big|_{t_0}^t - \int_{t_0}^t \partial_t \nabla \varphi_\infty \cdot \int_{\Omega_n} (\mathbf{x} - \hat{\mathbf{r}}) \rho \, d\sigma \, dt' \\ & + \int_{t_0}^t \nabla \varphi_\infty \cdot \int_{\partial\Omega_n} (\mathbf{x} - \hat{\mathbf{r}}) (\bar{\mathbf{n}} \cdot \mathbf{J} - \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho) \, d\sigma \, dt' \\ & + \int_{t_0}^t \nabla \varphi_\infty \cdot \partial_t \hat{\mathbf{r}} (\bar{\rho} - \bar{\rho}_\infty) \, dt'. \end{aligned}$$

Combining with (34.172), we obtain that

$$\bar{\mathcal{E}}_n(t) - \bar{\mathcal{E}}_\infty(t_0) = - \int_{t_0}^t \nabla \varphi_\infty \cdot \partial_t \hat{\mathbf{r}}(t) \bar{\rho}_\infty \, dt' + Q_1 - Q_{10}. \quad (34.177)$$

Using (34.146), (34.165), (34.164), (34.1), (34.149), we conclude that

$$\begin{aligned} |Q_{10}| \leq & C R_n \sup_{T_- \leq t \leq T_+} \int_{\Omega_n} |\rho| \, d\mathbf{x} \\ & + C (T_+ - T_-) R_n \sup_{T_- \leq t \leq T_+} |\hat{\mathbf{v}}| \int_{\partial\Omega_n} |\rho| \, d\sigma + C (T_+ - T_-) R_n \int_{\partial\Omega_n} |\mathbf{J}| \, d\sigma \\ & + C (T_+ - T_-) \sup_{T_- \leq t \leq T_+} |\hat{\mathbf{v}}| |\bar{\rho} - \bar{\rho}_\infty| \rightarrow 0. \end{aligned} \quad (34.178)$$

We obtain (34.170) and (34.171) from (34.177).

Remark 34.5.3 Equations (17.114) for the space components of the relativistic 4-vector are usually complemented (see, for instance, [27], [265], [248]) with the time component (17.115)

$$\frac{d}{dt} (Mc^2) = \mathbf{f} \cdot \mathbf{v}. \quad (34.179)$$

Formula (34.171), obviously, has the form of integrated equation (34.179).

We define the adjacent ergocenter $\mathbf{r}_n$ by the formula

$$\mathbf{r}_n(t) = \frac{1}{\bar{\mathcal{E}}_n} \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \mathbf{x} u_n \, d\mathbf{x}. \quad (34.180)$$

Lemma 34.5.2 *Let solutions of the NKG equation (34.128) concentrate at $\hat{\mathbf{r}}(t)$. Then the adjacent ergocenters $\mathbf{r}_n(t)$ of the solutions converge to $\hat{\mathbf{r}}(t)$ uniformly on the time interval $[T_-, T_+]$.*

Proof We infer from (34.151) and (34.180) that

$$\frac{1}{\bar{\mathcal{E}}_n} \int_{\Omega_n} (\mathbf{x} - \mathbf{r}_n) u_n \, d\mathbf{x} = \mathbf{r}_n - \mathbf{r}_n \frac{1}{\bar{\mathcal{E}}_n} \int_{\Omega_n} u_n \, d\mathbf{x} = 0. \quad (34.181)$$

From (34.149) we obtain that

$$\left| \int_{\Omega_n} (\mathbf{x} - \hat{\mathbf{r}}) u_n \, d\mathbf{x} \right| \leq R_n \int_{\Omega_n} |u_n| \, d\mathbf{x} \leq C R_n \rightarrow 0. \quad (34.182)$$

Using (34.152) we conclude that

$$|\hat{\mathbf{r}} - \mathbf{r}_n| = \frac{1}{\bar{\mathcal{E}}_n} \left| \int_{\Omega_n} (\mathbf{x} - \mathbf{r}_n) - (\mathbf{x} - \hat{\mathbf{r}}) u_n \, d\mathbf{x} \right| \leq C_1 R_n \rightarrow 0. \quad (34.183)$$

Lemma 34.5.3 *Let solutions $\psi = \psi_n$ of the NKG equation (34.128) concentrate at $\hat{\mathbf{r}}(t)$. Then*

$$|\mathbf{v}_n| = |\partial_t \mathbf{r}_n| \leq C_4, \quad (34.184)$$

and for any t_0 there exists a subsequence such that $\mathbf{v}_n(t_0)$ converges.

Proof Multiplying (34.136) by $(\mathbf{x} - \mathbf{r})$, we obtain that

$$\partial_t ((\mathbf{x} - \mathbf{r}) u) + u \partial_t \mathbf{r} = -(\mathbf{x} - \mathbf{r}) c^2 \nabla \cdot \mathbf{P} - (\mathbf{x} - \mathbf{r}) \nabla \varphi \cdot \mathbf{J}. \quad (34.185)$$

Integration over Ω_n yields equation

$$\begin{aligned} & \int_{\Omega_n} \partial_t ((\mathbf{x} - \mathbf{r}) u) \, d\mathbf{x} + \partial_t \mathbf{r} \int_{\Omega_n} u \, d\mathbf{x} \\ &= - \int_{\Omega_n} (\mathbf{x} - \mathbf{r}) c^2 \nabla \cdot \mathbf{P} \, d\mathbf{x} - \int_{\Omega_n} (\mathbf{x} - \mathbf{r}) \nabla \varphi \cdot \mathbf{J} \, d\mathbf{x}. \end{aligned}$$

We infer from the definition of the ergocenter $\mathbf{r}$ that

$$\int_{\Omega_n} \partial_t ((\mathbf{x} - \mathbf{r}) u) \, d\mathbf{x} + \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} u \, d\sigma = 0. \quad (34.186)$$

Therefore,

$$\begin{aligned} \partial_t \mathbf{r} u_n &= \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} u \, d\sigma - c^2 \int_{\Omega_n} (\mathbf{x} - \mathbf{r}) \nabla \cdot \mathbf{P} \, d\mathbf{x} \\ &\quad - \int_{\Omega_n} (\mathbf{x} - \mathbf{r}) \nabla \varphi \cdot \mathbf{J} \, d\mathbf{x}. \end{aligned}$$

Using (34.143), (34.152) and (34.183), we obtain the inequality

$$|\partial_t \mathbf{r}| \leq C \int_{\partial\Omega_n} (|u| + |\mathbf{P}|) \, d\sigma + \int_{\Omega_n} (|\mathbf{J}| + |\mathbf{P}|) \, d\mathbf{x}$$

yielding (34.184).

34.5.3 Proof of Theorem 34.5.1

We prove in this section that, if solutions to the NKG equation concentrate at a trajectory $\hat{\mathbf{r}}(t)$, then the limit restricted energy and the trajectory must satisfy Einstein's formula and the relativistic version of Newton's law. The proof is based on two facts. First, according to Lemma 34.5.2, the trajectory $\hat{\mathbf{r}}(t)$ is the limit of the adjacent ergocenters $\mathbf{r}_n(t)$. Second, the adjacent ergocenters satisfy equations (see Theorem 34.5.3 below) which yield in the limit the relativistic Newton's law and Einstein's formula.

Theorem 34.5.3 *Let solutions ψ_n of the NKG equation (34.128) concentrate at $\hat{\mathbf{r}}(t)$, and let $\mathbf{r}(t) = \mathbf{r}_n(t)$ be adjacent ergocenters defined by (34.180). Then for any $t_0, t \in [T_-, T_+]$*

$$\frac{1}{c^2} \bar{\mathcal{E}}_n(t) \partial_t \mathbf{r}(t) - \frac{1}{c^2} \bar{\mathcal{E}}_n(t_0) \partial_t \mathbf{r}(t_0) = - \int_{t_0}^t \bar{\rho}_\infty \nabla \varphi_\infty \, dt' + \delta_f \quad (34.187)$$

where $\delta_f \rightarrow 0$ uniformly on $[T_-, T_+]$.

Proof Integrating the momentum conservation equation (34.137) over $\Omega(\hat{\mathbf{r}}(t), R_n) = \Omega_n$, we obtain that

$$\begin{aligned} &\partial_t \int_{\Omega_n} \mathbf{P}(t) \, d\mathbf{x} - \int_{\partial\Omega_n} \mathbf{P}(t) \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \, d\sigma - \int_{\Omega_n} \mathbf{f} \, d\mathbf{x} \\ &+ \int_{\partial\Omega_n} \bar{\mathbf{n}} \mathcal{L}_{\text{KG}}(\psi) \, dt' \, d\mathbf{x} + \frac{\chi^2}{2m} \int_{\partial\Omega_n} \bar{n}_j (\partial_j \psi \nabla \psi^* + \partial_j \psi^* \nabla \psi) \, d\sigma = 0, \end{aligned}$$

where we use the summation convention. Now we integrate $\mathbf{P}(t)$ with respect to time:

$$\begin{aligned} \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \mathbf{P}(t) \, d\mathbf{x} - \int_{\Omega(\hat{\mathbf{r}}(t_0), R_n)} \mathbf{P}(t_0) \, d\mathbf{x} \\ + Q_0 - Q_{01} = \int_{t_0}^t \int_{\Omega_n} \mathbf{f} \, d\mathbf{x} dt', \end{aligned} \quad (34.188)$$

with

$$Q_0 = \int_{t_0}^t \int_{\partial\Omega_n} \bar{\mathbf{n}} \mathcal{L}_{\text{KG}}(\psi) \, d\sigma dt' + \frac{\chi^2}{m} \operatorname{Re} \int_{t_0}^t \int_{\partial\Omega_n} \bar{\mathbf{n}} \cdot \nabla \psi \nabla \psi^* \, d\sigma dt', \quad (34.189)$$

$$Q_{01} = \int_{t_0}^t \int_{\partial\Omega_n} \mathbf{P}(t') \cdot \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \, d\sigma dt'. \quad (34.190)$$

Using (34.166) and (34.150), we infer that

$$|Q_0| \leq C |t - t_0| \max_{T_- \leq s \leq T_+} \int_{\partial\Omega_n} (|\mathcal{L}_{\text{KG}}(\psi)| + \chi^2 |\nabla \psi|^2) \, d\sigma \rightarrow 0. \quad (34.191)$$

Using (34.163), (34.1) and (34.150), we conclude that

$$|Q_{01}| \leq |t - t_0| \max_{T_- \leq s \leq T_+} \int_{\partial\Omega_n} |\hat{\mathbf{v}}| |\mathbf{P}| \, d\sigma \rightarrow 0.$$

We obtain from the relation (34.185) the following expression for $\mathbf{P}$:

$$\begin{aligned} \mathbf{P} = \frac{1}{c^2} \partial_t ((\mathbf{x} - \mathbf{r}) u) + \frac{1}{c^2} u \partial_t \mathbf{r} + \partial_j (\mathbf{P}_j (\mathbf{x} - \mathbf{r})) \\ + \frac{1}{c^2} (\nabla \varphi \cdot \mathbf{J}) (\mathbf{x} - \mathbf{r}). \end{aligned} \quad (34.192)$$

We use this expression to write the first term in (34.188) as follows:

$$\int_{\Omega_n} \mathbf{P}(t) \, d\mathbf{x} = \frac{1}{c^2} \partial_t \mathbf{r} \int_{\Omega_n} u \, d\mathbf{x} + Q_{02} \quad (34.193)$$

where

$$\begin{aligned} Q_{02} = \int_{\Omega_n} \left(\frac{1}{c^2} \partial_t ((\mathbf{x} - \mathbf{r}) u) + \partial_j (\mathbf{P}_j (\mathbf{x} - \mathbf{r})) \right) d\mathbf{x} \\ + \frac{1}{c^2} \int_{\Omega_n} (\nabla \varphi \cdot \mathbf{J}) (\mathbf{x} - \mathbf{r}) \, d\mathbf{x}. \end{aligned} \quad (34.194)$$

Using (34.186), we obtain that

$$\begin{aligned} Q_{02} = & -\frac{1}{c^2} \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} u \, d\mathbf{x} \\ & + \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \bar{\mathbf{n}} \cdot \mathbf{P} \, d\mathbf{x} + \frac{1}{c^2} \int_{\Omega_n} (\mathbf{x} - \mathbf{r}) \nabla\varphi \cdot \mathbf{J} \, d\mathbf{x}. \end{aligned}$$

Using (34.183), (34.163), (34.166) and (34.150), we conclude that

$$|Q_{02}| \leq C R_n \int_{\partial\Omega_n} (|u| + |\mathbf{P}|) \, d\mathbf{x} + C R_n \max_{\Omega_n} |\nabla\varphi| \int_{\Omega_n} |\mathbf{J}| \, d\mathbf{x} \rightarrow 0.$$

Quite similarly to (34.193),

$$\int_{\Omega(\hat{\mathbf{r}}(t_0), R_n)} \mathbf{P}(t_0) \, d\mathbf{x} = \frac{1}{c^2} \partial_t \mathbf{r}(t_0) \bar{\mathcal{E}}(t_0) + Q_{012} \quad (34.195)$$

where $Q_{012} \rightarrow 0$. Now we write the last term in (34.188) in the form

$$\begin{aligned} \int_{t_0}^t \int_{\Omega_n} \mathbf{F} \, d\mathbf{x} dt' &= - \int_{t_0}^t \nabla\varphi_\infty \int_{\Omega_n} \rho_\infty \, d\mathbf{x} dt' + Q_{03}, \\ Q_{03} &= - \int_{t_0}^t \int_{\Omega_n} (\rho_n \nabla\varphi - \rho_\infty \nabla\varphi_\infty) \, d\mathbf{x}. \end{aligned} \quad (34.196)$$

Obviously,

$$\int_{\Omega_n} (\rho_n \nabla\varphi - \rho_\infty \nabla\varphi_\infty) \, d\mathbf{x} = \int_{\Omega_n} (\nabla\varphi - \nabla\varphi_\infty) \rho_n \, d\mathbf{x} + \nabla\varphi_\infty (\bar{\rho}_n - \bar{\rho}_\infty),$$

and by Lemma 34.5.1 and condition (34.144)

$$\begin{aligned} |Q_{03}| &\leq |T_+ - T_-| \max_{\hat{\Omega}_n} |\nabla\varphi - \nabla\varphi_\infty| \int_{\Omega_n} |\rho_n| \, d\mathbf{x} \\ &\quad + |T_+ - T_-| \max_t |\nabla\varphi_\infty| |\rho_n - \rho_\infty| \rightarrow 0. \end{aligned}$$

From (34.188), using (34.193), (34.195), (34.196), we obtain the following equation:

$$\begin{aligned} \frac{1}{c^2} \bar{\mathcal{E}}_n(t) \partial_t \mathbf{r}(t) + Q_{02} - \frac{1}{c^2} \bar{\mathcal{E}}_n(t_0) \partial_t \mathbf{r}(t_0) - Q_{01} - Q_{012} + Q_0 \\ = - \int_{t_0}^t \bar{\rho}_\infty \nabla\varphi_\infty dt' + Q_{03} \end{aligned}$$

which implies (34.187) with $\delta_f = Q_{03} - Q_0 + Q_{012} + Q_{01} - Q_{02}$.

Proof (Proof of Theorem 34.5.1) According to Lemma 34.5.2, the adjacent ergo-centers $\mathbf{r}_n(t)$ converge to $\hat{\mathbf{r}}(t)$. We take t_0 from Definition 34.5.2 and choose such a subsequence that, according to Lemma 34.5.3 and 34.5.1, $\partial_t \mathbf{r}_n(t_0) \rightarrow v_\infty$, $\bar{\rho}_n \rightarrow \bar{\rho}_\infty$.

We multiply (34.187) by $c^2/\bar{\mathcal{E}}_n(t)$ and obtain that

$$\partial_t \mathbf{r}_n(t) = \frac{c^2}{\bar{\mathcal{E}}_n} \int_{t_0}^t \mathbf{f}_\infty dt' + \frac{c^2}{\bar{\mathcal{E}}_n} \delta_f + \frac{1}{\bar{\mathcal{E}}_n} \bar{\mathcal{E}}_n(t_0) \partial_t \mathbf{r}_n(t_0). \quad (34.197)$$

Integration of the above equation yields the expression

$$\begin{aligned} & \mathbf{r}_n(t) - \mathbf{r}_n(t_0) \\ &= \int_{t_0}^t \frac{c^2}{\bar{\mathcal{E}}_n} \delta_f dt'' + \int_{t_0}^t \frac{c^2}{\bar{\mathcal{E}}_n} \int_{t_0}^{t''} \mathbf{f}_\infty dt' dt'' + \int_{t_0}^t \frac{1}{\bar{\mathcal{E}}_n(t'')} dt'' \bar{\mathcal{E}}_n(t_0) \partial_t \mathbf{r}_n(t_0). \end{aligned}$$

Observe that the sequence $1/\bar{\mathcal{E}}_n$ converges uniformly to $1/\bar{\mathcal{E}}_\infty$ according to Theorem 34.5.2 and condition (34.152); $\mathbf{r}_n(t)$ converges to $\hat{\mathbf{r}}(t)$, and δ_f converges to zero. Hence, taking the limit of the above, we obtain that

$$\hat{\mathbf{r}}(t) - \hat{\mathbf{r}}(t_0) = \int_{t_0}^t \frac{c^2}{\bar{\mathcal{E}}_\infty} \int_{t_0}^{t''} \mathbf{f}_\infty dt' dt'' + \int_{t_0}^t \frac{1}{\bar{\mathcal{E}}_\infty(t'')} dt'' \bar{\mathcal{E}}_\infty(t_0) v_\infty. \quad (34.198)$$

Taking the time derivative of the above equation, multiplying the result by $\bar{\mathcal{E}}_\infty(t)/c^2$ and taking the time derivative once more, we find that $\hat{\mathbf{r}}(t)$ satisfies (34.159), and $\hat{\mathbf{r}}(t_0) = \mathbf{r}_\infty(t_0)$, $\partial_t \hat{\mathbf{r}}(t_0) = v_\infty$.

Notice that Eq. (34.171) implies that

$$\partial_t \bar{\mathcal{E}}_\infty = \partial_t \hat{\mathbf{r}} \cdot \mathbf{f}_\infty. \quad (34.199)$$

Multiplying (34.159) by $2M\partial_t \hat{\mathbf{r}}$ where $M = \bar{\mathcal{E}}_\infty(t)/c^2$ and using (34.199), we obtain that

$$\partial_t (M\partial_t \hat{\mathbf{r}})^2 = 2M\partial_t \hat{\mathbf{r}} \cdot \mathbf{f}_\infty = 2c^2 M \partial_t M. \quad (34.200)$$

This equation implies after time integration the following relation:

$$M^2 - c^{-2} M^2 (\partial_t \hat{\mathbf{r}})^2 = M_0^2,$$

where M_0^2 is a constant of integration; consequently, we obtain (34.162). Note that $M_0 = \gamma^{-1}(t_0) c^{-2} \bar{\mathcal{E}}_\infty(t_0)$ is uniquely determined by the value $\bar{\mathcal{E}}_\infty(t_0)$ from (34.153). Hence the limit of the restricted energy $\bar{\mathcal{E}}_\infty(t)$ does not depend on particular subsequences $\partial_t \mathbf{r}_n(t_0)$, $\bar{\rho}_n$ and on $\bar{\rho}_\infty$. Based on this, we conclude that any subsequence $\bar{\mathcal{E}}_n(t)$ has the same limit, and the convergence in (34.170) holds for the given concentrating sequence.

Corollary 34.5.1 *Assume that $\partial_t^2 \hat{\mathbf{r}}$ is not identically zero on $[T_-, T_+]$. Then (34.168) holds for any subsequence of the concentrating sequence and the sequence $\bar{\rho}_n(t)$ converges to $\bar{\rho}_\infty$.*

Proof We have to prove that every convergent subsequence $\bar{\rho}_n(t_0)$ in Lemma 34.5.1 converges to the same limit. If we have a convergent subsequence $\bar{\rho}_n(t_0)$ which converges to $\bar{\rho}_\infty$, we obtain (34.162) and (34.159). According to formula (34.162), we can rewrite (34.159) in the form

$$M_0 \gamma \partial_t^2 \hat{\mathbf{r}} + M_0 \partial_t \gamma \partial_t \hat{\mathbf{r}} = -\bar{\rho}_\infty \nabla \varphi_\infty. \quad (34.201)$$

This relation uniquely determines $\bar{\rho}_\infty$ if $\nabla \varphi_\infty(t)$ is not identically zero. The left-hand side vanishes on an interval only if

$$\partial_t \hat{\mathbf{v}} + \hat{\mathbf{v}} \partial_t \ln \gamma = \mathbf{0}, \text{ where } \gamma = (1 - \hat{\mathbf{v}}^2/c^2)^{-1/2}. \quad (34.202)$$

The above equation implies that $\hat{\mathbf{v}}^2$ is a constant, and (34.201) implies that $\partial_t \hat{\mathbf{v}} = 0$ on the interval. Therefore, for the accelerated motion, $\nabla \varphi_\infty$ is not identically zero on the interval, and $\bar{\rho}_\infty$ is uniquely determined by $\hat{\mathbf{r}}, t_0$ and M_0 where M_0 is uniquely defined by (34.162) according to (34.153).

In the following Sect. 17.6.2 we present a class of examples where solutions of the NKG equation concentrate at a trajectory which describes an accelerated motion.

34.6 Rectilinear Accelerated Motion of a Wave-Corpuscle

We have proven in the preceding section that, if solutions of the NKG equation concentrate at a trajectory, then the trajectory of concentration and the energy satisfy the relativistic version of Newton's equation where the mass is defined by Einstein's formula. The results of the previous section are valid for general concentrating solutions of the NKG equation. In this section we present a specific class of examples for which the concentration assumptions can be explicitly verified. The constructions are explicit and provide examples of relativistic accelerated motion with the internal energy determined from the Klein–Gordon Lagrangian. To be able to construct an explicit example, we make the following simplifying assumptions: (i) the motion is rectilinear; (ii) the nonlinearity is logarithmic as in (14.41); (iii) the shape $|\psi|$ is fixed, namely it is Gaussian; (iv) the parameter $\zeta = \zeta_n$ in (34.141) satisfies condition $\zeta_n \rightarrow 0$. For a given rectilinear trajectory we find a sequence of parameters, potentials and solutions of the NKG equation which concentrate at the trajectory. Now we formulate the assumptions in detail.

Recall that the parameters c, m, q are fixed. We assume that the sequence $a = a_n, \chi = \chi_n$ satisfies the conditions (34.140), (34.141) and an additional condition

$$\zeta = \zeta_n = \frac{ac}{a} = \frac{\chi_n}{a_n mc} \rightarrow 0. \quad (34.203)$$

Trajectories $\hat{\mathbf{r}}(t)$ which describe *rectilinear* accelerated motion have the form

$$\hat{\mathbf{r}}(t) = (0, 0, r(t)), \quad -\infty < t < \infty. \quad (34.204)$$

We consider a fixed trajectory $r(t)$ such that corresponding velocity $v = \partial_t r$ is two times continuously differentiable and has uniformly bounded derivatives:

$$|v(t)| + |\partial_t v(t)| + |\partial_t^2 v(t)| \leq C, \quad -\infty < t < \infty. \quad (34.205)$$

We also impose a weaker version of the above restriction,

$$\sup_{\tau} (|\partial_{\tau} \beta| + |\partial_{\tau}^2 \beta|) \leq \hat{c}, \quad \text{where } \tau = \frac{c}{a} t, \quad (34.206)$$

β is the *normalized velocity*, namely $\beta = v/c$, $v = \partial_t r$, as in (17.163), this version is sufficient in many estimates. Since the parameter $a = a_n \rightarrow 0$, the assumption (34.206) is less restrictive than (34.205).

The velocity $v = \partial_t r$ is assumed to be smaller than the speed of light c , namely

$$|\beta(t)| \leq \hat{c}_1 < 1, \quad -\infty < t < \infty; \quad (34.207)$$

we assume also that the normalized velocity does not vanish:

$$|\beta(t)| \geq \check{\beta} > 0, \quad -\infty < t < \infty. \quad (34.208)$$

Here is the main result of this section.

Theorem 34.6.1 *For any trajectory $\hat{\mathbf{r}}(t) = (0, 0, r(t))$ where r satisfies (34.205), (34.206), (34.207), (34.208), for any T_- and T_+ there exists a sequence a_n, R_n, χ_n , and potentials φ_n such that the NKG equations are localized in $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$. There exists a sequence of solutions of the NKG equation which concentrates at $\hat{\mathbf{r}}$.*

Proof The statement follows from Theorem 34.6.4 which is proven at the end of this section. The potentials and solutions are described in the following subsections.

34.6.1 Reduction to One Dimension

When the external potential φ depends only on t and x_3 , the Eq. (34.128) in three-dimensional space with a logarithmic nonlinearity (14.42) can be reduced to a

problem in the one-dimensional space by the substitution (17.157), namely

$$\psi = \pi^{-1/2} a^{-1} \exp(-a^{-2}(x_1^2 + x_2^2)/2) \psi_{1D}(t, x_3) \quad (34.209)$$

where ψ_{1D} depends only on x_3 and t . The corresponding reduced 1D NKG equation for $\psi = \psi_{1D}$ with *one spatial variable* has the form of (17.158), namely

$$-c^{-2} \tilde{\partial}_t^2 \psi + \partial_3^2 \psi - G'_a(\psi^* \psi) \psi - \kappa_0^2 \psi = 0, \quad (34.210)$$

where the 1D logarithmic nonlinearity takes the form (17.159) and $\tilde{\partial}_t = \partial_t + \frac{iq}{\hbar} \varphi_{\text{ex}}$.

From now on, we write x instead of x_3 for notational simplicity. We look for the external potential φ in the form

$$\varphi(t, x) = \varphi_{\text{ac}}(t, x) + \varphi_{\text{b}}(t, x; \zeta), \quad (34.211)$$

where the *accelerating potential* φ_{ac} is *linear in y*, namely (17.165) holds:

$$\varphi_{\text{ac}} = \varphi_0(t) + \varphi'_{\text{ac}} y, \quad y = x - r, \quad (34.212)$$

and $\varphi_{\text{b}}(t, x; \zeta)$ is a small *balancing potential*. The coefficient $\varphi'_{\text{ac}} = \partial_x \varphi_{\text{ac}}$ is determined by the trajectory $r(t)$ according to the formula (17.166), namely

$$\partial_t(m\gamma v) + q \partial_x \varphi_{\text{ac}}(t, r) = 0, \quad (34.213)$$

which has the form of the *relativistic law of motion* (17.114). The potential φ_{ac} coincides in our construction with the limit potential φ_{∞} in (34.145). The coefficient $\varphi_0(t)$ can be prescribed as an arbitrary function with bounded derivatives.

According to (34.213), φ_{ac} directly relates to the acceleration of the charge, therefore we call it the “accelerating” potential, φ_{ac} does not depend on the small parameter $\zeta = \frac{ac}{a}$. The remaining part of the external potential φ in the NKG equation (34.210) is a small “balancing” potential φ_{b} which helps the charge to exactly preserve its form as it accelerates. Below we find a potential φ_{b} such that the Gaussian wave function with center $r(t)$ is a solution of (34.210) in the strip

$$\begin{aligned} \mathcal{E}(\theta_n) &= \{(t, x) : |x - r(t)| \leq \theta_n a_n\}, \\ \theta_n &\rightarrow \infty, \quad \theta_n a_n = R_n \rightarrow 0. \end{aligned} \quad (34.214)$$

The balancing potential vanishes asymptotically, that is, $\varphi_{\text{b}}(t, x; \zeta) \rightarrow 0$ as $\zeta \rightarrow 0$, and the forces it produces also become vanishingly small compared with the electric force $-q \partial_x \varphi_{\text{ac}}(x)$ in the strip $\mathcal{E}(\theta)$. Note that such an accelerated motion in the relativistic regime with a preserved shape is possible only with a properly chosen balancing potential φ_{b} .

34.6.2 Equation in a Moving Frame

As the first step of the construction of the potential φ_b , we rewrite the NKG equation (34.210) in a moving frame. We take $\hat{\mathbf{r}}(t)$ as the new origin and make the change of variables

$$\mathbf{x} = \hat{\mathbf{r}}(t) + \mathbf{y}, \quad x_3 = r(t) + y, \quad \psi(t, x) = \psi'(t, y), \quad v = \partial_t r. \quad (34.215)$$

The 1D NKG equation (34.210) then takes the form

$$\begin{aligned} -c^{-2} (\partial_t + iq\varphi/\chi - v\partial_y) (\partial_t + iq\varphi/\chi - v\partial_y) \psi \\ + \partial_y^2 \psi - G'_a (\psi^* \psi) \psi - a_C^{-2} \psi = 0. \end{aligned} \quad (34.216)$$

The 1D logarithmic nonlinearity $G'_a = G'_{a,1D}$ is defined by (17.159), and the electric potential φ has the form determined by (17.164), (17.165), namely

$$\varphi(t, x) = \varphi_{ac}(t, x) + \varphi_b(t, x; \zeta) \quad \varphi_{ac} = \varphi_0(t) + \varphi'_{ac} y, \quad y = x - r.$$

We assume that the solution $\psi(t, x)$ has the Gaussian form, namely

$$\begin{aligned} \psi = \psi_{1D} = \hat{\psi} \exp(i\hat{S}), \\ \hat{S} = \omega_0 c^{-2} \gamma v y - s(t) - S(t, y), \quad \omega_0 = c/a_C = mc^2/\chi \end{aligned} \quad (34.217)$$

where velocity $v(t)$ is a given function of time, and we explicitly define the real-valued function $\hat{\psi}$:

$$\hat{\psi} = a^{-1/2} \Psi(a^{-1} y) \quad (34.218)$$

where

$$\Psi(t, z) = \pi^{-1/4} e^{\sigma - z^2/2}, \quad (34.219)$$

with

$$\sigma = \sigma(t) = \ln \gamma^{-1/2}, \quad \gamma = (1 - \beta^2)^{-1/2}. \quad (34.220)$$

We define σ by the above formula (34.220) to satisfy Eq. (34.233) which arises in our analysis.

Substitution of (34.217) into (34.216) yields equation

$$\begin{aligned} -\frac{1}{c^2} \partial_t^2 \hat{\psi} + \left(\partial_y - i \partial_y S + i v \frac{\omega_0}{c^2} \gamma \right)^2 \hat{\psi} \\ - G'_a (|\hat{\psi}|^2) \hat{\psi} - \kappa_0^2 \hat{\psi} = 0, \end{aligned} \quad (34.221)$$

with the operator

$$\begin{aligned}\hat{\partial}_t = & \partial_t + i\partial_t(\gamma v) \frac{\omega_0}{c^2} y - i\partial_t s + \frac{iq\varphi_0}{\chi} - iv^2 \frac{\omega_0}{c^2} \gamma \\ & - i\partial_t S + iv\partial_y S + \frac{iq\varphi'_{ac}}{\chi} y + \frac{iq\varphi_b}{\chi} - v\partial_y.\end{aligned}$$

To eliminate the leading, i.e. independent of the small parameter ζ , terms in the above equation, we require that the following equation holds:

$$-\partial_t s + q\varphi_0/\chi - v^2\omega_0 c^{-2}\gamma = -\gamma\omega_0, \quad (34.222)$$

and the coefficient $\varphi'_{ac} = \partial_x \varphi_{ac}$ is determined by the trajectory $r(t)$ according to (17.166).

Observe that the expressions in (34.221) eliminated in view of Eqs. (34.222) and (17.166) do not depend on ζ . Note that the constant part $\varphi_0(t)$ of the accelerating potential can be prescribed arbitrarily in (17.165) since we can always choose the phase shift $s(t)$ so that the Eq. (34.222) holds. Equation (34.221) combined with Eqs. (34.222) and (17.166) can be transformed into the following equation:

$$\begin{aligned}-\frac{1}{c^2} \left(\partial_t - i\gamma \frac{c}{a_c} - i\partial_t S + iv\partial_y S + \frac{iq\varphi_b}{\chi} - v\partial_y \right)^2 \hat{\psi} \\ + \left(\partial_y - i\partial_y S + i\frac{1}{a_c} \frac{v}{c} \gamma \right)^2 \hat{\psi} - G'_a(|\hat{\psi}|^2) \hat{\psi} - \frac{1}{a_c^2} \hat{\psi} = 0.\end{aligned} \quad (34.223)$$

We will use the fact that, if $S = 0$ and $\varphi = 0$, the function Ψ in (34.219) with $\sigma = 0$ satisfies the equation

$$\partial_3^2 \Psi - G'_1(\Psi^2) \Psi = 0. \quad (34.224)$$

Then a direct computation based on the definition of γ shows that $\hat{\psi}$ given by (34.218) with $\sigma = 0$ provides a time-independent solution of equation (34.223). In the general case, φ_b can be considered as a perturbation of the trivial solution $\varphi = 0$. In the following sections, we find the potential φ_b and phase function S which are of order ζ^2 and satisfy Eq. (34.223).

