Igor Emri · Arkady Voloshin

Statics

Learning from Engineering Examples

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To Ilana and Vesna with love, you made it possible and worthwhile.

Arkady and Igor

Preface

Mathematical and physical theories cannot be used directly to solve real-life problems because models (theories) take into account only the most important physical quantities and cannot account for each particular characteristic of the given problem. The real-life engineering problems usually deal with objects of complicated geometrical configuration that cannot be easily modeled. Thus, simplifications are always required in order to apply the theory. The ability to simplify the real-life problems and represent them as solvable models is the most important skill of an engineer. Simplifications are commonly applied to the geometry of a real structure and to the selection of most important physical quantities that are of major importance to achieve an engineering solution. Because of these simplifications it is obvious that the analytical solution based on the assigned physical model does not represent the exact solution for the real problem, but at the best is a good engineering approximation.

This textbook of mechanics aims to teach the engineering students the ability to consider any problem and approach it in a systematic way that will allow creating a physical model of a real-life problem and arrive at the solution by writing and solving equations of equilibrium. In this process, the necessary simplifications will take place, and the complicated, real-life problem will be reduced to a manageable simple system that may be easily represented by its free body diagram and corresponding set of equilibrium equations.

Even though the field of mechanics of rigid bodies is well established and did not have any new developments in the last 100 years, we still have to develop better and easier ways for students to understand these basic elements and be capable to observe, understand, and simplify the existing problem. Thus, we enforce the concept of taking the real-world engineering examples and simplify them to become solvable by relatively simple means of the equations of equilibrium.

A set of equilibrium equations may be solved by "hand" or by any of the available computer tools. These tools rapidly change with new developments in computer science and engineering, which has nothing to do with the subject of this book. We therefore leave to the discretion of the instructor and student which of the available tools they may use for solving equations. These may be EXCEL, MATLAB, MATHEMATICA, or any other programs that may appear in the future.

Many of the currently available new texts are introducing so-called "computer" problems that are nothing more than using the analytical solution and substituting a range of variables to calculate the result. Our philosophy is that such an approach distracts students from comprehending the problem and leads them to rely on a numerical approach before developing a clear understanding of the mechanics. Therefore, we try not to emphasize such exercises.

Today's students are well versed in using computational software and thus just running a "do loop" to run the calculation through a range of values that does not contribute to the deeper understanding of the mechanics. Of course, we do not want to preclude students from using any computational software capable to ease the calculations necessary to get the result. For this purpose, a number of MATLAB routines are provided on the Springer website (http://extras.springer.com) that students may use to solve the linear system of equations of equilibrium. However, these routines still require from students the deep understanding of a given problem and ability to create the correct free body diagram. The MATLAB routines help to solve the system of equations, but do not solve the problem by themselves.

The implemented approach here will allow students to build a better understanding of the physical reality and ways to simplify it in order to create an acceptable engineering solution.

Ljubljana, Slovenia Igor Emri
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Introduction

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Contents

Nearly, each "Statics" book starts with the statement similar to this one: "the main objective of the book is to provide a student with a clear understanding of mechanics and develop the ability to analyze the problem in a simple and logical manner." It is indeed hard to argue with such a statement. And if any of the "Statics" books would "provide a clear understanding and ability to analyze the problem," this book would be unnecessary.

The underlying basic mechanics laws stay the same, but the way they are presented and interpreted is changing with time. The structures built 2–3 thousand years ago demonstrate that humans understood the underlying laws of mechanics even though they were not stated in the forms as they were postulated by Newton in seventeenth century. This clearly demonstrates that comprehending laws of nature is a continuing process, and this book represents one of the small steps in evolution of our understanding of nature.

1.1 General Approach

The intent of this book is to introduce and explain the basic concepts on which "Statics" is based utilizing real-life engineering examples. This is why you will see a large number of photographs of the real machinery and structures. The traditional "teaching" is substituted here by a "learning" approach. We show you a real problem, analyze it, simplify it, and develop a way to solve it. We cannot solve each problem exactly, but we will show you that thinking and

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simplification can help you to deal with many structural problems in a simple and reasonable way.

1.2 Review of the Contents

The scientific approach is empirical, based solely on observation and experiment. The book is built on the basis of fundamental laws of nature as were developed by Sir Isaac Newton, knowledge of vector algebra and common sense. Chapter 2 introduces the basic concepts and definitions; it explains the system of units used in engineering. Chapter 3 introduces a student to the very essential task—how to represent physical reality in a way that one may solve problems and get meaningful results. It teaches you how to identify the important features of the structure that should be included in your model and what may be omitted. This is extremely an important step since the obtained results will be useful only to the extent you understand and justify the simplifications introduced in the process of creating physical model and free body diagram.

Chapter 4 discusses how to find a resultant force replacing number of forces acting on a point in two and three dimensions. After we are comfortable with finding resultant, the main focus point of statics is discussed—how to analyze equilibrium of forces acting at a point on the plane and in the space.

Equilibrium of rigid bodies is the topic of Chap. 5. We introduce procedures to use the fundamental laws of nature in order to find the unknown parameters for a system in equilibrium. Two- and three-dimensional cases are discussed.

Chapter 6 introduces concepts of the center of gravity and centroids. In the same chapter, we discuss the problems of the water pressure on submerged surfaces. In Chap. 7, we will introduce concepts, assumptions, and rules necessary to classify structural elements. The following classes of structures are discussed: trusses, beams, frames, machines, and cables. We also introduce a procedure to calculate internal forces in various structural elements.

The following three Chaps. 8–10 introduce and discuss detailed procedures to solve the corresponding problems. There we develop a set of approaches one may use to calculate the internal forces and moments in a variety of structural elements introduced in Chap. 7.

Chapter 11 discusses how to deal with the problems that do not fall in any of simple classes of structures: trusses, beams, or cables. Here, we develop ways to solve such problems by disassembling a structure and to solve each constituent using the methods introduced in previous chapters.

Chapter 12 deals with the frictional forces that prevent relative movement of solid bodies in contact.

The home problems are divided into two distinct classes: real-life problems and problems represented via physical models, created mainly for methodological reasons. The first group of problem will mainly teach how to study a real-life problem, simplify it, and represent it as a “physical model,” which will allow using the solution techniques explained in the book. The second group of problems is mainly for learning the mathematical procedures and other tools necessary to solve the real-life problems represented by physical models. This process of modeling introduces inevitable errors; however, good engineering practice allows minimizing them and still getting solutions that will satisfy the real-life requirements. Some of the problems use the internationals system of units (SI units), whereas the others use the US customary units. Such approach will allow students to get familiar with both of the systems of units.

Problems identified by "*" are considered to be "challenging." They may be solved using the methods described in this book, but it will require extra effort from the student, thus they may be assigned for "extra" credit.

Appendix contains the basic information about the vectors, matrices, and the ways to manipulate them. It should be noted that the main purpose of this Appendix is to provide a refreshment of the rules on vectors and matrix algebra; it cannot serve as a tutorial.

Several routines written for MATLAB are available on the *extras.springer.com*. They allow students to simplify the process of calculations, but they still leave the burden of creating correct free body diagrams and writing appropriate equations of equilibrium. Their routines may be easily modified to solve a range of problems; they will decrease the calculation errors, but they will not solve a problem for you! You should always consider the MATLAB and other numerical procedures as a tool (e.g., pencil, calculator) and not as a problem solver.

Both authors are avid scuba divers and went to many scuba trips together (and still do). Since one cannot dive more than couple of hours per day, they used the rest of the time to discuss various topics of common interest. One of these was the way the course of "Statics" is taught at different universities. This was the start of a long and tedious work on this approach to comprehend "Statics."

We decided to deal with real problems as one may encounter in the real life and not with models and idealizations only, as it is customary in the many of the existing text books.

1.3 Conventions on Notations

In the course of this book, we usually use bold letters to define external forces, and we use capital letters to indicate the supports. For example, if A indicates support then we would use A, A_x, A_y, and A_z to define the components of the reaction forces. Distributed forces are usually defined by lower case q. Every time we use a summation sign $\sum$, it is assumed summation by all forces and/or moments. If we need to specify the details, $\sum_{i=1}^{N}$ is used.

Vectors are denoted in bold font and scalars in italics. Greek letters are used to define angles.

Vector or cross product is defined by x and scalar product is defined by a dot (·).

List of Symbols

a	Distance
b	Distance
c	Distance
A, B, C, D, E	Points
A, B, C, D, E, F, P, Q	Force reactions, magnitudes
A, B, C, D, E, F, P, Q, R	Force reactions, vectors
G	Center of gravity
q	Distributed forces
i, j, k	Unit vectors
M	Moment
O	Origin of coordinates
r	Position vector
x, y, z	Rectangular coordinates
α, β, γ, θ...	Angles

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2 Laws of Nature and Fundamental Concepts

Contents

In theory, there is no difference between theory and practice.
But, in practice, there is.

Jan L.A. van de Snepscheut

In this chapter you will learn:

- Basic concepts and definitions
- Fundamental laws of nature, as defined by Sir Isaac Newton
- Procedures to find a resultant force
- Mathematical definition of the moment
- How to calculate the moment of a couple
- How to find the projection of the moment of the axis
- System of units

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Everyday experience with objects and forces acting upon them shows that there is a definite relationship between the motion and the force. These relations were studied by Newton in the seventeenth century.[1] Based on the observation of the nature, he postulated basic relations describing interactions between forces, matter, space, and time. These physical quantities are fundamental quantities; they have to be accepted intuitively as *facts of nature*. In the scientific literature, there are many different ways to describe these quantities. Here, we are summarizing those that are appropriate for the purpose of this book.

Time is a quantity used to separate different stages of a process. In principle, evolution of any phenomenon may be used to measure time, providing that certain conditions are fulfilled. In order to introduce a unit of time, we have to consider a well-defined reference process, for example, swing of a pendulum, rotation of the Earth about its axis, and decay of a radioactive material. There are two commonly accepted independent approaches to define the time scale. The first one is based on the regularity of the celestial bodies' motions. The second one is based on the characteristic frequency of the electromagnetic radiation emitted or absorbed in quantum transitions between internal energy states of atoms.

Space can be considered as a boundless, three-dimensional extent in which objects and events occur and have relative positions and directions. The perception of space allows the concepts of position and geometry of a body. The fundamental elements of geometry are length, area, and volume. We have to define an appropriate unit to measure length, area, and volume. In principle, any distance may serve as the unit of length. Through the history, people used many different definitions for the unit of length. According to an international agreement, today we are using the unit of length called meter. Originally, the length unit called *meter* was defined as one ten millions of the distance from the North Pole to the equator on the meridian running through Paris. The latest definition of a *meter* is the length of the path traveled by light in a vacuum during a time interval of 1/299,792,458 of a second. Until recently, the English units of lengths were defined in terms of the imperial standard yard, which was the distance between the two lines on a bronze bar made in 1845. Because the imperial standard yard was shrinking at the rate of 1.5 millionths of an inch per year the United States adopted a copy of the international prototype meter as the national standard of length in 1889.

Matter is a substance that constitutes the observable universe and, together with energy, forms the basis of all objective phenomena. The main building blocks of matter are atoms. Matter has several states: gas, liquid, solid states, and plasma. Each state exhibits properties that distinguish it from the others. Moreover, these general states can be subdivided into groups according to particular types of properties listed in the periodic table. Matter exists in and occupies the space. Matter that occupies a specific space (volume) is called *body*. When distribution of matter in a given volume and its shape does not change in time, we talk about rigid bodies. Quantity of matter in a unit volume is called *density*. Inherent and

[1] Sir Isaac Newton, **Principia,** Vol. I The Motion of Bodies, University of California Press, 1962.

permanent property of matter is inertia, which causes a body to resist any change in its condition of rest or motion. The *mass* of a body is a measure of its inertia. It is also commonly taken as a measure of the amount of matter contained in a body. Each body possesses gravitation, the property by which it attracts every other body. The gravitation is still not fully understood. Gravitational attraction is *force* acting between bodies. When one of the bodies is a celestial body (e.g., Earth or Jupiter), this force is called the *weight* of the second body. Of our interest will be weight of the bodies located on the Earth. Hence, the weight is the force exerted on the matter by gravitational attraction of the Earth. This force is proportional to the mass of two bodies and inversely proportional to the square of the distance between them. Therefore, since the shape of the Earth is not a perfect sphere, the weight of a body varies from place to place. In contrast, the mass of the body remains constant regardless of its location, assuming that the velocity of body is significantly smaller than the velocity of light (which is the framework of the Newton's mechanics). The weight of a satellite launched into space, for example, decreases as it travels away from the Earth. Its mass, however, stays the same.

The mass of a body is a measure of its inertia. It is also used as a measure of the amount of matter contained in a body.

Weight is a force created by gravitational field, acting on each and every particle constituting a body. The origin of gravitation is still not completely understood.

Weight of a body can be viewed as the force acting on a point, called *center of gravity*.

The metric unit of mass is kilogram (kg) defined by a solid cylinder of platinum-iridium alloy maintained at constant temperature in Sevres, France. English unit of mass (pound mass) is one of the oldest units of weight. Its current definition is based on the metric standard of mass.

Classical mechanics addresses the interrelation between matter, space, force, and time as observed and postulated by Newton. In this book, we discuss systems that are not time dependent, thus only the conditions under which bodies are and remain at rest. This part of mechanics is called *Statics*.

Statics deals with time-independent problems, thus time does not appear in static problems.

2.1 Forces and Newton's Laws

Some of the important concepts of forces, their effect on bodies and the governing laws are discussed below.

2.1.1 Newton's Laws

When a cup of tea is served, one has to apply a force to the cup that is equal to its weight, to keep it in place, i.e., the two forces are in equilibrium.

At this point, we are ready to make our first abstraction and assume that the cup (or any other observed body) can be represented as a particle, which has no size, but the same mass and weight as the original body (in our case the cup). Furthermore, weight of the body, i.e., of the particle, can be viewed as the force acting at a particular point of space. Definition and location of this point, called *center of gravity*, will be discussed in detail later (Chap. 6).

Interaction of the cup and the hand, shown in Fig. 2.1, can be therefore viewed as an interaction of two forces acting on a point, as shown in Fig. 2.2.

These two forces, commonly called *action* and *reaction*, are equal in magnitude and opposite in direction. They represent one of the basic laws of nature, postulated by Sir Isaac Newton as the *Third* Fundamental Law:

> *To every action there is always opposed and equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.*

> Third Newton's Law: "To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts".

Fig. 2.1 Equilibrium of the cup's weight and the force acting on the cup, generated by the hand

Fig. 2.2 Two forces, the weight of the cup (**W**) and the force generated by the hand (**F**), are in equilibrium and are acting at the point ***A***

Since action and reaction are equal in magnitude and opposite in direction, the resultant force acting on the particle (body) is equal to zero. The particle (our cup) therefore will not change its state of motion. If it was at rest at the moment of observation, it will maintain its state, i.e., its location in space will not change.

To move the cup up, one has to increase the force acting upon it—force generated by hand should be larger than the body's weight and therefore the resultant force will not be zero anymore. Quantitative observations of this and similar examples show that application of an excessive force to an object causes its acceleration that is directly proportional to the resultant force. The proportionality constant happens to be reciprocal to the mass of the body. Sir Isaac Newton summarized this phenomenon as the *Second* Fundamental Law of nature:

The change of motion is proportional to the resultant force and is made in the direction of the line in which this force is acting.

Second Newton's Law: "The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which the force is impressed".

In other words, if the resultant force acting on a particle is not zero the particle will have an acceleration proportional to the magnitude of the resultant force and in the direction of this force. Mathematically, this can be expressed as

$$\mathbf{a} = \frac{1}{m}\mathbf{F} \tag{2.1}$$

or in more common form

$$\mathbf{F} = m\mathbf{a} \tag{2.2}$$

where m is a mass, **F** an acting force, and **a** the resulting acceleration of the body. Acting resulting force **F** and corresponding acceleration **a** act in the same direction.

They are *vectors*, having the same unit vector, which defines their direction (Appendix).

Acceleration is defined as a change of particle's velocity per unit time.

Newton's definition of mass reads: "The quantity of matter is the measure of the same, arising from its density and bulk conjointly" (see footnote 1).

Mass can be defined as a quantity of matter occupying a given volume. It is a measure of resistance to its change in motion.

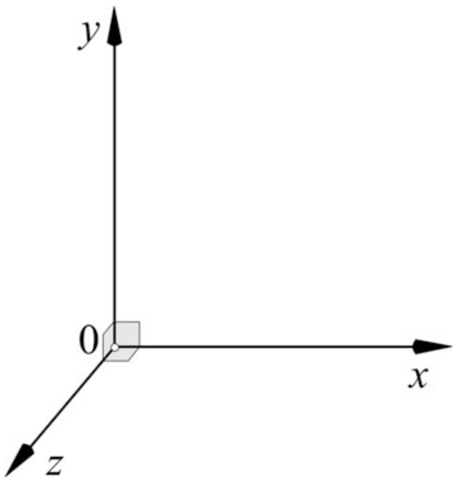
From the Second Fundamental Law follows that if no external force acts on a body, its acceleration is equal to zero. This means that its state of motion will not be altered. Newton summarized this observation as the *First* Fundamental Law:

> *Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces acting upon it.*

First Newton's Law: "Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it".

In addition to the three Fundamental Laws above, one can observe some other facts related to forces acting upon bodies.

The right-hand coordinate system is used throughout this book. This system is commonly used in mechanics and is shown below.



2.1.2 Internal and External Forces

The forces acting on a body can be separated into two distinct groups: external and internal. The simplest way to distinguish between them is to assume that any force that acts upon the object of interest is an *external* force. Any force exerted by one part of the object on another part of the same object is an *internal* force.

External forces may be contact and noncontact forces. Noncontact forces are gravitational and electromagnetic forces, whereas contact forces result from the contact (interaction) between bodies.

The following example will make such distinction obvious. Let us say that a car is stuck in mud, one person is pushing on the rear bumper of the car, while the driver presses the gas pedal. Let us consider the car and the driver as our object. The force which the person is applying on the rear bumper is an external force with respect to our object (car and driver), while the force driver applies to the pedal is an internal force.

It will be shown later that it is convenient to use the same coordinate system for representation of both, internal and external forces.

2.1.3 Principle of Transmissibility

An experiment shows that the effect of *external* force on the state of motion of a rigid body will remain the same if the location of the point where force is acting is moved along its line of action. For example, the effect of the external force on a rigid body, applied at point A, is the same as if a force of the same magnitude and direction is applied at another point B along the line of action of the first force (Fig. 2.3). However, it should be emphasized that moving the point of external force application will change distribution of internal forces.

> It is convenient to use the same coordinate system for representation of internal and external forces.

> The effect of *external* force on the state of motion of a rigid body will remain the same if the location of the point where force is acting is moved along its line of action.

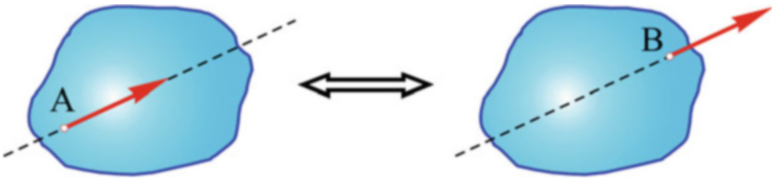


Fig. 2.3 Principle of transmissibility

2.2 Two Forces Acting Upon a Body

Let us consider what happens when two forces are acting on a rigid body. We will discuss the equilibrium case only—the basis for solving problems in statics. Also, the concepts of couple of forces, parallelogram of forces, and transmissibility principle will be introduced.

2.2.1 Equilibrium Pair of Forces

Two forces of equal magnitude and opposite in direction that are acting upon a body along the same line are called an *equilibrium pair of forces* as shown in Fig. 2.4. The resultant force of the equilibrium pair of forces is equal to zero.

When equilibrium pair of forces is acting upon a body, two extreme situations may be observed. *First*, when two forces are sufficiently small or, equivalently, the body is sufficiently rigid, the location and geometry of the body will remain unchanged. *Second*, when the two forces are sufficiently large, or body is not sufficiently rigid, the body will be deformed. In both cases, however, the body will retain the same location.

Let us concentrate on a study of a special case: no motion—*Statics*.

Adding (or subtracting) an equilibrium pair of forces, of any magnitude and direction to a nonrigid body, will change its geometry and internal forces. However, it will not alter its initial state of motion.

At present, we are not interested in the deformation of a body due to applied external forces, thus from here on we will use the concept of rigid *bodies*. Rigid body is the one in which distance between each and every point of the body remains constant at all times. Adding (or subtracting) an equilibrium pair of forces, of any magnitude and direction to a rigid body, will not change its geometry and its state of motion.

Fig. 2.4 Equilibrium pair of forces

> Adding (or subtracting) an equilibrium pair of forces will not alter the state of motion of a body.

> Rigid body is the one in which distance between each and every points of the body remains constant at all times.

2.2.2 Parallelogram of Forces

An experiment shows that the effect of two forces, acting at the same point on a rigid body, may be replaced by only one force, whose direction and magnitude is determined by the rule of parallelogram, as shown in Fig. 2.5. This force is commonly called the *resultant*, and is the vector sum of these two forces. Using vector algebra, this can be expressed as

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2$$

Two forces acting at the same point and their resultant are always in the same plane; thus, the problem of force summation may be considered as a two-dimensional problem.

> A body, acted on by two forces simultaneously, will describe the diagonal of a parallelogram in the same time as it would describe the sides by those forces separately (see footnote 1).

Let us introduce the coordinate system as shown in Fig. 2.6.

In two-dimensional space, each vector has two components: in x and y direction as follows:

$$\begin{aligned} \mathbf{F}_1 &= F_{1x} \cdot \mathbf{i} + F_{1y} \cdot \mathbf{j} \\ \mathbf{F}_2 &= F_{2x} \cdot \mathbf{i} + F_{2y} \cdot \mathbf{j} \\ \mathbf{R} &= \mathbf{F}_1 + \mathbf{F}_2 = F_{1x} \cdot \mathbf{i} + F_{1y} \cdot \mathbf{j} + F_{2x} \cdot \mathbf{i} + F_{2y} \cdot \mathbf{j} \\ &= (F_{1x} + F_{2x}) \cdot \mathbf{i} + \left(F_{1y} + F_{2y}\right) \cdot \mathbf{j} = R_x \cdot \mathbf{i} + R_y \cdot \mathbf{j} \end{aligned}$$

where **i** and **j** the unit vectors along the coordinates axes x and y and F_x and F_y are projections of vector **F** on these axes.

The parallelogram of forces (the axiom defining the effect of two forces acting upon a body) provides a basis for the rule of vector summation.

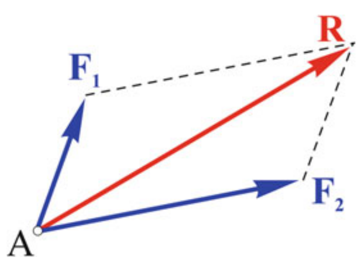


Fig. 2.5 Parallelogram of forces defining magnitude and direction of the resulting force, **R**, which replaces two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, acting at a point A

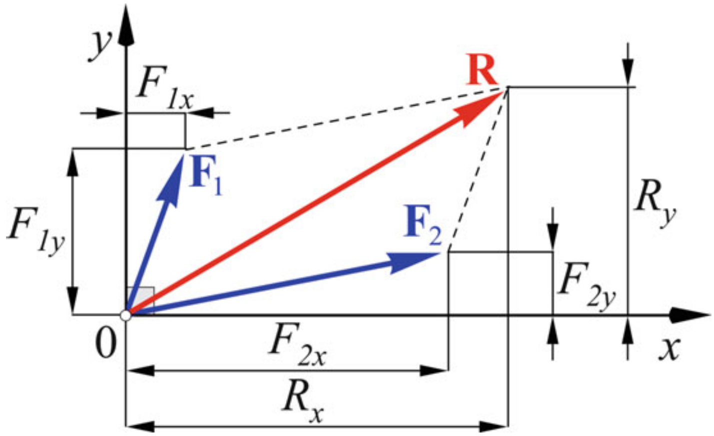


Fig. 2.6 Summation rule

The generalized rule for vector summation in three dimensions is given in the Appendix and may be summarized as

$$\begin{aligned}\mathbf{R} &= \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} + \ldots + \ \mathbf{F_N} \\ &= (F_{1x} + F_{2x} + F_{3x} + \ldots F_{Nx})\mathbf{i} + \left(F_{1y} + F_{2y} + F_{3y} + \ldots + F_{Ny}\right)\mathbf{j} \\ &\quad + (F_{1z} + F_{2z} + F_{3z} + \ldots F_{Nz})\mathbf{k} \\ &= R_x\mathbf{i} + R_x\mathbf{j} + R_x\mathbf{k}\end{aligned}$$

Example 2.1 The hydraulic cylinder exerts a 300 N force **F** on the car at A. This force makes an angle of 40° with the horizontal axis. Express **F** in terms of its scalar components (Fig. 2.7).

Solution In two-dimensional space, each vector has two components. We will represent force **F** as

$$\mathbf{F} = F_x \cdot \mathbf{i} + F_y \cdot \mathbf{j}$$

where F_x and F_y are projections of vector **F** on the x and y axes.

Thus, the magnitude of F_x

$$F_x = |F| \cos 40^\circ = 300 \cdot \cos 40^\circ = 230 \text{ N}$$

Similarly, the magnitude of F_y

Fig. 2.7 The car supported by the hydraulic cylinder

$$F_y = |F| \sin 40^\circ = 300 \cdot \sin 40^\circ = 192.8 \text{ N}$$

After you determined the vector components, it is a good idea to check your results to make sure that they give the correct magnitude of the original force.

$$|F| = \sqrt{(192.8)^2 + (230)^2} = 300 \text{ N}$$

Example 2.2 Force $\mathbf{F}_1 = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{z}$ [N] and force $\mathbf{F}_2 = -3\mathbf{i} + 2\mathbf{j} - 2\mathbf{z}$ [N] are acting at the point A. Find the resultant force.

Solution To find a resultant force, we will use a generalized rule for vector summation.

$$\begin{aligned}\mathbf{R} &= \mathbf{F_1} + \mathbf{F_2} = (F_{1x} + F_{2x})\mathbf{i} + (F_{1y} + F_{2y})\mathbf{j} + (F_{1z} + F_{2z})\mathbf{k} \\ &= (3 - 3)\boldsymbol{i} + (-4 + 2)\boldsymbol{j} + (2 - 2)\boldsymbol{k} = -2\boldsymbol{j}\,[\text{N}]\end{aligned}$$

Example 2.3 Force $\mathbf{F}_1 = -8\mathbf{i} + 12\mathbf{j} - 7\mathbf{z}$ [N] and force $\mathbf{F}_2 = -2\mathbf{i} - 7\mathbf{j} - 5\mathbf{z}$ [N] are acting at the point A. Find the resultant force.

Solution To find a resultant force, we will use a generalized rule for vector summation.

$$\begin{aligned}\mathbf{R} &= \mathbf{F_1} + \mathbf{F_2} = (F_{1x} + F_{2x})\mathbf{i} + (F_{1y} + F_{2y})\mathbf{j} + (F_{1z} + F_{2z})\mathbf{k} \\ &= (-8 - 2)\boldsymbol{i} + (12 - 7)\boldsymbol{j} + (-7 - 5)\boldsymbol{k} \\ &= -10\boldsymbol{i} + 5\boldsymbol{j} - 12\boldsymbol{k}\,[\text{N}]\end{aligned}$$

2.2.3 Couple of Forces

Two parallel forces of equal magnitude and opposite direction not acting along the same line we call a *couple of forces* or shortly, a *couple*. An experiment demonstrates that a *couple of forces* acting on a rigid body causes its rotation around the axis perpendicular to the plane defined by those two forces. We will discuss this subject in more detail in Chap. 5. At this point, it is important to recognize that there is no rotation without a couple of forces acting upon a body. Action of a couple is commonly represented as a physical quantity called *moment*.

Couple of forces are two parallel forces of equal magnitude and opposite direction not acting along the same line.

Figure 2.8 shows a person applying a couple of forces on a swivel chair. This couple, called *moment*, is represented as a vector **M** perpendicular to the plane defined by the couple of forces **F** and −**F**. It has a tendency to rotate the chair as shown in Fig. 2.8.

2.2.3.1 Definition of a Moment

Since the action of a couple, called a moment, is a derived physical quantity, we need its proper mathematical definition.

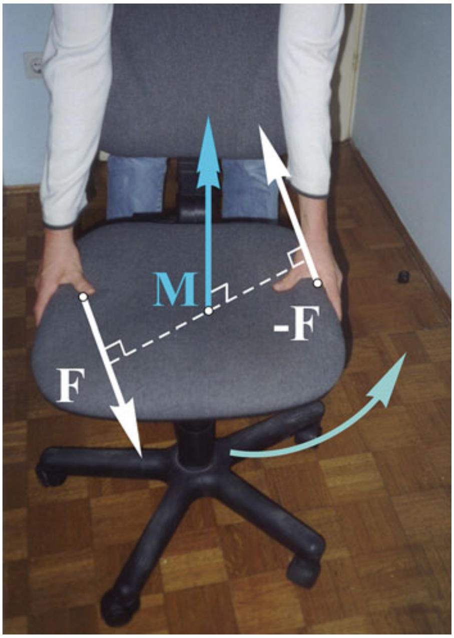


Fig. 2.8 Couple of forces

The moment is defined as

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} \tag{2.3}$$

where **r** is a space vector defining any point along the line of action of force **F**, **r** is also known as a position vector. The unit of a moment is Nm. Both vectors, **r** and **F**, can be represented in terms of their rectangular components,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{2.4}$$

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \tag{2.5}$$

Using the rule of a vector product (Appendix), one can find components of the moment vector.

$$\mathbf{M} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = \left(F_z y - F_y z\right)\mathbf{i} + (F_x z - F_z x)\mathbf{j} + \left(F_y x - F_x y\right)\mathbf{k} \tag{2.6}$$

Hence,

$$M_x = F_z y - F_y z$$

$$M_y = F_x z - F_z x$$

$$M_z = F_y x - F_x y$$

The magnitude of the moment is (Appendix)

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2} = F \cdot r \cdot \sin\varphi = F \cdot d \tag{2.7}$$

where F and r are magnitudes of force and space vector, φ is the angle between the two vectors, and d is the perpendicular distance between the origin of the coordinate system and the force's line of the action, as shown in Fig. 2.9. This perpendicular distance d is called the *moment arm*. Increasing either the moment arm or the force magnitude will increase the magnitude of the moment.

Moments are derived physical quantities. They result from the action of couple of forces.

The direction of the moment vector is defined by its unit vector $\mathbf{e}_\mathrm{M}$

$$\mathbf{e}_\mathrm{M} = \frac{\mathbf{M}}{M} = \frac{M_x}{M} \cdot \mathbf{i} + \frac{M_y}{M} \cdot \mathbf{j} + \frac{M_z}{M} \cdot \mathbf{k} = \cos\alpha \cdot \mathbf{i} + \cos\beta \cdot \mathbf{j} + \cos\gamma \cdot \mathbf{k}$$

where angles α, β, and γ are the angles between the vector **M** and the axes of the coordinate system.

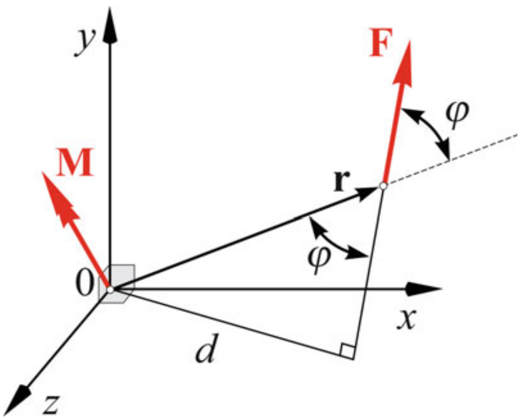


Fig. 2.9 Moment **M** of the force **F** about the point O

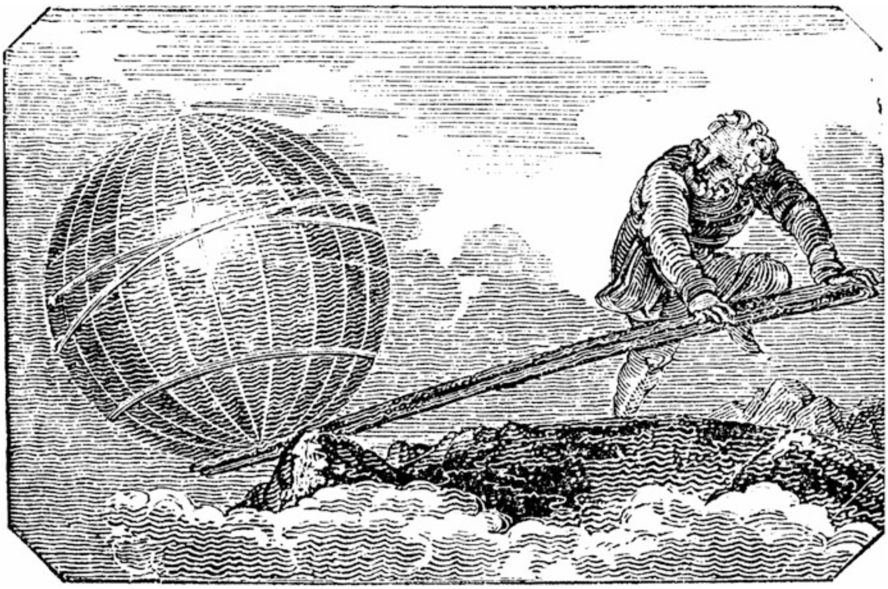


Fig. 2.10 Archimedes moving the Earth

There is an historical anecdote emphasizing the importance of the moments. When ancient Greek mathematician–inventor Archimedes (c. 287–212 BCE) wrote "Give me but one firm spot on which to stand, and I will move the Earth," he obviously knew that by using a long enough moment arm he can apply moment equal to the moment created by the weight of the Earth applied relatively close to the point of support (Fig. 2.10).

Technically, the Earth does not have a weight, only the mass, since the weight is defined as a force of the attraction between the Earth and the object.

Direction of the moment vector is perpendicular to the plane defined by vectors **r** and **F**. It is important to note that cross product of two vectors is *not* commutative, i.e., $\mathbf{r} \times \mathbf{F} \neq \mathbf{F} \times \mathbf{r}$.

Cross product of two vectors is not commutative, i.e., $\mathbf{r} \times \mathbf{F} \neq \mathbf{F} \times \mathbf{r}$.

Mathematical Corner

Prove that **M** is perpendicular to the plane defined by **r** and **F**.

For **M** to be perpendicular to the plane defined by **r** and **F** it should be perpendicular to each one, hence

$$\mathbf{M} \cdot \mathbf{r} = 0$$
$$\mathbf{M} \cdot \mathbf{F} = 0$$

Using moment components from (2.6), we get

$$\begin{aligned}\mathbf{M} \cdot \mathbf{r} &= \left(F_z y - F_y z\right)x + (F_x z - F_z x)y + \left(F_y x - F_x y\right)z \\ &= F_z yx - F_y zx + F_x zy - F_z xy + F_y xz - F_x yz \\ &= F_z yx - F_z yx + F_x zy - F_x zy + F_y xz - F_y xz \\ &= 0\end{aligned}$$

$$\begin{aligned}\mathbf{M} \cdot \mathbf{F} &= \left(F_z y - F_y z\right)F_x + (F_x z - F_z x)F_y + \left(F_y x - F_x y\right)F_z \\ &= F_x F_z y - F_x F_y z + F_y F_x z - F_y F_z x + F_z F_y x - F_z F_x y \\ &= F_x F_z y - F_x F_z y + F_x F_y z - F_x F_y z + F_z F_y x - F_z F_y x \\ &= 0\end{aligned}$$

The above expressions prove that the moment is perpendicular to the plane defined by the two vectors: **F** and **r**.

It is important to note that the moment of force **F** about point O does not depend on vector **r** defining a particular point on the line of action of force **F**. Vector **r** is a vector defining any point along this line; hence,

$$\mathbf{M} = \mathbf{r}_\mathrm{A} \times \mathbf{F} = \mathbf{r}_\mathrm{B} \times \mathbf{F}$$

To prove this, let us consider force **F**, acting at point A, as shown in Fig. 2.11. According to the definition, (2.3), we have

$$\mathbf{M} = \mathbf{r}_A \times \mathbf{F}$$

Let us choose another point on the force's line action, point B. The vector $\mathbf{r}_A$ can be represented as a sum (Fig. 2.11)

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{BA}$$

By substituting this expression in (2.3), one gets

$$\begin{aligned}\mathbf{M} &= \mathbf{r}_A \times \mathbf{F} = (\mathbf{r}_B + \mathbf{r}_{BA}) \times \mathbf{F} = \mathbf{r}_B \times \mathbf{F} + \mathbf{r}_{BA} \times \mathbf{F} \\ &= \mathbf{r}_B \times \mathbf{F}\end{aligned}$$

Vectors $\mathbf{r}_{BA}$ and **F** are parallel to one another. Since the cross product of two parallel vectors is zero by definition, their vector product is equal to zero.

$$\mathbf{r}_{BA} \times \mathbf{F} = 0$$

This is an important observation since it allows one to place the position vector **r** at any point along the line of force action and not only at the point of force application. This will be useful for solving problems, as it will be shown later.

> Moment of a force is NOT dependent on the location of the position vector along the force's line of action (Fig. 2.11).

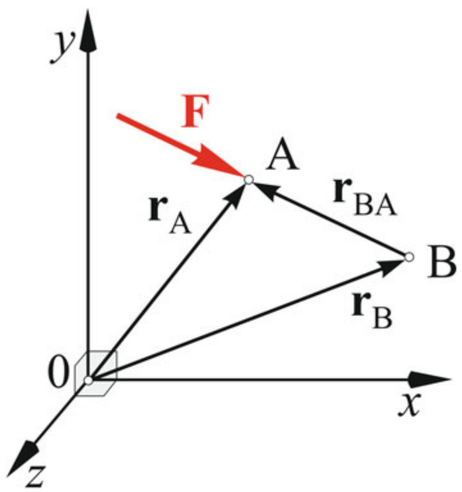


Fig. 2.11 Moment of the force **F** about point O

2.2.3.2 Varignon's Theorem

French mathematician Varignon (1654–1722) developed a concept called principle of moments, often referred as Varignon's theorem. It states that the moment of force **R** about a point is equal to the sum of the moments of the force's components ($\mathbf{F}_1, \mathbf{F}_2$) about the same point, Fig. 2.12. Therefore, the moment **M** of force **R** can be represented as

$$\mathbf{M} = \mathbf{r} \times \mathbf{R} = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2$$

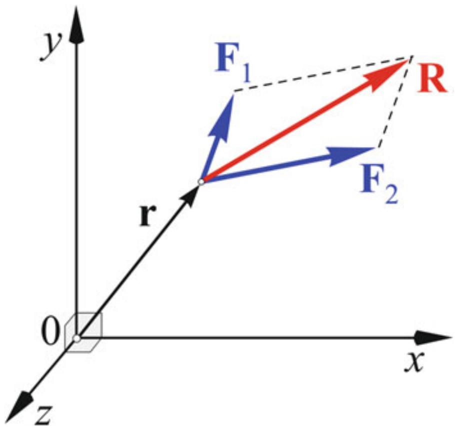


Fig. 2.12 Principle of moments

2.2.3.3 Moment of a Couple

Now, we are ready to apply the above definition to calculate the moment caused by a couple acting upon a rigid body, Fig. 2.13. The moment of a couple is equal to the sum of moments imposed by each of the forces constituting the couple. Hence,

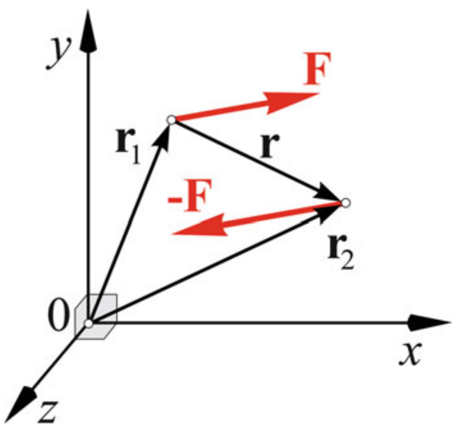


Fig. 2.13 Moment of a couple

$$\mathbf{M} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2$$

Since $\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{r}$, and $\mathbf{F}_1 = -\mathbf{F}_2 = -\mathbf{F}$, we obtain

$$\begin{aligned}\mathbf{M} &= \mathbf{r}_1 \times (-\mathbf{F}) + (\mathbf{r}_1 + \mathbf{r}) \times \mathbf{F} = (-\mathbf{r}_1 + \mathbf{r}_1 + \mathbf{r}) \times \mathbf{F} \\ &= \mathbf{r} \times \mathbf{F}\end{aligned} \tag{2.8}$$

This equation shows that the moment of a couple of forces ($\mathbf{F}_1 = -\mathbf{F}_2$) does not depend on their position relative to the origin of the coordinate system, but only on the relative position vector between the two forces. In other words, the vector $\mathbf{r}$ may run from any point along the line of action of the first force to any point along the line of action of the second force.

Thus, the moment of a couple is a free-floating vector, i.e., it has NO point of application; it may be moved to any point.

It should be noted that since the force may be moved along its line of action (principle of transmissibility), then vector $\mathbf{r}$ is a vector that runs from any point along the line of action of the force $\mathbf{F}_1$ to any point along the line of action of the force $\mathbf{F}_2$.

The magnitude of moment $\mathbf{M}$ is equal to the product of the magnitude of force $\mathbf{F}$ and the perpendicular distance between the two forces (see (2.7) and Fig. 2.9), $M = Fd$.

The moment of a force about a point is equal to the sum of the moments of the force's components about the same point.

The moment of a couple is a free-floating vector. It may be moved in parallel to any point.

2.2.3.4 Moment of a Force About an Axis

There are some engineering applications where solid bodies are subjected to rotation about the given axis. In this case, we may need to find the moment of an external force with respect to a given axis.

In the above sections, we discussed how to calculate the moment of any force with respect to a given point. Here, we will introduce a new concept of the *moment of a force about an axis*.

Let us consider moment $\mathbf{M}$ created by force $\mathbf{F}$ about a point O, as shown in Fig. 2.14. As defined above, the resulting moment $\mathbf{M}$ is

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

Let us define direction of arbitrary axis AB by unit vector, $\boldsymbol{\lambda}_{AB}$. The moment $\mathbf{M}_{AB}$ of force $\mathbf{F}$ about axis AB is a projection of moment $\mathbf{M}$ on axis AB (Fig. 2.14). As it is known from the vector algebra (Appendix), in order to find the magnitude of the

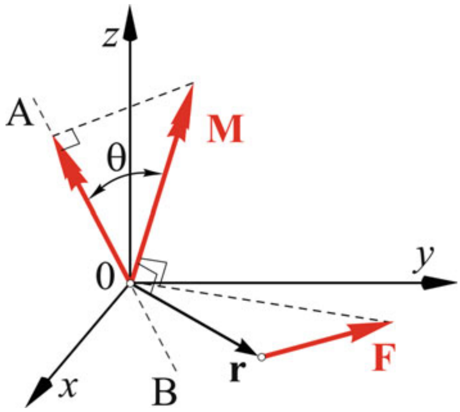


Fig. 2.14 Moment of a force about an axis AB

projection of one vector onto direction of another vector, one has to use a scalar product of the two vectors. Thus, magnitude M_{AB} of the projection of a moment **M** on direction AB, defined by vector $\boldsymbol{\lambda}_{\mathrm{AB}}$ may be calculated as

$$M_{\mathrm{AB}} = \boldsymbol{\lambda}_{\mathbf{AB}} \cdot \mathbf{M} = M \cdot \cos\ \theta$$

Using the rules of matrix algebra and the relationship $\mathbf{M} = \mathbf{r} \times \mathbf{F}$, the above expression may be written as

$$\begin{aligned} M_{\mathrm{AB}} = \boldsymbol{\lambda}_{\mathrm{AB}} \cdot (\mathbf{r} \times \mathbf{F}) &= \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \\ &= \lambda_x\left(yF_z - zF_y\right) - \lambda_y(xF_z - zF_x) + \lambda_z\left(xF_y - yF_x\right) \end{aligned} \tag{2.9}$$

where λ_x, λ_y, and λ_z are the components of the units vector (direction cosines of the vector) $\boldsymbol{\lambda}_{\mathrm{AB}}$, x, y, and z are the coordinates of any point along the line of action of force **F** and F_x, F_y, and F_z are the components of vector **F**.

Example 2.4 Force $\mathbf{F} = 8\mathbf{i} - 3\mathbf{j} + 6\mathbf{z}$ [N] is acting at the point A [1, −3, −2] m. Find the moment of this force about the point B [4, 7, 1] m.

Solution The moment is defined as

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

where **r** is a space vector defining any point along the line of action of force **F**. Let select **r** from the point B to point A. From the definition of the moment, we can select point A as any point along the line of action of the force F. Thus,

$$\mathbf{r} = (1-4)\mathbf{i} + y(-3-7)\mathbf{j} + (-2-1)\mathbf{k} = -3\mathbf{i} - \ 10\mathbf{j} - 3\mathbf{k}$$

and

$$\begin{aligned}\mathbf{M} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \\ &= \left(F_z y - F_y z\right)\mathbf{i} + (F_x z - F_z x)\mathbf{j} + \left(F_y x - F_x y\right)\mathbf{k} \\ &= \left[\left(6\cdot(-10) - (-3)\cdot(-3)\right]i + \left[\left(8\cdot(-3) - 6\cdot(-3)\right)\right]j \\ &\quad + [(-3)\cdot(-3) - 8\cdot(-10)]k \\ &= -69i - 6j + 89k \ \ [\mathrm{N\,m}]\end{aligned}$$

Example 2.5 Force $\mathbf{F} = 5\mathbf{i} + \mathbf{j} + 6\mathbf{z}$ [N] is acting at the point A [2, 3, 2] m. Find the projection of the moment of this force about the axis through the coordinate system origin defined by a unit vector $\lambda = -4\mathbf{i} + 3\mathbf{j} - 2\mathbf{z}$.

Solution Using the rule a vector product, one can find components of the moment vector.

$$\mathbf{M} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = \left(F_z y - F_y z\right)\mathbf{i} + (F_x z - F_z x)\mathbf{j} + \left(F_y x - F_x y\right)\mathbf{k}$$

To find a projection of a moment onto a given line, we can use the (2.9).

$$M_{\mathrm{AB}} = \lambda_{\mathrm{AB}} \cdot (\boldsymbol{r} \times \boldsymbol{F}) = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} -4 & 3 & -2 \\ 2 & 3 & 2 \\ 5 & 1 & 6 \end{vmatrix} = -44.0\ \mathrm{Nm}$$

It is obvious that when the vector, representing moment **M**, is parallel to the axis AB, its projection is equal to the magnitude of the moment **M**. When vector **M** is perpendicular to axis AB, its projection is equal to zero.

2.3 System of Units

To express quantities of matter, force, time, or length, we need to define the corresponding units. Historically different systems of units were developed.

Here, we are going to use: the SI (*Systeme International d' Unites*), which uses mass as a measure of matter, and US Customary Units System based on the weight as a measure of matter. Due to globalization of the economy, it is very important to use a proper system of units. When data are presented in different units, they have to

Table 2.1 System of units

Physical quantity	SI units	US customary units	Conversion factor
Length	m (meter)	in (inch)	1 in = 0.02540 m
		ft (foot)	1 ft = 0.3048 m
		mi (mile)	1 mi = 1609 m
Mass	kg (kilogram)	slug (slug)	1 slug = 14.59 kg
Time	s (second)	s (second)	–
Angle	rad (radian)	rad (radian)	–
Area	m^2 (square meter)	ft^2 (square foot)	1 ft^2 = 0.0929 m^2
		in^2 (square inch)	1 in^2 = 0.6452×10^{-3} m^2
Volume			
Solids	m^3 (cubic meter)	ft^3 (cubic foot)	1 ft^3 = 0.0283 m^3
		in^3 (cubic inch)	1 in^3 = 0.1639×10^{-4} m^3
Liquids	L (Liter)	gal (gallon)	1 gal = 3.785 L
	(1 L = $10^{-3}m^3$)	qt (quart)	1 qt = 0.9464 L
Velocity	m/s	ft/s	1 ft/s = 0.3048 m/s
		in/s	1 in/s = 0.0254 m/s
		mph (mile/hour)	1 mph = 0.4470 m/s
Acceleration	m/s^2	ft/s^2	1 ft/s^2 = 0.3048 m/s^2
		in/s^2	1 in/s^2 = 0.0254 m/s^2
Density	kg/m^3		
Force	N (Newton)	lb (pound)	1 lb = 4.448 N
	1 N = 1 kg m/s^2	oz (ounce)	1 oz = 0.2780 N
Moment	N m	lb ft	1 lb ft = 1.356 N m
Pressure, stress	Pa (Pascal)	lb/ft^2	1 lb/ft^2 = 47.88 Pa
	1 Pa = 1 N/m^2	lb/in^2 (psi)	1 psi = 6895 Pa
Work, Energy	J (Joule)	ft lb	1 lb ft = 1.356 J
	1 J = 1 N m		
Power	W (Watt)	ft lb/s	1 ft lb/s = 1.356 W
	1 W = 1 J/s	hp (horse power)	1 hp = 745.7 W

be converted into a single system. For example, the failure of the space probe sent to Mars in 1999 was due to the mistake of not converting the distance measured in meters to inches, which were used in calculation of the orbit. However, we will put emphasis on using SI system that is common in science and technology.

Example 2.6 Convert the weight of a car W = 4000 lb to appropriate SI unit.

Solution From Table 2.1, 1 lb = 4.448 N, thus W = 4000 lb = 4000 lb · 4.448 N/lb = 17,790 N = 17.79 kN.

Example 2.7 Convert the moment of 56.8 lb ft to SI units.

Solution From Table 2.1, 1 lb = 4.448 N and 1 ft = 0.3048 m, thus 56.8 lb ft = 56.8 lb · 4.448 (N/lb) · 0.3048 (m/ft) = 77.0 N m.

2.4 Numerical Calculations

Calculations of results are usually done by calculator: simple handheld or sophisticated PC. These devices have a capability to calculate the results with many significant digits. However, we have to remember that we are dealing with engineering problems that rely on the measured data. This data are measured with a particular error, usually about 1–5 %, therefore using three significant digits (4—for values starting with 1) provides the sufficient accuracy (higher than 99 %) for majority of engineering applications. Thus, three or four digit numbers usually represent the results.

> When doing computation, we use equal sign even though we are rounding the result to 3–4 significant digits, which is strictly mathematically incorrect; however, it is commonly used in the engineering practice.

Example 2.8 Convert the weight of 45,900 N to appropriate US Customary System of units.

Solution From Table 2.1, the conversion factor is 1 lb = 4.448 N, or 1 N = 1/4.448 lb, thus 45,900 N = 45,900/4.448 lb = 10,320 lb.

Example 2.9 A car is going at a speed of 90 mph (not on the US highway, but in Germany, where there is no speed limit). Convert this speed to the metric units (m/s). Also, show the result in the unit of "km/h."

Solution From Table 2.1, the conversion factor is 1 mph = 0.4470 m/s, thus 90 mph = $90 \cdot 0.4470$ m/s = 40.2 m/s, 40.2 m/s = $40.2 \cdot 10^{-3} \cdot 3600$ = 144.8 km/h.

What We Have Learned?

Basic concepts and definitions
In this chapter, we discussed basic physical quantities: matter, space, time, and force that are accepted intuitively as the *facts of nature*.

Fundamental Laws of Nature, as defined by Sir Isaac Newton
Three Newton's Laws and Axioms describing the effect of forces on a rigid body define the interrelation between above discussed physical quantities.

Procedures to find a resultant force
The rule of finding a resultant force is based on the parallelogram of forces axiom. To find resultant force acting on a rigid body, we have to represent each force through its orthogonal components and calculate the sum of all components.

$$\begin{aligned}\mathbf{R} &= \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} + \ldots + \ \mathbf{F_N} \\ &= (F_{1x} + F_{2x} + F_{3x} + \ldots F_{Nx})\mathbf{i} + \left(F_{1y} + F_{2y} + F_{3y} + \ldots + F_{Ny}\right)\mathbf{j} \\ &\quad + (F_{1z} + F_{2z} + F_{3z} + \ldots F_{Nz})\mathbf{k} \\ &= R_x\mathbf{i} + R_x\mathbf{j} + R_x\mathbf{k}\end{aligned}$$

Mathematical definition of the moment

The moment is a derived physical quantity describing the effect of a force couple on a body. It is defined as a vector product of the direction and the force vectors.

$$\mathbf{M} = \mathbf{r} \ \times \mathbf{F} == \left(F_z y - F_y z\right)\mathbf{i} + (F_x z - F_z x)\mathbf{j} + \left(F_y x - F_x y\right)\mathbf{k}$$

How to calculate the moment of a couple

The moment of a couple is a vector product of vector **r** pointing from any point along the line of action of the first force toward any point along the line of action of the second force.

How to calculate the projection of the moment on the axis

The projection of the moment on the axis defined by its unit vector **λ** can be calculated by using the following expression:

$$M_{\mathrm{AB}} = \boldsymbol{\lambda}_{\mathrm{AB}} \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \lambda_x\left(yF_z - zF_y\right) - \lambda_y(xF_z - zF_x) + \lambda_z\left(xF_y - yF_x\right)$$

System of units

We discussed two system of units used in the United States, Europe, and the rest of the world. It is extremely important to use a consistent system of units.

2.5 Problems

2.1 A 750 lb force is acting at 30° to the horizontal at point A. What are the horizontal and vertical components of this force?

2.2 Force $\mathbf{F}_1 = \ 6\mathbf{i} - \ 3\mathbf{j} + \mathbf{k}$[lb] and force $\mathbf{F}_2 = \ -11\mathbf{i} - \ 8\mathbf{j} - \ 5\mathbf{k}$[lb] are acting on the point C. Determine the resultant force.

2.3 Force $\mathbf{F} = -3\mathbf{i} + 7\mathbf{j} - 4\mathbf{z}$ [N] is acting at the point A [−4, −7, 3] m. Find the moment of this force about the point B [−3, 4, −2] m.

2.4 Force $\mathbf{F} \ = \ -4\mathbf{i} \ + 5\mathbf{j} \ + 11\mathbf{z}$ [N] is acting at the point A [−2, 5, 4] m. Find the projection of the moment of this force about the line through the coordinate system origin defined by a unit vector $\lambda = 4\mathbf{i} \ - 3\mathbf{j} \ + 2\mathbf{z}$.

2.5 Show that when moment **M** is parallel to axis AB, its projection is equal to the magnitude of moment **M**.
2.6 Show that when moment **M** is perpendicular to axis AB, its projection on this axis is equal to zero.
2.7 How many liters of gasoline are in one gallon? What is a price of the liter of gasoline today?
2.8 Convert the mass of 24 slugs to SI units.
2.9 A wooden block has a size of $30 \times 50 \times 70$ mm^3. If the density of the wood is 900 kg/m^3, determine the weight of this block in pounds and Newtons.
2.10 The weight of a person is 180 lb, express his weight in Newtons.
2.11 The pressure measured in a tire was 35 psi. Convert this value to SI units.
2.12 The new Ford has an engine rated at 120 hp, while the new Audi has an engine rated at 95 kW. Which car has more powerful engine?
2.13 Express the power of 250 kW in hp units.
2.14 A scuba tank is under pressure of 3000 psi, express it in SI units.

From Reality to a Free Body Diagram 3

Contents

Beauty of style and harmony and grace and good rhythm depend on Simplicity

Plato

A theory has only the alternatives of being wrong or right.
A model has a third possibility: it may be right but irrelevant

Manfred Eigen

In this chapter you will learn:

- How to single out the structure element of interest from its surrounding
- How to identify the key parameters of a system
- How to build a corresponding physical model
- How to draw a free body diagram of a system

In Chap. 2 we discussed laws of nature as they were observed and postulated by Sir Isaac Newton. Laws of nature were defined on a macroscale using idealization of physical reality. Forces were represented as vectors acting at a point. In this chapter

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we will develop ways to represent reality by correspondent physical models, so that the “laws of nature” can be conveniently applied.

3.1 Mechanical Systems

In everyday life we encounter mechanical systems, which usually consist of a number of structural elements. They have to be properly designed so that they can sustain loads applied upon them during their lifetime. In order to design any particular structural element we have to know magnitudes and locations of the forces acting upon it. These forces comprise loads and supports. To calculate these forces structural element has to be singled out and analyzed. This analysis is the most important part in solving problems in mechanics.

A common step in the required idealization process is an assumption that the observed structural element acts as a rigid body. Structural element may be modeled as a rigid body if the change of its geometry due to forces acting upon it is negligible. Geometry of this element should be simplified so that it will resemble an original structural element and, at the same time, allow the application of equilibrium equations to be discussed in Chaps. 4 and 5. All external loads will be represented by forces and moments, whereas supports will be represented by the corresponding symbols. A rigid body with a simplified geometry, applied loads, and symbolically represented supports is called a *physical model* of an observed structural element. As a matter of fact, drawing a physical model is essentially a step to simplify the structure so that it will belong to a certain group of problems, for which procedure to obtain the solution is known. Without doing this we cannot solve the problem.

> To be a good, experienced engineer means having a good judgment on what can be considered as being negligible.

> Structural element may be modeled as a *rigid body* if the changes of its geometry due to forces acting upon it are negligible.

> Externally applied loads are forces and moments.

There is only one step between the physical model and the corresponding free body diagram (FBD). This step is substitution of the symbols representing supports with the corresponding reactive forces and moments. Sketch of the idealized rigid body with forces and moments, representing all loads and supports, is called a *free body diagram*.

> Supports are commonly represented by symbols.

All loads and supports acting upon a rigid body are called the external forces and moments. The principle of transmissibility, discussed in Sect. 2.1.3, allows moving any force along its line of action. Any moment, applied upon a rigid body, can be positioned at any location since moments are free-floating vectors (see Sect. 2.2.3.3).

Thus, the effect of external forces and moments does not depend upon the rigid body's geometry, but only on their positions and direction of action. You usually see the whole structure, but the goal is to design each element separately. A selected element should be separated from the rest of the structure, while the rest of the structure will represent the loads and supports acting upon it.

> Any moment, applied upon a rigid body, can be positioned at any location since moments are free-floating vectors.

Let us discuss this process using a swing (Fig. 3.1a) as an example. First step in the idealization process is to define the structural element of interest, i.e., to decide which structural element you want to design. The next step is to identify which parts of the remaining system act as supports and which as loads. Based on this information, the FBD of the structural element of interest will be created. Since we are dealing with problems in statics, the corresponding equations of equilibrium will be derived and solved, as it will be discussed in the following chapters.

> Sketch of an idealized rigid body with forces and moments, representing all loads and supports, is called a *free body diagram*.

The process of simplification consists of two steps. In the first step we single out the structural element under consideration and define the supports and the loads. In this process, we simplify geometry of the structural element, replace supports by appropriate symbols, and represent the loads as forces and moments. The result of this process we call the "physical model" of the structural element. In the second step we represent the action of the supports by corresponding forces and moments called "reactions."

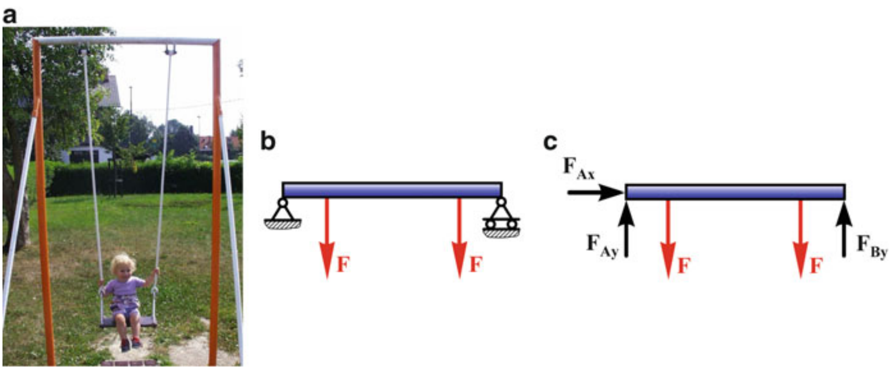


Fig. 3.1 Child on a swing: design of the upper bar. (**a**) Physical reality, (**b**) physical model, (**c**) free body diagram

A rigid body with simplified geometry, applied loads, and symbolically represented supports is called a *physical model* of an observed structural element.

Design of the upper bar. Assume that our task is to design the upper bar, thus it will become a structural element of interest. The two legs in this case act as a support and the child as a load. The corresponding physical model is shown in Fig. 3.1b. Since the front pair of legs does not extend to the top of the swing, we will be able to discuss this as a plane problem. The weight of the child is applied to the upper bar via two vertical ropes; their effect is represented as two point forces (Fig. 3.1.b). Since one can solve for only three unknowns in a plane problem (as will be discussed in Chap. 4) we will simplify the supports and represent them as a pivot (on the left side) and as a roller (on the right side). We need to be aware that this simplification will introduce an error into the solution.

In the next step we will represent the supports by forces and moments (Fig. 3.1c). In our case supports do not introduce any moments; therefore, they must be represented by forces only, as shown in Fig. 3.1c. The procedure of solving for unknown forces will be introduced later in Chap. 4.

It should be emphasized that this process of designing a FBD is one of the most important steps in solving problems in statics.

Design of the seat. If our goal is to design the seat of the swing, it will become a rigid body of interest. The child will be the load and the rest of the system will act as a support (Fig. 3.2a). Therefore, the physical model will consist of the seat supported by the cables (Fig. 3.2b). Next step is to represent the supports by reaction forces and create a free body diagram (Fig. 3.2c).

The example discussed above was a simple two-dimensional case where the loads were represented as forces acting at a point. However, depending on the objects' geometry, it can be modeled as a two-dimensional or three-dimensional

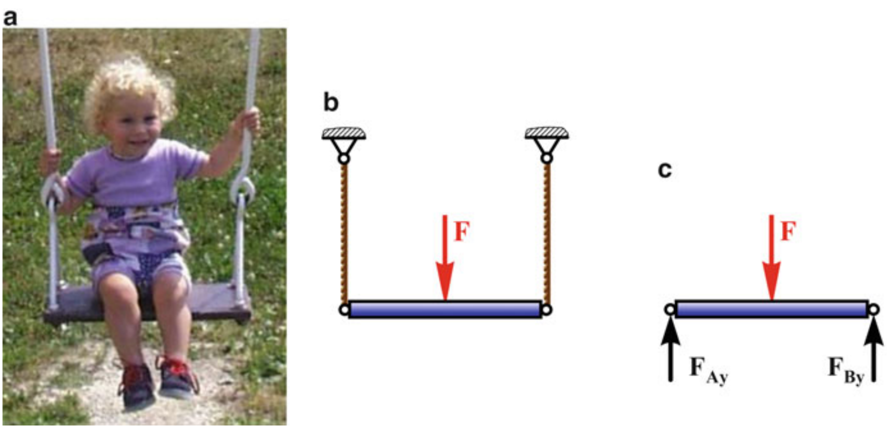


Fig. 3.2 The child on the swing. (**a**) Physical reality, (**b**) physical model, (**c**) free body diagram

problem. In addition, we need to introduce proper representation of forces and supports acting on the structural element of interest. In the above example, the child was represented as a force acting at a point; however, in some cases it would be more appropriate to model it as a distributed load.

> It should be emphasized that the process of designing a FBD is one of the *most* important steps in solving problems in statics.

3.2 Loads

Part of a system acting as a load should be represented (modeled) by one or more forces. Generally, external forces can be divided into concentrated and distributed forces, i.e., forces acting at a point, or along a line or an area. Let us consider several examples of loads. Figure 3.3a shows a system consisting of two persons standing on a bench. If we consider the bench as a structural element of interest, the two persons represent the load. The latter may be modeled as two concentrated forces acting at **A** and **B**, as shown in Fig. 3.3b.

In another case (Fig. 3.4a), a person is lying on a bench. Here we cannot use a concentrated force to represent the effect of a person on the bench, so we will use a distributed load, as shown in Fig. 3.4b. This distributed load may be uniform, as

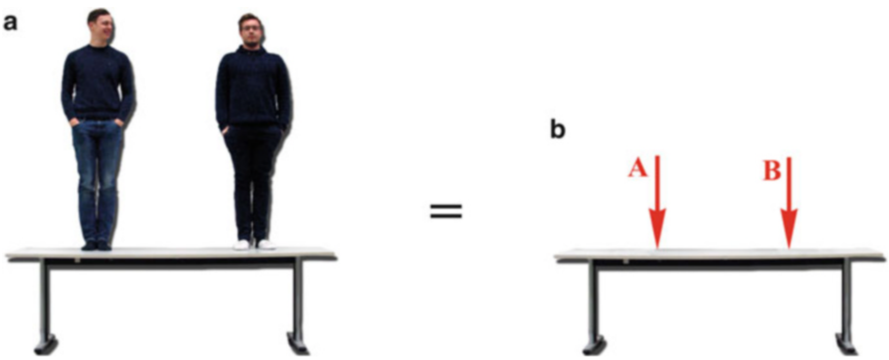


Fig. 3.3 Two boys standing on a bench

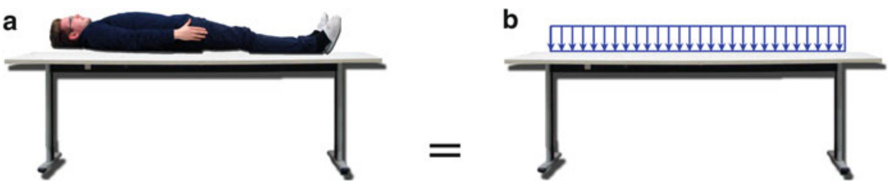


Fig. 3.4 A boy lying on a bench

Fig. 3.5 (**a**) A person riding a bike. (**b**) Forces acting on a bike

shown in Fig. 3.4b, or it may resemble actual distribution of the load if this information is available. Most often we assume that the load is distributed uniformly. As long as we are dealing with equilibrium of rigid bodies, distributed load can be further simplified and replaced by a concentrated force acting at the load's center of gravity. The latter will be discussed in detail in Chap. 6. This simplification is not permissible when we are concerned with the internal forces, which will be discussed in Chap. 9.

From examples shown in Figs. 3.3 and 3.4 we may conclude that decision on how to represent the load (as concentrated or distributed) depends on the size of the contact area between the two bodies and not on the size of the body (observed structure) itself.

This is further demonstrated in the following example. Consider a system consisting of a person on a bike (Fig. 3.5a). Bike, as the structural element of interest, is loaded by the weight of a person. Five concentrated forces represent this load (Fig. 3.5b).

Decision how to represent the load, i.e., as concentrated or distributed, depends on the size of the contact area between the two bodies.

As mentioned above the decision whether to consider the load as concentrated or as distributed is influenced by the size of contact area.

3.3 Supports and Free Body Diagrams

The function of a support is to prevent movements of the observed rigid body. In the orthogonal coordinate system, all possible movements of a rigid body can be represented by three orthogonal components of translation and three orthogonal components of rotation. Thus, all supports can be characterized according to their ability to prevent particular components of the rigid body's motion. For example, to prevent a component of rotation we have to apply the corresponding reactive moment; similarly, to prevent a component of translation we have to apply the related reactive force. Thus, each of the two ropes that support the seat (Fig. 3.2a) prevents only the movement in the vertical direction. This means that the effect of ropes (the support of the seat) can be represented by the symbol for one reactive force in the vertical direction (Fig. 3.2c).

Let us consider now the upper bar of the swing as the structural element of interest (Fig. 3.1a). The two supporting legs prevent horizontal and vertical movement of the bar. Their effect is represented by the symbol (Fig. 3.1b) that allows rotation about its center, but prohibits any other motion in any direction.

These loads that represent the effect of supports on the rest of the rigid body are called *reactions*. The supports can be identified by the number of reactions they impose on a rigid body. Equivalently we may say that supports of a rigid body can be identified by the number of the orthogonal movements and rotations they are preventing.

Reactions are moments and forces that are *preventing* any rigid body motion.

It has to be mentioned that supports, which provide the same motion restriction, i.e., they impose the same reactions on a rigid body, may represent different engineering implementations. For example, Fig. 3.6a shows a caster wheel under a chair. This wheel provides support for the chair and restricts the chair's motion in vertical direction. The corresponding symbol is represented in Fig. 3.6c. The same symbol will also represent completely different engineering structure a bridge support (Fig. 3.6b).

Depending on a structure's geometry and spatial distribution of acting forces and supports, it can be modeled as two- or three-dimensional system, as discussed below.

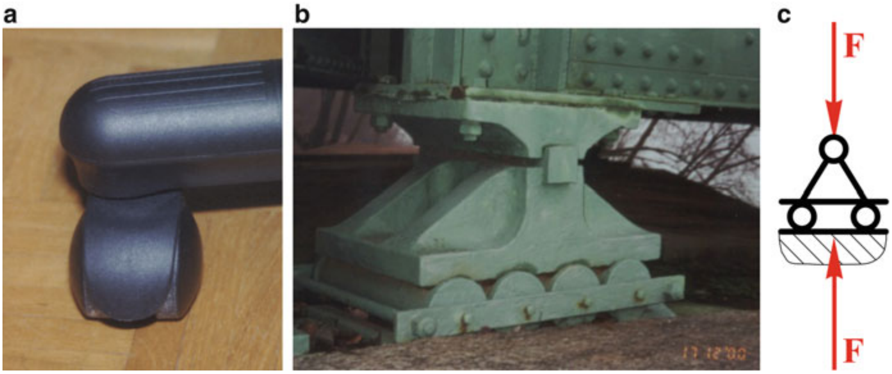


Fig. 3.6 (**a**) Caster wheel under a chair. (**b**) A bridge support. (**c**) A symbol representing caster wheel and a bridge support

3.3.1 Two-Dimensional Systems

All real structures are three-dimensional; however, in many situations some of them may be treated as the two-dimensional problems, especially if they have a plane of symmetry. In this section we will discuss conditions under which some problems may be treated as two-dimensional ones. It is important to utilize engineering judgment in order to simplify the structure.

> All real structures are three-dimensional; however, in many situations some of them may be treated as two-dimensional problems, especially if they have a plane of symmetry.

Consider a jutting roof over a door (Fig. 3.7a). The roof consists of three beams (on the picture one of them cannot be seen), which are built into the wall (commonly called cantilever beams). The beams are also supported by hangers. Assume that the goal is to design the cantilever beam that is loaded by the roof and the snow.

We may treat the beam as a two-dimensional case assuming that the load is uniformly distributed over the whole area of the roof, and that each of the cantilever beams carries one-third of the load, as shown in Fig. 3.7b. Since the roof is built into the wall, its movement in horizontal and vertical directions is restricted, as well as any rotation about the attachment to the wall. The physical model of the roof is shown in Fig. 3.7c. The distributed load q represents 1/3 of the total load acting on the roof normalized by the length of the beam. Force **F** represents the effect of the hanger on the beam. The associated FBD is shown in Fig. 3.7d.

Similar approach can be applied to a person sitting on a chair (Fig. 3.8a). Here the chair is the structural element of interest and the weight of a person is the external load. Since this system has a plane of symmetry, we may consider it as a two-dimensional problem, assuming that two side legs are carrying ½ of the

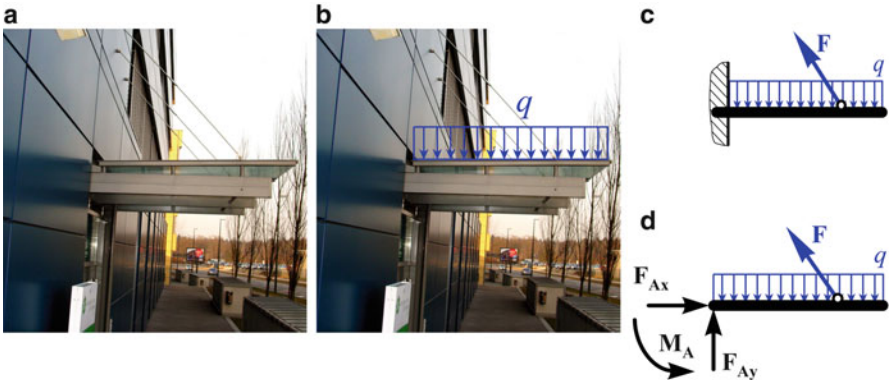


Fig. 3.7 (**a**) A jutting roof over a door. (**b**) Cantilever beam loaded with 1/3 of the roof weight. (**c**) Physical model of the cantilever beam. (**d**) Free body diagram of the cantilever beam

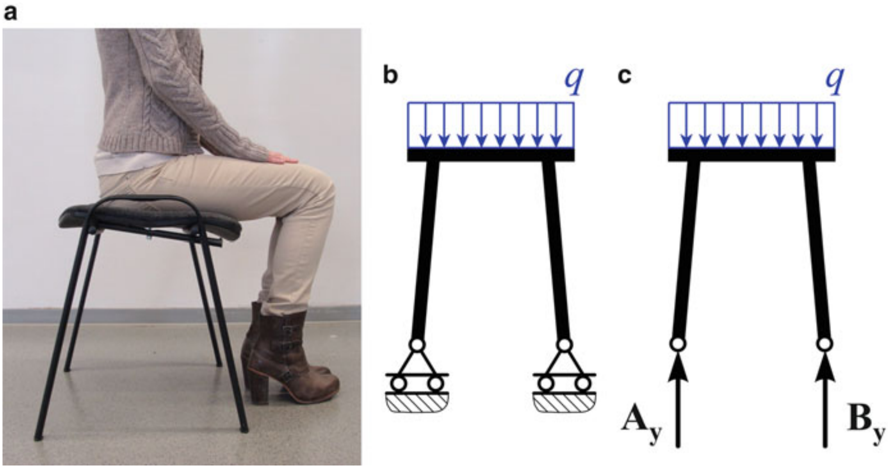


Fig. 3.8 Person sitting on a chair. (**a**) Physical reality, (**b**) physical model, (**c**) free body diagram

person's weight. If we will assume that friction between the floor and the chair legs may be neglected, the physical model can be drawn as shown in Fig. 3.8b. Here q represents half of the body weight divided by the width of the seat. The corresponding FBD is represented in Fig. 3.8c.

Another example of a three-dimensional system is a staircase (Fig. 3.9a), loaded by its own weight (distributed load q) and by the weight of a person (concentrated load **P**). The top and the bottom of this staircase are welded to supporting beams to prevent its tilting. However, if the task is to design a beam supporting the segment of stairs where the man is standing, we may simplify the problem and represent it as a two-dimensional case. To make it solvable in the framework of statics of rigid bodies, we need to make further simplifications, and represent the physical model as shown in Fig. 3.9b. We should be aware that such simplifications bring in errors and make analytical predictions less accurate.

However, defining the simplest physical model that sufficiently well represents the structure of our interest is one of the key engineering challenges, which follows the famous Einstein's quote "make things as simple as possible, but not simpler." The free body diagram derived from the simplified physical model has three unknown forces as shown in Fig. 3.9c.

> Defining the simplest physical model that sufficiently well represents the structure of our interest is one of the key engineering challenges, which follows the famous Einstein's quote: "Make things as simple as possible, but not simpler."

Different symbols are used to represent supports when dealing with various structures. However, in 2D problems there are essentially only four possible combinations of reactive forces and moments, which are summarized in the Table 3.1. We are going to discuss some of them here.

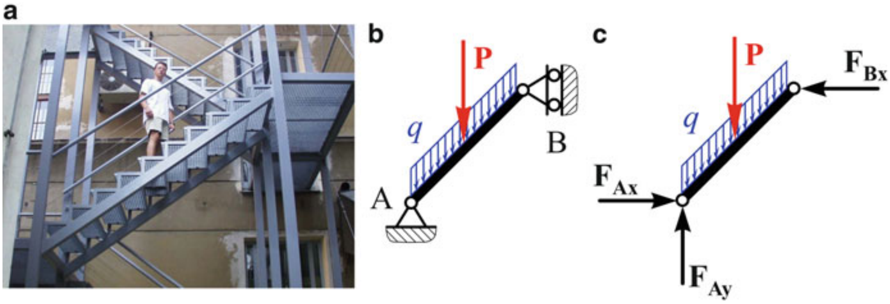


Fig. 3.9 A staircase loaded by its own weight and by the weight of a person (concentrated load P). (**a**) Physical reality, (**b**) physical model, (**c**) free body diagram

Let us start with a bridge support shown in Fig. 3.10a. The first step is to define a coordinate system, e.g., as shown in Fig. 3.10a. This support prevents any movement in a vertical direction, while it allows a free movement in a horizontal direction. Usually such supports are represented by a symbol shown in Fig. 3.10b, which corresponds to the case 1 in the Table 3.1. The effect of this support on a body (i.e., structure of our interest) is represented by the unknown reaction force F_y (Fig. 3.10c).

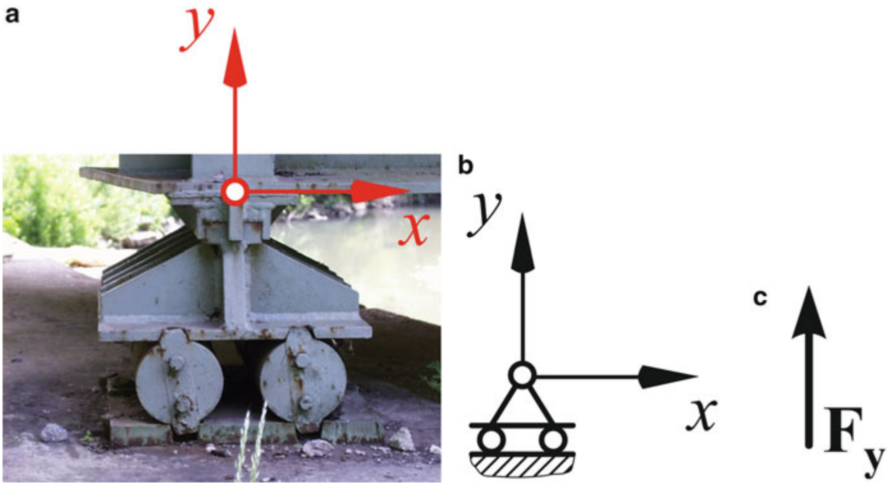
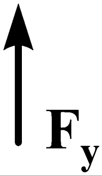
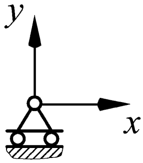
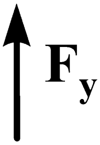
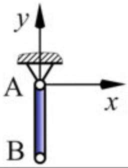
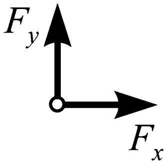
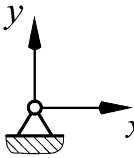
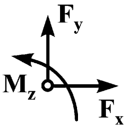
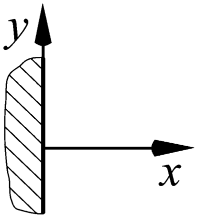
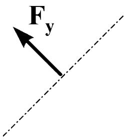
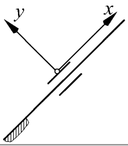


Fig. 3.10 (**a**) Bridge support. (**b**) Symbol. (**c**) Reaction

Supports are distinguished by the number of degrees of freedom that they restrict. Therefore, completely different structure may be represented by the same symbol, as it is shown in Fig. 3.11a. Here we model the chair wheel as a two-dimensional structure. It is possible to do so if there are no external forces

Table 3.1 Reactions and associated symbols

Case	Reactions	Symbol
1		
2		
3		
4		
5		

acting in the direction of wheel's axis. Such an external force would generate the reaction and thus prevent us to treat this problem as a two-dimensional. Thus, if such a force is absent, the only reaction will be vertical force F_y, as shown in Fig. 3.11c. Its symbol and reactions are the same as in the previous example (Fig. 3.10a), where the real structure is completely different.

Let us further consider a pulley suspended from a hook (Fig. 3.12a). The hook allows a free rotation of the pulley. The only reaction which pulley generates is the force acting along the pulley. This type of support may be represented by a symbol as shown in Fig. 3.12b. The reaction is the force acting in the direction of the pulley (Fig. 3.12c). This support is represented by the case 2 in Table 3.1.

In the next example, a pin that is used to support a cardan joint (Fig. 3.13a) is usually considered to be a frictionless. It is represented by a symbol shown in

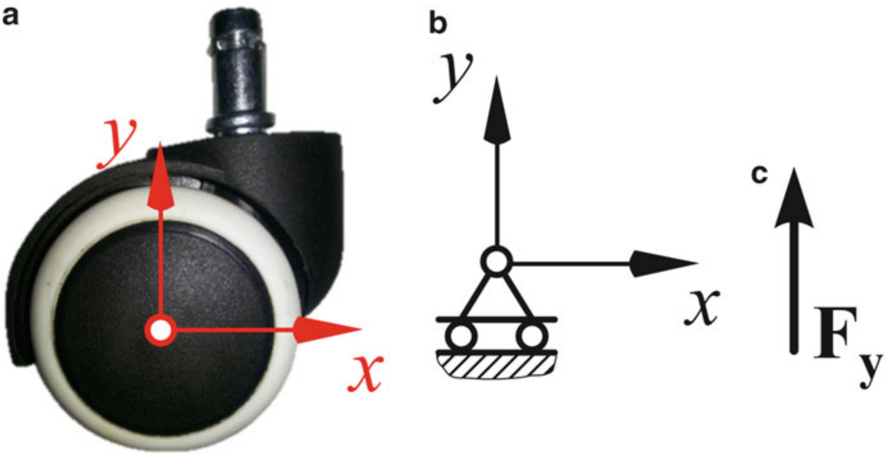


Fig. 3.11 (**a**) Chair wheel. (**b**) Symbol. (**c**) Reaction

Fig. 3.13b. Such a pin prevents motion in any direction; therefore, we need to show the two components of the unknown reaction force (Fig. 3.13c). This support is represented by the case 3 in Table 3.1.

A built into the wall beam, similar to the one shown in Fig. 3.14a, is a three-dimensional structure. However, in many cases, we may simplify it and treat it as a two-dimensional case. Such a support is called a fixed one and is represented by the symbol shown in Fig. 3.14b. This support prevents motion and rotation in any direction; therefore, we have to show two force components and the moment as reactions (Fig. 3.14c). This support is represented by the case 4 in Table 3.1.

It should be mentioned that not all supports have a generally accepted symbols, e.g., the collar on a frictionless rod (case 5) does not have a commonly accepted symbol. A possible symbol is shown in Table 3.1 as case 5. Collar on frictionless rod or any other sliding element along the straight line will generate reaction force that is always perpendicular to the line of sliding.

3.3.2 Three-Dimensional Systems

Many engineering systems cannot be modeled as the two-dimensional cases. The necessity for a three-dimensional consideration can be either due to geometry of a structural element or due to loads imposed upon it. One of such examples is a structure supporting a basketball board with a ring attached to it (Fig. 3.15a). The structure is loaded by the weight of the board and the ring (and sometimes a basketball player who is hanging on it), which can be represented as a single concentrated force. The structural element of interest is a three-dimensional object

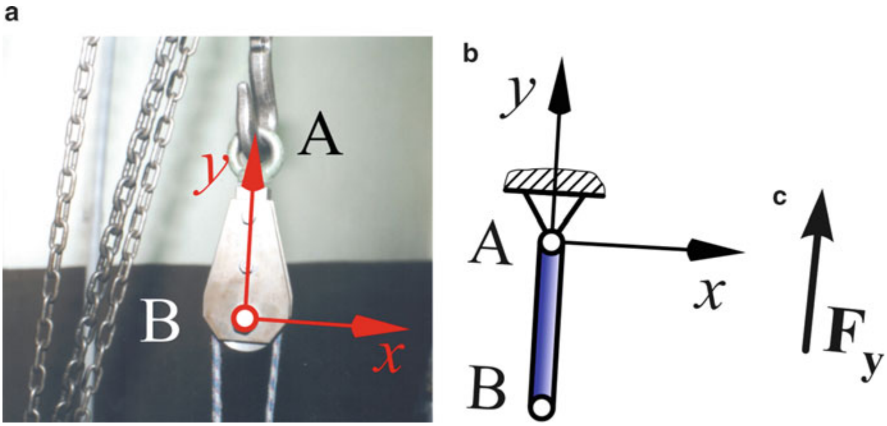


Fig. 3.12 (**a**) Hook. (**b**) Symbol. (**c**) Reaction

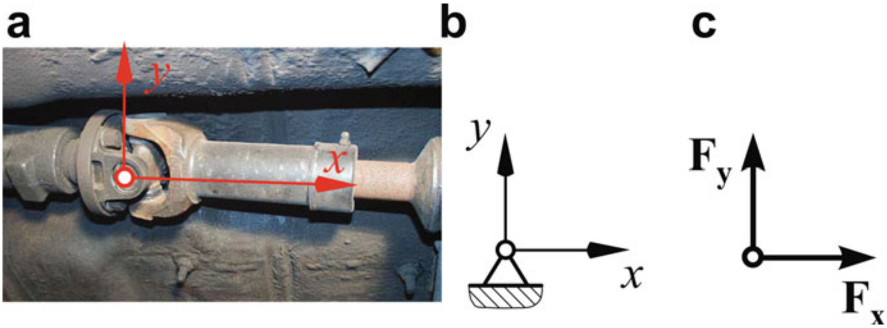


Fig. 3.13 (**a**) Frictionless pin. (**b**) Symbol. (**c**) Reaction

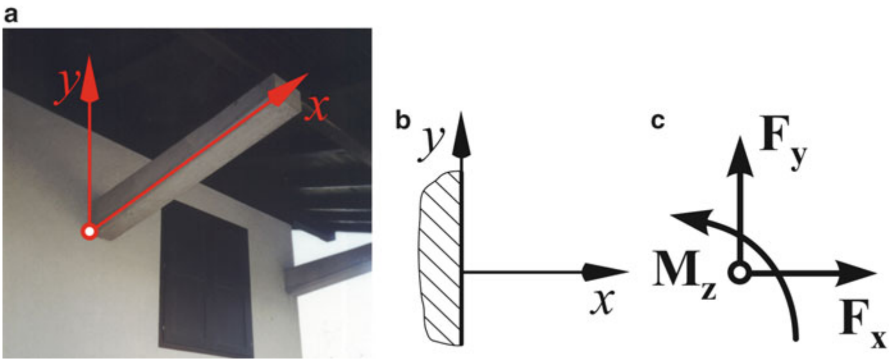


Fig. 3.14 (**a**) Fixed support. (**b**) Symbol. (**c**) Reaction

supported by three legs. The reaction forces acting at the leg-to-ground contact and the load do not belong to one plane. Thus, this system cannot be treated as a 2D case. In the process of generating a physical model we have assumed that the two

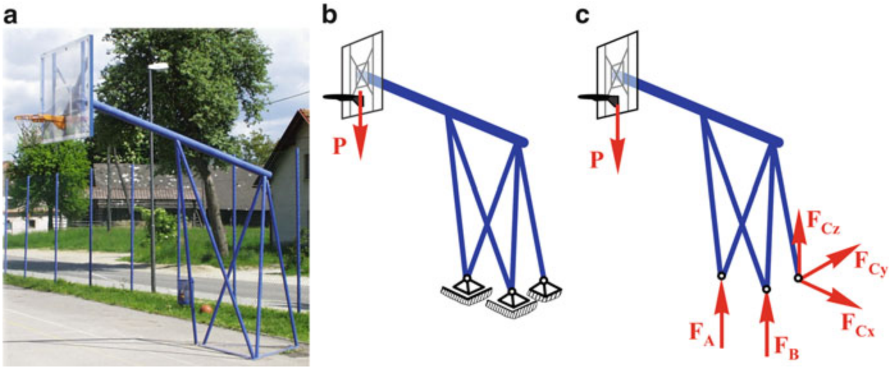


Fig. 3.15 (**a**) Reality. (**b**) Physical model. (**c**) Free body diagram

front legs are supported by supports that prevent motion in vertical direction only (these are 3D analog to the 2D case 1 support), whereas the rear leg has the support that prevents all three translations (this is a 3D analog to 2D case 3 support). This physical model is shown in Fig. 3.15b and the corresponding FBD in Fig. 3.15c. Hence, the assumptions we have made lead to five unknown reactions, which means that one degree of freedom is not restricted. In this case this would be a rotation around the axis that is perpendicular to the ground.

> In the process of simplification from 3D to 2D we always introduce an error, which in some cases may be too large to be neglected.

This shortcoming may be solved by replacing one of the front supports with the one that would additionally restrict one of the horizontal translations. However, in reality it is not easy to construct such a support. This example nicely demonstrates that assumed physical models could never represent the reality in full; our analytical predictions are therefore as good as the assumptions.

It should be mentioned that one need to make a further simplification in order to solve this problem. For example, we may further assume that the supporting structure consists of a beam and a truss (to be discussed in Chaps. 8 and 9).

> Physical models can never represent the reality in full; our analytical predictions are therefore only as good as the assumptions.

The example below shows direction signs (Fig. 3.16a). In this case the supporting structure holding the signs in place could be modeled as a 2D case. However, the external load on the structure, which comprises $\mathbf{W}_2$ (weight of the horizontal bar), $\mathbf{W}_3$ (weight of the vertical bar), the signs total weight $\mathbf{W}_1$ all acting in a vertical direction, and the effect of the wind $\mathbf{F}_w$, blowing in a horizontal direction. Thus, this system cannot be represented as a 2D case. The corresponding 3D physical model and the FBD are shown in Fig. 3.16b, c.

From the above examples, we have learned that some structures cannot be modeled as two-dimensional systems due to their geometry and/or loads acting upon them. For example, the sign structure as such (Fig. 3.16a) could be modeled as a two-dimensional structure if we would neglect the effect of wind.

In Fig. 3.15b we have used symbols for supports that represent 3D analogs for 2D case-1 and case-3 supports. However, in general there are no widely accepted symbols to represent a variety of different 3D supports; therefore, we will show only pictures of some real structures and the associated reactions.

Let us consider a beam that is built into the wall (Fig. 3.17a). We have modeled this beam as a 2D problem by assuming that there are no external loads in z direction (see Fig. 3.14a). However, such a beam may sustain load in any direction, thus we will treat it as a three-dimensional structure this time. The built in support prevents the beam from any rotation and translation, thus we need to deal with three unknown force components and three unknown moment components, total of six reaction components (Fig. 3.17b).

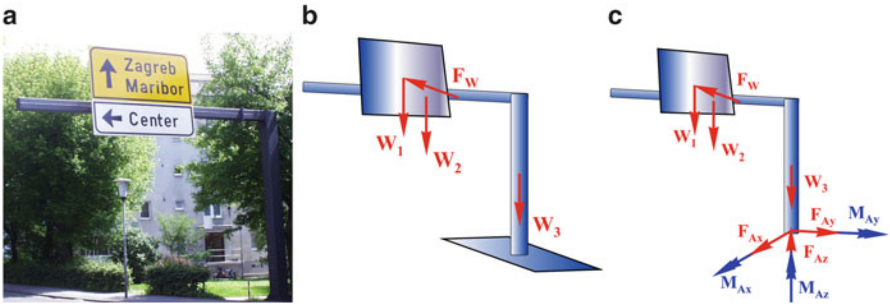


Fig. 3.16 (**a**) Road signs. (**b**) Physical model. (**c**) Free body diagram

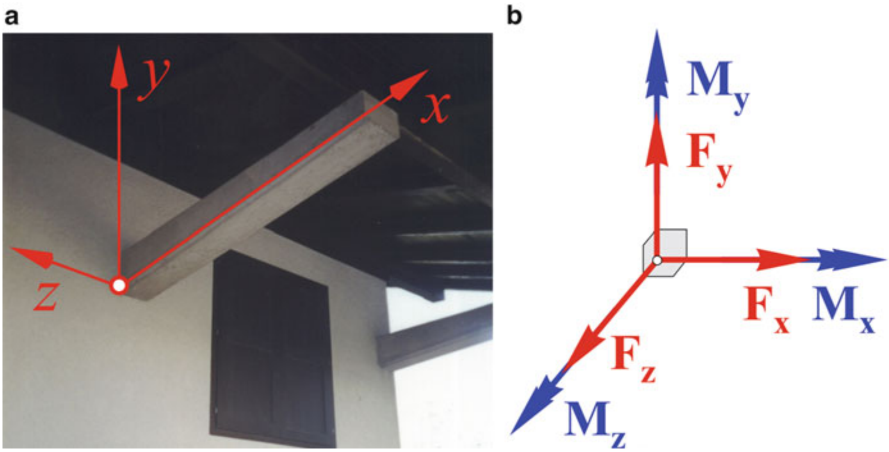


Fig. 3.17 (**a**) Fixed support. (**b**) Reactions

A universal joint, also known as universal coupling, U-joint, Cardan joint, Hardy-Spicer joint, or Hooke's joint, is commonly used in a car transmission (Fig. 3.18a). It resists rotation along its axis, but it is free to bend in directions perpendicular to the axis. The reactions are three force components and one moment component (Fig. 3.18b).

A door hinge (Fig. 3.19a), one of the most widely used supports, prevents three translations and two rotations. The reactions are shown in Fig. 3.19b.

A ball joint is used in many engineering, as well as in some bioengineering structures. The example shown in Fig. 3.20a represents a roentgenogram of an artificial hip joint. Its role is to allow for a free rotation in any direction, but to prevent any translation. The reaction components, which are in this case three forces, are shown in Fig. 3.20b.

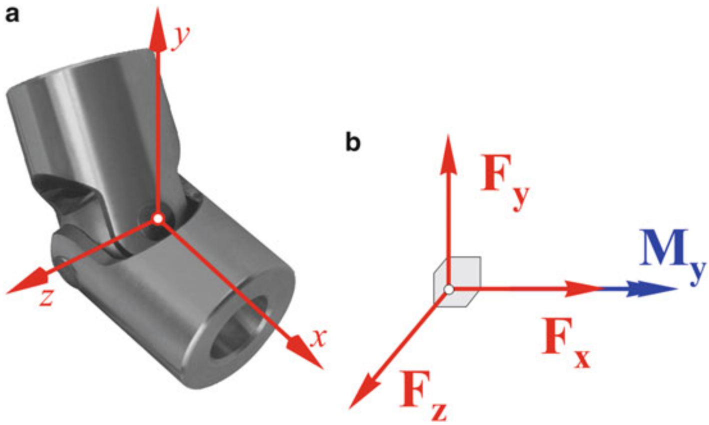


Fig. 3.18 (**a**) Universal joint. (**b**) Reactions

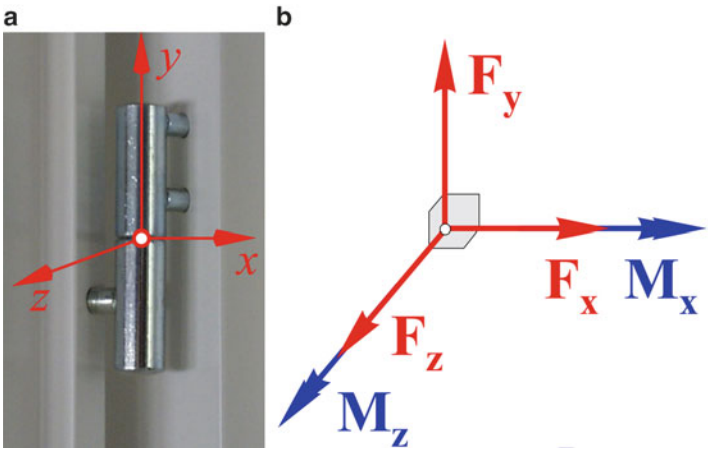


Fig. 3.19 (**a**) Hinge. (**b**) Reactions

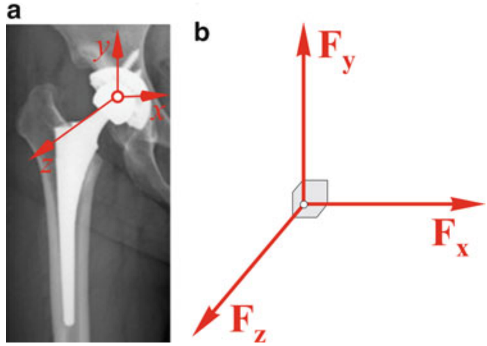


Fig. 3.20 (**a**) Ball joint. (**b**) Reactions

We have shown several examples of most common types of 3D supports. Each of the shown force-moment combinations (reactions) may have different engineering implementations. Each particular case should be studied and analyzed.

Guidelines and Recipes for Creating a Free Body Diagram

- Define an element of interest, i.e., choose the structural element you want to design.
- Identify which parts of the remaining system act as supports and which as loads.
- Represent the structural element of interest by a rigid body with a simplified geometry.
- Substitute parts of structure that act as loads, and external loads, such as wind, by the appropriate forces and moments.
- Create the physical model consisting of rigid body, supports, and loads.
- Substitute supports by associated reaction forces and moments.
- Draw the corresponding free body diagram.

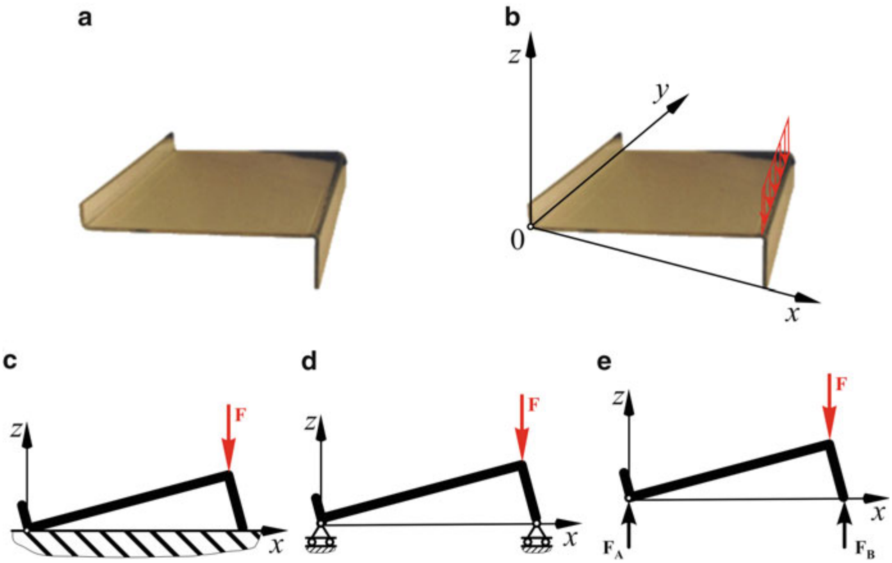


Fig. 3.21 (**a**) Picture of the stand. (**b**) Selection of coordinate system. (**c**) Reduction to a 2D case. (**d**) Physical model. (**e**) Free body diagram

Example 3.1 It is required to design a calendar stand (Fig. 3.21a) that will resist the weight of a person accidentally seating on it. The task is to draw a proper physical model and associated FBD.

Solution One of the important steps in the design process is to define the external loads and their distributions. The first step is to select the coordinate system (Fig. 3.21b). Let us assume that the load is uniformly distributed along the *y*-axis as shown with red arrows in Fig. 3.21b. Here it is assumed that the weight of a person is distributed along the edge of the calendar stand, which is the most severe case. It may appear that we oversimplified the loading situation; however, this leads to a simple 2D case, which is easily solved. Figure 3.21c shows a 2D sketch of the calendar stand. The distributed load applied by a person now becomes a concentrated force acting at the edge, which is the most severe case leading to a safer design of the stand. Now, we can show a physical model of the stand (Fig. 3.21d), assuming that there is no friction between the stand and the surface. The associated FBD is shown in Fig. 3.21e.

Any assumption we make in the process of simplification should lead to a *safer design* (overdesign), i.e., resulting structure will be stronger than it is required.

Example 3.2 It is required to design a canopy support (Fig. 3.22a). The task is to draw a proper physical model and associated FBD.

Solution The roof and possible snow (not shown on the photo) represent external load on the structure of interest. We will assume that four horizontal bars supported by four vertical poles carry this load. Due to symmetry of the system we will further assume that one-half of the load is carried by two vertical poles and the connecting horizontal bar. This assumption will lead to overdesigned horizontal bar. However, by doing this we can analyze the structure as a 2D problem. Each vertical pole is built into the ground. Such a support prevents all possible movements. However to be able to solve this problem within the framework of this course, we have to make further simplifications and use supports as shown in Fig. 3.22b. This assumption will lead to overdesign of the vertical poles. Corresponding FBD is presented in Fig. 3.22c.

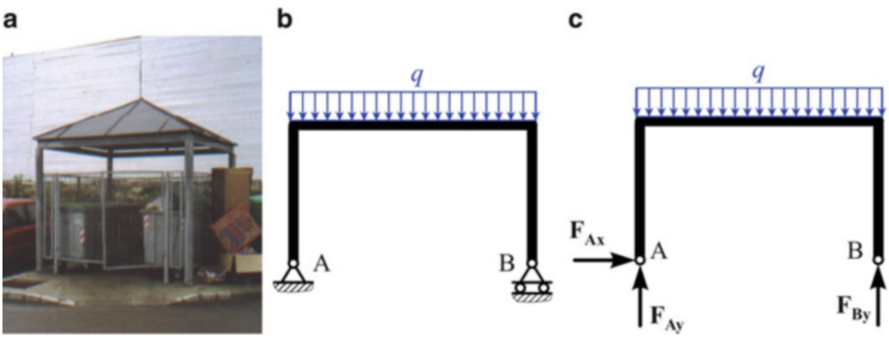


Fig. 3.22 (**a**) Canopy. (**b**) Physical model. (**c**) Free body diagram

Example 3.3 Draw a physical model and FBD of a coat hanger (Fig. 3.23a).

Solution A coat hanger is loaded by the weight of pants and a jacket. We assume that the weight of the pants and the jacket are uniformly distributed, and that the hanger is supported by a hinge on its top. The physical model is shown in Fig. 3.23b and the corresponding free body diagram in Fig. 3.23c.

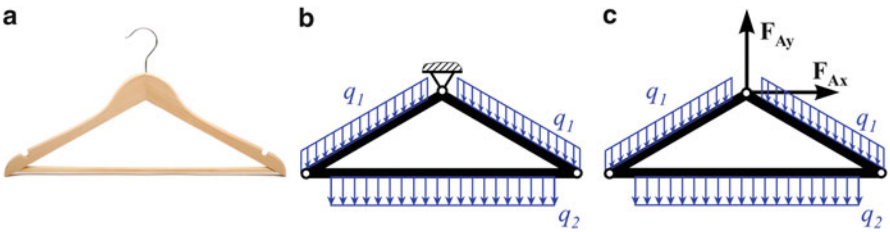


Fig. 3.23 (**a**) Hanger. (**b**) Physical model. (**c**) Free body diagram

What We Have Learned?

How to extract the system of interest from its surroundings
A structural element of interest should be represented by a rigid body of appropriately simplified geometry. It means that it should resemble the original geometry in its key features and at the same time it should provide the possibility to model its geometry easily.

How to represent the key parameters of a system
The key parameters of systems are loads and supports. Identify which parts of the system act as supports and which as loads. The observed structural element could be additionally loaded by the wind, snow, etc. Loads are represented either as distributed or concentrated forces, and moments. Supports are represented by reactions (forces and moments) preventing motion of the structural element of interest. Forces prevent the structural element's translations and moments its rotations.

How to create a corresponding physical model
A physical model consists of a rigid body with simplified geometry, loads, and properly selected supports.

How to create a free body diagram
A FBD is created from a physical model by replacing its supports by the corresponding reactions.

3.4 Problems

3.1 You have to analyze a structure supporting two traffic lights and a traffic sign. Draw a physical model and FBD of the structural system shown in Fig. P3.1. Assume that the weights of the supporting structure, of the traffic lights, and of the sign are known. Further assume that the size of the traffic light is small enough that the wind has a negligible effect.

Fig. P3.1 Traffic light

How would you address the problem if the effect of wind may not be neglected?

3.2 The task is to analyze a structure supporting five highway signs (Fig. P3.2). Draw a physical model and FBD of the structure, assuming that the wind is blowing in the direction of 30° to the highway signs.

Fig. P3.2 Highway signs

3.3 A canopy in Fig. P3.3 consists of number of semi-circular ribs designed to support the weight of the roof and occasional snow. Draw the physical model and FBD of a rib.

Fig. P3.3 Canopy

3.4 A cloth rack in a dormitory is loaded by a number of cloth items as seen in Fig. P3.4. Draw the physical model and FBD of the rack.

Fig. P3.4 Closet

3.5 A roof in Fig. P3.5 is supported by a rectangular structure consisting of two supporting pillars and a horizontal beam. Draw a physical model and FBD of the beam.

Fig. P3.5 House entry

3.6 Draw a physical model and FBD required for the analysis of the supporting pillars at the house entry shown in Fig. P3.5.

3.7 Draw a physical model and FBD required for the analysis of the beams supporting the wine barrels (Fig. P3.6).

Fig. P3.6 Wine barrels

3.8 Draw a physical model and FBD required for the analysis of the lamp pole shown in Fig. P3.7.

Fig. P3.7 Lamp pole

3.9 Draw a physical model and FBD of the bus-stop cover shown in Fig. P3.8.

Fig. P3.8 Bus stop cover

3.10 Draw a physical model and FBD of a C clamp (Fig. P3.9).

Fig. P3.9 C clamp

3.11 Draw a physical model and FBD of a door handle (Fig. P3.10).

Fig. P3.10 Door handle

3.12 Draw a physical model and FBD for one of the supporting arms of the car jack, shown in Fig. P3.11.

Fig. P3.11 Car jack

3.13 For the car jack, shown in Fig. P3.11, draw a physical model and FBD of the screw.

3.14 Draw a physical model and FBD of the left vertical column supporting the bridge (Fig. P3.12).

Fig. P3.12 Bridge

3.15 A terrace cover (Fig. P3.13) is supported by a number of semi-curricular ribs. Draw a physical model and FBD of a rib.

Fig. P3.13 Terrace cover

3.16 Draw a physical model and FBD of the frame of the swing (Fig. P3.14).

Fig. P3.14 Swing set

3.17 Draw a physical model and FBD of the horizontal beam of the bridge shown in Fig. P3.12.

3.18 A person is pulling a rope attached to the middle of a bar fixed between two pillars (Fig. P3.15). Draw a physical model and FBD of the bar. Neglect the weight of the bar.

Fig. P3.15 Fixed bar

3.19 Solve problem 3.18 taking into account the weight of the bar.

3.20 Draw a physical model and FBD of the front loader's arm (Fig. P3.16).

Fig. P3.16 Front loader

3.21 A bike brake consists of two pieces AOB and COD (Fig. P3.17). Draw a physical model and associated FBD for structural element AOB.

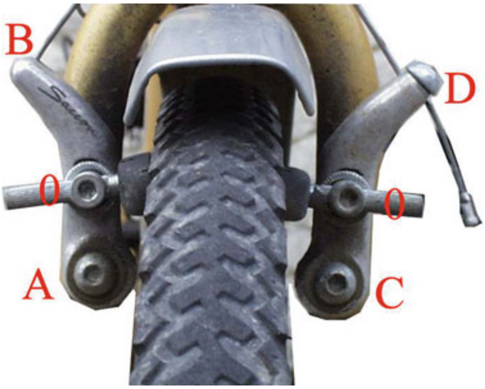


Fig. P3.17 The brake

3.22 Draw a physical model and associated FBD of the earth mover arm shown in Fig. P3.18.

Fig. P3.18 Earth mover

3.23 Draw a physical model and associated FBD of the cable supporting a street lamp (Fig. P3.19). Assume that the weight of the cable is negligible compared to the lamp.

Fig. P3.19 Street lamp

3.24 Draw physical model and associated FBD of a basketball stand (Fig. P3.20).

Fig. P3.20 Basketball stand

Resultant and Equilibrium of Forces Acting at a Point

4

Contents

Experience is the name that everyone gives to their mistakes.

Oscar Wilde

In this chapter you will learn:

- How to find a resultant force
- How to analyze an equilibrium of forces acting at a point in plane and in space

This chapter deals with forces acting at a point. This, however, does not imply that the rigid body has to be small. Sometimes *lines of action* of all forces applied to a body intersect at the same point. Thus, the effect of these forces may be studied as if they would act at a point. Such forces are called *concurrent*. Any two nonparallel in-plane forces acting on a rigid body will always intersect in a point. When two forces are parallel and are opposite in direction, they are called a *couple of forces* (Sect. 2.2.2) and they form a moment that tends to rotate the body. The effect of moments on rigid bodies will be discussed in Chap. 5. When a moment is small, in engineering practice we often neglect its effect and consider forces as if they are acting at a point.

I. Emri, A. Voloshin, *Statics*, DOI 10.1007/978-1-4939-2101-0_4

In this chapter we will discuss the procedures to find the *resultant* of any number of concurrent forces. Resultant is a force that has the same effect on the body as the original set of forces acting at that point. First we will consider in-plane forces, and then generalize the approach to three-dimensional cases.

4.1 Resultant and Equilibrium of In-Plane Forces

There are often situations where several forces act on a small area. In these cases we may represent a rigid body as a point, and consider all acting forces as being concurrent. Such an example is an electric pole with a number of wires attached to it (Fig. 4.1a). We want to find the resulting force acting on the pole due to the forces imposed by the attached wires.

We further assume that the forces imposed by wires are concurrent and belong to the same plane (Fig. 4.1b). In order to use vector algebra, we need to introduce a coordinate system, as shown in Fig. 4.2.

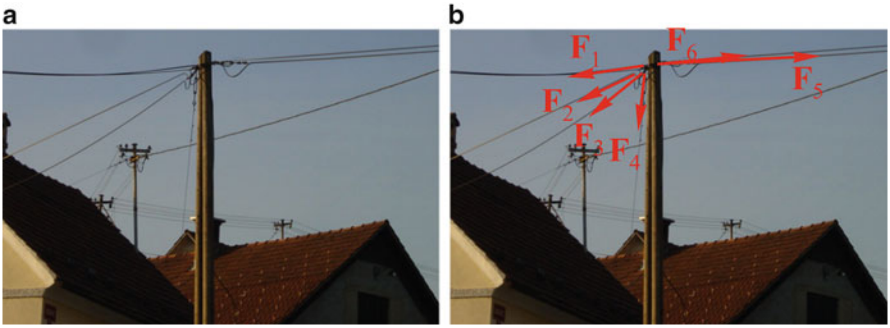
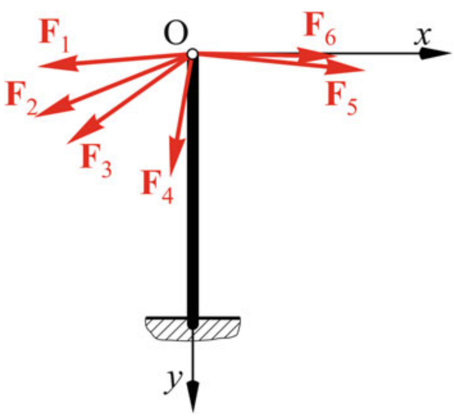


Fig. 4.1 Electrical pole with wires. (**a**) Photograph, (**b**) superimposed force vectors acting on the pole

Fig. 4.2 Forces acting in the wires



The process of making assumptions is an essential aspect of any engineering solution, since the results will be only as good as the assumptions made.

4.1.1 Resultant of Forces

Using the principle ("axiom of parallelogram of forces," explained in Sect. 2.1.3) that any two forces can be replaced by their resultant force, we can find the resultant for any number of concurrent forces (i.e., acting at the same point).

Let us consider an example of three forces acting at point O (Fig. 4.3a). Starting with any two forces, say $\mathbf{F}_1$ and $\mathbf{F}_2$, we can replace them by resultant $\mathbf{R}_{12}$, as shown in Fig. 4.3a. Obtained resultant $\mathbf{R}_{12}$ and another force, say $\mathbf{F}_3$, can be further replaced by $\mathbf{R}_{123}$. By using this procedure we can obtain the resultant of any number of concurrent forces.

The resultant may be found by repeated application of the parallelogram of force axiom.

On basis of the above procedure one can develop a polygon rule for the addition of forces. All forces have to be arranged in a "tip to tail" manner. We start with a force; next we move the second force parallel to itself such that the "tail" of this force meets the "tip" of the previous force. We follow this procedure for all forces. The resultant force connects the "tail" of the first force with "tip" of the last force, as shown in Fig. 4.3b. The sketch showing all forces for which we are trying to find a resultant is called "force diagram."

Resultant is a force that has the same effect on a body as the original set of forces acting at that point.

The same result will be obtained by using vector algebra (see Appendix). In an orthogonal Cartesian coordinate system, each force may be represented as the sum of its two orthogonal components. For example, any force **F** (Fig. 4.4) can be

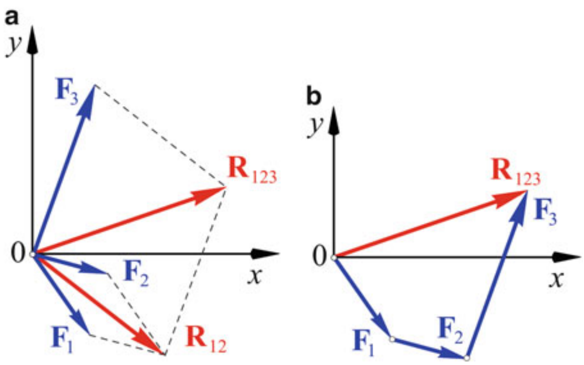


Fig. 4.3 Resultant of three forces acting on a point

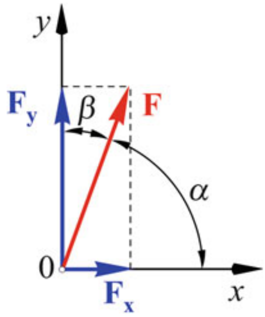


Fig. 4.4 Orthogonal components of a force

represented as the sum of two forces $\mathbf{F}_x$ and $\mathbf{F}_y$, acting in the directions of x- and y-axis, respectively,

$$\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y = F_x\mathbf{i} + F_y\mathbf{j} \tag{4.1a}$$

where

$$F_x = F \cdot \cos\alpha$$

$$F_y = F \cdot \cos\beta$$

Similarly, we can write the following equations for each force in Fig. 4.3b,

$$\mathbf{F}_1 = \mathbf{F}_{1x} + \mathbf{F}_{1y} = F_{1x}\mathbf{i} + F_{1y}\mathbf{j} \tag{4.1b}$$

$$\mathbf{F}_2 = \mathbf{F}_{2x} + \mathbf{F}_{2y} = F_{2x}\mathbf{i} + F_{2y}\mathbf{j} \tag{4.1c}$$

$$\mathbf{F}_3 = \mathbf{F}_{3x} + \mathbf{F}_{3y} = F_{3x}\mathbf{i} + F_{3y}\mathbf{j} \tag{4.1d}$$

The resulting force **R** is obtained as

$$\mathbf{R} = \sum_{n=1}^{n=3}\mathbf{F}_n = \left(\sum_{n=1}^{n=3}F_{nx}\right)\mathbf{i} + \left(\sum_{n=1}^{n=3}F_{ny}\right)\mathbf{j} \tag{4.2}$$

For arbitrary number of forces, say N, the above equation becomes,

$$\mathbf{R} = \sum_{n=1}^{n=N}\mathbf{F}_n = \left(\sum_{n=1}^{n=N}F_{nx}\right)\mathbf{i} + \left(\sum_{n=1}^{n=N}F_{ny}\right)\mathbf{j} = R_x\mathbf{i} + R_y\mathbf{j} \tag{4.3}$$

The resultant force may be also represented as the product of the force magnitude and the unit vector that defines the direction of the force action.

$$\mathbf{R} = R \cdot \mathbf{e} = R \cdot \left(e_x\mathbf{i} + e_y\mathbf{j}\right)$$

where the magnitude R can be calculated as

$$R = \sqrt{R_x^2 + R_y^2}$$

and its direction as

$$e_x = \frac{R_x}{R} \text{ and } e_y = \frac{R_y}{R}$$

Here e_x and e_y are cosines between the direction of the force and coordinate axes (Appendix). They are the components of a unit vector in 2D.

> The force diagram represents the forces for which we want to find a resultant force.

4.1.2 Equilibrium of Forces

When a resultant force is equal to zero all the concurrent forces are in equilibrium, thus the rigid body is in equilibrium. In such cases, according to the Newton's First Law (Chap. 2), the status of the body's motion will remain unchanged; i.e., it either moves with a constant velocity or has a zero velocity, depending on its status at the time of observation.

> Parameters defining the status of a body at the moment we start the observation are commonly called *initial conditions*.

This condition is mathematically expressed as $\mathbf{R} = 0$, or in scalar form

$$R_x = \sum_{n=1}^{n=N} F_{nx} = 0 \tag{4.4}$$

and

$$R_y = \sum_{n=1}^{n=N} F_{ny} = 0 \tag{4.5}$$

The equations (4.4) and (4.5) are called equilibrium equations. Since there are only two equations of equilibrium in 2D, we can solve them for only two unknown values. Those may be any variables describing the state of rigid body equilibrium.

Each force can be described in two ways: either as magnitude F and direction α (defining the unit vector **e**) or as its two orthogonal components F_x and F_y. According to the rules of vector algebra (Appendix) the relations between these four parameters representing a force are (Fig. 4.4):

$$F_x = F \cos \alpha \tag{4.6a}$$

$$F_y = F \cos \beta = F \sin \alpha \tag{4.6b}$$

$$F = \sqrt{F_x^2 + F_y^2} \tag{4.6c}$$

$$\tan \alpha = \frac{F_y}{F_x} \tag{4.6d}$$

It should be noted that (4.6c) contains a square root, i.e., we cannot solve it for, let say, F_x, even if F and F_y are known, since the result will have an uncertainty in sign, which defines the direction of the force action.

For finding a resultant, one should use formulae (4.4) and (4.5).

When an equation contains a square root the result has an uncertainty in the sign.

There are four different possibilities to represent a force acting in a plane as shown in Table 4.1.

Table 4.1 Four possibilities to define a force in 2D

1	2	3	4
F	F_x	F_x	F_y
α	F_y	α	α

When dealing with a problem that involves a large number of forces acting at a point, it may be convenient to organize the forces in a table form and utilize the convenience of available spreadsheets.

For the demonstration we will consider again the problem of an electric pole. Its physical model is illustrated in Fig. 4.2. Let us use the information provided in Table 4.2. Force $\mathbf{F}_1$ has two known orthogonal components, force $\mathbf{F}_2$ is given by the magnitude and the direction, force $\mathbf{F}_3$ is represented by its horizontal component and the direction of the force, while $\mathbf{F}_4$ is defined via its vertical component and the direction of the force action.

Thus, for each force one has to provide any two pieces of information, as shown in Table 4.2. In the next step we have to calculate the missing values for force components F_x and F_y, since they are needed for the calculation of the resultant. This can be done by using (4.6). The result of this process is summarized in Table 4.3.

Table 4.2 What is known about the forces acting on the pole

Force ID	F_x (N)	F_y (N)	Alpha (deg)	Magnitude (N)
1	−350	6		
2			150	300
3	100		140	
4		250	95	
5	450			500
6		6		400

After calculating the components of the resultant, it is also possible to calculate the magnitude and its direction, using (4.6). The results are $R_x = 391$ N and $R_y = 749$ N. When calculating unknown component using (4.6c) we have an uncertainty in the sign of the component. However, from the observation of the physical problem it is clear that all wires are in tension, thus we selected the positive sign.

Table 4.3 Calculation of the resultant force

Force	F_x (N)	F_y (N)	Alpha (deg)	Magnitude (N)
1	−350	6		
2	−260	150	150	300
3	−100	119.2	140	
4	251	250	95	
5	450	218		500
6	400	6		400
Resultant	391	749		

Example 4.1 A lamp is suspended by two cables above the street, as shown in Fig. 4.5a. The tension of the left cable was measured to be 724 N and in the right one 737 N. Determine the resultant of these two forces.

Solution

Step 1. Draw a physical model.

Drawing of the physical model requires the following:

(a) Knowledge of the geometry

(b) Assumption that the cables are straight

The corresponding physical model is derived from the image of the actual street lamp indicating geometry and assumptions (Fig. 4.5b). The measured distances are: $h_1 = 10$ m, $h_2 = 9$ m, $h_3 = 12$ m, $a = 13$ m, and $b = 15$ m.

Step 2. Draw a force diagram.

Utilizing the above information, we draw a force diagram of the two forces of interest (Fig. 4.6). The weight of the lamp is not shown in this diagram since it is not a part of the current problem.

Angles α and β are calculated as

$$\tan\alpha = \frac{h_3 - h_2}{b} = 0.2; \quad \text{i.e., } \alpha = 11.3^\circ$$

$$\tan(180 - \beta) = \frac{h_1 - h_2}{a} = 0.0769; \quad \text{i.e. } \beta = 175.6^\circ$$

It should be noted that we did not use a free body diagram in this case because we were not dealing with an equilibrium problem, but rather were

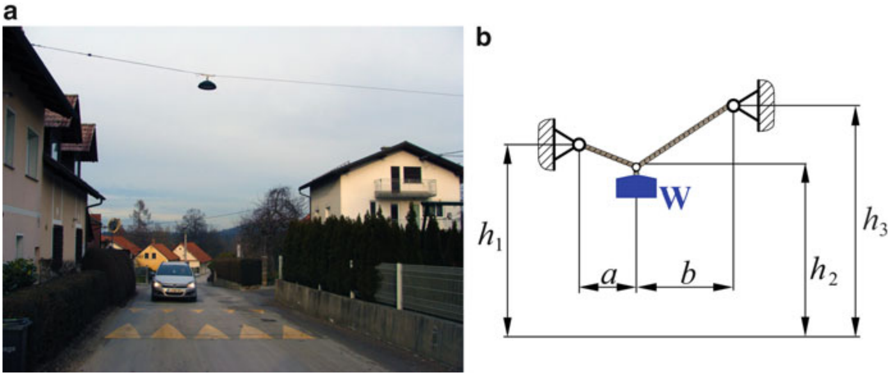


Fig. 4.5 (**a**) Street lamp. (**b**) Physical model of the lamp

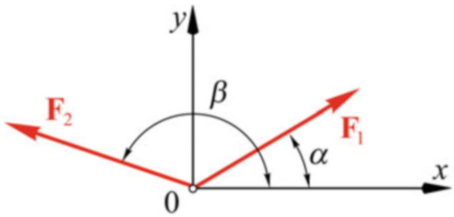


Fig. 4.6 Force diagram of the street lamp

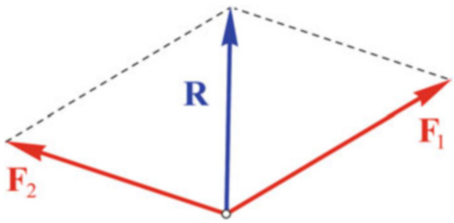


Fig. 4.7 Force resultant

looking for the resultant of two forces. This required the construction of a force diagram only.

Step 3. Find the resultant.

The resultant force may be calculated by using one of the following approaches.

(a) Graphical Solution

Figure 4.6 should be redrawn to scale in order to reflect the correct geometry. The solution is achieved by application of the rule of force parallelogram. Figure 4.7 shows the result.

The graphical solution provides a fast and simple determination of the magnitude and direction of the resultant (even if it is drawn by a freehand). In addition, such approach helps to develop an engineering intuition. However, the accuracy of obtained results is typically low. From Fig. 4.7 we obtained the magnitude of resultant to be 200 N and we determined that it acts in the vertical direction.

(b) Numerical Solution—"Hand" calculation
Since we have only two forces to deal with, one may directly use (4.6) to calculate the force components as

$$F_{1x} = F_1 \cos \alpha = 723\,\text{N}$$

$$F_{2x} = F_2 \cos \beta = -722\,\text{N}$$

$$F_{1y} = F_1 \sin \alpha = 144.4\,\text{N}$$

$$F_{2y} = F_2 \sin \beta = 55.5\,\text{N}$$

Summation of the components above provides the resultant force components:

$$R_x = F_{1x} + F_{2x} = 0.847\,\text{N}$$

$$R_y = F_{1y} + F_{2y} = 199.9\,\text{N}$$

If our assumptions and measurements were accurate, the resulting component R_x would be equal to zero because forces $\mathbf{F}_1$ and $\mathbf{F}_2$ were generated by the weight of the lamp, which acts in the vertical direction. However, due to inevitable errors in measurements and assumptions, the calculated resultant has an inherent error, about 0.4 % of the resultant. Such an error is acceptable in the majority of engineering calculations.

The result may be also represented as a magnitude and direction:

$$R = \sqrt{R_x^2 + R_y^2} = 200\,\text{N}$$

$$\tan \alpha = \frac{R_y}{R_x} = 236; \quad \text{i.e. } \alpha = 89.8^\circ$$

The angle should be 90° since the resultant force (weight of the lamp) acts in the vertical direction. The small discrepancy comes from the errors in the measured distances and forces.

(c) Use of MATLAB functions
Use the program "resultantPoint2D" to calculate the resultant for the system of forces shown in Fig. 4.6. The procedure is as follows.

Start the "MATLAB," select the program "resultantPoint2D" and run it. The first dialog box will ask to:

Enter Number of forces acting on a particle—*press OK to continue—enter* **2** and *click OK*

Since the magnitude and direction of each force is given, select
Magnitude and direction
Since force magnitude is given, select
Force magnitude and direction
The following dialogs will ask to enter the magnitude and direction for each of the two forces, enter the magnitude in Newtons and the angle in degrees from the *x*-axis. Thus, for first force and its directions you will input

737 11.3 and *click OK*

for the second

724 175.6 and *click OK*

The result will be calculated to be: $R_x = 0.847$ N and $R_y = 200$ N which corresponds to the numbers you got in the previous section. Obviously, it is much easier to use the developed software, than to calculate the values by using a calculator. To get the correct result you need only to draw the Force Diagram and input the proper values into the program.

> When solving an engineering problem, one always starts with several assumptions to create a model and uses measured quantities, assuming that they are true values. The input data and the results based on this model will always have an inherent error.

Example 4.2 Let us consider the problem shown in Fig. 4.5. Assume the weight of the lamp is 200 N and the task is to determine the forces acting in each cable.

Solution Using the same assumptions we arrive to the same physical model as shown in Fig. 4.5b.
In this problem, we are dealing with the equilibrium of three forces acting at point O. The force diagram, which now became a free body diagram of point O, is displayed in Fig. 4.8.

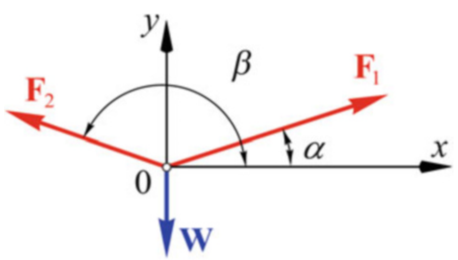


Fig. 4.8 Free body diagram

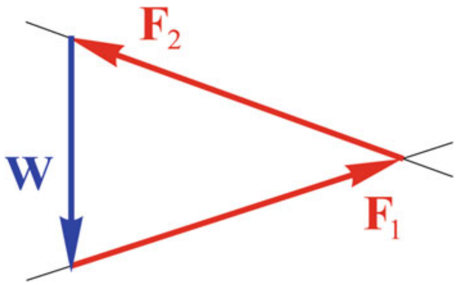


Fig. 4.9 Polygon of forces

(a) Graphical Solution
Figure 4.8 should be redrawn to scale in order to reflect the correct geometry. Solution is achieved by application of the force polygon rule. The three forces need to be in equilibrium; therefore, the polygon of forces should be closed, i.e., the resultant force is equal to zero. We start with known force **W**. From the "tip" of force **W** we draw a line parallel to the direction of force $\mathbf{F}_1$, and from the "tail" of force **W**—the line parallel to the direction of force $\mathbf{F}_2$, as shown in Fig. 4.9. Magnitudes of F_1 and F_2 can be measured from the drawing.

(b) Numerical Solution
Since the system is in equilibrium, the sum of all forces acting on point O should be equal to zero. Mathematically, this is expressed as

$$\mathbf{R} = 0, \quad \text{or in scalar form}$$

$$R_x = \sum_{n=1}^{n=3} F_{nx} = 0 \tag{4.7a}$$

and

$$R_y = \sum_{n=1}^{n=3} F_{ny} = 0 \tag{4.7b}$$

In this case

$$F_{1x} = F_1 \cos \alpha$$
$$F_{1y} = F_1 \sin \alpha$$
$$F_{2x} = F_2 \cos \beta$$
$$F_{2y} = F_2 \sin \beta$$
$$W_x = 0$$
$$W_y = -W$$

Substituting the above expressions into equations of equilibrium (4.7a) and (4.7b) yields two scalar equations with two unknowns F_1 and F_2

$$F_1 \cos \alpha + F_2 \cos \beta + 0 = 0$$

$$F_1 \sin \alpha + F_2 \sin \beta - W = 0$$

The solution is $F_1 = 737$ N and $F_2 = 725$ N. Those results are slightly different from the values used in Example 4.1. The reason is that in this case we started the problem with the measured weight of the lamp and the assumption that it is acting in the vertical direction only, while in the previous case we started with the measured forces in the two cables. Such a mismatch is an inherent part of the engineering work. Solving a problem with different assumptions leads to a slightly different result.

(c) Use of MATLAB functions

The easiest way to solve for an unknown weight is to use the MATLAB routine "equilibriumPoint2D." Start the MATLAB and run the "equilibriumPoint2D." The dialog explaining how to input the data will appear. Enter the number of forces acting on the point, 3 in our case. Next, you will have to enter the data for each force. You will be asked to provide the magnitude of the force and values for the projections on the x and y directions. Since this information is needed to define the force's direction, you have to provide the length of each component. The input is shown in the table below:

Force number	1	2	3
Force magnitude (N)	X	X	200
X component length (m)	−13	15	0
Y component length (m)	1	3	−1

The routine will calculate the unknown values and the result will be presented as a free body diagram on the sketch and numerically in the MATLAB "Command Window." The first unknown is 724 N, and the second 737 N. These values are close to the ones calculated by using graphics or calculator approach.

What is the difference between a force diagram and a free body diagram?

Free body diagram shows ALL forces acting on a body which may contribute to its mobility. In the case of statics, these forces should be in equilibrium, i.e., the resultant should be equal to zero.

Force diagrams are representations of the SELECTED forces acting on a body. If one selects to show all forces, the force diagram becomes a free body diagram.

Guidelines and Recipes for Solving 2D Problems When All Forces Are Concurrent

- Draw a physical model of the problem.
  Drawing a physical model means representing the simplified geometry of a structural element under consideration, supports, and all forces acting on a body in a coordinate system.
- Replace the structural element by a point.
- Represent the effect of the supports by the reactive forces.
- Create a free body diagram (FBD).
- Write equations of equilibrium.
- Find unknown values.

You may use one of these approaches:

Graphical approach requires drawing a FBD to the scale and use of the basic principle of the *force parallelogram*, as discussed above (Fig. 4.3).

Numerical approach requires representation of forces as a vector components, and writing and solving of the corresponding equilibrium equations. This procedure may be easily programmed. The sample **MATLAB** routines may be downloaded from the http://extras.springer.com.

4.1.3 Problems

4.1 A car, weight $P = 15$ kN is parked on a slope of 10°. Calculate the parallel and normal to the slope components of the force acting between the car and ground. Consider car as a point.

Fig. P4.1

4.2 Force vector **F** forms a 30° angle with the *y*-axis. Determine its components in the coordinate system $x' - y'$.

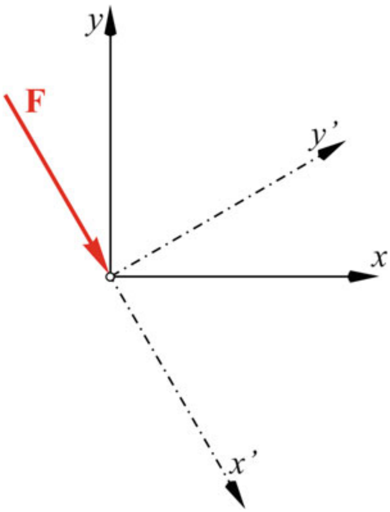


Fig. P4.2

4.3 Determine the magnitude and direction of the resultant of the forces shown. The force of 200 lb makes an angle of 45° and the force of 300 lb makes an angle of 60° with the vertical axis.

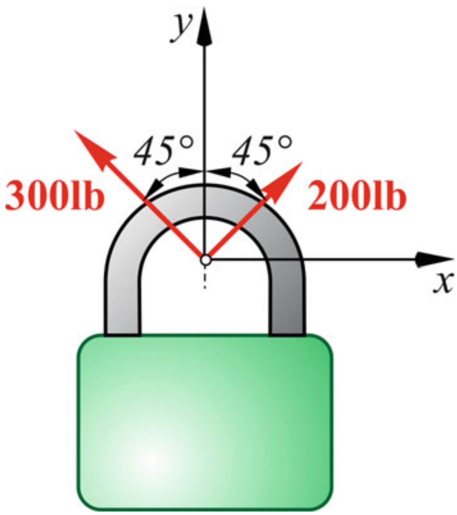


Fig. P4.3

4.4 Two forces are acting on point C: A = 12**i** + 32**j** and B = 4**i** − 21**j**. Determine the resultant of these forces.

4.5 Determine the resultant of the three forces shown.

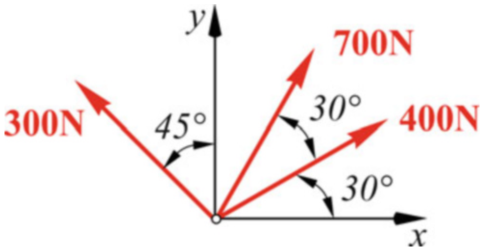


Fig. P4.5

4.6 Two water skiers are being pulled by a boat. The tension in rope A is 200 lb and in rope B is 300 lb. Determine the resultant of these two forces.

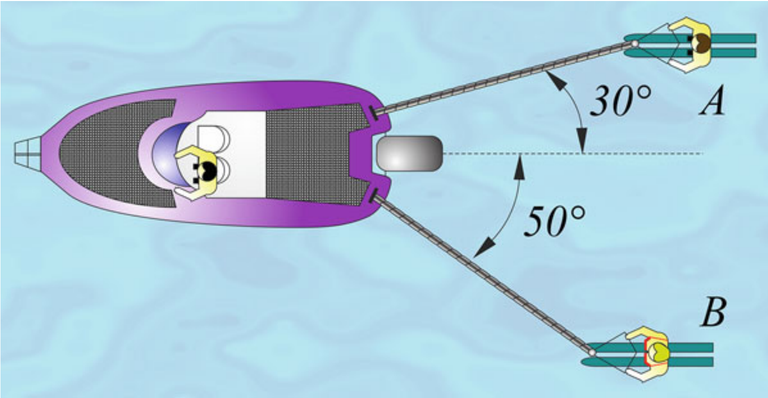


Fig. P4.6

4.7 Find the resultant of the three forces shown.

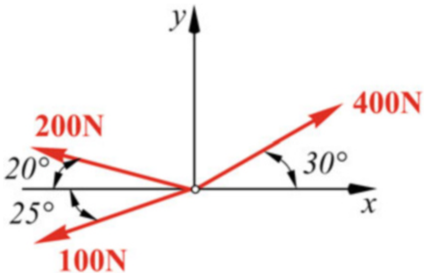


Fig. P4.7

4.8 Find the resultant of force 200 and 400 N (Fig. P4.7).

4.9 Find the resultant of force 100 and 400 N (Fig. P4.7).

4.10 Hip is loaded by body weight of 90 lb (1/2 body weight while standing). Calculate the component of this force in the directions along the femoral neck. Direction of the femoral neck is 60° to the vertical direction.

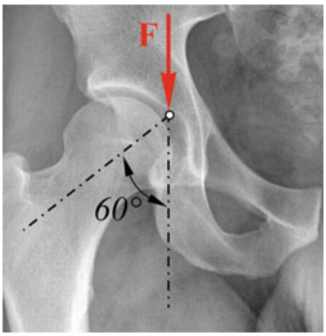


Fig. P4.10

4.11 The boat tows three water skiers. Skier A makes an angle of 30° with the direction of the boat ride, skier B makes an angle of 5°, while skier C makes a negative angle of 15°. Assume that the magnitude of a force that is acting on each skier is 200 lb. Calculate the magnitude of the resultant force and its angle relative to the direction of the boat ride.