34.6.3 Equations for Auxiliary Phases

We introduce in this subsection two auxiliary phases and reduce the problem of determination of the potential φ_b and the phase S to a first-order partial differential equation for a single unknown phase. Solution of this equation can be reduced to integration along characteristics, hence we are able to fulfil a rather detailed mathematical analysis.

It is convenient to introduce in the NKG equation (34.128) and formula (34.209) the following rescaled dimensionless variables $\mathbf{z}, \tau$:

$$\tau = \frac{c}{a}t, \quad \mathbf{z} = \frac{\zeta}{a_C}\mathbf{y} = \frac{1}{a}\mathbf{y}. \quad (34.225)$$

In particular, in the 1D NKG equation (34.210) and in (34.223)

$$\tau = \frac{c}{a}t, \quad z_3 = z = \frac{\zeta}{a_C}y. \quad (34.226)$$

We introduce now auxiliary phases Z and Φ :

$$Z = \zeta \partial_z S, \quad (34.227)$$

$$\Phi = -\zeta \partial_\tau S + \zeta \beta \partial_z S + \frac{q a_C \varphi_b}{c \chi}; \quad (34.228)$$

these phases will be our new unknown variables. Obviously, if we find Z and Φ , we can find S by integration in z and setting $S = 0$ at $z = 0$, see (34.305). After that, φ_b can be found from (34.228). Consequently, to find a small φ_b , we need to find small Z, Φ .

Equation (34.223) takes the following form:

$$\begin{aligned} & -(\zeta \partial_\tau + i\Phi - i\gamma - \beta \zeta \partial_z)^2 \Psi \\ & + (\zeta \partial_z - iZ + i\beta \gamma)^2 \Psi - \zeta^2 G'_1(\Psi^2) \Psi - \Psi = 0 \end{aligned} \quad (34.229)$$

where Ψ is explicitly given by (34.219). We look for a solution of (34.229) in the strip $\mathcal{E}(\theta)$ in the space-time:

$$\mathcal{E}(\theta) = \{(\tau, z) : -\infty < \tau < \infty, \quad |z| < \theta\}. \quad (34.230)$$

We expand (34.229) with respect to Φ, Z and rewrite Eq. (34.229) in the form

$$\begin{aligned} Q\Psi - i\Phi(\zeta \partial_\tau - i\gamma - \beta \zeta \partial_z)\Psi - i(\zeta \partial_\tau - i\gamma - \beta \zeta \partial_z)(\Phi\Psi) \\ + \Phi^2\Psi + iZ(\zeta \partial_z + i\beta \gamma)\Psi + i(\zeta \partial_z + i\beta \gamma)(Z\Psi) - Z^2\Psi = 0, \end{aligned} \quad (34.231)$$

where we denote by $Q\Psi$ the term which does not involve Φ and Z explicitly:

$$\begin{aligned} Q &= -\zeta^2 G'_1(\Psi^2) - 1 \\ &+ \frac{1}{\Psi} \left(-(\zeta \partial_\tau - i\gamma - \beta \zeta \partial_z)^2 \Psi + (\zeta \partial_z + i\beta \gamma)^2 \Psi \right). \end{aligned} \quad (34.232)$$

Using (34.219) and (34.220), we conclude that the imaginary part of Q is zero:

$$\text{Im } Q = 2\zeta\gamma\partial_\tau\sigma + \zeta\partial_\tau\gamma = 0. \quad (34.233)$$

Hence, $Q = \text{Re } Q$ can be written in the form

$$Q = \zeta^2 \left(-\partial_\tau^2 \Psi + \partial_\tau \beta \partial_z \Psi + 2\beta \partial_\tau \partial_z \Psi + \gamma^{-2} \partial_z^2 \Psi - G'_1(\Psi^2) \Psi \right) / \Psi. \quad (34.234)$$

Now we show explicitly the dependence of Q on z . Using (17.159), we easily verify that the function Ψ in (34.219) satisfies the following equation similar to (14.19):

$$\partial_z^2 \Psi - G'_a(\Psi \Psi^*) \Psi = 2\sigma \Psi. \quad (34.235)$$

Taking into account (34.235) and (34.219), we see that

$$\begin{aligned} Q &= \zeta^2 \left[-(\partial_\tau^2 \sigma + (\partial_\tau \sigma)^2) + 2\sigma \right] \\ &\quad - \zeta^2 \left[z \partial_\tau \beta + 2\beta z \partial_\tau \sigma + \beta^2 \zeta^2 (z^2 - 1) \right]. \end{aligned} \quad (34.236)$$

Now we rewrite the complex equation (34.231) as a system of two real equations. The real part of (34.231) divided by Ψ yields the following quadratic equation

$$Q - 2\gamma\Phi + \Phi^2 - 2\beta\gamma Z - Z^2 = 0. \quad (34.237)$$

The solution Z which is small for small Φ , Q is given by the formula

$$Z = \Theta(\Phi) = -\beta\gamma + (\Phi^2 - 2\gamma\Phi + \beta^2\gamma^2 + Q)^{1/2} \beta / |\beta|. \quad (34.238)$$

The imaginary part of (34.231) divided by $\zeta\Psi$ yields equation

$$-2\Phi(\partial_\tau - \beta\partial_z) \ln \Psi - (\partial_\tau - \beta\partial_z)\Phi + \partial_z Z + 2Z\partial_z \ln \Psi = 0 \quad (34.239)$$

where the coefficients are expressed in terms of Ψ defined by (34.219):

$$\ln \Psi = \sigma - z^2/2, \quad \partial_z \ln \Psi = -z, \quad \partial_\tau \ln \Psi = \partial_\tau \sigma. \quad (34.240)$$

To determine a *small* solution Φ of (34.239), (34.238) in the strip $\mathcal{E}$, we impose the condition

$$\Phi = 0 \text{ if } z = 0, \quad -\infty < \tau < \infty. \quad (34.241)$$

A solution Φ of the Eq. (34.239), where $Z = \Theta(\Phi)$ satisfies (34.238), is a solution of the following quasilinear first-order equation

$$\partial_\tau \Phi - \beta \partial_z \Phi - \Theta_\Phi(\Phi) \partial_z \Phi = -2\Phi \zeta (\partial_\tau - \beta \partial_z) \ln \Psi + 2\Theta \partial_z \ln \Psi + \Theta_z, \quad (34.242)$$

where Θ_Φ and Θ_z are the partial derivatives of $\Theta(\Phi)$ (Θ depends on z via Q) with Φ subjected to condition (34.241).

34.6.4 Construction and Properties of the Auxiliary Potential

To prove the existence of a solution in a wide enough strip $\mathcal{E}$ and study its properties, we use the method of characteristics. We introduce the characteristic equations for (34.242):

$$dz/ds = -\Theta_\Phi(\Phi) - \beta(\tau), \quad (34.243)$$

$$d\tau/ds = 1, \quad (34.244)$$

$$d\Phi/ds = -2\Phi(\partial_\tau - \beta\partial_z)\ln\Psi + 2\Theta(\Phi)\partial_z\ln\Psi + \Theta_z, \quad (34.245)$$

with the initial data

$$\tau_{s=0} = \tau_0, \quad z_{s=0} = 0. \quad (34.246)$$

From condition (34.241) on the line $z = 0$ we derive the initial condition

$$\Phi_{s=0} = 0. \quad (34.247)$$

The quantity Φ in the above equations is an independent variable which coincides with $\Phi = \Phi(\tau, z)$ on the characteristic curves:

$$\Phi = \Phi(\tau(\tau_0, s), z(\tau_0, s)). \quad (34.248)$$

Equation (34.244) can be solved explicitly:

$$\tau = \tau_0 + s.$$

Often, in a standard abuse of notation, we will write $\Phi(\tau_0, s)$ for the solution of (34.243)–(34.245) which is found without using the formula (34.248). The function $\Theta(\Phi)$ and its partial derivatives Θ_z and Θ_Φ are determined by (34.238) and are well-defined and smooth as long as

$$D(\tau, z, \Phi) = \Phi^2 - 2\gamma(\tau)\Phi + \beta^2\gamma^2(\tau) + Q(\tau, z) > 0. \quad (34.249)$$

Then formula (34.238) determines the function Θ as an analytic function of Q, Φ . Taking into account (34.206), we see that the right-hand side of the system (34.243)–(34.245) is a two times continuously differentiable function of variables τ, z, Φ in the domain in $\mathbb{R}^3$ defined by the inequality (34.249).

From classical theorem on the existence, uniqueness and regular dependence on the initial data and parameters of solutions of the Cauchy problem for a system

of ordinary differential equations (see, for instance, [62]), we readily obtain the following statement.

Lemma 34.6.1 *Let the initial data*

$$\tau_{s=0} = \tau_0, \quad z_{s=0} = z_0, \quad \Phi_{s=0} = \Phi_0 \quad (34.250)$$

be such that the function D defined by (34.249) is positive, namely $D(\tau_0, z_0, \Phi_0) > 0$. Then the system (34.243), (34.245) with the initial condition (34.250) has a unique solution $\tau(s), z(s), \Phi(s)$ defined on a maximal (finite or infinite) interval $s_- < s < s_+$. If the value $s_{\pm}$ is finite, then

$$\text{either } \lim_{s \rightarrow s_{\pm}} (|\Phi|^2 + z^2) = \infty \quad \text{or} \quad \lim_{s \rightarrow s_{\pm}} D(\tau, z, \Phi) = 0. \quad (34.251)$$

The solution is a twice continuously differentiable function of s , of the initial data τ_0, z_0, Φ_0 , and of the parameter ζ , and we have an explicit expression $\tau(s) = \tau_0 + s$.

34.6.4.1 Properties of the Characteristic Curves

The characteristic system (34.243)–(34.245), (34.247) involves the function $Z = \Theta(\Phi)$ defined by (34.238). We give sufficient conditions on the variables which ensure that (34.249) holds.

Proposition 34.6.1 *If*

$$|\Phi| \leq \check{\beta}^2/4 \quad (34.252)$$

and

$$Q \geq -\check{\beta}^2/4 \quad (34.253)$$

where $\check{\beta} > 0$ is the parameter from (34.208), then

$$D(\tau, z, \Phi) \geq \Phi^2 + \check{\beta}^2/4, \quad (34.254)$$

and (34.249) holds. Conditions (34.252) and (34.253) are fulfilled if

$$\zeta^{1/3} |z| \leq 1, \quad (34.255)$$

$$\zeta^2 \sup_{\tau} |\partial_{\tau}^2 \sigma + (\partial_{\tau} \sigma)^2 - 2\sigma - \beta^2 \zeta^2| \quad (34.256)$$

$$+ \zeta^{4/3} \sup_{\tau} |\partial_{\tau} \beta + 2\beta \partial_{\tau} \sigma| + \zeta^{10/3} \leq \check{\beta}^2/8.$$

Proof If (34.252) and (34.253) hold, we take into account that $\gamma \geq 1$ and conclude that

$$\Phi^2 - 2\gamma\Phi + Q + \beta^2\gamma^2 \geq \Phi^2 + \check{\beta}^2/4 > 0. \quad (34.257)$$

According to (34.236), condition (34.253) takes the form

$$\zeta^2 (\partial_\tau^2 \sigma + (\partial_\tau \sigma)^2 - 2\sigma - \beta^2 \zeta^2) + z \zeta^2 (\partial_\tau \beta + 2\beta \partial_\tau \sigma) + \zeta^4 z^2 \beta^2 \leq \check{\beta}^2 / 4.$$

Hence (34.253) is fulfilled if (34.255), (34.256) hold.

Therefore, $\Theta(\Phi)$ is a regular function of Φ which satisfies inequality (34.252) in the strip $\Xi(\theta)$ defined by (34.230) as long as ζ satisfies (34.255), (34.256).

It is convenient to introduce the following notation. Let 2_+ be a fixed number which is arbitrarily close to 2 and 1_+ be arbitrarily close to 1, and the numbers satisfy the inequality

$$1 < 2_+/2 < 1_+ < 2. \quad (34.258)$$

Below we obtain estimates of solutions of the characteristic equations in the strip $\Xi(\bar{\theta})$,

$$\bar{\theta} = \bar{\theta}(\zeta) = \ln^{1/2_+} \zeta^{-1}. \quad (34.259)$$

It is also convenient to introduce the following functions:

$$b(\tau) = \beta^{-1}(\tau) - \beta(\tau), \quad B(\tau_0, s) = \int_{\tau_0}^{\tau_0+s} b(\tau) d\tau. \quad (34.260)$$

Using the inequality

$$0 < \hat{\epsilon}_1^{-1} - \hat{\epsilon}_1 \leq |\beta^{-1} - \beta| = (\beta^{-1} - \beta) \beta / |\beta| \leq \check{\beta}^{-1} - \check{\beta} \quad (34.261)$$

where $\hat{\epsilon}_1, \check{\beta}$ are parameters in (34.207), (34.208), we obtain that

$$(\hat{\epsilon}_1^{-1} - \hat{\epsilon}_1) |s| \leq B(\tau_0, s) \leq |s| (\check{\beta}^{-1} - \check{\beta}). \quad (34.262)$$

Lemma 34.6.2 *Let $\zeta \leq 1/C_0$ where C_0 is sufficiently large. Let τ, z, Φ be a solution to (34.243)–(34.247), and assume that $z(s), \Phi(s)$ are defined on an interval $s_{1-} < s < s_{1+}$ with values in the strip $\Xi(\bar{\theta})$ and that the following estimate holds:*

$$|\Phi(\tau(s), z(s))| \leq |\zeta| \quad \text{for } s_{1-} < s < s_{1+}. \quad (34.263)$$

Then on the same interval

$$0 < (\hat{\epsilon}_1^{-1} - \hat{\epsilon}_1) / 2 \leq |dz/ds| \leq 2 (\check{\beta}^{-1} - \check{\beta}), \quad (34.264)$$

$$(\hat{\epsilon}_1^{-1} - \hat{\epsilon}_1) |s| / 2 \leq |z| \leq 2 |s| (\check{\beta}^{-1} - \check{\beta}), \quad (34.265)$$

and there exist constants $C > 0$, $C_5 > 0$ such that we have:

$$|z - B(\tau_0, s)| \leq C\zeta |s|, \quad (34.266)$$

$$|\Phi| \leq C_5\zeta^2 (|z|^2 + |z|) e^{|z|^2}. \quad (34.267)$$

Proof If (34.263) holds, conditions (34.255), (34.256) are satisfied for $\zeta \leq 1/C_0$, implying that (34.253) and (34.254) are fulfilled. Equation (34.243) can be written in the form

$$dz/ds = \beta^{-1} - \beta - \Theta_\Phi^1(\Phi) \quad (34.268)$$

where

$$\begin{aligned} \Theta_\Phi^1 = \Theta_\Phi(\Phi) + \beta^{-1} &= \frac{\Phi}{((\Phi - \gamma)^2 - 1 + Q)^{1/2}} \frac{\beta}{|\beta|} \\ &+ \frac{\Phi^2 - 2\gamma\Phi + Q}{\left(((\Phi - \gamma)^2 - 1 + Q)^{1/2} + \gamma|\beta| \right) ((\Phi - \gamma)^2 - 1 + Q)^{1/2}}. \end{aligned} \quad (34.269)$$

Evidently, Θ_Φ^1 has the form

$$\Theta_\Phi^1 = \Phi U_1 + Q U_2, \quad (34.270)$$

where U_1 and U_2 are algebraic expressions which are analytic if (34.254) is fulfilled, and consequently they are bounded. Using (34.236) with $|z| \leq \bar{\theta}$ and small ζ , we conclude that

$$\begin{aligned} |Q| &= \zeta^2 \left| \partial_\tau^2 \sigma + (\partial_\tau \sigma)^2 - 2\sigma - \beta^2 \zeta^2 + z(\partial_\tau \beta + 2\beta \partial_\tau \sigma) + \zeta^2 z^2 \beta^2 \right| \\ &\leq C_0 \zeta^2 (|z| + 1), \end{aligned} \quad (34.271)$$

and we obtain the following estimate:

$$|\Theta_\Phi^1| \leq C_1 \zeta \text{ for } s_{1-} < s < s_{1+}. \quad (34.272)$$

Hence, (34.243) implies that

$$|dz/ds - \beta^{-1} + \beta| \leq C\zeta^{2-\delta}. \quad (34.273)$$

We take ζ so small that $2C\zeta \leq \min(\hat{\epsilon}_1^{-1} - \hat{\epsilon}_1, \check{\beta}^{-1} - \check{\beta})$, and using (34.261) we obtain (34.264). Then integration implies (34.265) and (34.266).

Observe that Eq. (34.245) can be written in the form

$$\begin{aligned} \frac{d\Phi}{ds} &= -2\Phi \partial_\tau \ln \Psi + 2(\Theta(\Phi) - \Phi \Theta_\Phi(\Phi)) \partial_z \ln \Psi \\ &+ 2\Phi(\Theta_\Phi(\Phi) + \beta) \partial_z \ln \Psi + \Theta_z, \end{aligned}$$

which, with the use of (34.243), can be rewritten as

$$\frac{d\Phi}{ds} = -2\Phi \partial_\tau \ln \Psi + 2(\Theta(\Phi) - \Phi \Theta_\Phi(\Phi)) \partial_z \ln \Psi - 2\Phi \frac{dz}{ds} \partial_z \ln \Psi + \Theta_z.$$

According to (34.253), $\beta^2 \gamma^2 + Q > 0$, and we rewrite (34.245) in the form

$$\begin{aligned} \frac{d\Phi}{ds} &= -\Phi \frac{d \ln \Psi^2}{ds} + 2\Theta^1(\Phi) \partial_z \ln \Psi^2 \\ &\quad - 2 \left(\beta\gamma - (\beta^2 \gamma^2 + Q)^{1/2} \beta / |\beta| \right) \partial_z \ln \Psi^2 + \Theta_z \end{aligned} \quad (34.274)$$

where

$$\Theta^1(\Phi) = \Theta(\Phi) - \Phi \Theta_\Phi(\Phi) + \beta\gamma - (\beta^2 \gamma^2 + Q)^{1/2} \beta / |\beta|. \quad (34.275)$$

Multiplying by Ψ^2 , we obtain that

$$\frac{d(\Psi^2 \Phi)}{ds} = 2\Theta_1(\Phi) z - 2 \left(\beta\gamma - (\beta^2 \gamma^2 + Q)^{1/2} \right) z + \Theta_z \Psi^2. \quad (34.276)$$

According to (34.238),

$$\begin{aligned} &\frac{\beta}{|\beta|} \Theta^1(\Phi) \\ &= - \frac{\Phi^2}{((\Phi - \gamma)^2 - 1 + Q)^{1/2}} \frac{(\beta^2 \gamma^2 + Q)^{1/2}}{(\beta^2 \gamma^2 + Q)^{1/2} + ((\Phi - \gamma)^2 - 1 + Q)^{1/2}} \\ &\quad - \frac{\gamma \Phi^2 (\Phi - 2\gamma)}{\left[(\beta^2 \gamma^2 + Q)^{1/2} + ((\Phi - \gamma)^2 - 1 + Q)^{1/2} \right]^2 ((\Phi - \gamma)^2 - 1 + Q)^{1/2}}. \end{aligned} \quad (34.277)$$

Hence, taking into account (34.254), we obtain that

$$|\Theta^1(\Phi)| \leq C_1 \Phi^2.$$

To estimate remaining terms in (34.276), we note that in $\mathcal{E}(\bar{\theta})$

$$\left| \beta\gamma - (\beta^2 \gamma^2 + Q)^{1/2} \right| \leq C |Q| \leq C' \zeta^2 |z|. \quad (34.278)$$

Using (34.254), we obtain in $\mathcal{E}(\bar{\theta})$ the estimate

$$|\Theta_z(\Phi)| = \frac{|\zeta^2 (\partial_\tau \beta + 2\beta \partial_\tau \sigma) + 2\zeta^4 z \beta^2|}{2(\Phi^2 - 2\gamma\Phi + \beta^2 \gamma^2 + Q)^{1/2}} \leq C_1 \zeta^2. \quad (34.279)$$

The above estimates, combined with (34.276) and (34.279), yield

$$\left| \frac{d(\Psi^2 \Phi)}{ds} \right| \leq C_2 \Phi^2 |z| + C'' \zeta^2 |z| + C_3 \zeta^2. \quad (34.280)$$

Using (34.265), we obtain that

$$|\Psi^2 \Phi| \leq C_4 \zeta^2 s^2 + C_3 \zeta^2 |s|,$$

and, using (34.265) and the definition of Ψ , we obtain (34.267).

Theorem 34.6.2 *Let δ be an arbitrary small fixed number satisfying $0 < \delta < \frac{2_+-2}{32}$, and suppose that $\zeta \leq 1/C_0$ is sufficiently small. Let $s_{2-} < s < s_{2+}$ be a maximal interval on which a solution τ, z, Φ of (34.243)–(34.247) is such that $z(s), \tau(s)$ takes values in the strip $\Xi(\bar{\theta})$ where $\bar{\theta} = \ln^{1/2+} \zeta^{-1}$ and $|\Phi(s)| \leq \zeta$. Then*

$$z(s_{2+}) = \bar{\theta} \beta / |\beta|, \quad z(s_{2-}) = -\bar{\theta} \beta / |\beta|, \quad \bar{\theta} = \ln^{1/2+} \zeta^{-1}, \quad (34.281)$$

and $z(s), \Phi(s)$ satisfy inequalities (34.254), (34.265) on this interval as well as the inequality

$$|\Phi| \leq \zeta^{2-\delta}, \quad \left| \frac{d\Phi}{ds} \right| \leq C_7 \zeta^{2-\delta}. \quad (34.282)$$

The constants C_7, C_0 depend on δ but do not depend on τ_0, ζ .

Proof Consider an interval (s_{1-}, s_{1+}) satisfying the conditions of Lemma 34.6.2. According to Lemma 34.6.1, such an interval exists. Consider the maximal interval (s_{1-}, s_{1+}) such that $z(s), \Phi(s)$ takes values in the strip $\Xi(\bar{\theta})$ and the conditions of Lemma 34.6.2 are fulfilled, and denote the maximal value of s_{1+} by s_{2+} and the minimal value of s_{1-} by s_{2-} . According to Lemma 34.6.2, the inequality (34.254) is satisfied on every (s_{1-}, s_{1+}) , hence it is satisfied on (s_{2-}, s_{2+}) . On such an interval (34.267) is fulfilled, and taking into account that $|z| \leq \ln^{1/2+} \zeta^{-1}$, we obtain (34.282). Note that (34.267) implies that

$$|\Phi| \leq 2C_5 \zeta^2 \ln^{2/2+} \zeta^{-1} \exp \ln^{2/2+} \zeta^{-1} \leq \zeta^{2-\delta} / 2 \quad (34.283)$$

if ζ is small enough to satisfy

$$C_5 \ln^{2/2+} \zeta^{-1} \exp(\ln^{2/2+} \zeta^{-1} - \delta \ln \zeta^{-1}) \leq 1/2.$$

Hence (34.263) holds under the above condition on ζ , which evidently does not depend on τ_0 . We assert that (34.281) is fulfilled. Indeed, assume the contrary. If

$$|z(s_{2+})| < \ln^{1/2+} \zeta^{-1},$$

we can extend the solution to a small interval with $|s| > |s_{2+}|$. The solution stays in $\mathcal{E}(\bar{\theta})$ by continuity, and (34.283) implies (34.263) by continuity for small $s - s_{2+}$. This contradicts the assumption that s_{2+} is maximal. The quantity s_{2-} can be treated similarly yielding (34.281). The first inequality (34.282) follows from (34.283). To obtain the inequality for the derivative, we use (34.280) and (34.283) as follows:

$$|d\Phi/ds| \leq C_5 s \zeta^{4-2\delta} + C_3 \zeta^2 + C_6 |\Phi| \leq C_7 \zeta^{2-\delta}.$$

We denote by $s_{2\pm}(\tau_0)$ the value of $s_{2\pm}$ given by (34.281) which was found in Theorem 34.6.2. According to (34.264) and the implicit function theorem, the function $s_{2\pm}(\tau_0)$ is differentiable. Let us introduce a set

$$\mathcal{E}'(\bar{\theta}) = \{(\tau_0, s) \in \mathbb{R}^2 : \tau_0 \in \mathbb{R}, \quad s_{2-} \leq s \leq s_{2+}\}, \quad (34.284)$$

where according to (34.265)

$$C^{-1} \ln^{1/2+} \zeta^{-1} \leq |s_{2\pm}| \leq C \ln^{1/2+} \zeta^{-1}. \quad (34.285)$$

34.6.4.2 Properties of the Auxiliary Potentials

Based on the properties of solutions to the characteristic equations, we establish here properties of solutions to the quasilinear equation (34.242). Notice that at $\zeta = 0$ the characteristic equations are linear, and in the case of small ζ they can be considered in $\mathcal{E}(\bar{\theta})$ as a small perturbation.

We make use below of C^l norms of a function of two variables defined as follows:

$$\begin{aligned} \|u\|_{C^l(\Omega)} &= \sup_{y \in \Omega} \sum_{|\alpha| \leq l} |\partial^\alpha u(y)| \quad \text{where } y = (y_1, y_2), \\ \partial^\alpha &= \partial_1^{\alpha_1} \partial_2^{\alpha_2}, \quad \alpha = (\alpha_1, \alpha_2), \quad |\alpha| = \alpha_1 + \alpha_2. \end{aligned} \quad (34.286)$$

If $\Omega = \mathcal{E}'(\bar{\theta})$, we set in the above formula $y = (\tau_0, s)$, whereas if $\Omega = \mathcal{E}(\bar{\theta})$ we set $y = (\tau, z)$. Let $B(\tau_0, s)$ be defined by (34.260). Obviously,

$$|\partial^\alpha B(\tau_0, s)| \leq C, \quad 1 \leq |\alpha| \leq 2. \quad (34.287)$$

Lemma 34.6.3 *Under the conditions of Theorem 34.6.2, the functions $z(\tau_0, s; \zeta)$ and $\Phi(\tau_0, s; \zeta)$ satisfy in $\mathcal{E}'(\bar{\theta})$, $\bar{\theta} = \ln^{1/2} \zeta^{-1/2+}$, the following estimates:*

$$\|z - B\|_{C^l(\mathcal{E}'(\bar{\theta}))} \leq C \zeta^{2-\delta l - \delta}, \quad l = 0, 1, 2; \quad (34.288)$$

$$\|\Phi(\tau_0, s)\|_{C^l(\mathcal{E}'(\bar{\theta}))} \leq C \zeta^{2-\delta l - \delta}, \quad l = 0, 1, 2. \quad (34.289)$$

Proof The derivation of the above estimates is straightforward but tedious, and we present only the principal steps. The estimates are derived by induction in l . For $l = 0$ we use Theorem 34.6.2, and then inequality (34.289) follows from (34.282). According to (34.282), inequality (34.263) is fulfilled, and then inequality (34.288) for $l = 0$ follows from (34.266) and (34.265). Consider now $l > 0$ assuming that (34.288), (34.289) hold for $l - 1$. Equations (34.268), (34.274) can be written in the form

$$\frac{d(z - B)}{ds} = -\Phi U_{11} - Q U_{12}, \quad (34.290)$$

$$\frac{d\Phi}{ds} = -\Phi (\partial_\tau \sigma - z U_{20}) + \Phi^2 U_{21} z + Q U_{22} z + \zeta^2 U_{23} z + \zeta^2 U_{24}, \quad (34.291)$$

where U_{ij} are algebraic functions of variables Φ, Q, γ, β . These variables are bounded, and the derivatives of γ, β, σ are bounded as well by (34.206). The derivatives of the solutions, namely $z' = \partial^\alpha z$, $\Phi' = \partial^\alpha \Phi$, $|\alpha| = l$, satisfy the equations obtained by application of ∂^α to (34.268), (34.274). Since (34.254) is fulfilled, the coefficients U_{ij} and their derivatives with respect to Φ, Q, γ, β are bounded. The only unbounded variables are z and s according to (34.265), and their upper bounds are, respectively, $\bar{\theta}$ and $C\bar{\theta}$. Observe that z enters U_{11} , U_{12} , and U_{21} only through Q . The derivatives of U_{ij} or of Q up to l -th order cannot involve powers of z higher than z^{2l} . Since Q given by (34.236) involves the factor ζ^2 and $|z| \leq \bar{\theta}$ in $\mathcal{E}'(\bar{\theta})$, the derivatives of Q of order l with respect to z or τ are smaller than $\zeta^{2-\delta}$ for small ζ , and the derivatives of U_{ij} with respect to z or τ are smaller than $\zeta^{2-l\delta}$. We apply the Leibnitz formula to the derivatives of (34.290) and (34.291). Using the induction assumption

$$\|\Phi\|_{C^{l-1}(\mathcal{E}'(\bar{\theta}))} \leq C\zeta^{2-l\delta}, \quad \|z - B\|_{C^{l-1}(\mathcal{E}'(\bar{\theta}))} \leq C\zeta^{2-l\delta},$$

and the notation (34.260), we can write the equations in the form

$$\frac{dz'}{ds} = \partial^\alpha b + U'_{11} \Phi' + U'_{12} z' + U_{13,\alpha}, \quad (34.292)$$

$$\frac{d\Phi'}{ds} = U'_{21} \Phi' + U'_{22} z' + U_{23,\alpha}, \quad (34.293)$$

where

$$\begin{aligned} |U'_{11}| &\leq C_1, & U'_{12} &\leq \zeta^{2-\delta}, & U_{13,\alpha} &\leq \zeta^{2-l\delta}, \\ |U'_{21}| &\leq C_0 \bar{\theta}, & |U'_{22}| &\leq C\zeta^{2-l\delta}, & U_{23,\alpha} &\leq C\zeta^{2-l\delta}. \end{aligned} \quad (34.294)$$

The initial data for $z' = \partial^\alpha z$, $\Phi' = \partial^\alpha \Phi$ can also be analyzed by induction, since $\partial_{\tau_0}^i \Phi_{s=0} = 0$, $\partial_{\tau_0}^i z_{s=0} = 0$ and $\partial_s^j \partial_{\tau_0}^i \Phi$ can be expressed from the equation for $\partial_s^{j-1} \partial_{\tau_0}^i \Phi$. Note that $\partial^\alpha b = \partial_\tau^{|\alpha|} b = \partial_s^{|\alpha|} b$ and (34.292) can be rewritten in the form

$$\begin{aligned} \frac{d}{ds} (z' - \partial_s^{|\alpha|-1} b) &= U'_{11} \Phi' \\ &+ U'_{12} (z' - \partial_s^{|\alpha|-1} b) + U'_{12} \partial_s^{|\alpha|-1} b + U_{13, \alpha}. \end{aligned} \quad (34.295)$$

Hence, we get from the induction assumptions estimates of the initial data for small ζ

$$|\partial^\alpha z_{s=0} - \partial_s^{|\alpha|-1} b_{s=0}| \leq C_1 \zeta^{2-l\delta}, \quad |\partial^\alpha \Phi_{s=0}| \leq C_2 \zeta^{2-l\delta}, \quad |\alpha| = l.$$

We easily obtain from the system (34.295), (34.293) with small ζ the following estimate:

$$\begin{aligned} &|z' - \partial_s^{|\alpha|-1} b| + |\Phi'| \\ &\leq C_3 \zeta^{2-l\delta} \exp(C'_0 \bar{\theta} s) \leq C_3 \zeta^{2-l\delta} \exp(C'_0 \ln^{2/2+} \zeta^{-1}) \leq \zeta^{2-l\delta-\delta} \end{aligned} \quad (34.296)$$

which implies the desired (34.288) and (34.289).

Notice that the solutions of the characteristic equations determine the function $\Phi(\tau(\tau_0, s), z(\tau_0, s))$ on the characteristic curves as a function of parameters τ_0, s . The characteristic equations (34.243)–(34.247) also determine a function $\Pi(\tau_0, s) = (\tau(\tau_0, s), z(\tau_0, s))$ in the (τ, z) -plane. To obtain the function Φ of independent variables (τ, z) , we have to find the inverse of Π , that is, $(\tau_0, s) = \Pi^{-1}(\tau, z)$, in a strip about the line $z = 0$.

Lemma 34.6.4 *The image of the mapping Π contains the strip $\Xi(\bar{\theta})$ if $\zeta \leq 1/C$ for a sufficiently large constant C .*

Proof The mapping Π maps the straight line $\{\tau_0, s = 0\}$ onto the straight line $\{\tau, z = 0\}$. The straight line $\{\tau_0 = \tau_{00}, s\}$ is mapped onto the curve $\tau = \tau_{00} + s, z = z(\tau_{00}, s)$. This curve intersects the straight line $|z_0| = \bar{\theta}$ and, consequently, any straight line $z = z_0$ with $|z_0| \leq \bar{\theta}$. This intersection is transversal according to (34.264). Hence the point of intersection $\tau = p(\tau_{00}, z_0)$ continuously and differentiably depends on τ_{00} . Formula $\tau = \tau_{00} + s$ implies that $p(\tau_{00}, z_0) \rightarrow \pm\infty$ as $\tau_{00} \rightarrow \pm\infty$. Since $p(\tau_{00}, z_0)$ is a continuous function, it takes all intermediate values on the straight line, therefore the image of the mapping Π contains every straight line $z = z_0$ with $|z_0| \leq \bar{\theta}$.

Now we want to prove that the mapping Π is one-to-one on $\Xi(\bar{\theta})$ and that its inverse has uniformly bounded derivatives in the strip $\Xi(\bar{\theta})$. The characteristic system depends on the small parameter ζ , $\Pi(\tau_0, s) = \Pi(\tau_0, s; \zeta)$. The mapping differential is given by the matrix

$$\Pi'(\tau_0, s; \zeta) = \begin{pmatrix} \partial\tau/\partial\tau_0 & \partial\tau/\partial s \\ \partial z/\partial\tau_0 & \partial z/\partial s \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \partial z/\partial\tau_0 & \partial z/\partial s \end{pmatrix}. \quad (34.297)$$

If the matrix determinant is not zero, the inverse is given by the formula

$$\Pi'^{-1}(\tau_0, s; \zeta) = \frac{1}{\partial z/\partial s - \partial z/\partial \tau_0} \begin{pmatrix} \partial z/\partial s & -1 \\ -\partial z/\partial \tau_0 & 1 \end{pmatrix}.$$

For $\zeta = 0$ we obtain from (34.243)–(34.245) a simpler system for the resulting approximation $\mathring{\Phi}$:

$$\frac{d\tau}{ds} = 1, \quad dz/ds = (\beta^{-1} - \beta)(\tau_0 + s), \quad (34.298)$$

$$\frac{d\mathring{\Phi}}{ds} = -2\mathring{\Phi}(\partial_\tau - \beta\partial_z) \ln \Psi - 2\beta^{-1}\mathring{\Phi}\partial_z \ln \Psi. \quad (34.299)$$

The solution of (34.298) is given by the formula

$$\tau = \tau_0 + s, \quad z = z_0 = B(\tau_0, s), \quad (34.300)$$

where $B(\tau_0, s)$, $b(\tau)$ are given by (34.260). The differential of $\Pi(\tau_0, s; 0)$ is given by the matrix

$$\Pi'(\tau_0, s; 0) = \begin{pmatrix} 1 & 1 \\ b(\tau_0 + s) - b(\tau_0) & b(\tau_0 + s) \end{pmatrix} \quad (34.301)$$

with the determinant $\det \Pi(\tau_0, s; 0) = b(\tau_0)$. The matrix is invertible according to (34.261), and the inverse matrix is uniformly bounded.

When $\zeta > 0$ is small, we consider the system (34.243)–(34.245), (34.247) as a perturbation of the system with $\zeta = 0$, and the differential $\Pi'(\tau_0, s; \zeta)$ is also a small perturbation of $\Pi'(\tau_0, s; 0)$.

Lemma 34.6.5 *Let the conditions of Theorem 34.6.2 be satisfied. Then in $\Xi(\bar{\theta})$*

$$|\Pi'(\tau_0, s; \zeta) - \Pi'(\tau_0, s; 0)| \leq C\zeta^{2-\delta} \quad (34.302)$$

for $|\zeta| \leq 1/C_0$, the matrices $\Pi'(\tau_0, s; \zeta)$ are invertible, the inverse matrices $\Pi'^{-1}(\tau_0, s; \zeta)$ have continuously differentiable elements and their derivatives are uniformly bounded in $\Xi(\bar{\theta})$. The mapping Π is one-to-one from $\Xi'(\bar{\theta})$ to $\Xi(\bar{\theta})$, the mappings Π and Π^{-1} are two times differentiable with uniformly bounded derivatives.

Proof We use Lemma 34.6.3 and infer from (34.288) with $l = 1$ that

$$|\partial z/\partial \tau_0 - \partial z_0/\partial \tau_0| + |\partial z/\partial s - \partial z_0/\partial s| \leq C\zeta^{2-\delta}.$$

This inequality implies inequality (34.302). Inequality (34.302), in turn, implies that $\Pi'(\zeta) = \Pi'(\tau_0, s; \zeta)$ is a uniformly small perturbation of the invertible matrix (34.301) $\Pi'(0)$, hence $\Pi'^{-1}(\zeta)$ is a uniformly small perturbation of the matrix

$\Pi'^{-1}(0)$. The matrices $\Pi'(\zeta)$ and $\Pi'^{-1}(\zeta)$ have continuously differentiable entries. Since $|\Pi'^{-1}(0)|$ is uniformly bounded, and the derivatives of the entries of $\Pi'(0)$ are uniformly bounded and continuous in $\mathcal{E}(\bar{\theta})$, we conclude that the derivatives of entries of $\Pi'^{-1}(0)$ are uniformly bounded and continuous in $\mathcal{E}(\bar{\theta})$ as well. The mapping $\Pi(\zeta)$ is one-to-one since it is a local diffeomorphism between $\mathcal{E}'(\bar{\theta})$ and $\mathcal{E}(\bar{\theta})$, and the image is a simply connected domain. Since Π is two times continuously differentiable with uniformly bounded derivatives, and $\Pi'^{-1}(\zeta)$ is uniformly bounded, the inverse mapping $\Pi^{-1}(\zeta)$ is two times continuously differentiable with uniformly bounded derivatives.

Theorem 34.6.3 *For any $\delta > 0$ there exists a C_0 such that, if $\zeta \leq 1/C_0$, then there exists a solution $\Phi(\tau, z)$ of the quasilinear equation (34.242) defined in $\mathcal{E}(\bar{\theta})$. This solution is twice continuously differentiable in the strip $\mathcal{E}(\bar{\theta})$, and its derivatives are uniformly bounded and small for $\zeta \leq 1/C_0$, namely*

$$\|\Phi(\tau, z)\|_{C^2(\mathcal{E}(\bar{\theta}))} \leq C_1 \zeta^{2-3\delta}. \quad (34.303)$$

Proof The solution $\Phi(\tau, z)$ is defined by formula (34.248), which can be written in the form

$$\Phi(\tau, z; \zeta) = \Phi(\Pi^{-1}(\tau_0, s; \zeta), \zeta). \quad (34.304)$$

The function $\Phi(\tau, z; \zeta)$ is well-defined in $\mathcal{E}(\bar{\theta})$ according to Lemmas 34.6.4 and 34.6.5. Its differentiability properties follow from the properties of $\Phi(\tau_0, s; \zeta)$ described in (34.289) and the properties of $\Pi^{-1}(\tau_0, s; \zeta)$ described in Lemma 34.6.5. It is a solution to (34.242) according to the construction of $\Phi(\tau_0, s; \zeta)$ as a solution of (34.243)–(34.245).

The second auxiliary phase Z is given by the formula (34.238). Since (34.254) holds in $\mathcal{E}(\bar{\theta})$, $Z(\tau, z)$ is also twice continuously differentiable in the strip $\mathcal{E}(\bar{\theta})$ and has uniformly bounded derivatives. The potential φ_b and phase S can be found from (34.227), (34.228), namely

$$S = \zeta^{-1} \int_0^z Z(\tau, z_1) dz_1 = \zeta^{-1} \int_0^z \Theta(\Phi) dz_1, \quad (34.305)$$

and

$$\varphi_b = \frac{mc^2}{q} \left(\Phi + \partial_\tau \int_0^z \Theta(\Phi) dz_1 - \beta \Theta(\Phi) \right). \quad (34.306)$$

Lemma 34.6.6 *The phase function S defined by (34.305) and the potential φ_b defined by (34.306) satisfy the estimates*

$$\|S\|_{C^2(\mathcal{E}(\bar{\theta}))} \leq C_1 \zeta^{1-4\delta}, \quad (34.307)$$

$$\|\varphi_b\|_{C^1(\mathcal{E}(\bar{\theta}))} \leq C_2 \zeta^{2-4\delta}. \quad (34.308)$$

Proof The above estimates follow from (34.303) and the boundedness in $\mathcal{E}'(\bar{\theta})$ of derivatives of the function $\Theta(\Phi)$ which enters representations (34.305) and (34.306).

34.6.5 Verification of the Concentration Conditions

In this section we fix an interval $T_- \leq t \leq T_+$ and define sequences $a = a_n$, $\zeta = \zeta_n$, $R = R_n$. We verify then that the NKG equation is localized at the trajectory $\hat{\mathbf{r}}(t) = (0, 0, r(t))$ and that solutions ψ defined by (17.157), (34.217) concentrate at $\hat{\mathbf{r}}(t)$ as in Theorem 34.6.4.

Proposition 34.6.2 *Let $\varphi(t)$ be defined by (34.211)–(34.212), where φ_b is given by (34.306), and $\varphi_0(t)$ satisfy (34.321). Let $\varphi_\infty = \varphi_{ac}(t)$ be given by (34.212), (17.166), $R_n = \bar{\theta}a_n \rightarrow 0$. Let also*

$$a_C \leq Ca^2. \quad (34.309)$$

Then φ_n satisfy (34.143) and (34.144), and condition (34.146) holds.

Proof Note that $\varphi(t, x)$ does not depend on x_1, x_2 , and the estimates in $\hat{\Omega}(\hat{\mathbf{r}}(t), R_n)$ follow from the estimates in $\mathcal{E}(\bar{\theta})$:

$$\begin{aligned} \max_{\hat{\Omega}(\hat{\mathbf{r}}(t), R_n)} |\varphi(t, x)| &\leq \sup_{\tau, z \in \mathcal{E}(\bar{\theta})} |\varphi(\tau, z)|, \\ \max_{\hat{\Omega}(\hat{\mathbf{r}}(t), R_n)} |\partial_t \varphi(t, x)| &\leq \frac{c}{a} \sup_{\tau, z \in \mathcal{E}(\bar{\theta})} |\partial_\tau \varphi|, \\ \max_{\hat{\Omega}(\hat{\mathbf{r}}(t), R_n)} |\nabla \varphi(t, x)| &\leq \frac{1}{a} \sup_{\tau, z \in \mathcal{E}(\bar{\theta})} |\nabla_z \varphi|. \end{aligned} \quad (34.310)$$

According to (17.164)

$$|\varphi(t, x)| \leq |\varphi_0| + |\varphi'_{ac}| |y| + |\varphi_b| \leq |\varphi_0| + |\varphi'_{ac}| R_n + |\varphi_b|.$$

Since φ'_{ac} is defined by (17.166), we conclude using (34.308), (34.321) that $|\varphi| \leq C_1$ in $\Omega(\hat{\mathbf{r}}(t), R_n)$. Using (34.309), we obtain estimates of derivatives:

$$\begin{aligned} |\partial_t \varphi(t, x)| &\leq |\partial_t \varphi_0| + |\partial_t \varphi'_{ac}| |y| + ca^{-1} |\partial_\tau \varphi_b| \\ &\leq C + C_1 a^{-1} \zeta^{2-4\delta} \leq C + C_2 a^{1-4\delta} \leq C_3, \end{aligned} \quad (34.311)$$

$$|\nabla_x \varphi(t, x)| \leq |\varphi'_{ac}| + a^{-1} |\nabla_z \varphi_b| \leq C' + C'_1 a^{-1} \zeta^{2-4\delta} \leq C'_3. \quad (34.312)$$

Hence, (34.143) holds. According to (34.145), (17.164) and (17.165), $\varphi - \varphi_\infty = \varphi_b$. Using (34.308), (34.310), and observing that (34.309) implies $\zeta \leq Ca$, we conclude

that

$$\begin{aligned} |\varphi_b(t, x)| + |\nabla_{0,x} \varphi_b(t, x)| &\leq \sup_{\tau, z \in \Xi(\theta)} (|\varphi_b(\tau, z)| + \frac{1}{a} |\partial_\tau \varphi_b| + \frac{1}{a} |\nabla_z \varphi_b|) \\ &\leq C_4 a^{-1} \zeta^{2-4\delta} \rightarrow 0, \end{aligned}$$

yielding relations (34.144). To obtain (34.146), we use (34.205) and (17.166).

The solution of the NKG equation is given by the formulas (34.217), (17.157) where $s(t)$ is given by (34.222) with condition $s(0) = 0$.

Proposition 34.6.3 *Let ψ be defined by (17.157), (34.217) where s, S are defined by (34.222), (17.166), let $R_n = \theta a_n$. Then in $\hat{\Omega}(\hat{\mathbf{r}}(t), R_n)$*

$$|\partial_t \psi|^2 + |\nabla \psi|^2 \leq C_1 a_C^{-2} \psi^2, \quad (34.313)$$

$$G(|\psi|^2) \leq C_2 \zeta^{2-\delta} a_C^{-2} \psi^2, \quad (34.314)$$

where

$$\psi = \hat{\psi}(\mathbf{y}/a) = \hat{\psi}(\mathbf{z}) = \pi^{-3/4} a^{-3/2} e^{-|\mathbf{y}|^2 a^{-2}/2}.$$

Proof According to (17.157) and (34.217), $\psi = e^{i\hat{S}(t, (x-r))} \hat{\psi}$ with $\hat{\psi} = e^\sigma \hat{\psi}(\mathbf{x} - \hat{\mathbf{r}})$. For such solutions we use the change of variables (34.215) and relations (17.166) and (34.222), and we obtain similarly to (34.221) and (34.223) that

$$|\partial_0 \psi|^2 = a_C^{-2} \zeta^2 \left| \partial_\tau \hat{\psi} - \beta \partial \hat{\psi} / \partial z_3 \right|^2 + a_C^{-2} |\gamma - \zeta \partial_\tau S + \zeta \beta \partial_z S|^2 \left| \hat{\psi} \right|^2,$$

$$\begin{aligned} |\nabla_x \psi|^2 &= a^{-2} \left| \partial \hat{\psi} / \partial z_1 \right|^2 + a^{-2} \left| \partial \hat{\psi} / \partial z_2 \right|^2 \\ &+ a^{-2} \left| \partial \hat{\psi} / \partial z_3 \right|^2 + a_C^{-2} (-Z + \beta \gamma)^2 \left| \hat{\psi} \right|^2 \\ &= a^{-2} \left| \nabla_z \hat{\psi} \right|^2 + a_C^{-2} (-Z + \beta \gamma)^2 \left| \hat{\psi} \right|^2. \end{aligned}$$

Note that in $\hat{\Omega}(\hat{\mathbf{r}}(t), R_n)$

$$\left| \partial \hat{\psi} / \partial z_i \right|^2 \leq C (1 + R_n^2) \left| \hat{\psi} \right|^2. \quad (34.315)$$

Using (34.307) and (34.315), we get (34.313). According to (14.42),

$$G(|\psi|^2) = -a^{-2} e^{2\sigma} \hat{\psi}^2(z) [-\mathbf{z}^2 + 2\sigma + \ln \pi^{3/2} + 2],$$

yielding the desired inequality (34.314).

Proposition 34.6.4 *Let ψ be defined by (17.157), (34.217) where s, S are defined by (34.222), (17.166). Then the energy density is given by the following formula:*

$$\begin{aligned} u(\hat{\mathbf{r}} + a\mathbf{z}) &= \frac{mc^2}{2} \left[\zeta^2 (\sigma + \beta z)^2 + 2\zeta^2 \mathbf{z}^2 - \zeta^2 (2\sigma + \ln \pi^{3/2} + 2) \right] e^{2\sigma} \dot{\psi}^2(z) \\ &\quad + \frac{mc^2}{2} \left[(\Phi - \gamma)^2 + (\beta\gamma - Z)^2 + 1 \right] e^{2\sigma} \dot{\psi}^2(z). \end{aligned} \quad (34.316)$$

Proof Using the change of variables (34.215), (34.225) (34.227), (34.228), we obtain similarly to (34.221) and (34.223) that

$$|\tilde{\partial}_t \psi|^2 = \frac{\zeta^2 c^2}{a_C^2} \left| (\partial_\tau - \beta \partial_z) \dot{\psi} \right|^2 + \frac{c^2}{a_C^2} (-\gamma + \Phi)^2 \left| \dot{\psi} \right|^2.$$

The energy density in τ, z variables has the following form:

$$\begin{aligned} u(\hat{\mathbf{r}} + a\mathbf{z}) &= \frac{\chi^2}{2m} \left[\frac{\zeta^2}{a_C^2} \left| (\partial_\tau - \beta \partial_z) \dot{\psi} \right|^2 + \frac{\zeta^2}{a_C^2} \left(\left| \nabla_z \dot{\psi} \right|^2 + G \left(\left| \dot{\psi} \right|^2 \right) \right) \right] \\ &\quad + \frac{\chi^2}{2m} \left[\frac{1}{a_C^2} (-\gamma + \Phi)^2 \left| \dot{\psi} \right|^2 + \frac{1}{a_C^2} (\beta\gamma - Z)^2 \left| \dot{\psi} \right|^2 + \frac{m^2 c^2}{\chi^2} \left| \dot{\psi} \right|^2 \right]. \end{aligned}$$

Using (34.219) we obtain (34.316).

Proposition 34.6.5 *Let ζ_n and a_n be related by the formula*

$$\ln \zeta^{-1} = \ln^{1+} a^{-1}. \quad (34.317)$$

Then conditions (34.150) and (34.149) are satisfied.

Proof To obtain (34.149), we use Proposition 34.6.3:

$$\begin{aligned} a_C^2 \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \left| \nabla_{0,x} \psi(t, x) \right|^2 + |G(|\psi(t, x)|^2)| \, d\mathbf{x} &+ \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} |\psi(t, x)|^2 \, d\mathbf{x} \\ &\leq C_2 \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \dot{\psi}^2 \, d\mathbf{x} = C_2 a^{-3} \int_{\Omega(0, R_n)} \dot{\psi}^2(y/a) \, d^3 y \\ &\leq C_4 \int_{\mathbb{R}^3} \exp(-|z|^2) \, dz \leq C_5. \end{aligned}$$

To obtain (34.150), we once again use Proposition 34.6.3

$$\begin{aligned}
& \int_{\partial\Omega(\hat{\mathbf{r}}(t), R_n)} (a_C^2 |\nabla_{0,x} \psi(t, x)|^2 + a_C^2 |G(|\psi(t, x)|^2)| + |\psi(t, x)|^2) d\sigma \\
& \leq \int_{\partial\Omega(\hat{\mathbf{r}}(t), R_n)} C_2 \psi^2 d\sigma = \int_{\partial\Omega(\hat{\mathbf{r}}(t), R_n)} C_2 \psi^2 d\sigma \\
& = C_3 a^2 a^{-3} \int_{|z|=\bar{\theta}} e^{-|z|^2} d\sigma = C_4 \exp(\ln a^{-1} - \ln^{2/2+} \zeta^{-1}).
\end{aligned}$$

Condition (34.317) implies that $\ln a^{-1} = \ln^{1/1+} \zeta^{-1}$, and by (34.258) $2/2_+ > 1/1_+$. Hence

$$\exp(\ln a^{-1} - \ln^{2/2+} \zeta^{-1}) \rightarrow 0,$$

and (34.150) holds.

Proposition 34.6.6 *Let ψ' be defined by (17.157), (34.217), where s, S are defined by (34.222), (17.166), $R_n = \bar{\theta} a_n \rightarrow 0$. Then the following inequality holds in $\Xi(\bar{\theta})$:*

$$\left| u(\hat{\mathbf{r}} + a\mathbf{z}) - \gamma m c^2 \psi^2(\mathbf{z}) \right| \leq C \zeta^{2-\delta} \psi^2(\mathbf{z}). \quad (34.318)$$

Conditions (34.170), (34.152) are satisfied with $\bar{\mathcal{E}}_\infty(t) = \gamma m c^2$. The following estimate holds:

$$\left| \rho - q \psi^2(\mathbf{z}) \right| \leq C \zeta^{2-\delta} \psi^2(\mathbf{z}),$$

and (34.168) holds as well with $\bar{\rho}_\infty = q$.

Proof According to (34.316), since $\gamma^2 + \beta^2 \gamma^2 + 1 = 2\gamma^2$, we have

$$\begin{aligned}
u(\hat{\mathbf{r}} + a\mathbf{z}) &= m c^2 \gamma^2 e^{2\sigma} \psi^2(z) \\
&+ \frac{m c^2}{2} ((-\gamma + \Phi)^2 - \gamma^2 + (\beta\gamma - Z)^2 - \beta^2 \gamma^2) e^{2\sigma} \psi^2(\mathbf{z}) \\
&+ \frac{m c^2}{2} \zeta^2 [(\sigma + \beta z)^2 + 2\mathbf{z}^2 - (2\sigma + \ln \pi^{3/2} + 2)] e^{2\sigma} \psi^2(\mathbf{z}).
\end{aligned}$$

Using (34.238) and (34.289), we conclude that

$$\begin{aligned}
& \frac{m c^2}{2} \left| ((-\gamma + \Phi)^2 - \gamma^2 + (\beta\gamma - Z)^2 - \beta^2 \gamma^2) e^{2\sigma} \psi^2(\mathbf{z}) \right| \\
& \leq C_0 (|\Phi| (|\Phi| + \gamma) + |Z| (|Z| + |\beta| \gamma)) \psi^2(\mathbf{z}) \\
& \leq C'_0 |\Phi| \psi^2(\mathbf{z}) \leq C''_0 \zeta^{2-\delta} \psi^2(\mathbf{z}).
\end{aligned}$$

One can easily verify that in $\mathcal{E}(\bar{\theta})$

$$\begin{aligned} \frac{mc^2}{2} \zeta^2 \left| [(\sigma + \beta z)^2 + 2\mathbf{z}^2 - (2\sigma + \ln \pi^{3/2} + 2)] e^{2\sigma} \psi^2(z) \right| \\ \leq C_1 \zeta^2 (|\mathbf{z}|^2 + 1) \psi^2(z) \leq C'_1 \zeta^{2-\delta} \psi^2(\mathbf{z}), \end{aligned}$$

hence (34.318) holds. Let us estimate now the energy

$$\bar{\mathcal{E}}_n(t) = \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} u \, d\mathbf{x} = a^3 \int_{\mathcal{E}(\bar{\theta})} u(\hat{\mathbf{r}} + a\mathbf{z}) \, d^3z.$$

Note that

$$\begin{aligned} a^3 \int_{\mathcal{E}(\bar{\theta})} \psi^2(\mathbf{z}) \, d\mathbf{z} &= \pi^{-3/2} \int_{|z| \leq \bar{\theta}} e^{-|z|^2} \, d\mathbf{z} \\ &= 1 - \pi^{-3/2} \int_{|z| \geq \bar{\theta}} e^{-|z|^2} \, d\mathbf{z}. \end{aligned} \quad (34.319)$$

Using (34.318) and the above formula, we conclude that

$$\bar{\mathcal{E}}_n(t) \rightarrow \gamma mc^2 \quad (34.320)$$

uniformly for all t , and we get (34.170). Since $\gamma mc^2 > 0$, we get the energy estimate from below of the form (34.152). To express ρ , we use (11.132) and in (τ, z) -variables obtain that

$$\rho = -\frac{\chi q}{mc^2} |\psi|^2 \operatorname{Im} \frac{\tilde{\partial}_t \psi}{\psi} = -\frac{\chi q}{mc^2} e^{2\sigma} \psi^2 \frac{c}{a_C} (\Phi - \gamma) = q e^{2\sigma} \psi^2 \gamma - q e^{2\sigma} \Phi \psi^2.$$

Using (34.303), (34.220) and (34.319), we obtain (34.168).

Theorem 34.6.4 *Let the trajectory $r(t)$ satisfy relations (34.206)–(34.208), and let $\varphi(t)$ be defined by (17.164)–(17.166), where φ_b is given by (34.306), and $\varphi_0(t)$ is a given function which satisfies*

$$|\varphi_0| + |\partial_t \varphi_0| + |\partial_t^2 \varphi_0| \leq C. \quad (34.321)$$

Suppose also ψ to be of the form (17.157) with ψ_{1D} defined by (34.217) where the phases $s(t)$, $S(t, y)$ are given by (34.222), (34.305). Let $a_n \rightarrow 0$, ζ_n satisfy relations (34.317), and

$$\theta_n = \bar{\theta} = \ln^{1/2+} \zeta_n^{-1}, \quad \chi_n = mca_{C,n} = mc\zeta_n a_n, \quad R_n = \theta_n a_n. \quad (34.322)$$

Then ψ is a solution of the NKG equation which concentrates at the trajectory $\hat{\mathbf{r}}(t) = (0, 0, r(t))$.

Proof According to (34.317), $a^{-1} = \exp \ln^{1/1+} \zeta^{-1}$, hence

$$\begin{aligned} R_n &= \ln^{1/2+} \zeta_n^{-1} \exp(-\ln^{1/1+} \zeta_n^{-1}) \\ &= \exp(\ln \ln^{1/2+} \zeta_n^{-1} - \ln^{1/1+} \zeta_n^{-1}) \rightarrow 0, \end{aligned}$$

implying that the contraction condition (34.4) holds and we can define the concentrating neighborhood $\hat{\mathcal{S}}(\hat{\mathbf{r}}, R_n)$ by (34.3). Note that

$$a_c/a^2 = \zeta/a = \zeta \exp \ln^{1/1+} \zeta^{-1} = \exp(\ln^{1/1+} \zeta^{-1} - \ln \zeta^{-1}) \leq C,$$

implying that inequality (34.309) holds. The conditions of Definition 34.5.1 are satisfied according to Proposition 34.6.2. The conditions of Definition 34.5.2 are satisfied as well according to Propositions 34.6.5 and 34.6.6. Hence $\psi = \psi_n$ is a solution of the NKG equation which concentrates at the trajectory $\hat{\mathbf{r}}(t)$.

Remark 34.6.1 The accelerating force $-q\nabla\varphi$ defined by (17.166) is of order 1, whereas by (34.308) the balancing force $-q\varphi_b$ is of order $\zeta^{2-4\delta}$. Hence the balancing force, while preserving the shape $|\psi|$, is vanishingly small compared with the accelerating force.

Remark 34.6.2 Similarly to solutions of the form (17.157), (34.217), (34.219), we can introduce one more parameter $\gamma_0 = \gamma(t_0)$ and modify (34.219), (34.220) as follows:

$$\Psi(t, z) = \pi^{-1/4} e^{\sigma - \gamma_0^2 z^2/2}, \quad \sigma = \ln \gamma^{-1/2} - \ln \gamma_0^{-1/2}. \quad (34.323)$$

Such a modification introduces a fixed contraction which coincides with the Lorentz contraction at $t = t_0$. All the analysis is quite similar and the corresponding solutions concentrate at $\hat{\mathbf{r}}(t)$. In the case where $r(t) = vt$ for $t < 0$ such a modification with $t_0 = -1$ allows us to obtain solutions which coincide with the free uniform solutions with zero electric potential from Sect. 15.2 for $t < 0$ when they have a constant velocity v and can accelerate at positive times.

34.6.5.1 On the Robustness of the Lorentz Contraction in Accelerated Motion

Here we discuss the robustness of the Lorentz contraction in accelerated motion. Let us consider a rectilinear motion with a trajectory $r(t)$ such that velocity takes a constant value v_0 for $t < 0$ and a different constant value $v_1 \neq v_0$ for $t \geq T_1 > 0$. Consequently, there has to be a non-zero acceleration in the interval $0 < t < T_1$. Let us look at a solution ψ of the form (17.157), (34.217), (34.219) with Ψ of the form (34.323) and the parameter $\gamma_0 = (1 - v_0^2/c^2)^{-1/2}$. Such a solution has a Gaussian shape

$$|\psi| = \pi^{-3/4} \gamma_0^{1/2} \gamma^{-1/2} e^{-z_1^2/2 - z_2^2/2 - \gamma_0^2 z_3^2/2}$$

with the velocity dependent factor $\gamma_0^{1/2}\gamma^{-1/2}$ which does not depend on ζ . When $t < -1$, we have $\varphi = 0$, and ψ is given in $\Omega(\hat{\mathbf{r}}(t), \theta_n a_n)$ by the same formula (15.25) as the solution for a free charge, with a Gaussian shape

$$|\psi| = \pi^{-3/4} e^{-z_1^2/2 - z_2^2/2 - \gamma_0^2 z_3^2/2}.$$

When $t \geq T_1$, the motion is uniform with velocity v_1 , the accelerating part φ_{ac} of the potential is zero, and the balancing potential φ_b , though not zero when $t \geq T_0$, is of order $\zeta^{2-4\delta}$ in $\Omega(\hat{\mathbf{r}}(t), \theta_n a_n)$ and is vanishingly small as $\zeta \rightarrow 0$. For $t \geq T_1$ the Gaussian

$$|\psi| = \pi^{-3/4} \gamma_0^{1/2} \gamma_1^{-1/2} e^{-z_1^2/2 - z_2^2/2 - \gamma_0^2 z_3^2/2}$$

has the same Lorentz contraction factor γ_0 , but a different amplitude $\gamma_0^{1/2} \gamma_1^{-1/2}$. The principal part

$$\omega_0 c^{-2} \gamma v y - \omega_0 t / \gamma$$

of the phase $\hat{S}$ in (34.217) is the same as in (15.25), hence it involves the Lorentz contraction with the factor γ . Consequently, the principal part of the solution for $t \geq T_1$ involves components with different values of the contraction factor. Therefore, after the acceleration ended, *the principal part of the solution, while translating with the constant velocity v_1 , cannot be obtained by the Lorentz transformation from the solution for a free charge with the original velocity v_0* , whereas the phase can be. Thus, in general, the transition caused by the external force from velocity v_0 to velocity v_1 cannot be reduced to the Lorentz transformation. The fixed Lorentz contraction γ_0 of the Gaussian shape factor is preserved, while velocity changes thanks to the external electric force which causes the acceleration of the charge and also results in the change of the amplitude of the charge distribution. Observe that Theorem 34.5.1 can be applied to the considered example, and Einstein's formula is applicable at all times with the same rest mass m . In particular, formula $M = m\gamma$ implies that $M = m\gamma_0$ for $t < 0$ and $M = m\gamma_1$ for $t > T_1$. This observation shows that Einstein's formula holds all the time and fully applies to accelerating regimes, but the Lorentz contraction formula can be applied only to some characteristics of an accelerating charge distribution. Such a difference is specific to accelerating regimes and is in sharp contrast with the case of a global uniform motion without external forces. Note that the resulting change of shape in the moving frame is compensated by a vanishingly small, but still non-zero electric force which vanishes only in the limit as $\zeta \rightarrow 0$.

34.6.6 Concentration of Solutions of a Linear NKG Equation

The results of Sect. 34.5 on concentrating solutions are directly applicable to solutions of the linear NKG equation by setting $G = 0$. In this case Theorem 34.5.1 can be

applied, and, if solutions of the linear NKG equations concentrate at a trajectory, the trajectory must satisfy the relativistic point equations. The size parameter a is now not involved in the equation, but rather describes the localization of a sequence of solutions of the linear equations. *The significant difference is that the linear KG equations do not have global localized solutions* as described in Sects. 14.1 and 15.2. Consequently, one cannot simply apply the relativistic Einstein argument for the 4-vector of the global energy-momentum to the linear KG equation. As to the results of Sect. 17.6.2, they can be modified for the linear case as follows. For a given trajectory $r(t)$ the wave function ψ is defined by the same formulas (34.217), (17.157). The identity (34.224) cannot be used, and in Q defined by (34.234) the term $\zeta^2 \partial_z^2 \Psi$ will not cancel with G' , resulting in replacement of the term $\beta^2 \zeta^2 (z^2 - 1)$ in (34.236) by the term $\gamma^{-2} \zeta^2 (z^2 - 1)$; the term $2\zeta^2 \sigma$ in (34.236) now is absent. These arguments show that Q is a small perturbation of order ζ^2 . Since we do not use the structural details of Q , but only its smallness, all the estimates in Sect. 17.6.2 can be carried out with this modification. Hence, a small (of order $\zeta^{2-4\delta}$) balancing potential φ_b exists in the linear case as well. Analyzing estimates made in Proposition 34.6.6, one finds that the nonlinearity produces a vanishing contribution to E_∞ . Thus the following modification of Theorem 34.6.4 holds:

Theorem 34.6.5 *Let trajectory $r(t)$ satisfy (34.206)–(34.208), and let the NKG equation be linear, namely $G' = 0$ in (34.128) and (11.121), (11.142). Suppose ψ is of the form (17.157) with ψ_{1D} defined by (34.217) where the phases $s(t)$, $S(t, y)$ are given by (34.222), (34.305). Let $a_n \rightarrow 0$, ζ_n satisfy (34.317), χ_n , R_n , θ_n be defined by (34.322), and $\varphi(t)$ be defined by (17.164)–(17.166) where φ_b is given by (34.306) and $\varphi_0(t)$ is a given function satisfying (34.321). Then ψ is a solution of the linear KG equation which concentrates at the trajectory $\hat{\mathbf{r}}(t) = (0, 0, r(t))$.*

Chapter 35

Energy Functionals and Nonlinear Eigenvalue Problems

In this chapter we treat mathematical aspects of our studies which concern energy functionals, their critical points and corresponding nonlinear eigenvalue problems. The chapter includes two sections. In the first section we study the extremal properties of the logarithmic nonlinearity and their impact on the stability properties of solutions to the NLS equation. We also give mathematical details of the derivation of the Planck-Einstein formula relating the critical energy levels with the frequencies of multi-harmonic solutions. In the second section, we rigorously derive classical Hydrogen spectrum based on the mathematical analysis of the nonlinear eigenvalue problem for a balanced charge in the Coulomb field.