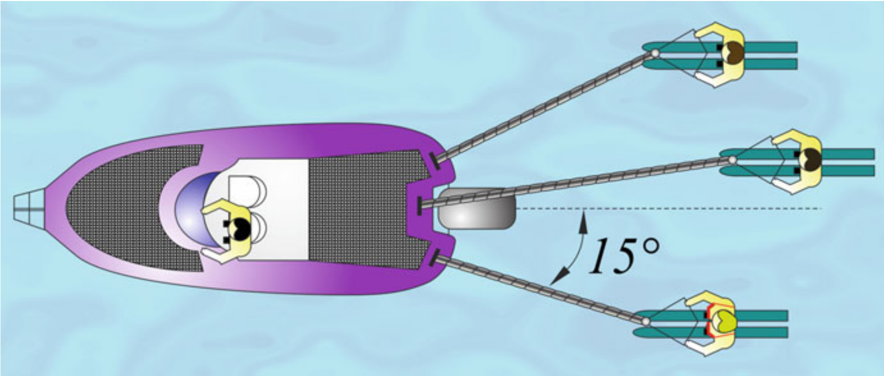


Fig. P4.11

4.12 Calculate the resultant of the three forces acting at a point.

$$\mathbf{F}_1 = -12\mathbf{i} + 5\mathbf{j}, \quad \mathbf{F}_2 = -2\mathbf{i} + 7\mathbf{j} \quad \text{and} \quad \mathbf{F}_3 = 4\mathbf{i} + 8\mathbf{j}$$

4.13 200-N force P is acting on a frame. Calculate the components of P along struts AC and BC for the case $\alpha = 30°$ and $\beta = 45°$.

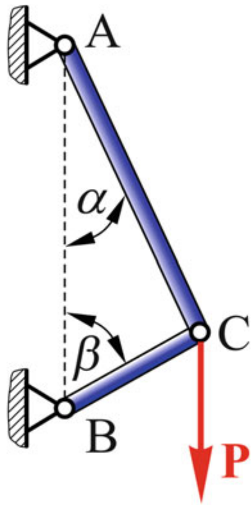


Fig. P4.13

4.14 10-N force Q is acting on a frame. Calculate the components of Q along struts AB and BC for the case of $\alpha = 30°$ and $\beta = 45°$.

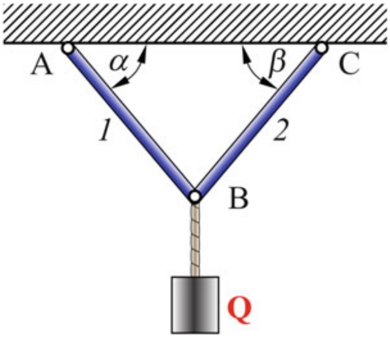


Fig. P4.14

4.15 60-lb force Q is acting on a frame. Calculate the components of Q along struts KN and KM for the case of $\alpha = 30°$ and $\beta = 60°$.

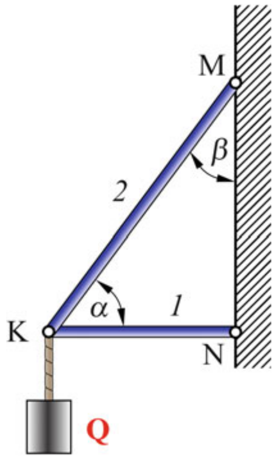


Fig. P4.15

4.16 Force $F = 80\mathbf{i} + 120\mathbf{j}$ N is applied to a joint of two links. Resolve this force into two components, one along the upper link and one along the lower link. Assume that the upper link makes an angle of 45° and lower link makes an angle of 30° with the horizontal axis.

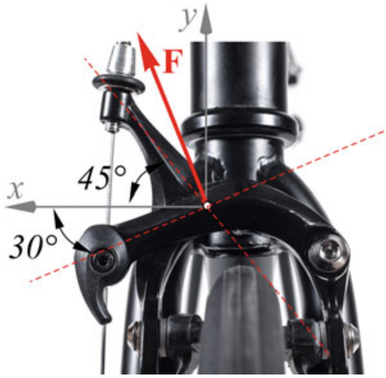


Fig. P4.16

4.17 Consider the part of a structure that is loaded by force $Q = 200$ N. Assuming an angle $\alpha = 45°$, compute the horizontal component of force Q.

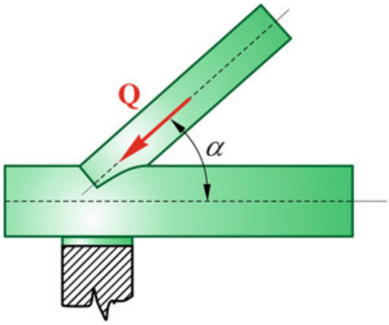


Fig. P4.17

4.18 Three springs are attached to the central point and loaded by the forces as shown. The system is in the state of equilibrium. What are the values of angles α and β?

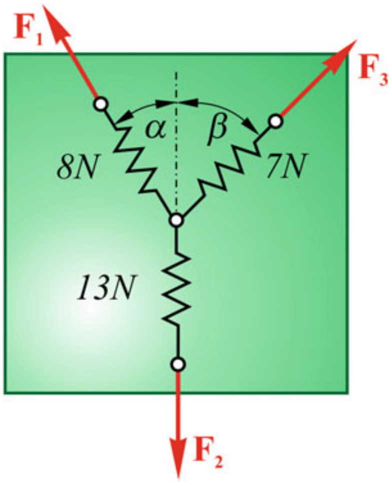


Fig. P4.18

4.19 Determine forces acting in each bar using the graphical approach (Fig. P4.19). Assume $Q = 1000$ N, $\alpha = 30°$, and $\beta = 60°$.

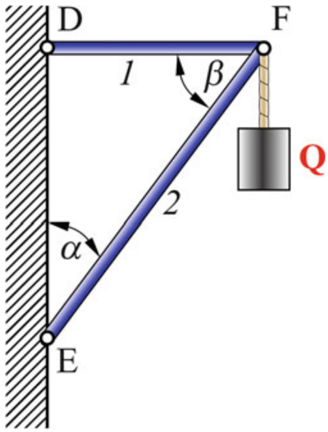


Fig. P4.19

4.20 Determine forces in each bar (Fig. P4.19) when $Q = 1000$ N, $\alpha = 45°$, and $\beta = 45°$.

4.21 Determine forces in each bar when $Q = 1000$ N, $\alpha = 30°$, and $\beta = 60°$ (Fig. P.4.19). Use equations of equilibrium.

4.22 The weight $\mathbf{M}_1$ is supported by a rod and is held in the position shown by weight $\mathbf{M}_2$. Determine weight $\mathbf{M}_2$ and the tension in the rod.

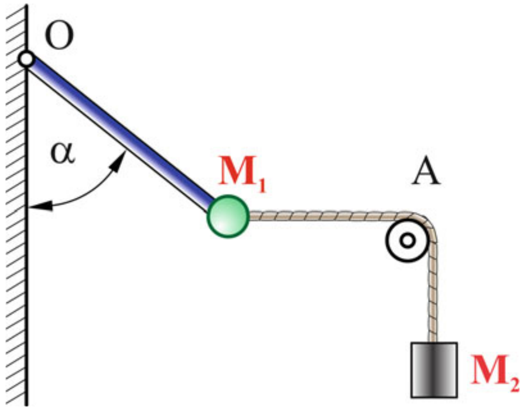


Fig. P4.22

4.23 Determine forces acting in rods KM and KN. Use $Q = 1000$ N, $\alpha = 60°$, and $\beta = 30°$.

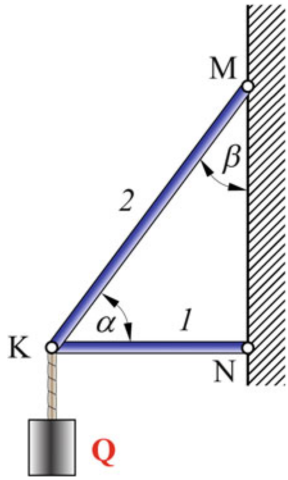


Fig. P4.23

4.24 A street light is suspended by two cables. Find the tension in each cable, if weight **P** of the light is 50 N, length of cable AB = BC = 6 m, and distance BD = 0.1 m.

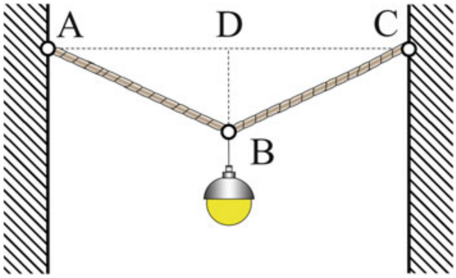


Fig. P4.24

4.25 A streetlight (weight 300 N) is supported by two bars. AC = 1.2 m and BC = 1.5 m. Find the forces acting in each bar.

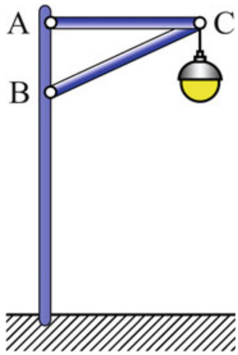


Fig. P4.25

4.26 A lamp (weight $W = 20$ N) is suspended by two cables as shown. The angle $\alpha = 60°$ and the angle $\beta = 135°$. Determine the tension in each cable.

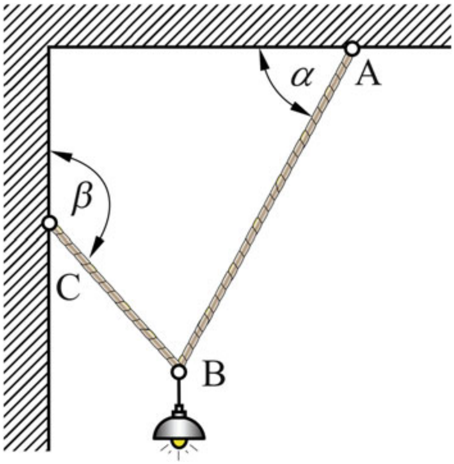


Fig. P4.26

4.27 Two water skiers are being pulled by a boat with a force of 450 lb. Determine the tension in each rope.

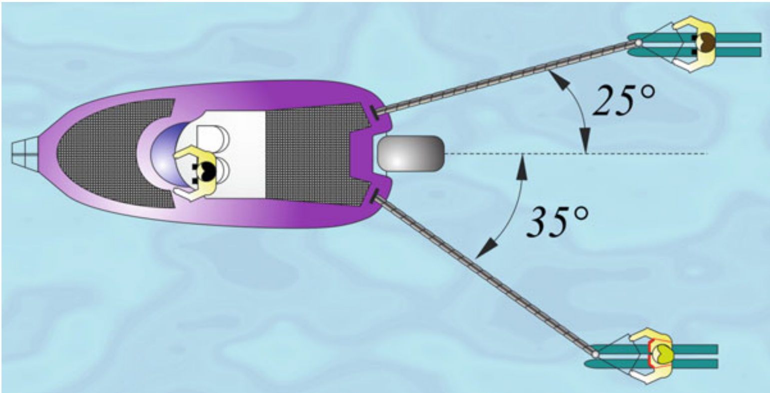


Fig. P4.27

4.28 Barrel $P = 2000$ N is supported by cable CB and boom AB. For the angles shown, find the tension in the cable and the force acting along the boom.

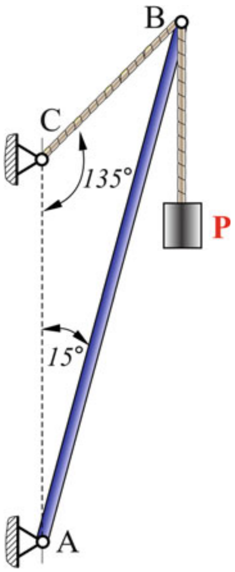


Fig. P4.28

4.29* Three masses suspended by a rope AEB are in state of equilibrium. Masses C and **D** each have weight **P** and mass attached at E weights 1.5 **P**, distance l is given. What is the distance x?

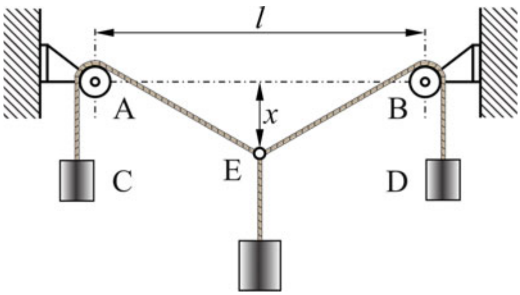


Fig. P4.29

4.30 A hot air balloon is moored to point O by a weightless cable of length *l*. The lift force acting on the balloon is Q. The wind keeps it at the location shown. Calculate the tension T in the cable and the wind force W. Assume that the wind is acting in the horizontal direction and the angle of the cable with the horizontal surface is α.

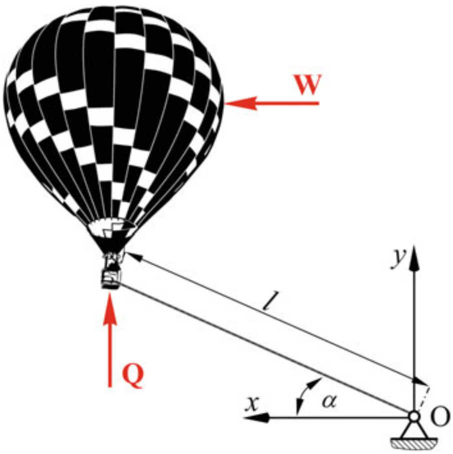


Fig. P4.30

4.31 Load $P = 3$ kN is supported by cable AD. Determine the forces in booms AB and AC.

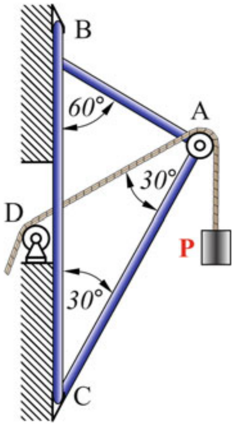


Fig. P4.31

4.32 Ball B (weight **P**) is suspended from point A and is touching the surface of the sphere with radius R. Knowing that line ACO is vertical, determine the tension in rope AB and the force exerted by ball B on the sphere. Neglect the size of the ball B. Use $l = 30$ cm and $R = d = 20$ cm.

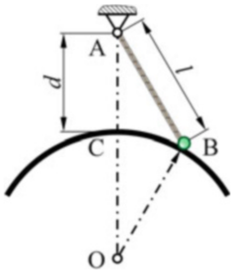


Fig. P4.32

4.33 A disc (weight 60 N) is supported by two frictionless surfaces. Angle ABC = 90°. Determine the forces between the disc and each surface using the graphical approach. How the result depends on the radius of the disc?

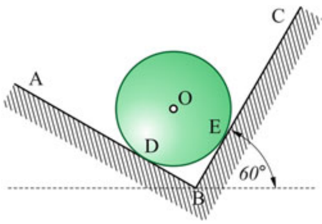


Fig. P4.33

4.34 A 10 N ball is supported by an incline and a cable making an angle α with the vertical direction. Knowing the force in the cable equal to 5 N, determine the angle α and the force exerted by the ball on the incline. Consider the ball as a particle.

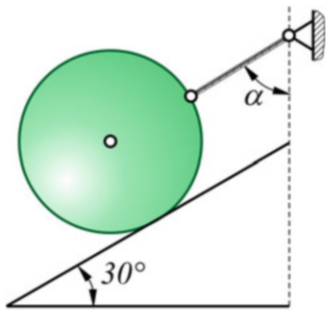


Fig. P4.34

4.35 Disc O (weight 10 N) is supported by two frictionless planes which are perpendicular one to another. Determine the force exerted by the disc on each plane. Solve problem analytically.

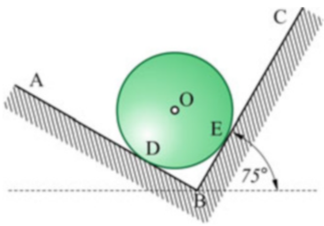


Fig. P4.35

4.36* A cylinder is suspended by two cables of the length l. Its weight is 2 $\mathbf{P}$ and radius is r. The distance between the points where the cables contact the cylinder is b. Calculate the tension in each cable.

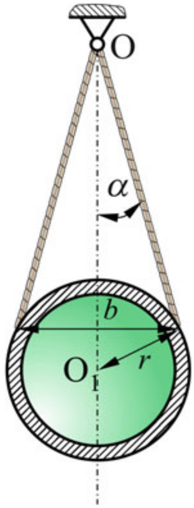


Fig. P4.36

4.37* Block A (weight **P**) is on a smooth surface making angle α with the horizontal. It is held in the state of equilibrium by weight **M** and the spring, as shown. The spring has stiffness "c." What is the elongation Δ of the spring? AB is horizontal. The force exerted by the spring is $c \cdot \Delta$.

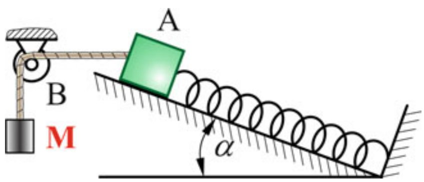


Fig. P4.37

4.38 Box B has a weight **P**. It is in equilibrium on a frictionless incline. Cable BC is in the horizontal direction. Determine the tension in cable AB and pressure on the surface. What must be the weight of D to lift box B from the surface?

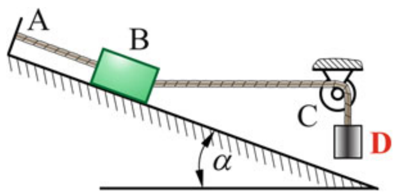


Fig. P4.38

4.39* A weightless triangle can freely rotate about the horizontal axis at A. Box M (weight **P**) is attached to BC. The system is in equilibrium. What are the axial loads in bars AB and AC? The values of angles α and β are given; the angle at A is 2α and AB = AC.

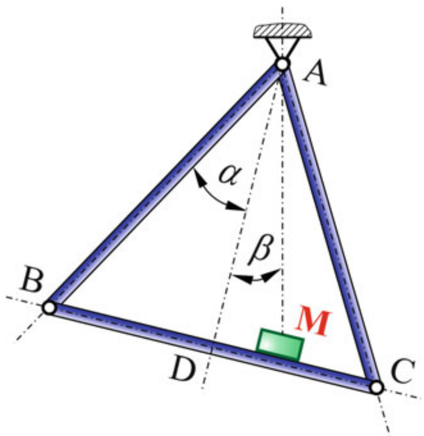


Fig. P4.39

4.40 Cylinder M (weight **P**) is held in equilibrium on a frictionless surface by weight **Q**. Determine force **N** between the surface and cylinder M. What is the value of angle α if $P = 10$ N and $Q = 8$ N.

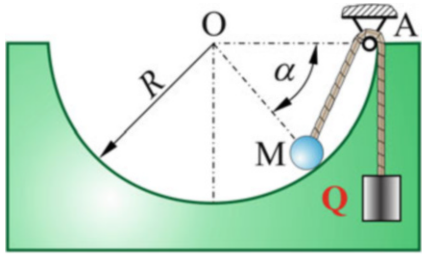


Fig. P4.40

4.41 Homogeneous cylinder A has a weight **P** and radius r. It is placed on the surface of cylinder B (radius R) and is held in equilibrium by rope CD $= l$. Determine the tension in rope CD and the force between the two cylinders.

It is suggested to use the graphical approach for the solution of this problem.

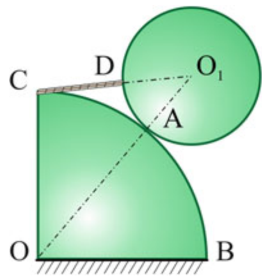


Fig. P4.41

4.42* Smooth ring A can glide along circular path AB. Determine angle φ as a function of **P** and **Q** for the system to be in equilibrium.

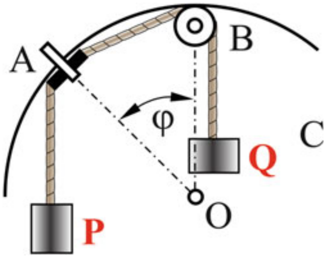


Fig. P4.42

4.2 Resultant and Equilibrium of Forces Acting in Space

Though there are many engineering problems in which forces are acting in a plane, as discussed above, there are even more cases when forces are acting in three dimensions. The goal is to consider the conditions necessary for equilibrium of all forces acting at a point or to find the resultant of the selected forces. We will discuss both cases using an example of an antenna tower supported by three cables (Fig. 4.10).

Let us assume that all forces are acting at the same point. As we have discussed, such an assumption is usually acceptable whenever distances between acting forces are small comparatively to the size of a structure. Using this assumption, we may represent forces in cables, as force vectors, acting at a point at the top of the tower (Fig. 4.10b). Creation of the corresponding physical model representing the real structure is probably the most important step in the process of solving engineering problems.

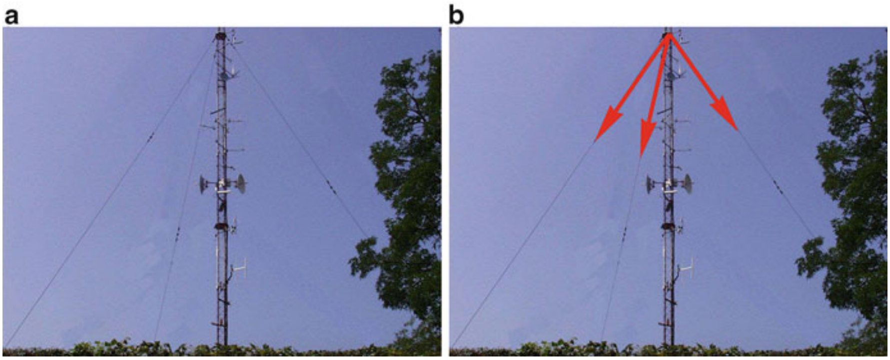


Fig. 4.10 (**a**) Antenna tower supported by cables, (**b**) superimposed force vectors acting in the wires

It is important to emphasize that any solution obtained is as good as the assumptions we have made when defining a physical model.

We are now ready to address the problem of finding the resultant of the selected forces and to analyze the conditions to be fulfilled to ensure the equilibrium of forces acting in the point.

4.2.1 Resultant of Forces

Let us find the resultant of the forces generated by three cables. The forces are shown in a 3D Cartesian coordinate system whose origin was placed at point O, on the base of the tower (Fig. 4.11a). Appropriate dimensions of the structure are shown in meters. The forces acting in each wire were measured to be $F_1 = 12$ kN, $F_2 = 12.3$ kN, $F_3 = 14.2$ kN.

All parallel vectors representing various physical quantities have the same unit vector.

Since two concurrent forces always belong to one plane, we may use the basic principal of parallelogram of forces, as demonstrated in Sect. 4.1, to find their resultant. Doing this in three dimensions is a tedious and not very accurate procedure. The numerical approach is therefore more convenient.

In the orthogonal Cartesian coordinate system, each force may be represented through its three orthogonal components

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \tag{4.8}$$

or as the product of its magnitude F and its unit vector $\mathbf{e}$. The unit vector defines the orientation and direction in space (Appendix).

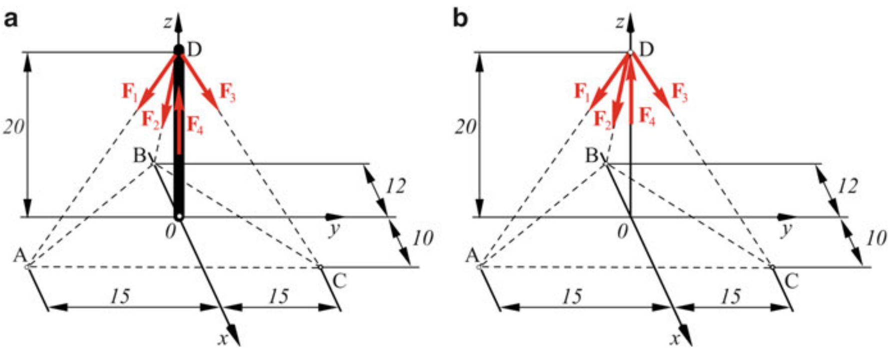


Fig. 4.11 (**a**) Cable forces acting on the tower. (**b**) Free body diagram of the point D

$$\mathbf{F} = F\mathbf{e} = F(e_x\mathbf{i} + e_y\mathbf{j} + e_z\mathbf{k}) = Fe_x\mathbf{i} + Fe_y\mathbf{j} + Fe_z\mathbf{k} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \quad (4.9)$$

Therefore, the forces generated by cables may be expressed as

$$\mathbf{F_1} = F_1(e_{1x}\mathbf{i} + e_{1y}\mathbf{j} + e_{1z}\mathbf{k}) = F_{1x}\mathbf{i} + F_{1y}\mathbf{j} + F_{1z}\mathbf{k}$$
$$\mathbf{F_2} = F_2(e_{2x}\mathbf{i} + e_{2y}\mathbf{j} + e_{2z}\mathbf{k}) = F_{2x}\mathbf{i} + F_{2y}\mathbf{j} + F_{2z}\mathbf{k}$$
$$\mathbf{F_3} = F_3(e_{3x}\mathbf{i} + e_{3y}\mathbf{j} + e_{3z}\mathbf{k}) = F_{3x}\mathbf{i} + F_{3y}\mathbf{j} + F_{3z}\mathbf{k}$$

The resulting force is then obtained as a sum of three forces,

$$\mathbf{R} = \sum_{n=1}^{n=3}\mathbf{F}_n = \left(\sum_{n=1}^{n=3}F_{nx}\right)\mathbf{i} + \left(\sum_{n=1}^{n=3}F_{ny}\right)\mathbf{j} + \left(\sum_{n=1}^{n=3}F_{nz}\right)\mathbf{k}$$

As in 2D case we can generalize the above equation to any number of forces, say N,

$$\mathbf{R} = \sum_{n=1}^{n=N}\mathbf{F}_n = \left(\sum_{n=1}^{n=N}F_{nx}\right)\mathbf{i} + \left(\sum_{n=1}^{n=N}F_{ny}\right)\mathbf{j} + \left(\sum_{n=1}^{n=N}F_{nz}\right)\mathbf{k}$$
$$= R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k} \quad (4.10)$$

Let us calculate the resultant of the three forces acting on the tower. The geometry of the structure and the magnitudes of forces are known (Fig. 4.11a). This information can be used to define force components. From the geometry of the structure, we first define the unit vectors for each of the forces.

To find a unit vector for a force, say, $\mathbf{F}_1$ (Fig. 4.11a) we have to define the coordinates of points A (x_A, y_A, z_A) and D (x_D, y_D, z_D) and use the following equation (Appendix)

$$\mathbf{e}_1 = \frac{(x_A - x_D)\mathbf{i} + (y_A - y_D)\mathbf{j} + (z_A - z_D)\mathbf{k}}{\sqrt{(x_A - x_D)^2 + (y_A - y_D)^2 + (z_A - z_D)^2}} = e_{1x}\mathbf{i} + e_{1y}\mathbf{j} + e_{1z}\mathbf{k}$$

Similarly, we obtain the unit vectors for forces $\mathbf{F}_2$, and $\mathbf{F}_3$

$$\mathbf{e}_2 = \frac{(x_B - x_D)\mathbf{i} + (y_B - y_D)\mathbf{j} + (z_B - z_D)\mathbf{k}}{\sqrt{(x_B - x_D)^2 + (y_B - y_D)^2 + (z_B - z_D)^2}} = e_{2x}\mathbf{i} + e_{2y}\mathbf{j} + e_{2z}\mathbf{k}$$

$$\mathbf{e}_3 = \frac{(x_C - x_D)\mathbf{i} + (y_C - y_D)\mathbf{j} + (z_C - z_D)\mathbf{k}}{\sqrt{(x_C - x_D)^2 + (y_C - y_D)^2 + (z_C - z_D)^2}} = e_{3x}\mathbf{i} + e_{3y}\mathbf{j} + e_{3z}\mathbf{k}$$

The coordinates of points A, B, C, and D are summarized in the following table (see Fig. 4.11a).

Point	x	y	z
A	10	−15	0
B	−12	0	0
C	10	15	0
D	0	0	20

Substituting these values into above equations will provide values for the components of the unit vectors for each force.

Unit vector	x	y	z
e_1	0.371	−0.557	−0.743
e_2	−0.515	0.0	−0.858
e_3	0.371	0.557	−0.743

Now we can use (4.10) to find the resultant force. One approach is to use these equations directly and perform all needed calculations by hand; another is to use MATLAB routines. Both approaches are shown below.

(a) "Hand" calculation
If directions and magnitudes of all forces acting in cables are known (i.e., $F_1 = 12$ kN, $F_2 = 12.3$ kN, $F_3 = 14.2$ kN), we can calculate their components using (4.9) and their corresponding unit vectors, as shown above. The resultant of those forces can be calculated from (4.10).

This process can be simplified by using Table 4.4. Columns 2, 3, and 4 show the unit vector components, while the fifth one displays the corresponding force magnitudes. The following three columns show the orthogonal components of each force. To obtain the force component in a particular direction, we multiply the force magnitude by the corresponding component of the unit vector according to (4.9).

For example,

$F_x = F_1 e_{1x}$, etc.

Summation of all force components provides the resultant

$$\mathbf{R} = (3.39\mathbf{i} + 1.230\mathbf{j} - 30.0\mathbf{k})\ \mathrm{N}$$

(b) Use MATLAB routine
Start the MATLAB and run the "resultantPoint3D." A dialog will come up asking to "Enter number of forces acting on a point:"—enter 3 in our case.

Table 4.4 Calculation of the resultant force

Force	e_x	e_y	e_z	Magnitude	F_x	F_y	F_z
1	0.371	−0.557	−0.743	12.0	4.45	−6.68	−8.92
2	−0.515	0.0	−0.858	12.3	−6.33	0.0	−10.55
3	0.371	0.557	−0.743	14.2	5.27	7.91	−10.55
Resultant					3.39	1.230	−30.0

Next, you will have to enter the data for each force. You will be given a choice: "components" or "magnitude and three projections of the force's line of action." Since we know for each force its magnitude and projections of its line of action, select the "magnitude and line of action," i.e., we will be providing three projections for the force direction. Now, enter the data for each force. The result as calculated by MATLAB routine is: $R_x = 3.40$, $R_y = 1.226$ and $R_z = -30.0$. It should be noted that the small difference from the results calculated in the table above are due to the rounding errors; since in each step of "hand" calculation we were rounding the result to three significant digits.

4.2.2 Equilibrium of Forces

To consider equilibrium of a point in space we have to draw a free body diagram of this point showing *all* forces acting on that point, in our example—point **D** (Fig. 4.11a). In addition to the three forces in cables, we have also a force generated by the tower.

According to the Second Newton's Law, point D will be in equilibrium when the resultant of all forces acting upon it is equal to zero. Hence, $\mathbf{R} = 0$, or in scalar form

$$
\begin{aligned}
R_x &= \sum_{n=1}^{n=N} F_{nx} = 0 \\
R_y &= \sum_{n=1}^{n=N} F_{ny} = 0 \qquad (4.11) \\
R_z &= \sum_{n=1}^{n=N} F_{nz} = 0
\end{aligned}
$$

The (4.11) are called the equilibrium equations.

If forces are represented in a component form, they can be directly used in (4.10) and (4.11). However, when solving a real-life problem, this is rarely the case. Usually, we know the geometry of a structure, as well as the force magnitudes, since they can be measured.

Let us now consider the example of the tower with the cables mentioned above. The goal is to find unknown force $\mathbf{F}_4$ imposed by the tower at point D. This force and the three cable forces are considered to be external forces acting on the body of interest, which is point D. The corresponding free body diagram, Fig. 4.11b, shows these forces.

The forces will be in equilibrium if they fulfill (4.11). The goal is to find unknown force $\mathbf{F}_4$. The equilibrium equations can be written utilizing the data from Table 4.4.

$$R_x = \sum_{n=1}^{n=4} F_{nx} = F_{1x} + F_{2x} + F_{3x} + F_{4x} = 4.45 - 6.33 + 5.27 + F_{4x} = 0$$

$$R_y = \sum_{n=1}^{n=N} F_{ny} = F_{1y} + F_{2y} + F_{3y} + F_{4y} = -6.68 + 0.0 + 7.91 + F_{4y} = 0$$

$$R_z = \sum_{n=1}^{n=N} F_{nz} = F_{1z} + F_{2z} + F_{3z} + F_{4z} = -8.92 - 10.55 - 10.55 + F_{4z} = 0$$

We obtain three equations with three unknowns. The solution represents three orthogonal components of force $\mathbf{F}_4$,

$$\mathbf{F_4} = (-3.39\mathbf{i} - 1.230\mathbf{j} + 30\mathbf{k})\ \mathrm{N}$$

The magnitude of the force is:

$$F_4 = \sqrt{(-3.39)^2 + (-1.230)^2 + (30.0)^2} = 30.2\ \ \mathrm{N}$$

Direction of the force is given by its unit vector $\mathbf{e}_4$, whose components are:

$$e_{4x} = \cos\alpha = \frac{F_{4x}}{F_4} = -0.1122$$

$$e_{4y} = \cos\alpha = \frac{F_{4x}}{F_4} = -0.0407$$

$$e_{4z} = \cos\gamma = \frac{F_{4z}}{F_4} = 0.993$$

where α, β, and γ are the angles between force vector $\mathbf{F}_4$ and x-, y-, and z-axis, respectively.

Procedure to use the MATLAB routines to solve for unknown forces in 3D
Start the MATLAB and run the "*equilibriumPoint3D*." The dialog will appear explaining how to input the data. Enter the number of forces acting on a point, i.e., "4" in our case. Next dialog will ask to input the coordinates of the point. Enter the known values, if not given, enter 0, 0, 0. Next, you will be asked to enter data for each force. You will have to provide the magnitude and the directions of each force vector. For direction, enter the coordinates of any point the force is pointed to. From the given geometry of the structure, shown in Fig. 4.11, we can enter the values for the force direction components in the x, y, and z directions. The input is shown in the table below:

Force	F (N)	X (m)	Y (m)	Z (m)
1	12.0	10	−15	−20
2	12.3	−12	0	−20
3	14.2	10	15	−20
4	x	x	x	1

The solution will appear on the screen as:
The value of the first unknown is: 30.23
The value of the second unknown is: −0.1134
The value of the third unknown is: −0.0408

Since we entered the amplitude of force #4 as the first unknown, the calculated force $\mathbf{F_4}$ magnitude is 30.2 N. We entered x and y as unknown lengths components, however for z we arbitrary prescribed value of 1, since three components of the force directions are interconnected—the sum of their squares is equal to the squared length. We have to normalize the calculated components by this length. Let us calculate the correct components of the unit vector $\mathbf{e}_4$ along the fourth force direction as shown below.

$$e_x = -0.1134/\sqrt{1 + (0.1134)^2 + (0.0408)^2} = -0.1126$$

$$e_y = -0.0408/\sqrt{1 + (0.1134)^2 + (0.0408)^2} = -0.0405$$

$$e_z = 1.0/\sqrt{1 + (0.1134)^2 + (0.0408)^2} = 0.993$$

These values are very close to the values calculated by hand, the differences can be attributed to the rounding errors.

The schematic of the four forces in the space is shown on the screen.

Guidelines and Recipes for Solving 3-D Problems

- Draw a physical model of the problem.
  Drawing a physical model means representing the simplified geometry of the structural element under consideration, the supports, and all forces acting on a body in the coordinate system.
- Substitute the supports by reactive forces and replace the structural element by a point.
- Create a free body diagram (FBD).
- Find unknown values.

Represent forces as vector components, and write and solve the corresponding equilibrium equations. This procedure may be easily

(continued)

programmed. The example of the **MATLAB** routines may be downloaded from http://extras.springer.com.

4.2.3 Problems

4.43 Cable OB is 25 m long and the tension in that cable is 500 N. What are the *x*, *y*, and *z* components of the force exerted by cable OB on point B?

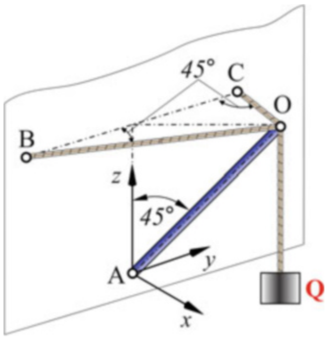


Fig. P4.43

4.44 Pole OA is 35 m long and the compression force in that pole is 900 N (Fig. P4.43). What are the *x*, *y*, and *z* components of the force exerted by pole OA on point A?

4.45 A tensile force of magnitude 600 N acts on point C. What are the *x*, *y*, and *z* components of this force?

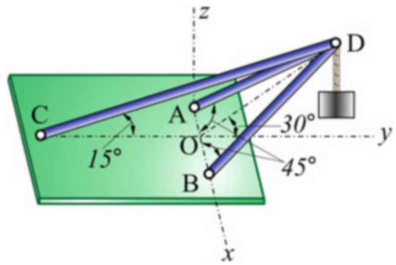


Fig. P4.45

4.46 Determine the resultant of three forces acting at a point. The forces are: $\mathbf{F}_1 = 12\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}$ [lb], $\mathbf{F}_2 = -2\mathbf{i} + 7\mathbf{j} - 3\mathbf{k}$ [lb], and $\mathbf{F}_3 = 4\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$ [lb].

4.47 Determine the resultant of three forces acting at a point. The forces are:

$$\mathbf{F}_1 = -9\mathbf{i} - 2\mathbf{j} - 14\mathbf{k}[\text{N}], \quad \mathbf{F}_2 = -2\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}[\text{N}], \quad \text{and}$$

$$\mathbf{F}_3 = 11\mathbf{i} + 6\mathbf{j} + 17\mathbf{k}[\text{N}].$$

4.48 Determine the resultant of (a) forces F_1 and F_4; (b) forces F_2 and F_3. Use the following as the force magnitudes: $F_1 = 3\text{p[N]}$, $F_2 = \text{p[N]}$, $F_3 = \text{p[N]}$, and $F_4 = 2\text{p[N]}$.

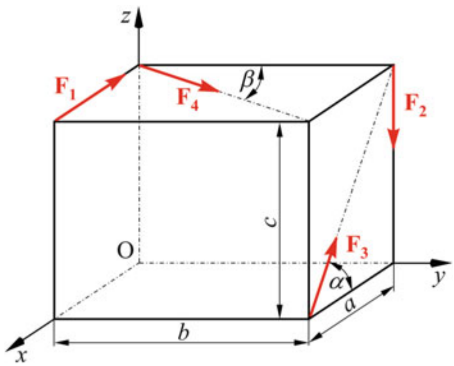


Fig. P4.48

4.49 Two identical poles AB and AC, each making the same angle with the ground, support two horizontal cables AD and AE. The tension in each cable is 50 N. Determine the forces in the poles, if the plane BAC divides angle DAE in half.

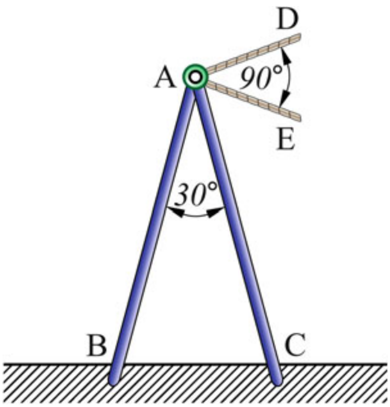


Fig. P4.49

4.50 Load $Q = 10$ N is supported by rod AO and by two horizontal cables OB and OC. Determine the axial force in AO and tension in the cables.

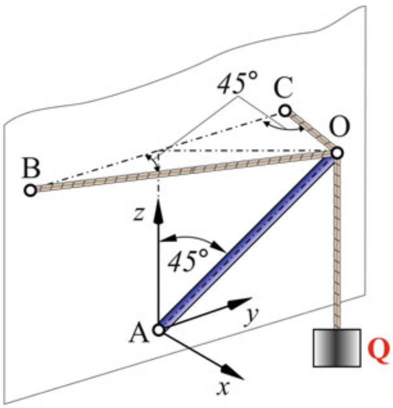


Fig. P4.50

4.51 Load **P** (weight = 300 N) is supported by rods AB, AC and cable AD. Plane ABC is horizontal, angle CBA = angle BCA = 60°, angle EAD = 30°. Determine the forces in each rod and in the cable.

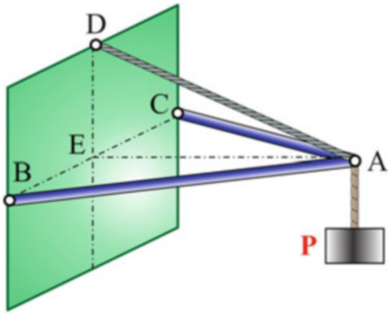


Fig. P4.51

4.52 Determine the forces in rod AB and the cables AD and AC when the system is loaded by the force $Q = 30$ N. AB = 70.7 cm, AC = 40 cm, AD = 50 cm. Plane of rectangular CADE is horizontal, planes V and W are perpendicular.

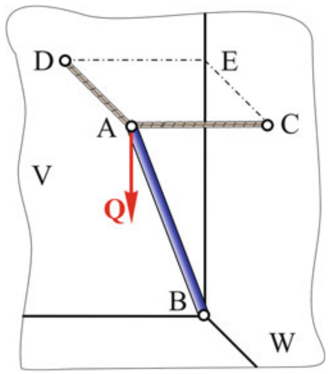


Fig. P4.52

4.53 Determine forces in cable AB and the rods AD and AC that are loaded by force $Q = 80$ N, when AB = 122.5 cm, AC = AD = CD = 100 cm, CK = KD and triangle ACD is in the horizontal plane.

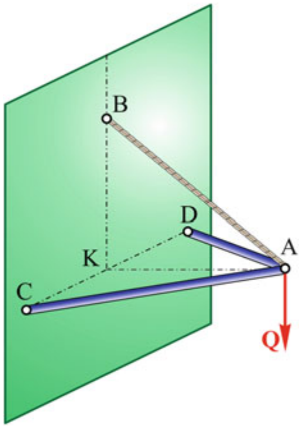


Fig. P4.53

4.54 Point D is loaded by weight $W = 300$ N and is supported by the rods as shown. Determine the forces in links AD, BD, and CD.

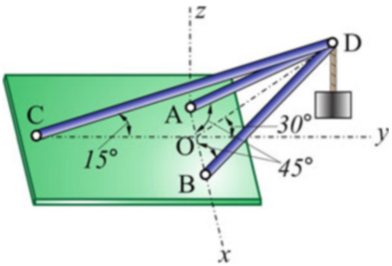


Fig. P4.54

4.55 Determine forces in pole AB and cables AC and AD that support two electrical wires. The wires are perpendicular to one another, are in the horizontal plane, and each one is under the given tension T. Angle CBD = 90° and $\varphi = 120°$.

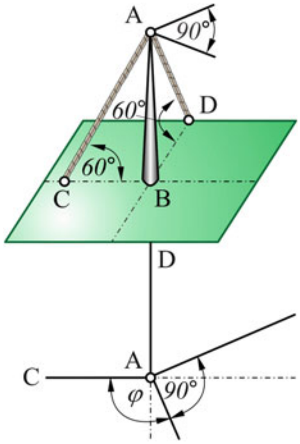


Fig. P4.55

4.56 For Problem 4.55 derive the dependence of the force in cable AC from the angle φ between 90 and 270° and plot it.

4.57 Weight $P = 3$ kN is lifted by winch E as shown. Determine the axial forces in each leg. Cable ED and each leg are making angle of 60° with the horizontal plane, AB = BC = AC.

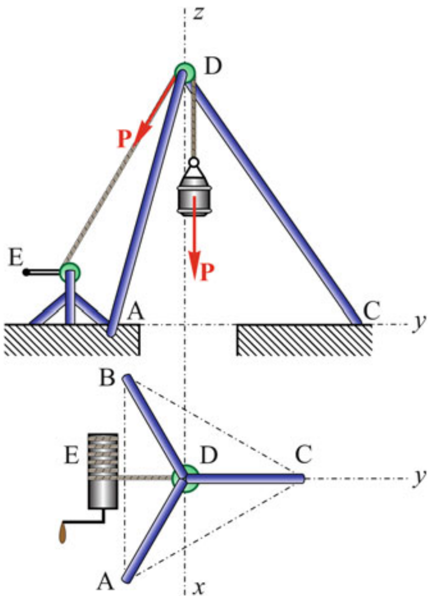


Fig. P4.57

4.58 A barrel (weight **Q**) is suspended by three rods of equal length. OA = OB = OC = 1 m. Point D is at $x = y = z = 0.100$ m. Determine the force in each rod.

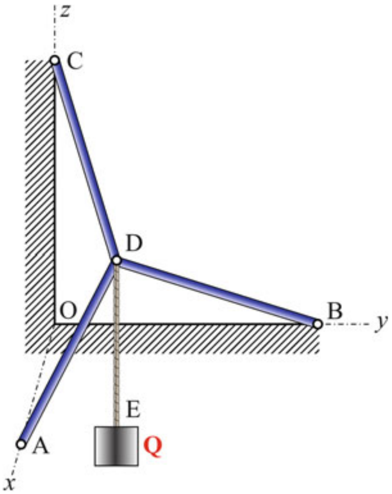


Fig. P4.58

4.59 Sphere A has a volume of 0.7 m^3 and weights 5 kN. It is hold underwater by three equidistant anchors B, C, and D planted at the same depth. Determine tension in each cable if each makes angle of 45° with the vertical direction. The specific weight of water $\gamma = 10$ kN/m^3.

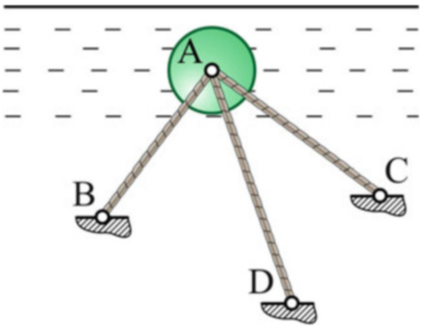


Fig. P4.59

4.60 Weight $P = 3$ kN is lifted by a cable as shown. Pulley D is mounted on a tripod. Each leg of the tripod makes an angle of 60° with the horizontal plane and AB = BC = AC. Determine the forces in each leg. *Note that Fig.* P4.60 *shows the view in a vertical plane only.*

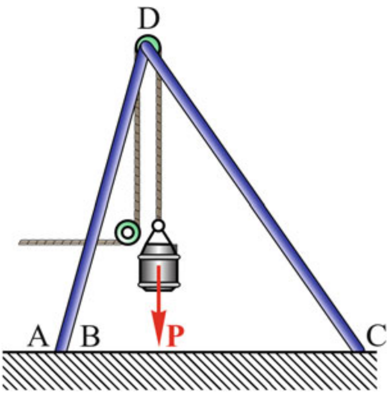


Fig. P4.60

4.61 A square plate (weight **P**) is suspended by three cables. Determine the tension in each cable, if plane OAB is normal to the plane of the plate, angle BOA = 90°, point C is in the middle of the side and OA = OB. Assume that the weight of the plate is acting along the vertical line passing through point O.

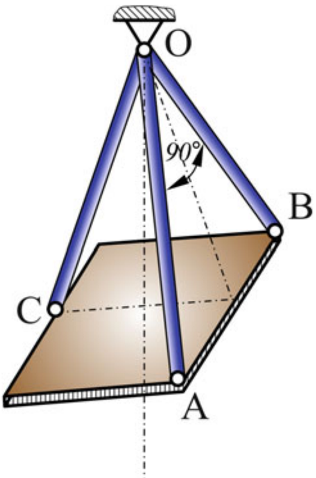


Fig. P4.61

4.62 Force $P = 2$ kN is acting in vertical plane CDE. Determine forces in each link when $\alpha = 30°$, $\beta = 45°$, $\gamma = 30°$.

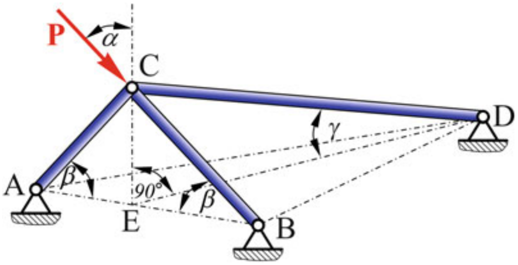


Fig. P4.62

4.63 Weight $K = 15$ kN is suspended by cable FBK. Links BC and BD are in the horizontal plane and are making angles of 45° with line BE. Determine the forces in links BC, BD, and AB if angle $\alpha = 45°$ and $\gamma = 30°$.

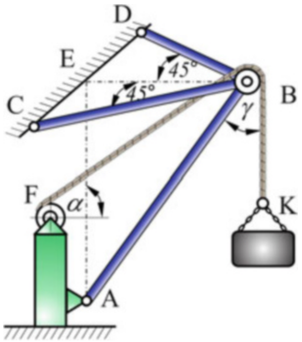


Fig. P4.63

4.64 Sphere M (weight **W**) is held by two cables AM and BM on a frictionless plane that makes angle α with a horizontal plane. Determine the force between the sphere and the plane, and the tension in each cable when $\alpha = 60°$, $\beta = 30°$, $\gamma = 45°$, and $W = 10$ N.

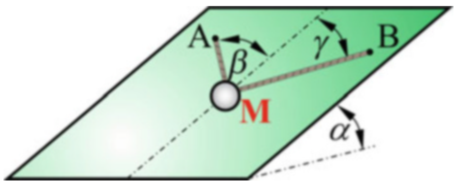


Fig. P4.64

4.65 A crane consists of two booms AB = BC supported by horizontal cable BE. Cable KDB supports load $P = 10$ kN. Angle ABC = 30° and AD = DC. Plane ABC makes an angle 60° with the horizontal plane. Determine the forces in members AB, BC, and BE.

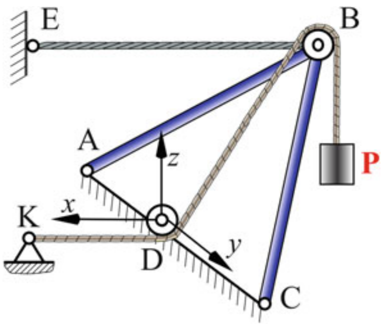


Fig. P4.65

What We Have Learned?

Procedures to find the resultant force

Resultant force is equal to the vector sum of all forces acting at a point

$$\mathbf{R} = \sum_{n=1}^{n=N} \mathbf{F}_n = \left(\sum_{n=1}^{n=N} F_{nx}\right)\mathbf{i} + \left(\sum_{n=1}^{n=N} F_{ny}\right)\mathbf{j} + \left(\sum_{n=1}^{n=N} F_{nz}\right)\mathbf{k} = R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k}$$

We have to express each force through its components and sum up the appropriate components of all forces (Fig. 4.12). Magnitude and direction angles can be obtained from

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

$$\cos\alpha = \frac{R_x}{R}$$

$$\cos\beta = \frac{R_y}{R}$$

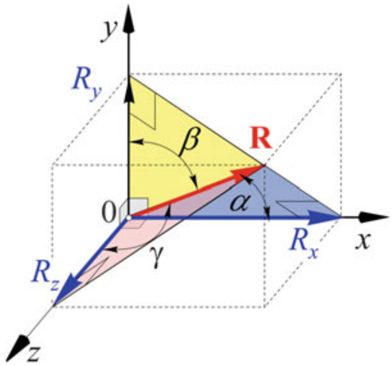
$$\cos\gamma = \frac{R_z}{R}$$

Procedures to analyze equilibrium of forces

A body, represented by a point, is in equilibrium when the sum of all forces (resultant) is equal to zero, hence

$$\mathbf{R} = \sum_{n=1}^{n=N} \mathbf{F}_n = \left(\sum_{n=1}^{n=N} F_{nx}\right)\mathbf{i} + \left(\sum_{n=1}^{n=N} F_{ny}\right)\mathbf{j} + \left(\sum_{n=1}^{n=N} F_{nz}\right)\mathbf{k} = 0$$

Fig. 4.12 Three-dimensional force **R** and its components



This means that each component separately should be equal to zero

$$\sum_{n=1}^{n=N} F_{nx} = 0, \quad \sum_{n=1}^{n=N} F_{ny} = 0, \quad \sum_{n=1}^{n=N} F_{nz} = 0$$

> When all forces are acting in a single plane, one component, usually z, is commonly set to be zero.

4.3 Review Problems

4.66 Calculate resultant of the three forces acting at a point.

$$\mathbf{F}_1 = 2\mathbf{i} - 3\mathbf{j}, \quad \mathbf{F}_2 = -3\mathbf{i} + 2\mathbf{j}, \quad \text{and} \quad \mathbf{F}_3 = -3\mathbf{i} + 7\mathbf{j}$$

4.67 Determine the magnitude and direction of the resultant of the forces exerted by supports CD and CE on stake C. Forces are A = 1 kN, D = 1.5 kN, and E = 2 kN, the coordinates are A (1, 0.5, 0), B (0.2, 0.5, 2), C (1.2, 8, 0), D (0.6, 8, 1.8), and E (0.4, 7.5, 2.2).

Fig. P4.67

4.68 A boat tows three water skiers. Skier **A** makes an angle of 30° with the direction of the boat ride, while skier B makes an angle of 5°. Skier C makes a negative angle of 15°. The boat pulls the skiers with the force of 300 N. The force that is acting on skier B is 80 N. Calculate the magnitude of the forces acting on skiers A and B.

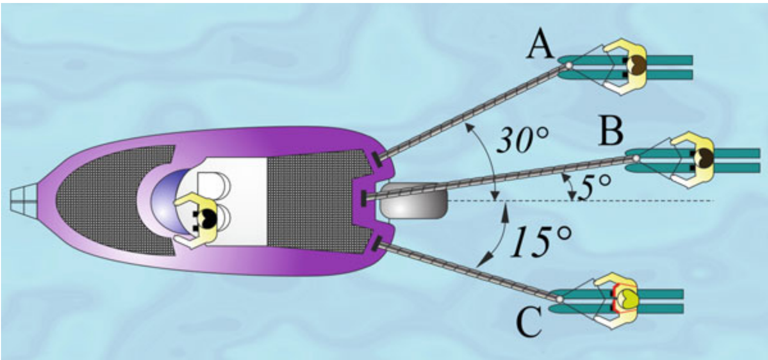


Fig. P4.68

4.69 Slender rod AB (weight **W** = 160 N, length $l = 1.2$ m) is held in the state of equilibrium by two cables AC and BC. Determine the tension in each cable. AC = BC = 1 m.

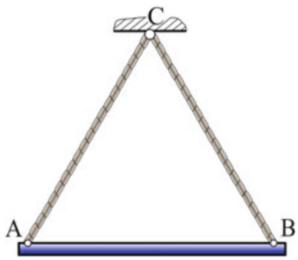


Fig. P4.69

4.70 Crane BAC is loaded by force **F** and load **M**. Determine the axial load in each leg, AB and AC.

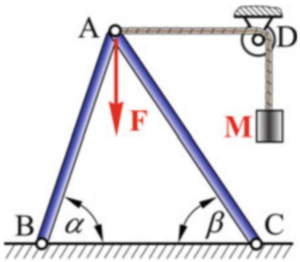


Fig. P4.70

4.71 Block M (weight **P**) is suspended by two identical springs. Without the block the springs are not taut and are in the horizontal direction. Determine weight **P** that results in $x = 8$ cm, when $a = 6$ cm and spring's stiffness $c = 200$ N/m. The force generated by a spring is equal to $c \cdot \Delta$, where Δ is the spring's elongation.

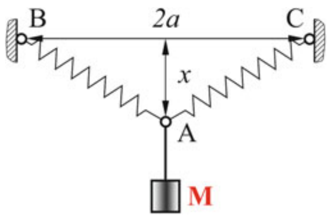


Fig. P4.71

4.72 Weight $\mathbf{M}_1$ is supported by the rod and is held in position shown by weight $\mathbf{M}_2$. Determine weight $\mathbf{M}_2$ and tension in the rod.

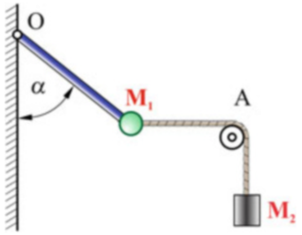


Fig. P4.72

4.73 Two cables tied together at B and loaded as shown below. Knowing that $Q = 100$ N, $\alpha = 45°$, and $\beta = 60°$, determine the tension in cable AB and the magnitude of load P.

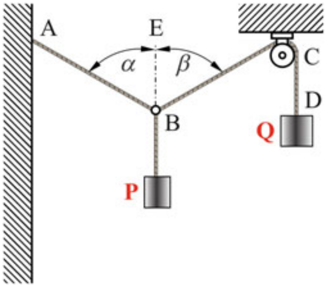


Fig. P4.73

4.74 A ball is suspended by two cables as shown. The weight of the ball is 40 N. Determine the tension in the cables. Consider the ball as a particle.

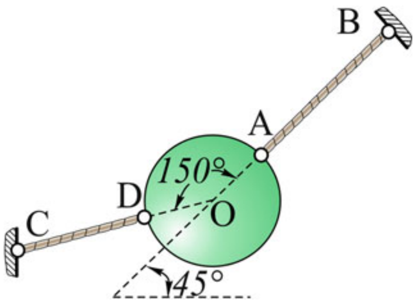


Fig. P4.74

4.75 Weight $Q = 5$ kN is supported by the system of rods and cables as shown. AB = AE = AF = 2 m. Determine the forces in cable BC and rod AC.

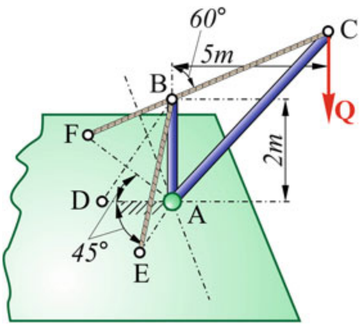


Fig. P4.75

4.76 For problem 4.75 calculate forces in cables BE, BF and rod AB, if the tension in the cable BC is given to be 14.4 kN.

4.77 Tea kettle E (weight 10 lb) is suspended as shown. The supporting legs have the same length and the angles between the legs are identical. Determine the forces in each leg. Each leg makes an angle of 30° with rope BE.

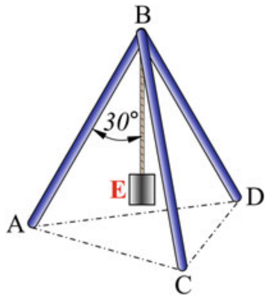


Fig. P4.77

Equilibrium of Rigid Bodies

5

Contents

Make things as simple as possible, but not simpler.

Albert Einstein

In this chapter you will learn:

- How to reduce a system of nonconcurrent forces into the system of forces acting at a single point
- Procedures to find a resultant force and a resultant moment
- Procedures to analyze equilibrium of forces acting on a rigid body in a plane and in a space

In Chap. 4, we discussed equilibrium of objects loaded by concurrent forces. Such objects were modeled as a particle (point). In these cases, the principle of parallelogram of forces may be applied to find the resultant force. Setting the resultant force equal to zero, according to the First Newton's law, leads to equilibrium equations.

I. Emri, A. Voloshin, *Statics*, DOI 10.1007/978-1-4939-2101-0_5

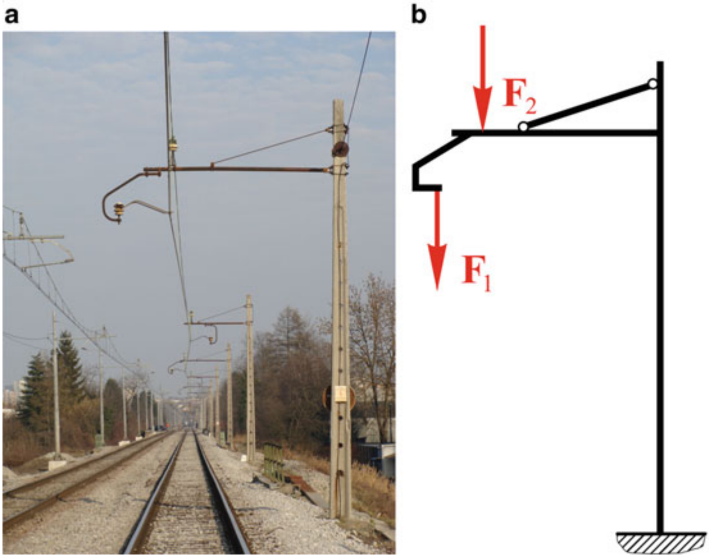


Fig. 5.1 (**a**) Pole supporting electrical wires. (**b**) Set of nonconcurrent forces

In reality, however, there are many situations when a body cannot be modeled as a particle, i.e., forces acting upon it are not concurrent, Fig. 5.1a. The forces acting on the pole are shown schematically in Fig. 5.1b. In these cases, we cannot directly apply the basic principles discussed in Chap. 2. We therefore need to develop a rule for moving a force to a point that is not located on its line of action, and use the concept of a moment in order to fulfill the equilibrium condition.

In the real world, all bodies have a size and a shape. In some cases, we may model a body as a particle and use the methods developed in the Chap. 4 to solve for unknown forces. However, in majority of cases we have to account for the size of a body, and the fact that the applied forces are not concurrent.

5.1 Force–Moment Systems

Since we do not model a rigid body as a particle, it becomes important to account for the location of the external forces acting on the body. Below we discuss how to account for the location of a force and the effect of moving it to a different position.

5.1.1 Moving a Force to an Arbitrary Point

Assume that there is only one force $\mathbf{F}_1$ acting on a rigid body at a point A, as shown in Fig. 5.2a. The goal is to develop a rule on how to move the force $\mathbf{F}_1$ to another arbitrary selected point B. One of the basic axioms postulated by Newton says that

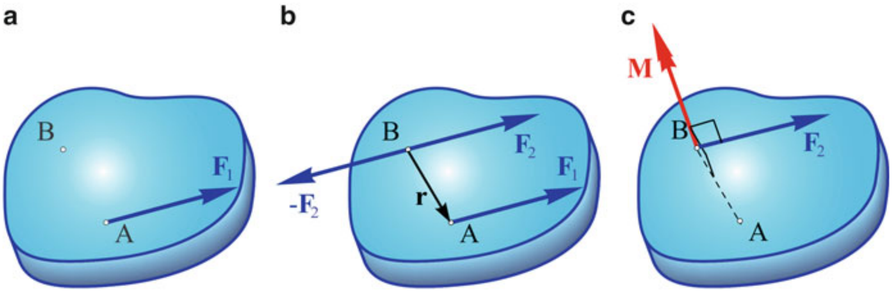


Fig. 5.2 (**a**)–(**c**) Rule for moving a force to an arbitrary point

the state of the motion of a rigid body will not be changed by adding or subtracting an equilibrium pair of forces (Sect. 2.2.1). According to this rule, we may add an equilibrium pair of forces to an arbitrary selected point B without altering the state of motion of the body. Let the pair of forces $\mathbf{F}_2$ and $-\mathbf{F}_2$ be of the same magnitude as force $\mathbf{F}_1$ and be parallel to it, as displayed in Fig. 5.2b.

Forces $\mathbf{F}_1$ and $-\mathbf{F}_2$ form a couple (Sect. 2.2.3) which may be shown as a free vector $\mathbf{M}$ perpendicular to the plane defined by these two forces. This process leaves us with the force $\mathbf{F}_2$, being equal to $\mathbf{F}_1$, and the moment vector $\mathbf{M}$, as shown in Fig. 5.2c. Therefore, when moving a force to another point, we need to add a couple of forces (moment) to preserve the state of motion[1] of the body. The moment is created by applying a couple of forces acting at the original and destination points. Moment, i.e., action of a couple, happens to be a free vector (see Sect. 2.2.3.3), meaning that it has *no* definite point of application. In other words, it may be positioned at any point of a rigid body without changing its state of motion. Based on this procedure, a force can be moved to any point of the body as needed.

To move a force to another point and preserve the state of motion, we need to add a couple of forces (called moment), acting at the original and the destination point.

This process will create a new "equivalent force system," consisting of the force and the moment (couple of forces), whose effect on the motion of the body is exactly the same as of the original force. As it will be shown later, this is not the case when dealing with internal forces and moments.

Any two force systems are *equivalent* if they produce the same effect on a rigid body motion.

[1] Body loaded by a force will accelerate according to the Second Newton's Law.

Moment (action of a couple) happens to be a free vector. It has *no* definite point of action. In other words, it can be positioned at any point of a rigid body without changing its state of motion.

Guidelines and Recipes for Moving a Force to an Arbitrary Point

- Apply an equilibrium pair of forces, which are equal in magnitude and are parallel to the force you are moving, at the point where you want to move the force, as shown in Fig. 5.2b.
- Calculate the moment of the couple consisting of forces $\mathbf{F}_1$ and $-\mathbf{F}_2$ as

$$\mathbf{M} = \mathbf{r} \times \mathbf{F_1}$$

where $\mathbf{r}$ is a vector from any point along the force $\mathbf{F}_2$ to any point along the line of action of the force $\mathbf{F}_1$.

- Represent the effect of the couple by this moment.
- Apply the original force at the new point.

Example 5.1 Worker is pushing the crate at point A with coordinates [1.2, 1.5, 0.0] m (Fig. 5.3a). The force between the hand and the crate is 350 N. Replace this force with an equivalent force system by moving the force to the point B with coordinates [0.6, 0.7, 0] m. The coordinates of point C is [2.3, 1.7, 1.0] m.

When calculating the moment of a couple of forces, $\mathbf{r}$ is the space vector connecting any two points along the lines of action of the two forces.

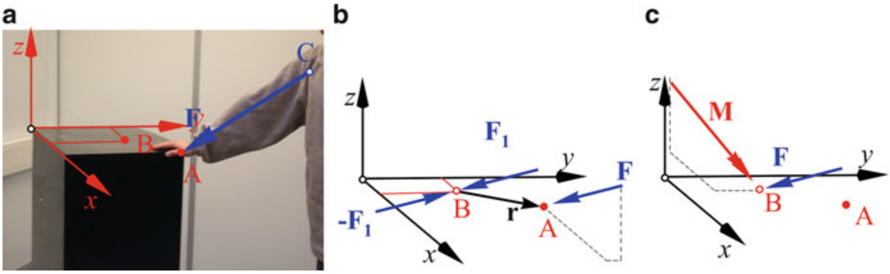


Fig. 5.3 (**a**) Worker pushing the crate. (**b**) Add two forces. (**c**) Resultant force and moment

Solution Force **F** can be represented as a product of its magnitude F and the unit vector **e**, $\mathbf{F} = F \cdot \mathbf{e}$

$$e = \frac{(x_A - x_C)\mathbf{i} + (y_A - y_C)\mathbf{j} + (z_A - z_C)\mathbf{k}}{\sqrt{(x_A - x_C)^2 + (y_A - y_C)^2 + (z_A - z_C)^2}}$$

Substituting the values, we will get

$$\mathbf{e} = \frac{(1.2 - 2.3)\mathbf{i} + (1.5 - 1.7)\mathbf{j} + (0 - 1.0)\mathbf{k}}{\sqrt{(-1.1)^2 + (-0.2)^2 + (-1.0)^2}} = -0.733\mathbf{i} - 0.133\mathbf{j} - 0.667\mathbf{k}.$$

Therefore,

$$\mathbf{F} = 350(-0.733\mathbf{i} - 0.133\mathbf{j} - 0.667\mathbf{k}) = (-257\mathbf{i} - 46.6\mathbf{j} - 233\mathbf{k})\ \mathrm{N}$$

Now, we can follow the procedure as described above.

1. Apply two forces $\mathbf{F}_1 = \mathbf{F}$ and $-\mathbf{F}_1$ at the point B (Fig. 5.3b)
2. Calculate the moment of the couple created by the original force **F**, acting at the point A, and force $-\mathbf{F}_1$, acting at the point B, as $\mathbf{M} = \mathbf{r} \times \mathbf{F}$, where **r** defines location of the point A relative to point B. As you may remember from the definition of the moment of a couple, **r** is a space vector connecting any point along the line of action of the force **F** with any point along the line of action of the force $-\mathbf{F}_1$ and pointing from the line of action of the force $-\mathbf{F}_1$ toward force **F**. In our case, **r** is given as

$$\begin{aligned}\mathbf{r} &= (x_A - x_B)\mathbf{i} + (y_A - y_B)\mathbf{j} + (z_A - z_B)\mathbf{k} \\ &= (1.2 - 0.6)\mathbf{i} + (1.5 - 0.7)\mathbf{j} + (0 - 0)\mathbf{k} = 0.6\mathbf{i} + 0.8\mathbf{j} + 0\mathbf{k}\end{aligned}$$

From (2.6) we can calculate the moment as

$$\mathbf{M} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.6 & 0.8 & 0 \\ -257 & -46.6 & -233 \end{vmatrix} = -186.4\mathbf{i} + 139.8\mathbf{j} + 177.6\mathbf{k} \ \ [\text{Nm}]$$

3. Now, we obtained an equivalent force system consisting of the force **F** of the same magnitude and direction acting at the point B and moment **M** (Fig. 5.3c).

5.1.2 Reduction of a System of Forces

Any system of forces and moments acting on a rigid body may be replaced by an equivalent force and a moment called *resultant force* and *resultant moment*. Consider a rigid body loaded by number of moments ($\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_\mathbf{K}$) and forces ($\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_\mathbf{N}$) at various locations (Fig. 5.4a). Each force may be moved to a new location providing that a moment is added as described above (Sect. 5.1.1). Let us move all forces to an arbitrary point **O**. Transferring each force to the point **O** will result in a system of concurrent forces ($\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots, \mathbf{F}_\mathbf{N}$) and associated moments ($\mathbf{M}_{\mathbf{F}1}, \mathbf{M}_{\mathbf{F}2}, \mathbf{M}_{\mathbf{F}3}, \ldots, \mathbf{M}_{\mathbf{FN}}$) appearing due to the translation of forces (Fig. 5.4b). Since moments are free vectors, we can place them at the same point **O**. Adding original moments to those generated by force translation, we get a resultant moment **M**.

$$\mathbf{M} = \sum_{i=1}^{K} \mathbf{M}_i + \sum_{i=1}^{N} \mathbf{M}_{\mathbf{F}i} \tag{5.1}$$

where K—number of original moments acting on a body, N—number of forces acting on a body.

Since all forces acting on a body now are concurrent, their resultant may be found as

$$\mathbf{R} = \sum_{i=1}^{N} \mathbf{F}_i \tag{5.2}$$

Thus, we reduced all forces and moments acting on a rigid body to the resultant force **F** and moment **M** as shown in Fig. 5.4c.

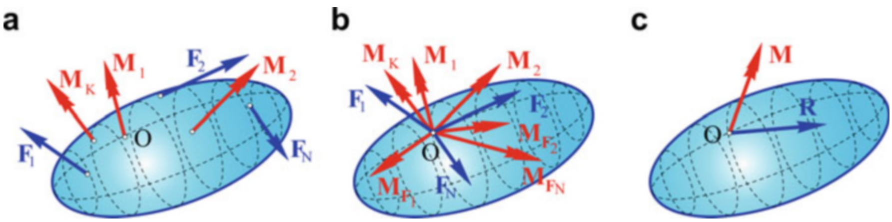


Fig. 5.4 Reduction of a system of forces and moments to a resultant force–moment

Example 5.2 Three forces are acting at points A (2, 4, 6) m, B (−2, 3, −1) m, and C (1, −2, 4) m. The forces are

$$\mathbf{F_1} = 12\mathbf{i} - 3\mathbf{j} + 6\mathbf{k} \ \ [\mathrm{N}]$$
$$\mathbf{F_2} = -4\mathbf{i} + \mathbf{7}\,\mathbf{j} - 4\mathbf{k} \ \ [\mathrm{N}]$$
$$\mathbf{F_3} = 2\mathbf{i} - 8\mathbf{j} + 9\mathbf{k} \ \ [\mathrm{N}]$$

Find the resultant force and the resultant moment (magnitude and direction) if the resultant force is acting at

(a) Point A
(b) Point C

Solution ("hand" calculation)

(a) Resultant force can be calculated using (5.2)

$$\mathbf{R} = \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} = 10\mathbf{i} - 4\mathbf{j} + 11\mathbf{k} \ \ [\mathrm{N}]$$

Moving force $\mathbf{F}_2$ from B to A will result in a free moment.

$$\mathbf{M_2}{=}\mathbf{r_{AB}}{\times}\mathbf{F_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -1 & -7 \\ -4 & 7 & -4 \end{vmatrix} = 53\mathbf{i} + 12\mathbf{j} - 32\mathbf{k} \ \ [\mathrm{Nm}]$$

Moving force $\mathbf{F}_3$ from C to A will result in a free moment.

$$\mathbf{M_3} = \mathbf{r_{AC}} \times \mathbf{F_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -6 & -2 \\ 2 & -8 & 9 \end{vmatrix} = -70\mathbf{i} + 5\mathbf{j} + 20\mathbf{k} \ \ [\mathrm{Nm}]$$

The resultant moment is: $\mathbf{M} = \mathbf{M}_2 + \mathbf{M}_3 = -17\mathbf{i} + 17\mathbf{j} - 12\mathbf{k}$ [N m]. Its magnitude is $M = \sqrt{M_x^2 + M_y^2 + M_z^2} = 26.9$ Nm and the corresponding unit vector is

$$\mathbf{e} = \frac{M_x}{M}\mathbf{i} + \frac{M_y}{M}\mathbf{j} + \frac{M_z}{M}\mathbf{k} = -0.632\mathbf{i} + 0.632\mathbf{j} - 0.446\mathbf{k}.$$

(b) Resultant force will be the same as in the case (a)

$$\mathbf{R} = \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} = 10\mathbf{i} - 4\mathbf{j} + 11\mathbf{k} \ \ [\mathrm{N}]$$

Moving force $\mathbf{F}_1$ from the point A to point C will create a free moment.

$$\mathbf{M_1} = \mathbf{r_{CA}} \times \mathbf{F_1} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 6 & 2 \\ 12 & -3 & 6 \end{vmatrix} = 42\mathbf{i} + 18\mathbf{j} - 75\mathbf{k} \ \text{[Nm]}$$

Similarly, moving force $\mathbf{F}_2$ from B to C will create a free moment.

$$\mathbf{M_2} = \mathbf{r_{CB}} \times \mathbf{F_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 5 & -5 \\ -4 & 7 & -4 \end{vmatrix} = 15\mathbf{i} + 8\mathbf{j} - 1\mathbf{k} \ \text{[Nm]}$$

Now, the resultant moment is: $\mathbf{M} = 57\mathbf{i} + 26\mathbf{j} - 76\,\mathbf{k}$ [Nm]. Its magnitude is 98.5 Nm and the corresponding unit vector is $\mathbf{e} = 0.579\mathbf{i} + 0.264\mathbf{j} - 0.772\,\mathbf{k}$.

The obtained resultant force is the same in both cases, while the resultant moments are different, both in the magnitude and direction. It should be noted that since two moments are different (direction and magnitude) in both cases, the angles between the resultant force and the resultant moment are also different. We have three different cases here: (a) three forces, (b) a resultant force at the point A and a moment, and (c) the same resultant force at point C and a different moment. However, the effect of each one of the three loading cases on the state of motion of the rigid body is the same.

Solution (*MATLAB calculation for the case a*) To solve for the resultant force and moment, you may use the MATLAB routine "*resultantBody3D*". Run it, and follow the prompts.

Enter number of forces: 3
Enter number of moments: 0 (there are no external moments)
Enter location of the point: 2, 4, 6 (x, y, z of point A)

Now, we have to prepare data for input. The magnitude of force #1 is: $\sqrt{12^2 + 3^2 + 6^2} = 13.75$ [N] and its line of action components are: 12, −3, 6. Next, enter the coordinates of any point along its line of action. Since it is given that this force acts at point (2, 4, 6)—enter these values.

For force # 2 the magnitude is 9 [N], components are: −4, 7, −4 [N] and the point is: −2, 3, −1 [m].

Force # 3 has magnitude of 12.21 [N], components are: 2, −8, 9 [N] and it acts at the point: 1, −2, 4 [m].

The solution is:

*Equivalent resultant Force: 10.00 * i – 4.00 * j + 11.00 * k*
*Equivalent resultant Moment: –17.01 * i + 17.00 * j – 11.99 * k*

These are the same values as calculated above. Small differences are due to the rounding errors.

5.1.3 Special Case of a Force-Couple System

Any number of external forces and moments acting on a rigid body may be reduced to only one force and one moment. Magnitude and direction of the resulting moment depends on the point to which the force was moved (see Example 5.2). The fact that the magnitude of the resultant moment depends on the point where the forces were moved to, raises the question—is there a location such that the resultant moment will become zero?

To answer this question, let us consider again what happens when we move a single force to a new location. A force appears at the new location and a couple is created. As it was shown already, mathematically the effect of a couple is represented by a moment, which is a vector resulting from the cross product of a radius vector, **r**, with a force vector, **F**. Result of such a product is a vector **M** *perpendicular* to the plane defined by vectors **r** and **F**. Now, it is obvious, that only the force and the couple that are *perpendicular* to one another may be reduced to force only (i.e., the resultant moment is zero).

Let us consider a case when the loading consists of a moment and force perpendicular to each other. Without loss of generality, we may select a coordinate system as desired. Let's select a coordinate system such that the force **F** will act in x–y plane and the moment **M** along z-axis (Fig. 5.5). This force–moment system may be reduced to one force only by moving the force **F** to the new location A, defined by the space vector **r**. Moving the force **F** away from its original location will generate a moment, acting in z-direction. The point A, defined by **r**, may be chosen so that the newly generated moment will be equal in magnitude and opposite in direction to the original moment **M**. Thus, by moving the force **F** to the point A, the new force-couple system will consist of the force **F** only, since the resultant moment

$$\mathbf{M}_A = \mathbf{M} + \mathbf{r} \times (-\mathbf{F}) = 0 \text{ only when}$$

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

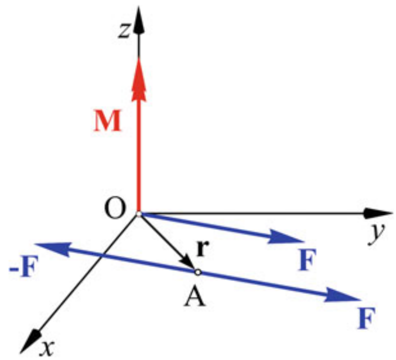


Fig. 5.5 Force-couple system

From this equation, we can find the components of the vector **r** defining the position of the point A. Since vector **r** belongs to the plane *x*–*y*, it may be represented as

$$\mathbf{r} = x_A\mathbf{i} + y_A\mathbf{j}$$

Let us rewrite the above equation in the matrix form

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_{\mathrm{A}} & y_{\mathrm{A}} & 0 \\ F_x & F_y & 0 \end{vmatrix} = \left(x_{\mathrm{A}}F_y - y_{\mathrm{A}}F_x\right)\mathbf{k} = M\mathbf{k} \tag{5.3}$$

Since **M** has only one nonzero component in the *z*-direction, the above equation is reduced to a scalar equation of a line in *x*–*y* plane.

$$x_{\mathrm{A}}F_y - y_{\mathrm{A}}F_x = M$$

This is equation of a line representing a locus of points where we can move the force **F** in order to reduce the force-couple system to one force only. It is should not be a surprise, since one can move force along its line of action without changing the effect of this force of a rigid body. To draw this line, one can find its intersection with x- and y-axes by setting first $x_{\mathrm{A}} = 0$ and solving for

$$y_{\mathrm{A}} = -M/F_x \tag{5.4}$$

then, setting $y_{\mathrm{A}} = 0$ that will result in a second point of intersection

$$x_{\mathrm{A}} = M/F_y \tag{5.5}$$

> To reduce a force-couple system to one force only, the force and couple must be perpendicular to each other.

Example 5.3 Two workers are moving a box as shown in Fig. 5.6a. The worker on the right applies a couple of forces ($F_1 = F_2 = 50$ N) and that on the left pushes the box with a force of 100 N. The distance between the lines of action for forces $\mathbf{F_1}$ and $\mathbf{F_2}$ is 0.8 m. The line of action for the force F_3 passes from the point [0, 0, 0] to the point [0.8, 0.5, 0.0] m in the coordinate system shown in Fig. 5.6a. Reduce this system of forces (a) to force-couple system at A and, (b) if possible, to a single force.

> Two vectors **A** and **B** are perpendicular to each other when their dot product is equal to zero:
>
> $$\mathbf{A} \cdot \mathbf{B} = 0$$

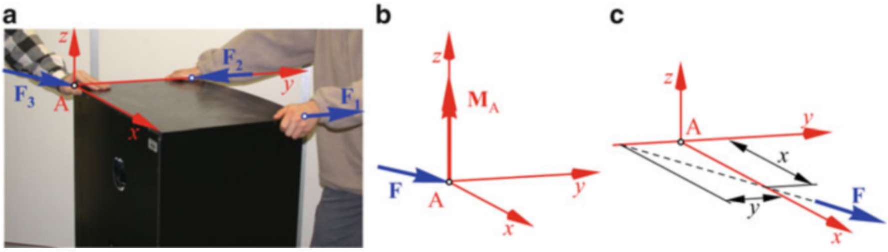


Fig. 5.6 (**a**) Workers moving the box. (**b**) The equivalent force-couple system. (**c**) The resultant force

Solution Let's show all forces applied to the box in the rectangular coordinate system as shown in Fig. 5.6a. Each force can be represented as a product of its magnitude and its unit vector. In our case,

$$\mathbf{F}_1 = 50\mathbf{j}\ \text{N}$$

$$\mathbf{F}_2 = -50\mathbf{j}\ \text{N}$$

Forces $\mathbf{F}_1$ and $\mathbf{F}_2$ represent a couple

$$\mathbf{M}_A = \mathbf{r}_1 \times \mathbf{F}_1 = 0.8\mathbf{i} \times 50\mathbf{j} = 40\mathbf{k}$$

where $\mathbf{r}_1 = 0.8\mathbf{i}$ is the vector pointing from force $\mathbf{F}_2$ to force $\mathbf{F}_1$.

(a) The force-couple system at A equivalent to the given system of forces will consist of a resultant force $\mathbf{F}$ and a couple $\mathbf{M_A}$.

$$\mathbf{F} = \sum_{i=1}^{2} \mathbf{F}_i = 50\mathbf{j} + (-50\mathbf{j}) + 84.8\mathbf{i} + 53.0\mathbf{j} = (84.8\mathbf{i} + 53.0\mathbf{j})\ \text{N}$$

$$\mathbf{M}_A = 40\mathbf{k}$$

Thus, the equivalent force-couple system at A is

$$\mathbf{F} = (84.8\mathbf{i} + 53.0\mathbf{j})\ \text{N}$$

and

$$\mathbf{M}_A = (40\mathbf{k})\ \text{N m} \quad (\text{Figure 5.6b}).$$

(b) Now, the force-couple system consists of the couple $\mathbf{M_A}$ and force $\mathbf{F}$. Since they are perpendicular their scalar product is equal to zero, $\mathbf{M_A} \cdot \mathbf{F} = \mathbf{0}$. We can move the resultant force **F** to another location (x, y) in order to reduce the force-couple system to one force only. Using (5.3), we will get

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = (x\mathbf{i} + y\mathbf{j}) \times (84.8\mathbf{i} + 53.0\mathbf{j}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & 0 \\ 84.8 & 53.0 & 0 \end{vmatrix}$$

$$= (53.0x - 84.8y)\mathbf{k} = 40\mathbf{k}$$

or in the scalar form

$$53.0x - 84.8y = 40.0$$

To find the intersection of the line of action of a single resultant force with x-axis and y-axis, we use (5.4) and (5.5) and obtain

$$x = 0.755 \text{ m} \quad \text{and} \quad y = -0.472 \text{ m}$$

The position of the resultant force is shown in Fig. 5.6c.

5.1.4 General Case of a Force-Couple System

Generally, the vectors of a resultant force and resultant moment are not perpendicular to one another. Let's represent the moment vector **M** as a sum of two vectors, one—perpendicular ($\mathbf{M_N}$) and one—parallel ($\mathbf{M_P}$) to the resultant force **R** acting on the point A (Fig. 5.7a). The couple $\mathbf{M_N}$ and resultant force **R** are perpendicular to one another and thus may be substituted by a single force **R** acting along a new line of action (Fig. 5.7b). Thus, we now have the force **R** acting along the new line of action, called *axis of wrench*, which is parallel to its original line of action. The

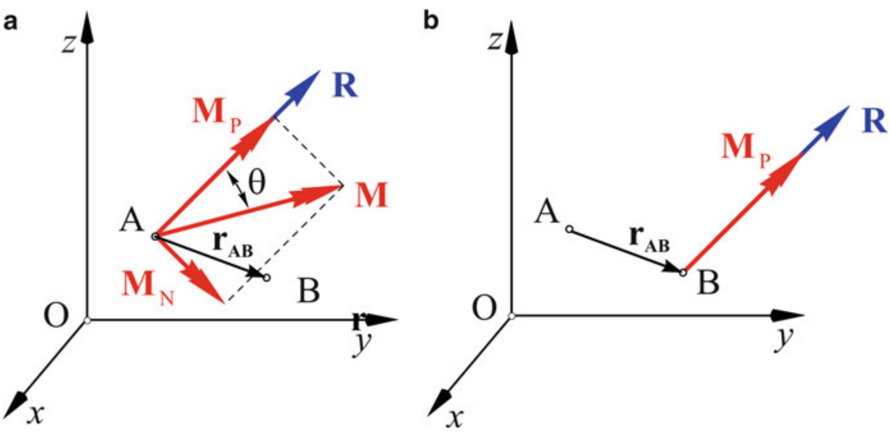


Fig. 5.7 General case of force-couple system

position of a point along the new line of action that is parallel to **R** may be found from the following relationship.

$$\mathbf{r_{A/B}} \times \mathbf{R} + \mathbf{M_N} = \mathbf{0}$$

Such system is called a *wrench*, since its action may be described as a combination of a push (or pull) and twist about the axis of the push (or pull). When the force and moment vectors have the same direction, the wrench is defined as positive.

The axis of the wrench may be defined by a position vector **r** of a point along this axis.

The preceding paragraphs showed that any system of external forces and moments acting on a rigid body might be reduced to an equivalent force-couple system or even to a single force in some cases. The resultant force causes translation of a rigid body, while the moment causes rotation.

Example 5.4 Two friends are moving a box as shown in Fig. 5.8. The coordinate system is shown. The friend from the left applies a couple of forces that create a moment $\mathbf{M} = 40\mathbf{i} + 35\mathbf{j} + 40\mathbf{k}$ Nm and that on the right pushes with the force $\mathbf{F} = -20\mathbf{i} + 15\mathbf{j} - 25\mathbf{k}$ N applied at the point A (0.7, 0.1, 0.2) m. Determine (a) the wrench force and moment and (b) the point, where the axis of the wrench intersects the x–z plane.

Solution Let's resolve the moment vector **M** as a sum of two vectors, one—perpendicular ($\mathbf{M_N}$) and one—parallel ($\mathbf{M_P}$) to the force **F**.

Fig. 5.8 Two friends are moving a box

The projection of vector **M** on direction of the force **F** may be figured out as following. The dot product **M·F** will give the projection of the magnitude of the vector M on the direction of the vector **F**. This direction **λ** of the vector **F** is

$$\boldsymbol{\lambda} = \frac{\mathbf{F}}{|\mathbf{F}|} = \frac{-20\mathbf{i} + 15\mathbf{j} - 25\mathbf{k}}{35.4} = -0.565\mathbf{i} + 0.424\mathbf{j} - 0.706\mathbf{k}$$

By multiplying this projection of the magnitude by the unit vector in the direction of the force **F** will result in the $\mathbf{M_P}$. Thus,

$$\mathbf{M}_\mathrm{P} = (\mathbf{M} \cdot \boldsymbol{\lambda})\,\boldsymbol{\lambda} \;= 20.3\mathbf{i} - 15.26\mathbf{j} + 25.4\mathbf{k}\ \text{ Nm}$$

The perpendicular component of the moment **M** is

$$\mathbf{M_N} = \mathbf{M} - \mathbf{M_P} = 19.7\mathbf{i} + 15.3\mathbf{j} + 14.6\mathbf{k}\ \text{ Nm}$$

Now, we can reduce the force **F** and $\mathbf{M_N}$ to one force only since they are mutually perpendicular (Sect. 5.1.3). Let's define the point B where the wrench interests the *x*–*z* plane.

$\mathbf{r_B} = \mathrm{x}\mathbf{i} + \mathrm{z}\mathbf{k}$; therefore, we can find the coordinates of the wrench intersect from $(\mathbf{r_A} - \mathbf{r_B}) \times \mathbf{F} = \mathbf{M_N}$

$$[(0.7 - x)\mathbf{i} + 0.1\mathbf{j} + (0.2 - z)\mathbf{k}] \times (-20\mathbf{i} + 15\mathbf{j} - 25\mathbf{k}) = 19.7\mathbf{i} + 50.3\mathbf{j} + 14.6\mathbf{k}$$

$$x = -0.133\,\mathrm{m} \quad \text{and} \quad z \;=\; 1.673\,\mathrm{m}$$

5.1.5 Moment of a Force About an Axis

Let us again consider force acting on a rigid body as we did in Sect. 5.1.1. Moving it to an arbitrary point resulted in introduction of a vector quantity—moment **M**. Sometimes, it is necessary to determine the moment of a force not about a point, but rather about a given axis or specific direction. We define the moment $\mathbf{M}_\mathrm{AB}$ about the axis AB as the projection of the moment **M** of the force **F** on the axis AB (Fig. 5.9).

Consider a door hinged along an axis AB and pushed by a force **F** (Fig. 5.9). First, we calculate moment of the force **F** about any point along the axis AB as $\mathbf{M} = \mathbf{r} \times \mathbf{F}$ and next find projection of this vector on the direction AB; in other words, a component of the vector M on direction AB. It is easily done by using a dot product. Thus, $M_\mathrm{AB} = \mathbf{M} \cdot \mathbf{AB} = (\mathbf{r} \times \mathbf{F}) \cdot \mathbf{AB}$. **AB** is a unit vector along axis AB. This may be expressed as

$$M_\mathrm{AB} = \begin{vmatrix} \mathrm{AB}_x & \mathrm{AB}_y & \mathrm{AB}_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

Fig. 5.9 Hinged door with the applied force

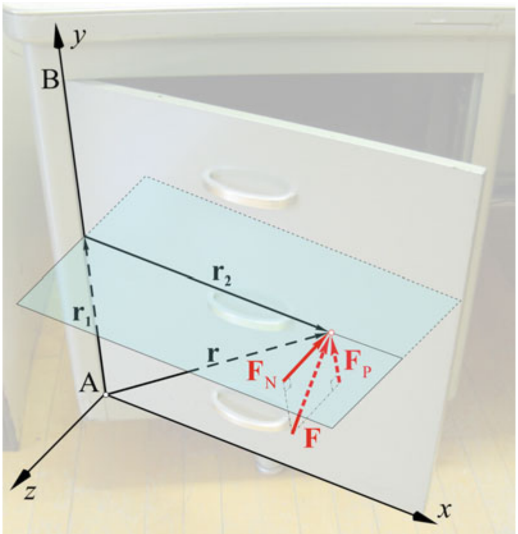


Fig. 5.10 Applied force split into components parallel to the axis AB and component lying in the plane perpendicular to AB

where AB_x, AB_y, and AB_z are direction cosines of the given axis AB. x, y, and z are the coordinates of any point along vector **F** and F_x, F_y, and F_z are components of force **F**.

Such expression is called mixed triple product of **r**, **F**, and **AB**.

Let us resolve force **F** into two rectangular components $\mathbf{F_p}$ along AB and $\mathbf{F_n}$ lying in the plane perpendicular to **AB**, the same for the vector $\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$ (Fig. 5.10).

$$M_{AB} = (\mathbf{r} \times \mathbf{F}) \cdot \mathbf{AB} = \left[(\mathbf{r_1} + \mathbf{r_2}) \times \left(\mathbf{F_p} + \mathbf{F_n}\right)\right] \cdot \mathbf{AB}$$
$$= \left(\mathbf{r_1} \times \mathbf{F_p}\right) \cdot \mathbf{AB} + (\mathbf{r_1} \times \mathbf{F_n}) \cdot \mathbf{AB} + \left(\mathbf{r_2} \times \mathbf{F_p}\right) \cdot \mathbf{AB} + (\mathbf{r_2} \times \mathbf{F_n}) \cdot \mathbf{AB}$$

First three mixed triple products are equal to zero since all three vectors are "coplanar," i.e., they belong to the same plane, thus

$$M_{AB} = (\mathbf{r_2} \times \mathbf{F_n}) \cdot \mathbf{AB}$$

The term $(\mathbf{r_2} \times \mathbf{F_n})$ is a vector parallel to the axis AB, and it represents the moment of the $\mathbf{F_n}$ about the point where AB intersects the plane. The other force component $\mathbf{F_p}$ has no tendency to rotate the body around axis AB since it is parallel to this axis.

Example 5.5 For the hinged door (Fig. 5.9), the force $\mathbf{F} = 20\ (3\mathbf{i} + 5\mathbf{j} - 4\mathbf{k})$ N is acting at the point (40, −10, 5) cm; point A's coordinates are (0, −30, 0) cm and B's coordinates are (0, 0, 0). Determine the moment of the force **F** about the AB.

Solution Axis AB may be defined by a unit vector along direction AB as

$$\mathbf{AB} = (0\mathbf{i} + 30\mathbf{j} + 0\mathbf{k})/30 = \mathbf{j}$$

and $\mathbf{r} = 40\mathbf{i} - 10\mathbf{j} + 5\mathbf{k}$, thus

$$M_{AB} = (\boldsymbol{r} \times \boldsymbol{F}) \cdot \mathbf{AB} = [(40\mathbf{i} - 10\mathbf{j} + 5\mathbf{k}) \times 20(3\mathbf{i} + 5\mathbf{j} - 4\mathbf{k})] \cdot (\mathbf{j})$$

$$M_{AB} = 3500\ \ \mathrm{M \cdot cm.}$$

5.1.6 Problems

5.1 A force $\mathbf{F} = 12\mathbf{i} + 21\mathbf{j} - 15\mathbf{k}$ acts on a point A (4, −7, 9). Calculate the moment of the force about the origin of the coordinate system.

5.2 A bucket B with a person weights 200 lb. Determine the moment of this force about the elbow A. Use AB = 20 ft and assume that it makes angle of 10° with the horizontal.

Fig. P5.2

5.3 A person (W = 600 N) is climbing a tree. He is supported by a rope that makes angle of 30° with the vertical axis of the tree. Assume that his weight is applied at the point of the rope attachment. His left leg makes an angle of 80° with the vertical axis. Determine the moment of the force exerted by his weight about the left leg contact with the tree. The distance between the point of the rope attachment and the left leg contact with the tree is 1.1 m.

Fig. P5.3

5.4 Use the parameters of the problem 5.3 to determine the moment of the force exerted by the rope about the left leg contact with the tree.

5.5 Determine the moment of the forces $\mathbf{F}_1$ and $\mathbf{F}_4$ about point O. Use the following as the force magnitudes: $F_1 = 3p$ and $F_4 = 2p$.

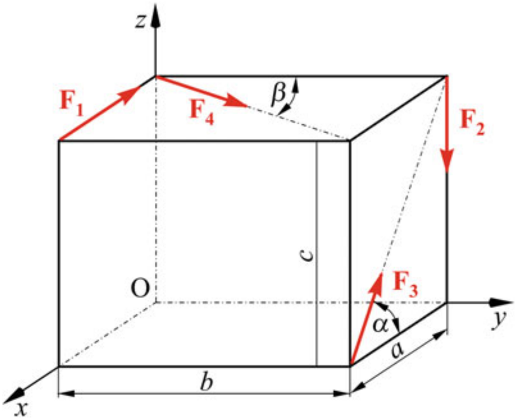


Fig. P5.5

5.6 Determine the moment of the forces $\mathbf{F}_2$ and $\mathbf{F}_3$ about point O (Fig. 5.5). Use the following as force magnitudes: $F_2 = p$, $F_3 = p$. Use $b = 0.1$ m, $c = 0.2$ m, $p = 10$ N, and $\alpha = 30°$.

5.7 Calculate the moment of a force $\mathbf{F}$ about the point L. DL = KL = CL = a.

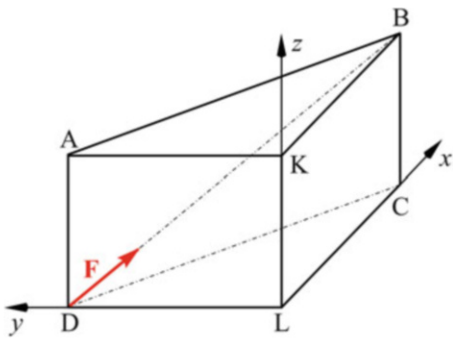


Fig. P5.7

5.8 Replace the force $\mathbf{F} = 5\mathbf{i} - 3\mathbf{j} + 8\mathbf{z}$ acting at the point A (2, −1, 4) by the equivalent force system at the point B(−3, 2, 5).

5.9 Replace the force $\mathbf{F} = -2\mathbf{i} - 7\mathbf{j} + 3\mathbf{z}$ acting at the point A (−4, 1, −2) by the equivalent force system at the point B(4, 1, −6).

5.10 Replace the force $\mathbf{F} = -4\mathbf{i} + 3\mathbf{j} - 5\mathbf{z}$ acting at the point A (1, 7, −3) by the equivalent force system at the point B(−2, −2, 7).

5.11 Four forces: $\mathbf{F_A} = 4\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$, $\mathbf{F_B} = -2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$, $\mathbf{F_C} = -4\mathbf{i} - \mathbf{j} - 5\mathbf{k}$, and $\mathbf{F_D} = \mathbf{i} - 7\mathbf{j} + 6\mathbf{k}$ are acting at the points A (1, 4, 2), B (−3, 0, 4), C (2, −3, −2), and D (−3, 0, 0). Find the resultant force and resultant moment if the resultant force is acting at the point B.

5.12 Calculate the moment of the force (800 N) about the *z*-axis a basketball player applied on the rim when he hangs from it. The player placed his hand 30° off the *x*-axis, the radius of the rim is 25 cm, and the distance from the *z*-axis to the rim's center (d) is 30 cm.

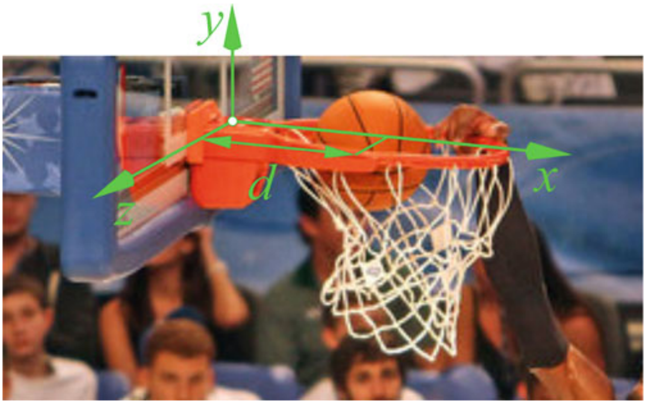


Fig. P5.12

5.13 A pyramid with a square base (size "a") is acted upon by the forces **Q** and **F**. Determine the resultant force and its moment about the point A, edge OA, and the diagonal BD.

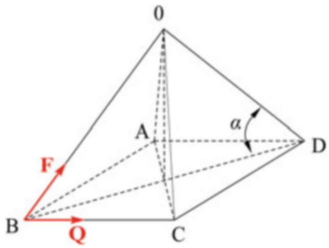


Fig. P5.13

5.14 The cube with a side of 20 cm is acted upon the system of forces as shown. Magnitude of each force is equal to 10 N. Replace these forces with an equivalent force-couple system at the point A.

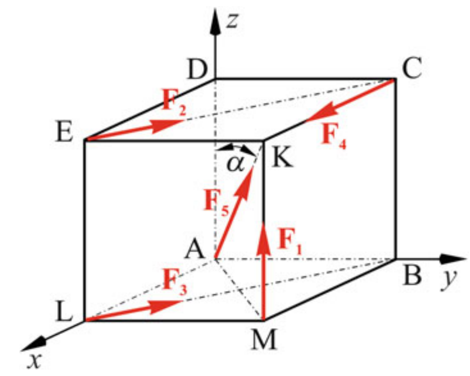


Fig. P5.14

5.15 The force-couple system consisting of the force (0, 3, 4) kN and the moment (−1, 2, 5) kN m is acting at the point A (1, 2, 3). Replace this system with the equivalent force-couple system at the point B (2, −3, 4).

5.16 The forces $\mathbf{F}_1 = (0, 1)$ N, $\mathbf{F}_2 = (1, -1)$ N, and $\mathbf{F}_3 = (2, 2)$ N are acting at the points A (1, 1), B (2, 1), and C (−1, 0), respectively. Reduce the system to a single resultant force **R** and determine its line of action.

5.17 Reduce the forces $P_1 = 4$ kN, $P_2 = 6$ kN, $P_3 = 8$ kN, and $P_4 = 10$ kN to the resultant force and moment about the point O. All the dimensions are in meters.

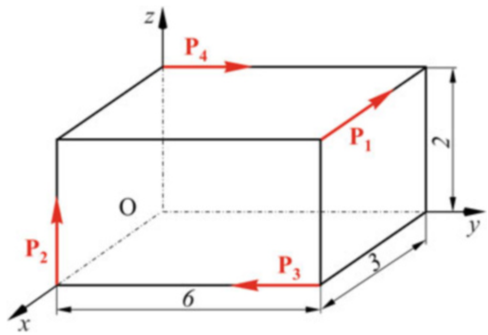


Fig. P5.17

5.18 The box is loaded by the forces $\mathbf{F_A} = 4\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$, acting at A (0, 1, 1), $\mathbf{F_B} = -4\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$, acting at B (1, 0, 1), and $\mathbf{F_C} = 4\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$, acting at C (1 ,−1, 1). Reduce these forces to a force-couple system. Is it possible to reduce them to one force only?

5.19 A crate is loaded by a force (−2, 12, −4) kN acting at the point A (−1, 6, −2) and the moment (12, 4, 6) kN m. Is it possible to reduce this force-couple system to one force only? If yes, where this force will cross the XY plane?

5.20 A power transmission system consists of three pulleys—A and B have belts acting in the horizontal direction and C has a belt acting in the vertical direction. The diameters are 0.15, 0.60, and 0.25 m (from left to right). The tensions in the belts are 500, 1500, and 1000 N. The distance AD = 0.25 m, DB = 0.15 m, and BC = 0.40 m. The whole assembly is supported by a frictionless journal bearing D. Reduce all forces exerted by the belts to the force-couple acting at the point D (neglect the size of the bearing).

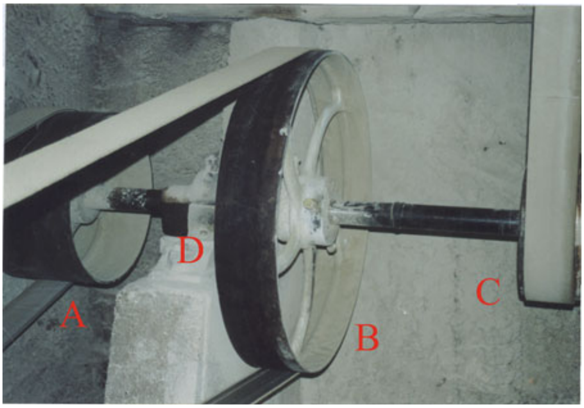


Fig. P5.20

5.21 Replace the resultant force $\mathbf{F} = -4\mathbf{i} + 3\mathbf{j} - 5\mathbf{z}$ **N** and couple $\mathbf{M} = 12\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$ Nm by the equivalent wrench.

5.22 For the given force $\mathbf{R} = 12\mathrm{i} - 4\mathbf{j} + 9\mathbf{k}$ **N** acting at the point O (2, 1, 4) and the couple $\mathbf{M} = -2\mathbf{i} - 5\mathbf{j} + 12\mathbf{k}$ Nm determine the equivalent wrench and find where the line of action of the wrench's force intersects the x–z plane.

5.23 Three forces are acting on the body: $\mathbf{F}_1 = \mathbf{F}_2 = -8\mathbf{i} - 9\mathbf{j} + 5\mathbf{k}$ N and $\mathbf{F}_3 = 18\mathbf{i} + 11\mathbf{j} - 6\mathbf{k}$ N. Calculate the resultant force and moment. The force $\mathbf{F}_1$ acting at the point A (1, 0, 0), $\mathbf{F}_2$ at the point B (1, 2, 0), and $\mathbf{F}_3$ at the point C(2, 2, 0). Determine the equivalent wrench.

5.24 Consider the force $\mathbf{F} = 4\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$ N acting at the origin and couple $\mathbf{M} = -8\mathrm{i} - 9\mathbf{j} + 5\mathbf{k}$ N. Represent this force and moment by a wrench and determine where the line of action of $\mathbf{F}$ intersects the y–z plane.

5.2 Equilibrium in Two Dimensions

All problems in nature are three-dimensional since we are living in a three-dimensional world. However, in many cases, the structure may have a plane of symmetry, i.e., both in geometry and loading. In such case, the equilibrium of the structure may be treated as a two-dimensional problem, assuming that the external loads are acting in the same plane. In addition, the load distribution must be known. Such example is shown in Fig. 5.11a depicting a gymnast on the beam, the physical

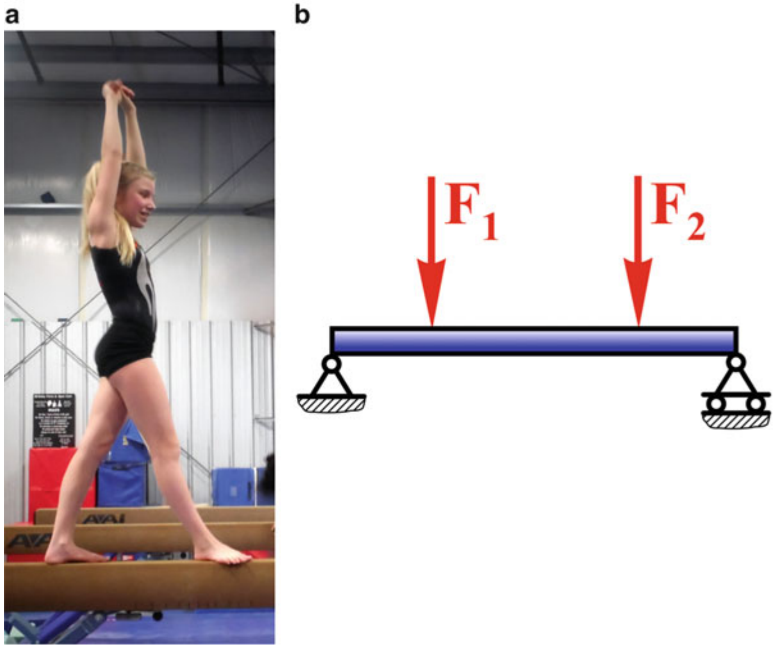


Fig. 5.11 (**a**) Gymnast on the beam. (**b**) Physical model

model is shown in Fig. 5.11b. The beam has symmetry along its longitudinal axis in the vertical plane. The load (the weight of the gymnast) is acting in the same vertical plane.

> Solving real problems in two dimensions requires simplifications that introduce errors and reduce the accuracy of the solution. The allowed degree of simplification relies on the engineering intuition and is based on experience.

Another less obvious example of the problem which is usually modeled as a two-dimensional case is shown in Fig. 5.12a, its physical model is shown in Fig. 5.12b. The chair and the person have a common plane of symmetry assuming the weight of the person is equally distributed among all four legs of the chair. If the structural elements of interest are the legs of the chair, the problem may be modeled as a two-dimensional case.

Therefore, the problems that satisfy the following conditions may be treated as two-dimensional cases:

- The objects are having a plane of symmetry.
- All external forces and reactions are acting in this plane.
- All external moments are being perpendicular to this plane.

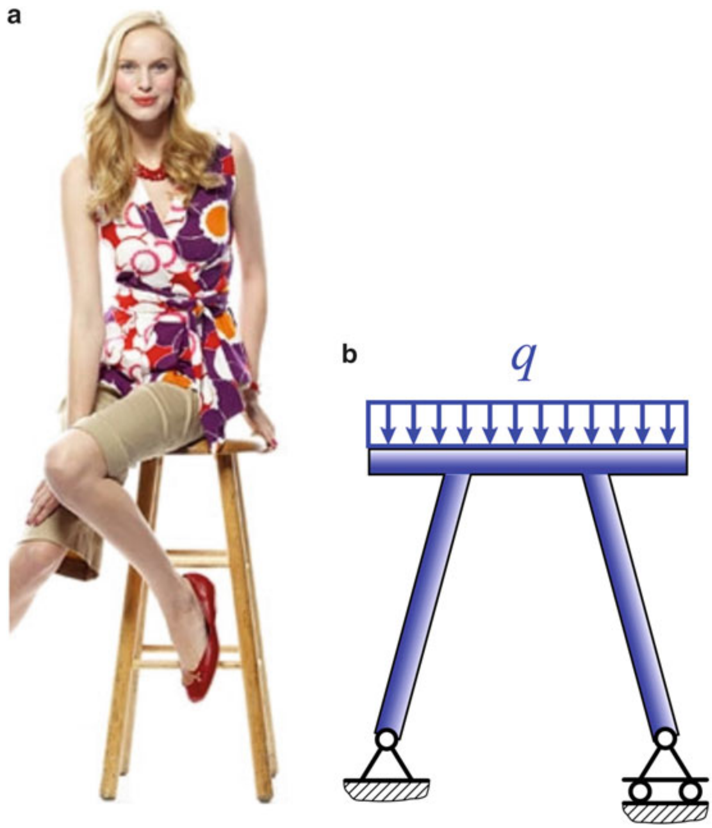


Fig. 5.12 (**a**) Person on the chair. (**b**) Physical model

In both examples above, we are dealing with modeling of real-life problems. Solving the real problems requires simplifications that introduce errors and thus reduce the accuracy of the solution. The allowed degree of simplification relies on the engineering intuition and is based on experience.

In Sect. 5.1, we showed that any force system may be reduced to a single resultant force and a single resultant moment. According to the First Newton's law, the system is in equilibrium when the resultant force and resultant moment are equal to zero.

$$\mathbf{R} = \sum_{i=1}^{N} \mathbf{F}_i = 0 \quad \text{and} \quad \mathbf{M} = \sum_{i=1}^{N} \mathbf{M}_i = 0 \tag{5.6}$$

Or in the scalar form

$$\begin{aligned} \sum F_x &= 0 \qquad & \sum M_x &= 0 \\ \sum F_y &= 0 \qquad & \sum M_y &= 0 \\ \sum F_z &= 0 \qquad & \sum M_z &= 0 \end{aligned} \tag{5.7}$$

> There are ONLY three independent equations of equilibrium in a two-dimensional case.

In the case of two-dimensional problems, we rewrite the equations of the equilibrium (5.7). Let's assume that all forces are acting in the plane *x*–*y*, thus each force has only two components and the moment has only one.

$$\begin{aligned} R_x &= \sum F_x = 0 \\ R_y &= \sum F_y = 0 \\ M &= \sum M_z = 0 \end{aligned} \tag{5.8}$$

while the rest of equations in (5.7) are automatically satisfied.

The three equations may be solved for maximum of three unknowns. The above set of three equations is not the unique way of satisfying the equilibrium condition.

> The problem may be treated as two-dimensional when:
>
> - Objects are having a plane of symmetry.
> - External forces and reactions are acting in this plane.
> - External moments are being perpendicular to this plane.

There are additional ways in which three equations of equilibrium in two dimensions may be expressed. Let us start with the set of equilibrium equations (5.8) describing the loading schematically shown in Fig. 5.13a. Resolve force **R** into components $\mathbf{R_x}$ and $\mathbf{R_y}$ (Fig. 5.13b). Next, let us move the force $\mathbf{R_y}$ to a new location B. By doing so, we have to introduce an additional moment $M_B = d{\cdot}R_y$, where d is the distance between the lines of actions of the forces at the points A and B (Fig. 5.13c). Now, we have the forces R_x and R_y and moments M_B and M_A. The equilibrium requires that the resultant force and the resultant moment be equal to zero. Thus, $M_B = 0$, but it is equal to zero only when $R_y = 0$. Therefore, there is no need to write an explicit equation $R_y = 0$, and we may use the alternate set of equilibrium equations.

This means that instead of the three equations (5.8), an alternative set of equilibrium equations can be used, assuming that the points A and B *do not* coincide.

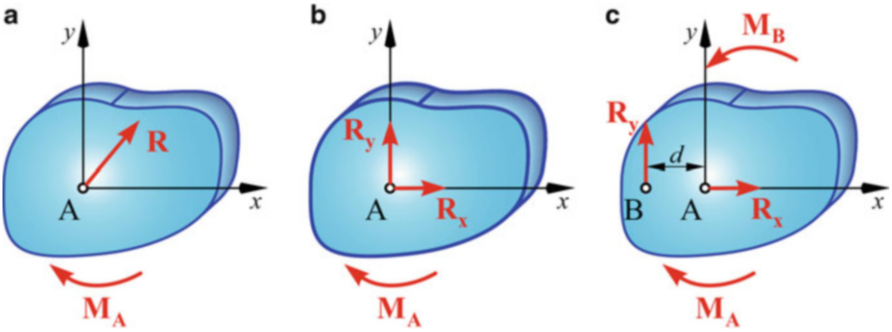


Fig. 5.13 Schematic of equilibrium equations: (**a**) Force and Moment; (**b**) Moment and two force components; (**c**) Two Moments and one force component

$$\begin{aligned}\sum F_x &= 0\\ \sum M_{\mathrm{A}} &= 0\\ \sum M_{\mathrm{B}} &= 0\end{aligned} \tag{5.9}$$

Similarly, it may be shown that the first equilibrium equation in (5.9) may be replaced by another moment equation.

$$\begin{aligned}\sum M_{\mathrm{C}} &= 0\\ \sum M_{\mathrm{A}} &= 0\\ \sum M_{\mathrm{B}} &= 0\end{aligned} \tag{5.10}$$

where the subscripts A, B, and C mean sum of moments about the corresponding points. It should be stressed that all three points A, B, and C should not belong to the same straight line.

It is important to note that only three of the above equations (5.8), (5.9), and (5.10) are linearly independent; therefore, we can solve them for three unknowns only. Any combination of those equations can be used to guarantee the equilibrium of a structural element. The only reason to use any combination other than (5.8) is to reduce the effort required to solve them. However, usually the simplest way is to write three independent linear equations (5.8) and to solve them using any of the available computational tools, such as MATLAB, etc.

Example 5.6 The driver applied pressure to the brake pedal (Fig. 5.14a) with the force of 300 N. Determine the tension in the cable CD and the reaction force at the frictionless pin B. All dimensions are in centimeters.

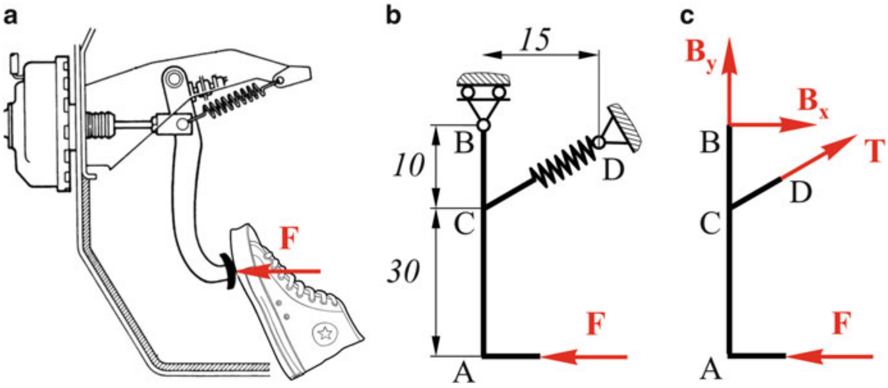


Fig. 5.14 (**a**) Brake pedal. (**b**) Physical model. (**c**) Free body diagram

Solution From Fig. 5.14a a physical model is drawn as shown in Fig. 5.14b. A free body diagram of the brake pedal is shown in Fig. 5.14c. The cable CD exerts a tension **T** on the brake in the direction of the cable, i.e., the direction of the tension **T** is known. At the pin B neither the magnitude nor the direction of the reaction force are known. Force **F** is known, since its magnitude is given and the direction may be extracted from the picture. Let's place the origin of coordinate system at the point A and express each force as a vector.

$$\mathbf{F} = -300\mathbf{i}$$

$$\mathbf{T} = T(15\mathbf{i} + 10\mathbf{j})/18.03$$

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j}$$

For the system to be in equilibrium, the following conditions should apply: $\sum \mathbf{F} = 0$ and $\sum \mathbf{M} = 0$

or in scalar form

$$\sum F_x = 0.832T - 300 + B_x = 0$$

$$\sum F_y = 0.555T + B_y = 0$$

$$\sum M_B = 300 \cdot 40 - 0.832T \cdot 10 = 0$$

From the last equation we can get T = 1442 N. This result is substituted into the first and second equations to yield $B_y = -800$ N and $B_x = -900$ N.

The negative values of the force components indicate that the force **B** components act in direction opposite to the shown in Fig. 5.14c.

Using the sum of moments about point A rather than point B would create a set of three simultaneous equations with three unknowns that could be solved by a long process of elimination or by using a numerical tool, like MATLAB.

You may use the MATLAB routine *equilibriumBody2D.m* (http://extras.springer.com) to solve this problem. Start MATLAB, run the routine, and enter all pertinent information.

Example 5.7 A bike rider applied the front hand brakes (Fig. 5.15a) with the force of 100 N. Consider only half of the brake assembly as a structural element. Determine the force applied to the tire and the reaction force at pin B.

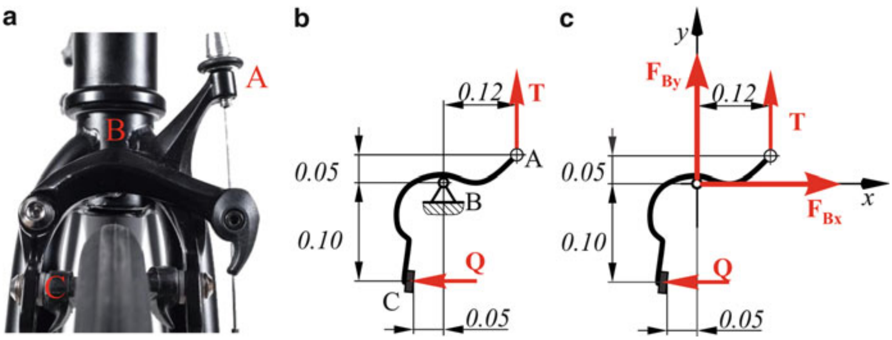


Fig. 5.15 (**a**) Bike brake assembly. (**b**) Physical model. (**c**) Free body diagram

Solution Since the hand brake is applied to the system attached to the flexible cable, the force acting on the point A (Fig. 5.15a) is equal to the force applied to the brake handle. From Fig. 5.15a a physical model is drawn as shown in Fig. 5.15b. All dimensions are in meters.

This is a two-dimensional problem, it has three unknowns: two components of the reaction force **B** and the horizontal force **Q** acting from the tire on the bracket. Let us set a coordinate system at the point B (Fig. 5.15c). Now, we have to express each of the forces through its rectangular components and use them in the equations of equilibrium. The force **T** is acting in the vertical direction; therefore, its components are $T_y = 100$ N and $T_x = 0$. The force **B** has two unknown components B_x and B_y. Now, we are ready to write the equations of equilibrium.

$$\sum F_x = 0 \Rightarrow B_x - Q = 0$$
$$\sum F_y = 0 \Rightarrow B_y + T_y = B_y + 100 = 0$$
$$\sum M_B = 0 \Rightarrow 0.12 \cdot T_y - 0.1 \cdot Q = 0.12 \cdot 100 - 0.1 \cdot Q = 0$$

Solution of the above set of three equations results in

$$B_y = -100 \ N$$
$$B_x = 120 \ N$$
$$Q = 120 \ N$$

MATLAB Solution This problem may also be solved by using the MATLAB routine *equilibriumBody2D.m*. Start the MATLAB and execute this routine. The box will appear explaining how to use this routine and how to input the data. Read it and click OK. Next, dialog box will ask you to input number of forces and moments. In this problem, we have three forces and zero moments, enter these values and click OK. Next, box will ask to input the following information for each force (input values are shown in *italic*). Let us assign to the force **T** number 1, force **B** number 2, and force **Q** number 3. Below is input for the force #1.

Magnitude (force)……………… *100*
X component of the line of action……*0*
Y component of the line of action……*1*
X coordinate of force vector …*0.12* (i.e., point of force application)
Y coordinate of force vector ……*0.05*

Click OK and fill the data for the rest of the forces, enter x for any unknown value.

Input for the force #2:

Magnitude (force)……………………*X*
X component of line of action……… *X*
Y component of line of action………*1*
X coordinate of force vector ………*0*
Y coordinate of force vector ……*0*

It should be noted that since we do not know the magnitude and direction of the force **B**, we set its magnitude as unknown and, let say, direction *x* as unknown. For direction y, we can enter any value since the vector is defined by only two parameters and the third one is dependent on the values of the first two.

And finally, for the third force we input the following information:

Magnitude (force)……………………*X*
X component of line of action……… *−1*
Y component of line of action………*0*
X coordinate of force vector ……… *−0.05*
Y coordinate of force vector ……… *−0.1*

Click OK and the solution will appear as:

Force #	Magnitude	*X* component	*Y* component
1	100.0	0.000	100.0
2	156.2	120.0	−100.0
3	120.0	−120.0	0.000

The result will be also shown on a sketch showing the locations, magnitudes, and directions of all forces.

Guidelines and Recipes for Solving an Equilibrium Problem in Two Dimensions

- Select a coordinate system.
- Draw a physical model.
- Show all reactions, external forces, and moments.
- Create a free body diagram.
- Represent all forces and moments in vector notation.
- Write two vector equations of equilibrium and represent them through the corresponding three scalar equations.
- Solve the system of scalar equations for three unknowns.

5.2.1 Two-Force Members

Sometimes, structural elements are loaded by only two forces. Consider an example of a bar connecting the ski lift chair to the cable (Fig. 5.16a). Assuming a frictionless pin attachment at the top, the physical model may be represented as shown in Fig. 5.16b and the corresponding free body diagram in Fig. 5.16c. For this system to be in equilibrium, the moment of forces **F** and **W** about any axis should be equal to zero. Taking the sum of moments, for example, about the point A leads to the conclusion that the line of action of force **W** must pass through point A. Similarly, taking the sum of moments about point B leads to the conclusion that the line of

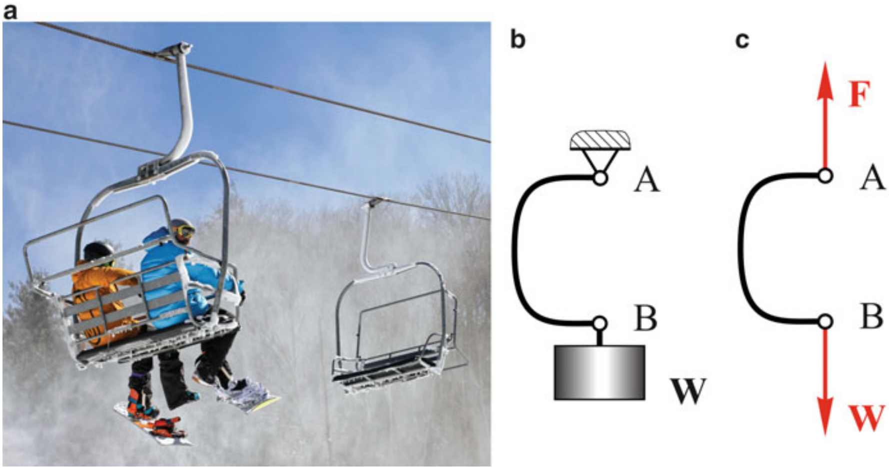


Fig. 5.16 (**a**) Lift chair. (**b**) Physical model of the bar. (**c**) Free body diagram

action of force **F** must pass through point B. This is possible only if both forces have the same line of action, or in other words, are acting along the same line. By taking the sum of forces along this line of action, we conclude that they have to have the same magnitude, but act in opposite directions (Fig. 5.16c). These two forces represent an *equilibrium pair of forces*, as described in Chap. 2. It is important to recognize the two-force members in order to simplify solutions of certain problems.

Two-force body in equilibrium is loaded by two forces with the same line of action, same magnitude, but acting in the opposite directions.

5.2.2 Three-Force Members

When only three coplanar forces are acting on a body, it is called a three-force body. For such a body to be in equilibrium, the lines of action of the three forces must be concurrent (Fig. 5.17a), i.e., intersect in one point, or the forces must be parallel and coplanar (Fig. 5.17b).

When a body loaded with three coplanar, nonparallel forces is in equilibrium, the three forces have to intersect at the same point.

Let us consider the case shown in Fig. 5.17a. Assume that the two of the forces, e.g., $\mathbf{F}_1$ and $\mathbf{F}_2$ will intersect in point A. For a body to be in equilibrium, the sum of the moments must be equal to zero. Moments of forces $\mathbf{F}_1$ and $\mathbf{F}_2$ with respect to the point A are always equal to zero. Thus, the moment of force $\mathbf{F}_3$ with respect to the same point has to be equal to zero. This is the case only when the line of action of force $\mathbf{F}_3$ passes through the same point A. Therefore, we effectively reduced the

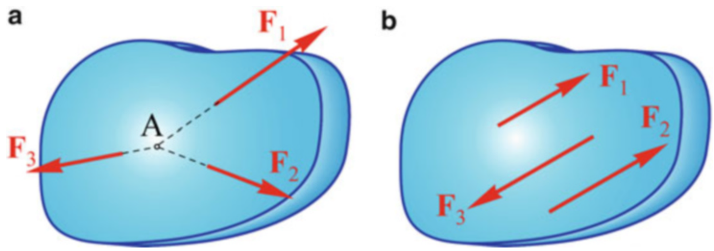


Fig. 5.17 (**a**) Concurrent forces. (**b**) Parallel forces

problem of a body loaded by three forces to a problem of the forces acting at a point A. Now, we have only two equations of equilibrium $\sum F_x = 0$ and $\sum F_y = 0$.

In the case when three forces are parallel to one another, equilibrium is possible only when one of the forces is in the opposite direction to the other two, and it is located between the two. The location of the middle force is dependent on the magnitudes of the other two forces. The seesaw (Fig. 5.18a) is a good example of such case; its free body diagram is shown in Fig. 5.18b.

Example 5.8 Use the three-force body approach to determine the force which

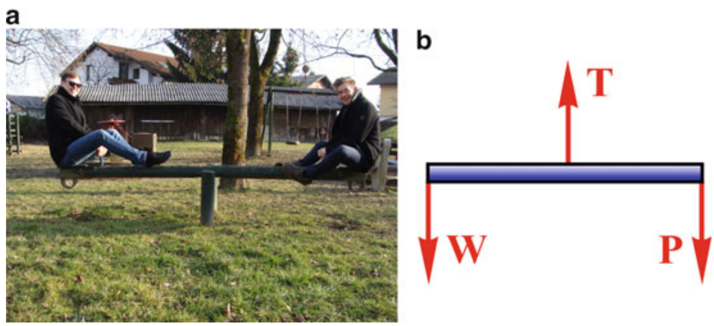


Fig. 5.18 (**a**) Seesaw. (**b**) Free body diagram

worker has to apply to the rope in order to keep the box, weighting 150 N, in equilibrium at the distance of 3 m from the wall (Fig. 5.19a). All dimensions are in meters.

Solution The free body diagram of the box is shown in Fig. 5.19b. We place the origin of the coordinate system at point O, which is the point where all three forces have to intersect (this is a three-force body). Now, we can use the same procedure as we used for solving the equilibrium problems for a point in two dimensions (Sect. 4.2.2).

The equilibrium equations are:

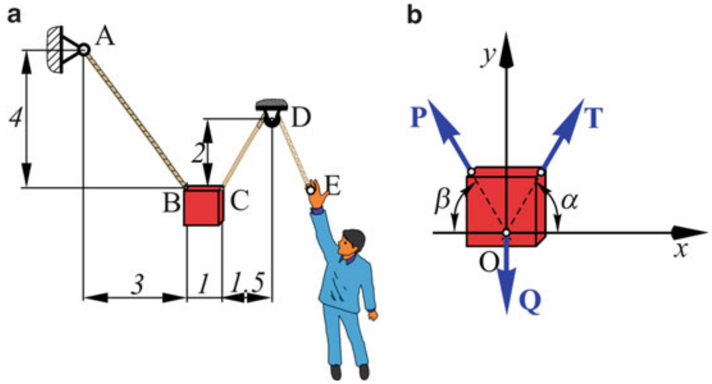


Fig. 5.19 (**a**) Physical model. (**b**) Free body diagram

$$\sum F_x = 0 \Rightarrow -P \cdot \cos\beta + T \cdot \cos\alpha = 0$$
$$\sum F_y = 0 \Rightarrow P \cdot \sin\beta + T \cdot \sin\alpha - Q = 0$$

The angles may be determined from the geometry:

$$\alpha = \tan^{-1}\left(\frac{2}{1.5}\right) = 53.1^\circ \quad \text{and} \quad \beta = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^\circ$$

The solution is: $P = 93.8$ N and $T = 93.8$ N. The worker has to apply force of 93.8 N in order to keep the box at the desired location.

Example 5.9 A box is loaded by forces A = 20 N and B = 50 N as shown in Fig. 5.20a. Determine the position and magnitude of the force C to keep the box in equilibrium. All dimensions are in meters.

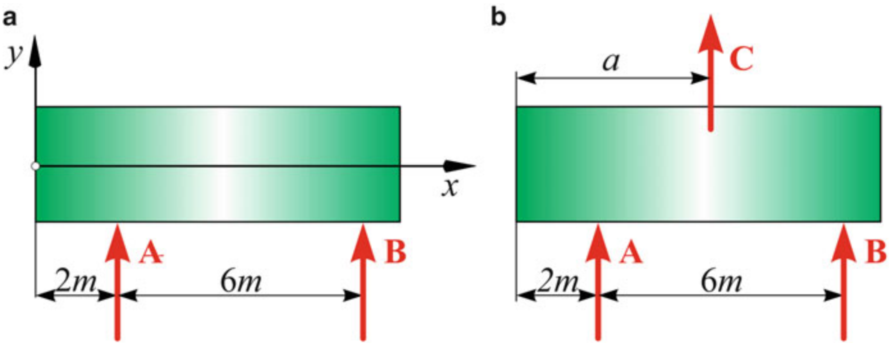


Fig. 5.20 (**a**) Box. (**b**) Free body diagram

Solution The free body diagram of the box is shown in Fig. 5.20b. Since the two known forces are parallel to one another, it is obvious that the third force should be also parallel to them in order to keep the box in equilibrium. We place the unknown force C at distance "a" from the coordinate system origin. The equations of equilibrium are

$$\sum F_x = 0$$

$$\sum F_y = 0 \Rightarrow A + B + C = 20 + 50 + C = 0 \Rightarrow C = -70 \text{ N}$$
$$\sum M_o = 0 \Rightarrow A \cdot 2 + B \cdot 8 + C \cdot a \Rightarrow a = 6.29 \text{ m}$$

The first equitation is satisfied since all forces are in the y-direction.

It is obvious that the position of the force C is between the forces **A** and B.

You may also use the MATLAB routine *equilibriumBody2D.m* to solve this problem. Start the MATLAB and run the *equilibriumBody2D.m*. The dialog box will appear with general explanation on how to use this procedure. Read it and click OK. The next dialog will ask you to input number of forces and number of moments acting in this problem. You input *3*—for number of forces and *0*—for number of moments. Now, you will be asked to input all relevant information for each of the three forces acting on the body. Let us input data for force **A** as force #1, **B** #2, and **C** #3. Remember to input ***x*** for any unknown value. The following table summarizes the input.

Force number	1	2	3
Magnitude	20	50	x
x component of the line of action	0	0	x
y component of the line of action	1	1	1
x coordinate of force application	2	6	x
y coordinate of force application	0	0	0

When a body loaded with three coplanar, parallel forces is in equilibrium, one of them should be opposite in direction and located between the other two forces.

We input x for unknown values of the third force magnitude, and x for its x component of the line of action (since we do not know the direction of this force). For its y component of the line of action, we input an arbitrary value (*1* in this case), since each force in two dimension is defined by two parameters. These could be its x and y components or magnitude and one of the components of the line of action. The result will appear as shown below.

Load # 3 the unknown parameter #1 is: −70.0000
Load # 3 the unknown parameter #2 is: 0.0000
Load # 3 the unknown parameter #4 is: 6.2857

Force #	Magnitude	X component	Y component
1	20.000	0.000	20.000
2	50.000	0.000	50.000
3	70.000	0.000	−70.000

First three lines provide the values of three unknowns we defined by x. The first parameter is the force magnitude. It has a negative sign since we prescribed its y-direction as +1, which is up and the correct direction is down. The second unknown is equal to zero, which is the x component of the line of action for the force **C**. The third is 6.29, which is the x coordinate of the force **C** application. Next three lines represent each force as a magnitude and as its x and y components. The MATLAB also draws a schematic of the forces as they applied to a rigid body.

Example 5.10 The ball is attached to the wall by cable AC (Fig. 5.21a). Assume that the weight of the ball is **P**, determine the tension **T** in the rope, and the reaction **R** at the wall/ball contact.

Solution The free body diagram of the ball is shown in Fig. 5.21b. Three forces are acting on the ball: its weight **P**, reaction from the wall **R**, and tension from the cable **T**. Since it is a three-body problem, all forces have to intersect at one point, O in this case, in order to keep the box in equilibrium. The equations of equilibrium are

$$\sum F_y = T\cos\alpha - P = 0$$

$$T = P/\cos\alpha$$

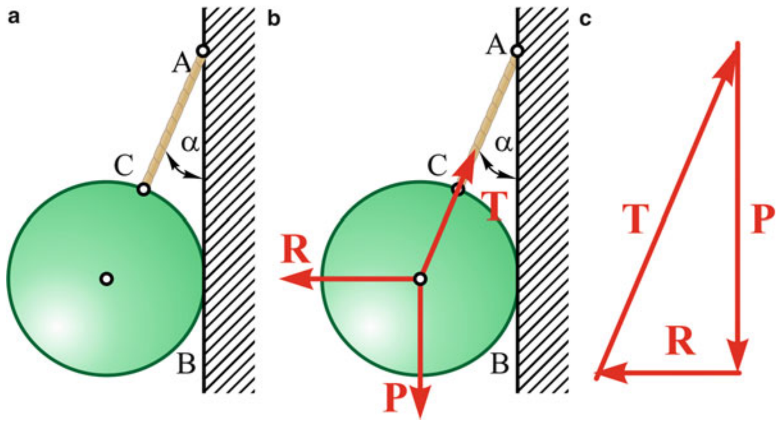


Fig. 5.21 The ball attached to the wall. (**a**) Physical model, (**b**) Free Body Diagram, (**c**) Graphical method

$$\sum F_x = T \sin \alpha - R = 0$$

$$R = T \sin \alpha = P \sin \alpha / \cos \alpha = P \tan \alpha$$

Alternatively, we can use the graphical method and draw a triangle of forces (Fig. 5.21c). Use the "sin" law to create the following relationships:

$$\frac{T}{\sin 90} = \frac{P}{\sin (90 - \alpha)} = \frac{R}{\sin \alpha}$$

$$T = P / \cos \alpha$$

and

$$R = T \sin \alpha = P \sin \alpha / \cos \alpha = P \tan \alpha$$

5.2.3 Problems

5.25 Determine force **F** to keep the wheelbarrow ($P = 80$ N) in equilibrium if $Q = 450$ N. All dimensions are in mm.

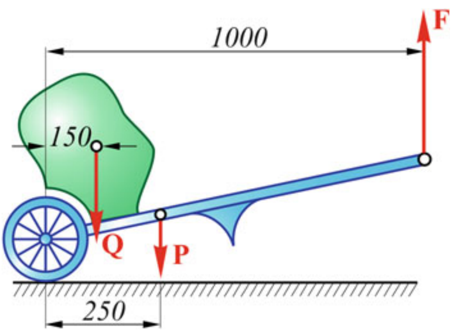


Fig. P5.25

5.26 Person ($W = 180$ lb) is standing on one foot. What is the contact force between the ground and heel? The dimensions are in inches.

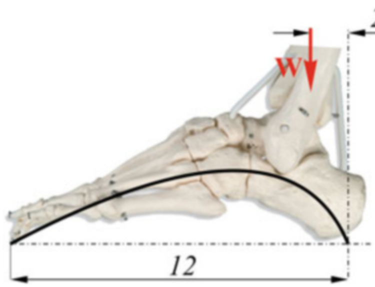


Fig. P5.26

5.27 Car is located on the bridge, length AB = 30 m. Loads on each axel are $Q_1 = 12$ kN, $Q_2 = 18$ kN, the distance between the axels 3 m. Determine distance x that will result in equal vertical components of the reactions at **A** and **B**.