35.1 Properties of the NLS with Logarithmic Nonlinearity

35.1.1 *Gaussian Shape as a Global Minimum of Energy*

We proved in Sect. 17.4.1 that the logarithmic nonlinearity defined by (14.39) is singled out if we require that the Planck–Einstein frequency-energy relation holds exactly. Let us look further into the properties of the logarithmic nonlinearity.

We already observed that the equilibrium equation with the logarithmic nonlinearity admits the Gaussian function as an exact solution. This solution is a critical point of the energy functional, and we show below that the Gaussian function provides a unique (modulo translations) minimum of the energy functional on functions subjected to the charge normalization condition. This fact was already found in [40]. As we mentioned in Sects. 17.2.6 and 16.3.3, this property is related to the stability of the wave-corpuscle based on the Gaussian ground state. Here we consider these extremal properties of the Gaussian in more detail.

Let us consider the energy $\mathcal{E}(\psi) = E_0(\psi)$ defined by (17.53) in the case of a single particle with $\varphi_{\text{ex}} = 0$ and G defined for $a = 1$ by (14.40), namely

$$\mathcal{E}(\psi) = \frac{\chi^2}{2m_\ell} \int \left\{ |\nabla \psi|^2 - |\psi|^2 \ln |\psi|^2 + |\psi|^2 \left(\ln \frac{1}{\pi^{3/2}} - 2 \right) \right\} d\mathbf{x}. \quad (35.1)$$

Of course, the case of general $a > 0$ can be considered similarly. In accordance with (17.7), we consider this functional on a set $\mathcal{E}$ defined by

$$\mathcal{E} = \left\{ \psi \in H^1(\mathbb{R}^3) : \|\psi\|^2 = \int |\psi|^2 d\mathbf{x} = 1 \right\}, \quad (35.2)$$

where H^1 is the Sobolev space with the norm

$$\|\psi\|_{H^1}^2 = \int |\psi|^2 + |\nabla \psi|^2 d\mathbf{x}. \quad (35.3)$$

Using the Gross inequality, which is often called the logarithmic Sobolev inequality, we show in the following theorem that the Gaussian function defined by (14.38), namely

$$\psi_g = C_g e^{-|\mathbf{x}|^2/2}, \quad C_g = \pi^{-3/4},$$

provides the global minimum of the energy $\mathcal{E}(\psi)$, and that all the global minima belong to the set

$$\Omega = \left\{ \psi : \psi = e^{i\theta} \psi_g(\cdot - \mathbf{r}), \quad \mathbf{r} \in \mathbb{R}^3, \quad \theta \in \mathbb{R} \right\} \quad (35.4)$$

obtained from ψ_g by spatial translations and gauge factor multiplication. Another proof of this statement is given in [58].

Theorem 35.1.1 *Let $a = 1$. Then, for any $\psi \in \mathcal{E}$,*

$$\mathcal{E}(\psi) \geq \mathcal{E}(\psi_g) = \frac{\chi^2}{2m_\ell}, \quad (35.5)$$

and the equality holds only for $\psi \in \Omega$.

Proof For simplicity we set here $\frac{\chi^2}{m_\ell} = 1$. The value of the energy $\mathcal{E}(\psi)$ on ψ_g can be found explicitly:

$$\mathcal{E}(\psi_g) = \frac{1}{2}. \quad (35.6)$$

Note that the Euler equation for constrained critical points of $\mathcal{E}(\psi)$ has the form

$$\lambda \psi + \nabla^2 \psi = \left(-\ln(|\psi|^2) + \left(\ln \frac{1}{\pi^{3/2}} - 3 \right) \right) \psi, \quad (35.7)$$

where λ is Lagrange multiplier, and $\psi = \psi_g$ satisfies this equation. Now we show that ψ_g provides the global minimum of $\mathcal{E}(\psi)$ defined by (35.1) for real-valued ψ . We will use the following logarithmic Sobolev inequality:

$$\int_{\mathbb{R}^d} \psi^2 \ln \psi^2 \, d\mathbf{x} \leq \frac{d}{2} \ln \left[\frac{2}{\pi d e} \int_{\mathbb{R}^d} |\nabla \psi|^2 \, d\mathbf{x} \right] \quad (35.8)$$

for functions from $H^1(\mathbb{R}^d)$, where the equality holds only for Gaussian functions (see [83]). Hence,

$$\begin{aligned} \mathcal{E}(\psi) &\geq \frac{1}{2} \int \left[|\nabla \psi|^2 + |\psi|^2 \left(\ln \frac{1}{\pi^{3/2}} - 2 \right) \right] d\mathbf{x} - \frac{3}{4} \ln \left[\frac{2}{3\pi e} \int_{\mathbb{R}^d} |\nabla \psi|^2 \, d\mathbf{x} \right] \\ &= \frac{1}{2} \left\{ \int |\nabla \psi|^2 \, d\mathbf{x} - \frac{1}{2} - \frac{3}{2} \ln \left[\frac{2}{3} \int_{\mathbb{R}^d} |\nabla \psi|^2 \, d\mathbf{x} \right] \right\}. \end{aligned} \quad (35.9)$$

To find $\int_{\mathbb{R}^d} |\nabla \psi|^2 \, d\mathbf{x}$, we consider the Euler equation (35.7) for the minimizer ψ . Using the Pohozaev identity applied to (35.7), we obtain:

$$\frac{1}{2} \int \frac{1}{2} G(|\psi|^2) \, d\mathbf{x} = \lambda \|\psi\|^2 - \frac{1}{6} \int |\nabla \psi|^2 \, d\mathbf{x}, \quad (35.10)$$

where

$$G(|\psi|^2) = -|\psi|^2 \ln |\psi|^2 + |\psi|^2 \left(\ln \frac{1}{\pi^{3/2}} - 2 \right).$$

Multiplying Eq. (35.7) by ψ and integrating the result, we obtain that

$$\lambda \|\psi\|^2 - \mathcal{E}(\psi) = -\frac{1}{2} \int |\psi|^2 \, d\mathbf{x} = -\frac{1}{2}. \quad (35.11)$$

Adding (35.10) and (35.11), we obtain:

$$-\frac{1}{2} \int |\nabla \psi|^2 \, d\mathbf{x} = -\frac{1}{2} - \frac{1}{6} \int |\nabla \psi|^2 \, d\mathbf{x},$$

hence

$$\int |\nabla \psi|^2 \, d\mathbf{x} = \frac{3}{2}.$$

Substitution of the above expression into the inequality (35.9) implies that $\mathcal{E}(\psi) \geq \frac{1}{2}$, where equality holds only for Gaussian functions. Since ψ is a minimizer, the inequality holds for any real $\psi \in \mathcal{E}$. Let us consider now a complex minimizer $\psi = u + iv$ such that $\mathcal{E}(\psi) \leq \frac{1}{2}$. Notice that the following elementary inequality holds:

$$|\nabla (u^2 + v^2)^{1/2}|^2 \leq |\nabla u|^2 + |\nabla v|^2$$

if $u^2 + v^2 \neq 0$, and the equality is possible only if $v \nabla u = u \nabla v$. The above inequality implies in particular that $(u^2 + v^2)^{1/2} \in H^1$. Applying once more the Euclidean logarithmic Sobolev inequality for real functions, we see that

$$\frac{1}{2} \geq \mathcal{E}(\psi) \geq \mathcal{E}\left((u^2 + v^2)^{1/2}\right).$$

For such a minimizer, $(u^2 + v^2)^{1/2}$ must coincide with a Gaussian function, and the equality $v \nabla u = u \nabla v$ must hold almost everywhere. Since

$$\nabla \ln u = \frac{\nabla u}{u} = \frac{\nabla v}{v} = \nabla \ln v$$

for non-zero u and v , we conclude that $\ln u - \ln v$ is a constant, and the complex minimizer ψ is obtained from a Gaussian function via multiplication by a gauge factor $e^{i\theta}$.

35.1.2 Orbital Stability

We discuss here some dynamical issues concerning the logarithmic nonlinearity closely related to results in [58], [60]. Roughly speaking, if the initial data are close to the set Ω obtained from the Gaussian by translations and gauge transformations and defined by (35.4), then the solution stays close to it. Now we give precise formulations. The following theorem directly follows from results of Cazenave and Lions [61].

Theorem 35.1.2 *Let Ω_ϵ be the ϵ -neighborhood in the H^1 norm of the set Ω ,*

$$\Omega_\epsilon = \left\{ \psi \in \mathcal{E} : \inf_{v \in \Omega} \|\psi - v\|_{H^1} \leq \epsilon \right\}, \quad (35.12)$$

where $\mathcal{E}$ is defined by (35.2). Let M_δ be the energy sublevel set of the Gaussian ψ_g :

$$M_\delta = \left\{ \psi \in \mathcal{E} : \mathcal{E}(\psi) \leq \mathcal{E}(\psi_g) + \delta \right\}. \quad (35.13)$$

Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that $M_\delta \subset \Omega_\epsilon$.

Proof Assume the contrary, namely that there exist $\epsilon > 0$ and a sequence $\psi_j \in \mathcal{E}$ such that $\mathcal{E}(\psi_j) \rightarrow \mathcal{E}(\psi_g)$, and $\inf_{v \in \Omega} \|\psi_j - v\|_{H^1} \geq \epsilon$. According to Theorem II.1 and Remark II.3 in [61], a subsequence $\psi_j(\cdot - \mathbf{r}_j)$ is relatively compact in H^1 . Hence there is a subsequence that converges in H^1 to a global minimizer which belongs to Ω . This contradicts the assumption $\epsilon > 0$.

Let us consider now the one-particle equation (17.45) with the logarithmic nonlinearity in the absence of external fields (a free particle):

$$i\chi\partial_t\psi = -\frac{\chi^2}{2m}\nabla^2\psi + \frac{\chi^2}{2m}(-\ln(|\psi|^2/C_g^2) - 3)\psi. \quad (35.14)$$

The existence and the uniqueness of solutions to the initial value problem for this equation is proven in [60]. Since the logarithm has a singularity at $\psi = 0$, to describe classes of functions for which the problem is well-posed, we need to introduce special spaces. Following Cazenave [59], we introduce the functions

$$A(|\psi|) = -|\psi|^2 \ln |\psi|^2 \text{ if } 0 \leq |\psi| \leq e^{-3}, \quad (35.15)$$

$$A(|\psi|) = 3|\psi|^2 + 4e^{-3}|\psi| - e^{-3} \text{ if } |\psi| \geq e^{-3},$$

$$B(|\psi|) = |\psi|^2 \ln |\psi|^2 + A(|\psi|). \quad (35.16)$$

So defined A is a *convex* C^1 function (one time continuously differentiable) on the real line, this function is of class C^2 (twice continuously differentiable) for $|\psi| \neq 0$. The function $B(|\psi|)$ vanishes for small $|\psi|$, and $\psi B(|\psi|)/|\psi|^2$ is of class C^1 and satisfies a uniform Lipschitz condition. Let A^* be the convex conjugate function of A , it is also convex, belongs to the class C^1 and is positive except at zero. We introduce the following Banach spaces:

$$X = \{\psi \in L^1(\mathbb{R}^3) : A(|\psi|) \in L^1(\mathbb{R}^3)\}, \quad (35.17)$$

$$X' = \{\psi \in L^1(\mathbb{R}^3) : A^*(|\psi|) \in L^1(\mathbb{R}^3)\},$$

where $L^1(\mathbb{R}^3)$ is the space of Lebesgue integrable functions. Let

$$W = H^1(\mathbb{R}^3) \cap X, \quad W^* = H^1(\mathbb{R}^3) + X', \quad (35.18)$$

where $H^1(\mathbb{R}^3)$ is the Sobolev space with the norm (35.3). Note that M_δ is bounded in W . By Lemma 9.3.2 from [59], the function $\psi \rightarrow \psi \ln |\psi|^2$ is continuous from X to X' and is bounded on bounded sets.

According to [58], [59], for any $\psi_0 \in W$ there exists a unique solution $\psi \in C(\mathbb{R}, W) \cap C^1(\mathbb{R}, W^*)$ of (35.14) with initial data $\psi(0) = \psi_0$; we use the notation $C(\mathbb{R}, W)$ for the space of continuous functions on $\mathbb{R}$ with values in W . Properties of solutions of (35.14) are described in [58], [59], in particular $\mathcal{E} \cap W$ and M_δ are invariant sets. According to [61], the solution ψ_g is *orbitally stable*, namely the invariance of M_δ and Theorem 35.1.2 imply that if $\psi(0, \cdot) \in M_\delta$ then $\psi(0, t) \in \Omega_\epsilon$ for $t \geq 0$. Consequently, small initial perturbations of the Gaussian shape do not cause its large perturbations in the course of time evolution given by (35.14), but they may cause considerable spatial shifts of the entire wave function. Such a time evolution of charges in an EM field is described in Sect. 16.1.5. Note that if the field is electric, and the potential $\varphi_{\text{ex}}(t, x)$ is linear, one can rewrite Eq. (16.43) in the

moving frame with the origin at $\mathbf{r}(t)$ using a corresponding phase correction for ψ as in (16.63) (see Sect. 16.3.2 for details). As a result, one obtains an equivalent equation of the form (16.43) with $\varphi_{\text{ex}}(t, \mathbf{x}) = 0$, namely Eq. (35.14). The wave-corpuscle solution (16.63) after the change of variables turns into the Gaussian ψ_g . Therefore, using Theorem 35.1.2, we conclude that the wave-corpuscle solutions of (16.43) constructed in Sect. 16.1 are orbitally stable.

35.1.3 The Planck–Einstein Formula for Multiharmonic Solutions

Here we give mathematically rigorous statements concerning fulfillment of the Planck–Einstein formula for multi-harmonic states which were considered in Sect. 17.4. Here we restrict ourselves to the case where all magnetic fields are absent: $\mathbf{A}_\ell = 0$.

First we consider the energy $\mathring{\mathcal{E}}$ defined by (17.51) on the set $\mathcal{E}$ of normalized functions defined by (35.2) and give a sufficient condition for boundedness of the energy from below.

Theorem 35.1.3 *Let $G_\ell(s)$ be a continuously differentiable function (a function of class C^1) which has subcritical growth, namely there exists $\delta > 0$ and C such that*

$$\left| \int G_\ell(|\psi|^2) \, d\mathbf{x} \right| \leq C + C \|\psi\|_{H^1}^{2-\delta} \text{ for all } \psi \in \mathcal{E}, \quad \ell = 1, \dots, N. \quad (35.19)$$

Then the functional $\mathring{\mathcal{E}}$ defined by (17.51) is bounded from below on the set $\mathcal{E}^N = \mathcal{E} \times \dots \times \mathcal{E}$, and boundedness of $\mathring{\mathcal{E}}$ implies boundedness of $\|\psi_\ell\|_{H^1}$, $\ell = 1, \dots, N$. Boundedness of $\mathring{\mathcal{E}}$ on $\mathcal{E}^N$ is equivalent to boundedness of all functionals $\mathbf{E}_{0\ell}$ defined by (17.53), $\ell = 1, \dots, N$.

Proof We use formula (17.55) and note that on $\mathcal{E}$, according to (35.19),

$$\int \{|\nabla \psi_\ell|^2 + G_\ell(|\psi_\ell|^2)\} \, d\mathbf{x} \geq \frac{1}{2} \|\psi\|_{H^1}^2 - C_1. \quad (35.20)$$

Now we estimate the remaining terms in (17.55). According to (12.19),

$$\int |\psi_\ell|^2 \varphi_{\ell'} \, d\mathbf{x} = q_{\ell'} \int \int |\psi_\ell(\mathbf{x})|^2 \frac{1}{|\mathbf{x} - \mathbf{y}|} |\psi_{\ell'}(\mathbf{y})|^2 \, d\mathbf{x} d\mathbf{y}.$$

We split the domain of integration into $|\mathbf{x} - \mathbf{y}| \geq \theta$ and $|\mathbf{x} - \mathbf{y}| < \theta$ with an arbitrarily small θ . Then using the Sobolev imbedding theorem, we obtain the following inequality:

$$\begin{aligned} & \int |\psi_\ell(\mathbf{x})|^2 \frac{1}{|\mathbf{x} - \mathbf{y}|} |\psi_{\ell'}(\mathbf{y})|^2 d\mathbf{x} d\mathbf{y} \\ & \leq C_\epsilon \|\psi_\ell\|^2 \|\psi_{\ell'}\|^2 + \epsilon \|\psi_\ell\|_{H^1} \|\psi_{\ell'}\|_{H^1} \|\psi_\ell\| \|\psi_{\ell'}\|, \end{aligned} \quad (35.21)$$

where ϵ can be chosen arbitrarily small. Using this inequality and (35.20) to estimate the right-hand side of (17.55) from below, we conclude that $\hat{\mathcal{E}}$ is bounded from below on $\mathcal{E}^N$, and boundedness of $\hat{\mathcal{E}}$ implies boundedness of $\|\psi_\ell\|_{H^1}$ and boundedness of all functionals $\mathbf{E}_{0\ell}$. If all $\mathbf{E}_{0\ell}$ are bounded, (35.20) implies boundedness of all norms $\|\psi_\ell\|_{H^1}$, and $\hat{\mathcal{E}}$ is bounded on $\mathcal{E}^N$.

Theorem 35.1.4 *Let every $G_\ell(s)$, $\ell = 1, \dots, N$, be the logarithmic function defined by (14.42) with $a = a_\ell$. Then the functional $\hat{\mathcal{E}}$ defined by (17.51) is bounded from below on the set $\mathcal{E}^N$. Boundedness of $\hat{\mathcal{E}}$ or boundedness of all $\{\mathbf{E}_{0\ell}\}_{\ell=1}^N$ implies for every component of $\{\psi_\ell\}_{\ell=1}^N$ boundedness of $\|\psi_\ell\|_{H^1}$ and $\|\psi_\ell\|_X$, where the space X is defined by (35.17).*

Proof The proof of the boundedness from below is similar to the proof of Theorem 35.1.3: we use (35.21), and instead of (35.19) we use (35.8). Using the boundedness of $\mathcal{E}$ or $\{\mathbf{E}_{0\ell}\}_{\ell=1}^N$ and (35.8), we infer the boundedness in H^1 , and from the boundedness of $\int G_\ell(|\psi_\ell|^2) d\mathbf{x}$ we infer the boundedness of $\|\psi_\ell\|_X$.

Theorem 35.1.5 *Let $\{\hat{\psi}_\ell^\sigma\}_{\ell=1}^N \in \mathcal{E}^N$ be a set of multiharmonic solutions of the nonlinear eigenvalue problem (17.50), (17.47), the index σ labeling the solutions. Let $\{\omega_\ell^\sigma\}_{\ell=1}^N$ and $\{\mathbf{E}_{0\ell}^\sigma\}_{\ell=1}^N$ be the frequencies and the energies of the solutions respectively. Then the components of any two solutions $\hat{\psi}_\ell^\sigma = \hat{\psi}_\ell$ and $\hat{\psi}_\ell^{\sigma_1} = \hat{\psi}_\ell'$ satisfy the Planck–Einstein relation (17.57).*

Proof According to Theorem 35.1.4, the boundedness of $\mathbf{E}_{0\ell}^\sigma$ for every given σ implies that functions ψ_ℓ^σ are bounded in H^1 and in X . Let us consider the left-hand side of (17.50) and denote it by $F(\psi_\ell)$. Similarly to (35.21), we obtain that for any test function f

$$\left| \int \psi_\ell f \varphi_{\ell'} d\mathbf{x} \right| \leq C \|\psi_\ell\|_{H^1} \|f\|_{H^1},$$

where C does not depend on f . Hence the multiplication by $\varphi_{\ell'}$ is a bounded operator from H^1 to H^{-1} . Using this fact to obtain continuous dependence of the term $\varphi_{\ell'} \psi_\ell$ on ψ_ℓ and using Lemma 9.3.3 in [59] for remaining terms in $F(\psi_\ell)$, we conclude that $F(\psi_\ell)$ depends continuously in the space W^* defined in (35.18) on $\psi_\ell \in H^1 \cap X$, and it is bounded in W^* . Since

$$\int F(\psi_\ell) \psi_\ell d\mathbf{x} = \chi \omega_\ell \int |\psi_\ell|^2 d\mathbf{x} - \mathbf{E}_{0\ell} + \frac{\chi^2}{2(a_\ell)^2 m_\ell} \int |\psi_\ell|^2 d\mathbf{x},$$

Equation (17.59) holds for smooth rapidly decaying functions ψ_ℓ , and we can use in a standard way that smooth functions with a compact support are dense in $H^1 \cap X$ and obtain (17.59) written for the logarithmic nonlinearity in the form

$$\chi\omega_\ell^\sigma - \mathbf{E}_{0\ell}^\sigma = -\frac{\chi^2}{2(a_\ell)^2 m_\ell} \int |\psi_\ell^\sigma|^2 \, d\mathbf{x}$$

for functions ψ_ℓ^σ from $H^1 \cap X$. The right-hand side does not depend on σ thanks to the normalization condition (17.7), and we obtain (17.57).

35.2 Mathematical Aspects of the Hydrogen Model for Balanced Charges

We have already described in Sect. 17.5.1 basic properties of the Hydrogen atom model. Here we mathematically rigorously prove that the frequencies of the time-harmonic solutions of balanced charge equations which model the electron in the Coulomb field of the proton converge to classical Hydrogen spectrum. The proof is based on the reduction of the nonlinear eigenvalue problem to a variational problem with a constraint.

The difficulties in the proofs stem from the lack of compactness; to overcome the difficulties we apply with certain modifications the approach developed in [38], [39].

35.2.1 The Variational Problem for a Charge in the Coulomb Field

Recall that a balanced charge in the Coulomb field is described by the solution $\Psi = \Psi_1$ of the eigenvalue problem (17.94) subjected to the normalization condition $\|\Psi\| = 1$. This equation coincides with the critical point of the energy functional $\mathcal{E}_{\text{Cb}}(\Psi)$. In this section we consider in detail *radial* critical points of the functional $\mathcal{E}_{\text{Cb}}(\Psi)$ defined by (17.93) and establish its basic properties. In this and the following sections we look for *real solutions* $\Psi = \Psi^*$, and all function spaces involve only real functions. In what follows, we explicitly take into account the dependence of the nonlinearity on the size parameter a (or on the related parameter κ), namely the nonlinearity has the form

$$\begin{aligned} G'_{/\kappa}(|\Psi|^2) &= \kappa^2 G'_1(\kappa^{-3}|\Psi|^2), \\ G_{/\kappa}(|\Psi|^2) &= \kappa^5 G_1(\kappa^{-3}|\Psi|^2), \quad \kappa = \frac{a_1}{a}, \end{aligned}$$

where, for consistency with the notation (14.27) where $a = \hat{a}_1/\kappa$, we use the notation “/ κ ” to denote the dependence of rescaled G on κ , that is,

$$/\kappa = 1/\kappa.$$

We also use the notation

$$\hat{a}_1 = \frac{\chi^2}{q^2 m_1}$$

for the Bohr radius as in Sect. 17.5.1. Note that the dependence on $\hat{a}_1$ and a is factored out after the rescaling, and that $G_{/\kappa}$ in the above formula depends only on their ratio κ . We consider the case of small κ , namely

$$\kappa = \frac{\hat{a}_1}{a} \ll 1, \quad (35.22)$$

where the electron size parameter a is much larger than the Bohr radius $\hat{a}_1$. To treat technical difficulties related to the singularity of the logarithm at zero, we use, together with the logarithmic nonlinearity given by (14.41), a regularized logarithmic nonlinearity defined as follows:

$$G'_{/\kappa, \xi}(|\psi|^2) = -\kappa^2 \ln_{+, \xi}(\kappa^{-3}|\psi|^2/C_g^2) - 3\kappa^2, \quad (35.23)$$

where $\xi \leq 0$,

$$\ln_{+, \xi}(s) = \max(\ln(s), \xi) \quad \text{for } s > 0, \quad \ln_{+, \xi}(0) = \xi. \quad (35.24)$$

For $\xi = -\infty$ we set $\ln_{+, -\infty}(s) = \ln(s)$. The function $\ln_{+, \xi}(s)$ with a finite ξ is bounded for bounded s and satisfies the Lipschitz condition for $s \geq 0$. Obviously,

$$-\ln_{+, \xi}(s) \leq -\ln_{+, \xi'}(s) \quad \text{for } \xi > \xi' \geq -\infty. \quad (35.25)$$

Now we describe the basic properties of $G'_{/\kappa, \xi}$ and its integral $G_{/\kappa, \xi}$,

$$G_{1, \xi}(s) = \int_0^s G'_{1, \xi}(s') ds', \quad G_{/\kappa, \xi}(s) = -\kappa^5 G_{1, \xi}(\kappa^{-3}s). \quad (35.26)$$

Proposition 35.2.1 *The function $G_{/\kappa, \xi}$ with $-\infty \leq \xi \leq 0$ can be written in a form*

$$G_{/\kappa, \xi}(s) = -\kappa^2 s \left(\ln_{+, \xi}(\kappa^{-3}s/C_g^2) + 2 + C_g^2 e^{\xi - \ln_{+, \xi} \kappa^{-3}s} \right) \quad (35.27)$$

where

$$e^{\xi - \ln_{+, \xi} s} \leq \min\left(1, \frac{e^\xi}{s}\right) \leq 1, \quad (35.28)$$

and $G_{/\kappa, \xi}$ for $0 \geq \xi \geq \xi' \geq -\infty$ satisfies the following inequalities:

$$\begin{aligned} G_{/\kappa, \xi}(s) &\leq G_{/\kappa, \xi'}(s) \leq G_{/\kappa, -\infty}(s) \\ &= -\kappa^2 s \ln(s) - \kappa^2 s \left(\ln \kappa^{-3}/C_g^2 + 2 \right), \end{aligned} \quad (35.29)$$

$$|G_{1,\xi}(|\Psi|^2) - G_{1,-\infty}(|\Psi|^2)| \leq C e^{\xi/4} |\Psi|. \quad (35.30)$$

Proof Elementary computation shows that for $s \geq C_g^2 e^\xi$

$$G_{1,\xi}(s) = -s \ln(s/C_g^2) - s(C_g^2 e^{\xi - \ln s} + 2), \quad (35.31)$$

and

$$G_{1,\xi}(s) = -\xi s - 3s \text{ for } s \leq e^\xi C_g^2. \quad (35.32)$$

Hence

$$G_{1,\xi}(s) = -s(\ln_{+,\xi}(s/C_g^2) + 2 + C_g^2 e^{\xi - \ln_{+,\xi} s}), \quad (35.33)$$

and (35.27) holds. Using (35.25), we derive (35.29) from (35.26). Obviously,

$$G_{1,\xi}(s) - G_{1,-\infty}(s) = -e^\xi \text{ for } s \geq e^\xi C_g^2,$$

hence

$$|G_{1,\xi}(|\Psi|^2) - G_{1,-\infty}(|\Psi|^2)| \leq e^{\xi/2} |\Psi| / C_g \text{ for } |\Psi|^2 \geq e^\xi C_g^2. \quad (35.34)$$

We have for Ψ satisfying $|\Psi|^2 \leq e^\xi C_g^2$ the following inequalities:

$$\begin{aligned} |G_{1,\xi}(|\Psi|^2) - G_{1,-\infty}(|\Psi|^2)| &= |\Psi|^2 (\xi - \ln |\Psi|^2 / C_g^2)| \\ &\leq e^{\xi/4} C_g |\Psi|^{3/2} |\ln |\Psi|^2 / C_g^2| \leq C_1 e^{\xi/4} C_g |\Psi|. \end{aligned} \quad (35.35)$$

From (35.34) and (35.35) we obtain (35.30).

The nonlinear eigenvalue problem with the Coulomb potential and general ξ has the form similar to (17.87):

$$\omega \Psi + \frac{1}{2} \nabla^2 \Psi + \frac{1}{|\mathbf{x}|} \Psi = \frac{1}{2} G'_{/\kappa,\xi}(|\Psi|^2) \Psi, \quad (35.36)$$

where the dimensionless spectral parameter $\omega = \omega_{\kappa,\xi}$ is given by the formula

$$\omega = \frac{\chi \hat{a}_1}{q^2} \omega^1. \quad (35.37)$$

Obviously, this equation coincides with (17.94) if $\xi = -\infty$. Note that Eq. (35.36) can be obtained as the Euler equation by variation of the energy functional (17.93) which we write in the form

$$\mathcal{E}_{\kappa,\xi}(\Psi) = \mathcal{E}_0(\Psi) + \mathcal{G}_{\kappa,\xi}(\Psi), \quad (35.38)$$

where $\mathcal{E}_0$ is the quadratic energy functional:

$$\mathcal{E}_0 = \int \left(\frac{1}{2} |\nabla \Psi|^2 - \frac{1}{|\mathbf{x}|} |\Psi|^2 \right) d\mathbf{x}, \quad (35.39)$$

and $\mathcal{G}_{\kappa, \xi}$ is the nonlinear functional:

$$\begin{aligned} \mathcal{G}_{\kappa, \xi}(\Psi) \\ = -\frac{\kappa^2}{2} \int \left(\ln_{+, \xi} \left(\kappa^{-3} |\Psi|^2 / C_g^2 \right) + 2 + C_g^2 e^{\xi - \ln_{+, \xi} \kappa^{-3} |\Psi|^2} \right) |\Psi|^2 d\mathbf{x}. \end{aligned} \quad (35.40)$$

As always, we assume the charge normalization constraint

$$\|\Psi\|^2 = \int |\Psi|^2 d\mathbf{x} = 1, \quad (35.41)$$

and denote similarly to (35.2) the set of radial functions (which depend only on $|\mathbf{x}|$) by

$$\mathcal{E}_{\text{rad}} = \{ \Psi \in H_{\text{rad}}^1(\mathbb{R}^3) : \|\Psi\|^2 = 1 \}. \quad (35.42)$$

Obviously, for $\Psi \in \mathcal{E} \cap X$ where X is defined by (35.17),

$$\begin{aligned} \mathcal{G}_{\kappa}(\Psi) &= \mathcal{G}_{\kappa, -\infty}(\Psi) \\ &= -\frac{\kappa^2}{2} \int (|\Psi|^2 \ln |\Psi|^2) d\mathbf{x} + \frac{\kappa^2}{2} \left(\ln \frac{1}{\pi^{3/2}} - 2 + 3 \ln \kappa \right). \end{aligned} \quad (35.43)$$

The spectral parameter ω is the Lagrange multiplier, and it relates to corresponding critical energy levels $E_n^{\kappa, \xi}$ of $\mathcal{E}_{\kappa, \xi}$ on $\mathcal{E} \cap X$ by formula (17.63), from which, using (35.28), we obtain the following inequality:

$$0 < E_n^{\kappa, \xi} - \omega_{\kappa, \xi, n} \leq \frac{\kappa^2}{2}. \quad (35.44)$$

Let us introduce the function

$$g_{\kappa, \xi}(\Psi) = G'_{/\kappa, \xi}(|\Psi|^2)\Psi = \left(-\kappa^2 \ln_{+, \xi} \left(\kappa^{-3} |\Psi|^2 / C_g^2 \right) - 3\kappa^2 \right) \Psi, \quad (35.45)$$

where $\kappa > 0$, $\xi \leq 0$, $\Psi \in \mathbb{C}$. General properties of the functional $\mathcal{E}_{\kappa, \xi}(\Psi)$ with the regularized logarithm are described in the following statements.

Lemma 35.2.1 *The functional $\mathcal{E}_{\kappa, \xi}(\Psi)$ defined on $\mathcal{E}_{\text{rad}}$ by (35.38), (35.33) and (35.40) with $-\infty < \xi \leq 0$ has the following properties: (i) it is bounded from below on $\mathcal{E}$ uniformly in ξ . (ii) If $\Psi \in \mathcal{E}$, $\kappa \leq 1$, and $\mathcal{E}_{\kappa, \xi}(\Psi) \leq C$, then $\|\Psi\|_{H^1} \leq C'$ where C' depends only on C . (iii) The functional $\mathcal{E}_{\kappa, \xi}(\Psi)$ restricted to $\mathcal{E}_{\text{rad}}$ is of class C^1 with respect to the H^1 norm. (iv)*

$$\mathcal{G}_{\kappa,\xi}(\Psi) \geq \mathcal{G}_{\kappa,0}(\Psi) \geq -C''\kappa^2(1 + |\ln \kappa|) \text{ if } \|\Psi\|_{H^1} \leq C'.$$

Proof Boundedness from below follows from (35.9), (35.29) and the inequality

$$\int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}|} |\Psi|^2 d\mathbf{x} \leq C_0 \|\Psi\| \|\nabla \Psi\|. \quad (35.46)$$

From the boundedness of $\mathcal{E}_{\kappa,\xi}(\Psi)$, (35.9) and (35.29), we obtain boundedness of $\|\nabla \Psi\|$ which on $\mathcal{E}$ implies boundedness in H^1 . To prove (iii), observe that by (35.23)

$$g_{\kappa,\xi}(\Psi) = \kappa^2(-\xi - 3)\Psi + g_{\kappa,\xi}^1(|\Psi|^2),$$

where $g_{\kappa,\xi}^1(|\Psi|^2)$ is identically zero for small $|\Psi|$, and it has less than quadratic growth for $|\Psi| \rightarrow \infty$. According to Theorem A.VI in [38], then the functional $\mathcal{G}_{\kappa,\xi}(\Psi) + \kappa^2(\xi + 3)\|\Psi\|^2$ is of class C^1 , hence $\mathcal{G}_{\kappa,\xi}(\Psi)$ is of class C^1 too. To obtain (iv), we use (35.29) and observe that $\mathcal{G}_{\kappa,0}(\Psi)$ can be estimated in terms of $\|\Psi\|_{L^6}$ which, in turn, can be estimated by $\|\Psi\|_{H^1}$.

Lemma 35.2.2 *Let a sequence of radial functions $\Psi_j \in \mathcal{E}_{\text{rad}}$ satisfy (35.47). Then: (i) a subsequence of Ψ_j converges in $L^p(\mathbb{R}^3)$ and almost everywhere to Ψ_∞ if $2 < p < 6$; (ii) $\Psi_j(\mathbf{x}) \rightarrow 0$ as $|\mathbf{x}| \rightarrow \infty$ uniformly in j .*

Proof The estimate (35.47) together with the normalization condition imply the uniform boundedness of Ψ_j in the Sobolev space $H^1(\mathbb{R}^3)$. According to Theorem A.I' of [38], the space of radial functions $H_{\text{rad}}^1(\mathbb{R}^3)$ is compactly imbedded into $L^p(\mathbb{R}^3)$ if $2 < p < 6$. Hence, a subsequence $\Psi_j(\mathbf{x})$ converges to $\Psi_\infty(\mathbf{x})$ strongly in $L^p(\mathbb{R}^3)$ and almost everywhere. Statement (ii) follows from Radial Lemma A.II in [38].

The functional $\mathcal{E}_{\kappa,\xi}(\Psi)$ with the regularized logarithm satisfies the Palais–Smale condition similar to condition (P-S⁺) in [39]:

Theorem 35.2.1 *Let $\xi > -\infty$. Let a sequence $\Psi_j \in \mathcal{E}_{\text{rad}}$ have the following properties:*

$$\mathcal{E}_{\kappa,\xi}(\Psi_j) \leq -\beta, \quad \beta > 0, \quad (35.47)$$

and

$$F\Psi_j = \omega_j\Psi_j + \frac{1}{2}\nabla^2\Psi_j + \frac{1}{|\mathbf{x}|}\Psi_j - \frac{1}{2}g_{\kappa,\xi}(\Psi_j) \rightarrow 0 \quad (35.48)$$

strongly in $H^{-1}(\mathbb{R}^3)$. Then the sequence contains a subsequence which converges strongly in $H_{\text{rad}}^1(\mathbb{R}^3)$.

Proof The proof is similar to the treatment of radial solutions in [38], [225]. We only sketch the main steps. Since (35.46) holds, (35.47), (35.29) and (35.8) imply the boundedness of Ψ_j in $H^1(\mathbb{R}^3)$. Using Lemma 35.2.2, we conclude that we can choose a subsequence such that $\Psi_j \rightarrow \Psi_\infty$ in $L^5(\mathbb{R}^3)$ and $L^{8/3}(\mathbb{R}^3)$, it converges weakly in $H^{-1}(\mathbb{R}^3)$ and almost everywhere in $\mathbb{R}^3$. Multiplying (35.48) by Ψ_j in $L^2(\mathbb{R}^3) = H$ with inner product $\langle \cdot, \cdot \rangle$, we obtain that

$$\begin{aligned} \omega_j \langle \Psi_j, \Psi_j \rangle - \frac{1}{2} \langle \nabla \Psi_j, \nabla \Psi_j \rangle - \frac{1}{2} \langle (G'_{/\kappa, \xi} |\Psi_j|^2) \Psi_j, \Psi_j \rangle \\ = \langle F \Psi_j, \Psi_j \rangle - \left\langle \frac{1}{|\mathbf{x}|} \Psi_j, \Psi_j \right\rangle, \end{aligned} \quad (35.49)$$

where F is the left-hand side of (17.50). We rewrite (35.49) in the form

$$\omega_j \langle \Psi_j, \Psi_j \rangle = \langle F \Psi_j, \Psi_j \rangle + \mathcal{E}_{\kappa, \xi}(\Psi_j) - \int G_{/\kappa, \xi}(|\Psi|^2) - G'_{/\kappa, \xi}(|\Psi|^2) |\Psi|^2 \, d\mathbf{x}.$$

Using (35.28), we obtain for $\Psi \in \mathcal{E}_{\text{rad}}$ that

$$0 < \int G_{/\kappa, \xi}(|\Psi|^2) - G'_{/\kappa, \xi}(|\Psi|^2) |\Psi|^2 \, d\mathbf{x} \leq \frac{\kappa^2}{2}. \quad (35.50)$$

Since $\langle F \Psi_j, \Psi_j \rangle \rightarrow 0$, (35.47) and (35.50) imply that we can choose a subsequence Ψ_j with $\omega_j \rightarrow \omega_\infty$, where $\omega_\infty \leq -\beta$. Multiplying (35.48) by a smooth test function with a compact support and passing to the limit, we obtain in a standard way that the limit $\Psi_\infty \in H^1(\mathbb{R}^3)$ satisfies (35.36) with $\omega = \omega_\infty$.

Now we want to prove that $\Psi_\infty \in \mathcal{E}$, and the convergence in $H^1_{\text{rad}}(\mathbb{R}^3)$ is strong. We rewrite (35.49) in the form

$$\begin{aligned} \omega_j \langle \Psi_j, \Psi_j \rangle - \frac{1}{2} \langle \nabla \Psi_j, \nabla \Psi_j \rangle - \frac{1}{2} \langle (G_{/\kappa, \xi}(|\Psi_j|^2) - \kappa^2 \xi) \Psi_j, \Psi_j \rangle + \kappa^2 \xi \langle \Psi_j, \Psi_j \rangle \\ = \langle F \Psi_j, \Psi_j \rangle - \left\langle \frac{1}{|\mathbf{x}|} \Psi_j, \Psi_j \right\rangle. \end{aligned} \quad (35.51)$$

We have $\langle F \Psi_j, \Psi_j \rangle \rightarrow 0$,

$$\omega_j \langle \Psi_j, \Psi_j \rangle + \kappa^2 \xi \langle \Psi_j, \Psi_j \rangle = \omega_j + \kappa^2 \xi \rightarrow \omega_\infty + \kappa^2 \xi.$$

Since $G_{/\kappa, \xi}(|\Psi|^2) - \kappa^2 \xi$ is identically zero for small $|\Psi|^2$ and grows slower than $|\Psi|^3$ as $|\Psi| \rightarrow \infty$, and $|\Psi_j|^5$ are converging in $L^1(\mathbb{R}^3)$, we can apply the compactness Lemma of Strauss (see Theorem A.I in [38]) and obtain

$$\langle (G_{/\kappa, \xi}(|\Psi_j|^2) - \kappa^2 \xi) \Psi_j, \Psi_j \rangle \rightarrow \langle (G_{/\kappa, \xi}(|\Psi_\infty|^2) - \kappa^2 \xi) \Psi_\infty, \Psi_\infty \rangle.$$

Using the Hölder inequality, we obtain that the quadratic mapping $\Psi \rightarrow \frac{1}{|\mathbf{x}|} |\Psi|^2$ is continuous from $L^5(\mathbb{R}^3) \cap L^{8/3}(\mathbb{R}^3)$ into $L^1(\mathbb{R}^3)$. Therefore

$$\left\langle \frac{1}{|\mathbf{x}|} \Psi_j, \Psi_j \right\rangle \rightarrow \left\langle \frac{1}{|\mathbf{x}|} \Psi_\infty, \Psi_\infty \right\rangle.$$

We denote

$$M_0 = \lim_{j \rightarrow \infty} \inf \langle \Psi_j, \Psi_j \rangle = 1, \quad M_1 = \lim_{j \rightarrow \infty} \inf \langle \nabla \Psi_j, \nabla \Psi_j \rangle.$$

We infer from (35.51) passing to the limit that

$$\begin{aligned} (\omega_\infty + \kappa^2 \xi) M_0 - \frac{1}{2} M_1 - \frac{1}{2} \langle (G_{/\kappa, \xi}(|\Psi_\infty|^2) - \kappa^2 \xi) \Psi_\infty, \Psi_\infty \rangle \\ = - \left\langle \frac{1}{|\mathbf{x}|} \Psi_\infty, \Psi_\infty \right\rangle. \end{aligned} \quad (35.52)$$

Multiplying (35.36) with $\omega = \omega_\infty$ by Ψ in H , we obtain that

$$\begin{aligned} \omega_\infty \langle \Psi_\infty, \Psi_\infty \rangle - \frac{1}{2} \langle \nabla \Psi_\infty, \nabla \Psi_\infty \rangle - \frac{1}{2} \langle (G_{/\kappa, \xi}(|\Psi_\infty|^2) - \kappa^2 \xi) \Psi_\infty, \Psi_\infty \rangle \\ + \kappa^2 \xi \langle \Psi_\infty, \Psi_\infty \rangle = \langle F \Psi_\infty, \Psi_\infty \rangle - \left\langle \frac{1}{|\mathbf{x}|} \Psi_\infty, \Psi_\infty \right\rangle. \end{aligned}$$

Comparing with (35.52), we observe that

$$(\omega_\infty + \kappa^2 \xi) M_0 - \frac{1}{2} M_1 = (\omega_\infty + \kappa^2 \xi) \langle \Psi_\infty, \Psi_\infty \rangle - \frac{1}{2} \langle \nabla \Psi_\infty, \nabla \Psi_\infty \rangle. \quad (35.53)$$

The weak convergence in $H_{\text{rad}}^1(\mathbb{R}^3)$ to Ψ_∞ implies that $M_0 \geq \|\Psi_\infty\|^2$, $M_1 \geq \|\nabla \Psi_\infty\|^2$. Since $(\omega_\infty + \kappa^2 \xi) < 0$, (35.53) is possible only if $M_0 = \|\Psi_\infty\|^2$, $M_1 = \|\nabla \Psi_\infty\|^2$. These equalities imply that the weak convergence in H^1 of the subsequence to Ψ_∞ is strong.

35.2.2 Nonlinear Eigenvalues for a Charge in the Coulomb Field

We prove in this section that, if κ is small enough, the nonlinear eigenvalue problem (35.36) with the logarithmic nonlinearity has solutions with eigenvalues which are close to the eigenvalues of the corresponding linear problem. We look for *real-valued* solutions. As a first step, we prove the existence of such solutions of the

problem with the regularized logarithm $\ln_{+,\xi}$. As the second step, we pass to the limit as $\xi \rightarrow -\infty$ and obtain solutions of the eigenvalue problem with the original logarithmic nonlinearity.

Consider the linear Schrödinger operator $\mathcal{O}$ with the Coulomb potential which corresponds to $\mathcal{E}_0(\Psi)$, that is,

$$\mathcal{O}\Psi = -\frac{1}{2}\nabla^2\Psi - \frac{1}{|\mathbf{x}|}\Psi. \quad (35.54)$$

Note that $\mathcal{O}$ has the following well known negative eigenvalues

$$E_n^0 = \omega_{0,n} = -\frac{1}{2n^2}, \quad n = 1, 2, \dots, \quad (35.55)$$

which coincide with the negative eigenvalues of the operator $\mathcal{O}_{\text{rad}}$ obtained by the restriction of $\mathcal{O}$ to radial functions (without counting their multiplicity) and are equal to the negative critical energy levels of $\mathcal{E}_0(\Psi)$. Every eigenvalue of the radial problem

$$\mathcal{O}_{\text{rad}}\Psi_n^0 = \omega_{0,n}\Psi_n^0$$

is simple, and corresponding eigenfunctions $\Psi_n^0(r)$ given by well-known explicit formulas decay exponentially as $r \rightarrow \infty$; for example, the linear ground state is given by the formula

$$\Psi_1^0(\mathbf{x}) = \frac{1}{\pi^{1/2}}e^{-|\mathbf{x}|}, \quad E_1^0 = -\frac{1}{2}.$$

Let us denote by L_n^0 the eigenspace of $\mathcal{O}_{\text{rad}}$ which corresponds to the eigenvalue E_n^0 and consists of radial functions, we denote by $L_n^- \subset H_{\text{rad}}^1$ the finite-dimensional invariant subspace of $\mathcal{O}_{\text{rad}}$ which corresponds to eigenvalues $\lambda \leq E_n^0$, and the orthogonal in L^2 to L_n^- infinite-dimensional subspace of $H_{\text{rad}}^1(\mathbb{R}^3)$ is denoted by L_n^+ . We have the following orthogonal decomposition:

$$H_{\text{rad}}^1(\mathbb{R}^3) = L_n^- + L_n^+ = L_{n-1}^- + L_n^0 + L_n^+.$$

We look for the eigenvalues $\omega = \omega_{\kappa,\xi,n} < 0$ of the nonlinear eigenvalue problem (35.36) which can be considered as a perturbation of the eigenvalues E_n^0 . The minimizer of $\mathcal{E}_{\kappa,-\infty}(\Psi)$ and the corresponding energy level $\omega_{\kappa,-\infty,1}$ was found in [40]. It is given by the formula

$$\Psi_{\kappa,-\infty,1}(\mathbf{x}) = \kappa^{3/2}C_g e^{-|\kappa\mathbf{x}|^2/2} C e^{-|\mathbf{x}|}, \quad \omega_{\kappa,1} = -\frac{1}{2} - \frac{1}{2}\kappa^2 \ln C^2, \quad (35.56)$$

where

$$C^{-2} = \frac{4}{\pi^{1/2}} \left[\frac{3}{4} \sqrt{\pi} e - \frac{3}{4} \sqrt{\pi} \operatorname{erf}(1) e - \frac{1}{2} \right], \quad \frac{1}{2} \ln C^2 \simeq 1.868.$$

Hence the nonlinear correction for the ground energy level is of order κ^2 .

Since the Palais–Smale condition is satisfied according to Theorem 35.2.1, the existence of an infinite sequence of discrete negative energy values of $\mathcal{E}_{\kappa,\xi}(\Psi)$ can be proven exactly along the lines of [38]. But we want to show in addition that the eigenvalues of the nonlinear problem with small κ are close to the eigenvalues of the linear problem (this is not surprising, since all the eigenvalues of the linear problem for radial functions are simple). Therefore, we modify the construction of critical points of the functional $\mathcal{E}_{\kappa,\xi}$ on $\mathcal{E}$.

We will use the following deformation lemma, which is a minor modification and reformulation in our notation of Lemma 5 from [38] (with essentially the same proof).

Lemma 35.2.3 *Suppose that the functional $\mathcal{E}_{\kappa,\xi} \in C^1(\mathcal{E}_{\text{rad}}, \mathbb{R})$ satisfies the Palais–Smale condition as in Theorem 35.2.1. Suppose also that $E'' < 0$ and that the segment $E' \leq E \leq E''$ does not contain critical values of $\mathcal{E}_{\kappa,\xi}$. Let $\bar{\epsilon} > 0$ be an arbitrary small number. Then there exists $0 < \epsilon < \bar{\epsilon}$ and a mapping $\eta \in C(\mathcal{E}_{\text{rad}}, \mathcal{E}_{\text{rad}})$ such that*

- (i) $\eta(u) = u$ for $u \in \mathcal{E}_{\text{rad}}$ with $\mathcal{E}_{\kappa,\xi}(u) \leq E' - \bar{\epsilon}$ or $\mathcal{E}_{\kappa,\xi}(u) \geq E'' + \bar{\epsilon}$.
- (ii) η is a homeomorphism $\mathcal{E}_{\text{rad}} \rightarrow \mathcal{E}_{\text{rad}}$ and it is odd if $\mathcal{E}_{\kappa,\xi}$ is even.
- (iii) $\mathcal{E}_{\kappa,\xi}(\eta(u)) \leq \mathcal{E}_{\kappa,\xi}(u)$.
- (iv) $\eta(\{u \in \mathcal{E}_{\text{rad}} : \mathcal{E}_{\kappa,\xi}(u) \leq E + \epsilon\}) \subset \{u \in \mathcal{E}_{\text{rad}} : \mathcal{E}_{\kappa,\xi}(u) \leq E - \epsilon\}$ for $E \in [E', E'']$.

To prove the existence when κ is small of critical points of $\mathcal{E}_{\kappa,\xi}(\Psi)$ on the infinite-dimensional sphere $\mathcal{E}_{\text{rad}}$ with critical values $E_n^{\kappa,\xi}$ near E_n^0 , $n = 2, 3, \dots$, we slightly modify the construction from [38]. As a first step, we define a finite-dimensional domain $B = B_n \subset L_n^-$ which lies in $\mathcal{E}_{\text{rad}}$ near the eigenfunction Ψ_n^0 of $\mathcal{O}_{\text{rad}}$ as follows:

$$B_n = \left\{ \Psi : \Psi = (1 - R^2)^{1/2} \Psi_n^0 + v, \quad v \in L_{n-1}^-, \quad \|v\| = R, \quad 0 \leq R \leq R_0 \right\}, \quad (35.57)$$

where $R_0 < 1/2$ is a small fixed number, for simplicity we fix $R_0 = 1/3$. The domain B_n is diffeomorphic to an $(n-1)$ -dimensional ball. The boundary of B_n is the $(n-2)$ -dimensional sphere

$$\partial B_n = \left\{ \Psi : \Psi = (1 - R_0^2)^{1/2} \Psi_n^0 + v, \quad v \in L_{n-1}^-, \quad \|v\| = R_0 \right\}. \quad (35.58)$$

We consider continuous functions Γ defined on B_n with values in $\mathcal{E}$. We consider the class $\{\Gamma\}_n$ of continuous mappings $\Gamma : B_n \rightarrow \mathcal{E}_{\text{rad}}$ such that Γ restricted to ∂B_n is the identity,

$$\Gamma(\Psi) = \Psi \text{ if } \Psi \in \partial B_n. \quad (35.59)$$

The n -th min-max energy level $E_{\kappa,\xi,n}$ is defined as follows:

$$E_{\kappa,\xi,n} = \inf_{\Gamma \in \{\Gamma\}_n} \max_{\Psi \in B_n} \mathcal{E}_{\kappa,\xi}(\Gamma(\Psi)). \quad (35.60)$$

Lemma 35.2.4 *Let $\xi \leq 0$, let $n = 2, 3, \dots$, $0 < \kappa \leq 1$. Then $E_{\kappa,\xi,n}$ defined by (35.60) lies in the interval $I_n = [E', E'']$ with*

$$E' = E_n^0 - C_{3n}\kappa^2(1 + |\ln \kappa|), \quad E'' = E_n^0 + C_{1n}\kappa^2, \quad (35.61)$$

where C_{1n} and C_{3n} depend only on n .

Proof Since we can take a particular function $\Gamma(\Psi) = \Gamma_0(\Psi) = \Psi_n^0 + v$ for $\Psi \in B_n$ as in (35.57), we have

$$\begin{aligned} E_{\kappa,\xi,n} &\leq \max_{\Psi \in B_n} \mathcal{E}_{\kappa,\xi}(\Psi) \leq \mathcal{E}_0(\Psi_n^0) + \max_{B_n} \mathcal{G}_{\kappa,\xi}(\Psi) \\ &= E_n^0 + \max_{B_n} \mathcal{G}_{\kappa,\xi}(\Psi). \end{aligned} \quad (35.62)$$

Note that B_n lies in a ball in a finite-dimensional subspace L_n^- in H^1 , the basis of L_n^- consists of exponentially decaying functions with bounded first derivatives, hence B_n is compact in $H^1(\mathbb{R}^3)$ and in $L^p(\mathbb{R}^3)$ for every $p \geq 1$. Therefore,

$$\|\Psi\|_{H^1} + \|\Psi\|_{L^3} + \|\Psi\|_{L^{4/3}} \leq C_{0,n} \text{ for } \Psi \in B_n.$$

Hence, using (35.29) and power estimates for $\Psi^2 \ln \Psi^2$, we conclude that

$$\begin{aligned} \mathcal{G}_{\kappa,\xi}(\Psi) &\leq \mathcal{G}_{\kappa,-\infty}(\Psi) \\ &\leq C\kappa^2 \left(\|\Psi\|_{L^3}^3 + \|\Psi\|_{L^{4/3}}^{4/3} \right) - \kappa^2 (\ln \kappa^{-3}/C_g^2 + 2) \|\Psi\|^2, \end{aligned}$$

and we obtain that

$$\max_{B_n} \mathcal{G}_{\kappa,\xi}(\Psi) \leq C_{1n}\kappa^2 \quad (35.63)$$

where C_{1n} depends only on n . Using $\Gamma_0(\Psi)$ and the estimate (35.62) of $\mathcal{E}_{\kappa,\xi}(\Psi)$, we observe that

$$E_{\kappa,\xi,n} = \inf_{\Gamma \in \{\Gamma\}_n, \mathcal{E}_{\kappa,\xi}(\Gamma(B_n)) \leq C_{2n}} \max_{\Psi \in B_n} \mathcal{E}_{\kappa,\xi}(\Gamma(\Psi)) \quad (35.64)$$

where $C_{2n} = C_{1n}\kappa^2 + E_n^0$. Inequality (35.63) provides E'' in (35.61).

Now we estimate $E_{\kappa,\xi,n}$ from below. Mappings $\Gamma \in \{\Gamma\}_n$ which satisfy (35.59) have the following topological property: the set $\Gamma(B_n)$ must intersect the subspace L_{n-1}^+ . Indeed, assume that the contrary is true, and for some Γ there is no intersection of $\Gamma(\Psi)$ with L_{n-1}^+ , that is, $\|\Pi_{n-1}^- \Gamma(\Psi)\| \neq 0$ for all $\Psi \in B_n$ where Π_{n-1}^- is the orthoprojection onto the finite-dimensional subspace L_{n-1}^- . Then we could consider the following function on the ball $\|v\| \leq R_0$ in L_{n-1}^- :

$$\Gamma_1(v) = \frac{R_0}{N(v)} \Pi_{n-1}^- \Gamma(v),$$

where $N(v) = \|\Pi_{n-1}^- \Gamma(v)\|$. This function is continuous on the ball $\{\|v\| \leq R_0\}$ (since the denominator $N(v)$ does not vanish), and $\Gamma_1(v) = v$ if $\|v\| = R_0$, and it also has the property $\|\Gamma_1(v)\| = R_0$ if $\|v\| \leq R_0$. A function with such properties is called a retraction of the ball $\{\|v\| \leq R_0\}$ in the $(n-1)$ -dimensional space L_{n-1}^- onto its boundary $\{\|v\| = R_0\}$. It is a well-known fact from algebraic topology that such a retraction cannot exist. Therefore, for every Γ , we must have $\|\Pi_{n-1}^- \Gamma(\Psi_0)\| = 0$ for some $\Psi_0 \in B$. Therefore there exists a $\Psi_0 \in B_n$ such that

$$\Gamma(\Psi_0) = (1 - R^2)^{1/2} \Psi_n^0 + w, \quad w \in L_n^+, \quad \|w\|^2 = R^2 \leq 1. \quad (35.65)$$

To obtain an estimate of $E_{\kappa,\xi,n}$ from below, note that (35.65) implies

$$\begin{aligned} \mathcal{E}_0(\Gamma(\Psi_0)) &= (1 - R^2) \mathcal{E}_0(\Psi_n^0) + \mathcal{E}_0(w) \\ &\geq (1 - R^2) E_n^0 + E_n^0 \|w\|^2 \geq E_n^0. \end{aligned}$$

Therefore, we obtain for $\Gamma \in \{\Gamma\}_n$, $\mathcal{E}_{\kappa,\xi}(\Gamma(B_n)) \leq C_{2n}$ that

$$\mathcal{E}_{\kappa,\xi}(\Gamma(\Psi_0)) \geq E_n^0 + \min_{\Psi \in \mathcal{E}, \mathcal{E}_{\kappa,\xi}(u) \leq C_{2n}} \mathcal{G}_{\kappa,\xi}(u).$$

Inequality $\mathcal{E}_{\kappa,\xi}(\Gamma(B_n)) \leq C_{2n}$ implies by Lemma 35.2.1 the boundedness $\Gamma(B_n)$ in H^1 . Using point (iv) in Lemma 35.2.1, we conclude that

$$E_{\kappa,\xi,n} \geq E_n^0 - \kappa^2 C_{3n} (1 + |\ln \kappa|). \quad (35.66)$$

Combining (35.66), (35.62) and (35.63), we obtain (35.61).

Now we prove the existence of solutions of the eigenvalue problem with the regularized logarithm.

Theorem 35.2.2 *Let $-\infty < \xi \leq 0$. Let $n \geq 2$ be an integer, and $\kappa \leq 1$ be so small that*

$$\frac{1}{9} (E_{n-1}^0 - E_n^0) + C_{1n}\kappa^2 + C_{3n}\kappa^2 (1 + |\ln \kappa|) < 0, \quad (35.67)$$

where E_n^0 are defined by (35.55) and C_{1n} is the constant from (35.63). Then

- (i) the interval I_n defined by (35.61) contains a critical value $E_n^{\kappa, \xi}$ of $\mathcal{E}_{\kappa, \xi}$;
- (ii) there exists a solution $\Psi_n \in \mathcal{E}_{\text{rad}}$ of (35.36) with eigenvalue $\omega = \omega_{\kappa, \xi, n}$ such that $|E_n^{\kappa, \xi} - \omega_{\kappa, \xi, n}| \leq \frac{\kappa^2}{2}$ which satisfies

$$|\omega_{\kappa, \xi, n} - \omega_{0, n}| \leq C_n \kappa^2 (1 + |\ln \kappa|). \quad (35.68)$$

Proof We follow here the lines of [38] with minor modifications. Point (ii) follows from point (i) and (35.50). To prove (i), assume the contrary, namely let n and κ satisfy all the conditions and assume that the interval I_n defined by (35.61) does not contain critical points of $\mathcal{E}_{\kappa, \xi}$. We apply then Lemma 35.2.3 where we take

$$\bar{\epsilon} \leq \frac{1}{2} \left| \frac{1}{9} (E_{n-1}^0 - E_n^0) + C_{1n}\kappa^2 + C_{3n}\kappa^2 (1 + |\ln \kappa|) \right|,$$

with $E' = E_n^0 - \kappa^2 C_{3n} (1 + |\ln \kappa|)$ and find a mapping $\eta : \mathcal{E}_{\text{rad}} \rightarrow \mathcal{E}_{\text{rad}}$. By Lemma 35.2.4, the value $E_{\kappa, \xi, n}$ defined by (35.60) or (35.64) lies inside I_n . By point (iv) of Lemma 35.2.3

$$\eta \left(\{u \in \mathcal{E}_{\text{rad}} : \mathcal{E}_{\kappa, \xi}(u) \leq E_{\kappa, \xi, n} + \epsilon\} \right) \subset \{u \in \mathcal{E}_{\text{rad}} : \mathcal{E}_{\kappa, \xi}(u) \leq E_{\kappa, \xi, n} - \epsilon\}.$$

We can find $\Gamma \in \{\Gamma\}_n$ such that $\max_{B_n} \Gamma(\Psi) \leq E_{\kappa, \xi, n} + \epsilon/2$. If we consider $\Gamma'(\Psi) = \eta(\Gamma(\Psi))$, we get $\max_{B_n} \Gamma'(\Psi) \leq E_{\kappa, \xi, n} - \epsilon/2$. To infer from this inequality a contradiction with definition (35.60), we have to check that $\Gamma' \in \{\Gamma\}_n$, namely to check (35.59). Note that on ∂B_n

$$\mathcal{E}_0 \left((1 - R_0^2)^{1/2} \Psi_n^0 + v \right) = (1 - R_0^2) E_n^0 + \mathcal{E}_0(v) \leq E_n^0 + R_0^2 (E_{n-1}^0 - E_n^0),$$

hence

$$\begin{aligned} & \mathcal{E}_{\kappa, \xi} \left((1 - R_0^2)^{1/2} \Psi_n^0 + v \right) \\ & \leq E_n^0 + R_0^2 (E_{n-1}^0 - E_n^0) + \mathcal{G}_{\kappa, \xi} \left((1 - R_0^2)^{1/2} \Psi_n^0 + v \right). \end{aligned}$$

Therefore,

$$\mathcal{E}_{\kappa,\xi}|_{\partial B_n} \leq E_n^0 + R_0^2 (E_{n-1}^0 - E_n^0) + \max_{\partial B_n} (\mathcal{G}_{\kappa,\xi}(\Psi)), \quad (35.69)$$

and, since $R_0^2 = 1/9$, we conclude using (35.63) and (35.67) that

$$\mathcal{E}_{\kappa,\xi}|_{\partial B_n} \leq E_n^0 + R_0^2 (E_{n-1}^0 - E_n^0) + C_{1n}\kappa^2 \leq E' - \bar{\epsilon}, \quad (35.70)$$

which implies that $\eta(\Psi) = \Psi$ on ∂B_n by point (i) of Lemma 35.2.3. Hence, $\Gamma' \in \{\Gamma\}_n$, and we obtain a contradiction with (35.60). So we conclude that I_n contains a critical value corresponding to a critical point Ψ' . Since $\mathcal{E}_{\kappa,\xi}$ is differentiable at Ψ' , this critical point provides a solution to (35.36), and (35.68) follows from (35.44).

Now we show that solutions of (35.36) decay exponentially as $r = |\mathbf{x}| \rightarrow \infty$.