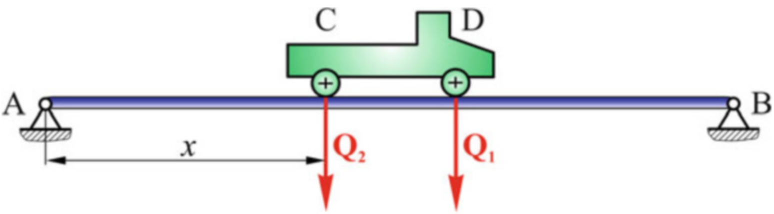


Fig. P5.27

5.28 Calculate the force acting on the axis of the rear wheel of the bicycle when the weight of the rider **W** is 500 N. The distance between the front and rear axles is 1.2 m, and the distance between the rear axis and the seat is 0.3 m. Neglect the bicycle weight.

Fig. P5.28

5.29 The front-end loader has total weight of 50 kN. Assume that the weight is applied at the midline between the front and rear wheels. By activating the cylinder, the loader is capable of lifting the front wheel from the ground by pushing the bucket against the ground. What is the force necessary to apply between the bucket and ground to lift the front wheel from the ground?

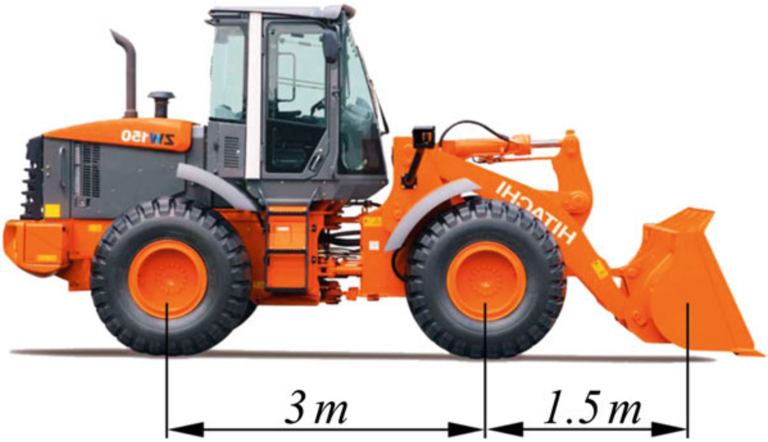


Fig. P5.29

5.30 Calculate the load acting on the barrel at D, if $a = 50$ cm, $b = 10$ cm, and $M = 50$ N.

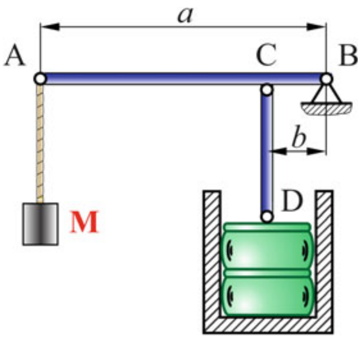


Fig. P5.30

5.31 A free standing bridge is supported by two columns at C and D. Its weight is 1.5 kN/m. Determine the maximum length l so that the bridge will not flip over when a truck will ride over it. Truck exerts load of 20 kN on the front axle and 25 kN on the rear. Distance between the axles is 3 m and 2d = 8 m.

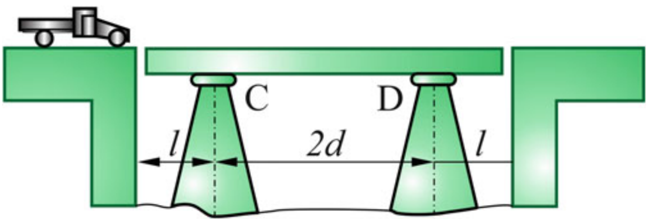


Fig. P5.31

5.32 Homogeneous rod OA is loaded by the barrel **Q** and is held in equilibrium by a block F. What is the length l of OA for the minimum value of **F** to keep the system in equilibrium, if $OB = b$ and rod OA weights q N/m?

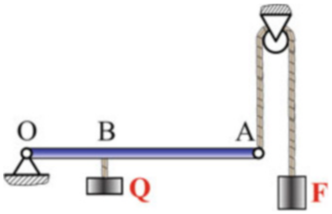


Fig. P5.32

5.33 The homogeneous rod AB, length 10 in. and weight 90 lb is suspended by two springs. Stiffness of the spring DL is twice of the spring EF. (Stiffness is the force needed to extend the spring by 1 cm). Determine the load **P** that has to be placed at point K to insure that the rod is horizontal. AD = BE = 2 in., DK = 1 in. When springs are unloaded, they have the same length.

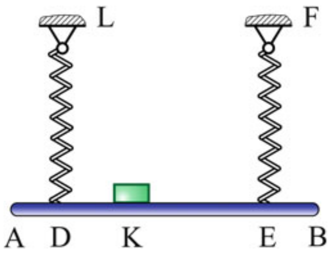


Fig. P5.33

5.34 Horizontal beam AB is loaded by two weights **C** (50 N) and **D** (10 N). At what value of x, the reaction force at A is twice the reaction force at B.

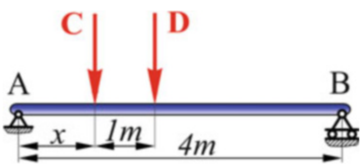


Fig. P5.34

5.35 The power transmission system consists of three pulleys: $P_1 = 20$ kN, $P_2 = 34$ kN, and $P_3 = 12$ kN, all acting in the downward vertical direction. At what distance "x" one should install the pulley $\mathbf{P_2}$ so that the reaction forces at **A** and **B** will be the same? Neglect the weight of the axis.

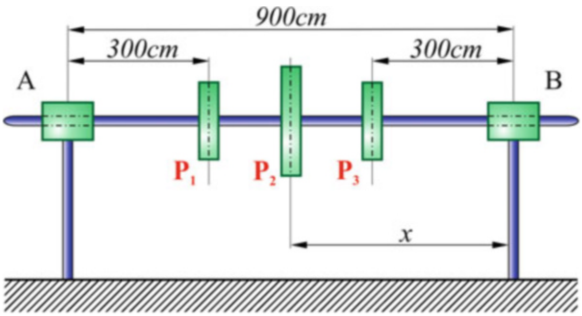


Fig. P5.35

5.36 Determine the reactions at A and B as function of the position of the cart C along the crane. Weight of the crane **W** = 60 kN, weight of the cart with the load **P** = 40 kN. AC/AB = 0.8. Model crane as a two-dimensional structure.

Fig. P5.36

5.37 Beam AB (weight 150 N) is loaded by the weight **Q** = 750 N located at the distance of 5 cm from the point A. The weight **P** = 200 N. What should be the length of the beam AB for system to be in equilibrium?

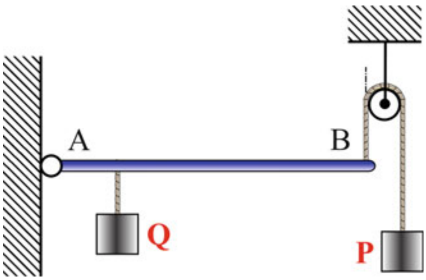


Fig. P5.37

5.38 Horizontal beam AB (length 10 m) is loaded as shown. AC = CD = DE = EF = FB = 2 m. Where should be located support so the beam will be in equilibrium?

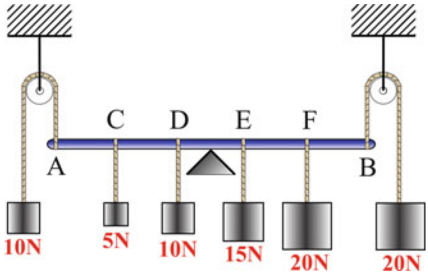


Fig. P5.38

Determine the reactions at A and B for the loads shown in the problems 5.39–5.42.

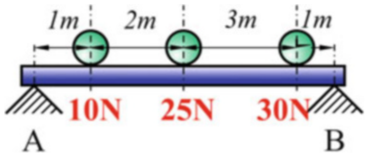


Fig. P5.39

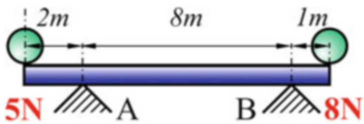


Fig. P5.40

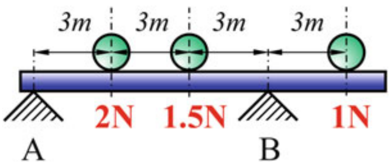


Fig. P5.41

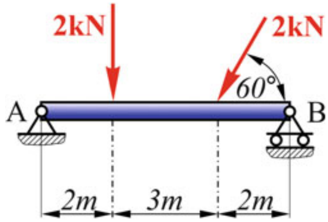


Fig. P5.42

5.43 Safety valve A (diameter d = 4 cm) of the pressure vessel is attached by the link AB to the bar CD (weight of the bar CD = 12 N). Determine the weight **Q** that will allow the valve to open when inside pressure will reach 8 N/cm^2. CD = 60 cm, BC = 6 cm.

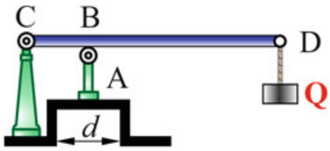


Fig. P5.43

5.44 Three barrels are kept in equilibrium by system of pulleys as shown. Determine the relationship between the barrels having weights $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$, and the tension in the cables. A radius of each pulley is given as r_1, r_2, r_3, r_4, and r_5.

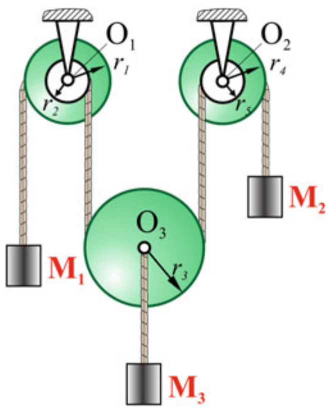


Fig. P5.44

5.45 Crane (weight $D = 500$ kN) is riding along the rail AB (weight $R = 30$ kN) with a load $P = 10$ kN. Horizontal distance between the left support A and mass center of the crane is 20 m, distance between the mass center and the load **P** is 30 m and rail AB is 70 m. Determine the reactions at A and B.

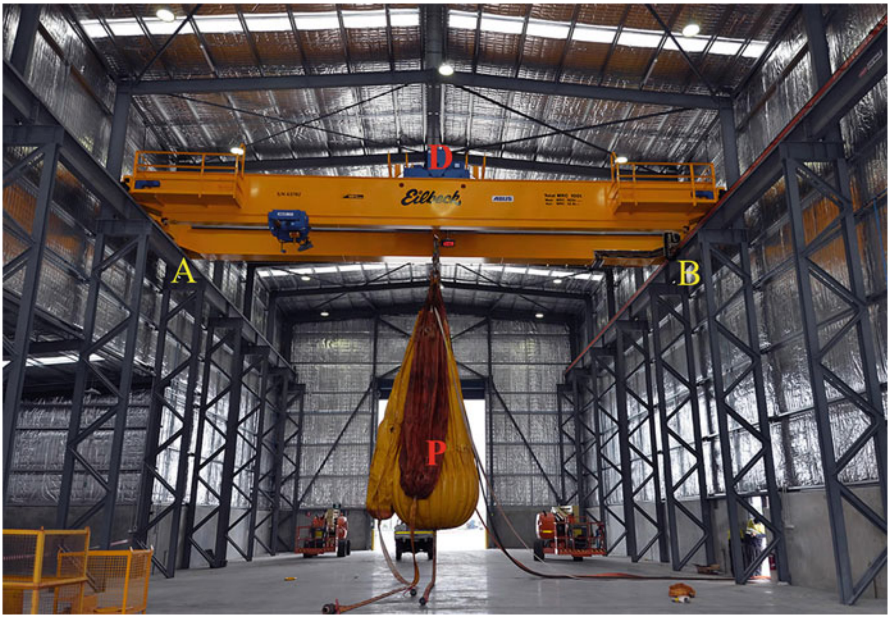


Fig. P5.45

5.46 Helicopter ($W = 10$ kN) made a crash landing on the crane. What is the tension in the vertical cable system that holds the crane in equilibrium? Weight of the horizontal boom $P = 500$ N, its length is 10 m, helicopter landed 1 m from its end. The distance between the vertical cables and the crane is 1 m.

Fig. P5.46

5.47 Power transmission tower is loaded by horizontal force **P** and by tension from the electrical wire attached to the pulley *D*. Weight of the tower is **G** and is acting along the midline *mn*. The counter balance has weight **M**. Determine the vertical component of the reaction at *A*.

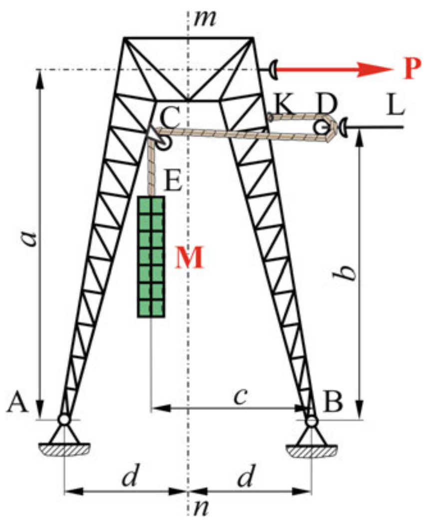


Fig. P5.47

5.48 30 kN pipe with radius $R = 1$ m is supported by walls as shown. Determine the forces exerted by the pipe on the wall at points A and B. Use $l = 1.6$ m.

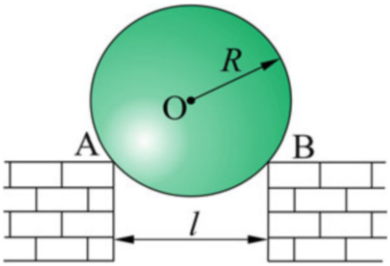


Fig. P5.48

5.49 Horizontal beam BC (weight $W = 600$ N) is built into the wall as shown. Determine the reactions at A and B if it is loaded by the weight $P = 5$ kN.

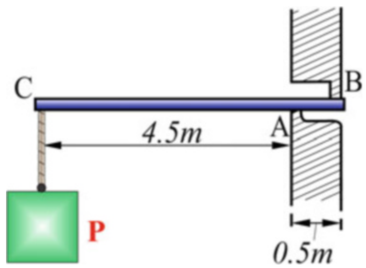


Fig. P5.49

5.50 The railroad crane mounted on the cart is riding along the tracks as shown. Weight of the cart applied at A is 20 kN, weight of the crane applied at C is 5 kN, weight of the counter balance *D* is 30 kN, and weight of the link FG applied at H is 7 kN. Determine the largest load **Q** that can be safely hoisted by the crane.

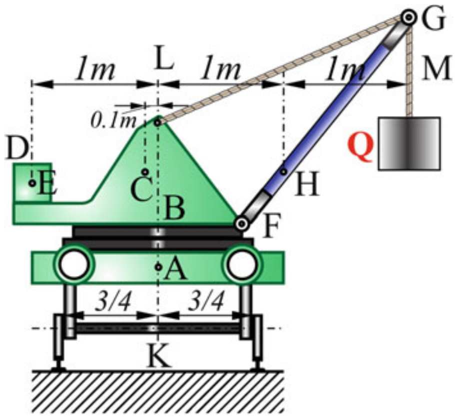


Fig. P5.50

5.51 Weight of the crane $P_1 = 500$ kN. It is designed to lift load $P_2 = 250$ kN. What should be the minimum weight of the counter ballast **Q** and the maximum distance *x* to assure that the crane will be stable with and without load $\mathbf{P_2}$?

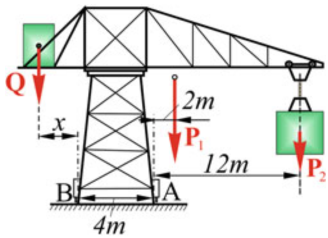


Fig. P5.51

5.52 A crane A moves along the rail *B* to place load **Q** in the opening on the right. Determine **P** (weight of the crane *A* and the holding column *D*) so that load $Q = 12$ kN will not lift the wheel *E* from the rail.

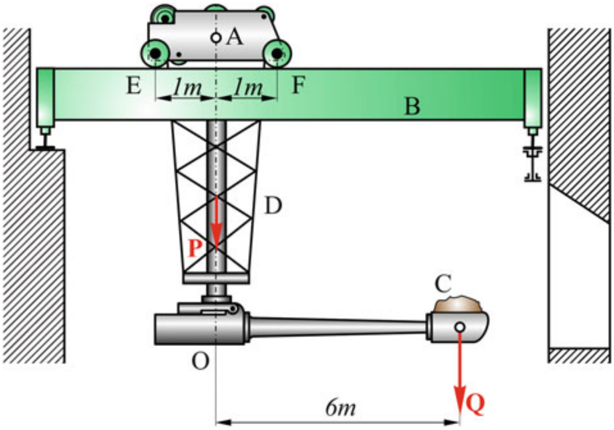


Fig. P5.52

5.53 Crane's weight of 35 kN is applied at the point A. The base of the crane is a square block (2 × 2 m) of concrete with specific weight of 40 kN/m^3. What should be the base height so the crane will be able to safely lift load $P = 40$ kN?

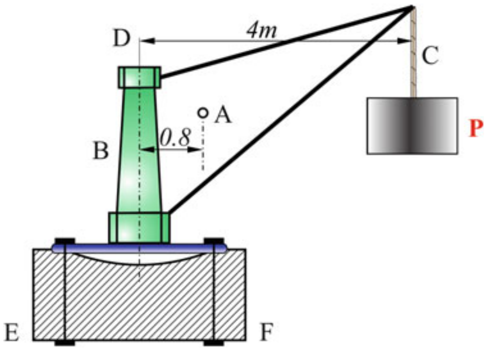


Fig. P5.53

5.54 The slender rod AB (weight $\mathbf{W} = 200$ N and length 10 m) is suspended by two cables and loaded by the weight $P = 100$ N. Determine the tension in each cables if AE = 3 m.

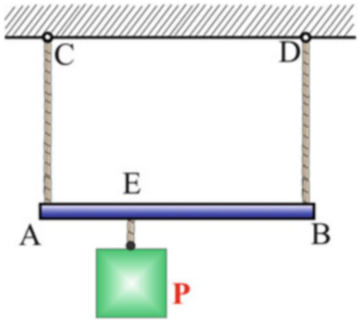


Fig. P5.54

5.55 Helicopter ($P = 4000$ lb) is hanging in the air. Assume that the lift $F_1 = 0.04$ P. Determine **d**, the distance between the axis of the main propeller and center of gravity for the craft and the lift force $\mathbf{F}_2$. $L = 15$ ft.

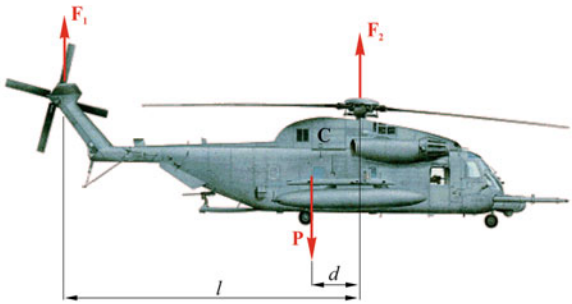


Fig. P5.55

5.56 The weight of the boom AB ($P = 3$ kN) is acting at point C. Determine the tension in the cable AD and the components of the reaction at B, if $Q = 10$ kN, $\alpha = \beta = 20°$, BC = 1/3 AB.

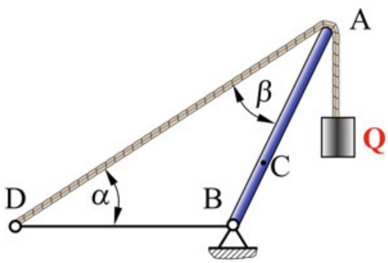


Fig. P5.56

5.57 The slender homogeneous rod AB (weight $W = 100$ N, length l) is held in the state of equilibrium by two cables AC and BC. Determine the tension in each cable. Consider equilibrium of the rod AB. Use AC = BC = l = 1 m.

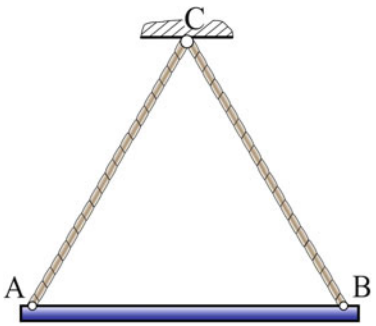


Fig. P5.57

5.58 The link loaded as shown. Determine the force **P** and reaction at A, when $P_1 = 50$ N, $P_2 = 100$ N, and $P_3 = 200$ N. All dimensions are in cm.

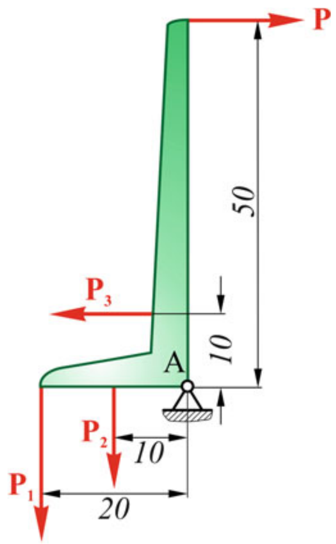


Fig. P5.58

5.59 Boom AD is loaded by load **M** and is held in equilibrium by cable BC. The weight of the boom AD $W = 2$ kN, $\alpha = 30°$, and the tension in the cable is 25 kN. Determine the weight of the load **M**, if AB = AC = 0.7 AD.

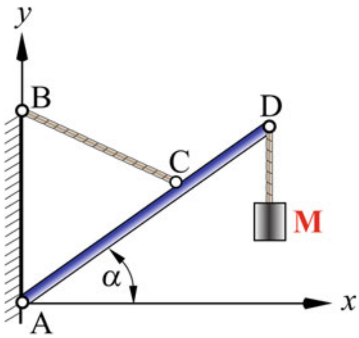


Fig. P5.59

5.60 Box (weight **P**) is lifted along the incline by block **M**. What should be the weight of the block **M** so the box will flip over when it is in the position shown?

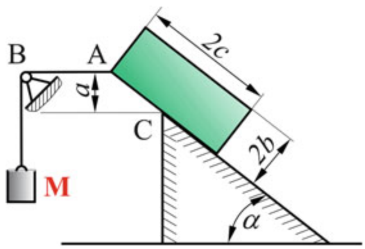


Fig. P5.60

5.61 The ball *M* (weight **Q** and radius *a*) and the weight **P** are suspended by the cables as shown. Determine the angle φ when the system is in equilibrium. Use $OM = b$.

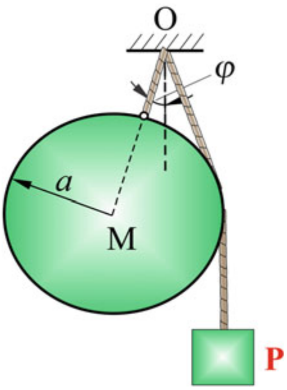


Fig. P5.61

5.62 The part of the gym exercise equipment is shown in Fig. P5.62. The tension in the cable is 500 N. The pulley is attached via frictionless bracket to the 0.1 m tall vertical holder AB. Calculate the horizontal components of the reaction forces acting at points A and B. The upper part of the cable is horizontal, while the left part makes angle of 10° with the vertical direction. The diameter of the pulley is 0.3 m. The distance between the center of the pulley and the holder AB is 0.25 m.

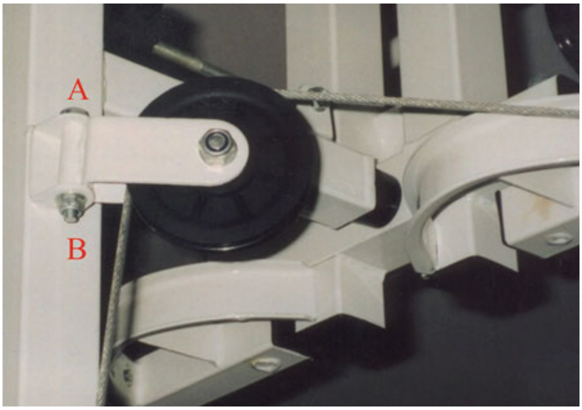


Fig. P5.62

5.63 The beam rests upon support B. Its weight of 300 N is acting at C. Determine the reactions at points A and B.

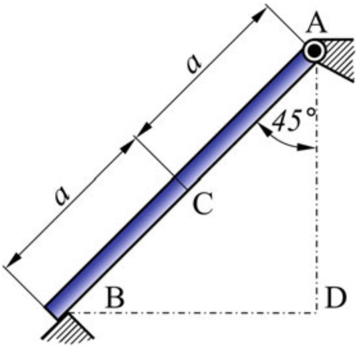


Fig. P5.63

5.64 A 30 kN roller is pulled by the horizontal force **P** over the frictionless obstacle *C*. Determine the force **P** required to move the roller over the obstacle. Force from the obstacle to the roller is perpendicular to the surface of the roller.

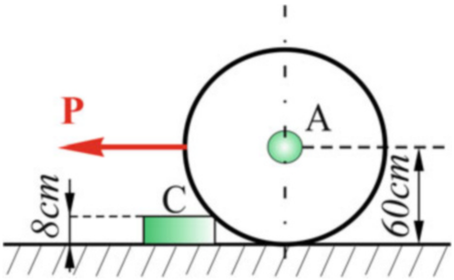


Fig. P5.64

5.65 The cover of the aircraft engine is supported by two links AB and BC. Determine the forces in each link, if cover weight is 10 kN and it is acting along the line passing through the point B, $\alpha = 25°$ and $\beta = 40°$.

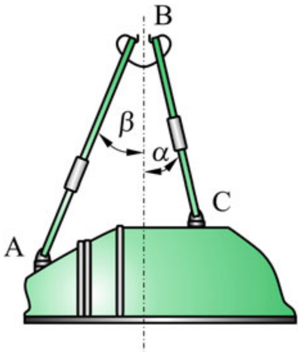


Fig. P5.65

5.66 The beam AB is held by the link CD as shown. Determine the reactions at A and D when $F = 15$ kN.

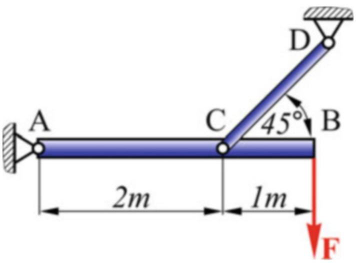


Fig. P5.66

5.67 The weightless beam AB is supported as shown and loaded by force $P = 10$ kN. Determine the reactions at A and B.

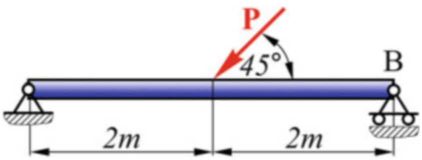


Fig. P5.67

5.68 The weightless beam AB is supported as shown and loaded by force $P = 10$ kN. Determine the reactions at A and B.

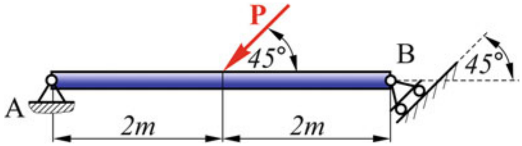


Fig. P5.68

5.69 The weightless beam CB is supported as shown and loaded by force $F = 5$ kN and $M = 10$ k N m. Determine the reactions at A and B, if $a = 3$ m, $l = 6$ m, and $\alpha = 20^0$.

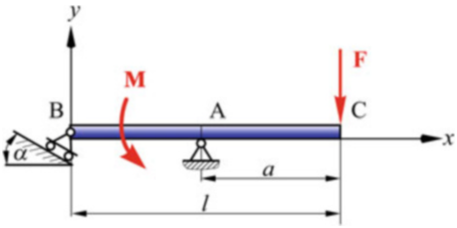


Fig. P5.69

5.70 The weightless beam AB is supported by the cable and loaded by force $F = 5$ kN and $M = 10$ k N m. Determine the reactions at A and C, if $a = 2$ m, $l = 4$ m, and $\alpha = 60°$.

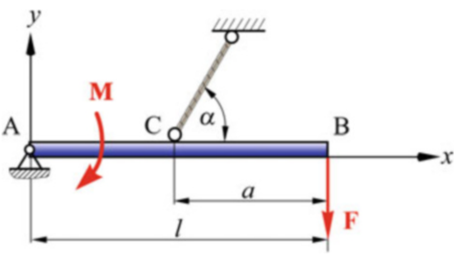


Fig. P5.70

5.71 Beam AB is loaded by the weight **P**. Determine the tension in the cable BC as function of the position "x".

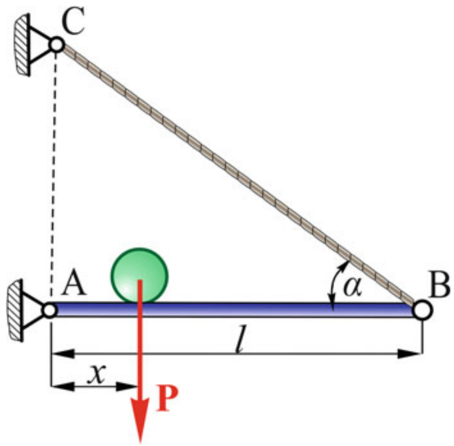


Fig. P5.71

5.72 Rigid link ABC weights 80 N, its weight is applied at point E which is 20 cm from the vertical line BD. Determine angle φ if system is in equilibrium. $P_1 = 300$ N, $P_2 = 100$ N, AB = 40 cm, BC = 80 cm.

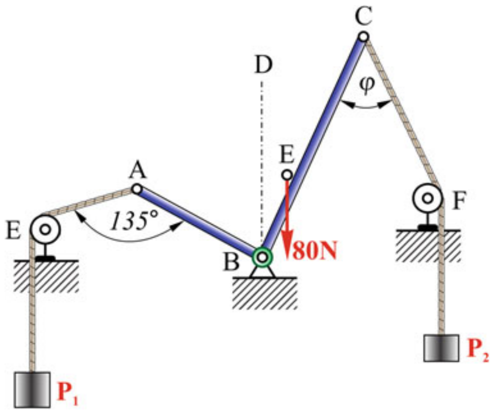


Fig. P5.72

5.73 The beam AB, length l, is loaded by a box, weight **M**. It leans on the smooth vertical wall at A and on the corner C. Determine the reactions at A and C, and length AC when the beam AB is in equilibrium.

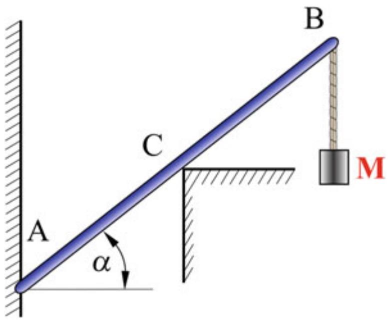


Fig. P5.73

5.74 Cube, weight **P** is loaded by horizontal force **Q**. Determine the minimum force **Q** necessary to start flipping the box over the point A.

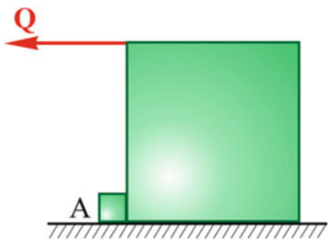


Fig. P5.74

5.75 A slender rod AB is held in equilibrium as shown. Determine the reactions at A, knowing that the weight of AB is 10 N.

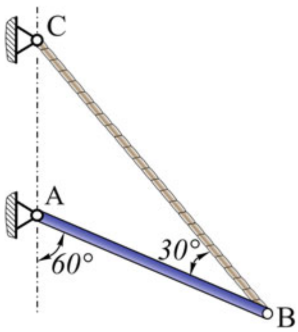


Fig. P5.75

5.76 The bar AB (weight **W**) is held open by the rope BCD. Determine the tension in the rope as function of the angle φ, when AB = AC.

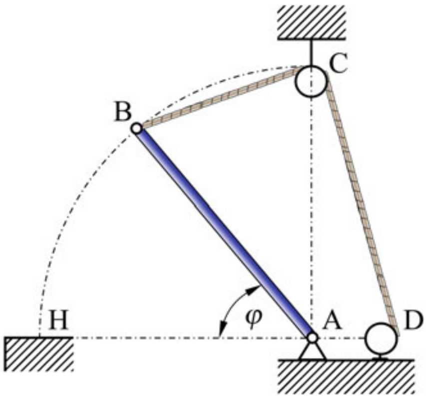


Fig. P5.76

5.77 A slender rod of length 2 m and weight 100 N is held in equilibrium as shown with one end against a frictionless wall and the other end attached to a rope BC. Determine the distance AC, tension in the rope BC, and the reaction at point A.

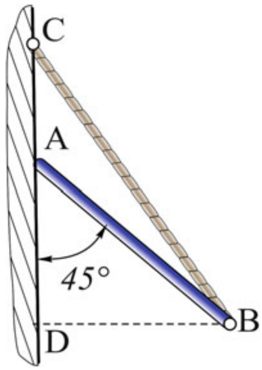


Fig. P5.77

5.78 Two masses are attached to B by the cables BC and BD and they are in a static equilibrium. Determine the angle α, if load C is **P**, load D is 2**P**, and weight of the boom AB is 2**P**. AB = AD.

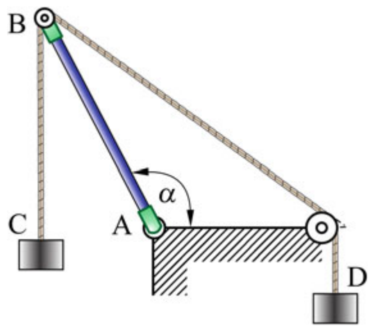


Fig. P5.78

5.79 Weightless plate ABCD (AB = a and BC = b) is attached by the pin at B and supported by the frictionless wall at A. Determine the reactions at A and B, when the weight **M** is suspended at C.

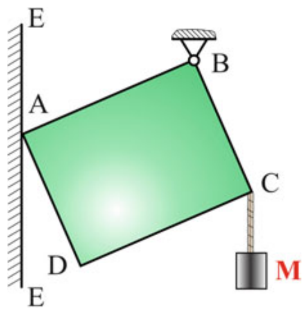


Fig. P5.79

5.80 The crane is lifting a crate having weight **M**. Determine the reactions at O and tension in the cable BC, when AB = OB = 2 m and OA = 3.5 m.

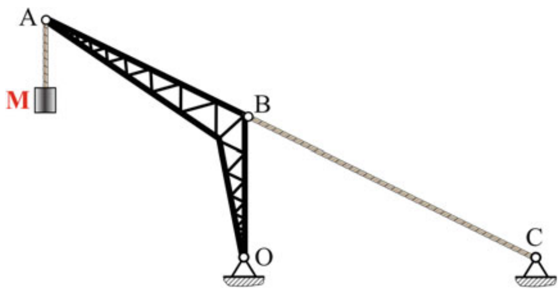


Fig. P5.80

5.81 Rod CB (weight **W**) is supported by two frictionless walls as shown. Determine reactions at C and B and angle θ, when the rod is in equilibrium.

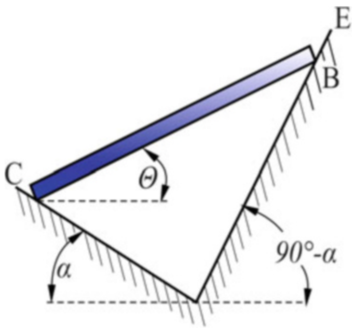


Fig. P5.81

5.82 Rod CB (weight 200 N) leans on the frictionless wall AB. It is held in equilibrium by the cable AC. Determine the tension in the cable AC and reactions at B and C.

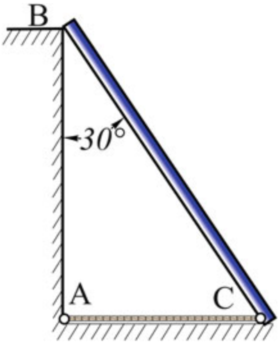


Fig. P5.82

5.83 Rod CB (weight 200 N and length $l = 4$ m) leans on the frictionless wall AB, 3 m tall. It is held in equilibrium by the cable AC. Determine the tension in the cable AC and reactions at B and C.

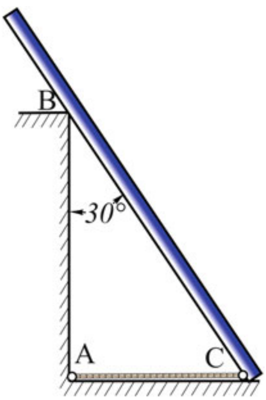


Fig. P5.83

5.84 Rod AB is supported in a frictionless device as shown. AB = 3 m, CB = 0.5 m, BD = 1 m, and weight of AB is 10 N. Determine the reactions at points B, C, and D.

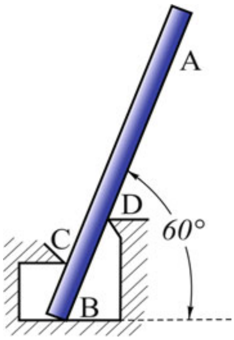


Fig. P5.84

5.85 The homogeneous rod AB (weight = 10 N) is supported by a smooth floor and an incline as shown. Determine the reactions at A and B and the mass P.

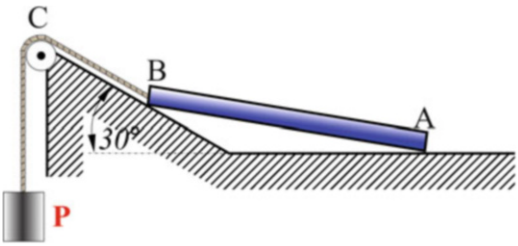


Fig. P5.85

5.86 The horizontal rod is suspended by three ropes. Its weight is 5 N and it is applied at the point D. Determine tension in each rope.

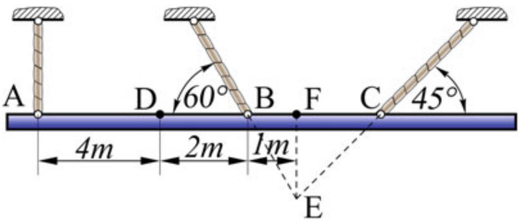


Fig. P5.86

5.87 The homogeneous plate AB ($W = 12$ kN) is supported by the wall at A and frictionless support at B. Determine the reactions at A and B, when $\sin\alpha = 0.5$.

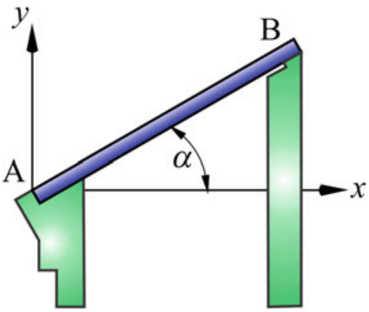


Fig. P5.87

5.88 The ladder AB (weight $W = 100$ N) is supported by a smooth wall at B and a step at A. A person (weight $F = 750$ N) stands on it at the point D that is at the 1/3 of the ladder's length from the bottom. Determine the reactions at A and B.

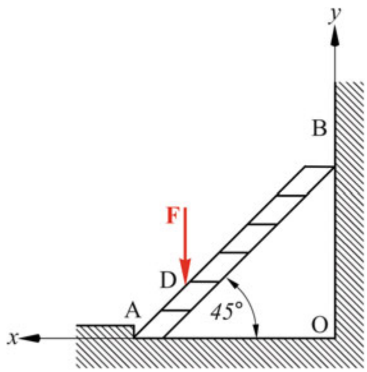


Fig. P5.88

5.89 A uniform ladder AB (length 16 m, weight 300 N) is suspended by the rope BC. A person (weight $G = 600$ N) is standing at the point D. Determine the tension in the rope BC and reaction at A.

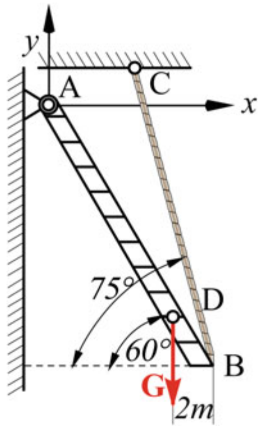


Fig. P5.89

5.90 Weightless plate BCED is loaded as shown and is held in equilibrium by cable AE. Diagonal BE is in the horizontal direction. Determine reactions at C and tension in the cable. $P_1 = 50$ N, $P_2 = 200$ N, angle BED = 60°, and BD = 4ED.

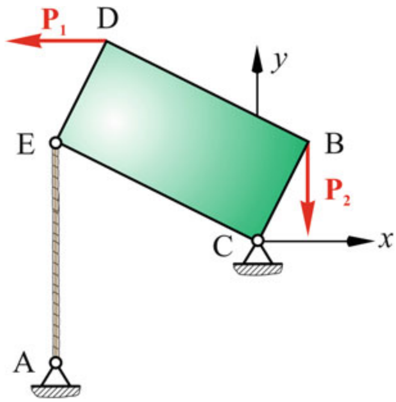


Fig. P5.90

5.91 A 20 kN cylinder Q is suspended from the homogeneous rod AB (weight 6 kN) and kept in the equilibrium by a barrel (weight **G**). Determine weight **G** and reaction at A, when BD = AB/4.

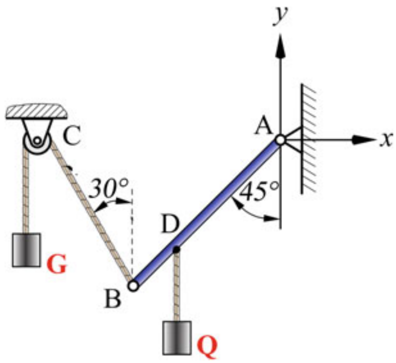


Fig. P5.91

5.92 The homogeneous L-shaped element ABC is suspended by the cable AD. Link BC is twice as long as the link AB. Determine the angle α.

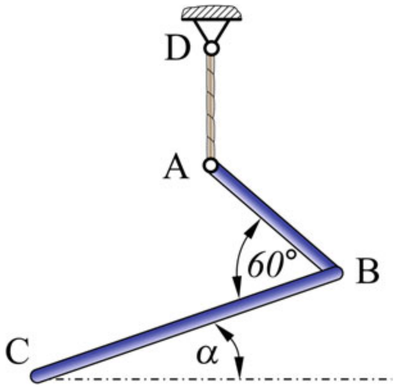


Fig. P5.92

5.93* The homogeneous plate ABD is attached to the floor by the pivot A. The weight of the unit area of plate is q and is applied at the point C, CA/CK = 2 when AK that is perpendicular to BD. What is the maximum value for angle α to keep the plate from flipping over the point A when force **P** applied as shown?

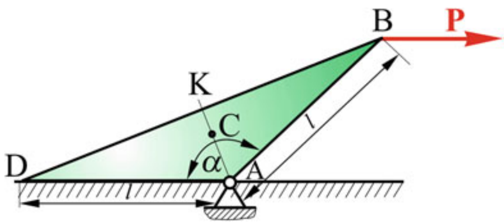


Fig. P5.93

5.3 Equilibrium in Three Dimensions

As it was shown in Sect. 5.1, a general three-dimensional system of forces and moments can be reduced to a resultant force–moment system. The condition of equilibrium may be expressed as shown in (5.6) and (5.7). In most problems, the scalar equations will be convenient to use and we can solve for no more than six unknowns.

$$\sum F_x = 0 \qquad \sum M_x = 0$$
$$\sum F_y = 0 \qquad \sum M_y = 0$$
$$\sum F_z = 0 \qquad \sum M_z = 0$$

This system of equations implies that the set of external forces and moments, which satisfy the above equations will not impose any motion to the rigid body, i.e., the rigid body is in the state of equilibrium in accordance with the First Newton's Law. These equations are called the *equations of equilibrium*.

The procedure for solution is very similar to the procedure outlined for a two-dimensional case, as seen below.

Guidelines and Recipes for Solving Equilibrium Problems in Three Dimensions

- Draw a physical model.
- Create a free body diagram; show all reaction and applied forces and moments.
- Select a coordinate system.
- Represent all forces and moments in vector notation.
- Write two vector equations of equilibrium and corresponding six scalar equations.
- Solve the system of equations.

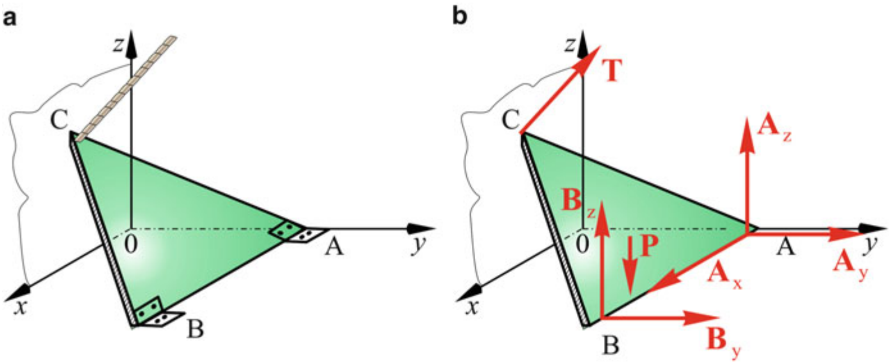


Fig. 5.22 (**a**) Physical model. (**b**) FBD

> Two vectors **A** and **B** are perpendicular one to another when their dot product is equal to zero: **A**·**B** = **C**.

Example 5.11 The homogeneous plate ABC (Fig. 5.22a) is suspended by a cable CD and attached to the floor by two hinges. The weight **P** of the plate is 10 kN, AB = BC = AC = 2 m. The plate is making an angle θ of 30° with the floor. The coordinates of the point D are (0, 2, 2) m. Determine the tension in the cable and the reactions at each of the hinges. The hinges do not exert couples on the plate and the hinge B does not exert force in *x*-direction.

Solution Let's draw a free body diagram shown in Fig. 5.22b. There are six unknown values to solve for: A_x, A_y, A_z, B_y, B_z, and T. The direction of the tension in the cable is known from the geometry. You may recall that in the general three-dimensional case one can write six equations of equilibrium and to solve for six unknowns. Force **P** has only one nonzero component, P_z. Let us represent all forces and reactions as vectors.

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$
$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$
$$\mathbf{T} = T_x\mathbf{i} + T_y\mathbf{j} + T_z\mathbf{k}$$
$$\mathbf{P} = P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}$$

Since direction of the force T is known, let's find its unit vector. From the geometry, point C coordinates are (1, 0, √3/2), thus unit vector for force T is $\boldsymbol{\lambda} = -0.399\mathbf{i} + 0.798\mathbf{j} + 0.452\mathbf{k}$ and $\mathbf{T} = \mathrm{T}\,\boldsymbol{\lambda}$.

Now, we can write the equations of equilibrium.

$$\sum F_x = A_x + T_x = 0$$

$$\sum F_y = A_y + B_y + T_y = 0$$

$$\sum F_z = A_z + B_z + T_z - P = 0$$

$$\sum M_A = \boldsymbol{r}_{C/A} \times \boldsymbol{T} + \boldsymbol{r}_{B/A} \times \boldsymbol{B} + \boldsymbol{r}_{P/A} \times \boldsymbol{P} = 0$$

Define vectors

$$\boldsymbol{r}_{C/A} = \mathbf{i} - 1.5\mathbf{j} + 0.866\mathbf{k}$$

$$\boldsymbol{r}_{B/A} = 2\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

$$\boldsymbol{r}_{P/A} = \mathbf{i} - 0.5\mathbf{j} + 0.578\mathbf{k}$$

Let us expand the moment equation

$$\sum M_A = (-1.369T + 5)\mathbf{i} + (-2B_z - 0.798T + 10)\mathbf{j} + \left(2B_y + 0.2T\right)\mathbf{k} = 0$$

into three scalar components.

i: $-1.369\mathrm{T} + 5 = 0$
j: $-2\ \mathrm{B}_z - 0.798\mathrm{T} + 10 = 0$
k: $2\ \mathrm{B}_y + 0.2\mathrm{T} = 0$

From here, $\mathrm{B}_y = -0.365$ kN, $\mathrm{B}_z = 3.54$ kN, and $T = 3.65$ kN. Substituting these values in the first two equations will lead to $\mathrm{A}_z = 4.81$ kN, $\mathrm{A}_y = -2.55$ kN, and $\mathrm{A}_x = 1.457$ kN.

You may also use the MATLAB routine *equilibriumBody3D.m* to solve this problem. Start the MATLAB and run the *equilibriumBody3D.m*. The dialog box will appear with general explanation on how to use this procedure. Read it and click OK. The next dialog will ask you to input number of forces and number of moments acting in this problem. You have to input 4—for number of forces and 0—for number of moments. Now, you will be asked to input the relevant information for each of the four forces acting on the body. Let us input data for the force **A** as force #1, **B** #2 and **T** #3, **P** #4.

The following table summarizes the input.

Force ID number	1	2	3	4
Magnitude (force)	*x*	*x*	*x*	10
Unit vector: *X* component (length)	*x*	0	−0.399	0
Unit vector: *Y* component (length)	*x*	*x*	0.798	0
Unit vector: *Z* component (length)	1	1	0.452	−1
x coordinate of force application	0	2	1	1
y coordinate of force application	1.5	1.5	0	1
z coordinate of force application	0	0	0.866	0.578

The solution is shown below.

For load # 1 the unknown parameter # 1 is: 5.6321
For load # 1 the unknown parameter # 2 is: 0.3032
For load # 1 the unknown parameter # 3 is: −0.5307
For load # 2 the unknown parameter # 1 is: 3.5623
For load # 2 the unknown parameter # 3 is: −0.1028
For load # 3 the unknown parameter # 1 is: 3.6527

The above list summarizes the results for each value we entered as an unknown (x), while each force acting on the pad is presented below through its rectangular components and magnitude.

Force # 1: F = (1.457)i + (−2.550)j + (4.806)k Mag = 5.63
Force # 2: F = (0.000)i + (−0.364)j + (3.544)k Mag = 3.56
Force # 3: F = (−1.457)i + (2.914)j + (1.651)k Mag = 3.63
Force # 4: F = (0.000)i + (0.000)j + (−10.000)k Mag = 10.00

Compare these forces to the results we were getting by a manual solution and you will see that we are getting the same results with less computational efforts. Again, remember that this routine makes calculations easier, but it does not replace the need for you to create a correct free body diagram.

What We Have Learned?

How to move a force to an arbitrary point
To move a force to an arbitrary point and preserve the state of equilibrium, one has to add a couple of forces.

How to reduce a system of nonconcurrent forces into a system of forces acting at the single point
To reduce a system of nonconcurrent forces to an equivalent system acting at the particular point, one has to move each force to this point and add appropriate moment.

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_A & y_A & 0 \\ F_x & F_y & 0 \end{vmatrix} = \left(x_A F_y - y_A F_x\right)\mathbf{k} = M\mathbf{k}$$

Procedures to find a resultant force and a resultant moment

To find the resultant force and the moment for a system of nonconcurrent forces, one has to move each force to the selected point. Add all forces using the appropriate rule for adding concurrent forces to get a resultant force. All moments should be added also, using the appropriate rules of vector algebra, to create a resultant moment.

$$\mathbf{R} = \sum \mathbf{F}$$

$$\mathbf{M}_r = \sum \mathbf{M}$$

Procedures to analyze equilibrium of forces acting on a rigid body in a plane and in space

Reduce the system of forces and moments to one force and one moment, i.e., find the resultant force and moment and set them equal to zero. The obtained system of equations may be solved for three unknowns in the plane case and for six unknowns in the 3D case.

$$\begin{aligned} \sum F_x &= 0 & \sum M_x &= 0 \\ \sum F_y &= 0 & \sum M_y &= 0 \\ \sum F_z &= 0 & \sum M_z &= 0 \end{aligned}$$

5.3.1 Problems

5.94 The 4×6 m homogeneous plate ($W = 40$ N) is suspended by three vertical rods attached at A (6, 1, 0), B (5, 4, 0), and C (0, 3, 0). Determine the force acting in each rod.

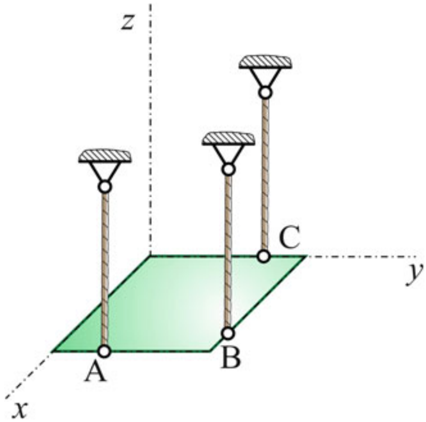


Fig. P5.94

5.95 The ladder ABCD is leaning against frictionless wall and floor. Determine reactions at A, B, C and D. Weight of the ladder **Q** is acting at its center, a and h are given.

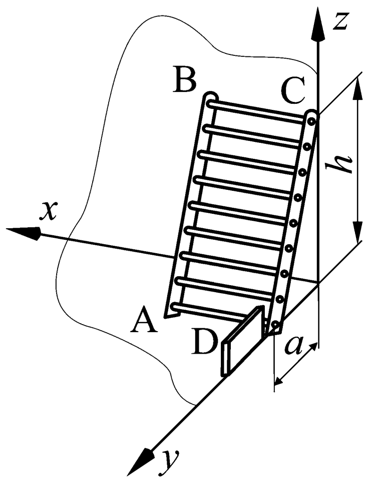


Fig. P5.95

5.96 Pole AB is held in the vertical position by symmetrically placed cables. The angle between each adjacent pair of cables is 60°, the tension in each cable is 30 N, and the pole weights 80 N. Determine the pressure pole AB is exerting on the bottom surface.

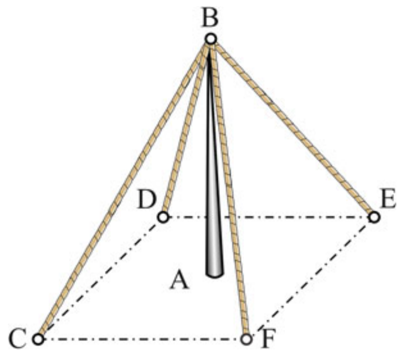


Fig. P5.96

5.97 Ship steering system consists of the solid bar ACD attached to the rudder F (weight $P = 30$ N). Water resistance force $Q = 300$ N acts in the negative x-direction. Force **S** acts at the angle $\beta = 120^o$. Determine the magnitude of the force **S** to keep the system in equilibrium. Also determine reactions at A and B. AK = 30 cm, AB = 60 cm, CB = 20 cm, CD = 60 cm, EK = 20 cm, EF = 10 cm.

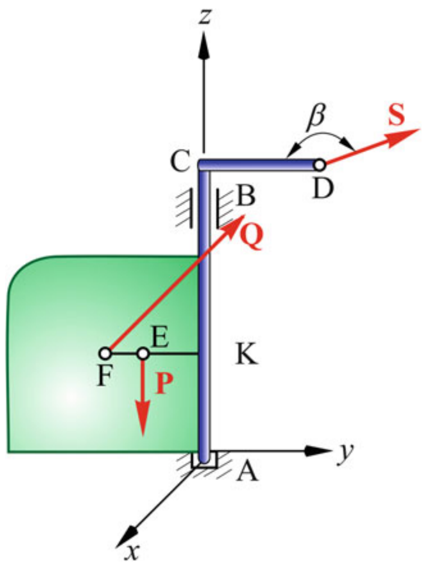


Fig. P5.97

5.98 Square plate ABCD (weight $G = 10$ kN) is supported by the ball-and-socket joint A, by the cable at B, and by the links CN and CK at C. Barrel (weight $M = 30$ kN) is suspended at B. Determine the reaction forces at N, K, A, and tension in the cable BS. Use AB = AK = a.

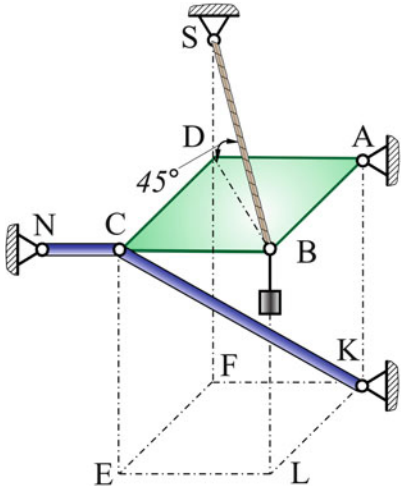


Fig. P5.98

5.99. The plate ABC (weight $W = 20$ kN) leans against the wall and makes an angle of 30° with the horizontal plane. There is no friction. Hinges do not have any force components in x-direction. Determine the reactions at A, B, and C, if AB = BC = AC = a.

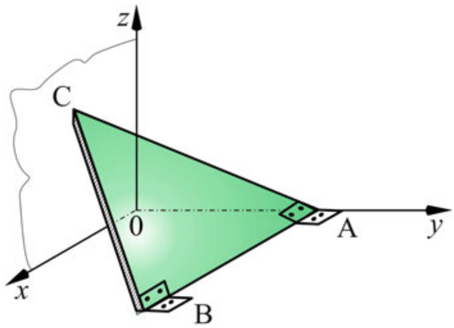


Fig. P5.99

5.100 The homogeneous rectangular plate ABCD weights $W = 500$ N and is held in the place by force **F** acting perpendicular to the plane of the plate. Determine the magnitude of the force **F** and the reactions at bearings A and B. Use $\alpha = 20°$, AD = 0.3 m, CE = 0.2 m, and DC = 0.6 m. The bearings are frictionless and they do not carry load in the y-direction.

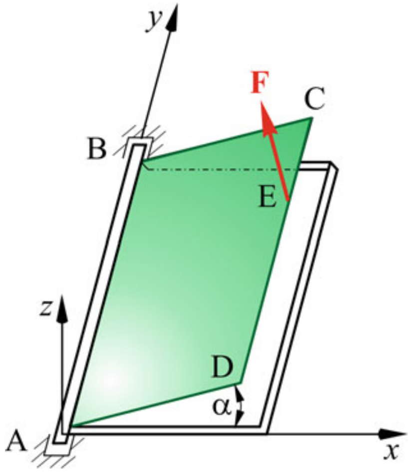


Fig. P5.100

5.101 The nonhomogeneous rectangular plate ABCD weights $W = 500$ N (Fig. P5.100). Its weight **W** acts at the point (0.114, 0.200, 0.096) m. The plate is held in the place by force $\mathbf{F} = -123\mathbf{i} + 147\mathbf{k}$, it is acting at the point (0.230, 0.400, 0.190). Determine the reactions at bearings A and B. Use AD = 0.3 m, CE = 0.2 m, DC = 0.6 m. The bearings are frictionless, and they do not carry load in the y-direction.

5.102 The square homogeneous plate ABC (weight $P = 4$ kN) is supported by the ball-and-socket at A and by three rigid links. Determine the reactions at A and the forces in each link. Use AB = EF = AD = a.

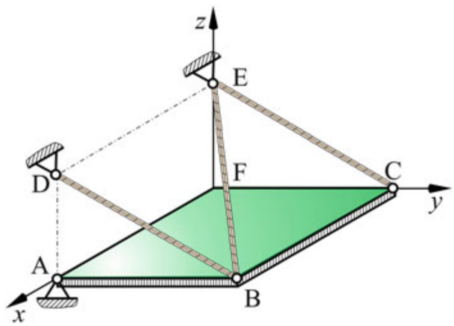


Fig. P5.102

5.103 The crankshaft is supported by bearings at A and B. The gear C has a radius $R = 20$ cm. Determine force **Q** and reactions at the bearings A and B, if the force $F = 40$ kN is acting in the plane YOZ and makes an angle $\alpha = 30°$ with the vertical axis. ED = 15 cm, a = 15 cm, b = 20 cm, c = 25 cm.

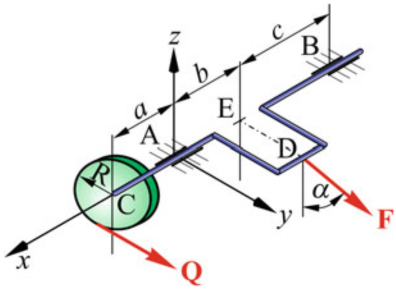


Fig. P5.103

5.104 The crankshaft is supported by bearings at A and B. The force $P = 30$ N is acting at point C in a plane parallel to the plane ZY at the middle of the part EG, and it makes angle $\alpha = 10^\circ$ with the vertical axis. Plane DECGF forms an angle $\varphi = 60^\circ$ with the horizontal plane. Reactions provided by bearings A and B are only forces in the directions y and z. Determine the moment needed to be applied to the crankshaft to keep the system in equilibrium. Determine reactions at A and B using DE = GF = 20 cm, FB = AD = FD = 40 cm. Neglect the weight of the crankshaft.

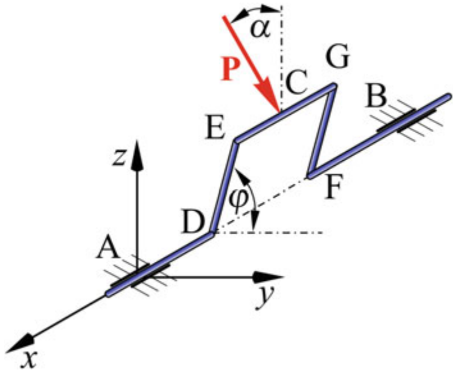


Fig. P5.104

5.105 The bent rod ACDE, supported by the ball-and-socket joint at A, the bearing at B and the cable EK, is acted upon by a force $P = 100$ N and a couple $M = 110$ N m at point E. Cable EK is in the x-direction. Determine the reactions at A and B and the tension in the cable EK. AC = DE = 0.5 m, BC = 0.4 m, BD = 0.2 m.

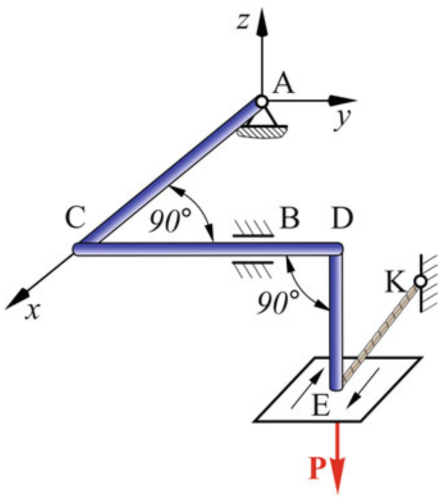


Fig. P5.105

5.106 The bent rod ADEK is supported by three bearings and is loaded by the forces $A = 100$ N and $Q = 200$ N. Determine the reactions at A, B, and C. AD = EC = 20 cm, BD = 10 cm, ED = 25 cm, and EK = 30 cm.

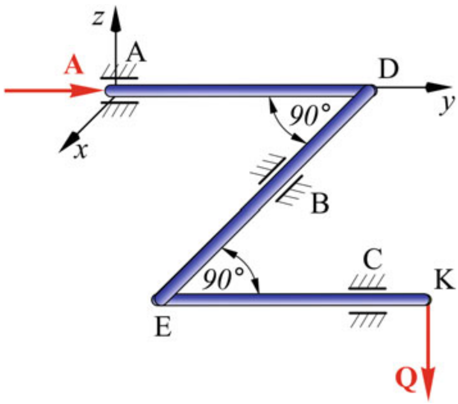


Fig. P5.106

5.107 The boom AB (weight **P**) is supported by a ball-and-socket joint at A and is loaded by the load **Q**. Determine the reactions at A and the tension in each cable. Use AB = $3b$, $\alpha = 45^0$, AD = AE = DC = b, AF = $2b$.

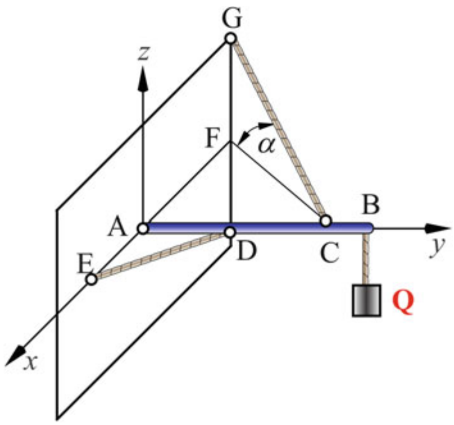


Fig. P5.107

5.108 A 5-m homogeneous bar AL (weight $P = 2$ kN) is held by the ball-and-socket joint at A and two cables DF and DC. Each cable makes angle $\beta = 30^\circ$ with the horizontal plane. For the loading shown, determine the reaction forces at A and the tension in each cable. $M = 4$ kN, LK = 1 m, AD = 2 m, $\alpha_1 = 45^\circ$, $\alpha_2 = 60^\circ$.

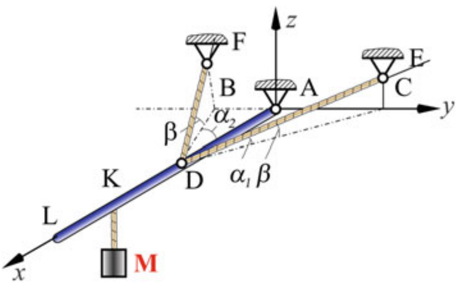


Fig. P5.108

5.109 The 4-m pole OA (weight 2.1 kN) is supported by the toe A and bearing at B. Cables OC, OD, and OE are under tension $T_c = T_d = 55$ N and $T_e = 25$ N. AB = 1.2 m, OA = 4 m, $\alpha = \beta = 60^\circ$, $\gamma = 30^o$. Determine the reaction forces at A and B.

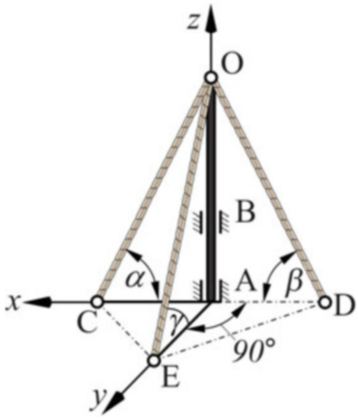


Fig. P5.109

Distributed Forces: Center of Gravity and Centroids

6

Contents

Example isn't another way to teach, it is the only way to teach.

Albert Einstein

In this chapter you will learn:

- Difference between concentrated and distributed forces
- How to replace the effect of a distributed force by a concentrated force
- About centers of gravity and centroids
- How to find centers of gravity and centroids of various bodies

In previous chapters, we have learned about the equilibrium of rigid bodies loaded by concentrated forces and moments. In this chapter, we will discuss the difference

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between the concentrated forces, i.e., force acting on a body at a given point, and the distributed forces. It is obvious that the notion of a concentrated force is a simplification of the load acting on a small area. This may be a valid representation when the area of the contact between the load and the body is small. However, when the contact area is not small, relative to the rigid body size, we have to account for the actual force distribution.

6.1 Distributed Forces and Rigid Body

When the contact area between a load and a rigid body is large, it is not obvious where to position the line of action of the force representing the distributed load. For example, Fig. 6.1 shows a train riding on a rail. On a free-body diagram, we have to show the effect of the train on the rail, but where should we apply its weight **W**? As we already know, the placement of this force will have a direct effect on the values of calculated reaction forces between the rail and the supporting columns. Thus, we have to develop a procedure that will allow substituting the effect of the distributed load (train) on supports by a concentrated force.

> Replacement of a distributed load by a concentrated force is permissible only when dealing with the equilibrium of a rigid body loaded by external forces and reactions.

Fig. 6.1 Train riding on a rail

6.2 Center of Gravity of a Flat Plate

Any rigid body may be considered to consist of a number of particles, each having a weight, $d\mathbf{W}$ directed toward the center of the Earth. For bodies, that are significantly smaller than the Earth, gravitational forces acting on each particle can be considered parallel. The resultant of these parallel forces is equal to the weight of the body $\mathbf{W}$,

$$\mathbf{W} = \int_{\text{volume}} d\mathbf{W} \tag{6.1}$$

The moment $\mathbf{M}$ of the resultant and the sum of the moments of all gravitational forces acting on the particles, with respect to the same point, should be equal.

$$\mathbf{M} = \int_{\text{volume}} d\mathbf{M} = \int_{\text{volume}} \mathbf{r} \times d\mathbf{W} = \mathbf{r}_{\mathrm{G}} \times \mathbf{W} \tag{6.2}$$

where $\mathbf{r}_{\mathrm{G}}$ defines the location of the center of gravity.

From the above, the location $\mathbf{r}_{\mathrm{G}}$ of the line of action of the resultant can be determined by calculating the sum of moments for parallel forces.

> The force exerted by the Earth, due to gravitation, on a particle or body is defined as its *weight*.

Mathematical Corner

Derivation of the center of gravity equations.

We will start with (6.2)

$$\mathbf{M} = \int_{\text{volume}} d\mathbf{M} = \int_{\text{volume}} \mathbf{r} \times d\mathbf{W} = \mathbf{r}_{\mathrm{G}} \times \mathbf{W}$$

where $d\mathbf{M} = \mathbf{r} \times d\mathbf{W}$ may be expressed as

$$d\mathbf{M} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ dW_x & dW_y & dW_z \end{bmatrix}$$
$$= \left(y \cdot dW_z - z \cdot dW_y\right)\mathbf{i} + \left(z \cdot dW_x - x \cdot dW_z\right)\mathbf{j} + \left(x \cdot dW_y - y \cdot dW_x\right)\mathbf{k}$$

Similarly, we may write

(continued)

$$\mathbf{M} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_G & y_G & z_G \\ W_x & W_y & W_z \end{bmatrix}$$
$$= \left(y_G \cdot W_z - z_G \cdot W_y\right)\mathbf{i} + \left(z_G \cdot W_x - x_G \cdot W_z\right)\mathbf{j} + \left(x_G \cdot W_y - y_G \cdot W_x\right)\mathbf{k}$$

Integrating $d\mathbf{M}$ and equating it to $\mathbf{M}$ will result in the following expression:

$$\left(\int_V y \cdot dW_z - \int_V z \cdot dW_y\right)\mathbf{i} + \left(\int_V z \cdot dW_x - \int_V x \cdot dW_z\right)\mathbf{j} + \left(\int_V x \cdot dW_y - \int_V y \cdot dW_x\right)\mathbf{k}$$
$$= \left(y_G \cdot W_z - z_G \cdot W_y\right)\mathbf{i} + \left(z_G \cdot W_x - x_G \cdot W_z\right)\mathbf{j} + \left(x_G \cdot W_y - y_G \cdot W_x\right)\mathbf{k}$$

Two vectors are equal if their corresponding components are equal. Therefore, we get three equations for three unknown coordinates defining the center of gravity.

$$\int_V y \cdot dW_z - \int_V z \cdot dW_y = y_G \cdot W_z - z_G \cdot W_y$$

$$\int_V z \cdot dW_x - \int_V x \cdot dW_z = z_G \cdot W_x - x_G \cdot W_z$$

$$\int_V x \cdot dW_y - \int_V y \cdot dW_x = x_G \cdot W_y - y_G \cdot W_x$$

Vectors $\mathbf{W}$ and $d\mathbf{W}$ are parallel, therefore we can express them as

$$\mathbf{W} = \mathbf{e} \cdot W$$
$$d\mathbf{W} = \mathbf{e} \cdot dW$$

where $\mathbf{e}$ may be expressed as $\mathbf{e} = e_x \cdot \mathbf{i} + e_y \cdot \mathbf{j} + e_z \cdot \mathbf{k}$, therefore the above three equations defining the center of gravity may be rewritten as

$$e_z \int_V y \cdot dW - e_y \int_V z \cdot dW = e_z y_G \cdot W - e_y z_G \cdot W$$

$$e_x \int_V z \cdot dW - e_z \int_V x \cdot dW = e_x z_G \cdot W - e_z x_G \cdot W$$

$$e_y \int_V x \cdot dW - e_x \int_V y \cdot dW = e_y x_G \cdot W - e_x y_G \cdot W$$

Mathematical Corner

Rewriting the above set of equations, we obtain

$$e_z\left(\int_V y\cdot dW - y_G\cdot W\right) = e_y\left(\int_V z\cdot dW - z_G\cdot W\right)$$

$$e_x\left(\int_V z\cdot dW - z_G\cdot W\right) = e_z\left(\int_V x\cdot dW - x_G\cdot W\right)$$

$$e_y\left(\int_V x\cdot dW - x_G\cdot W\right) = e_x\left(\int_V y\cdot dW - y_G\cdot W\right)$$

The equity is true only if the values in parentheses are equal to zero. This leads us to three linearly independent equations.

$$\int_V x\cdot dW - x_G\cdot W = 0$$

$$\int_V y\cdot dW - y_G\cdot W = 0$$

$$\int_V z\cdot dW - z_G\cdot W = 0$$

From these equations the coordinates of the center of gravity are

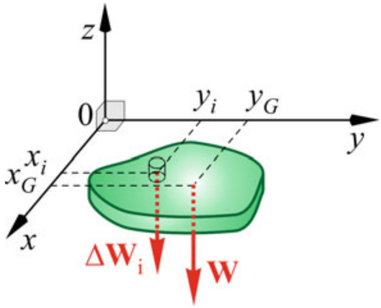
$$y_G = \frac{\int_V x\cdot dW}{W}$$

$$x_G = \frac{\int_V y\cdot dW}{W}$$

$$z_G = \frac{\int_V z\cdot dW}{W}$$

We will start with a two-dimensional body (Fig. 6.2) in a *xoy* plane. This body may be represented as an assembly of n small elements. Let's define location of

Fig. 6.2 Locating of the center of gravity for a two-dimensional rigid body



each element by its coordinates x_i and y_i, while the gravitational force exerted by the Earth on it is defined as $\Delta \mathbf{W}_i$, where i is the number of an element. The resultant force (weight) of all these elements is defined by

$$\mathbf{W} = \sum_{\mathrm{i}} \Delta \mathbf{W}_{\mathrm{i}}$$

The location of the resultant force (line of action) may be determined by summing the moments of each $\Delta \mathbf{W} i$ about both axes and equating them to the moment of the resultant $\mathbf{W}$ about the same axes.

About x-axis

$$\sum_{\mathrm{i}} \Delta \mathbf{M}_{\mathrm{i}} = \sum_{\mathrm{i}} \Delta \mathbf{W}_{\mathrm{i}} \cdot \mathrm{y}_i = \mathbf{W} \cdot y_{\mathrm{G}} \tag{6.3}$$

and about y-axis,

$$\sum_{i} \Delta \boldsymbol{M}_i = \sum_{i} \Delta \mathbf{W}_{\mathrm{i}} \cdot x_i = \mathbf{W} \cdot x_{\mathrm{G}} \tag{6.4}$$

Here, x_{G} and y_{G} are the coordinates of the application of the resultant force $\mathbf{W}$. This point defines the center of gravity of the two-dimensional body.

We may increase the number of particles representing our body and at the same time decrease their size to obtain an infinite number of infinitesimally small particles. In this case, the following expressions will describe the weight and location of the center of gravity.

$$\mathbf{W} = \int d\mathbf{W} \tag{6.5}$$

$$\int y\, d\mathbf{W} = \mathbf{W} \cdot y_{\mathrm{G}} \tag{6.6}$$

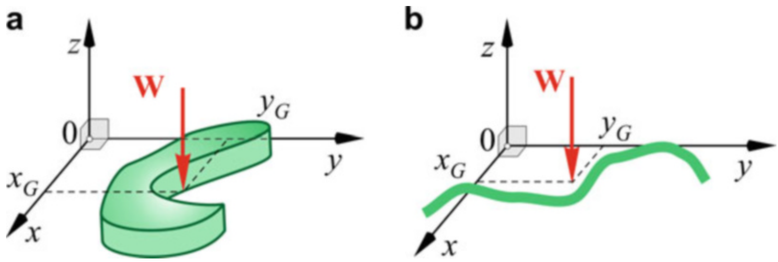


Fig. 6.3 Center of gravity for bodies with different shapes (**a**) and for wires (**b**)

$$\int x\,d\mathbf{W} = \mathbf{W} \cdot x_{\mathrm{G}} \tag{6.7}$$

The above equations were derived on the basis of a physical concept that the sum of moments of number of forces affects a body as the moment of their resultant. The same equations may be derived more generally as shown in the "Mathematical Corner" above.

Since we did not place any restrictions on the shape of a two-dimensional rigid body, one may consider bodies with cutouts (Fig. 6.3a) and wires (Fig. 6.3b). It is clear that the location of the center of gravity may be *outside* the body.

6.3 Centroids

Let's assume that a body (Fig. 6.3a) has a constant thickness t and is made of homogeneous material (i.e., its physical properties are *not* the function of the location) with a specific weight $\boldsymbol{\gamma}$ (weight per unit volume). The weight of an element $\Delta\mathbf{W}_i$, which occupies volume $\Delta V_i = t\Delta A_i$ may be expressed as

$$\Delta\mathbf{W}_i = t \cdot \boldsymbol{\gamma} \cdot \Delta A_i \tag{6.8}$$

In the limit, equation (6.8) becomes

$$d\mathbf{W} = t \cdot \boldsymbol{\gamma} \cdot dA \tag{6.9}$$

Substituting expression for $d\mathbf{W}$ into (6.5)–(6.7), one gets

$$\int_{\text{area}} x \cdot dA = A \cdot x_{\mathrm{G}} \tag{6.10}$$

$$\int_{\text{area}} y \cdot dA = A \cdot y_G \tag{6.11}$$

where A is the area of the top surface of a rigid body. It should be noted that the integration over volume is reduced here to the integration by area, since the thickness of the body is constant. The values of x_G and y_G obtained from (6.10) and (6.11) define the location of the centroid of the area of the top surface of the rigid body we consider.

Similar equations can be derived for determination of the centroid of a flat wire (i.e., all points of the wire belong to the same plane).

$$\int_{\text{contour}} y \cdot dl = L \cdot y_G \tag{6.12}$$

$$\int_{\text{contour}} x \cdot dl = L \cdot x_G \tag{6.13}$$

where dl is the length of the infinitesimal element of the wire and L is the total length of the wire.

It is important to understand the difference between the centroid and center of gravity of a body. The centroid is purely geometrical characteristic of a body. Both locations coincide only when a body has uniform thickness and is homogeneous.

The integrals $\int y\,dA$ and $\int x\,dA$ are called the *first moments* of the area with respect to the x- and y-axis.

$$Q_y = \int_{\text{area}} x \cdot dA$$

and

$$Q_x = \int_{\text{area}} y \cdot dA$$

Thus, one can get the location of the centroid by dividing the *first moment* of the area by the area.

$$x_G = \frac{Q_y}{A}$$
$$y_G = \frac{Q_x}{A} \tag{6.14}$$

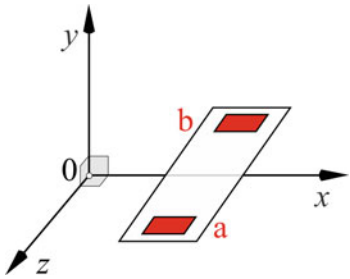


Fig. 6.4 Axis of symmetry

Area is a scalar quantity and is always positive, while the associated location of the centroid, relative to the selected coordinate system, may be positive or negative, thus the *first moment* of the area may be positive, negative, or zero. If it is zero, it means that the centroid coincides with the coordinate axis. When an area has an axis of symmetry, let say axis x, the centroid is on this axis, since for each element "a" one may find a corresponding element "b" on the other side of the symmetry axis (Fig. 6.4). It is obvious, that if an area has two axes of symmetry the centroid must be on their intersection. This allows us to find centroids of a number of common shapes, having an axis of symmetry, such as circles, squares, and rectangles, without calculations. The same is true for bodies with openings and wires.

It should be emphasized that the above discussion applies only to the bodies whose geometry may be analytically modeled. For all other cases, we have to simplify the geometry so it may be modeled analytically.

6.3.1 Centroids by Integration

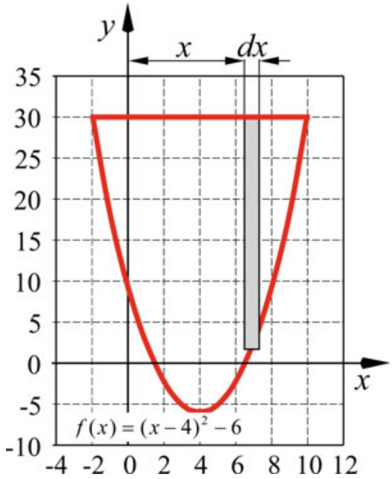
The centroid of an area bound by a curve, expressed by an analytical function, may be calculated by direct integration of (6.10) and (6.11). A double integration is necessary for calculation of the first moment. However, by choosing an element of the area for which the distance from its centroid to the axis is known, one may reduce the problem to a single integration. The following example illustrates such approach.

Example 6.1 Find the centroid of an area defined by the curve $f(x) = (x-4)^2 - 6$ and the horizontal line $y = b$, when $-2 < x < 10$ cm.

Solution Let's choose a vertical rectangular strip (Fig. 6.5) at location x of width dx. Its height is $b - f(x)$, where b is a y value of the top horizontal line. The centroid of the rectangle is located at the intersection of its diagonals. Its coordinates are:

$$x = x$$
$$y = (b - f(x))/2 + f(x) = (b + f(x))/2$$

Fig. 6.5 Direct integration



The area of this rectangular element is

$$dA = (b - f(x))\,dx$$

The location of the centroid x_G may be calculated from (6.11) by direct single integration. By using boundary $x = -2$ or $x = 10$, from the equation $f(x)$ of the line one may get $b = 30$.

$$x_G = \frac{\int\limits_{\text{area}} x dA}{A}$$

$$A = \int\limits_{-2}^{10} \left(20 - x^2 + 8x\right) dx = \left(20x - \frac{x^3}{3} + 4x^2\right)\Bigg|_{-2}^{10}$$

$$= 20 \cdot 12 - \frac{1}{3} \cdot (1000 + 8) + 4 \cdot (100 - 4) = 288$$

and

$$x_G = \frac{\int\limits_{-2}^{10} \left(20 - x^2 + 8x\right) \cdot x dx}{A} = \frac{10x^2 - \frac{x^4}{4} + \frac{8}{3}x^3\Big|_{-2}^{10}}{A} = \frac{1152}{288} = 4$$

This result may be predicted from the fact that the curve is symmetrical about vertical axis passing through $x_G = 4$ cm. Now, let's calculate the location of y_G

using (6.11). Consider again the vertical strip, shown in Fig. 6.5, its vertical coordinate and area were defined above, thus we will substitute it in the equation and integrate by dx.

$$y_G = \frac{\int\limits_{\text{area}} y\,dA}{A} = \frac{\int\limits_{-2}^{10} \frac{b+f(x)}{2} \cdot (b - f(x))dx}{288} = \frac{\frac{1}{2}\int\limits_{-2}^{10} \left(b^2 - f^2(x)\right)dx}{288} = 15.60$$

The result is $y_G = 15.60$ cm.

The integration may be easily done by using MATLAB routines. The website "extras.springer.com" has a file named "*centroidIntegr.m*" with the necessary information.

Example 6.2 Find the centroid of a rectangle of width b and height h (Fig. 6.6).

Solution Since the rectangle has two axes of symmetry, the centroid should be located at the intersection of its diagonals. However, we will use integration to show this.

Let us choose a vertical strip, as shown, which is located at the distance x from the origin of the coordinate system and has an infinitesimal width of dx. The first moment Q_y of this strip about the y-axis is

$$Q_y = \int\limits_{\text{area}} x\,dA = \int\limits_0^b x \cdot (h\,dx) = h\left[\frac{x^2}{2}\right]_0^b = \frac{b^2 h}{2}$$

and $A = b \cdot h$.

From (6.14) we obtain

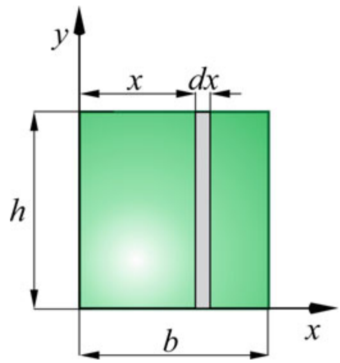


Fig. 6.6 Centroid of a rectangle

$$x_G = \frac{Q_y}{A} = \frac{b}{2}$$

as we have predicted from the symmetry of the rectangle. In the similar manner, we can calculate the value of y_G to be equal to $h/2$.

The first moment for a rectangle about the axis parallel to its base is equal to the base times square of the height divided by two.

Let us call the side of the rectangle along the axis about which we are calculating the 1st moment a base, while the other side we will call a height. Thus, the first moment for a rectangle about the axis parallel to the base is equal to the product of its base by square of height divided by two.

Example 6.3 Find the centroid of a uniform wire in a shape of a quarter of a circle with radius R (Fig. 6.7).

Solution We will use the formulae (6.12) and (6.13). Let us select an infinitesimal element of the length $dl = R \cdot d\theta$ at the location defined by an angle θ, as shown in Fig. 6.7. Centroid of this element is

$$x = R \cdot \cos\theta \quad \text{and} \quad y = R \cdot \sin\theta$$

The total length of the quarter arc is

$$L = \frac{\pi R}{2}$$

Thus, the centroid can be calculated from the following expression:

$$x_G = \frac{\int_0^L x \cdot dl}{L} = \frac{\int_0^{\pi/2} R\cos\theta \cdot R \cdot d\theta}{L} = \frac{R^2}{\pi R/2} \cdot \int_0^{\pi/2} \cos\theta \cdot d\theta = \frac{2R}{\pi}$$

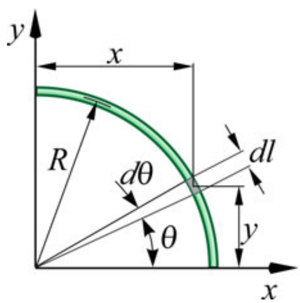


Fig. 6.7 Centroid of a wire

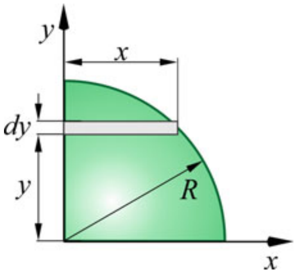


Fig. 6.8 Centroid of a quarter circle

and in the similar manner we can find that

$$y_G = \frac{2R}{\pi}$$

It should be noted that $x_G = y_G$ since the structure (wire) has the axis of symmetry at 45°.

Example 6.4 Find the location of a centroid of the quarter circle (Fig. 6.8).

Solution Quarter of a circle has an axis of symmetry at 45°. Thus, $x_G = y_G$. Let us use a horizontal strip located at the distance y and having thickness of dy. The equation of the circle is

$$x^2 + y^2 = R^2$$

and the limits defining the quarter circle are $0 < x < R$ and $0 < y < R$.

The area of the strip is $dA = x \cdot dy$, where

$$x = \sqrt{R^2 - y^2}$$

Thus, the vertical coordinate of the centroid of the quarter circle can be calculated by direct integration

$$y_G \cdot A = \int_0^R y \cdot dA = \int_0^R y \cdot x \cdot dy = \int_0^R y \cdot \sqrt{R^2 - y^2} \cdot dy = \frac{R^3}{3}$$

Since the area of the quarter circle is

$$A = \frac{\pi R^2}{4}$$

the location of the centroid in the y direction is

$$y_{\mathrm{G}} = \frac{4R}{3\pi}$$

Due to the symmetry we have $x_{\mathrm{G}} = \dfrac{4R}{3\pi}$.

The shapes discussed in the above examples belong to basic elements, some of which are listed in Table 6.1.

6.3.2 Centroids of Composite Bodies

Determination of the centroids by integration may become a rather tedious task for complicated body shapes. Very often, an area may be divided into a number of basic elements, like rectangle, triangles, circles, and others, whose centroid coordinates may be easily obtained by using integration. The centroids of such elements are summarized in Table 6.1. Many structures are built from those simple shapes. For example, the structure in Fig. 6.9 is composed of two rectangles, a triangle, and a circular cutout. Their boundaries are shown as dotted lines. Since we know the centroid location of each constituent part, we may rewrite (6.3) and (6.4) utilizing (6.8) to get the centroid of the whole assembly.

$$x_{\mathrm{G}} = \frac{\sum \Delta W_i \cdot x_i}{W} = \frac{\sum t \cdot \gamma \cdot \Delta A_i \cdot x_i}{t \cdot \gamma \cdot A} = \frac{\sum \Delta A_i \cdot x_i}{A} \tag{6.15}$$

$$y_{\mathrm{G}} = \frac{\sum \Delta W_i \cdot y_i}{W} = \frac{\sum t \cdot \gamma \cdot \Delta A_i \cdot y_i}{t \cdot \gamma \cdot A} = \frac{\sum \Delta A_i \cdot y_i}{A} \tag{6.16}$$

Let us introduce a common coordinate system as shown in (Fig. 6.9) and define each constituent part by letters A, B, C, and D. All dimensions are in mm; the diameter of a circular cutout is equal to 2 mm. It is convenient to organize all data in a table, e.g., Table 6.2.

From the Table 6.2, we can calculate

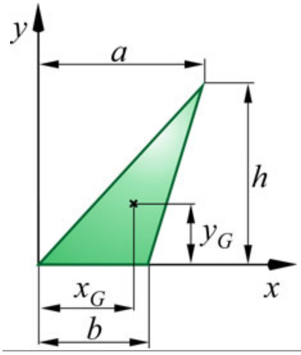
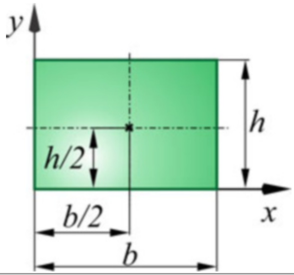
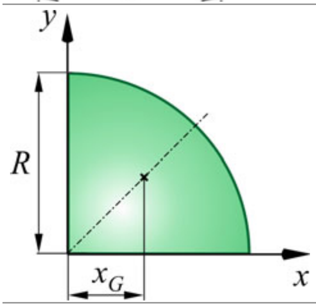
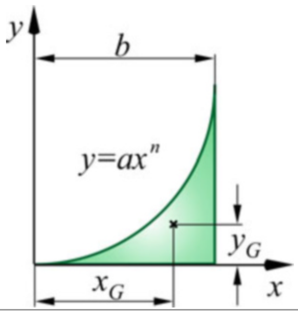
$x_{\mathrm{G}} = 3410/321 = 10.62$ mm and $y_{\mathrm{G}} = 4529/321 = 14.08$ mm.

One should be careful to use a proper sign for an area (negative, if it is a cutout) and proper sign for centroid coordinates of each component in a *common* coordinate system. Table 6.1 contains examples of common shapes with their areas and centroids for reference.

This problem may be solved much easier by using a simple MATLAB routine, located at "extras.springer.com."

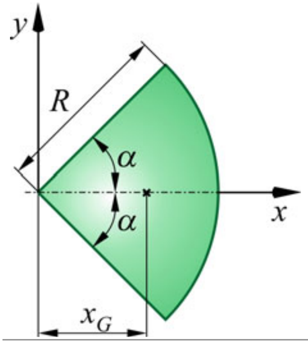
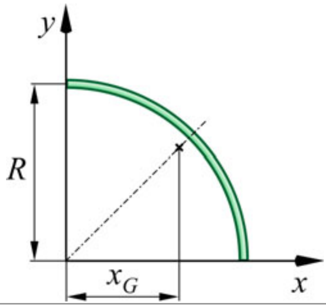
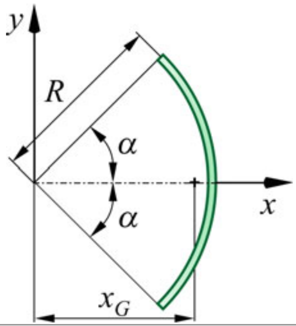
Start the MATLAB and select the file "*centroid.m*". It will ask you to prepare the input data in a particular format for calculations. Follow the prompts and enter the data for this problem as

Table 6.1 Centroids of basic elements

Element	Area	x_G	y_G
	$\frac{b \cdot h}{2}$	$\frac{a+b}{3}$	$\frac{h}{3}$
	$b \cdot h$	$\frac{b}{2}$	$\frac{h}{2}$
	$\frac{\pi R^2}{4}$	$\frac{4R}{3\pi}$	$\frac{4R}{3\pi}$
	$\frac{a\,b^{n+1}}{n+1}$	$\frac{(n+1)b}{n+2}$	$\frac{(n+1)ab^n}{4n+2}$

(continued)

Table 6.1 (continued)

Element	Area	x_G	y_G
	αR^2	$\frac{2R\sin\alpha}{3\alpha}$	0
Element	**Length**	**x_G**	**y_G**
	$\frac{\pi R}{2}$	$\frac{2R}{\pi}$	$\frac{2R}{\pi}$
	$2\alpha R$	$\frac{R\sin\alpha}{\alpha}$	0

1	9	−21	−108
2	8	12	108
3	15	9	108
4	15	6	-0.785

Each raw has a data for a particular constituent area A, B, C, and D, very similar to Table 6.2. Save this data in the file "*centdata.m*" and run the routine "*centroid.m*". It will use the data from the file "*centdata.m*" and calculate for you the values for $x_G = 10.62$ mm, $y_G = 14.08$ mm, and area $= 321$ mm^2. Compare them to the

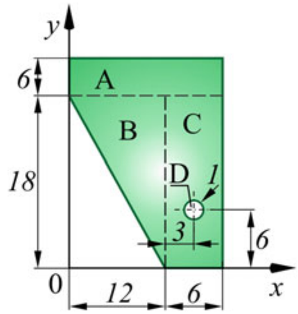


Fig. 6.9 Composite area

Table 6.2 Centroid of a composite body

Region	A (mm^2)	x_G (mm)	y_G (mm)	$A \cdot x_G$ (mm^3)	$A \cdot y_G$ (mm^3)
A	$6 \cdot 18$	9	$18+3$	972	2268
B	$½ \cdot 12 \cdot 18$	$2/3 \cdot 12$	$2/3 \cdot 18$	864	1296
C	$6 \cdot 18$	$12+3$	$½ \cdot 18$	1620	972
D	-0.785	$12+3$	6	-11.18	-4.71
Total	323			3444	4531

results presented in Table 6.2. You may use this routine for any problem dealing with location of the centroid of a composite two-dimensional body.

Example 6.5 Find the center of gravity of a triangle structure (Fig. 6.10) made of a homogeneous wire. The wire-specific weight per unit length is 5 N/m.

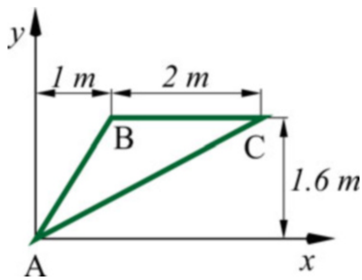


Fig. 6.10 Triangle Structure

Solution Since the structure is made from a wire of uniform-specific weight, its center of gravity coincides with its centroid. Therefore, we will proceed with calculations for the centroid.

The lengths of the segments are:

$$L_{AB} = \sqrt{1^2 + 1.6^2} = 1.89\,\mathrm{m}$$
$$L_{BC} = 2.00\,\mathrm{m}$$
$$L_{CA} = \sqrt{3^2 + 1.6^2} = 3.40\,\mathrm{m}$$

The total length of the wire is equal to the sum of its constituent parts, i.e., $L = L_{AB} + L_{BC} + L_{CA} = 7.29\,\text{m}$.

Now, let us find the coordinates of the centroid of each segment. They are located at the geometrical center of each part, i.e.,

$$x_{AB} = \frac{1}{2} = 0.5\,\text{m} \quad \text{and} \quad y_{AB} = \frac{1.6}{2} = 0.8\,\text{m}$$

$$x_{BC} = \left(1 + \frac{2}{2}\right) = 2.0\,\text{m} \quad \text{and} \quad y_{BC} = 1.6\,\text{m}$$

$$x_{CA} = \frac{1+2}{2} = 1.5\,\text{m} \quad \text{and} \quad y_{CA} = \frac{1.6}{2} = 0.8\,\text{m}$$

Let us organize all the data in Table 6.3:

Table 6.3 Information for determining the centroid coordinates

Segment	L (m)	x_G (m)	y_G (m)	$L \cdot x_G$ (m^2)	$L \cdot y_G$ (m^2)
AB	1.89	0.50	0.80	0.945	1.51
BC	2.00	2.00	1.60	4.00	3.20
CA	3.40	1.50	0.80	5.10	2.72
Total	7.29			10.05	7.43

By using (6.15) and (6.16) and replacing area A with length L, we calculate the centroid location.

$$x_G = \frac{\sum \Delta L_i \cdot x_i}{L} = \frac{10.05}{7.29} = 1.38\,\text{m}$$

$$y_G = \frac{\sum \Delta L_i \cdot y_i}{L} = \frac{7.43}{7.29} = 1.02\,\text{m}$$

Guidelines and Recipes to Calculate the Centroid Location for Composite Bodies

- Select a common coordinate system.
- Divide the body into number of parts with simple geometry, so that for each part the location of its centroid is known.
- Identify the centroid location for each part in the common coordinate system and its corresponding area or length.

(continued)

- Assemble all data in a table and calculate the location of the centroid by using appropriate formulae. Alternatively, use MATLAB routine "*centroid*".

6.3.3 Problems

6.1 Determine the centroid of an area bounded by the x-axis, line $x = 8$, and curve $y^2 = 2x$.

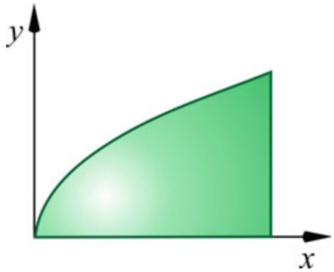


Fig. P6.1

6.2 Determine the centroid of a triangle by integration.

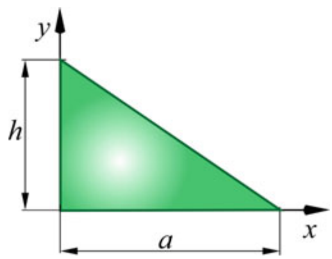


Fig. P6.2

6.3 Find the centroid of the area under one-half cycle of a sine curve with an amplitude "a".

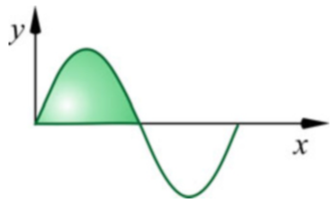


Fig. P6.3

6.4 Find the centroid of an area bounded by a parabola $y = 4x^2$ and line $y = x$.

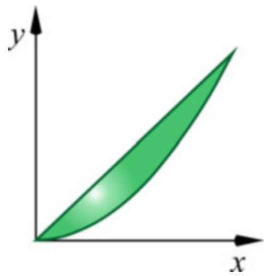


Fig. P6.4

6.5 Find the centroid of a circular arc defined by radius R and angle α.

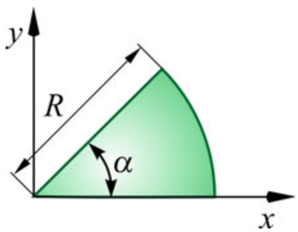


Fig. P6.5

6.6 Locate the centroid of a half-circle rod.

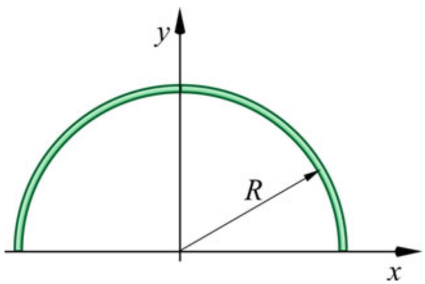


Fig. P6.6

6.7 The line $y = x^2/5$ is limited between $x = 0$ and $x = 5$. Locate the x coordinate of its centroid.

6.8 Find the centroid C of an area bound by the half circle AOB (radius R) and two lines of the equal length AD = DB. Use OD = 3R.

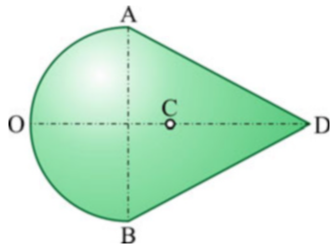


Fig. P6.8

6.9 Find centroid C of a circular segment. OA = OB = 40 cm and the angle AOB = 60°.

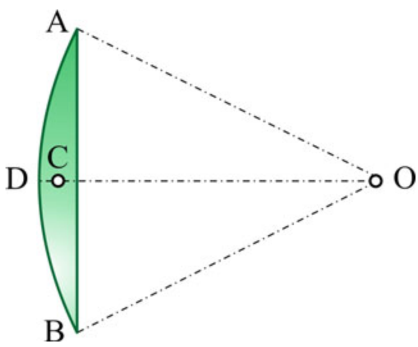


Fig. P6.9

6.10 Find the centroid of a ring segment.

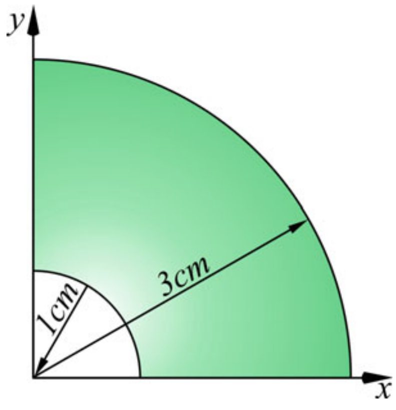


Fig. P6.10

6.11 A wing of a WWII British plane has an elliptical shape. Its maximum width is 2 m and length is 6 m. Find the location of the wing's centroid.

Fig. P6.11

6.12 Find the centroid of the area shown. $a = 80$ mm, $b = 60$ mm, $d = 10$ mm.

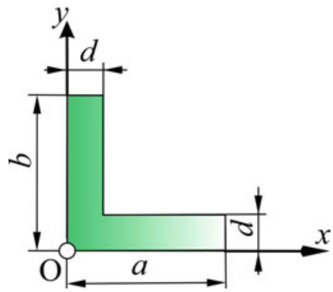


Fig. P6.12

6.13 Find the centroid of the area shown. $a = 40$ mm, $h = 60$ mm, $b = d = 10$ mm.

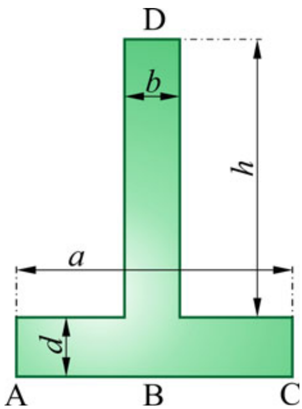


Fig. P6.13

6.14 Find the centroid of the area shown. Use $a = 20$ cm.

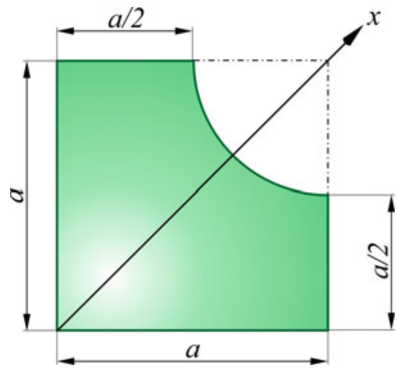


Fig. P6.14

6.15 Find the centroid of the bent rod below. All dimensions are in cm.

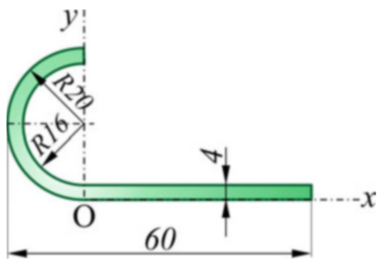


Fig. P6.15

6.16 Determine the centroid of the area shown. All dimensions are in cm.

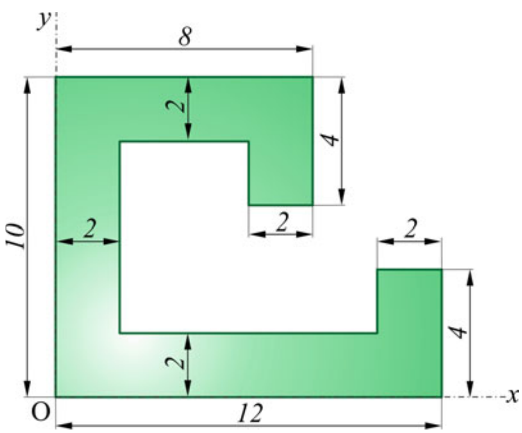


Fig. P6.16

6.17 Find the centroid of the area shown. All dimensions are in cm.

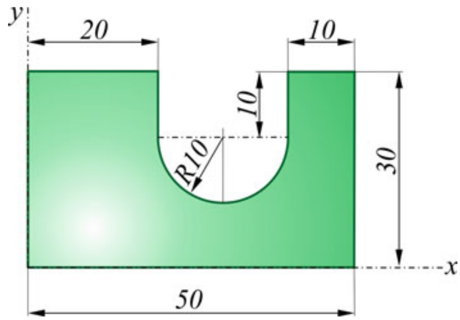


Fig. P6.17

6.18 Find the centroid of the area shown. All dimensions are in cm.

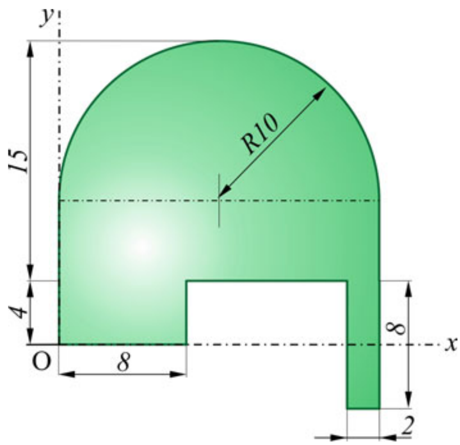


Fig. P6.18

6.19 Find the centroid of the area shown. All dimensions are in cm.

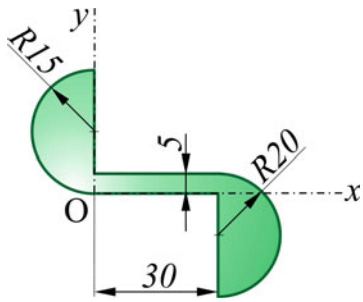


Fig. P6.19

6.20 Find the centroid of the area shown. All dimensions are in inches.

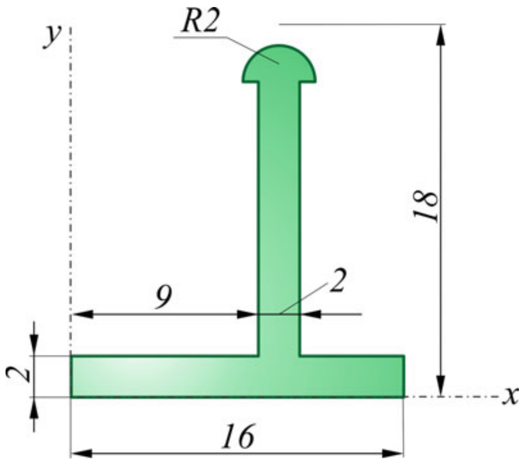


Fig. P6.20

6.21 Find the centroid of the beam section. Use $a = 80$ mm, $b = 60$ mm, $d = 10$ mm, $h = 100$ mm.

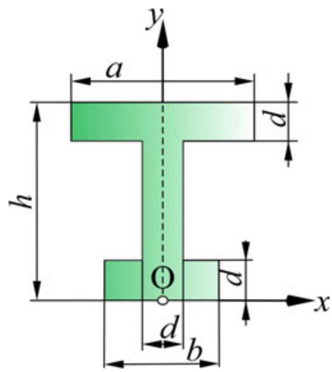


Fig. P6.21

6.22 Find the centroid of the area shown. Use $a = 100$ mm, $b = 60$ mm, $d = 10$ mm, $h = 80$ mm.

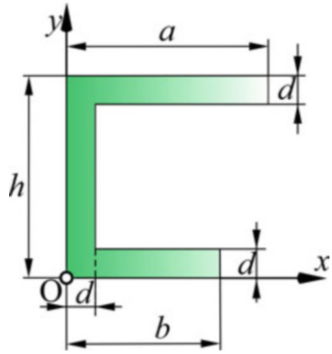


Fig. P6.22

6.23 Locate the centroid of a plane area ADCO. Use OA = OB = a, CB = $a/3$.

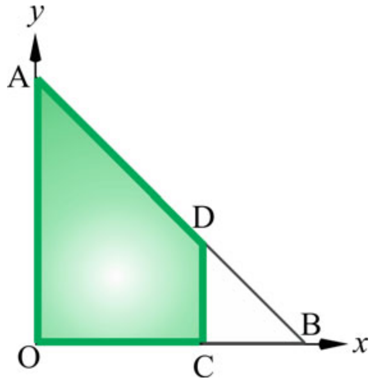


Fig. P6.23

6.24 Locate the centroid of the area shown. $AB_1 = a$, $AB = 2a$, $AD_1 = a$, $AD = 2a$.

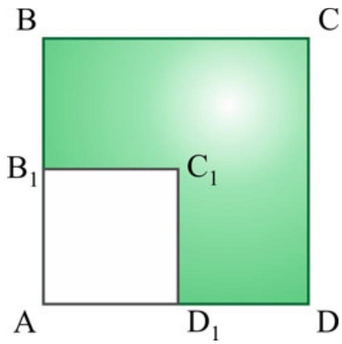


Fig. P6.24

6.25 Locate the centroid of the area shown. $R = 4a$, OA = OB = $3a$.

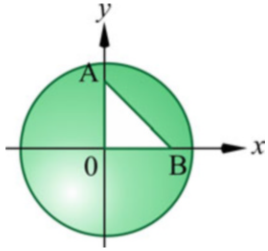


Fig. P6.25

6.26 Locate the centroid of the area EBAD. AD = 15 in., CD = 30 in., EC = 9 in.

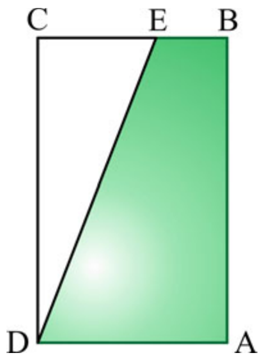


Fig. P6.26

6.27–6.34 Find the centroid of the plane areas shown. All dimensions are in cm.

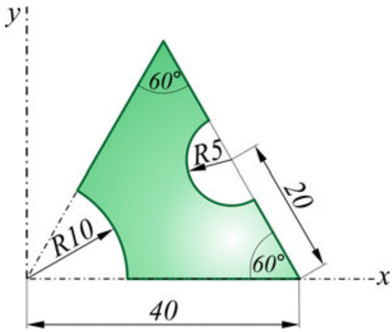


Fig. P6.27

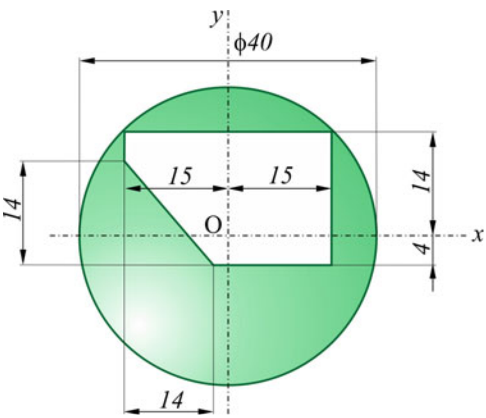


Fig. P6.28

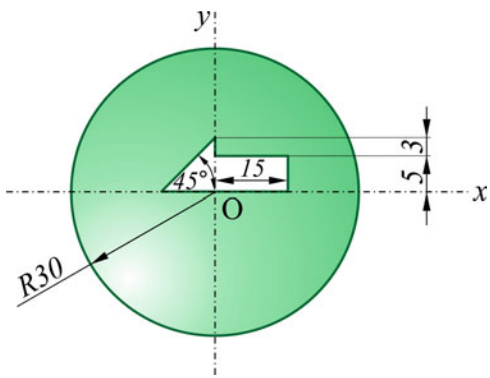


Fig. P6.29

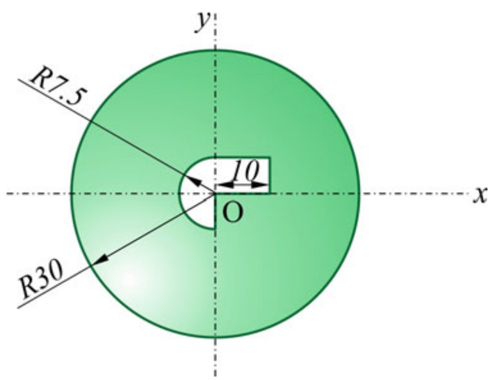


Fig. P6.30

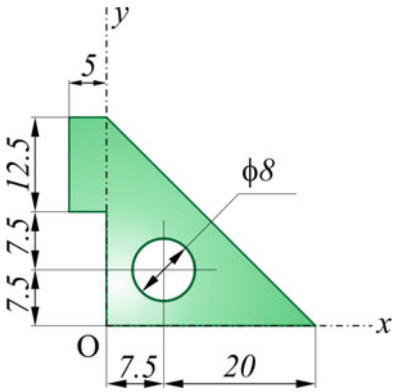


Fig. P6.31

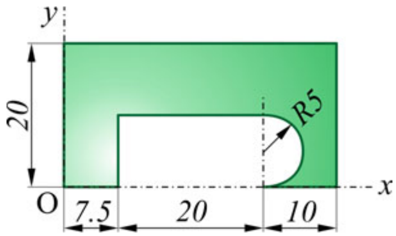


Fig. P6.32

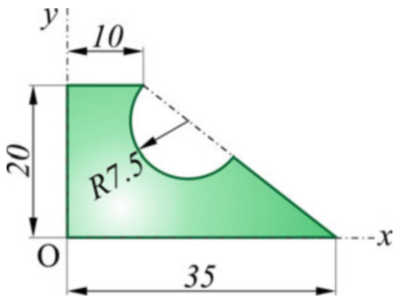


Fig. P6.33

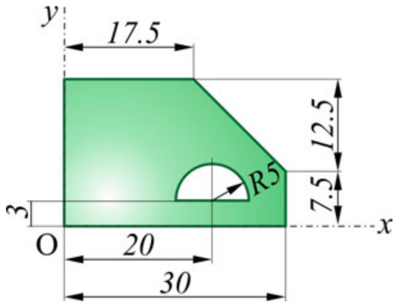


Fig. P6.34

6.35 Locate the centroid of the homogeneous wire below.

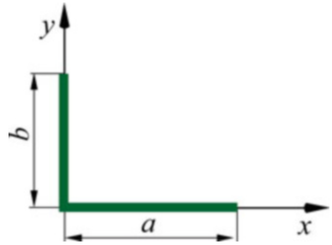


Fig. P6.35

6.36 Locate the centroid of a homogeneous wire, $a = 8$ cm and $b = 6$ cm.

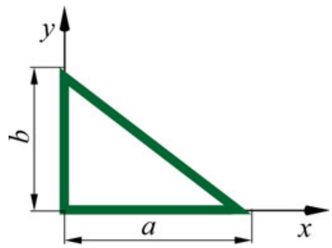


Fig. P6.36

6.37 Locate the centroid of the homogeneous wire below, $a = 80$ mm and $\alpha = 45°$.

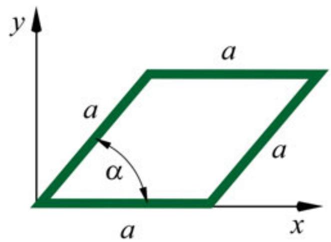


Fig. P6.37

6.38 Locate the centroid of a homogeneous wire, $a = 6$ in.

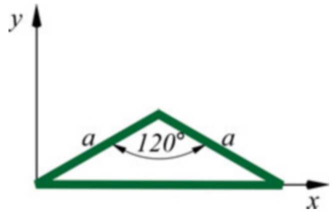


Fig. P6.38

6.39 Locate the centroid of the homogeneous wire shown when $a = 4$ m.

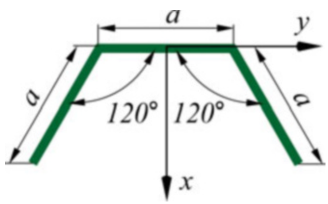


Fig. P6.39

6.40 Locate the centroid of the homogeneous wire shown. Dimensions are in ft.

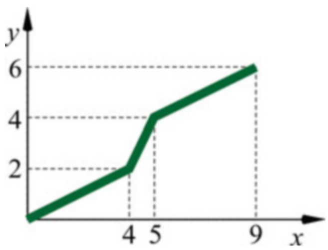


Fig. P6.40

6.41–6.47 The following structures are constructed from a set of homogeneous bars. Locate the centroids of the structures. All dimensions are in m.

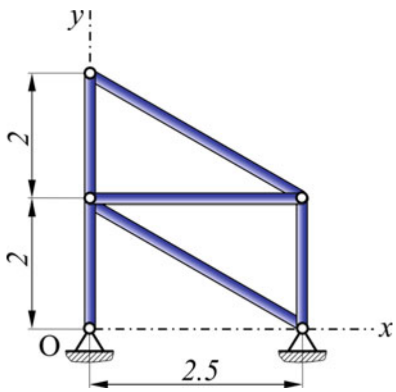


Fig. P6.41

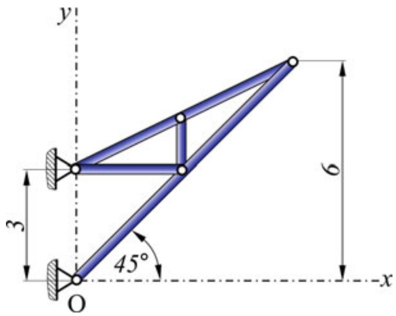


Fig. P6.42

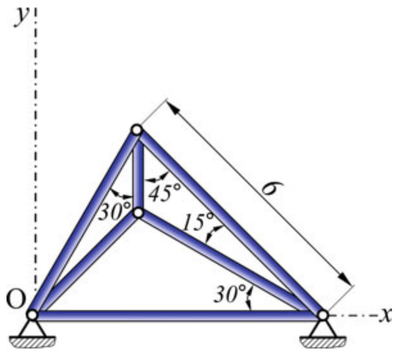


Fig. P6.43

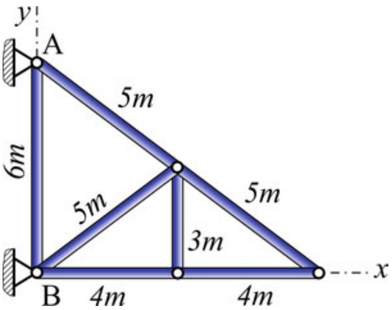


Fig. P6.44

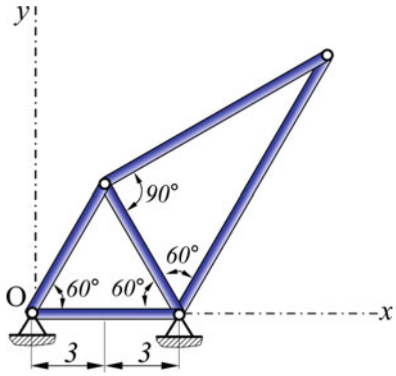


Fig. P6.45

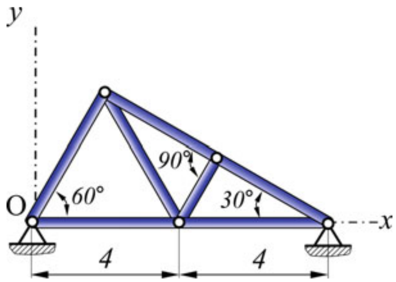


Fig. P6.46

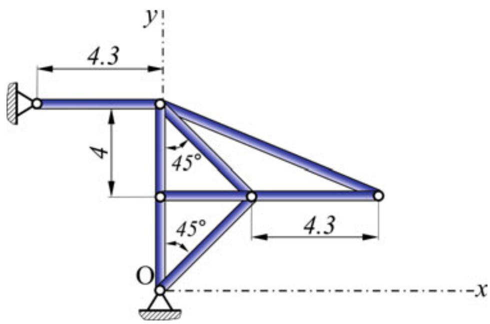


Fig. P6.47

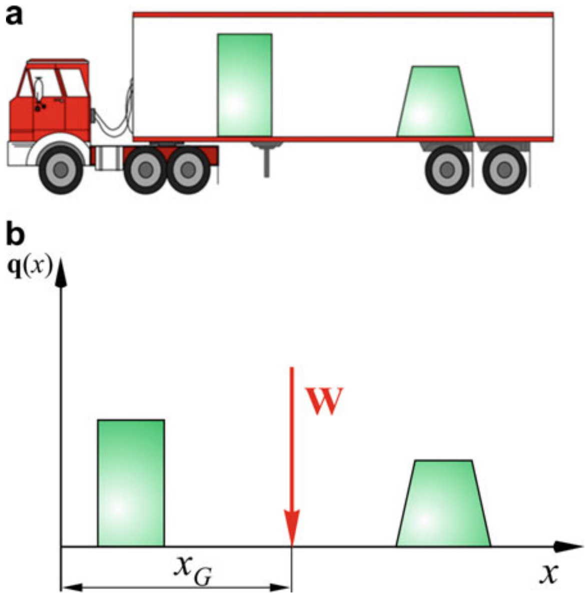


Fig. 6.11 (**a**) Truck loaded with two crates. (**b**) Truck's load

6.4 Distributed Loads

Until now, we considered forces that were applied at the relatively small areas. Those forces were represented as the concentrated forces; however, in many cases such approximation is not valid. Consider a two-dimensional case of a truck loaded by an oddly shaped cargo (Fig. 6.11a). In order to calculate the reactions, we will need to know where the resultant force of the cargo is acting.

Each crate exerts a force over an area. It is called a distributed force. We may consider the distributed force to consist of an infinite number of concentrated forces, where each force can be considered to act on an infinitesimal part of the support. Assuming that all those forces are parallel and are acting in the vertical direction, we may find their resultant and the point of its application, as explained in Sect. 6.2. The distributed load $\mathbf{q}$ is defined as load per unit area $\mathbf{q}(A)$ or unit length $\mathbf{q}(x)$. The latter is used for two-dimensional problems. Assuming the symmetry of the truck, we may represent the load as a two-dimensional problem as shown in Fig. 6.11b. The resultant force may be calculated as

$$\mathbf{W} = \int\limits_{\text{length}} \mathbf{q}(x)dx$$

and is applied at x_G, the centroid of the load distribution diagram (Fig. 6.11b)

> It has to be emphasized that substitution of a distributed load by its resultant may be applied *only* to the task of finding reactions. It cannot be used for calculation of internal forces in a structure since the substitution of the distributed load by a concentrated one will change the distribution of the internal forces.

$$x_G = \frac{\int xq(x)dx}{W}$$

It should be noted that there is no need to calculate the location of y_G since the resultant force is acting in the vertical direction (weight) and, according to the principle of transmissibility (Chap. 2), it may be applied anywhere along its line of action. Thus, to calculate the reactions one may use resultant force $\mathbf{W}$ applied at x_G instead of the distributed load.

It has to be emphasized that such substitutions may be applied only to the tasks of finding reactions. It cannot be used for calculation of internal forces in a structure since it is obvious that the substitution of the distributed load by a concentrated one will change the distribution of the internal forces. For example, if you will push something with a hand or with a needle using the same force, the global effect will be the same. However, the local effect of the applied force will be obviously different.

6.4.1 Effect of the Fluid Pressure

One of the interesting examples of distributed forces is the calculation of the reactions due to a fluid pressure on a structure. The force due to the pressure always acts normal to the surface; this fact may be used to solve problems dealing with the effect of the fluid pressure on structures.

Force due to fluid pressure is always acting in the direction perpendicular to the surface. Thus, the direction of the resulting force is determined by the spatial orientation of the surface upon which the pressure is acting.

Consider a plate submerged in water (Fig. 6.12). The pressure acting at any point under the water, due to the weight of the water, is directly proportional to the depth of the point and to the specific weight γ of the fluid. Thus, at the depth h the pressure p is

$$p = h \cdot \gamma \tag{6.17}$$

Let's consider an underwater gate AB shown in Fig. 6.13a and calculate the resultant force acting on it. The water pressure on the gate may be represented as a distributed load. Its magnitude increases linearly with the depth. The pressure distribution along gate AB is schematically shown in Fig. 6.13b. At point A the pressure is equal to $p_A = h_A \cdot \gamma$, while at point B the pressure is $p_B = h_B \cdot \gamma$.

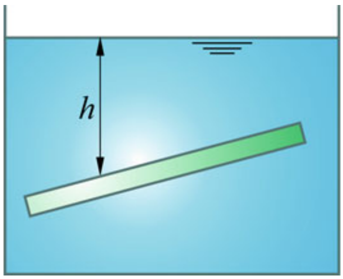


Fig. 6.12 Submerged surface

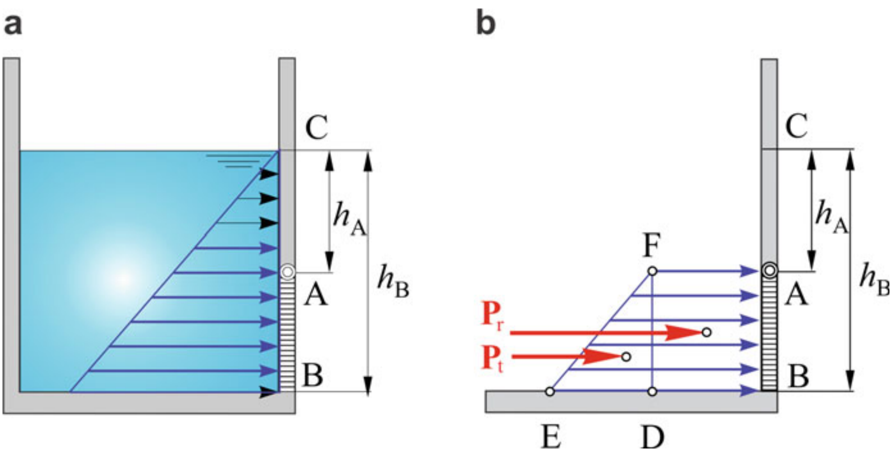


Fig. 6.13 Underwater gate. (**a**) Schematic. (**b**) Resultant of the water pressure

One can find the resultant of the distributed load acting on gate AB by representing it as rectangle ABDF and triangle DEF. The effect of the rectangle can be represented by a resultant force $\mathbf{P}_r$. The direction of this force is normal to the surface it is acting upon, and its magnitude is

$$P_{\mathrm{r}} = h_{\mathrm{A}} \cdot (h_{\mathrm{B}} - h_{\mathrm{A}}) \cdot \gamma \cdot d$$

where d is the width of the gate.

The resultant $\mathbf{P}_{\mathrm{r}}$ is acting at the centroid of rectangle ABDF. The effect of triangle DEF can be represented by force $\mathbf{P}_{\mathrm{t}}$. Its magnitude is

$$P_{\mathrm{t}} = \frac{1}{2}(h_{\mathrm{B}} - h_{\mathrm{A}}) \cdot (h_{\mathrm{B}} - h_{\mathrm{A}}) \cdot \gamma \cdot d$$

where d is the width of the gate. The resultant $\mathbf{P}_{\mathrm{t}}$ is acting at the centroid of triangle DEF. Thus, we represented the effect of the distributed force due to the water pressure by two concentrated forces. Using such approach one can calculate the effect of the water pressure on a gate, as shown in the following example.

Example 6.6 To keep the water level constant, gate AC (width $d = 2$ m) can pivot around point B located at distance a from the bottom (Fig. 6.14a). Find the water level h if $a = 0.5$ m. The specific weight of the water $\gamma = 9810\ \mathrm{N/m^3}$.

Solution Let's show the distributed load acting on the gate by a black line (Fig. 6.14b). The gate will turn clockwise when the moment about point B, created by the distributed load acting on part AB, will become smaller than the moment created by the distributed load acting on part BD. The distributed force acting on the part AB may be represented by a triangle (F_1) and rectangle (F_2), while the force acting on part BD may be represented by a triangle (F_3). As we discussed above, the

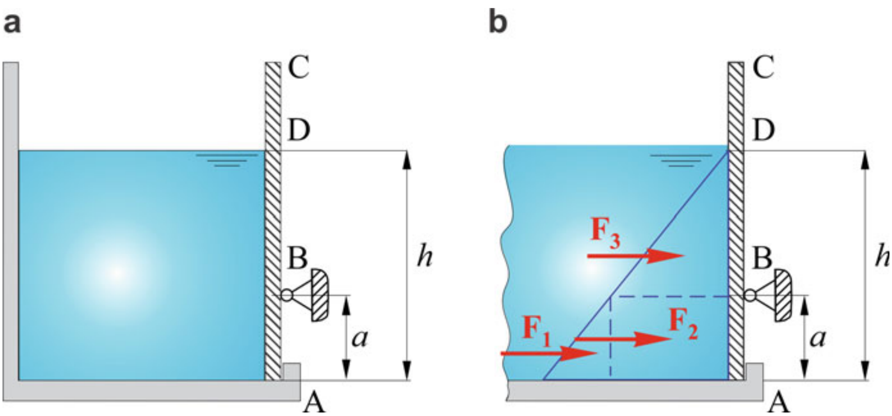


Fig. 6.14 Self-regulated underwater gate. (**a**) Schematic. (**b**) Calculation of the resultant

effect of the distributed load acting on the part AB may be represented by two concentrated forces $\mathbf{F}_1$ and $\mathbf{F}_2$, and on part BD by force $\mathbf{F}_3$ as shown in Fig. 6.14b.

To calculate magnitudes of these forces, we need the values of the pressure at points A, B, and D:

$$P_D = 0,\ P_B = \gamma \cdot (h - a),\ \text{and}\ P_A = \gamma \cdot h.$$

The magnitudes of these forces are represented by the areas of the rectangle and the two triangles. The magnitude of the first force is $F_1 = \frac{1}{2} \cdot (P_A - P_B) \cdot a \cdot d$, and it is applied at the centroid of the triangle, i.e., at a point $a/3$ from the bottom. The second force is $F_2 = P_B \cdot a \cdot d$, and it is applied at the centroid of the rectangle, i.e., at a point $a/3$ from the bottom. Finally, the magnitude of the third one is $F_3 = \frac{1}{2} \cdot P_B \cdot (h - a) \cdot d$, and it is applied at the centroid of the triangle, i.e., at the distance of $(h - a)/3$ from point B.

The water level, at which the gait will start to open, may be determined from the equilibrium equation (sum of moments about point B)

$$\sum M_B = F_1 \cdot \frac{2}{3} \cdot a + F_2 \cdot \frac{a}{2} - F_3 \cdot \frac{(h - a)}{3} = 0$$

Substituting the known values, we will get one equation of third order with one unknown h:

$$(h - 0.5)^3 - 0.75 \cdot (h - 0.5) - 0.25 = 0$$

Its solution yields three roots 0, 0, and 1.5. Since $h = 0$ is a trivial solution, i.e., there is no water, the correct result is $h = 1.5$ m.

Guidelines and Recipes for Finding the Magnitude and Point of Application of the Resultant of a Distributed Load

- Find the resultant load $\mathbf{W}$ by integrating the function representing the distributed load.

$$\mathbf{W} = \int_{\text{length}} \mathbf{q}(x)dx$$

- Find centroid of the area representing the distributed load. This centroid is the point of the application of the resultant load.

$$x_G = \frac{\int xq(x)dx}{W}$$

(continued)

6.4.2 Problems

6.48 Determine the magnitude and point of application of the resultant of a distributed load.

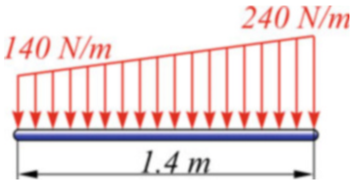


Fig. P6.48

6.49 The turtle's shape may be approximated as suggested by the dotted lines. Assume that the turtle's specific weight, $\gamma = 10\ \text{kN/m}^2$. Treat the turtle as a two-dimensional body. Replace the effect of the turtle's weight by the corresponding resultant force.

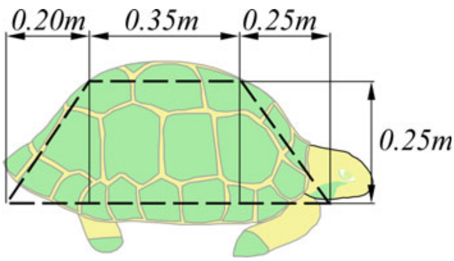


Fig. P6.49

6.50 Replace the distributed load by a resultant force.

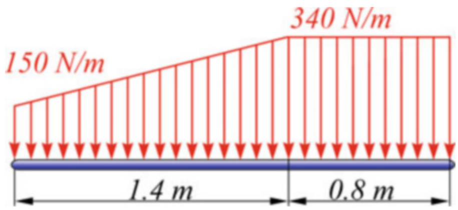


Fig. P6.50

6.51 Replace the distributed load by a resultant force.

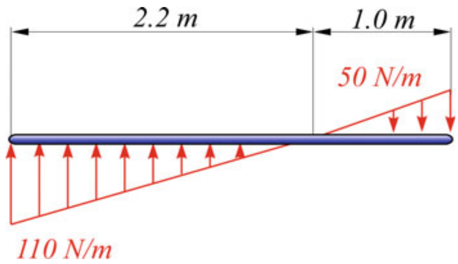


Fig. P6.51

6.52 Replace the distributed load by a resultant force.

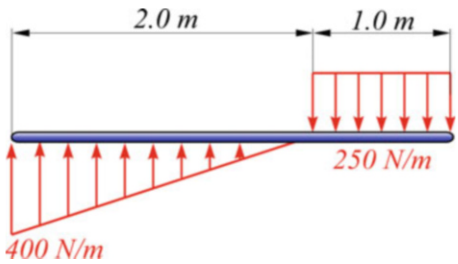


Fig. P6.52

6.53 Boxes are placed on a flat platform. Each box weights 10 N and has width of 0.20 m. Determine the location and magnitude of the resultant force.

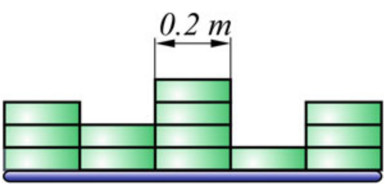


Fig. P6.53

6.54 Beam AB is built into a wall of thickness a. Assume that reaction forces are distributed as shown. Determine an equivalent force system for the beam. Use $q_1 = 20$ N/m, $q_2 = 10$ N/m, and $a = 0.4$ m.

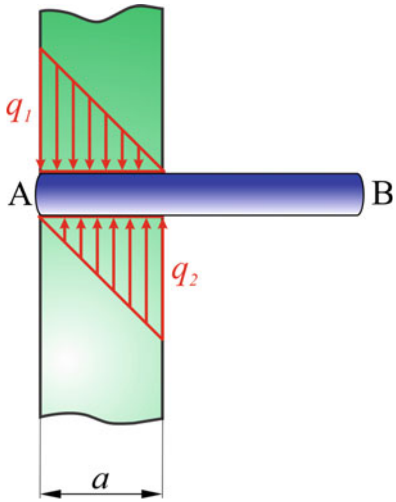


Fig. P6.54

6.55 Determine reactions at the structure supports.

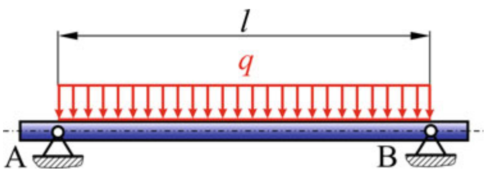


Fig. P6.55

6.56 Determine reactions at the wall.

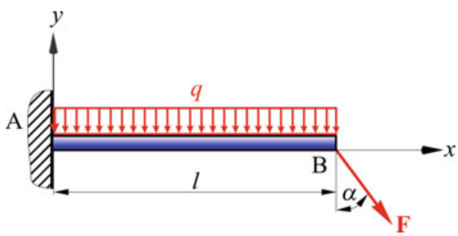


Fig. P6.56

6.57 Determine reactions at the wall. Use $F = 100$ lb, $M = 500$ lb ft, $q = 50$ lb/ft, $a = 3$ ft, and $l = 6$ ft.

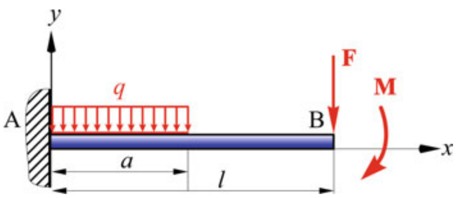


Fig. P6.57

6.58 Bracket ACDB is loaded by distributed load $q = 20$ kN/m. Determine reaction forces at the supports. Use $a = 6$ m, $b = 2$ m, angle $\alpha = 45°$.

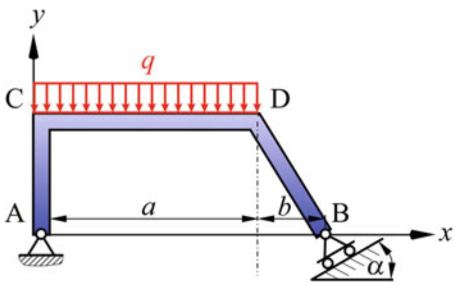


Fig. 6.58

6.59 Determine reactions at the structure supports. It is loaded by the distributed load $q = 0.5$ kN/m and force $P = 3$ kN. Use $h = 6$ m and $l = 2$ m.