Lemma 35.2.5 *Let $n \leq n_0$. Let $0 < \kappa \leq \kappa_0$, where κ_0 is small enough for conditions of Theorem 35.2.2 to be fulfilled for $n \leq n_0$. Let $\bar{\xi} \leq 0$ satisfy*

$$1 - \frac{\kappa^2}{4}|\bar{\xi}| + \frac{3\kappa^2}{2} < 0. \quad (35.71)$$

Let $p = \frac{\kappa}{3}|\bar{\xi}|^{1/2}$. Let $\Psi_n \in \mathcal{E}_{\text{rad}}$ be the solution of (35.36) with the eigenvalue $\omega = \omega_{\kappa,\xi,n}$ which is described in Theorem 35.2.2 with $\xi \leq \bar{\xi}$. Then there exist positive constants R_1 and C which depend only on n_0 , κ and $\bar{\xi}$, such that for $\xi \leq \bar{\xi}$, $n \leq n_0$

$$|\Psi_n(r)| \leq C''e^{-pr} \text{ for } r \geq R_1. \quad (35.72)$$

Proof Since $\|\Psi_n\|_{H^1}$ are bounded uniformly in $\xi \leq 0$ and $\kappa \leq 1$, we can use the following result of Strauss (see Lemma A.II in [38]):

$$|\Psi_n(r)| \leq C'_0 r^{-1} \|\Psi_n\|_{H^1} \text{ for } r \geq \bar{R}, \quad (35.73)$$

where C'_0 and $\bar{R}$ are absolute constants, $\bar{R} \geq 1$. We apply this inequality to radial solutions $\Psi = \Psi_n$ of (35.36) with $\omega = \omega_{\kappa,\xi,n} < 0$ and obtain

$$|\Psi_n(r)| \leq C_0 r^{-1} \text{ for } r \geq \bar{R}, \quad (35.74)$$

where, according to Lemma 35.2.1, the constant C_0 is bounded uniformly in $\kappa \leq 1$, $\xi \leq 0$ and n . We write (35.36) in the form

$$\frac{1}{2} \nabla^2 \Psi + V_{\kappa,\xi} \Psi = 0, \quad (35.75)$$

where

$$V_{\kappa,\xi}(\mathbf{x}) = \frac{1}{|\mathbf{x}|} - \frac{1}{2} G'_{/\kappa,\xi}(|\Psi(\mathbf{x})|^2) + \omega.$$

According to (35.74), using monotonicity of the function $\ln_{+,\xi}(s)$ with respect to s , we obtain for $|\mathbf{x}| \geq R \geq \bar{R} \geq 1$ that

$$V_{\kappa,\xi}(\mathbf{x}) \leq \frac{1}{R} + \frac{\kappa^2}{2} (\ln_{+,\xi}(\kappa^{-3} R^{-2} C_0^2 / C_g^2) + 3) + \omega.$$

Note that

$$V_{\kappa,\xi}(\mathbf{x}) \leq \frac{1}{R} + \frac{\kappa^2}{2} (\xi + 3) + \omega \text{ if } \ln(\kappa^{-3} R^{-2} C_0^2 / C_g^2) \leq \xi, \quad R \geq \bar{R}.$$

Therefore, for any $0 < \kappa \leq \kappa_0$, $\bar{\xi} \leq 0$, we can find $R_1 \geq \bar{R}$ such that

$$\ln(\kappa^{-3} R_1^{-2} C_0^2 / C_g^2) \leq \bar{\xi},$$

and if $|\bar{\xi}|$ is so large that (35.71) holds, we obtain for $\xi = \bar{\xi}$ that

$$V_{\kappa,\xi}(\mathbf{x}) \leq -\frac{\kappa^2}{4} |\bar{\xi}| + \omega < 0 \text{ for } |\mathbf{x}| \geq R_1. \quad (35.76)$$

According to definition (35.24), $V_{\kappa,\xi}(\mathbf{x})$ is a monotonically increasing function of ξ , therefore this inequality holds for all $\xi \leq \bar{\xi}$.

Note that $\Psi(\mathbf{x})$ cannot change sign for $|\mathbf{x}| > R_1$. Indeed, if $\Psi(\mathbf{x}) = 0$ at $|\mathbf{x}| = R'_1 > R_1$, we could multiply (35.75) by Ψ and integrate over $\{|\mathbf{x}| > R'_1\}$ which would yield

$$\int_{\{|\mathbf{x}| > R'_1\}} \left(\frac{1}{2} |\nabla \Psi|^2 - V_{\kappa,\xi}(\mathbf{x}) |\Psi|^2 \right) d\mathbf{x} = 0,$$

and $\Psi(\mathbf{x}) = 0$ for all $|\mathbf{x}| > R'_1$, and this is impossible for a radial solution of (35.36) with $\|\Psi\| = 1$. So we assume that $\Psi(\mathbf{x}) > 0$ for $|\mathbf{x}| > R_1$ (the case of negative Ψ is similar). We compare the solution Ψ of this equation in the domain $|\mathbf{x}| > R$ with the function

$$Y = C_Y \frac{1}{r} e^{-pr}, \quad p > 0, \quad C_Y > 0,$$

which is a solution of the equation

$$\nabla^2 Y - p^2 Y = 0.$$

Subtracting half this equation from (35.75), we obtain:

$$\frac{1}{2} \nabla^2 (\Psi - Y) + V_{\kappa, \xi} (\Psi - Y) + \left(V_{\kappa, \xi} + \frac{1}{2} p^2 \right) Y = 0. \quad (35.77)$$

We choose $p^2 = \frac{\kappa^2}{9} |\bar{\xi}|$, and (35.76) implies that

$$\left(V_{\kappa, \xi} + \frac{1}{2} p^2 \right) < \omega < 0 \text{ for } |\mathbf{x}| \geq R_1. \quad (35.78)$$

We choose $C_Y > 0$ so that

$$C_Y e^{-p R_1} > C_0,$$

where C_0 is the same as in (35.74). We assert that

$$\Psi(\mathbf{x}) - Y(\mathbf{x}) \leq 0 \text{ for } |\mathbf{x}| \geq R_1. \quad (35.79)$$

This is true at $|\mathbf{x}| = R_1$ according to the choice of C_Y , and the limit of $\Psi - Y$ as $|\mathbf{x}| \rightarrow \infty$ is zero. Assume now that (35.79) does not hold for some $|\mathbf{x}| > R_1$. Then $\Psi(\mathbf{x}) - Y(\mathbf{x})$ must have a local positive maximum at some $|\mathbf{x}| = R'_1 > R_1$, at this point $\nabla^2 (\Psi - Y) \leq 0$, by (35.78) $(V_{\kappa, \xi} + p^2) Y < 0$ and $V_{\kappa, \xi} (\Psi - Y) \leq 0$ by (35.76). These inequalities contradict (35.77), hence (35.79) holds. Inequality (35.72) follows from (35.79).

Now we prove the main result of this section for the nonlinear eigenvalue problem with the original logarithmic inequality which corresponds to $\xi = -\infty$.

Theorem 35.2.3 *Let $\xi = -\infty$, $n_0 > 0$, and let an integer n satisfy the inequality $2 \leq n \leq n_0$, and $\kappa \leq 1$ be so small that (35.67) holds. Then there exists a solution $\Psi_n \in \mathcal{E}_{\text{rad}} \cap X$ of (35.36) with an eigenvalue $\omega = \omega_{\kappa, n}$ and with the energy $E_n^\kappa = \omega_{\kappa, n} + \frac{\kappa^2}{2}$ which satisfies the inequality*

$$|\omega_{\kappa, n} - \omega_{0, n}| \leq C_n \kappa^2 (1 + |\ln \kappa|). \quad (35.80)$$

The solution Ψ_n decays superexponentially as $|\mathbf{x}| \rightarrow \infty$, namely (35.72), where R_1 depends on p , holds for an arbitrary large p .

Proof We fix n and consider a sequence $\xi_m \rightarrow -\infty$. We have a sequence of solutions $\Psi_{n, m} \in \mathcal{E}_{\text{rad}}$ of (35.36) with $\omega_{\kappa, \xi_m, n}$ satisfying (35.68) which exists according to Theorem 35.2.2 with $\xi = \xi_m$. Note that, according to (35.61), $\mathcal{E}_\kappa(\Psi_{n, m}) \in [E', E'']$ are bounded uniformly in m . Therefore $\Psi_{n, m}$ are bounded in H^1 . Let us fix $\bar{\xi}$ which satisfies (35.71) and take m such that $\xi_m \leq \bar{\xi}$. All $\Psi_{n, m}$ satisfy (35.72). We can find a subsequence of $\Psi_{n, m}$ which converges to $\Psi_{n, \infty} \in \mathcal{E}_{\text{rad}}$ weakly in H^1 , strongly in $L^5(\{|\mathbf{x}| \leq R\}) \cap L^1(\{|\mathbf{x}| \leq R\})$ for every $R > 0$ and almost everywhere in $\mathbb{R}^3$;

we also choose a subsequence so that $\omega_{\kappa, \xi_m, n} \rightarrow \omega_{\kappa, -\infty, n} \in [E', E''] - \frac{\kappa^2}{2}$. We obtain that $\nabla^2 \Psi_{n, m}$ converges weakly to $\nabla^2 \Psi_{n, \infty} \in \mathcal{E}_{\text{rad}}$. Now we show that $\Psi_{n, \infty}$ is a solution of (35.36) with $\omega = \omega_{\kappa, n} = \omega_{\kappa, -\infty, n}$ in the sense of distributions. The restriction of $g_{\kappa, \xi_m}(\Psi_{n, m}(\mathbf{x}))$ to the ball $\{|\mathbf{x}| \leq R\}$ converges to $g_{\kappa, -\infty}(\Psi_{n, \infty}(\mathbf{x}))$ in $L^2(\{|\mathbf{x}| \leq R\})$. Multiplying (35.36) with $\xi = \xi_m$ by an infinitely smooth test function f with compact support and integrating, we observe that we can pass to the limit in every term of the equation, obtaining that $\Psi_{n, \infty}$ is a solution of (35.36). Now we show that $\Psi_{n, \infty} \in \mathcal{E}_{\text{rad}} \cap X$. Since the estimate (35.72) is uniform in $\xi_m \leq \bar{\xi}$, and $|G_{\kappa, -\infty}(C''e^{-pr})|$ is integrable over $\mathbb{R}^3$, we easily derive from the dominated convergence theorem that

$$\int_{|\mathbf{x}| \geq R_1} G_{\kappa, \xi_m}(\Psi_{n, m}) \, d\mathbf{x} \rightarrow \int_{|\mathbf{x}| \geq R_1} G_{\kappa, -\infty}(\Psi_{n, \infty}) \, d\mathbf{x}. \quad (35.81)$$

Note also that

$$\int_{|\mathbf{x}| \leq R_1} G_{\kappa, \xi_m}(\Psi_{n, m}) \, d\mathbf{x} \rightarrow \int_{|\mathbf{x}| \leq R_1} G_{\kappa, -\infty}(\Psi_{n, \infty}) \, d\mathbf{x} \quad (35.82)$$

as $m \rightarrow \infty$. Indeed,

$$\begin{aligned} & G_{\kappa, \xi_m}(|\Psi_{n, m}|^2) - G_{\kappa, -\infty}(|\Psi_{n, \infty}|^2) \\ &= G_{\kappa, \xi_m}(|\Psi_{n, m}|^2) - G_{\kappa, -\infty}(|\Psi_{n, m}|^2) \\ &+ G_{\kappa, -\infty}(|\Psi_{n, m}|^2) - G_{\kappa, -\infty}(|\Psi_{n, \infty}|^2). \end{aligned}$$

Note that $G_{\kappa, -\infty}(|\Psi|^2)$ defined by (14.40) satisfies the following Lipschitz condition:

$$\begin{aligned} & |G_{\kappa, -\infty}(|\Psi_{n, m}|^2) - G_{\kappa, -\infty}(|\Psi_{n, \infty}|^2)| \\ & \leq C \|\Psi_{n, m} - \Psi_{n, \infty}\| (1 + |\Psi_{n, m}|^2 + |\Psi_{n, \infty}|^2). \end{aligned}$$

Hence

$$\begin{aligned} & \int_{|\mathbf{x}| \leq R} |G_{\kappa, -\infty}(|\Psi_{n, m}|^2) - G_{\kappa, -\infty}(|\Psi_{n, \infty}|^2)| \, d\mathbf{x} \quad (35.83) \\ & \leq C' \|\Psi_{n, m} - \Psi_{n, \infty}\| (1 + \|\Psi_{n, m}\|_{L^4}^2 + \|\Psi_{n, \infty}\|_{L^4}^2) \\ & \quad + C' \|\Psi_{n, m} - \Psi_{n, \infty}\|_{L^1(|\mathbf{x}| \leq R)}. \end{aligned}$$

From (35.30) we obtain the inequality

$$\int_{|\mathbf{x}| \leq R} |G_{\kappa, \xi_m}(|\Psi_{n, m}|^2) - G_{\kappa, -\infty}(|\Psi_{n, m}|^2)| \, d\mathbf{x} \leq C e^{\xi_m/4} \int_{|\mathbf{x}| \leq R} |\Psi_{n, m}| \, d\mathbf{x}. \quad (35.84)$$

Using (35.83), (35.84) and convergence of $\Psi_{n,m}$ in $L^1(\{|x| \leq R\})$, we obtain (35.82). From (35.81) and (35.82) we infer that

$$\int_{\mathbb{R}^3} G_{\kappa, \xi_m}(\Psi_{n,m}) d\mathbf{x} \rightarrow \int_{\mathbb{R}^3} G_{\kappa, -\infty}(\Psi_{n,\infty}) d\mathbf{x}. \quad (35.85)$$

This implies that $\Psi_{n,\infty} \in X$ with X defined by (35.17). From (35.85) and the weak convergence in H^1 we infer that

$$\mathcal{E}_\kappa(\Psi_{n,\infty}) \leq \lim_{m \rightarrow \infty} \sup \mathcal{E}_\kappa(\Psi_{n,m}).$$

Since $\mathcal{E}_\kappa(\Psi_{n,m})$ are bounded uniformly, $\mathcal{E}_\kappa(\Psi_{n,\infty})$ is bounded, hence $\Psi_{n,\infty} \in W$. Therefore, (35.36) holds in $H^{-1} + X'$. Similarly to (35.85), we obtain that $\|\Psi_{n,m}\|^2 \rightarrow \|\Psi_{n,\infty}\|^2 = 1$, hence $\Psi_{n,\infty} \in \mathcal{E}_{\text{rad}}$. Since we can choose $\bar{\xi}$ arbitrarily large, p in (35.72) can also be taken arbitrarily large, and from the convergence almost everywhere we derive that $\Psi_{n,\infty}(r)$ decays superexponentially.

The following theorem shows, roughly speaking, that there is no eigenvalues ω of the nonlinear problem in the gaps between small neighborhoods of the eigenvalues of the linear problem.

Theorem 35.2.4 *Let ω satisfy the inequalities*

$$-\frac{1}{2} \leq \omega_{0,n} + \delta < \omega < \omega_{0,n+1} - \delta < 0, \quad (35.86)$$

where $\delta > 0$. There exists a $C_4(n) > 0$ such that if

$$\delta \geq C_4(n) \kappa^2 (1 + |\ln \kappa|), \quad (35.87)$$

then the set of (radial or non-radial) solutions $\Psi \in \mathcal{E} \cap X$ of Eq. (35.36) with such ω is empty.

Proof Consider in $L^2(\mathbb{R}^3)$ a linear operator $\mathcal{O}_\omega$ (the classical Schrödinger operator with the Coulomb potential shifted by ω)

$$\mathcal{O}_\omega \Psi = \mathcal{O} \Psi - \omega \Psi = -\omega \Psi - \frac{1}{2} \nabla^2 \Psi - \frac{1}{|x|} \Psi.$$

Consider a solution of (35.36) written in the form

$$\mathcal{O}_\omega \Psi = -\frac{1}{2} g_\kappa(\Psi) \quad (35.88)$$

with ω in the gap between two consecutive points of the spectrum of $\mathcal{O}$. Multiplying (35.36) (where $\xi = -\infty$) in $H = L^2(\mathbb{R}^3)$ by $\Psi \in \mathcal{E} \cap X$ we obtain

$$\langle \mathcal{O}_\omega \Psi, \Psi \rangle = -\frac{1}{2} \langle g_\kappa(\Psi), \Psi \rangle, \quad (35.89)$$

where $g_\kappa(\Psi)$ is defined in (35.45), $\langle u, v \rangle$ coincides with the scalar product in $H = L^2(\mathbb{R}^3)$ and determines the duality between H^1 and H^{-1} and X and X' in (35.17). Note that $g_\kappa(\Psi) \in X'$ according to Lemma 9.3.3 in [59], hence the expressions are well-defined. Since

$$\int \frac{1}{|\mathbf{x}|} |\Psi|^2 d\mathbf{x} \leq C_\epsilon \|\Psi\|^2 + \epsilon \|\Psi\|_{H^1} \|\Psi\| \text{ for } \Psi \in H^1$$

with arbitrary small $\epsilon > 0$, we have

$$\langle \mathcal{O}_\omega \Psi, \Psi \rangle \geq \frac{1}{4} \|\Psi\|_{H^1}^2 - C'_\epsilon \|\Psi\|^2.$$

Using (35.8), we estimate $\langle g_\kappa(\Psi), \Psi \rangle$, and we obtain for solutions of (35.36) the following inequality:

$$\frac{1}{4} \|\Psi\|_{H^1}^2 - C'' \|\Psi\|^2 \leq \frac{3}{2} \kappa^2 \ln \int |\nabla \Psi|^2 d\mathbf{x}.$$

Hence $\|\Psi\|_{H^1} \leq M$ for solutions $\Psi \in \mathcal{E} \cap X$.

Let $\check{E}_n^0 \subset H^1(\mathbb{R}^3)$ be the finite-dimensional space with the orthonormal basis of eigenfunctions $\check{\Psi}_j$ of the operator $\mathcal{O}_0 = \mathcal{O}$ with negative eigenvalues $\check{\omega}_j \leq \omega_{0,n}$ (since we are not restricted to radial functions, the eigenvalues $\check{\omega}_j \leq \check{\omega}_{j+1}$ may have high multiplicity and $\check{\omega}_n < \omega_{0,n}$ for $n > 2$). Let Π_n be the orthoprojection in $L^2(\mathbb{R}^3)$ onto $\check{E}_n^0$. According to (35.86), we have the following inequality:

$$\langle (\mathcal{O}_\omega - \mathcal{O}_\omega \Pi_n) \Psi, \Psi \rangle \geq (\omega_{0,n+1} - \omega) \|\Psi\|^2.$$

Since the eigenfunctions $\check{\Psi}_j$ of $\mathcal{O}$ are bounded and continuous, and they decay exponentially, we see that the operators

$$\Pi_n \Psi = \sum_{j=1}^n \langle \Psi, \check{\Psi}_j \rangle \check{\Psi}_j, \quad \mathcal{O}_\omega \Pi_n \Psi = \sum_{j=1}^n (\check{\omega}_j - \omega) \langle \Psi, \check{\Psi}_j \rangle \check{\Psi}_j$$

are bounded from $L^p(\mathbb{R}^3)$ to $L^{p'}(\mathbb{R}^3)$ for any given $p, p', \infty \geq p \geq 1, \infty \geq p' \geq 1$. Solutions $\Psi \in \mathcal{E} \cap X'$ are bounded in H^1 and, by Sobolev imbedding, in $L^6(\mathbb{R}^3)$. Since

$$|g_\kappa(\Psi)| \leq C \kappa^2 (1 + |\ln \kappa|) (|\Psi|^{3/2} + |\Psi|^{1/2}), \quad (35.90)$$

it follows that

$$\|\Pi_n g_\kappa(\Psi)\| \leq C'' \kappa^2 (1 + |\ln \kappa|). \quad (35.91)$$

Multiplying (35.36) by $\Psi - \Pi_n \Psi$, we obtain that

$$\langle \mathcal{O}_\omega \Psi, (\Psi - \Pi_n \Psi) \rangle = -\frac{1}{2} \langle g_\kappa(\Psi), (\Psi - \Pi_n \Psi) \rangle.$$

Therefore

$$\begin{aligned} (\omega_{0,n+1} - \omega) \|(1 - \Pi_n) \Psi\|^2 &\leq -\frac{1}{2} \langle g_\kappa(\Psi), (\Psi - \Pi_n \Psi) \rangle \\ &\leq -\frac{1}{2} \int g_\kappa(\Psi) \Psi d\mathbf{x} + \frac{1}{2} \|\Pi_n g_\kappa(\Psi)\| \|\Psi\|. \end{aligned} \quad (35.92)$$

Using (35.8) and (35.91) to estimate the right-hand side, we obtain the inequality

$$(\omega_{0,n+1} - \omega) \|(1 - \Pi_n) \Psi\|^2 \leq C_1 \kappa^2 (1 + |\ln \kappa|). \quad (35.93)$$

Multiplying (35.88) by $\Pi_n \Psi$, we obtain that

$$\begin{aligned} |\omega_{0,n} - \omega| \|\Pi_n \Psi\|^2 &\leq |\langle \mathcal{O}_\omega \Psi, \Pi_n \Psi \rangle| = \frac{1}{2} |\langle \Pi_n g_\kappa(\Psi), \Pi_n \Psi \rangle| \\ &\leq \frac{1}{2} \|\Pi_n g_\kappa(\Psi)\| \leq C_2 \kappa^2 (1 + |\ln \kappa|). \end{aligned} \quad (35.94)$$

From (35.93) and (35.94), using (35.86), we obtain that

$$\delta \|\Psi\|^2 \leq C_3 \kappa^2 (1 + |\ln \kappa|).$$

If we take in (35.87) $C_4(n) = 2C_3$, we obtain a contradiction with $\|\Psi\| = 1$. Hence, a solution of (35.36) with such ω does not exist.

Remark 35.2.1 Theorem 35.2.3 states that for every energy level of the linear Schrödinger Hydrogen operator there exists an energy level of the nonlinear Hydrogen equation with the Coulomb potential with an estimate of difference of order $\sim \kappa^2 \ln \kappa$. Based on (35.37), we see that if $\chi = \hbar$ where $\hbar$ is the Planck constant, and $\kappa = 0$, then the formula for $\omega^1 = \omega_n^1$ coincide with the classical expression for Hydrogen frequencies, therefore the natural choice for parameter χ is $\chi = \hbar$.

Remark 35.2.2 A theorem similar to Theorem 35.2.3 (without superexponential decay of solutions) can be proven not only for the logarithmic nonlinearity, but for more general subcritical nonlinearities which involve a small parameter κ . Moreover, if corresponding functions g_κ are of class C^1 and their derivatives satisfy natural growth conditions, then a standard bifurcation analysis shows that every E_n^0 generates a branch $E_{\kappa,n}$ of solutions $\Psi_n \in \mathcal{E}_{\text{rad}}$ of (35.36) for small κ .

Remark 35.2.3 In Sects. 16.3 and 34.1 we assumed in the macroscopic case the electron size a to be very small, and in this section we assume $\kappa = \hat{a}_1/a$ to be very small, which seem to contradict one another. But a closer look shows that, in fact, both the cases are consistent with one another, if we assume the possibility $\hat{a}_1 \rightarrow 0$. To show that this assumption is quite reasonable, we look at the values of physical constants. Note that a reasonable macroscopic scale is $R_{\text{ex}} \sim 10^{-3}m$ and the Bohr radius $\hat{a}_1 \sim 5.3 \times 10^{-11}m$. The assumptions in Sect. 16.3 $a^2/R_{\text{ex}}^2 \ll 1$ and here $\hat{a}_1/a \ll 1$ are compatible if $\hat{a}_1 \ll a \ll R_{\text{ex}}$. Since $\hat{a}_1/R_{\text{ex}} \sim 5.3 \times 10^{-8}$, we consider this assumption as reasonable.

Remark 35.2.4 In the case when the energy functional with the Coulomb potential $\mathcal{E}_{\text{Cb}}(\Psi)$ is considered not only on radial functions, the eigenvalues of the linear problem are not simple and have very high degeneracy. Note that if we start from the relativistic equation (17.98) and look for ω -harmonic solutions, we arrive at a nonlinear perturbation of the so-called *relativistic Schrödinger equation*, see [291]. The spectrum of this linear problem has a fine structure, and the degeneracy with respect to the total angular momentum disappears. If we fix the value of the z -component L_z of the angular momentum L , for example by looking for solutions of the form $\Psi_1 = e^{im_z\theta} \Psi_{10}(r, z)$ with a fixed integer m_z , one can eliminate the remaining degeneracy in the linear problem. This provides a ground to expect that the lower energy levels of the nonlinear relativistic Schrödinger equation can be approximated by the lower energy levels of the linear relativistic Schrödinger equation if κ is small.

Remark 35.2.5 Note that the decoupled equation (17.88) with $b = 0$ for the proton has a unique ground state solution (gaussian) which corresponds to the minimum E_1^p of energy (see Theorem 35.1.1) and has a sequence of solutions with higher critical energy levels E_j^p , $j \geq 2$ (see [58]). Since the ground state is unique, $E_j^p > E_1^p$ if $j > 1$. The logarithmic nonlinearity $G'_2(|\psi|^2) = -\kappa_p^2 \ln(\kappa_p^{-3}|\psi|^2/C_g^2) - 3$ for the proton involves the parameter κ_p which is different from the parameter κ for the electron. A simple rescaling and use of (17.65) shows that dependence of $E_n^p(\kappa_p)$ on κ_p is very simple: $E_n^p(\kappa_p) = \kappa_p^2 E_n^p(1)$. There are physical motivations based on experimental estimates of the size of the proton to assume that κ_p is large, $\kappa_p \gg 1$. So we can assume that the gap $E_j^p(\kappa_p) - E_1^p(\kappa_p) \geq (E_j^p(1) - E_1^p(1)) \kappa_p^2$ is larger than 1 for $j \geq 2$. (Numerically calculated values of $E_2^p(1) - E_1^p(1)$ are given in [41], they are of order 1). At the same time, $\omega_{0,n}$ given by (35.55) satisfy inequality $-\frac{1}{2} \leq \omega_{0,n} < 0$. Consequently, since the total energy $\mathcal{E}$ defined by (17.79) of the system (17.72)–(17.74) equals the sum of energies of the electron and proton, one may expect for small b and κ and large κ_p that the *lower* negative energy levels of the functional $\mathcal{E}$ cannot originate from $E_j^p + \omega_{0,n}$ with $j \geq 2$ and must branch from $E_1^p + \omega_{0,n}$ as b varies in the vicinity of 0. Hence we expect that, if b and κ are small, and κ_p is large, the lower negative energy levels of the functional $\mathcal{E}$ which corresponds to the system (17.72)–(17.74) are close to lower discrete energy levels of $\mathcal{E}_{\text{Cb}}$ which are described in Theorem 35.2.3.

Part VI

Appendices

Chapter 36

Elementary Momentum Equation Derivation for NKG

For completeness, as an alternative to applying (11.92) or (11.139), we present here a direct verification of the energy and momentum conservation laws (34.136) and (34.137) for the NKG equation with $\mathbf{A} = \mathbf{0}$ which is considered in Sect. 34.3. To verify the energy conservation equation, we multiply (34.128) by $\tilde{\partial}_t^* \psi^*$ where $\tilde{\partial}_t$ is defined by (11.123) and add the conjugate. We obtain the following identity:

$$\begin{aligned} & -c^{-2} \partial_t \left(\tilde{\partial}_t \psi \tilde{\partial}_t^* \psi^* \right) + \nabla \left(\nabla \psi \tilde{\partial}_t^* \psi^* + \nabla \psi^* \tilde{\partial}_t \psi \right) \\ & - \left(\nabla \psi \nabla \tilde{\partial}_t^* \psi^* + \nabla \psi^* \nabla \tilde{\partial}_t \psi \right) - \partial_t G(\psi^* \psi) - \kappa_0^2 \partial_t \psi \psi^* = 0, \end{aligned}$$

and we rewrite it in the form

$$\begin{aligned} & \partial_t \left(c^{-2} \left(\tilde{\partial}_t \psi \tilde{\partial}_t^* \psi^* \right) + \nabla \psi \nabla \psi^* + G(\psi^* \psi) + \kappa_0^2 \psi \psi^* \right) \\ & = \nabla \left(\nabla \psi \tilde{\partial}_t^* \psi^* + \nabla \psi^* \tilde{\partial}_t \psi \right) - 2m\chi^{-2} \nabla \varphi \cdot \mathbf{J}. \end{aligned} \quad (36.1)$$

Finally, using the definition of $\tilde{\partial}_t$, $\mathbf{J}$ and $\mathbf{P}$, we rewrite it in the form of the energy conservation equation (34.136).

To obtain the momentum conservation equation (34.137), we multiply Eq. (34.128) by $\nabla \psi^*$, and add the conjugate. Then using the identity $v \tilde{\partial}_t u + u \tilde{\partial}_t^* v = \partial_t(uv)$, we obtain

$$\begin{aligned} & -\frac{1}{c^2} \partial_t \left(\tilde{\partial}_t \psi \nabla \psi^* \right) + \frac{1}{c^2} \tilde{\partial}_t \psi \tilde{\partial}_t^* \nabla \psi^* + \tilde{\nabla}^2 \psi \tilde{\nabla} \psi^* - G'(\psi^* \psi) \psi \tilde{\nabla} \psi^* \\ & - \kappa_0^2 \psi \tilde{\nabla} \psi^* = 0. \end{aligned}$$

Obviously, $\tilde{\partial}_t(\nabla \psi) = \nabla \tilde{\partial}_t \psi - \frac{i q}{\chi} \nabla \varphi \psi$. Therefore, we arrive at the equation

$$\begin{aligned}
& \partial_t \mathbf{P} + \frac{\chi^2}{2m} \frac{1}{c^2} \left(\tilde{\partial}_t \psi \left(\nabla \tilde{\partial}_t^* \psi + \frac{i q}{\chi} \nabla \varphi \psi^* \right) \right) \\
& + \frac{\chi^2}{2m} \frac{1}{c^2} \left(\tilde{\partial}_t^* \psi^* \left(\nabla \tilde{\partial}_t \psi - \frac{i q}{\chi} \nabla \varphi \psi \right) \right) \\
& + \frac{\chi^2}{2m} (\nabla^2 \psi \nabla \psi^* + \nabla^2 \psi^* \nabla \psi) - \frac{\chi^2}{2m} \nabla (G(\psi^* \psi) + \kappa_0^2 \psi^* \psi) = 0.
\end{aligned} \tag{36.2}$$

Substituting the identity

$$(\nabla^2 \psi \nabla \psi^* + \nabla^2 \psi^* \nabla \psi) = \partial_j \cdot (\partial_j \psi \nabla \psi^* + \partial_j \psi^* \nabla \psi) - \nabla (\nabla \psi \cdot \nabla \psi^*), \tag{36.3}$$

where we use the summation convention, into (36.2), we obtain the *momentum conservation law* for l -th component of $\mathbf{P}$ in the following form

$$\begin{aligned}
& \partial_t \mathbf{P}_l + \frac{\chi^2}{2m} \frac{1}{c^2} \partial_l \left(\tilde{\partial}_t \psi \tilde{\partial}_t^* \psi \right) + \rho \nabla \varphi - \frac{\chi^2}{2m} \partial_l (G(\psi^* \psi) + \kappa_0^2 \psi^* \psi) \\
& + \frac{\chi^2}{2m} \left(\sum_j \partial_j \left(\tilde{\nabla}_j \psi \tilde{\nabla}_l \psi^* + \tilde{\nabla}_j \psi^* \tilde{\nabla}_l \psi \right) - \partial_l \sum_j \left(\tilde{\nabla}_j \psi \tilde{\nabla}_j \psi^* \right) \right) = 0.
\end{aligned}$$

Finally, using (11.121), we rewrite the above equation in the form (34.137).

Chapter 37

Fourier Transforms and Green Functions

The polar coordinates representation of the Laplace operator in $\mathbb{R}^3$, [92, 8], [321, 3.6], is

$$\Delta = \Delta_r + \frac{1}{r^2} \Delta_s = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \frac{1}{r^2} \Delta_s, \quad \mathbf{x} \in \mathbb{R}^3, \quad r = |\mathbf{x}|, \quad (37.1)$$

where Δ_s is the Laplace operator on the unit sphere $\mathbb{S}^2$.

The Green function of the Helmholtz equation has the form, [321, (5,59)],

$$(\kappa^2 - \Delta)^{-1} \delta(\mathbf{x}) = \frac{e^{-\kappa|\mathbf{x}|}}{4\pi |\mathbf{x}|}, \quad \mathbf{x} \in \mathbb{R}^3, \quad \kappa \geq 0. \quad (37.2)$$

In particular,

$$-\Delta \frac{1}{4\pi |\mathbf{x}|} = -4\pi \delta(\mathbf{x}), \quad (37.3)$$

where $\delta(\mathbf{x})$ is the Dirac delta-function. In the case of a variable potential w , the Green function $G_w(\mathbf{x}, \mathbf{y}) = (-\Delta + w)^{-1}(\mathbf{x}, \mathbf{y})$ is defined as the fundamental solution to the equation

$$(-\Delta + w) G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}). \quad (37.4)$$

If $w(\mathbf{x})$ is a real-valued function satisfying the inequality

$$0 \leq w(\mathbf{x}) \leq w_\infty < \infty, \quad \mathbf{x} \in \mathbb{R}^3, \quad (37.5)$$

its Green function G_w satisfies the following inequalities

$$\begin{aligned} \frac{e^{-\sqrt{w_\infty}|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|} &\leq (-\Delta + w_\infty)^{-1}(\mathbf{x}, \mathbf{y}) \leq (-\Delta + w)^{-1}(\mathbf{x}, \mathbf{y}) \\ &\leq (-\Delta)^{-1}(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}, \end{aligned} \quad (37.6)$$

which follow from the Feynman–Kac formula for the heat kernel, [257, 8.2], associated with the operator $-\Delta + w$.

The Fourier transform of a complex function $f(\mathbf{x})$ is defined by the formula,

$$\hat{f}(\mathbf{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) d\mathbf{x}. \quad (37.7)$$

In the particular case of a radial function $f(\mathbf{x}) = f(|\mathbf{x}|)$, its Fourier transform $\hat{f}(\mathbf{k}) = \hat{f}(|\mathbf{k}|)$ is a radial function as well, namely, [92, 41], [139, B5], [321, 3.6]

$$\hat{f}(\mathbf{k}) = \hat{f}(|\mathbf{k}|) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty f(r) \frac{\sin(r|\mathbf{k}|)}{|\mathbf{k}|} r dr. \quad (37.8)$$

Using the theory of analytic continuation of distributions (generalized functions), one can establish the following important formula for the Fourier transform of $\frac{1}{|\mathbf{x}|^{3-\lambda}}$ that holds for any complex λ , [92, 32], [130, II.3.3], [140, 6.1]

$$\left[\frac{1}{2^{\lambda/2} \Gamma\left(\frac{\lambda}{2}\right)} \frac{1}{|\mathbf{x}|^{3-\lambda}} \right]^\wedge = \frac{1}{2^{(3-\lambda)/2} \Gamma\left(\frac{3-\lambda}{2}\right)} \frac{1}{|\mathbf{k}|^\lambda}, \quad \mathbf{x}, \mathbf{k} \in \mathbb{R}^3. \quad (37.9)$$

The Green function for the operator $(-\Delta)^{-\frac{s}{2}}$, known also as the *Riesz potential*, has the following representation, [140, 6.1.1]

$$(-\Delta)^{-\frac{s}{2}}(\mathbf{x}, \mathbf{y}) = 2^{-s} \pi^{-\frac{3}{2}} \frac{\Gamma\left(\frac{3-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \frac{1}{|\mathbf{x}-\mathbf{y}|^{3-s}}, \quad (37.10)$$

where Γ is the Gauss Gamma function. In particular, for two cases important to us, $s = 2$ and $s = 4$, formula (37.10) implies that

$$(-\Delta)^{-1}(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi} \frac{1}{|\mathbf{x}-\mathbf{y}|}, \quad (37.11)$$

$$(-\Delta)^{-2}(\mathbf{x}, \mathbf{y}) = -\frac{1}{8\pi} |\mathbf{x}-\mathbf{y}|. \quad (37.12)$$

Chapter 38

Splitting of a Field into Gradient and Sphere-Tangent Parts

Here we describe the splitting of a general vector field into sphere-tangent and gradient parts, which was used to derive the properties of wave-corpuscles in Sect. 16.1.

Theorem 38.0.5 (Field splitting) *Any continuously differentiable vector field $\mathbf{V}(\mathbf{y})$ can be uniquely split into a gradient field ∇P and a sphere-tangent field $\check{\mathbf{V}}$:*

$$\mathbf{V} = \nabla P + \check{\mathbf{V}}, \quad (38.1)$$

where P is continuously differentiable, $\check{\mathbf{V}}$ is continuous and satisfies the orthogonality condition (16.15), namely $\check{\mathbf{V}} \cdot \mathbf{y} = 0$.

Proof To determine P , we multiply (38.1) by $\mathbf{y}$, observe that $\mathbf{y} \cdot \check{\mathbf{V}} = 0$, and obtain the following equation:

$$\mathbf{y} \cdot \mathbf{V} = \mathbf{y} \cdot \nabla P.$$

The directional derivative $\mathbf{y} \cdot \nabla$ can be written as $r \partial_r$; using the notation $\mathbf{y} = \boldsymbol{\Omega} r$ where $|\boldsymbol{\Omega}| = 1$, $\boldsymbol{\Omega}$ is the angular variable with values on the unit sphere, and $r = |\mathbf{y}|$, we rewrite the above equation in the form

$$\boldsymbol{\Omega} \cdot \mathbf{V} = \partial_r P(\boldsymbol{\Omega} r).$$

Note that P is defined from the above equation up to a function of the angular variable $\boldsymbol{\Omega}$; since such a function which is continuous at the origin must be a constant, P is defined uniquely modulo constants (and ∇P is defined uniquely). We can find P by integration:

$$\begin{aligned}
 P(\mathbf{y}) &= \int_0^{|\mathbf{y}|} \boldsymbol{\Omega} \cdot \mathbf{V}(\boldsymbol{\Omega} r) \, dr \\
 &= \int_0^{|\mathbf{y}|} \frac{1}{|\mathbf{y}|} \mathbf{y} \cdot \mathbf{V}\left(\frac{1}{|\mathbf{y}|} \mathbf{y} r\right) \, dr = \int_0^1 \mathbf{y} \cdot \mathbf{V}(\mathbf{y} s) \, ds.
 \end{aligned} \tag{38.2}$$

If the potential P is defined by (38.2) then $\mathbf{y} \cdot \check{\mathbf{V}} = \mathbf{y} \cdot \mathbf{V} - \mathbf{y} \cdot \nabla P$ satisfies (16.15). Note that the continuous differentiability of $P(\mathbf{y})$ follows from the continuous differentiability of $\mathbf{V}$, hence ∇P and $\check{\mathbf{V}}$ are continuous.

To obtain an explicit expression for P , we assume that $\mathbf{V}$ and P are expanded into series of homogeneous expressions:

$$\mathbf{V}(\mathbf{y}) = \sum_j \mathbf{V}_j(\mathbf{y}), \quad P = \sum_j P_j(\mathbf{y}), \quad \check{\mathbf{V}}(\mathbf{y}) = \sum_j \check{\mathbf{V}}_j(\mathbf{y}), \tag{38.3}$$

where $P_j, \check{\mathbf{V}}_j$ satisfy the homogeneity condition

$$\mathbf{V}_j(\zeta \mathbf{y}) = \zeta^j \mathbf{A}_j(\mathbf{y}), \quad P_j(\zeta \mathbf{y}) = \zeta^j P_j(\mathbf{y}).$$

For a j -homogeneous $\mathbf{V}_j$

$$P_{j+1}(\mathbf{y}) = \int_0^{|\mathbf{y}|} \frac{1}{|\mathbf{y}|} \mathbf{y} \cdot \mathbf{V}_j\left(\frac{1}{|\mathbf{y}|} \mathbf{y} r\right) \, dr = \int_0^{|\mathbf{y}|} \frac{1}{|\mathbf{y}|} \frac{1}{|\mathbf{y}|^j} \mathbf{y} \cdot \mathbf{V}_j(\mathbf{y}) r^j \, dr,$$

hence

$$P_{j+1}(\mathbf{y}) = \frac{1}{j+1} \mathbf{y} \cdot \mathbf{V}_j(\mathbf{y}). \tag{38.4}$$

In particular, the zero order $\mathbf{V}_0$ corresponds to

$$P_1(\mathbf{y}) = \mathbf{V}(0) \cdot \mathbf{y}, \quad \check{\mathbf{V}}_0(\mathbf{y}) = 0. \tag{38.5}$$

The first order $\mathbf{V}_1$ corresponds to

$$P_2(\mathbf{y}) = \frac{1}{2} \mathbf{V}_1(\mathbf{y}) \cdot \mathbf{y}, \tag{38.6}$$

and

$$\nabla P_2(\mathbf{y}) = \frac{1}{2} \mathbf{V}_1(\mathbf{y}) + \frac{1}{2} \mathbf{V}_1^T(\mathbf{y}) \tag{38.7}$$

where the matrix $\mathbf{V}_1^T$ is the transpose of $\mathbf{V}_1$. Obviously, $\nabla P_2(\mathbf{y})$ coincides with the symmetric part of the linear transformation $\mathbf{V}_1(\mathbf{y})$, and

$$\check{\mathbf{V}}_1(\mathbf{y}) = \frac{1}{2} \mathbf{V}_1(\mathbf{y}) - \frac{1}{2} \mathbf{V}_1^T(\mathbf{y}) \tag{38.8}$$

coincides with the anti-symmetric part. For higher values of j we use (38.4):

$$\nabla P_{j+1}(\mathbf{y}) = \frac{1}{j+1} \nabla (\mathbf{y} \cdot \mathbf{V}_j(\mathbf{y})).$$

Using vector calculus, we obtain

$$\begin{aligned} \nabla (\mathbf{y} \cdot \mathbf{V}_j(\mathbf{y})) &= (\mathbf{y} \cdot \nabla) \mathbf{V}_j + (\mathbf{V}_j \cdot \nabla) \mathbf{y} + \mathbf{y} \times (\nabla \times \mathbf{V}_j) + \mathbf{V}_j \times (\nabla \times \mathbf{y}) \\ &= (\mathbf{y} \cdot \nabla) \mathbf{V}_j + \mathbf{V}_j + \mathbf{y} \times (\nabla \times \mathbf{V}_j), \end{aligned}$$

where, by Euler's identity for homogeneous functions,

$$(\mathbf{y} \cdot \nabla) \mathbf{V}_j(\mathbf{y}) = j \mathbf{V}_j(\mathbf{y}).$$

Hence

$$\begin{aligned} \nabla P_{j+1}(\mathbf{y}) &= \frac{1}{j+1} \nabla (\mathbf{y} \cdot \mathbf{V}_j(\mathbf{y})) = \frac{1}{j+1} ((\mathbf{y} \cdot \nabla) \mathbf{V}_j + \mathbf{V}_j + \mathbf{y} \times (\nabla \times \mathbf{V}_j)) \\ &= \frac{1}{j+1} (j \mathbf{V}_j + \mathbf{V}_j + \mathbf{y} \times (\nabla \times \mathbf{V}_j)) = \mathbf{V}_j + \frac{1}{j+1} \mathbf{y} \times (\nabla \times \mathbf{V}_j), \\ \check{\mathbf{V}}_j(\mathbf{y}) &= \mathbf{V}_j(\mathbf{y}) - \nabla P_{j+1}(\mathbf{y}) = -\frac{1}{j+1} \mathbf{y} \times (\nabla \times \mathbf{V}_j). \end{aligned} \quad (38.9)$$

Now we consider the zero divergence condition (16.22) imposed on a sphere-tangent part of the field. According to (38.9)

$$\nabla \cdot \check{\mathbf{V}}_j(\mathbf{y}) = -\frac{1}{j+1} \nabla \cdot (\mathbf{y} \times (\nabla \times \mathbf{V}_j)).$$

Using vector algebra, we obtain that

$$\begin{aligned} \nabla \cdot (\mathbf{y} \times (\nabla \times \mathbf{V}_j)) &= (\nabla \times \mathbf{V}_j) \cdot (\nabla \times \mathbf{y}) - \mathbf{y} \cdot (\nabla \times (\nabla \times \mathbf{V}_j)) \\ &= -\mathbf{y} \cdot (\nabla \times (\nabla \times \mathbf{V}_j)) = \mathbf{y} \cdot (\nabla^2 \mathbf{V}_j - \nabla (\nabla \cdot \mathbf{V}_j)). \end{aligned}$$

Hence condition (16.22) can be written in the form

$$\mathbf{y} \cdot (\nabla \times (\nabla \times \mathbf{V}_j)) = 0. \quad (38.10)$$

Chapter 39

Hamilton–Jacobi Theory

We consider in this section certain aspects of the Hamilton–Jacobi theory which play a very important role in the Schrödinger and de Broglie wave theories as well as in the Bohmian mechanics. The Hamilton–Jacobi theory provides valuable insights in our theory as well, particularly in the construction of a special particle-like wave which we refer to as a “*wave-corpuscle*”.

In quantum mechanics, the Hamilton–Jacobi equation serves as a constructive basis relating via the “quantization” a classical ensemble of point particles to the single phase function and ultimately to the single wave function for the entire system of many charges.

The Hamilton–Jacobi theory is based on a deep analogy between geometrical optics, operating with successive wavefronts and the corresponding bundles of rays, on one hand, and on the other hand Lagrangian analytical mechanics, operating with trajectories (orbits, extremals) and the corresponding action functional. There are many excellent expositions of the Hamilton–Jacobi theory emphasizing its different features and points of view, [264, 15, 16], [129, 16–32], [133, II.6], [134, III.7, IV.9–10], [8, 46], [287, 2], [212, VIII], [176, 2]. Every of those points of view articulates a valuable insight into geometrical optics and Lagrangian mechanics. The analogy between geometrical optics and Lagrangian analytical mechanics was a key ingredient of Schrödinger’s wave mechanics, as he pointed out in [296, p. ix]:”

“The Hamiltonian analogy of mechanics to optics (pp. 13–18) is an analogy to geometrical optics, since to the path of the representative point in configuration space there corresponds on the optical side the light ray, which is only rigorously defined in terms of geometrical optics. The undulatory elaboration of the optical picture (pp. 19–30) leads to the surrender of the idea of the path of the system, as soon as the dimensions of the path are not great in comparison with the wave-length (pp. 25–26). Only when they are so does the idea of the path remain, and with it classical mechanics as an approximation (pp. 20–24, 41–44); whereas for “micro-mechanical” motions the fundamental equations of mechanics are just as useless as geometrical optics is for the treatment of diffraction problems. In analogy with the latter case, a wave equation in configuration space must replace the fundamental equations of mechanics.”

A concise quantitative representation of the optical-mechanical analogy is provided by V. Arnold in [8, 46.B].

Optics	Mechanics
Optical medium	Extended configuration space $\{(\mathbf{q}, t)\}$
Fermat's principle	Hamilton's principle $\delta \int L \, dt = 0$
Indicatrices	Lagrangian L
Rays	Trajectories $\mathbf{q}(t)$
Normal slowness vector $\mathbf{p}$ of the front	Momentum $\mathbf{p}$
Expression of $\mathbf{p}$ in terms of velocity of the ray, $\dot{\mathbf{q}}$	Legendre transformation
1-form $\mathbf{p} \, d\mathbf{q}$	1-form $\mathbf{p} \, d\mathbf{q} - H \, dt$

We start by fixing the notation and supposing that there is a system with n degrees of freedom described by generalized coordinates $q = \{q_r, r = 1, \dots, n\}$ and the corresponding velocities $\dot{q} = \{\dot{q}_r, r = 1, \dots, n\}$. This system is governed by the Lagrangian $L(q, \dot{q}, t)$, and its evolution can be found based on the principle of least (stationary) action:

$$\delta S = \delta \int_{t_0}^{t_1} L(q, \dot{q}, t) \, dt = 0, \quad \delta q(t_1) = \delta q(t_2) = 0. \quad (39.1)$$

Consequently, the evolution of the system state $q(t)$ is described by the *Euler–Lagrange (EL) equations*

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \right) - \frac{\partial L}{\partial q_r} = 0, \quad \text{where } \dot{q}_r = \frac{dq_r}{dt}, \quad r = 1, \dots, n. \quad (39.2)$$

We refer to solutions to the EL equations (39.2) as *trajectories (also called orbits or extremals) in the configuration space*.

An alternative important way of representing the system evolution is by the *Hamilton equations*. Namely, one introduces first the momenta

$$p_r = \frac{\partial L}{\partial \dot{q}_r}, \quad p = \{p_r, r = 1, \dots, n\}, \quad (39.3)$$

and then the so-called *Hamiltonian* $H(q, p, t)$ by the *Legendre transformation* for $\dot{q}$

$$H(q, p, t) = \sum_j p_j \dot{q}_j - L(q, \dot{q}, t). \quad (39.4)$$

Then under assumptions of sufficient smoothness of the Lagrangian L and the non-degeneracy condition

$$\det \left\{ \frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_s} \right\} \neq 0, \quad (39.5)$$

the relations (39.3) between p and $\dot{q}$ can be equivalently represented by

$$\dot{q}_r = \frac{\partial H}{\partial p_r}. \quad (39.6)$$

The Lagrangian and its Hamiltonian satisfy the following identities

$$\frac{\partial H}{\partial q_r} = -\frac{\partial L}{\partial q_r}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \quad (39.7)$$

Using the relations (39.3) and (39.6) we can rewrite the EL equations (39.2) in the form of the famous *Hamilton equations*

$$\frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r}, \quad (39.8)$$

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \quad r = 1, \dots, n. \quad (39.9)$$

The first set of equations (39.8) can be viewed as the nontrivial one and the second set of equations (39.9) can be viewed as merely a change of variables (39.6) between p and $\dot{q}$. Similarly to the case of EL equations, we refer to solutions of Hamilton equations (39.8)–(39.9) as *trajectories in the phase space*.

A significant idea in the calculus of variation due to Weierstrass was to consider a bundle of trajectories rather than a single one, [133, 6.1]. That idea combined with the optical-mechanical analogy has been a basis for further development of the Hamilton–Jacobi theory by K. Caratheodory in the form of the “*complete figure of the calculus of variations*” in the configuration space, [287, 2.1–4], [133, 6.1]. In a nutshell, this complete figure features a one-parameter family of hypersurfaces (representing successive wave-fronts in geometric optics) which are intersected in a special manner by the corresponding bundle (congruence, ensemble) of curves (representing a bundle of rays in geometric optics). A concise presentation of the Caratheodory approach is as follows, [287, 2.1–4]. The hypersurfaces are naturally described in terms of a sufficiently smooth function $S(q, t)$ by equations

$$S(q, t) = \Sigma, \quad (39.10)$$

where the constant Σ takes real values. The first significant contact between the function $S(q, t)$ and the Lagrangian theory of trajectories is made by identifying the gradient $\frac{\partial S}{\partial q}$ with the Lagrangian momentum p , namely

$$\frac{\partial S}{\partial q} = p = \frac{\partial L}{\partial \dot{q}}. \quad (39.11)$$

Viewing the second equation (39.11) as an implicit relation between p and $\dot{q}$, we invert it, obtaining the relation (39.6). Combining then the relation (39.6) with the first equation (39.11), we obtain the following equations

$$\dot{q} = v(q, t), \text{ where } v(q, t) = \frac{\partial H}{\partial p} \left(q, \frac{\partial S(q, t)}{\partial q}, t \right), \quad (39.12)$$

describing a bundle (congruence) of trajectories associated with the function $S(q, t)$. Observe that it is the fundamental relation (39.11) that yields the time dependent velocity field $v(q, t)$ in (39.12), which in turn generates the whole bundle of trajectories as its integral curves. The rise of the velocity field rather than velocity associated with a single trajectory is a key feature of this approach.

Notice that so far no significant assumption has been made about the function $S(q, t)$, for the relations (39.11) generate velocity $v(q, t)$ for any sufficiently smooth function $S(q, t)$. The significant assumption we impose on $S(q, t)$ relates it to the system Lagrangian by requiring hypersurfaces $S(q, t) = \Sigma$ with different Σ to be “*geodesically equidistant*” where the “*geodesical distance*” is defined in terms of the system action function. More precisely, the *geodesical distance* $W(q_0, t_0; q_1, t_1)$ between two points (q_0, t_0) and (q_1, t_1) in the space-time is defined as *Hamilton’s principal (characteristic) function*, [287, 2.11], [264, 15.5], [134, IV.9.2.2], that is,

$$W(q_0, t_0; q_1, t_1) = \int_{t_0}^{t_1} L(q, \dot{q}, t) dt, \quad (39.13)$$

where trajectory $q = q(t)$ is a solution to the EL equation (39.2) satisfying

$$q(t_0) = q_0, \quad q(t_1) = q_1. \quad (39.14)$$

Then there is a fundamental theorem, [287, 2.2], stating that *in order for a family of hypersurfaces (39.11) to be geodesically equidistant with respect to the Lagrangian $L(q, \dot{q}, t)$, it is necessary and sufficient that $S(q, t)$ is a solution of the following Hamilton–Jacobi equation*

$$\frac{\partial S}{\partial t} + H \left(q, \frac{\partial S}{\partial q}, t \right) = 0, \quad (39.15)$$

where $H(q, p, t)$ is the Hamiltonian defined by (39.4), and the quantity $p = \frac{\partial S}{\partial q}$ for such an S is called the *geodesic gradient*. The significance of the Hamilton–Jacobi equation to a large extent is related to this theorem. The optical counterpart of the Hamilton–Jacobi equation (39.15) is known as the *eikonal equation* and the function S is called the *eikonal (optical distance)*, [50, III.3.3.1].

One can also consider the initial value (Cauchy) problem for the Hamilton–Jacobi equation, [8, 46.D], [287, 2.8], [134, III.7.2.4, IV.10.1.4], that is,

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0, \quad S(q, t_0) = S_0(q) \quad (39.16)$$

where $S_0(q)$ is the initial condition. This Cauchy problem has a unique solution that can be obtained by the *method of characteristics* as follows, [8, 46.D], [134, 2.8], [134, III.7.2.4, IV.10.1.4]. First, one solves the equation

$$\frac{\partial S_0(q)}{\partial q} = p_0(q) = \frac{\partial L(q, v, t_0)}{\partial v} \quad (39.17)$$

for v to find out the velocity field $v_0(q)$ similar to the one in relation (39.11), that is,

$$v_0(q) = \frac{\partial H}{\partial p}\left(q, \frac{\partial S_0(q)}{\partial q}, t_0\right). \quad (39.18)$$

Then, solving the following initial value problem for the EL evolution equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0, \quad q_r(t_0) = \tilde{q}_r \quad \dot{q}_r(t_0) = (v_0)_r(\tilde{q}), \quad (39.19)$$

we find a unique bundle $Q(\tilde{q}, t)$ of trajectories (characteristics) associated with the Lagrangian L and the initial value function $S_0(q)$. Notice that every characteristic trajectory $Q(\tilde{q}, t)$ is a solution to EL equations (39.19) which is naturally parametrized by the initial data $q(t_0) = \tilde{q}$ and satisfies

$$Q(\tilde{q}, t_0) = \tilde{q}, \quad \dot{Q}(\tilde{q}, t_0) = v_0(\tilde{q}), \quad (39.20)$$

with $v_0(q)$ defined by (39.18).

Now, using the constructed characteristics $Q(\tilde{q}, t)$, we can represent the solution to the Cauchy problem by the following formula

$$S(q, t) = S_0(q_0) + \int_{t_0}^t L(Q, \dot{Q}, \tau) d\tau, \quad \text{where } q_0 = q_0(q, t), \quad (39.21)$$

with $Q(t) = Q(\tilde{q}, t)$ being a characteristic trajectory and $\tilde{q}$ satisfying

$$Q(\tilde{q}, t) = q. \quad (39.22)$$

Solving equation (39.22) for $\tilde{q}$, we obtain its solution $\tilde{q} = \tilde{q}(q, t)$. Then the function $q_0(q, t)$ in relations (39.21) is equal to the $\tilde{q}(q, t)$ defined by Eq. (39.22), that is

$$q_0(q, t) = \tilde{q}(q, t). \quad (39.23)$$

Chapter 40

Point Charges in a Spatially Homogeneous Electric Field

We consider here an application of the Hamilton–Jacobi theory to the dynamics of a point charge q in a spatially homogeneous electric field $\mathbf{E}_0(t)$. The dynamics of such a point charge is governed by the Lagrangian, [264, 15.9, 16.6], [137, 1.5],

$$L_p(\mathbf{x}, \dot{\mathbf{x}}, t) = \frac{m\dot{\mathbf{x}}^2(t)}{2} - q\varphi_{\text{ex}}(t, \mathbf{x}(t)), \quad \dot{\mathbf{x}}(t) = \mathbf{v}(t) = \frac{d\mathbf{x}(t)}{dt} \quad (40.1)$$

where $\mathbf{v}(t)$ is the point charge velocity, and $\varphi_{\text{ex}}(t, \mathbf{x})$ is the electrostatic potential corresponding to the electric field $\mathbf{E}_0(t)$, namely

$$\varphi_{\text{ex}}(t, \mathbf{x}) = \varphi_0(t) - \mathbf{E}_0(t) \cdot \mathbf{x}. \quad (40.2)$$

The Hamiltonian H_p corresponding to L_p is given by the formula

$$H_p(\mathbf{x}, \mathbf{p}, t) = \frac{\mathbf{p}^2}{2m} + q\varphi_{\text{ex}}(t, \mathbf{x}), \quad \mathbf{p} = m\dot{\mathbf{x}} = m\mathbf{v}. \quad (40.3)$$

The Euler–Lagrange (EL) equation associated with the Lagrangian L_p turns into the Newton equation with the Lorentz force $q\mathbf{E}_0(t)$, that is,

$$m \frac{d^2\mathbf{x}(t)}{dt^2} = q\mathbf{E}_0(t). \quad (40.4)$$

The initial conditions

$$\mathbf{x}(0) = \mathbf{x}_0, \quad \frac{d\mathbf{x}}{dt}(0) = \mathbf{v}_0, \quad (40.5)$$

identifies a unique solution $\mathbf{x}(t) = \mathbf{x}(t; \mathbf{x}_0, \mathbf{v}_0)$ to the EL equation (40.4); this solution is called a *trajectory*.

Following the optical-mechanical analogy, we introduce the following bundle (congruence, ensemble) of trajectories satisfying equations (40.4)–(40.5)

$$\{\mathbf{x}(t; \mathbf{x}_0, \mathbf{v}_0) : \mathbf{x}_0 \in \mathbb{R}^3\}, \text{ where } \mathbf{v}_0 \in \mathbb{R}^3 \text{ is a fixed vector.} \quad (40.6)$$

In other words, the above bundle of trajectories is parametrized by the initial position vector $\mathbf{x}_0$ with a fixed initial velocity vector $\mathbf{v}_0$. For the purpose of reference, we also pick and fix a position vector $\mathbf{r}_0$ and the corresponding unique trajectory $\mathbf{r}(t) = \mathbf{x}(t; \mathbf{r}_0, \mathbf{v}_0)$ in the bundle (40.6), that is,

$$m \frac{d^2 \mathbf{r}(t)}{dt^2} = q \mathbf{E}_0(t), \quad \mathbf{r}(0) = \mathbf{r}_0, \quad \frac{d\mathbf{r}}{dt}(0) = \mathbf{v}_0. \quad (40.7)$$

Notice that the following explicit representation holds for the trajectory $\mathbf{x}(t; \mathbf{x}_0, \mathbf{v}_0)$ satisfying equations (40.4)–(40.5):

$$\mathbf{v}(t) = \mathbf{v}(t; \mathbf{v}_0) = \mathbf{v}_0 + w_0(t), \quad w_0(t) = \int_0^t \frac{q}{m} \mathbf{E}_0(\tau) d\tau, \quad (40.8)$$

$$\mathbf{x}(t) = \mathbf{x}(t; \mathbf{x}_0, \mathbf{v}_0) = \mathbf{x}_0 + \int_0^t \mathbf{v}(\tau) d\tau = \mathbf{x}_0 + \mathbf{v}_0 t + \int_0^t w_0(\tau) d\tau. \quad (40.9)$$

Observe also that

$$\mathbf{x}(t; \mathbf{x}_0, \mathbf{v}_0) = (\mathbf{x}_0 - \mathbf{r}_0) + \mathbf{r}(t), \quad (40.10)$$

suggesting that the bundle of trajectories $\mathbf{x}(t; \mathbf{x}_0, \mathbf{v}_0)$ can be viewed as representing the motion of a solid body moving with velocity $\mathbf{v}(t; \mathbf{v}_0)$ defined by (40.8).

Having defined the bundle of trajectories $\mathbf{x}(t; \mathbf{x}_0, \mathbf{v}_0)$, we introduce now a related Cauchy (initial value) problem for the corresponding Hamilton–Jacobi equation (39.16) [8, 45.D], [134, III.2.4, IV.10.1]:

$$\partial_t S = -H_p(\nabla_{\mathbf{x}} S, \mathbf{x}, t), \quad S(0, \mathbf{x}) = S_0(\mathbf{x}) = m \mathbf{v}_0 \cdot (\mathbf{x} - \mathbf{r}_0). \quad (40.11)$$

The unique solution $S(t, \mathbf{x})$ to this problem described by Eqs. (39.21)–(39.23) can be expressed in terms of the point trajectory $\mathbf{r}(t)$ as follows

$$S(t, \mathbf{x}) = m \mathbf{v}(t) \cdot (\mathbf{x} - \mathbf{r}(t)) + s_p(t), \quad \mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}, \quad (40.12)$$

where

$$s_p(t) = \int_0^t L_p d\tau = \int_0^t \left[\frac{m \mathbf{v}^2(\tau)}{2} - q \varphi_{\text{ex}}(t, \mathbf{r}(\tau)) \right] d\tau. \quad (40.13)$$

Notice that the point charge action function $s_p(t)$ can be written in the form

$$s_p(t) = W_p(\mathbf{r}_0, 0; t, \mathbf{r}(t), t), \quad (40.14)$$

where $W(t_1, \mathbf{r}_1; t_0, \mathbf{r}_0)$ is the *Hamilton principal (characteristic)* function defined by (39.13)–(39.14).

Using evolution equations (40.4)–(40.5) and integration by parts, we transform the function $S(t, \mathbf{x})$ defined by equations (40.12)–(40.13) into the following alternative form:

$$S(t, \mathbf{x}) = m [\mathbf{v}(t) \cdot \mathbf{x} - \mathbf{v}_0 \cdot \mathbf{r}_0] - \int_0^t \left[\frac{m\mathbf{v}^2(\tau)}{2} + q\varphi_0(\tau) \right] d\tau. \quad (40.15)$$

Evidently, the above Hamilton–Jacobi action function $S(t, \mathbf{x})$ is linear in $\mathbf{x}$ and can be associated with the phase function of a plane wave. Observe also that the fundamental momentum and velocity fields $\mathbf{p}(t, \mathbf{x})$ and $\mathbf{v}(\mathbf{x}, t)$ defined by equations (39.11)–(39.12) and associated with this action function $S(t, \mathbf{x})$ are spatially homogeneous and are equal, respectively, to

$$\mathbf{p}(\mathbf{x}, t) = \nabla_{\mathbf{x}} S(\mathbf{x}, t) = \mathbf{p}(t) = m\mathbf{v}(t) \quad (40.16)$$

and

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v}(t) = \mathbf{v}(t; \mathbf{v}_0). \quad (40.17)$$

The formulas (40.16)–(40.17) suggest once again that the action function $S(t, \mathbf{x})$ defined by (40.12) represents the motion of an effectively solid body with velocity $\mathbf{v}(t; \mathbf{v}_0)$ defined by (40.8).

Chapter 41

Statistical and Wave Viewpoints in Hamilton–Jacobi Theory

The key geometrical and geometric optics concepts of the Hamilton–Jacobi theory are bundles of trajectories and fields, namely the action function $S(\mathbf{x}, t)$, the momentum field $\mathbf{p}(\mathbf{x}, t) = \nabla_{\mathbf{x}} S(\mathbf{x}, t)$ and the velocity field $\mathbf{v}(\mathbf{x}, t) = \frac{\partial H}{\partial \mathbf{p}}(\mathbf{x}, \nabla_{\mathbf{x}} S(\mathbf{x}, t), t)$. The development of those concepts allows for a natural association of a bundle of trajectories and the corresponding fields with an ensemble of identical particles. Namely, let us consider following [176, 2.3–2.6] a single particle governed by the Lagrangian

$$L_p(\mathbf{x}, \dot{\mathbf{x}}, t) = \frac{m\dot{\mathbf{x}}^2(t)}{2} - V(\mathbf{x}, t), \quad \dot{\mathbf{x}}(t) = \mathbf{v}(t) = \frac{d\mathbf{x}(t)}{dt}, \quad (41.1)$$

with the corresponding Hamiltonian H_p and the momentum $\mathbf{p}$ defined by

$$H_p(\mathbf{x}, \mathbf{p}, t) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}, t), \quad \mathbf{p} = m\dot{\mathbf{x}} = m\mathbf{v}. \quad (41.2)$$

We consider then a function $S(\mathbf{x}, t)$ solving the Cauchy problem for the Hamilton–Jacobi equation (39.16), that is,

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V(t, \mathbf{r}) = 0, \quad S(\mathbf{x}, 0) = S_0(\mathbf{x}). \quad (41.3)$$

Being given such a function $S(\mathbf{x}, t)$, we associate with it a fictitious Gibbs ensemble of identical noninteracting particles which differ from one another only by their initial locations and velocities. The motion of these particles is completely determined by trajectories associated with $S(\mathbf{x}, t)$ and the corresponding momentum and velocity fields

$$\mathbf{p}(\mathbf{x}, t) = \nabla_{\mathbf{x}} S(\mathbf{x}, t), \quad \mathbf{v}(\mathbf{x}, t) = \frac{\partial H}{\partial \mathbf{p}}(\mathbf{x}, \nabla_{\mathbf{x}} S(\mathbf{x}, t), t) = \frac{1}{m} \nabla_{\mathbf{x}} S(\mathbf{x}, t). \quad (41.4)$$

We assume then that the density of particles per unit volume in the volume element $d\mathbf{x}$ surrounding the point $\mathbf{x}$ at a time t is given by a function $\rho(\mathbf{x}, t) \geq 0$. With the Liouville theorem in mind and in view of the equation (41.4) for velocity field $\mathbf{v}(\mathbf{x}, t)$, we require the particle density $\rho(\mathbf{x}, t)$ to satisfy the following local equation of conservation:

$$\partial_t \rho + \nabla \cdot \left(\rho \frac{\nabla_{\mathbf{x}} S}{m} \right) = 0. \quad (41.5)$$

Consequently, the constructed Gibbs ensemble of particles is described by functions $S(\mathbf{x}, t)$ and $\rho(\mathbf{x}, t)$ satisfying equations (41.3) and (41.5) which form the following system of partially coupled equations:

$$\partial_t S + \frac{(\nabla_{\mathbf{x}} S)^2}{2m} + V(\mathbf{x}, t) = 0, \quad (41.6)$$

$$\partial_t \rho + \nabla \cdot \left(\rho \frac{\nabla_{\mathbf{x}} S}{m} \right) = 0, \quad (41.7)$$

subject to the initial conditions $S_0(\mathbf{x})$ and $\rho_0(\mathbf{x})$. Observe that the choice of $S_0(\mathbf{x})$ fixes the solution $S(\mathbf{x}, t)$ and the system dynamics which consequently is independent of $\rho_0(\mathbf{x})$. The system (41.6)–(41.7) can be considered as a field theoretic formulation of classical statistical mechanics.