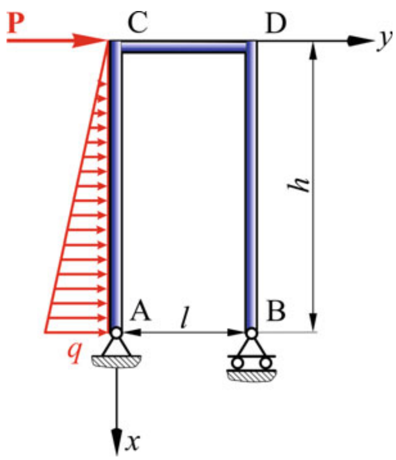


Fig. P6.59

6.60 Determine reactions at the beam supports. $l = 6$ m, $q = 3$ N/m, angle $\alpha = 30°$.

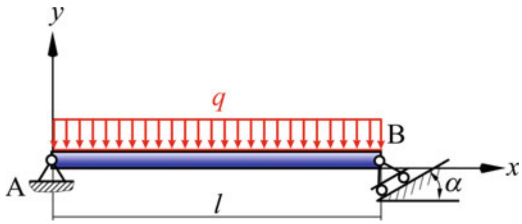


Fig. P6.60

6.61 Determine reactions at A and B, when beam is loaded by distributed load $q = 4$ kN/m and force $F = 16$ kN. Use $a = 4$ m, $l = 8$ m, and $\alpha = 30°$.

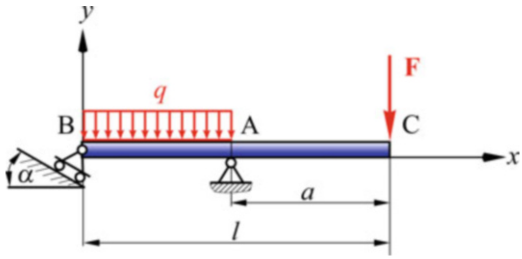


Fig. P6.61

6.62 Determine reactions at A and B. Use $q = 8$ N/m, $a = 4$ m, $l = 10$ m, $M = 20$ Nm, and angle $\alpha = 45°$.

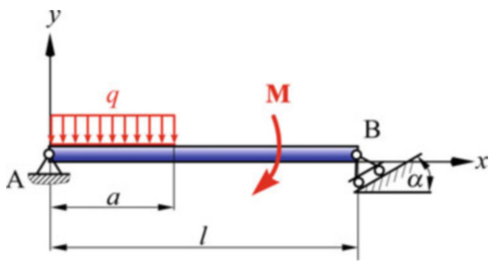


Fig. P6.62

6.63 Determine reactions at A and B. Use $q = 10$ N/m, $F = 20$ N, $a = 4$ m, $l = 12$ m, and angle $\alpha = 45°$.

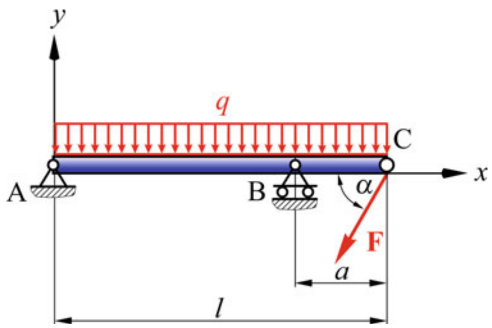


Fig. P6.63

6.64 Determine reactions at support A and tension in the cable. Use $q = 4$ lb/in., $a = 30$ in., $l = 70$ in., and $\alpha = 60°$.

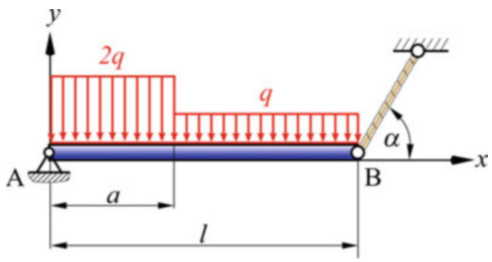


Fig. P6.64

6.65 Determine reactions at beam supports. Use $q = 10$ N/m, $a = 3$ m, $b = 2$ m, $l = 7$ m, and $\alpha = 30°$.

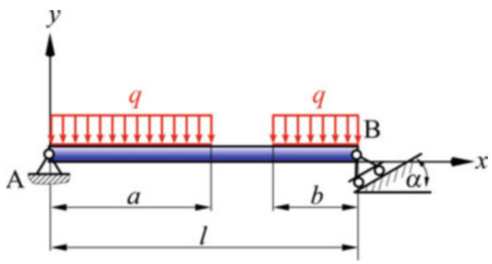


Fig. P6.65

6.66 Determine reactions at the beam supports. Use $q = 80$ lb/ft, $a = 4$ ft, $l = 8$ ft, and $\alpha = 30°$.

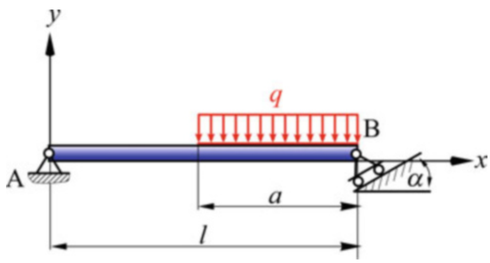


Fig. P6.66

6.67 Determine reactions at support A and tension in cable C. Use $q = 40$ N/m, $a = 4$ m, $l = 8$ m, and $\alpha = 45°$.

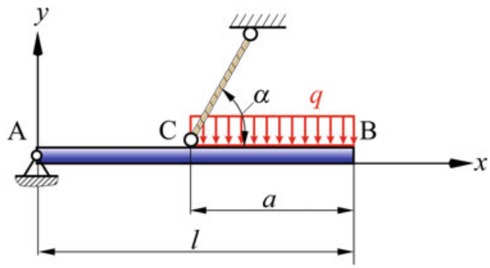


Fig. P6.67

6.68 Determine reactions at A, when beam AB is loaded by distributed load $q = 60$ N/m. Use $a = 3$ m and $l = 7$ m.

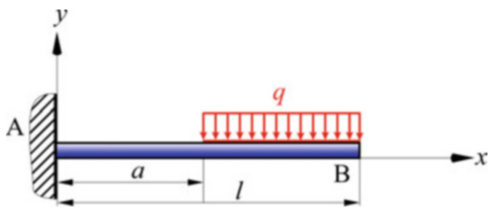


Fig. P6.68

6.69 Determine the magnitude of M so that reactions at A and B are equal. Use $q = 2$ N/m, $a = 4$ m, $l = 8$ m, and $\alpha = 45°$.

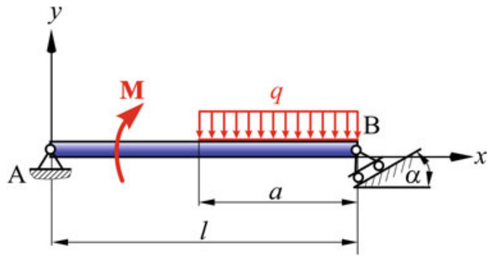


Fig. P6.69

6.70 Rod AB has weight $P = 200$ N and length 2 m. It is loaded by a distributed load $q = 25$ N/m. Determine reactions at A when $\alpha = 45°$.

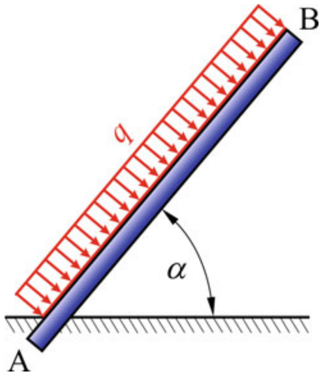


Fig. P6.70

6.71 Determine distance a so that vertical reactions at A and B are equal. Use $q = 2$ kN/m, $F = 5$ kN, $b = 1$ m, $l = 6$ m, and $\alpha = 60°$.

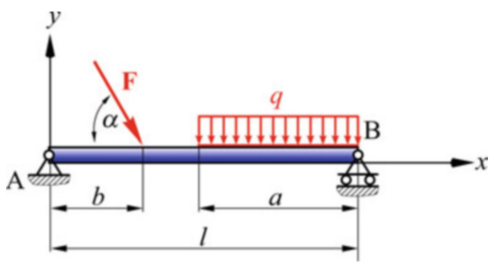


Fig. P6.71

6.72 Find the resultant force magnitude and its line of action on a 750-m long dam. The water depth is 50 m, its specific weight is 9800 N/m^3, and $\varphi = 60°$.

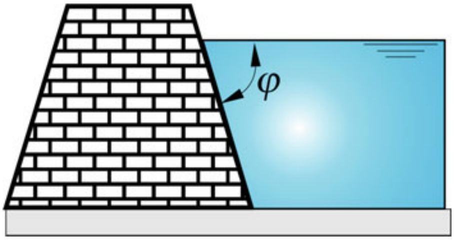


Fig. P6.72

6.73 Solid gate ABC can freely rotate about pin B. It is closed as long as water level is above a specific height h. When the level drops below this height, the door will swing clockwise and the water will flow out. Find the minimum height h to keep the gate closed. Use angle $\alpha = 45°$ and specific weight of the water $\gamma = 9800$ N/m^3.

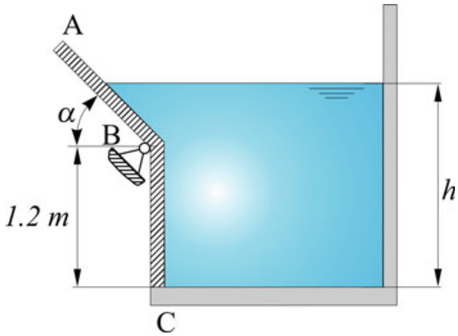


Fig. P6.73

6.74 Find the value of the angle α in Problem 6.73 to keep water at the level of 2.3 m.

6.75 A brick wall (2 m tall) is built to support the pressure of ground water from its right side. Find the width "*a*" necessary to keep the wall from tipping around point A. Specific weight of bricks is 25 kN/m^3 and of water is -10 kN/m^3.

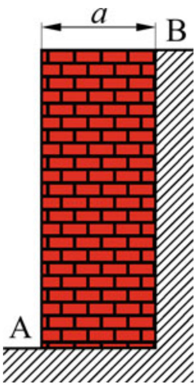


Fig. P6.75

6.76 Gate AB can freely rotate about pin O. It is closed as long as water level is below a specific height H. When the level rises above this height, the door will swing clockwise and the water will pour out. Find the minimum height H to keep the gate closed. Use angle $\alpha = 60°$, $h = 2$ m and specific weight of water $\gamma = 10$ kN/m^3.

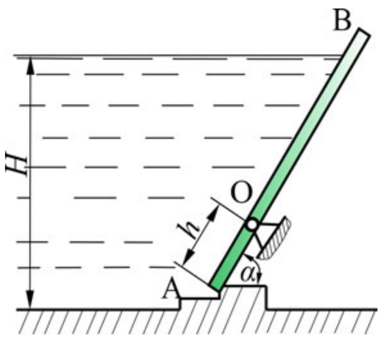


Fig. P6.76

6.5 Center of Gravity and Centroid of Bodies

Many engineering problems may be treated as two-dimensional problems. However, some structure elements with more complex geometries need a three-dimensional treatment. In this section, we will discuss the methodology to find the center of gravity and centroid for such elements.

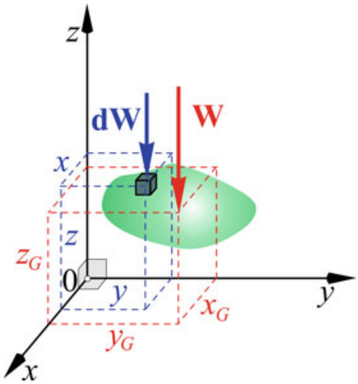


Fig. 6.15 Center of gravity

6.5.1 Center of Gravity

Mathematical procedure to derive the coordinates of the center of gravity was presented in the *Mathematical Corner* above. Here, we show another way to derive the same equations by extending the special case—center of gravity for a flat plate.

Let's choose a coordinate system with the *z*-axis parallel to the weight of a body, as shown in Fig. 6.15. The total weight **W** of the body can be represented as a sum (integral) of weights of infinitesimally small elements $d\mathbf{W}$ comprising the body.

$$\mathbf{W} = \int_{\text{volume}} d\mathbf{W} \tag{6.18}$$

Since we selected the coordinate system with the *z*-axis parallel to the weight (Fig. 6.15), the weight may be treated as a scalar quantity because its direction is known.

$$\mathbf{W} = 0 \cdot \mathbf{i} + 0 \cdot \mathbf{j} - W \cdot \mathbf{k}$$

The moment of the body (weight W) should be equal to the sum (integral) of moments of the body's constituent elements with respect to the x and y axes.

$$W \cdot x_{\mathrm{G}} = \int_{\text{volume}} x dW \tag{6.19}$$

$$W \cdot y_{\mathrm{G}} = \int_{\text{volume}} y dW \tag{6.20}$$

where x_{G} and y_{G} are the coordinates of the center of gravity.

Now, let us rotate the body and the coordinate system by 90° about, let say, x-axis (Fig. 6.16) so it is now parallel to the y-axis, i.e.,

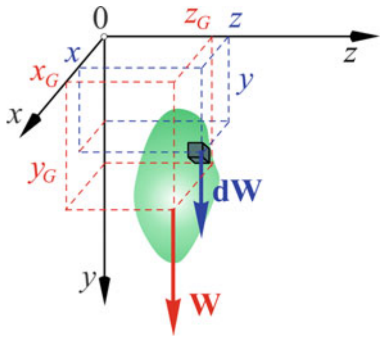


Fig. 6.16 Center of gravity for a three-dimensional body in a new coordinate system

$$\mathbf{W} = 0 \cdot \mathbf{i} + W \cdot \mathbf{j} + 0 \cdot \mathbf{k}$$

Now, we can calculate the moment of the weight and equate it to the sum of moments of all elements with respect to x-axis.

$$W \cdot z_G = \int_{\text{volume}} z dW \tag{6.21}$$

where z_G is the third coordinate of the center of gravity. From (6.19), (6.20), and (6.21) we can calculate the location of the center of gravity of the given body as:

$$\begin{aligned} x_G &= \frac{\int_{\text{volume}} x dW}{W} \\ y_G &= \frac{\int_{\text{volume}} y dW}{W} \\ z_G &= \frac{\int_{\text{volume}} z dW}{W} \end{aligned} \tag{6.22}$$

The obtained equations are the same as the derived in the *Mathematical Corner*.

6.5.2 Centroids

When a body is made of a homogeneous material, i.e., it has a constant specific weight γ at any point, the weight of each element of the body can be represented as $dW = dV \cdot \gamma$ and the total weight as $W = V \cdot \gamma$.

By substituting these relations into (6.19)–(6.21), we get

$$V \cdot x = \int_{\text{volume}} x dV$$
$$V \cdot y = \int_{\text{volume}} y dV \qquad (6.23)$$
$$V \cdot z = \int_{\text{volume}} z dV$$

where x_G, y_G, and z_G represent location of the body's *centroid*. In this case, the center of gravity of the body is located at the same point as the centroid. It is clear that if the density of the body is not constant, as we assumed in this case, the locations of centroid and center of gravity will not necessarily coincide.

> The locations of centroid and center of gravity will always coincide in a homogeneous body.

Example 6.7
Calculate the location of a cone's centroid (Fig. 6.17). The height of the cone is h, and the radius is R.

Fig. 6.17 The cone

Solution Let us select a coordinate system as shown in Fig. 6.18. The centroid should be on the axis of symmetry, which is in this case the x-axis. Therefore, only location of centroid along this axis has to be calculated using (6.23). First, we have to find the volume of the cone. Let us select a thin plate parallel to the cone's base located at distance x from yz plane. The volume of this plate is

$$dV = \pi r^2 dx$$

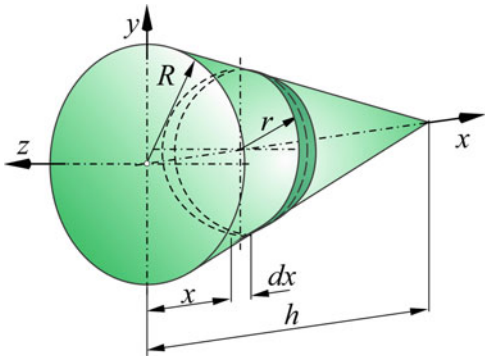


Fig. 6.18 Geometry of the cone

where

$$r = \frac{R(h - x)}{h}$$

dx is the thickness of the plate, and r is its radius.

Integrating this expression along the x-axis will result in the volume of the cone

$$V = \int_0^h \pi \cdot r^2 \cdot dx = \int_0^h \frac{\pi \cdot R^2 \cdot (h - x)^2}{h^2} dx = \frac{\pi \cdot R^2 \cdot h}{3}$$

From (6.23) we obtain

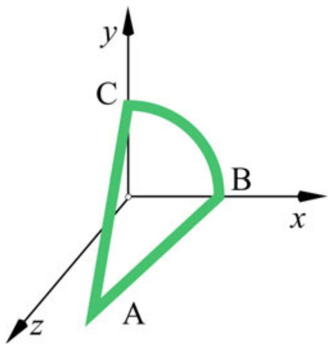
$$V \cdot x_G = \int_0^h x \cdot dV = \int_0^h x \cdot \pi \cdot r^2 \cdot dx = \int_0^h x \cdot \frac{\pi \cdot R^2 \cdot (h - x)^2}{h^2} dx = \frac{\pi \cdot R^2 \cdot h^2}{12}$$

Thus, the location of centroid is at $x_G = h/4$.

Example 6.8 Find the location of a centroid of a uniform, 3D structure shown in Fig. 6.19. Point A is located at (2.0, 0.0, 5.0) m, and the radius of quarter circle BC is 3.0 m.

Solution We will use the approach of the composite bodies. The structure consists of three segments: AB and AC are straight lines, and BC is a quarter circle. The centroid of a straight line is located at its geometrical center, while the centroid of a quarter circle is at location $x_G = y_G = 2R/\pi$, as was shown in Example 6.3. The location of the centroid can be calculated as

Fig. 6.19 Centroid of a line



$$x_G = \frac{\sum_{i=1}^{3} x_{Gi} \cdot L_i}{L}$$

$$y_G = \frac{\sum_{i=1}^{3} y_{Gi} \cdot L_i}{L}$$

$$z_G = \frac{\sum_{i=1}^{3} z_{Gi} \cdot L_i}{L}$$

where centroids of each segment are shown in the table below

Segment	L	x_G	y_G	z_G	$x_G \cdot L$	$y_G \cdot L$	$z_G \cdot L$
AB	5.10	2.5	0.0	2.5	12.75	0.0	12.75
BC	4.71	1.910	1.910	0.0	9.00	9.00	0.0
AC	6.16	1.0	1.5	2.5	6.16	9.24	15.4
	15.97				27.9	18.24	28.2

Thus,

$$x_G = \frac{\sum_{i=1}^{3} x_{Gi} \cdot L_i}{L} = \frac{27.9}{15.97} = 1.747\,\text{m}$$

$$y_G = \frac{\sum_{i=1}^{3} y_{Gi} \cdot L_i}{L} = \frac{18.24}{15.97} = 1.142\,\text{m}$$

$$z_G = \frac{\sum_{i=1}^{3} z_{Gi} \cdot L_i}{L} = \frac{28.2}{15.97} = 1.766\,\text{m}$$

6.5.3 Theorems of Pappus

Many geometrical shapes, which are used in engineering practice, can be generated by revolving a plane curve or a flat surface about an axis. Greek mathematician Pappus of Alexandria (*circa* 290–350 CE) derived the theorem for calculating the surface area or volume created by revolving a plane curve or an area. They apply to the curves and areas that do not intersect the axis of rotation. He showed that the surface and the volume are related to the distance traveled by their centroids. These theorems are sometimes called Pappus–Guldinus. Guldinus (1577–1643) was a Swiss mathematician, who had rediscovered these theorems, but was not able to give a satisfactory proof of them.

Theorem 1 *The surface area created by revolution of a plane curve about the axis belonging to the same plane is equal to the length of the curve multiplied by the distance traveled by its centroid.*

The line AB is revolved about the x-axis (Fig. 6.20) by an angle of 2π. The differential length dl generates a surface of $2\pi y dl$. Thus, the entire area S generated after rotation of 2π is

$$S = \int_A^B 2\pi y dl = 2\pi \int_A^B y dl$$

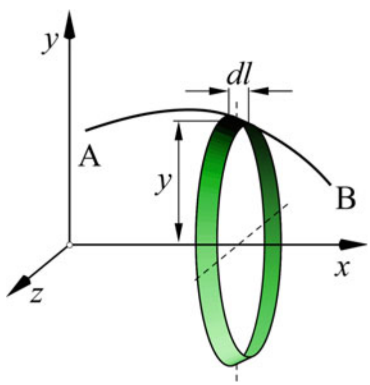


Fig. 6.20 Surface generated by a rotating line

As it was shown above, (6.12), the integral in the above equation is

$$\int_{\text{contour}} ydl = L \cdot y_{\text{G}}$$

Therefore, we have

$$S = 2\pi L y_{\text{G}} \tag{6.24}$$

where is the distances traveled by the centroid of the curve during the 2π revolution around the x-axis.

Theorem 2 *The volume of a body created by revolution of a plane area about the axis belonging to the same plane is equal to the area multiplied by the distance traveled by its centroid.*

Plane area A is revolving about the x-axis (Fig. 6.21) by an angle of 2π. Let's consider differential element dS. Rotation of element dS will generate a toroid with volume $dV = 2\pi y\, dS$. Thus, volume V generated by rotation of the whole area A by 2π is

$$V = \int_{\text{area}} 2\pi y\; dS = 2\pi \int_A y\, dS$$

The above integral can be represented as (see (6.12))

$$\int_A y\, dS = A y_{\text{G}}$$

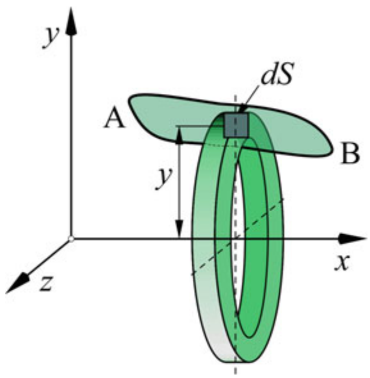


Fig. 6.21 Rotating a plane area

where y_G is the centroid of area A. The volume of the body may be expressed as

$$V = 2\pi A y_G \tag{6.25}$$

Here, $2\pi y_G$ is the distance traveled by the centroid of the plane area.

If the curve or the area is revolved through an angle φ less than 2π, the resulted area or volume can be found by substituting 2π by the angle φ. Thus, (6.24) and (6.25) become

$$A = \varphi L y_G \tag{6.26}$$

and

$$V = \varphi A y_G \tag{6.27}$$

Example 6.9 Calculate the surface area generated by revolving a line ABC shown in Fig. 6.22 about y-axis by the angle of 2π. The dimensions are: $L_{AB} = 1.2$ m, $L_{BC} = 2$ m, and angle $\varphi = 45°$.

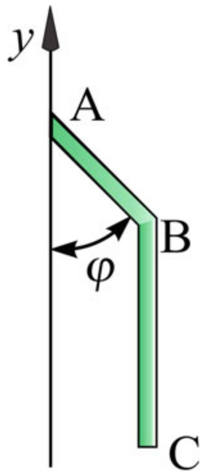


Fig. 6.22 Rotating line ABC

Solution Two surfaces will be generated by revolving line ABC about the y-axis: one by segment AB, and the second-by segment BC. The centroid of the segment AB is

$$x_{AB} = \frac{L_{AB} \cdot \sin\varphi}{2}$$

and for segment BC

$$x_{BC} = L_{AB} \cdot \sin\varphi.$$

Using (6.24) and combining both surfaces, we will get

$$A = 2\pi L_{AB} x_{AB} + 2\pi L_{BC} x_{BC} = 2\pi \cdot 1.2 \cdot 0.6 \cdot \sin 45^\circ + 2\pi \cdot 2 \cdot 1.2 \cdot \sin 45^\circ$$
$$= 13.86\,\mathrm{m}^2$$

Example 6.10 Calculate the area of the surface generated by rotation of a quarter circle shown in Fig. 6.23 about y-axis by angle $\pi/2$. The radius of quarter circle $R = 2\,\mathrm{m}$.

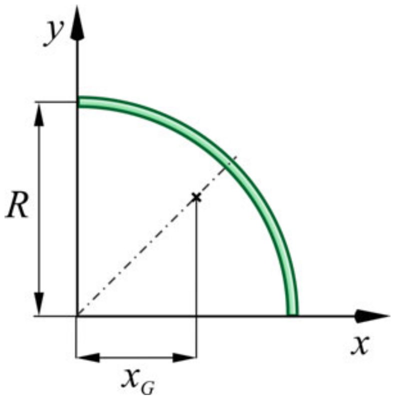


Fig. 6.23 Rotating quarter circle

Solution The centroid of the quarter circle is defined in Table 6.1, and is shown in Fig. 6.23, $x_G = 2R/\pi$. We can use (6.26) to calculate the area of the surface generated by rotation of the quarter circle,

$$A = \varphi L x_G = \frac{\pi}{2} \cdot \frac{2\pi R}{4} \cdot \frac{2R}{\pi} = \frac{\pi R^2}{2} = 6.29\,\mathrm{m}^2$$

Example 6.11 Find the volume of the body created by rotation of a triangle by angle about the y-axis using Pappus theorem. The geometry of the triangle shown in Fig. 6.24 is defined as:

$$a = 20\,\mathrm{cm}, \quad b = 10\,\mathrm{cm}, \quad \text{and} \quad h = 18\,\mathrm{cm}$$

Solution The volume may be calculated using (6.25),

$$V = 2\pi A x_G$$

where A is the area of the triangle and x_G is the location of the centroid in x direction.

Fig. 6.24 Rotating triangle

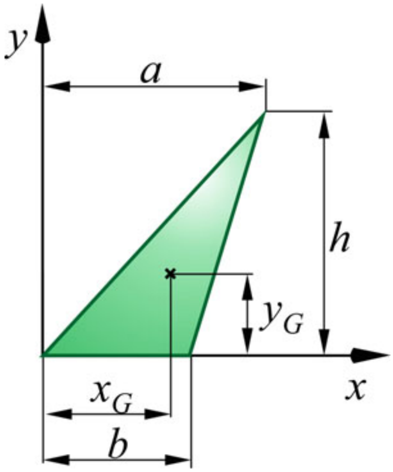
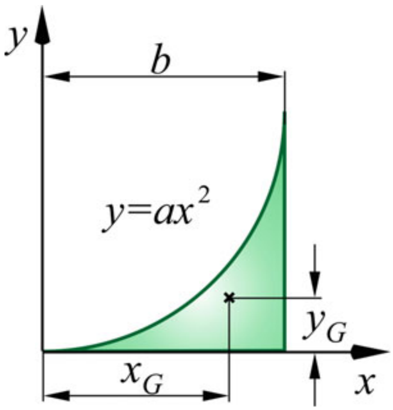


Fig. 6.25 Rotating body



The area of the triangle is $A = bh/2$. Location of its centroid, as given in Table 6.1, is $x_G = (a + b)/3$. Thus, the volume of the body created by rotating the triangle around y-axis is

$$V = 2\pi A x_G = 2\pi \frac{bh(a+b)}{2 \cdot 3} = \pi \frac{10 \cdot 18 \cdot 30}{3} = 5650\,\text{cm}^3$$

Example 6.12 Find the volume of the body created by rotation of the area defined by a parabola and lines $y = 0$ and $x = b$ around the x-axis (Fig. 6.25).

Solution The volume is calculated by (6.25),

$$V = 2\pi A y_G$$

where $A = ab^3/3$ is the area of the parabolic triangle and $y_G = 3ab^2/10$ is the location of the centroid in y direction. The values are taken from Table 6.1. Thus, the volume of the body created by rotating the parabolic triangle around y-axis is

$$V = 2\pi A y_G = 2\pi \frac{ab^3}{3} \cdot \frac{3ab^2}{10} = \frac{\pi a^2 b^5}{5}.$$

What We Have Learned?

The difference between the concentrated and distributed loads
The *concentrated* loads act at a point, while the *distributed* loads act over an area. In reality, there are no concentrated loads. However, when size of the contact area is small compared to the structural element, the load may be considered as a concentrated load. When dealing with the equilibrium of a rigid body only, all distributed loads may be replaced by the equivalent concentrated loads.

About centers of gravity and centroids
Center of gravity is a point where the total weight (resultant) is acting. The effect of the resultant on rigid body equilibrium is the same as of the distributed load.
Centroid is the geometrical center of a body, which depends on body's geometry only. For homogeneous bodies, the location of the centroid and the center of gravity coincide.

How to find center of gravity and centroids of various bodies
Center of gravity is defined by equating the effect of the distributed load on the body equilibrium with the effect of its resultant. Equations (6.22) and (6.12)–(6.13) are used to calculate the location of the center of gravity and the centroid of 2D and 3D solid bodies and wires.

How to replace the effect of a distributed force by a concentrated force
To find the concentrated force equivalentType equation here. to a distributed load generated by the weight of a body, we have to integrate over the body volume to find its total weight (force). The location of the concentrated load is in the center of gravity of a body, which is defined by (6.22).

6.5.4 Problems

6.77 Find the center of gravity of a wire. Weight of the wire per unit length is 10 N/m.

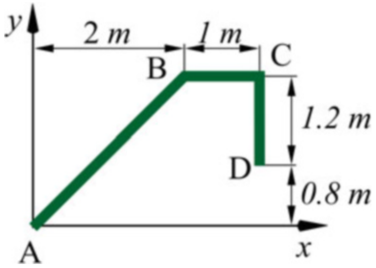


Fig. P6.77

6.78 Find the center of gravity of the dam. The left side is made of sand with specific gravity of 1.8 kN/m^3, and the right side is a concrete with specific gravity of 24 kN/m^3. Assume the depth of the dam is 1 m.

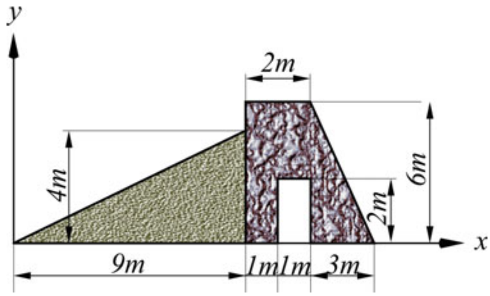


Fig. P6.78

6.79 Find the centroid of the structure made of two quarter-circles, each having radius R, and a straight piece of a wire connecting them.

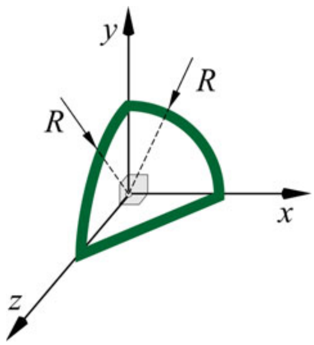


Fig. P6.79

6.80 Find the centroid of a structure made of homogeneous bars. The length of each bar is 40 cm, except the bars 9 and 10, their length is 20 cm each.

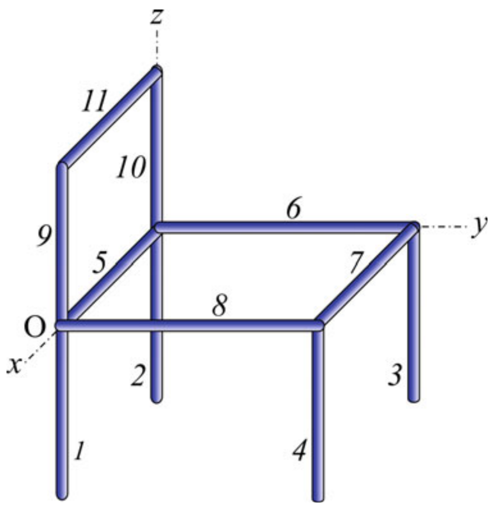


Fig. P6.80

6.81 Find the centroid of the structure made of five homogeneous rods. The length of each road is 30 cm, and it makes angle of 45° with the plane ABCD.

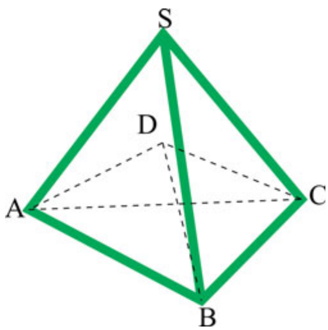


Fig. P6.81

6.82 Determine the volume of a body obtained by revolving the area between a line and the y-axis about the y-axis. The dimensions are in m.

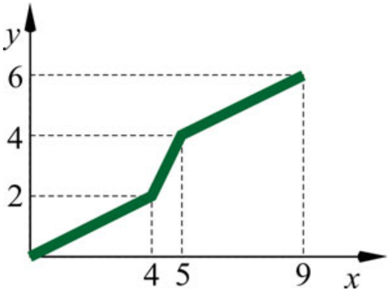


Fig. P6.82

6.83 Determine the area of the surface obtained by revolving the area between a line and the y-axis about the y-axis. The dimensions are in m.

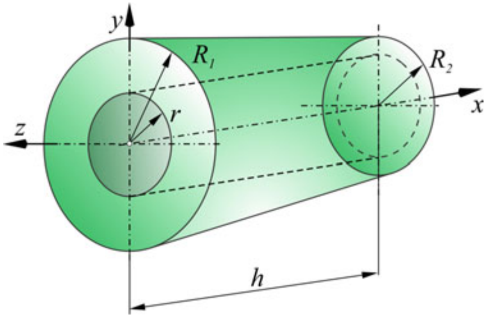


Fig. P6.83

6.84 Find the total surface area of a body using Pappus theorem. Height h is 100 cm, internal radius r is 20 cm, the outside radius at base R_1 is 60 cm, and at top R_2 is 30 cm.

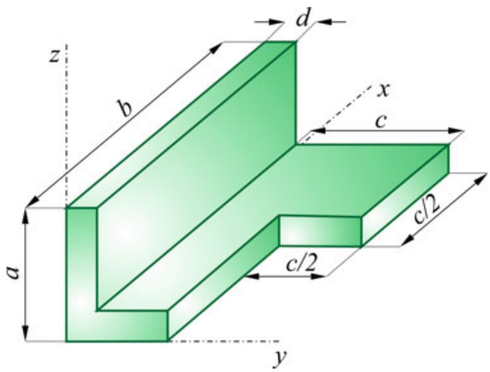


Fig. P6.84

6.85 Calculate the volume of the body shown in Fig. P6.84 using Pappus theorem.

6.86 Locate the centroid of a homogeneous body. Use $a = 40$ cm, $b = 140$ cm, $c = 60$ cm, $d = 10$ cm.

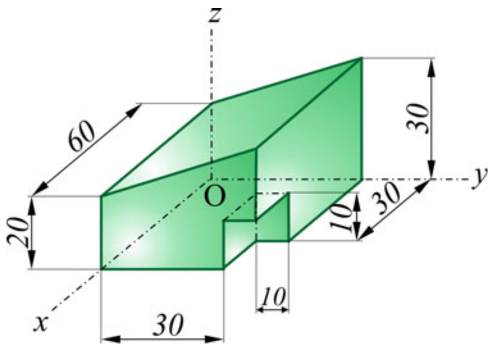


Fig. P6.86

6.87 Locate the centroid of a homogeneous body. All dimensions are in cm.

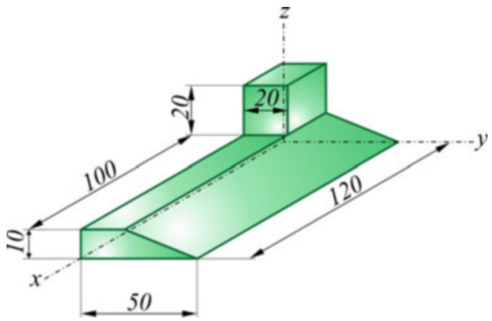


Fig. P6.87

6.88 Locate the centroid of a homogeneous body. All dimensions are in cm.

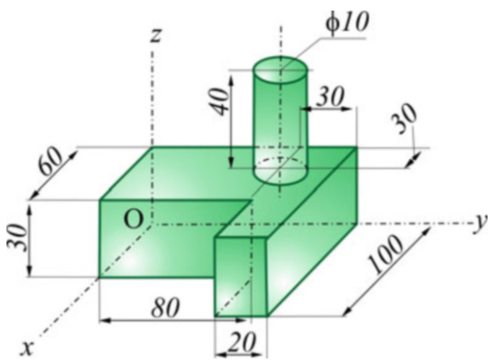


Fig. P6.88

6.89 Locate the centroid of a homogeneous body. All dimensions are in cm.

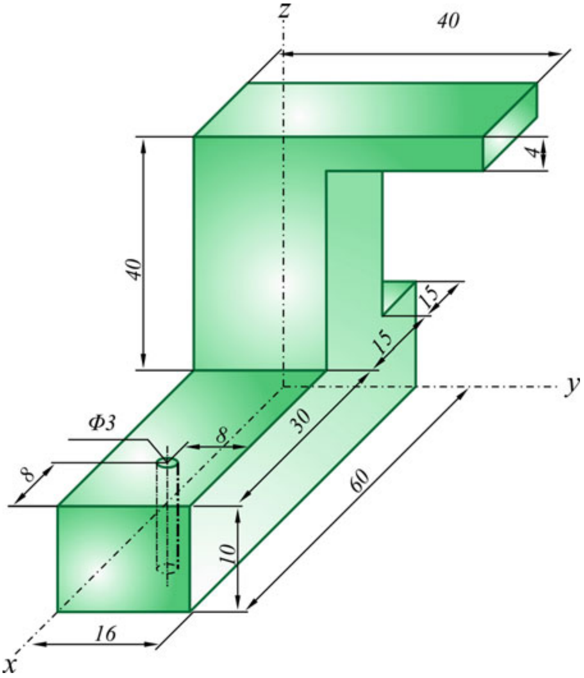


Fig. P6.89

6.90 Determine the centroid of a bridge support. All dimensions are in cm.

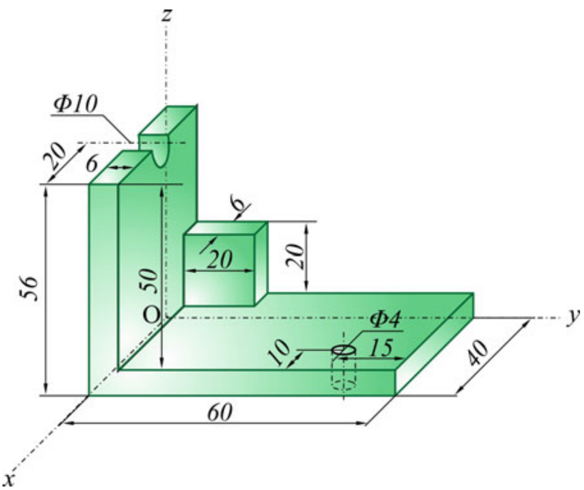


Fig. P6.90

6.91 A sphere (radius $2a$) contains a spherical cutout with radius a centered at O_1 $(0, a/2, a/2)$. Find the coordinates of the centroid of this body.

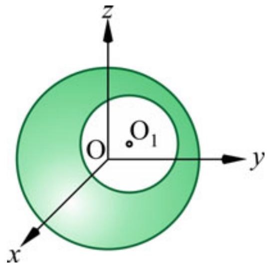


Fig. P6.91

6.92 Centroid of a homogeneous body assembled from a cube with base "a" and a prism (height b) is located at plane ABCD. Determine b.

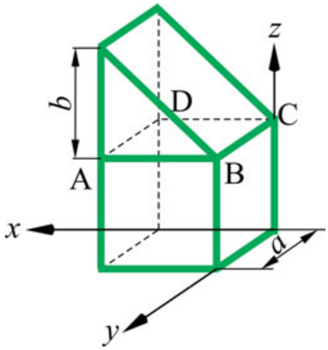


Fig. P6.92

Classification of Structural Elements 7

Contents

Not everything that can be counted counts, and not everything that counts can be counted

—Albert Einstein

In this chapter you will learn:

- When a structural element can be considered as a truss, frame, or machine
- When a structural element can be considered as a beam
- When a structural element can be considered as a cable
- How to find internal forces

In previous chapters, we were dealing with equilibrium of objects modeled either as particles (points) or rigid bodies depending on their size and loading conditions. As we already know the actual geometry of a rigid body has no effect on its equilibrium. The latter depends only on the moments and the relative location of the forces acting upon the body.

In this chapter, we will introduce concepts, assumptions, and rules necessary to classify structural elements. The following classes of structural elements will be introduced: trusses, beams, frames, machines, and cables. We will also discuss the procedure to calculate internal forces in various structural elements.

I. Emri, A. Voloshin, *Statics*, DOI 10.1007/978-1-4939-2101-0_7

There is no general approach for solving all possible structures. Historically, methods were developed for solving a particular class of structural elements. Whenever a structure is being analyzed, one of the first questions that arise is to what class its elements belong. As a matter of fact, drawing a physical model is essentially a step to simplify the structure so that it will belong to a certain group of problems, for which the procedure to obtain the solution is known.

7.1 Types of Structural Elements

Observation of the nature and man-made objects reveals that usually they are not made from a single homogeneous piece. In many instances, the objects are made from a number of constituents. As a rule, the constituent elements are more homogeneous and have simpler forms than the whole object. Even human being may be considered as constructed of a number of different systems, like musculo-skeletal, cardiovascular, etc. Each one of those is significantly simpler to model than the human being as a whole. Though the engineering structures are much simpler than the human body, they still need to be represented and modeled as an assembly of their constituents. For example, even such sophisticated engineering system as an airplane, which is made of different materials and structural elements (Fig. 7.1), can be represented as an assembly of thin plates, straight and curved elements, wheels, wires, etc.

Because the whole objects, like an airplane or car, usually cannot be analyzed due to their complexity, here we will develop approaches to represent complex

Fig. 7.1 An airplane, a complex engineering structure

objects as a sum of their simplified constituent parts. The simplification allows to represent each of the constituent parts as an element belonging to a particular class of structural elements for which the procedure to find the solution is known.

> The simplification allows to represent a structure elements as an element belonging to a particular class of structural elements for which the procedure to find the solution is known.

Now we are ready to discuss the assumptions used in the process of creating a physical model. These assumptions are needed in order to reduce the problem to a case that is possible to solve. Historically, methods were developed for solving different classes of problems such as beams, trusses, cables, plates, and shells. We will discuss requirements that a structural element has to fulfill to be assigned to a certain class of structural elements.

7.1.1 Truss Members

We will discuss truss members using an example of a high voltage transmission tower (Fig. 7.2a). It is loaded by tension of wires applied at different points. The tower below is built from straight, slender bars connected one to another at their ends. We will assume that members of this structure are connected by joints that may be represented as frictionless pins. We will also assume that the weight of each member is negligible compared to the forces it carries. We shall further assume that loads are applied only at the joints. When these assumptions are fulfilled, each member is subjected either to tensile or compressive load along its longitudinal axis. Such elements are called *truss elements*, and the structure built from these elements is called a *truss*. Using these assumptions, we may model the tower as a truss. A segment of this model is shown in Fig. 7.2b.

To find the forces acting on each member, we will have to disassemble the tower and consider equilibrium of each member separately. Since there are only two forces acting on an element of a truss, it is considered two-force elements (Fig. 7.2c).

> A structural element may be considered as a truss member if:
>
> - It is a slender, straight element.
> - It is connected to other members only at its ends with a frictionless pin.
> - It is loaded by forces at its ends only.

> Truss members are never loaded by moments.

It should be noted that in reality the connections between the structural elements are not frictionless pins, but either bolts or welded joints. Such connections will

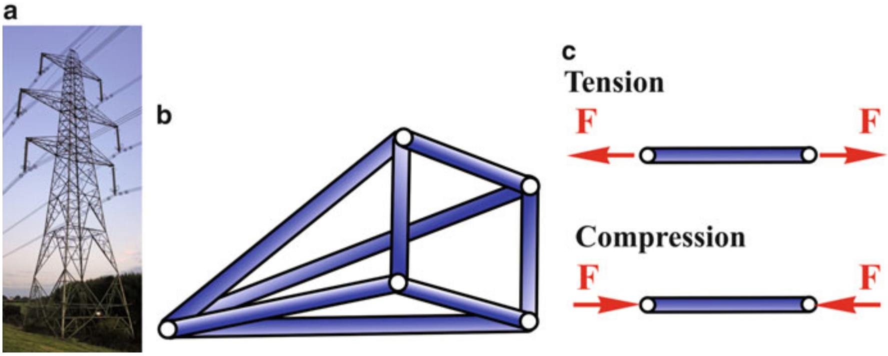


Fig. 7.2 (**a**) High voltage tower. (**b**) Segment of the truss. (**c**) Two-force member

introduce bending moments. The theory developed to solve trusses does not account for these moments. Therefore, the results obtained by modeling such structure as a truss will contain errors. These errors need to be accounted for in the design process by utilizing appropriate safety factor.

7.1.2 Beams

In many engineering applications, slender structural elements are loaded by moments or by forces positioned not only at joints, but also at some other locations. Such members cannot be modeled as truss elements. The same is true when structural elements are not straight. Slender elements that are curved or loaded by moments or forces acting not only at the joints are called beams.

> Bending of a structural element is always a result of a moment.

Let us consider an example, a flowerpot supported by a holder (Fig. 7.3a). The physical model of the holder is shown in Fig. 7.3b. It is modeled as a curved beam, loaded by a distributed load.

> A structural element may be considered a beam if:
>
> - It is a slender, straight, or curved element.
> - It may be connected to other members at any point.
> - It can be loaded at any point by forces and moments.

Beams are important parts of structures called *frames*. They will be discussed later in Chap. 8.

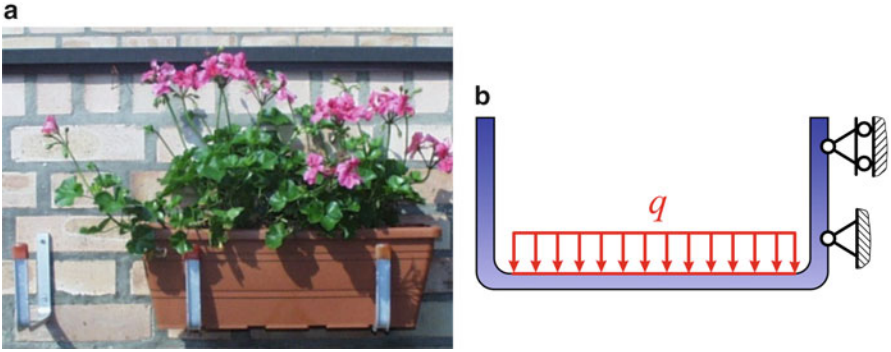


Fig. 7.3 (**a**) Flowerpot suspended by a holder. (**b**) Physical model of the holder

It should be noted that a straight beam loaded only by forces acting at its ends would behave as a truss member. It means that a truss member is a special case of a straight beam.

7.1.3 Cables

If a structural element becomes very thin comparatively to its length, it will not be able to support compressive loads or moments. It will be able to carry tensile loads only. Such structural elements are called *cables*. Cables are considered as rigid bodies only when loaded by tensile forces. They cannot sustain any compressive or bending loads, i.e., they are considered to be ideally flexible. A cable loaded only by an axial tensile force will behave as a truss member. When loading is applied in any other than the axial direction, the cable will change its geometry to accommodate the load. Cable can be connected to other member of a structure at its ends only, but it may be loaded by force at any point. Cables cannot be loaded by moment. Figure 7.4a depicts a cable loaded by its own weight, i.e., by a distributed load. The cable loaded by a concentrated force, which is significantly larger than its own weight, is shown in Fig. 7.4b.

A structural element may be considered a cable if:

- It is a slender element.
- It is rigid in axial direction, in tension only.
- It is ideally flexible.
- It is connected to other members at ends only.
- It can be loaded at any point by any force, but not by a moment.

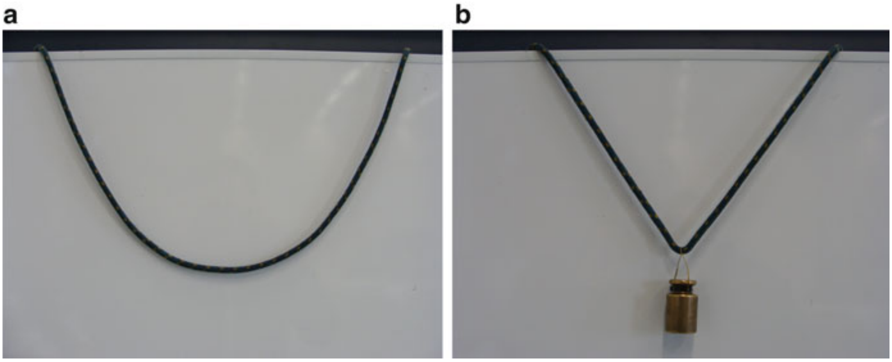


Fig. 7.4 Cables

In both cases, the cable assumes the shape imposed by the load. It should be noted here that the solution procedures are quite different for the cables loaded by concentrated vs. distributed loads, as it will be shown in Chap. 10.

Geometry of a cable depends on the applied load.

Cables can support tensile loads only.

7.2 Internal Forces

Loads and supports represent external forces and moments acting on a matter (material) comprising the body. Matter from which a body is built has to sustain these external forces and moments. Forces acting on the matter at a particular location inside the body are called *internal forces*. These forces appear in the body due to the external forces according to the Newton's Third Law, and they resist the change of the body's geometry. Knowledge of the internal forces distribution is necessary for design of an element, i.e., choice of geometry and material.
Let us consider a free body diagram of the rigid body loaded by external forces and moments, as shown in Fig. 7.5a.

To analyze the internal forces, we should utilize the principle of equilibrium. The external forces and moments have to satisfy the equations of equilibrium:

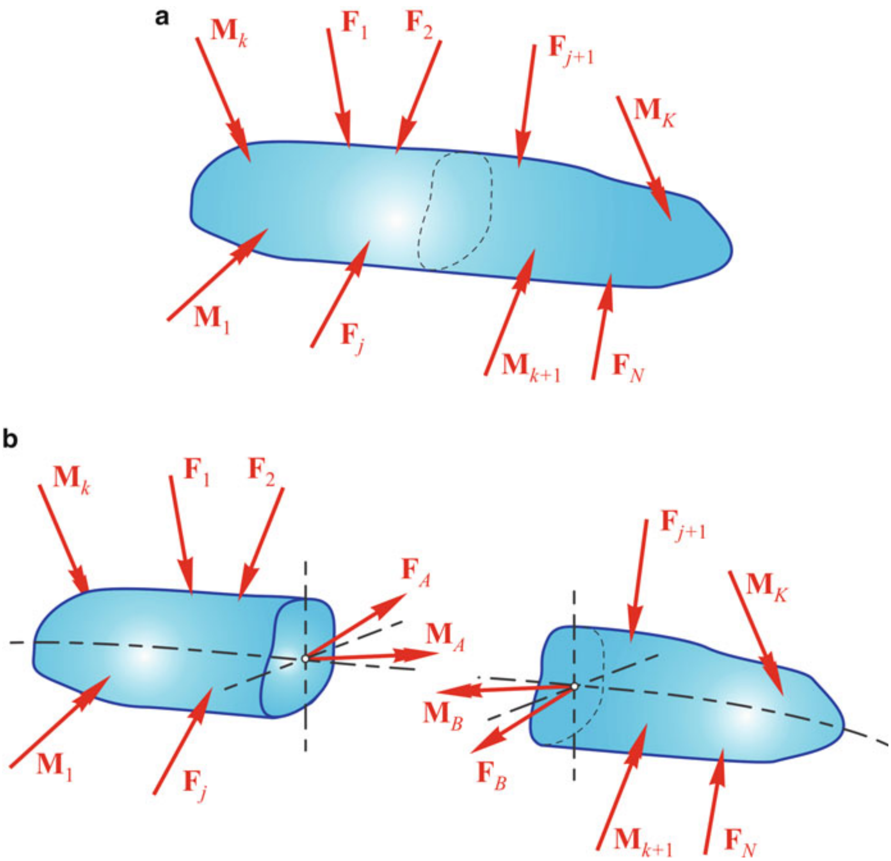


Fig. 7.5 (**a**) Free body diagram of a rigid body. (**b**) Body cut in two

$$\sum_{i=1}^{N}\mathbf{F}_i = 0$$
$$\sum_{i=1}^{K}\mathbf{M}_i = 0 \tag{7.1}$$

Let us mentally cut the body into two parts A and B at an arbitrary location (Fig. 7.5b). Since the original rigid body was in equilibrium, each part should be in equilibrium as well. Let us analyze each part as a separate body.

To keep each part in the equilibrium, we should add a force and moment, as shown in Fig. 7.5b. That together with external loads acting on this part will satisfy the equations of equilibrium. The force and the moment, that we have added, are the resultant force and resultant moment of the internal forces acting on the surface of the cut.

The equations of equilibrium for part A are

$$\begin{aligned} \mathbf{F}_{\mathrm{A}} + \sum_{i=1}^{j} \mathbf{F}_i &= 0 \\ \mathbf{M}_{\mathrm{A}} + \sum_{i=1}^{k} \mathbf{M}_i &= 0 \end{aligned} \tag{7.2}$$

and for part B

$$\begin{aligned} \mathbf{F}_{\mathrm{B}} + \sum_{i=j+1}^{N} \mathbf{F}_i &= 0 \\ \mathbf{M}_{\mathrm{B}} + \sum_{i=k+1}^{K} \mathbf{M}_i &= 0 \end{aligned} \tag{7.3}$$

The summations in (7.1) may be split into two ranges to obtain

$$\begin{aligned} \sum_{i=1}^{j} \mathbf{F}_i + \sum_{i=j+1}^{N} \mathbf{F}_i &= 0 \\ \sum_{i=1}^{k} \mathbf{M}_i + \sum_{i=k+1}^{K} \mathbf{M}_i &= 0 \end{aligned}$$

From here it follows that

$$\begin{aligned} \sum_{i=1}^{j} \mathbf{F}_i &= -\sum_{i=j+1}^{N} \mathbf{F}_i \\ \sum_{i=1}^{k} \mathbf{M}_i &= -\sum_{i=k+1}^{K} \mathbf{M}_i \end{aligned}$$

and therefore $\mathbf{F}_{\mathrm{A}} = -\mathbf{F}_{\mathrm{B}}$ and $\mathbf{M}_{\mathrm{A}} = -\mathbf{M}_{\mathrm{B}}$. It means that the internal force and moment acting on the part A are equal in magnitude and opposite in direction to the internal force and moment acting on part B, as could be predicted from the Newton's Third Law. Thus, internal force and moment in this cross section can be calculated either from (7.2) or (7.3).

Example 7.1 The beam shown in Fig. 7.6a is loaded by a vertical force $P = 12$ kN. Determine the internal force and moment acting at point C when $a = 0.1$ m.

Solution Cut the beam at point C, apply the unknown internal force and moment at this section and consider the equilibrium of the right hand part of the structure. The free body diagram is shown in Fig. 7.6b. We will consider equilibrium of the right

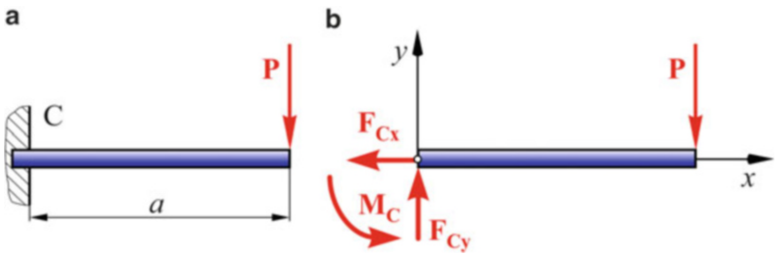


Fig. 7.6 (**a**) Physical model of the beam. (**b**) Free body diagram

hand part of the beam since all external loads acting on it are known. The unknown internal force is represented through its two orthogonal components: C_x and C_y as it is shown on the free body diagram. Directions of the unknown forces and moments can be assigned arbitrary. The solution will show if the direction was chosen correctly, positive sign indicates that the direction was assumed correctly. The equations of equilibrium are (we are taking the sum of the moments about point C):

$$\sum F_x = -F_{Cx} = 0$$
$$\sum F_y = F_{Cy} - P = 0$$
$$\sum M = M_{\mathrm{C}} - Pa = 0$$

Solving these equations and using given values of **P** and a result in $F_{Cx} = 0$, $F_{Cy} = 12$ kN, and $M_C = 1.2$ kN m. It should be noted that the positive values indicate that the unknown forces and the moment act in the directions as it was shown in the free body diagram.

Example 7.2 The ladder, schematically shown in Fig. 7.7a, is loaded by a vertical force $P = 810$ N. Determine the internal force and moment acting at the point D, when $a = 0.2$ m.

Solution Before cutting the supporting leg at point D, we have to find the reactions at point A. Assume that the ladder is on the frictionless floor. The corresponding free body diagram is shown in Fig. 7.7b. Let us consider the equilibrium of the ladder. Since we need to find only the reaction at support A, we will use only one equation of equilibrium. Let us write the sum of moments about support B, this will eliminate the unknown reaction B from the equation.

$$\sum M = -A_y \cdot 3a + P \cdot a = 0$$

Solution is $A_y = \frac{1}{3}P$. Now, we are ready to cut the leg at point D and apply the unknown internal force and moment, as shown in Fig. 7.7c.

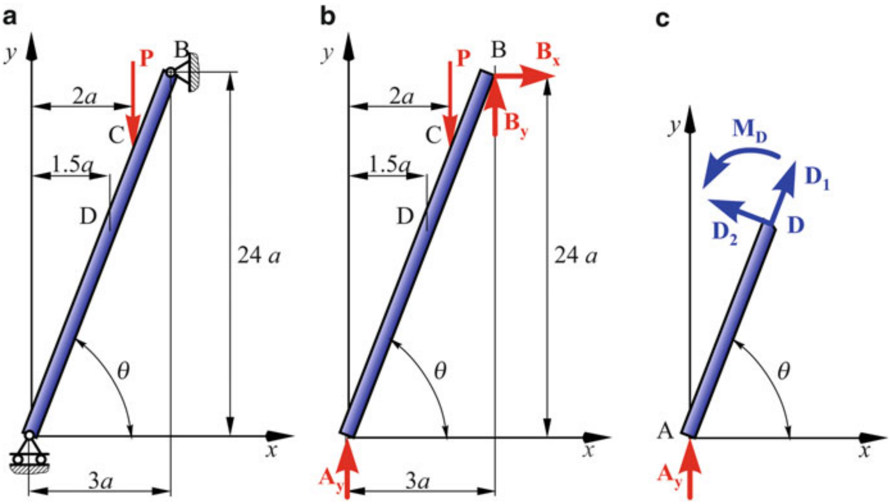


Fig. 7.7 (**a**) The ladder. (**b**) Free body diagram. (**c**) Free body diagram of the section of interest

We will represent the unknown internal force by its two orthogonal components, one along the member (D_1) and one in the perpendicular direction (D_2). From Fig. 7.7a we can find the angle $\theta = a\tan(8) = 82.9°$ and the length $\mathrm{AD} = 1.5a/\cos\theta$. The equations of equilibrium for the part AD are:

$$\sum F_x = D_1 \cdot \cos\theta - D_2 \cdot \sin\theta = 0$$

$$\sum F_y = D_1 \cdot \sin\theta + D_2 \cdot \cos\theta + A_y = 0$$

$$\sum M = M_D - A_y \cdot AD \cdot \cos\theta = M_D - A_y \cdot 1.5a = 0$$

Solution of this system of three equations is

$$D_1 = -268\,\mathrm{N}$$
$$D_2 = -33.4\,\mathrm{N}$$
$$M_\mathrm{D} = 81\,\mathrm{N\,m}$$

The negative sign means that the directions of both force components D_1 and D_2 should be reversed. The magnitude of the internal force $D = \sqrt{D_1^2 + D_2^2} = 270$ N.

Guidelines and Recipes for Determination of Internal Forces and Moments

- Find the unknown reactions.
- Cut the rigid body at the selected location.
- Add unknown internal force and moment to any part.
- Write and solve the equations of equilibrium for this part.

What We Have Learned?

When a structural element may be considered as a truss member
A structural element should be modeled as a truss member when it is a straight, slender body loaded by the axial forces acting at its ends only. It should be connected to the other members at its ends by frictionless pins only.

When a structural element may be considered as a beam
A structural element should be modeled as a beam when it is a slender, straight, or curved element, connected to the other members at any point, and it is loaded by moments or by forces acting at any point.

When a structural element may be considered as a cable
A structural element should be modeled as a cable when it is a slender element, rigid in the axial direction and very (ideally) flexible. It should be connected to other members at both ends only. It may be loaded at any point by forces, but not by moments.

How to find internal forces
Internal forces and moments are obtained by cutting a rigid body at a selected location, adding an unknown internal force and a moment and applying the equilibrium equations.

7.2.1 Problems

7.1 Four hangers are ready to support a 500 N beam. Determine the internal forces and moments at the base of hanger A. Assume that the load of the beam is uniformly distributed and the hanger's width is 20 cm.

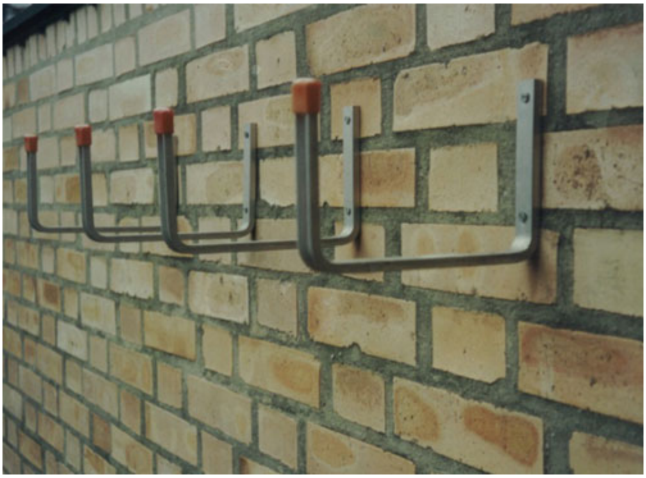


Fig. P7.1

7.2 The weight of homogeneous member DE is 300 N, it is acting at the middle of span DE. Force $Q = 100$ N. Determine the internal forces and moments in the member DE, just above the point A. Use $AB = 3$ m, $BC = 1$ m, $AD = 1/4DE$, and $\alpha = 30°$.

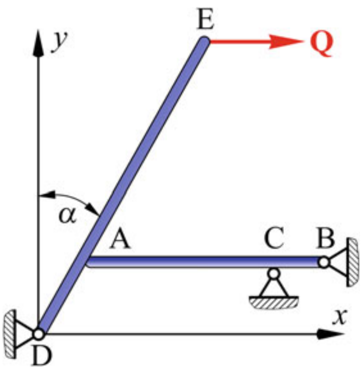


Fig. P7.2

7.3. Rod AB is supported by rod CD; attached weight $Q = 200$ N. Determine the internal forces and moments in member CD, just below point E. Angle $\varphi = 60°$ and $l = 60$ cm.

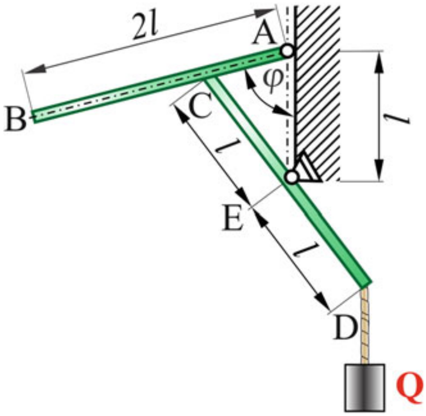


Fig. P7.3

7.4 Homogeneous rod Fig. P7.3 AB (weight $W = 100$ N) is supported by rod CD; attached weight $Q = 200$ N. Determine the internal forces and moments in member BC, just to the left of point C. Angle $\varphi = 60°$ and $l = 60$ cm.

7.5 Rod AB is supported by rod CD and loaded by force $F = 50$ N. Determine the internal forces and moments in the member AB, just to the right of point C.

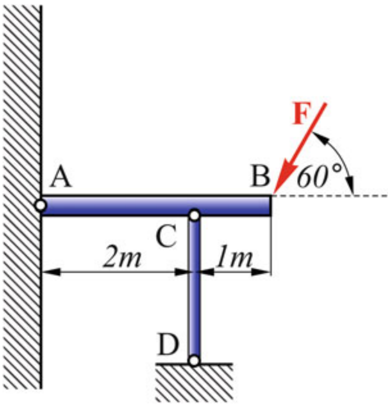


Fig. P7.5

7.6 Rod AB (length $l = 50$ cm) is supported as shown below and loaded by force **P** and moment **M**. Determine the internal forces and moment in member AB, just to the right of point F. Use $a = 20$ cm, $b = 30$ cm, $P = 30$ N, and $M = 200$ N m.

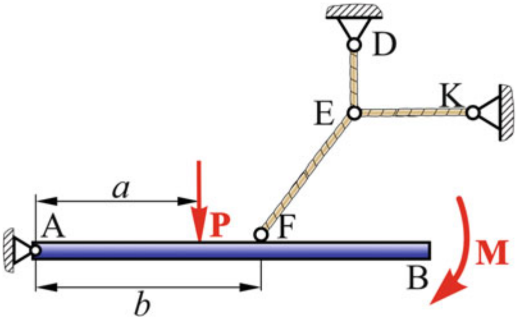


Fig. P7.6

7.7 Member AOB is attached to cable CD and is loaded by force $P = 100$ N and counterweight $B = 300$ N. Determine the internal forces and moment at the section just to the left of point O. Use $AO = OB = 0.5$ m, $OC = CB$, $\alpha = 30°$, $\gamma = 60°$, and force **P** is acting normal to AO.

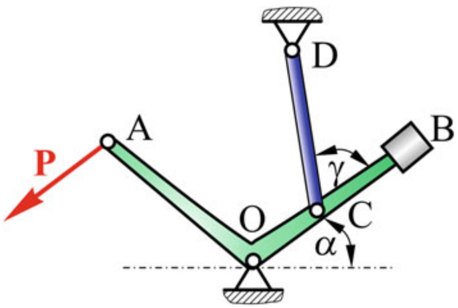


Fig. P7.7

7.8. Weightless beam AB leans on beam CD and is loaded by force $P = 120$ N. Determine the internal forces and moment in member AB, just above point B_1. Use $BB_1 = 1.2$ m.

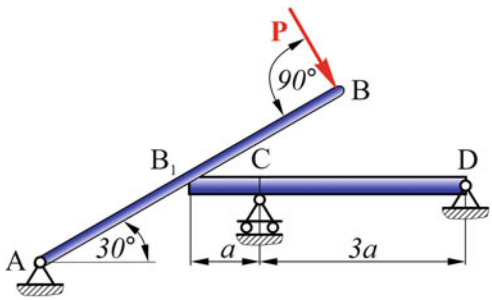


Fig. P7.8

7.9 Disk C is held in equilibrium by cable AC and loaded by a homogeneous bar AB. Determine the internal forces and moment at the point just above of the contact between the bar and the disc. Weight of bar AB is 16 N and $r = 0.5$ m.

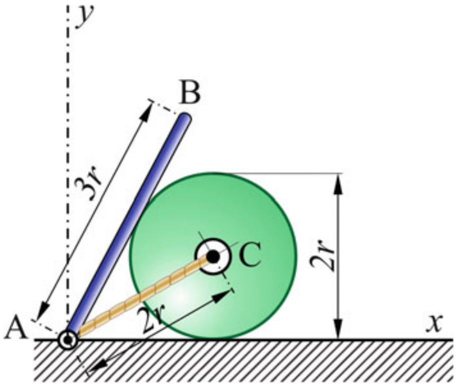


Fig. P7.9

7.10 Safety valve A of the pressure vessel is attached by link AB to homogeneous bar CD (weight of bar $CD = 12$ N). Determine the internal forces and moment at the point just to the right of point B. $CD = 60$ cm, $BC = 6$ cm, and $Q = 20$ N.

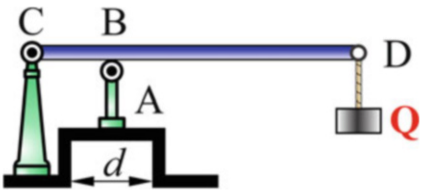


Fig. P7.10

Analysis of Truss Structures

8

Contents

I find that the harder I work, the more luck I seem to have

—Thomas Jefferson

Do, or do not. There is no "try"

—Yoda

In this chapter you will learn:

- What is truss
- What is compound truss
- How to use different techniques to find the internal forces in trusses

As we described in Chap. 7, there are several main types of structural elements. In this chapter, we will discuss ways to find the internal forces acting in structures. Structures may be spatial or planar. For example, if all of the structure members and loads belong to the same plane, we will call such a structure planar.

Some of the structures may be modeled as a truss, others as a frame, or a mechanism. In this chapter, we will discuss how to solve for internal forces in truss structures.

I. Emri, A. Voloshin, *Statics*, DOI 10.1007/978-1-4939-2101-0_8

Truss is a structure that is built only from two-force elements (truss members) connected at the ends one to another (joints) in order to create a desired shape. The external loads may be applied only at the joints.

Trusses are built to support external loads and prevent any movements.

We will use the principles of equilibrium of forces and moments to find forces acting in each member of a structure.

One should use "common sense" in the process of idealization and modeling of real objects. An exact analysis of the system may require very sophisticated analysis tools, which are beyond the scope of this book. However, any system may be simplified and modeled so that a lot of useful information can be obtained by using methods discussed in this book. Let us consider, e.g., a racing car (Fig. 8.1). Observation of the object leads to the conclusion that it has some kind of symmetry, not in a pure mathematical sense, but rather in approximate, engineering sense. Well, the car is not exactly symmetrical since, e.g., the steering wheel is on the left side and the glove compartment is on the right. However, such non-symmetry is not significant for the analysis of structural capacity of the car. We will make our first simplification by assuming the car to be a symmetrical structure. We will assume that the plane of symmetry cuts the car into left and right halves. In such a case, one can reduce the three-dimensional problem into a two-dimensional one. Let us now consider the plane view of the car (Fig. 8.1)

> *Truss* is a structure that is built only from two-force elements (truss members) connected one to another in order to create a desired shape.

Now, we make the next simplification and represent the car as a system of uniform, straight members built to support the load of all essential car components. To design the size and material of those components, we have to find a way to estimate the forces that will be carried by each member of the structure. We also assume that the members of this structure are straight, rigid members interconnected by frictionless pins that are loaded at these joints only. Figure 8.2 shows an overlay of the two-dimensional view of the car with the simplified model of the car as a truss (solid lines).

> Truss members are connected by frictionless pins at their ends and are loaded at the pins only. The weight of the member is usually neglected.

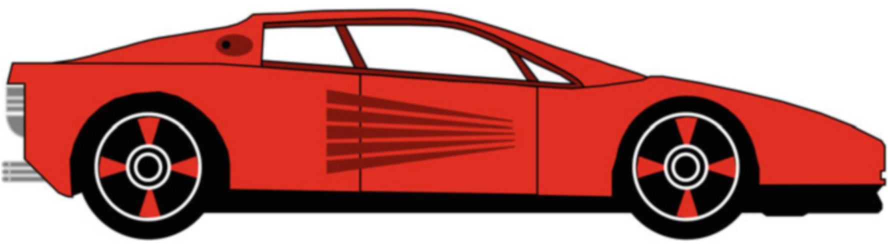


Fig. 8.1 Your favorite car

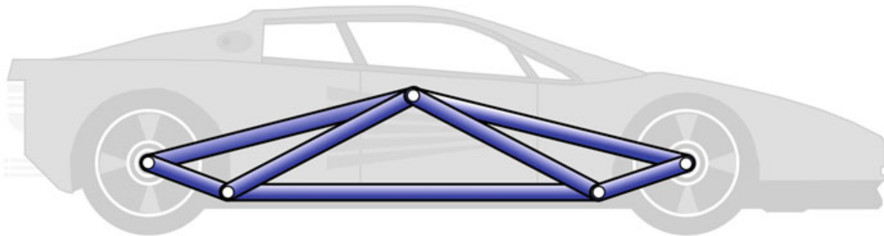
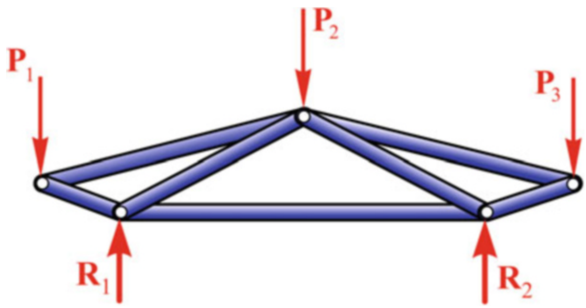


Fig. 8.2 Model of the car as a truss

Fig. 8.3 Truss (physical model of the car)



Next, let us apply all external loads and reactions. We assumed that the external loads are applied to the pins (joints) only; it means that we neglect the weight of the truss members and, therefore, each member is subjected to axial load only (Fig. 8.3).

The truss representing our car is called a *rigid* truss, since removing any one member will destroy its rigidity, and therefore it will become a *mechanism*. Adding extra members will result in an *over-rigid*[1] (over-constrained) truss.

For a three-dimensional *rigid* truss, we can develop a simple relationship between the number of members and joints on the basis of equilibrium conditions. We can consider a truss as a three-dimensional assembly of m members connected by p pins. If the structure is in the static equilibrium, then each pin should be in equilibrium as well. For each pin, we can write three equilibrium equations, thus the total number of independent equations is equal to $3p$. Each truss element has one unknown internal force, i.e., if the truss consists of m elements, there are m unknown internal forces. To keep a three-dimensional structure in a static equilibrium, there will be additional r unknown reaction components. Value of r is dependent on the type and the number of supports. It is equal to 3 for the two-dimensional structures and to 6 for three-dimensional structures. Thus, to

[1] *Over-rigid* trusses will be studied in the strength of material courses since deformations must be considered to get the forces in members and this is beyond this course.

ensure the equilibrium of a three-dimensional truss, the following equation should be satisfied:

$$3p = m + r \tag{8.1}$$

For a two-dimensional case, (8.1) will take the form

$$2p = m + r \tag{8.2}$$

To ensure the equilibrium of a 3D truss, the following equation should be satisfied:

$$3p = m + r$$

For a two-dimensional case, the equation will take the form

$$2p = m + r$$

where m—number of elements, p—number of pins, $r = 3$ in a two-dimensional structure, and $r = 6$ for a three-dimensional structure.

If the left hand side of (8.1) or (8.2) is larger than the right hand side, it means that the truss is a *mechanism*, while in the opposite case the truss is *over-rigid*.

Above equations cannot be used to verify that the truss under consideration is a rigid one, since (8.1) and (8.2) are only the necessary conditions for a solvable system of equilibrium equations, but not sufficient. The example (Fig. 8.4) shows two trusses satisfying the above conditions; however, truss (a) is obviously not rigid, while truss (b) is a *rigid* one.

Since any structure composed from three members forming a triangle is inherently a rigid one, it is often used to construct a truss. One can add two new members to the existing joints to form a new triangle. Thus by adding new members, a truss of any desired shape may be constructed. The truss that has been built using only triangular additions is called a *simple* truss.

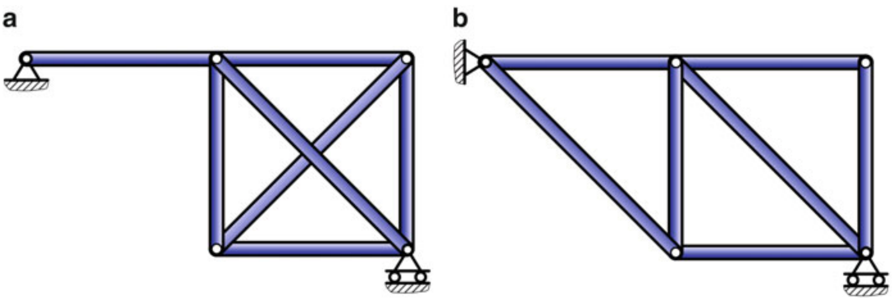


Fig. 8.4 Nonrigid (**a**) and rigid (**b**) trusses

Truss constructed from triangles is called *simple* truss.

It is usually assumed that a truss member does not have weight; however, if this is not a valid assumption, the total weight of the member may be divided by two and applied at its ends. Thus, each member is loaded only at the ends and may be considered as a two-force member. As we already know, the two-force member will be in equilibrium only when those two forces are acting in opposite directions, are equal in magnitude, and are collinear. When a force tends to extend a member, it will be called a *tensile* force; when it tends to shorten a member, the force will be called a *compressive* force. To distinguish between these two types of forces, we assign to the *tensile* force a positive sign and to the *compressive* a negative sign.

Tensile forces are marked as *positive* ones, while compressive as *negative* ones.

The internal forces in truss members can be determined by using method of joints or sections. Both methods are based on the principle of equilibrium, i.e., that the sum of all forces and sum of all moments acting on the truss has to be equal to zero.

$$\sum \mathbf{F} = 0$$

$$\sum \mathbf{M} = 0$$

Since the whole structure is in equilibrium, each part of it also has to be in equilibrium. This suggests a simple way to solve for unknown internal forces in truss members. We can "disassemble" a truss into smaller parts, more convenient for the solution, and by using equilibrium equations to solve for the unknown forces. This procedure will be discussed in detail in the following sections.

Trusses shown in Figs. 8.3 and 8.4b represent *simple* trusses. They are usually constructed by adding two new members to the existing truss to create a new joint and a new triangle. We can build a truss of any size and shape using this procedure.

We will start with two-dimensional trusses and later, in Sect. 8.1.4, will discuss space structures.

8.1 Method of Joints

This method is based on extracting (cutting) joints out of the truss one after another and considering their equilibrium. For each two-dimensional joint, we can write two equations of equilibrium. Even though each joint is in the state of equilibrium, not each joint we can solve directly since there may be more unknown forces acting on the joint than there are equations of equilibrium. Thus, we should use our judgment on where to cut because it should be not more than two unknown forces acting on the joint of interest! Proper selection of the cutting order will allow solving each joint separately.

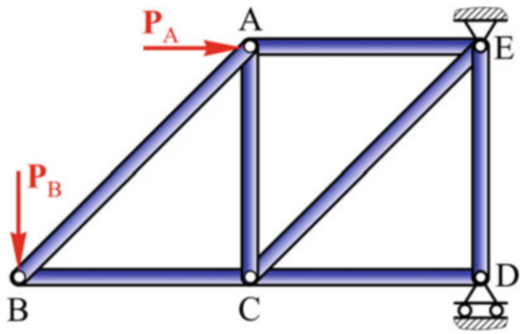


Fig. 8.5 Physical model of a truss

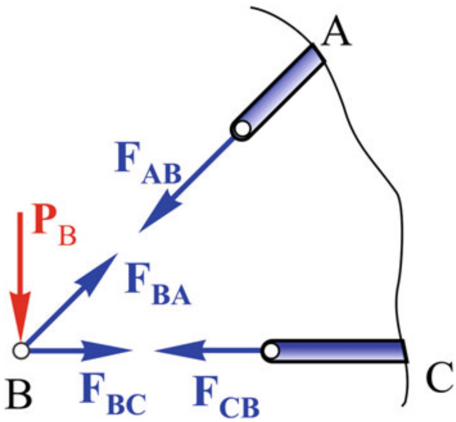


Fig. 8.6 Forces acting on joint B

Let us consider a two-dimensional truss (Fig. 8.5) loaded by external forces $\mathbf{P}_A$ and $\mathbf{P}_B$. We want to find the forces acting on the joint B, so let's cut off joint B and consider its equilibrium.

Figure 8.6 shows forces acting on joint B. They comprise the external force $\mathbf{P}_B$ and two unknown forces $\mathbf{F}_{BA}$ and $\mathbf{F}_{BC}$ imposed by two truss members acting upon the joint B. These forces, $\mathbf{F}_{BA}$ and $\mathbf{F}_{BC}$, are due to the action of the truss members on the joint. Directions of these forces are chosen so that the truss members are assumed to be under tensile loading. The lines of action of the internal forces are along the corresponding truss members AB and BC; thus, there are only two unknowns (magnitudes of the internal forces). Since we are dealing with the concurrent system of forces, there are only two equations of equilibrium.

When a force is directed away from the joint (pin) the corresponding member is under tensile load; when the force is directed toward the joint (pin) the truss member is under compression.

$$\sum F_x = 0 \tag{8.3}$$

$$\sum F_y = 0 \tag{8.4}$$

The solution of a system of two equations with two unknowns is a straightforward procedure. The same procedure should be applied to all joints until we determine all unknown internal forces acting in all truss members.

Let us again consider a two-dimensional truss (Fig. 8.5), but this time it is loaded only by force $\mathbf{P}_A$ and force $\mathbf{P}_B = 0$. Here, we have a situation where pin B cannot be in equilibrium unless the forces in each member are zero. This is obvious if you write two equation of equilibrium ((8.3) and (8.4)) for the joint B. Thus, the truss joint with two noncollinear members and no external load *always* has zero force in each member, and these members are called zero-force members.

Guidelines and Recipes for Finding the Internal Forces in Truss Members using the Method of Joints

- Create a physical model of the truss.
- Draw a free body diagram of the truss.
- Solve for the unknown reactions.
- Choose a joint with no more than two unknown internal forces and disassemble it.
- Draw a free body diagram of this joint and solve for the unknown internal forces.
- Repeat for each joint.

Example 8.1 A truck is traveling over a bridge (Fig. 8.7). Determine the internal forces in each member of the bridge when the truck is passing the middle of the bridge.

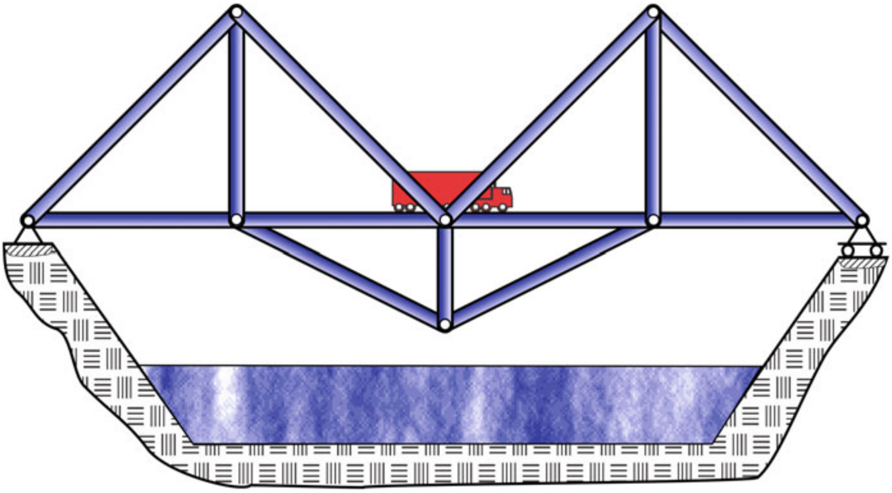


Fig. 8.7 Bridge and truck

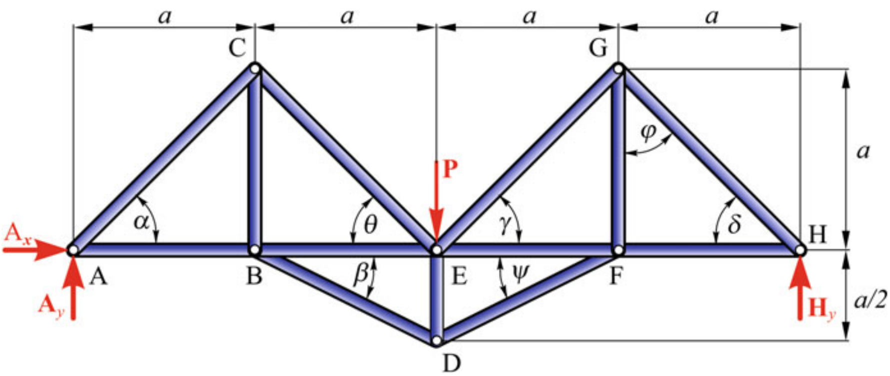


Fig. 8.8 FBD of the bridge

Solution First, we have to create a physical model of the structure. Let us assume that the left side of the bridge is supported by frictionless pin, while rollers support the right end. Since the task is to find internal forces in the members comprising the bridge, we may assume that the truck is much smaller than the bridge members, thus its effect may be represented as a concentrated load. Therefore, the bridge may be modeled as a truss.

Now, we can draw a free body diagram of the bridge (Fig. 8.8) and determine the external reactions at joints A and H. Do we have to do this? Yes, since each joint is subjected to more than *two* unknown forces, we need to know external reactions before we will cut out the joints for analysis. The free body diagram is shown in Fig. 8.8. The equations of equilibrium are

$$\sum \mathbf{F} = 0 = \mathbf{F}_{\mathrm{A}} + \mathbf{F}_{\mathrm{H}} + \mathbf{P}$$

$$\sum \mathbf{M} = 0 = \mathbf{M}(\mathbf{F}_{\mathrm{A}}) + \mathbf{M}(\mathbf{F}_{\mathrm{H}}) + \mathbf{M}(\mathbf{P})$$

These equations of equilibrium may be expressed in a scalar form and solved as shown below.

$$\sum F_x = A_x = 0$$
$$\sum M_{\mathrm{A}} = H_y \cdot 4a - P \cdot 2a = 0 \Rightarrow H_y = P/2$$
$$\sum F_y = A_y - P + H_y = 0 \Rightarrow A_y = P/2$$

Assuming the total weight of the car is 16 kN, we get

$$A_y = H_y = 8 \text{ kN}$$

Now, we have to select a joint that has only two unknown forces, say the joint A. Free body diagram of the joint is shown in Fig. 8.9. It is a common practice to assume that the truss members are under the tension while solving equilibrium equation for a selected joint. Thus, if force is directed away from the pin, it means tension in the corresponding member; when the force is directed toward the pin, it means that the truss member is under compression. However, we do not know in advance the magnitude and the direction of the forces acting upon the joint. It should be noted that the order of the subscripts used for the member forces is of no significance, thus force $\mathbf{F}_{\mathrm{AB}} = \mathbf{F}_{\mathrm{BA}}$.

It should he noted that getting a negative value from solution of the equilibrium equations means that we have chosen the wrong direction of the unknown force. Therefore, setting the unknown internal forces as tensile (positive), as shown in Fig. 8.9, will yield proper signs of the unknown forces. It means that the positive result will indicate tension in the truss member, while the negative result—compression.

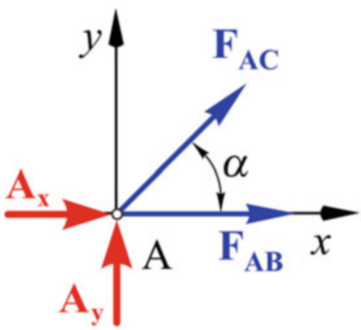


Fig. 8.9 FBD of joint A

Select the x–y axes as shown in Fig. 8.9 and write the equation of equilibrium.

$$\sum \mathbf{F} = 0 = \mathbf{F}_{AC} + \mathbf{F}_{AB} + \mathbf{A}$$

In the above equation, we included all the forces acting on joint A; they are shown as vectors. To solve this equation, we have to separate them into x and y components and find two unknown values—the magnitudes of forces $\mathbf{F}_{AC}$ and $\mathbf{F}_{AB}$. Their directions are known—they coincide with members AC and AB.

$$\sum F_y = A_y + F_{AC} \cdot \sin\alpha = 0$$

It should be noted that the order of subscripts used for the member forces in this and other examples is of no significance, thus force $\mathbf{F}_{AC}$ and $\mathbf{F}_{CA}$ refer to the same force.

Angle α may be calculated from the known geometry of the bridge; it is 45°. Solution of the above equation will give

$$F_{AC} = -11.31\,\text{kN}$$

The negative sign indicates that the member AC is a compression member rather than a tension member as was our initial guess.
Now, we can substitute this value in the following equation:

$$\sum F_x = A_x + F_{AC} \cdot \cos\alpha + F_{AB} = 0$$

and find that $F_{AB} = 8.00\,\text{kN}$. Positive sign indicates that the initial guess was correct and the member AB is in tension.

The next joint, which has only two unknown forces, is joint C. Let us cut it away, draw a free body diagram (Fig. 8.10), and write equations of equilibrium. They will include all three forces acting on the joint: $\mathbf{F}_{CA}$, $\mathbf{F}_{CB}$, and $\mathbf{F}_{CE}$. The directions of all

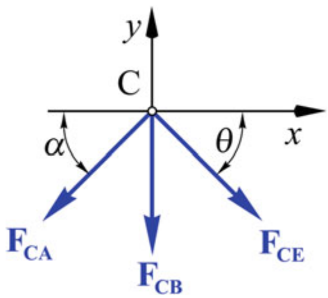


Fig. 8.10 FBD of joint C

forces are known and the magnitude of the force $\mathbf{F}_{AC}$ was found in the previous step. Thus, we have two unknown values–magnitudes of forces $\mathbf{F}_{CB}$ and $\mathbf{F}_{CE}$. As we already know, in a two-dimensional case of concurrent system of forces we can solve for only two unknown values, and this is exactly the case. Equations of the equilibrium are

$$\sum F_x = F_{CE} \cdot \cos\theta - F_{AC} \cdot \cos\alpha = 0, \quad \text{where} \quad \alpha = 45^\circ$$

$$\sum F_y = -F_{AC} \cdot \sin\alpha - F_{CB} - F_{CE} \cdot \sin\theta = 0, \quad \text{where} \quad \theta = 45^\circ$$

The solution of this system of equations is:

$$F_{CE} = -11.31\,\text{kN} \quad \text{and} \quad F_{CB} = 15.99\,\text{kN}$$

The negative sign in the first result indicates that the member CE is a compression rather than a tension member, as was our initial guess. The second result indicates that the initial guess (tension) was correct.

Next, we will solve for forces in members connected at joint B. Four members are connected together here, and the free body diagram is shown in Fig. 8.11. The values of the internal forces are known for two of them (AB and BC). It is left to solve for two unknown values—magnitudes of forces $\mathbf{F}_{BE}$ and $\mathbf{F}_{BD}$. From the bridge geometry, we may calculate angle $\beta = 26.6^\circ$.

The equilibrium equation in y direction is

$$\sum F_y = F_{BC} - F_{BD} \cdot \sin\beta = 0$$

From here

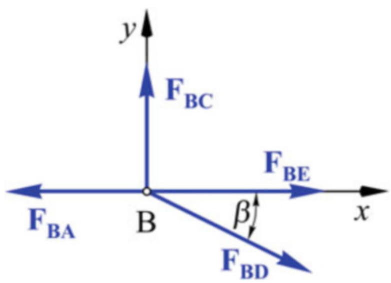
$$F_{BD} = 35.7\ \text{kN}$$

In x direction, we have

$$\sum F_x = F_{BD} \cdot \cos\beta + F_{BE} - F_{AB} = 0$$

From here

Fig. 8.11 FBD of joint B



$$F_{BE} = -23.9 \text{ kN}$$

i.e., it is in compression.

We move now to the joint E; however, its FBD (Fig. 8.12) has three unknown forces $\mathbf{F}_{ED}$, $\mathbf{F}_{EF}$, and $\mathbf{F}_{EG}$. Since this is a concurrent system of forces, one can use only two equations of equilibrium. Thus, we must choose another joint, and come back to this one after the number of unknown will be reduced to two.

Thus, we have to choose another joint from the rest of the joints D, F, G, or H. Only joint H has two unknown force values (Fig. 8.13).

The equations of equilibrium for the joint H are:

$$\sum F_y = F_{HG} \cdot \sin\delta + H_y = 0, \text{ where } \delta = 45^\circ \text{ and}$$

$$\sum F_x = -F_{HF} - F_{HG} \cdot \cos\delta = 0$$

The solution is

$$F_{HG} = -11.31\,\text{kN and } F_{HF} = 8.00\,\text{kN}.$$

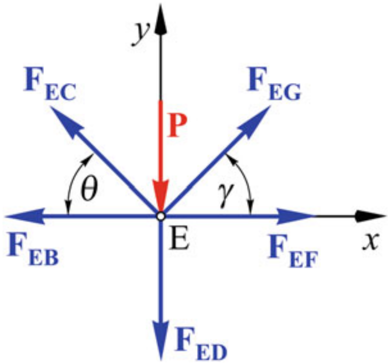


Fig. 8.12 FBD of joint E

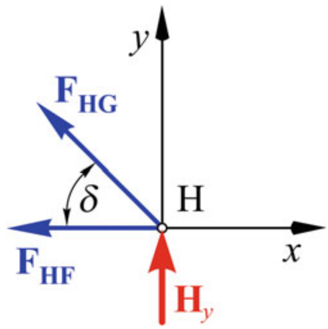


Fig. 8.13 FBD of joint H

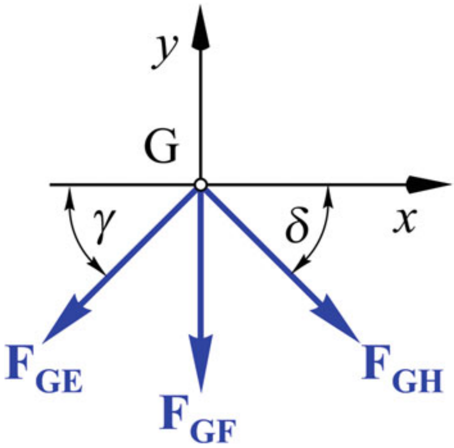


Fig. 8.14 FBD of joint G

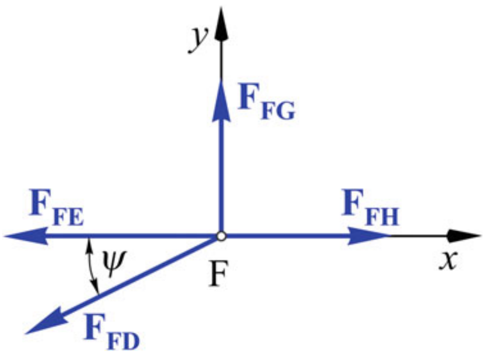


Fig. 8.15 FBD of joint F

Now, we move to the joint G. Its FBD is shown in Fig. 8.14. The equations of equilibrium are:

$$\sum F_y = -F_{\mathrm{GH}} \cdot \sin\delta - F_{\mathrm{GF}} - F_{\mathrm{GE}} \cdot \sin\gamma = 0, \;\; \text{where} \;\; \gamma = \; 45^\circ \;\; \text{and}$$

$$\sum F_x = F_{\mathrm{GH}} \cdot \cos\delta - F_{\mathrm{GE}} \cdot \cos\gamma = 0$$

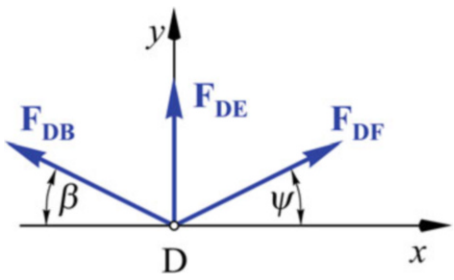
The result is

$$F_{\mathrm{GE}} = -11.31 \;\; \mathrm{kN}, \text{and } F_{\mathrm{GF}} = 16.00 \;\; \mathrm{kN}.$$

We proceed to joint F, which now has only two unknown forces (Fig. 8.15). The equations of equilibrium for the joint F are:

$$\sum F_y = F_{\mathrm{FG}} - F_{\mathrm{FD}} \cdot \sin\psi = 0, \;\; \text{and} \;\; \text{where} \;\; \psi = 26.6^\circ$$

Fig. 8.16 FBD of joint D



$$\sum F_x = F_{FH} - F_{FE} - F_{FD} \cdot \cos\psi = 0$$

The solution is

$$F_{FE} = -23.9\,\text{kN and } F_{FD} = 35.7\,\text{kN}.$$

Finally, we proceed to joint D, which has only one unknown force, $\mathbf{F}_{DE}$ (Fig. 8.16). The sum of forces in y direction is:

$$\sum F_y = F_{DE} + F_{DF} \cdot \sin\psi + F_{DB} \cdot \sin\beta = 0$$

From here we obtain

$$F_{DE} = -32.00\,\text{kN}$$

The remaining joint is joint E, which we could not solve earlier since it had too many unknown forces. However, since we have solved all other joints there are no unknown forces left. Thus, the equilibrium in the joint E should be fulfilled. This joint can be used to check the correctness of the results. Let us write the equations of equilibrium, and substitute the values for all forces acting at the joint E (Fig. 8.12). Both equations must be satisfied.

$$\sum F_y = F_{EG} \cdot \sin\gamma - P + F_{EC} \sin\theta - F_{ED}$$
$$- 11.31 \cdot \sin 45^\circ - 16.00 - 11.31 \cdot \sin 45^\circ + 32.00 = 0$$

and

$$\sum F_x = F_{EF} + F_{EG} \cdot \cos\gamma - F_{EC} \cdot \cos\theta - F_{EB}$$
$$- 23.9 - 11.31 \cdot \cos 45^\circ + 11.31 \cdot \cos 45^\circ + 23.9 = 0$$

Both equations are satisfied, thus we solved the previous joints correctly. Since the suggestion to show all unknown forces as positive (tensile) is not a standardized rule, it is practical to indicate the mode of loading (tensile or compressive) by a letter T for tension and C for compression, and leaving out the sign.

Thus, the results are:

$$F_{AB} = 8.00 \text{ kN (T)}, \quad F_{AC} = 11.31 \text{kN (C)}, \quad F_{BC} = 15.99 \text{kN (T)}$$

$$F_{CE} = 11.31 \text{ kN (C)}, \quad F_{BD} = 35.7 \text{kN (T)}, \quad F_{BE} = 23.9 \text{kN (C)}$$

$$F_{GH} = 11.31 \text{ kN (C)}, \quad F_{FH} = 8.00 \text{kN (T)}, \quad F_{GE} = 11.31 \text{kN (C)}$$

$$F_{GF} = 15.99 \text{ kN (T)}, \quad F_{FE} = 23.9 \text{kN (C)}, \quad F_{FD} = 35.7 \text{kN (T)} \text{ and}$$

$$F_{DE} = 32.00 \text{kN (C)}$$

Each of the above equations may be solved by using the MATLAB routine *equilbriumPoint2D.m* provided at the site extras.springer.com. Let us use this routine to solve, for example, the forces acting at the joint A. Start MATLAB and run *equilbriumPoint2D*. The first box will explain how to use the routine. Read it and press OK. The next box will ask to input the total number of forces acting on a given point. You have to enter 3, since the force Ax is already known to be zero. Let us assign an ID number to each force as shown in Table 8.1. In the following box, you will have to input data for each force.

Table 8.1 Data for MATLAB solution

Force number (name)	1 ($\mathbf{A}_y$)	2 (F_{AB})	3 (F_{AC})
Magnitude (force)	8	x	x
X component (length)	0	1	1
Y component (length)	1	0	1

The solution for the problem will be shown as:

The value of the first unknown is: 8.0000.
The value of the second unknown is: −11.3137.

It means that the first unknown, which we set to be magnitude of the force $F_{AB} = 8.0$ kN and the second unknown, which we set to be force F_{AC} is equal to −11.32 kN. The negative sign indicates that the member AC is under the compressive force.

Example 8.2 Houses in old Amsterdam have very narrow staircases, thus people used an outside lifting system (Fig. 8.17a) that is schematically shown in Fig. 8.17b. We have to find forces in each member of the system when a man is lifting a box weighting 400 N. All dimensions are in meters. The lifting system is

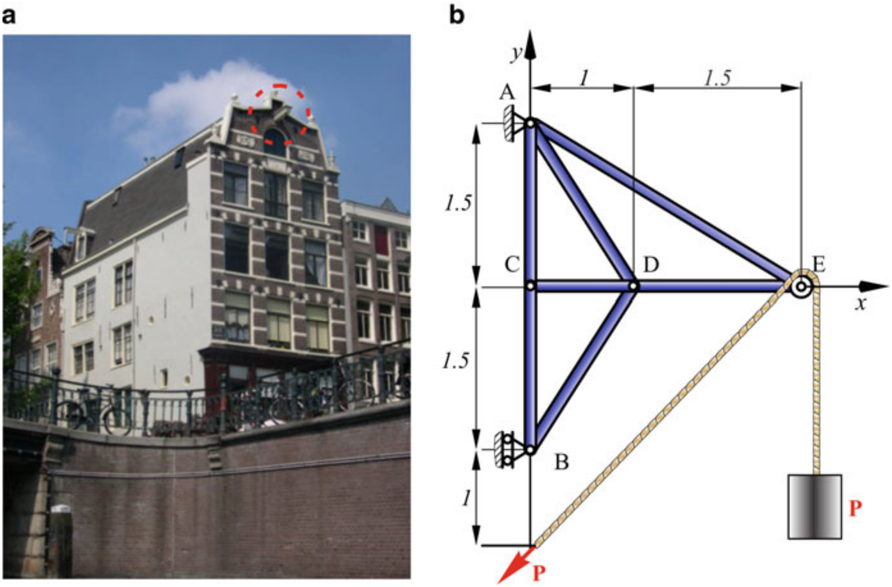


Fig. 8.17 (**a**) House with the outside lifting device. Lifting system on the top floor. (**b**) Schematic of the lifting system

connected to the wall by a pin at the point A and is supported by a frictionless roller at the point B.

Solution To solve this problem, we have to create a physical model of the shown structure. Since there is no friction at point B, we can place a roller at this point. The diameter of pulley E is small relative to the whole structure, thus we will assume its diameter equal to zero. Utilizing these assumptions, we can draw a free body diagram of the structure (Fig. 8.18). Each joint in this structure has more than two members with unknown forces, thus we have to solve for the reactions first, and use them for the solution of the problem. Let's write three equations of equilibrium and solve for unknown values of A_x, A_y, and B_x. The angle θ between the rope and the bar DE (Fig. 8.18) is defined as:

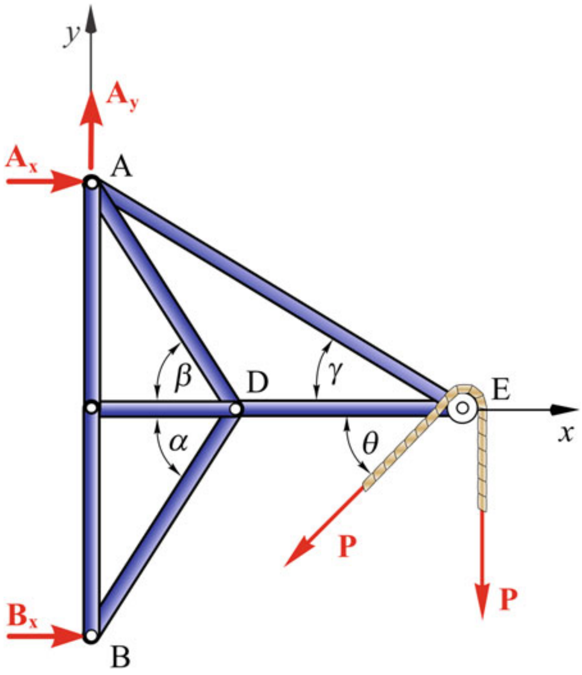
$$\tan\theta = \frac{2.5}{2.5} = 1; \quad \Rightarrow \theta = 45^\circ = \frac{\pi}{4}$$

The three equations of equilibrium are:

$$\sum F_x = B_x + A_x - P \cdot \cos\theta = 0$$

$$\sum F_y = A_y - P \cdot \sin\theta - P = 0, \text{ and}$$

Fig. 8.18 FBD of the lifting system



$$\sum M_A = B_x \cdot AB - P \cdot CE - P \cdot \sin\theta \cdot CE - P \cdot \cos\theta \cdot AC = 0$$

We selected to write the sum of moments about the point A to exclude the unknown forces acting at this point.

The solution is: $A_x = -428\,\text{N}$, $A_y = 683\,\text{N}$, and $B_x = 711\,\text{N}$. The positive values obtained for A_y and B_x confirm the original assumption concerning direction of those forces. Negative sign for A_x indicates that the reaction A_x acts in the opposite direction. We will continue to use this reaction as indicated on the free body diagram and use its values as a negative number. You may get this solution either by solving the above system of three equations manually or by using the provided MATLAB routine *equilbriumPoint2D.m* (extras.springer.com).

Now, we can use the method of joints. Let us start with the joint B (Fig. 8.19). Three forces act on this joint, magnitudes of two of them are unknown, F_{BC} and F_{BD}. We can write two equations of equilibrium:

$$\sum F_x = B_x + F_{BD} \cdot \cos\alpha = 0$$

$$\sum F_y = F_{BC} + F_{BD} \cdot \sin\alpha = 0$$

where α is the angle between members BD and CD. From the geometry of structure, we find $\tan\alpha = 1.5/1$ and $\alpha = 56.3°$.

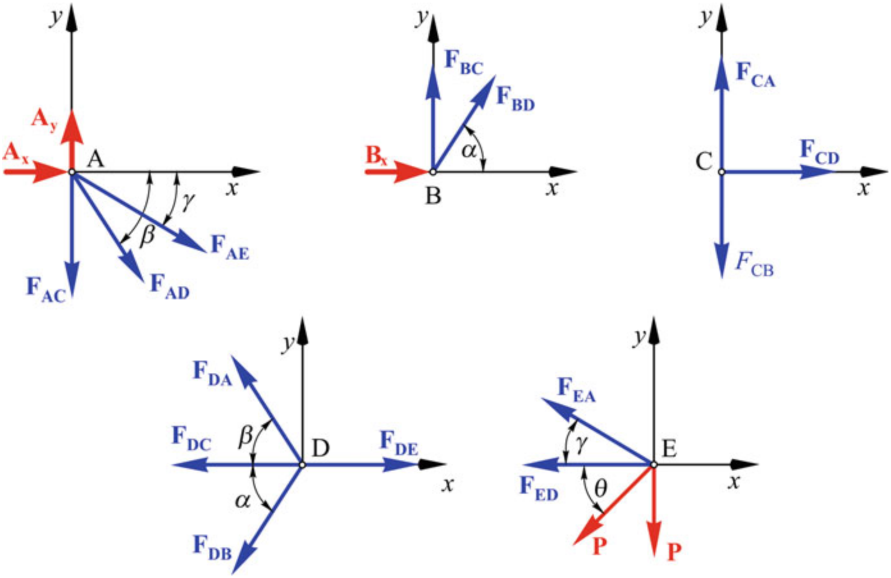


Fig. 8.19 FBD of joints A, B, C, D, and E

The solution results in $F_{\text{BD}} = -1281\,\text{N}$ and $F_{\text{BC}} = 1066\,\text{N}$, negative sign means that the force acting in the member BC is a compressive force.

The next joint where there are no more than two members with unknown forces appears to be the joint C shown in Fig. 8.19. Using the same procedure as for the joint B, we can write two equilibrium equations:

$$\sum F_x = F_{\text{CD}} = 0$$
$$\sum F_y = F_{\text{CA}} - F_{\text{CB}} = 0$$

and solve them for $F_{\text{CA}} = 1066$ N and $F_{\text{CD}} = 0\,\text{N}$. Next, we can proceed to joint D and solve the following two equations of equilibrium.

$$\sum F_x = F_{\text{DE}} - F_{\text{DC}} - F_{\text{DA}} \cdot \cos\beta - F_{\text{DB}} \cdot \cos\alpha = 0$$

$$\sum F_y = F_{\text{DA}} \cdot \sin\beta - F_{\text{DB}} \cdot \sin\alpha = 0$$

where α is the angle between the truss members DB and DC, and $\beta = \alpha$ is the angle between the truss members AD and CD.

The resulting forces are: $F_{\text{DE}} = -1422\,\text{N}$ and $F_{\text{DA}} = -1281\,\text{N}$, negative value indicates that forces in the members DA and DE are compressive forces. The last unknown force in member AE can be calculated from the equations of equilibrium for the joint E:

$$\sum F_x = -F_{\mathrm{ED}} - F_{\mathrm{EA}} \cdot \cos\gamma - P \cdot \cos\theta = 0$$
$$\sum F_y = F_{\mathrm{EA}} \cdot \sin\gamma - P - P \cdot \sin\theta = 0$$

where $\tan\gamma = 1.5/2.5$, thus $\gamma = 31°$ and $\theta = 45°$.

The unknown force in the member AE may be calculated from any of the above two equations, since all other forces are known. Solving the first equation yields force value of 1329 N, while from the second equation we will get value of 1326 N. The difference is due to the rounding errors.

We did not utilize joint A for solution, thus it may be used to verify our calculations. The equilibrium equations for this joint are:

$$\sum F_x = F_{\mathrm{AE}} \cdot \cos\gamma + F_{\mathrm{AD}} \cdot \cos\beta + A_x = 0$$
$$\sum F_y = A_y - F_{\mathrm{AE}} \cdot \sin\gamma - F_{\mathrm{AD}} \cdot \sin\beta - F_{\mathrm{AC}} = 0$$

The left hand side of these equations should be equal to zero if we will substitute appropriate values for all forces. The substitution yields for the first equation the value of 0.4196 and for the second −1.752. These numbers are very small when compared to the forces acting in the joints, thus our solution is correct.

> If the truss has a joint where only two truss members are connected (two unknown forces) we may start the determination of internal forces without solving the equilibrium equations for reactions.

All of the above equations may be solved by the MATLAB routines (extras.springer.com). This will save considerable amount of time and possible calculation errors. You nevertheless have to draw a *correct* free body diagram for each joint of interest. Let us consider some special cases when the joint is free from the external load.

(a) Truss joint with two collinear members. Sum of forces in the direction of members is equal to zero, thus the axial forces are equal, $F_{\mathrm{AC}} = F_{\mathrm{AB}}$ (Fig. 8.20a).
(b) Truss joint with two noncollinear members. Sum of forces in the direction y is equal to zero, thus $F_{\mathrm{BD}} = 0$. Therefore, F_{BE} must be equal to zero too. Thus, both axial forces in this case are equal to zero (Fig. 8.20b).
(c) Truss joint with two collinear members and one noncollinear member. Sum of forces in the direction y is equal to zero, thus $F_{\mathrm{CM}} = 0$. Sum of forces in the direction x is equal to zero, thus $F_{\mathrm{CN}} = F_{\mathrm{CK}}$ (Fig. 8.20c).
(d) Trust joint with four members lying in two intersecting straight lines. Summation of forces in y direction results in $F_{\mathrm{DT}} = F_{\mathrm{DS}}$, thus the magnitudes of $\mathbf{F}_{\mathrm{DP}}$ and $\mathbf{F}_{\mathrm{DQ}}$ should be equal as well.

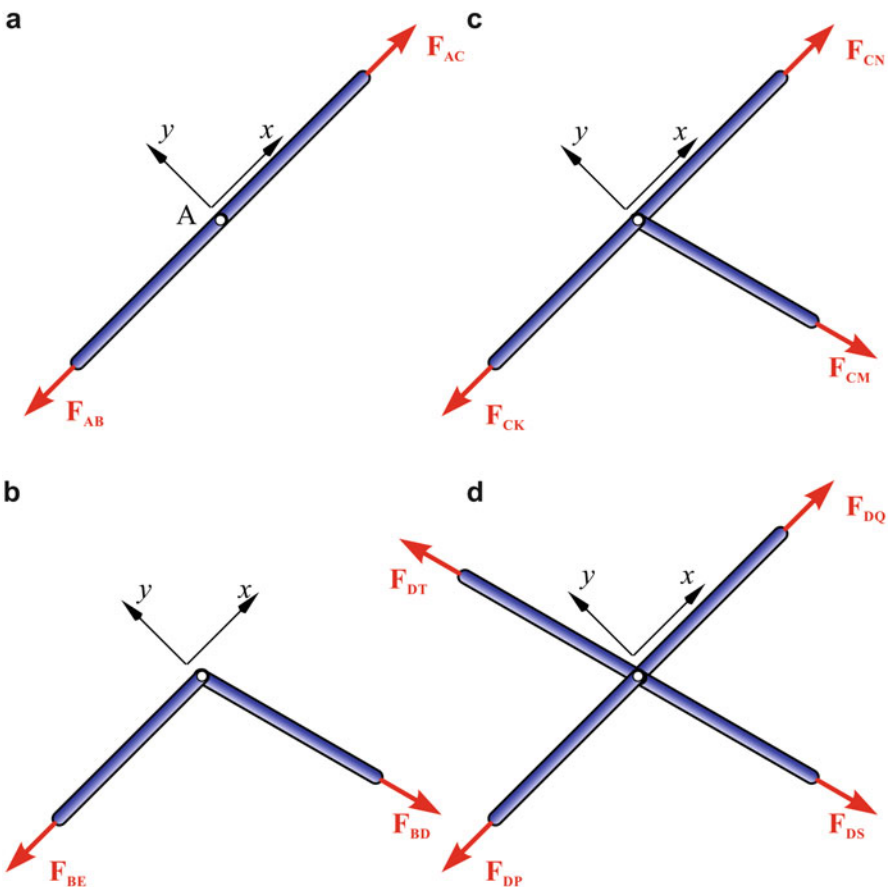


Fig. 8.20 Special cases of joints physical models: (**a**) Two collinear members, (**b**) Two noncollinear members, (**c**) Two collinear members and one noncollinear member, and (**d**) Joint with four members

8.1.1 Problems

8.1–8.3 Use method of joints to calculate the force in each member of the truss shown. Use $F = 80$ N and $a = 20$ cm.

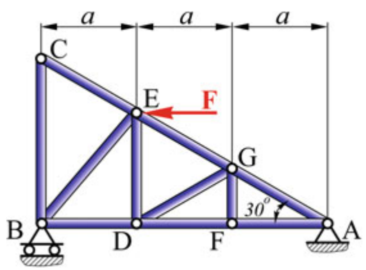


Fig. P8.1

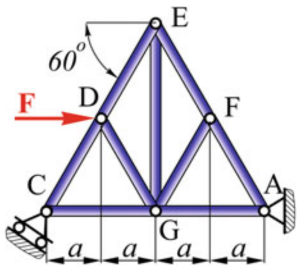


Fig. P8.2

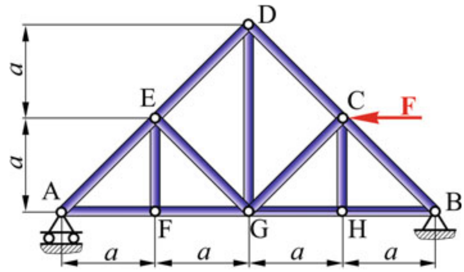


Fig. P8.3

8.4 Determine the force in truss members 1–6. Use $P = 60$ N. Each trust member is 80 cm long.

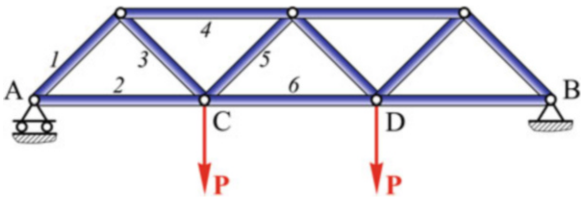


Fig. P8.4

8.5 Determine the force in each member of the truss. Use $F = 20$ N and $a = 20$ cm.

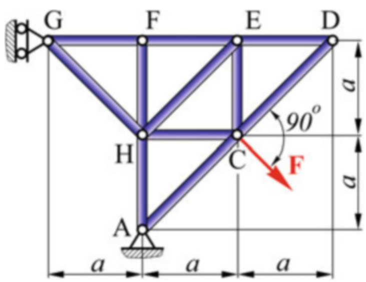


Fig. P8.5

8.6 Determine the force in each member of the truss. Use $F = 10$ N and $a = 20$ cm.

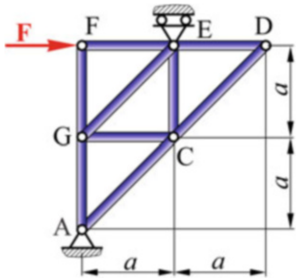


Fig. P8.6

8.7 Determine the force in each member of the truss. Use $F = 40$ N and $a = 20$ cm.

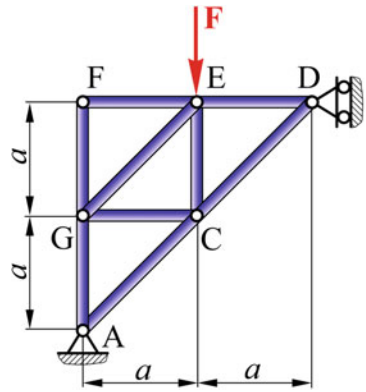


Fig. P8.7

8.8 Determine the force in each member of the truss. Use $F = 10$ N and $a = 20$ cm.

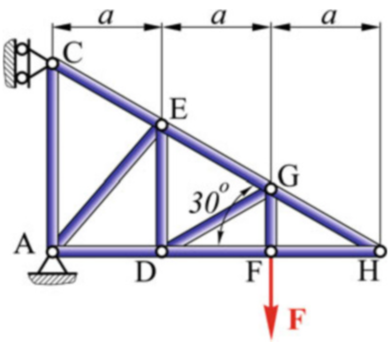


Fig. P8.8

8.9 Determine the force in each member of the truss. Use $F = 40$ N and $a = 20$ cm.

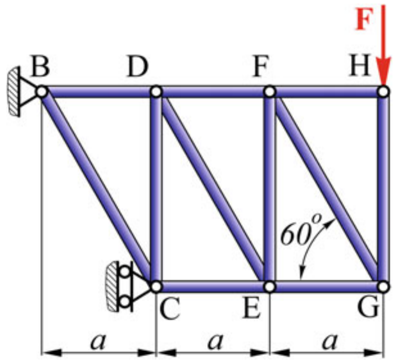


Fig. P8.9

8.10 Determine the force in each member of the truss. Use $F = 20$ lb and $a = 10$ in.

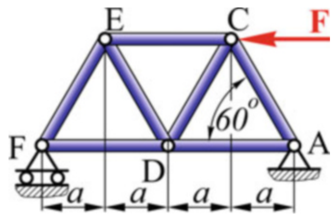


Fig. P8.10

8.11 Determine the force in each member of the truss. Use $F = 30$ N and $a = 10$ cm.

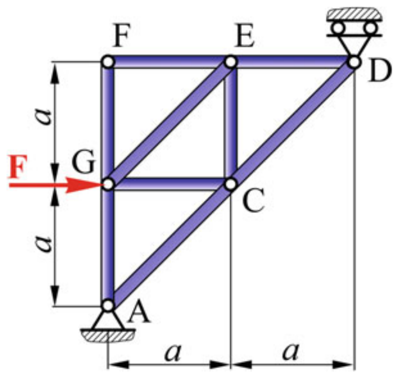


Fig. P8.11

8.12 Determine the force in each member of the truss. Use $F = 10$ N and $a = 20$ cm.

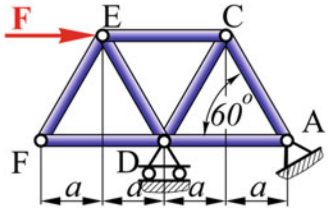


Fig. P8.12

8.13 Determine the force in each member of the truss. Use $F = 60$ lb and $a = 10$ ft.

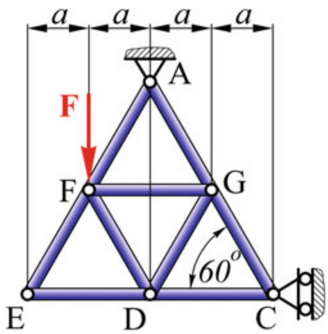


Fig. P8.13

8.14 Determine the force in each member of the truss. Use $F = 40$ N and $a = 20$ cm.

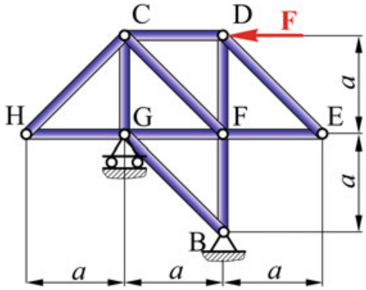


Fig. P8.14

8.15 Determine the force in each member of the truss. Use $F = 20$ N and $a = 30$ cm.

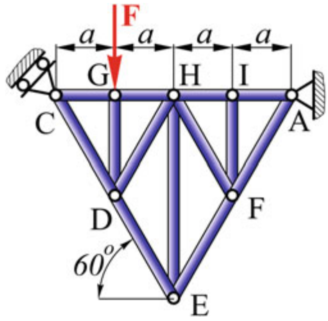


Fig. P8.15

8.16 Determine the force in each member of the truss. Use $F = 10$ N and $a = 20$ cm.

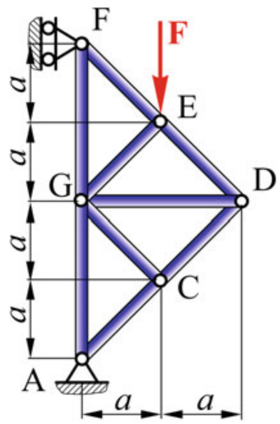


Fig. P8.16

8.17 Determine the force in each member of the truss. Use $F = 60$ N and $a = 10$ cm.

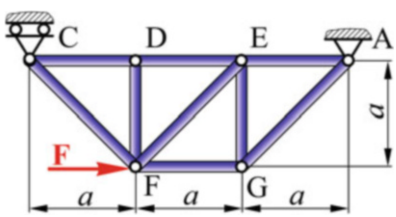


Fig. P8.17

8.18 Determine the force in each member of the truss. Use $F = 40$ N and $a = 10$ cm. *Hint: reaction at B is along the member BD. Why?*

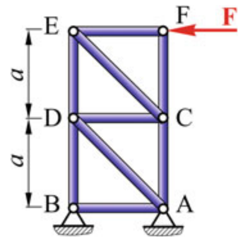


Fig. P8.18

8.19 Determine the force in each member of the truss. Use $F = 30$ N and $a = 10$ cm.

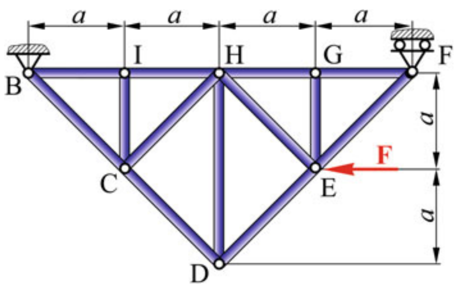


Fig. P8.19

8.20 Determine the force in each member of the truss.

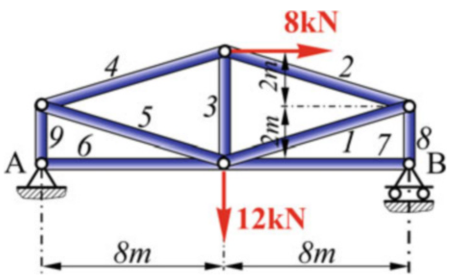


Fig. P8.20

8.21 Determine the force in each member of the truss.

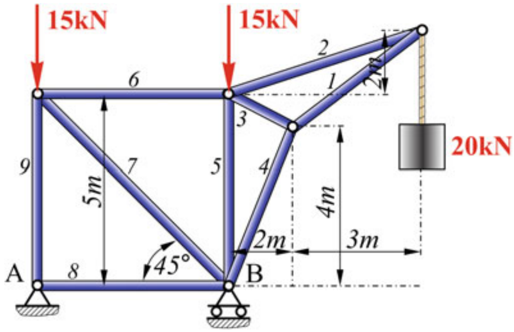


Fig. P8.21

8.22 Determine the forces in each member of the truss. Use $P_1 = 10$ N, $P_2 = P_3 =$ 30 N, and $a = 30$ cm.

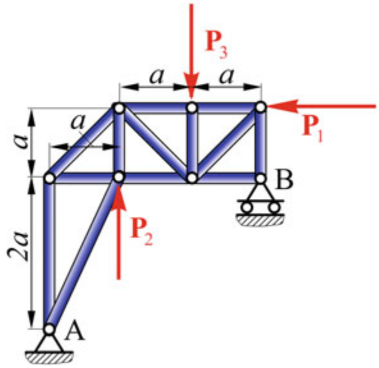


Fig. P8.22

8.23 Determine the forces in each member of the truss when $P_1 = 10$ N, $P_2 = 40$ N, $P_3 = 30$ N, and $a = 30$ cm

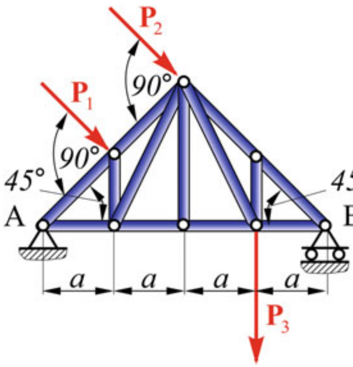


Fig. P8.23

8.24 Determine forces in each member of the truss.

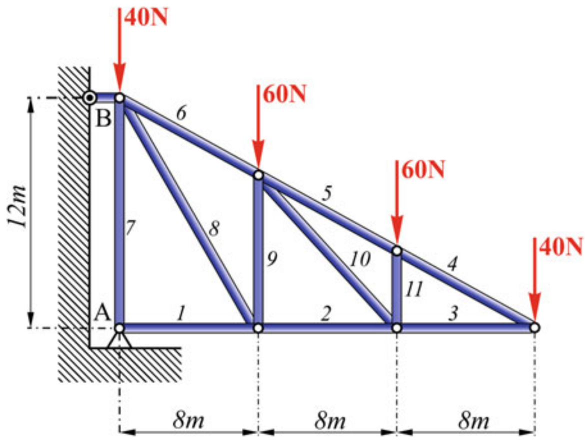


Fig. P8.24

8.25 Determine the forces in each member of the truss.

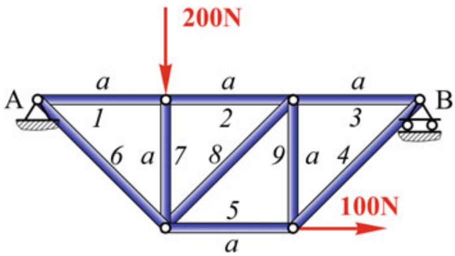


Fig. P8.25

8.26 Determine the forces in each member of the truss.

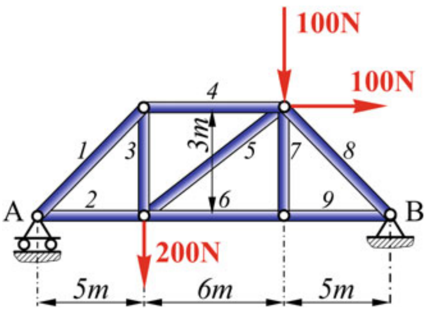


Fig. P8.26

8.27 Determine the forces in each member of the truss.

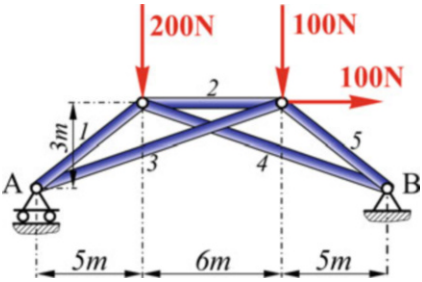


Fig. P8.27

8.2 Method of Sections

The method of joints allows finding the internal forces acting in each member of the truss. However, in some applications we need forces in a limited number of members only. To solve for the internal force in a selected member, we may have to proceed through a large number of joints since the method of joints requires a *step-by-step* procedure. Therefore, we will introduce an approach that will allow the direct calculation of the needed forces.

If an entire truss is in equilibrium, then any part of the truss should be in equilibrium as well. Thus, if we are looking for a force acting in a particular member, we can divide the structure into two parts by an imaginary cut and consider equilibrium of each part separately.

For example, let us consider the truss shown in Fig. 8.21a. Assume that only the forces in members EC, FC, and FD are of interest. By using the method of joints, we would have to solve each of the joints I, G, H, F, and E until we would get to the members of interest. Alternatively, we could start on the left hand side, but we would have to solve for the reactions first, and then to solve the joints A, B, C, and D.

Using the method of sections, we may divide the truss into two parts by an imaginary cut through members EC, FC, and FD, as shown in Fig. 8.21a by the red dashed line "c-c." In order to keep each part in equilibrium, we must add the internal forces acting in the cut members. Now, we can draw a free body diagram of each part separately (Fig. 8.21b). Since each member is a straight two-force body, the internal forces are acting along the members (Fig. 8.21b).

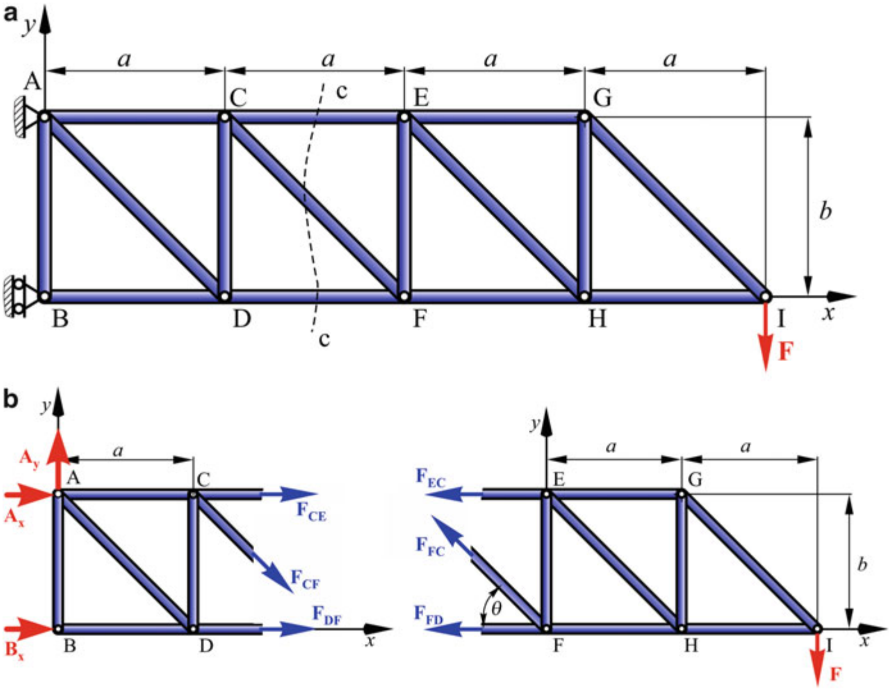


Fig. 8.21 (**a**) Physical model of the structure with an imaginary cut along c-c. (**b**) Free body diagrams for each part of the truss

The force acting at the left side of a removed member is equal in magnitude and opposite in direction to its counterpart acting at the right side of the member, i.e., $\mathbf{F}_{EC} = \mathbf{F}_{CE}$ and $\mathbf{F}_{FC} = \mathbf{F}_{CF}$.

Now, we can consider equilibrium, e.g., of the right side of the structure and write the appropriate equilibrium equations. If, e.g., only $\mathbf{F}_{FC}$ is required, we can use the equilibrium condition that the sum of forces in y direction equal zero and get the desired result by solving one equation only. On the other hand, if only force $\mathbf{F}_{EC}$ is of interest, we may use the sum of moments about the point F. It is important to note that since we chose the right side of the structure, there is no need to solve for reaction forces acting at the points A and B. Thus, this approach of sections can save a lot of valuable time and efforts while looking for a force in a particular member of a truss.

It should be mentioned that the cut should involve not more than three members with unknown internal forces. If one cuts in such a way that there are more than three unknown internal forces, it is not possible to solve the problem since there are only three independent equations of equilibrium in two dimensions.

If it is possible, cut through not more than three members with unknown forces, since we can write only three equations of equilibrium in 2D.

Example 8.3 Calculate the internal forces acting in the truss members CE, CF, and DF using the method of sections (Fig. 8.21). Use $a = 2\,\text{m}$, $b = 1\,\text{m}$, and $F = 200\,\text{N}$.

Solution Cut the truss along the section c-c, as shown in Fig. 8.21a and consider the equilibrium of the right hand side. Let define the unknown internal forces as positive. The free body diagram of the right hand side and the associated coordinate system are shown in Fig. 8.21b. It is a two-dimensional case of a rigid body loaded by several forces. The equations of equilibrium are

When using the method of sections, one should carefully select which equations of equilibrium to use (sum of forces or moments). Proper selection may lead to one unknown force per equation.

$$\sum F_x = -F_{\text{EC}} - F_{\text{FD}} - F_{\text{FC}} \cdot \cos\theta = 0$$
$$\sum F_y = F_{\text{FC}} \cdot \sin\theta - F = 0$$
$$\sum M_F = F_{\text{EC}} \cdot b - F \cdot 2 \cdot a = 0$$

where $\tan\theta = b/a$, thus $\theta = 26.6°$.

Solution of this system yields

$$F_{\text{FC}} = 447\,\text{N}, \; F_{\text{EC}} = 800\,\text{N}, \; \text{and } F_{\text{FD}} = -1200\,\text{N}.$$

The negative value indicates that the truss member FD is under compression, while members FC and EC are under tensile loading.

Guidelines and Recipes for Finding the Internal Forces in Truss Members Using the Method of Sections

- Select a cut through the truss members of interest.
- Draw a free body diagram of one of the two parts.
- Write the equilibrium equations and solve for the unknown forces.

8.2.1 Problems

8.28 Determine the forces in members CD, CF, and GF. Use $F = 40$ N and $a = 20$ cm.

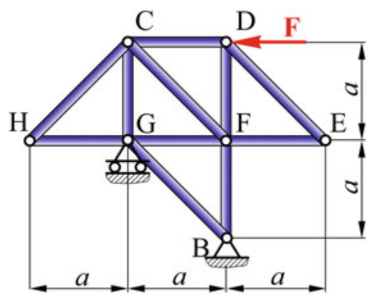


Fig. P8.28

8.29 Determine the forces in members DF, ED, and CE. Use $F = 20$ N and $a = 20$ cm.

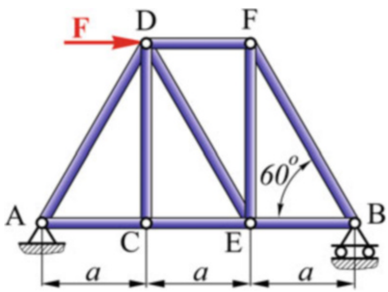


Fig. P8.29

8.30 Determine the forces in members ED, EH, and GH, when $F = 40$ lb and $a = 2$ ft. *Hint: reaction at B is along the member CB. Why?*

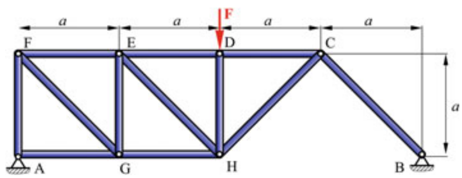


Fig. P8.30

8.31 Determine the forces in members KL, FL, and FG. Use $F_1 = 20$ N, $F_2 = 30$ N, $F_3 = 40$ N, $F_4 = 10$ N, $F_5 = 60$ N, $a = 20$ cm.

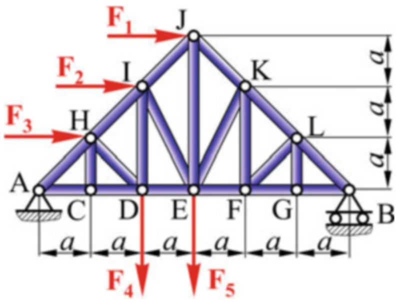


Fig. P8.31

8.32 Determine the forces in members KL, FL, and FG. Use $F_1 = 20$ N, $F_2 = 30$ N, $F_3 = 40$ N, $F_4 = 10$ N, $F_5 = 60$ N, $a = 20$ cm.

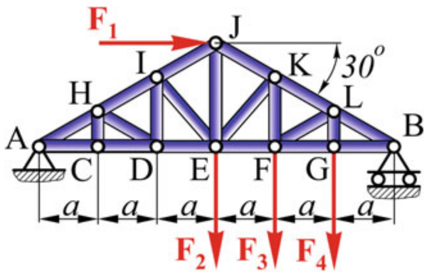


Fig. P8.32

8.33 Determine the forces in members DC, BC, and AB. Length of each member $a = 30$ cm and $P = 20$ kN.

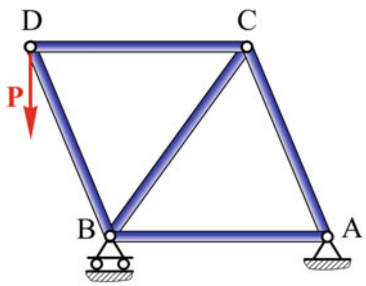


Fig. P8.33

8.34 Determine the forces in members HG, CD, and DH. Use $F_1 = 20$ lb, $F_2 = 30$ lb, $F_3 = 10$ lb.

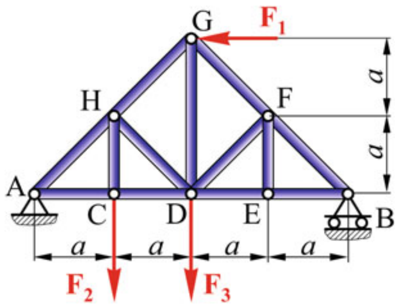


Fig. P8.34

8.35 Determine the forces in members IJ, ID, and CD. Use $F_1 = 20$ lb, $F_2 = 30$ lb, $F_3 = 10$ lb, and $F_4 = 40$ lb.

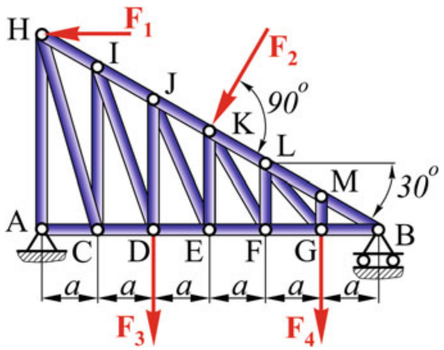


Fig. P8.35

8.36 Determine the forces in members IH, IE, and ED. Use $F_1 = 20$ N, $F_2 = 30$ N, $F_3 = 40$ N, $F_4 = 10$ N, $F_5 = 60$ N, $a = 20$ cm.