Notice also that the so defined ensemble prescribing the momentum $\mathbf{p} = \mathbf{p}(\mathbf{x}, t)$ to every point $\mathbf{x}$ as in Eq.(41.4) is called “pure” Then the corresponding phase space distribution function $f(\mathbf{x}, \mathbf{p}, t)$ and its initial values $f_0(\mathbf{x}, \mathbf{p})$ are of the special form

$$\begin{aligned} f(\mathbf{x}, \mathbf{p}, t) &= \rho(\mathbf{x}, t) \delta(\mathbf{p} - \nabla_{\mathbf{x}} S(\mathbf{x}, t)), \\ f_0(\mathbf{x}, \mathbf{p}) &= \rho_0(\mathbf{x}) \delta(\mathbf{p} - \nabla_{\mathbf{x}} S_0(\mathbf{x})), \end{aligned} \quad (41.8)$$

and their perfect consistency with the system (41.6)–(41.7) can be verified.

Quite remarkably, the equations of the system (41.6)–(41.7) are the EL equations for a system with the following Lagrangian density:

$$L(\rho, \dot{S}, \nabla_{\mathbf{x}} S, \mathbf{x}, t) = -\rho \left[\partial_t S + \frac{(\nabla_{\mathbf{x}} S)^2}{2m} + V(\mathbf{x}, t) \right]. \quad (41.9)$$

Notice that

$$\frac{\partial L}{\partial \partial_t S} = -\rho, \quad (41.10)$$

implying that S and $-\rho$ can be treated as canonically conjugate “position” and “momentum”.

For the purpose of comparison with quantum mechanics, one can introduce the classical “wavefunction” by

$$\psi(\mathbf{x}, t) = R(\mathbf{x}, t) \exp \left\{ i \frac{S(\mathbf{x}, t)}{\hbar} \right\}, \quad (41.11)$$

where $R = \sqrt{\rho}$, and $\hbar$ is the Planck constant. Using this field variable, the Lagrangian density (41.9) can be transformed into

$$L = \frac{i\hbar}{2} (\psi^* \partial_t \psi - \psi \partial_t \psi^*) - \frac{\hbar^2 \psi^* \psi}{2m} \left(\frac{\psi^{-1} \nabla \psi - \psi^{*-1} \nabla \psi^*}{2i} \right)^2 - V \psi^* \psi. \quad (41.12)$$

The corresponding EL equations are

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \nabla_{\mathbf{x}}^2 \psi + V \psi + \frac{\hbar^2}{2m} \frac{\nabla_{\mathbf{x}}^2 |\psi|}{|\psi|} \psi, \quad |\psi| = R, \quad (41.13)$$

and its complex conjugate. The obtained EL equations are equivalent to EL equations (41.6)–(41.7).

The details of the above exact field representation for the classical particle dynamics originally introduced in [293], [294], [284], [285] can be found in [176, 2.6].

Chapter 42

Almost Periodic Functions and Their Time-Averages

We provide here some very basic information on real and complex-valued almost periodic (a.p.) functions following [66, 3.1]. In fact, we consider for simplicity's sake the class $\mathcal{A}_1(\mathbb{R}, \mathbb{C})$ of a.p. functions of the form

$$f(t) = \sum_{s=1}^{\infty} f_s e^{-i\omega_s t}, \quad -\infty < t < \infty, \quad (42.1)$$

where the exponents ω_s are real numbers, and the amplitudes f_s are complex numbers such that

$$|f| = \sum_{s=1}^{\infty} |f_s| < \infty. \quad (42.2)$$

We refer to the set of exponents $\Lambda_f = \{\omega_s\}$ as the Fourier spectrum of the function f and to the numbers f_s as its Fourier coefficients. The class $\mathcal{A}_1(\mathbb{R}, \mathbb{C})$ is a Banach algebra, that is, (i) it is a linear space, and (ii) for any f and g in $\mathcal{A}_1(\mathbb{R}, \mathbb{C})$ the product fg is in $\mathcal{A}_1(\mathbb{R}, \mathbb{C})$ as well, and $|fg| \leq |f| |g|$. The derivatives of a.p. $f(t)$ satisfy the relation

$$\partial_t^r f(t) = \sum_{s=1}^{\infty} (-i\omega_s)^r f_s e^{-i\omega_s t}, \quad (42.3)$$

provided

$$\sum_{s=1}^{\infty} |\omega_s|^r |f_s| < \infty. \quad (42.4)$$

Every a.p. function f in $\mathcal{A}_1(\mathbb{R}, \mathbb{C})$ is assigned the *time average (mean)* value $\langle f \rangle$ defined by

$$\langle f \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t) dt. \quad (42.5)$$

The mean value has the fundamental property

$$\langle f(t) e^{i\theta t} \rangle = \begin{cases} f_s & \text{if } \theta = \omega_s \\ 0 & \text{otherwise.} \end{cases} \quad (42.6)$$

It is convenient in some applications to view a real a.p. function $f(t)$ as the real part of a complex-valued function $\mathbf{f}(t)$ written in the following form

$$\mathbf{f}(t) = \sum_{\omega \in \Lambda_f} f_\omega e^{-i\omega t}, \quad (42.7)$$

$$\text{where } f_\omega = \langle f(t) e^{i\omega t} \rangle = \langle f \cos(\omega t) \rangle + i \langle f \sin(\omega t) \rangle.$$

The set Λ_f in (42.7) is a finite or countable set of non-negative frequencies $\omega \geq 0$, and we refer to it as the frequency spectrum of f . Then the corresponding real-valued a.p. function has the following representation:

$$f(t) = \text{Re} \{ \mathbf{f}(t) \} = \frac{1}{2} \sum_{\omega \in \Lambda_f} (f_\omega e^{-i\omega t} + f_\omega^* e^{i\omega t}). \quad (42.8)$$

Evidently, any real a.p. function can be represented in the form (42.7), (42.8). Observe that for any a.p. complex-valued function $\mathbf{f}$ and $\mathbf{g}$ of the form (42.7) we have

$$\langle \text{Re} \{ \mathbf{f}(t) \} \text{Re} \{ \mathbf{g}(t) \} \rangle = \begin{cases} \frac{1}{2} \sum_{\omega \in \Lambda_f \cap \Lambda_g} \text{Re} \{ f_\omega g_\omega^* \} & \text{if } \Lambda_f \cap \Lambda_g \neq \emptyset, \\ 0 & \text{if } \Lambda_f \cap \Lambda_g = \emptyset. \end{cases} \quad (42.9)$$

The identity (42.9) readily follows from formula (42.7). Obviously, if the frequency spectra Λ_f^+ and Λ_g^+ have no common frequencies, the time-average $\langle \text{Re} \{ f \} \text{Re} \{ g \} \rangle$ is identically zero. If f and g are real-valued, the relation (42.9) implies that

$$\langle f(t) g(t) \rangle = \begin{cases} \frac{1}{2} \sum_{\omega \in \Lambda_f \cap \Lambda_g} \text{Re} \{ f_\omega g_\omega^* \} & \text{if } \Lambda_f \cap \Lambda_g \neq \emptyset, \\ 0 & \text{if } \Lambda_f \cap \Lambda_g = \emptyset \end{cases}. \quad (42.10)$$

A particular case of the formula (42.9), where the frequency spectra of f and g consist of the same single frequency ω , is customarily used in electrodynamics for time-harmonic fields, [263, 11.2], [314, 2.20].

For any a.p. function f with frequency spectrum Λ_f , one can introduce the smallest additive group in the set of all real numbers that contains all frequencies ω from Λ_f . This group is called the module of f and is denoted by $\text{mod}(f)$, [66, 4.6]. It is easy to see that $\text{mod}(f)$ consists of all real numbers of the form

$$\sum_{j=1}^s m_j \omega_j \text{ where } m_j \text{ are integers, } \omega_j \in \Lambda_f \text{ and } s \text{ is a natural number.}$$

Chapter 43

Vector Formulas

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \quad (43.1)$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} \quad (43.2)$$

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c}) (\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d}) (\mathbf{b} \cdot \mathbf{c}) \quad (43.3)$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla (\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a} \quad (43.4)$$

$$\nabla \cdot (\psi \mathbf{a}) = \mathbf{a} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{a} \quad (43.5)$$

$$\nabla \times (\psi \mathbf{a}) = \nabla \psi \times \mathbf{a} + \psi \nabla \times \mathbf{a} \quad (43.6)$$

$$\nabla (\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) \quad (43.7)$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}) \quad (43.8)$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a} (\nabla \cdot \mathbf{b}) - \mathbf{b} (\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla) \mathbf{a} - (\mathbf{a} \cdot \nabla) \mathbf{b} \quad (43.9)$$

$$\nabla \cdot \mathbf{x} = 3, \quad \nabla \times \mathbf{x} = 0 \quad (43.10)$$

$$\nabla \cdot [\hat{\mathbf{x}} f(r)] = \frac{2}{r} f(r) + \frac{\partial f}{\partial r}, \quad \nabla \times (\hat{\mathbf{x}} f(r)) = 0, \quad (43.11)$$

$$\text{where } \hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}, \quad r = |\mathbf{x}|$$

$$(\mathbf{a} \cdot \nabla) (\hat{\mathbf{x}} f(r)) = \frac{f(r)}{r} [\mathbf{a} - \hat{\mathbf{x}} (\mathbf{a} \cdot \hat{\mathbf{x}})] + \hat{\mathbf{x}} (\mathbf{a} \cdot \hat{\mathbf{x}}) \frac{\partial f}{\partial r} \quad (43.12)$$

$$\nabla (\mathbf{x} \cdot \mathbf{a}) = \mathbf{a} + \mathbf{x} (\nabla \cdot \mathbf{a}) + \mathbf{i} (\mathbf{L} \times \mathbf{a}), \text{ where } \mathbf{L} = -\mathbf{i} (\mathbf{x} \times \nabla) \quad (43.13)$$

$$\mathbf{u} \times (\mathbf{a} \times \mathbf{u}) = \mathbf{a} - (\mathbf{a} \cdot \mathbf{u}) \mathbf{u}, \quad |\mathbf{u}| = 1, \quad (43.14)$$

$$\begin{aligned} [\mathbf{u} \times (\mathbf{a} \times \mathbf{u})] \times (\mathbf{u} \times \mathbf{b}) &= [(\mathbf{a} \times \mathbf{u}) \times \mathbf{u}] \times (\mathbf{b} \times \mathbf{u}) \\ &= [\mathbf{a} \cdot \mathbf{b} - (\mathbf{a} \cdot \mathbf{u}) (\mathbf{u} \cdot \mathbf{b})] \mathbf{u} \\ &= [(\mathbf{a} \times \mathbf{u}) \cdot (\mathbf{b} \times \mathbf{u})] \mathbf{u}, \quad |\mathbf{u}| = 1. \end{aligned} \quad (43.15)$$

$$\frac{1}{4\pi} \int_{|\mathbf{x}|=1} [\mathbf{a} \cdot \mathbf{b} - (\mathbf{a} \cdot \hat{\mathbf{x}}) (\hat{\mathbf{x}} \cdot \mathbf{b})] d\sigma = \frac{2}{3} \mathbf{a} \cdot \mathbf{b}, \quad \hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}. \quad (43.16)$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}). \quad (43.17)$$

$$(\nabla \times \mathbf{A})^2 + (\nabla \cdot \mathbf{A})^2 = \nabla \mathbf{A} \cdot \nabla \mathbf{A} + \nabla \cdot (\mathbf{A} (\nabla \cdot \mathbf{A})) - \nabla \cdot ((\mathbf{A} \cdot \nabla) \mathbf{A}). \quad (43.18)$$

43.1 Integral Identities

Let Ω be a bounded domain in three-dimensional space $\mathbb{R}^3$ with a Lipschitz-continuous boundary $\partial\Omega$. Then the following versions of the Green's theorem (formula) hold, [136, 2.2–2.3],

$$\int_{\Omega} [\mathbf{u} \cdot \nabla \varphi + (\nabla \cdot \mathbf{u}) \varphi] d\mathbf{x} = \int_{\partial\Omega} (\mathbf{u} \cdot \mathbf{n}) \varphi dS, \quad (43.19)$$

$$\int_{\Omega} [(\nabla \times \mathbf{u}) \cdot \mathbf{v} - \mathbf{u} \cdot (\nabla \times \mathbf{v})] d\mathbf{x} = \int_{\partial\Omega} (\mathbf{u} \times \mathbf{n}) \cdot \mathbf{v} dS, \quad (43.20)$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x})$ and $\mathbf{v} = \mathbf{v}(\mathbf{x})$ are $\mathbb{R}^3$ -vector-valued fields, $\varphi = \varphi(\mathbf{x})$ is a scalar-valued field, and $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is normal unit vector to the boundary $\partial\Omega$ at point $\mathbf{x}$ pointing into the exterior of the domain Ω . Notice that in view of (43.1), the expression $(\mathbf{u} \times \mathbf{n}) \cdot \mathbf{v}$ in the right hand side can be recast in the following manifestly skew-symmetric form with respect to $\mathbf{u}$ and $\mathbf{v}$:

$$(\mathbf{u} \times \mathbf{n}) \cdot \mathbf{v} = \mathbf{n} \cdot (\mathbf{v} \times \mathbf{u}) = \mathbf{n} \cdot (\mathbf{v}_{\tau} \times \mathbf{u}_{\tau}), \quad (43.21)$$

where $\mathbf{v}_{\tau} = \mathbf{v}_{\tau}(\mathbf{x})$ is the component of the field $\mathbf{v}$ tangential to the boundary $\partial\Omega$ at the point $\mathbf{x}$ defined by

$$\mathbf{v}_{\tau} = \mathbf{v} - \mathbf{n} (\mathbf{n} \cdot \mathbf{v}). \quad (43.22)$$

Chapter 44

The Helmholtz Decomposition

The *Helmholtz (Helmholtz–Hodge) decomposition* states that any 3-dimensional field $\mathbf{F}(\mathbf{x})$ defined over the entire three-dimensional space $\mathbb{R}^3$ can be uniquely split into its *longitudinal (irrotational, lamellar, curl-free)* part $\mathbf{F}_L(\mathbf{x})$ and *transversal (solenoidal, divergence-free)* part $\mathbf{F}_T(\mathbf{x})$, [324, App.1], [7, 3.9], [142, 20], [332, 2.1.2]. More precisely, under the assumption that the field $\mathbf{F}(\mathbf{x})$ is sufficiently smooth and decays sufficiently fast at infinity together with its derivatives, it can be decomposed as follows:

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_L(\mathbf{x}) + \mathbf{F}_T(\mathbf{x}) \quad (44.1)$$

where

$$\mathbf{F}_L(\mathbf{x}) = -\nabla\phi(\mathbf{x}) \quad (44.2)$$

is the *longitudinal (irrotational, lamellar, curl-free)*, and

$$\mathbf{F}_T(\mathbf{x}) = \nabla \times \mathbf{V}(\mathbf{x}), \quad (44.3)$$

is the *transversal (solenoidal, divergence-free)* component. The scalar function $\phi(\mathbf{x})$ and vector function $\mathbf{V}(\mathbf{x})$ are represented by the following integrals:

$$\phi(\mathbf{x}) = \frac{1}{4\pi} \int \frac{\nabla \cdot \mathbf{F}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}', \quad \mathbf{V}(\mathbf{x}) = \frac{1}{4\pi} \int \frac{\nabla \times \mathbf{F}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'. \quad (44.4)$$

The above formulas show that the transformations (projections) $F \rightarrow F_L$ and $F \rightarrow F_T$ are nonlocal. We refer to the scalar potential $\phi(\mathbf{x})$ as the longitudinal (scalar potential) and to the vector potential $\mathbf{V}(\mathbf{x})$ as the transversal (vector potential). Note that the longitudinal and transversal components are orthogonal, since for smooth and decaying fields

$$\int \nabla\phi(\mathbf{x}) \cdot (\nabla \times \mathbf{V}(\mathbf{x})) d\mathbf{x} = 0. \quad (44.5)$$

Justifying the names “irrotational” and “solenoidal”, the field components $\mathbf{F}_L(\mathbf{x})$ and $\mathbf{F}_T(\mathbf{x})$ evidently satisfy the following equalities:

$$\nabla \times \mathbf{F}_L(\mathbf{x}) = 0, \quad \nabla \cdot \mathbf{F}_T(\mathbf{x}) = 0. \quad (44.6)$$

Notice also that the vector function $\mathbf{V}(\mathbf{x})$ is divergence free, that is,

$$\nabla \cdot \mathbf{V}(\mathbf{x}) = 0. \quad (44.7)$$

Observe also that formulas (44.1) and (44.4) show that a vector field $\mathbf{F}(\mathbf{x})$ can be uniquely recovered from its divergence $\nabla \cdot \mathbf{F}(\mathbf{x})$ and curl $\nabla \times \mathbf{F}(\mathbf{x})$.

One way to prove the Helmholtz decomposition is as follows, [221, 5.70]. Let us introduce the following vector field

$$\mathbf{G}(\mathbf{x}) = \frac{1}{4\pi} \int \frac{\mathbf{F}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'. \quad (44.8)$$

Applying $\nabla_{\mathbf{x}}^2$ to the equality (44.8) defining $\mathbf{G}(\mathbf{x})$ and using formula (37.3), that is,

$$\Delta_{\mathbf{x}} \frac{1}{|\mathbf{x} - \mathbf{x}'|} = -4\pi\delta(\mathbf{x} - \mathbf{x}'), \quad \Delta_{\mathbf{x}} = \nabla_{\mathbf{x}}^2, \quad (44.9)$$

we find that $\mathbf{G}(\mathbf{x})$ satisfies the Poisson equation

$$\nabla_{\mathbf{x}}^2 \mathbf{G}(\mathbf{x}) = \frac{1}{4\pi} \int \Delta_{\mathbf{x}} \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \mathbf{F}(\mathbf{x}') d\mathbf{x}' = -\mathbf{F}(\mathbf{x}). \quad (44.10)$$

Applying identity (43.4) to the field $\mathbf{G}$, we get

$$\nabla \times (\nabla \times \mathbf{G}) = \nabla (\nabla \cdot \mathbf{G}) - \nabla^2 \mathbf{G}. \quad (44.11)$$

Combining relations (44.10) and (44.11), we obtain the desired Helmholtz decomposition (44.1) in terms of the field $\mathbf{G}(\mathbf{x})$, namely

$$\mathbf{F}(\mathbf{x}) = -\nabla (\nabla \cdot \mathbf{G}(\mathbf{x})) + \nabla \times (\nabla \times \mathbf{G}(\mathbf{x})). \quad (44.12)$$

Consequently, the following representations holds for the longitudinal and transversal components:

$$\mathbf{F}_L(\mathbf{x}) = -\nabla (\nabla \cdot \mathbf{G}(\mathbf{x})), \quad \mathbf{F}_T(\mathbf{x}) = \nabla \times (\nabla \times \mathbf{G}(\mathbf{x})). \quad (44.13)$$

We can find the divergence and the curl of the vector field $\mathbf{G}(\mathbf{x})$, namely

$$\begin{aligned}\nabla \cdot \mathbf{G}(\mathbf{x}) &= \int \nabla_{\mathbf{x}} \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \cdot \mathbf{F}(\mathbf{x}') \, d\mathbf{x}' \\ &= - \int \nabla_{\mathbf{x}'} \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \cdot \mathbf{F}(\mathbf{x}') \, d\mathbf{x}' = \int \frac{1}{|\mathbf{x} - \mathbf{x}'|} \nabla_{\mathbf{x}'} \cdot \mathbf{F}(\mathbf{x}') \, d\mathbf{x}',\end{aligned}\quad (44.14)$$

and similarly

$$\begin{aligned}\nabla \times \mathbf{G}(\mathbf{x}) &= \int \nabla_{\mathbf{x}} \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \times \mathbf{F}(\mathbf{x}') \, d\mathbf{x}' \\ &= \int \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \nabla_{\mathbf{x}'} \times \mathbf{F}(\mathbf{x}') \, d\mathbf{x}'.\end{aligned}\quad (44.15)$$

Hence we obtain equalities (44.4).

The Helmholtz decomposition (44.1)–(44.4) can also be recast into operator form. Indeed, it follows from (44.10) that

$$\mathbf{G}(\mathbf{x}) = [(-\Delta)^{-1} \mathbf{F}](\mathbf{x}). \quad (44.16)$$

Using formulas (44.13), we introduce projection operators $\mathcal{P}_L$ and $\mathcal{P}_T$ on, respectively, longitudinal and transversal fields as follows

$$\begin{aligned}\mathbf{F}_L &= \mathcal{P}_L \mathbf{F} = -\nabla (\nabla \cdot (-\Delta)^{-1} \mathbf{F}), \\ \mathbf{F}_T &= \mathcal{P}_T \mathbf{F} = \nabla \times (\nabla \times (-\Delta)^{-1} \mathbf{F}).\end{aligned}\quad (44.17)$$

The projection nature and yet another justification of the terms “longitudinal” and “transversal” becomes evident if we consider the Fourier transform $\check{\mathbf{F}}(\mathbf{k})$ with respect to the space variable $\mathbf{x}$ of the field $\mathbf{F}(\mathbf{x})$:

$$\check{\mathbf{F}}(\mathbf{k}) = [\mathcal{F}\mathbf{F}](\mathbf{k}) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-i\mathbf{k} \cdot \mathbf{x}} \mathbf{F}(\mathbf{x}) \, d\mathbf{x}. \quad (44.18)$$

Then the Fourier transforms $\check{\mathcal{P}}_L$ and $\check{\mathcal{P}}_T$ of the projection operators $\mathcal{P}_L$ and $\mathcal{P}_T$ defined by (44.17) take the form

$$\begin{aligned}\check{\mathcal{P}}_L \check{\mathbf{F}}(\mathbf{k}) &= (\hat{\mathbf{k}} \cdot \check{\mathbf{F}}(\mathbf{k})) \hat{\mathbf{k}}, \\ \check{\mathcal{P}}_T \check{\mathbf{F}}(\mathbf{k}) &= -\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \check{\mathbf{F}}(\mathbf{k})) = (\mathbb{I} - \check{\mathcal{P}}_L) \check{\mathbf{F}}(\mathbf{k}),\end{aligned}\quad (44.19)$$

where

$$\hat{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|}, \quad \text{and } \mathbb{I} \text{ is the identity operator.} \quad (44.20)$$

One can readily see from the above formulas that the longitudinal projection operator $\check{\mathcal{P}}_L \check{\mathbf{F}}(\mathbf{k})$ is the orthonormal projection of the vector $\check{\mathbf{F}}(\mathbf{k})$ onto the unit wavevector $\hat{\mathbf{k}}$, whereas the transversal projection operator $\check{\mathcal{P}}_T$ projects the vector $\check{\mathbf{F}}(\mathbf{k})$ orthonormally onto the plane orthogonal (transversal) to $\hat{\mathbf{k}}$.

The following Helmholtz decomposition of the matrix (dyadic) delta function $\delta_{ij} \delta(\mathbf{x})$ into its longitudinal and transversal components holds, [34], [325, 3.20], [324, App. 1], [192, 5.10]:

$$\delta_{ij} \delta(\mathbf{x}) = \delta_{Lij}(\mathbf{x}) + \delta_{Tij}(\mathbf{x}), \quad (44.21)$$

where

$$\delta_{Lij}(\mathbf{x}) = -\frac{1}{4\pi} \partial_i \partial_j \left(\frac{1}{|\mathbf{x}|} \right) = \frac{1}{3} \delta_{ij} \delta(\mathbf{x}) - \frac{3x_i x_j - |\mathbf{x}|^2 \delta_{ij}}{4\pi |\mathbf{x}|^5}, \quad (44.22)$$

$$\delta_{Tij} = \frac{2}{3} \delta_{ij} \delta(\mathbf{x}) + \frac{3x_i x_j - |\mathbf{x}|^2 \delta_{ij}}{4\pi |\mathbf{x}|^5}. \quad (44.23)$$

The Helmholtz decomposition (44.1)–(44.4) for fields over the entire space can also be extended to domains with boundaries, see [324, App. 1], [325, 3.22–3.23], [71, IX.A.3], [136, 3.3].

Chapter 45

Gaussian Wave Packets

Following [35, App. G] and [176, 4.7.1], we summarize here the basic properties of the Gaussian wave packet solution $\psi(t, \mathbf{x})$ to the Schrödinger equation for the free particle

$$i\hbar\partial_t\psi(t, \mathbf{x}) = -\frac{\hbar^2}{2m}\nabla^2\psi(t, \mathbf{x}), \quad (45.1)$$

assuming the initial state $\psi(0, \mathbf{x})$ of the wave function $\psi(t, \mathbf{x})$ at $t = 0$ to be the minimum uncertainty wave packet, that is, [243, 3.2],

$$\psi(0, \mathbf{x}) = \frac{1}{(2\pi\sigma_0^2)^{\frac{3}{4}}} \exp\left\{i\mathbf{k}_0 \cdot \mathbf{x} - \frac{\mathbf{x}^2}{4\sigma_0^2}\right\}, \text{ where } \mathbf{k}_0 = \frac{\mathbf{p}_0}{\hbar}, \quad (45.2)$$

where

$$\mathbf{p}_0 = \hbar\mathbf{k}_0 = \int_{\mathbb{R}^3} \psi^*(0, \mathbf{x}) \frac{\hbar}{i} \nabla \psi(0, \mathbf{x}) \, d\mathbf{x} \quad (45.3)$$

is the free particle momentum and the dispersion σ_0 satisfies

$$\sigma_0^2 = \int_{\mathbb{R}^3} \mathbf{x}^2 |\psi(0, \mathbf{x})|^2 \, d\mathbf{x}. \quad (45.4)$$

Notice that

$$\int_{\mathbb{R}^3} |\psi_0(\mathbf{x})|^2 \, d\mathbf{x} = 1, \quad \int_{\mathbb{R}^3} \mathbf{x} |\psi(0, \mathbf{x})|^2 \, d\mathbf{x} = \mathbf{0}. \quad (45.5)$$

The dispersion σ_0 can be interpreted as the width of the initial wave packet $\psi(0, \mathbf{x})$ defined by Eq. (45.2), and $\mathbf{p}_0$ as its momentum. According to the second equality in (45.5), the center of the initial wave packet $\psi(0, \mathbf{x})$ is at $\mathbf{0}$.

The solution $\psi(t, \mathbf{x})$ to the free particle Schrödinger equation (45.1) is the following Gaussian wave packet:

$$\psi(t, \mathbf{x}) = \frac{1}{(2\pi s_t^2)^{\frac{3}{4}}} \exp \left\{ i\mathbf{k}_0 \cdot \left(\mathbf{x} - \frac{1}{2}\mathbf{u}t \right) - \frac{(\mathbf{x} - \mathbf{u}t)^2}{4\sigma_t} \right\} \quad (45.6)$$

where

$$s_t = \sigma_0 \left(1 + \frac{i\hbar t}{2m\sigma_0^2} \right), \quad \mathbf{u} = \frac{\hbar\mathbf{k}_0}{m} = \frac{\mathbf{p}_0}{m}. \quad (45.7)$$

The momentum space representation $\hat{\psi}(t, \mathbf{p})$ can be defined as the Fourier transform of $\psi(t, \mathbf{x})$, [243, 10.5], [176, 4.7.1]:

$$\hat{\psi}(t, \mathbf{p}) = (2\pi\hbar)^{-\frac{3}{2}} \int \exp \left\{ -i\frac{\mathbf{p} \cdot \mathbf{x}}{\hbar} \right\} \psi(t, \mathbf{x}) d\mathbf{x}, \quad (45.8)$$

with the inverse transform

$$\psi(t, \mathbf{x}) = (2\pi\hbar)^{-\frac{3}{2}} \int \exp \left\{ i\frac{\mathbf{p} \cdot \mathbf{x}}{\hbar} \right\} \hat{\psi}(t, \mathbf{p}) d\mathbf{p}. \quad (45.9)$$

For the Gaussian wave packet defined by (45.6), we have

$$\hat{\psi}(t, \mathbf{p}) = \left(\frac{\sigma_0}{\hbar} \sqrt{\frac{2}{\pi}} \right)^{\frac{3}{2}} \exp \left\{ -\sigma_0^2 \left(\frac{\mathbf{p}}{\hbar} - \mathbf{k}_0 \right)^2 - i\frac{\mathbf{p}^2}{2m\hbar}t \right\}, \quad (45.10)$$

implying

$$\left| \hat{\psi}(t, \mathbf{p}) \right|^2 = (2\pi\sigma_p^2)^{-\frac{3}{2}} \exp \left\{ -\frac{(\mathbf{p} - \hbar\mathbf{k}_0)^2}{2\sigma_p^2} \right\} \quad (45.11)$$

where the dispersion σ_p is defined by

$$\sigma_p = \frac{\hbar}{2\sigma_0}. \quad (45.12)$$

It is evident from relations (45.10)–(45.12) that the momentum-space representation $\hat{\psi}(t, \mathbf{p})$ maintains its width and does not spread with time.

In the original space, we have from (45.6)

$$|\psi(t, \mathbf{x})|^2 = (2\pi\sigma_t^2)^{-\frac{3}{2}} \exp \left\{ -\frac{(\mathbf{x} - \mathbf{u}t)^2}{2\sigma_t^2} \right\} \quad (45.13)$$

where the dispersion σ_t is defined by

$$\sigma_t = |s_t| = \sigma_0 \left[1 + \left(\frac{\hbar t}{2m\sigma_0^2} \right)^2 \right]^{\frac{1}{2}}. \quad (45.14)$$

From (45.11) and (45.13) we readily obtain that

$$\int |\psi(t, \mathbf{x})|^2 d\mathbf{x} = 1, \quad \int |\hat{\psi}(t, \mathbf{p})|^2 d\mathbf{p} = 1. \quad (45.15)$$

As to the uncertainty at time t , the equalities (45.12) and (45.14) imply that

$$\sigma_t \sigma_p = \frac{\hbar}{2} \left[1 + \left(\frac{\hbar t}{2m\sigma_0^2} \right)^2 \right]^{\frac{1}{2}}, \quad (45.16)$$

and for $t \rightarrow \infty$

$$\sigma_t = \frac{\hbar t}{2m\sigma_0} (1 + o(1)), \quad \frac{d\sigma_t}{dt} \rightarrow \frac{\hbar}{2m\sigma_0} = \frac{\sigma_p}{m}. \quad (45.17)$$

In particular, the dispersion speed $\frac{d\sigma_t}{dt}$ is constant asymptotically for $t \rightarrow \infty$. Equality (45.16) shows that the minimum uncertainty for the Gaussian wave packet (45.6) is achieved only at $t = 0$, and the dispersion σ_t grows linearly in time. To put in some numbers, for electrons the dispersion speed $\frac{d\sigma_t}{dt}$ is of the order of 1 cm s^{-1} for $\sigma_0 = 0.1 \text{ cm}$, but it is of the order of 100 km s^{-1} for $\sigma_0 = 5 \text{ \AA}$, [35, App. G].

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