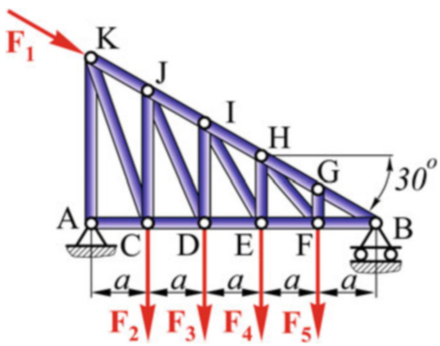


Fig. P8.36

8.37 Determine the forces in members GF, FC, and CD. Use $F_1 = 20$ N, $F_2 = 30$ N. *Hint: reaction at B is along the member HB. Why?*

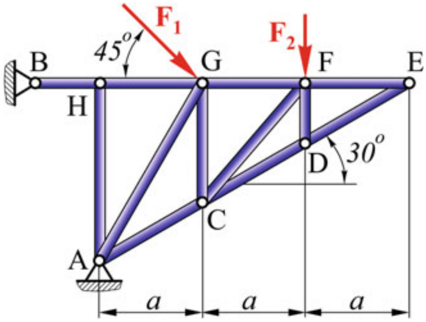


Fig. P8.37

8.38 Determine the forces in members IJ, JB, and EB. Use $F_1 = 10$ N, $F_2 = 20$ N, and $F_3 = 30$ N.

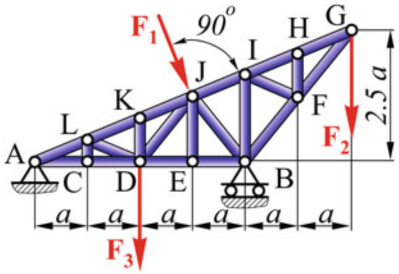


Fig. P8.38

8.39 Determine the forces in members DH, CD, and GH. Use $F_1 = 50$ N, $F_2 = 20$ N, and $F_3 = 10$ N.

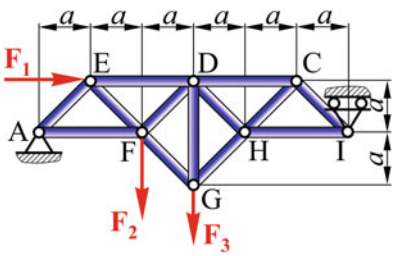


Fig. P8.39

8.40 Determine the forces in members IJ, IE, and EF. Use $F_1 = 10$ N, $F_2 = 20$ N, and $F_3 = 30$ N.

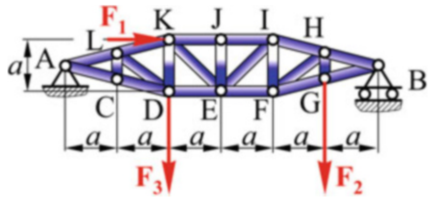


Fig. P8.40

8.41 Determine the forces in members HG, HK, and JK. Use $F_1 = 40$ N, $F_2 = 20$ N, $F_3 = F_4 = 30$ N, $a = 10$ cm, and $h = 12$ cm.

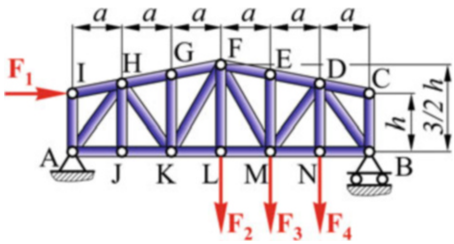


Fig. P8.41

8.42 Determine the forces in members IJ, IE, and EF. Use $F_1 = 40$ N, $F_2 = 10$ N, $F_3 = 30$ N, $h = 20$ cm, $a = 15$ cm.

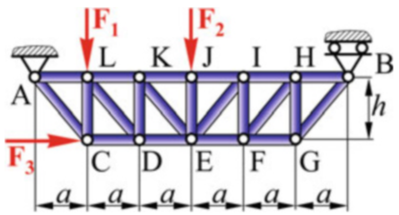


Fig. P8.42

8.43 Determine the forces in members HI, HD, and ED. Use $F_1 = 20$ N, $F_2 = 30$ N, $F_3 = 40$ N, $F_4 = 10$ N, $h = 20$ cm, $a = 15$ cm.

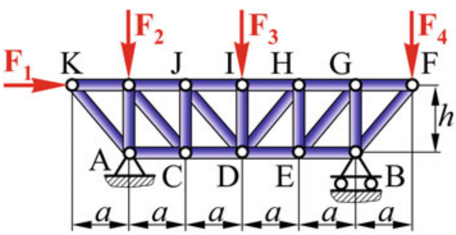


Fig. P8.43

8.44 Determine the forces in members AE, AF, and BF. Use $F_1 = F_2 = F_3 = 20$ N, $F_4 = F_5 = 40$ N, $a = 10$ cm.

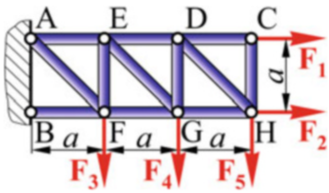


Fig. P8.44

8.45 Determine the forces in members IJ, IB, and BD. Use $F_1 = 50$ N, $F_2 = 30$ N, $F_3 = 20$ N, $a = 10$ cm, $h = 16$ cm.

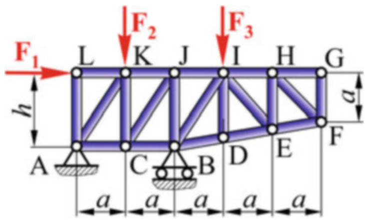


Fig. P8.45

8.46 Determine the forces in members attached to the support A. Use $P_1 = P_2 = 20$ N, $P_3 = 40$ N, $a = 10$ cm.

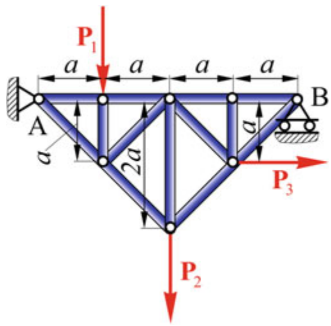


Fig. P8.46

8.47 Determine the forces in members PR and ED. Use $F_1 = F_2 = F_3 = 40$ N, $F_4 = F_5 = 10$ N, $a = 10$ cm.

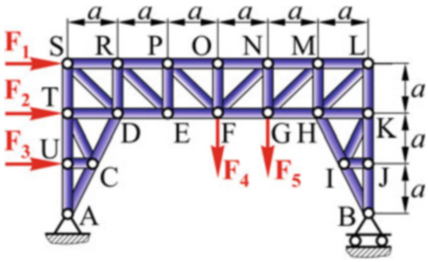


Fig. P8.47

8.48 Determine the forces in members EF and KJ. Use $F_1 = F_2 = F_3 = 20$ N, $F_4 = 40$ N, $a = 10$ cm, $h = 12$ cm.

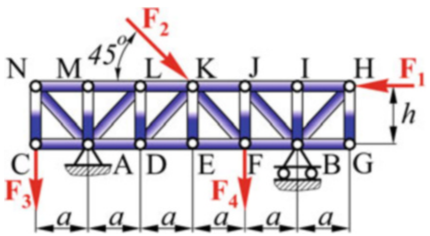


Fig. P8.48

8.49 Determine the forces in members EF and LM (Fig. P8.49). Use $F_1 = F_2 = F_3 = 20$ N, $F_4 = 40$ N, $a = 10$ cm.

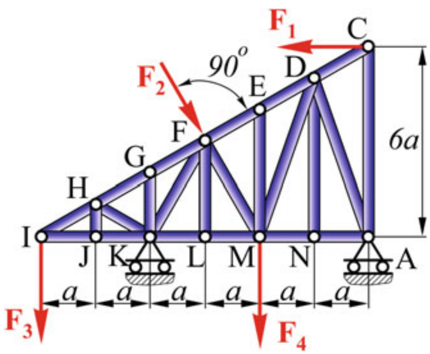


Fig. P8.49

8.50 Determine the forces in members IH and BF (Fig. P8.50). Use $F_1 = 20$ N, $F_2 = 30$ N, $F_3 = 40$ N, $F_4 = 10$ N.

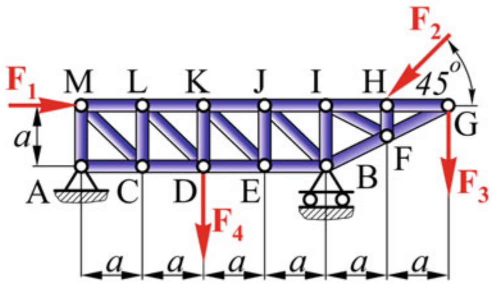


Fig. P8.50

8.51 Calculate the forces in members GF and IF of Fig. P8.50.

8.52 Determine the forces in members ML and MC of Fig. P8.50.

8.53 Determine the forces in members GF and KF (Fig. P8.53). Use $F_1 = 20$ N, $F_2 = 30$ N, $F_3 = 40$ N, $F_4 = 10$ N, $a = 20$ cm, and $h = 30$ cm.

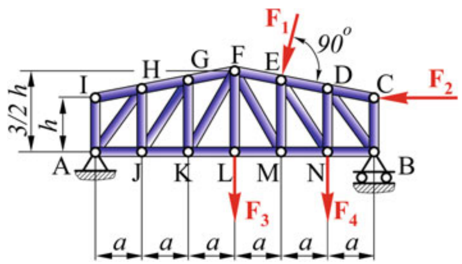


Fig. P8.53

8.54 Determine the forces in members GF and HI. Use $F_1 = 20$ N, $F_2 = 30$ N, $F_3 = 40$ N, $a = 20$ cm.

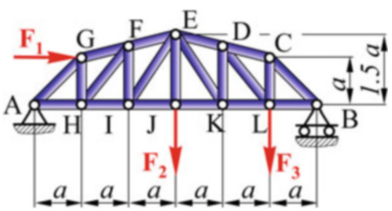


Fig. P8.54

8.55 Determine the forces in members EF and IE (Fig. P8.54). Use $F_1 = 20$ N, $F_2 = 30$ N, $F_3 = 40$ N, $a = 15$ cm.

8.56 Determine the forces in members IH and GF. Use $F_1 = F_2 = F_3 = 20$ N, $a = 10$ cm, and $h = 15$ cm.

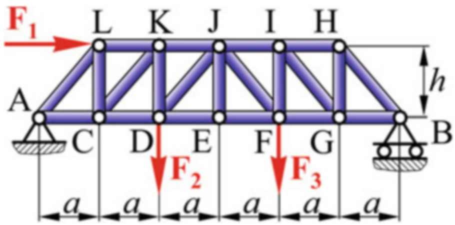


Fig. P8.56

8.57 Determine the forces in members IH and IL. Use $F_1 = F_2 = F_4 = 40$ N, $F_3 = 20$ N, $a = 10$ cm.

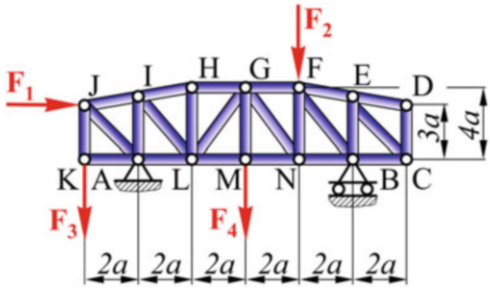


Fig. P8.57

8.58 Determine the forces in members NB and FE (Fig. P8.57). Use $F_1 = F_2 = F_3 = 20$ N, $F_4 = 40$ N, $a = 10$ cm.

8.59 Determine the forces in members ED and GF. Use $F_1 = 30$ lb, $F_2 = 40$ lb, $a = 3$ in.

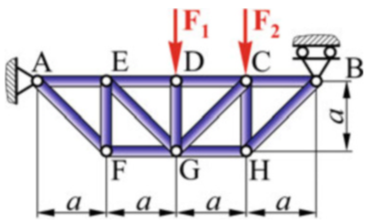


Fig. P8.59

8.60 Determine the forces in members BC and BH (Fig. P8.59). Use $F_1 = 50$ lb, $F_2 = 20$ lb, $a = 4$ in.

8.61 Effect of wind on a bridge is represented by forces $P_1 = P_2 = P_3 = P_4 = 30$ kN acting perpendicular to members 1, 2, and 3 (they are of the same length). Determine the reactions at A and B and the forces in members 2, 8, and 18.

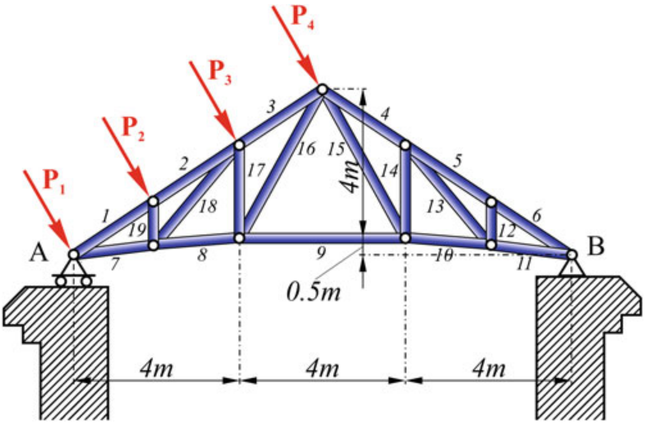


Fig. P8.61

In Problems 8.62 and 8.63, use any method: joints or sections.

8.62 Determine the force in truss members 1–9 when $P = 50$ N and each horizontal and vertical member is 40 cm long.

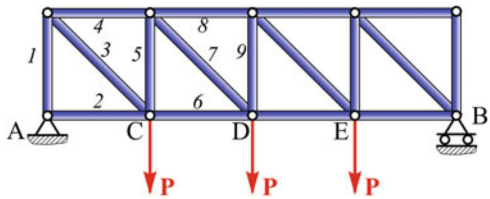


Fig. P8.62

8.63 Determine the force in each member of the truss.

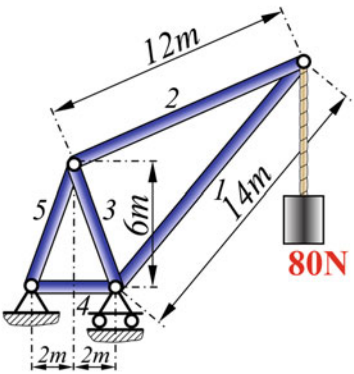


Fig. P8.63

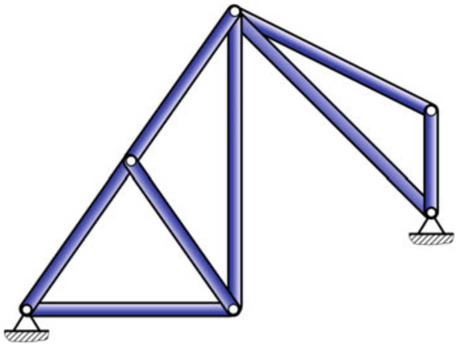
8.3 Compound Trusses

Until now, we have studied only simple trusses that were constructed by adding two new members to the existing truss to form a triangle. However, we can build a truss by connecting number of simple trusses, as shown in Fig. 8.22. This truss will lose its rigidity if it is removed from its supports. The equations of equilibrium for the unknown reaction cannot be solved. However, we still can solve for the forces in each member since this truss would satisfy the relationship.

$$2p = m + r$$

where m is number of members connected by p pins, and r is the number of reactions. It should be noted that in the case of a simple truss, the value of r is equal to three, since this is a number of unknowns one can solve for in a two-dimensional case. In our case, Fig. 8.22, $r = 4$. To solve this problem, one has either to write the complete set of equilibrium equations for each pin or separate the compound truss into simple trusses. The solution is demonstrated in Example 8.4.

Fig. 8.22 Compound truss



Example 8.4 Consider the compound truss shown in Fig. 8.23a. Assume that it is loaded by a vertical force $P = 2500$ N. All dimensions are in meters. Determine the internal forces in members BD and CD.

Solution The free body diagram of the compound truss is shown in Fig. 8.23b. Since both supports are of a pin type, we have four unknown reaction components: A_x, A_y, E_x, and E_y. The available equations of equilibrium allow solving only for three unknown reaction components. Therefore, we need to disassemble the truss into two structures and use the equations of equilibrium for each of them. Figure 8.23c shows the free body diagrams of each of the structures. It should be noted that the internal force **D** is represented via its two components D_x and D_y. Since it is an internal force, we show it as acting in opposite directions at the point D for each part of the structure. If you put the structure back, sum of these forces should be equal to zero. Now, we have two rigid bodies. For each one, we can write

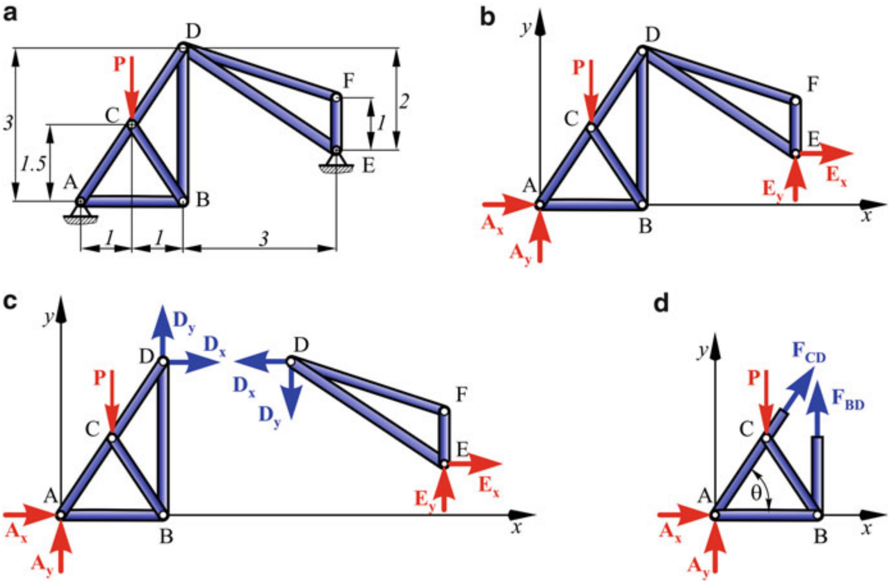


Fig. 8.23 (**a**) Compound truss. (**b**) Free body diagram. (**c**) Free body diagrams for each of the structures. (**d**) Cut across members BD and CD

three equations of equilibrium, thus we will have a total of six equations. From the other hand, there are six unknowns: A_x, A_y, D_x, D_y, E_x, and E_y.

$$\sum F_x = A_x + D_x = 0$$
$$\sum F_y = A_y + D_y - P = 0$$
$$\sum M_D = A_x \cdot 3 - A_y \cdot 2 + P \cdot 1 = 0$$

These are equations of equilibrium for the left part of the truss.

Similarly, we can write the equations of equilibrium for the right side part of the truss.

$$\sum F_x = E_x - D_x = 0$$
$$\sum F_y = E_y - D_y = 0$$
$$\sum M_D = E_x \cdot 2 + E_y \cdot 3 = 0$$

The solution yields the unknown reaction components:

$$A_x = \frac{3}{13}P, \quad A_y = \frac{11}{13}P, \quad E_x = -\frac{3}{13}P, \quad E_y = \frac{2}{13}P$$

Now, we can use the method of section and cut across members BD and CD, as shown in Fig. 8.23d. The corresponding equations of equilibrium are:

$$\sum F_x = A_x + F_{\mathrm{CD}} \cos\theta = 0$$
$$\sum F_y = A_y - P + F_{\mathrm{BD}} + F_{\mathrm{CD}} \sin\theta = 0$$

The solution yields

$$F_{\mathrm{CD}} = -\frac{3P}{13 \cdot \cos\theta} \quad \text{and} \quad F_{\mathrm{BD}} = \frac{2P}{13} + \frac{3P \cdot \sin\theta}{13 \cdot \cos\theta}$$

Substituting values for the load $P = 2500$ N and angle $\theta = 56.3°$ will result in $F_{\mathrm{CD}} = -1040$ N and $F_{\mathrm{BD}} = 1250$ N. The negative value of the internal force F_{CD} means that member CD is under compressive load.

8.4 Space Trusses

Some truss structures are built from the straight members that do not belong to the same plane, such structure is called *space truss*. Since each member is straight, it can be modeled as a two-force member and use the same approach of joints or sections to find the unknown forces. We assume that the loading is applied at joints only and that the joints are capable of resisting forces in any direction, but do not support any moments. These joints are called ball-and-socket joint. The weight of a member is frequently neglected. However, when it is of a significant value compared to the external forces, we can apply it as an external load at the both ends of a member. Half of the member's weight is usually applied at each end of the member. Simple space trusses are constructed from tetrahedrons (Fig. 8.24a), which are spatial equivalents of a plane triangle. Figure 8.24b shows a space truss constructed by joining two tetrahedrons. As we can see, addition of a new joint results in creation of three new members, thus the relationship between the number

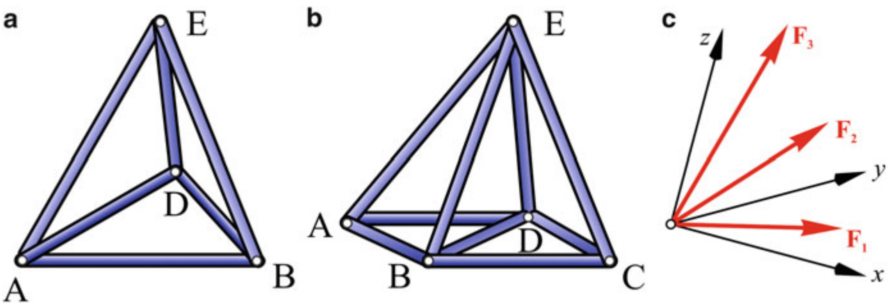


Fig. 8.24 (**a**) Tetrahedron. (**b**) Space truss. (**c**) Free body diagram of joint A

of joints p, number of members m and number of reactions r may be expressed as $3p = m + r$.

The free body diagram of each joint consists of a number of concurrent forces. Figure 8.24c shows the free body diagram of the joint A. Such system of forces is in equilibrium if the resultant of all forces is equal to zero, i.e., we have three scalar equations of equilibrium per joint. Thus, for a simple space truss we have $3p$ equations of equilibrium, which have to be equal to the number of unknown internal forces plus the number of unknown reactions. This condition will create a statically determinate truss, i.e., a simple space statically determinate truss may have only six unknown reaction components. If there are more than six reaction components, the truss is over-constrained and it is statically indeterminate.

When the forces in all members have to be calculated, we may use the method of joints. Since only three equations of equilibrium for each joint may be written, we will have the total of three times the number of joints equations. Those equations can be solved for unknown forces using any appropriate mathematical tools. When it is required to find the forces in only some of the members, we can use the method of sections. The selected section should have no more than six unknown forces acting on the isolated part of the truss.

Example 8.5 We have to find the internal forces in the three bars supporting the glass roof, as shown in Fig. 8.25a. Assume that all the bars are attached to the roof and to the base support (point A) by frictionless pins. The total weight of the glass

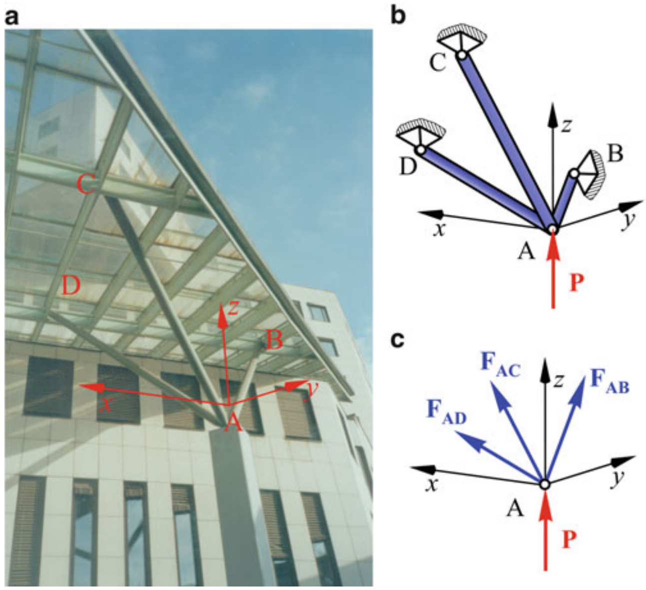


Fig. 8.25 (**a**) Glass roof. (**b**) Physical model of the 3D truss. (**c**) Free body diagram

roof supported by the bars is $P = 15$ kN. The coordinates of the attachment points are: A (0, 0, 0), B (−2, −3, 2), C (3, 6, 2), and D (3, −3, 2). The dimensions are in meters.

Solution We are dealing with a three-dimensional truss. The physical model is shown in Fig. 8.25b. The load (weight of the roof) is supported at point A, and it is in the vertical direction. Since there are only three truss members at the point of support (A), we can use the method of joints. Let us select (cut out) the joint A and draw a free body diagram as shown in Fig. 8.25c. We can write three equations of equilibrium. There are three unknown values: the magnitudes of the internal forces because the direction of each force is known. Let us calculate the unit vector for each truss member.

$$\begin{aligned}\mathbf{e}_{AB} &= \frac{x_B - x_A}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\mathbf{i} \\ &\quad + \frac{y_B - y_A}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\mathbf{j} \\ &\quad + \frac{z_B - z_A}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\mathbf{k} \\ &= \frac{-2}{\sqrt{(-2-0)^2 + (-3-0)^2 + (2-0)^2}}\mathbf{i} + \frac{-3}{\sqrt{17}}\mathbf{j} + \frac{2}{\sqrt{17}}\mathbf{k} \\ &= \frac{-2}{\sqrt{17}}\mathbf{i} + \frac{-3}{\sqrt{17}}\mathbf{j} + \frac{2}{\sqrt{17}}\mathbf{k}\end{aligned}$$

Similarly, we can find

$$\begin{aligned}\mathbf{e}_{AC} &= \frac{3}{\sqrt{49}}\mathbf{i} + \frac{6}{\sqrt{49}}\mathbf{j} + \frac{2}{\sqrt{49}}\mathbf{k} \\ \mathbf{e}_{AD} &= \frac{3}{\sqrt{22}}\mathbf{i} + \frac{-3}{\sqrt{22}}\mathbf{j} + \frac{2}{\sqrt{22}}\mathbf{k}\end{aligned}$$

Since the sum of all forces should be equal to zero, we can write three corresponding equations of equilibrium

$$\begin{aligned}\sum F_x &= F_{AB} \cdot \left(\frac{-2}{\sqrt{17}}\right) + F_{AC} \cdot \left(\frac{3}{\sqrt{49}}\right) + F_{AD} \cdot \left(\frac{3}{\sqrt{22}}\right) = 0 \\ \sum F_y &= F_{AB} \cdot \left(\frac{-3}{\sqrt{17}}\right) + F_{AC} \cdot \left(\frac{6}{\sqrt{49}}\right) + F_{AD} \cdot \left(\frac{-3}{\sqrt{22}}\right) = 0 \\ \sum F_z &= F_{AB} \cdot \left(\frac{2}{\sqrt{17}}\right) + F_{AC} \cdot \left(\frac{2}{\sqrt{49}}\right) + F_{AD} \cdot \left(\frac{2}{\sqrt{22}}\right) + P = 0\end{aligned}$$

The solution yields

$$F_{\text{AB}} = -18.55\,\text{kN},\, F_{\text{AC}} = -17.50\,\text{kN},\text{ and } F_{\text{AD}} = -2.35\,\text{kN}.$$

The minus sign shows that truss member is under compression forces.

Example 8.6 A force $P = 200$ N is acting in the plane ABCD and its line of action makes angle of 45° with vertical line AC. Angle EAK = angle FBM = angle NDB = 90°, AE = AK, BF = BM, and ND = BD. Determine the forces in each of the six members.

Solution This is a three-dimensional truss. Its physical model is shown in Fig. 8.26a. We have to identify a joint that is loaded by no more than three unknown forces. Joint A is such a joint. Let us cut it out and consider its equilibrium. First, we draw the free body diagram as shown in Fig. 8.26b. We can write three equations of equilibrium. There are three unknown values: the magnitudes of the internal forces, since the direction of each force is known. Let us express force in each member as a vector.

$$\boldsymbol{F}_{\text{AK}} = F_{\text{AK}}(\boldsymbol{i}\cdot\cos 45^\circ - \boldsymbol{k}\cdot\cos 45^\circ)$$
$$\boldsymbol{F}_{\text{AE}} = F_{\text{AE}}(-\boldsymbol{i}\cdot\cos 45^\circ - \boldsymbol{k}\cdot\cos 45^\circ)$$
$$\boldsymbol{F}_{\text{AB}} = F_{\text{AB}}(\boldsymbol{j})$$
$$\boldsymbol{P} = P(\boldsymbol{j}\cdot\cos 45^\circ - \boldsymbol{k}\cdot\cos 45^\circ)$$

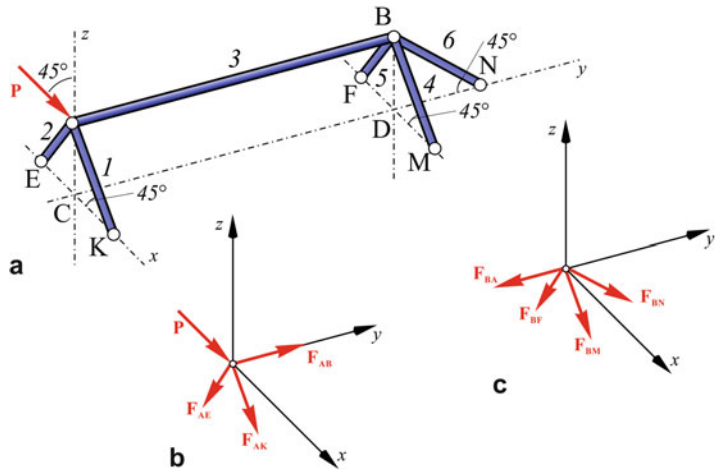


Fig. 8.26 (**a**) Physical model. (**b**) FBD of joint A. (**c**) FBD of joint B

The corresponding equations of equilibrium are

$$\sum F_x = F_{AK} \cdot \cos 45^\circ - F_{AE} \cdot \cos 45^\circ = 0$$
$$\sum F_y = F_{AB} + P \cdot \cos 45^\circ = 0$$
$$\sum F_z = -F_{AK} \cdot \cos 45^\circ - F_{AE} \cdot \cos 45^\circ - P \cdot \cos 45^\circ = 0$$

Solving these three scalar equations result in

$$F_{AB} = -P \cdot \cos 45^o = -141.4\,\text{N}$$
$$F_{AK} = F_{AE} = -P/2 = -100.0\,\text{N}$$

Now consider joint B, since AB is a two-force cos member, we know the force it will apply to the joint B (Fig. 8.26c). Thus, there are only three unknown forces acting at this joint. The direction of each force is known, since each member is a two-force member. We will write the equations of equilibrium and solve for unknown values of these forces.

Let us express force in each member as a vector.

$$\boldsymbol{F}_{BN} = F_{BN}(\boldsymbol{j} \cdot \cos 45^\circ - \boldsymbol{k} \cdot \cos 45^\circ)$$
$$\boldsymbol{F}_{BM} = F_{BM}(\boldsymbol{i} \cdot \cos 45^\circ - \boldsymbol{k} \cdot \cos 45^\circ)$$
$$\boldsymbol{F}_{BF} = F_{BF}(-\boldsymbol{i} \cdot \cos 45^\circ - \boldsymbol{k} \cdot \cos 45^\circ)$$
$$\boldsymbol{F}_{BA} = -F_{BA}(\boldsymbol{j})$$

The corresponding equations of equilibrium are

$$\sum F_x = F_{BM} \cdot \cos 45^\circ - F_{BF} \cdot \cos 45^\circ = 0$$
$$\sum F_y = F_{BN} \cdot \cos 45^\circ - F_{BA} = 0$$
$$\sum F_z = -F_{BN} \cdot \cos 45^\circ - F_{BM} \cdot \cos 45^\circ - F_{BF} \cdot \cos 45^\circ = 0$$

Solving these three scalar equations result in

$$F_{BM} = F_{BF} = P/2 = 100.0\text{ N}$$
$$F_{BN} = -P = -200.0\text{ N}$$

Guidelines and Recipes for Finding the Internal Forces in Three-Dimensional Truss Members Using the Method of Joints

- Create the physical model of the truss.
- Draw the free body diagram of the truss.
- Solve for the unknown reactions.
- Choose and disassemble a joint, upon which not more than three unknown internal forces are acting.
- Draw the free body diagram of this joint and solve for unknown internal forces.
- Repeat for each joint.

What We Have Learned?
How to solve trusses
Truss is a structure that is built from only two-force elements connected at the ends one to another in order to create a desired shape. To solve for unknown forces, we can use the method of joints or method of sections.

8.4.1 Problems

8.64 The construction crane was designed to carry maximum load **W**. Calculate the internal forces acting in the truss members AB, AC, and AD attached to point of loading A. Use $W = 10{,}000$ N, $a = 1$ m, $b = 2$ m, $c = 4$ m, and $h = 3$ m.

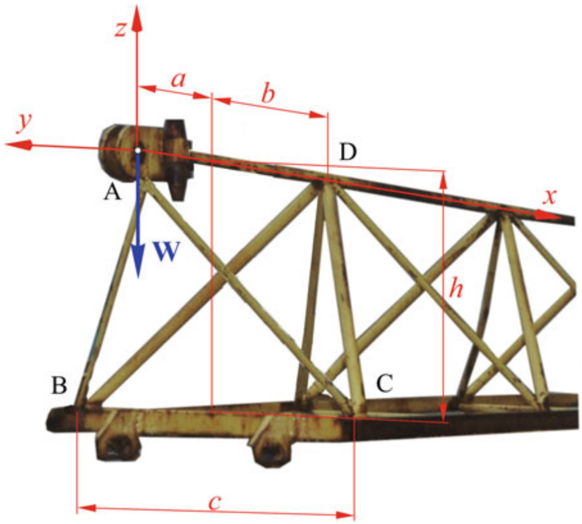


Fig. P8.64 Construction crane

8.65 Six bars are connected and loaded as shown. The direction of force **Q** is along the diagonal LD. Determine the reactions at B, D, L, H, and forces in each bar when $P = 250$ N, $Q = 100$ N, and $AB = AD = AL = 50$ cm.

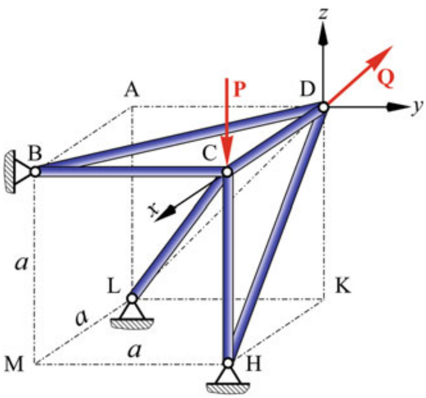


Fig. P8.65

8.66 The truss consists of six members loaded by forces $Q = 3$ N and $F = 5$ N. Determine the forces in each member. $AB = BC = BE$.

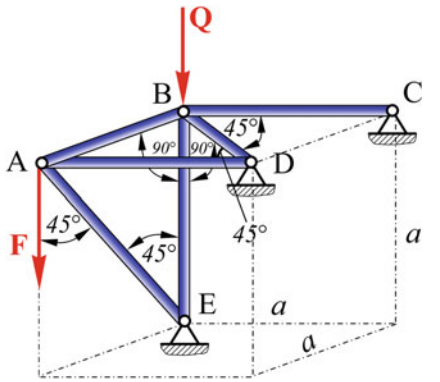


Fig. P8.66

8.67 Determine forces in each truss member, when $P = 12$ kN, $a = 20$ cm, $b = 40$ cm, $c = 50$ cm, and $d = 10$ cm. Force P is acting along the line AB.

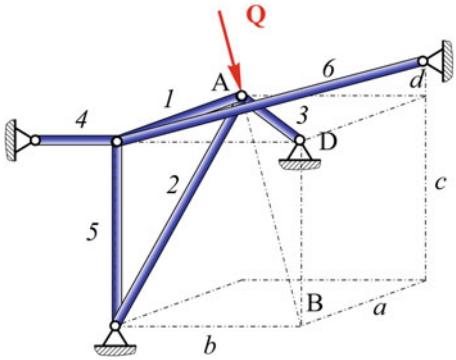


Fig. P8.67

8.68 Determine forces in each truss member, when $P = 10$ N, $Q = 6$ N, and $a = 10$ cm. Force **Q** is acting along the line DB.

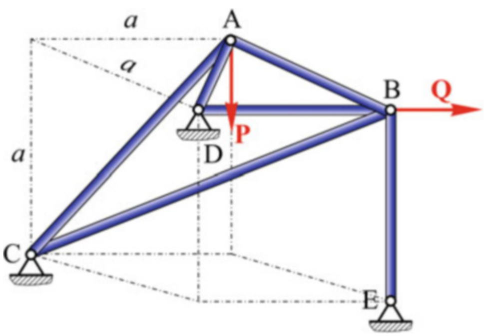


Fig. P8.68

8.69 Determine forces in each truss member, when $P = 100$ N, $Q = 200$ N, $a = 20$ cm, $b = 40$ cm, and $c = 50$ cm. Force **P** is acting along the line AB and force **Q**—along the line DE.

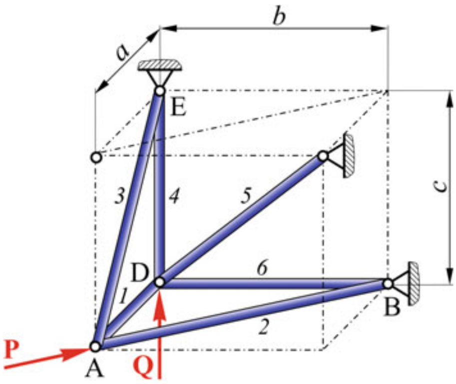


Fig. P8.69

8.70 The space truss is loaded by the vertical force $P = 20$ N. Planes of the triangles DAE and FBG are perpendicular to the plane of the triangle ABC. Determine the forces in each bar. The angle BAC = angle ABC = 30° and angle AED = ADE = BFG = BGF = 60°. $AB = 3BG$.

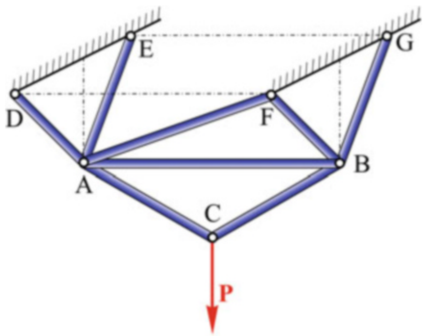


Fig. P8.70

8.71 Determine forces in each truss member, when $P = 100$ N, $Q = 50$ N, $a = 20$ cm. Force **P** is acting along the member CH and force **Q**—along the member CD.

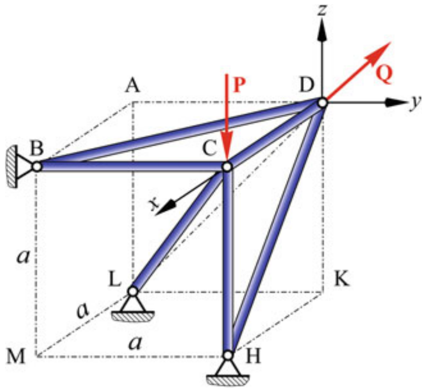


Fig. P8.71

8.72 Determine forces in each truss member, when $P = 40$ lb, $Q = 50$ lb, $a = b = 20$ in., $c = 40$ in. Force **P** is acting along the direction AB and force **Q**—along the member DE.

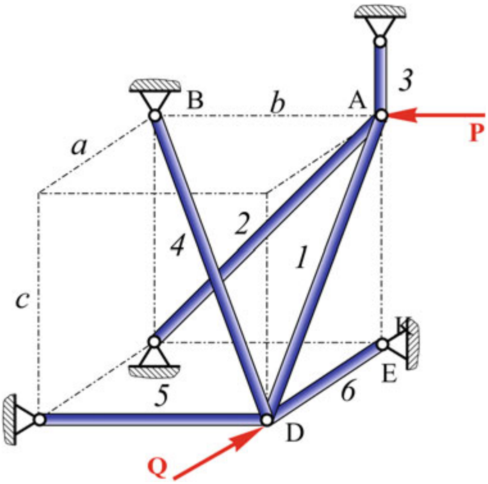


Fig. P8.72

8.73 Determine forces in each truss member, when $P=4$ kN, $Q=2$ kN, $a=20$ cm, $b=c=30$ cm. Force **P** is acting along member AB and force **Q** along the line DE.

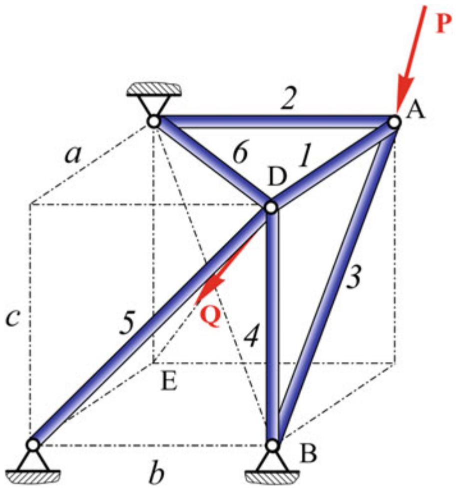


Fig. P8.73

8.74 Determine forces in each truss member, when $P=400$ lb, $Q=100$ lb, $a=b=c=30$ in. Force **P** is acting along line AB and force **Q** along member DE.

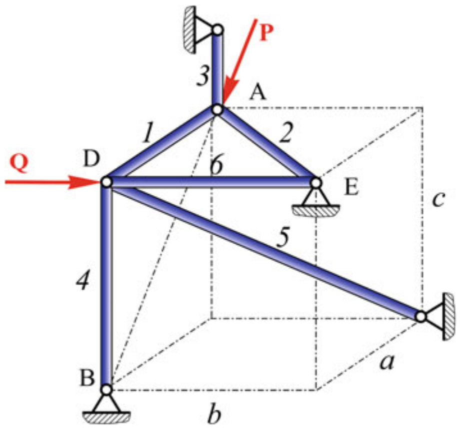


Fig. P8.74

8.75 Determine forces in each truss member, when $P = 60$ N, $Q = 10$ N, $a = 20$ cm, $b = 40$ cm, and $c = 30$ cm.

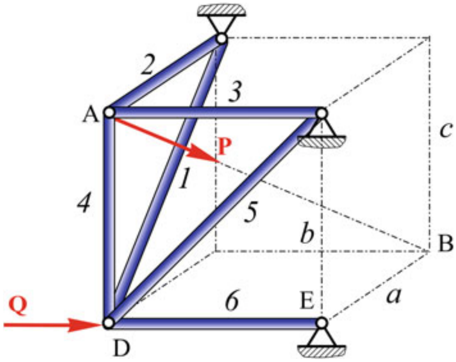


Fig. P8.75

8.76 Determine forces in each truss member, when $P = 400$ N, $a = 30$ cm, $b = c = 50$ cm, $d = 10$ cm. Force **P** is acting along the line AB.

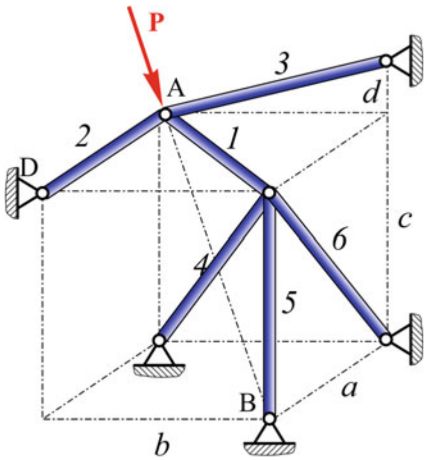


Fig. P8.76

8.77 Joints A, B, C, and D define a square in the horizontal plane. Joints E and B belong to the same vertical line and $BE = AB$. The truss is loaded by a vertical force $\mathbf{P} = 60$ kN. Determine forces in each truss member.

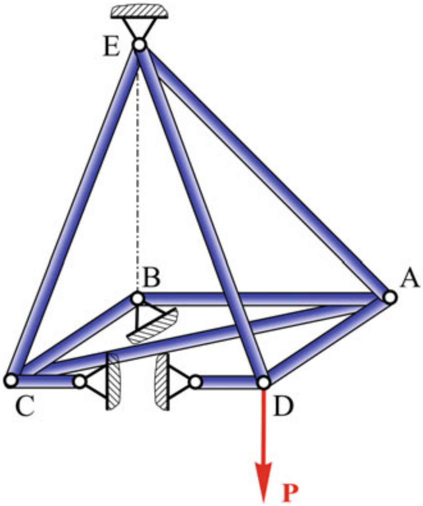


Fig. P8.77

Beams 9

Contents

Few things are harder to put up with than a good example

—Mark Twain

In this chapter you will learn:

- How to calculate internal forces and moments in a straight beam
- How to calculate internal forces and moments in a piece-wise straight beam
- How to calculate internal forces and moments in a curved beam
- How to create a diagram of bending moments and internal forces using intuitive approach
- How to use superposition principle to draw the diagrams of internal forces and moments

I. Emri, A. Voloshin, *Statics*, DOI 10.1007/978-1-4939-2101-0_9

In Chap. 7, we have introduced three classes of structural elements: truss members, beams, and cables. For each class, we have discussed the requirements that structural element has to meet in order to belong to a particular class. In Chap. 8, we have discussed trusses, and frames. Frames and mechanisms (Chap. 11) are built from truss members and beams.

Here, we will study techniques to calculate and to present graphically the distribution of internal forces and moments in various types of beams along their axis.

Beams are the most common structural elements in engineering applications. They are capable of bearing any type of load. Beams can be classified according to their geometry as: straight, piece-wise straight, or curved. When a beam and a loading are in the same plane, the problem may be considered as two-dimensional. In all other cases, it has to be treated as a three-dimensional problem.

> When a beam and a loading are in the same plane, the problem may be considered as two-dimensional. In all other cases, it has to be treated as a three-dimensional problem.

Consider a free body diagram of a beam loaded by an arbitrary set of forces and moments. Remember that we are interested in the internal forces and moments acting at an arbitrary location of the beam. As discussed in Chap. 7, in order to find the internal forces and moments, we have to cut the structure at a selected location, apply unknown internal force and moment, and consider equilibrium of this part. In engineering practice, we are interested in distribution of internal forces and moments along the beam's longitudinal axis to identify the most critical cross section. Since we are looking for the distribution of internal forces and moments along the beam axis, we have to express them analytically as a function of the external loads and location along the beam axis. To do this, we have to divide the beam into a number of *regions*, within which there are no changes in external loads and beam geometry.

> *Region* is a segment of a beam within which there are no changes in external loads or the beam geometry.

When we are calculating internal forces and moments, we cannot change the location of external forces and moments or substitute the distributed load by its equivalent concentrated load. Such change would affect the distribution of internal forces and moments. One need to remember that the principle of force transmissibility and the fact that a moment is a free-floating vector and may be moved to any location may be applied only for equilibrium of rigid bodies, and NOT when we determine internal forces and moments!

> When examining internal forces, the location of external moments and forces cannot be moved.

Internal and external forces and moments have to be represented in the same coordinate system.

The principle of force transmissibility, and the fact that a moment is a free-floating vector and may be moved to any location, may be applied only for equilibrium of rigid bodies and NOT when the internal forces are determined!

Internal forces and moments are reactions to external forces and moments, as follows from the Third Newton's Law. Therefore, they have to be represented in the same coordinate system. From the Third Newton's Law follows also that the changes in internal forces and moments along the axis of a beam are possible only if there is a change in external loading. Therefore, "no action—no reaction."

Next two sections describe the procedure for drawing diagrams of internal forces and moments for straight beams loaded by variety of loads.

9.1 Selection of Coordinate System for the Internal Forces and Moments and Sign Convention

Before analyzing internal forces and moments, we have to select a coordinate system and define convention for the internal forces and moment signs. Let us start with a straight beam loaded by an arbitrary set of external loads as shown in Fig. 9.1a.

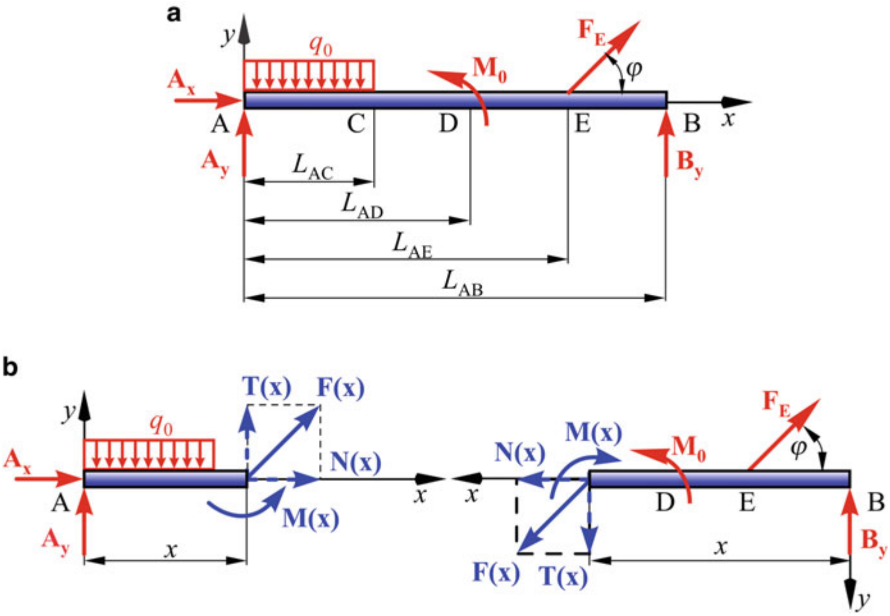


Fig. 9.1 (**a**) Free body diagram of a beam with arbitrary loading. (**b**) Internal forces and moments

In order to determine internal forces and moments, as discussed in Chap. 7, we need to cut the structural element at the location of our interest. Therefore, we will cut this beam at an arbitrary location x, which may be measured from the left or, equivalently, from the right side of the beam, as shown in Fig. 9.1b.

To keep the beam in equilibrium after the cut, we have to apply the internal forces and moments. The force, $F(x)$, and the moment, $M(x)$, added on the left, and on the right the internal forces and moments that are equal in magnitude and opposite in sign are added, i.e., they form pair of forces and pair of moments, respectively (see Chap. 2 for details). Each of the two internal forces may be represented via their two components: one along the axis of the beam, $N(x)$, and another normal to the axis, $T(x)$. Both forces are shown as dashed lines to indicated that they are components of $F(x)$ with which the force may be replaced.

The internal forces and moments acting at the left side of the cut should be equal in magnitude and opposite in direction to the ones acting on the right hand side of the same cut, as illustrated in Fig. 9.1b. This derives from the Third Newton's Law. In other words, we inverted the direction of the internal forces and moments when applied them on the right side of the cut. In order to have these internal forces and moment positive, the coordinate system should be inverted as well, see Fig. 9.1b.

The coordinate system and direction of internal forces and moments, shown in Fig. 9.1b, we will consider as *positive internal forces and moments*.

We will follow this agreement throughout this textbook. You may see other sign conventions used in different books. There is no physical reason to select this or another coordinate system and sign agreement. However, the same convention should be followed throughout the course of solving a problem.

> It is essential to use the same coordinate system for the external and internal forces and moments. In our case, the right hand system, which is commonly used in mechanics, is applied. The coordinate system of internal forces and moments is always located at the point, where we cut the beam. The coordinate system "travels" along the axis of the beam.

The coordinate system of internal forces and moments is always located at the point, where we cut the beam. Hence, one may say that the coordinate system of internal forces and moments "travels" along the axis of the beam.

9.2 Straight Beams

We will start with a free body diagram of a straight beam loaded by reaction forces A_x, A_y, and B_y, external concentrated force F_E, distributed force $q(x) = q_0$, and moment $M_D = M_0$. Angle φ indicates the direction of force F_E (Fig. 9.1a). The x-axis of Cartesian coordinate system is aligned with longitudinal axis of the beam.

> We will assume that the magnitudes of q_0, M_0, and F_E are such that both reactions A_y and B_y are positive.

It is a common practice that directions of all reactions are selected such that they have the positive sign in a chosen coordinate system.

The beam will be in equilibrium when

$$\begin{aligned}
&\sum F_x = A_x + F_E \cdot \cos\varphi = 0 \\
&\sum F_y = A_y - q_0 \cdot L_{AC} + F_E \cdot \sin\varphi + B_y = 0 \\
&\sum M_i = -q_0 \cdot \frac{L_{AC}^2}{2} + M_0 + L_{AE} \cdot F_E \sin\varphi + L_{AB} \cdot B_y = 0
\end{aligned} \tag{9.1}$$

The sum of moments is taken with respect to point A. From the above equilibrium equations, we obtain the reactions.

$$\begin{aligned}
A_x &= -F_E \cdot \cos\varphi \\
A_y &= q_0 \cdot L_{AC} - F_E \cdot \sin\varphi - B_y \\
B_y &= \frac{1}{L_{AB}}\left(q_0 \cdot \frac{L_{AC}^2}{2} - M_0 - L_{AE} \cdot F_E \cdot \sin\varphi\right)
\end{aligned} \tag{9.2}$$

> The internal force and the moment always point in opposite direction of the external force and moment. In the selected coordinate system, this means that internal forces and moments will have sign opposite to that of the external forces and moments acting upon the rigid body.

Please note that reaction A_x is negative, which means that A_x acts in an opposite direction as it was assumed in the free body diagram at the beginning, i.e., it acts to the left. We will not change its direction in the free body diagram; therefore, from here on we will need to take into account that A_x has the negative sign. From (9.2) we see that the sign of the reactions and consequently distribution of internal forces and moments depends on the magnitudes of q_0, M_0, and F_E. Let us choose their magnitudes such that both reactions A_y and B_y are positive!

Please note that choosing other possible magnitudes of q_0, M_0, and F_E will result in completely different distributions of internal forces and moments. Analyzing these different possibilities could be a very instructive homework exercise.

The beam in Fig. 9.1a consists of four regions: AC, CD, DE, and EB. The beginning and the end of each region define discontinuity or change in the loading. The AC region is defined by the beginning and the end of the continuous load; the CD region is defined by the end of the continuous load and the point of action of moment $\mathbf{M_D} = \mathbf{M}_0$; similarly, the regions DE and EB are defined between the points of action of moment $\mathbf{M}_D$, force $\mathbf{F}_E$, and the end of the beam, respectively.

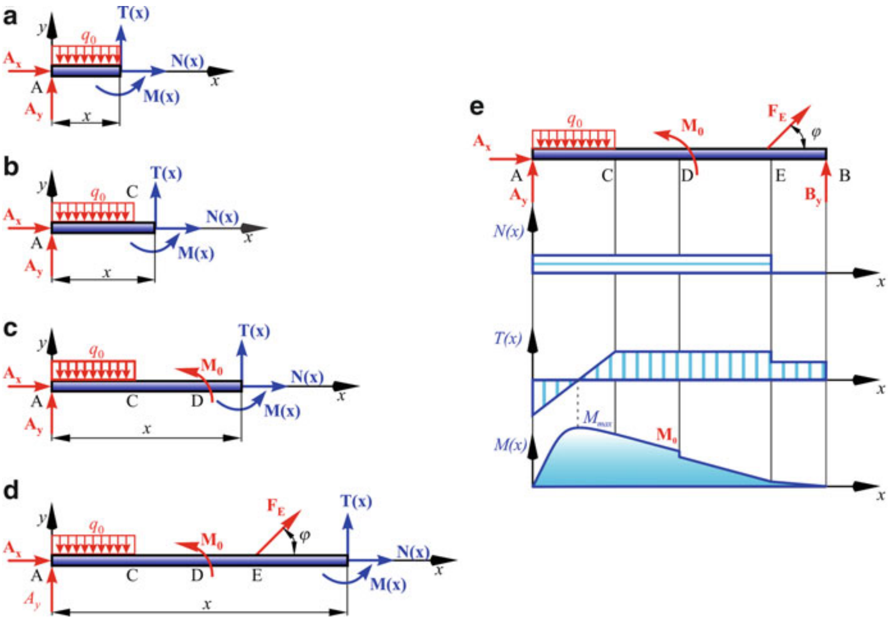


Fig. 9.2 (**a**) Region AC. (**b**) Region CD. (**c**) Region DE. (**d**) Region EB. (**e**) Diagrams of the internal forces and moments

> A beam may be subdivided into regions. Beginning and end of each region defines discontinuity or change in the loading.

Let us proceed with determination of distributions $N(x)$, $T(x)$, and $M(x)$ within each of the above mentioned regions.

Region AC

We cut the beam at location x between A and C, $0 \leq x < L_{AC}$ (Fig. 9.2a), and write the equilibrium equations

$$\begin{aligned} \sum F_x &= A_x + N(x) = 0 \\ \sum F_y &= A_y - q_0 \cdot x + T(x) = 0 \\ \sum M &= M(x) + q_0 \cdot \frac{x^2}{2} - A_y \cdot x = 0 \end{aligned} \tag{9.3}$$

It is convenient to take the sum of the moments about the cut point. From (9.3) we can find distributions of the internal forces and the moment within the region AC.

$$\begin{aligned} N(x) &= -A_x \\ T(x) &= q_0 \cdot x - A_y \\ M(x) &= -q_0 \cdot \frac{x^2}{2} + A_y \cdot x \end{aligned} \tag{9.4}$$

Taking into account reactions displayed in (9.2), we obtain the distributions of the axial and shear forces and the moment within region AC in an explicit form:

$$N(x) = F_\mathrm{E} \cdot \cos\varphi$$

$$T(x) = q_0 \cdot x + \frac{1}{L_\mathrm{AB}}\left(q_0 \cdot \frac{L_\mathrm{AC}^2}{2} - M_0 - L_\mathrm{AE} \cdot F_\mathrm{E} \cdot \sin\varphi\right) + F_\mathrm{E} \cdot \sin\varphi - q_0 \cdot L_\mathrm{AC}$$

$$M(x) = -q_0 \cdot \frac{x^2}{2} + \left[q_0 \cdot L_\mathrm{AC} - F_\mathrm{E} \cdot \sin\varphi - \frac{1}{L_\mathrm{AB}}\left(q_0 \cdot \frac{L_\mathrm{AC}^2}{2} - M_0 - L_\mathrm{AE} \cdot F_\mathrm{E} \cdot \sin\varphi\right)\right] \cdot x$$

It is important to stress that one obtains exactly the same distribution of the internal forces and the moment by considering equilibrium equations for the complementary part of the rigid body to the right of the cut.

Region CD

Now, we cut the beam at location x between C and D, as shown in Fig. 9.2b. To cover region CD, x needs to change as follows $L_\mathrm{AC} \leq x < L_\mathrm{AD}$.

The corresponding equilibrium equations are

$$\begin{aligned} \sum F_x &= A_x + N(x) = 0 \\ \sum F_y &= A_y - q_0 \cdot L_\mathrm{AC} + T(x) = 0 \\ \sum M &= M(x) + q_0 \cdot L_\mathrm{AC} \cdot \left(x - \frac{L_\mathrm{AC}}{2}\right) - A_y \cdot x = 0 \end{aligned} \tag{9.5}$$

The sum of the moments was again taken relative to the cut point. From (9.5) we find distributions of the internal forces and the moment within the region CD

$$\begin{aligned} N(x) &= -A_x \\ T(x) &= -A_y + q_0 \cdot L_\mathrm{AC} \\ M(x) &= \left(A_y - q_0 \cdot L_\mathrm{AC}\right) \cdot x + q_0 \cdot \frac{L_\mathrm{AC}^2}{2} \end{aligned} \tag{9.6}$$

After taking into account reaction forces (9.2), we find the explicit expressions of the internal forces and the moment within the region CD

$$N(x) = F_E \cdot \cos\varphi$$
$$T(x) = F_E \cdot \sin\varphi + \frac{1}{L_{AB}}\left(q_0 \cdot \frac{L_{AC}^2}{2} - M_0 - L_{AE} \cdot F_E \cdot \sin\varphi\right)$$
$$M(x) = -\left[F_E \cdot \sin\varphi + \frac{1}{L_{AB}}\left(q_0 \cdot \frac{L_{AC}^2}{2} - M_0 - L_{AE} \cdot F_E \cdot \sin\varphi\right)\right] \cdot x + q_0 \cdot \frac{L_{AC}^2}{2}$$

Region DE

We cut the beam at location x between D and E; hence, $L_{AD} \leq x < L_{AE}$ (Fig. 9.2c), and write the equilibrium equations

$$\begin{aligned} &\sum F_x = A_x + N(x) = 0 \\ &\sum F_y = A_y - q_0 \cdot L_{AC} + T(x) = 0 \\ &\sum M = M(x) + M_0 + q_0 \cdot L_{AC} \cdot \left(x - \frac{L_{AC}}{2}\right) - A_y \cdot x = 0 \end{aligned} \tag{9.7}$$

From (9.7) we find distribution of the internal forces and the moment within region DE:

$$\begin{aligned} N(x) &= -A_x \\ T(x) &= -A_y + q_0 \cdot L_{AC} \\ M(x) &= \left(A_y - q_0 \cdot L_{AC}\right) \cdot x - M_0 + q_0 \cdot \frac{L_{AC}^2}{2} \end{aligned} \tag{9.8}$$

Again, after substitution of reaction forces given with (9.2) we obtain the explicate expressions for the internal forces and the moment within region DE.

$$N(x) = F_E \cdot \cos\varphi$$
$$T(x) = F_E \cdot \sin\varphi + \frac{1}{L_{AB}}\left(q_0 \cdot \frac{L_{AC}^2}{2} - M_0 - L_{AE} \cdot F_E \cdot \sin\varphi\right)$$
$$M(x) = -\left[F_E \cdot \sin\varphi + \frac{1}{L_{AB}}\left(q_0 \cdot \frac{L_{AC}^2}{2} - M_0 - L_{AE} \cdot F_E \cdot \sin\varphi\right)\right] \cdot x - M_0 + q_0 \cdot \frac{L_{AC}^2}{2}$$

Region EB

Again, we cut the beam at location x between E and B; hence, $L_{AE} \leq x < L_{AB}$ (Fig. 9.2d), and write the equilibrium equations

$$\sum F_x = A_x + N(x) + F_E \cdot \cos\varphi = 0$$
$$\sum F_y = A_y - q_0 \cdot L_{AC} + F_E \cdot \sin\varphi + T(x) = 0$$
$$\sum M = M(x) - (x - L_{AE}) \cdot F_E \cdot \sin\varphi + M_0 + q_0 \cdot L_{AC} \cdot \left(x - \frac{L_{AC}}{2}\right) - A_y \cdot x = 0 \quad (9.9)$$

Thus, distributions of the internal forces and the moment within the region DE are

$$N(x) = -A_x - F_E \cdot \cos\varphi$$
$$T(x) = -A_y + q_0 \cdot L_{AC} - F_E \cdot \sin\varphi$$
$$M(x) = \left(F_E \cdot \sin\varphi - q_0 \cdot L_{AC} + A_y\right) \cdot x - L_{AE} \cdot F_E \cdot \sin\varphi - M_0 + q_0 \cdot \frac{L_{AC}^2}{2} \quad (9.10)$$

Taking into account (9.2), we find the explicate form of the internal forces and the moment within region EB.

$$N(x) = 0$$
$$T(x) = \frac{1}{L_{AB}}\left(q_0 \cdot \frac{L_{AC}^2}{2} - M_0 - L_{AE} \cdot F_E \cdot \sin\varphi\right)$$
$$M(x) = -\frac{1}{L_{AB}}\left(q_0 \cdot \frac{L_{AC}^2}{2} - M_0 - L_{AE} \cdot F_E \cdot \sin\varphi\right) \cdot x - L_{AE} \cdot F_E \cdot \sin\varphi - M_0 + q_0 \cdot \frac{L_{AC}^2}{2}$$

Diagrams of the internal forces and moments are shown and discussed in following section.

9.2.1 Diagrams of Internal Forces and Moments

Now, we are ready to draw the diagrams of the internal forces and moments (Fig. 9.1a) by using (9.4), (9.6), (9.8), and (9.10) and their explicit forms. The diagrams of the internal axial force, $N(x)$, the shearing force, $T(x)$, and the bending moment, $M(x)$, are shown in Fig. (9.2e). We will discuss each diagram one after another.

It is appropriate to stress again that the diagrams in Fig. 9.2e correspond to the assumption that the magnitudes of q_0, M_0, and F_E are chosen such that both reactions A_y and B_y are positive! For other combinations of q_0, M_0, and F_E, the distributions of the internal forces and moments will look differently!

Distribution of internal forces and moments depends on the magnitude and direction of the external forces (loads and reactions) acting upon the beam.

Axial Force N(x)

At point A, there is an external reaction force $A_x = -F_\mathrm{E} \cdot \cos\varphi$, which is negative. According to the Third Newton's Law, the internal force must be equal in magnitude and opposite in sign (opposite in direction). The external and internal forces together form an equilibrium pair of forces (see Chap. 2). Hence, within the first region AC the internal axial force will be constant and positive, $N(x) = F_\mathrm{E} \cdot \cos\varphi$.

Any change in internal forces and moments is always equivalent to the magnitude of external load at that point, and opposite in sign—Third Newton's Law!

Within the next two regions, i.e., CD and DE there is no external force acting on the beam; therefore, the internal axial force will be unchanged and constant up to point E, where an external force $F_\mathrm{E} \cdot \cos\varphi$ is acting in positive direction. According to the Third Newton's Law, the corresponding change of axial internal force will be equal in magnitude and opposite in sign, i.e., negative $(-F_\mathrm{E} \cdot \cos\varphi)$. Consequently, the internal force within the last region EB will be equal to zero.

We may double-check if the axial internal force in region EB is indeed equal to zero by starting the analysis of the internal axial forces from the right hand side of the beam. Remember, in this case we need to invert the coordinate system as shown in Fig. 9.1b and to maintain the same positive coordinate system used in analyzing the internal forces from the left hand side of the beam.

In the inverted coordinate system, the signs of the external forces (loads and reactions) are changed, i.e., all positive external forces become negative and all negative become positive.

At point B, there is no external force acting on the beam in axial direction. Hence, according to the Third Newton's Law the internal force must be also equal to zero. This means that there will be no axial force within the region EB. At point E, there is an external force F_E, which component in axial direction has (in the inverted coordinated system) negative sign, i.e., $-F_\mathrm{E} \cdot \cos\varphi$. According to the Third Newton's Law, the corresponding internal force will be of equal magnitude and opposite sign, hence positive $(F_\mathrm{E} \cdot \cos\varphi)$. Since within regions DE, CD, and AC there are no external axial forces, the internal axial force fill remains constant and unchanged till the end of region AC, i.e., point A, where the external reaction A_x is acting. In the original coordinate system, this reaction was negative; therefore, in the inverted coordinate system it will become positive. The corresponding internal force will therefore be negative, which will bring diagram $N(x)$ to zero.

> If we analyze the internal forces from the right hand side of a beam, we need to invert the coordinate system as shown in Fig. 9.1b.

Let us recall that the sum of all external forces (loads and reactions) and all internal forces in any direction must be equal to zero, as shown in Fig. 9.2e.

> The sum of all external forces (loads and reactions) and all internal forces in any direction must be always equal to zero.

From this discussion we may conclude that we indeed obtain the same result in the original and the inverted coordinate systems. Therefore, when solving problems, we may use any of the two approaches.

Shear Force T(x)

At point A, there is an external reaction force A_y, which we assumed to be positive (sign of A_y depends on magnitudes of q_0, M_0, and F_E). According to the Third Newton's Law, the corresponding internal force will be equal in magnitude and opposite in sign, hence negative. Within the first region AC, the external load is changing linearly due to the constant distributed load (more about this will be discussed in Sect. 9.2.2). At the support A, the external reaction force is positive. According to the Third Newton's Law, the corresponding internal force will be negative and will be increasing linearly, i.e., $T(x) = q_0 \cdot x - A_y$, to become positive at the end of the first region, where $x = L_{AC}$. The internal shear force will change its sign at the point, where $x = A_y/q_0$.

Let us remember again that any change in internal forces and moment along the beam axis can happen only if there is a change in external loading.

> Any change in internal forces or moment along the beam axis can happen only if there is a change in external loading!

At the end of the third region, at point E, an external positive force, $F_E \cdot \sin\varphi$, is acting in y-direction. According to the Third Newton's Law, this external force generates equivalent internal shear force of opposite direction (negative sign). Consequently, the positive internal shear force at point E will be reduced by $(-F_E \cdot \sin\varphi)$ and then will remain constant throughout region EB. At the end of region EB, at point B, there is again an external force, reaction B_y, which is positive. The corresponding internal force will be therefore negative and will bring the internal shear force down to zero, as shown in Fig. 9.2e. Let us recall again that the sum of all external forces (loads and reactions) and all internal forces in any direction must be always equal to zero. This means that all diagrams should start and finish with zero at both ends of the beam.

> Sum of all external and internal forces in any direction must always be equal to zero!

Let us now briefly analyze the shear forces in the inverted coordinate system; see Fig. 9.1b, which we use if we start analyzing the internal forces from the right hand side of the beam.

At point B, the beam is loaded with the external reaction force B_y, which is positive in the original coordinate system (see Fig. 9.1b) and negative in the inverted coordinate system. Hence, the corresponding internal force will be positive with the magnitude equal to B_y. Within the region EB there are no external forces; therefore, the internal shear force will remain constant throughout the region.

At point E, we have another external force $F_E \sin \varphi$, which is negative in the inverted coordinate system. Thus, the corresponding internal force will be positive. Consequently, at this point the internal force is increased for the same value, i.e., $F_E \sin \varphi$, and remains constant throughout region CD.

At point C, we enter region AC where a constant distributed force q_0 is acting. Its cumulative magnitude will therefore change linearly, $Q = q_0 \cdot x$. In inverted coordinate system, the distributed force acts in positive direction, which means that the corresponding internal force will be negative and its magnitude will increase linearly. Consequently, the internal shear force will start to decrease linearly and become negative at point A where it should be equal in magnitude to the external reaction A_y, which is in the inverted coordinate system negative. The corresponding internal force will be therefore positive and will bring the internal shear-force at point A to zero, as it should be, because the sum of all external forces and all internal forces in any direction should be equal to zero.

> Only external forces acting at a point can generate discontinuity ("jump") in the distribution of internal forces!

Moment M(x)

We start again at the left hand side of the beam at point A. Since there is no external moment acting on the beam at this point, the internal moment will be equal to zero. Within region AC, the internal moment is changing as a quadratic (parabolic) function of the form $M(x) = -q_0 x^2/2 + A_y \cdot x$, see (9.4). This means that the moment will increase and reach its maximum at the point, where $dM(x)/dx = 0$, or at $x = A_y/q_0$. From (9.4) we see that this happens to be exactly at the point where the shear forces are equal to zero, i.e., $T(x) = 0$, as shown in Fig. 9.2e. We will discuss this in more details later.

At point C, where region AC ends and region CD starts, the magnitude of the internal moment will be the same for both regions, i.e., $M(x = L_{AC}) = -q_0 \cdot L_{AC}^2/2 + A_y \cdot L_{AC}$. This also follows from the Third Newton's Law. Since at point C there is no external moment acting on the beam, there cannot be any change in the internal moment. Within region CD, internal moment is changing linearly, as seen from (9.6) up to point D, where the external positive moment M_0 is acting. This is the end of region CD and beginning of region DE. In Sect. 9.2.2, we will discuss how to draw diagrams of internal forces and moment without using equations that define their distribution within individual regions.

Only an external moment can cause discontinuity ("jump") in distribution of the internal moments.

According to the Third Newton's Law, positive external moment M_0 will generate a negative internal moment, which will be equal in magnitude, i.e., $-M_0$. Consequently, the internal moment in point D will be decreased for the same magnitude, as shown in Fig. 9.2e. Within region DE, the internal moment is again changing linearly as shown in (9.8). Note that the slope at which moment is changing within region DE is the same as slope within region CD. The reasons for it we will discuss in Sect. 9.2.2. Region DE ends at point E, where external force F_E is acting and the last region EB starts. By introducing $x = L_{AE}$ into (9.8) and (9.10), we again find that at point E there is no change in the internal moment, as it should be according to the Third Newton's Law, because there is no external moment acting on the beam at this point.

Within the last region BE, the moment is again changing linearly according to (9.10) to become zero at point B, where $x = L_{AB}$.

In principle, we could obtain the same diagram of internal moments using the inverted coordinate system (see Fig. 9.1b), which need to be used if we start solving the problem from the right hand side. In case of the moments, without writing the equilibrium equations for the inverted coordinate system, or using principles that will be discussed in Sect. 9.2.2, this becomes quite cumbersome. Therefore, we will refrain from doing this. However, the reader could write the equilibrium equations for the moments in the inverted coordinate system and then draw the diagrams. This could be a nice homework.

Let us proceed with solving an example.

Example 9.1 Draw a diagram of internal moments and forces in the shelf supporting a radio and five books. The weight of the radio is 30 N and the weights of books are 3, 6, 9, 12, and 15 N, respectively (Fig. 9.3a), where $a = 0.25\,\text{m}$, $b = 0.18\,\text{m}$, $c = 0.20\,\text{m}$, and $L = 0.80\,\text{m}$.

Solution The physical model of the shelf is shown in Fig. 9.3b. We will approximate the weight of the books by a linearly distributed function $q_1(x) = 1440x$ [N/m] and the weight of the radio by a $q_2(x) = q_2 = 150$ [N/m]. The corresponding free body diagram is shown in Fig. 9.3c.

We can find the reactions from the equations of equilibrium

$$\sum_i F_{ix} = A_x = 0$$

$$\sum_i F_{iy} = A_y - \frac{1}{2}\cdot q_1(a)\cdot a - q_2\cdot c + B_y = 0$$

$$\sum_i M_i^A = -\frac{1}{2}\cdot q_1(a)\cdot a\cdot\frac{2}{3}\cdot a - q_2\cdot c\cdot\left(a + b + \frac{c}{2}\right) + B_y\cdot L = 0$$

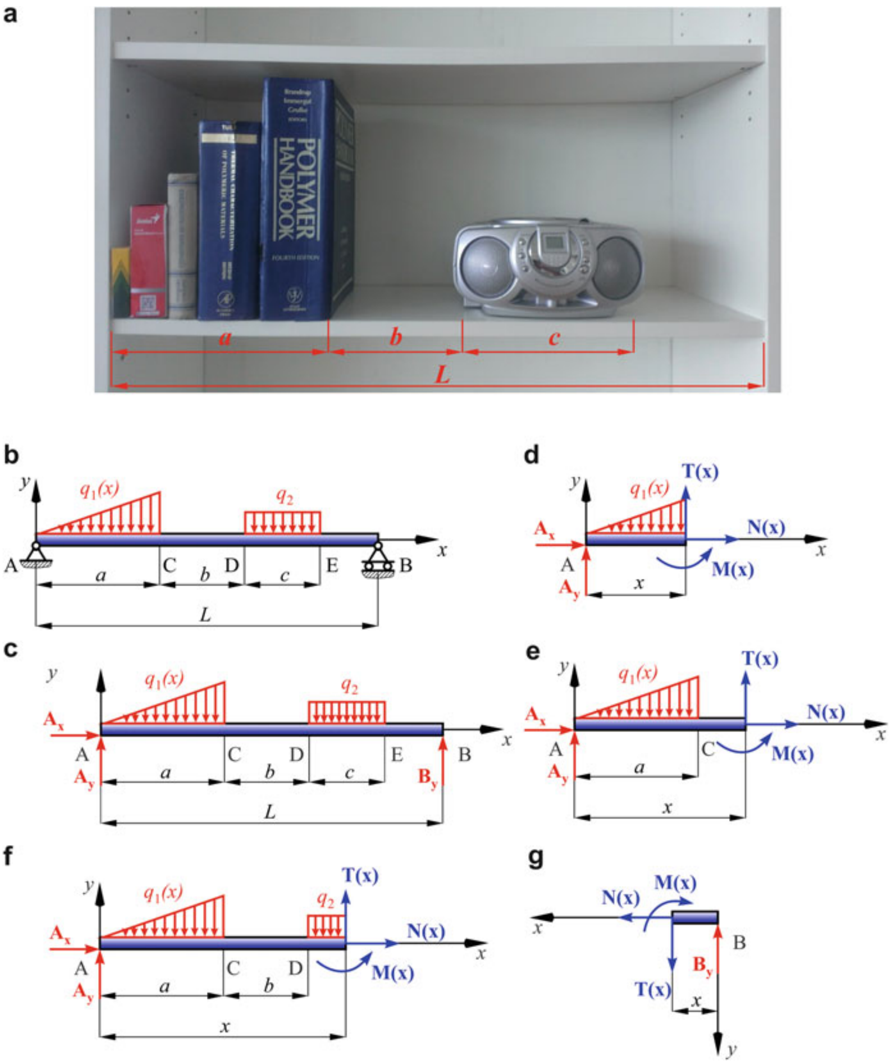


Fig. 9.3 (**a**) The shelf supporting five books and a radio. (**b**) The physical model of the shelf supporting five books and a radio. (**c**) The free body diagram of the shelf supporting five books and a radio. (**d**) Internal forces and moments in Region AC. (**e**) Internal forces and moments in Region CD. (**f**) Internal forces and moments in Region DE. (**g**) Internal forces and moments in Region EB

Hence,

$$A_x = 0$$
$$B_y = \frac{1}{L} \cdot \left[\frac{1}{3} \cdot q_1(a) \cdot a^2 + q_2 \cdot c \cdot \left(a + b + \frac{c}{2}\right)\right]$$
$$A_y = \frac{1}{2} \cdot q_1(a) \cdot a + q_2 \cdot c - B_y$$

By substituting the appropriate numerical values, we obtain

$$A_x = 0.0\,\mathrm{N}$$
$$B_y = 29.3\,\mathrm{N}$$
$$A_y = 45.8\,\mathrm{N}$$

Now, we can proceed and define the internal forces and moments. The beam in Fig. 9.3c consists of four regions: AC, CD, DE, and EB. The axial load is equal to zero in all the regions since reaction $A_x = 0$. Now, we will derive expressions for internal load and moment for each of the four regions.

Region AC
We cut the beam at location x between A and C, $0 \leq x < a$ (Fig. 9.3d), and write the equilibrium equations for the left part of the beam.

$$\sum_i F_{iy} = A_y - \frac{1}{2} q_1(x) \cdot x + T(x) = 0,$$
$$\sum_i M_i^x = M(x) + \frac{1}{2} q_1(x) \cdot x \cdot \frac{x}{3} - A_y \cdot x = 0$$

Thus, the distributions of the internal shear forces and moments within region AC are

$$T(x) = -45.8 + 720 \cdot x^2\,\mathrm{N}$$
$$M(x) = (-240 \cdot x^3 + 45.8 \cdot x)\,\mathrm{N\,m}$$

Region CD
Within the region
$a \leq x < (a + b)$ (Fig. 9.3e), we are again considering the left part of the beam.

$$\sum_i F_{iy} = A_y - \frac{1}{2} q_1(a) \cdot a + T(x) = 0$$
$$\sum_i M_i^x = M(x) - \frac{1}{2} q_1(a) \cdot a \cdot \left(x - \frac{2 \cdot a}{3}\right) - A_y \cdot x = 0$$

The distributions of the internal forces and moments within region CD are

$$T(x) = -0.75\,\mathrm{N},$$
$$M(x) = (0.75 \cdot x + 7.52)\,\mathrm{N\,m}$$

Region DE
$(a+b) \leq x < (a+b+c)$ (Fig. 9.3f). We are considering the left part of the beam.

$$\sum_i F_{iy} = A_y - \frac{1}{2} q_1(a) \cdot a - q_2 \cdot (x-a-b) + T(x) = 0$$
$$\sum_i M_i^x = M(x) + q_2 \cdot (x-a-b) \cdot \frac{(x-a-b)}{2} + \frac{1}{2} q_1(a) \cdot a \cdot \left(x - \frac{2 \cdot a}{3}\right) - A_y \cdot x = 0$$

The distributions of the internal forces and the moment within region DE are

$$T(x) = (-65.3 + 150 \cdot x)\,\mathrm{N}$$
$$M(x) = (-75 \cdot x^2 + 65.3 \cdot x - 6.35)\,\mathrm{N\,m}$$

Region EB
To simplify the equilibrium equations, we are considering here the right part of the beam, $0 < x < (L-a-b-c)$ (Fig. 9.3g). In order to do that, we need to invert the coordinate system as demonstrated in Fig. 9.1b.

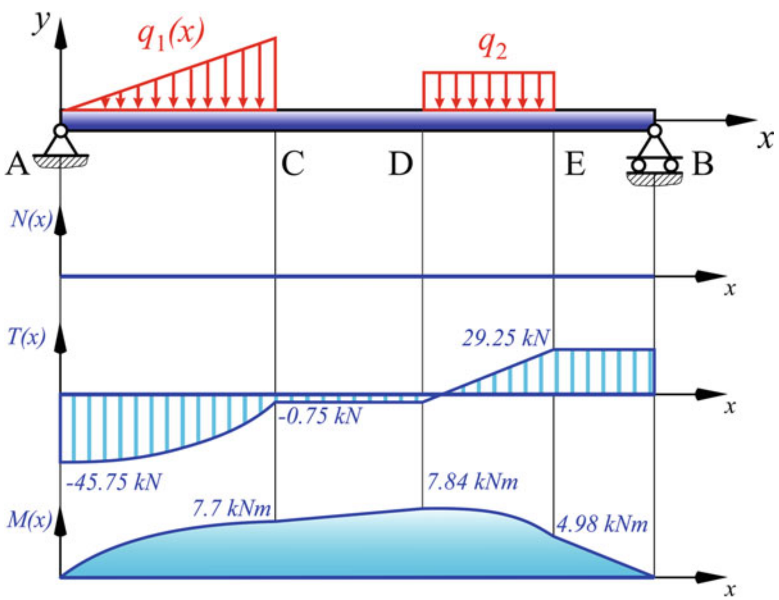


Fig. 9.4 The diagrams of the internal shear force and bending moment for all regions of the beam

$$\sum_i F_{iy} = T(x) - B_y = 0$$
$$\sum_i M_i^x = M(x) - B_y \cdot x = 0$$

The distributions of the internal forces and moments within region EB are (Fig. 9.4)

$$T(x) = 29.3\,\text{N}$$
$$M(x) = 29.3 \cdot x\,\text{N m}$$

9.2.2 Relationship Between the Distributed Load, Shear Force, and Bending Moment – Rules for Drawing Diagrams of Internal Forces and Moment Intuitively

In the previous section, we learned that according to the Third Newton's Law, any change in internal forces could be a result from the external loads only. We learned also that a continuous load causes continuous changes in the internal forces and that the force acting at a point will cause a "jump," i.e., discontinuity, in the internal forces. In the case of internal moments, we similarly observed that diagram of internal moments will be continuous if there are no external moments acting on the observed beam, and that moment acting at a point will cause a "jump," i.e., discontinuity, in internal moment.

However, at this point we do not have a general rule how external forces, continuous and those acting at a point, affect the distribution of internal moments. The goal of this section is to discuss these interrelations and derive ten "intuitive Rules" for drawing diagrams of internal forces and moment without writing equilibrium equitations for each region along the beam.

Let us assume that a segment of a beam is loaded by an arbitrary distributed load $q(x)$ only, as shown in Fig. 9.5. We will have to determine internal forces and moments at location x.

Internal Shear Forces

The internal shear force $T(x)$ is obtained from the equilibrium equation in y direction

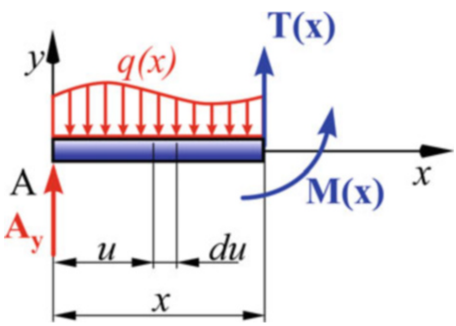


Fig. 9.5 Segment of a beam with distributed load

$$A_y - \int_0^x q(u)du + T(x) = 0 \tag{9.11}$$

In (9.11), u is a running variable denoting the position of segment du within interval $[0, x]$. From the above equation we can find shear force $T(x)$

$$T(x) = \int_0^x q(u)du - A_y \tag{9.12}$$

Within a region loaded with a distributed load, the shear force will be equal to the integral of the corresponding distributed load, and will act in the opposite direction (opposite sign)

The magnitude of the shear force is proportional to the integral of distributed load $q(x)$ and acts in the opposite direction. In our case, the distributed load has a negative sign; therefore, the corresponding internal shear force resulting from the distributed load will be positive, as seen in (9.12).

> Derivative dy/dx of function $y = f(x)$ represents a slope (rate) at which $f(x)$ changes its magnitude.

Let us recall from mathematics that derivative dy/dx of function $y = f(x)$ represents a slope (rate) at which $f(x)$ changes its magnitude. Therefore, knowing derivative $dT(x)/dx$ could be a very useful information. Unfortunately deriving (9.12), which is a definite integral, is not a straightforward procedure. We need to use the rule for derivative of definite integrals, shown in *Mathematical Corner*, to obtain

$$\frac{dT(x)}{dx} = q(x) \tag{9.13}$$

Mathematical Corner

How to take a derivative of the definite integral.

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,u)du = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}[f(x,u)]du + f[x, b(x)] \cdot \frac{db(x)}{dx} - f[x, a(x)] \cdot \frac{da(x)}{dx}$$

In our case, $f(x,u) = q(u)$, $b(x) = x$, $a(x) = 0$; thus,

$$\frac{d}{dx}T(x) = \int_0^x \frac{\partial}{\partial x} q(u)du + q(x) \cdot \frac{dx}{dx} - q(x) \cdot \frac{d0}{dx} = q(x)$$

Hence, the derivative of the internal shear forces $T(x)$ within the observed region is equal to the magnitude of the corresponding distributed (continuous) load. From mathematics we know that (**i**) when the first derivative (slope) of $T(x)$ is positive the $T(x)$ will be an increasing function, whereas (**ii**) if the derivative (slope) is negative the $T(x)$ will be a decreasing function. Finally, (**iii**) when the derivative (slope) is equal to zero the distribution of shear forces within the region will be constant.

From (9.12) it is derived that (**iv**) when the distributed load within the region is a constant, $q(x) = \text{const.}$, the shear force will change within the observed region linearly. Finally, (**v**) when the distributed load changes linearly the shear force $T(x)$ within the region will be a quadratic function.

It is worthwhile to stress that when within the observed region the beam is loaded with a distributed load in the axial direction, the same rules apply as for the shear forces.

If a beam is loaded with a distributed load, $p(x)$, in the axial direction, then the very same rules apply as in the case of shear forces.

Internal Moments

Let us proceed with the equilibrium equation for moments with respect to the point, where the beam was cut.

$$M(x) + \int_0^x (x - u) \cdot q(u) \cdot du - A_y \cdot x = 0$$

From the equilibrium equation we find

$$M(x) = -\int_0^x (x - u) \cdot q(u) \cdot du + A_y \cdot x \tag{9.14}$$

Using the rule of derivatives for definite integrals, shown in the *Mathematical Corner*, we obtain

$$\frac{dM(x)}{dx} = -\int_0^x q(u)du + A_y \tag{9.15}$$

Thus, derivative of internal moment is equal to magnitude of the corresponding internal shear force, c.f., (9.13), and is opposite in sign. Therefore,

$$\frac{dM(x)}{dx} = -T(x) \tag{9.16}$$

Using (9.16), we may derive several useful rules, similar to those derived for shear forces. Again, from mathematics we need to remember that the sign of a function derivative defines if function is increasing or decreasing. The positive sign indicates that the function is increasing, whereas the negative sign indicates that it is decreasing.

In a view of this, we may conclude that (**vi**) when within an observed region the internal shear force is negative, the derivative of the internal moment will be positive, i.e., $dM(x)/dx > 0$ and consequently the distribution of internal moment, $M(x)$, will be an increasing function. Similarly, (**vii**) when the internal shear force is positive, then derivative of the internal moment will be negative, $dM(x)/dx < 0$ and distribution of internal moment will be a decreasing function. Finally, (**viii**) when the internal shear force within a region is zero, i.e., $dM(x)/dx = 0$, then internal moment $M(x)$ will be a constant.

As before, we may formulate two additional rules that are very useful when drawing diagrams of internal forces and a moment. (**ix**) If the internal shear force within the observed region is a constant, $T(x) = \text{const}$, then distribution of the corresponding internal moment, $M(x)$, will be a linear function. Similarly, (**x**) when the internal shear force within the observed region changes as a linear function, the corresponding internal moment, $M(x)$, will change as a quadratic function.

Internal Forces and Moments Within the Region

To determine the shear force change between points 1 and 2, we need to integrate (9.13)

$$\frac{dT(x)}{dx} = q(x)$$

$$\int_{T_1}^{T_2} dT(x) = \int_{L_1}^{L_2} q(x)dx$$

and

$$T_2 - T_1 = \int_{L_1}^{L_2} q(x)dx \tag{9.17}$$

Equation (9.17) shows that the change of the internal shear force between two locations is equal to the cumulative contribution of the distributed load between the two locations.

Similarly, we can integrate (9.16) to obtain an expression for the internal moment

$$M_2 - M_1 = -\int_{L_1}^{L_2} T(x)dx \tag{9.18}$$

This equation shows that a change of the internal moment between two locations is opposite in sign and equal to the cumulative contribution of the internal shear forces between these two points.

Utilizing the derived ten "Intuitive Rules", (i)-(x), we are ready to draw the diagrams of the internal forces and moment *intuitively* without writing equilibrium equations for the each region along the beam.

Rules for drawing diagrams of internal forces and moment "intuitively":

(i) If within the observed region $dT(x)/dx > 0$, then $T(x)$ will be an increasing function.
(ii) If within the observed region $dT(x)/dx < 0$, then $T(x)$ will be a decreasing function.
(iii) If within the observed region $dT(x)/dx = 0$, then the internal shear force will be constant, i.e., $T(x) = const.$
(iv) If within the observed region $dT(x)/dx = const.$, then internal shear force $T(x)$ will change linearly.
(v) If within the observed region $q(x)$ is a linear function, then $T(x)$ will be a quadratic function.
(vi) If within an observed region $Tx < 0$, and therefore $dM(x)dx > 0$, then the internal moment, $M(x)$, will be an increasing function.
(vii) If within an observed region $Tx > 0$ and consequently $dM(x)dx < 0$ then the internal moment, $M(x)$, will be a decreasing function.
(viii) If within an observed region $dM(x)dx = 0$, then $M(x)$ will be a constant.
(ix) If within an observed region the internal shear force is a constant $dM(x)$ $dx = const$ then the internal moment, $M(x)$, will be a linear function.
(x) If within an observed region the internal shear force, $T(x)$, changes linearly, the corresponding internal moment, $M(x)$, will change as a quadratic function.

9.2.3 Intuitive Drawing of Internal Forces and Moment Diagrams

Utilizing the rules developed in previous section, we can draw the diagrams of internal forces and moments "*intuitively*" without writing the equilibrium equations for each region of a beam.

Internal forces are reactions to external loads, thus their directions are always opposite to that of the external loads (Third Newton's Law). Any change in the external forces and moments (loads) will result in the equivalent change of the internal forces and moments that will always be equal in magnitude and opposite in direction (sign).

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This approach of intuitive drawing of the diagrams of internal forces and moment is demonstrated in Example 9.2 below.

Example 9.2 Draw the diagrams of internal shear forces and bending moment of a bench loaded, as shown in Fig. 9.6a, by using the *intuitive approach*. Solve the equilibrium equations for external forces, but do not write equations of equilibrium for each region. The weight of the child on the left is $P_1 = 50$ lb, while the weight of the child on the right is $P_2 = 90$ lb. The dimensions are in "in".

Solution We will treat this problem as a 2D case since all external forces are acting in the same vertical plane, and the bench is supported by two legs, which may be replaced with two forces acting in the same plane as external forces. Since we have only three equilibrium equations for a plane case, we will simplify the supports and draw the physical model of the bench as shown in Fig. 9.6b. We will represent the weight of each child by a concentrated force. It should be mentioned here that from experiments it is known that such simplification results in over design.

Engineering practice requires that any assumption or simplification of a structural part will result in its over design, meaning the part will be safer.

The reactions are calculated from the equilibrium equations for the free body diagram (Fig. 9.6c). The results are $A_y = 62$ lb and $D_y = 78$ lb. Now, we can start to draw diagrams of the internal forces and moment.

Diagram of Forces

There is no axial internal force because $A_x = 0$. Since there are two external forces acting along the axis of the beam (bench), it should be divided into three regions: AB, BC, and CD. We start at the left hand side of the bench and "travel" along the bench (beam) axis, and apply the *intuitive rules* explained in the previous section. Reaction A_y is positive, so the corresponding internal force should be of the same magnitude and have the opposite (negative) sign, as shown in Fig. 9.6d. As there is no distributed load acting on a beam, according to (9.13) the shear force has to be a constant in all three regions (*intuitive rule* (iii)). At point B, we encounter external force $\mathbf{P_1}$ acting in negative direction. The change in the internal force at this point should be positive and equal to the magnitude of $\mathbf{P_1}$. At point C, we encounter force $\mathbf{P_2}$, which causes a change of the internal force to positive direction. At this point, the internal shear force changes its sign. Finally, we arrive to the end of the bench, where the external reaction force D_y is acting. Since D_y is positive, the

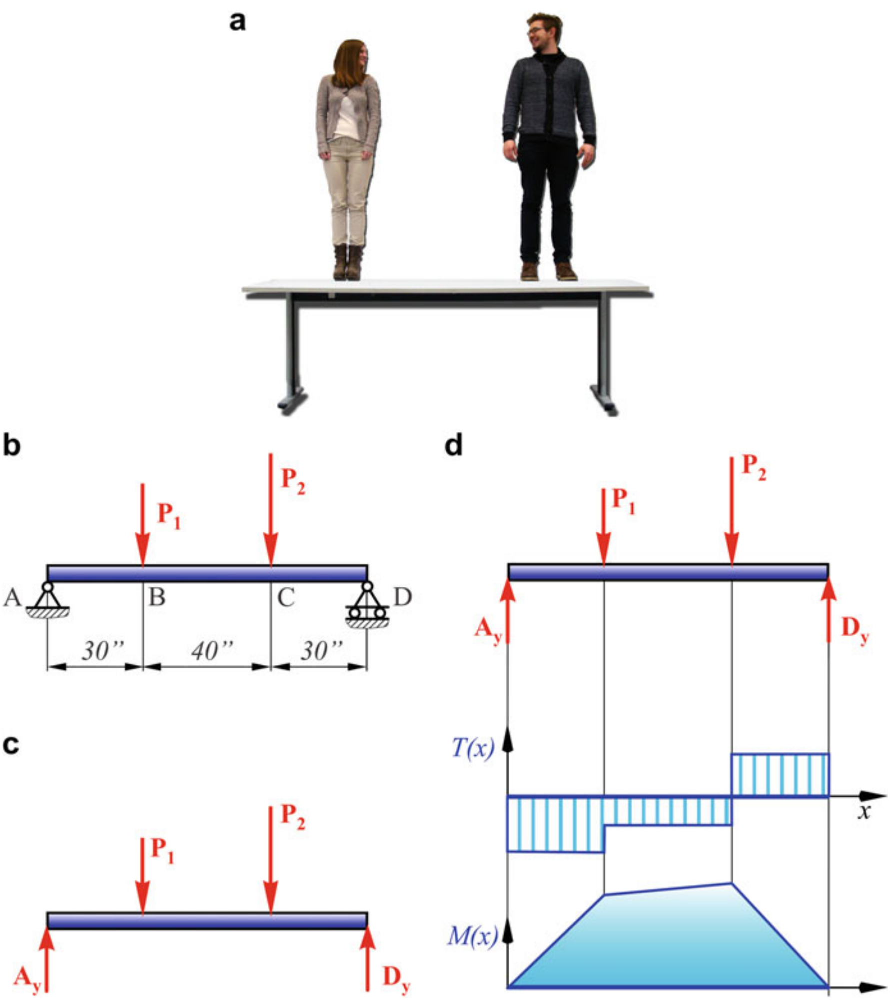


Fig. 9.6 (**a**) Two persons standing on a bench. (**b**) Physical model of the bench. (**c**) Free body diagram of the bench. (**d**) Qualitative diagrams of the bending moment and transverse (shear) forces

corresponding change of the internal force should be negative and equal in magnitude. At this point, the diagram of the internal shear forces should be closed, because there are no forces outside of the bench, and the sum of all the internal and external forces must be equal to zero. Thus, the diagram of the shear forces looks as shown in Fig. 9.6d.

Diagram of Moments

Now, we draw the diagram of moments, and start from the left hand side of the bench again. Since there is no external moment at the left end of the bench, we start the moment diagram from zero. Within the first region, the shear force is constant and negative; therefore, $dM(x)/dx > 0$, and consequently the internal moment in this region will be an increasing linear function, see (9.16), and the *intuitive rules* (vi) and (ix). In the second region, the slope of the moment diagram should be also positive but smaller because the slope is equal to the shear force magnitude, (9.16).

If the magnitude of $T(x)$ is smaller, then the slope of $M(x)$ will be smaller, and vice versa.

Within the last region, the shear force is positive, thus the slope of the moment $M(x)$ must be negative and become zero at the right hand side of the beam. At the point where the shear force changes the sign, $T(x) = 0$, the moment diagram should have an extreme value, (9.16). This follows from the mathematical rule, which says that a function has an extreme at the point, where its first derivative is equal to zero. The diagram of bending moments is shown in Fig. 9.6d.

> Within the region of a beam the slope of $M(x)$ is equal to the corresponding shear force magnitude.

> If the magnitude of $T(x)$ is larger, then the slope of $M(x)$ will be larger, and vice versa.

Example 9.3 Draw diagrams of internal shear forces and bending moments for a beam loaded by its own weight (1500 N) and the weight of a boy (300 N) (Fig. 9.7a). Do not write equations of equilibrium for each region; use intuitive approach to draw the required diagrams.

Solution This is a cantilever beam, loaded by a distributed and concentrated load. Assuming that the boy does not touch the ground, we will create a physical model as shown in Fig. 9.7b and the related free body diagram as shown in Fig. 9.7c. Examination of the free body diagram suggests that there are two regions: AC and CB. From the equilibrium equations of the external forces we find that the reactions are:

$$M_A = 4950\,\mathrm{N\,m}$$

$$A_y = 1800\,\mathrm{N}$$

Diagram of Forces

We start at the left hand side of the beam. At point A, the internal shear (transverse) force will be negative because it has to be equal in magnitude and opposite in sign to reaction A_y, which is positive (Third Newton's Law). Within region AC, the beam is loaded with a constant distributed load. According to (9.13) and the *intuitive rules* (i) and (iv), the internal shear force will be a linear function with a positive slope. From (9.17) we can calculate the shear force magnitude at location C.

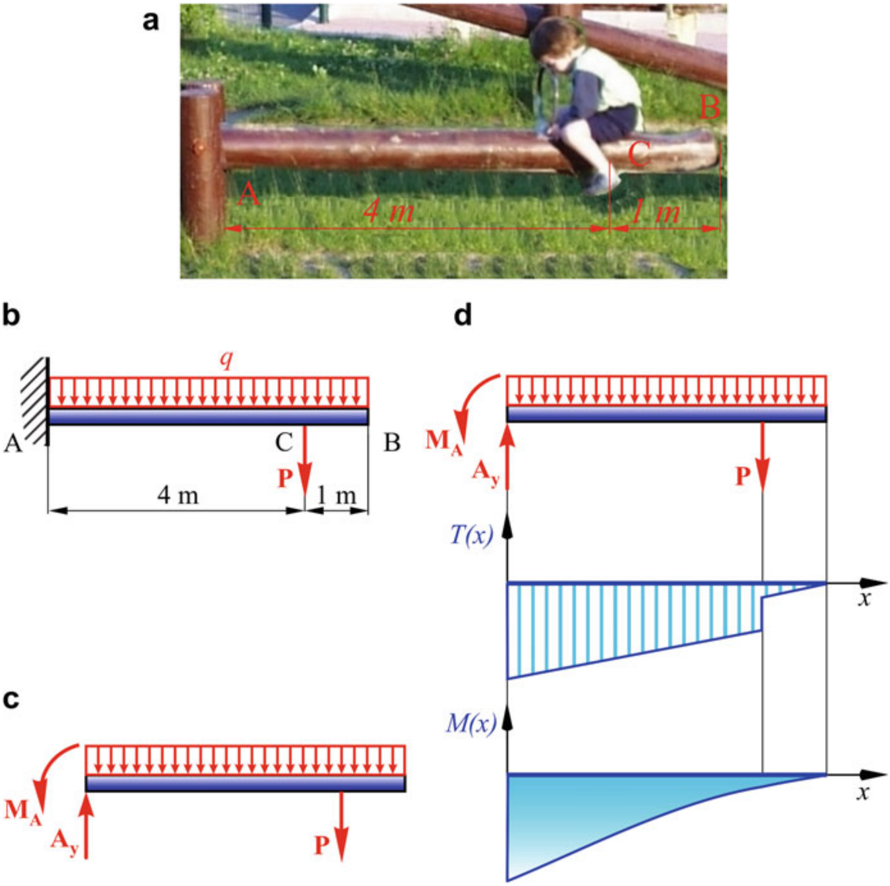


Fig. 9.7 (**a**) A boy on the cantilever beam. (**b**) Physical model. (**c**) Free body diagram. (**d**) Diagrams of the internal bending moment and shear force

$$T_C = T_A + \int_0^4 300dx = -1800 + 1200 = -600\,\mathrm{N}$$

At point C, where the first region ends and the second starts, the external force **P** is acting in negative direction. The related change in the internal shear force will be therefore positive and equal to 300 N. Consequently, the magnitude of internal shear force will change from – 600 to –300 N, Fig. 9.7d.

Within region CB, there is a distributed load of the same magnitude as in region AC. Therefore, the internal shear force will have the same positive slope as before. Since there is no external force acting at the end of the beam and the sum of all external and internal forces must be equal to zero, the internal shear force in point B must be zero. This also follows from the equation below.

$$T_B = T_C + \int_0^1 300dx = -300 + 300 = 0.0\,\mathrm{N}$$

Diagram of Moments

The internal moment at point A is equal and opposite to the external moment M_A, which is positive. Related internal moment at point A will be therefore negative, as shown in Fig. 9.7d. Within region AC, the internal shear force is negative and a linear function. Hence, the internal moment should be represented as a parabolic function, the *intuitive rule* (x), and have a positive slope, (9.16) and the *intuitive rule* (vi). The magnitude of the moment at point C may be calculated from (9.18)

$$M_C = M_A - (-4800) = -4950 + 4800 = -150\,\mathrm{N\,m}$$

In region CB, the internal shear force is again changing linearly; therefore, the internal moment must approach zero parabolically at the right hand side of the beam. The change in the value of the moment between points C and B is equal to the area under the shear force diagram. Both diagrams are shown in Fig. 9.7d.

9.2.4 Problems

9.1–9.20 Draw diagrams of internal moments and forces. The values of external concentrated forces, distributed forces and moments are given in the table below. Related physical models are numbered accordingly and shown below. Location of the coordinate system is in some cases given and in some it is not.

n	F (kN)	M (kN m)	q (kN/m)	q_1 (kN/m)	q_2 (kN/m)	a (m)	b (m)	c (m)	l (m)	α (°)
9.1	–	–	2	–	–	1	2	1.5	–	–
9.2	–	–	4	–	–	–	–	–	4	–
9.3	–	–	3	–	–	2	3	2	–	–
9.4	–	–	5	–	–	2	3	–	–	–
9.5	–	–	–	2	4	–	–	–	5	–
9.6	–	–	3	–	–	–	–	–	5	30
9.7	8	–	3	–	–	4	2	–	7	60
9.8	–	–	4	–	–	4	2	–	7	30
9.9	8	–	2	–	–	4	–	–	8	30
9.10	–	6	4	–	–	2	–	–	5	30
9.11	5	10	–	–	–	2	–	–	4	30
9.12	–	–	5	–	–	4	–	–	8	30
9.13	4	6	–	–	–	4	–	–	8	60

(continued)

n	F (kN)	M (kN m)	q (kN/m)	q_1 (kN/m)	q_2 (kN/m)	a (m)	b (m)	c (m)	l (m)	α (°)
9.14	–	10	3	–	–	4	–	–	8	30
9.15	–	–	4	–	–	2	–	–	4	60
9.16	–	–	3	–	–	2	–	–	5	60
9.17	8	–	4	–	–	1	–	–	4	60
9.18	6	–	2	–	–	–	–	–	4	30
9.19	–	–	5	–	–	2	–	–	4	–
9.20	4	2	2	–	–	2	–	–	4	–

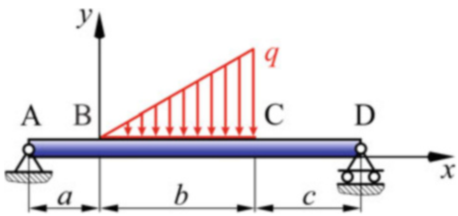


Fig. P9.1

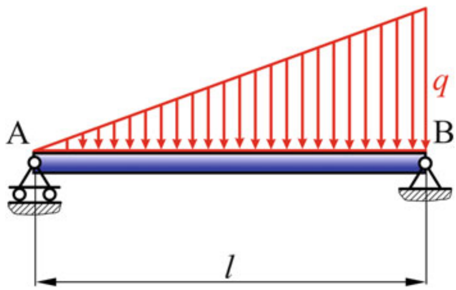


Fig. P9.2

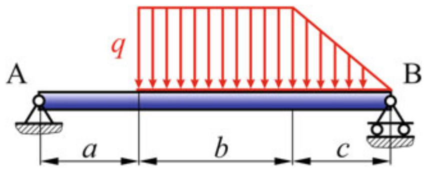


Fig. P9.3

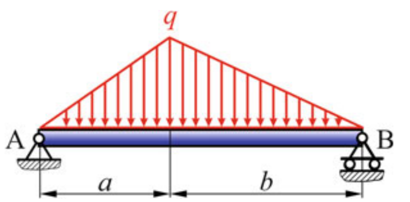


Fig. P9.4

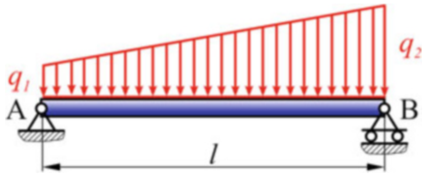


Fig. P9.5

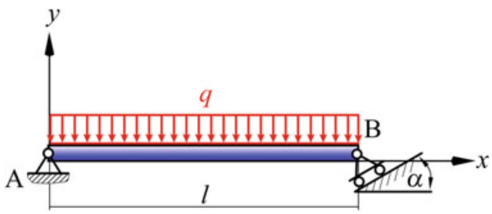


Fig. P9.6

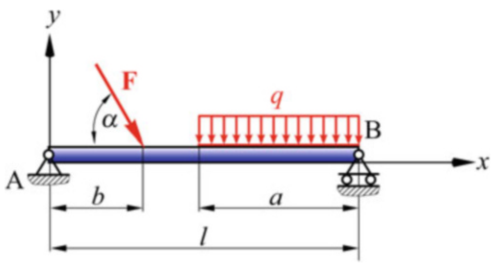


Fig. P9.7

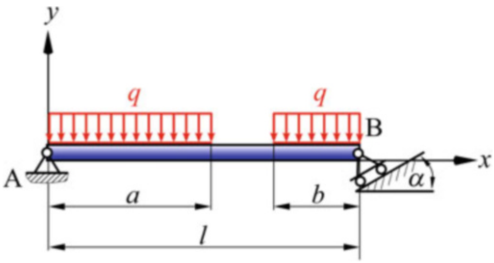


Fig. P9.8

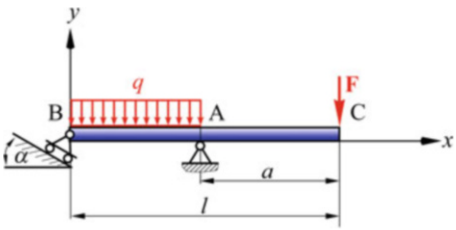


Fig. P9.9

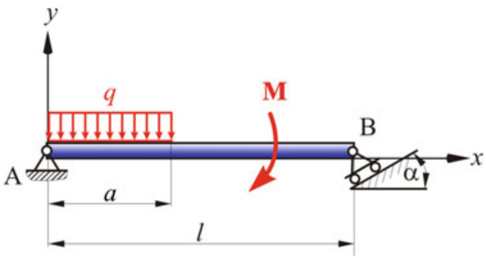


Fig. P9.10

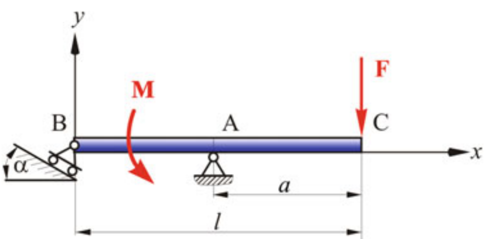


Fig. P9.11

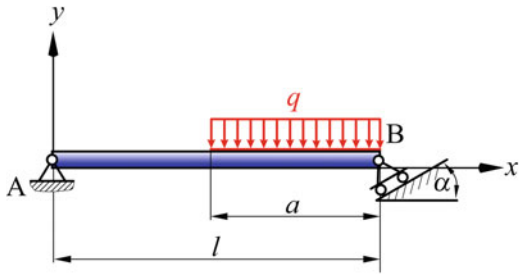


Fig. P9.12

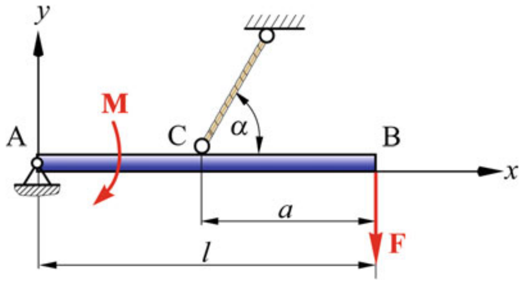


Fig. P9.13

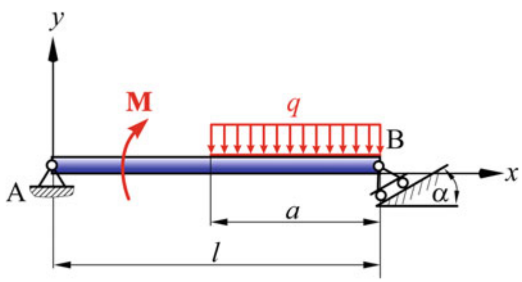


Fig. P9.14

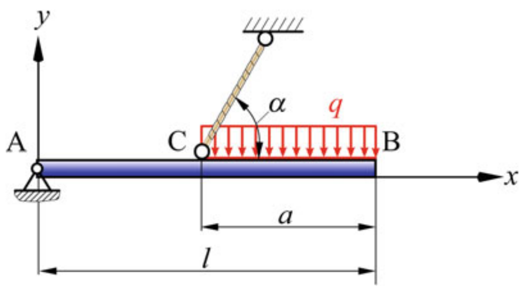


Fig. P9.15

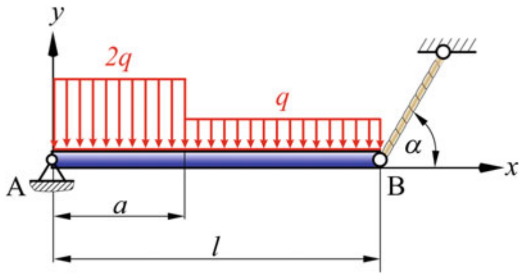


Fig. P9.16

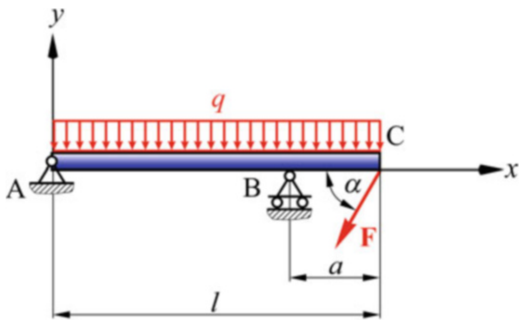


Fig. P9.17

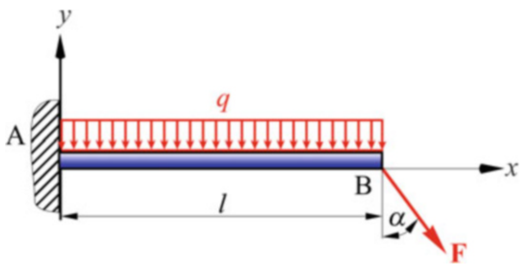


Fig. P9.18

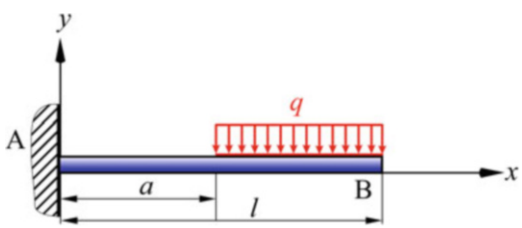


Fig. P9.19

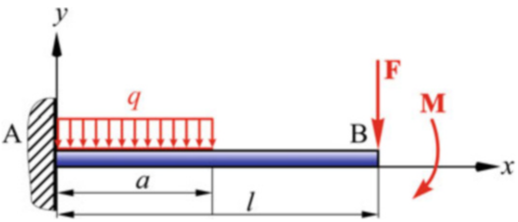


Fig. P9.20

9.21–9.32 Draw diagrams of internal moments and forces. The values of external concentrated forces, distributed forces and moments are given in the table below. Corresponding physical models are numbered accordingly and shown below. Location of the coordinate system is in some cases given and in some it is not.

N	*F* (kN)	*W* (kN)	*M* (kNm)	*q* (kN/m)	*a* (m)	*b* (m)	*c* (m)	*α* (°)
9.21	6	–	3	–	2	3	–	–
9.22	–	–	–	6	2	3	–	–
9.23	4	–	–	6	5	1	–	–
9.24	9	–	–	4	3	3	1.5	45
9.25	10	–	4	–	3	3	2	30
9.26	–	–	10	–	3	5	–	–
9.27	–	–	2	2	3	2	6	30
9.28	2	–	–	4	4	4	–	30
9.29	8	4	5	2	1	–	–	–
9.30	–	12	10	4	1	–	–	–
9.31	6	4	10	4	1	–	–	–
9.32	2	4	6	–	1	–	–	30

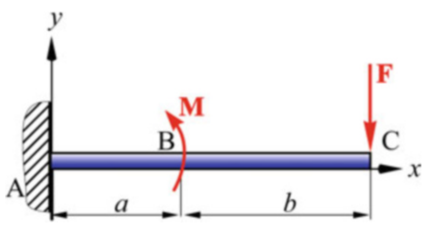


Fig. P9.21

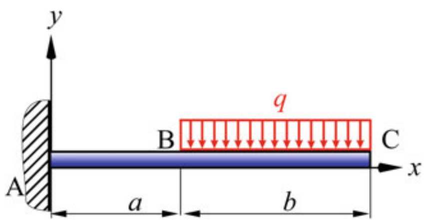


Fig. P9.22

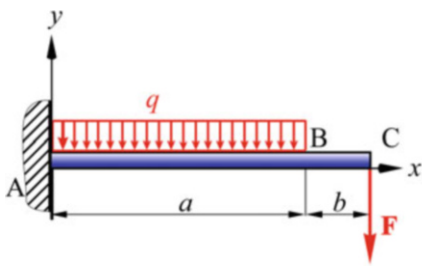


Fig. P9.23

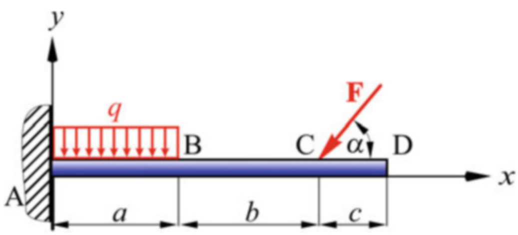


Fig. P9.24

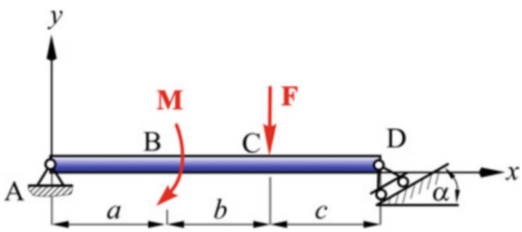


Fig. P9.25

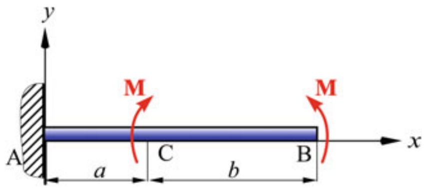


Fig. P9.26

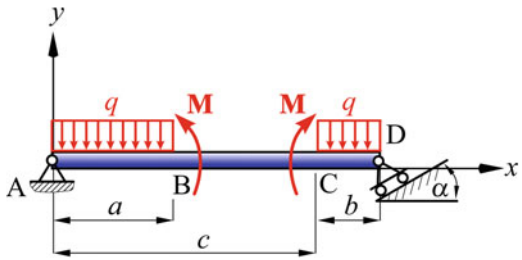


Fig. P9.27

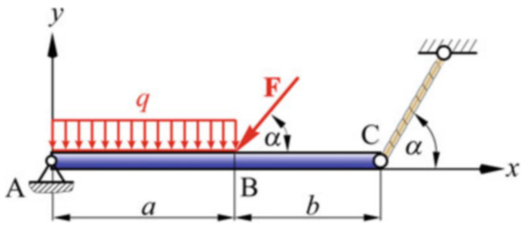


Fig. P9.28

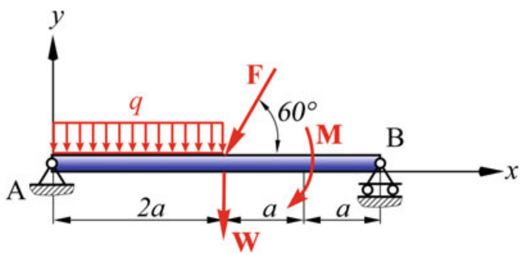


Fig. P9.29

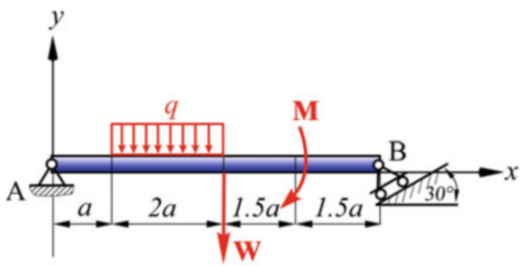


Fig. P9.30

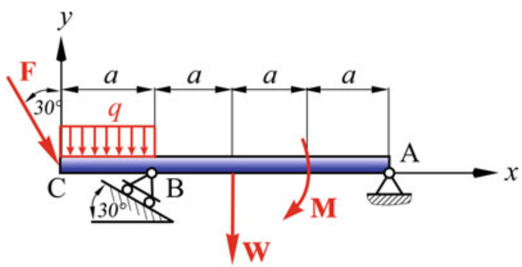


Fig. P9.31

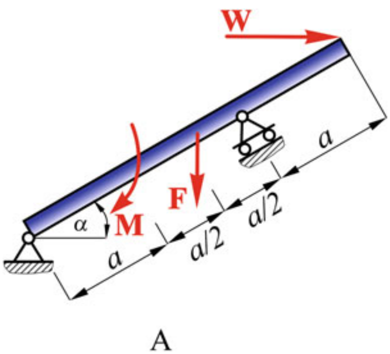


Fig. P9.32

9.3 Curved Beams

In engineering practice, there are structures that cannot be modeled as straight or piece-wise straight beams. In order to solve these structures, their shape (geometry) should be possible to model by an analytical function. Most common are structures that can be modeled as a part of a circle.

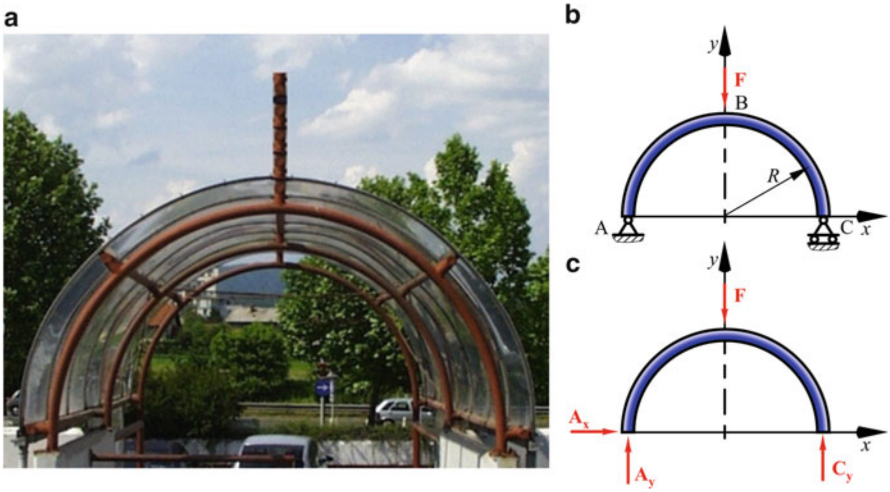


Fig. 9.8 (**a**) Shelter. (**b**) Physical model. (**c**) Free body diagram

Let us consider a curved beam supporting the roof of the shelter shown in Fig. 9.8a, which can be modeled as a half circle with radius R, supported by pin A and roller C, and loaded by vertical force **F** at point B, as shown in Fig. 9.8b.

Figure 9.8c shows the related free body diagram. By writing and solving the equilibrium equations of the external forces, we obtain the reactions:

$$A_x = 0$$
$$A_y = C_y = F/2$$

Since we have only one external force acting at point B, the beam can be divided into two regions: AB and BC.

Region AB

To obtain the internal forces and moment, we need to cut the beam at an arbitrary location within region AB. We need to remember that the coordinate system of internal forces and moments "travels" along the axis of the beam, which means that the direction of internal forces will change from point to point. Since the beam's axis has a shape of a circle, we will use the polar coordinate system, as shown in Fig. 9.9a.

The location of the cut is defined by angle φ. As we know, the internal force may be represented through its two components: normal internal force N, which is tangential to the beam, and the internal transverse force T that is perpendicular to N, i.e., it acts in radial direction. Internal bending moment M is acting counter-clockwise as shown in Fig. 9.9a.

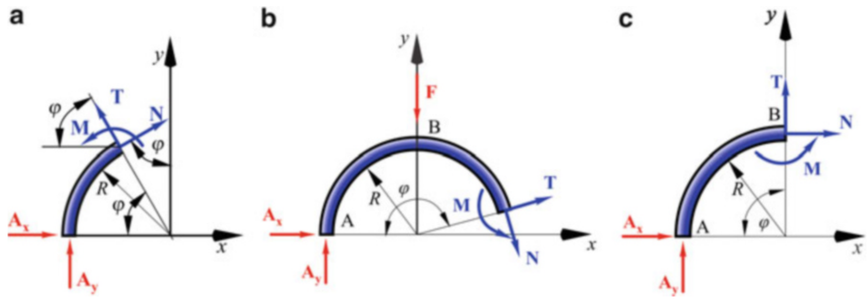


Fig. 9.9 (**a**) Region AB. (**b**) Region BC. (**c**) End of Region AB

The equilibrium equations are

$$\sum F_x = A_x - T\cos\varphi + N\sin\varphi = 0$$
$$\sum F_y = A_y + T\sin\varphi + N\cos\varphi = 0$$
$$\sum M = M + A_x R\sin\varphi - A_y R(1 - \cos\varphi) = 0$$

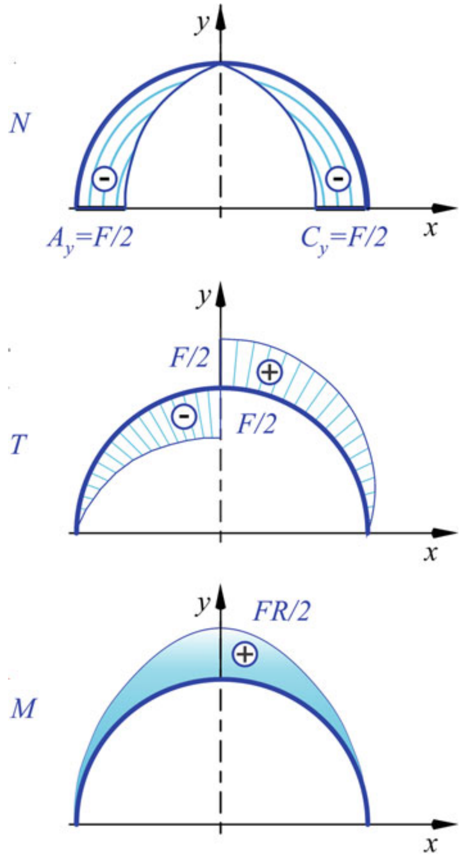
By multiplying the first equation by $\sin\varphi$, second by $\cos\varphi$, then summing them up and substituting values of A_x and $A_{y,}$ we obtain the expression for normal force N. Similarly, by multiplying the first equation by $\cos\varphi$, second by $\sin\varphi$, and subtracting them we obtain expression for shear force T. The expression for internal moment M is obtained simply by substituting values of A_x and A_y. Hence,

$$N = -\frac{F}{2}\cos\varphi$$
$$T = -\frac{F}{2}\sin\varphi$$
$$M = \frac{F}{2}R(1 - \cos\varphi)$$

The expressions above represent the diagrams of the normal and transverse forces, and the bending moment within region AB (Fig. 9.10). Complete region AB is covered when φ changes from 0 to $\pi/2$, hence $\varphi \in [0, \pi/2]$.

The diagrams shown in Fig. 9.10 can also be drawn by using the intuitive approach. Let us start at point A. The internal normal force at point A should be equal and opposite in direction to external force A_y. Since A_y is positive, the internal normal force will be negative. Since there is no distributed load acting within region AB, the internal normal force should be constant, see the *intuitive rule* (iii). However, due to the continuous changing of the beam's geometry, the normal force direction is also changing relative to the direction of the external load. Thus, its magnitude will continuously change (diminish) as well. At point B, direction of the internal normal force will become horizontal; therefore, its magnitude will

Fig. 9.10 Diagrams of axial force, shear force, and bending moment



become zero. Similarly, the shear force is zero at point A and will gradually increase to become $F/2$ at point B. The internal moment is an integral of the shear force (9.18). Since the shear force within region AB is negative, the internal moment will be an increasing function with its maximum at point B where the shear force changes its sign, (9.16), and the *intuitive rule* (vi).

Region BC

For region BC, shown in Fig. 9.9b, we will skip writing the equilibrium equations and draw the diagrams for the internal forces and bending moment using the intuitive approach only.

Let us proceed with the normal (axial) forces first. External reaction force C_y is acting at the end of region BC. It acts in normal (axial) direction. If the beam would be straight, the normal (axial) force would be constant throughout the beam because within both regions AB and BC there is no external force acting in normal (axial) direction. Hence, the magnitude of the internal normal (axial) force is changing merely because of changing geometry of the beam. As a result, the angle between the external forces and the internal normal force is changing along the axis of the beam.

At point B, where the first region AB ends and the second region BC start, the internal normal (axial) force is again (still) zero since there is no external load in horizontal direction. Progressing from point B toward point C, the magnitude of the internal normal (axial) force gradually increases, and keeps the same negative sign, i.e., the same direction of action as in region AB. This has to be so because there is no external "reason" that would change direction of the internal axial force action.

When we reach point C, where the external reaction force C_y is acting, the change in axial force magnitude will be equal to magnitude of C_y. Since C_y acts upward, the related internal force will act in opposite downward direction. This means that the change of the internal force will be positive, see the direction of $N(x)$ in Fig. 9.9b. In addition, at the end of the beam the diagram has to become zero because the sum of all external and internal forces in any direction must be zero.

Let us now proceed with the internal shear forces. At point B, where region AB ends and region BC starts there is an external force acting downward. According to the Third Newton's Law, the corresponding internal force will act upward. The internal coordinate system at the end of region AB is shown in Fig. 9.9c. The change of the internal force, caused by external force **F,** will be therefore positive. Consequently, the internal transverse force at point B changes from negative, $-F/2$, to positive $F/2$ value. Since in region BC there is no distributed load, the transverse internal force should be constant throughout the region. However, due to the changing geometry of the beam the transverse force gradually diminishes and becomes zero at point C. At point C, there are no external forces acting in horizontal direction; hence, related internal force should be also zero.

The internal moment at point B remains the same since there are no external moments acting at this point. Within region BC $dM(x)/dx < 0$ therefore, according to the *intuitive rule* (vii), internal moment $M(x)$ will be a decreasing function and will diminish as we approach point C. At point C, the internal moment is zero since there are no external moments acting at the pin. The final diagrams are shown in Fig. 9.10.

9.4 Piece-Wise Straight and Curved Beams

Geometry of real engineering structures is often complex. However, it usually may be represented as a composition of a number of straight and/or curved beams rigidly attached to one another. A simple example of a structure that can be modeled by piece-wise straight and curved beams is shown in Fig. 9.11a. This structure can be represented by two straight (AB and BC) and one curved (CD) beams rigidly attached to one another. First, we draw a free body diagram, as shown in Fig. 9.11b, and solve for reactions A_x, A_y, and D_y.

Geometry of real engineering structures is often complex. However, it usually may be represented as a composition of a number of straight and/or curved beams rigidly attached to one another.

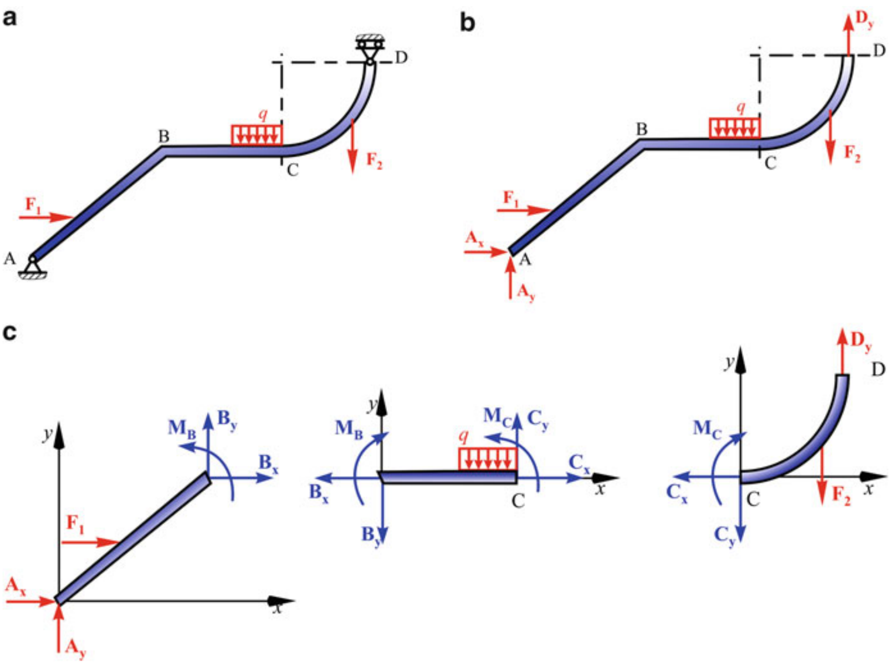


Fig. 9.11 (**a**) Piece-wise straight and curved beam. (**b**) Free body diagram of a piece-wise straight and curved beam. (**c**) Piece-wise presentation of the beam

Now, we cut the structure into three parts AB, BC, and CD as shown in Fig. 9.11c. To keep each part in equilibrium, we have to apply internal forces and moments at the points of the cuts. It should be noted that the internal forces and moment acting on part AB at point B, i.e., B_x, B_y, and M_B, should be equal in magnitude and opposite in direction to the internal forces and moment acting on part BC at the same point B. Similarly, the internal forces and moment acting on part BC in point C, i.e., C_x, C_y, and M_c, should be equal in magnitude and opposite in direction to the internal forces and moment acting on part CD at point C, see Fig. 9.11c.

To solve for unknown forces and moments at the location of the cut, we have to start with the beam that is attached to one of the supports. If we would start with the beam in the middle, it would be impossible to solve for unknown forces and moments because there will be more unknowns than equations of equilibrium.

Now, consider equilibrium of part AB and solve for unknown reactions B_x, B_y, and M_B. Next, consider equilibrium of the second part BC and solve for C_x, C_y, and M_c. The latter could be obtained also from the equilibrium equations for part CD. Thus, these equilibrium equations could be used to check the correctness of the obtained results.

The unknown forces and moments could be obtained simultaneously by writing nine equations of equilibrium for all three parts at once. These nine equations will have nine unknowns, A_x, A_y, B_x, B_y, M_B, C_x, C_y, M_C, and D_y. MATLAB or similar software package may be used to solve such system.

From this point on we can solve for internal forces and moments in each part separately using the rules and procedures developed for the straight and curved beams in Sects. 9.1 and 9.2, respectively.

When solving for the internal forces and moment in each part of a structure, the forces and moments acting at the cuts, should be considered as **external loads**!

When solving for internal forces and moment in each part of a structure, the obtained forces and moments (in this case A_x, A_y, B_x, B_y, M_B, C_x, C_y, M_c, and D_y), acting at the cuts, should be considered as **external loads**.

Example 9.4 Draw diagrams of internal forces and bending moments for the staircase shown in Fig. 9.12a. The weight of a person is $W = 800$ N and the weight of a box is $G = 2000$ N. All dimensions are shown in Fig. 9.12b.

Solution The first step is to construct physical model that will appropriately represent the staircase, Fig. 9.12b. At point A, we assume that the structure is supported by a pin, while at point C by a roller. The effect of the person's weight is represented by concentrated force W and the effect of the box by uniformly distributed load $q = G/0.8$ N/m. The related free body diagram of the staircase is shown in Fig. 9.12c. From the equilibrium equations we find the reactions:

$$A_x = 0\,\text{N}$$
$$A_y = 800\,\text{N}$$
$$C_y = 2000\,\text{N}$$

Now, we split the structure in two parts: AB and BC, as shown in Fig. 9.12d, and add unknown internal forces and moments, B_x, B_y, and M_B, to keep each part in equilibrium.

We can obtain the unknown forces and moment B_x, B_y, and M_B either from the equilibrium equations for part AB or for part BC. Let us consider the part BC:

$$\sum F_x = -B_x = 0$$
$$\sum F_y = -B_y - G + C_y = 0$$
$$\sum M = -M_B - 0.6 \cdot G + 1 \cdot C_y = 0$$

The sum of the moments was taken about point B. Solving for unknown values leads to

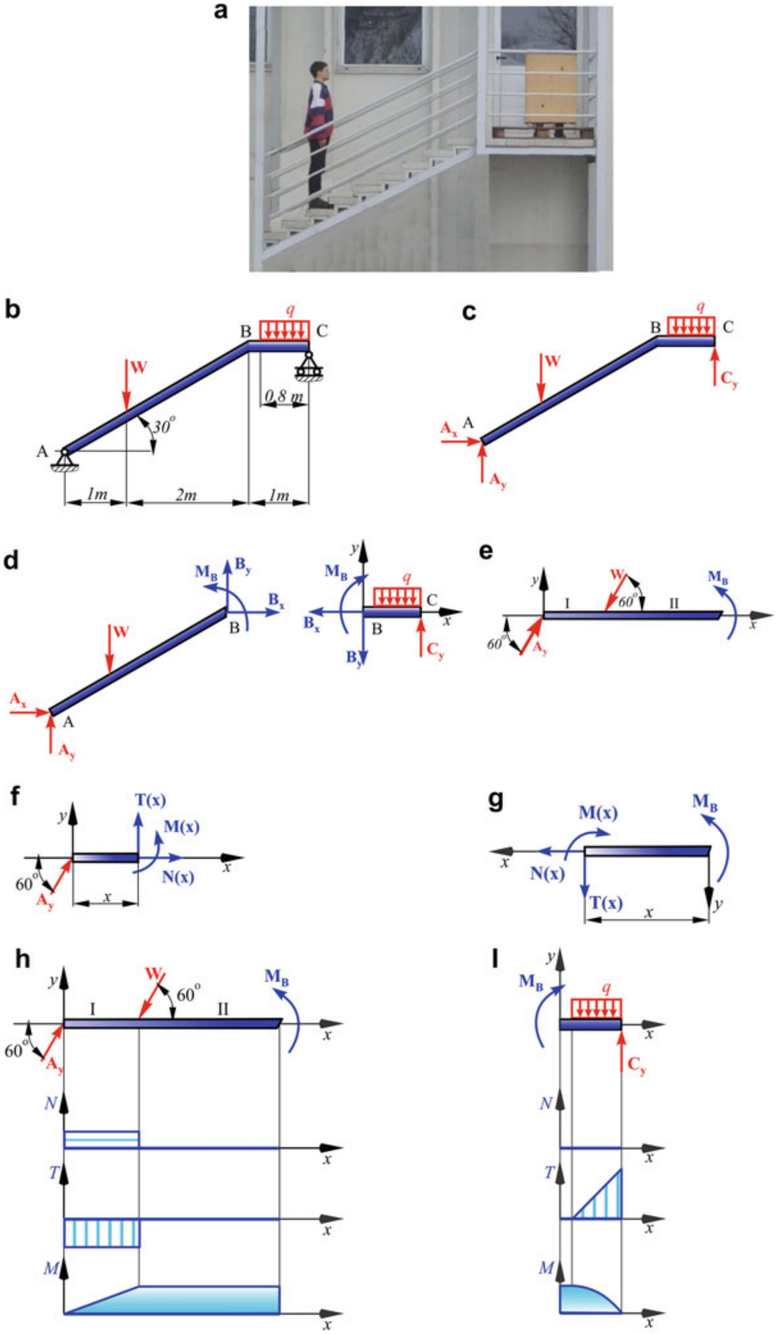


Fig. 9.12 (**a**) Staircase loaded by a person and a box. (**b**) Physical model of the staircase. (**c**) Free body diagram of the staircase. (**d**) Free body diagrams of the parts AB and BC. (**e**) Free body diagram of part AB. (**f**) Free body diagram of Region I (part AB). (**g**) Free body diagram of Region II (part AB), analyzed from the right hand side. (**h**) Distributions of internal forces and moment in part AB of the staircase. (**i**) Diagrams of normal and transverse forces and bending moments in part BC

$$B_x = 0$$

$$B_y = 0 \text{ and}$$

$$M_B = 800 \text{ N m}$$

Interestingly, we find that for these loading conditions at point B there are no forces in axial and transverse direction.

Now, we are ready to start solving for internal forces and moments in both parts.

Part AB

Since we are looking for the distribution of normal and shear forces along the beam's axis, we will orient the coordinate system along the longitudinal axis of part AB. Part AB has two regions, i.e., Region I and Region II, as shown in Fig. 9.12e.

To obtain the distribution of the internal forces and moment, we need to cut the beam in each of the two regions.

Within Region I, we cut the beam at distance x, as shown in Fig. 9.12f.

The equations of equilibrium are

$$\sum F_x = A_y \cos 60 + N = 0$$

$$\sum F_y = A_y \sin 60 + T = 0$$

$$\sum M = M - x \cdot A_y \sin 60 = 0$$

When solving the right part of a beam, we need to flip the directions of the internal forces and moment.

However, we should NOT change the direction of the external forces and moments.

Solving for unknown forces and moment yields

$$N = -400\,\text{N}$$

$$T = -693\,\text{N and}$$

$$M = 693x\,\text{N m}$$

Next, we cut the beam at any point within Region II. For a change, we will consider the right part of the beam, Fig. 9.12g. In order to do that, we need to invert the coordinate system of the internal forces and moment as explained in Sect. 9.1. The z-axis is now pointing into the plane, which means that the positive moment has the direction of a clock.

We need to stress that direction of external forces and moments remains unchanged! Hence, in this case the direction of M_B will remain unchanged!

The equilibrium equations for Region II are:

$$\sum F_x = N = 0$$

$$\sum F_y = T = 0$$

$$\sum M = +M - M_{\mathrm{B}} = 0$$

Solving for unknown internal forces and moment yields

$$N = 0\,\mathrm{N}$$

$$T = 0\,\mathrm{N} \;\; \text{and}$$

$$M = 800\,\mathrm{N\,m}$$

Diagrams showing the distributions of the internal forces and moment in part AB of the staircase are given in Fig. 9.12h.

Intuitive Drawing of the Internal Forces and Moment in Part AB
Diagrams of the internal forces and moment could be drawn without explicitly writing equilibrium equations, but simply by using intuitive procedures outlined in Sect. 9.2.2.

We start the intuitive drawing at the left hand side of beam AB in Region I. At point A, the external force A_y has two components (i) one in axial (normal) direction, $A_y \cdot \cos 60^\circ$, and another, (ii) in transverse (shear) direction, $A_y \cdot \sin 60^\circ$. According to the Third Newton's Law, the internal forces at this point should be equal in magnitude and opposite in direction to these two external forces. The external axial force $A_y \cdot \cos 60^\circ$ acts in positive direction; hence, the corresponding internal axial force will be negative. Similarly, it is true for the transverse (shear) force. The external force $A_y \cdot \sin 60^\circ$ acts in positive direction; hence, the related internal shear force will be negative. Within Region I, there are no external distributed loads; therefore, the internal axial and shear force will be constant. The internal moment at point A will be zero because the beam is supported by a pin. Internal shear force within Region I is constant; hence, according to the *intuitive rule* (ix), the internal moment will change as a linear function (for details, see Sect. 9.2.2). Since $dM/dx = -T(x) > 0$ we find that $M(x)$ needs to be an increasing function.

We will start the analysis of the second region at the right hand side of the beam, at point B and proceed to the left. This means that the coordinate system should be inverted as discussed above. In this case, the z-axis points into the plane, which means that the positive moments act clockwise.

Since there are no forces at point B, both internal forces (axial and shear) are zero. The internal moment at point B will be equal in magnitude and opposite in direction as moment M_{B}. In the inverted coordinate system, M_{B} is negative; therefore, the related internal moment will be positive. The shear force in Region II is zero, according to the *intuitive rule* (viii) (Sect. 9.2.2). $M(x)$ will be constant, as shown in Fig. 9.12h. The diagrams for N, T, and M for part AB are given in Fig. 9.12h.

Part BC
To draw the diagrams of the internal forces and moment for Part BC, we will use the intuitive approach only. There are two regions: Region I, extending from the left end up to the beginning of distributed load, and Region II, covering section of the beam loaded with a distributed load, as shown in Figs. 9.12d, and i.

We will analyze Region I from left to right. Since $B_x = 0$ and $B_y = 0$, both internal forces (shear and normal) are zero, $T = 0$ and $N = 0$. Moment M_B (considered as an external moment) is negative (it acts clockwise); the internal moment at point B is therefore positive and constant within the region. This is so according to the *intuitive rule* (viii) because the shear force within the region is zero, i.e., $dM/dx = -T(x) = 0$.

We will continue analyzing the internal forces and moments in part BC from left toward right.

Within Region II, the internal normal force will remain zero because there are no new axial forces acting at the beam. However, in transverse (shear) direction Region II is loaded with constant distributed load $\boldsymbol{q}$. Therefore, $dT/dx = q = \text{const.} > 0$, and according to the *intuitive rules* (i) and (iv) described in Sect. 9.2.2, the transverse force will change linearly as an increasing function up to point C where the external force C_y acts. According to the Third Newton's Law, the related change of the internal shear force at point C should be equal to C_y and opposite in sign. The external force C_y is positive, the corresponding internal force will be therefore negative and the diagram of internal shear forces will go down to zero, as it should be, because the sum of all the external and internal forces acting on a beam should be equal to zero.

The internal moment at the beginning of Region II should have the same value as the moment at the end of Region I. This is so because there are NO external moments at this location, which would cause a discontinuous "jump" in the distribution of the internal moments. Similarly, the internal moment at the end of the beam (point C) should be equal to zero since there is no external moment acting at this point. Within Region II, the moment is equal to the integral of the transverse force, and opposite in sign, thus it should decrease as a parabola. The diagrams of the internal forces and moment are shown in Fig. 9.12i.

9.4.1 Problems

9.33–9.46 Draw the diagrams of internal moments and forces. The values of external concentrated forces, distributed forces and moments are given in the table below. Related physical models are numbered accordingly and shown below.

n	F (kN)	W (kN)	M_0 (kN m)	q (kN/m)	a (m)	α (°)
9.33	10	–	6	5	1	–
9.34	6	–	4	4	1	–
9.35	2	–	4	1	0.25	–
9.36	4	6	8	4	1	–
9.37	6	4	2	–	1	30
9.38	6	–	8	2	1	45
9.39	8		10	4	1	30
9.40	4	–	3	2	1	30
9.41	2	–	6	4	1	30
9.42	6		8	4	1	45
9.43	4	–	6	2	1	45
9.44	6	–	10	2	1	30
9.45	4	–	2	2	1	30
9.46	6	–	8	4	1	20

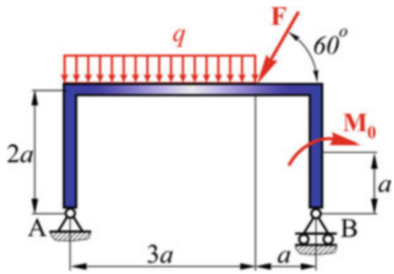


Fig. P9.33

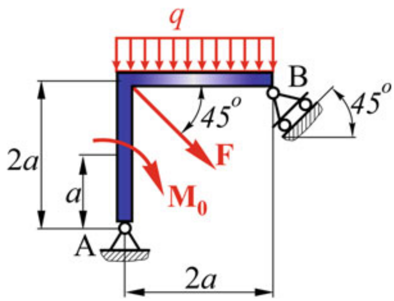


Fig. P9.34

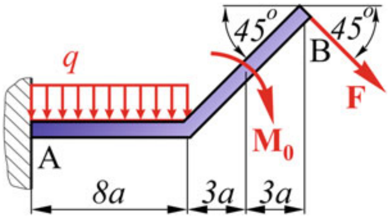


Fig. P9.35

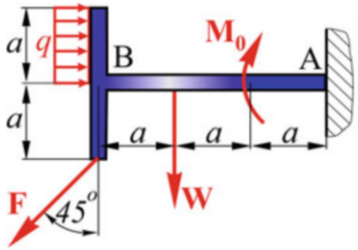


Fig. P9.36

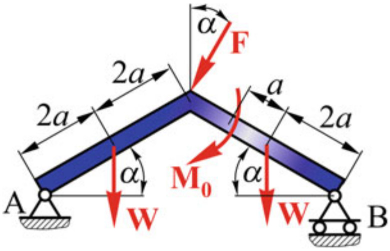


Fig. P9.37

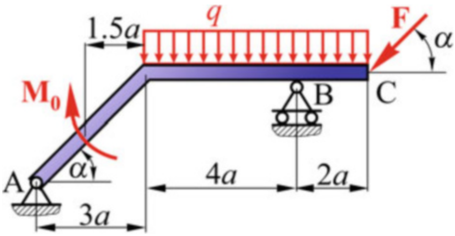


Fig. P9.38

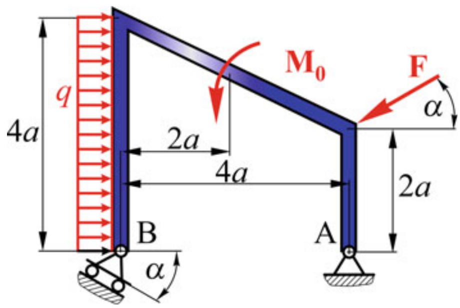


Fig. P9.39

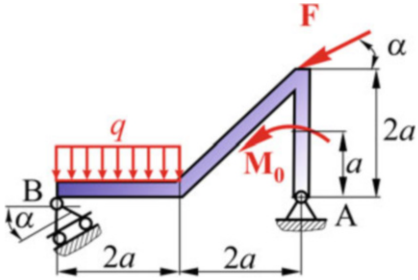


Fig. P9.40

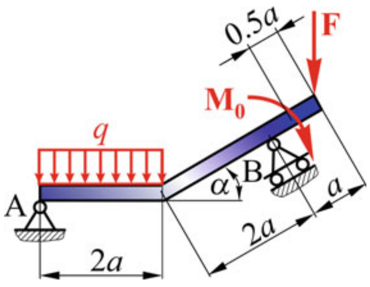


Fig. P9.41

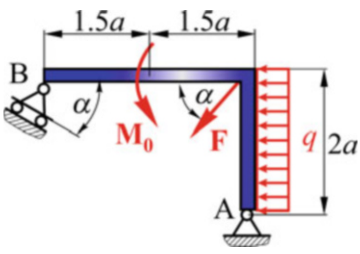


Fig. P9.42

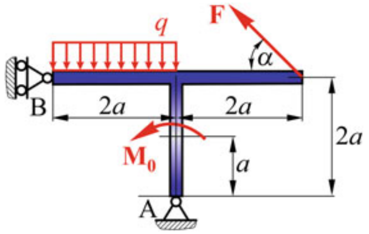


Fig. P9.43

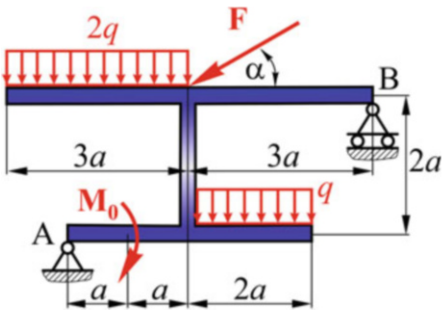


Fig. P9.44

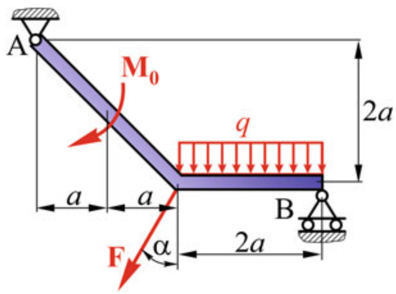


Fig. P9.45

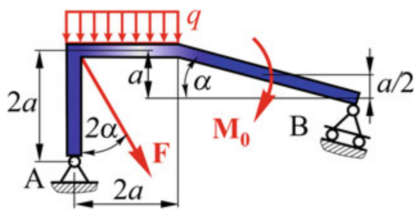


Fig. P9.46

9.5 Superposition Principle

Distribution of internal forces and moments in beams loaded by only one concentrated or distributed load may be easily obtained by using the intuitive approach, or else, using preprepared solutions for common types of beams and loading cases. However, when the number of external loads is large the determination of the internal forces and moments could become a tedious job. In this chapter, we present an approach called superposition principle, which to some extend simplifies the solving procedure.

The number of external forces and moments acting on a structure may be viewed as a sum of individual loads and moments each acting on a structure separately. Since we are dealing here with rigid structures, it is possible to use the principle of superposition to determine the internal forces and moments resulting from the large number of loads by summing up the internal forces and moments resulting from each individual load separately. Preprepared solutions for a several different loading cases are shown in Table 9.1. For each case the corresponding reactions and internal forces and moments are given in Table 9.2.

We demonstrate applicability of the superposition principle on an example of a beam loaded by concentrated force **F** and uniformly distributed load **q**, shown in Fig. 9.13a.

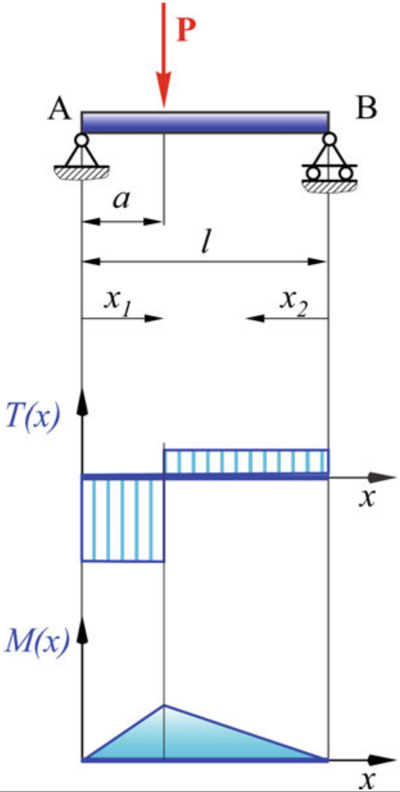
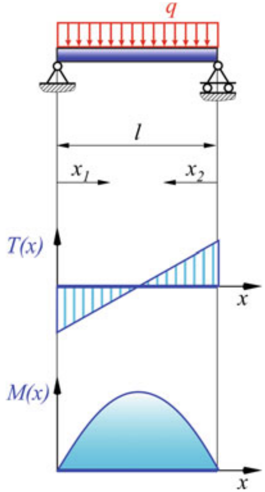
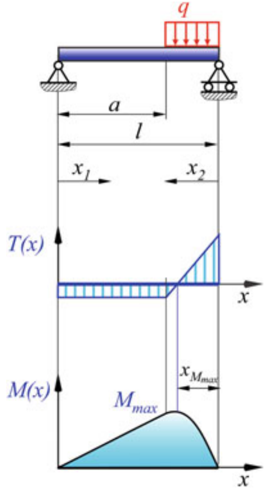
The loaded beam may be viewed as a sum of two beams of the same length and geometry (Fig. 9.13b), one loaded by the concentrated force **F**, and another by the distributed load **q**, as shown in Fig. 9.13b.

For each of the two beams, we will draw the distribution of the internal forces and moments using the preprepared solutions Case 1 and Case 2, presented in Table 9.1. Instead, one could easily draw these "basic" diagrams of the internal forces and moments using the intuitive approach introduced in Sect. 9.2.2.

The left part of Fig. 9.13c shows the distribution of the internal forces and moment arising from the concentrated load, while the central part of Fig. 9.13c shows the same information caused by the distributed load. According to the principle of superposition, we can add the diagrams of internal shear and normal forces caused by the concentrated load **F** to the corresponding diagrams belonging to the distributed load q. In this case, there are no normal (axial) forces; hence, we need to deal with shear forces only. The resulting distribution of the shear forces is obtained by point-by-point summing up the two shear force distributions. The result is shown on the right side of Fig. 9.13c. We draw the diagram of the internal moments again by adding the internal moments caused by the concentrated load and those generated by the distributed load at each selected location. The result is again shown on the right hand side of Fig. 9.13c. It should be noted that the resulting diagrams of the internal moment and forces were drawn by assuming that the $q \cdot a < F$ (See Case 1 and Case 2 in Tables 9.1 and 9.2).

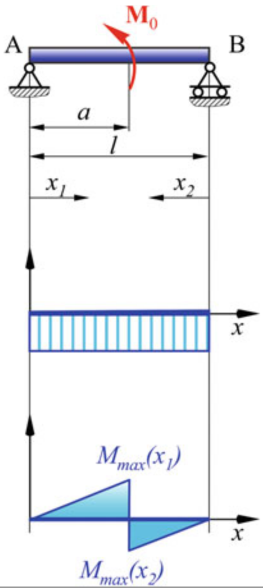
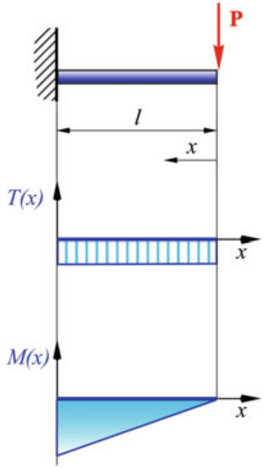
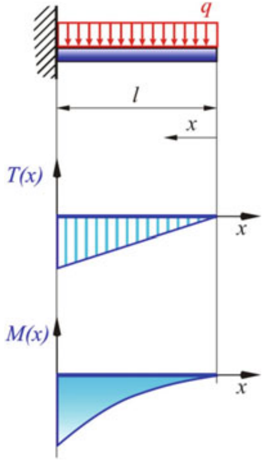
Example 9.5 Consider a beam loaded by two forces $\mathbf{P_1}$ and $\mathbf{P_2}$ and distributed load q (Fig. 9.14a). Find the distribution of internal forces and moments using principle

Table 9.1 Various loading and beams configurations

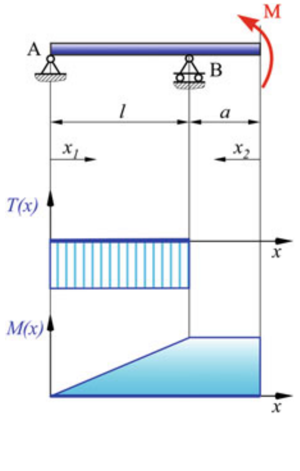
Case 1	Case 2	Case 3
		

(continued)

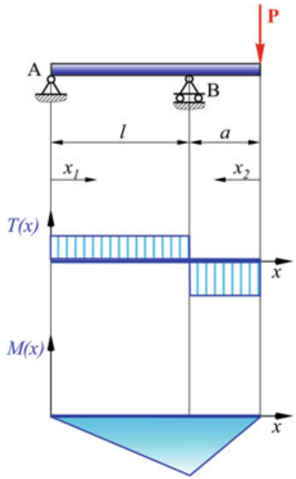
Table 9.1 (continued)

Case 4	Case 5	Case 6
		

Case 7



Case 8



Case 9

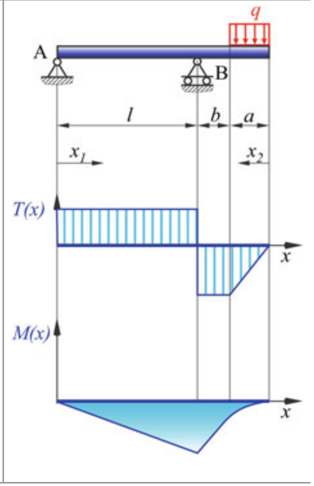


Table 9.2 Reactions, internal forces and moments corresponding to Cases shown in Table 9.1

Case	Reactions	Region 1	Region 2	Region 3
1	$F_A = P\frac{(l-a)}{l}$ $F_B = P\frac{a}{l}$	$T(x_1) = -P\frac{(l-a)}{l}$ $M(x_1) = P\frac{(l-a)}{l}x_1$ $M_{max} = P\frac{(l-a)}{l}a$	$T(x_2) = P\frac{a}{l}$ $M(x_2) = P\frac{a}{l}x_2$	
2	$F_A = \frac{ql}{2}$ $F_B = \frac{ql}{2}$	$T(x_1) = -\frac{ql}{2} + qx_1$ $M(x_1) = \frac{ql}{2}x_1 - qx_1\frac{x_1}{2}$ $M_{max} = \frac{ql^2}{8}$	$T(x_2) = \frac{ql}{2} - qx_2$ $M(x_2) = \frac{ql}{2}x_2 - qx_2\frac{x_2}{2}$	
3	$F_A = \frac{q(l-a)^2}{2l}$ $F_B = \frac{q(l-a)\left(\frac{l}{2}+\frac{a}{2}\right)}{l}$	$T(x_1) = -\frac{q(l-a)^2}{2l}$ $M(x_1) = \frac{q(l-a)^2}{2l}x_1$	$T(x_2) = \frac{q(l-a)(l+a)}{2l} - qx_2$ $M(x_2) = \frac{q(l-a)(l+a)}{2l}x_2 - qx_2\frac{x_2}{2}$ $M_{max} = \frac{q(l^2-a^2)^2}{8l^2}$ $x_2(M_{max}) = \frac{l^2-a^2}{2l}$	
4	$F_A = \frac{M_0}{l}$ $F_B = -\frac{M_0}{l}$	$T(x_1) = -\frac{M_0}{l}$ $M(x_1) = \frac{M_0}{l}x_1$ $M_{max}(x_1) = M_0\frac{a}{l}$	$T(x_2) = -\frac{M_0}{l}$ $M(x_2) = -\frac{M_0}{l}x_2$ $M_{max}(x_2) = -M_0\frac{l-a}{l}$	

5	$F_{\mathrm{A}} = P \;\; M_{\mathrm{A}} = Pl$	$T(x) = -P \;\; M(x) = P \cdot (x-1)$ $M_{\max} = -Pl$		
6	$F_{\mathrm{A}} = ql$ $M_{\mathrm{A}} = q\frac{l^2}{2}$	$T(x) = -q(x-1)$ $M(x) = -q\frac{(x-1)^2}{2}$ $M_{\max} = -q\frac{l^2}{2}$		
7	$F_{\mathrm{A}} = \frac{M}{l}$ $F_{\mathrm{B}} = -\frac{M}{l}$	$T(x_1) = -\frac{M}{l}$ $M(x_1) = \frac{M}{l}x_1$	$T(x_2) = 0$ $M(x_2) = M$ $M_{\max} = M$	
8	$F_{\mathrm{A}} = -P\frac{a}{l}$ $F_{\mathrm{B}} = P\frac{(l+a)}{l}$	$T(x_1) = P\frac{a}{l}$ $M(x_1) = -P\frac{a}{l}x_1$ $M_{\max} = -Pa$	$T(x_2) = -P$ $M(x_2) = -Px_2$	
9	$F_{\mathrm{A}} = -\frac{qa\left(\frac{a}{2}+b\right)}{l}$ $F_{\mathrm{B}} = \frac{qa\left(\frac{a}{2}+b+l\right)}{l}$	$T(x_1) = \frac{qa\left(\frac{a}{2}+b\right)}{l}$ $M(x_1) = -\frac{qa\left(\frac{a}{2}+b\right)x_1}{l}$ $M_{\max} = -q \cdot a \cdot \left(\frac{a}{2}+b\right)$	$a \le x_2 \le a+b$ $T(x_2) = -qa$ $M(x_2) = -qa\left(x_2 - \frac{a}{2}\right)$	$0 \le x_2 \le a$ $T(x_2) = -qx_2$ $M(x_2) = -qx_2\frac{x_2}{2}$

of superposition. Assume that $P_1 = 10$ kN, $P_2 = 20$ kN, $q = 2$ kN/m, $L_1 = 2$ m, $L_2 = 3$ m, and $L_3 = 1$ m.

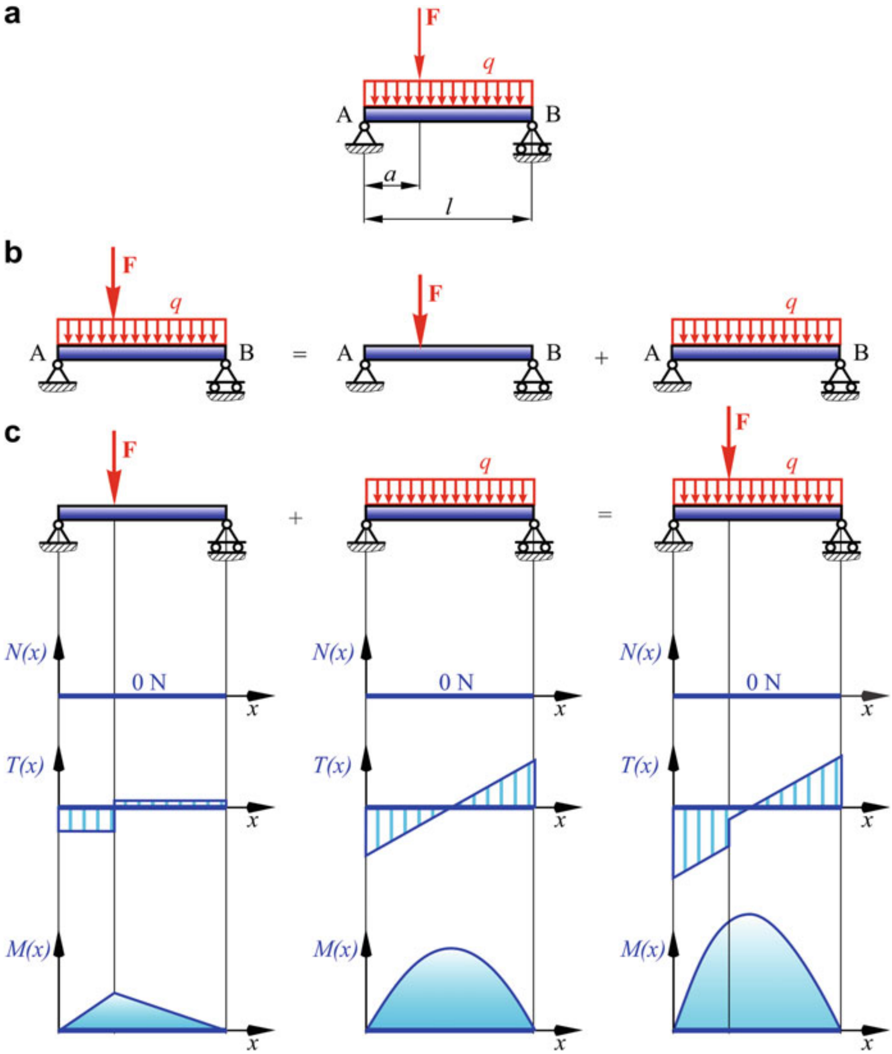


Fig. 9.13 (**a**) A beam loaded by concentrated and distributed loads. (**b**) Superposition of two loads. (**c**) Diagrams of the internal forces and moments for two loads

Solution Loading of the beam can be represented as a sum of three single loads, two concentrated loads and one distributed load, as shown in Fig. 9.14b. Distributions of the internal forces and moment for each of this simple cases may be obtained by using the *intuitive approach*, or simply by using preprepared solutions shown in Tables 9.1 and 9.2. We will use the latter approach.

Solutions for the internal forces and moment for each individual loading we find in Tables 9.1 and 9.2. The result is shown as a sum of two Case 1 and one Case 2 loads on the left of Fig. 9.14c. Now, we can add the corresponding diagrams for the normal forces, shear forces, and bending moments, respectively. The resulted "cumulative" diagrams are shown on the right of Fig. 9.14c.

Example 9.6 Consider a cantilever beam loaded by two concentrated loads $\mathbf{P_1}$ and $\mathbf{P_2}$ and distributed load q (Fig. 9.15a). Find the distribution of the internal forces and moment using the principle of superposition.

Use $P_1 = 10$ kN, $P_2 = 20$ kN, $q = 5$ kN/m. Force P_1 makes angle of 60° with the beam's axis.

Solution Loading can be represented as a sum of individual loads, as shown in Fig. 9.15b. Force $\mathbf{P_1}$ is represented as a sum of its vertical $P_{1v} = P\sin 60°$, and horizontal $P_{1h} = P\cos 60°$ components.

For the three loading cases (first, second, and fourth), we find solutions for internal forces and moment in Tables 9.1 and 9.2. Distributions are shown in Fig. 9.15c. However, for the third loading case, where the cantilever beam is loaded with external force P_{1h} acting in axial direction, we do not find the preprepared solution. Therefore, we will use the *intuitive approach* to draw related diagrams of the internal forces and moment.

Since there is only one external force P_{1h} acting to the left (i.e., it has negative sign), there will be only one (positive) reaction force A_x, acting to the right. The two forces have the same magnitude. Since there are no external moments and external shear forces, there are no internal shear forces and no internal moment. The only internal force is a normal force acting in axial direction. The external reaction force in point A is positive; therefore, the corresponding internal axial force is negative and its magnitude equal to A_x, as shown in Fig. 9.15c. The internal axial force is constant up to the point where negative external force P_{1h} acts. This force generates a positive change in the internal axial force and brings the diagram to zero. From this point on there are no external forces therefore all internal forces must be zero. The result is shown in Fig. 9.15c.

Now, we are ready to add the individual diagrams for the normal forces, transverse forces, and bending moment in order to obtain the solution for our problem. The resulted diagrams are shown in Fig. 9.15d.

Example 9.7 Consider a beam with an overhang loaded by concentrated load $\mathbf{P}$, distributed load q, and moment $\mathbf{M}$ (Fig. 9.16a). Find the distribution of internal forces and moments using principle of superposition. Use $P = 10$ kN, $M = 20$ kN m, $q = 5$ kN/m.

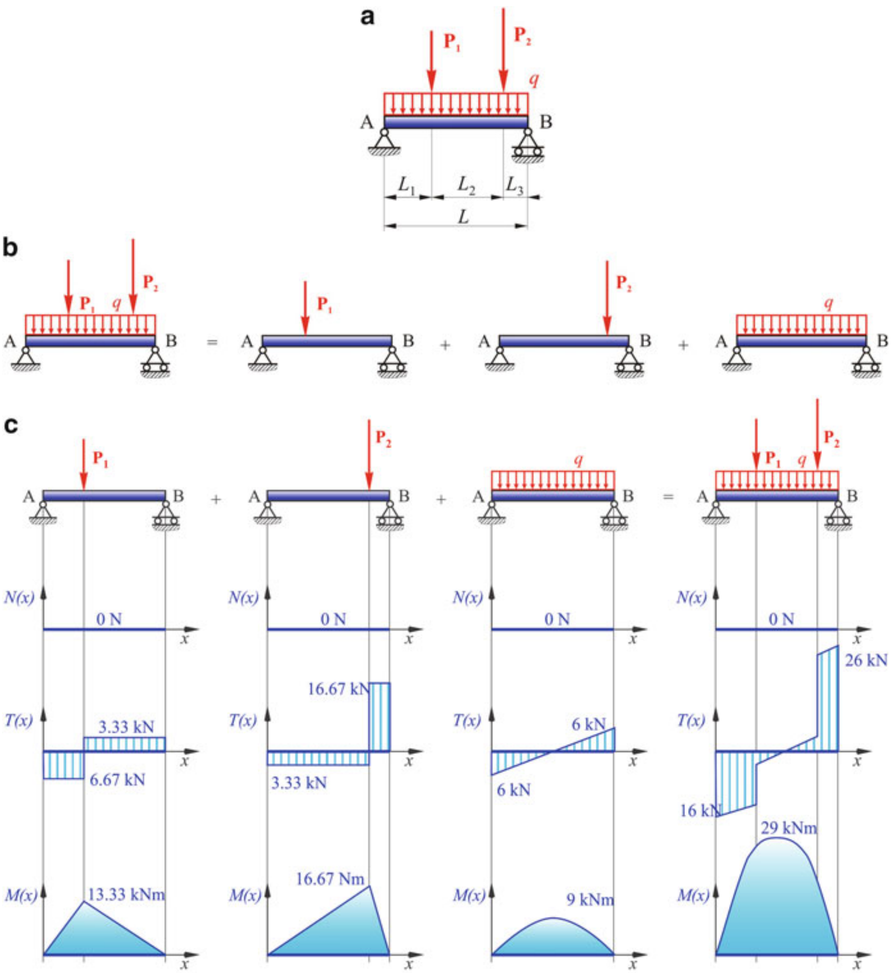


Fig. 9.14 (**a**) Physical model of the beam. (**b**) Superposition of loads. (**c**) Diagrams of the internal forces and moments

Solution The beam is exposed to the load consisting of the concentrated load, moment, and distributed load. Since Table 9.1 does not show the distributed load similar to the one in Fig. 9.16a, one can compose this loading by using a sum of two distributed loads as shown in Fig. 9.16b.

Diagrams of the internal forces and moment for each case, and distributions of the internal forces and moment for the case when all loads act simultaneously, are shown in Fig. 9.16c.

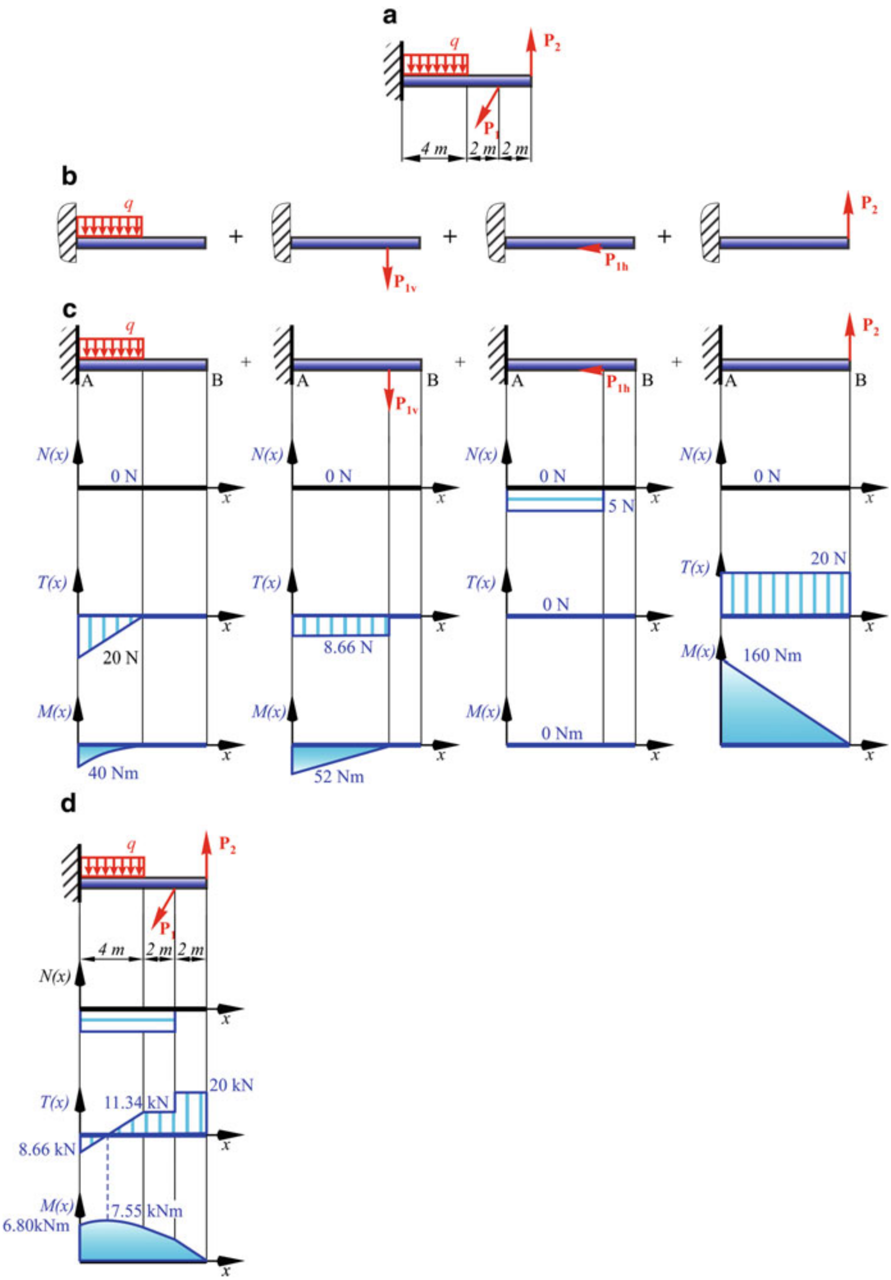


Fig. 9.15 (**a**) Cantilever beam. (**b**) Superposition of four loads. (**c**) Diagrams of the internal forces and moments for each load. (**d**) Diagrams of the internal forces and moments

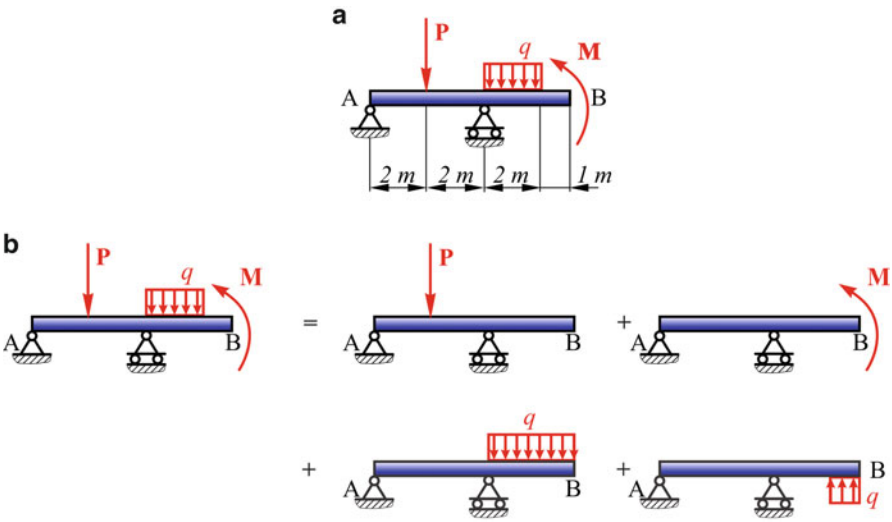


Fig. 9.16 (**a**) Beam with the overhang. (**b**) Superposition of the external loads. (**c**) Diagrams of the internal forces and moments

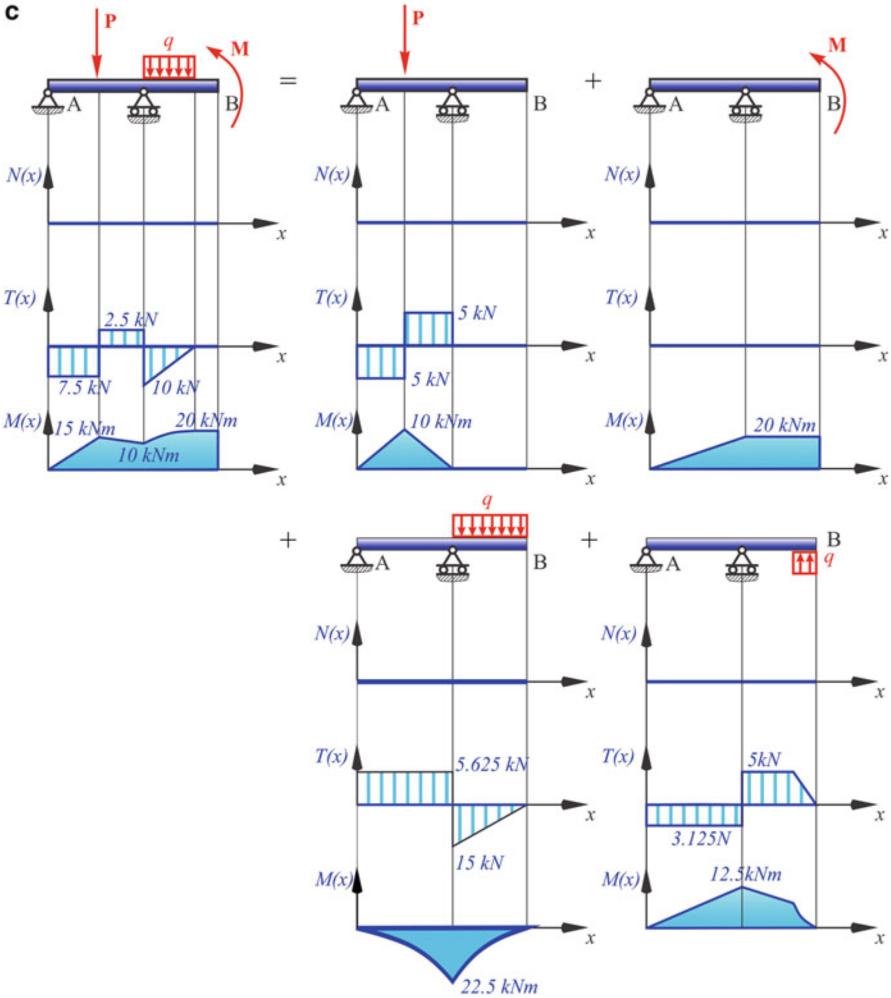


Fig. 9.16 (continued)

Guidelines and Recipes for Drawing Diagrams

- Whenever an external force or moment is acting on a rigid body, the change of the corresponding internal force or moment should be of the same magnitude and have opposite sign, i.e., should act in the opposite direction.
- Region is a segment of a beam within which there are no discontinuities in external loads and/or the beam's geometry.
- An internal force within a region (shear and/or normal) is constant if there is no continuous load in the same direction.

(continued)

- If a region is loaded with a constant continuous load, then the corresponding internal force will change linearly. If $dT(x)/dx = q(x) > 0$, then $T(x)$ will be an increasing function, whereas if $dT(x)/dx = q(x) < 0$, then $T(x)$ will be a decreasing function.
- A type of the function describing the distribution of shear forces is given with the integral of the function describing distributed load.
- The internal moment within a region will be an increasing function when related internal shear force is negative, $T(x) < 0$, whereas it will be a decreasing function when $T(x) > 0$.
- When distributed load, $q(x)$, is constant (and acts downward), the shear force is a linearly increasing function, and the moment diagram is parabolic. Similarly, when $q(x)$ acts upward the shear force will be a decreasing function, i.e., it will have a negative slope and moment diagram will be parabolic.
- If a distributed load can be represented as a first-order function (linear function), the shear force diagram will be represented by a function of the second order and the moment diagram by a function of the third order. The same logic follows for the higher order of the distributed load.

What We Have Learned?

How to calculate internal forces and moments in straight beams
After determining reactions, we divide the beam into regions within which there are no discontinuous changes in the external loading. Next, we cut the beam anywhere within each region. To keep the part in equilibrium, we add at the cut unknown internal normal and shear forces and moment. Writing and solving equilibrium equations results in expressions for distribution of the internal forces and moment within that region.

How to calculate internal forces in a piece-wise straight or curved beam
When a structure consists of a number of straight and/or curved parts, we divide it into basic elements: straight and curved. At the point of division, we add the appropriate internal forces and moments. These forces and moments are determined from corresponding equilibrium equations. From now on the internal forces and moments in each part are determined as described above.

How to create diagrams of bending moments and internal forces using an intuitive approach
There are few basic rules that allow us to draw diagrams of internal forces and moment without explicitly writing equilibrium equations for the internal forces and moments. We are using the Third Newton's Law, which means that the change in the internal forces is always opposite in direction and equal in magnitude to the acting external force. We can draw the diagrams of internal forces and bending moments using the following rules.

- Whenever a concentrated external force is acting, it causes a change of the internal force equal in magnitude and opposite in direction to that force.
- Within a region of a beam where no distributed load is acting all internal forces will be constant.
- If there is a uniformly distributed load within a region of a beam, the corresponding internal forces will change linearly.
- Whenever there is an arbitrary distributed external load within a region of a beam, the corresponding internal force will change as an integral of this load.
- The slope of the diagram of bending moments is opposite in sign to the corresponding transverse force.

How to use superposition principle to draw the diagrams of internal forces and moments
Represent a complex external load as a sum of simpler loads for which the distribution of the internal forces and moments is known, such as shown in Table 9.1. Add-up (point-by-point) the corresponding diagrams for internal forces and moments to get the resultant diagram.

9.6 Problems

9.47 Calculate and draw distributions of internal forces and bending moment on the roof of a bus stop shelter (Fig. P9.47).

(a) Assume that the width of the roof is 1.5 m and its weight is 900 N. The weight of the gutter (500 N) is applied at the right hand side of the roof.
(b) Assume that the width of the roof is 5 ft and its weight is 200 lb. The weight of the gutter (100 lb) is applied at the right hand side of the roof.

Fig. P9.47 Bus stop shelter

9.48 Use an intuitive approach (do not write equations for each region) to solve the problem shown in Fig. P9.47.

9.49 Calculate and draw distributions of the internal forces and bending moments in the vertical support of the bus stop shelter for problem shown in Fig. P9.47.

9.50 Use an intuitive approach (do not write equations for each region) to solve the problem shown in Fig. P9.51.

9.51 Calculate and draw distributions of the internal forces and bending moment in a beam which supports the roof of a garage, Fig. P9.51. Make the appropriate assumptions about the beam's supports.

(a) Assume that the width of the garage is 4 m and the weight of the roof is 10,000 N. The 400 N box is suspended at 1.5 m from the right side of the garage (Fig. P9.51).

(b) Assume that the width of the garage is 12 ft, weight of the roof is 2000 lb. The 100 lb box is suspended at 5 ft from the right side of the garage.

Fig. P9.51 Roof of a garage

9.52 Use an intuitive approach (do not write equations for each region) to solve the problem shown in Fig. P9.53.

9.53 Calculate the internal forces and bending moment in a bridge, Fig. P9.53, if a heavy rain would wash away the supporting ground on both ends of the bridge. Assume that in such case the bridge is supported by two columns only.

(a) The weight of the bridge is 5000 kN and of a truck is 500 kN. Assume that the distance between the supporting columns is 10 m and the left and right overhangs are 4 and 5 m, respectively.

(b) The weight of the bridge is 10^6 lb and of the truck is 10^5 lb. Assume that the distance between the supporting columns is 30 ft and the left and right overhangs are 14 and 20 ft, respectively.

Fig. P9.53 Truck on the bridge

9.54 Use an intuitive approach (do not write equations for each region) to solve problem shown in Fig. P9.55.

9.55 Draw diagrams of the internal moment and forces in a shelf supporting stacks of CDs, Fig. P9.55.

(a) The weight of each tall stack is 20 N and the weight of the short stack is 5 N (Fig. P9.55). Assume that the width of each stack is 15 cm and the space between the tall stacks is 15 cm as well. The space between the long and short stack is 3 cm.

(b) The weight of each tall stack is 5 lb and the weight of the short stack is 1 lb. Assume that the width of each stack is 6 in. and the space between the tall stacks is 6 in. as well. The space between the long and short stack is 1 in.

Fig. 9.55 Shelf with compact disk stacks

9.56 Use an intuitive approach (do not write equations for each region) to solve Problem 9.57

9.57 Draw diagrams of the internal forces and moments in a shelf loaded as shown in Fig. P9.57. Assume that the shelf is supported by a pin on the left and is supported by a roller on the right.

(a) Weight of the book is 10 N and of the box 40 N, $a = b = 10$ cm, $c = 30$ cm, and $d = 20$ cm.

(b) Weight of the book is 2 lb and of the box 10 lb, $a = b = 4$ in., $c = 12$ in., and d 8 in.

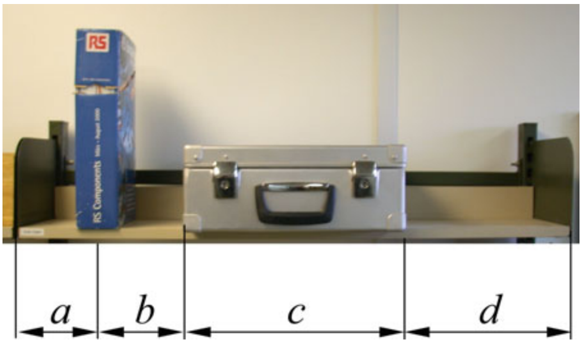


Fig. P9.57 Shelf with a box and book

9.58 Use an intuitive approach (do not write equations for each region) to solve the problem shown in Fig. P9.59.

9.59 Draw diagrams of the internal forces and moment in a shelf loaded as shown in Fig. P9.59. Assume that the shelf is supported by a pin on the left and is supported by a roller on the right.

(a) Weight of the left empty bottle is 3 N, full bottle 10 N, the box 30 N, $a = b = c = d = e = 10$ cm, and $c = 30$ cm.

(b) Weight of the left empty bottle is 1 lb, full bottle 4 lb, the box 10 lb, $a = b = c = d = e = 4$, and $c = 10$ in.

Fig. P9.59 Shelf with a box and two bottles

9.60 Use an intuitive approach (do not write equations for each region) to solve the Problem 9.13.

9.61 Derive equations for the internal forces and bending moment for the loading Case 1 shown in Table 9.1. Draw the appropriate diagrams.

9.62 Derive equations for the internal forces and bending moment for the loading Case 2 shown in Table 9.1. Draw the appropriate diagrams.

9.63 Derive equations for the internal forces and bending moment for the loading Case 3 shown in Table 9.1. Draw the appropriate diagrams.

9.64 Derive equations for internal forces and bending moment for the loading Case 4 shown in Table 9.1. Draw the appropriate diagrams.

9.65 Derive equations for the internal forces and bending moment for the loading Case 5 shown in Table 9.1. Draw the appropriate diagrams.

9.66 Derive equations for the internal forces and bending moment for the loading Case 6 shown in Table 9.1. Draw the appropriate diagrams.

9.67 Derive equations for the internal forces and bending moment for the loading Case 7 shown in Table 9.1. Draw the appropriate diagrams.

9.68 Derive equations for the internal forces and bending moments for the loading Case 8 shown in Table 9.1. Draw the appropriate diagrams.

9.69 Derive equations for the internal forces and bending moment for the loading Case 9 shown in Table 9.1. Draw the appropriate diagrams.

9.70 Derive equations for the internal forces and bending moment for the vertical straight beam shown in Fig. P9.70. The weight of the board is 400 N. Neglect the weight of the beam itself. Draw the appropriate diagrams.

Fig. P9.70 Basketball stand

9.71 Derive equations for the internal forces and bending moment for a straight beam that extends under the angle shown in Fig. P9.70. The weight of the board is 400 N. (a) Neglect the weight of the beam itself. (b) Consider that the weight of the beam is 100 N/m. Dimensions of the basketball stand define from the experience. Draw the appropriate diagrams.

Cables 10

Contents

You may be disappointed if you fail, but you are doomed if you don't try.

—Beverly Sills

In this chapter you will learn:

- How to calculate the internal forces in cables loaded by concentrated forces?
- How to calculate the internal forces in cables loaded by distributed forces?
- How to determine the geometry of cables for different loading modes?
- How to determine the length of a cable for different loading modes?
- How to determine axial force at the any point of a cable?
- How to determine maximal deflection (sag) of a cable?
- How to determine location of a sag?
- How to determine length of a cable?

I. Emri, A. Voloshin, *Statics*, DOI 10.1007/978-1-4939-2101-0_10

In Chap. 7 we have introduced three classes of structural elements: truss members, beams, and cables. For each class we have discussed the requirements that a structural element has to meet in order to belong to a particular class. Here we will study techniques to calculate the internal forces and geometry of cables loaded by concentrated and distributed forces.

"Cables" are structural elements that can sustain tensile loads only. Wires, chains, ropes, and cables are typical representatives of "cables." For simplicity reason we will use the word "cable" to indicate all previously mentioned structural elements. These structures behave as rigid bodies only when loaded by tensile forces. When loading is applied in any other than the axial direction, a cable will change its geometry to accommodate the load, hence, geometry of the cable is "load dependent" and, therefore, not known a priori. Therefore, procedures for solving cables are quite different for cables loaded by concentrated forces in comparison to those loaded by distributed forces.

> Cables are structural elements that can sustain tensile loads only.

In this chapter we will consider situations where all external forces are acting in the same plane, which is most common situation. Such loading arrangement allows treating cables as 2D problem.

10.1 Cables Loaded by Concentrated Forces

There are situations when a cable is loaded by one or more concentrated forces whose magnitudes are significantly larger than the weight of a cable. Such situation is schematically shown in Fig. 10.1a, where a cable is loaded by two concentrated forces $\mathbf{F_1}$ and $\mathbf{F_2}$, located at distances l_1 and l_2 from the left support, respectively. The distance between the two supports is $l_{AB} = l_3$, and support B is higher than support A by distance h.

When magnitudes of concentrated forces are significantly larger than the weight of a cable, the weight of the cable may be neglected, and individual segments of the cable may be assumed to be straight. We will further assume that a cable at the point of application of external force behaves as a joint. We need to stress that this assumption leads to under-design of the cable in the vicinity of acting forces. In practice we overcome this problem by using an appropriate safety factor.

> When magnitudes of concentrated forces are significantly larger than the weight of a cable, the weight of the cable may be neglected, and individual segments of the cable may be assumed to be straight.

> Assumption that each point where a force is applied acts as a joint leads to under-designing of the structure.

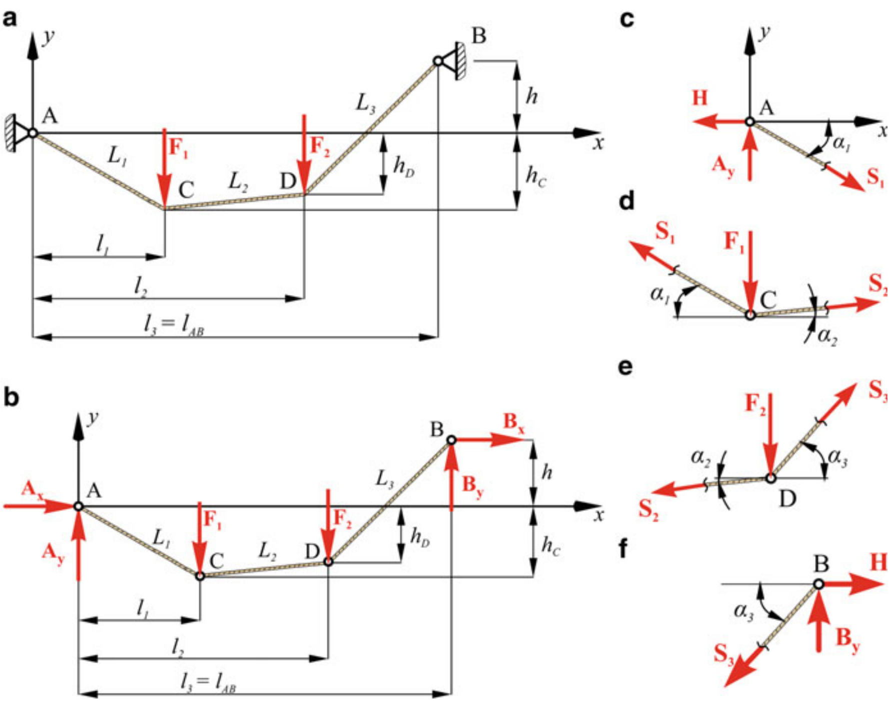


Fig. 10.1 (**a**) Cable loaded by two concentrated forces. (**b**) Free body diagram of a cable loaded by two concentrated forces. (**c**) Free body diagram of joint A. (**d**) Free body diagram of joint C. (**e**) Free body diagram of joint D. (**f**) Free body diagram of support B

By taking into account these assumptions we may model each segment of a cable as a truss member. The free body diagram of such cable structure is shown in Fig. 10.1b.

> With the procedures presented in this chapter we cannot predict internal forces at the location of external forces action. In practice we overcome this problem by using an appropriate safety factor.

> When all external loads are acting in vertical direction then the horizontal forces in both supports are equal in magnitude and opposite in sign, i.e., $A_x = -B_x = -H$

It should be noted that this system is externally statically underdetermined, since there are four unknown reactions A_x, A_y, B_x, and B_y, and only three equations of equilibrium.

$$\sum_i F_{ix} = A_x + B_x = 0 \tag{10.1a}$$

$$\sum_i F_{iy} = A_y + B_y - F_1 - F_2 = 0 \tag{10.1b}$$

$$\sum_i M_{iA} = B_y \cdot l_3 - B_x \cdot h - F_1 \cdot l_1 - F_2 \cdot l_2 = 0 \tag{10.1c}$$

From the first equation we obtain,

$$A_x = -B_x = -H \tag{10.2}$$

We assume that there are no internal bending moments and/or transversal forces in a cable.

where H is as yet an unknown horizontal component of the reaction forces at points A and B. Hence, when all external loads are acting in vertical direction then the horizontal forces in both supports are equal in magnitude and opposite in sign. As we will see later, this is very important information when solving for internal forces in cable structures.

In case when we have N external concentrated forces, the second and the third equilibrium equation may be generalized as,

$$\sum_i F_{iy} = A_y + B_y - \sum_{i=1}^{i=N} F_i = 0 \tag{10.3}$$

and

$$\sum_i M_{i\mathrm{A}} = B_y \cdot l_{\mathrm{AB}} - H \cdot h - \sum_{i=1}^{i=N} l_i \cdot F_i = 0 \tag{10.4}$$

In order to solve the problem we need to address the system as a whole, i.e., consider the equilibrium of external and internal forces simultaneously. The internal tensile forces in each of the three segments AC, CD, and DB are unknown. However, it is known that they act along the longitudinal axis of each corresponding segment. Also, it is important to mention that in cables loaded with concentrated forces there are no internal bending moments and/or transversal forces. It should be noted, however, that this assumption is not valid at the point of action of an external concentrated load; consequently the obtained solution leads to under-design of cables in the vicinity of acting external forces. As said before, in practice we overcome this problem by using an appropriate safety factor.

This assumption is not valid at the point of action of an external concentrated load; consequently the obtained solution leads to under-design of the cable in the vicinity of acting forces.

Since all internal forces act in axial direction and all segments of the cable may be considered to be straight one may solve for internal tensile forces in each segment by using the approach of joints introduced in Chap. 8.

In our case cable structure has four joints; therefore, we can write eight independent equations of equilibrium, i.e., two equilibrium equations for each joint. Unfortunately, geometry of a cable is not known because it depends on location, magnitude, and direction of external forces. We therefore need some additional information to make the problem solvable.

Let us analyze how many additional parameters we need to define the geometry of a cable structure, if we assume that horizontal, l_{AB}, and vertical, h, distance between the two supports and locations where the external forces are acting, in our case l_1 and l_2, are known.

After carefully analyzing geometry of the cable structure, shown in Fig. 10.1a, b, we find that geometry of a cable structure is defined if in addition to the location of external forces and supports, we would know at least one angle defining direction of one of the internal forces acting to the left or right of each of the external force. Hence, we need to know as many additional parameters as there are external forces acting on a cable, hence, in our case two. However, if all external forces act in vertical direction we need only one additional parameter.

> The geometry of a cable structure is defined if, in addition to the location of external forces and supports, as many additional parameters related to the geometry of the cable, forces in the supports, or internal forces, in the cable, as there are external forces acting on the cable are known.

> However, in case when all external forces act vertically we need only one additional parameter.

Since in our case all external loads act in vertical direction we have only nine unknowns, i.e., three reactions, H, A_y, B_y, three internal forces, S_1, S_2, S_3, and three parameters defining geometry of the cable structure, α_1, α_2, α_3. At the same time we have only eight independent equilibrium equations, i.e., two for each joint. Hence, we have one unknown more than the number of equilibrium equations, which means that cable structures are underdetermined and consequently unstable. Their geometry will change as soon as magnitude or direction of external forces changes.

> All cable structures are underdetermined and consequently unstable. Their geometry will change as soon as magnitude or direction of external forces changes.

To solve cable structures we need additional information, such as length of a cable L, or a force in a given segment of the cable, or one of the components of reaction forces, or vertical deflection at a given point, or any other parameter related to geometry or internal forces of a cable.

For a demonstration let us assume that additional information is vertical deflection at point C where external force $\mathbf{F_1}$ acts, Fig. 10.1a, and solve for unknown internal forces. This will be discussed as Example 10.1.

Example 10.1 Calculate internal forces and angles for each segment of a cable loaded by two forces as shown in Fig. 10.1a, b. Assume that the following parameters are given: F_1, F_2, l_1, l_2, l_3, h and vertical deflection, h_C, at the point of action of external force $\mathbf{F_1}$. For numerical evaluation use $h_C = 2\,\text{m}$, $F_1 = 30\,\text{N}$, $F_2 = 10\,\text{N}$, $l_1 = 3\,\text{m}$, $l_2 = 6\,\text{m}$, $l_3 = 10\,\text{m}$, and $h = 1\,\text{m}$.

Solution The free body diagram is shown in Fig. 10.1b. From the equilibrium equations (10.1a), (10.1b), (10.1c), we find,

$$B_y = \frac{1}{l_3}(Hh + F_1 l_1 + F_2 l_2), \quad \text{and}$$

$$A_y = F_1 + F_2 - \frac{1}{l_3}(Hh + F_1 l_1 + F_2 l_2)$$

To determine internal forces we shall use the method of joints introduced in Chap. 8. Let us start with joint A, Fig. 10.1c, and consider its equilibrium. We can write two equations of equilibrium

$$\sum F_x = -H + S_1 \cos\alpha_1 = 0$$

$$\sum F_y = A_y - S_1 \sin\alpha_1 = 0$$

Here angle α_1 is known since we know deflection h_C under the force $\mathbf{F_1}$ and its distance l_1 from support A,

$$\tan\alpha_1 = \frac{h_C}{l_1} = \frac{2}{3}, \quad \text{and}$$

$$\alpha_1 = \tan^{-1}\left(\frac{h_C}{l_1}\right) \cong 33.69^\circ$$

From the above equilibrium equations we find the general rule that the reaction force in a support is always equal to the tensile force in the cable. Considering that in general we could have N external forces and $(N+1)$ sections of the cable, we find,

$$A = S_1 = \sqrt{A_x^2 + A_y^2} = \sqrt{H^2 + A_y^2} = \sqrt{H^2 + S_{1y}^2} \tag{10.5}$$

$$B = S_{N+1} = \sqrt{B_x^2 + B_y^2} = \sqrt{H^2 + B_y^2} = \sqrt{H^2 + S_{(N+1)y}^2} \tag{10.6}$$

From there it also follows,

$$A_y = S_{1y} = S_1 \sin\alpha_1 = H \cdot \tan\alpha_1 \tag{10.7}$$

$$B_y = S_{(N+1)y} = S_{(N+1)} \sin\alpha_{(N+1)} = H \cdot \tan\alpha_{(N+1)} \tag{10.8}$$

In this particular case we have,

$$A_y = H \tan\alpha_1 = H\frac{h_C}{l_1}$$

Considering in addition that $A_y = F_1 + F_2 - \frac{1}{l_3}(Hh + F_1 l_1 + F_2 l_2)$ we obtain the relation from H, and A_y. Hence

$$H = \frac{l_1[F_1(l_3 - l_1) + F_2(l_3 - l_2)]}{l_3 h_C + l_1 h} \cong 32.6\text{N}$$

and

$$A_y = 21.7\text{N}$$

Now we may calculate the internal force within the first section of the cable,

$$S_1 = \frac{H}{\cos\alpha_1} \cong 39.2\text{N}$$

We proceed now and write the equilibrium equations for joint C shown in Fig. 10.1d,

$$\sum F_x = -S_1 \cos\alpha_1 + S_2 \cos\alpha_2 = 0$$

$$\sum F_y = S_1 \sin\alpha_1 + S_2 \sin\alpha_2 - F_1 = 0$$

From the first equilibrium equation we find

$$S_1 \cos\alpha_1 = S_2 \cos\alpha_2 = H$$

which means that at any point along the cable the magnitude of horizontal component of the internal force is equal to H,

$$S_i \cos\alpha_i = H \tag{10.9}$$

Considering (10.5–10.9) we find also,

$$S_i = \sqrt{H^2 + S_{iy}^2} \tag{10.10}$$

$$\tan\alpha_i = \frac{S_{iy}}{H}, \quad \text{and} \tag{10.11}$$

$$S_i = H\sqrt{1 + \tan^2\alpha_i} \tag{10.12}$$

Using the second equilibrium equation and (10.12) we obtain,

$$S_2 \sin\alpha_2 = F_1 - H \cdot \tan\alpha_1 = F_1 - H\frac{h_C}{l_1}$$

$$\tan\alpha_2 = \frac{F_1}{H} - \frac{h_C}{l_1}$$

and

$$\alpha_2 = \tan^{-1}\left(\frac{F_1}{H} - \frac{h_c}{l_1}\right) \cong 14.23^\circ$$

From (10.9) we can calculate tensile force in the second section of the rope,

$$S_2 = \frac{H}{\cos\alpha_2} \cong 33.6\text{N}$$

Equilibrium equations for point D (Fig. 10.1e) are

$$\sum F_x = -S_2\cos\alpha_2 + S_3\cos\alpha_3 = 0$$

$$\sum F_y = -S_2\sin\alpha_2 + S_3\sin\alpha_3 - F_2 = 0$$

From the first equilibrium equation for joint D we find again that the horizontal component of the internal forces is equal to H,

$$S_2\cos\alpha_2 = S_3\cos\alpha_3 = H$$

which confirms (10.9). From the second equilibrium equation for point D we find,

$$S_3\sin\alpha_3 = F_2 + F_1 - H\frac{h_C}{l_1}$$

and considering (10.12) we find

$$\tan\alpha_3 = \frac{F_2}{H} + \frac{F_1}{H} - \frac{h_C}{l_1}, \quad \text{and}$$

$$\alpha_3 = \tan^{-1}\left(\frac{F_2}{H} + \frac{F_1}{H} - \frac{h_C}{l_1}\right) \cong 29.3°$$

Finally, we write the equilibrium equations for joint B (Fig. 10.1f)

$$\sum F_x = -S_3 \cos\alpha_3 + H = 0$$

$$\sum F_y = -S_3 \sin\alpha_3 + B_y = 0$$

Using second equilibrium equation for joint B and (10.12) we obtain

> Horizontal component of internal forces at any point of a cable is equal to H when all external loads are acting in vertical direction,
>
> $$S_i \cos\alpha_i = H$$

$$\tan\alpha_3 = \frac{B_y}{H}, \quad \text{and}$$

$$\alpha_3 = \tan^{-1}\left(\frac{B_y}{H}\right)$$

Since α_3 is known already we could calculate B_y,

$$B_y = H \cdot \tan\alpha_3 = \frac{1}{l_3}(Hh + F_1 l_1 + F_2 l_2) \cong 18.26\text{N}$$

From (10.9) we find the axial force in the last section of the cable.

$$S_3 = \frac{H}{\cos\alpha_3} \cong 37.4\text{N}$$

We see that unknown quantities may be calculated from different equations, which gives us possibility to check the correctness of the obtained results.

> We have confirmed that the horizontal component of the internal force in any segment of a cable is constant and equal to the horizontal component at each support.

Example 10.2 Consider again the cable loaded by two forces $F_1 = 500\,\text{N}$, acting at $l_1 = 3\,\text{m}$, and $F_2 = 1000\,\text{N}$, acting at $l_2 = 6\,\text{m}$, as shown in Fig. 10.1a. The distance between the two supports is $l_{AB} = 12\,\text{m}$, and the right support is $h = 1\,\text{m}$ higher than the left one. Determine the length of the cable that we need to use if the horizontal force in two supports should not exceed $H = 300\,\text{N}$. Determine also the maximal tensile force in the cable and its lowest point. The corresponding free body diagram is shown in Fig. 10.2a.

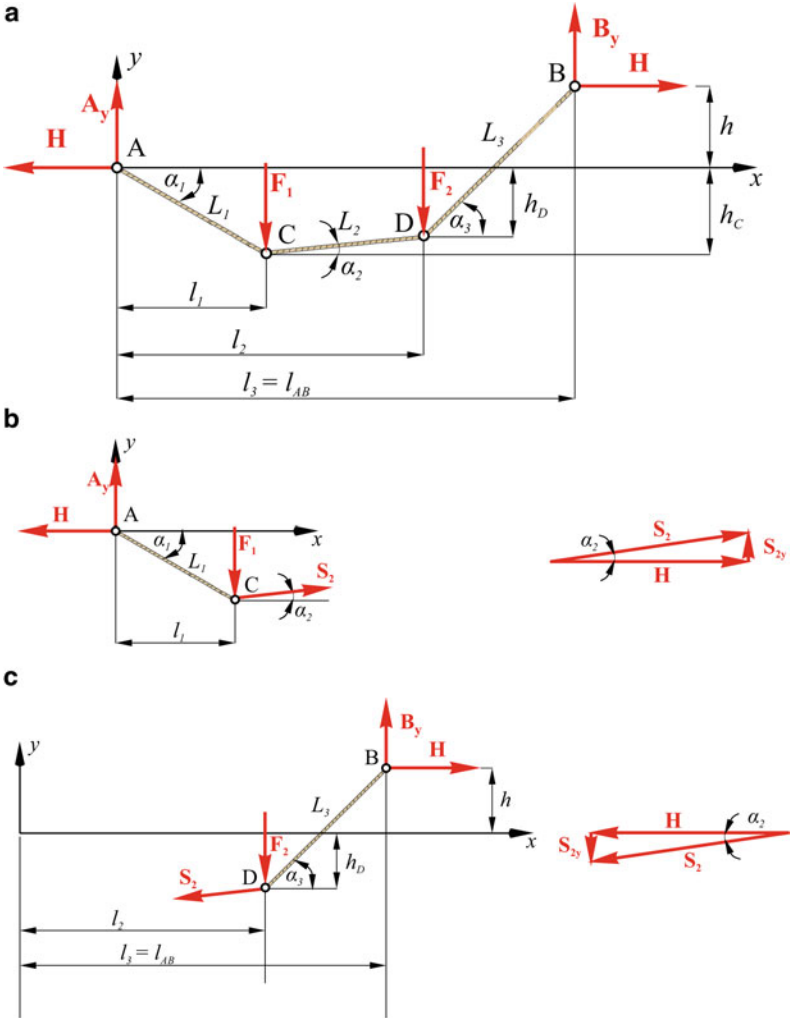


Fig. 10.2 (**a**) Free body diagram of a cable analyzed in Example 10.2. (**b**) Free body diagram of the left segment of the cable, which is cut within the second region. (**c**) Free body diagram of the right segment of the cable, which is cut within the second region

Solution Since horizontal force **H** is known we may use equilibrium equation for moments to determine both reaction forces $\mathbf{A_y}$ and $\mathbf{B_y}$. Both reaction forces we find by writing equilibrium equations for moments relative to supports B and A, respectively.

$$\sum_i M_{iB} = -A_y \cdot l_{AB} - H \cdot h + F_1 \cdot (l_{AB} - l_1) + F_2 \cdot (l_{AB} - l_2) = 0$$

$$A_y = \frac{1}{l_{AB}}[-H \cdot h + F_1 \cdot (l_{AB} - l_1) + F_2 \cdot (l_{AB} - l_2)], \quad \text{and}$$

$$A_y = \frac{1}{12}[-300 \cdot 1 + 500 \cdot (12 - 3) + 1000 \cdot (12 - 6)] = 850\,\text{N}$$

Similarly we find the vertical reaction force in support A,

$$\sum_i M_{iA} = B_y \cdot l_{AB} - H \cdot h - F_1 \cdot l_1 - F_2 \cdot l_2 = 0$$

$$B_y = \frac{1}{l_{AB}}[H \cdot h + F_1 \cdot l_1 + F_2 \cdot l_2]$$

$$B_y = \frac{1}{12}[300 \cdot 1 + 500 \cdot 3 + 1000 \cdot 6] = 650\text{N}$$

Using (10.5) we find the reaction force in a support A and the internal force in the first segment of the cable,

$$A = S_1 = \sqrt{A_y^2 + H^2} = \sqrt{850^2 + 300^2} \cong 901\,\text{N}$$

Similarly we find

$$B = S_3 = \sqrt{B_y^2 + H^2} = \sqrt{650^2 + 300^2} \cong 716\,\text{N}$$

Angle α_1 we can calculate from (10.9),

$$\alpha_1 = \cos^{-1}\left(\frac{H}{S_1}\right) = \cos^{-1}\left(\frac{300}{901.4}\right) \cong 70.6^\circ$$

We could determine all remaining internal forces and angles by using previously developed equations. However, to demonstrate another way of solving cables loaded with concentrated forces we will introduce another approach, which is very useful when cables are loaded with many concentrated forces.

We will cut the cable within the second region, as shown in Fig. 10.2b.

Form the equilibrium equation in y-direction we obtain,

$$\sum F_y = A_y - F_1 + S_{2y} = 0, \quad \text{and}$$

$$S_{2y} = F_1 - A_y = 500 - 850 = -350\,\text{N}$$

Negative sign indicates that $\mathbf{S_{2y}}$ acts downwards. Above equation may be generalized as

$$S_{ky} = \sum_{i=1}^{k-1} F_i - A_y \tag{10.13}$$

Hence, sum of the external forces acting to the left of the observed section is equal to the sum of the vertical reaction in the left support and vertical component of the internal force in the observed section of the cable.

Angle α_2 and S_2 may be obtained from, Fig. 10.2b, and (10.10) and (10.11),

$$\tan\alpha_2 = \frac{S_{2y}}{H}, \quad \text{and} \quad S_2 = \sqrt{S_{2y}^2 + H^2} \text{ hence,}$$

$$\alpha_2 = \tan^{-1}\left(\frac{S_{2y}}{H}\right) = \tan^{-1}\left(\frac{-350}{300}\right) \cong -49.4^\circ$$

$$S_2 = \sqrt{S_{2y}^2 + H^2} = \sqrt{(-350)^2 + 300^2} \cong 461\,\text{N}$$

Internal force S_3 and α_3 may be obtained again from (10.13), (10.10), and (10.11)

$$S_{3y} = F_1 + F_2 - A_y = 500 + 1000 - 850 = 650\,\text{N}$$

$$\alpha_3 = \tan^{-1}\left(\frac{S_{3y}}{H}\right) = \tan^{-1}\left(\frac{650}{300}\right) \cong 65.2^\circ$$

$$S_3 = \sqrt{S_{3y}^2 + H^2} = \sqrt{650^2 + 300^2} \cong 716\,\text{N}$$

Hence, the largest internal force appears at support A.

If we analyze the right segment of the cable structure, shown in Fig. 10.2c, we can derive similar rule as shown in (10.13) for support B.

Form the equilibrium equation in y-direction we obtain,

$$\sum F_y = B_y - F_2 - S_{2y} = 0, \quad \text{and}$$

$$S_{2y} = B_y - F_2$$

which may be generalized as,

$$S_{ky} = B_y - \sum_{i=k}^{N} F_i \tag{10.14}$$

where N denotes number of external concentrated forces.

By equating (10.13) and (10.14) we obtain,

$$\sum_{i=1}^{k-1} F_i - A_y = B_y - \sum_{i=k}^{N} F_i$$

and

$$B_y + A_y - \sum_{i=1}^{N} F_i = 0$$

which is identical to (10.3) representing generalized equilibrium equation for the external forces, as it should be.

Let us now calculate the sags at the joints where the external forces are acting. From Fig. 10.2a we see that,

$$h_{\mathrm{C}} = l_1 \cdot \tan \alpha_1 = 3 \cdot \tan(70.56^\circ) \cong 8.52\,\mathrm{m}$$

Similarly we calculate for joint D,

$$h_{\mathrm{D}} = h_{\mathrm{C}} - (l_2 - l_1) \tan \alpha_2 \cong 8.5 - (6 - 3) \cdot \tan(-49.4) \cong 12.0\,\mathrm{m}$$

We find that the largest sag appears at joint D where the external force F_2 is acting.

We still need to calculate the length of the cable. From Fig. 10.2a we see that the length of the cable may be expressed as,

$$L = \frac{l_1}{\cos \alpha_1} + \frac{l_2 - l_1}{\cos \alpha_2} + \frac{l_3 - l_2}{\cos \alpha_3}$$

After inserting numerical values we obtain,

$$L \cong \frac{3}{\cos(70.56^\circ)} + \frac{6 - 3}{\cos(-49.4)} + \frac{12 - 6}{\cos(65.22^\circ)} \cong 27.9\,\mathrm{m}$$

For N external forces above equation may be generalized as,

$$L = \sum_{i=1}^{N+1} \frac{l_i - l_{i-1}}{\cos(\alpha_i)} \tag{10.15}$$

where $l_0 = 0$, and $l_{N+1} = l_{\mathrm{AB}}$ is the horizontal distance between the two supports.

Guidelines and Recipes for Cables Loaded with Concentrated Forces

- For each of the N external forces F_i define location l_i of the line of action relative to the left support.
- Write equilibrium equations for the external forces, i.e.,

$$\sum_i F_{iy} = A_y + B_y - \sum_{i=1}^{i=N} F_i = 0$$

$$\sum_i M_{iA} = B_y \cdot l_{AB} - H \cdot h - \sum_{i=1}^{i=N} l_i.F_i = 0$$

Calculate internal forces in all sections of the cable, S_k, and its geometry, α_k, h_k, and L using:

$$A_y = S_{1y} = S_1 \sin \alpha_1 = H \cdot \tan \alpha_1$$

$$B_y = S_{(N+1)y} = S_{(N+1)} \sin \alpha_{(N+1)} = H \cdot \tan \alpha_{(N+1)}$$

$$S_k \cos \alpha_k = H, \quad S_k = H\sqrt{1 + \tan^2 \alpha_k}$$

$$S_{ky} = \sum_{i=1}^{k-1} F_i - A_y; \quad S_{ky} = B_y - \sum_{i=k}^{N} F_i; \quad \tan \alpha_k = \frac{S_{ky}}{H}$$

$$h_k = l_k \cdot \tan \alpha_k; \quad \text{and} \quad L = \sum_{i=1}^{N+1} \frac{l_i - l_{i-1}}{\cos(\alpha_i)}$$

10.2 Cables with Distributed Loads

Cables are very often exposed to their weight only, which may be considered as uniformly distributed load. If the cable has a weight Q and length L, then the distributed load per cable unit length will be $p = Q/L$ [N/m or lb/ft]. An example of such cable structure is shown in Fig. 10.3a, where we assumed that the distance between the two supports is $l_{AB} = l$, and support B is h higher than support A. The origin of the coordinate system we place at the lowest point of the cable, as shown in Fig. 10.3a. The corresponding free body diagram is shown in Fig. 10.3b. From the equilibrium equations of external forces we again find that the horizontal

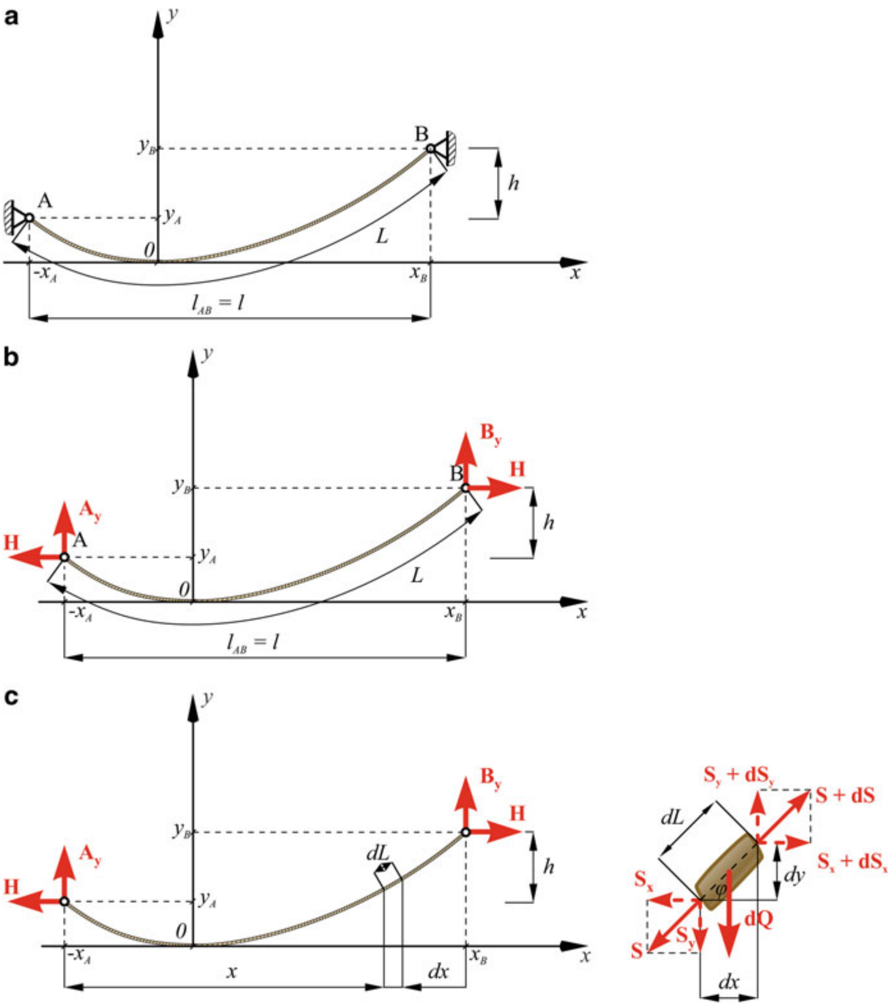


Fig. 10.3 (**a**) Physical model of a cable loaded with its weight. (**b**) Free body diagram of a cable loaded with its weight. (**c**) Free body diagram of the cable and of the infinitesimal cable segment

components of the reactions at supports A and B are equal in magnitude and opposite in direction, hence, $-A_x = B_x = H$.

It should be noted that geometry of the cable is not known a priori, it is again a function of the load, i.e., cable weight, and location of both supports. To determine the geometry of the cable we have to take into account the equilibrium equations for the internal forces.

To do that we consider equilibrium of an infinitesimal segment of cable dx, located at distance x, as shown in Fig. 10.3c, where dQ indicates the weight of the infinitesimal segment, while S and $S + dS$ denote the two axial forces acting in tangential direction, which may be replaced with two corresponding orthogonal components S_x and S_y, and $S_x + dS_x$ and $S_y + dS_y$, respectively.

Geometry of cables is not known a priori. It depends on the cable weight and location of both supports.

Equilibrium equations for the infinitesimal segment of the cable are:

$$\sum F_x = -S_x + S_x + dS_x = 0 \tag{10.16}$$

$$\sum F_y = -S_y + S_y + dS_y - dQ = 0 \tag{10.17}$$

From the first equilibrium equation we find that $dS_x = 0$, and after the integration,

$$S_x = H = \text{const.} \tag{10.18}$$

Similarly, as in the case of cables loaded by concentrated loads, we find that the horizontal component of the internal force at any point along the cable is constant and equal to the horizontal forces in both supports,

From the second equation we obtain,

$$dS_y = dQ \tag{10.19}$$

Since both forces, S_y and Q, are functions of x we may not integrate both sides of the equation independently. We need to express S_y and Q as functions of x first.

Horizontal component of the internal force at any point of the cable is constant and equal to H.

Vertical component of the internal force at any point of the cable is equal to $$S_y(x) = H\frac{dy}{dx}$$

In any selected coordinate system geometry of a cable may be described as a function, $y = f(x)$. From differential calculus we know that first derivative of a function $f(x)$ represents its slope at selected point x, Fig. 10.3c. At any point x we may write,

$$\frac{dy}{dx} = f'(x) = \frac{S_y}{S_x} = \tan\varphi \tag{10.20}$$

Since $S_x = H = \text{const.}$, we find

$$S_y = S_x \frac{dy}{dx} = H \frac{dy}{dx} \tag{10.21}$$

and after differentiating both sides of the equation we obtain,

$$\frac{dS_y}{dx} = H \frac{d^2y}{dx^2} \tag{10.22}$$

and therefrom

$$dS_y = H \frac{d^2y}{dx^2} dx \tag{10.23}$$

Combining (10.19) and (10.23), we obtain a second order differential equation which describes geometry of cables exposed to continuous load,

$$H \frac{d^2y}{dx^2} dx = dQ \tag{10.24}$$

Equation (10.24) represents the second order differential equation, which defines geometry of the cable exposed to continuous load.

> When the length of a cable is not much larger than the distance between the two supports one may assume that the weight of the cable is distributed uniformly along the horizontal distance between the two supports

Before solving the above equation we still have to express dQ as function of x. It turns out that there are two common ways of doing this, depending on (1) the length of a cable and (2) the way in which a cable is loaded. Let us discuss these situations in more details.

> In case of suspension bridges we may again assume that the external load is distributed uniformly along the horizontal distance between the two supports.

In many engineering applications cables are very taut and the length of a cable is not much larger than the distance between the two supports, i.e., $L \approx l$, where L is the length of a cable, and l is the horizontal distance between two supports. In such

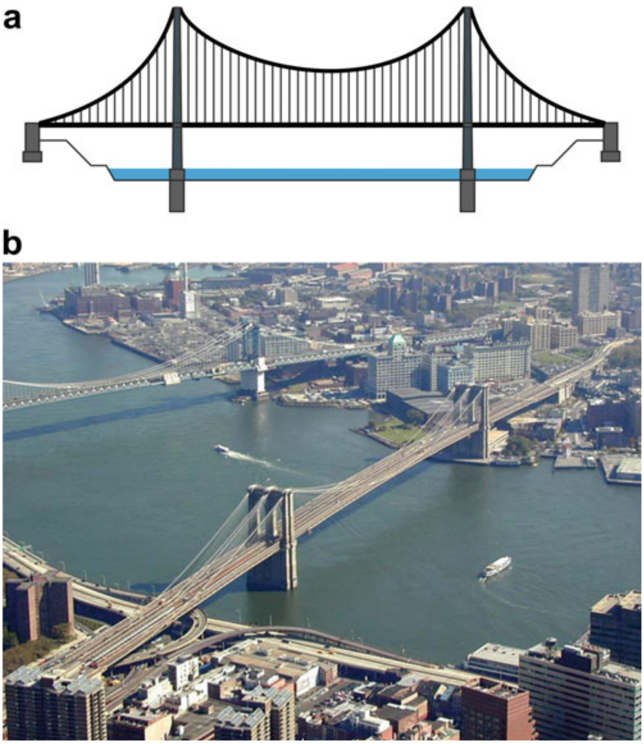


Fig. 10.4 (**a**) Cables loaded with hanging load. (**b**) Brooklyn Suspension Bridge

cases we may make another simplification and assume that cable's weight is distributed uniformly along the horizontal distance between the two supports. In this case the continuous load may be expressed as $q = pL/l$.

Another situation when one may assume that a load is distributed along the distance between the two cable supports are suspension bridges. Here the load to which a cable is exposed is hanging on a cable, as schematically shown in Fig. 10.4a. The world famous Brooklyn Suspension Bridge is a good example, Fig. 10.4b.

In both described situations the load to which the cables are exposed can be expressed as

$$dQ = q \cdot dx \tag{10.25}$$

This assumption leads to the so-called *parabolic solution* of (10.24), as discussed below. We have to be aware that such simplification brings in an additional (small) error. In real situations we overcome this shortcoming with an appropriate safety factor.

10.2.1 Parabolic Solution

The parabolic solution of (10.24) is obtained when a distributed load may be expressed as $dQ = q \cdot dx$, as shown in (10.25). Physical model of such cable is shown in Fig. 10.5a, where we assumed that the distributed load is constant along the x-axis. The corresponding free body diagram is shown in Fig. 10.5b. The origin of coordinate system we place into the lowest point of the cable. As we will see later, such location of coordinate system provides the simplest boundary conditions, from which we will need to determine the integration constants.

> The horizontal component of the internal force in any segment of a cable is constant and equal to the horizontal component at each support.

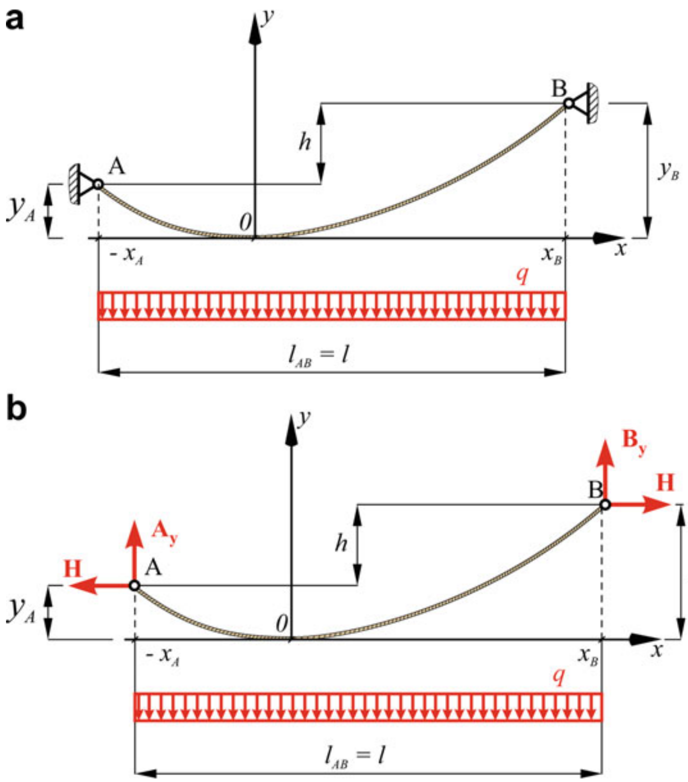


Fig. 10.5 (**a**) Physical model of a cable loaded with distributed load constant along the horizontal distance between the two supports. (**b**) Free body diagram of a cable loaded with a distributed load constant along the horizontal distance between the two supports

When a load is distributed along the distance between the two supports the equilibrium equations for the complete cable structure are:

$$\sum_i F_{iy} = A_y + B_y - ql = 0 \tag{10.26a}$$

$$\sum_i M_{iA} = l \cdot B_y - h \cdot H - \frac{ql^2}{2} = 0 \tag{10.26b}$$

By inserting (10.25) into (10.24) we obtain a simple second order differential equation,

$$H\frac{d^2y}{dx^2}dx = q \cdot dx$$

Since H and q are constants the equation may be simplified as,

$$\frac{d^2y}{dx^2} = \frac{q}{H} \tag{10.27}$$

The solution of this equation is obtained through a double integration. After the first integration we have,

$$dy = \left(\frac{q}{H}x + C_1\right)dx \tag{10.28}$$

and after the second,

$$y = \frac{q}{2H}x^2 + C_1x + C_2 \tag{10.29}$$

When an external load is distributed uniformly along the horizontal distance between the two supports, the geometry of the cable may be described with a parabola.

$$y = \frac{q}{2H}x^2 + C_1x + C_2$$

Equation (10.29) represents a function, which describes geometry of cables where we may assume that distributed load is constant along the distance between the two supports. Equation (10.29) is a parabola; therefore, this solution is called a *parabolic solution*.

Maximal internal force in a cable will appear at the support that has higher elevation. Its magnitude will be equal to the reaction force in the pertaining support and opposite in direction.

Constants C_1 and C_2 may be determined from boundary conditions, which depend on the location of the selected coordinate system and elevation and location of the two supports. However, independently of the location of the two supports, all parabolas have a common point, which is their minimum. Hence, the simplest solution we obtain if we place the origin of coordinate system into the lowest point of the cable, as shown in Fig. 10.5.

If the origin of the coordinate system is placed as shown in Fig. 10.5 the boundary conditions are independent of the location of the two supports, hence,

$$x = 0; \quad y = 0 \tag{10.30a}$$

$$x = 0; \quad y' = 0 \tag{10.30b}$$

By applying the two boundary conditions to (10.29) and its derivative, $y' = (q/H)x + C_1$, we find that both integration constants are equal to zero,

$$0 = \frac{q}{2H} \cdot 0^2 + C_1 \cdot 0 + C_2 \Rightarrow C_2 = 0 \tag{10.31a}$$

and

$$0 = \frac{q}{H} \cdot 0 + C_1 \Rightarrow C_1 = 0 \tag{10.31b}$$

By positioning the origin of a coordinate system into the vertex of the parabola leads to the simplest parabola, which describes the geometry of the cable, i.e.,

$$y = qx^2/2H$$

Hence, by positioning the origin of the coordinate system into the minimum of the parabola, known as *vertex*, leads to the simplest parabola describing geometry of the cable that is loaded with continuous load distributed along the horizontal distance between the two supports.

$$y = \frac{q}{2H} \cdot x^2 \tag{10.32}$$

10.2.1.1 Internal Tensile Force, *S*(*x*)

Tensile force $S(x)$ at any point along a cable is equal to the vector sum of its two components, see Fig. 10.3c, Tensile force $S(x)$ at any point of a cable is equal to the vector sum of the two components in vertical and horizontal direction, and it acts in tangential direction.

$$S(x) = \sqrt{H^2 + (qx)^2}$$

$$S(x) = \sqrt{S_x^2 + S_y^2} = \sqrt{H^2 + S_y^2}$$

If x_A and y_A and/or x_B and y_B are known we can calculate horizontal force H from (10.32),

$$H = \frac{qx_A^2}{2y_A} = \frac{qx_B^2}{2y_B} \tag{10.33}$$

Horizontal component of the internal force H is constant along the entire cable.

Taking into account that $S_y = H(dy/dx)$, (10.21), we find $S(x) = H\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

By utilizing (10.32) and finding its derivative, $dy/dx = qx/H$, we find the tensile (tangential) force at any point along the cable,

$$S(x) = H\sqrt{1 + \left(\frac{qx}{H}\right)^2} = \sqrt{H^2 + (qx)^2} \tag{10.34a}$$

and its both components,

$$S_x(x) = H = \text{const}, \quad \text{and} \tag{10.34b}$$

$$S_y(x) = H \cdot \frac{dy}{dx} = H \cdot \frac{qx}{H} = qx \tag{10.34c}$$

Vertical component of the internal force at any point is equal to

$$S_y(x) = qx$$

The very same solution we may obtain from the equilibrium of a segment of a cable, which starts at the lowest point of the cable where the internal force acts horizontally and is equal to H. Let us consider the right part of the cable as shown in Fig. 10.6.

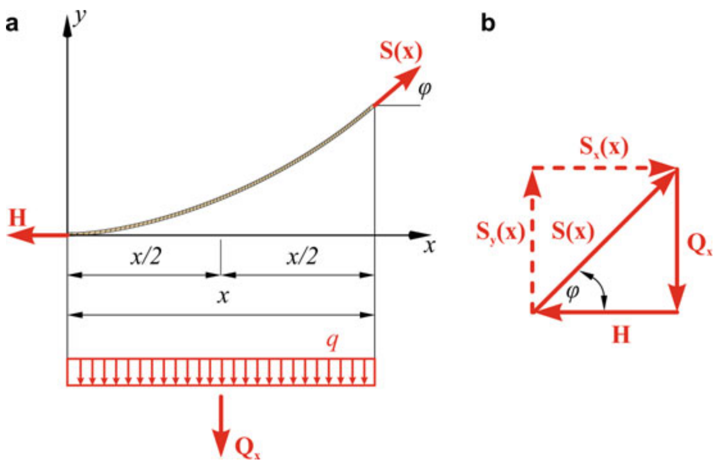


Fig. 10.6 (**a**) Free body diagram of a cable segment located to the right of the lowest point and (**b**) the equilibrium of the external and internal forces

An internal force in a cable at the point of the largest deflection acts horizontally and is equal to H.

The vertical component of the internal force at any point of the cable to the right or left of the lowest point will be equal to the weight of the cable section located to the right or left of the lowest point.

At point x the internal force is equal to $S(x)$, while at the lowest point of the cable it is equal to H. The weight of the cable segment within a distance x is $Q_x = q \cdot x$ and it is positioned at the middle since the continuous load is distributed along the distance x. The external and two internal forces need to be in equilibrium, as shown in Fig. 10.6b. Since H and Q_x are perpendicular $S(x)$ may be expressed as in (10.34a), (10.34b), (10.34c).

From Fig. 10.6 we see that the vertical component of the internal force at any point of the cable to the right or left of the lowest point will be equal to the weight of the cable section located to the right or left of the lowest point, as shown in (10.34c).

However, usually we are interested in the maximal tensile force acting in a cable. From (10.34a), (10.34b), (10.34c) we see that $S(x)$ increases with x and assumes the smallest value at $x = 0$. This can be also shown by setting the first derivative of (10.34a), (10.34b), (10.34c) to zero,

$$\frac{dS(x)}{dx} = \frac{q \cdot x}{\sqrt{1 + \left(\frac{q}{H}x\right)^2}} = 0$$

Hence, $S(x)$ has minimum at $x = 0$. From (10.34a), (10.34b), (10.34c) we find that the tensile force in the cable at this point is equal to the horizontal component, $S(x = 0) = H$. Thus, $S(x)$ will have the largest magnitude at the support that is positioned further away from the origin of the coordinate system. This will be at the support which is higher. Hence,

$$S_{\max} = S(x_k) = H\sqrt{1 + \left(\frac{qx_k}{H}\right)^2} = \sqrt{H^2 + (qx_k)^2} \tag{10.35a}$$

> The maximal internal force in the cable will appear at the support that has higher elevation. Its magnitude will be equal to the reaction force in the pertaining support and opposite in direction.

where $k = A$, for $y_A > y_B$, and $k = B$, for $y_B > y_A$. The reaction forces in both supports are,

$$A_x = B_x = H = \text{const}, \quad \text{and} \tag{10.35b}$$

$$A_y = H \cdot \frac{qx_A}{h} \tag{10.35c}$$

$$B_y = H \cdot \frac{qx_B}{h} \tag{10.35d}$$

To summarize, the maximal internal force in the cable will be equal to the reaction force in the support that is higher.

In case when the two supports are at the same level then $x_A = x_B$, and the two results are identical.

10.2.1.2 Length of the Cable, *L*

Length L of a cable is another important parameter that is required when we build cable structures. From Fig. 10.3c we see that a differential segment of a cable may be expressed as,

$$dL = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \tag{10.36a}$$

Utilizing (10.32), i.e., $y = qx^2/(2H)$, we find that $dy/dx = qx/H$, and can rewrite (10.36a) as

$$dL = \sqrt{1 + \left(\frac{qx}{H}\right)^2}\,dx \tag{10.36b}$$

We obtain the length of the cable between two points along the cable, say T_1 and T_2,

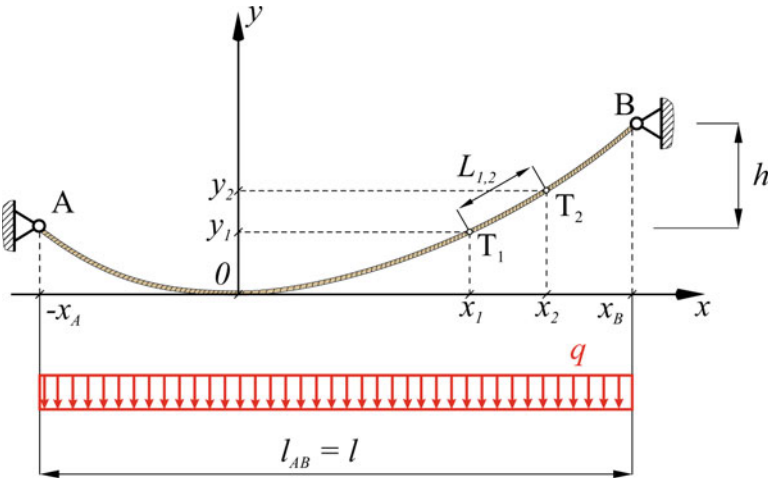


Fig. 10.7 Length $L_{1,2}$ of the cable between two arbitrary points T_1 and T_2 along the cable

as shown in Fig. 10.7, by integrating the above equation within the interval $x \in [x_1, x_2]$. Hence,

$$L_{1,2} = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{qx}{H}\right)^2} dx \tag{10.36c}$$

We can solve the above integral by using the binomial theorem and expand the radical in an infinite series (see, e.g., http://en.wikipedia.org/wiki/Binomial_theorem).

$$\sqrt{1+\lambda} = 1 + \frac{1}{2}\lambda - \frac{1}{8}\lambda^2 + \cdots$$

> By using the binomial theorem, one obtains simplified solution for the length of a cable, which is sufficiently good for all engineering applications, providing that we have taken appropriate number of terms in the series. Most often two terms are sufficient.

By considering that in our case $\lambda = (qx/H)^2$, we may rewrite (10.36c) as

$$L_{1,2} = \int_{x_1}^{x_2} \left[1 + \frac{1}{2}\left(\frac{qx}{H}\right)^2 - \frac{1}{8}\left(\frac{qx}{H}\right)^4 + \cdots\right] \tag{10.37}$$

The solution of the corresponding indefinite integral is given in the *Mathematical Corner I*,

$$I = \int \left[1 + \frac{1}{2}\left(\frac{qx}{H}\right)^2 - \frac{1}{8}\left(\frac{qx}{H}\right)^4 + \cdots\right] dx = x\left(1 + \frac{1}{6}\left(\frac{qx}{H}\right)^2 - \frac{1}{40}\left(\frac{qx}{H}\right)^4 + \cdots\right)$$

By utilizing the above indefinite integral solution we obtain,

$$L_{1,2} = \left[x\left(1 + \frac{1}{6}\left(\frac{qx}{H}\right)^2 - \frac{1}{40}\left(\frac{qx}{H}\right)^4 + \cdots\right)\right]_{x_1}^{x_2} \tag{10.38a}$$

and

$$L_{1,2} = x_2\left(1 + \frac{1}{6}\left(\frac{qx_2}{H}\right)^2 - \frac{1}{40}\left(\frac{qx_2}{H}\right)^4 + \cdots\right) - x_1\left(1 + \frac{1}{6}\left(\frac{qx_1}{H}\right)^2 - \frac{1}{40}\left(\frac{qx_1}{H}\right)^4 + \cdots\right) \tag{10.38b}$$

Equation (10.38b) defines the length of the cable between two arbitrary points along the cable. If we are interested in total length L of the cable we have to place T_1 into support A, $x_1 = -x_A$, and T_2 into support B, $x_2 = x_B$.

$$L = x_B\left(1 + \frac{1}{6}\left(\frac{qx_B}{H}\right)^2 - \frac{1}{40}\left(\frac{qx_B}{H}\right)^4 + \cdots\right) + x_A\left(1 + \frac{1}{6}\left(\frac{qx_A}{H}\right)^2 - \frac{1}{40}\left(\frac{qx_A}{H}\right)^4 + \cdots\right) \tag{10.38c}$$

If the two supports are at the same elevation we have $x_A = x_B = l/2$, and

$$L = l\left(1 + \frac{1}{6}\left(\frac{ql}{2H}\right)^2 - \frac{1}{40}\left(\frac{ql}{2H}\right)^4 + \cdots\right) \tag{10.38d}$$

Equation (10.38b) may be reorganized by considering (10.32), i.e., $y = qx^2/2H$, to obtain,

$$\frac{qx_1}{H} = \frac{2y_1}{x_1}, \text{ and } \frac{qx_2}{H} = \frac{2y_2}{x_2} \tag{10.39}$$

Using (10.39) (10.38b) and (10.38c) may be rewritten as,

$$L_{1,2} = x_2\left(1 + \frac{2}{3}\left(\frac{y_2}{x_2}\right)^2 - \frac{2}{5}\left(\frac{y_2}{x_2}\right)^4 + \cdots\right) - x_1\left(1 + \frac{2}{3}\left(\frac{2y_1}{x_1}\right)^2 - \frac{2}{5}\left(\frac{2y_1}{x_1}\right)^4 + \cdots\right) \tag{10.40a}$$

and,

$$L = x_B\left(1 + \frac{2}{3}\left(\frac{y_B}{x_B}\right)^2 - \frac{2}{5}\left(\frac{y_B}{x_B}\right)^4 + \cdots\right) + x_A\left(1 + \frac{2}{3}\left(\frac{y_A}{x_A}\right)^2 - \frac{2}{5}\left(\frac{y_A}{x_A}\right)^4 + \cdots\right) \tag{10.40b}$$

This series converges rapidly for $y/x < 0.5$; in most cases this ratio is much smaller, and it is sufficient to take only first two terms. Of course, more terms may be used if required.

When the two supports are at the same elevation we have $x_A = x_B = l/2$, $y_A = y_B = f$, and $y_A/y_B = 2f/l$. In a view of this, (10.40a) becomes

$$L = l\left[1 + \frac{2}{3}\left(\frac{2f}{l}\right)^2 - \frac{2}{5}\left(\frac{2f}{l}\right)^4 + \cdots\right] \tag{10.40c}$$

Equations (10.40a), (10.40b), (10.40c) allow an easy way to calculate length L of a cable, between two points along the cable.

Mathematical Corner I: Solution of the Integral Corresponding to equation (10.37)

The corresponding indefinite integral is,

$$I = \int \left[1 + \frac{1}{2}\left(\frac{qx}{H}\right)^2 - \frac{1}{8}\left(\frac{qx}{H}\right)^4 + \cdots\right] dx \tag{i – 1}$$

which may be expressed as a sum of integrals

$$I = \int dx + \int \frac{1}{2}\left(\frac{qx}{H}\right)^2 dx - \int \frac{1}{8}\left(\frac{qx}{H}\right)^4 dx + \cdots \tag{i – 2}$$

that, except the first term, have a common indefinite integral,

$$\int \left(\frac{qx}{H}\right)^n dx = \frac{H}{q(n+1)}\left(\frac{qx}{H}\right)^{n+1} \tag{i – 3}$$

By utilizing this, we find the solution of the above indefinite integral,

$$I = \left[x + \frac{H}{6q}\left(\frac{qx}{H}\right)^3 - \frac{H}{40q}\left(\frac{qx}{H}\right)^5 + \cdots\right], \text{ and} \tag{i – 4}$$

$$I = x\left(1 + \frac{1}{6}\left(\frac{qx}{H}\right)^2 - \frac{1}{40}\left(\frac{qx}{H}\right)^4 + \cdots\right) \tag{i – 5}$$

10.2.1.3 Sag of the Cable, *f*

We have derived the simplest form of (10.32), by describing the geometry of cables loaded with a continuous load distributed along the horizontal distance between the two supports. This form of the equation is obtained when the coordinate system is positioned at the lowest point of the cable. Unfortunately, the location of the cable lowest point is known a priori only when the two supports are at the same elevation. When the two supports are not at the same elevation the location of the lowest point, i.e., the location of the coordinate system, is NOT known. Equivalently, we may say that the locations of two supports x_A and x_B, relative to the origin of the coordinate system placed at the parabola's vertex (minimum), is not known!

To determine the location of the cable's lowest point (sag f, or vertex of the parabola) we consider the equilibrium of the cable shown in Fig. 10.5b, and the segment of the cable located to the right of the coordinate system origin. The segment is shown in Fig. 10.8.

We write equilibrium equation for moments for the entire cable structure,

$$\sum_i M_{i\mathrm{A}} = -\frac{q \cdot l^2}{2} + B_y \cdot l - H \cdot h = 0 \tag{10.41a}$$

and force equilibrium for the segment for y-direction,

$$\sum_i F_{iy} = -Q_\mathrm{B} + B_y = 0 \tag{10.41b}$$

From (10.41b) we find

$$B_y = Q_\mathrm{B} = qx_\mathrm{B} \tag{10.42a}$$

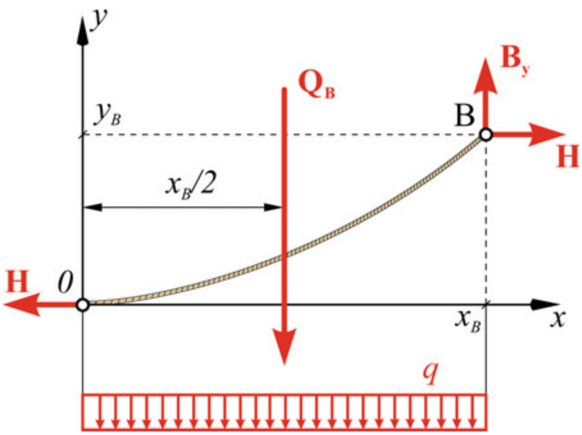


Fig. 10.8 Free body diagram of the segment of a cable shown in Fig. 10.5b, located to the right of the coordinate system origin

We similarly find,

$$A_y = Q_A = qx_A \tag{10.42b}$$

By inserting (10.42a) into (10.41a) we get

$$-\frac{q \cdot l^2}{2} + q \cdot x_B \cdot l - H \cdot h = 0 \tag{10.43}$$

Solving (10.42a), (10.42b) for x_B yields,

$$x_B = \frac{l}{2} + \frac{Hh}{ql}, \quad \text{and} \tag{10.44a}$$

$$x_A = l - x_B = \frac{l}{2} - \frac{Hh}{ql} \tag{10.44b}$$

In case when the vertical locations of both supports are known we can use (10.32), $y = qx^2/(2H)$ to obtain horizontal locations of the two supports,

$$y_A = \frac{qx^2}{2H} = f_A, \quad \text{and} \quad y_B = \frac{q(l - x_A)^2}{2H} = f_A + h$$

From both equations we express $2H/q$ and obtain

$$\frac{2H}{q} = \frac{x_A^2}{f_A} = \frac{(l - x_A)^2}{f_A + h}$$

which reduces to $\frac{h}{f_A} x_A^2 + 2lx_A - l^2 = 0$, and

$$x_A = l\frac{f_A}{h}\left(\sqrt{1 + \frac{h}{f_A}} - 1\right) \tag{10.44c}$$

where we have used only positive solutions of the quadratic equation. Location of the support we find from $x_B = l - x_A$.

> Maximal deflection f (sag) of a cable is always located at the distance
>
> $$x_A = |l/2 - Hh/(ql)|$$
>
> from the left support.

Equations (10.44a), (10.44b), (10.44c) define locations of the two supports relative to the coordinate system origin. By considering the symmetry there are four different possibilities for the locations of the two supports relative to the origin of the coordinate system. This is shown in Fig. 10.9.

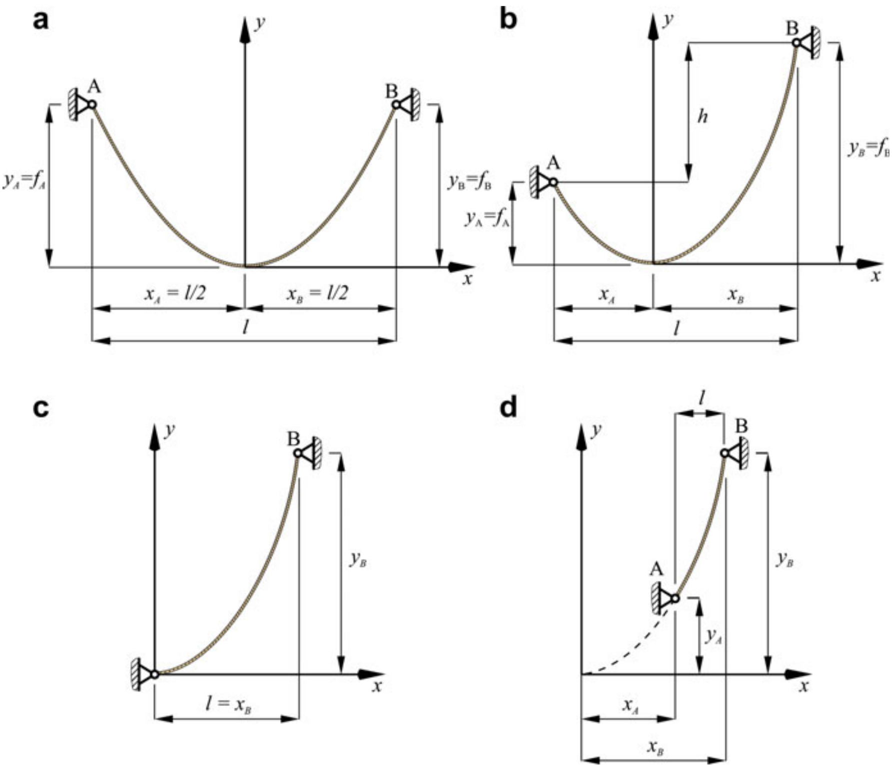


Fig. 10.9 Four different possibilities for the location of the two supports

The situation denoted as *Cases (b)* is most common; *Case (a)* is the simplest, whereas *Case (c)* and *Case (d)* are two special cases that are possible if the cable fulfills certain specific condition.

Case (a) and Case (b):

From Fig. 10.9 we may conclude that a cable will have a sag f, only when two supports A and B are located to the left and to the right of the lowest point, i.e., to the left and right of the coordinate system origin, shown as Case (a) and Case (b) in Fig. 10.9. This condition is fulfilled when

$$x_A = \frac{Hh}{ql} - \frac{l}{2} < 0, \quad \text{or}$$

$$\frac{l^2}{h} > \frac{2H}{q} \tag{10.45}$$

A cable will have a sag f, only when the supports are located to the left and to the right of the lowest point. This condition is fulfilled when $(l^2/h) > (2H/q)$, which is the most common situation.

Condition in (10.45) is always fulfilled when $h \to 0$, and will not be fulfilled only when one of the two supports is significantly higher than the other and/or when the horizontal reaction force H is very large.

By using (10.32), and (10.44a), (10.44b), (10.44c) we can calculate the sag f, relative to support A or to support B,

$$f_{\mathrm{A}} = y_{\mathrm{A}} = \frac{q}{2H} x_{\mathrm{A}}^2 = \frac{q}{2H}\left(\frac{l}{2} - \frac{Hh}{ql}\right)^2 \tag{10.46a}$$

$$f_{\mathrm{B}} = y_{\mathrm{B}} = \frac{q}{2H} x_{\mathrm{B}}^2 = \frac{q}{2H}\left(\frac{l}{2} + \frac{Hh}{ql}\right)^2 \tag{10.46b}$$

When the two supports are at the same elevation, i.e., $h = 0$, we have,

$$f = f_{\mathrm{A}} = f_{\mathrm{B}} = \frac{ql^2}{8H} \tag{10.47}$$

Case (c):
Case (c) in Fig. 10.9 is a special case when

$$x_{\mathrm{A}} = 0, \text{ and } x_{\mathrm{B}} = l \tag{10.48a}$$

Both conditions lead to the same result. We will have the situation denoted as Case (c) when the following condition is fulfilled:

$$\frac{l^2}{h} = \frac{2H}{q} \tag{10.48b}$$

Hence, Case (c) will be present then and only then when (10.48b) is fulfilled. Hence there is only one possible combination l, h, H, and q when this is possible; therefore, in reality it will be encountered very seldom.

Case (d):
Case (d) in Fig. 10.9 will be observed in reality when

$$x_{\mathrm{A}} > 0 \tag{10.49a}$$

or

$$\frac{2H}{q} > \frac{l^2}{h} \tag{10.49b}$$

Such situation will appear when one of the supports is much higher than the other and the horizontal reactions in both supports are very large, hence, $H \to \infty$.

Guidelines and Recipes for Cables with Distributed Loads—Parabolic Solution

- Place coordinate system into the lowest point of the cable and denote location of the two supports as (x_A, y_A) and (x_B, y_B)
- Write equilibrium equations for the external forces, i.e.,

$$\sum_i F_{iy} = A_y + B_y - ql = 0; \quad \sum_i M_{iA} = l \cdot B_y - h \cdot H - \frac{ql^2}{2} = 0$$

Calculate reactions in the supports A and B and their locations (x_A, y_A) and (x_B, y_B), internal force $S(x)$, sag, f of the cable, and its length L by using:

$$x_A = \frac{l}{2} - \frac{Hh}{ql}; \text{ or } \quad x_A = l\frac{f_A}{h}\left(\sqrt{1 + \frac{h}{f_A}} - 1\right); \; x_B = l - x_A$$

$$y(x) = f(x) = \frac{qx^2}{2H}; \; y_A = f_A = \frac{qx_A^2}{2H}; \; y_B = f_B = \frac{qx_B^2}{2H}$$

$$A_y = qx_A; \; B_y = qx_B; \; A = \sqrt{H^2 + \left(qx_A^2\right)}; \; B = \sqrt{H^2 + \left(qx_B^2\right)}$$

$$S(x) = \sqrt{H^2 + (qx)^2}, \text{ and}$$

$$L = x_B\left(1 + \frac{1}{6}\left(\frac{qx_B}{H}\right)^2 - \frac{1}{40}\left(\frac{qx_B}{H}\right)^4 + \cdots\right) + x_A\left(1 + \frac{1}{6}\left(\frac{qx_A}{H}\right)^2 - \frac{1}{40}\left(\frac{qx_A}{H}\right)^4 + \cdots\right)$$

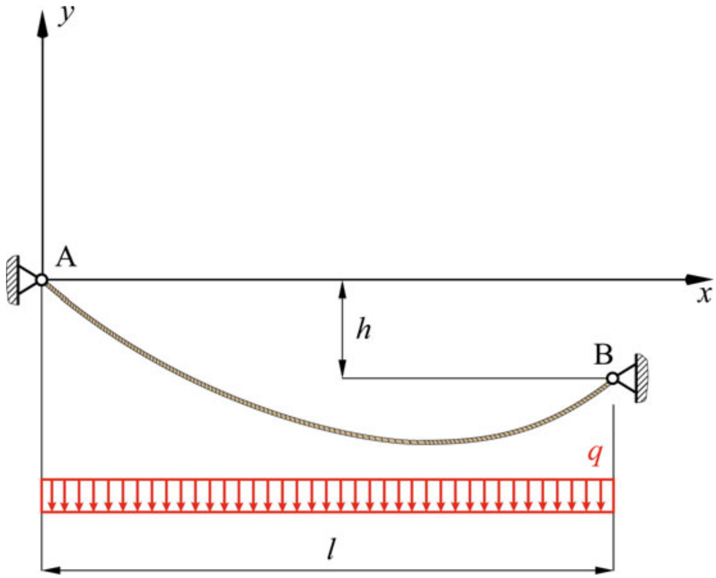


Fig. 10.10 Cable geometry

Example 10.2 Derive the equation describing geometry of a cable in a coordinate system located at the left support as shown in Fig. 10.10. Calculate the maximum force in the cable, its sag f_A relative to support A, and the location of the sag $x(f_A)$. The span between the two supports is $l = 10\,\text{m}$, weight of the cable per unit distance between the two supports is $q = 10\,\text{N/m}$, $l/h = 10$, and $H = 100\,\text{N}$.

Solution We will start from (10.29),

$$y = \frac{q}{2H}x^2 + C_1 x + C_2$$

For the selected coordinate system the boundary conditions are,

$$\begin{aligned} x &= 0; \quad y = 0 \\ x &= l; \quad y = -h \end{aligned}$$

From the first boundary condition we find that $C_2 = 0$. However, the second integration constant is in this case

$$C_1 = -\left(\frac{h}{l} + \frac{q}{2H}l\right)$$

The equation describing the geometry of the cable in the selected coordinate system is then,

$$y = \frac{q}{2H}x^2 - \left(\frac{h}{l} + \frac{q}{2H}l\right)x$$

which is the first result that we are looking for. We see that the relation is indeed more complicated as that presented in (10.32).

The internal force in the cable at any location x is given by (10.34a), (10.34b), and (10.34c),

$$S(x) = \sqrt{S_x^2 + S_y^2} = H\sqrt{1 + (y')^2}$$

Considering the first derivative

$$y' = \frac{q}{H}x - \frac{h}{l} - \frac{q}{2H}l$$

we find,

$$S(x) = H\sqrt{1 + \left(\frac{q}{H}x - \frac{h}{l} - \frac{q}{2H}l\right)^2}$$

Which is again much more complicated equation. Since the left support is higher than the right one the maximum force in the cable will appear at the left support, i.e., $S_{\max} = S(x = 0)$. Hence,

$$S_{\max} = H\sqrt{1 + \left(\frac{h}{l} + \frac{q}{2H}l\right)^2}$$

and

$$S_{\max} = 100\sqrt{1 + \left(0.1 + \frac{10.10}{2 \cdot 100}\right)^2}\,\mathrm{N} \simeq 116.6\mathrm{N}$$

The location of the sag may be obtained also by setting the first derivative of the function describing the geometry of the cable to zero,

$$y' = \frac{q}{H}x - \frac{h}{l} - \frac{q}{2H}l = 0$$

which leads to

$$x(f_{\mathrm{A}}) = \frac{H}{q}\left(\frac{h}{l} + \frac{q}{2H}l\right) = \frac{l}{2} + \frac{Hh}{ql}$$

and

$$x(f_{\mathrm{A}}) = \left(\frac{10}{2} + \frac{100}{10 \cdot 10}\right) = 6\,\mathrm{m}$$

As expected, the obtained expression for $x(f_{\mathrm{A}})$ is the same as that in (10.43), where we need to take into account that h is in this case negative.

The sag f_{A} we obtain by inserting $x(f_{\mathrm{A}})$ into the equation describing geometry of the cable,

$$f_{\mathrm{A}} = y[x(f_{\mathrm{A}})] = \frac{q}{2H}x(f_{\mathrm{A}})^2 - \left(\frac{h}{l} + \frac{ql}{2H}\right)x(f_{\mathrm{A}}) = -\frac{q}{2H}\left(\frac{l}{2} + \frac{Hh}{ql}\right)^2$$

and

$$f_{\mathrm{A}} = -\frac{10}{2 \cdot 100}\left(\frac{10}{2} + \frac{100}{10 \cdot 10}\right)^2 = -1.8\,\mathrm{m}$$

Again, the obtained expression for the sag is the same as that in (10.46a), where we need to take into account that h is now negative. Alternatively, if we take into account the symmetry the above expression must be the same, and it is, as that in (10.46b) for the support B. Negative sign indicates position of the sag in the selected coordinate system.

From this example one may learn that selection of the coordinate system defines boundary conditions and consequently complexity of the form of equations, which describes the geometry of the structural element, in our case cables.

10.2.2 Hyperbolic Solution

The hyperbolic solution of (10.24), i.e.,

$$H\frac{d^2y}{dx^2}dx = dQ$$

is obtained when a continuous load is distributed along the length of a cable, which is usually its weight, and the cable assumes the shape of hyperbola, also known as catenary. In this case line of action of the cable weight is at $L/2$, which location is not known. Therefore we can write the equilibrium equations for external forces for y-direction only:

$$\sum_i F_{iy} = A_y + B_y - p \cdot L = 0 \tag{10.50}$$

By considering that p is weight of the cable per unit length, [N/m or lb/ft], we can write (10.24) as

$$H \frac{d^2 y}{dx^2} dx = dQ = p \cdot dL \tag{10.51a}$$

where

$$dL = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)} dx$$

Now, (10.51a) may be changed to

$$\frac{d^2 y}{dx^2} = \frac{p}{H} \sqrt{1 + \left(\frac{dy}{dx}\right)} \tag{10.51b}$$

The solution of the obtained differential equation presents the geometry of the cable, commonly known as catenary. The solution of (10.51a), (10.51b), given in *Mathematical Corner II*, is

$$y = \frac{H}{p} \cosh\left(\frac{p}{H}\ x + C_1\right) + C_2 \tag{10.52a}$$

and its derivative is

$$y' = \sinh\left(\frac{px}{H} + C_1\right) \tag{10.52b}$$

Hence, cables exposed to their weight will assume the shape of a hyperbolic function, which is commonly called *catenary*.

Hyperbolic solution is obtained when a continuous load is distributed along the length of a cable. In this case the cable assumes the shape known as *catenary*.

Similarly as for the parabolic solution, constants C_1 and C_2 are obtained from the boundary conditions, which also depend on the location of the coordinate system origin and position of the two supports. We will again place the coordinate system into the lowest point of the cable, as shown in Fig. 10.11a.

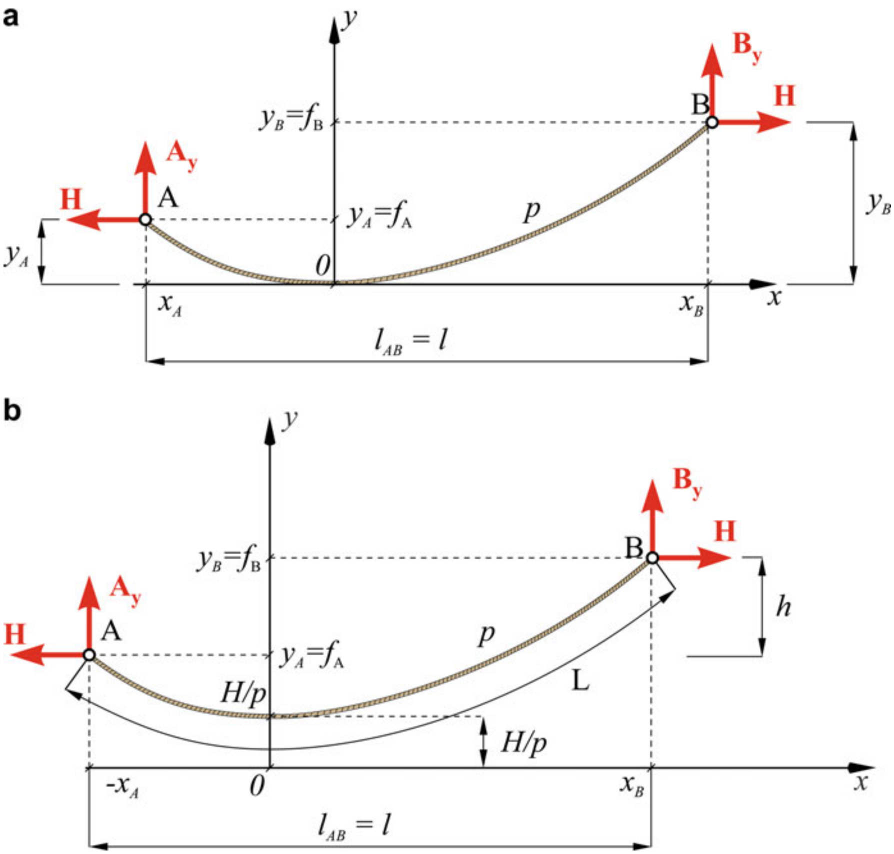


Fig. 10.11 (**a**) Free body diagram of a cable loaded with its weight, p. (**b**) Free body diagram of a cable loaded with its weight p in the coordinate system with the origin H/h below the lowest point of the cable

Mathematical Corner II: Solution of equations (10.51a), (10.51b)

Differential equation

$$\frac{d^2y}{dx^2} = \frac{p}{H}\sqrt{1+\left(\frac{dy}{dx}\right)} \qquad \text{(ii} - 1\text{)}$$

may be solved by introducing substitutions,

$$v = \frac{dy}{dx}; \quad \text{and} \quad \frac{dv}{dx} = \frac{d^2y}{dx^2} \qquad \text{(ii} - 2\text{)}$$

which leads to

(continued)

$\frac{dv}{dx} = \frac{p}{H}\sqrt{1+v^2}$, and after separation of variables to,

$$\frac{dv}{\sqrt{1+v^2}} = \frac{p}{H}dx \qquad \text{(ii} - 3)$$

Obtained equation may be solved simply by integrating both sides of the equation,

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{p}{H}dx + C_1. \qquad \text{(ii} - 4)$$

From the table of integrals (e.g., http://integral-table.com) we find the solution of both indefinite integrals,

$\sinh^{-1}(v) = \frac{p}{H}x + C_1$, which leads to $v = \sinh\left(\frac{p}{H}x + C_1\right)$.

Taking into account that, $v = dy/dx$, we find

$$dy = \sinh\left(\frac{p}{H}x + C_1\right)dx \qquad \text{(ii} - 5)$$

After the integration, we obtain,

$$\int dv = \int \sinh\left(\frac{p}{H}x + C_1\right)dx + C_2 \qquad \text{(ii} - 6)$$

To solve the right side integral we need to introduce substitution

$$\chi = (px/H) + C_1 \qquad \text{(ii} - 7)$$

which leads to

$dx = (H/p)d\chi$ and further

$$\int dy = \frac{H}{p}\int (\sinh\chi)d\chi + C_2 \qquad \text{(ii} - 8)$$

Now we can integrate both sides and obtain the equation of catenary

$$y = \frac{H}{p}\cosh\left(\frac{p}{H}x + C_1\right) + C_2 \qquad \text{(ii} - 9)$$

The cable exposed to its weight only assumes the shape of hyperbolic function also known as *catenary*.

In this case the boundary conditions are the same as for the case of parabolic solution (10.30a), (10.30b), i.e.,

$$x = 0; \quad y = 0 \tag{10.53a}$$

$$x = 0; \quad y' = 0 \tag{10.53b}$$

If we place the coordinate system into the lowest point of the cable the equation of the catenary is:

$$y = \frac{H}{p}\cosh\left(\frac{p}{H}x\right) - \frac{H}{p}$$

Whereas, if we "shift" the coordinate system H/h below the lowest point, the equation obtains its simplest form:

$$y_s = \frac{H}{p}\cosh\left(\frac{p}{H}x\right)$$

Applying the above boundary conditions to (10.52a) and (10.52b), respectively, yields

$$\frac{H}{p}\cosh\left(\frac{p}{H}\cdot 0 + C_1\right) + C_2 = 0 \tag{10.54a}$$

and

$$\sinh(C_1) = 0 \tag{10.54b}$$

Taking into account that $\sinh(z) = (e^z - e^{-z})/2$, and $\cosh(z) = (e^z + e^{-z})/2$, we find the expressions for the two integration constants:

$$C_1 = 0, \quad \text{and} \quad C_2 = -\frac{H}{p} \tag{10.54c}$$

Introducing above expressions into (10.52a) we obtain the equation of the *catenary*,

$$y = \frac{H}{p}\cosh\left(\frac{p}{H}x\right) - \frac{H}{p} \tag{10.55}$$

Equation (10.55) describes geometry of the cable, in the coordinate system which origin is placed at the lowest point of the cable. In this case the second integration constant C_2 is unfortunately not zero, which makes the equation slightly more complicated.

Let's see if we could change the location of the coordinate system so as to get rid of the constant C_2. Such coordinate system one obtains if the origin of the coordinate system is placed H/h below the lowest point of the cable, as it is shown in Fig. 10.11b

$$x = 0; \quad y = H/p \tag{10.56a}$$

$$x = 0; \quad y^{'} = 0 \tag{10.56b}$$

By applying these modified boundary conditions into (10.52a) and (10.52b), respectively, yields

$$\frac{H}{p} = \frac{H}{p}\cosh\left(\frac{p}{H} \cdot 0 + C_1\right) + C_2 \tag{10.57a}$$

and

$$\sinh(C_1) = 0, \tag{10.57b}$$

From (10.57a), (10.57b) we find that by taking into account that in "shifted" coordinate system both constants are zero, $C_1 = 0$, and $C_2 = 0$, the equation for the *catenary* obtains its simplest form

$$y_s = \frac{H}{p}\cosh\left(\frac{p}{H}x\right) \tag{10.58}$$

We have used the subscript "s" to distinguish this solution from the previous one.

10.2.2.1 Internal Tensile Force, *S*(*x*)

As in the case of parabolic solution, the internal tensile force at $S(x)$ any point along the cable is equal to the vector sum of the two components. Hence,

$$S(x) = \sqrt{S_x^2 + S_y^2} = H\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Since $y' = \sinh(px/H)$ for both coordinate systems, we find the relation for the vertical component of the internal force,

$$S_y(x) = H\sinh\left(\frac{px}{H}\right) \tag{10.59a}$$

and the tensile force at any point of the catenary,

$$S(x) = H\sqrt{1 + \left(\sinh\frac{px}{H}\right)^2} = H\cosh\frac{px}{H} \tag{10.59b}$$

Equations (10.59a) and (10.59b) define the internal force in the cable at any point along the cable. However, usually we are interested in the maximal force. We see again that when $x = 0$ we obtain the minimal tensile force $S(x = 0) = H$.

As x increases $S(x)$ will increase and will have again its maximal value at the higher support,

$$S_{\max} = H\cosh\frac{px_k}{H} \tag{10.60a}$$

where $k = A$, for $y_A > y_B$, and $k = B$, for $y_B > y_A$.

The vertical reaction forces in both supports are,

$$A_y = H\sinh\left(\frac{px_A}{H}\right) \quad \text{and} \quad B_y = H\sinh\left(\frac{px_B}{H}\right) \tag{10.60b}$$

whereas

$$A = H\cosh\left(\frac{px_A}{H}\right), \quad \text{and} \quad B = H\cosh\left(\frac{px_B}{H}\right) \tag{10.60c}$$

The magnitude of the maximal tensile force in cable $S_{\max}$ will be equal to the magnitude of the reaction force at the support that is located at the higher elevation.

In case when both supports are at the same elevation we have $x = l/2$, and

$$S_{\max} = A = B = H\cosh\frac{pl}{2H}, \quad \text{and} \tag{10.60d}$$

$$A_y = B_y = H \cdot \sinh\left(\frac{pl}{2H}\right) \tag{10.60e}$$

All above equations require H to be known. In reality we often know (allowable) sag of the cable, i.e., $y_A = f_A$, distance between supports l, and the elevation difference h. In this case we may use (10.55),

$$y_A = f_A = \frac{H}{p}\left(\cosh\left(\frac{px_A}{H}\right) - 1\right), \quad \text{and}$$

$$y_B = f_A + h = \frac{H}{p}\left(\cosh\left(\frac{p(l - x_A)}{H}\right) - 1\right)$$

The two equations may be rewritten as,

$$\cosh\left(\frac{px_A}{H}\right) = \frac{pf_A}{H} + 1, \quad \text{and}$$

$$\cosh\left(\frac{pl}{H} - \frac{px_A}{H}\right) = \frac{p(f_A + h)}{H} + 1$$

Since $\cosh^2(x) - \sinh^2(x) = 1$, we can rewrite the first equation as

$$\sinh\left(\frac{px_A}{H}\right) = \sqrt{\cosh^2\left(\frac{px_A}{H}\right) - 1} = \sqrt{\left(\frac{pf_A}{H} + 1\right)^2 - 1}$$

The second equation may be reorganized, by taking into account that $\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$

$$\cosh\left(\frac{pl}{H}\right)\cosh\left(\frac{px_A}{H}\right) - \sinh\left(\frac{pl}{H}\right)\sinh\left(\frac{px_A}{H}\right) = \frac{p(f_A + h)}{H} + 1$$

After reorganization we find the transcendental equation, which has to be solved numerically,

$$H = (pf_A + H) \cdot \cosh\left(\frac{pl}{H}\right) - \left(\sqrt{(pf_A)^2 + 2pf_A}\right) \cdot \sinh\left(\frac{pl}{H}\right) - p(f_A + h) \tag{10.61}$$

10.2.2.2 Length of the Cable, *L*

As in the cases of parabolic solution, we find the length of the catenary

$$dL = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx$$

By considering that in the both coordinate systems $y' = \sinh(px/H)$, we obtain

$$dL = \sqrt{1 + \sinh^2\left(\frac{px}{H}\right)}dx = \cosh\left(\frac{px}{H}\right)dx$$

The length of the catenary $L_{1,2}$, between two arbitrary points T_1 and T_2, Fig. 10.12, we obtain by integrating the equation over the interval $x \in [x_1, x_2]$,

$$L_{1,2} = \int_{x_1}^{x_2} \cosh\left(\frac{px}{H}\right)dx \tag{10.62}$$

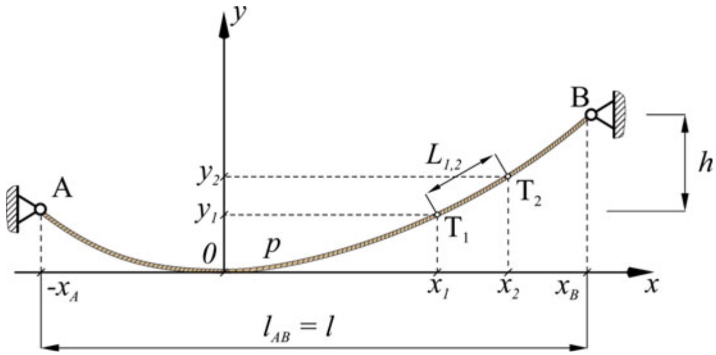


Fig. 10.12 Length of the cable $L_{1,2}$, between the two arbitrary points T_1 and T_2 along the cable

Solution of the above integral is given in *Mathematical Corner III*. The length of the segment between the two arbitrary points T_1 and T_2 is

$$L_{1,2} = [I]_{x_1}^{x_2} = \frac{H}{p}\left[\sinh\left(\frac{px_2}{H}\right) - \sinh\left(\frac{px_1}{H}\right)\right] \tag{10.63a}$$

If we are interested in the total length of cable L, we have to place T_1 into support A, $x_1 = -x_A$, and T_2 into support B, $x_2 = x_B$, to obtain,

$$L = \frac{H}{p}\left[\sinh\left(\frac{px_B}{H}\right) + \sinh\left(\frac{px_A}{H}\right)\right] \tag{10.63b}$$

When the two supports are at the same elevation the distance between the supports A and B is called the *span*. In such cases we have $x_A = x_B = l/2$, and the length of the catenary becomes,

$$L = \frac{2H}{p}\sinh\left(\frac{pl}{2H}\right) \tag{10.63c}$$

Mathematical Corner III: Solution of equation (10.62)

We have to solve the integral, (10.62), i.e.,

$$L_{1,2} = \int_{x_1}^{x_2} \cosh\left(\frac{px}{H}\right)dx \tag{iii − 1}$$

(continued)

First, we have to solve the corresponding indefinite integral I

$$I = \int \cosh\left(\frac{px}{H}\right) dx \qquad \text{(iii} - 2)$$

By introducing the substitution $z = (px/H)$, and $dx = (H/p)dz$ and consulting the table of integrals (e.g., http://integral-table.com) we find that

$$I = \frac{H}{p}\int (\cosh z)dz = \frac{H}{p}\sinh z = \frac{H}{p}\sinh\left(\frac{px}{H}\right) \qquad \text{(iii} - 3)$$

By utilizing the above solution we obtain the length of the cable by introducing the integration limits and considering that $\sinh(-x) = -\sinh(x)$,

$$L_{1,2} = [I]_{x_1}^{x_2} = \frac{H}{p}\left[\sinh\left(\frac{px_2}{H}\right) - \sinh\left(\frac{px_1}{H}\right)\right] \qquad \text{(iii} - 4)$$

10.2.2.3 Sag of the Catenary, *f*

When two supports are at the same elevation the sag of a catenary may be determined from (10.55),

$$f = f_A = f_B = y\left(x = \frac{l}{2}\right) = \frac{H}{p}\cosh\left(\frac{pl}{2H}\right) - \frac{H}{p} \qquad (10.64)$$

However, when the two supports are not at the same elevation the location of the catenary's largest deflection, where we have positioned the coordinate system, is NOT known! To determine the location of the cable lowest point we consider that $x_B + x_A = l$ and $y_B - y_A = h$, as shown in Fig. 10.13.

By using (10.55) and above conditions we find,

$$\frac{H}{p}\cosh\left(\frac{p}{H}(1 - x_A)\right) - \frac{H}{p}\cosh\left(\frac{p}{H}x_A\right) = h \qquad (10.65)$$

Solution of this equation is given in *Mathematical Corner IV*. We have,

$$x_A = \frac{H}{p}\text{arccos}h\left\{\frac{ph}{2H} \pm \sqrt{\frac{\left[\cosh\left(\frac{pl}{H}\right) + 1\right]}{\left[\cosh\left(\frac{pl}{H}\right) - 1\right]}\left(\frac{ph}{2H}\right)^2 + \left[\cosh\left(\frac{pl}{H}\right) + 1\right]}\right\} \qquad (10.66)$$

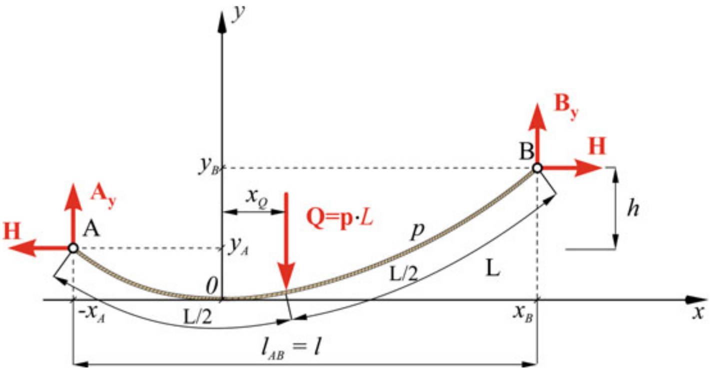


Fig. 10.13 Free body diagram of a cable shown exposed to continuous load distributed along the length of the cable

By knowing location of the coordinate system we may find the sag of the cable relative to support A and B by introducing (10.66) and $x_B = l - x_A$, respectively, into (10.55),

$$f_A = \frac{H}{p}\cosh\left(\frac{p}{H}x_A\right) - \frac{H}{p} \tag{10.67a}$$

$$f_B = \frac{H}{p}\cosh\left(\frac{p}{H}(l - x_A)\right) - \frac{H}{p} \tag{10.67b}$$

Equations (10.66) and (10.67a), (10.67b) are quite cumbersome; therefore, we usually use the parabolic solution to determine the location of the lowest point. However, the question arises how big is the difference between the parabolic and hyperbolic solutions.

We examine this in continuation in Example 10.3.

Mathematical Corner IV: Solution of equation (10.65)

We start from (10.65), i.e.,

$$\frac{H}{p}\cosh\left(\frac{p}{H}(l - x_A)\right) - \frac{H}{p}\cosh\left(\frac{p}{H}x_A\right) = h \tag{iv − 1}$$

Taking into account that $\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$, see, e.g., (http://en.m.wikipedia.org/wiki/Hyperbolic_function), we obtain

(continued)

$$\cosh\left(\frac{pl}{H}\right)\cosh\left(\frac{px_A}{H}\right) - \sinh\left(\frac{pl}{H}\right)\sinh\left(\frac{px_A}{H}\right) - \cosh\left(\frac{p}{H}x_A\right) = \frac{ph}{H} \quad (\text{iv} - 2)$$

By considering that $\cosh^2(x) - \sinh^2(x) = 1$ and introducing $\lambda = \cosh(px_A/H)$ we find

$$\left(\sqrt{\lambda^2 - 1}\right) = \frac{\left[\cosh\left(\frac{pl}{H}\right) - 1\right]}{\sinh\left(\frac{pl}{H}\right)} \cdot \lambda - \frac{ph}{H\sinh\left(\frac{pl}{H}\right)} \quad (\text{iv} - 3)$$

Denoting

$$a = \frac{\left[\cosh\left(\frac{pl}{H}\right) - 1\right]}{\sinh\left(\frac{pl}{H}\right)}, \text{ and } b = \frac{ph}{H\sinh\left(\frac{pl}{H}\right)} \quad (\text{iv} - 4)$$

we obtain a simple relation,

$$\left(\sqrt{\lambda^2 - 1}\right) = a \cdot \lambda - b \quad (\text{iv} - 5)$$

which yields quadratic equation

$$\left(a^2 - 1\right)\lambda^2 - 2ab\lambda + \left(b^2 + 1\right) = 0 \quad (\text{iv} - 6)$$

The solution of the above quadratic equation is $\lambda_{1,2} = \frac{2ab \pm \sqrt{4a^2b^2 - 4(a^2-1)\left(b^2+1\right)}}{2(a^2-1)}$, which may be rearranged as

$$\lambda_{1,2} = \frac{ab \pm \sqrt{b^2 - a^2 + 1}}{(a^2 - 1)} \quad (\text{iv} - 7)$$

By using the explicate expressions for a and b, (iv-4), we may express the parameters in (iv-7) as

$$ab = \frac{ph}{H}\frac{\left[\cosh\left(\frac{pl}{H}\right) - 1\right]}{\sinh^2\left(\frac{pl}{H}\right)} = \frac{ph}{H}\frac{\sqrt{1 + \sinh^2\left(\frac{pl}{H}\right)} - 1}{\sinh^2\left(\frac{pl}{H}\right)} \quad (\text{iv} - 8)$$

(continued)

$$b^2 - a^2 + 1 = \frac{\left(\frac{ph}{H}\right)^2}{\sinh^2\left(\frac{pl}{H}\right)} - \frac{\left[\cosh\left(\frac{pl}{H}\right) - 1\right]^2}{\sinh^2\left(\frac{pl}{H}\right)} + 1$$

$$= \frac{\left(\frac{ph}{H}\right)^2 - 2 + 2\cosh\left(\frac{pl}{H}\right)}{\sinh^2\left(\frac{pl}{H}\right)} \quad \text{(iv} - 9)$$

$$a^2 - 1 = \frac{\left[\cosh\left(\frac{pl}{H}\right) - 1\right]^2}{\sinh^2\left(\frac{pl}{H}\right)} - 1 = \frac{2 - 2\cosh\left(\frac{pl}{H}\right)}{\sinh^2\left(\frac{pl}{H}\right)} \quad \text{(iv} - 10)$$

By utilizing (iv-8), (iv-9), and (iv-10) we may rewrite (iv-7) as

$$\lambda_{1,2} = \frac{\frac{ph}{H}\frac{\left[\cosh\left(\frac{pl}{H}\right)-1\right]}{\sinh^2\left(\frac{pl}{H}\right)} \pm \sqrt{\frac{\left(\frac{ph}{H}\right)^2 - 2 + 2\cosh\left(\frac{pl}{H}\right)}{\sinh^2\left(\frac{pl}{H}\right)}}}{\frac{2-2\cosh\left(\frac{pl}{H}\right)}{\sinh^2\left(\frac{pl}{H}\right)}} \quad \text{(iv} - 11)$$

which reduces to

$$\lambda_{1,2} = -\frac{ph}{2H} \pm \frac{\sinh\left(\frac{pl}{H}\right)\sqrt{\left[\left(\frac{ph}{2H}\right)^2 - \left[1 - \cosh\left(\frac{pl}{H}\right)\right]\right]}}{1 - \cosh\left(\frac{pl}{H}\right)} \quad \text{(iv} - 12)$$

or alternatively,

$$\lambda_{1,2} = -\frac{ph}{2H} \pm \sqrt{\frac{\left[\cosh\left(\frac{pl}{H}\right) + 1\right]}{\left[\cosh\left(\frac{pl}{H}\right) - 1\right]}\left(\frac{ph}{2h}\right)^2 + \left[\cosh\left(\frac{pl}{H}\right) + 1\right]} \quad \text{(iv} - 13)$$

By taking into account (iv-6) (λ) = cosh(p × A/H) and considering that $\cosh(-x) = \cosh(x)$, we find the location of the maximal catenary deflection (sag).

$$x_A = \frac{H}{p} arcosh\left\{\frac{ph}{2H} \pm \sqrt{\frac{\left[\cosh\left(\frac{pl}{H}\right) + 1\right]}{\left[\cosh\left(\frac{pl}{H}\right) - 1\right]}\left(\frac{ph}{2H}\right)^2 + \left[\cosh\left(\frac{pl}{H}\right) + 1\right]}\right\}$$
$$\text{(iv} - 14)$$

Guidelines and Recipes for Cables with Distributed Loads—Hyperbolic Solution

Place a coordinate system into the lowest point of a cable and denote the locations of two supports as $(x_{\mathrm{A}}, y_{\mathrm{A}})$ and $(x_{\mathrm{B}}, y_{\mathrm{B}})$

Calculate the reactions in supports A and B and their locations, $(x_{\mathrm{A}}, y_{\mathrm{A}})$ and $(x_{\mathrm{B}}, y_{\mathrm{B}})$, sag of the cable f, internal force $S(x)$, and the cable's length L by using:

$$x_{\mathrm{A}} = \frac{H}{p} arcosh\left\{\frac{ph}{2H} \pm \sqrt{\frac{\left[\cosh\left(\frac{pl}{H}\right)+1\right]}{\left[\cosh\left(\frac{pl}{H}\right)-1\right]}\left(\frac{ph}{2H}\right)^2 + \left[\cosh\left(\frac{pl}{H}\right)+1\right]}\right\}$$

$$x_{\mathrm{B}} = l - x_{\mathrm{A}}$$

$$y_{\mathrm{A}} = f_{\mathrm{A}} = \frac{H}{p}\left(\cosh\left(\frac{px_{\mathrm{A}}}{H}\right) - 1\right); \quad y_{\mathrm{B}} = f_{\mathrm{A}} + h$$

$$A_y = H\sinh\left(\frac{px_{\mathrm{A}}}{H}\right); \quad B_y = H\sinh\left(\frac{px_{\mathrm{B}}}{H}\right)$$

$$A = H\cosh\left(\frac{px_{\mathrm{A}}}{H}\right), \text{ and } B = H\cosh\left(\frac{px_{\mathrm{B}}}{H}\right)$$

$$S(x) = \sqrt{S_x^2 + S_y^2} + \sqrt{H^2 + \left(H\sinh\left(\frac{px}{H}\right)\right)^2} = H\cosh\left(\frac{px}{H}\right)$$

$$L = \frac{H}{p}\left[\sinh\left(\frac{px_{\mathrm{B}}}{H}\right) + \sinh\left(\frac{px_{\mathrm{A}}}{H}\right)\right]$$

Relation between H and f_{A}:

$$H = (pf_{\mathrm{A}} + H)\cdot\cosh\left(\frac{pl}{H}\right) - \left(\sqrt{(pf_{\mathrm{A}})^2 + 2pf_{\mathrm{A}}}\right)\cdot\sinh\left(\frac{pl}{H}\right) - p(f_{\mathrm{A}} + h)$$

Example 10.3 A cable with weight $p = 8\,\mathrm{N/m}$ hangs between two supports A and B as shown in Fig. 10.14a. The span between the two supports is $l = 1000\,\mathrm{m}$, and the sag is $f = y_A = y_B = 200\,\mathrm{m}$. Determine the maximal internal force in the cable and its length. Compare the results obtained by using parabolic and hyperbolic solutions.

Solution

(a) Parabolic Solution

The corresponding free body diagram is shown in Fig. 10.14b. In this case the weight of the cable is distributed along the horizontal distance between two supports; therefore, the total external load may be directly calculated. Assuming that $q \cong p$ we find,

$$Q = q \cdot l = 8000\,\mathrm{N}$$

In this case, due to symmetry, the vertical reaction forces in both supports will be

$$A_y = B_y = \frac{Q}{2} = 4000\mathrm{N}$$

By using (10.47), $f = ql^2/(8H)$, we can calculate the horizontal force in the cable,

$$H_{\mathrm{par}} = \frac{ql^2}{8f} = \frac{8 \cdot 1000^2}{8 \cdot 200} = 5000\,\mathrm{N}$$

and the maximal force which will be equal to the reaction forces in the supports,

$$S_{\mathrm{max,par}} = A = B = \sqrt{A_y^2 + H^2} = \sqrt{4000^2 + 5000^2} \cong 6400\,\mathrm{N}$$

The length of the cable we obtain from (10.40c),

$$L = l\left[1 + \frac{2}{3}\left(\frac{2f}{l}\right)^2 - \frac{2}{5}\left(\frac{2f}{l}\right)^4 + \cdots\right], \text{ hence}$$

$$L_{\mathrm{par}} = 1000 \cdot \left[1 + \frac{2}{3}\left(\frac{2 \cdot 200}{1000}\right)^2 - \frac{2}{5}\left(\frac{2 \cdot 200}{1000}\right)^4 + \cdots\right] \cong 1096\,\mathrm{m}$$

We have used subscript "*par*" to indicate parabolic solutions.

(b) Hyperbolic Solution

The free body diagram corresponding to hyperbolic solution is shown in Fig. 10.14c.

Now the weight of the cable is distributed along its length which is not known; therefore, reactions in both supports cannot be determined directly. First we need to determine horizontal force H, using (10.64),

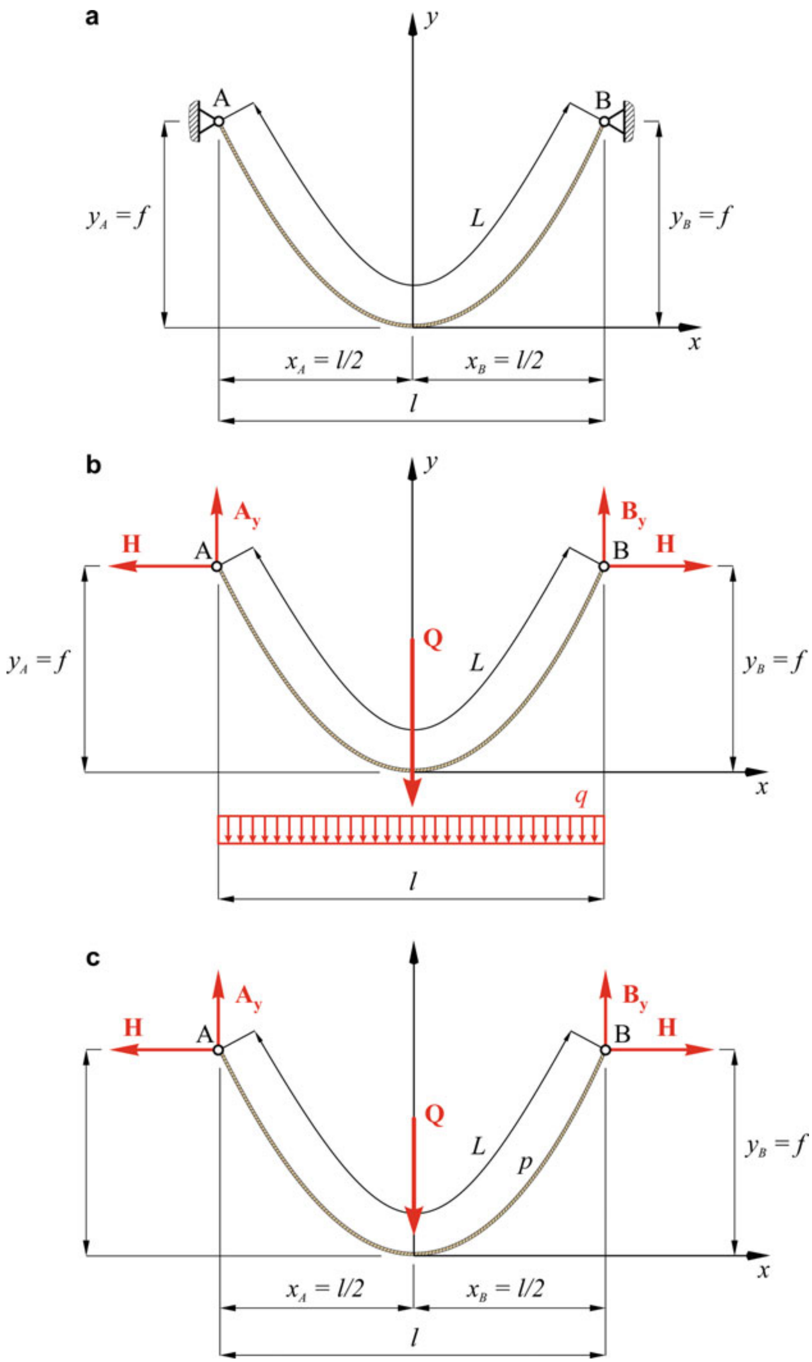


Fig. 10.14 (**a**) Physical model of a cable loaded with its weight. (**b**) Free body diagram of a cable loaded with continuous load distributed along the horizontal distance between two supports. (**c**) Free body diagram of a cable loaded with its weight

$$f = \frac{H}{p}\cosh\left(\frac{pl}{2H}\right) - \frac{H}{p}$$

This equation is transcendental and has no close solution, hence, we have to solve it numerically. After inserting numerical values and reorganizing the equation we obtain,

$$\frac{H}{8} + 200 = \frac{H}{8}\cosh\left(\frac{4000}{H}\right), \quad \text{and}$$

$$\frac{4}{10}\cdot\frac{4000}{H} = \cosh\left(\frac{4000}{H}\right) - 1 \tag{10.68a}$$

By introducing $\chi = 4000/H$ we obtain form of the equation that is easy to solve iteratively,

$$\frac{10}{4}\cdot[\cosh(\chi) - 1] = \chi \tag{10.68b}$$

The solution is given in *Mathematical Corner V*,

$$H_{\text{hyp}} = \frac{4000}{\chi} = \frac{4000}{0.762} = 5250\,\text{N}$$

By knowing H we can calculate the length of the cable by using (10.63c),

$$L_{\text{hyp}} = \frac{2H}{p}\sinh\left(\frac{pl}{2H}\right) = \frac{2\cdot 5250}{8}\sinh\left(\frac{8\cdot 1000}{2\cdot 5250}\right) \cong 1100\,\text{m}$$

Now we are ready to calculate the vertical reaction forces in supports A and B,

$$A_y = B_y = \frac{p\cdot L}{2} = \frac{8\cdot 1099.7}{2} \cong 4400\,\text{N}$$

and maximal internal force in the cable, which is equal to the reaction forces in supports A and B,

$$S_{\text{max, hyp}} = A = B = \sqrt{A_y^2 + H^2} = \sqrt{4398.8^2 + 5247^2} \cong 6850\,\text{N}$$

We have used subscript "*hyp*" to indicate hyperbolic solutions.

(c) Comparison of Parabolic and Hyperbolic Solutions

By comparing the absolute and relative differences between parabolic and hyperbolic solutions for H, S_{max}, and L we find,

$$\Delta H = H_{\text{par}} - H_{\text{hyp}} = (5000 - 5250)\,\text{N} = -250\,\text{N}$$

$$\delta_H = \frac{\Delta H}{H_{\text{hyp}}} \cdot 100\,\% = \frac{-250}{5250} \cdot 100\,\% \cong -4.76\,\%$$

$$\Delta S_{\max} = S_{\max,\text{par}} - S_{\max,\text{hyp}} = (6400 - 6850)\,\text{N} \cong -450\,\text{N}$$

$$\delta_S = \frac{\Delta S_{\max}}{S_{\max,\text{hyp}}} \cdot 100\,\% = \frac{-450}{6850} \cdot 100\,\% \cong -6.57\,\%$$

$$\Delta L = L_{\text{par}} - L_{\text{hyp}} = (1096 - 1100)\,\text{m} = -4.00\,\text{m}$$

$$\delta_L = \frac{\Delta L}{L_{\text{hyp}}} \cdot 100\,\% = \frac{-4.00}{1100} \cdot 100\,\% \cong -0.364\,\%$$

We have learned an important fact that the parabolic solution underestimates all values; however, the error is less than 10 %. When designing cable structures this discrepancy has to be taken into account by an appropriate safety factor.

Mathematical Corner V: Numerical Solution of equations (10.68a), (10.68b)

We start from (10.68a), (10.68b),

$$\frac{10}{4} \cdot \left[\cosh\left(\chi_{i+1}\right) - 1\right] = \chi_i \qquad (\text{v} - 1)$$

i	1	2	3	4	5	6
χ_i	1.0	0.8670	0.8105	0.7849	0.7730	0.7674
χ_{i+1}	0.8670	0.8105	0.7849	0.7730	0.7674	0.7648
i	7	8	9	10	11	12
χ_i	0.7648	0.7635	0.7629	0.7626	0.7625	0.7624
χ_{i+1}	0.7635	0.7629	0.7626	0.7625	0.7624	0.7624

The solution of the above transcendental equation is

$$\chi = 0.762 \qquad (\text{v} - 2)$$

and after inserting into $\chi = 4000/H$

$$H = \frac{4000}{\chi} = \frac{4000}{0.7624} = 5250\,\text{N} \qquad (\text{v} - 3)$$

Example 10.4 A cable with the weight $p = 50\,\text{N/m}$ hangs between two supports A and B, which have different elevation. B is $h = 10\,\text{m}$ higher than A, as shown in Fig. 10.15a. The horizontal distance between the supports is $l = 80\,\text{m}$, and the sag is $f_A = y_A = 20\,\text{m}$. Determine the maximal internal force in the cable, its length and location of the sag. Compare the results obtained with the parabolic and hyperbolic solutions.

Solution

(a) Parabolic Solution

In case of the parabolic solution we assume that the load of the cable is distributed along the horizontal distance between the two supports; therefore, the total external load may be directly calculated. Free body diagram is shown in Fig. 10.15b.

By assuming that $q \cong p$ we find,

$$Q = q \cdot l = q \cdot x_A + q \cdot x_B = Q_A + Q_B = 4000\,\text{N}$$

Since vertical locations of both supports are known we can use (10.32), $y = qx^2/(2H)$ to obtain locations of the two supports,

$$y_A = \frac{qx_A^2}{2H} = f, \quad \text{and} \quad y_B = \frac{q(l - x_A)^2}{2H} = f + h$$

From both equations we express $2H/q$ and obtain

$$\frac{2H}{q} = \frac{x_A^2}{f} = \frac{(l - x_A)^2}{f + h}$$

which reduces to $\frac{h}{f}x_A^2 + 2lx_A - l^2 = 0$, and

$$x_A^2 + 320x_A - 12800 = 0$$

Quadratic equation has two solutions $x_{A1} = 36.0$ and $x_{A2} = 356$, out of which only the first one has correct physical meaning. Hence,

$$x_{A,\text{par}} = 36.0\,\text{m}, \quad \text{and} \quad x_{B,\text{par}} = l - x_A = 80 - 36.0 = 44.0\text{m}$$

By using (10.32), $y_A = \frac{qx_A^2}{2H} = f$, we can calculate the horizontal force in the cable,

$$H_{\text{par}} = \frac{qx_A^2}{2f} = \frac{50 \cdot 36^2}{2 \cdot 20} = 1620\,\text{N}$$

By knowing x_A and x_B we can calculate the vertical reactions in supports A and B by using (10.42a), (10.42b),

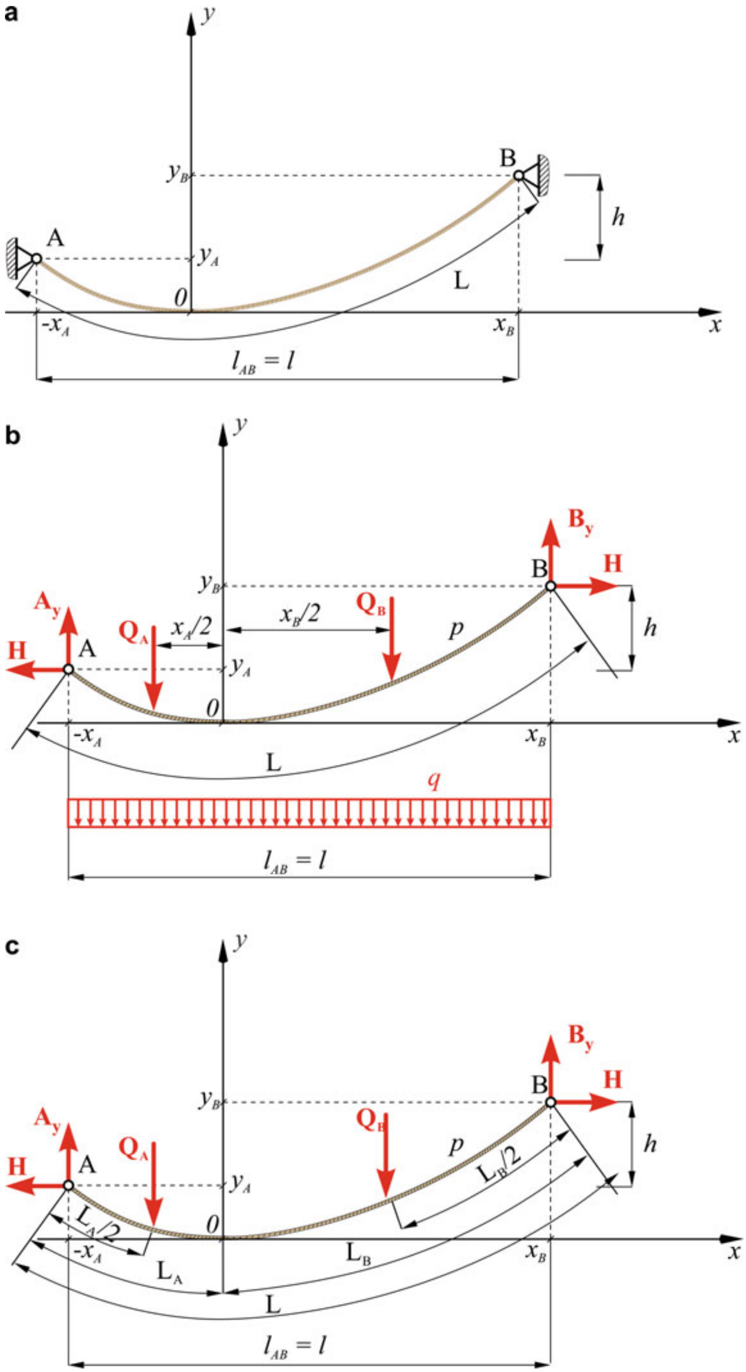


Fig. 10.15 (**a**) Physical model of a cable loaded with its weight. (**b**) Free body diagram of a cable loaded with a continuous load distributed along the horizontal distance between two supports. (**c**) Free body diagram of a cable, which supports are at different elevations, is loaded with its weight

$$A_{y,\text{par}} = Q_\text{A} = qx_\text{A} = 50 \cdot 36.0 = 1800\,\text{N, and}$$
$$B_{y,\text{par}} = Q_\text{B} = qx_\text{B} = 50 \cdot 44.0 = 2200\,\text{N}$$

The maximal force will appear at the support which is higher, i.e., at support B,

$$S_{\max,\text{par}} = B = \sqrt{B_y^2 + H^2} = \sqrt{2200^2 + 1620^2} \cong 2730\,\text{N}$$

The length of the cable we obtain from (10.38c),

$$\begin{aligned} L_\text{par} &= x_\text{B}\left(1 + \frac{1}{6}\left(\frac{qx_\text{B}}{H}\right)^2 - \frac{1}{40}\left(\frac{qx_\text{B}}{H}\right)^4 + \cdots\right) \\ &\quad + x_\text{A}\left(1 + \frac{1}{6}\left(\frac{qx_\text{A}}{H}\right)^2 - \frac{1}{40}\left(\frac{qx_\text{A}}{H}\right)^4 + \cdots\right) \\ &= L_\text{B} + L_\text{A} \end{aligned}$$

By inserting numerical values we find,

$$L_{\text{B,par}} = 44.0 \cdot \left(1 + \frac{1}{6}\left(\frac{50 \cdot 44.0}{1620}\right)^2 - \frac{1}{40}\left(\frac{50 \cdot 44.0}{1616}\right)^4 + \cdots\right) = 53.8\,\text{m}$$

$$L_{\text{A,par}} = 36.0 \cdot \left(1 + \frac{1}{6}\left(\frac{50 \cdot 36.0}{1616}\right)^2 - \frac{1}{40}\left(\frac{50 \cdot 36.0}{1616}\right)^4 + \cdots\right) = 42.0\,\text{m}$$

and

$$L_\text{par} = 53.8\,\text{m} + 42\,\text{m} = 95.8\,\text{m}$$

We have used again subscript "*par*" to indicate the parabolic solutions.
(b) Hyperbolic Solution
A free body diagram corresponding to the hyperbolic solution is shown in Fig. 10.15c.

The weight of the cable is distributed along its length, which is not known; therefore, the reactions in both supports cannot be determined directly. Since vertical locations of both supports are known we can use (10.55),

$$y = \frac{H}{p}\cosh\left(\frac{px}{H}\right) - \frac{H}{p}$$

to obtain the locations of the two supports x_A and x_B and the horizontal force **H**,

$$y_{\mathrm{A}} = f = \frac{H}{p}\left(\cosh\left(\frac{p x_{\mathrm{A}}}{H}\right) - 1\right), \text{ and}$$

$$y_{\mathrm{B}} = f + h = \frac{H}{p}\left[\cosh\left(\frac{p(l - x_{\mathrm{A}})}{H}\right) - 1\right]$$

The two equations may be rewritten as,

$$\cosh\left(\frac{p x_{\mathrm{A}}}{H}\right) = \left(\frac{f \cdot p}{H} + 1\right) = \left(\frac{1000 + H}{H}\right), \text{ and}$$

$$\cosh\left(\frac{pl}{H} - \frac{p x_{\mathrm{A}}}{H}\right) = \left(\frac{(f + h)p}{H} + 1\right) = \left(\frac{1500 + H}{H}\right)$$

Using $\cosh^2(x) - \sinh^2(x) = 1$, and first equation we find,

$$\sinh\left(\frac{p x_{\mathrm{A}}}{H}\right) = \sqrt{\cosh^2\left(\frac{p x_{\mathrm{A}}}{H}\right) - 1} = \sqrt{\left(\frac{f \cdot p}{H} + 1\right)^2 - 1}$$

and after inserting numerical values,

$$\sinh\left(\frac{p x_{\mathrm{A}}}{H}\right) = \sqrt{\left(\frac{1000}{H} + 1\right)^2 - 1} = \frac{20}{H}\sqrt{5H + 2500}$$

Now, by using the second equation and taking into account that $\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$, we obtain,

$$\cosh\left(\frac{pl}{H}\right)\cosh\left(\frac{p x_A}{H}\right) - \sinh\left(\frac{pl}{H}\right)\sinh\left(\frac{p x_A}{H}\right) = \frac{(f + h)p}{H} + 1$$

and

$$\cosh\left(\frac{4000}{H}\right) \cdot \left(\frac{1000 + H}{H}\right) - \sinh\left(\frac{4000}{H}\right) \cdot \frac{20}{H} \cdot \sqrt{5H + 2500} = \frac{1500 + H}{H}$$

After reorganization we find transcendental equation, which need to be solved numerically,

$$H = (1000 + H)\cosh\left(\frac{4000}{H}\right) - 100\sqrt{H + 500} \cdot \sinh\left(\frac{4000}{H}\right) - 1500$$

To solve it we need to write a small program by using MATLAB or similar. The result is,

$$H_{\mathrm{hyp}} = 1795\,\mathrm{N}$$

Now we can calculate the locations of supports A and B by using

$$\cosh\left(\frac{px_{\mathrm{A}}}{H}\right) = \left(\frac{1000 + H}{H}\right) = \frac{1000 + 1795}{1795} \cong 1.56, \quad \text{and}$$

$$x_{\mathrm{A,\,hyp}} = \frac{H}{p}\cosh^{-1}\left(\frac{1000 + H}{H}\right) = \frac{1795}{50}\cosh^{-1}(1.56) = 36.4\mathrm{m}$$

$$x_{\mathrm{B,\,hyp}} = l - x_{\mathrm{A}} = 80 - 36.4 = 43.6\,\mathrm{m}$$

The maximal internal force will appear at support B and can be calculated from (10.60a)

$$S_{\mathrm{max,\,hyp}} = H\cosh\frac{px_{\mathrm{B}}}{H} = 1795 \cdot \cosh\left(\frac{50 \cdot 43.6}{1795}\right) \cong 3290\,\mathrm{N}$$

We may also calculate the vertical components of the reactions in supports A and B from (10.60b)

$$A_{y,\,\mathrm{hyp}} = H\sinh\left(\frac{px_{\mathrm{A}}}{H}\right) = 1795 \cdot \sinh\left(\frac{50 \cdot 36.4}{1795}\right) \cong 2150\,\mathrm{N}$$

and

$$B_{y,\,hyp} = H\sinh\left(\frac{px_{\mathrm{B}}}{H}\right)A_y = 1795 \cdot \sinh\left(\frac{50 \cdot 43.59}{1795}\right) \cong 2760\,\mathrm{N}$$

The length of the cable we find from (10.63b),

$$L_{\mathrm{hyp}} = \frac{H}{p}\left[\sinh\left(\frac{px_{\mathrm{B}}}{H}\right) + \sinh\left(\frac{px_{\mathrm{A}}}{H}\right)\right] = L_{\mathrm{B,\,hyp}} + L_{\mathrm{A,\,hyp}}$$

$$L_{\mathrm{A,\,hyp}} = \frac{1795}{50} \cdot \sinh\left(\frac{50 \cdot 36.41}{1795}\right) \cong 43.0\,\mathrm{m}$$

$$L_{\mathrm{hyp}} = \frac{1795}{50} \cdot \sinh\left(\frac{50 \cdot 43.59}{1795}\right) \cong 55.2\,\mathrm{m}$$

$$L_{\mathrm{hyp}} = L_{\mathrm{B,\,hyp}} + L_{\mathrm{A,\,hyp}} = 98.2\,\mathrm{m}$$

We have used subscript “hyp” to indicate hyperbolic solutions.

(c) Comparison of Parabolic and Hyperbolic Solutions
By comparing the absolute and relative differences between parabolic and hyperbolic solutions for H, $S_{\max}$, and L we find,

$$\Delta H = H_{\text{par}} - H_{\text{hyp}} = (1616.4 - 1795)\,\text{N} = -178.6\,\text{N}$$

$$\delta_H = \frac{\Delta H}{H_{\text{hyp}}} \cdot 100\,\% = \frac{-178.6}{1795} \cdot 100\,\% \cong -9.9\,\%$$

$$\Delta S_{\max} = S_{\max,\,\text{par}} - S_{\max,\,\text{hyp}} = (2730 - 3290)\,\text{N} \cong -560\,\text{N}$$

$$\delta_S = \frac{\Delta S_{\max}}{S_{\max,\,\text{hyp}}} \cdot 100\,\% = \frac{-560}{3290} \cdot 100\,\% \cong -17.02\,\%$$

$$\Delta L = L_{\text{par}} - L_{\text{hyp}} = (95.8 - 98.2)\,\text{m} = -2.4\,\text{m}$$

$$\delta_L = \frac{\Delta L}{L_{\text{hyp}}} \cdot 100\,\% = \frac{-2.4}{98.2} \cdot 100\,\% \cong -2.44\,\%$$

In this case the discrepancy is much larger than in the previous one, which is mainly due to the larger sag of the cable. In general parabolic solutions become less accurate when the sag of the cable is more than 20 % of the horizontal distance between the two supports.

Example 10.5 Figure 10.16 shows a cable loaded with continuous load $q = 10\,\text{kN/m}$, distributed along the distance between two supports. Distance

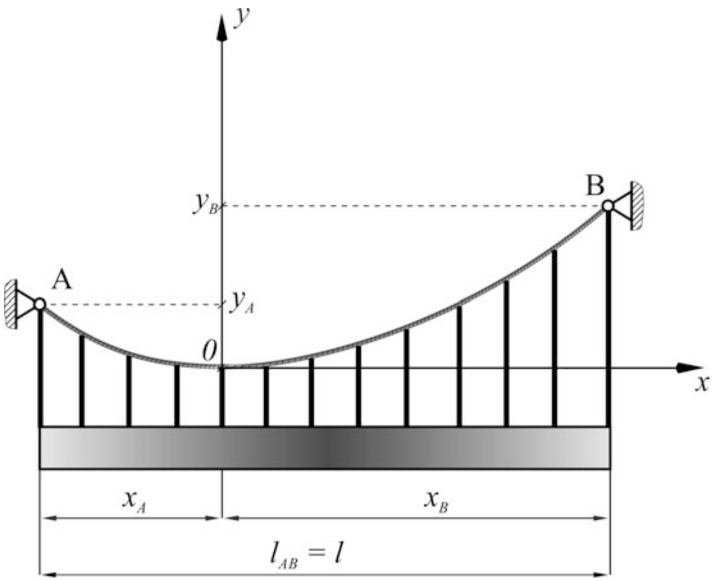


Fig. 10.16 Physical model of a cable loaded by a distributed load

between the supports is $l = 50\,\text{m}$. Location of the two supports relative to the lowest point is known, $y_A = 15\,\text{m}$ and $y_B = 21\,\text{m}$. Determine the reaction forces in both supports and the sag location.

Solution Since vertical locations of both supports are known we can use (10.32) to obtain the locations of the two supports,

$$y_A = \frac{qx_A^2}{2H}, \text{and } y_B = \frac{q(l - x_A)^2}{2H}$$

From both equations we obtain

$$\frac{2H}{q} = \frac{x_A^2}{y_A} = \frac{(l - x_A)^2}{y_B}$$

which can be reduced to

$$x_A^2 + 250x_A - 6250 = 0$$

The quadratic equation has two solutions out of which only one has the correct physical meaning. Hence,

$$x_A = 22.9\,\text{m}, \quad \text{and} \quad x_B = l - x_A = 50 - 22.9 = 27.1\,\text{m}$$

By using (10.32), $y_A = \frac{qx_A^2}{2H}$, we can calculate the horizontal force in the cable,

$$H = \frac{qx_A^2}{2y_A} = \frac{10 \cdot 22.90^2}{2 \cdot 15} = 174.8\,\text{kN}$$

By knowing x_A and x_B we can calculate vertical reactions in supports A and B by using (10.42a), (10.42b),

$$\begin{aligned} A_y &= qx_A = 10 \cdot 22.90 = 229\,\text{kN, and} \\ B_Y &= qx_B = 10 \cdot 27.1 = 271\,\text{kN} \end{aligned}$$

The maximal force will appear at the support which is higher, i.e., at support B,

$$A = \sqrt{H^2 + A_y^2} = \sqrt{174.8^2 + 229^2} \cong 288\,\text{kN}$$

$$B = \sqrt{H^2 + B_y^2} = \sqrt{174.8^2 + 271^2} \cong 323\,\text{kN}$$

What We Have Learned?

How to calculate internal forces in cables loaded with concentrated forces

When external concentrated loads are much larger than the weight of the cable we may neglect the latter and assume that individual segments of the cable are straight, and solve the problem by using the methodology developed for solving truss structures. We place the coordinate system into the left support and define location l_i of the lines of action of external forces F_i relative to the left support. In this case equilibrium equations may be written as

$$\sum_i F_{iy} = A_y + B_y - \sum_{i=1}^{i=N} F_i = 0; \quad \sum_i M_{iA} = B_y \cdot l_{AB} - H \cdot h - \sum_{i=1}^{i=N} l_i \cdot F_i = 0$$

Internal forces in all sections of the cable S_k, and its geometry: α_k, h_k, and L may be calculated by using equations:

$$A_y = S_{1y} = S_1 \sin\alpha_1 = H \cdot \tan\alpha_1 \; B_y = S_{(N+1)y} = S_{(N+1)} \sin\alpha_{(N+1)} = H \cdot \tan\alpha_{(N+1)}$$

$$S_k \cos\alpha_k = H; \quad S_k = H\sqrt{1 + \tan^2\alpha_k}$$

$$S_{ky} = \sum_{i=1}^{k-1} F_i - A_y; \quad S_{ky} = B_y - \sum_{i=k}^{N} F_i; \quad \tan\alpha_k = \frac{S_{ky}}{H}$$

$$h_k = l_k \cdot \tan\alpha_k; \quad \text{and} \quad L = \sum_{i=1}^{N+1} \frac{l_i - l_{i-l}}{\cos(\alpha_i)}$$

How to calculate internal forces in cables loaded with continuous load distributed along the horizontal distance between two supports—parabolic solution
When the load of a cable is distributed continuously along the horizontal distance between two supports the cable assumes the shape of a parabola. By placing the coordinate system into the lowest point of the cable its geometry may be described with a simple formula $y = qx^2/(2H)$. In this case the equilibrium equations for the external forces are:

$$\sum_i F_{iy} = A_y + B_y - ql = 0; \quad \sum_i M_{i\mathrm{A}} = l \cdot B_y - h \cdot H - \frac{ql^2}{2} = 0$$

The locations of the two supports $(x_\mathrm{A}, y_\mathrm{A})$ and $(x_\mathrm{B}, y_\mathrm{B})$ may be obtained from

$$x_\mathrm{A} = \frac{l}{2} - \frac{Hh}{ql}, \quad \text{or} \quad x_\mathrm{A} = l\frac{f_\mathrm{A}}{h}\left(\sqrt{1 + \frac{h}{f_\mathrm{A}}} - 1\right); \quad x_\mathrm{B} = l - x_\mathrm{A}$$

$$\text{and,} \quad y_\mathrm{A} = f_\mathrm{A} = \frac{qx_\mathrm{A}^2}{2H}; \quad y_\mathrm{B} = f_\mathrm{B} = \frac{qx_\mathrm{B}^2}{2H}$$

We find the reactions acting in supports A and B from

$$A_y = qx_\mathrm{A}; \quad B_y = qx_\mathrm{B}; \quad A = \sqrt{H^2 + \left(qx_\mathrm{A}^2\right)}; \quad B = \sqrt{H^2 + \left(qx_\mathrm{B}^2\right)}$$

Whereas, the internal force in the cable may be calculated from

$$S(x) = \sqrt{H^2 + (qx)^2}, \text{ where } H = \frac{ql^2}{h} \cdot \left[\frac{1}{2} - \frac{f_A}{h}\left(\sqrt{1 + \frac{h}{f_A}} - 1\right)\right]$$

The length of the cable we find from

$$L = x_B\left(1 + \frac{1}{6}\left(\frac{qx_B}{H}\right)^2 - \frac{1}{40}\left(\frac{qx_B}{H}\right)^4 + \cdots\right) + x_A\left(1 + \frac{1}{6}\left(\frac{qx_A}{H}\right)^2 - \frac{1}{40}\left(\frac{qx_A}{H}\right)^4 + \cdots\right), \text{ or}$$

$$L = x_B\left(1 + \frac{2}{3}\left(\frac{y_B}{x_B}\right)^2 - \frac{2}{5}\left(\frac{y_B}{x_B}\right)^4 + \cdots\right) + x_A\left(1 + \frac{2}{3}\left(\frac{y_A}{x_A}\right)^2 - \frac{2}{5}\left(\frac{y_A}{x_A}\right)^4 + \cdots\right)$$

How to calculate the internal forces in cables loaded with continuous load distributed along the cable's length—hyperbolic solution

When the load of the cable is distributed continuously along its length the cable assumes the shape of a hyperbola, commonly called catenary. By placing the coordinate system into the lowest point the geometry may be described as

$$y = \frac{H}{p}\cosh\left(\frac{p}{H}x\right) - \frac{H}{p}$$

In this case the line of action of the cable's weight is at $L/2$, which location is not known. Therefore we can write the equilibrium equations for external forces for the y-direction only:

$$\sum_i F_{iy} = A_y + B_y - p \cdot L = 0$$

The locations of two supports (x_A, y_A) and (x_B, y_B) may be obtained from

$$x_A = \frac{H}{p}\operatorname{arccosh}\left\{\frac{ph}{2H} \pm \sqrt{\frac{\left[\cosh\left(\frac{pl}{H}\right) + 1\right]}{\left[\cosh\left(\frac{pl}{H}\right) - 1\right]}\left(\frac{ph}{2H}\right)^2 + \left[\cosh\left(\frac{pl}{H}\right) + 1\right]}\right\}$$

$$x_B = l - x_A$$

and

$$y_A = f_A = \frac{H}{p}\left(\cosh\left(\frac{px_A}{H}\right) - 1\right); \quad y_B = f_A + h$$

The reactions acting in supports A and B we find from

$$A_y = H\sinh\left(\frac{px_A}{H}\right); \quad B_y = H\sinh\left(\frac{px_B}{H}\right), \quad \text{and}$$

$$A = H\cosh\left(\frac{px_A}{H}\right), \quad \text{and} \quad B = H\cosh\left(\frac{px_B}{H}\right)$$

Whereas, the internal force in the cable may be calculated from

$$S(x) = \sqrt{S_x^2 + S_y^2} = \sqrt{H^2 + \left(H\sinh\left(\frac{px}{H}\right)\right)^2} = H\cosh\left(\frac{px}{H}\right)$$

The length of the cable we find from

$$L = \frac{H}{p}\left[\sinh\left(\frac{px_B}{H}\right) + \sinh\left(\frac{px_A}{H}\right)\right]$$

The relation between H and f_A is given by a transcendental equation. It should be solved numerically:

$$H = (pf_A + H)\cdot\cosh\left(\frac{pl}{H}\right) - \left(\sqrt{(pf_A)^2 + 2pf_A}\right)\cdot\sinh\left(\frac{pl}{H}\right) - p(f_A + h)$$

10.3 Problems

10.1 Calculate the internal forces in a 13 m long cable holding the street sign shown in Fig. P10.1. Its weight is 80 N. The distance between the attachment points is 12 m. They are at the same height from the street level. The sign is located 5 m from the left attachment point.

Fig. P10.1 Street sign

10.2 Find the location and magnitude of the maximum tensile force in the cable loaded by the corn cobs, as shown in Fig. P10.2. Assume that the total weight of the cobs is 500 N.

Fig. P10.2 Corn cobs on the cable

10.3 A cable is attached to supports A and B, Fig. P10.3. The distance between the two points is $l = 80$ m, and the length of the cable is $L = 81$ m. Cable's weight per unit length is $q = 8$ N/m. Determine the maximal tensile force, S_{max}, and sag of the cable f.

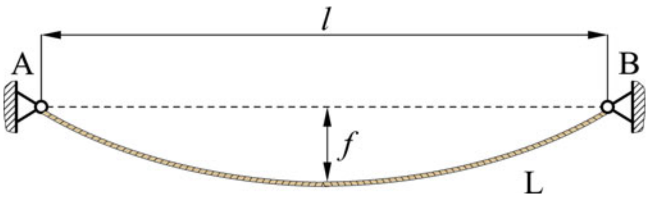


Fig. P10.3

10.4 A cable is attached to two supports A and B, Fig. P10.4. The distance between the two points in vertical direction is $h = 6$ m, and in horizontal direction $l = 30$ m. The cable's weight per unit length is $q = 4$ N/m. Determine the location of the maximum sag of the cable f, length of the cable L, and the maximal tensile force S_{max}, in the cable.

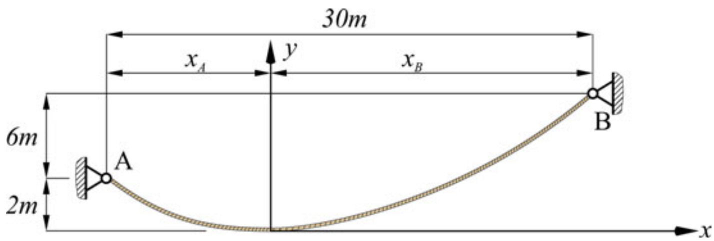


Fig. P10.4

10.5 A cable is attached to support A, passes over a small pulley at support B, and supports load **G**, Fig. P10.5. The cable's weight per unit length is $q = 8$ N/m. By knowing that the distance between A and B is $l = 8$ m, and the sag of the cable is $f = 0.3$ m, determine the magnitude of load **G**, the slope of the cable at support B, and the total length of the cable L from A to B.

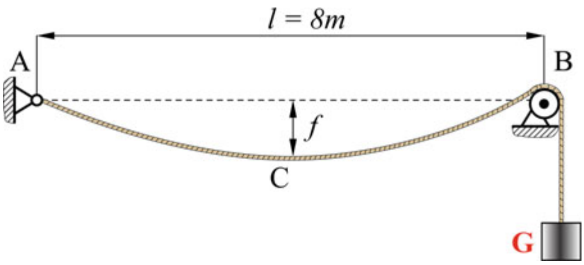


Fig. P10.5

10.6 A cable is attached to supports A and B, Fig. P10.6. The cable supports three vertical loads at points C, D, and E, as shown in Fig. P10.6. The external loads are: $F_1 = 400$ N, $F_2 = 600$ N, and $F_3 = 500$ N. Determine the reactions at points A and B, and the maximum tensile force, S_{max}, in the cable.

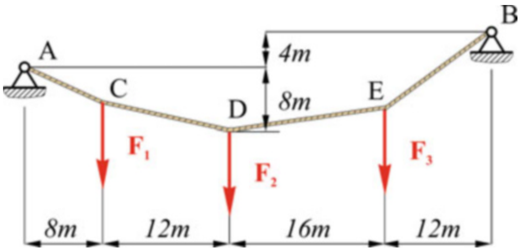


Fig. P10.6

10.7 A cable is attached to supports A and B, and supports three vertical loads from the points indicated in Fig. P10.7: $F_1 = 150$ N, $F_2 = 300$ N, and $F_3 = 500$ N. The sag of the cable at point C is $f = 12$ m. Determine the reactions at points A and B, and the sags of the cable at points B and D.

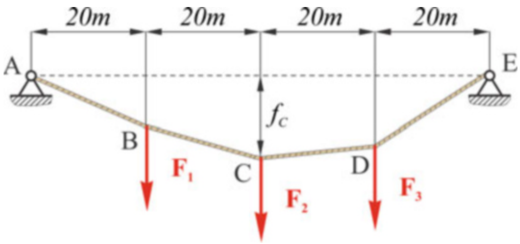


Fig. P10.7

10.8 A cable is attached to supports A and B, and supports three vertical loads as indicated in Fig. P10.8: $F_1 = 200$ N, $F_2 = 50$ N, and $F_3 = 200$ N. The sag of the cable at point C is $f = 12$ m. Determine the reactions at points A and B, and the maximal tensile force S_{max} in the cable.

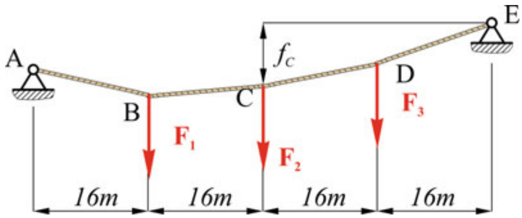


Fig. P10.8

10.9 A cable is attached to supports A and B, and supports two vertical loads from the points indicated in Fig. P10.9: $F_1 = 100$ N and $F_2 = 150$ N. Horizontal component of the tension force in the cable is equal to $H = 200$ N. Determine the reactions at points A and B, the length of cable L, and tensile force S in each section of the cable.

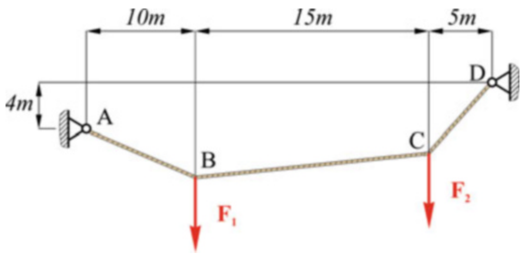


Fig. P10.9

10.10 A cable is attached to supports A and B, Fig. P10.10. The distance between A and B is $l = 100$ m, and the sag of the cable is $f = 5$ m. The weight per unit length of the cable is $q = 12$ N/m. Determine the length, and the maximal tensile force in the cable.

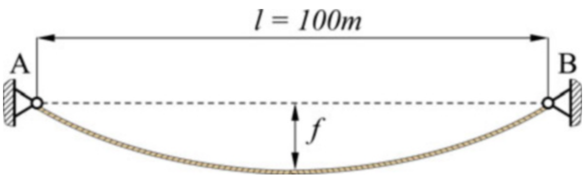


Fig. P10.10

10.11 A cable is attached to supports A and B, Fig. P10.11. The distance between two points in vertical direction is 10 m, the sag of the cable is 5 m, and its length is $L = 200$ m. By knowing that the weight per unit length is $q = 5$ N/m, determine distance l between points A and B, and the maximal tensile force in the cable.

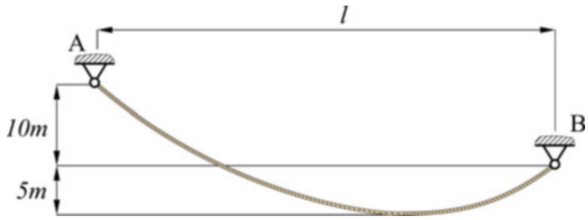


Fig. P10.11

10.12 A cable is attached to supports A and B, Fig. P10.12. The distance between supports A and B in vertical direction is 20 m, the sag of the cable is 10 m, and its length is $L = 150$ m. The weight per unit length of the cable is $q = 20$ N/m. Determine distance l between points A and B, and the magnitudes of tension forces at points A and B.

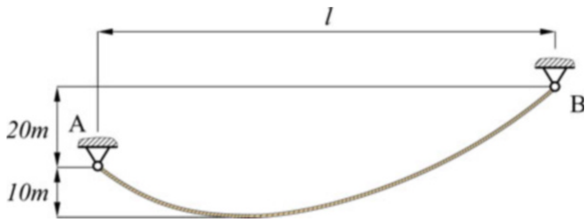


Fig. P10.12

10.13 Weights of the attached blocks are $G = 50$ N and $P = 20$ N, see Fig. P10.13. Determine the magnitude of force **F** required to maintain equilibrium in the position shown. In addition determine the tension of the cable at points A and D.

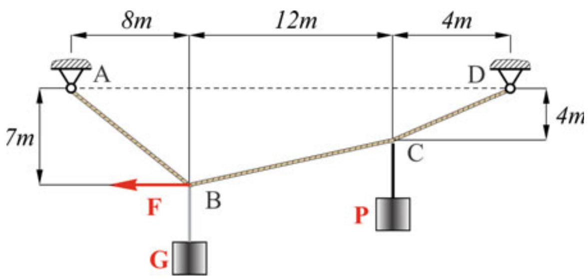


Fig. P10.13

10.14 By knowing that $F_2 = 60$ N, determine the magnitude of forces F_1 and F_3 required to maintain the system in equilibrium, see Fig. P10.14. In addition determine the tension force in the cable at points A and E

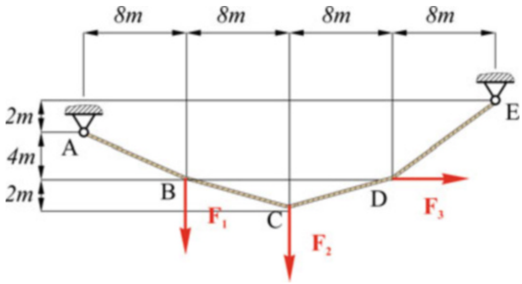


Fig. P10.14

10.15 A cable is attached to two supports A and B, Fig. P10.15. The span and the sag of the cable are $l = 50$ m and $f = 30$ m, respectively. The cable's weight per unit length is $q = 4$ N/m. Determine the length of cable L, maximal tensional force $S_{\max}$, and angle, α, *slope of the cable* at support B.

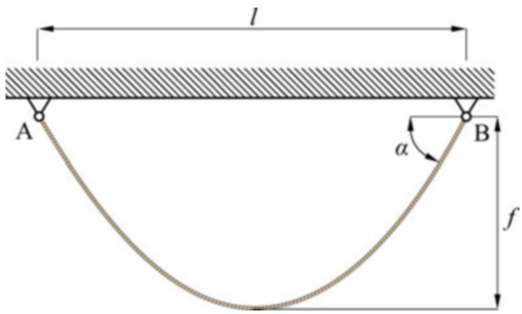


Fig. P10.15

10.16 A very long cable ABDE rests on the rough horizontal surface up to point B, as shown in Fig. P10.16. By knowing that the weight per unit length of the cable is 4 N/m, determine magnitude of force **F** required to keep the cable in equilibrium.

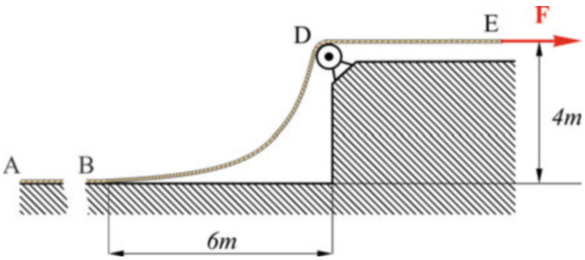


Fig. P10.16

Compound Structures 11

Contents

Anything that happens, happens.
Anything that, in happening, causes something else to happen, causes something else to happen. Anything that, in happening, causes itself to happen again, happens again.
It doesn't all necessarily happen chronologically, though.

Douglas Adams, Mostly Harmless

In this chapter you will learn:

- How to identify a compound structure
- How to disassemble it into known types of structures
- How to solve such structure
- When a compound structure is modeled as a frame
- When a compound structure is modeled as a mechanism

In the previous Chaps. 8, 9, and 10 we developed methodologies for solving different types of structures and structural elements. We have introduced three different procedures to deal with trusses, beams, and cables. In the process of developing a physical model we have to represent physical reality in such a way that it will fit in one of these three groups. By doing so, we will be able to utilize the developed "standard" procedures to solve for unknown forces and moments.

I. Emri, A. Voloshin, *Statics*, DOI 10.1007/978-1-4939-2101-0_11

In reality, we have many situations where a structure does not belong to any of three groups; however, it may be treated as a structure compounded of trusses, beams, and cables.

In this chapter we will show how compound structures could be treated by using the previously developed procedures. We will demonstrate this procedure using an example.

11.1 Introduction

Let us consider a compound structure schematically represented in Fig. 11.1a. This structure comprises three known types of structural elements: truss, beam, and cable. The goal is to determine value of the forces in the cables KE and ED when the system is in equilibrium and also to determine the reactions at A. The free body diagram is shown in Fig. 11.1b. As may be seen from the free body diagram, this structure has four unknown reactions: two at point A, one at each of the points D and K. Thus, it cannot be solved directly for unknown reactions.

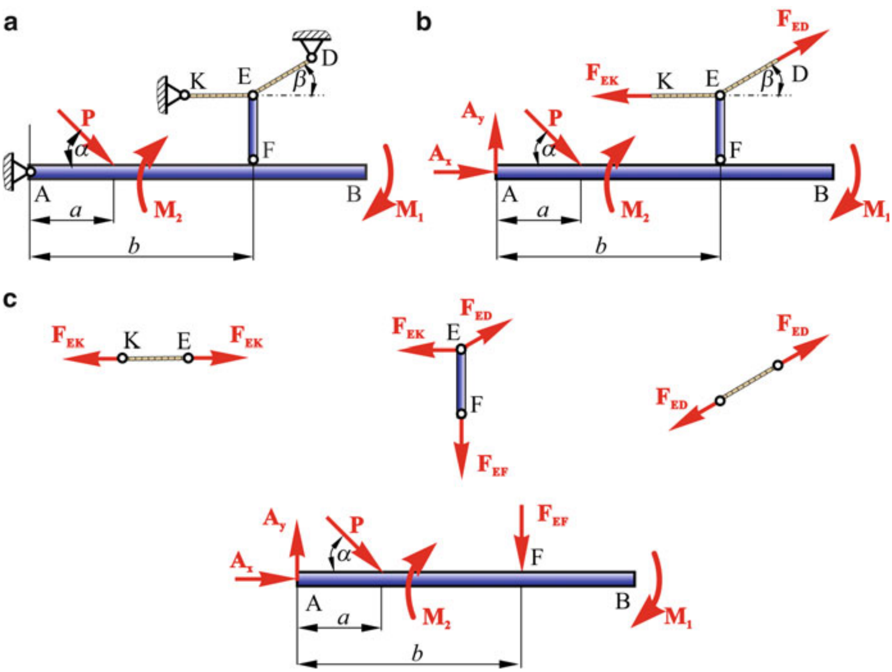


Fig. 11.1 (**a**) Compound structure. (**b**) Free body diagram. (**c**) Disassembly of the compound structure

We will solve this structure by separating it into three constituent parts: part I, part II, and part III. Part I belongs to the type of structures called cables, part II beams, and part III trusses.

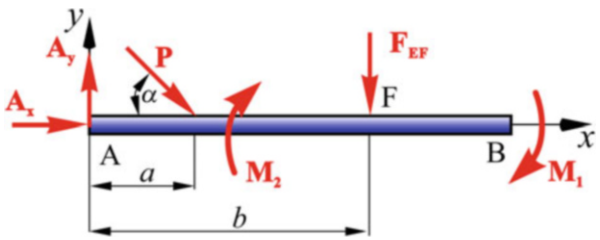
To keep each part in equilibrium, we introduce the interaction forces (equilibrium pair of forces) at each points of separation, as it is shown in Fig. 11.1c.

These interaction forces were added in points F and E. At each of these points we add only one pair of forces since their line of action is known and it is along the axis of the member. Interaction forces are external forces when we consider equilibrium of each part. Now, we can solve each part separately using procedures described in Chaps. 8, 9 and 10.

It should be mentioned that the compound structure does not have to comprise all three types of structural elements: truss, beam, and cable. It may contain any mixture of these.

Let's consider each part of the disassembled structure and count the number of unknown forces. Here is the list: A_x, A_y, F_{EF}, F_{EK}, and F_{ED}. Thus, there are five unknown forces. By considering the equilibrium of the part AB we can write three equation of equilibrium and by considering equilibrium of the point E we can write another two equations of equilibrium, total five equations that will allow us to solve for five unknowns.

Let us consider equilibrium of the part AB.



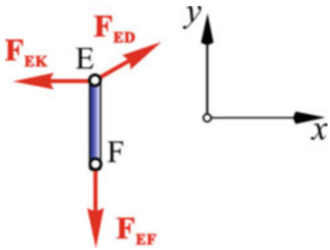
Since the member EF is a two-force member, the force $\mathbf{F}_{\text{FE}}$ is acting along the member EF. We may write the following three equations

$$\sum M_{\text{A}} = F_{\text{FE}} \cdot b - P \cdot a \sin\alpha - M_1 - M_2 = 0$$

$$\sum F_x = A_x + P \cdot \cos\alpha = 0$$

$$\sum F_y = A_y - P \cdot \sin\alpha + F_{\text{EF}} = 0$$

Next, let's consider equilibrium of point E and write two more equations of equilibrium



$$\sum F_x = -F_{\mathrm{EK}} + F_{\mathrm{ED}} \cdot \cos\beta = 0 \quad \text{and}$$

$$\sum F_y = F_{\mathrm{ED}} \cdot \sin\beta - F_{\mathrm{FE}} = 0$$

Therefore, we have five equations that can be solved for five unknowns.

Example 11.1 Determine the force between the drum N and link DE, if $P = 250$ N, AC = 18 cm, BC = 6 cm, $a = 6$ cm, and $b = 12$ cm. Neglect weight of the rods (Fig. 11.2a).

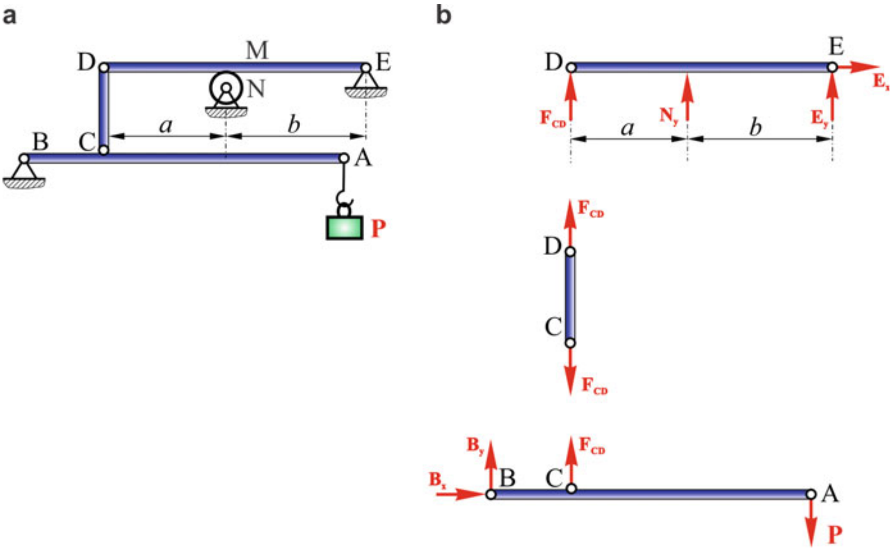


Fig. 11.2 (**a**) Compound structure. (**b**) Disassembly of the compound structure

Solution Since this system may be considered as consisting of two beam elements: DE and AB and one truss element: CD, we will consider equilibrium of each of these constituents separately. The disassembled system with the interaction forces: C_y and D_y is shown in Fig. 11.2b.

By considering the equilibrium of the member AB we can write

$$\sum M_B = F_{\text{CD}} \cdot BC - P \cdot \text{AB} = 0$$

and solve for $F_{\text{CD}} = 1000$ N.

Since member CD is a two-force truss member, the $F_{\text{CD}} = 1000$ N is acting at both the ends of the member CD. Let us define the force between the drum and the member DE as N_y. From the equilibrium of the beam member DE

$$\sum M_E = F_{\text{CD}} \cdot \text{DE} - N_y \cdot \text{NE} = 0$$

we can solve for $N_y = 1500$ N.

Example 11.2 Determine the reaction forces at A and B, also determine the internal force at C tension in the cable DE. Use $q = 5$ lb/ft, $a = 5$ ft, $l = 9$ ft, $b = 4$ ft, and $\alpha = 30°$.

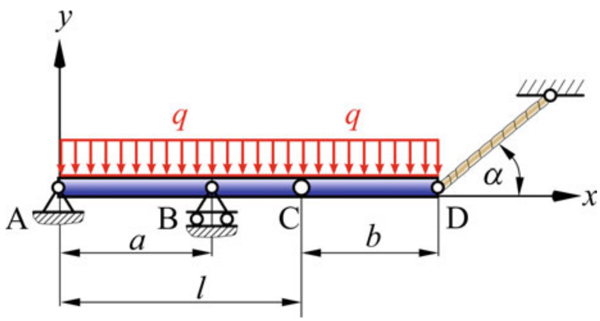


Fig. 11.3a Compound structure

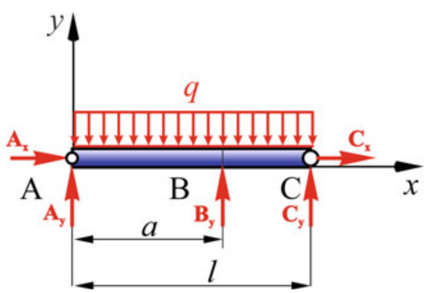


Fig. 11.3b Free body diagram of the part AC

Solution DE is a cable, thus the direction of the force DE is known to be along the cable (Fig. 11.3a).

Let us disassemble the structure, remove it from its supports, show the reactions and consider equilibrium of each part. We will start from the part AC (Fig. 11.3b).

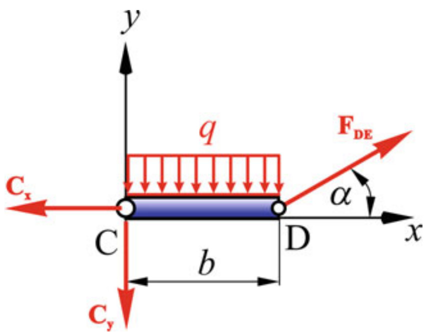


Fig. 11.3c Free body diagram of the part CD

We can write three equations of equilibrium for the part AC

$$\sum F_x = A_x + C_x = 0$$

$$\sum F_y = A_y + B_y + C_y - q \cdot l = 0$$

$$\sum M_A = -q \cdot \frac{l^2}{2} + B_y \cdot a + C_y \cdot l = 0$$

Thus, $A_x = -C_x$.

Next, let us consider the equilibrium of the part CD and write the equations of equilibrium (Fig. 11.3c)

$$\sum F_x = F_{\mathrm{DE}} \cos\alpha - C_x = 0$$

$$\sum F_y = -C_y - q \cdot l + F_{\mathrm{DE}} \sin\alpha = 0$$

$$\sum M_C = -q \cdot \frac{b^2}{2} + F_{\mathrm{DE}} \sin\alpha \cdot b = 0$$

From the above equation, we can get $F_{\mathrm{DE}} = 20$ lb.

From the sum of forces in y direction we can get $C_y = -35$ lb and from the sum of forces in x direction $C_x = 17.32$ lb.

Using these values for the equation of equilibrium we derived for the part AC we can solve for the rest of the unknowns.

$A_x = -17.32$ lb.

$B_y = 103.5$ lb and $A_y = -23.5$ lb.

11.1.1 Problems

11.1 Weightless beam AB is supported by link DC and loaded by force $F = 10$ kN. Determine the reaction at **A** and the force in link CD.

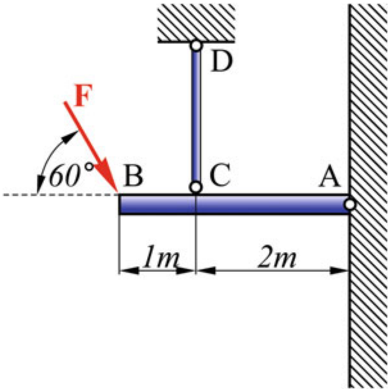


Fig. P11.1

11.2 Homogeneous beam AB is supported by homogeneous beam CD. Weight of beam AB is 4 kN and of CD is 3 kN. The loads at points M and N are equal to 8 kN each. Determine the reactions at A, B, E, and D.

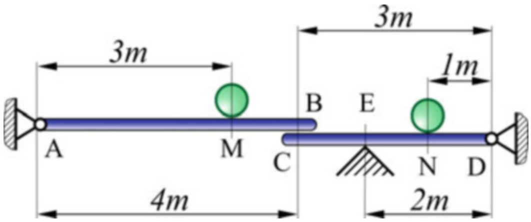


Fig. P11.2

11.3 A bridge consists of two segments connected at point A by a hinge. The weights of each segment (40 kN) are applied at D and E, respectively. The bridge is loaded by a truck represented by force $P = 60$ kN. Determine the reactions at B, C and force at A.

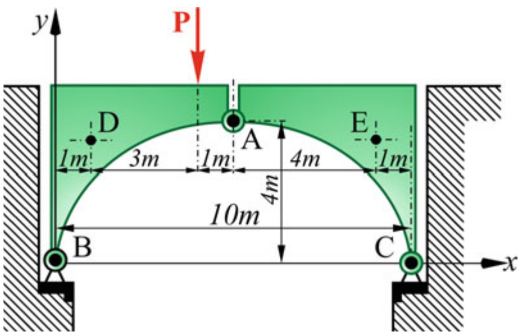


Fig. P11.3

11.4 Rod AB is loaded by force $P = 20$ N and moment $M = 4$ N m. Determine the tension in link DF and the components of all forces acting on member AB. Use $AF = 40$ cm, $a = 30$ cm, $b = 20$ cm, $\alpha = 45°$.

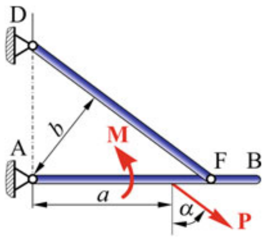


Fig. P11.4

11.5 Ladder ACB consists of segments AC and CB connected by a hinge at C. It supports a person (weight $P = 650$ N) standing at D. Weight of each segment of the ladder is $W = 80$ N and is applied at its corresponding center. Cable EF holds the ladder in place. Determine the reactions at A, B, C and tension in cable EF.

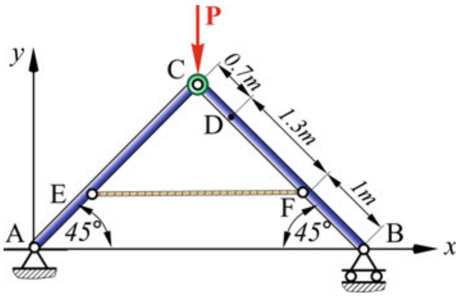


Fig. P11.5

11.6 A bridge consists of two identical segments M and N connected by links making an angle of 45° with the horizontal line. Determine the axial forces in each link. The load $P = 100$ kN.

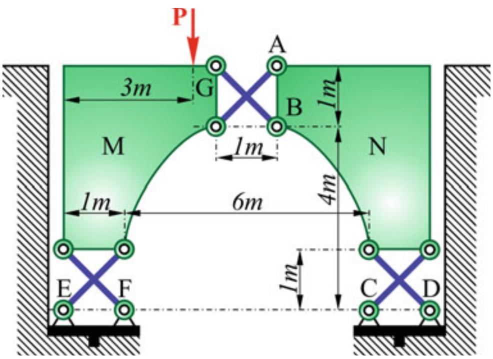


Fig. P11.6

11.7 A crane is supported by two pin connected beams, AC and CB. The crane's weight $W = 60$ kN is acting at point D. For load $P = 10$ kN, determine the reactions at A and B.

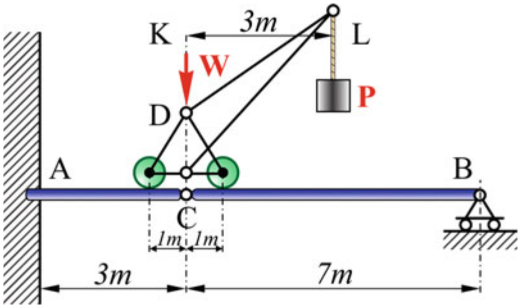


Fig. P11.7

11.8 Load $P = 12$ kN is located in the middle of member ML. Determine the load at each wheel. Use $AB = 2BC$, $DE = 0.75\ EF$, and $GK = 0.8$ KN.

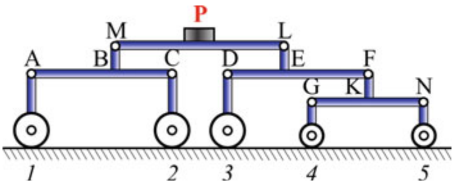


Fig. P11.8

11.9 Weight $P = 50$ N is applied at point B of horizontal rod AB having weight $Q = 10$ N that is acting at E. Determine reactions at A and C.

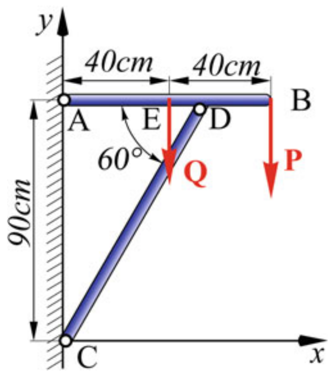


Fig. P11.9

11.10 Two homogeneous bars are connected by a pin at C and loaded by force **Q**. The weight of each bar is **P**. Determine the reactions at A and B. Use $Q = 80$ N, $P = 20$ N, $b = 40$ cm, and $d = 80$ cm.

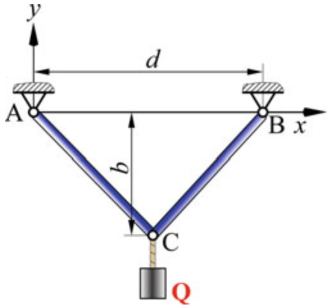


Fig. P11.10

11.11 Two homogeneous bars are connected as shown and loaded by loads $P_1 = 40$ kN, $P_2 = 40$ kN, and $Q = 100$ kN. Determine the reactions at A and B. Use $AC = BD$, $AE = EC$, and $BF = FD$.

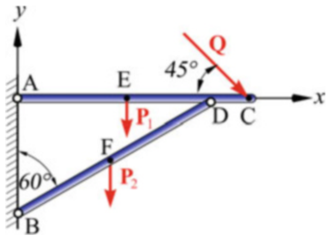


Fig. P11.11

11.12* A system of bars is loaded by weight M. Determine the axial forces in each bar. AD is parallel to BC and AB to CD.

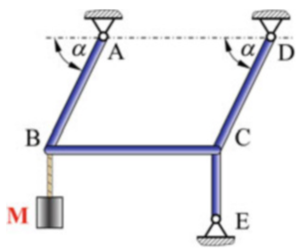


Fig. P11.12

11.13 Bars AB and CD support weight $P = 100$ N. The weight of bar AB is 30 N and is applied at point E, while the weight of bar CD is 20 N and is applied at point F. Determine the reactions at A and C. Use $AB = CD = 1.0$ m, $AE = 0.35$ m, $CF = 0.5$ m, $\alpha = 60°$, and $\beta = 35°$.

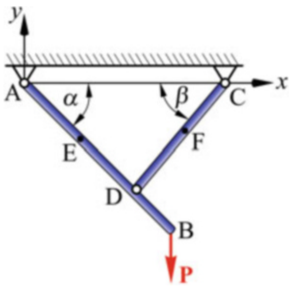


Fig. P11.13

11.14 Identical bars AB and BC connected by frictionless pin at B (each weight **P** is 30 N) are held in the horizontal position by cables ED and EC. Determine the tension in each cable and the reaction forces at A and B. $\alpha = 2\beta = 60°$.

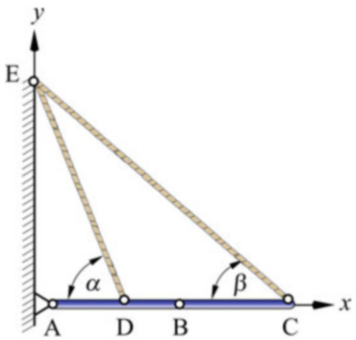


Fig. P11.14

11.15 Determine the reaction forces and moments at A, B, and C. Use $q = 10$ N/m, $F = 40$ N, $a = 5$ m, $b = 2$ m, $l = 10$ m, $\alpha = 30°$.

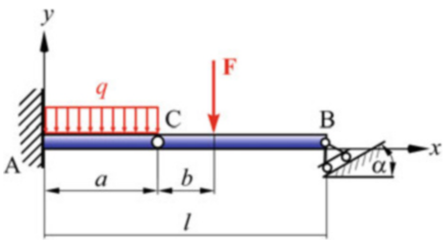


Fig. P11.15

11.16 Determine the reaction forces and moments at A, B, and C. Use $M = 10$ N m, $a = 3$ m, $l = 5$ m, $\alpha = 30°$.

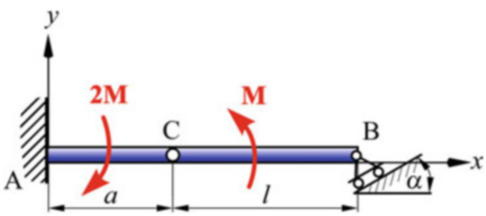


Fig. P11.16

11.17 Horizontal bar AB is attached to vertical bar AC and supported by bar DE. Bar AC is built into the ground at C and supported by bar FG. All bars are connected by pins. Determine axial load in bars FG and DE, and reaction at point C. Use $Q = 100$ N.

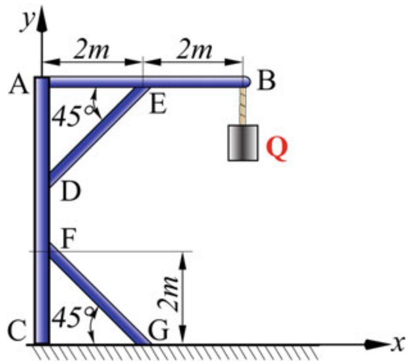


Fig. P11.17

11.18 The disk (weight $Q = 200$ N) is logged between a vertical wall and member AB (weight $P = 100$ N). Assume that there is no friction in the system. $AB = l$, $BD = 2/3\, l$ and angle CAB is $\alpha = 60°$. Determine the reaction forces and moment at A, C, and D.

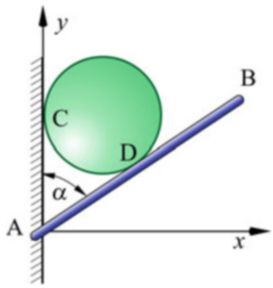


Fig. P11.18

11.19 Two homogeneous beams are connected at B. Weight of each beam is $W = 30$ N and length is $a = 40$ cm. Determine the reactions at A and C and the pin B. Force $Q = 10$ N is acting at the middle of BC and is perpendicular to it.

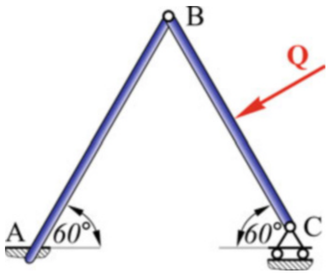


Fig. P11.19

11.20 Weight of bar AB is 20 N and BD is 40 N. The length $BE = 2/3BD = 15$ cm and $AC = 2/3AB = 12$ cm, $\alpha = 30°$. Assuming that there is no friction at E and C, determine the reactions at A, C, and E.

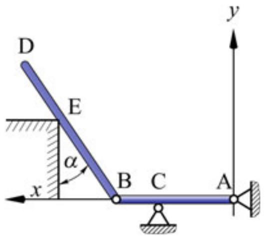


Fig. P11.20

11.21 Determine the reaction forces at A, B, and C. $q = 100$ N/m, $a = l = 2b = 40$ cm, $\alpha = 30°$.

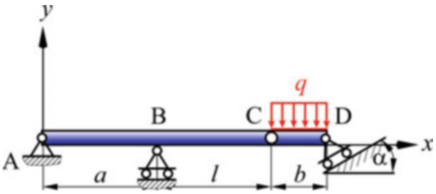


Fig. P11.21

11.22 Determine the reaction forces at A, B, and C. Use $q = 400$ N/m, $a = l = 2b = 20$ cm, $\alpha = 30°$.

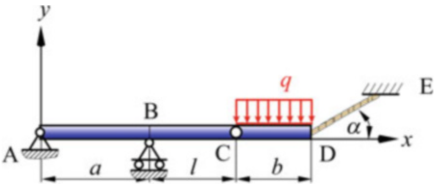


Fig. P11.22

11.23 Homogeneous bar AB (weight $Q = 200$ N) leans on frictionless support E. Homogeneous bar CD (weight $W = 400$ N) is loaded by weight $P = 100$ N. $CB = 4$ m, $BD = 1$ m, $BE = 0.830$ m. Angle $\alpha = 45°$. Determine the reactions at A, C, and E.

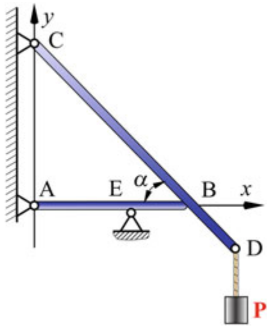


Fig. P11.23

11.24 Member AB, weight 300 N, is leaning on frictionless support C and supporting member DE, weight 400 N. Determine the reaction forces at B, C, and D and contact force at point A. Use $Q = 60$ N, $AB = 3$ m, $BC = 1$ m, $AD = 1/3AE$, and $\alpha = 30°$.

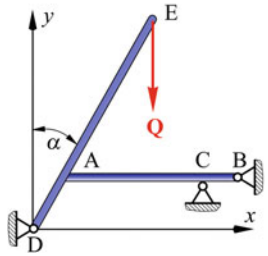


Fig. P11.24

11.25 The weight of member AC is $P = 800$ N and member CK is $Q = 160$ N. $AC = 4$ m, $BC = 1$ m. Member CK makes an angle of 45° with the horizontal axis. Determine the reaction forces at B, C, and K. What is the axial force in bar AD?

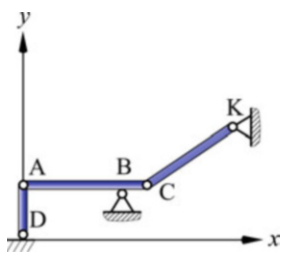


Fig. P11.25

11.26 The weight of frictionless bar AB is 60 N and of bar BD is 100 N. The system is loaded by force $F = 100$ N. $AB = 4.5$ m, $BC = 1$ m, $AE = 1$ m, $\alpha = 45°$. Determine the reactions at A, C, and D.

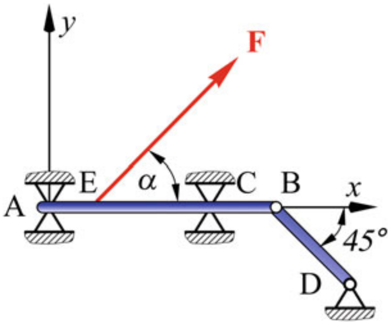


Fig. P11.26

11.27 Three homogeneous bars are connected as shown below. Force $Q = 60$ N is applied at the center of member CD. Determine the components of forces acting on members AB and CD. Bar AB weights 30 N, bars BC and CD each weight 10 N. $AB = 4$ m and $KB = 1$ m. K is a frictionless support.

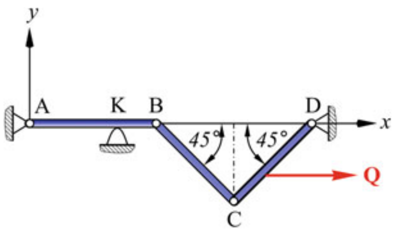


Fig. P11.27

11.2 Frames

In Sect. 8.1 we considered structures that can be modeled as trusses, i.e., they consist of only two-force members. *Frame* is similar to truss, but it has to contain at least one element which is not a two-force member (beam element). Therefore frame has at least one three-force member.

When a truss member is loaded by an external force not acting at a joint, we have a structure called a *frame*. An example of a frame is shown in Fig. 11.4a. This frame has three external reactions, which can be determined by treating the entire frame as a rigid structure. In order to analyze the frame we have to disassemble it and to draw a free body diagrams for each member of the frame (Fig. 11.4b).

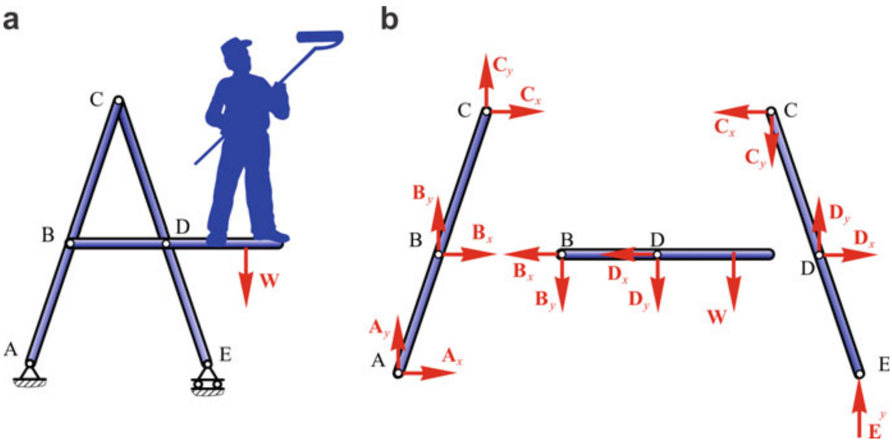


Fig. 11.4 (**a**) Painter on the bench. (**b**) Free body diagrams

Frame is similar to truss, but it contains at least one element that is not a two-force member (i.e., a beam element). Frame and truss are built to support external loads and prevent any motions.

We separate all members of the frame and show internal forces acting from one member on another. At each joint we have a pair of equal and opposite forces. Since they are internal forces with respect to the whole structure, their sum should be equal to zero. Each one of the three members is not a two-force body, thus we do not know the force direction and have to show its two unknown components. The reaction force **E** has only one component in vertical direction, since we assume frictionless support. The weight of the painter **W** is acting downward. Since we have three rigid bodies in equilibrium, we can write nine equations of equilibrium. The total number of unknown forces, both internal and reactions, is also equal to nine, thus we can solve for all unknown forces. Although we could treat the whole structure as a rigid body and solve for three unknown reactions, those equations will not be independent from the previous set of nine. If we solve all disassembled segments, we will get values for all internal forces and all reactions. Thus, a free body diagram of the whole assembly is not necessary for solving for reactions.

There is no need to use the equilibrium equations for the whole structure, since these equations would be linearly dependent on the equilibrium equations of the constituent members. Thus, they would not provide any new information.

Example 11.3 Find the reactions at points A and B if the weight of the hoof and the load applied at D is 20 kN.

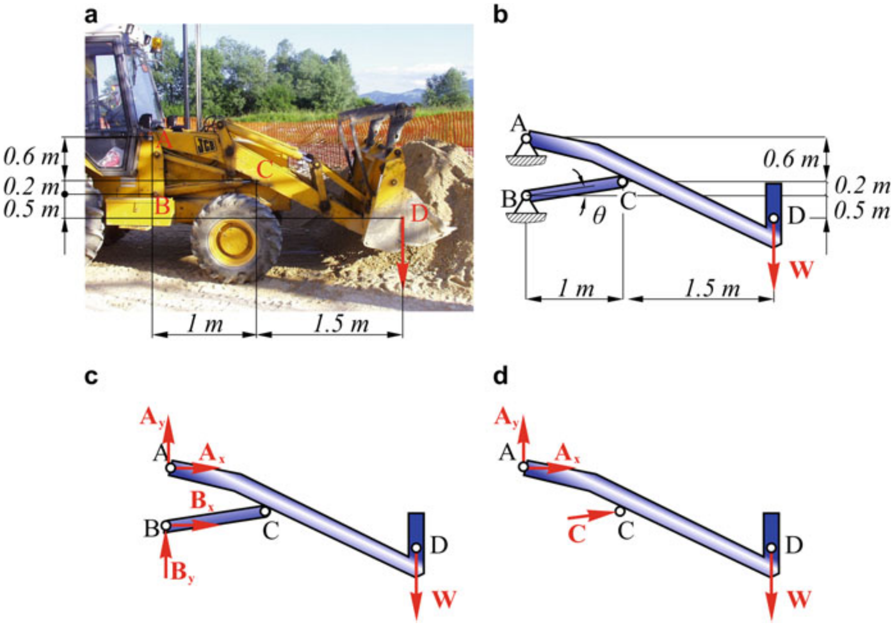


Fig. 11.5 (**a**) Front-end loader. (**b**) Physical model. (**c**) Free body diagram. (**d**) Free body diagram of ACD

Solution Physical model of the system is shown in Fig. 11.5b.

Let's create a free body diagram of the model. It is obvious that there are four unknown reactions (Fig. 11.5c). Therefore, we will have to disassemble the system and consider equilibrium of each part separately. It should be noted that the part BC is a two-force member, thus we know the direction of the internal force at B. From the free body diagram of the part ACD (Fig. 11.5d) we can write three equations of equilibrium.

Let us define the angle between the link BC and the x-axis as θ.

$$\sum F_x = A_x + C\cos\theta = 0$$
$$\sum F_y = A_y + C\sin\theta - W = 0$$
$$\sum M_A = C\cos\theta \cdot (a+b) + C\sin\theta \cdot d - W \cdot (d+\ell) = 0$$

From geometry (Fig. 11.5b) we find angle $\theta = \text{arc tan } 0,2 = 11,3$. Solution of the above set of three equations with three unknown results is C = 51 kN, $A_x = -50$ kN, and $A_y = 10$ kN. Since part BC is a two-force member, the reaction at point B is equal to the force C. Its components are components of the force C.

$$C_x = C\cos\theta = 50 \text{ kN} \;\text{ and }\; C_y = C\sin\theta = 9.99 \text{ kN}$$

Example 11.4 Frame (Fig. 11.6) is loaded by a horizontal force $P = 100$ lb at point E. Determine the reactions at A and B. Use $a = 6$ ft, $b = 5$ ft, and $c = 7.5$ ft.

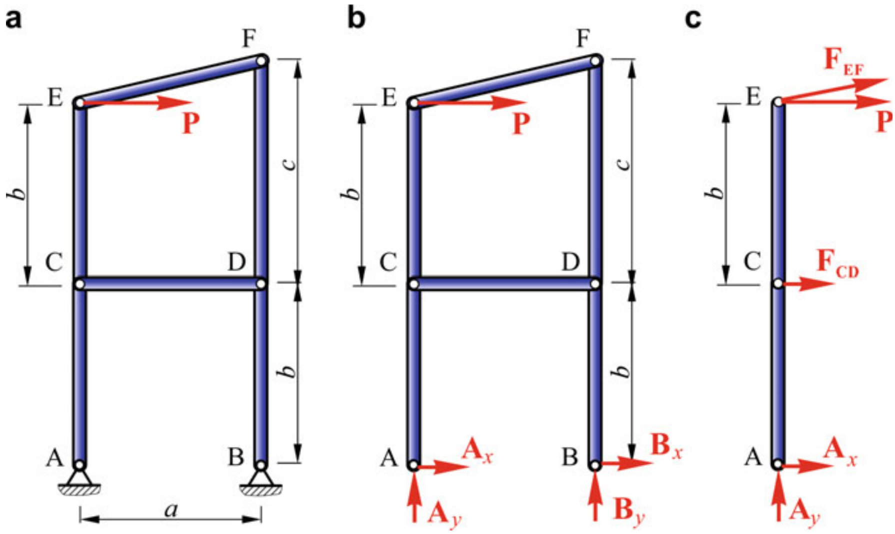


Fig. 11.6 (**a**) Frame. (**b**) Free body diagram of the frame. (**c**) Free body diagram of the member ACE

Solution Free body diagram of this frame is shown in Fig. 11.6b. It has four unknown reactions, thus it is not possible to find them only from the equations of equilibrium of the whole frame. We will have to disassemble it and consider equilibrium of each part separately.

Let's start with the whole structure and write three equations of equilibrium

$$\sum F_x = A_x + B_x + P = 0$$
$$\sum F_y = A_y + B_y = 0$$
$$\sum M_B = A_y \cdot a + P \cdot 2b = 0$$

From the above we can get

$$A_y = -B_y$$

$$B_x = -A_x - P$$

and

$$A_y = -\frac{2bP}{a}$$

Now let's consider the equilibrium of the member ACE, its free body diagram of the member ACE is shown in Fig. 11.6c.

It should be noted that the links CD and EF are two-force members, thus we know the direction of the internal forces acting on the member AE at the points C and E.

From the geometry of the frame we can calculate the angle α from the following

$$\tan \alpha = (c - b)/a$$

Thus $\alpha = 22.6°$.

Let's write the equations of equilibrium for the member ACE.

$$\sum F_x = A_x + F_{\mathrm{CD}} + F_{\mathrm{EF}} \cos\alpha + P = 0$$
$$\sum F_y = A_y + F_{\mathrm{EF}} \sin\alpha = 0$$
$$\sum M_A = F_{\mathrm{CD}} \cdot b + P \cdot 2b + F_{\mathrm{EF}} \cdot 2b \cos\alpha = 0$$

From here we can find

$$F_{\mathrm{EF}} = -\frac{A_y}{\sin\alpha} = \frac{2bP}{a \cdot \sin\alpha}$$
$$F_{\mathrm{CD}} = -\left[2P + \frac{4bP \cdot \cos\alpha}{a \cdot \sin\alpha}\right]$$

and

$$A_x = -\left[P - 2P - \frac{4bP \cdot \cos\alpha}{a \cdot \sin\alpha} + \frac{2bP \cdot \cos\alpha}{a \cdot \sin\alpha}\right]$$

Substitute the known values to get

$$A_x = 500\,\mathrm{lb}$$
$$A_y = -166.7\,\mathrm{lb}$$
$$B_y = 166.7\,\mathrm{lb}$$
$$B_x = -P - A_x = -600\,\mathrm{lb}$$

Guidelines and Recipes for Finding Forces Acting at the Frame Joints

- Disassemble the frame at the joints.
- Show two unknown force components (2D case) or three force components (3D case) at each joint.
- Draw the free body diagram for each member.
- Write the equilibrium equations.
- Solve for the unknown forces.

11.2.1 Problems

11.28 Determine the components of forces acting on members AB and CD. Weight of each member is 40 N and length is 20 cm.

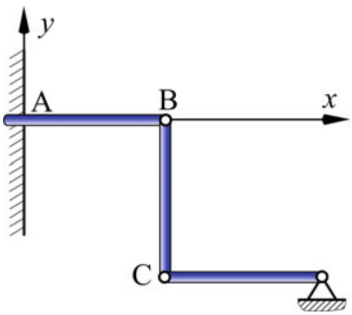


Fig. P11.28

11.29 Determine the reactions at A and the forces in links EG and HK, CE = EH = HD = a; $P = 120$ N.

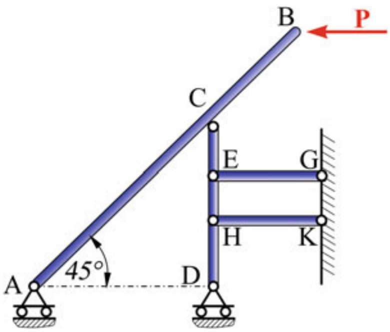


Fig. P11.29

11.30 At what distance from the point A one should place support O in order to generate tensile force of 10 kN at M. $F = 0.6$ kN, AB = 4 m, CQ = 2 m, DQ = 0.2 m.

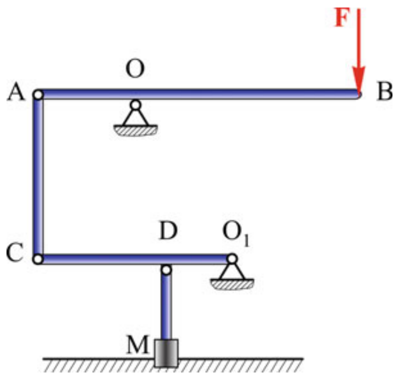


Fig. P11.30

11.31 A bridge consists of two pin connected horizontal plates supported by rods. The bridge is carrying load **P**. Determine the force acting in each rod and reaction forces at A. Use $P = 250$ kN, AC = 3 m, AB = 12 m, $a = 4$ m, and $\alpha = 45°$.

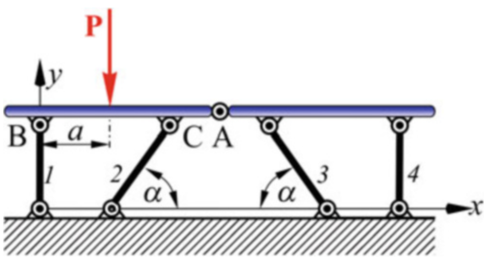


Fig. P11.31

11.32 Determine the forces exerted on bar AB (length $2a$) and bar BC. Weight of the homogeneous bar AB is **P** and of the homogeneous bar BC is **Q**.

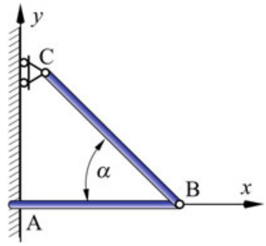


Fig. P11.32

11.33 Two identical bars AB and BC (weight **P** and length a) are loaded by the moment **M**. Determine reactions at A, B, and C.

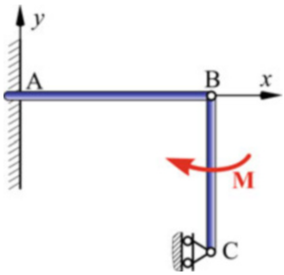


Fig. P11.33

11.34 Two bars AB and CD (each has weight $P = 50$ kN and length a) are loaded by the force $F = 200$ kN. Determine reactions at A, D, and C.

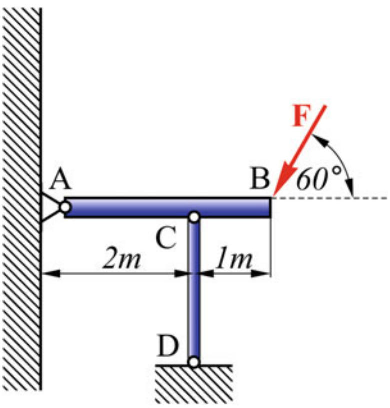


Fig. P11.34

11.35 The structure is loaded by a force $P = 200$ N. Determine the reaction force components at A and B.

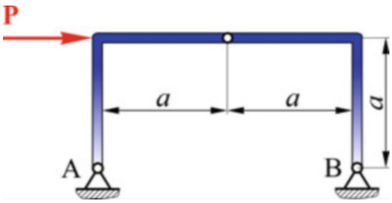


Fig. P11.35

11.3 Mechanisms

Mechanism is a structure designed to transmit and modify forces. It contains moving parts and allows motion of one part of the mechanism relative to another. In the analysis presented here, we will not consider inertial forces, which will be dealt with in the dynamics course.

Frames that are designed to transmit and change forces, and are capable of motion are called *mechanisms*. The main difference between the frame and mechanism is that frames are rigid structures, while mechanisms are not. However, both frames and mechanisms consist of rigid elements. The method for solving frames, as described above, is also applicable for solution of mechanisms. Since the geometry of a mechanism is not fixed, the solution should be done for a particular configuration. It should be noted that for different configurations, we usually would obtain different values for internal forces. The approach for analysis of a mechanism can be shown by using a hole puncher (Fig. 11.7a).

> *Mechanism is a structure designed to transmit and modify forces. It contains moving parts and allows motion of one part of the mechanism relative to another.*

Figure 11.7b shows free body diagrams for each member of this mechanism. In order to maintain equilibrium, force **P** applied to the handle should create a moment about point A equal and opposite in direction to the moment created by the force **C**. As a result, we have two equal and opposite forces acting on the plunger, shown as forces **C**. Consider, for example, equilibrium of the upper member. By taking sum of moments about point A we may solve for unknown force **C**. During operation the geometry of the mechanism is changing; therefore, the internal forces will change accordingly. In the design process we have to consider the worst-case scenario, i.e., when the forces in members are at their maximum. The geometry for the worst case may be different for different members. For complicated structures we will have to

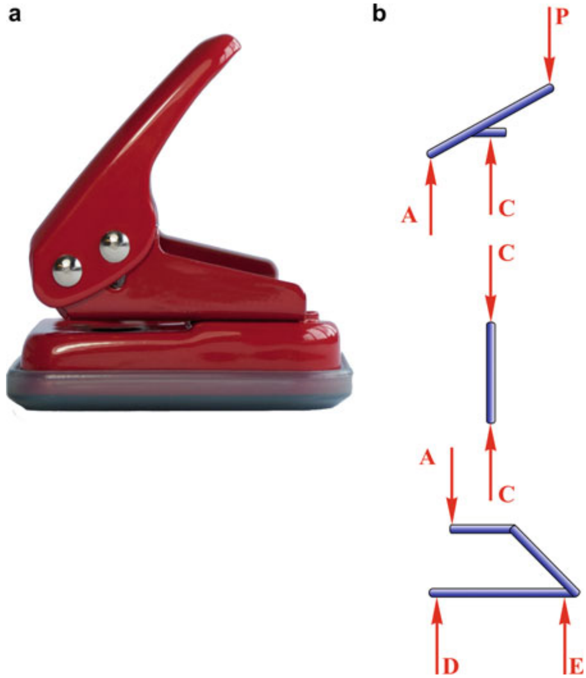


Fig. 11.7 (**a**) The hole puncher. (**b**) FBDs of the hole puncher parts

consider more free body diagrams; however, the process of solution will remain the same. As in any static problem, the number of unknown forces should be equal to the number of independent equations of equilibrium.

Example 11.5 The frictionless mechanism (Fig. 11.8a) is loaded by moment $\mathbf{M_1}$. Determine moment $\mathbf{M_2}$ necessary to keep the mechanism in equilibrium. $OA = 2\,PB = 2a$, $\alpha = 90°$, and $\beta = 60°$. Neglect the weight of the links.

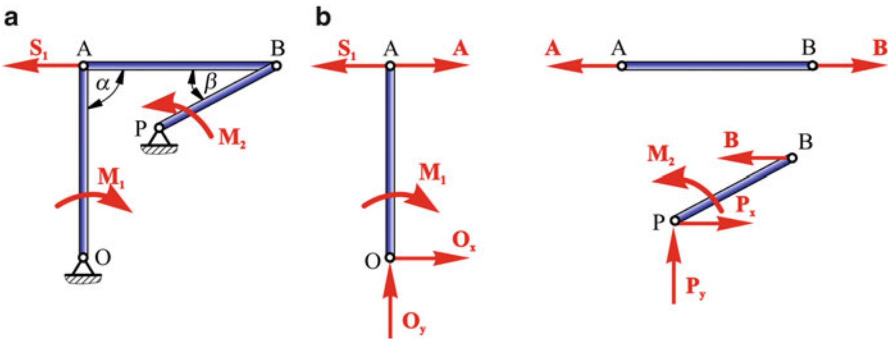


Fig. 11.8 (**a**) The mechanism. (**b**) FBD of the mechanism's components

Solution Since the mechanism is in equilibrium, each link is in equilibrium as well. Let us disassemble the structure (Fig. 11.8b) and consider equilibrium of each link. At each joint we will show the internal forces. Of course, they have to be equal in magnitude and should act in the opposite directions so that their sum for each joint will be equal to zero. Since link AB is a two-force member and it is in a state of equilibrium, **A** = **B**. Because we are not interested in the reaction at the supports, let us use the sum of moments about point O for link OA and the sum of moments about point P for link PB.

For link OA: $\sum M_o = A \cdot 2a + M_1 = 0$

For link PB: $\sum M_P = -B \cdot a \cdot \sin\beta - M_2 = 0$

Since A = B the solution of the above equations will be

$$\frac{M_1}{2a} = \frac{M_2}{a \cdot \sin\beta}$$

Or substituting the given values we will get

$$M_2 = \frac{\sqrt{3}}{4} M_1$$

Guidelines and Recipes for Finding Forces Acting in Mechanisms

- Create a physical model.
- Disassemble the mechanism at joints.
- Show two unknown force components (2D case) or three force components (3D case) at each joint.
- Draw a free body diagram for each member.
- Write equilibrium equations.
- Solve for the unknown forces.

11.3.1 Problems

11.36 A mechanism is loaded by forces **P** and **Q**, force **P** is acting at the center of link AB and **Q** is applied at point E. CD = 3CE. What is the relationship between forces **P** and **Q** for the system to be in equilibrium? Determine the reaction at *D*.

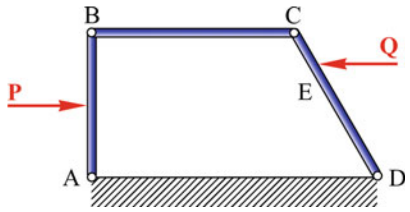


Fig. P11.36

11.37 Determine the force applied to block M when $P = 5$ kN, $OE = 100$ cm, $OA = 10$ cm. Force **P** is perpendicular to EO, $\alpha = 60°$ and $\beta = 45°$.

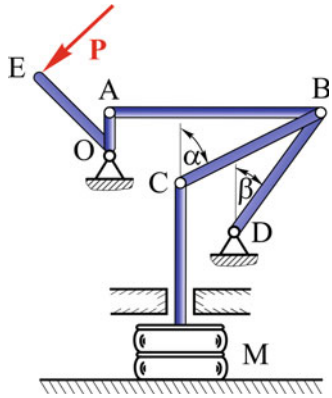


Fig. P11.37

11.38 Bracket ABC is held in equilibrium by horizontal force **Q**. Bar DKE is suspended by cable BKC, passing through frictionless pulley K, and loaded by force **P**. The angles $\alpha = 30°$ and $\beta = 45°$. The distances are $EK = KD = OK = 0.1$ m, $OA = 0.2$ m, KB ⊥ BO, and KC ⊥ CO. Determine the magnitude of force **Q**.

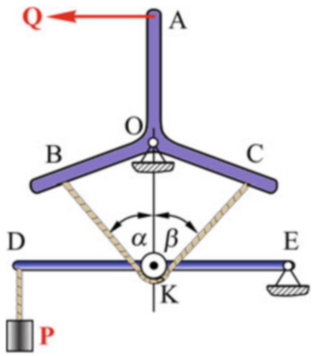


Fig. P11.38

11.39 Determine the force which is compressing barrel M, when force $P = 60$ kN and is acting perpendicular to AO. Use $AO = 2$ m, $OB = 0.4$ m, link BC is horizontal and divides the angle ECD = 160° in half.

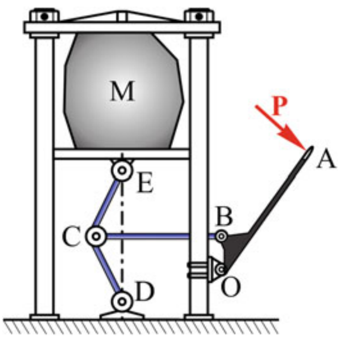


Fig. P11.39

11.40 A mechanism is loaded by forces **F** and **Q**. Determine the value of force **Q** to keep the system in equilibrium, when $OA = OB = 30$ cm, $PC = PD = 40$ cm. Angle $\alpha = 60°$ and $\beta = 30°$ and force $F = 20$ N.

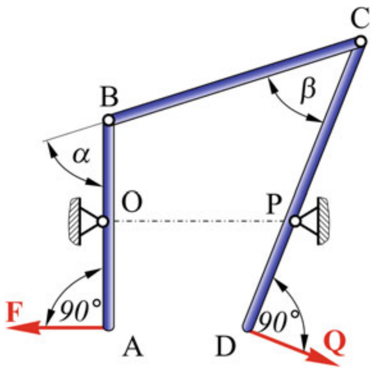


Fig. P11.40

11.41 Bar AB is attached to collar B that may move freely along CD. CD is loaded by moment $M = 450$ lb ft. Determine moment $\mathbf{M_1}$ to keep the system in equilibrium if $AB = 1$ ft and $\alpha = 30°$. Also, determine reactions at A and C.

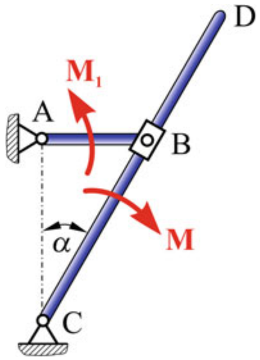


Fig. P11.41

11.42 A system of gears is loaded by moments $M_1 = 30$ lb ft and $M_2 = 70$ lb ft. Determine moment $\mathbf{M_3}$. The radii of A, B, and C are $r_1 = 0.6$ ft, $r_2 = 1.0$ ft, and $r_3 = 0.4$ ft.

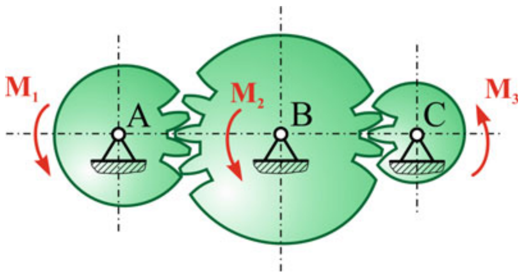


Fig. P11.42

11.43 A system of gears (Fig. P11.42) is loaded by moments $M_1 = 10$ N m, $M_2 = 20$ N m, and $M_3 = 15$ N m. Radii of A and C are $r_1 = 0.4$ m and $r_3 = 0.2$ m. Determine r_2 when the system of gears is in equilibrium.

11.44 Box Q, weighing 20 kN, is lifted by a pair of tongs CAE and DBF. Use $OC = OD = 60$ cm, $OK = 10$ cm. Determine the axial force in the member AB. Assume that all parts have no weight.

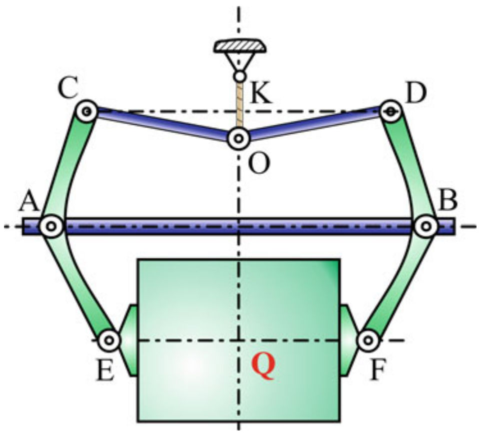


Fig. P11.44

11.45 A worker applies forces of magnitude $Q = 20$ N to the handles of a crimping tool. $CD = DE = 10$ mm. Determine the crimping force exerted by this tool. Dimensions are in mm. BK is a two-force member.

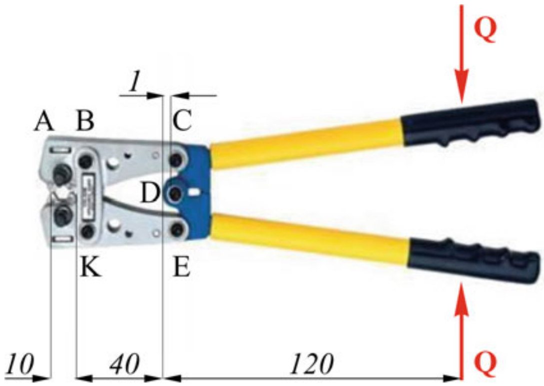


Fig. P11.45

11.46 Determine the magnitude of the gripping forces produced when two forces ($Q = 20$ N) are applied to the locking plier. Dimensions are in mm.

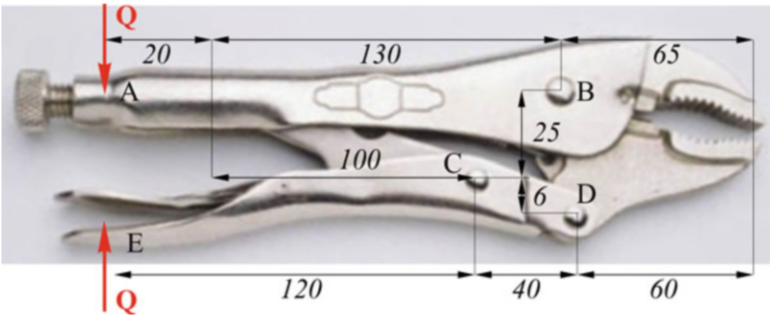


Fig. P11.46

11.47 Find the force acting on body T loaded via the system of bars. Force $P = 100$ N. Links DC = CE = MK = MN and each makes an angle $\alpha = 5°$ with the vertical direction, while links BC = BM and each makes an angle $\alpha = 5°$ with the horizontal direction.

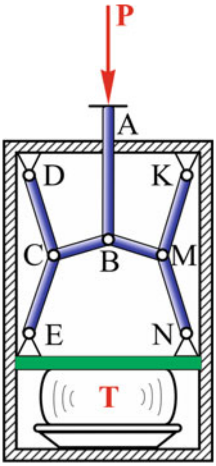


Fig. P11.47

11.48 A crank and slider mechanism is activated by moment $M = 150$ N m. Determine force **P** to keep the mechanism from moving. $OA = 0.40$ m, $\alpha = 30°$, and $\beta = 60°$.

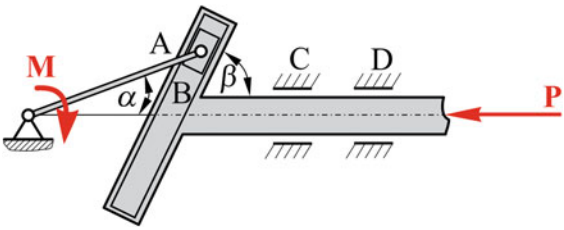


Fig. P11.48

11.49* Three bars are loaded by vertical forces **P** and **Q**. Angles $\alpha = \beta = \gamma$. Determine the axial forces in each bar and relationship between P and Q for this system to be in equilibrium.

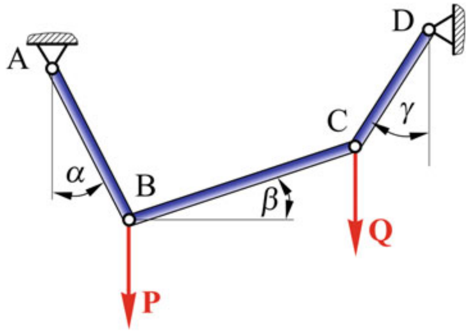


Fig. P11.49

11.50 Mechanism is loaded by force $F = 200$ N. Determine moment **M** to keep it in equilibrium if $EO = 40$ cm, $\alpha = 45°$, and $\beta = 90°$.

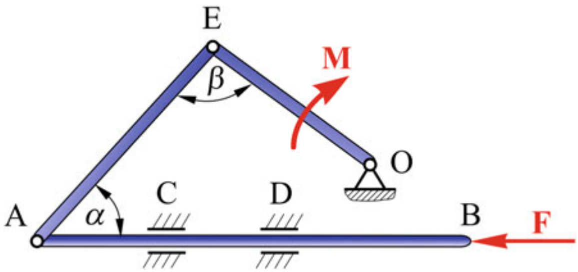


Fig. P11.50

11.51 The press is activated by rotation of link OA. At the instance shown, force $P = 100$ N is acting on joint B. Determine the force acting on the body M, when $\alpha = 45°$ and $\beta = 30°$.

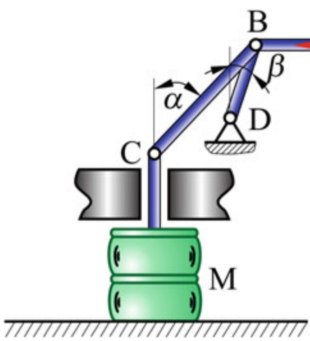


Fig. P11.51

11.52 Horizontal link AB of the ratchet rack is kept in a place by force $Q = 200$ N. Determine force **P** to move link AB to the right. **P** is acting perpendicular to link DE. $EC = 3\ DC$, $\varphi = 30°$, and $\psi = 60°$.

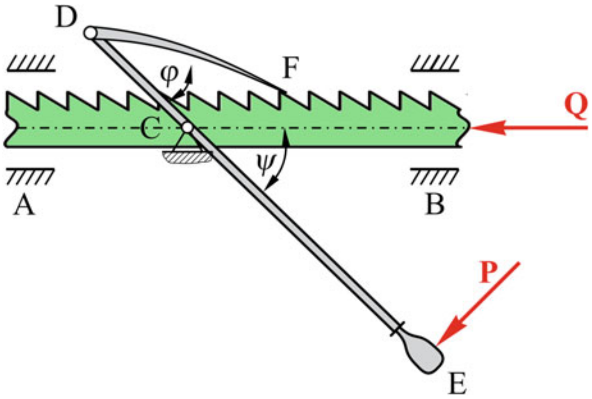


Fig. P11.52

What Have We Learned?

How to separate a compound structure into structures of known types

A structure is separated into constituent parts that belong to the known types of structures: cables, beams, trusses.

To keep each part in equilibrium, we introduce the interaction forces (equilibrium pair of forces) at each point of separation. After this, we can solve each part separately using the procedures appropriate to each known type of structures.

How to solve frames

Frame is similar to truss, but it has to contain at least one element that is not a two-force member (beam element). To solve for unknown forces, the frame should be disassembled by adding internal forces at the points of separation. Then we write equations of equilibrium for all constituent elements and solve for the unknown forces. To solve for the internal forces we should use method appropriate for each structural element.

How to solve mechanisms

Mechanism is a structure designed to transmit and modify forces. It contains moving parts and allows motion of one part of the mechanism relative to another. After selecting a desired position of the mechanism, we may use the same approach as for the frames. Depending on the task, we may need to repeat the solution for several different positions. For each position, we have to draw a new free body diagram.

Friction 12

Contents

An experiment is a question which science poses to Nature, and a measurement is the recording of Nature's answer.

Max Planck

In this chapter you will learn:

- About the origin of friction
- About the friction between solid bodies
- What is the difference between Static and Kinetic friction
- How to apply laws of friction to wedges
- Principles of rolling friction
- How to apply laws of friction to friction bearings
- How to apply laws of friction to ropes and belts

I. Emri, A. Voloshin, *Statics*, DOI 10.1007/978-1-4939-2101-0_12

In previous chapters we have discussed the conditions needed to prevent motion of a rigid body (assuming that the body was at rest at the beginning of observation). First Newton Law teaches us that we do not need any force to keep a body in motion at a constant velocity. However, everyday experience demonstrates that this is not the case when bodies are in contact with each other. For example, if we want to pull a table from one part of a room to another, we have to use a force in order to move it with a constant velocity. This force is used to overcome the so-called "friction force" generated at the interface of two bodies that are in contact, in this case the table's legs and the floor. From the experience we know that to move a table will require a larger force if somebody would sit on it. The additional weight will increase the normal force (contact force) that acts between the two contact surfaces. From such observations we may conclude that friction is a resistance encountered when one body moves relative to another body with which it is in a contact. To keep two bodies in contact we need a normal force, which is often called the contact force. Within this textbook we will analyze friction between solid (rigid) bodies only.

> We encounter friction forces everywhere where objects come into contact with each other.

> Friction is a resistance encountered when one body moves, or tries to move, relative to another body with which it is in contact.

12.1 Introduction

We encounter friction force everywhere where objects come into contact with each other. The friction force acts always in the direction *opposite* to the way an object wants to slide. Some friction forces, such as the traction needed to walk without slipping, are beneficial; however, in many engineering applications friction is not desirable. For example, about 20 % of a car engine power is consumed by overcoming friction forces between the moving parts!

> Friction force always acts in the direction *opposite* to the direction an object slides or impends to slide.

The rules of sliding friction were first discovered by Leonardo da Vinci (1452–1519), and much later rediscovered and further developed to the level presented here by Charles-Augustin de Coulomb in 1785. Coulomb investigated the influence of three main factors on friction: the nature of materials in contact, size of the surface area in contact, and the normal force. Coulomb also considered the influence of sliding velocity, temperature, and humidity on the nature of friction. He also indicated the distinction between static and kinetic friction. Further historical prospective may be found in Encyclopedia Britannica, Wikipedia, or other resources available on Internet and in literature.

Three simple experimental facts characterize the friction between rigid bodies sliding relative to each other. *First*, the amount of friction is independent of the area of contact. For example, if a brick is pulled along a table, the friction force is the same whether the brick is lying flat or standing on end. *Second*, the friction force is proportional to the normal load that presses the surfaces together. *Third*, the friction force is generated in the plane of contact, which determines its line of action that is perpendicular to the normal force.

> The amount of friction is independent of the area of contact. It is proportional to the normal load that presses two surfaces together.

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12.2 Friction Between Solid Bodies

In general there are two types of friction between solid bodies: (a) non-lubricated also called *dry friction* or *Coulomb friction* and (b) *lubricated friction*. The theory (model) presented and discussed here works well for dry friction, whereas for the cases when the contact surfaces are lubricated its applicability is limited and one has to use more advanced tribological models. In this chapter we will discuss the models related to *dry friction* only.

> In general there are two types of friction between rigid bodies: (a) non-lubricated or *dry friction* and (b) *lubricated friction*.

> The model discussed here works well for dry friction; for lubricated contact surfaces its applicability is limited and one has to use more advanced tribological models.

Dry friction between solid bodies is extremely complicated physical phenomena. It depends mainly on surface roughness and type of the materials that are in contact. When two surfaces move relative to each other the interaction between the two surfaces leads to elastic and plastic deformations of the surface layers of the contacting bodies, microfractures and interaction with wear particles. Latest research results show that dry friction involves even excitations of electrons and phonons, chemical reactions, and transfer of particles from one body to another.

At the same time, it is astonishing that it is possible to formulate a very simple law for dry friction, which is sufficient for many engineering applications. This simple law is discussed below.

Let us consider a simple case when a box of weight $\mathbf{F_g}$, which we will consider as a rigid body, is lying on a horizontal flat surface. We want to move the box by pushing it in horizontal direction with force **F**, as shown in Fig. 12.1a. Figure 12.1b shows the corresponding physical model, Fig. 12.1c shows the free body diagram, and Fig. 12.1d shows equilibrium of the external forces and reactions.

The friction force is generated at the contact surface of two bodies; therefore, it is obvious (see Fig. 12.1c) that the line of action of the friction force $\mathbf{F}_f$ does not pass through the mass center of the body. Consequently the external force, **F**, and the friction force, $\mathbf{F}_f$, form a couple of forces (moment) that tend to rotate the rigid body (the box). To compensate this moment the line of action of the normal force **N** is shifted in the direction of the intended movement of the body, Fig. 12.1c, d. Force **N** together with the external normal force $\mathbf{F}_g$ forms a couple of forces that compensates the moment of the first couple, Fig. 12.1c.

External force **F** acting parallel to the contact surfaces and friction force $\mathbf{F_f}$ form a couple of forces that tends to rotate the rigid body.

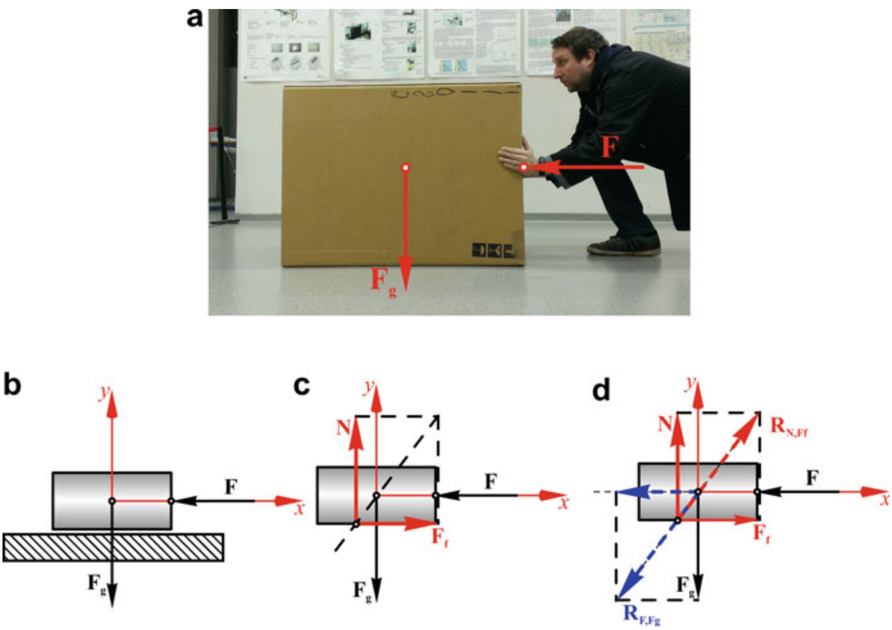


Fig. 12.1 (**a**) A person pushing a box. (**b**) Physical model of the box. (**c**) Free body diagram of the box. (**d**) Equilibrium of external forces and reactions

This moment is compensated by the shift of the line of action of reacting normal force N, which together with external normal force $\mathbf{F}_g$ form another couple of forces that compensates the moment of the first couple.

Friction force $\mathbf{F}_f$ acts at the interface of two surfaces in contact and does not pass through the mass center of the body. Consequently an external force and the friction force form a couple of forces that tend to rotate the body.

Hence, the two moments are equal in magnitude and opposite in sign and their resultant is equal to zero. This means that the resultant of the external forces,

$$\mathbf{R}_{F,F_g} = \mathbf{F} + \mathbf{F}_g = -F \cdot \mathbf{i} - F_g \cdot \mathbf{j}$$

and the resultant of the reactions,

$$\mathbf{R}_{N,F_f} = \mathbf{F}_f + \mathbf{N} = F_f \cdot \mathbf{i} + N \cdot \mathbf{j}$$

form a pair of forces and their sum is equal to zero, as shown in Fig. 12.1d

$$\mathbf{R}_{N,F_f} + \mathbf{R}_{F,F_g} = 0$$

The resultant of external forces and the resultant of reactions on a body form a pair of forces and their sum is equal to zero.

Now, let us analyze the case when we slowly increase external force $\mathbf{F}$ with which we push the box. As we increase the external force we realize that the box will not move until force $\mathbf{F}$ reaches a certain "threshold magnitude." This is shown in Fig. 12.2, where the magnitude of external force $\mathbf{F}$ is displayed on abscissa (x-axis) and the corresponding friction force $\mathbf{F}_f$ is displayed on ordinate (y-axis).

The resultant of all external forces, including the weight of a body, and reaction force from the ground, comprising friction and normal force, form a pair of forces that are in equilibrium, Fig. 12.1d.

From a diagram (Fig. 12.2) we may analyze the specific characteristic of dry friction. As we slowly increase external force $\mathbf{F}$, we observe that the corresponding friction force $\mathbf{F}_f$ increases linearly and consistently with Third Newton's Law. It takes place until the external force reaches certain threshold value at which the box suddenly starts to move from the state of rest. This threshold force is called *static friction force* $\mathbf{F}_f^s$. Behavior of the friction force up to point $\mathbf{F}_f = \mathbf{F}_f^s$ may be summarized as

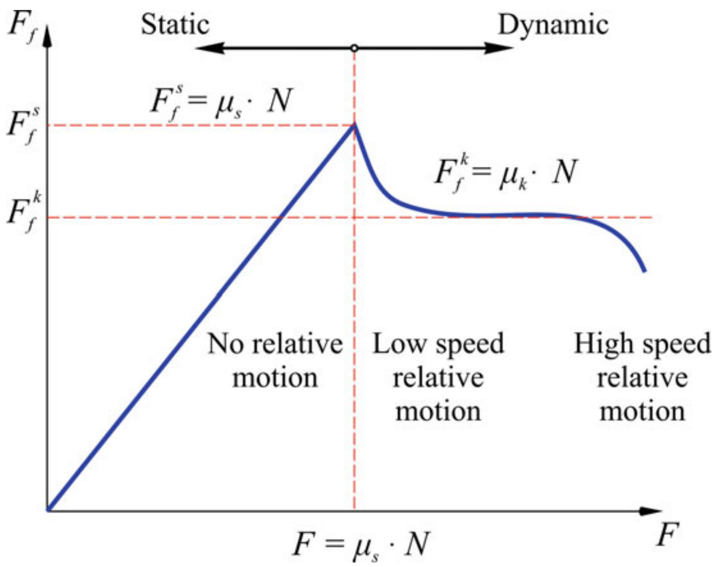


Fig. 12.2 Static and kinetic (dynamic) friction forces

> The maximum friction force that can appear at the interface of two bodies is called a *static friction force* $\mathbf{F}_f^s$.

$$F_f\left(F \leq F_f^s\right) = F \tag{12.1}$$

As soon as a rigid body moves, the force required to keep the rigid body (in our case box) in movement at a constant velocity $\mathbf{F}_f^k$ is smaller than the threshold force $\mathbf{F}_f^s$. Hence, $F_f^k \leq F_f^s$. This friction force is called *kinetic-friction force* (sometimes called *dynamic-friction force*). We further observe that kinetic-friction force $\mathbf{F}_f^k$ is in the wide range of velocities independent of applied external force **F**! So, as soon as $F > F_f^s$, the friction force will be constant in the wide range of velocities,

> As long as external force **F** acting on a body parallel to the contact surface between the two bodies is smaller than *static friction force* $\mathbf{F}_f^s$, friction force $\mathbf{F}_f$ will be of the same magnitude as the external force. Hence, $F_f\left(F \leq F_f^s\right) = F$.

$$F_f^k = F_f(v > 0) = \mu_k \cdot N = \text{const.} \tag{12.2}$$

Consequently, as we increase the external force beyond $\mathbf{F}_f^s$ the rigid body will start to move with an acceleration according to the Second Newton's Law: $a = \left(F - F_f^k\right)/m$, where m is the mass of the rigid body. Hence, in order to keep the rigid body in motion at a constant velocity we need to push the body with a constant force equal to kinetic-friction force $\mathbf{F} = \mathbf{F}_f^k$. Experiments show that kinetic-friction force $\mathbf{F}_f^k$ stays nearly constant as long as the relative sliding velocity is small. At higher relative velocities the kinetic friction force usually decreases as shown in Fig. 12.2.

> When external force **F** acting on a body parallel to the contact surfaces between the two bodies is larger than *static friction force* $\mathbf{F}_f^s$, friction force $\mathbf{F}_f$ will be constant within the wide range of velocities, $F_f(v > 0) = F_f^k = \mu_k \cdot N$

Friction force always opposes the motion or attempted motion of one surface relative to another surface. One may also say that the friction force always opposes the resultant force that attempts to move the body. Value of the friction depends on the nature and texture of both surfaces and magnitude of the contact (normal) force pushing the two surfaces together.

> Friction force always opposes the motion or attempted motion of one surface relative to another surface.

In our case normal force **N** is the reaction to weight $\mathbf{F_g}$ of the box. The *static friction force* may be expressed as

$$F_f^s = \mu_s \cdot N \tag{12.3}$$

whereas the *kinetic-friction force* as

$$F_f^k = \mu_k \cdot N \tag{12.4}$$

> The characteristic of dry friction $\mu_s > \mu_k$ may lead to a phenomenon known as stick-slip effect.

Here μ_s and μ_k are *static* and *kinetic* friction coefficients. A few examples of static and kinetic friction coefficients for different materials in contact are shown in Table 12.1 (please note that given values are just representative values).

12.2.1 Stick-Slip Effect

Most often static friction force is larger than the corresponding kinetic friction force, i.e., $F_f^s > F_f^k$. This is the special characteristic of Coulomb friction, as we commonly call dry friction between rigid bodies. It leads to the phenomenon called *stick-slip* effect. The effect may be demonstrated with a simple experiment shown in Fig. 12.3a.

A wooden block with mass *m* is attached to an elastic rubber band and placed on a flat horizontal surface, i.e., table. Figure 12.3b, c shows the corresponding

Table 12.1 Examples of static, μ_s, and kinetic, μ_k, friction coefficients

Materials in contact	μ_s	μ_k
Tire-on-dry pavement	0.9	0.8
Tire-on-wet pavement	0.8	0.7
Glass-on-glass	0.9	0.4
Wood-on-metal	0.7	0.5
Metal-on-metal (dry)	0.6	0.4
Wood-on-wood	0.6	0.4
Smooth tire-on-wet pavement	0.5	0.4
Metal-on-metal (lubricated)	0.1	0.05
Metal-on-ice	0.1	0.05
Metal-on-Teflon	0.05	0.05
Teflon-on-Teflon	0.04	0.04

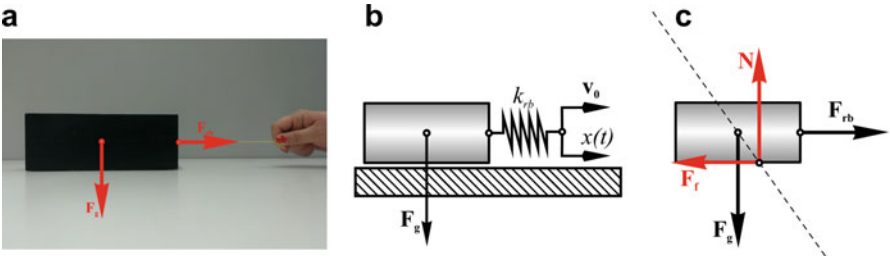


Fig. 12.3 Demonstration of the stick-slip effect. (**a**) A wooden block pulled by a rubber band. (**b**) Physical model of a wooden block pulled by a rubber band. (**c**) Free body diagram of a wooden block pulled by a rubber band

physical model and free body diagram. We start slowly pulling the block with the help of the rubber band with a small constant velocity v_0. In the physical model Fig. 12.3b, the rubber band is represented as a spring. The rubber band will deform and the force will gradually increase according to

$$F_{rb} = k_{rb} \cdot x(t) = k_{rb} \cdot v_0 \cdot t$$

where k_{rb} is the stiffness of the rubber band and $x(t) = v_0 t$ is an increasing deformation of the rubber band (as long as the body is in a state of rest). As long as the external force generated by the rubber band, $\mathbf{F}_{rb}$, is smaller than $\mathbf{F}_f^s$, i.e., $F_{rb} < F_f^s$, the wooden block will remain in a state of rest. However, when the force of the deformed rubber band reaches the threshold force, which is the static friction force, $\mathbf{F}_{rb} = \mathbf{F}_f^s$, the body will start to move.

Stick-slip effect is one of the key problems in the development of high precision manufacturing systems, such as CNC machines.

To overcome the stick-slip effect we have to use materials with the same static and kinetic friction coefficients, $\mu_s = \mu_k$, such as Teflon, Table 12.1.

As soon as the wooden block starts to move, the resisting friction force will drop to become kinetic-friction force $\mathbf{F}_f^k$. As a result, at this moment we have the excessive resultant force acting upon the body $F = F_{rb} - F_f^k = F_f^s - F_f^k$. Consequently, according to the Second Newton's Law, the wooden block will start to accelerate, i.e., $a = \left(F_f^s - F_f^k\right)/m$. As a result, the velocity of the wooden block will increase and may become larger than v_0 with which we pull the rubber band. Therefore, the deformation of the rubber band will decrease and the force in the rubber band will become smaller than the static friction force $F_{rb} < F_f^s$, forcing the body to stop. Since we are pulling the rubber band with a constant velocity its deformation will increase again, and the force in the rubber band will again supersede the threshold static friction force, $F_{rb} > F_f^s$, causing the "jump" of the wooden block. This process will be repeated, and it will result in a "jump-like" movement of the wooden block. Observed phenomenon is known as the *stick-slip* effect.

Stick-slip effect is instrumental mechanism for functioning of all stringed instruments, such as violin.

Antonio Stradivari violin of 1703 on exhibit at the museum in Berlin

As friction itself, the stick-slip effect is one of the key problems in many engineering applications. For example, stick-slip effect causes dynamic errors in CNC machine tools and is one of the main problems in the development of high precision manufacturing machines. On the other hand the stick-slip effect may be very beneficial, and without its existence our life would be different. The most vivid examples are stringed instruments.

12.2.1.1 Stick-Slip Effect of String Instruments

The stick-slip effect is instrumental in functioning of all string instruments, such as violin. The action of a bow that drives the strings is a regular cycle of stick-slip-stick-slip. Let us analyze one such cycle. When the bow is placed over the strings the relative velocity between the two is zero and the corresponding friction coefficient between the two is *static friction coefficient* μ_s. The bow sticks to a string and drags the string along, until the internal resultant force in the deformed string reaches the threshold value of the *static friction force*. At this point, the string breaks free of the bow and then slides past it easily with a very little friction, thanks to the low kinetic-friction coefficient μ_k (players put rosin on the bow to have a large difference between static, μ_s, and kinetic, μ_k, friction coefficients). The string doesn't stop when it passes the original (static) position because its momentum carries it on until it eventually stops and reverses direction. At this point, it catches on the bow again, the static friction reigns, and the cycle begins once more.

Various string instruments on display at the Museo de Arte Popular in Mexico City

Since the friction force depends on the normal contact force generated by a player and the force in the deformed string depends on the location where the bow is placed on a string, playing string instruments is very difficult.

String instruments require materials that have a large difference between the static, μ_s, and kinetic, μ_k, friction coefficient. To enhance the difference between the two friction coefficients players put rosin on the strings of their instruments.

12.3 Angles of Friction

In the previous section we have analyzed the simplest situation when a rigid body was laying on a flat horizontal surface and was loaded with its own weight and one external horizontal force. The line of action was parallel to the friction force. We have used this simple case to introduce the difference between the static and kinetic (dynamic) friction forces, and explain the phenomenon known as the stick-slip effect.

However, in reality bodies are most often exposed to more than one external force, including their own weight. The most general situation we can have is when n external forces are acting on a rigid body, $\{\mathbf{F}_i,\ \mathbf{r}_i; i = 1, 2, \ldots, n\}$, where each vector $\boldsymbol{r}_i$ defines the location of the point of action of external forces. We have assumed that $\mathbf{F}_n = \mathbf{F}_{g.}$ as shown in Fig. 12.4a. For the clarity we show only vector $\boldsymbol{r}_i$, which defines the point of action of force $\mathbf{F}_i$.

Let us assume that the external forces do not create a couple of forces (a moment). In such cases, as we have previously learned, all external forces may be replaced by the one resultant force,

$$\mathbf{R} = \sum_{i=1}^{i=n} \mathbf{F}_i = R_x\mathbf{i} + R_y\mathbf{j} \tag{12.5}$$

Its line of action may be determined from the equation by defining the resultant external moment

When more than one external force is acting on a body, we have to find the resultant force of the external forces. It includes the weight of the body.

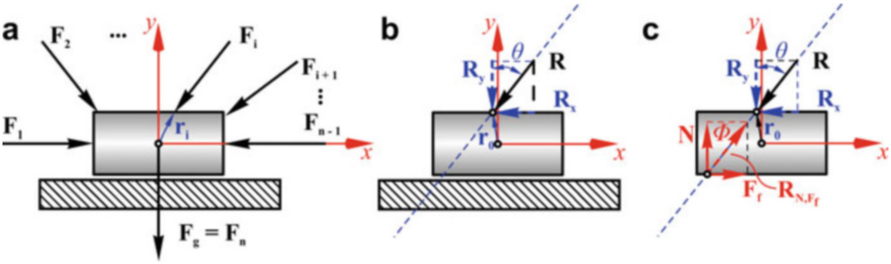


Fig. 12.4 Rigid body loaded with n-forces. (**a**) Physical model of a rigid body. (**b**) Physical model of a rigid body loaded with the resultant of n-forces. (**c**) Free body diagram of the rigid body loaded with the resultant force

$$\mathbf{M}_R = \mathbf{r}_0 \times \mathbf{R} = \sum_{i=1}^{i=n} \mathbf{r}_i \times \mathbf{F}_i = 0 \tag{12.6}$$

where vector $\mathbf{r}_0$ defines line of action of resultant force $\mathbf{R}$, which in general does not go through the center of mass of the body.

> The coordinate system should be placed in such a way that one of the coordinates, say, x-coordinate will be parallel to the contact surface (parallel to the line of action of the friction force), and y-axis will be in the direction normal to the contact surface.

Here the coordinate system was placed into the center of the mass of the body, so that x-coordinate is parallel to the contact surface (parallel to the line of action of the friction force), and y-axis is normal to the contact surface (Fig. 12.4a).

The resultant force may be expressed as a sum of its two components, one acting parallel to contact surfaces $\mathbf{R}_x$, and another that acts normal to contact surfaces $\mathbf{R}_y$, ((12.5) and Fig. 12.4b). Angle θ, which determines the line of action of resultant external force $\mathbf{R}$, may be obtained from its two components

$$\tan\theta = R_x/R_y \tag{12.7}$$

Figure 12.4c displays the corresponding free body diagram, where we also show reaction normal force $\mathbf{N}$ and friction force $\mathbf{F}_f$. The magnitudes of these two forces may be obtained from the force equilibrium equations in y and x directions, respectively. From the equilibrium equations in y-direction we find,

$$N = R_y \tag{12.8a}$$

> As long as $R_x \le F_f^s$, the magnitude of the friction force will be equal to R_x, thus $F_f\left(R_x \le F_f^s\right) = R_x$.

We have to remember that a rigid body will be in a state of rest as long as $R_x \le F_f^s$, where $\mathbf{F}_f^s$ is the static friction force, which is the largest friction force possible between the two selected surfaces in contact. Let us assume that the condition $R_x \le F_f^s$ is fulfilled. In this case we find from the equilibrium equation in x direction

$$F_f\left(R_x \le F_f^s\right) = R_x \tag{12.8b}$$

When $R_x \leq F_f^s$, the shift of normal force **N** is defined by the intersection of the line of action of the resultant external force and the line of action of the friction force.

The line of action of normal force **N** is shifted again in order to compensate the moment generated by friction force $\mathbf{F}_f$, and $\mathbf{R}_x$ component of the resultant external force; the two (always) form a couple of forces that tends to rotate a rigid body. The line of action of the resultant reaction force

$$\mathbf{R}_{N,F_f} = \mathbf{N} + \mathbf{F}_f = F_f\mathbf{i} + N\mathbf{j} \tag{12.9}$$

is defined by angle ϕ, commonly called *angle of friction*. The angle of friction may be obtained from Fig. 12.4c.

$$\tan\phi = F_f/N \tag{12.10}$$

From (12.8) we may conclude that as long as $R_x \leq F_f^s$, the line of action of resultant external force **R** must be equal to the line of action of resultant of reacting forces, $\mathbf{R}_{N,F_f}$,

$$\phi = \theta\left(R_x \leq F_f^s\right) \tag{12.11}$$

Moreover, the two resultant forces **R** and $\mathbf{R}_{N,F_f}$ form an equilibrium pair of forces, i.e., $\mathbf{R}_{N,F_f} + \mathbf{R} = 0$, as shown in Fig. 12.4c. From Fig. 12.4c we see that the shift of normal force **N** is defined by the intersection of the line of action of resultant external force **R** and the line of action of friction force $\mathbf{F}_f$.

A body will be in state of equilibrium as long as $\mathbf{R}_{N,F_f} + \mathbf{R} = 0$.

When the horizontal component of the resultant external force is smaller than the static friction force, $R_x < F_f^s$, then the resultant of external forces and the resultant of reactions form a pair of forces. The forces are equal in magnitude and opposite in direction.

12.3.1 Angles of Static and Kinetic Friction

At the beginning of this chapter we have learned that the largest friction force that is possible between two surfaces in contact is the *static friction force* $\mathbf{F}_f^s$ (Fig. 12.2). Let us define the largest angle of friction

$$\tan\phi_s = F_f^s/N \tag{12.12}$$

which we will call *angle of static friction* ϕ_s. By combining (12.3) and (12.12) we get that the angle of static friction is defined by the static friction coefficient.

$$\tan \phi_s = \mu_s \tag{12.13}$$

Similarly we may define the *angle of kinetic friction*, ϕ_k,

$$\tan \phi_k = F_f^k / N = \mu_k \tag{12.14}$$

Furthermore, if we take into consideration relation (12.11), we obtain a simple methodology for determining static friction coefficient μ_s from the line of action of the external resultant force **R** at which the body starts to move (slide),

$$\mu_s = \tan \phi_s = F_f^s / N = \tan \theta_s = R_x / R_y \tag{12.15}$$

Static friction coefficient μ_s may be determined from the line of action of external resultant force **R** at which the body starts to move.

The easiest way to modify the line of action of the resultant external force is by placing the body exposed only to its own weight on a plane with the inclination that can be changed.

From the angle of the plane's inclination at which the body starts to slide we can calculate the static friction coefficient by using (12.15).

12.3.1.1 Friction of a Rigid Body on a Slope

Let us consider a rigid body that is exposed only to its own weight $\mathbf{F}_g$ and is laying on a plane inclined by angle θ which may be changed as shown in Fig. 12.5a.

One may immediately see that the angle of the plane's inclination θ is the angle that is defining the line of action of the external resultant force, as discussed previously.

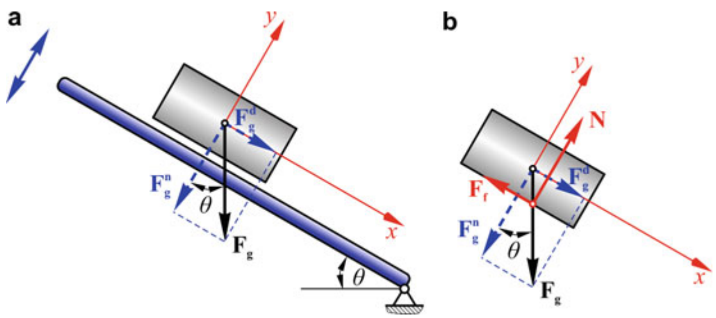


Fig. 12.5 A rigid body on a slope: (**a**) Physical model of the rigid body; (**b**) free body diagram of the rigid body on a slope

Static friction coefficient μ_s between two surfaces may be determined from the angle at which the body starts to slide. In such experiments the weight and the size of the body are not important.

Weight of the rigid body $\mathbf{F_g}$ may be split into two components, $\mathbf{F}_\mathrm{g}^\mathrm{n}$, which acts normal to the contact surface between the rigid body and the slope, and $\mathbf{F}_\mathrm{g}^\mathrm{d}$, which acts parallel to the slope and tries to move the rigid body down the slope. The corresponding free body diagram is shown in Fig. 12.5b. There are also shown two reaction forces: normal force $\mathbf{N}$ and friction force $\mathbf{F_f}$. The line of action of normal force $\mathbf{N}$ is shifted to form a couple of forces with external force $\mathbf{F}_\mathrm{g}^\mathrm{n}$. The moment of these two forces compensates the moment of the couple of forces $\mathbf{F}_\mathrm{g}^\mathrm{d}$ and $\mathbf{F_f}$.

As long as $\theta < \theta_s$ a body will not move along the plane, which means that such slopes are "self-locking," hence, we need an additional force to move the body down the slope.

By slowly increasing the angle of the plane inclination θ, we may find angle θ_s at which the body will start moving down the plane. According to (12.15) this angle of inclination defines the static friction coefficient, $\mu_s = \tan\theta_s$. This means that the friction coefficient between two surfaces may be determined from the angle at which the body starts to slide. In such experiments the weight and the size of the body are not important!

From this experiment we may also conclude that as long as $\theta < \theta_s$ the body will not move along the slope. It means that such slopes are "self-locking," so we need to apply an additional external force to move the body down the slope. This information is very important for designing wedges, axial bearings, etc.

Guidelines and Recipes for Problems Involving Dry Friction

- Friction force always acts in *opposite* direction to the direction an object slides or wants to slide.
- Draw a free body diagram of the body under consideration and place a coordinate system such that one of the coordinates, say x-axis, will be parallel to the friction contact surfaces. If several bodies are involved, draw a free body diagram for each of them following the same procedure.
- Find the resultant of the external forces acting on the observed body (including weight of the body), and present it via its components in x and y directions, $\mathbf{R} = \sum_{i=1}^{i=n}\mathbf{F}_i = R_x\mathbf{i} + R_y\mathbf{j}$.
- The reaction exerted by a surface on a free body consists of two components, $\mathbf{R}_{N,F_f} = \mathbf{N} + \mathbf{F}_f = F_f\mathbf{i} + N\mathbf{j}$, where $\mathbf{N}$ is a normal force and $\mathbf{F_f}$ is a tangential *friction force*. The reaction forces are obtained from the equilibrium equations.

(continued)

- The two resultant forces, $\mathbf{R}$ and $\mathbf{R}_{N,F_f}$, form a couple of forces, i.e., $\mathbf{R}_{N,F_f} + \mathbf{R} = 0$
  - *No motion* will occur as long as R_x does not exceed the maximum value of friction force $R_x < \left(F_f^s = N \cdot \mu_s\right)$, where μ_s is the *coefficient of static friction.*
  - For $R_x < F_f^s$, the friction force will always be equal to R_x, hence $F_f\left(R_x < F_f^s\right) = R_x$.
  - *A motion* will occur if the value of R_x is larger than the value of F_f^s. As the motion takes place the friction force drops to $F_f^k = N \cdot \mu_k$, where μ_k is the *coefficient of kinetic friction*.
- A body that is placed on an incline with angle θ and exposed to its own weight will start sliding when $\tan\theta \geq \mu_s$, where μ_s is the coefficient of static friction.

Example 12.1 A car of weight $\mathbf{F}_g$ is parked on a slope with inclination φ (Fig. 12.6a). Determine the conditions at which the car will slide along the incline and at which it will rollover. Assume that the static friction coefficient between the

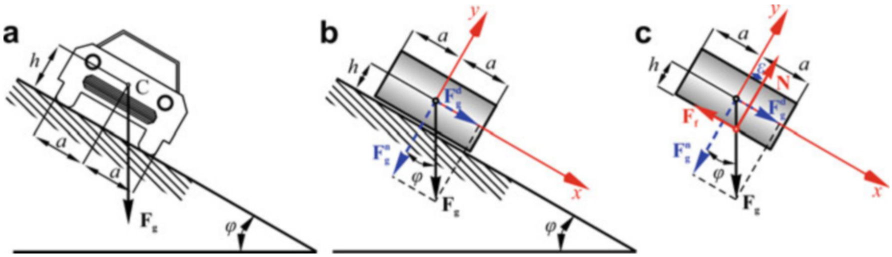


Fig. 12.6 Car standing on a slope: (**a**) schematic drawing; (**b**) physical model of a car standing on a slope; (**c**) free body diagram of a car standing on a slope

tires and the slope is μ_s. The location of the car's center of gravity C and the distance between the wheels are shown in Fig. 12.6a.

Solution The physical model and free body diagram are shown in Fig. 12.6b, c, respectively. Weight $\mathbf{F}_g$ of the car may be split into two components, contact force $\mathbf{F}_{\mathrm{g}}^{\mathrm{n}}$, which acts normal to the slope, and $\mathbf{F}_{\mathrm{g}}^{\mathrm{d}}$, which acts parallel to the slope and pushes the car to slide down the slope,

$$F_g^n = F_g \cdot \cos\varphi \tag{12.16a}$$

and

$$F_g^d = F_g \cdot \sin\varphi \tag{12.16b}$$

Condition at which the car will slide along the slope:
The fastest and easiest way to determine the condition at which the car will slide along the slope is to use (12.15). The car is loaded with its own weight only; therefore, angle of the slope inclination φ is the angle defining the line of action of external (resultant) force $\mathbf{F}_g$. According to (12.15) the car will start sliding down the slope when

$$\mu_s = \tan\varphi \tag{12.17}$$

It means that the condition at which the sliding starts is independent of the weight of the car. Let us recall once again that μ_s depends only on the type of materials and roughness of the two surfaces being in contact. This explains why quality of tires in Formula 1 racing plays such an important role and why replacing tires during the race makes sense. The importance of choosing hiking shoes with a proper sole material is another example where the static friction coefficient plays an essential role.

Condition at which a car will rollover:
It is obvious that as soon as a car slides it cannot rollover, except if it hits a barrier which, however, is not envisioned in this task. Therefore, in our case rolling over can happen only before the car starts sliding. From the free body diagram in Fig. 12.6c we see that the couple of forces that tends to rotate the car consists of $\mathbf{F}_{\mathrm{g}}^{\mathrm{d}}$ and $\mathbf{F}_{\mathbf{f}}$. Let us recall that $F_f\left(F_g^d \le F_f^s\right) = F_g^d$, and as soon as $F_g^d > F_f^s$, the car will slide along the slope. Hence, the largest "rolling moment" that is possible is

$$M_{\mathrm{roll}} = h \cdot F_f^s$$

where h is a perpendicular distance between $\mathbf{F}_{\mathrm{g}}^{\mathrm{d}}$ and $\mathbf{F}_{\mathbf{f}}$.

This “rollover moment” is compensated by the shift of the line of action of normal force **N** that we find from the equilibrium equation in y-direction,

$$N = F_g^n = F_g \cdot \cos\varphi$$

Stability of a car will be better if the center of its gravity is as low as possible and if the distance between the two wheels (track) is as large as possible.

Forces **N** and $\mathbf{F}_\mathrm{g}^\mathrm{n}$ form a couple of forces that prevents the car to rollover. The corresponding “stabilizing moment” is

$$M_{\mathrm{stab}} = \varepsilon \cdot N$$

where ε is the magnitude of the normal force shift, as shown in Fig. 12.6c. The car will start to rollover when $M_{\mathrm{roll}} \geq M_{\mathrm{stab}}$,

$$\left(M_{\mathrm{roll}} = h \cdot F_f^s\right) \geq (M_{\mathrm{stab}} = \varepsilon \cdot N) \quad (12.18a)$$

When the size of the contact surface goes to zero, i.e., $\varepsilon \to 0$, we do not need external moment for rolling a body, $M_{roll}(\varepsilon = 0) = 0$! *We have just (re)invented the wheel!*

The magnitude of the “stabilizing moment” depends on the magnitude of the normal force and its horizontal shift. Thus its maximum value is limited by the size of the contact surfaces.
This is a very important observation, because it tells us that when the size of the contact surface goes to zero, i.e., $\varepsilon \to 0$, we do not need any external moment for rolling a body!

From Fig. 12.6c we see that the largest possible shift of the normal force is $\varepsilon_{\max} = a$, which defines the largest stabilizing moment $M_{\mathrm{stab}}^{\max} = a \cdot N$. By considering that $F_f^s = \mu_s \cdot N$, we find the condition at which the car will rollover,

$$\mu_s \geq \frac{a}{h}. \quad (12.18b)$$

So, the car will rollover when static friction coefficient μ_s will be larger than the ratio between the half of the distance between the wheels (track) a and the vertical distance of the car’s center of gravity from the ground h. For the most modern cars $a/h \gg 1$ and therefore at the normal conditions they will not rollover.
From this we may also conclude that stability of a car will be better if center of its gravity h is as low as possible and if distance between the two wheels (track) $2a$ is as large as possible.

Finally, let us consider a "border situation" when a car is at the point to slip down the slope and at the same time it is at the point to rollover. The condition for this situation we find by combining (12.7) and (12.8),

$$\tan\varphi = \frac{a}{h}. \tag{12.19}$$

Equation (12.9) defines the largest slope on which a car may stand without slipping or rolling over. Hence, whenever $\tan\varphi > a/h$ the car will either slip or rollover, depending on the magnitude of static friction coefficient μ_s. If $\mu_s < \tan\varphi$ the car will slip, and if $\mu_s > \tan\varphi$ the car will rollover.

Condition $\tan\varphi = a/h$ defines the largest slope on which a car will stand without slipping or rolling over.

Example 12.2

A heavy wooden box (weight $\mathbf{F}_g$) is located on a plane inclined at angle φ. External force $\mathbf{F}$ is acting on the box parallel to the incline (Fig. 12.7a). Let's assume that the static friction coefficient between the contact surfaces is μ_s, whereas the kinetic friction coefficient is μ_k and $\mu_s \gg \mu_k$. Let us analyze the following:

(a) Between which values we need to maintain the magnitude of force $\mathbf{F}$ to keep the box in the state of rest, i.e.,

$$\mathbf{F}_{\text{min}} < \mathbf{F} < \mathbf{F}_{\text{max}}?$$

(b) How large should be force $\mathbf{F}$ to move the box up the incline, $\mathbf{F}^k_{\text{up}}$, and down the incline, $\mathbf{F}^k_{\text{down}}$, with a constant velocity?

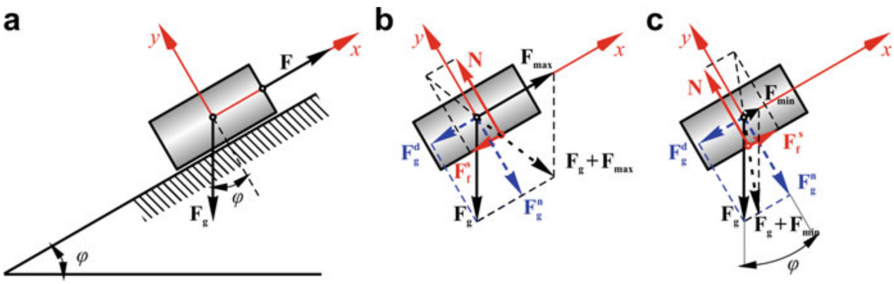


Fig. 12.7 (**a**) Wooden box pulled with the force F. (**b** and **c**) Free body diagrams of a wooden box with tendency to move (**b**) up and (**c**) down

Solution

(a) *Determination of limits* $\mathbf{F}_{\min} < \mathbf{F} < \mathbf{F}_{\max}$

To determine the range of the external force magnitude required to keep the box in a state of rest, i.e., $\mathbf{F}_{\min} < \mathbf{F} < \mathbf{F}_{\max}$, we have to consider two extreme cases.

$F_{\max}$ is the magnitude of applied force just before the box will start moving up the slope, whereas $F_{\min}$ is the magnitude of applied force just before the box will start moving down the slope. The two cases are shown in Fig. 12.7b that shows the corresponding free body diagrams. The friction force always opposes the direction of relative motion, or tendency to motion; therefore, in these two cases the direction of the friction force will be different. In the first case the friction force will point down the slope, whereas in the second case it will point up the slope, as demonstrated in Fig. 12.7b. In both cases the magnitude of the friction force will assume its extreme value, $F_f = F_f^s$.

The weight of box $\mathbf{F}_g$ may be split into two components, $\mathbf{F}_{\mathrm{g}}^{\mathrm{n}}$, which acts normal to the slope, and $\mathbf{F}_{\mathrm{g}}^{\mathrm{d}}$, which acts parallel to the slope. $\mathbf{F}_{\mathrm{g}}^{\mathrm{d}}$ tries to make the box slide down the slope and will be called the dynamic component,

$$F_g^n = F_g \cdot \cos\varphi \tag{12.20a}$$

$$F_g^d = F_g \cdot \sin\varphi \tag{12.20b}$$

From Fig. 12.7b we see that the resultant of all external forces (including the weight of the body) and the reaction force of the ground ($\mathbf{N} + \mathbf{F}_f$) form an equilibrium pair of forces.

The equilibrium equation in y direction for both cases is the same. So,

$$N = F_g^n = F_g \cdot \cos\varphi \tag{12.21}$$

Now we can calculate the magnitude of the static friction force,

$$F_f^s = \mu_s \cdot N = \mu_s \cdot F_g \cdot \cos\varphi \tag{12.22}$$

We obtain $F_{\min}$ and $F_{\max}$ from the equilibrium equations in x direction. For the case when the box has tendency to move up we have

$$\sum_i F_{x,i} = F_{\max} - F_g^d - F_f^s = 0 \tag{12.23a}$$

whereas for the case when it tends to move down the sign of the static friction force will change,

$$\sum_i F_{x,i} = F_{\min} - F_g^d + F_f^s = 0 \tag{12.23b}$$

By combining (12.20), (12.21), and (12.23) we find

$$F_{\max} = F_g \cdot (\sin\varphi + \mu_s \cdot \cos\varphi), \quad \text{and} \tag{12.24a}$$

$$F_{\min} = F_g \cdot (\sin\varphi - \mu_s \cdot \cos\varphi) \tag{12.24b}$$

The result for $F_{\min}$ requires some further analysis. From (12.24b) we see that depending on the inclination of the plane and the magnitude of the static friction coefficient, $F_{\min}$ can be positive, equal to zero, or even negative! Particularly interesting is the situation when $F_{\min} \le 0$. This will happen when

$$\sin\varphi - \mu_s \cdot \cos\varphi \le 0$$

Since the static friction coefficient may be expressed as the *angle of static friction* (12.13), $\mu_s = \tan\phi_s$, we obtain that $F_{\min} \le 0$ when

$$\tan\phi_s \ge \tan\varphi \tag{12.25}$$

Inclination slopes φ that are smaller than the *angle of static friction* $\varphi < \phi_s$ are called *self-locking*.

i.e., when the *angle of static friction* is larger than the inclination angle of the slope. The inclination angles that fulfill this condition we will call *self-locking*!

From (12.24) we can calculate an answer to the first task:

$$F_g \cdot (\sin\varphi - \mu_s \cdot \cos\varphi) \le F \le F_g \cdot (\sin\varphi + \mu_s \cdot \cos\varphi)$$

Let us discuss the second task.

(b) *Determination of limits* $\mathbf{F}_{\text{up}}^k$ and $\mathbf{F}_{\text{down}}^k$

To initiate the box movement up the incline, we first need to increase the magnitude of force **F** to the level required to overcome the static friction force defined in the previous section, hence

$$F_{max} = F_g \cdot (\sin\varphi + \mu_s \cdot \cos\varphi)$$

As soon as the box starts to move the friction coefficient will decrease from static friction coefficient μ_s to kinetic-friction coefficient μ_k. So, we can determine F_{up}^k from the following equation:

$$F_{up}^k = F_g \cdot (\sin\varphi + \mu_k \cdot \cos\varphi) \tag{12.26}$$

When $\mu_s \gg \mu_k$ it could happen that while we are capable of keeping a load on a slope in the state of rest, we will not be able to provide enough force to maintain its movement down the slope at a constant velocity!

Let us emphasize again that $F^k_{up} < F_{max}$, which means that as soon as the box starts to move force **F** has to be decreased to maintain its velocity constant.

We observe quite a different situation when we start decreasing the magnitude of **F**, which brings into dominance the dynamic component of the box weight, F^d_g. Consequently, the box at some point assumes tendency to move down the slope. This "switch" will happen when the magnitude of force **F** will become equal to the dynamic component of the box weight, i.e., when $F^d_g = F$. At this point friction plays no role. However, this is also the point when the friction force changes its direction of action and starts acting up the slope. Upon further decreasing of the magnitude of force **F** the friction will reach the minimum.

$$F_{min} = F_g \cdot (\sin\varphi - \mu_s \cdot \cos\varphi)$$

F_{min} is the smallest magnitude of force **F** that maintains the box in the state of rest. At this point the box will start moving down the slope and static friction coefficient μ_s will switch to kinetic-friction coefficient μ_k, which is always smaller than μ_s.

$$F^k_{down} = F_g \cdot (\sin\varphi - \mu_k \cdot \cos\varphi) \tag{12.27}$$

That means that as soon as the box starts moving down the slope the magnitude of force **F** (which acts up the slope) has to be increased to maintain its constant velocity, $F^k_{down} > F_{min}$.

Since the difference between μ_s and μ_k can be quite large, see Table 12.1, it could happen that while we are capable of keeping the load in the state of rest, we will not be able to provide enough force to maintain its movement down the slope at a constant velocity.

12.4 Wedges

Wedges are commonly used to elevate heavy objects such as heavy machinery in an industrial environment. Figure 12.8a shows an example of the use of a wedge to lift object A with weight $\mathbf{F}_g$, which rests against vertical wall D. For the lifting of the object we use two wedges B and C with the same angle of inclination φ to assure that the object will keep its original orientation, Fig. 12.8a. Usually the weight of wedges may be neglected. We will assume that static and kinetic friction

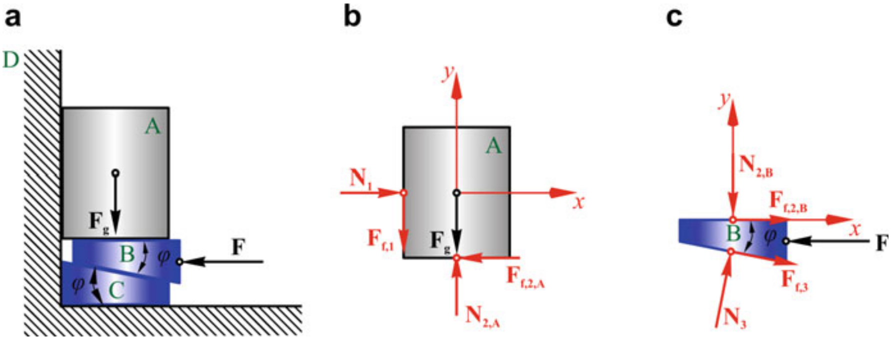


Fig. 12.8 (**a**) Physical model of a wedge for lifting an object; (**b**) Free body diagram of object; (**c**) Free body diagram of wedge B

coefficients μ_s and μ_k are the same for all contact surfaces. The wedge inclination angle should be selected so that the wedge will be self-locking, i.e., $\tan\varphi < \mu_s$.

Figure 12.8b, c shows the free body diagrams for object A and for wedge B that include the reaction and friction forces. It is important to show the direction of friction forces correctly. The direction of the friction force depends on the assumed direction of the impending motion.

The direction of the friction force depends on the assumed direction of the impending motion. It cannot be guessed!

Friction force $\mathbf{F}_{f,2,A}$ that is acting on object A, Fig. 12.8b, and friction force $\mathbf{F}_{f,2,B}$ that is acting on wedge B, Fig. 12.8c, should be of the same magnitude and opposite in direction, $\mathbf{F}_{f,2,A} = -\mathbf{F}_{f,2,B}$, hence they form a *pair of forces*. Since wedge B moves to the left, friction forces $\mathbf{F}_{f,2,B}$ and $\mathbf{F}_{f,3}$ should oppose its movement and act to the right as shown in Fig. 12.8c. The corresponding friction force $\mathbf{F}_{f,2,A}$ that is acting on object A will therefore act in opposite direction, i.e., to the left. The same is true for normal reaction forces $\mathbf{N}_{2,A}$ and $\mathbf{N}_{2,B}$; they also need to form a pair of forces. Hence $\mathbf{N}_{2,B} = -\mathbf{N}_{2,A}$. Normal force N_3 acts up because the top wedge presses the lower wedge down. Object A presses against the wall, i.e., to the left; therefore, reaction normal force $\mathbf{N}_1$ has to act to the right. Since object A will move up, friction force $\mathbf{F}_{f,1}$, generated between the wall and the object, will act down, opposing its movement. Unknown forces can be determined from the equilibrium equations.

Guidelines and Recipes for Applying the Laws of Friction to Wedges

- Draw a free body diagram of all wedges and of all other bodies involved.
- Determine the maximum static friction force at each contact surface.
- Friction forces always act opposite to the direction of the wedge's or body's relative motion or its impending motion.

(continued)

- Determine unknown forces from the equilibrium equations.

Example 12.3 Determine the smallest force **F** required to raise object A (Fig. 12.8). $\mathbf{F}_g$, φ, and friction coefficients μ_s and μ_k are known.

Solution The smallest force required to start moving object A upwards is the static friction force. From then on, the smallest force required to maintain its movement at a constant velocity is the force needed to overcome the kinetic friction force.

The magnitude of all friction forces may be expressed in terms of corresponding normal forces $\mathbf{N}_\mathrm{i}$, $N_{2,B} = -N_{2,A} = -N_2$, and N_3 only when the motion is impending. Hence, $F^s_{f,1} = N_1 \cdot \mu_s$, $F^s_{f,2,A} = N_2 \cdot \mu_s$, $F^s_{f,2,B} = N_2 \cdot \mu_s$, and $F^s_{f,3} = N_3 \cdot \mu_s$.

Force **F** may be obtained from the equilibrium equations of the forces for object A and wedge B,

$$\sum_i F^A_{i,x} = N_1 - N_2 \cdot \mu_s = 0$$

$$\sum_i F^A_{i,y} = -N_1 \cdot \mu_s + N_2 - F_g = 0$$

$$\sum_i F^B_{i,x} = -F + N_2 \cdot \mu_s + N_3 \cdot \mu_s \cdot \cos\varphi + N_3 \cdot \sin\varphi = 0$$

$$\sum_i F^B_{i,y} = -N_2 - N_3 \cdot \mu_s \cdot \sin\varphi + N_3 \cdot \cos\varphi = 0$$

From the first two equilibrium equations we find,

$$N_1 = F_g \cdot \frac{\mu_s}{\left(1 - \mu_s^2\right)}, \quad \text{and}$$

$$N_2 = \frac{F_g}{\left(1 - \mu_s^2\right)}$$

From the fourth equilibrium equation we then obtain,

$$N_3 = \frac{F_g}{\left(\cos\varphi - \mu_s \cdot \sin\varphi\right)\left(1 - \mu_s^2\right)}$$

Finally, from the third equation we find the expression for the initial force that is required to overcome the static friction forces,

$$F^s = F_g \cdot \left[\frac{\mu_s}{\left(1 - \mu_s^2\right)} + \frac{\left(\mu_s \cdot \cos\varphi + \sin\varphi\right)}{\left(\cos\varphi - \mu_s \sin\varphi\right)\left(1 - \mu_s^2\right)}\right]$$

The force required for the continuous movement upwards with a constant velocity of object A we obtain by replacing the static friction coefficient with the kinetic-friction coefficient,

$$F^k = F_g \cdot \left[\frac{\mu_k}{\left(1 - \mu_k^2\right)} + \frac{\left(\mu_k \cdot \cos\varphi + \sin\varphi\right)}{\left(\cos\varphi - \mu_k \sin\varphi\right)\left(1 - \mu_k^2\right)}\right]$$

Example 12.4 Determine the smallest force **F** required: (a) to raise and (b) to lower object B at a constant velocity shown in Fig. 12.9a. The weight of the object B is $F_g = 10$ kN, the angle of the wedge inclination is $\alpha = 10°$, and the kinetic friction coefficient between all the surfaces $\mu_k = 0.3$.

Solution The smallest force required to start moving object B upwards is equal to the sum of static friction forces. From then on, the smallest force required to maintain its movement at a constant velocity is the force needed to overcome kinetic-friction forces. In this example we need to determine a force required to

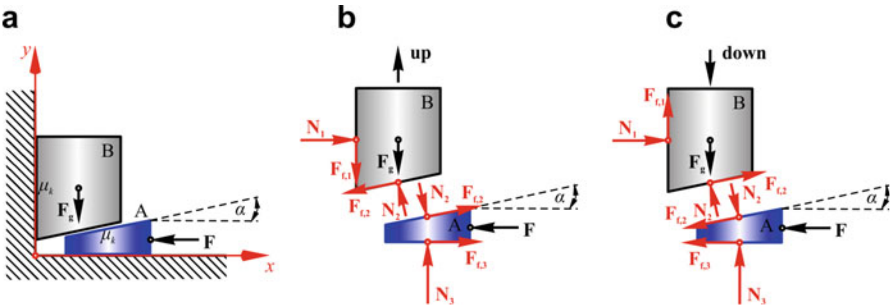


Fig. 12.9 (**a**) Physical model of a wedge used to raise or lower an object; (**b**) Free body diagram of the object and the wedge for raising the object; (**c**) Free body diagram of the object and the wedge for lowering the object

overcome kinetic friction force. Figure 12.9b, c shows the free body diagrams for the cases when object B is moving upwards and downwards, respectively.

(a) *Object B is moving upwards*
Let us first write equilibrium equations for the case when object B is moving upwards. The corresponding free body diagram is shown in Fig. 12.9b. For object B we have

$$N_1 - N_2 \cdot \sin\alpha - F_{f,2} \cdot \cos\alpha = 0$$

$$-F_{f,1} + N_2 \cdot \cos\alpha - F_{f,2} \cdot \sin\alpha - F_g = 0$$

and for object A,

$$-F + N_2 \cdot \sin\alpha + F_{f,2} \cdot \cos\alpha + F_{f,3} = 0$$

$$-N_2 \cdot \cos\alpha + F_{f,2} \cdot \sin\alpha + N_3 = 0$$

The friction forces are given as,

$$F_{f,1} = \mu_k \cdot N_1, \quad F_{f,2} = \mu_k \cdot N_2, \quad \text{and} \quad F_{f,3} = \mu_k \cdot N_3$$

From the above set of equations we find the solution for the case when object B is moving upwards,

$$N_{1,\mathrm{up}} = F_g \frac{\sin\alpha + \mu_k \cos\alpha}{\left(1 - \mu_k^2\right)\cos\alpha - 2\mu_k \sin\alpha}$$

$$N_{2,\mathrm{up}} = \frac{F_g}{\left(1 - \mu_k^2\right)\cos\alpha - 2\mu_k \sin\alpha}$$

$$N_{3,\mathrm{up}} = F_g \frac{\cos\alpha - \mu_k \sin\alpha}{\left(1 - \mu_k^2\right)\cos\alpha - 2\mu_k \sin\alpha}$$

and

$$F_{\mathrm{up}} = F_g \frac{\left(1 - \mu_k^2\right)\sin\alpha + 2\mu_k \cos\alpha}{\left(1 - \mu_k^2\right)\cos\alpha - 2\mu_k \sin\alpha} = F_g \frac{\left(1 - \mu_k^2\right)\tan\alpha + 2\mu_k}{\left(1 - \mu_k^2\right) - 2\mu_k \tan\alpha}$$

By expressing the kinetic friction coefficient as *angle of kinetic friction,* and using (12.14), $\tan\phi_k = \mu_k$, we may express the above equations for F_{up} as

$$F_{\text{up}} = F_g \frac{(1 - \tan^2\phi_k)\tan\alpha + 2\tan\phi_k}{(1 - \tan^2\phi_k) - 2\tan\phi_k \cdot \tan\alpha} = F_g \frac{\tan\alpha + \tan 2\phi_k}{1 - \tan 2\phi_k \cdot \tan\alpha}$$
$$= F_g \tan(\alpha + 2\phi_k)$$

After inserting numerical values we obtain

$$\phi_k = 16.7^\circ, \quad \text{and} \quad F_{up} = 9.46\ \text{kN}$$

It is interesting to note that since $\phi_k > \alpha$ the wedge will be self-locking, which means that for lowering body B force **F** will need to change its sign (direction), as will be shown later.

(b) *Object B is moving downwards*
Similarly we obtain equilibrium equations for the case when object B is moving downwards.

$$N_1 - N_2 \cdot \sin\alpha + F_{f,2} \cdot \cos\alpha = 0$$

$$F_{f,1} + N_2 \cdot \cos\alpha + F_{f,2} \cdot \sin\alpha - F_g = 0$$

The equations for object A,

$$-F + N_2 \cdot \sin\alpha - F_{f,2} \cdot \cos\alpha - F_{f,3} = 0$$

$$-N_2 \cdot \cos\alpha - F_{f,2} \cdot \sin\alpha + N_3 = 0$$

As before, the friction forces may be expressed as

$$F_{f,1} = \mu_k \cdot N_1, \quad F_{f,2} = \mu_k \cdot N_2, \quad \text{and} \quad F_{f,3} = \mu_k \cdot N_3$$

By solving the obtained set of linear equations we get the solution for the case when object B is moving down,

$$N_{1,\text{down}} = F_g \frac{\sin\alpha - \mu_k \cos\alpha}{\left(1 - \mu_k^2\right)\cos\alpha + 2\mu_k \sin\alpha}$$

$$N_{2,\text{down}} = \frac{F_g}{\left(1 - \mu_k^2\right)\cos\alpha + 2\mu_k \sin\alpha}$$

$$N_{3,\text{down}} = F_g \frac{\cos\alpha + \mu_k \sin\alpha}{\left(1 - \mu_k^2\right)\cos\alpha + 2\mu_k \sin\alpha}$$

and

$$F_{\text{down}} = F_g \frac{\left(1 - \mu_k^2\right)\sin\alpha - 2\mu_k \cos\alpha}{\left(1 - \mu_k^2\right)\cos\alpha + 2\mu_k \sin\alpha} = F_g \frac{\left(1 - \mu_k^2\right)\tan\alpha - 2\mu_k}{\left(1 - \mu_k^2\right) + 2\mu_k \tan\alpha}$$

If we will express the kinetic friction coefficient again as *angle of kinetic friction* and use (12.14), $\tan\phi_k = \mu_k$, we may express the above equations for F_{down} as,

$$F_{\text{down}} = F_g \frac{(1 - \tan^2\phi_k)\tan\alpha - 2\tan\phi_k}{(1 - \tan^2\phi_k) + 2\tan\phi_k \cdot \tan\alpha} = F_g \frac{\tan\alpha - \tan 2\phi_k}{1 + \tan 2\phi_k \cdot \tan\alpha}$$
$$= F_g \tan(\alpha - 2\phi_k)$$

After inserting the numerical values we find,

$$F_{\text{down}} = -4.33 \text{ kN}$$

Hence, we need to pull the wedge out under object B in order to lower it. This will be true for $0 \le \alpha \le \tan^{-1}(\mu_k)$. After substituting by numerical values: $0 \le \alpha \le 33.4°$.

For rolling a perfectly round and rigid cylinder or wheel on a perfectly flat and rigid surface at a constant velocity we do not need any force and any energy input!

12.5 Rolling and Rolling Resistance

Engineering structures that very closely mimic ideal rolling conditions are trains where steel wheels are rolling over steel rails.

In Example 12.1 we have learned that rolling of a body is caused by rolling moment M_{roll}, generated by a couple of forces consisting of friction force F_f and a component of the external resultant force whose line of action is parallel to F_f. We have also learned that rolling of a body is prevented by shift ε of the line of action of normal force N along the contact surface between the body and ground, Fig. 12.10b. The rolling of the body happens when

$$M_{\text{roll}} \ge \varepsilon \cdot N \tag{12.28}$$

From the above equation we see that the size of the contact surface and consequently the magnitude of ε depend on geometry of the body. This means that by changing geometry of the body we can change ε and, as a result, the magnitude of M_{roll} required to rotate the body. In the extreme situation $M_{\text{roll}}(\varepsilon = 0) = 0$ (12.28).

Friction is a precondition for wheels rotation, hence, NO friction –NO rotation.

Let us consider a perfectly round rigid cylinder with diameter $2r_0$ and weight $\mathbf{F}_g$, which is placed on a perfectly flat rigid surface, as shown in Fig. 12.10a. Figure 12.10b shows the corresponding free body diagram. The static and kinetic friction coefficients are μ_s and μ_k, respectively. In this case the contact surface between the cylinder and the flat surface will be just a line or a point if we observe the contact in two dimensions, as shown in Fig. 12.10a. Therefore, the size of the contact surface is equal to zero, $\varepsilon = 0$, and $M_{\text{roll}}(\varepsilon = 0) = 0$.

> If the external force is larger than the static friction force, $F > F_f^s$, then the cylinder will slide and rotate simultaneously.

This means that for rolling a perfectly round and rigid cylinder or wheel over a perfectly flat and rigid surface at a constant velocity we do not need ANY external moment or force. In other words, in described ideal case rolling of a body with a constant velocity does not require any energy input! The external force in this case will immediately and continuously accelerate the cylinder. If the external force is larger than the static friction force, $F > F_f^s$, then the cylinder will slide and rotate simultaneously. It is worth mentioning that if there is no friction there would be no rotation! Static friction is therefore precondition for rotation of wheels. One may experience this in winter while driving a car over an icy road.

Engineering structures that very closely mimic described ideal rolling conditions are trains where steel wheels are rolling over steel rails. Next to trains are all wheels-based vehicles. Of course, rubber tires are far from being rigid, which means that the contact surface will not be zero (Fig. 12.11).

> As soon as there is a contact surface between a body and the ground, the line of action of the reaction force will be shifted in the direction of motion. The amount of the shift we call *coefficient of rolling resistance* e[μm].

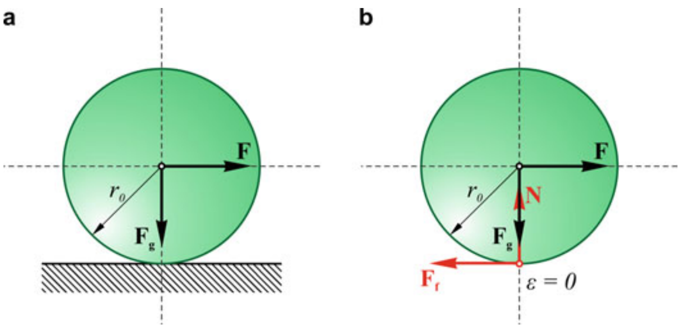


Fig. 12.10 (**a**) Rigid cylinder on a flat rigid surface; (**b**) Free body diagram of a rigid cylinder on a flat rigid surface

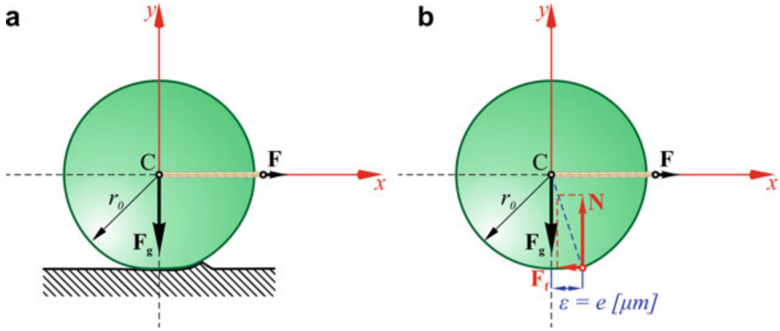


Fig. 12.11 (**a**) Non-rigid wheel on a flat deformable surface; (**b**) free body diagram of a non-rigid wheel on a flat deformable surface

In reality ideally rigid bodies do not exist. Due to the contact forces the wheel and the surface deform, as shown in Fig. 12.11a. This means that the contact surface between the wheel and the ground is NOT zero and, therefore, $\varepsilon = e \neq 0$ as schematically shown in the free body diagram (Fig. 12.11b).

Hence, as soon as there is a contact surface between the body and the ground, the line of action of the reaction force will be shifted in direction of the motion, forming with the external normal force a couple of forces whose moment tries to prevent rotation of the wheel (Fig. 12.11b). This moment is called *rolling resistance moment*. The amount of the shift of the line of action of normal force $\varepsilon = e$ [μm] caused by deformation of the wheel and the ground is called the *coefficient of rolling resistance*. *The coefficient of rolling resistance* as units of distance, and is usually measured in micrometers, μm, millimeters, mm or inches, in.

The *coefficient of rolling resistance e* has units of distance, and is usually measured in micrometers, μm, millimeters, *mm,* or inches, *in*.

From the equilibrium equation of moments about point C (Fig. 12.11b)

$$\sum_i M_i^C = N \cdot e - F \cdot r_0 = 0$$

we can find the force required to overcome the rolling resistance

$$F = N \cdot \frac{e}{r_0} = F_g \cdot \frac{e}{r_0} \tag{12.29}$$

Here we assumed that $N = F_g$, while e is the coefficient of the rolling resistance, and r_0 is the radius of the rolling body (cylinder or wheel).

To experience how the amount of deformation affects the rolling resistance you can perform a simple experiment. Find a flat empty parking space, put the gears of the car into neutral, and try to push the car a few meters (yards). Then deflate the tires for, let say 0.5 bars, and try to push the car again. You will experience the practical meaning of (12.29)!

Guidelines and Recipes for Rolling and Rolling Resistance

- The *rolling resistance* results from the deformation of both a wheel and the surface on which the wheel is rolling. As a result, reaction force **N** of the ground is shifted in the direction of the body motion. The distance of this shift e is known as the *coefficient of rolling resistance* and is usually expressed in millimeters or inches.
- The force required to overcome the rolling resistance is

$$F_{\text{roll}} = N \cdot \frac{e}{r_0}$$

where N is the reaction force of the ground, e is the coefficient of rolling-resistance and r_0 is the radius of the rolling body (cylinder or wheel).
- Draw the free body diagram of a rolling body, write the equilibrium equations, and solve for unknown forces.

Example 12.5 Cylinder with radius $r = 0.5$ m and weight $F_g = 100\,\text{N}$ is located on a slope inclined by angle $\alpha = 25°$. External force **F** is acting horizontal, as shown in Fig. 12.12a. The static friction coefficient between the contact surfaces is μ_s. Determine the minimal friction coefficient and magnitude of force **F** to keep the cylinder in the state of rest if (a) there is no rolling resistance and (b) if there is rolling-resistance coefficient $e = 10$ mm.

Solution The first step in solving the problem is drawing the free body diagram, which is shown in Fig. 12.12b, for the case (a) when there is no rolling resistance, and in Fig. 12.12c, for the case (b) when between the cylinder and the slope there is a coefficient of rolling resistance e. In both cases we obtain minimal force **F** when the cylinder has tendency of moving downwards. In all three figures the forces are not shown in scale.

(a) *No rolling resistance*
In this case the contact between the cylinder and the slope is a point and there will be no rolling resistance. Since it is a two-dimensional problem, we can write three equilibrium equations.

$$-F_g \sin\alpha + F\cos\alpha + F_f^s = 0$$

$$-F_g \cos\alpha - F\sin\alpha + N = 0$$

$$r \cdot F_f^s - r \cdot F = 0$$

and by assuming the impending motion, we can use the following relationship

$$F_f^s = \mu_s N$$

From the third equation we find $F_f^s = F$. By introducing this into the first equation we get

$$F = F_f^s = F_g \frac{\sin\alpha}{1 + \cos\alpha} = 22.2 \text{ N}$$

By introducing the result into the second equation we find

$$N = F_g = 100 \text{ N}$$

From the last equation we find the required static friction coefficient.

$$\mu_{s,\min} = \frac{\sin\alpha}{1 + \cos\alpha} = 0.222$$

(b) *With rolling resistance*
In this case reaction normal force N is shifted in the direction of the cylinder motion for distance $e = 10$ mm. We can write two equilibrium equations for

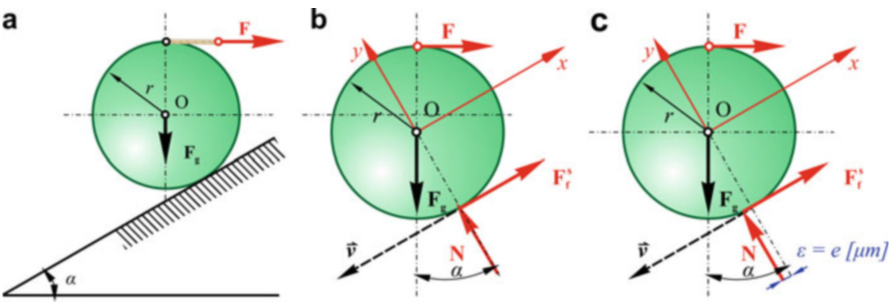


Fig. 12.12 (**a**) Physical model of a cylinder; (**b**) free body diagram of a cylinder (no rolling resistance); (**c**) free body diagram of a cylinder (with rolling resistance)

translation, one equilibrium equation for rotation, and one for the static friction force, assuming impending motion

$$-F_g \sin\alpha + F\cos\alpha + F_f^s = 0$$

$$-F_g \cos\alpha - F\sin\alpha + N = 0$$

$$r \cdot F_f^s - r \cdot F - e \cdot N = 0, \quad \text{and}$$

$$F_f^s = \mu_s N$$

Solving for N and F_f^s from the first two equilibrium equations and substituting them into the third equation we obtain,

$$r\left(F_g \sin\alpha - F\cos\alpha\right) - r.F - e\left(F_g \cos\alpha + F\sin\alpha\right) = 0$$

and

$$F = F_g \frac{\sin\alpha - \frac{e}{r}\cos\alpha}{\cos\alpha + 1 + \frac{e}{r}\sin\alpha} = 100\frac{\sin 25 - \frac{10}{500}\cos 25}{\cos 25 + 1 + \frac{10}{500}\sin 25} = 21.2\ \text{N}$$

Introducing this into the first two equilibrium equations we obtain

$$F_f^s = F_g\left(\sin\alpha - \frac{\sin\alpha - \frac{e}{r}\cos\alpha}{\cos\alpha + 1 + \frac{e}{r}\sin\alpha}\cos\alpha\right) = 23.1\ \text{N}$$

and

$$N = F_g\left(\cos\alpha + \frac{\sin\alpha - \frac{e}{r}\cos\alpha}{\cos\alpha + 1 + \frac{e}{r}\sin\alpha}\sin\alpha\right) = 99.5\ \text{N}$$

From the equation for the static friction force we obtain the required static friction coefficient,

$$\mu_s = \frac{\sin\alpha + e/r}{\cos\alpha + 1} = 0.232$$

From the obtained results we see that when we have rolling resistance force F required to maintain the cylinder in equilibrium is smaller than the rolling resistance force of no rolling resistance. This is an expected result. However, the required static friction coefficient to maintain the cylinder in static equilibrium is larger! From the above equation we see that as the rolling-resistance coefficient increases, the static friction coefficient must increase in order to maintain the equilibrium. In other words this means that if we drive with underinflated tires the breaking efficiency will be smaller than if the tires are inflated correctly.

Example 12.6 Weight $Q = 80$ kN hangs on a weightless rope, which is winded around the cylinder with radius $r_1 = 0.18$ m (Fig. 12.13a). This cylinder is attached to another cylinder with radius $r_2 = 0.35$ m, which serves as a brake. Braking is achieved by pressing a break shoe against it as shown in Fig. 12.13a. Static friction coefficient between the cylinder and shoe is $\mu_s = 0.4$ and dimensions of the braking mechanism are: $a = 0.4$ m, $b = 0.5$ m, and $c = 0.08$ m. Determine the required minimal force to keep the system in equilibrium and the reaction forces in supports O and A.

Solution The free body diagram is shown in Fig. 12.13b. The device consists of two rigid bodies that have to be in equilibrium. We obtain the minimal force just before weight Q starts to move downwards. For each body we can write three equilibrium equations, two for translation and one for rotation. In addition we have an equation for the friction force. For the first body we have

$$A_x + F_f = 0$$

$$A_y + N - F = 0$$

$$a \cdot N + c \cdot F_f - (a + b) \cdot F = 0$$

and similarly for the second body,

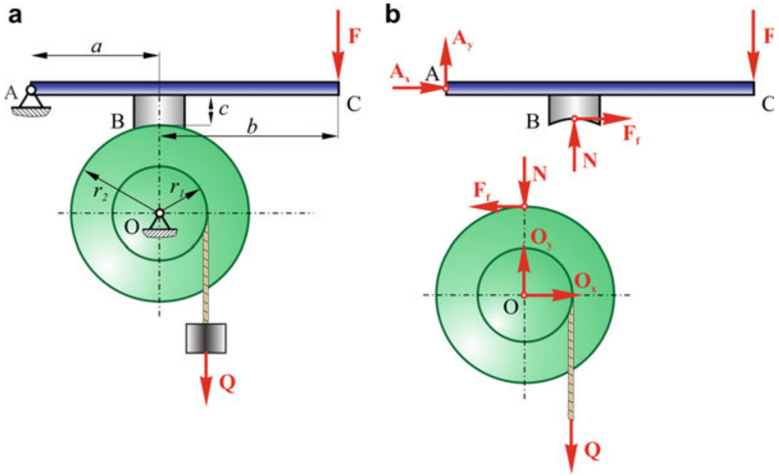


Fig. 12.13 (**a**) Physical model of the problem described in Example 12.6; (**b**) free body diagram of the problem described in Example 12.6

$$-F_f + O_x = 0$$

$$-N + O_y - Q = 0$$

$$r_2 \cdot F_f - r_1 \cdot Q = 0,$$

and for the friction force, assuming the motion is impending

$$F_f = \mu_s N$$

By solving the set of equations we can calculate the sought for unknowns. From the fifth and sixth equations we find

$$F_f = Q \cdot \frac{r_1}{r_2} = 41.1 \text{ kN}, \quad \text{and} \quad N = \frac{F_f}{\mu_s} = 102.8 \text{ kN}$$

Similarly, from the first and fourth equation we find

$$A_x = -F_f = -41.1 \text{ kN}, \quad \text{and} \quad O_x = F_f = 41.1 \text{ kN}$$

We may now calculate the sought for braking force from the third equation,

$$F = \frac{aN + cF_f}{a + b} = 49.4 \text{ kN}$$

Finally, we obtain the remaining two unknown forces from the second and fifth equation

$$A_y = F - N = -53.5 \text{ kN}, \quad \text{and}$$

$$O_y = Q + N = 182.8 \text{ kN}$$

12.6 Plain Bearings

Plain bearings are the simplest and the oldest type of bearings consisting of a bearing surface and a shaft that is in contact with the bearing surface, e.g., the shaft rotating in a hole. Plain bearings are known since invention of the wheel. The oldest wooden wheel dates back more than 5000 years (Fig. 12.14). It was found about 20 km south of Ljubljana, the capital of Slovenia in 2002. It was created in Chalcolithic period of Copper Age, 5150 BCE. The wheel has a radius of 70 cm and is made of ash and oak.

With the invention of solid plastics plain bearings became increasingly popular due to the dry-running lubrication-free behavior. Solid polymer plain bearings have low weight, corrosion resistant and are maintenance free. Plastic bearings are used from printers to cash registers in supermarkets. Other applications include agricultural and textile machinery, medical devices, food and packaging machines, car

Fig. 12.14 The oldest wooden wheel in the world found in 2002 in Ljubljana Marshes, Slovenia. Present location: Ljubljana City Museum, Ljubljana, Slovenia

seating, marine equipment, and many more. They are particularly important for equipment in food and pharmaceutical industry where use of oil lubricated bearings is not permitted. The physics behind their operation is dry friction.

> Plain bearings are known since the invention of the wheel. The oldest wooden wheel in the world was found about 20 km south of Ljubljana, the capital of Slovenia in 2002. It was created in Chalcolithic period of Copper Age, 5150 BCE.

Depending on the direction in which plain bearings carry the load they may be subdivided into two main groups: (a) Radial or journal bearings and (b) axial or trust bearings. Each of them will be discussed later.

12.6.1 Radial or Journal Bearings

A radial bearing carries external load **F** in the radial direction and is driven by external moment **M**, as schematically shown in Fig. 12.15a. Static and kinetic friction coefficients are μ_s and μ_k, respectively.

Radial plain bearings usually consist of an inner rotating shaft or an axis and outer supporting part, which is stationary. Of course, it can be the other way around, the axis can be fixed and the outer part, say wheel, mobile. We will analyze the first case.

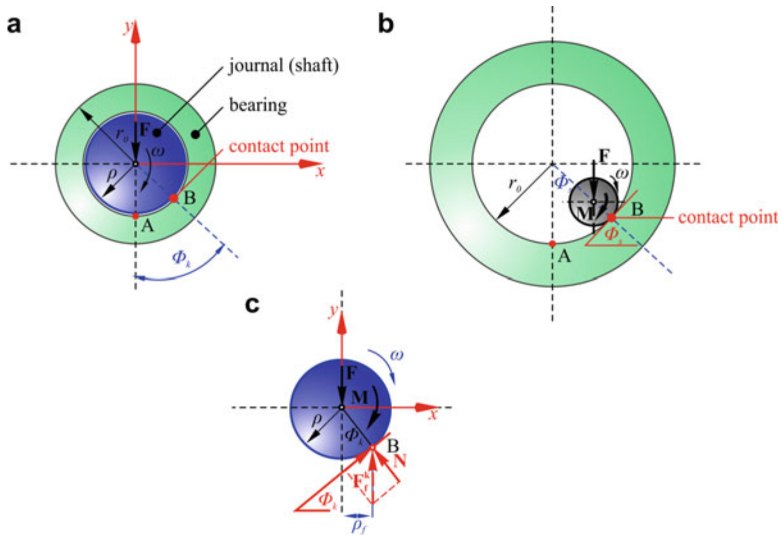


Fig. 12.15 (**a**) Schematic presentation of plain radial bearing; (**b**) schematics of the physical concept of the plain radial bearing. (**c**) Free body diagram of the shaft, which rotates at constant angular velocity ω

> Radial plain bearings most often consist of an inner rotating shaft or axis and outer supporting part, which is stationary, or the other way around, the axis can be fixed and the outer part, the wheel mobile.

In our analysis we will assume that both parts of the bearing are infinitely rigid. Thus, there will be no rolling resistance caused by deformation of any part. Let us further assume that the diameter of the outer nonmoving part is $2r_0$ and of the axis is 2ρ. The outer diameter $2r_0$ of a bearing is usually just slightly larger than 2ρ, hence $r_0 \cong \rho$. However, for easier understanding of the physical concept we will assume that $r_0 \gg \rho$, as shown in Fig. 12.15b.

Consider that the shaft is in the state of rest at point A (Fig. 12.15a) and starts rolling clockwise with constant angular velocity ω. As a result, the axis will start climbing against the outer part, which may be viewed as a slope with a changing inclination (Fig. 12.15b, c). At the certain point B the angle of the slope becomes equal to *angle of static friction* ϕ_s, i.e., $\tan \phi_s = \mu_s$. At this point the shaft will slip and the friction coefficient will drop from the static friction to kinetic friction coefficient, i.e., $\mu_s \to \mu_k$. At the same time angle of static friction ϕ_s will decrease to *angle of kinetic friction* ϕ_k, where $\tan \phi_k = \mu_k$. This situation is presented in Fig. 12.15c, which shows the free body diagram of the shaft that rotates at constant angular velocity ω.

External moment **M**, needed to maintain the rotation of the shaft at a constant angular velocity, is the moment required to overcome the friction resistance of the radial plain bearing. Let us determine the magnitude of moment M. The equilibrium equations for the selected coordinate system (Fig. 12.15c) are

$$\begin{aligned} &\sum_i F_{i,x} = -N \cdot \sin\phi_k + F_f^k \cdot \cos\phi_k = 0 \\ &\sum_i F_{i,y} = N \cdot \cos\phi_k + F_f^k \cdot \sin\phi_k - F = 0 \\ &\sum_i M_i = \rho \cdot F_f^k - M = 0 \end{aligned} \tag{12.30}$$

From the above equilibrium equations we obtain,

$$N = \frac{F}{\cos\phi_k + \mu_k \sin\phi_k}, \quad \text{and}$$

$$M = \rho \cdot F \cdot \frac{\mu_k}{\cos\phi_k + \mu_k \sin\phi_k} \tag{12.31a}$$

A rotating axis climbs against the outer part, which may be viewed as a slope with a changing inclination. The axis stops climbing when the inclination becomes equal to *angle of static friction* ϕ_s, $\tan\phi_s = \mu_s$

Angle of kinetic friction ϕ_k is usually small (good bearings should have small friction coefficient); therefore, we may assume that $\cos\phi_k \approx 1$, and $\mu_k \sin\phi_k \approx 0$, and obtain,

$$M \cong F \cdot \rho \cdot \mu_k \tag{12.31b}$$

Equation (12.31) defines moment **M**, required to maintain the rotation of the shaft at a constant angular velocity, i.e., to overcome the friction resistance of the radial plain bearing. Considering that $\mu_k = \tan\phi_k \approx \sin\phi_k$, (12.31) may be simplified even further (Fig. 12.15c),

$$M \cong F \cdot \rho \cdot \sin\phi_k = F \cdot \rho_f \tag{12.32}$$

and

$$\rho_f = \rho \cdot \sin\phi_k \approx \rho \cdot \mu_k. \tag{12.33a}$$

Radius ρ_f of a *circle of friction* is used as an independent parameter to define the characteristics of a dry friction bearing. The smaller is the radius ρ_f, the better is the bearing.

ρ_f is often called *radius of the circle of friction* and is used as an independent parameter to define the characteristics of dry friction bearings. The smaller is radius ρ_f, the better are the bearings. When the motion is impending, i.e., just before it starts, we have

$$\rho_f = \rho \cdot \sin\phi_s \approx \rho \cdot \mu_s. \tag{12.33b}$$

12.6.2 Axial or Trust Bearings

Axial bearings provide support in the axial direction. In general there are three types of axial bearings: (a) conical bearings, (b) end bearings, and (c) collar bearings. The last two are essentially special cases of conical bearings, which we will analyze first.

12.6.2.1 Conical Axial Bearings

An example of a conical bearing, which carries an external axial force **F**, is schematically shown in Fig. 12.16. Our goal is to determine external moment **M** required to overcome the friction resistance of the axial plain bearing and to rotate the axis at a constant angular velocity. We will call this moment an *operational moment*. Let us assume that static and kinetic friction coefficients are μ_s and μ_k, respectively.

The bearing carries load over conical-shaped area with inner radius r_1, outer radius r_2, and opening angle 2α, as shown in Fig. 12.16. We will use the cylindrical coordinate system and assume that the pressure between the surfaces of the contact is uniform and equal to

$$p = \frac{F}{\pi\left(r_2^2 - r_1^2\right)} = \text{const.} \tag{12.34}$$

The normal force acting on the differentially small area of the contact surface is then

$$dN = p \cdot dA \tag{12.35}$$

The kinetic friction force acting on dA will be $dF_f^k = \mu_k \cdot dN$, and the corresponding moment

$$dM = r \cdot dF_f^k = \mu_k \cdot r \cdot p \cdot dA$$

To obtain external moment M required to overcome the friction resistance of the axial bearing and to rotate the axis at a constant angular velocity we need to integrate (12.24) over the entire contact surface area A, hence

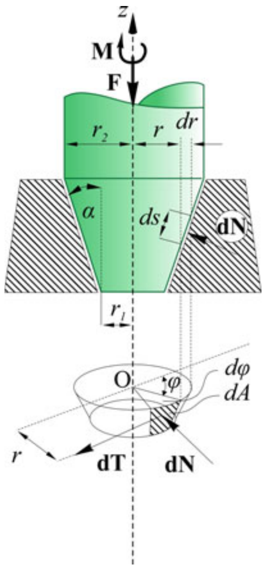


Fig. 12.16 Schematic presentation of a plain conical axial bearing

$$M = p\mu_k \iint_A r dA \tag{12.36}$$

By considering that (Fig. 12.16)

$$dA = (r \cdot d\varphi) \cdot ds = \frac{r}{\sin\alpha} \cdot dr \cdot d\varphi$$

we will cover the entire contact surface by integration of r from r_1 to r_2, and φ from 0 to 2π,

$$M = \frac{p\mu_k}{\sin\alpha} \int_0^{2\pi} \left[\int_{r_1}^{r_2} r^2 dr \right] d\varphi$$

After the integration and introduction of (12.34) we will find the moment required to rotate the axis at a constant angular velocity,

$$M = \frac{2\pi p\mu_k}{3\sin\alpha}\left(r_2^3 - r_1^3\right) = F \cdot \left[\frac{2\mu_k}{3\sin\alpha} \cdot \frac{\left(r_2^3 - r_1^3\right)}{\left(r_2^2 - r_1^2\right)}\right] \tag{12.37}$$

From (12.37) we see that conical bearings require quite large operational moments; therefore, they are not as widely used as radial bearings; however, they are often used in applications where rotation must be prevented.

12.6.2.2 End Bearings

Example of an end bearing is schematically shown in Fig. 12.17. The equation for the operational moment for an end bearing (moment required to rotate axis at a constant angular velocity) may be obtained from (12.37) by assuming that $r_1 = 0$, $r_2 = r$, and $\alpha = \pi/2$),

$$M = F \cdot \frac{2r\mu_k}{3} \tag{12.38}$$

From (12.38) we observe very interesting and important fact that $M(r \to 0) \to 0$. Hence, if we reduce the contact area the required operational moment will tend towards zero. Example of such bearing which carries the load **F** is schematically shown in Fig. 12.18. Torque **M** required to rotate the bearing at a constant angular

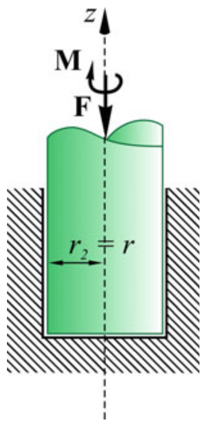


Fig. 12.17 Schematic presentation of plain axial end bearing

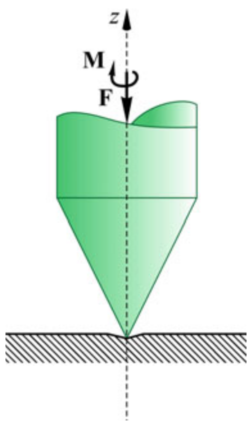


Fig. 12.18 Axial end bearing with $M(r \to 0) \to 0$

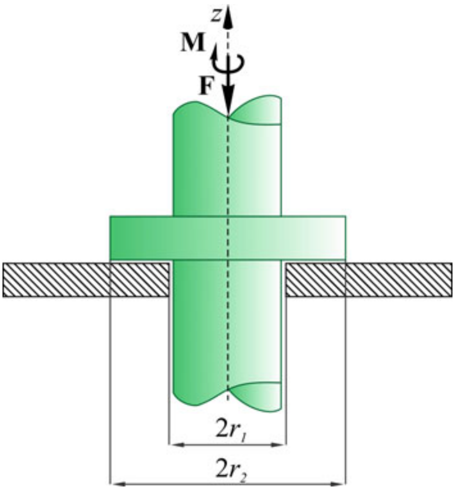


Fig. 12.19 Axial collar bearing

velocity will be very close to zero. In reality the contact area cannot be a point but some reasonably small circular area that will according to (12.38) require a very small torque. Such bearings are used in handmade mechanical watches and in many other mechanisms.

12.6.2.3 Collar Bearings

In the case of collar bearings, the friction forces develop between two ring-shaped areas that are in contact, as shown in Fig. 12.19.

The operational moment required to rotate the axis with a constant angular velocity may be obtained from (12.37) by letting $\alpha = \pi/2$,

$$M = \frac{2\pi p \mu_k}{3}\left(r_2^3 - r_1^3\right) = F \cdot \frac{2\mu_k\left(r_2^3 - r_1^3\right)}{3\left(r_2^2 - r_1^2\right)} \tag{12.39}$$

Collar bearings are usually used for relatively small axial forces only. However, similar engineering solutions are used in automotive industry as disk clutches, often called *friction disks*. Here the friction between two disks is beneficial and the magnitude of moment M defines the quality of the clutch. Since clutches has to transmit the moment there should be no relative rotation between the two disks. Therefore, the friction coefficient between the two disks will be static friction coefficient μ_s.

Guidelines and Recipes for Applying Laws of Friction to Bearings

For all types of bearings first draw corresponding free body diagram and determine resultant external load F that a bearing needs to carry.

Journal Bearings are used to provide lateral support to rotating shafts and axises.

- The moment required to maintain shaft rotation at a constant angular velocity is $M \cong F \cdot \rho \cdot \mu_k$, where F is the load carried by the shaft, ρ is the radius of the axis, and μ_k is the kinetic friction coefficient between the two surfaces.
- Characteristics of bearings are often expressed with parameter called *radius (circle) of friction*, $\rho_f = \rho \cdot \sin\phi_k \approx \rho \cdot \mu_k$.

Axial or Trust Bearings are used to provide axial support to rotating shafts and axis.

- For all types of axial bearings the moment required to maintain shaft rotation at a constant angular velocity may be obtained from the formula of conical bearings:

$$M = F \cdot \frac{2\mu_k}{3\sin\alpha} \cdot \frac{(r_2^3 - r_1^3)}{(r_2^2 - r_1^2)}$$

  (a) By letting $r_1 = 0$, $r_2 = r$, and $\alpha = \pi/2$ we describe the *end bearing*, and
  (b) By letting $\alpha = \pi/2$, we describe the *collar bearing*

Example 12.7 A hoist of a bridge crane (weight $F_g = 100$ kN) has four wheels and travels horizontally along the bridge and carries load $Q = 500$ kN, as shown in

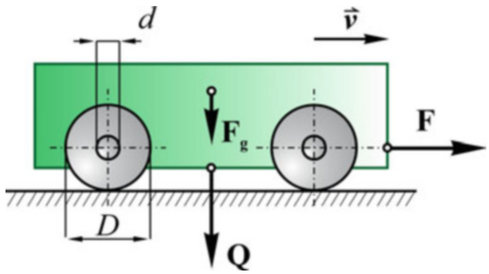


Fig. 12.20 Physical model of a hoist

Fig. 12.20. The diameter of the wheels is $D = 0.7$ m. The wheels are mounted to the hoist via radial bearings with diameter $d = 0.06$ m. The coefficient of the kinetic friction is $\mu_k = 0.1$, and coefficient of the rolling resistance between the wheels and the rails of the bridge crane is $e = 0.75$ mm. Calculate force **F** required to move the hoist with a constant velocity (Fig. 12.20).

Solution We will assume that external load **Q** and weight of the hoist $\mathbf{F_g}$ act along the same line of action and that each of the wheels carries one-quarter of the load, $(F_g + Q)/4$. By using (12.31b), $M \cong F \cdot \rho \cdot \mu_k$, for the radial bearings we can calculate the moment required for rotating each of the four wheels

$$M_w = \frac{F_g + Q}{4} \cdot \frac{d}{2} \cdot \mu_k$$

By using (12.28) we may calculate moment $\mathbf{M_r}$ required to overcome the rolling resistance of each of the four wheels,

$$M_r = \frac{F_g + Q}{4} \cdot e$$

Now, by using the equilibrium equation for moments we can calculate the force required to move the hoist with a constant velocity,

$$4(M_w + M_r) = F \cdot \frac{D}{2},$$

and obtain

$$F = \frac{8(M_w + M_r)}{D} = 6.43\,\text{kN}$$

Example 12.8 Figure 12.21 shows the physical model of a disk clutch with inner diameter $d_1 = 0.4$ m, and outer diameter $d_2 = 0.7$ m. The static friction coefficient between the contact surfaces is $\mu_s = 0.45$. Calculate contact force **F**, which is required to generate a uniform contact pressure, $p = 500$ kN/m^2, between two

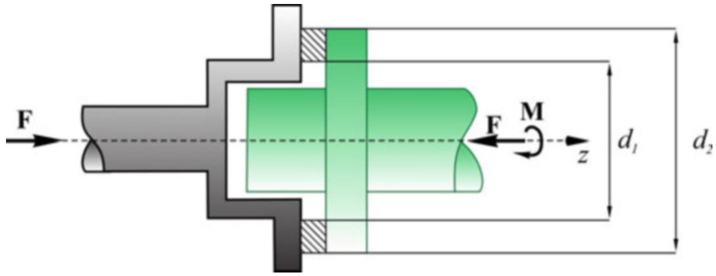


Fig. 12.21 Physical model of a clutch

ring-shaped equal areas and determine maximal moment **M** which the clutch can transmit.

Solution The maximal moment that the clutch can transmit may be calculated from (12.39) by replacing kinetic friction coefficient μ_k by static friction coefficient μ_s

$$M = \frac{2\pi p \mu_s}{3}\left(r_2^3 - r_1^3\right) = 16.4\,\text{kN/m}.$$

We calculate the required axial force from (12.34)

$$F = p\pi\left(r_2^2 - r_1^2\right) = 129.6\ \text{kN}$$

12.7 Belts and Ropes Friction

There are several situations when flexible belts, cables, or ropes are in contact with rigid cylinders. Examples are capstans and Halyard Winches on sailing boats (Fig. 12.22). From the experience we know that by winding several times a rope over the winch we can generate force $\mathbf{S_2}$, which will hold the sail in the required position by pulling the other side of the rope with relatively small force $\mathbf{S_1}$. This is because the friction force between the rope and the surface of the winch helps us to keep force S_2 in equilibrium.

The friction force between a rope and the surface of a cylinder helps us to keep a very large force $\mathbf{S_2}$ in equilibrium with small force $\mathbf{S_1}$.

Let us derive a model, which will relate two forces $\mathbf{S_1}$ and $\mathbf{S_2}$ as a function of an angle of the contact between a rope and winch φ, and static friction coefficient μ_s acting between the rope and the surface of the winch. In this derivation we will assume that winch is rigid and has cylindrical geometry, and that the rope is ideally

Fig. 12.22 Halyard Winch on a sailing boat

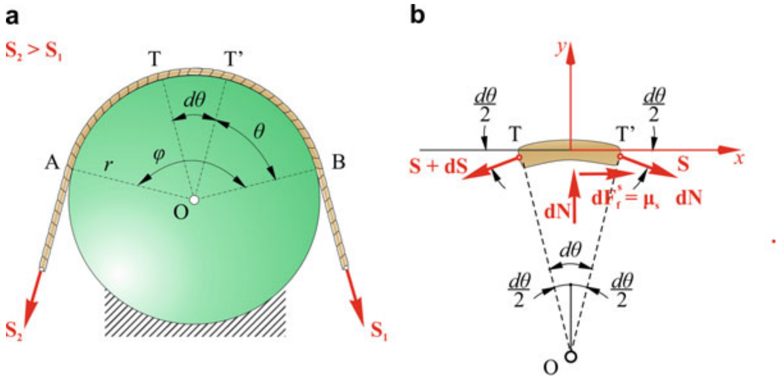


Fig. 12.23 (**a**) Physical model of a rope (belt) passing over a rigid cylinder. (**b**) Free body diagram of a small segment of the rope (belt)

flexible and non-deformable in axial direction. Physical model of such a case is shown in Fig. 12.23a, and the corresponding free body diagram in Fig. 12.23b.

To derive the relation between two forces $\mathbf{S_1}$ and $\mathbf{S_2}$ we assume that the cylinder (winch) is rigid, and that the rope is ideally flexible and non-deformable in the axial direction.

The rope and the cylinder are in contact between points A and B over angle φ as shown in Fig. 12.23a. The rope is loaded on both ends with forces $\mathbf{S_1}$ and $\mathbf{S_2}$, and has a tendency to move to the left. At an arbitrary angle θ we cut out an infinitesimally small segment of rope $d\theta$ and analyze its equilibrium. The free body diagram of this segment is shown in Fig. 12.23b.

Four external forces are acting on the infinitesimal rope segment: two axial forces, S on the right hand side, and $\mathbf{S} + \mathbf{dS}$ on the left; reaction force of the cylinder $\mathbf{dN}$; and friction force $\mathbf{dF}_{\mathrm{f}}^{\mathrm{s}}$. We may write two equilibrium equations for the forces and the equation for the friction force,

$$S \cdot \cos\left(\frac{d\theta}{2}\right) - (S + dS) \cdot \cos\left(\frac{d\theta}{2}\right) + dF_f^s = 0 \tag{12.40a}$$

$$-S \cdot \sin\left(\frac{d\theta}{2}\right) - (S + dS) \cdot \sin\left(\frac{d\theta}{2}\right) + dN = 0 \tag{12.40b}$$

$$dF_f^s = \mu_s \cdot dN \tag{12.40c}$$

> Angle of contact φ must be expressed in radians and may be larger than 2π. For example, if a rope is wrapped n times around a cylinder, φ is equal to $2\pi n$.

By taking into account the fact that $\cos(d\theta/2) \cong 1$ and $\sin(d\theta/2) \cong d\theta/2$ we can reduce the first two equations to

$$dF_f^s = dS \tag{12.41a}$$

$$dN = S \cdot d\theta + dS \cdot \frac{d\theta}{2} \cong S \cdot d\theta. \tag{12.41b}$$

In (12.41b) the second term on the right hand side may be neglected, $dS \cdot d\theta/2 \approx 0$, because it is an order of magnitude smaller than the first term. By introducing (12.41) into (12.40c) we obtain,

$$\frac{dS}{S} = \mu_s d\theta \tag{12.42}$$

> In case when there is a movement we need to replace the static friction coefficient μ_s with the kinetic friction coefficient μ_k.

By integrating between the limits, we find,

$$\int_{S_1}^{S_2} \frac{dS}{S} = \int_0^{\varphi} \mu_s d\theta$$

and after the integration

$$\ln S_2 - \ln S_1 = \mu_s \varphi$$

$$\ln \frac{S_2}{S_1} = \mu_s \varphi, \quad \text{or} \tag{12.43a}$$

$$S_2 = S_1 \cdot e^{\mu_s \varphi} \tag{12.43b}$$

> Equation (12.43) may be used to solve problems involving band breaks. In such problems the drum has a tendency to rotate whereas the band remains fixed.

The angle of contact φ must be expressed in radians, and it may be larger than 2π. For example, if the rope is wrapped n times around the cylinder, φ is equal to $2\pi n$ [rad].

In (12.43) we have assumed that the rope (belt) does not move relative to the cylinder. In case when there is a relative movement between the rope (belt) and the cylinder we need to replace static friction coefficient, μ_s with the kinetic friction coefficient μ_k.

The derived formulas apply equally to the problems involving flat belts passing over cylinders or cylindrical drums and to the problems involving ropes wrapped around a post, capstan, or winch. They can also be used to solve the problems involving band breaks. In such problems the drum has a tendency to rotate whereas the band remains fixed.

Equation (12.43) may also be used to solve problems of belt drives. In this case both the pulley and the belt rotate and our concern is to find under what conditions the belt will slip, i.e., move relative to the pulley.

12.7.1 V-Shaped Belts

Belts used in belt drives are often V-shaped with grove angle α, as shown in Fig. 12.24a. The corresponding free body diagrams of the V-shaped belt in z–y and x–y planes are shown in Fig. 12.24b. The contact between a belt and a pulley takes place along the side of a groove. In this case the equilibrium equations are

$$S \cdot \cos\left(\frac{d\theta}{2}\right) - (S + dS) \cdot \cos\left(\frac{d\theta}{2}\right) + 2dF_f^s = 0 \tag{12.44a}$$

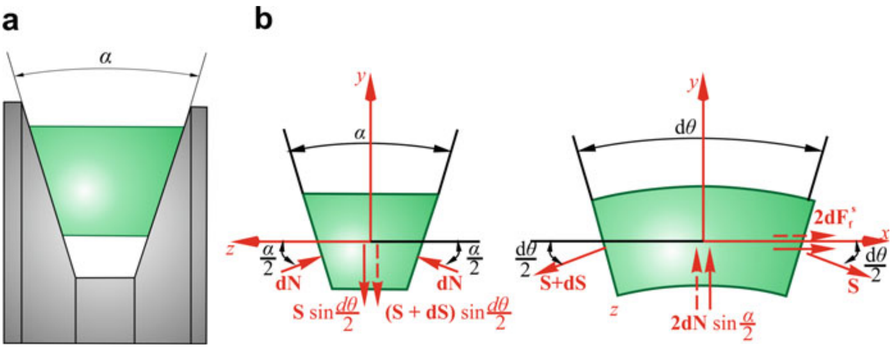


Fig. 12.24 (**a**) V-shaped belt with groove angle α; (**b**) free body diagram of a V-shaped belt shown in z–y and x–y plane

$$-S\cdot\,\sin\left(\frac{d\theta}{2}\right)-(S+dS)\cdot\,\sin\left(\frac{d\theta}{2}\right)+2dN\sin\left(\frac{\alpha}{2}\right)=0 \tag{12.44b}$$

$$dF_f^s=\mu_s\cdot dN \tag{12.44c}$$

By considering again that $\cos(d\theta/2)\cong 1$ and $\sin(d\theta/2)\cong d\theta/2$ we can reduce the first two equations to

$$dF_f^s=\frac{dS}{2},\quad \text{and} \tag{12.45a}$$

$$dN=\frac{S}{2\sin\left(\frac{\alpha}{2}\right)}\cdot d\theta \tag{12.45b}$$

By introducing (12.45) into (12.44c) we find

$$\frac{dS}{2}=\mu_s\cdot\frac{S}{2\sin\left(\frac{\alpha}{2}\right)}\cdot d\theta$$

and after the integration, by following the same procedure as before, we can obtain

$$\ln\frac{S_2}{S_1}=\frac{\mu_s\varphi}{\sin(\alpha/2)},\quad \text{or} \tag{12.46a}$$

$$S_2=S_1\cdot e^{\mu_s\varphi/\sin(\alpha/2)} \tag{12.46b}$$

The carrying capacity of V-shaped belt drastically increases when $\alpha\to 0$. The angle has to be adjusted to the axial strength of the belt and it is usually about $\alpha=40°$.

From (12.46) we can see that if $\alpha = \pi$ the relation reduces to solution for flat belts, (12.43b). On the other hand we see that the carrying capability will drastically increase when $\alpha \to 0$. In reality the size of α should be adjusted to the axial strength, and it is usually about $\alpha = 40°$.

Guidelines and Recipes for Belts and Ropes Friction

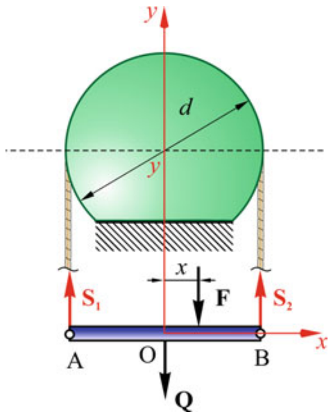
- Determine total angle of contact φ, between a rope/belt and cylinder expressed in radians and static friction coefficient μ_s.
- Use equation $S_2/S_1 = e^{\mu_s \varphi}$ to calculate the ratio between two forces acting on both sides of the rope.
- When the belt is V-shaped with groove angle α, use formula $S_2/S_1 = e^{\mu_s \varphi / \sin(\alpha/2)}$ instead.
- In case when the rope/belt is moving relative to the cylinder (as in the case of band breaks), replace the static friction coefficient μ_s with the kinetic friction coefficient μ_k.

Example 12.9 A flexible rope is wrapped around a fixed cylinder with diameter $d = 0.9$ m. The static friction coefficient between the rope and the cylinder is $\mu_s = 0.4$. The rope is fixed at both ends to the horizontal bar with weight $Q = 200$ N as shown in Fig. 12.25. Determine distance x we may move force $F = 250$ N before the rope will start to slide relative to the cylinder.

Solution As force **F** starts moving from point O towards point B the rope will get the tendency to move to the right. However, as long as the rope is not moving bar AB needs to be in equilibrium. For the bar we may write two equilibrium equations, one for the forces in y-direction and one for the moments,

$$S_1 + S_2 - Q - F = 0$$

Fig. 12.25 Free body diagram of the rope wrapped around the fixed cylinder



$$\frac{d}{2} \cdot (S_2 - S_1) - x \cdot F = 0$$

In this case the contact area between the rope and the cylinder is $\varphi = \pi$, and the relation between the two forces on both ends of the belt is,

$$S_2 = S_1 \cdot e^{\mu\pi} = 3.51 \cdot S_1$$

By applying the result to the first equation, we will get

$$S_1 = \frac{Q + F}{1 + e^{\mu\pi}} = \frac{200 + 250}{1 + 3.51} = 99.7 \text{ N}, \quad \text{and}$$

$$S_2 = Q + F - S_1 = 450 - 99.7 = 350 \text{ N}$$

From the second equation we can now calculate distance x,

$$x = \frac{d(S_2 - S_1)}{2F} = 0.451 \text{ m}$$

Example 12.10 Figure 12.26a shows the physical model of a band-break. The rope wrapped around a drum may be considered flexible. The friction coefficient between the drum and the rope is $\mu = 0.3$. The drum is exposed to external moment $M = 200$ Nm. Determine the magnitude of force **F**, which acts at the end of a lever to prevent the rotation of the drum if (a) the moment is acting clockwise and (b) counterclockwise. For both cases calculate reactions at the supports A and O.

Solution The rope-break consists of two structural elements: the rope wrapped around the drum and the lever supported at joint A and loaded with external braking force **F**. Hence, we need to construct two free body diagrams (Fig. 12.26b).

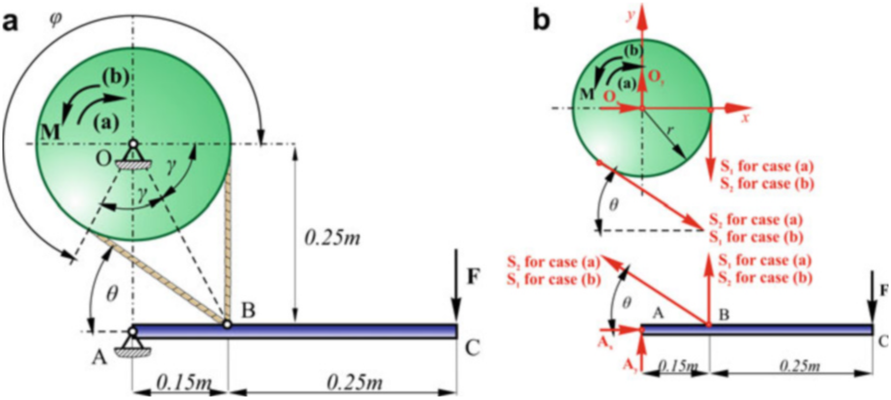


Fig. 12.26 (**a**) Physical model of a band-break. (**b**) Two free body diagrams of the rope-break

In order to determine the relation between forces S_2 and S_1 we need to determine the angle of the contact between the rope and the break drum. From Fig. 12.26a we find, $\tan\gamma = \frac{0.25}{0.15} = 1.67$; therefore, $\gamma = 1.03\ \text{rad} = 59^\circ$. This leads to

$$\varphi = (2\pi - 2\gamma) = 2\pi - 2\cdot 1.03 = 4.22\ \text{rad} = 242^\circ$$

The relation between the two forces acting on both sides of the rope may be expressed as

$$S_2 = S_1 \cdot \mathrm{e}^{\mu\varphi}$$

From the equilibrium equation of the moments for both cases, Fig. 12.26b, we obtain
$S_2 - S_1 = \frac{M}{r}$. By combing both equations, we find

$$S_1 = \frac{M}{r}\cdot\frac{1}{e^{\mu\varphi}-1} = 523\ \text{N}, \quad \text{and}$$

$$S_2 = \frac{e^{\mu\varphi}}{e^{\mu\varphi}-1} = 1856\ \text{N}$$

Now we can calculate the required force **F** acting at the end of the lever. Its magnitude will depend on direction of the external moment acting on the drum. Therefore we have to solve each case separately.

Case (a)—external moment is acting clockwise
When the moment is acting clockwise force S_2 will appear on the left of the drum and force S_1 on the right. Force F may be obtained from the equilibrium of the moments with respect to point A,

$$0.15 \cdot S_2 \cdot \sin\theta + 0.15 \cdot S_1 - (0.15 + 0.25) \cdot F = 0$$

Since $\theta = 2\gamma - 90^\circ = 28^\circ$ (Fig. 12.26a) we find

$$F = \frac{0.15 \cdot S_2 \cdot \sin\theta + 0.15 \cdot S_1}{0.15 + 0.25} = 523\ \text{N}$$

To determine the reaction forces at supports O and A, we have to write the equilibrium equations for the forces for both free body diagrams,

$$O_x + S_2 \cos\theta = 0$$

$$O_y - S_2 \sin\theta - S_1 = 0$$

$$A_x - S_2 \cos\theta = 0, \quad \text{and}$$

$$A_y + S_2 \sin\theta + S_1 - F = 0$$

From these equations we find:

$$O_x = -A_x = -1639\ \text{N}; \quad O_y = 1393\ \text{N}; \quad \text{and} \quad A_y = -870\ \text{N}$$

Case (b)—external moment is acting counterclockwise
When the moment is acting counterclockwise force $\mathbf{S_1}$ will act on the left of the drum and force $\mathbf{S_2}$ on the right. Force **F** may be obtained from equilibrium of the moments in respect to point A,

$$0.15 \cdot S_1 \cdot \sin\theta + 0.15 \cdot S_2 - (0.15 + 0.25) \cdot F = 0, \quad \text{and}$$

$$F = \frac{0.15 \cdot S_1 \cdot \sin\theta + 0.15 \cdot S_2}{0.15 + 0.25} = 788\ \text{N}$$

For the case (b) we write the similar equilibrium equations for the forces for both free body diagrams,

$$O_x + S_1 \cos\theta = 0$$

$$O_y - S_1 \sin\theta - S_2 = 0$$

$$A_x - S_1 \cos\theta = 0, \quad \text{and}$$

$$A_y + S_1 \sin\theta + S_2 - F = 0$$

From these equations we find:

$$O_x = -A_x = -462\ \text{N}; \quad O_y = 2100\ \text{N}; \quad \text{and} \quad A_y = -1313\ \text{N}$$

What We Have Learned?

The origin of friction

Friction is a resistance encountered when one body moves relative to another body with which it is in contact. Friction between solid bodies is extremely complicated physical phenomena. It depends mainly on surface roughness and type of the materials that are in contact. When two surfaces move relative to each other they encompass elastic and plastic deformations of the surface layers of the contacting bodies, microfractures and interaction with the wear particles. The latest research results show that dry friction involves even excitations of electrons and phonons, chemical reactions, and transfer of particles from one body to another. At the same time it is astonishing that it is possible to formulate a very simple phenomenological law for dry friction, which is sufficient for many engineering applications.

This simple law encompasses three important facts: *First*, the amount of friction is independent of the area of contact. *Second*, friction is proportional to the normal load that presses the surfaces together. *Third*, the friction force is generated in a plane of the contact, which determines its line of action that is perpendicular to the normal force and acts always in **opposite** direction to the way an object slides or impends to slide.

Friction between solid bodies

The friction force is generated at the contact surface of two bodies; therefore, the line of action of friction force $\mathbf{F}_f$ does not pass through the center of mass of the body. Consequently the component of the resultant of external forces $\mathbf{R}_x$ which acts in parallel to the contact surfaces and friction force $\mathbf{F}_f$ form a couple of forces that tends to rotate the rigid body. This moment is compensated by the shift of the line of action of reacting normal force **N**, which together with the normal component of the resultant of external forces $\mathbf{R}_y$, forms another couple of forces that compensates the moment of the first couple.

The resultant of the external forces, which includes the weight of the body, and the reactions from the ground (consisting of normal force and friction force) form an equilibrium pair of forces.

Difference between Static and Kinetic friction

The *static friction force* appears when there is no relative motion between the two surfaces in contact. It is expressed as

$$F_f^s = \mu_s \cdot N$$

The *kinetic friction force* appears when the surfaces in contact move relative to each other,

$$F_f^k = \mu_k \cdot N$$

Here μ_s and μ_k are *static* and *kinetic* friction coefficients, respectively, and **N** is the reaction of the ground force acting normal to the contact surfaces.

Static friction force $F_f^s = \mu_s \cdot N$ is the largest friction force that is possible between the two surfaces (bodies). As long as the component of the resultant of external forces R_x, acting in parallel to the contact surface between the two bodies is smaller than the static friction force F_f^s, $R_x \leq F_f^s$, friction force F_f will be of the same magnitude as the external force R_x, hence $F_f\left(R_x \leq F_f^s\right) = R_x$. When $R_x > F_f^s$ friction force F_f will be equal to kinetic friction force F_f^k and constant within wide range of relative velocities, $F_f(\mathrm{v} > 0) = F_f^k = \mu_k \cdot N$.

Static friction coefficient μ_s may be determined from the angle of the slope inclination θ_s, when a body starts to slide $\mu_s = \tan\theta_s$. The kinetic friction coefficient from angle θ_k, when a body slides with a constant velocity $\mu_k = \tan\theta_k$. For most materials $\mu_s > \mu_k$.

A body exposed to its own weight and standing on a slope inclined by angle φ will be in the state of rest when

$$F_g \cdot (\sin\varphi - \mu_s \cdot \cos\varphi) \leq F \leq F_g \cdot (\sin\varphi + \mu_s \cdot \cos\varphi)$$

How to apply laws of friction to wedges

Wedges are commonly used to raise heavy objects. For each wedge and all other bodies in contact we have to draw the free body diagram and determine the maximum static friction force at each of the contact surfaces. Friction forces always act opposite to the direction of a wedge's or body's relative or impending motion. Unknown forces are determined from the equilibrium equations, which we write for each body (wedge) separately.

Wedge inclination angle φ should be selected so that the wedge will be self-locking, i.e., $\tan\varphi < \mu_s$.

Principles of rolling friction

For rolling a perfectly round and rigid cylinder or a wheel on a perfectly flat and rigid surface at a constant velocity we do not need any force and any energy input. In this case the contact surface between the two bodies (e.g., cylinder and the flat surface) will be just a line or a point if we observe the contact in two dimensions.

The *rolling resistance* results from the deformation of the wheel and the surface on which a wheel is rolling. As a consequence reaction force **N** is shifted in the direction of the body motion. The distance of this shift e is known as the *coefficient of rolling resistance* and is expressed in the units of length, e.g., *millimeters or inches*. The force required to overcome the rolling resistance is

$$F_{roll} = N \cdot \frac{e}{r_0}$$

where N is the reaction of the ground, e is the coefficient of the rolling-resistance and r_0 is the radius of the rolling body (cylinder or wheel). After drawing the free body diagram of the rolling body, we find the solution by solving the equilibrium equations for the unknown forces.

Friction bearings
Plain or friction bearings are the simplest and the oldest type of bearings comprising just the bearing surface and shaft that are in contact with the bearing surface, e.g., a shaft rotating in a hole. Based on the direction of the load the bearings carry we may group them into (a) radial and (b) axial bearings. In all the cases we have to draw their free body diagrams and determine resultant external load **F** that the bearing has to carry.

Journal Bearings are used to provide lateral support to rotating shafts and axis. The moment required to maintain the rotation at a constant angular velocity is

$$M \cong F \cdot \rho \cdot \mu_k,$$

where F is the load carried by shaft, ρ is the radius of the axis, and μ_k is the kinetic friction coefficient between the two surfaces. The characteristics of bearings are often expressed by parameter called *radius (circle) of friction*,

$$\rho_f \approx \rho \cdot \mu_k$$

Axial or Trust Bearings are used to provide the axial support to rotating shafts and axis. For all types of axial bearings the moment required to maintain the shaft rotation at constant angular velocity may be obtained from the formula of conical bearings:

$$M = F \cdot \frac{2\mu_k}{3 \sin \alpha} \cdot \frac{\left(r_2^3 - r_1^3\right)}{\left(r_2^2 - r_1^2\right)}$$

(a) By letting $r_1 = 0$, $r_2 = r$, and $\alpha = \pi/2$, we obtain *End bearing* and
(b) By letting $\alpha = \pi/2$, we obtain *Collar bearing*

Friction of ropes and belts
There are several situations when flexible belts or cables are in contact with rigid cylinders. The friction force between the rope and the surface of the cylinder helps us to keep a very large force S_2 in equilibrium with a small force S_1. To determine the relation between the two forces S_2/S_1 we have to determine the total angle of contact, φ, between the rope (belt) and the cylinder expressed in radians, and static friction coefficient μ_s. The relation between the two forces may be calculated from,

$$\frac{S_2}{S_1} = e^{\mu_s \varphi}.$$

When a belt is V-shaped with a groove angle α, the relation is given as

$$\frac{S_2}{S_1} = e^{\mu_s \varphi / \sin(\alpha/2)}$$

In case when a rope (belt) is sliding relative to a cylinder, (as in the case of band breaks) the static friction coefficient μ_s has to be replaced with kinetic friction coefficient μ_k.

12.8 Problems

12.1–12.10 A disk is attached to the pin O and loaded as shown in Figs. P12.1–P12.10. Determine the minimum value of the force **P** needed to keep the system in equilibrium. The friction coefficient between the disk and the pusher is μ. The values of the external forces and geometrical parameters are given in the table below. The friction between the rope and the disk is neglected. In some figures are given geometrical data that are not needed for solving the problem.

Figure	Q (kN)	a (m)	b (m)	e (m)	α (°)	μ (/)
P12.1	8	0.6	0.2	0.1	30	0.2
P12.2	18	–	–	–	–	0.4
P12.3	10	0.3	0.2	0.1	–	0.25
P12.4	12	0.4	0.8	0.1	–	0.15
P12.5	18	0.4	0.6	0.1	–	0.3
P12.6	20	–	–	–	–	0.4
P12.7	26	0.8	1.6	0.3	60	0.4
P12.8	20	–	–	–	–	0.25
P12.9	18	–	–	–	–	0.2
P12.10	12	0.8	0.8	0.2	45	0.2

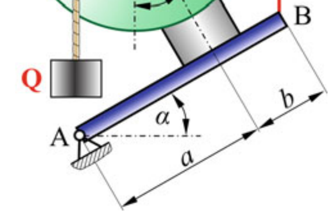


Fig. P12.1

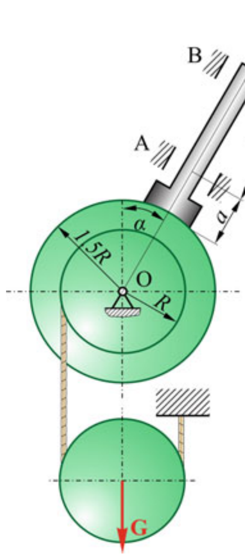


Fig. P12.2

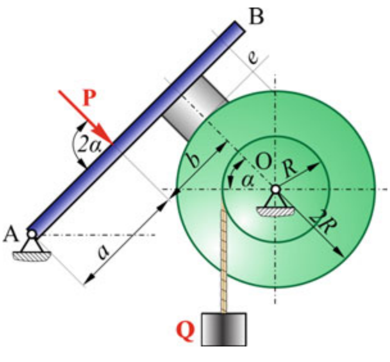


Fig. P12.3

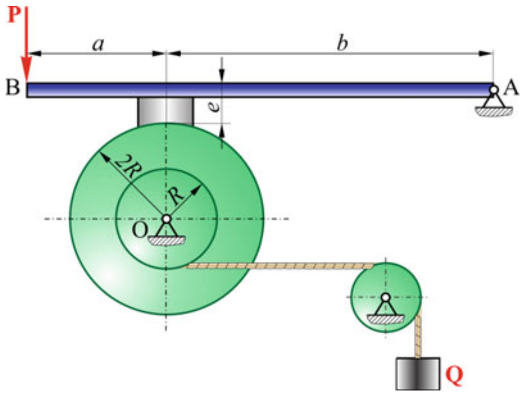


Fig. P12.4

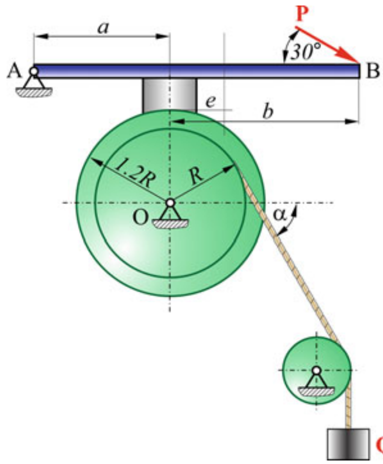


Fig. P12.5

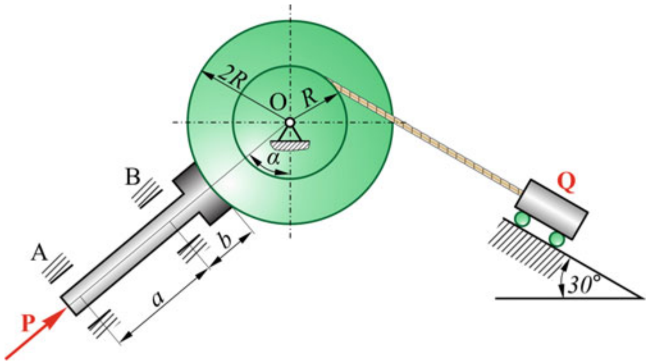


Fig. P12.6

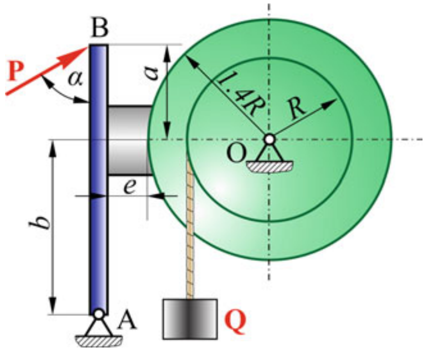


Fig. P12.7

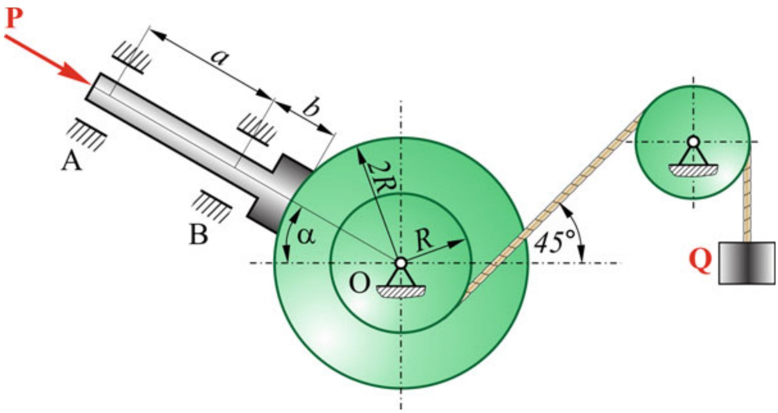


Fig. P12.8

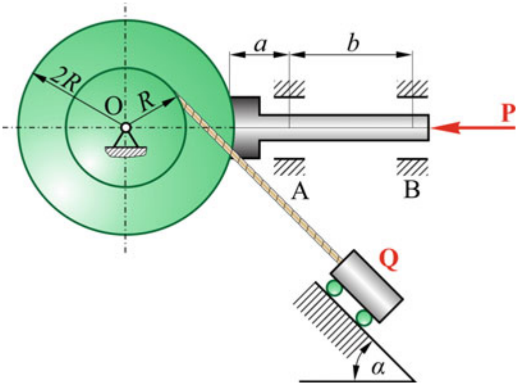


Fig. P12.9

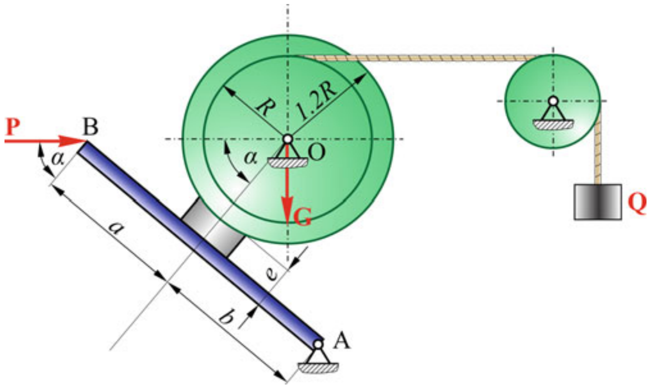


Fig. P12.10

12.11 A square block has weight **P**. The friction coefficient between the block and the horizontal plane is μ. Determine angle β at which minimal force **Q** should be applied to move the block. Determine the magnitude of minimal force **Q** as well.

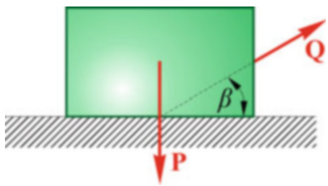


Fig. P12.11

12.12 Three blocks A, B, and C have weights of 20 kN, 60 kN, and 120 kN, respectively. They lie on an inclined plane. The angle between the inclined and horizontal plane is α. The blocks are connected by rigid cables.

The coefficients of friction between the blocks and the plane are $\mu_A = 0.2$, $\mu_B = 0.3$ and $\mu_C = 0.5$, respectively. Determine angle α at which the blocks move down at constant velocity, and the tension in the cables between blocks A and B, and B and C.

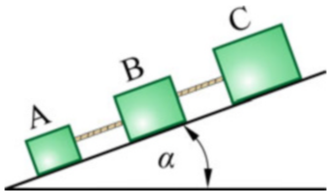


Fig. P12.12

12.13 Blocks A and B weight 120 N and 200 N, respectively. Force $P = 80$ N is applied to block A at the angle equal 30°. The coefficient of the friction between blocks A and B is $\mu_1 = 0.5$, between block B and the horizontal plane is $\mu_2 = 0.3$. Determine if the applied force will move blocks A and B?

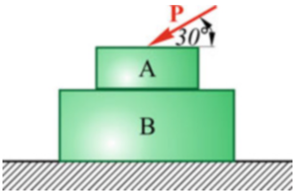


Fig. P12.13

12.14 Blocks A and B weight 120 N and 200 N, respectively. Force **P** acts parallel to the plane, which is inclined relative to the horizontal plane at the angle of 30°. The coefficient of friction between the blocks A and B is $\mu_1 = 0.7$, and between block B and plane C $\mu_2 = 0.2$. Investigate the equilibrium state of the system as function of the magnitude of force **P**?

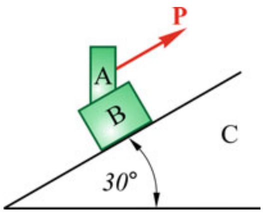


Fig. P12.14

12.15 Two blocks A and B weight 150 N and 300 N, respectively. They lie on a plane inclined at the angle of 30° relative to the horizontal plane and are connected by a rigid cable. The coefficients of friction between the blocks and the plane are $\mu_A = 0.5$ and $\mu_B = 0.8$, respectively. Check if blocks A and

B will move? Determine the internal force in the cable that connects blocks A and B.

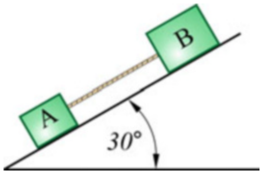


Fig. P12.15

12.16 Two blocks A and B weight 500 N and 400 N, respectively. Wedge C is placed between blocks A and B, its weight may be neglected. The coefficients of friction between block A, the horizontal plane and wedge C is $\mu_1 = 0.2$, and between block B, the horizontal plane and wedge C is $\mu_2 = 0.25$. Determine the value of force **Q**, at which one of the blocks moves and the value of the force of friction acting on the remaining motionless block. The angle of the wedge is $\varphi = \pi/8$. Analyze how the size of the wedge angle affects the magnitude of force **Q**.

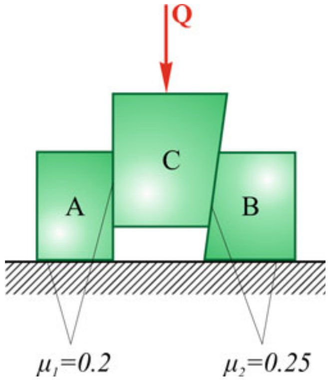


Fig. P12.16

12.17 Cylinder A is lying in the wedge of block B. The cylinder has weight **Q**. The coefficient of friction is μ. Determine the value of force **P** at which the cylinder will start to move in horizontal direction. Determine the value of angle θ at which cylinder A will start to move if exposed to force **P** that is equal to weight **Q**.

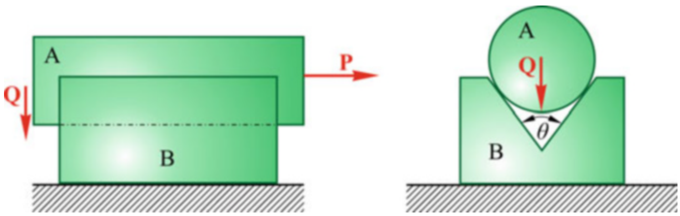


Fig. P12.17

12.18 A cylinder of weight **Q** leans on two motionless blocks A and B. The coefficient of friction at the contact surfaces is μ. Determine the magnitude of force **T** at which the cylinder will start to rotate. Determine the magnitude of force **T** at which angle θ of the cylinder will be self-blocked.

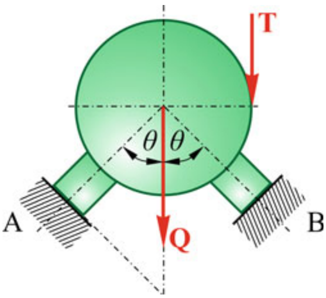


Fig. P12.18

12.19 Two forces, which generate the moment of 150 kN/m, are applied to a cylinder. The radius of the brake shoe is 0.3 m, and the coefficient of friction is 0.4. Determine the minimum value of force **Q** required to keep the system in equilibrium.

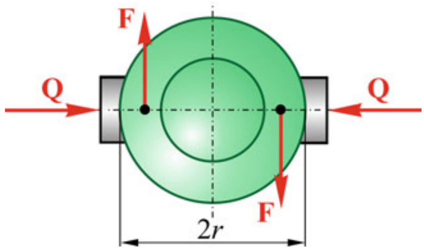


Fig. P12.19

12.20 Bar AB of length l leans against a support. The weight of the bar is **P**. Determine the minimum coefficient of static friction between the bar and the floor to maintain equilibrium. Assume $\alpha = 60°$ and $a = l/2$.

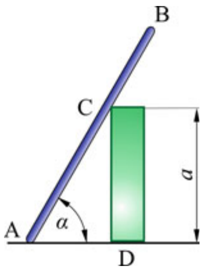


Fig. P12.20

12.21 Disk D with radius r and weight **W** is attached to pin O, which is located at distance e (Fig. P12.21) on the diameter of the disk and is loaded by moment **M** and a force from the pusher. Determine the minimum value of force **F** to keep the system in equilibrium. Friction coefficient between the disk and the pusher is μ.

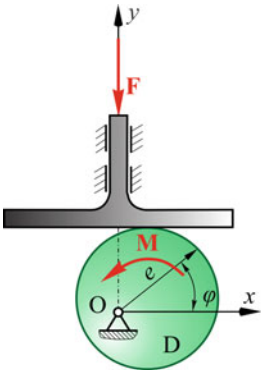


Fig. P12.21

12.22 A special friction based device consists of three parts A, B, and C and serves to lift tubes. A tube shown in figure weights **P**. The coefficient of friction between the blocks A and B and the parts is μ. Neglect friction between the block C and blocks A and B. Determine the minimum value of angle of wedges α to keep the system in equilibrium.

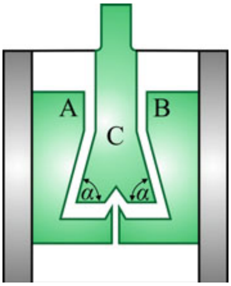


Fig. P12.22

12.23 Friction based device for lifting tubes consist of the parts A, B, and C that weight $\mathbf{G_A}$, $\mathbf{G_B}$, and $\mathbf{G_C}$. Assume that friction coefficient is μ for all contact surfaces and $\alpha = 15°$. Determine the minimum coefficient of friction μ to lift a tube that weights $\mathbf{G_T}$.

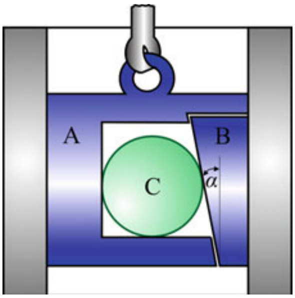


Fig. P12.23

12.24 Square blocks A and B weight **P** and **Q**, respectively. Their dimensions are shown in Fig. P12.24. The coefficient of friction between block A and the horizontal plane is μ. Block B is placed on a frictionless inclined plane.

(a) Determine the relationship between Q, P, α and μ needed to keep blocks A and B in equilibrium when the reaction normal force on block A is acting at point K. Determine the distance OK as well.

(b) Consider the case when the friction coefficient between block B and the inclined plane is μ

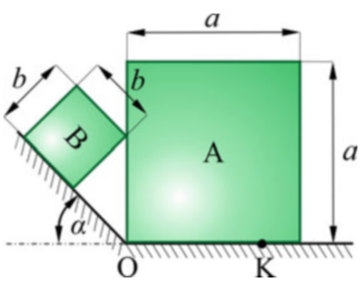


Fig. P12.24

12.25 Determine force **P** that will keep a disk in equilibrium if the coefficient of friction between the disk and the brake is 0.25, $F = 45$ kN, $a = 17$ cm, $b = 170$ cm, $d_1 = 10$ cm, and $d_2 = 60$ cm.

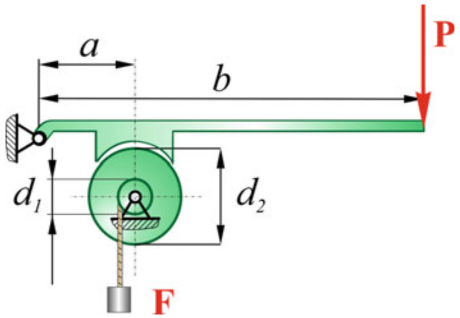


Fig. P12.25

12.26 Determine equal forces $\mathbf{F_1}$ and $\mathbf{F_2}$ that will keep a roller in equilibrium. The roller is loaded by a moment equal to 80 *Nm*. Assume that the coefficient of friction is 0.2, $a = 0.4$ m, $b = 0.4$ m, $d = 0.2$ m, and $l = 1$ m.

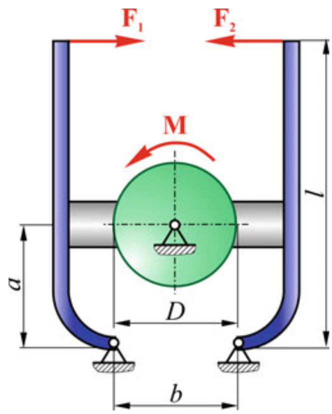


Fig. P12.26

12.27 Determine force **F** to keep box $P = 1$ kN in equilibrium. The coefficient of friction between cylinders A and B is 0.5 and ratio D/d is 2.

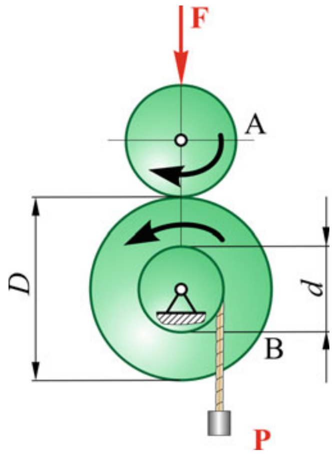


Fig. P12.27

12.28 Two cylinders with radii R_I and R_II are loaded as shown in Fig. P12.28. Assume $Q_1 = Q_2$. The coefficient of friction between the two cylinders is μ. Determine the magnitude of Q_1 to keep the system in equilibrium.

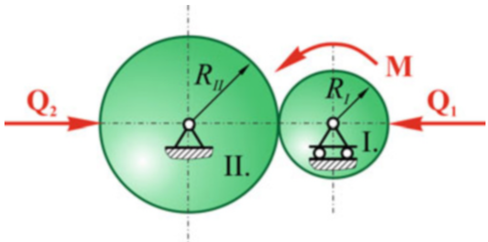


Fig. P12.28

12.29 Load **Q** is supported by two wedges A and B; their weights may be neglected. The coefficient of friction between block A and the ground is μ. (a) Determine the minimum force **P** to maintain load **Q** in equilibrium if there is no friction between blocks A and B. (b) How large should force **P** be if the friction coefficient between blocks A and B is $\mu/2$?

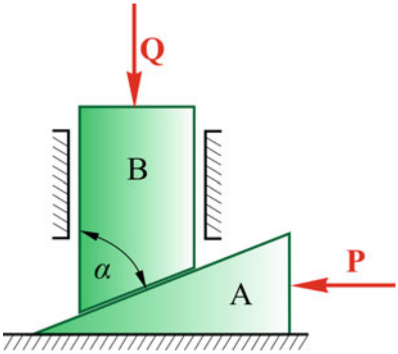


Fig. P12.29

12.30 Load **Q** is supported by two wedges A and B, which weights are $\mathbf{G_A}$ and $\mathbf{G_B}$. The coefficient of friction between blocks is μ_1 and between block A and the ground μ_2. Determine minimum force **P** to maintain load **Q** in equilibrium.

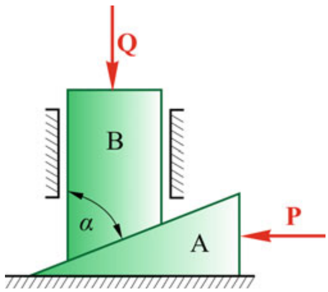


Fig. P12.30

12.31 A cylinder with radius r is moved along the horizontal plane by the force **Q**. The coefficient of the rolling friction is e. What should be the value of coefficient μ to assure that the cylinder will roll without slipping.

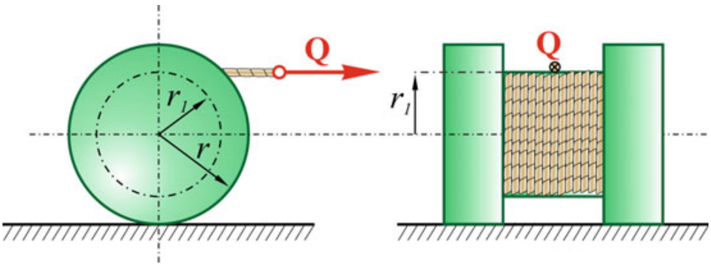


Fig. P12.31

12.32 A wagon moves with a constant velocity under force **P**. The weight of wagon is **Q,** the radius of each wheel is r and the coefficient of rolling resistance is e. Determine force **P** at which wheels of the wagon will start slipping. The coefficient of static friction is μ.

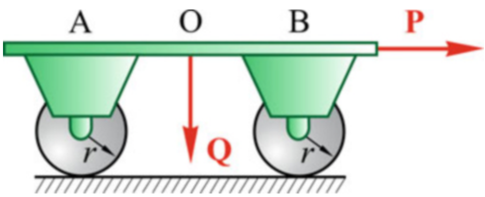


Fig. P12.32

12.33 Cylinder B and weight **D** are in static equilibrium. Determine the coefficient of the rolling resistance e and minimal static friction coefficient μ.

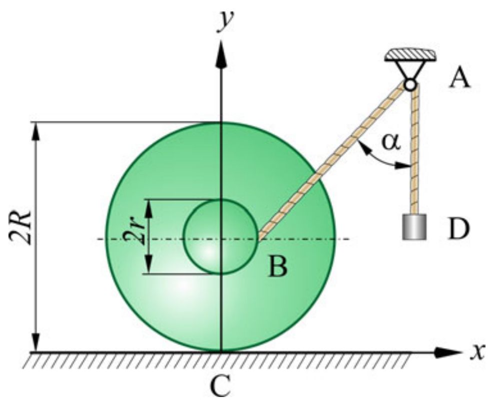


Fig. P12.33

12.34 Moment **M** is applied to link OA. What should be the tensile force in link OA to keep the system in equilibrium? Cylinder II is moving around cylinder I without sleeping. The coefficient of friction is μ.

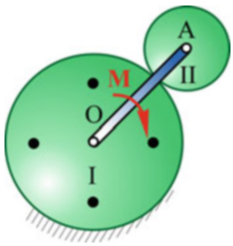


Fig. P12.34

12.35 Weightless bar O_1O_2 connects wheels I and II. Radii of the two wheels are r and their weights are **Q**. The coefficient of friction between the wheels and the surface is μ. The coefficient of the rolling resistance is e. Determine maximum moment **M** so that the system will be in static equilibrium.

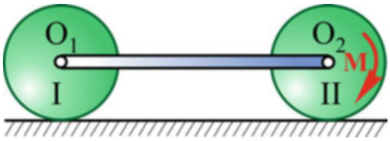


Fig. P12.35

12.36 A radial bearing carries external load $F = 20$ kN in the radial direction. The diameter of the shaft is 60 mm. Calculate the radius of the circle of friction and moment **M** that is required to maintain the rotation of the shaft at a constant angular velocity, $\mu_k = 0.03$.

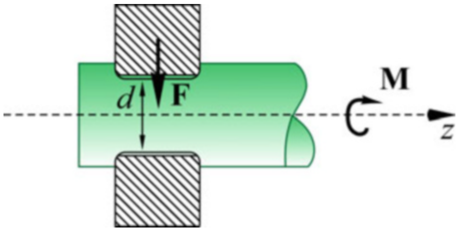


Fig. P12.36

12.37 An end bearing carries external load $F = 15$ kN. The diameter of the shaft is 50 mm. Calculate moment **M** required to rotate the axis at a constant angular velocity, $\mu_k = 0.02$.

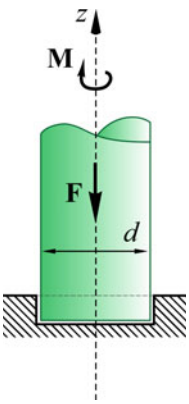


Fig. P12.37

12.38 An axial bearing with enclosing surface in the form of a ring carries external load $F = 50$ kN. The outer diameter of the ring is 120 mm, inner diameter is 90 mm. Calculate operational moment **M** required to rotate axis at a constant angular velocity, $\mu_k = 0.025$.

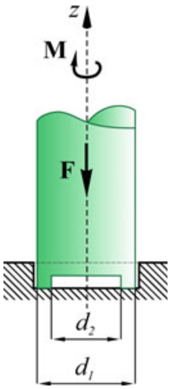


Fig. P12.38

12.39 A conical bearing carries external axial force **F**. Determine value of angle α at which *operational moment* **M**, that is required to overcome the friction resistance of the axial plain bearing and to rotate the axis at a constant angular velocity, will be minimal. The static and kinetic friction coefficients are μ_s and μ_k, respectively.

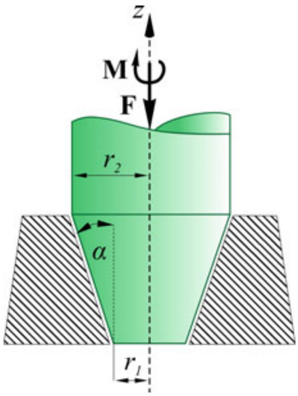


Fig. P12.39

12.40 Rope AB with length $l = 200$ m and specific weight $q = 90$ N/m is lying on the floor. We start to pull one end of the rope over frictionless wheel A located at height h as shown in Fig. P12.40. Calculate distance x where the rope touches the ground at the moment when it starts sliding against the floor. The friction coefficient between the rope and the floor is $\mu = 0.5$.

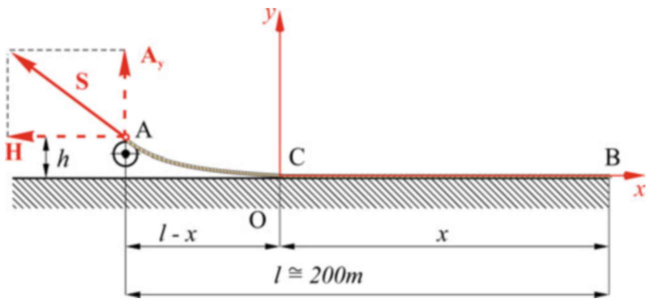


Fig. P12.40

12.41 A disk is attached to pin O and loaded with weight **Q** as shown in Fig. P12.41. Determine minimum value of force **P** to keep the system in equilibrium. The friction coefficient between the disk and the pusher is $\mu_1 = 0.6$. Directions of external forces and geometrical parameters are shown in Fig. P12.41. The friction between the rope and the cylinder with radius r is $\mu_2 = 0.4$.

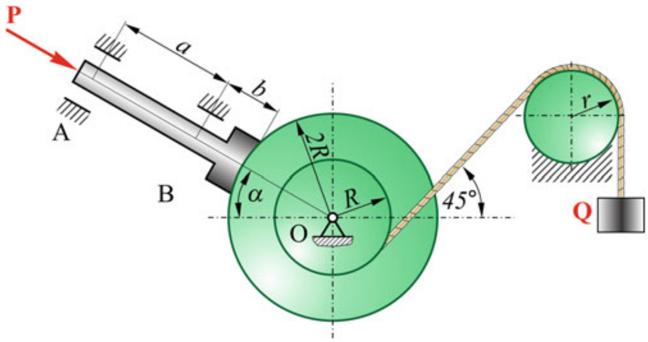


Fig. P12.41

12.42 A disk is attached to pin O and loaded with weight **Q** as shown in Fig. P12.42. Determine minimum value of force **P** to keep the system in equilibrium. The friction coefficient between the disk and the pusher is $\mu_1 = 0.5$. Directions of external forces and dimensions of the mechanism are shown in Fig. P12.42. The friction between the rope and the cylinder with radius r is $\mu_2 = 0.25$.

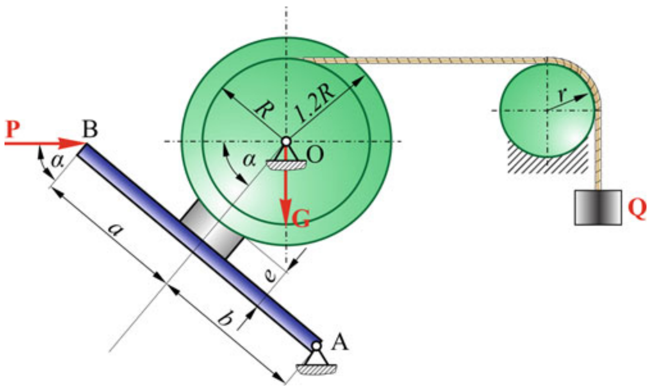


Fig. P12.42

12.43 A disk is attached to pin O and loaded with weight **Q** as shown in Fig. P12.43. Determine the minimum value of force **P** to keep the system in equilibrium. The friction coefficient between the disk and the pusher is $\mu_1 = 0.3$. Directions of external forces and dimensions of the mechanism are shown in Fig. P12.43. The friction between the rope and the cylinder with radius r is $\mu_2 = 0.2$.

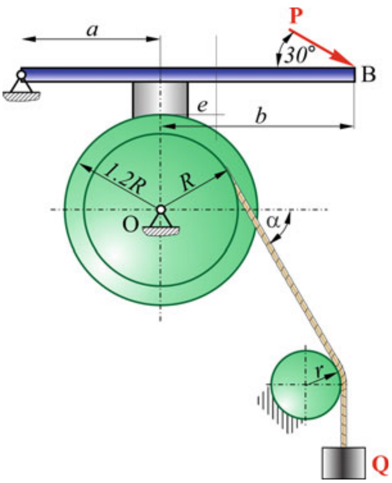


Fig. P12.43

Appendix

Vector **A** can be represented through its three orthogonal components

$$\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$$

where the unit vectors **i**, **j**, and **k** correspond to the coordinate axes in *x*, *y*, and *z* directions.

Its magnitude can be represented as

$$|\boldsymbol{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

And its unit vector

$$\boldsymbol{\lambda} = \frac{\boldsymbol{A}}{|\boldsymbol{A}|}$$

Thus, vector **A** may be represented also as

$$\boldsymbol{A} = |\boldsymbol{A}|\boldsymbol{\lambda}$$

Its direction cosines (cosine of the angle between the vector and corresponding axis) are

$$\lambda_x = \frac{A_x}{|\boldsymbol{A}|} \quad \lambda_y = \frac{A_y}{|\boldsymbol{A}|} \quad \lambda_z = \frac{A_z}{|\boldsymbol{A}|}$$

Two vectors are equal if there corresponding components are equal, i.e.,

$\mathbf{A} = \mathbf{B}$, when $A_x = B_x$, $A_y = B_y$, and $A_z = B_z$.

Vector addition

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\ \mathbf{i} + (A_y + B_y)\ \mathbf{j} + (A_z + B_z)\ \mathbf{k}$$

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Vector may be multiplied by a scalar

$$n\mathbf{A} = n\mathbf{A}_x\mathbf{i} + n\mathbf{A}_y\mathbf{j} + n\mathbf{A}_z\mathbf{k}$$

Cross product (vector product) is defined as

$$\mathbf{C} = \boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{\lambda}|\boldsymbol{A}||\boldsymbol{B}|\sin\alpha$$

where α is the angle between the vectors, **λ** is the unit vector perpendicular to the plane formed by the vectors **A** and **B**. Its direction is defined by the right hand rule.

From the definition of cross product follows

$$\boldsymbol{i} \times \boldsymbol{i} = 0 \quad \boldsymbol{j} \times \boldsymbol{j} = 0 \quad \boldsymbol{k} \times \boldsymbol{k} = 0$$
$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k} \quad \boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k} \quad \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$$

The rectangular components of the cross product are

$$C_x = A_yB_z - A_zB_y$$
$$C_y = A_zB_x - A_xB_z$$
$$C_z = A_xB_y - A_yB_x$$

Using a determinant

$$\boldsymbol{C} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{j} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Scalar or dot product of vectors **A** and **B** is defined as

$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\cos\theta$$

where θ is the angle between the vectors.

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z$$

From the definition of dot product follows

$$\boldsymbol{i} \cdot \boldsymbol{i} = 1 \quad \boldsymbol{j} \cdot \boldsymbol{j} = 1 \quad \boldsymbol{k} \cdot \boldsymbol{k} = 1$$
$$\boldsymbol{i} \cdot \boldsymbol{j} = 0 \quad \boldsymbol{j} \cdot \boldsymbol{k} = 0 \quad \boldsymbol{k} \cdot \boldsymbol{i} = 0$$

It should be noted that the main purpose of this Appendix is to provide a refreshment of the rules on vectors and matrix algebra; it cannot serve as a tutorial.

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